Perfect-Prismatic F-Crystals and p-adic Shtukas in Families

Anton Güthge, M. Sc.

Darmstadt

Vom Fachbereich Mathematik
der Technischen Universität Darmstadt
zur Erlangung des Grades eines
Doktors der Naturwisschenschaften
(Dr. rer. nat.)
genehmigte Dissertation
mündliche Prüfung: 2024

Advisor	Prof. Dr. Torsten Wedhorn
Second assessor	Prof. Dr. Alexander Ivanov

TECHNISCHE UNIVERSITÄT DARMSTADT

FACHBEREICH MATHEMATIK

Anton Güthge
Perfect-Prismatic F-Crystals and p-adic Shtukas in Families
Technische Universität Darmstadt

Jahr der Veröffentlichung der Dissertation auf TUprints: 2025
Tag der mündlichen Prüfung: 05.03.2024

Abstract

This is the version originally handed in as a Ph. D. Thesis. It contains numerous mistakes which will be addressed in the journal version.

We show an equivalence between the two categories in the title, thus establishing a link between Frobenius-linear objects of formal (schematic) and analytic (adic) nature. We will do this for arbitrary p-complete rings, with extra structure from arbitrary affine flat group schemes and without making use of the Frobenius-linear structure.

Contents

1 Introduction

v-locally trivial implies Zariski-locally trivial: A broad approach

For a fixed scheme F, an F-bundle on a scheme B is a scheme map $E \to B$ that locally on B looks like the projection $B \times F \to B$. A common question is then whether the category of F-bundles stays the same when changing what "locally" means in this context, i.e., when passing to a coarser or finer topology. For example, vector bundles of rank n are $\mathbb{A}^n$-bundles (with linear transition maps), and it doesn't matter if we ask for them to be trivializable in the Zariski- or fpqc topology: The two categories are equivalent. Before going into the precise definition of our objects of study, let's try to examine our main theorem from this point of view.

Our base scheme B will (for now) be $\mathrm{Spec}(R)$ for a perfect ring R of characteristic p, and we want to consider objects that locally look like $W(R)$. The category on the side of the equivalence using the coarser topology will consist of objects that are trivializable Zariski-locally on $\mathrm{Spec}(R)$. We give two equivalent definitions, the first one is most in line with the philosophy put forth in this introduction, while the second one is the one that is actually used. $(W(R), (p))$ being a henselian pair implies that they are equivalent.

Remark 1.1. The following pieces of data are equivalent:

1. A sheaf of $W(\mathcal{O}_{\mathrm{Spec}(R)})$-modules on $\mathrm{Spec}(R)$ that Zariski-locally[1] looks like a finite direct sum of copies of $W(\mathcal{O}_{\mathrm{Spec}(R)})$.

2. A vector bundle on $\mathrm{Spec}(W(R))$.

Now, of course, $W(R)$ is not isomorphic to $\mathrm{Spec}(R) \times F$ for any scheme F as the product would have to have characteristic p. But in many ways it behaves like a product "$R \otimes_{\mathbb{F}_p} W(\mathbb{F}_p)$" would (which doesn't even make sense as there is no map $\mathbb{F}_p \to W(\mathbb{F}_p) = \mathbb{Z}_p$). This can be made precise in the category of v-sheaves on affinoid perfectoids, where "$R \otimes_{\mathbb{F}_p} W(\mathbb{F}_p)$" has an interpretation as a literal product. This may suggest that v-sheaves are the natural context in which these Witt vector bundles should be studied. Let's take a closer look.

Adic spaces are a common generalization of schemes, formal schemes and rigid analytic spaces. Let's collect some useful facts about them:

- They have a nice subcategory consisting of so-called *affinoid perfectoids*, which carry a (very fine) topology called the v-topology. One can then define a functor from adic spaces to *v-sheaves on affinoid perfectoids*, or just *v-sheaves* for short, written as $X \mapsto X^\diamond$.

- The category of adic spaces (always assumed to be p-complete in this paper) has a final object $\mathrm{Spa}(\mathbb{Z}_p)$, and the functor from adic spaces to v-sheaves over $\mathrm{Spa}(\mathbb{Z}_p)^\diamond$ is fully-faithful for nice adic spaces.

[1] In fact, as we will see later, one could also ask only for triviality arc-locally at this point and the resulting category would be the same. In particular, this implies the same for the schematic v-topology. Do not confuse this with the more subtle results concerning the adic v-topology we proof in this paper! For us, the name v-topology always refers to the adic v-topology.

- For R perfect of characteristic p we have

$$\mathrm{Spa}(W(R))^\diamond \cong \mathrm{Spa}(R)^\diamond \times \mathrm{Spa}(\mathbb{Z}_p)^\diamond.$$

  This is the way in which the Witt vectors can be understood as a literal product of v-sheaves which we referenced above.

The finer topology on the other side of the equivalence is the following: We view R as a v-sheaf $\mathrm{Spa}(R)^\diamond$, and look at the site of v-sheaves over $\mathrm{Spa}(R)^\diamond$ with the topology generated by surjective maps. Now the v-sheaf $X^\diamond \times \mathrm{Spa}(\mathbb{Z}_p)^\diamond$ is not always representable, but it is for X affinoid perfectoid, which are by definition the representable v-sheaves and thus form a basis for the topology. For those, $X^\diamond \times \mathrm{Spa}(\mathbb{Z}_p)^\diamond$ is represented by an adic space usually called $\mathcal{Y}_{[0,\infty)}(X)$. With this, we can define the category on the other side.

Definition 1.2. An integral $\mathcal{Y}$-bundle on a quasi-compact v-sheaf $\mathcal{X}$ is a cover $X \to \mathcal{X}$ by an affinoid perfectoid X together with a vector bundle $\mathcal{E}$ on $\mathcal{Y}_{[0,\infty)}(X)$ and a descent datum of $\mathcal{E}$ over $\mathcal{X}$.

Remark 1.3. Note that $\mathcal{Y}_{[0,\infty)}(X)$ is not quasi-compact and that it is not at all clear whether every vector bundle on $\mathcal{Y}_{[0,\infty)}(X)$ is already trivializable on X. Even for $X = \mathrm{Spa}(K, K^+)$ a perfectoid field, I am uncertain if all vector bundles on $\mathcal{Y}_{[0,\infty)}(\mathrm{Spa}(K, K^+))$ are trivial, though our main result implies this a fortereori for vector bundlees on $\mathcal{Y}_{[0,\infty)}(K, K^+)$ that arise as part of the datum of an integral $\mathcal{Y}$-bundles on $\mathrm{Spa}(R^+)$ for a p-complete ring R^+. But keep in mind that our definition of $\mathcal{Y}$-bundles, which also seems the most sensible one with an eye towards applications, might not completely align with the idea of an object locally looking like a power of $\mathcal{Y}_{[0,\infty)}(X)$.

Now a special version of our main theorem reads as follows:

Theorem 1.4. *For R perfect of characteristic p, the natural functor from vector bundles on $W(R)$ to integral $\mathcal{Y}$-bundles on $\mathrm{Spa}(R)^\diamond$ is an equivalence of categories.*

Or, in less precise words, every Witt vector bundle that is v-locally trivial is already Zariski-locally trivial

Shtukas and Perfect-Prismatic F-Crystals

Let's now leave this standpoint of v-locally trivial implies Zariski-locally trivial for a bit and instead motivate the result starting from the categories in the title of this paper. From this point on, I assume familiarity with the language of adic spacees and v-sheaves. p-adic shtukas were first defined in [SW20] over perfectoid spaces:

Definition 1.5. Let (B, B^+) be a perfectoid Huber pair in characteristic p. A GL_n-*shtuka* over (B, B^+) with leg in some untilt $(B^\sharp, B^{+\sharp})$ consists of:

- A vector bundle $\mathcal{E}$ on $\mathcal{Y}_{[0,\infty)}(B, B^+) = \mathrm{Spa}(B, B^+) \dot{\times} \mathrm{Spa}(\mathbb{Z}_p)$.

- An isomorphism

$$\Phi\colon \varphi^*\mathcal{E}|_{\mathcal{Y}_{[0,\infty)}(B,B^+)-V(\xi)} \to \mathcal{E}|_{\mathcal{Y}_{[0,\infty)}(B,B^+)-V(\xi)},$$

  where $\varphi\colon \mathcal{Y}_{[0,\infty)} \to \mathcal{Y}_{[0,\infty)}$ is the Frobenius isomorphism induced from the Witt vector Frobenius $W(B^+) \to W(B^+)$ and $\xi \in A_{\mathrm{inf}}(B^{+\sharp}) = W(B^+)$ is a distinguished element cutting out $\mathrm{Spa}(B^\sharp, B^{+\sharp})$ from $\mathcal{Y}_{[0,\infty)}(B, B^+)$. Furthermore, we ask Φ to be meromorphic around ξ.

If $(B^\sharp, B^{+\sharp})$ also lives in characteristic p (i.e., it is equal to (B, B^+)) and one base changes $\mathcal{E}$ to $\mathcal{Y}_{(0,\infty)}(B, B^+)$, one arrives at a vector bundle on the Fargues-Fontaine curve. The notion of shtukas as defined above has a globalization from perfectoid spaces to arbitrary (not necessarily analytic) adic spaces $\mathcal{X}$, arriving at so-called *shtukas in families on* $\mathcal{X}$, either by v-descent from a cover $\mathrm{Spa}(B, B^+) \to \mathcal{X}$ by a perfectoid space or, equivalently, as a compatible families of shtukas on (B, B^+) for every map $\mathrm{Spa}(B, B^+) \to X$. This is done e.g. in [PR21] and it is the incarnation of shtukas that we will consider throughout.

On the other side of the equivalence, perfect-prismatic[2] F-crystals are a variant of the prismatic F-crystals from e.g. [BS23]: Instead of the (absolute) prismatic site over some p-complete ring A^+, one uses the (absolute) perfect prismatic site over A^+, which admits an equivalent definition as the site of all integral perfectoid A^+-algebras. If A^+ is integral perfectoid itself, the category of perfect-prismatic F-crystals is, just like the category of usual prismatic F-crystals, equivalent to the category of Breuil-Kisin-Fargues modules, an integral mixed-characteristic version of F-isocrystals:

Definition 1.6. Let A^+ be an integral perfectoid ring. A *BKF module* on A^+ is given by:

- A finite projective $A_{\mathrm{inf}}(A^+) = W(A^{+\flat})$-module $\mathcal{E}$.
- An isomorphism

$$\Phi\colon \varphi^*\mathcal{E}[1/d] \to \mathcal{E}[1/d],$$

  where $\varphi\colon A_{\mathrm{inf}}(A^+) \to A_{\mathrm{inf}}(A^+)$ is the Witt vector Frobenius and $d \in A_{\mathrm{inf}}(A^+)$ is a distinguished element such that $A_{\mathrm{inf}}(A^+)/d = A^+$.

So in particular, prismatic F-crystals and perfect-prismatic F-crystals agree on integral perfectoids. Where they differ is in the fact that the category of perfect-prismatic F-crystals on any p-complete ring is already completely determined by what happens on integral perfectoid rings via p-complete arc-descent. Thus, all the prismatic terminology introduced in [BS21] is not strictly necessary to define perfect-prismatic F-crystals: They can be completely described using the slightly more classical notions of integral perfectoid rings, BKF-modules and the p-complete arc topology.

[2] A hyphen is used to make clear these are to be understood as F-crystals on the perfect prismatic side – not F-crystals on the prismatic side which happen to be perfect in some sense.

The strong similarity between the two definitions above should be apparent even to someone reading them for the first time. And indeed, a number of equivalences between various Frobenius-linear objects of adic and schematic nature has been proven over the years, for example:

1. A bijection on isomorphism classes between isocrystals and vector bundles on the Fargues-Fontaine curve in [FF18], the latter of which has been shown to have a (simpler) adic description later.

2. Building on that, an equivalence between shtukas on (C, C°) and BKF modules on C° in [SW20] for C an algebraically closed non-archimedian field.

On the other hand, shtukas and BKF modules are objects living over fairly different kinds of spaces, as perfectoid spaces are *analytic* and integral perfectoids are *formal* in nature. This is also visible in the two equivalences above as these pass from objects on a (formal) scheme to objects on some version of analytic locus of the associated adic space. This shrinking of the space gives rise to numerous nuisances:

- Restrictions on the input space, e.g. that C be an algebraically closed field or that $C^+ = C^\circ$,

- Making it necessary to use the Frobenius structure in an essential way, i.e., one doesn't get an equivalence of the categories of vector bundles without such Frobenius structure,

- The equivalences are usually not exact; if they were, the main theorem of [Ans18] would have been two lines instead of many pages. Perhaps more crucially, it would hold for any affine flat group scheme and not just parahoric groups.

A way to get around this problem is to carry a formal scheme to the analytic world not by shrinking it, but by considering its associated v-sheaf on the category of perfectoid spaces in characteristic p.

3. In [Ans23], this technique is used to establish an equivalence between isocrystals on some algebraically closed field k in characteristic p and vector bundles on a "family of Fargues-Fontaine curves" over $\mathrm{Spa}(k)$.

4. In [PR21], Pappas-Rapoport aim to generalize this idea both to arbitrary schemes in characteristic p [PR21, Theorem 2.3.5] and fields in mixed characteristic [PR21, Proposition 2.3.8].

5. In their new article [GI23], Gleason and Ivanov obtain a classicality result for rational bundles (i.e., excluding $p = 0$) and, for shtukas with a leg in $p = 0$, also for integral ones.[3]

[3] This differs from our "rational" version of the equivalence in Proposition 4.15 in that they invert p in a meromorphic way, instead of just removing its vanishing from the adic space. I don't expect my non-meromorphic version to generalize past integral perfectoids, or even be relevant outside of proving my main theorem

We want to generalize 4. while eliminating the three nuisances mentioned above: Our equivalence will work for arbitrary integral perfectoids and even (trivially by descent) for arbitrary p-complete rings, it produces an exact tensor equivalence, so in particular, we get the relevant statement for $\mathcal{G}$-bundles for any affine flat group scheme for free, and without making use of the Frobenius structure at all:

Theorem 1.7. (Corollary 4.20) *Let R^+ be a complete topological ring carrying the Π-adic topology for some $\Pi \in R^+$ dividing p and let $\mathcal{G}/\mathbb{Z}_p$ be any affine flat group scheme. There is a canonical equivalence of groupoids*

$$\mathcal{G}\text{-}\mathcal{O}_{\mathbb{\Delta}}^{\mathrm{perf}}\text{-bundles on}\,\mathrm{Spd}(R^+) \to \text{integral } \mathcal{G}\text{-}\mathcal{Y}\text{-bundles on } \mathrm{Spd}(R^+).$$

The $\mathcal{O}_{\mathbb{\Delta}}^{\mathrm{perf}}$-bundles can be understood as "perfect-prismatic F-crystals without Frobenius structure", while integral $\mathcal{Y}$-bundles are "shtukas in families without Frobenius structure". Adding these Frobenius structures, we arrive at Theorem 4.29:

Theorem 1.8. *The functor defined above induces an equivalence of groupoids*

$$\text{perfect-prismatic } \mathcal{G}\text{-crystals on } R^+ \to \mathcal{G}\text{-shtukas with fixed leg on } \mathrm{Spd}(R^+).$$

For the rest of this introduction, we will only talk about the objects including Frobenius structure as their terminology should be more familiar to most readers, but all statements actually work in the generality of Corollary 4.20. Note the curious phenomenon that while the left side does not depend on the topology on R^+ (i.e., the choice of Π), one would think that the right side does. I do not have an explanation for this, as I only explicitly understand this equivalence for certain R^+ and Π.

Proof Strategy

While both [Ans23] and [PR21, Theorem 2.3.5] deal only with the (discrete) characteristic p case, our proof will be much closer in spirit to [PR21, Proposition 2.3.8], reducing everything to rings that are Π-complete for a non-zero divisor Π. More precisely, we will show the following:

Proposition 1.9. *(Lemma 4.21) Let S^+ be any p-complete ring. Then there exists a map $S^+ \to A^+$ which is simultaneously a p-complete arc-cover and a v-cover of the associated v-sheaves with A^+ a product of valuation rings with algebraically closed fields of fractions, ϖ-complete for a non-zero divisor $\varpi \in A^+$.*

As perfect-prismatic F-crystals descend along p-complete arc covers and shtukas descend along v-covers of v-sheaves, this allows us to reduce our statement to rings of the above form. The precise assumptions on A^+ imply that A^+ is an integral perfectoid and $(A, A^+) := (A^+[1/\varpi], A^+)$ is a perfectoid Huber pair, and in fact of a very simple kind, a so called *product of points*. This is where we are able to build the bridge between the adic and the schematic world.

The easy direction of the equivalence is to produce a family of shtukas from a perfect-prismatic F-crystal via base change. In particular, for A^+ as above and a BKF-module $(\mathcal{E}, \Phi)$ on A^+ (i.e. a vector bundle $\mathcal{E}$ on $A_{\inf}(A^+)$ + some Frobenius structure) we can restrict $\mathcal{E}$ along $\mathcal{Y}_{[0,\infty)}(A, A^+) \hookrightarrow \mathrm{Spa}(A_{\inf}(A^+))$ to obtain a shtuka on (A, A^+). This is just one member of the "family of shtukas", but it is important as $\mathcal{Y}_{[0,\infty)}(A, A^+)$ is dense in $\mathrm{Spa}(A_{\inf}(A^+))$ (for all this to work it is essential that we chose $\varpi \in A^+$ a non-zero divisor!), so all we need to do to construct our inverse functor is to extend $\mathcal{E}|_{\mathcal{Y}_{[0,\infty)}(A^+)}$ back to $A_{\inf}(A^+)$. We proceed in two steps: First, we extend our shtuka to $\mathcal{Y}_{[0,\infty]}(A, A^+) = \mathrm{Spa}(A_{\inf}(A^+) - V(p, [\varpi])$. Usually this is done using the Frobenius structure and some extra assumptions on (A, A^+), but instead we will get this by making use of all the other evaluations of the family of shtukas (i.e., on $\mathrm{Spa}(B, B^+)$ for any map to a perfectoid Huber pair$(A^+, A^+) \to (B, B^+)$) and some Sen-theory magic, heavily inspired by [PR21]. In the second step, using that A^+ is a product of points, we get the extension from $\mathcal{Y}_{[0,\infty]}$ to all of $\mathrm{Spa}(A_{\inf}(A^+))$ (or, equivalently, to $\mathrm{Spec}(A_{\inf}(A^+))$) by a result of Kedlaya, generalized by Gleason.

Structure of the paper

The first two chapters after the introduction serve to establish the basics for both the schematic and the adic side. All proofs are straightforward, but we introduce three new definitions: $\mathcal{O}_{\Delta}^{\mathrm{perf}}$-bundles in Definition + Proposition 2.9 and $\mathcal{Y}$-bundles[4] in Definition 3.7, which encode the underlying vector bundle data (i.e., without the Frobenius structure) of perfect-prismatic F-crystals (the third new definition) and shtukas in families, respectively. Notably, the $\mathcal{Y}$-bundles sit somewhere between the well-established notions of vector bundles and v-bundles on an adic space, but the exact relationship between those three notions is surprisingly subtle and remains something to watch out for throughout the paper.

All the interesting proofs are in section 4. The first three subsections tackle the extension from $\mathcal{Y}_{[0,\infty)}$ to $\mathcal{Y}_{[0,\infty]}$ and all the Sen-theory magic mentioned above: The first one contains Theorem 4.7 and Corollary 4.9, comparing descent data on adic spaces to certain Galois cocycles. This result will be combined with methods of Sen in the second subsection to prove the general descent result Theorem 4.10 giving a criterion for when effectivity of a certain pro-étale descent datum of vector bundles can be tested both generically and pointwise. Both of these results should be interesting in their own right. Theorem 4.10 then gets applied in the third subsection to show a classicality result on $\mathcal{Y}_{(0,\infty]}(A, A^+)$ which can be seen as a rational version of our main result, at least for those integral perfectoids arising as ring of integral elements of an affinoid perfectoid. The fourth subsection then deals with what happens in $p = [\varpi] = 0$, or in other words, the extension from $\mathcal{Y}_{[0,\infty]}(A, A^+)$ to $\mathrm{Spa}(A_{\inf})$, and contains the lengthy and fairly subtle Proposition 4.25 as well as the main theorem of the paper. The last subsection contains a rather straightforward result about comparing the Frobenius linear structures on either side.

[4]These already appeared under the name *classical v-bundles* in [GI23]

In the last section we take a look at perfect-prismatic F-crystals arising from prismatic cohomology, construct an orthogonal/symplectic structure on the middle cohomology groups via Poincaré duality, look at K3-surfaces and complete intersections of projective space as examples and finish with some remarks and speculations about stratifications arising from perfect prismatic F-crystals.

Notation and conventions

We use the notion of integral perfectoid rings from [BMS19]. We make the (at this point somewhat common) convention that we call an element ϖ in an integral perfectoid R^+ a *pseudo-uniformizer* if R^+ is ϖ-adically complete and $\varpi^p | p$, in line with the situation for perfectoid Huber pairs. Note that in general ϖ is not a pseudo-uniformizer in any literal sense, e.g. in a perfect ring of characteristic p it can always be chosen as 0.

We furthermore make use of the whole formalism of adic and perfectoid spaces from [SW20], always assumed to be p-complete. For (B, B^+) a perfectoid Huber pair in characteristic p, we will use $\mathrm{Spa}(B, B^+)$ and $\mathrm{Spd}(B, B^+)$ interchangeably. We call a v-sheaf *representable* if it is of this form. For any perfectoid Huber pair (A, A^+) with pseudouniformizer ϖ, we will make use of the following adic space:

$$\begin{aligned}
\mathrm{Spa}(A^+) \dot{\times} \mathrm{Spa}(\mathbb{Z}_p) &= \mathrm{Spa}(A_{\mathrm{inf}}(A^+)) \\
\mathrm{Spa}(A^+) \dot{\times} \mathrm{Spa}(\mathbb{Q}_p) &= \mathcal{Y}_{(0,\infty]}(A, A^+) = \mathrm{Spa}(A_{\mathrm{inf}}) - V(p) \\
\mathrm{Spa}(A, A^+) \dot{\times} \mathrm{Spa}(\mathbb{Z}_p) &= \mathcal{Y}_{[0,\infty)}(A, A^+) = \mathrm{Spa}(A_{\mathrm{inf}}) - V([\varpi]) \\
\mathrm{Spa}(A, A^+) \dot{\times} \mathrm{Spa}(\mathbb{Q}_p) &= \mathcal{Y}_{(0,\infty)}(A, A^+) = \mathrm{Spa}(A_{\mathrm{inf}}) - V(p[\varpi]) \\
& \quad \mathcal{Y}_{[0,\infty]}(A, A^+) = A_{\mathrm{inf}}(A^+) - V(p, [\varpi]).
\end{aligned}$$

Each of these carries a Frobenius endomorphism, which we will always call φ, while φ-linear maps will be named Φ. As demonstrated above, we will make use of the common convention $\mathrm{Spa}(A) \coloneqq \mathrm{Spa}(A, A^\circ)$ if and only if $A = A^\circ$ or $A = \mathbb{Q}_p$.

Remark 1.10. (Different uniformizers) We will use three different kinds of uniformizer, called Π, p and ϖ. For later reference, I want to clarify the role these are playing:

- p plays the role one would expect in this context: All our topological rings and rings of integral elements in a Huber pair are complete with a topology coarser than the p-adic topology.
- We want to formalize this idea of complete with a topology coarser than the p-adic one, while still being generated by single element. To do this, we introduce an element Π dividing p[5] so that our topological rings carry the Π-adic topology. Π can always be chosen as p, but in a ring where $p = 0$ it could also be any non-zero divisor. Any pseudo-uniformizer in an integral perfectoid in the sense above would be a suitable choice for Π.

[5] The relation $\Pi | p$ does not hold anymore after passing to A_{inf}

- Finally, we have ϖ, is much like Π, except that we always want it to be a non-zero divisor so that we can use it as the pseudo-unifomizer of a complete Huber pair. when both are present, it will generally divide Π[6].

Acknowledgments

First and Foremost, I want to thank my Ph.D. advisor Torsten Wedhorn who has taught and mentored me since my first contact with algebraic geometry 5 years ago. He encouraged me to study prismatic F-crystals even before there was an established notion of such things in the literature, and in this project in particular he found and helped to resolve many small and big mathematical errors and misconceptions. I also want to thank Christopher Lang and Ian Gleason for helpful conversations and the latter also for useful comments and corrections on the first version of this paper.

This project was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) TRR 326 Geometry and Arithmetic of Uniformized Structures, project number 444845124.

2 The schematic side

2.1 The perfect prismatic site and the Π-complete arc topology

The following variant of the arc topology was already considered in [BS21] (for $\Pi = p$) and in [Ito21] (on integral perfectoid rings).

Definition 2.1. (The Π-complete arc site) Let R^+ be a Π-complete ring for $\Pi \in R^+$ some element that divides p (think ϖ^p in an integral perfectoid) and let $\mathcal{C}$ be a class of Π-complete R^+-algebras, stable under Π-complete tensor products[7]. Then the Π-*complete arc topology* on $\mathcal{C}$ has covers given by maps $A^+ \to B^+$ which satisfy the arc lifting property with respect to Π-complete rank one valuation rings V, i.e.: For every map $A^+ \to V$ we must be able to find a commutative diagram

$$\begin{array}{ccc} B^+ & \longrightarrow & W \\ \uparrow & & \uparrow \\ A^+ & \longrightarrow & V \end{array}$$

with $V \to W$ a faithfully flat map (equivalently, an injective local homomorphism) of Π-complete rank one valuation rings.

We have made an implicit claim in the footnote which we still need to verify:

[6] Similar to how the situation is with Π and p, ϖ will divide p unless we pass to $\mathcal{Y}_{[0,\infty)}(B, B^+)$ or some variation thereof

[7] In our case, $\mathcal{C}$ will always be either the category of all Π-complete rings or of Π-complete integral perfectoids.

Lemma 2.2. *Let S^+ be an integral perfectoid with a pseudouniformizer $\varpi \in S^+$. Then the category of ϖ-complete integral perfectoid S^+-algebras is indeed closed under ϖ-complete tensor products.*

Proof. It is enough to show this for $\varpi = p$. Indeed, if $A \leftarrow B \rightarrow C$ is a diagram of integral perfectoid S^+-algebras, and we know their p-completed tensor product to be an integral perfectoid again, then the same must be true for its ϖ-completed version. So consider such a diagram. Then one has

$$(A \hat{\otimes}_B^L C)_{\mathrm{Perfd}} = A_{\mathrm{Perfd}} \hat{\otimes}^L_{B_{\mathrm{Perfd}}} C_{\mathrm{Perfd}} = A \hat{\otimes}_B^L C,$$

where the first identity is [BS21, Proposition 8.12] and the second one is [BS21, Corollary 8.13]. Now the left term is coconnective by [BS21, Lemma 8.4] and the right side is connective since both derived tensor products and derived completions preserve connectiveness. So the whole thing is discrete and thus, by applying [BS21, Corollary 8.13] again, a discrete perfectoid ring. □

Note that we have even shown that the category of Π-complete perfectoids is closed under *derived* tensor products. Even though we don't need this property explicitly, this kind of Tor-independence is essential for the theory, e.g. working in the background of the proof of perfectoid arc-descent, c.f. Lemma 2.8.

We use the language of prisms from [BS21]. Recall that a prism is called *perfect* if its Frobenius endomorphism is bijective. Perfect prisms are in equivalence to integral perfectoid rings by $(A, I) \mapsto A/I$.

Definition 2.3. The underlying category of the *perfect-prismatic site* over some p-complete ring R^+ is the restriction of the (absolute) prismatic site over R^+ to the sub-site containing only perfect prisms. Equivalently, it is the opposite of the category of integral perfectoid R^+-algebras.

We can equip it with two different topologies:

1. The *flat topology*, obtained by restriction of the topology on the usual prismatic site; here, covers are given by maps $(A, I) \to (B, J)$ such that $A \to B$ is (p, I)-completely faithfully flat.

2. We can also consider the *p-complete arc topology* on integral perfectoids, i.e., covers are given by maps $(A, I) \to (B, J)$ such that $A/I \to B/J$ is a p-complete arc cover.

As we will see in Lemma 2.5, the p-complete arc topology is finer (in fact, much finer) than the flat topology. We will use $(R^+)^{\mathrm{perf}}_{\Delta}$ to refer to the underlying category and will always explicitly mention which topology we put on it whenever it is relevant.

Remark 2.4. This is similar to the situation for the usual prismatic site, which is also defined as having the flat topology, but when working with finite locally free sheaves on it, the more natural choice of topology seems to be the quasi-syntomic topology, as used e.g. in [AB23].

Again, we made a claim in the definition which we should prove.

Lemma 2.5. *In the context of Definition 2.3, the p-complete arc topology is indeed finer than the flat topology.*

Proof. Let $(A, I) \to (B, J)$ be a (p, I)-completely faithfully flat map. Then certainly $A/I \to B/J$ is p-completely faithfully flat. Let $A/I \to \tilde{B}$ be a faithfully-flat model (i.e. a faithfully-flat map whose p-completion is our map $A/I \to B/J$). Then it is an arc cover as any faithfully flat map is, see e.g. Lemma 20.3.3 in [Ked]. Thus, we can lift any map to a rank one valuation ring, and after p-adically completing $\tilde{B}$, this is still possible with maps to p-complete rank one valuation rings. □

Definition 2.6. Let R^+ be a p-complete ring. By restricting their analogs from the non-perfect prismatic site, we get the following sheaves for the flat topology on $(R^+)^{\mathrm{perf}}_{\mathbb{\Delta}}$:

1. The *reduced structure sheaf*
$$\overline{\mathcal{O}}^{\mathrm{perf}}_{\mathbb{\Delta}} \colon (A, I) \mapsto A/I.$$
2. The (full) *structure sheaf*
$$\mathcal{O}^{\mathrm{perf}}_{\mathbb{\Delta}} \colon (A, I) \mapsto A.$$
3. a rational variant of the structure sheaf:
$$\mathcal{O}^{\mathrm{perf}}_{\mathbb{\Delta}}[1/d] \colon (A, I) \mapsto A[1/d].$$

These are should also be sheaves for the p-complete arc topology, but we will not need this explicitly. For $\mathcal{O}^{\mathrm{perf}}_{\mathbb{\Delta}}$, we get this for free from Lemma 2.8, while for $\mathcal{O}^{\mathrm{perf}}_{\mathbb{\Delta}}[1/d]$ it will follow from the proof of Proposition 2.21

2.2 Locally free sheaves on the perfect-prismatic site

Definition 2.7. Let R^+ be a p-complete ring. We say that a presheaf $\mathcal{E}$ of $\mathcal{O}^{\mathrm{perf}}_{\mathbb{\Delta}}$-modules on $(R^+)^{\mathrm{perf}}_{\mathbb{\Delta}}$ satisfies the *crystal condition* if for every map of perfect prisms $(A, I) \to (B, J)$ over R^+ the natural map
$$\mathcal{E}(A, I) \otimes_A B \to \mathcal{E}(B, J)$$
is an isomorphism.

The following result will be used at multiple points:

Lemma 2.8. *[Ito21, Theorem 1.1] Let R^+ be an integral perfectoid ring with pseudouniformizer $\varpi \in R^+$. Sending an integral perfectoid ϖ-complete R^+-algebra S^+ to the category of finite-projective $A_{\mathrm{inf}}(S^+)$-modules satisfies ϖ-complete arc descent.*

With this, one could define quasi-coherent prismatic crystals (as they do in [AB23]), but we will not do this.

Definition + Proposition 2.9. Let R^+ again be any p-complete ring. An $\mathcal{O}_{\Delta}^{\mathrm{perf}}$-*bundle* over R^+ is a presheaf $\mathcal{E}$ of $\mathcal{O}_{\Delta}^{\mathrm{perf}}$-modules on $(R^+)_{\Delta}^{\mathrm{perf}}$ subject to one of the following equivalent conditions:

1. $\mathcal{E}$ is a sheaf for the topology τ and finite locally free in the topology τ', for any choice $(\tau, \tau') \in \{(\mathrm{flat}, \mathrm{flat}), (\mathrm{arc}, \mathrm{arc}), (\mathrm{arc}, \mathrm{flat})\}$.
2. $\mathcal{E}$ sends any prism (A, I) to a finite-projective A-module and satisfies the crystal condition.

We equip the category of $\mathcal{O}_{\Delta}^{\mathrm{perf}}$-bundles with the exact structure coming from the flat topology – In a second we will see that, once again, this is equivalent to defining it through the arc topology.

Proof. First, every rule as in (2) is an arc sheaf by Lemma 2.8, and flat-locally free as it is Zariski-locally free on A, and a completed Zariski-localization of a perfect prism is again a perfect prism (see [BS21, Remark 2.16] for localization of delta rings) and the completed localization map will certainly be a (p, I)-complete flat cover.

Now every arc sheaf is a flat sheaf and every sheaf that is flat-locally free is also arc-locally free, so we get the other two variants in (1) from the one discussed above.

Finally, we have to show that a τ-sheaf $\mathcal{E}$ that is $\tau = \tau'$-locally free only admits values in finite projective modules and satisfies the crystal condition. The former is a direct consequence of Lemma 2.8 (or its variant for the flat topology, following from Lemma 2.5), the latter is a standard argument: Let $(A, I) \to (B, J)$ be a map in $(R^+)_{\Delta}^{\mathrm{perf}}$ and let $(A, I) \to (A', I')$ be a τ-cover trivializing $\mathcal{E}$. Write

$$\begin{aligned}(A'', I'') &= (A', I') \otimes_{(A,I)} (A', I') \\ (B', J') &= (A', I') \otimes_{(A,I)} (B, J) \\ (B'', J'') &= (A'', I'') \otimes_{(A,I)} (B, J).\end{aligned}$$

Then $\mathcal{E}$ is also free on all those. We now have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E}(A, I) \otimes_A B \arrow[r] \arrow[d] & \mathcal{E}(A', I') \otimes_{A'} B' \arrow[r] \arrow[d] & \mathcal{E}(A'', I'') \otimes_{A''} B'' \arrow[d] \\ 0 \arrow[r] & \mathcal{E}(B, J) \arrow[r] & \mathcal{E}(B', J') \arrow[r] & \mathcal{E}(B'', J''). \end{tikzcd}$$

The rows are exact by the sheaf condition (here it becomes relevant that $\tau = \tau'$), and the two vertical arrows on the right are isomorphisms because we already know $\mathcal{E}$ to be free on those prisms and because tensor products commute with finite products. The result follows by the five lemma. $\square$

Corollary 2.10. *For R^+ an integral perfectoid ring, there are mutually inverse exact tensor equivalences between $\mathcal{O}_{\mathbb{\Delta}}^{\mathrm{perf}}$-bundles on R^+ and finite projective $A_{\mathrm{inf}}(R^+)$-modules given by the obvious constructions, namely: evaluating the $\mathcal{O}_{\mathbb{\Delta}}^{\mathrm{perf}}$-bundle at $(A_{\mathrm{inf}}(R^+), (d))$ resp. sending an $A_{\mathrm{inf}}(R^+)$-module M to $(A, I) \mapsto M \otimes_{A_{\mathrm{inf}}(R^+)} A$.*

Proof. The equivalence of categories follows immediately from characterization (2) in Definition + Proposition 2.9. Compatibility with tensor products is clear. For exactness, we need to show that a map between $\mathcal{O}_{\mathbb{\Delta}}$-modules is surjective (i.e., surjective (p, I)-completely flat locally) if and only if it is surjective on global sections. This follows from Lemma 2.5 and the following lemma (which also shows that flat-exactness is equivalent to arc-exactness). $\square$

Lemma 2.11. *Let $R^+ \to S^+$ be a ϖ-complete arc cover between ϖ-complete integral perfectoid rings for some $\varpi \in R^+$ dividing p. Then a sequence of finite projective $A_{\mathrm{inf}}(R^+)$-modules*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is exact if and only if its base change to $A_{\mathrm{inf}}(S^+)$ is.

Proof. By finite-projectiveness, it is clear that exactness of

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

implies exactness of the base changed version. For the opposite direction, we want to apply [PR21, Lemma 2.3.6], which implies the statement of the theorem under the condition that (0) the sequence of finite-projective modules is actually a complex[8] (1) the map of rings $A_{\mathrm{inf}}(R^+) \to A_{\mathrm{inf}}(S^+)$ is injective and (2) that the induced map of schemes $\mathrm{Spec}(A_{\mathrm{inf}}(S^+)) \to \mathrm{Spec}(A_{\mathrm{inf}}(R^+))$ is surjective on closed points. Concerning (0), we have to show that if the pullback of the composition $\alpha\colon M_2 \to M_2 \to M_3$ is zero, the same is true for α itself. Since the question is local, we can assume that the M_i are free, so that the morphism between them is defined by a matrix with entries in the structure sheaf, which are 0 if their pullback is by (1).

Now, I claim that for both (1) and (2) it is enough to show the relevant statement for the map of rings $R^+ \to S^+$. Indeed, for (1) one obtains the map $A_{\mathrm{inf}}(R^+) \to A_{\mathrm{inf}}(S^+)$ by applying first the tilt functor and then the Witt vector functor, both of which preserve injections. For (2), note that as $A_{\mathrm{inf}}(R^+)$ is d-complete for a distinguished element $d \in A_{\mathrm{inf}}(R^+)$ such that $A_{\mathrm{inf}}(R^+)/d = R^+$, all closed points of $\mathrm{Spec}(A_{\mathrm{inf}}(R^+))$ actually lie in the closed subset $V(d) = \mathrm{Spec}(R^+)$, and their preimages automatically lie in $\mathrm{Spec}(S^+)$.

Now to prove (2), note that by ϖ-completeness of R^+, all closed points of R^+ must even lie in $\mathrm{Spec}(R^+/\operatorname{rad}(\varpi))$. But as $R^+ \to S^+$ is a ϖ-complete arc-cover, $R^+/\operatorname{rad}(\varpi) \to S^+/\operatorname{rad}(\varpi)$ must be an arc-cover in the usual sense, so the associated map of schemes is surjective on topological spaces.

[8]This argument shows that asking for the sequence to be a complex in loc. cit. is also unnecessary

For (1), let $a \in R^+$ be a non-zero element, which by reducedness of R^+ is not even nilpotent. This implies that there is some $x \in \mathrm{Spec}(R^+)$ such that $a \neq 0 \in \kappa(x)$. The arc lifting property now gives us a diagram

$$\begin{tikzcd} L & S^+ \arrow[l] \\ \kappa(x) \arrow[u] & R^+ \arrow[l] \arrow[u] \end{tikzcd}$$

with the left arrow injective, so the image of a in L and thus also in S^+ is non-zero. □

For the final descent result of this chapter, we need a result by Scholze-Česnavicius, a stronger version of which we will also prove later in Lemma 4.21.

Lemma 2.12. *[ČS21][9] Let R^+ be a Π-complete ring for some $\Pi \in R^+$ dividing p. Then R^+ has a Π-complete arc cover $R^+ \to A^+$ where A^+ is a product of Π-complete valuation rings with algebraically closed field of fractions. In particular, A^+ is integral perfectoid.*

Proposition 2.13. *Let R^+ be a Π-complete ring for some $\Pi \in R^+$ dividing p. Sending a Π-complete R^+-algebra S^+ to the category of $\mathcal{O}_{\Delta}^{\mathrm{perf}}$-bundles on S^+ satisfies Π-complete arc descent. This is compatible with the exact tensor structure, i.e. for a Π-complete arc cover $S^+ \to S'^+$, the equivalence of categories between $\mathcal{O}_{\Delta}^{\mathrm{perf}}$-modules on S^+ and those on S'^+ together with a descend datum can be upgraded to an exact tensor equivalence.*

Proof. It suffices to show this on the basis of the Π-complete arc topology specified in Lemma 2.12. In particular, we can assume that S^+ is perfectoid, so by Corollary 2.10 we just have to show descent for mapping S^+ to finite projective $A_{\mathrm{inf}}(S^+)$-modules. Also, by absolute integral closedness, there is some pseudouniformizer $\varpi \in S^+$ such that $\varpi^p = \Pi$. But then the ϖ-adic topology is the same as the Π-adic one, and Π-complete arc covers are the same as ϖ-complete ones, so we can apply Lemma 2.8 to conclude.

Compatibility with tensor products is clear. For compatibility with exactness, we need to show that a sequence of $A_{\mathrm{inf}}(S^+)$-modules is exact if and only if it is exact after pullback along a ϖ-complete arc cover $S^+ \to \widetilde{S}^+$, which we have shown in Lemma 2.11. □

2.3 Adding a group structure

Recall the Tannaka formalism over discrete valuation rings:

Theorem 2.14. *Let V be a regular ring of dimension ≤ 1 and let $\mathcal{G}$ be any affine flat group scheme over V. Then for any V-scheme X, there is an equivalence of categories between $\mathcal{G}$-torsors on X and exact tensor functors from the category of finite rank representations of $\mathcal{G}$ to the category of vector bundles on X.*

[9] in loc. cit., they only prove the p-complete version, but the Π-complete one can be done with the same proof. In any case, our finer version in Lemma 4.21 will also produce a Π-complete arc cover.

We will use this statement essentially as the definition of $\mathcal{G}$-bundle.

Definition 2.15. Let $\mathcal{G}$ be any affine flat group scheme of finite type over $\mathbb{Z}_p$ and let R^+ be a p-complete ring. A $\mathcal{G}$-$\mathcal{O}_{\Delta}^{\mathrm{perf}}$*-bundle* over R^+ is given by the following equivalent definitions:

1. an exact tensor functor from the category of finite rank representations of $\mathcal{G}$ to the category of $\mathcal{O}_{\Delta}^{\mathrm{perf}}$-bundles over R^+.

2. A rule sending each $(R^+ \to S^+) \ni (R_{\Delta}^{\mathrm{perf}})$ to a $\mathcal{G}$-bundle on $A_{\mathrm{inf}}(S^+)$, compatible with pullbacks.

Indeed, the two definitions are equivalent as $(R^+)_{\Delta}^{\mathrm{perf}}$ carries the (p, I)-complete flat topology and exactness can be checked (p, I)-complete flat-locally by Lemma 2.5 and Lemma 2.11.

Since our equivalences from the previous sections were exact, they immediately transfer to $\mathcal{G}$-$\mathcal{O}_{\Delta}^{\mathrm{perf}}$-bundles. In particular:

Proposition 2.16. *1. If R^+ is an integral perfectoid ring, the category of $\mathcal{G}$-$\mathcal{O}_{\Delta}^{\mathrm{perf}}$-bundles on R^+ is equivalent to the category of $\mathcal{G}$-torsors on its associated perfect prism $(A_{\mathrm{inf}}(R^+), d)$.*

2. Let R^+ be a Π-complete ring for some $\Pi \in R^+$ dividing p. The functor sending a Π-complete R^+-algebra S^+ to the groupoid of $\mathcal{G}$-$\mathcal{O}_{\Delta}^{\mathrm{perf}}$-bundles on S^+ satisfies Π-complete arc-descent.

2.4 Adding a Frobenius structure: perfect-prismatic F-crystals

Remark 2.17. Recall that each prism comes equipped with a Frobenius endomorphism φ, which assemble to a Frobenius isomorphism of sheaves of rings $\mathcal{O}_{\Delta}^{\mathrm{perf}} \to \mathcal{O}_{\Delta}^{\mathrm{perf}}$, also denoted φ.

We now define perfect-prismatic F-crystals, and immediately add a $\mathcal{G}$-structure as well.

Definition 2.18. Let R^+ be a p-complete ring and $\mathcal{G} \to \mathrm{Spec}(\mathbb{Z}_p)$ an affine flat group scheme.

1. A *perfect-prismatic $\mathcal{G}$-crystal* on R^+ is a $\mathcal{G}$-$\mathcal{O}_{\Delta}$-bundle $\mathcal{E}$ together with a Frobenius-linear isomorphism

$$\Phi\colon \varphi^*\mathcal{E} \otimes_{\mathcal{O}_{\Delta}^{\mathrm{perf}}} \mathcal{O}_{\Delta}^{\mathrm{perf}}[1/d] \xrightarrow{\sim} \mathcal{E} \otimes_{\mathcal{O}_{\Delta}^{\mathrm{perf}}} \mathcal{O}_{\Delta}^{\mathrm{perf}}[1/d]$$

2. A *perfect-prismatic F-crystal* is a perfect-prismatic GL_n-crystal for some n.

The comparison with BKF modules now follows immediately from Corollary 2.10:

Proposition 2.19. *Let R^+ be an integral perfectoid with associated perfect prism $(A,(d))$. Then the equivalence of categories from Proposition 2.16 induces an equivalence of categories between*

1. *perfect-prismatic F-crystals on R^+*
2. *BKF modules on R^+, i.e. finite projective $A_{\mathrm{inf}}(R^+)$-modules M together with an isomorphism*
$$\Phi\colon \varphi^*M[1/d] \to M[1/d].$$

Basically, when passing from a single finite locally free $A_{\mathrm{inf}}(A^+)$-module to a perfect-prismatic $\mathcal{G}$-crystal, one has to take three steps: Adding the $\mathcal{G}$ structure, adding the Frobenius structure and globalizing, i.e., replacing a single object (vector bundle with or without $\mathcal{G}$- and Frobenius structure) on $A_{\mathrm{inf}}(A^+)$ by a map sending any $A^+ \in (R^+)^{\mathrm{perf}}_{\mathbb{\Delta}}$ to such an object. By the exactness properties we already showed, we can choose in whichever order we apply them, leading to $\sharp S_3 = 6$ slightly different, but equivalent, variations of the definition. Let's write down the most important one explicitly:

Lemma 2.20. *Perfect-prismatic $\mathcal{G}$-crystals on R^+ are in equivalence to exact tensor functors from the category of finite rank $\mathcal{G}$-representations over $\mathbb{Z}_p$ to the category consisting of rules sending an $(R^+ \to S^+) \in (R^+)^{\mathrm{perf}}_{\mathbb{\Delta}}$ to a finite projective $A_{\mathrm{inf}}(S^+)$-module M together with an isomorphism*
$$\varphi^*M[1/d_{S^+}] \to M[1/d_{S^+}].$$

This allows us to reduce all questions about perfect-prismatic $\mathcal{G}$-crystals to perfect-prismatic $\mathcal{F}$-crystals plus the appropriate exactness property.

Proposition 2.21. *Let R^+ be a Π-complete ring for some $\Pi \in R^+$ dividing p and let $\mathcal{G}$ be an affine flat group scheme over $\mathbb{Z}_p$. Sending a Π-complete R^+-algebra S^+ to the groupoid of perfect-prismatic $\mathcal{G}$-crystals on S^+ satisfies Π-complete arc descent.*

Proof. As in Corollary 2.10, it suffices to check descent on the basis of the topology given by R^+-algebras S^+ which are products of Π-complete valuation rings with algebraically closed field of fractions. Also, using Lemma 2.20, it is enough to show the result for $\mathcal{G} = \mathrm{GL}_n$ and compatibility with exactness. Descent of the underlying finite projective $A_{\mathrm{inf}}(S^+)$-modules was the point of Proposition 2.13, so it suffices to show that we can also descend a φ-linear isomorphism $\Phi\colon \varphi^*\mathcal{E}[1/d] \to \mathcal{E}[1/d]$. This can be done exactly as in Theorem 4.29: Namely, we first multiply Φ by a power of d to obtain a morphism between the $\mathcal{O}^{\mathrm{perf}}_{\mathbb{\Delta}}$-bundles $\mathcal{E}$ and $\varphi^*\mathcal{E}$, which we can descend using Proposition 2.13, and divide by the same power of d again. We then use that the map on global sections associated to an arc cover is injective (shown in the proof of Lemma 2.11) to show that the descended morphism is still an isomorphism after inverting d.

The statement for the exact structure is clear from the relevant statement for $\mathcal{G}$-crystals. $\square$

3 The adic side

3.1 Generalities on the v-site

We now start dipping into the adic world, namely v-sheaves on the category Perf of perfectoid Huber pairs in characteristic p and certain locally free sheaves on such things.

Notation 3.1. (letters for v-sheaves) Just for this section, we use the following notational convention: We will denote a v-sheaf that comes with a structure map to $\mathbb{Z}_p$ with a normal letter with a diamond in the lower index, like $X_\diamond$. It may or may not (but in our applications always will) be given as $X^\diamond$ for some adic space X. In particular, whenever we have a map from a representable $\mathrm{Spa}(B, B^+) \to X_\diamond$, we get a canonical untilt $(B^\sharp, B^{+\sharp})$ supplied by the composition $\mathrm{Spa}(B, B^+) \to X_\diamond \to \mathrm{Spd}(\mathbb{Z}_p)$. In contrast, v-sheaves without a structure map to $\mathrm{Spd}(\mathbb{Z}_p)$ will be denoted by cursive letters such as $\mathcal{X}$.

Definition 3.2. (covers of v-sheaves) A map $\mathcal{X}' \to \mathcal{X}$ of v-sheaves is called a *v-cover* or simply *cover* if it is surjective in the sheaf-theoretic sense. Explicitly, (using that affinoid perfectoids are qcqs in the sheaf theoretic sense) for any map from a representable, i.e. an affinoid perfectoid in characteristic p, $\mathrm{Spa}(B, B^+) \to \mathcal{X}$, one has to be able to find a commutative diagram

$$\begin{array}{ccc} \mathrm{Spa}(B', B'^+) & \longrightarrow & \mathcal{X}' \\ \downarrow & & \downarrow \\ \mathrm{Spa}(B, B^+) & \longrightarrow & \mathcal{X} \end{array}$$

with $\mathrm{Spa}(B', B'^+)$ also representable and the left map being a v-cover of affinoid perfectoids.

Recall that a v-sheaf is called *small* if it admits a cover by a perfectoid space, which is really just a set-theoretic bound and holds for most v-sheaves arising in practice. For v-sheaves arising from certain p-complete rings one can even cover them by a single representable, which will be useful later:

Lemma 3.3. *Consider A^+ a complete topological ring carrying the Π-adic topology for some $\Pi \in A^+$ dividing p. Then $\mathrm{Spd}(A^+)$ is quasi-compact (in the sheaf theoretic sense). Equivalently, there is a v-cover by a representable*

$$\mathrm{Spa}(B, B^+) \to \mathrm{Spd}(A^+).$$

Proof. It is enough to find a cover of $\mathrm{Spd}(A^+)$ by a quasi-compact v-sheaf. We know from [Sch17, Chapter 15] that the v-sheaves associated to affinoid analytic adic spaces are spatial diamonds, so in particular quasi-compact in the sheaf-theoretic sense, so it suffices to find a cover by a finite number of these.

Consider the A^+-algebra

$$\tilde{A}^+ = A^+[[t]],$$

equipped with the (Π, t)-adic topology. Then $\mathrm{Spa}(\tilde{A}^+)$ is not an analytic adic space, but $X \coloneqq \mathrm{Spa}(\tilde{A}^+) - V(t, \Pi)$ is[10] and one quickly convinces oneself that $X^\diamond \to \mathrm{Spd}(A^+)$ still satisfies the lifting criterion of a v-cover: Indeed, if $(A^+, A^+) \to (B^\sharp, B^{+\sharp})$ is a map to a perfectoid Huber pair that we want to lift, we first produce a map $\tilde{A}^+ \to B^{+\sharp}$ by sending t to the pseudouniformizer of $B^\sharp$. This will still be continuous, and furthermore the image of t in $\mathrm{Spa}(B^\sharp, B^{+\sharp})$ vanishes nowhere, so the map $\mathrm{Spa}(B^\sharp, B^{+\sharp}) \to \mathrm{Spa}(\tilde{A}^+)$ factors through X.

Now X can be covered by the open subspaces

$$U_1 \coloneqq \mathrm{Spa}(\tilde{A}^+)_{\Pi \leq t \neq 0}, \quad U_2 \coloneqq \mathrm{Spa}(\tilde{A}^+)_{t \leq \Pi \neq 0},$$

which are rational subsets of $\mathrm{Spa}(\tilde{A}^+)$, so in particular affinoid. Thus

$$(U_1 \sqcup U_2)^\diamond \to X^\diamond \to \mathrm{Spd}(A^+)$$

is our desired cover. $\square$

Definition 3.4. Let $\mathcal{X}$ be a v-sheaf. We get certain sheaves of rings on the v-site $\mathrm{Perf}_{/\mathcal{X}}$ of characteristic p affinoid perfectoids over $\mathcal{X}$:

1. The (tilted) structure sheaf

$$\mathcal{O}^{\flat\,\mathrm{Perf}}_{/\mathcal{X}} \colon (\mathrm{Spa}(B, B^+) \to \mathcal{X}) \mapsto B$$

2. The $\mathcal{Y}$-sheaf

$$\mathcal{Y}^{\mathrm{Perf}}_{/\mathcal{X}} \colon (\mathrm{Spa}(B, B^+) \to \mathcal{X}) \mapsto \Gamma(\mathcal{Y}_{[0,\infty)}(B, B^+), \mathcal{O}_{\mathcal{Y}_{[0,\infty)}(B,B^+)}),$$

   See Remark 3.5 though.

If $\mathcal{X}$ comes equipped with a map to $\mathrm{Spd}(\mathbb{Z}_p)$, so we would call it $X_\diamond$ in line with Notation 3.1, we furthermore have:

3. The untilted structure sheaf

$$\mathcal{O}^{\sharp\,\mathrm{Perf}}_{/X_\diamond} \colon (\mathrm{Spa}(B, B^+) \to X_\diamond) \mapsto B^\sharp.$$

4. The canonical (untilted) ideal sheaf

$$\mathcal{I}^{\sharp\,\mathrm{Perf}}_{/X_\diamond}$$

   mapping some $\mathrm{Spa}(B, B^+) \to X_\diamond$ to the global sections of the ideal sheaf generated by ξ inside $\mathcal{Y}_{[0,\infty)}(B, B^+)$.

Remark 3.5. Note the strong similarity with the sheaves on the perfect-prismatic site from the first chapter. Also note the slight annoyance that $\mathcal{Y}_{[0,\infty)}(B, B^+)$ is not affinoid, so we cannot literally define the $\mathcal{Y}$-bundles below as finite locally free $\mathcal{Y}^{\mathrm{Perf}}_{/\mathcal{X}}$-modules. In fact, we only defined $\mathcal{Y}^{\mathrm{Perf}}_{/\mathcal{X}}$ for underlining the conceptual similarity with the prismatic site.

[10] This is completely analogous to $\mathrm{Spa}(A_{\mathrm{inf}})$ and $\mathcal{Y}_{[0,\infty]}$.

3.2 Locally free sheaves on the v-site: v-bundles and $\mathcal{Y}$-bundles

The following notion seems to be well-known, e.g. a special case is studied in [Heu22]:

Definition 3.6. Let $X_\diamond$ be a small v-sheaf over $\mathrm{Spd}(\mathbb{Z}_p)$ (in cases of interest to us, this will always be given by $X^\diamond$ for some adic space X). Recall v-descent of vector bundles on perfectoid spaces from [SW20, Lemma 17.1.8]. It implies that the following sets of data are equivalent, either of these will be called a *v-bundle $\mathcal{E}$ on $X_\diamond$*; The proofs of the equivalence are very similar to Definition + Proposition 2.9 and shall not be repeated here.

1. A finite locally free $\mathcal{O}^{\sharp\,\mathrm{Perf}}_{/X_\diamond}$-module,
2. A rule assigning to any $(B, B^+) \in \mathrm{Perf}$ and (B, B^+)-valued point $\alpha\colon \mathrm{Spa}(B, B^+) \to X_\diamond$ with associated untilt $(B^\sharp, B^{+\sharp})$ (supplied by the map to $\mathrm{Spd}(\mathbb{Z}_p)$) a vector bundle on $(B^\sharp, B^{+\sharp})$ which we will call $\alpha^*\mathcal{E}$, or, if α is clear from the context, $\mathcal{E}|_{\mathrm{Spa}(B^\sharp, B^{+\sharp})}$. This rule has to be compatible with pullbacks, more precisely: For any map $f\colon \mathrm{Spa}(A, A^+) \to \mathrm{Spa}(B, B^+)$ in $\mathrm{Perf}_{/X_\diamond}$, which also gives us a map $f^\sharp\colon \mathrm{Spa}(A^\sharp, A^{+\sharp}) \to \mathrm{Spa}(B^\sharp, B^{+\sharp})$, we want $f^{\sharp *}(\alpha^*\mathcal{E}) = (\alpha \circ f)^*\mathcal{E}$. Here $f^{\sharp *}$ just denotes the usual pullback of vector bundles along maps of adic spaces.
3. A 2-truncated v-hypercover
$$X'''^\diamond \mathrel{\substack{\to\\ \to\\ \to}} X''^\diamond \rightrightarrows X'^\diamond \to X_\diamond$$
by v-sheaves associated to not necessarily in characteristic p perfectoid spaces (note that we can always find this by smallness of $X_\diamond$), with the maps assumed to be compatible with the structure maps to $\mathrm{Spd}(\mathbb{Z}_p)$, together with a descent datum of vector bundles on
$$X''' \mathrel{\substack{\to\\ \to\\ \to}} X'' \rightrightarrows X'.$$

The second definition makes it clear that we can pull back v-bundles along maps of v-sheaves over $\mathrm{Spd}(\mathbb{Z}_p)$

Even more relevant to us will be a slightly different notion, which exists for v-sheaves of the form $\mathcal{X} \times \mathrm{Spd}(\mathbb{Z}_p)$ and $\mathcal{X} \times \mathrm{Spd}(\mathbb{Q}_p)$ only:

Definition 3.7. Let $\mathcal{X}$ be any small v-sheaf and set $\mathcal{S} = \mathbb{Z}_p$ or $\mathcal{S} = \mathbb{Q}_p$. Recall the v-descent of vector bundles on open subspaces of $\mathcal{Y}_{[0,\infty)}$, [SW20, Proposition 19.5.3]. As before, this implies that the following two sets of data are equivalent, either of which we will a *semi-classical v-bundle* on $\mathcal{X} \times \mathcal{S}$:

1. A rule associating to any map $\mathrm{Perf} \ni \mathrm{Spa}(B, B^+) \to \mathcal{X}$ a vector bundle on
$$\mathrm{Spa}(B, B^+) \dot{\times} \mathrm{Spa}(\mathcal{S}) = \begin{cases} \mathcal{Y}_{[0,\infty)}(B, B^+) & \text{if } \mathcal{S} = \mathbb{Z}_p \\ \mathcal{Y}_{(0,\infty)}(B, B^+) & \text{if } \mathcal{S} = \mathbb{Q}_p, \end{cases}$$
compatibly with pullbacks along maps $\mathrm{Spa}(A, A^+) \to \mathrm{Spa}(B, B^+)$ in $\mathrm{Perf}_{/\mathcal{X}}$ in the same way as for v-bundles.

2. A 2-truncated v-hypercover

$$X''' \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} X'' \rightrightarrows X' \to \mathcal{X}$$

by perfectoid spaces in characteristic p with a descent datum of finite projective modules on

$$X''' \dot{\times} \operatorname{Spa}(\mathcal{S}) \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} X'' \dot{\times} \operatorname{Spa}(\mathcal{S}) \rightrightarrows X' \dot{\times} \operatorname{Spa}(\mathcal{S}).$$

Again, the first definition gives us a pullback functor from rational/integral $\mathcal{Y}$-bundles along maps of v-sheaves $\mathcal{X} \to \mathcal{X}'$.

Remark 3.8. 1. (semi-classical v-bundles as full subcategory) We will construct a natural functor from semi-classical v-bundles to v-bundles in Construction 3.10 (2) and show in Proposition 3.12 (2) that it is fully-faithful. This allows us to understand semi-classical v-bundles as a full subcategory of v-bundles, as the terminology might already suggest.

2. (similar objects in [GI23] Gleason and Ivanov have considered similar objects, but under slightly different names: In loc.cit., what we call v-bundles are called *families of untilted vector bundles* with the category denoted $\mathrm{Vect}_v^{\mathcal{O}^\sharp}(X_\diamond)$. Similarly, our integral (resp. rational) $\mathcal{Y}$-bundles on some $\mathcal{X}$ is denoted $\mathrm{Vect}_{\mathcal{Y}}^{\mathrm{cl}}(\mathcal{X})$ (resp. $\mathrm{Vect}_{Y}^{\mathrm{cl}}(\mathcal{X})$), with cl standing for classical.

3. (about the term *semi-classical*) In my opinion, the term *classical v-bundle* should be reserved for those that come from actual vector bundles. Then:

   - Classical v-bundles would simply be vector bundles.
   - General v-bundles are objects on a v-sheaf (over $\mathbb{Z}_p$) that become classical after pullback along a v-cover.
   - Semi-classical v-bundles on $\mathcal{X} \times \operatorname{Spd}(\mathcal{S})$ become classical after pullback to $\operatorname{Spa}(B, B^+) \times \operatorname{Spd}(\mathcal{S})$ for a representable $\operatorname{Spa}(B, B^+)$. Thus, we only need to look at them v-locally in the first component of $\mathcal{X} \times \operatorname{Spd}(\mathcal{S})$, while in the second component, they already are classical.

   However, since the name classical vector bundle already exists with a different meaning in [GI23], I will avoid using it in this paper.

4. (Old terminology) In a former version of this paper, semi-classical v-bundles were called

Proposition 3.9. *The functors*

1. *associating to a small v-sheaf $X_\diamond$ over $\operatorname{Spd}(\mathbb{Z}_p)$ the category of v-bundles on $X_\diamond$,*

2. *associating to a small v-sheaf $\mathcal{X}$ the category of semi-classical v-bundles on $\mathcal{X} \times \operatorname{Spd}(\mathcal{S})$*

satisfy descent along covers of small v-sheaves (assumed to be compatible with the maps to $\mathrm{Spd}(\mathbb{Z}_p)$ *in the first case).*

Proof. By the already cited descent results [SW20, p. 17.1.8] and [SW20, p. 19.5.3] they are sheaves on affinoid perfectoids, which form a basis for the v-topology: Indeed, by definition every small v-sheaf has a cover by a perfectoid space in characteristic p, which in turn clearly has a cover by affinoid perfectoids. $\square$

In cases where this makes sense, there are base change functors from vector bundles to v-bundles, from $\mathcal{Y}$-bundles to v-bundles and from vector bundles bundles to $\mathcal{Y}$-bundles; The constructions work in somewhat different generalities, so let's name all three – A diagram of those functors will follow at the end of the subsection.

Construction 3.10. 1. (vector bundles to v-bundles) Let X be any adic space over $\mathbb{Z}_p$, and consider its associated v-sheaf $X^\diamond$. We get a functor from vector bundles on X to v-bundles on $X^\diamond$ as follows: Let $\mathcal{E}$ be a vector bundle on X. The (B, B^+)-valued points of $X^\diamond$ are by definition given by an untilt $(B^\sharp, B^{+\sharp})$ together with a map $\mathrm{Spa}(B^\sharp, B^{+\sharp}) \to X$. By definition of the map $X^\diamond \to \mathrm{Spa}(\mathbb{Z}_p)$, this untilt agrees with the one given by the composition

$$\mathrm{Spa}(B, B^+) \to X^\diamond \to \mathrm{Spd}(\mathbb{Z}_p).$$

Pullback of $\mathcal{E}$ along $\mathrm{Spa}(B^\sharp, B^{+\sharp}) \to X$ defines a rule for a v-sheaf, and compatibility with pullbacks is obvious by the definition.

2. ($\mathcal{Y}$-bundles to v-bundles) Now consider any small v-sheaf $\mathcal{X}$. In this context, there is a functor from integral (resp. rational) $\mathcal{Y}$-bundles on $\mathcal{X}$ to v-bundles on $Y_\diamond = \mathcal{X} \times \mathrm{Spd}(\mathcal{S})$ for $\mathcal{S} = \mathbb{Z}_p$ (resp. $\mathcal{S} = \mathbb{Q}_p$) (with the map to $\mathrm{Spd}(\mathbb{Z}_p)$ given by projection to the second component): Let $\mathcal{E}$ be such a $\mathcal{Y}$-bundle. We need to assign to every map from a representable $\mathrm{Perf} \ni T \to Y_\diamond$ a vector bundle on $T^\sharp$. To do this, choose a cover $X' \to \mathcal{X}$ by a perfectoid space in characteristic p. By the definition of $\mathcal{Y}$-bundles, this gives us a vector bundle $\mathcal{E}|_{X'}$ on $X' \dot{\times} \mathrm{Spa}(\mathcal{S})$ with descent information with respect to $Y_\diamond$. Consider the fiber product $T \times_{Y_\diamond} (X' \times \mathrm{Spd}(\mathcal{S}))$ and cover this in turn with a v-sheaf T' representable by a perfectoid space, which we are able to do by smallness. In total, we have constructed the following diagram of v-sheaves over $\mathrm{Spd}(\mathbb{Z}_p)$:

$$\begin{tikzcd} & T' \arrow[dl, two heads] \arrow[d, two heads] \\ T \arrow[d] & T \times_{Y_\diamond} (X' \times \mathrm{Spd}(\mathcal{S})) \arrow[l, two heads] \arrow[d] \\ Y_\diamond = \mathcal{X} \times \mathrm{Spd}(\mathcal{S}) & X' \times \mathrm{Spd}(\mathcal{S}) \arrow[l, two heads] \end{tikzcd}$$

The composition of the two right vertical maps gives us a map

$$T' \to (X' \times \mathrm{Spd}(\mathcal{S}))$$

of v-sheaves over $\mathrm{Spd}(\mathbb{Z}_p)$, corresponding to a map of adic spaces

$$T'^{\sharp} \to X' \dot\times \mathrm{Spa}(\mathcal{S}).$$

The crucial reason why we get to do this untilt is that T' is perfectoid. Pulling back $\mathcal{E}|_{X'}$ along this map gives us an actual vector bundle $\mathcal{E}_{T'^{\sharp}}$ on $T'^{\sharp}$. In a similar way, we can pull back the entire descent datum to a descent datum of $\mathcal{E}_{T'^{\sharp}}$ over $T^{\sharp}$, allowing us, via v-descent of vector bundles over perfectoids, to descend $\mathcal{E}_{T'^{\sharp}}$ down to $T^{\sharp}$.

3. (vector bundle to $\mathcal{Y}$-bundles) Now keep the assumption from (2) and assume furthermore that $Y_\diamond = \mathcal{X} \times \mathrm{Spd}(\mathcal{S})$ is given as $Y^\diamond$ for a suitable adic space Y – this is the case notably if $\mathcal{X}$ is either a perfectoid Huber pair $\mathrm{Spa}(B, B^+)$ in characteristic p or $\mathrm{Spd}(A^+, A^+)$ for an integral perfectoid A^+, equipped with the ϖ-adic topology for a suitable pseudouniformizer ϖ[11]. In this context we can associate to any vector bundle $\mathcal{E}$ on Y an integral/rational $\mathcal{Y}$-bundle on $\mathcal{X}$, very similarly to (1): Let $\mathrm{Spa}(B, B^+) \to \mathcal{X}$ be any map from a representable. Then we can simply associate to this map the pullback of $\mathcal{E}$ along

$$\mathrm{Spa}(B, B^+) \dot\times \mathrm{Spa}(\mathcal{S}) \to Y$$

obtained by untilting the map

$$\mathrm{Spa}(B, B^+) \times \mathrm{Spd}(\mathcal{S}) \to \mathcal{X} \times \mathrm{Spd}(\mathcal{S}).$$

4. (integral to rational $\mathcal{Y}$-bundles) For any small v-stack $\mathcal{X}$, there is also a functor from integral to rational $\mathcal{Y}$-bundles in the obvious way.

In cases where it makes sense, these constructions commute.

Remark 3.11. Let's try to summarize the situation in a single commutative diagram. To make sure all the categories make sense, we only consider the case $X_\diamond = X \dot\times \mathrm{Spa}(\mathcal{S})$ for $X = \mathrm{Spa}(A^+)$ integral perfectoid or $X = \mathrm{Spa}(B, B^+)$ affinoid perfectoid.

$$\begin{tikzcd}
\text{vector bundles on } X \dot\times \mathrm{Spa}(\mathbb{Z}_p) \arrow[r] \arrow[d, "\mathcal{F}_{\mathrm{vec}\to\mathcal{Y}}"] \arrow[dd, bend right=60, "\mathcal{F}_{\mathrm{vec}\to\mathrm{v}}"'] & \text{vector bundles on } X \dot\times \mathrm{Spa}(\mathbb{Q}_p) \arrow[d, "\mathcal{G}_{\mathrm{vec}\to\mathcal{Y}}"] \arrow[dd, bend left=60, "\mathcal{G}_{\mathrm{vec}\to\mathrm{v}}"] \\
\text{integral } \mathcal{Y}\text{-bundles on } X \arrow[r] \arrow[d, "\mathcal{F}_{\mathcal{Y}\to\mathrm{v}}"] & \text{rational } \mathcal{Y}\text{-bundles on } X \arrow[d, "\mathcal{G}_{\mathcal{Y}\to\mathrm{v}}"] \\
\text{v-bundles on } (X \dot\times \mathrm{Spa}(\mathbb{Z}_p))^\diamond \arrow[r] & \text{v-bundles on } (X \dot\times \mathrm{Spa}(\mathbb{Q}_p))^\diamond
\end{tikzcd}$$

[11] Note the following subtlety for $\mathcal{S} = \mathbb{Q}_p$: If A^+ is perfect of characteristic p with the discrete topology, then $Y = \mathrm{Spa}(W(A^+)[1/p], W(A^+))$; meanwhile, if A^+ carries the ϖ-adic topology for some pseudouniformizer ϖ which is a non-zero divisor, so that $(A^+[1/\varpi], A^+)$ is a perfectoid Huber pair, then $Y = \mathrm{Spa}(A^+) \dot\times \mathrm{Spa}(\mathbb{Q}_p) = \mathcal{Y}_{(0,\infty]}(A^+[1/\varpi], A^+)$; i.e., the inversion of p can happen both in a rational or analytic way, and which one of these happens might depend completely on the choice of ϖ!

The properties of the horizontal base change maps do not concern us here; concerning the vertical maps, we will see the following properties:

- $\mathcal{F}_{\mathrm{vec}\to\mathrm{v}}$ and $\mathcal{G}_{\mathrm{vec}\to\mathrm{v}}$: Fully-faithful if $X = \mathrm{Spa}(B, B^+)$ (Proposition 3.12 (1)).
- $\mathcal{F}_{\mathcal{Y}\to\mathrm{v}}$ and $\mathcal{G}_{\mathcal{Y}\to\mathrm{v}}$: Fully-faithful (Proposition 3.12 (2)).
- These four vertical errors ending in the bottom row are not essentially surjective, as the pullback of any non-trivial v-bundle on $\mathrm{Spd}(\mathcal{S})$ to the product $(X \dot{\times} \mathrm{Spa}(\mathcal{S}))^\diamond = X^\diamond \times \mathrm{Spd}(\mathcal{S})$ will not be in the essential image.
- $\mathcal{F}_{\mathrm{vec}\to\mathcal{Y}}$: Equivalence of categories (trivial if $X = \mathrm{Spa}(B, B^+)$, a slightly special case of our main result Theorem 4.18 if $X = \mathrm{Spa}(A^+)$).
- $\mathcal{G}_{\mathrm{vec}\to\mathcal{Y}}$ Equivalence of categories if $X = \mathrm{Spa}(B, B^+)$ (trivial). For $X = \mathrm{Spa}(A^+)$, the situation is more subtle. If A^+ arises as ring of integral elements in some perfectoid Huber pair (A, A^+), then it is also an equivalence of categories (Proposition 4.15). If A^+ is discrete, [GI23] shows an equivalence if the objects on both sides also carry a Shtuka structure (with leg in $p = 0$).

Proposition 3.12. *1. Let X be a sousperfectoid adic space. Then the functor from vector bundles on X to v-bundles on $X^\diamond$ is fully faithful, and an equivalence of categories if X is perfectoid.*

2. For any small v-sheaf $\mathcal{X}$, the functor from integral (resp. rational) $\mathcal{Y}$-bundles on $\mathcal{X}$ to v-bundles on $\mathcal{X} \times \mathrm{Spd}(\mathbb{Z}_p)$ (resp. $\mathcal{X} \times \mathrm{Spd}(\mathbb{Q}_p)$) is fully faithful.

3. If X is a perfectoid adic space, then the functor from vector bundles on $\mathcal{Y}_{[0,\infty)}(X)$ (resp. $\mathcal{Y}_{(0,\infty)}(X)$) to integral (resp. rational) $\mathcal{Y}$-bundles on X is an equivalence of categories.

Proof. 3. is clear for $X = \mathrm{Spa}(B, B^+)$ affinoid perfectoid because the v-site over $\mathrm{Spa}(B, B^+)$ has $\mathrm{Spa}(B, B^+)$ itself as a final object and our $\mathcal{Y}$-bundles are assumed to be compatible with pullback. The statement for general perfectoid spaces X follows from analytic gluing.

1. The equivalence for perfectoid X can be done the same way as the proof of (3), i.e., it is trivial for X affinoid perfectoid and the general version follows by gluing. Regarding the full-faithfulness for sousperfectoid spaces, recall the notion of v-completeness from [HK20, Definition 9.6]: A Huber pair (A, A^+) is called v-complete if the natural map

$$(A, A^+) \to (\check{A}, \check{A}^+) := \big(H^0(\mathrm{Spa}(A, A^+)_v, \mathcal{O}), H^0(\mathrm{Spa}(A, A^+)_v, \mathcal{O}^+\big)$$

is an isomorphism. In [HK20, Lemma 11.4 (a)], it is shown that any sousperfectoid ring is v-complete. Let $\mathcal{E}_1, \mathcal{E}_2$ be two vector bundles on X. Since the question is analytic-local on X, we can assume $X = \mathrm{Spa}(A, A^+)$ is affinoid (and still sousperfectoid, thus v-complete[12]) and the $\mathcal{E}_i$ are free. Now a morphism of vector bundles $\mathcal{E}_1 \to \mathcal{E}_2$ is simply given by a matrix with coefficients in $\Gamma(X, \mathcal{O}_X)$, while a morphism of v-bundles $\mathcal{E}_1 \to \mathcal{E}_2$ is given by a matrix with coefficients in $H^0(X_v, \mathcal{O})$ by description (1) in Definition 3.6. Fully-faithfullness now follows by v-completeness of X.

2. We first show the result in case $\mathcal{X} = \mathrm{Spd}(B, B^+)$ is representable. Then the functor from vector bundles to $\mathcal{Y}$-bundles is an equivalence by (3) and both

$$\mathrm{Spa}(B, B^+) \dot{\times} \mathrm{Spa}(\mathbb{Z}_p) = \mathcal{Y}_{[0,\infty)}(B, B^+)$$

and

$$\mathrm{Spa}(B, B^+) \dot{\times} \mathrm{Spa}(\mathbb{Q}_p) = \mathcal{Y}_{(0,\infty)}(B, B^+)$$

are sousperfectoid, so the functor from vector bundles to v-bundles is fully-faithful by (1). This formally implies full-faithfulness of the functor from $\mathcal{Y}$-bundles to v-bundles.

Now let $\mathcal{X}$ be a general small v-sheaf, let $\mathcal{E}_1, \mathcal{E}_2$ be two integral/rational $\mathcal{Y}$-bundles on $\mathcal{X}$, choose $\mathcal{S}$ as $\mathbb{Z}_p$ or $\mathbb{Q}_p$ accordingly, and let $\mathcal{E}_i^v$ be the associated v-bundles on $\mathcal{X} \times \mathrm{Spd}(\mathcal{S})$. Let $\alpha\colon \mathcal{E}_1 \to \mathcal{E}_2$ be a morphism of $\mathcal{Y}$-bundles and $\alpha^v\colon \mathcal{E}_1^v \to \mathcal{E}_2^v$ the associated map of v-bundles. Now for any map from an affinoid perfectoid $\mathrm{Spd}(B, B^+) \to \mathcal{X}$, pullback induces a map of v-bundles

$$\alpha^v|_{\mathrm{Spd}(B,B^+)\times \mathrm{Spd}(\mathcal{S})} \colon \mathcal{E}_1^v|_{\mathrm{Spd}(B,B^+)\times \mathrm{Spd}(\mathcal{S})} \to \mathcal{E}_2^v|_{\mathrm{Spd}(B,B^+)\times \mathrm{Spd}(\mathcal{S})},$$

which, by the case of $\mathcal{X}$ affinoid perfectoid just proved, corresponds to a unique map of $\mathcal{Y}$-bundles

$$\alpha|_{\mathrm{Spd}(B,B^+)} \colon \mathcal{E}_1|_{\mathrm{Spd}(B,B^+)} \to \mathcal{E}_2|_{\mathrm{Spd}(B,B^+)}.$$

Letting (B, B^+) vary, the $\alpha|_{\mathrm{Spd}(B,B^+)}$ are exactly the data needed to define a morphism of $\mathcal{Y}$-bundles, so we see that α is uniquely determined by α_v. This shows faithfulness. For fullness, by the case $\mathcal{X}$ affinoid perfectoid discussed above, it suffices to show that a morphism of v-bundles $\mathcal{E}_1^v \to \mathcal{E}_2^v$ is uniquley determined by its pullbacks to $\mathrm{Spd}(B, B^+) \times \mathrm{Spd}(\mathcal{S})$ for variying representables $\mathrm{Spd}(B, B^+)$. This follows from compatibility with pullbacks and the following straightforward lemma.

□

Lemma 3.13. *Let $\mathcal{X}$ be a v-sheaf, let $\mathrm{Spa}(B, B^+)$ be a not necessarily characteristic p affinoid perfectoid and let $\mathrm{Spa}(B, B^+)^\diamond \to \mathcal{X} \times \mathrm{Spd}(\mathcal{S})$ be a map of v-sheaves, compatible with the structure maps to $\mathrm{Spd}(\mathbb{Z}_p)$. Then there exists a factorization*

$$\mathrm{Spa}(B, B^+)^\diamond \to \mathrm{Spa}(B^\flat, B^{+\flat}) \times \mathrm{Spd}(\mathcal{S}) \to \mathcal{X} \times \mathrm{Spd}(\mathcal{S}),$$

[12] The reason we need sousperfectoidness instead of just v-completeness for the theorem is that as far as I know v-completeness is not known to be preserved by passing to open subspaces, see Remark 9.9 in loc. cit.

compatible with the structure maps to $\operatorname{Spd}(\mathbb{Z}_p)$ *(which in the middle, just as on the right, is given by projection to the second component).*

Proof. Disregarding the structure maps to $\operatorname{Spd}(\mathbb{Z}_p)$ for just a second, a map $\operatorname{Spd}(B, B^+) \to \mathcal{X} \times \operatorname{Spd}(\mathcal{S})$ is a $\operatorname{Spa}(B^\flat, B^{+\flat})$-valued point $x \in \mathcal{X}(B^\flat, B^{+\flat})$ together with an untilt $(B^{\flat\sharp}, B^{+\flat\sharp})$. Now compatibility with the structure map to $\operatorname{Spa}(\mathbb{Z}_p)$ just means that $(B^{\flat\sharp}, B^{+\flat\sharp})$ agrees with (B, B^+). This makes it clear that taking $\operatorname{Spa}(B, B^+) \to \operatorname{Spd}(B^\flat, B^{+\flat}) \times \operatorname{Spd}(\mathcal{S})$ the natural embedding and $\operatorname{Spd}(B^\flat, B^{+\flat}) \times \operatorname{Spd}(\mathcal{S}) \to \mathcal{X} \times \operatorname{Spd}(\mathcal{S})$ the map induced from the $\operatorname{Spa}(B^\flat, B^{+\flat})$-valued point x defines a factorization as desired. □

3.3 Adding a group structure

Definition + Proposition 3.14. For an affine flat group scheme $\mathcal{G}$ over $\mathbb{Z}_p$ and a v-sheaf $\mathcal{X}$, an integral *$\mathcal{G}$-$\mathcal{Y}$-bundle*[13] on $\mathcal{X}$ is one of the following equivalent pieces of data:

1. Associating to any representable $\operatorname{Spa}(B, B^+) \to \mathcal{X}$ a $\mathcal{G}$-bundle on
$$\mathcal{Y}_{[0,\infty)}(B, B^+) = \operatorname{Spa}(B, B^+) \dot{\times} \operatorname{Spa}(\mathbb{Z}_p).$$
2. An exact tensor functor from finite dimensional representations of $\mathcal{G}$ to integral $\mathcal{Y}$-bundles on $\mathcal{X}$.

The rest of this section will be used to show this equivalence.

Remark 3.15.
1. The equivalence also holds for rational $\mathcal{Y}$-bundles with the same proof.
2. I would think that for the reduction step to GL_n-torsors in the very beginning of [SW20, Proposition 19.5.3], one needs the statement of my proposition (at least for $\mathcal{X} = \operatorname{Spa}(B, B^+)$ perfectoid, to which we will reduce in the beginning of the proof). So maybe the statement is obvious or well-known in a way I'm not aware of.

We need a lemma to text exactness on points. This seems like it should be common knowledge, but I was not able to find it in the literature.

Lemma 3.16. *[Wed19, Proposition 7.51] Let (B, B^+) be a complete Huber pair and let $\mathfrak{m} \subset B$ be a maximal ideal. Then $\mathfrak{m}$ is closed and there exists $v \in \operatorname{Spa}(A)$ with* $\operatorname{supp} v = \mathfrak{m}$.

Lemma 3.17. *Let X be a reduced analytic adic space and consider a sequence of vector bundles*

$$(*) \qquad 0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0.$$

[13] One could of course also define $\mathcal{G}$-v-bundles, but this will not be necessary for us.

1. *$*$ is exact if and only if it is exact when pulled back to $(\kappa(x), \kappa^+(x)$ for every $x \in X$.*

2. *In particular, if X' is another reduced analytic adic space and $X' \to X$ is a map that is surjective on topological spaces, then $*$ is exact if and only if its pullback to X' is.*

Proof. The question is local on X, so we may assume $X = (B, B^+)$ is affinoid and the $\mathcal{E}_i$ correspond to finite projective B-modules. It is then well-known that $*$ is exact if and only if it is exact when base changed to $\kappa(x)$ for all closed points $x \in \operatorname{Spec}(B)$. Now for every such closed point x, corresponding to a maximal ideal $\mathfrak{m}$, Lemma 3.16 produces a point $x_{\mathrm{ad}} \in \operatorname{Spa}(B, B^+)$ such that $\operatorname{supp}(x_{\mathrm{ad}} = \mathfrak{m})$. By the definition of the residue fields in the schematic and the adic case, we thus get a field extension

$$\kappa(x) \to \kappa(x_{\mathrm{ad}}),$$

conpatible with the maps from B, which shows that $*$ base changed to $\kappa(x)$ is exact if and only if $*$ base changed to $\kappa(x_{\mathrm{ad}})$ is. Since this is possible for any closed point x, statement (1) follows. Statement (2) is a direct consequence. □

Proof. (of Definition + Proposition 3.14 Both pieces of data are functors associating to a finite-rank representation of $\mathcal{G}$ and a representable $\operatorname{Spa}(B, B^+) \to \mathcal{X}$ a vector bundle on $\mathcal{Y}_{[0,\infty)}(B, B^+)$. To show that the conditions imposed on either side are the same, we need to show that the notions of exactness on both sides align. More precisely, we need to show that if a sequence of integral $\mathcal{Y}$-bundles on some perfectoid $\operatorname{Spa}(B, B^+)$ is exact (i.e., it is exact after passage to a v-cover $\operatorname{Spa}(A, A^+) \to \operatorname{Spa}(B, B^+)$ and then analytically localizing on $\mathcal{Y}_{[0,\infty)}(A, A^+)$) then already the associated map of vector bundles on $\mathcal{Y}_{[0,\infty)}(B, B^+)$ is exact (i.e. surjective after just localizing analytically on $\mathcal{Y}_{[0,\infty)}(B, B^+)$).

So consider a short exact sequence

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$

of $\mathcal{Y}$-bundles on (B, B^+) and let $\operatorname{Spa}(A, A^+) \to \operatorname{Spa}(B, B^+)$ be a v-cover (which can be always chosen to be affinoid as $\operatorname{Spa}(B, B^+)$ is quasi-compact as a v-sheaf) such that the evaluation of our SES at $\operatorname{Spa}(A, A^+)$ is (analytic-locally) exact on $\mathcal{Y}_{[0,\infty)}(A, A^+)$. I claim that the underlying map of topological spaces

$$|\mathcal{Y}_{[0,\infty)}(A, A^+)| \to |\mathcal{Y}_{[0,\infty)}(B, B^+)|$$

is surjective. Indeed, as both adic spaces are analytic, we have

$$|\mathcal{Y}_{[0,\infty)}(A, A^+)| = |\mathcal{Y}_{[0,\infty)}(A, A^+)^\diamond| = |\operatorname{Spa}(A, A^+) \times \operatorname{Spd}(\mathbb{Z}_p)|,$$

and similarly for (B, B^+). But as

$$|\operatorname{Spa}(A, A^+)| \to |\operatorname{Spa}(B, B^+)|$$

is surjective by assumption, the same must be true after taking the product with $\operatorname{Spd}(\mathbb{Z}_p)$ on either side, showing the claim.

But both $\mathcal{Y}_{[0,\infty)}(A, A^+)$ and $\mathcal{Y}_{[0,\infty)}(B, B^+)$ are reduced, so applying Lemma 3.17, we can indeed check exactness on the latter by pullback to the former. □

3.4 Adding a Frobenius structure: shtukas

Let $\mathcal{G}$ still be an affine flat group scheme over $\mathbb{Z}_p$.

Definition 3.18. (Scholze-Weinstein shtukas) By a *$\mathcal{G}$-shtuka* over some affinoid perfectoid (R, R^+) of characteristic p we mean the data of

- An untilt $(R^\sharp, R^{+\sharp})$ over $\mathbb{Z}_p$ (so our shtukas will always be in the integral sense),
- A $\mathcal{G}$ torsor $\mathcal{E}$ over $\mathcal{Y}_{[0,\infty)}(R, R^+) = W(R^+) - V(\varpi)$,
- An isomorphism of $\mathcal{G}$ torsors $\Phi\colon \varphi^*\mathcal{E}|_{\mathcal{Y}_{[0,\infty)} - \mathrm{Spa}(R^\sharp, R^{+\sharp})} \to \mathcal{E}|_{\mathcal{Y}_{[0,\infty)} - \mathrm{Spa}(R^\sharp, R^{+\sharp})}$ that is meromorphic with respect to a distinguished element ξ cutting out $\mathrm{Spa}(R^\sharp, R^{+\sharp}) \subset \mathrm{Spa}(R, R^+) \dot{\times} \mathrm{Spa}(\mathbb{Z}_p)$.

A *shtuka with fixed leg* over some not necessarily characteristic p perfectoid Huber pair (B, B^+) is a shtuka over $(B^\flat, B^{+\flat})$ where the untilt $(B^{\flat\sharp}, B^{+\flat\sharp})$ that comes as part of the datum is given by (B, B^+).

Definition 3.19. Let $X_\diamond$ be a v-sheaf together with a map to $\mathrm{Spd}(\mathbb{Z}_p)$. A (family of) GL_n-shtuka(s) over $X_\diamond$ is given by

- for any map from a representable $\mathrm{Spa}(B, B^+) \to X_\diamond$ with associated untilt $\mathrm{Spa}(B^\sharp, B^{+\sharp})$, a GL_n-shtuka with fixed leg on $\mathrm{Spa}(B^\sharp, B^{+\sharp})$, compatibly with pullbacks.

Similarly, *$\mathcal{G}$-shtukas with fixed leg* over $X_\diamond$ is given by one of the following equivalent pieces of data:

- For any map from a representable $\mathrm{Spa}(B, B^+) \to X_\diamond$ with associated untilt $\mathrm{Spa}(B^\sharp, B^{+\sharp})$, a $\mathcal{G}$-shtuka with fixed leg on $\mathrm{Spa}(B^\sharp, B^{+\sharp})$, compatibly with pullbacks.
- An exact tensor equivalence from the category of $\mathcal{G}$-representations to the category of GL_n-shtukas on $X_\diamond$ for varying n.

Remark 3.20. Similarly to the prismatic $\mathcal{G}$-crystals discussed earlier, we have three "components" going into a (family of) mixed characteristic shtuka; Just that because $\mathcal{Y}_{[0,\infty)}(B, B^+)$ is not affinoid, we have to add the Frobenius structure *before* we pass to families of perfectoids, leaving us with 3 slightly varying definitions, as opposed to 6 on the schematic side (the first point of the definition above kind of contains 2 different variants as we can decide if we first add the F- or $\mathcal{G}$-structure).

Remark 3.21. $\mathcal{G}$-shtukas with fixed leg on $X^\diamond$ for X an adic space have a slightly simpler description: They are given by a rule associating to any map $\mathrm{Spa}(B, B^+) \to X$ from a (not necessarily characteristic p) affinoid perfectoid a $\mathcal{G}$-shtuka with fixed leg on (B, B^+).

Proposition 3.22. *Associating to a small v-sheaf $X_\diamond$ over $\mathrm{Spd}(\mathbb{Z}_p)$ the $\mathcal{G}$-shtukas with fixed legs on $X_\diamond$ satisfies descent along v-covers that are compatible with the structure maps to $\mathrm{Spd}(\mathbb{Z}_p)$.*

Proof. It is enough to show this on the basis given by affinoid perfectoids (not necessarily in characteristic p). Without fixing the legs, this is just the well-known statement that shtukas on affinoid-perfectoids satisfy v-descent shown in [SW20]. For obtaining the statement about fixed legs, we use that the covers are compatible with the maps to $\mathrm{Spd}(\mathbb{Z}_p)$ by assumption and that a map of affinoid perfectoids is a v-cover if and only if the associated map between diamonds is a cover of v-sheaves. $\square$

4 Equivalence of the two categories

4.1 Galois descent of vector bundles on adic spaces

In algebraic geometry, descent data along certain finite étale covers can be reformulated using Galois representation, and in fact this can be generalized to infinite Galois extensions. We want to study an analogue of this for adic spaces, i.e., with completions at various points. We will do this in somewhat more generality than necessary for our purposes, as it should be of interest for its own sake.

Throughout this subsection, fix:

- K any non-archimedian field with ring of integers $\mathcal{O}_K$ and (pseudo)uniformizer ϖ,
- B a reduced Tate[14] Banach K-algebra with ring of definition B_0. Note that the image of ϖ in B which we will also denote ϖ is automatically a pseudouniformizer as both K and B are Tate.
- L an infinite Galois extension of K and $\hat{L}$ its completion,
- $B_L = B \otimes_K L$ and $\hat{B}_L = B \hat{\otimes}_K \hat{L}$,
- $\Gamma = \mathrm{Gal}(L/K)$,
- $\mathcal{A}$ a non-commutative Banach Tate B-algebra, finitely presented as a B-module,
- $\mathcal{A}_L \coloneqq \mathcal{A} \otimes_B B_L$ and $\hat{\mathcal{A}}_L \coloneqq \mathcal{A} \hat{\otimes}_B \hat{B}_L = \mathcal{A} \otimes_B \hat{B}_L$[15].

The model case and the only one relevant for the sequel is $\mathcal{A} = \mathrm{End}_B(P)$ for a finite projective B-module P, so that $\mathcal{A}_L = \mathrm{End}_{B_L}(P \otimes_B B_L)$ and $\hat{\mathcal{A}}_L = \mathrm{End}_{\hat{B}_L}(P \otimes_B \hat{B}_L)$.

Let's start with some remarks on topology: As any finitely generated module over a topological ring, $\mathcal{A}$ carries a canonical topology induced from a quotient map of B-modules $B^k \twoheadrightarrow \mathcal{A}$. Using the linear nature of the topology on B, we can give this a more algebraic description:

[14] I don't think it is necessary to assume Tateness, but it helps with some topological considerations.

[15] Whenever we tensor with a finitely presented module, no completion is necessary: In fact, if R is a complete topological ring and M is a finitely presented module over R with the canonical topology defined below, M is automatically complete. This is clearly true for finite free modules M and follows for finitely presented ones as a finitely generated submodule of such a free module can be shown to be closed.

Lemma 4.1. *Let $\mathcal{A}_0 \subset \mathcal{A}$ be a finitely generated B_0-submodule, where $B_0 \subset B$ is a ring of definition, such that*

$$\mathcal{A}_0 \otimes_{B_0} B = \mathcal{A}.$$

Then a neighbourhood basis of 0 *in $\mathcal{A}$ is given by*

$$\varpi^n \mathcal{A}_0 \subset \mathcal{A}.$$

Proof. Choose a surjection $B_0^k \twoheadrightarrow \mathcal{A}_0$. As tensoring is right exact, this gives rise to a commutative diagram of B_0-modules with both horizontal maps surjective:

$$\begin{tikzcd} B_0^k \arrow[r, two heads, "p_0"] \arrow[d, hook] & \mathcal{A}_0 \arrow[d, hook] \\ B^k \arrow[r, two heads, "p"] & \mathcal{A} \end{tikzcd}$$

Now by definition of the topology on $\mathcal{A}$, a subset $0 \in U \subseteq \mathcal{A}$ is a neighbourhood of 0 if and only if its preimage $p^{-1}(U) \subseteq B^k$ is, which, as $B_0 \subset B$ is open, is the case if and only if $p^{-1}(U) \cap B_0^k$ is a neighbourhood of 0 in B_0^k. But a neighbourhood basis in B_0^k is given by the $\varpi^n B_0^k$, so in total $U \subseteq \mathcal{A}$ is a neighbourhood of zero if and only if $\varpi^n B_0^k \subseteq p^{-1}(U)$ for some $n \in \mathbb{N}$, which, by B_0-linearity and surjectivity of the upper arrow is the case if and only if $\varpi^n \mathcal{A}_0 \subset U$. □

Going in another direction, $B' := B \otimes_K K'$ for a finite subextension $K \subseteq K' \subset L$ as well as B_L and $\hat{B}_L$ carry canonical topologies: As B, K and K' are Tate, the same is true for their tensor product $B' = B \otimes_K K'$, in fact with pseudouniformizer given by the image of the pseudouniformizer $\varpi \in K$, and similarly for the other two cases. Now, putting this together with the construction from before, we get canonical topologies on both $\mathcal{A}_L$ and $\hat{\mathcal{A}}_L$, and the statement of Lemma 4.1 also holds in this context, so that the topology is obtained from the ϖ-adic topology on certain open submodules $\mathcal{A}_{L,0} = \mathcal{A}_0 \otimes_{B_0} B_{L,0}$ and $\hat{\mathcal{A}}_{L,0} = \mathcal{A}_0 \otimes_{B_0} \hat{B}_{L,0}$. Finally, we also get a topology on tensor powers like $B_L \otimes_B B_L$ (as all components are Tate) and their modules like $\mathcal{A} \otimes_B B_L \otimes_B B_L$, with topology given by the ϖ-adic topology on $\mathcal{A}_0 \otimes_{B_0} B_{L,0} \otimes_{B_0} B_{L,0}$, and similarly for the completed version.

The most important aspect of this construction is the following easy fact:

Remark 4.2. Both $B_L \subset \hat{B}_L$ and $\mathcal{A}_L \subset \hat{\mathcal{A}}_L$ are dense, and similarly for their tensor powers.

This is all that needed to be said about the topologies involved; let's turn to Galois actions now:

Construction 4.3. We get the following multiplicative actions of groups:

1. For any finite subextension $K \subseteq K' \subset L$, $B' := B \otimes_K K'$, there is a continuous action of $\mathrm{Gal}(K'/K)$ on

$$\mathcal{A}_{K'} := \mathcal{A} \otimes_B B' = (\mathcal{A} \otimes_B B) \otimes_K K'$$

given by

$$\mathrm{Gal}(K'/K) \times ((\mathcal{A} \otimes_B B) \otimes_K K') \ni (\sigma, \gamma \otimes c) \mapsto \gamma \otimes \sigma(c).$$

2. By essentially the same construction, $\Gamma = \mathrm{Gal}(L/K)$ acts continuously on $\mathcal{A}_L$.
3. By continuously extending this, we also get an action of Γ on $\hat{\mathcal{A}}_L$

We write each of these as $(\sigma, a) \mapsto \sigma(a)$.

Remark 4.4. The reason for this construction is that for $\sigma \in \Gamma$

$$\sigma^* \mathcal{A}_{K'} \to \mathcal{A}_{K'}, \quad a \mapsto \sigma(a)$$

is an isomorphism of non-commutative B'-algebras. In this way, the action constructed above allows us to substitute "morphisms of non-commutative B'-algebras $\sigma^* \mathcal{A}_{K'} \to \mathcal{A}_{K'}$" for "Endmorphisms of $\mathcal{A}_{K'}$", by composing with the inverse of the isomorphism above, and doing so without loss of information. The same is of course also true for the actions on $\mathcal{A}_L$ and $\hat{\mathcal{A}}_L$.

We now have everything needed to define the central objects of interest for this subsection. All tensor powers and completed tensor powers are taken either over B_0, which, for a B-module, is the same as tensoring over B because $B = B_0[1/\varpi]$.

Construction 4.5. Consider the following non-commutative B-algebras:

(1a) For a finite subextension $K \subseteq K' \subset L$, set $B' = B \otimes_K K'$ and let $C_{K'}$ be the algebra of maps of sets

$$\mathrm{Gal}(K'/K) \to \mathcal{A} \otimes_B B',$$

with addition and multiplication defined pointwise.

(1b) $D_{K'} \coloneqq \mathcal{A} \otimes_B B'^{\otimes 2}$ with K' and B' as before with its natural algebra structure.

(2a) C_L consisting of *continuous* maps

$$\Gamma = \mathrm{Gal}(L/K) \to \mathcal{A}_L^{\mathrm{disc}} = (\mathcal{A} \otimes_B B_L)^{\mathrm{disc}},$$

where Γ carries the standard (profinite) topology and $\mathcal{A}_L^{\mathrm{disc}}$ the discrete topology.

(2b) $D_L \coloneqq \mathcal{A} \otimes_B B_L^{\otimes 2}$.

(3a) $\hat{C}_L$ consisting of continuous maps of sets

$$\Gamma = \mathrm{Gal}(L/K) \to \hat{\mathcal{A}}_L = \mathcal{A} \otimes_B \hat{B}_L,$$

where now the right side carries the canonical topology.

(3b) $\hat{D}_L \coloneqq \mathcal{A} \otimes_B \hat{B}_L^{\hat{\otimes} 2}$.

These have the following extra structures:

- A canonical topology, in the case of the C's via the compact open topology, or, equivalently, as the codomain is metrizable and the domain is compact, with the topology induced from the supremum metric. [16]

- Subgroups $C_{K'}^{\mathrm{cocl}} \subset C_{K'}^{\times}$, $D_{K'}^{\mathrm{cocl}} \subset D_{K'}^{\times}$ etc. of the groups of units in the respective non-commutative rings $C_{K'}$, $D_{K'}$, etc. given by those elements that satisfy a certain cocycle condition:

  – For $D_{K'}$, this is meant as follows: For $\beta \in D_{K'} = \mathcal{A} \otimes_B B'^{\otimes 2}$ and $i \neq j \in \{1,2,3\}$, let $p_{ij}^*\beta \in \mathcal{A} \otimes_B B'^{\otimes 3}$ be the element inserting a 1 in the k'th copy of B', with k the unique element in $\{1,2,3\} - \{i,j\}$. Then β lies in $D_{K'}^{\mathrm{cocl}}$ if and only if it is invertible in $D_{K'}$ and

  $$p_{23}^*\alpha \cdot p_{12}^*\alpha = p_{13}^*\alpha.$$

  If $\mathcal{A} = \mathrm{End}_B(P)$ for a finite projective B-module P, the elements of $D_{K'}^{\mathrm{cocl}}$ are those $\beta \in D_{K'}^{\times}$ which, when viewed as an isomorphism of $B'^{\otimes 2}$-modules

  $$(P \otimes_B B') \otimes_B B' \cong P \otimes_B B'^{\otimes 2} \xrightarrow{\beta} P \otimes_B B'^{\otimes 2} \cong B' \otimes_B (P \otimes_B B'),$$

  satisfy the cocycle condition on $B'^{\otimes 3}$ in the usual sense of descent theory.

  – Similarly for D_L and $\hat{D}_L$ with $B'^{\otimes 3}$ replaced by $B_L^{\otimes 3}$ and $\hat{B}_L^{\hat{\otimes} 3}$.

  – For the C's, a map $\alpha\colon \sigma \mapsto \alpha(\sigma)$ is said to satisfy the cocycle condition if it takes values in units and it is a 1-cocycle in the cohomological sense, i.e. if

  $$\alpha(\sigma\tau) = \sigma(\alpha(\tau))\alpha(\sigma).$$

  Here $\alpha(\sigma)$ is specified by the function $\alpha\colon \mathrm{Gal}(K'/K) \to \mathcal{A}_{K'}$, while $\sigma(c)$ for any element $c \in \mathcal{A}_{K'}$, e.g. for $c = \alpha(\sigma)$, comes from the action defined in Construction 4.3.

- The underlying sets of the subgroups just defined are themselves the objects of certain groupoids:

  – For two elements $\beta, \beta' \in D_{K'}$, define

  $$\mathrm{Hom}_{D_{K'}^{\mathrm{cocl}}}(\beta,\beta') := \{d \in \mathcal{A}_{K'}^{\times} \mid p_1^*(d) \cdot \beta = \beta' \cdot p_2^*(d)\},$$

  where

  $$p_i^*\colon \mathcal{A} \otimes_B B' \to \mathcal{A} \otimes_B B'^{\otimes 2}$$

  is the map inserting a 1 in the $(i-1)$'th position. If again $\mathcal{A} = \mathrm{End}_B(P)$, then the $d \in (\mathrm{End}_B(P) \otimes_B B')^{\times} = \mathrm{Aut}_{B'}(P \otimes_B B')$ satisfying the above condition are exactly the isomorphisms of descent data.

[16] Note that in the definition of C_L as a topological B-algebra, the B_L-module $\mathcal{A}_L$ appears both with the discrete (for the underlying B-algebra) and the canonical (for the topological structure) topology.

- Similarly for D_L and $\hat{D}_L$.
- For two elements $\alpha, \alpha' \in C_{K'}^{\mathrm{cocl}}$ define

  $$\mathrm{Hom}_{C_{K'}^{\mathrm{cocl}}}(\alpha, \alpha') := \{c \in \mathcal{A}_{K'}^{\times} \mid \alpha(\sigma) = c^{-1}\alpha'(\sigma)\sigma(c) \text{ for all } \sigma \in \mathrm{Gal}(K'/K)\}.$$

  Thus, α and α' are isomorphic if and only if they are cohomologous as cocycles.
- Similarly for C_L and $\hat{C}_L$.

Remark 4.6.
In the language of group cohomology, one has

$$\begin{aligned} C_{K'} &= C^1(\mathrm{Gal}(K'/K), \mathcal{A}_{K'}), \\ C_{K'}^{\mathrm{cocl}} &= Z^1(\mathrm{Gal}(K'/K), \mathcal{A}_{K'}^{\times}), \\ C_{K'}^{\mathrm{cocl}}/\cong &= H^1(\mathrm{Gal}(K'/K), \mathcal{A}_{K'}^{\times}), \\ C_L &= C^{1,\mathrm{cont}}(\Gamma, \mathcal{A}_L^{\mathrm{disc}}), \\ C_{\hat{L}} &= C^{1,\mathrm{cont}}(\Gamma, \hat{\mathcal{A}}_L), \end{aligned}$$

and similarly for the other two Z^1's and H^1's. Note that the second-to-last equality only holds as non-commutative B-algebras and doesn't respect the topology!

After four pages of definitions, we can finally formulate the main theorem of this subsection

Theorem 4.7. *There are canonical isomorphisms*

$$C_{K'} \cong D_{K'}, \tag{1}$$

$$C_L \cong D_L, \tag{2}$$

$$\hat{C}_L \cong \hat{D}_L, \tag{3}$$

compatible with all the extra structures from above: That is, they are isomorphisms of non-commutative topological B-algebras, respecting the subsets given by elements satisfying the cocycle condition on either side and in fact they induce isomorphisms of groupoids on them.

Remark 4.8. If once again $\mathcal{A} = \mathrm{End}_B(P)$, so we can view the elements of the D's as descent data, then all descent data in $D_{K'}^{\mathrm{cocl}}$ and D_L^{cocl} are effective: for $D_{K'}$ this is just usual étale descent, while it follows for D_L since every $\beta \in D_L$ is already in the image of some $D_{K'}$. On the other hand, for an element $\beta \in \hat{D}_L^{\mathrm{cocl}}$ corresponding to some $\alpha \in \hat{C}_L^{\mathrm{cocl}}$, the following are equivalent:

1. The descent datum β is effective.
2. α is cohomologous to a cocycle vanishing on an open normal subgroup of Γ.

3. α is cohomologous to a cocycle lying in the image of $C_L \to \hat{C}_L$.

Indeed, (b)⇒(a) follows again from usual étale descent, (c)⇒(b) follows as the elements in C_L automatically vanish on an open normal subgroup by continuity. Finally, we get (a)⇒(c) because the functor from finite-projective B-modules to descent data as in $\hat{D}_L$ factors over the category of descent data as in D_L.

Proof. (1) is mostly classical, see e.g. [Sta20, 0CDQ]. We reproduce most of it for the sake of clarity. The normal basis theorem produces an isomorphism of K-algebras

$$\begin{aligned} K' \otimes_K K' &\cong \prod_{\sigma \in \mathrm{Gal}(K'/K)} K', \\ a \otimes b &\mapsto (a\sigma(b))_{\sigma \in \mathrm{Gal}(K'/K)}. \end{aligned}$$

As tensoring commutes with finite products, this induces an isomorphism of non-commutative B-algebras

$$D_{K'} = \mathcal{A} \otimes_B B'^{\otimes 2} \cong \prod_{\sigma \in \mathrm{Gal}(K'/K)} \mathcal{A} \otimes_B B' = C_{K'}.$$

In fact, it is even an isomorphism of topological non-commutative B-algebras. Indeed, let $B_0' = B_0 \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$ for $\mathcal{O}_{K'}$ the integral closure of $\mathcal{O}_K$ in K'. Then B_0' is a ring of definition of B', stable under the action of $\mathrm{Gal}(K'/K)$, and thus the isomorphism above restricts to an isomorphism of non-commutative B_0-algebras

$$C_{K',0} \coloneqq \mathcal{A}_0 \otimes_{B_0} B_0'^{\otimes 2} \cong \prod_{\sigma \in \mathrm{Gal}(K'/K)} \mathcal{A}_0 \otimes_{B_0} B_0' =: D_{K',0}$$

Now these are open sub-(topological non-commutative B_0-)algebras of $C_{K'}$ and $D_{K'}$, and in fact $\varpi^n C_{K',0} = \varpi^n D_{K',0}$, with equality because of the B_0-linearity, form a neighbourhood basis on either side. This shows equality of the topologies.

There is also an isomorphism of K-algebras

$$\begin{aligned} K' \otimes_K K' \otimes_K K' &\cong \prod_{(\sigma,\tau) \in \mathrm{Gal}(K'/K)^2} K', \\ a \otimes b \otimes c &\mapsto (a\sigma(b)\sigma(\tau(c)))_{\sigma,\tau \in \mathrm{Gal}(K'/K)}, \end{aligned}$$

resulting in an isomorphism of non-commutative B-algebras

$$\mathcal{A} \otimes_B B'^{\otimes 3} \cong \prod_{(\sigma,\tau) \in \mathrm{Gal}(K'/K)^2} \mathcal{A} \otimes_B B'.$$

In [Sta20, 0CDQ], they use this to prove the equivalence of the cocycle conditions on either side, which we will not reproduce here.

Finally, let's take a look at the groupoid structure. For $\alpha, \alpha' \in C_{K'}^{\mathrm{cocl}}$ corresponding to $\beta, \beta' \in D_{K'}^{\mathrm{cocl}}$, we have by definition

$$\mathrm{Hom}_{C_{K'}^{\mathrm{cocl}}}(\alpha, \alpha') = \{c \in \mathcal{A}_{K'}^{\times} \mid \alpha(\sigma) = c^{-1}\alpha'(\sigma)\sigma(c) \text{ for all } \sigma \in \mathrm{Gal}(K'/K)\}$$

and

$$\mathrm{Hom}_{D_{K}'^{\mathrm{cocl}}}(\beta, \beta') \coloneqq \{d \in \mathcal{A}_{K'}^{\times} \mid p_1^*(d) \cdot \beta = \beta' \cdot p_2^*(d)\}.$$

As composition on either side is simply given by multiplication inside $\mathcal{A}_{K'}^{\times}$, it suffices to show that the condition imposed on c agrees with the one imposed on d. This is clear from the fact that the compositions

$$\mathcal{A} \otimes_B B' \xrightarrow{p_i^*} \mathcal{A} \otimes_B B'^{\otimes 2} \xrightarrow{\sim} \prod_{\gamma \in \mathrm{Gal}(K'/K)\}} B'$$

are given by

$$d \mapsto \prod_{\gamma \in \mathrm{Gal}(K'/K)\}} d, \quad \text{resp. } d \mapsto \prod_{\gamma \in \mathrm{Gal}(K'/K)\}} \gamma(d)$$

for $i = 1$ resp. $i = 2$.

(1') This proves (1), but we want to state two more precise variants of the above properties that will be useful in the proof of (3). Firstly, the two continuous functions

$$\begin{aligned}
\epsilon_1 \colon C_{K'} &\to \mathcal{A} \otimes_B B'^{\otimes 3} \\
\alpha &\mapsto p_{23}^*\alpha \cdot p_{12}^*\alpha - p_{13}^*\alpha. \\
\epsilon_2 \colon D_{K'} &\to \mathcal{A} \otimes_B B'^{\otimes 3} \\
\beta &\mapsto \sigma(\alpha(\tau))\alpha(\sigma) - \alpha(\sigma\tau)
\end{aligned}$$

are equal, or more precisely, commute with the isomorphism $C_{K'} \cong D_{K'}$. The fact that the preimages of 0 for both agree (together with the fact that $C_{K'} \cong D_{K'}$ preserves units) just amounts to the compatibility with the cocycle condition. That the functions themselves agree can be understood as the fact that the isomorphism $C_{K'} \cong D_{K'}$ does not only preserve the cocycle condition, it also preserves the deviation from it. The second remark is of a very similar nature, but for morphisms. Fix elements $\alpha, \alpha' \in C_{K'}$ corresponding to $\beta, \beta' \in D_{K'}$. Then the functions

$$\begin{aligned}
\epsilon_3 \colon \mathcal{A}_{K'} &\to C_{K'} \\
c &\mapsto c\alpha(\sigma) - \alpha'(\sigma)\sigma(c) \\
\epsilon_4 \colon \mathcal{A}_{K'} &\to D_{K'} \\
d &\mapsto p_1^*(d) \cdot \beta - \beta' \cdot p_2^*(d)
\end{aligned}$$

are equal, i.e. commute with the isomorphism between the respective codomains. This can be seen as the isomorphism of groupoids also preserving how far any given object in $\mathcal{A}_{K'}$ is from being a morphism.

(2) Choose a compatible system $K \subset K_1 \subset K_2 \subset \cdots \subset L$ of finite subextensions such that $\cup_{n=1}^{\infty} K_n = L$ and let $B_n := B \otimes_K K_n$. Then we have isomorphisms of non-commutative B-algebras

$$\begin{aligned} D_L &= \mathcal{A} \otimes_B (\operatorname{colim}_n B_n)^{\otimes 2} \\ &= \mathcal{A} \otimes_B \operatorname{colim}_n (B_n^{\otimes 2}) \\ &= \operatorname{colim}_n (\mathcal{A} \otimes_B B_n^{\otimes 2}) \\ &= \operatorname{colim}_n \prod_{\sigma \in \operatorname{Gal}(K_n/K)} \mathcal{A} \otimes_B B_n \\ &= C_L. \end{aligned}$$

Indeed, in the second equality we just pull the colimit out of the two factors of $B_L \otimes_B B_L$, which is possible since every element of the tensor product is a finite sum of elementary tensors; similarly, for the third equality, we use that tensoring with $\mathcal{A}$ commutes with filtered colimits. The fourth inequality is taken straight from (1), while the last one uses that every continuous map from Γ to a discrete topological space factors over a finite quotient of Γ.

In fact, the colimit construction also takes care of the cocycle condition and the groupoid structure, but for the topology, one has to be a bit careful: Indeed, if we were to give C_L and D_L simply the colimit topology, one could take for increasing K_n smaller and smaller open neighbourhoods of 0 (each containing the previous one of course). The result will be an open neighbourhood in the colimit topology by definition, but not in C_L. So instead of considering the colimit topology, we will verify that this is a homeomorphism by hand, which can be done just the same way as in the proof of (1). Indeed, define as above $B_{L,0} = B_L \otimes_{\mathcal{O}_K} \mathcal{O}_L$ for $\mathcal{O}_L$ the integral closure of $\mathcal{O}_K$ in L. Then again $B_{L,0}$ is a ring of definition of B_L, stable under the action of Γ, and we get an isomorphism of non-commutative B_0-algebras

$$C_{L,0} := \mathcal{A}_0 \otimes_{B_0} B_{L,0}^{\otimes 2} \cong \operatorname{colim}_n \prod_{\operatorname{Gal}(K_n/K)} \mathcal{A}_0 \otimes_{B_0} B_{n,0} =: D_{L,0}.$$

Now $\varpi^n C_{L,0} \subset C_L$ and $\varpi^n D_{L,0} \subset D_L$ are neighbourhood bases of zero on either side, and they agree by B_0-linearity, thus showing the equivalence of topologies.

(3) the isomorphism $\hat{C}_L \cong \hat{D}_L$ of topological non-acommutative B-algebras follows from the isomorphism $C_L \cong D_L$ by passing to completions. Indeed, it is enough to show that $\hat{C}_L$ is complete as a topological group and that $C_L \subset \hat{C}_L$ is dense, and similarly for $D_L \subset \hat{D}_L$. Completeness is given by definition and denseness is Remark 4.2, except for the subtlety that we must be able to approximate a map $\Gamma \to \hat{\mathcal{A}}_L$ which is continuous for the analytic topology on $\hat{\mathcal{A}}_L$ by maps $\Gamma \to \mathcal{A}_L^{\mathrm{disc}}$.

To see this, note first that by an $\epsilon/2$-argument it is enough to approximate any continuous function $c\colon \Gamma \to \mathcal{A}_L$ by continuous functions $\Gamma \to \mathcal{A}_L^{\mathrm{disc}}$. We will now use the fact that Γ is profinite, which implies in particular that the image of c is compact, to approximate c by continuous maps $\Gamma \to \mathcal{A}_L$ with finite image: We cover $\mathrm{Im}(c)$ by finitely many balls $U_1, \ldots, U_n$ of radius ϵ, the preimage of each of which will be open in Γ, hence defined by a condition on finitely many factors in the representation of Γ as subspace of a product of discrete spaces. Taking the union of these finitely many finite sets of factors and dividing out all others, we obtain a finite (hence discrete) quotient Γ' of Γ with projection $p\colon \Gamma \to \Gamma'$ such that for every $g \in \Gamma$ there is some U_{i_g} such that $c(p^{-1}(g)) \subset U_{i_g}$. Picking some arbitrary $y_i \in U_i$ for each i and setting

$$c'\colon \Gamma \to \mathcal{A}_L, \qquad x \mapsto y_{i_g} \text{ if } x \in p^{-1}(g)$$

defines a continuous map that approximates c up to ϵ. As it has finite image, it is even continuous for the discrete topology on the right, which is what we had to show.

To show that the isomorphism $\hat{C}_L \cong \hat{D}_L$ is compatible with the subgroups of the unit group $\hat{C}_L^{\mathrm{cocl}}$ and $\hat{D}_L^{\mathrm{cocl}}$, consider an element $\alpha \in \hat{C}_L^\times$ and approximate it with a sequence $\alpha_n \in C_L$, each element defined over $B_n = B \otimes_K K_n$ for some finite subextension $K \subseteq K_n \subset L$. These correspond under the isomorphism $C_L \cong D_L$ to elements $\beta_n \in D_L$ which converge to the $\beta \in \hat{D}_L$ corresponding to α under the isomorphism $\hat{C}_L \cong \hat{D}_L$. Define maps $\epsilon_1\colon \hat{C}_L \to \mathcal{A} \otimes_B \hat{B}_L^{\otimes 3}$, $\epsilon_2\colon \hat{D}_L \to \mathcal{A} \otimes_B \hat{B}_L^{\otimes 3}$ as in (1'). Now:

$$\begin{aligned} &\alpha \text{ satisfies the cocycle condition} \\ \iff &\epsilon_1(\alpha) = 0 \\ \iff &\lim_{n\to\infty} \epsilon_1(\alpha_n) = 0 \\ \iff &\lim_{n\to\infty} \epsilon_2(\beta_n) = 0 \\ \iff &\epsilon_2(\beta) = 0 \\ \iff &\beta \text{ satisfies the cocycle condition,} \end{aligned}$$

with the middle equivalence coming from the observation in (1'). By applying entirely the same argument to ϵ_3 and ϵ_4, one sees that some $c \in \mathcal{A}_{\hat{L}}$ is a morphism in $C_{\hat{L}}^{\mathrm{cocl}}$ if and only if it is a morphism in $D_{\hat{L}}^{\mathrm{cocl}}$.

□

For citability, let's point out what is actually important in the above theorem.

Corollary 4.9. *Let $K, L, B, \hat{B}_L$ and Γ be as specified in the beginning of the subsection. Let $\hat{M}_L$ be a finite-projective $\hat{B}_L$-module that is obtained by base change from a finite projective B-module P. Then there is a bijection between*

1. *Descent data of $\hat{M}_L$ over B.*

2. *Continuous cocycles $\Gamma \to \mathrm{Aut}_{\hat{B}_L}(\hat{M}_L)$ up to being cohomologous.*

Under this equivalence, a descent datum is effective if and only if its corresponding continuous cocycle is cohomologous to a cocycle vanishing on an open normal subgroup of Γ*.*

Note that if the descent datum is effective, so that there exists a B-module M such that $\hat{M}_L = P \otimes_B \hat{B}_L$ with its descent datum is obtained from M via pullback, this does not mean that P is isomorphic to M. We just need P as a "model" of $P \otimes_B \hat{B}_L$ because all the cocycle theory established above needs a B-algebra such as $\mathcal{A} = \mathrm{End}_B(P)$ to be formulated.

Proof. The first part follows from the isomorphism of groupoids

$$\hat{C}_L^{\mathrm{cocl}} \cong \hat{D}_L^{\mathrm{cocl}}$$

in Theorem 4.7 (3) applied to $\mathcal{A} = \mathrm{End}_B(P)$. The last statement is Remark 4.8 (1) $\iff$ (2). □

4.2 A descent result via Sen operators

Heavily inspired by the proof of [PR21, Theorem 2.3.5], we want to use the results of [Sen93] and the translations proved in the last subsection to prove Theorem 4.10. Roughly speaking, it tells us when effectivity of a descent datum of vector bundles along certain pro-étale covers of adic spaces can be tested both generically and pointwise. Fix:

- K a discretely valued p-adic field,
- K_∞/K an infinite completely ramified Galois extension,
- $\Gamma := \mathrm{Gal}(K_\infty/K)$ the Galois group,
- $\hat{K}_\infty$ the completion,
- (B, B^+) a reduced Tate Huber pair over $(K, \mathcal{O}_K)$,
- $B_\infty := B \otimes_K K_\infty$, $B_\infty^+ := B^+ \otimes_{\mathcal{O}_K} \mathcal{O}_{K_\infty}$. These are non-complete topological rings.
- $(\hat{B}_\infty, \hat{B}_\infty^+) := (B, B^+) \hat{\otimes}_{(K, \mathcal{O}_K)} (\hat{K}_\infty, \hat{\mathcal{O}}_{K_\infty})$ the completed version,
- $\hat{M}_\infty$ a finite-projective $\hat{B}_\infty$-module,
- A completed descent datum of $\hat{M}_\infty$ over B, i.e., an isomorphism β between the two pullbacks of $\hat{M}_\infty$ to
$$\hat{B}_\infty \hat{\otimes}_B \hat{B}_\infty$$
satisfying a cocycle condition over
$$\hat{B}_\infty \hat{\otimes}_B \hat{B}_\infty \hat{\otimes}_B \hat{B}_\infty.$$

Theorem 4.10. *Assume there exists an open dense subset $U \subseteq \mathrm{Spa}(B, B^+)$ such that for each $x \in U$ with residue field $\kappa(x)$, the $\hat{\kappa}(x)_\infty = \kappa(x)\hat{\otimes}_B \hat{B}_\infty$-module*[17]

$$\hat{M}_\infty \otimes_{\hat{B}_\infty} \hat{\kappa}(x)_\infty,$$

with descend isomorphism β_x obtained from the one over B, descends to a $\kappa(x)$-module M_x. Then $\hat{M}_\infty$ with its descent datum descends to a B-module M.

Remark 4.11. 1. Beware that the descent datum is still very much given over all of $\mathrm{Spa}(B, B^+)$, it is just *effectivity* which we can be tested pointwise and generically.

2. Also note that if B is sousperfectoid and $\hat{B}_\infty$ is perfectoid (which is the situation where we will apply this result), then by Proposition 3.12 (1) the functor from finite projective B-modules to descent data on B_∞ is fully-faithful. For more general B and $\hat{B}_\infty$, I am not sure whether this remains true.

3. Full-faithfulness of the pullback functor for $\kappa(x)$ in place of B would also imply that for every x the isomorphism

$$(M \otimes_B \kappa(x)) \otimes_{\kappa(x)} \hat{\kappa}(x)_\infty \xrightarrow{\sim} M_x \otimes_{\kappa(x)} \hat{\kappa}(x)_\infty$$

together with the descent data on either side descend to an isomorphism of $\kappa(x)$-vector spaces

$$M \otimes_B \kappa(x) \xrightarrow{\sim} M_x,$$

so that our big descended module M would specialize to all the M_x. However, this is neither clear from our proof nor will it be relevant for us. For the purposes of the theorem, the pointwise descent is really used as a property, not as a datum.

We can assume that there is always a "model" of M_∞ over some finite subextension, by which we mean the following:

Lemma 4.12. *There is a finite subextension $K \subset K' \subset L$ and a finite projective $B' = B \otimes_K K'$-module P such that*

$$\hat{M}_\infty \cong P \otimes_{B'} \hat{B}_\infty.$$

Note that the isomorphism $\hat{M}_\infty \cong P \otimes_{B'} \hat{B}_\infty$ does not have anything to do with our descent isomorphism β, so it is still far from solving the descent problem.

Proof. As a first step, we want to find a finite projective B_∞-module P_∞ whose completion (in other words, its base change to $\hat{B}_\infty$) is isomorphic to $\hat{M}_\infty$. [BC22, Corollary 2.1.22 (c)] gives us such a P_∞, under the condition that B_∞ contains a non-unital open subring B_∞^I such that:

1. B_∞^I is *henselian* as a non-unital ring. This property is discussed in [BC22, p. 2.1.1], but a sufficient condition for a non-unital ring to be henselian is if it can be realized as the ideal of a henselian pair.

[17] As in the previous subsection, we don't need to complete our tensor products when the module is finitely presented.

2. The subspace topology on B_∞^I is linear, i.e. admits a neighbourhood basis of 0 consisting of ideals in B_∞^I.

We will show that these properties are satisfied for $B_\infty^I := \varpi B_{\infty,0}$, where $\varpi \in B$ is a pseudouniformizer (it will then automatically be a pseudouniformizer for B_∞) and $B_{\infty,0} \subset B_\infty$ is a ring of definition, which we assume to be obtained by base change from a ring of definition B_0 of B. Now $\varpi B_{\infty,0}$ clearly satisfies the second property above. For the henselian property, note that $(B_0, \varpi B_0)$ is a henselian pair since B_0 is ϖ-adically complete. But $B_0 \to B_{\infty,0}$ is the base change of the integral morphism $\mathcal{O}_K \to \mathcal{O}_{K_\infty}$ and hence and integral morphism itself, and henselian pairs get preserved by integral morphisms, so $(B_{\infty,0}, \varpi B_{\infty,0})$ is a henselian pair and thus $\varpi B_{\infty,0}$ is henselian in the above sense.

This proves the existence of P_∞. Going further, we have an isomorphism

$$B_\infty = \varinjlim_{K \subseteq K_n \subset L} B \otimes_K K_n,$$

with the colimit running over all finite subextensions. This translates into a filtered colimit of the (equivalence classes of) finite projective modules, with the transition maps given by base change: As each finite projective B_∞-module can be described by finitely many generators and relations, there must be some $B_n := B \otimes_K K_n$ where all or these relations can be defined. Thus we find a B_n and a finite projective B_n-module P which after base change to B_∞ becomes isomorphic to P_∞, and after completing becomes isomorphic to $\hat{M}_\infty$, as desired. □

Remark 4.13. Note that in order to descend $\hat{M}_\infty$ down to B it is actually enough just to descend it to $B_n := B \otimes_K K_n$ for some finite subextension $K \subseteq K_n \subset K_\infty$: Once we have descended our $\hat{M}_\infty$ with its descent datum down to B_n, we can use étale descent to descend it all the way down to B. Combining this with Lemma 4.12, we can from now on replace K by K_n and B by B_n. This allows us to assume that $\hat{M}_\infty$ is actually obtained as a completed base change of a finite projective B-modules P.

In §2.3 [Sen93], Sen developed a theory to test triviality of a cocycle on open normal subgroups by a vanishing of a certain endomorphism, in fact in much more generality than is needed here. Let's briefly sketch his construction. Let $\alpha\colon \Gamma \to \mathrm{End}_{\hat{B}_\infty}(\hat{M}_\infty)$ be a continuous cocycle. One then finds some finite subextension $K \subseteq K' \subset K_\infty$ and a continuous cocycle $\rho\colon \mathrm{Gal}(K_\infty/K') \to \mathrm{End}_B(P) \otimes_B K'$ such that the composition

$$\mathrm{Gal}(K_\infty/K') \xrightarrow{\rho} \mathrm{End}_B(P) \otimes_B K' \hookrightarrow \mathrm{End}_{\hat{B}_\infty}(\hat{M}_\infty)$$

is cohomologous to $\alpha|_{\mathrm{Gal}(K_\infty/K')}$, i.e. $\rho(\sigma) = m^{-1}\alpha(\sigma)\sigma(m)$ for some $m \in \mathrm{Aut}_{\hat{B}_\infty}(\hat{M}_\infty)$ and all $\sigma \in \mathrm{Gal}(K_\infty/K')$. One then defines an operator

$$\varphi := \varphi_\alpha := \lim_{\mathrm{Gal}(K_\infty/K') \ni \sigma \to 1} \frac{m \log \rho(\sigma) m^{-1}}{\log \chi(\sigma)},$$

where log is the p-adic logarithm and $\chi(\sigma)\colon \mathrm{Gal}(K_\infty/K) \xrightarrow{\sim} \mathbb{Z}_p^*$ is the p-adic cyclotomic character.

Theorem 4.14. (Sen) *This construction enjoys the following properties:*

- φ *exists and is independent of the choices made.*
- φ *is compatible with base change in* B.
- $\varphi = 0$ *if and only if* α *is cohomologous to the trivial cocycle on some open normal subgroup of* Γ.

The first and third property are proven by Sen in [Sen93], the second one is clear from the construction.

Proof. (of Theorem 4.10): Let α be the cocycle associated to our descent isomorphism β via Corollary 4.9, and consider its Sen operator φ_α. By our assumption, we know that in any $x \in U$, the base changed datum $(\hat{M}_\infty \otimes_B \hat{\kappa}(x)_\infty, \beta_x)$ descends to some M_x. But this effectivity implies that the cocycle α is trivial on an open normal subgroup by the second part of Corollary 4.9 and hence by Theorem 4.14, $\varphi_{\alpha_x} = (\varphi_\alpha)_x$ vanishes. So φ_α is an endomorphism of a finite projective module vanishing pointwise on an open dense subset of the adic space associated to a reduced algebra, and it thus vanishes identically. Another application of Theorem 4.14 now implies that α is cohomologous to zero on an open normal subgroup of Γ and thus the reverse direction of the second part of Corollary 4.9 implies that descent is effective. $\square$

4.3 The equivalence away from $p = 0$

The goal of this subsection is to use the Sen formalism established above to prove the following proposition, which can be regarded as a rational version of our main theorem under extra assumptions on A^+[18]:

Proposition 4.15. *Let* A^+ *be an integral perfectoid topological ring with pseudouniformizer* ϖ *such that* $(A, A^+) := (A^+[1/\varpi], A^+)$ *is a perfectoid Huber pair. Then the functor*

$$\text{vector bundles on } \operatorname{Spa}(A^+) \dot{\times} \operatorname{Spa}(\mathbb{Q}_p) \longrightarrow \text{rational } \mathcal{Y}\text{-bundles on } \operatorname{Spd}(A^+),$$

is an equivalence of categories.

Proof. Full-faithfulness follows formally from full-faithfulness of the functors from vector bundles to v-bundles ($\operatorname{Spa}(A^+) \dot{\times} \operatorname{Spa}(\mathbb{Q}_p) = \mathcal{Y}_{(0,\infty]}(A, A^+)$ is sousperfectoid) and from $\mathcal{Y}$-bundles to v-bundles, Proposition 3.12 (1) and (2), respectively. For essential surjectivity, we use our Sen theoretic formalism. $\mathcal{Y}_{(0,\infty]}(A, A^+)$ has an analytic cover by

$$U^m := \operatorname{Spa}(B^m, B^{m+}) := \mathcal{Y}_{(0,\infty]}(A, A^+)_{[\varpi^\flat] \leq p^m \neq 0} \to \mathcal{Y}_{(0,\infty]}(A, A^+)$$

[18] This extra assumption guarantees that p gets inverted in a transcendental way in $\operatorname{Spa}(A^+) \dot{\times} \operatorname{Spa}(\mathbb{Q}_p)$. Dropping this assumption requires some subtle reflections on meromorphicity conditions as in [GI23]; in [Ans23], those reflections are somewhat simpler as only single points $\operatorname{Spec}(k)$ are considered there.

with
$$B^m = \mathrm{Spa}(W(R^+)\langle \frac{[\varpi^\flat]}{p^m} \rangle.$$
In the notation of [SW20], $U^m = \mathcal{Y}_{[1/m,\infty]}$. In turn, every U^m has a pro-étale cover by an affinoid perfectoid
$$U^m_\infty = \mathrm{Spa}(\hat{B}^m_\infty, \hat{B}^{m+}_\infty) \to U^m$$
with
$$\hat{B}^{m+}_\infty := B^{m+} \hat{\otimes}_{\mathbb{Z}_p} \hat{\mathcal{O}}_\infty, \quad \hat{B}^m_\infty := \hat{B}^{m+}_\infty[1/p],$$
where as usual $\hat{\mathcal{O}}_\infty = \mathbb{Z}_p[p^{1/p^\infty}]^{\wedge p}$.

By full-faithfulness of the functor to v-bundles, Proposition 3.12 (2), it is enough to show that if $\mathcal{E}$ is a v-bundle on $\mathcal{Y}_{(0,\infty]}(A, A^+)$ that lies in the essential image of the functor from $\mathcal{Y}$-bundles, then it also lies in the essential image of the functor from vector bundles. Now we can evaluate the v-bundle $\mathcal{E}$ at the affinoid perfectoid U^m_∞ to get an actual vector bundle $\hat{M}^m_\infty = \mathcal{E}|_{U^m_\infty}$ on U^m_∞ with (completed) descent information over U^m, in other words: A finite projective $\hat{B}^m_\infty$-module, also denoted $\hat{M}^m_\infty$ with (completed) descent information over B^m. By Theorem 4.10, it is enough to show the descent is effective on the open dense subset $U^m - \{[\varpi^\flat] = 0\}$. But as $\mathcal{E}$ lies in the essential image of the functor from rational $\mathcal{Y}$-bundles, we can simply evaluate it at (A, A^+) to get an actual vector bundle on $\mathcal{Y}_{(0,\infty)}(A, A^+)$, and $\mathcal{Y}_{(0,\infty)}(A, A^+) \cap U^m = U^m - \{[\varpi^\flat] = 0\}$, so $\hat{M}_\infty$ descends to a finite projective B-modules M. The v-bundle associated to M is of course still $\mathcal{E}|_{U^m}$, and using full-faithfulness of the functor from vector bundles to v-bundles for sousperfectoids again, they glue for varying m to a single vector bundle over $\mathcal{Y}_{(0,\infty]}(A, A^+)$ whose associated v-bundle is $\mathcal{E}$. $\square$

Remark 4.16. Note that we have only really used that we can test effectivity of the descent *generically*; on the other hand, in [PR21], it was only used that we can test effectivity *pointwise*.

4.4 Finishing the proof

Let's finally formulate and proof our main theorem:

Construction 4.17. Let R^+ be any topological ring such that (R^+, R^+) is a Huber pair (i.e. R^+ is p-complete by convention). Then there is a functor
$$\mathcal{O}^{\mathrm{perf}}_{\mathbb{\Delta}}\text{-bundles on } R^+ \to \text{integral } \mathcal{Y}\text{-bundles on } \mathrm{Spd}(R^+)$$
defined as follows: For any map from a representable $\mathrm{Spa}(B, B^+) \to \mathrm{Spd}(R^+)$, we get an untilt $(B^\sharp, B^{+\sharp})$ with a map $R^+ \to B^{+\sharp}$. As $B^{+\sharp}$ itself is integral perfectoid, an $\mathcal{O}^{\mathrm{perf}}_{\mathbb{\Delta}}$-bundle defines a vector bundle on $A_{\mathrm{inf}}(B^{+\sharp}) = W(B^+)$. Base changing this along the map of adic spaces
$$\mathcal{Y}_{[0,\infty)}(B, B^+) \to \mathrm{Spa}(A_{\mathrm{inf}}(B^+))$$
gives us a vector bundle on the former, which, upon varying (B, B^+), is exactly what we need to define a $\mathcal{Y}$-bundle on $\mathrm{Spd}(R^+)$.

Theorem 4.18. *Let R^+ be a complete topological ring carrying the Π-adic topology for some $\Pi \in R^+$ dividing p. Then the functor*

$$\mathcal{O}_{\mathbb{A}}^{\mathrm{perf}}\text{-bundles on } R^+ \to \text{integral } \mathcal{Y}\text{-bundles on } \operatorname{Spd}(R^+)$$

constructed above is an exact tensor equivalence. In particular, if R^+ is integral perfectoid, finite projective $A_{\mathrm{inf}}(R^+)$-modules are in equivalence to integral $\mathcal{Y}$-bundles on $\operatorname{Spd}(R^+)$ by Corollary 2.10.

Remark 4.19. Since the left side does not depend on the choice of topology on R^+, i.e., the choice of Π, neither does the right side. This came as quite a surprise to me, and frustratingly I haven't been able to come up with a more direct way to see this, even in simple cases.

Because of the exactness and the equivalence of the two definitions in Definition 2.15 resp. Definition + Proposition 3.14, we will get the following for free:

Corollary 4.20. *For any affine flat group scheme $\mathcal{G}/\mathbb{Z}_p$, the construction above defines an equivalence of groupoids*

$$\mathcal{G}\text{-}\mathcal{O}_{\mathbb{A}}^{\mathrm{perf}}\text{-bundles on } R^+ \to \text{integral } \mathcal{G}\text{-}\mathcal{Y}\text{-bundles on } \operatorname{Spd}(R^+).$$

To prove this, we first need a result to reduce the situation to products of points. This is the refinement of [ČS21, Lemma 2.2.3] alluded to in section 1.

Lemma 4.21. *Let R^+ be as in Theorem 4.18. Then there is a topological R^+-algebra $R^+ \to A^+$ such that:*

1. *A^+ is a product of valuation rings V_i with algebraically closed field of fractions.*
2. *A^+ is a complete topological ring carrying the $(\varpi_i)_{i\in I}$-adic topology, where $\varpi_i \in V_i$ is some non-zero divisor dividing Π. In particular, A^+ is a Π-adically complete integral perfectoid.*
3. *$R^+ \to A^+$ is a Π-complete arc cover of rings in the sense of Definition 2.1.*
4. *$\operatorname{Spd}(A^+) \to \operatorname{Spd}(R^+)$ is a v-cover of v-sheaves.*

In fact, for the obvious definition of (schematic) Π-complete v-topology on Π-complete rings, the proof even shows that $R^+ \to A^+$ is a Π-complete v-cover, but we only work with the Π-complete arc topology on the schematic side as this seems to be the most natural choice.

Proof. Consider the set[19] I of equivalence classes of maps $R^+ \to V$ with V a Π-complete valuation ring. In each of these equivalence classes we can find a smallest valuation ring by taking any representative $v \colon R^+ \to V$ and considering

$$\operatorname{Frac}(\operatorname{Im}(v)) \cap V \subset \operatorname{Frac}(V),$$

[19] The argument for finding the smallest valuation ring in each equivalence class just below shows that each such class has a representative which is $\kappa := \max(\omega, |R^+|)$-small, so the equivalence classes indeed form a set, not a proper class.

with $\mathrm{Im}(v)$ the image of v.

For every $i \in I$ let $\tilde{V}_i$ be the Π-completion of the absolute integral closure of the associated smallest valuation ring as above. If Π gets mapped to a non-zero element in $\tilde{V}_i$, we can set $V_i = \tilde{V}_i$ and $\varpi_i \in V_i$ the image of Π. If Π gets mapped to zero[20], then this choice would not satisfy (2), so we define V_i as the t-adic completion of the absolute integral closure of

$$\tilde{V}_i + t\,\mathrm{Frac}(\tilde{V}_i)[[t]] \subset \mathrm{Frac}(\tilde{V}_i)[[t]]$$

and set $\varpi_i := t$. The essential aspect of this construction is that it introduces a new element t which has smaller valuation than any element in $\tilde{V}_i$. Indeed, it can be easily read off from the construction that t is divided by any non-zero element of $\tilde{V}_i$. The result is a valuation ring with all the properties we demand and such that

$$V_i / \mathrm{rad}(\varpi_i) = \tilde{V}_i,$$

so we get a splitting

$$(*) \qquad \prod_i \tilde{V}_i \to \prod_i V_I \to \prod_i \tilde{V}_i,$$

continuous for the Π-adic resp. $(\varpi_i)_i$-adic topology.

So far we have produced a topological R^+-algebra $A^+ = \prod_i V_i$ satisfying properties (1) and (2) (for the in particular statement in (2), note that we can replace ϖ_i by a p'th power root by absolute integral closedness to get an actual pseudouniformizer without changing the topology). For (3), note that every map $R^+ \to V$ to a Π-complete valuation ring V (so in particular to one of rank one) factors over some $\tilde{V}_i$, at least after exchanging V for its Π-completed absolute integral closure, which is an extension of valuation rings. Via projection to the i'th component we get a factorization over $R^+ \to \prod_i \tilde{V}_i$, and precomposing this with our projection $(*)$ we get a lift to $\prod_i V_i$, showing that $R^+ \to A^+$ is indeed a Π-complete arc cover.

Finally, for (4) it suffices to show that for all maps to products of points in the adic spaces sense

$$(R^+, R^+) \to \left((\prod_{j \in J} C_j^+)[((\varpi_j)_{j \in J})^{-1}], \prod_{j \in J} C_j^+ \right) =: (S, S^+)$$

with $C_j^+[1/\varpi_j]$ algebraically closed there exists a lift of maps of Huber pairs

$$\begin{tikzcd}
(A^+, A^+) \arrow[dr, dotted] & \\
(R^+, R^+) \arrow[u] \arrow[r] & (S, S^+).
\end{tikzcd}$$

[20] as $\Pi | p$, this can only happen if $p = 0$ in $\tilde{V}_i$

By what we have seen in (3), we can lift every map $R^+ \to C_j^+$ to a map of rings $\prod_i \tilde{V}_i \to C_j^+$, producing a map of pairs of rings

$$(\prod_i \tilde{V}_i, \prod_i \tilde{V}_i) \to (S^+, S^+) \to (S, S^+).$$

To see that this map is continuous, note that the topology on $\prod_i \tilde{V}_i$ is generated by the image of Π. Hence we need to show that $(\varpi_j)_j \in S^+$ divides a power of the image of Π under the composition $R^+ \to A^+ \to S^+$, which is of course the same as the image of Π under the map $R^+ \to S^+$. But this is exactly the condition for $R^+ \to S^+$ to be continuous. Finally, precomposing with the section in $(*)$ once again gives our desired lift to $(\prod_i V_i, \prod_i V_i) = (A^+, A^+)$. $\square$

Remark 4.22. Note that for R^+ perfectoid, the categories in Theorem 4.18 depend only on the tilt $R^{+\flat}$ (this will change in the next subsection when we also need to consider the position of the leg). Thus, we could henceforth replace A^+ by its tilt $A^{+\flat}$ which is perfect of characteristic p, and conceptually this is probably the right way to view it. But since this does not seem to simplify the proofs, we will not do it.

The following was proved by Gleason, building on a result by Kedlaya:

Lemma 4.23. ([Gle21, Proposition 2.1.17]) *Let A^+ be as in the previous Lemma. Then every vector bundle on $\mathcal{Y}_{[0,\infty]}(A, A^+)$ is free. In particular, as*

$$\Gamma(\mathcal{Y}_{[0,\infty]}(A, A^+), \mathcal{O}_{\mathcal{Y}_{[0,\infty]}(A,A^+)}) = A_{\mathrm{inf}}(A^+),$$

the base change functor from vector bundles on $\mathrm{Spa}(A_{\mathrm{inf}}(A^+))$ *to those on* $\mathcal{Y}_{[0,\infty]}(A, A^+)$ *is a (non-exact) equivalence of categories.*

Construction 4.24. Now let A^+ be any integral perfectoid, equipped with and complete with respect to the ϖ-adic topology for some non-zero divisor $\varpi \in A^+$ dividing p. We want to construct a functor from the category of integral $\mathcal{Y}$-bundles on $\mathrm{Spd}(A^+)$ to the category of vector bundles on the adic space $\mathcal{Y}_{[0,\infty]}(A, A^+)$, where of course $A := A^+[1/\varpi]$. Let $\mathcal{E}$ be such an integral $\mathcal{Y}$-bundle. We get

1. a rational $\mathcal{Y}$-bundles $\mathcal{E}_{(0,\infty]}$ on $\mathrm{Spd}(A^+)$,
2. an integral $\mathcal{Y}$-bundle $\mathcal{E}_{[0,\infty)}$ on $\mathrm{Spd}(A, A^+)$,
3. an isomorphism of rational $\mathcal{Y}$-bundles on $\mathrm{Spd}(A, A^+)$ between the relevant base changes of $\mathcal{E}_{(0,\infty]}$ and $\mathcal{E}_{[0,\infty)}$

by, respectively, Construction 3.10 (4), pullback of $\mathcal{Y}$ bundles along maps of v-sheaves and compatibility of those two constructions, mentioned in the end of Construction 3.10. By separately passing all these through the equivalences of categories in Proposition 3.12 (3) and Proposition 4.15, we get:

1. A vector bundle on $\mathcal{Y}_{(0,\infty]}(A, A^+)$,
2. A vector bundle on $\mathcal{Y}_{[0,\infty)}(A, A^+)$,
3. An isomorphism between the two restrictions to $\mathcal{Y}_{(0,\infty)}(A, A^+)$,

which by the usual formalism of gluing vector bundles on adic spaces form a vector bundle on $\mathcal{Y}_{[0,\infty]}(A, A^+)$.

Proposition 4.25. *Let A^+ be again as in Lemma 4.21. Then the functor just constructed is an equivalence of categories.*

Before embarking on the proof, let's quickly remind ourselves of the notation for rational subsets used in e.g. [SW20]: For $X = \mathrm{Spa}(A, A^+)$ an affinoid adic space, $T = \{t_1, \ldots, t_n\} \subset A$ such that $TA^+ \subseteq A$ is open and any $s \in A$, one has the rational subset

$$X_{T \leq s \neq 0} = \mathrm{Spa}\left(A\left\langle \frac{T}{s} \right\rangle, A\left\langle \frac{T}{s} \right\rangle^+\right)$$

with

$$A\left\langle \frac{T}{s} \right\rangle^+ = A^+[\frac{t_1}{s}, \ldots, \frac{t_n}{s}]^\wedge, \qquad A\left\langle \frac{T}{s} \right\rangle = A\left\langle \frac{T}{s} \right\rangle^+ [1/s].$$

Note that this notation is very slightly abusive, as $A\left\langle \frac{T}{s} \right\rangle^+$ depends on A^+, not A.

Proof. Let's first do essential surjectivity: By Lemma 4.23, every vector bundle on $\mathcal{Y}_{[0,\infty]}(A, A^+)$ is free, so it is isomorphic to the image of the free integral $\mathcal{Y}$-bundle of the same rank.

Full-faithfullness is more subtle. After some reformulation steps regarding the v-locality of the situation, we will reduce it to invariance of global sections under removal of a codimension two subset. Write

$$X = \mathrm{Spa}(W(A^+))$$

and

$$Y = \mathcal{Y}_{[0,\infty]}(A, A^+) = X - V(p, \varpi).$$

First note that the functor from integral $\mathcal{Y}$-bundles on $\mathrm{Spd}(A^+)$ to v-bundles on $\mathrm{Spd}(W(A^+))$ is fully faithful (because it always is by Proposition 3.12(2)), and the same is true for the functor from vector bundles on Y to v-bundles on Y (by Proposition 3.12(1) and sousperfectoidness of Y).

We get a commutative diagram of categories

$$\begin{tikzcd}
\text{integral } \mathcal{Y}\text{-bundles on } \mathrm{Spd}(A^+) \arrow[r] \arrow[d, hook] & \text{vector bundles on } Y \arrow[d, hook] \\
\text{v-bundles on } X^\diamond \arrow[r] & \text{v-bundles on } Y^\diamond
\end{tikzcd}$$

with the top arrow the functor we are interested in, the bottom arrow being the restriction map of v-bundles and fully-faithful vertical arrows. Thus to test full-faithfulness for the top arrow we can test it for the bottom arrow for all elements lying in the essential image of the left downward arrow. By passing to internal homs, it suffices to prove the following:

Reformulation 1. For any v-bundle $\mathcal{E}$ on $X^\diamond$ lying in the essential image of the functor from integral $\mathcal{Y}$-bundles[21] on $\mathrm{Spd}(A^+)$, the restriction map

$$H_v^0(X, \mathcal{E}) \to H_v^0(Y, \mathcal{E})$$

is a bijection.

This can be clearly tested v-locally, more precisely: Given a v-cover $\tilde{X} \to X$ defining a pullback square

$$\begin{tikzcd} \tilde{Y} \arrow[r] \arrow[d] & \tilde{X} \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

and $\mathcal{E}$ as in the reformulation above, we need to show that the restriction functor

$$H_0^v(\tilde{X}, \mathcal{E}|_{\tilde{X}}) \to H_0^v(\tilde{Y}, \mathcal{E}|_{\tilde{Y}})$$

is a bijection. The reason why this is useful is that we can choose $\tilde{X}$ to be of the form $\mathrm{Spa}(\mathbb{Z}_p)\dot{\times}T$ for some perfectoid space T. Then $\mathcal{E}|_{\tilde{X}}$ lying in the essential image of the functor from integral $\mathcal{Y}$-bundles implies that it actually lies in the essential image of the functor from vector bundles (by Proposition 3.12(3)) and both $\tilde{X}$ and $\tilde{Y}$ will be sousperfectoid, so the functors from vector bundles to v-bundles are fully-faithful, and we can reformulate our problem again as

Reformulation 2. For $\tilde{X}$ chosen of the form above and any vector bundle $\mathcal{E}$ on $\tilde{X}$, the restriction map

$$H_0(\tilde{X}, \mathcal{E}) \to H_0(\tilde{Y}, \mathcal{E})$$

is a bijection.

Now this is something we can finally calculate. Construct $\tilde{X} \to X$ as follows: Define $\tilde{A}^+ := A^+[[t^{1/p^\infty}]]$. Then $\mathrm{Spd}(\tilde{A}^+) \to X = \mathrm{Spd}(A^+)$ is a v-cover of v-sheaves. Now $\mathrm{Spa}(\tilde{A}^+)$ is not analytic, but $\mathrm{Spa}(\tilde{A}^+) - V(\varpi, t)$ is, and it still covers X — This is exactly as in the proof of Lemma 3.3, just with added p-power roots of t. Now

$$U := \mathrm{Spa}(\tilde{A}^+) - V(\varpi, t).$$

has a rational open cover by the two affinoid perfectoid subsets $U_{t \le \varpi \ne 0}$ and $U_{\varpi \le t \ne 0}$. Define

[21] It seems plausible that this condition could be unnecessary. Proving this would require us to choose the cover $\tilde{X}$ below as a perfectoid space, instead of $\mathrm{Spa}(\mathbb{Z}_p)\dot{\times}T$ for a perfectoid space T.

$$\tilde{X} := (U_{t\leq\varpi\neq 0} \sqcup U_{\varpi\leq t\neq 0}) \dot{\times} \operatorname{Spa}(\mathbb{Z}_p) = U_{t\leq\varpi\neq 0} \dot{\times} \operatorname{Spa}(\mathbb{Z}_p) \sqcup U_{\varpi\leq t\neq 0} \dot{\times} \operatorname{Spa}(\mathbb{Z}_p).$$

Now $\tilde{Y}$ is obtained from $\tilde{X}$ by removing $V(p,\varpi)$. As ϖ vanishes nowhere on

$$U_{t\leq\varpi\neq 0} \dot{\times} \operatorname{Spa}(\mathbb{Z}_p) \subset \tilde{X},$$

this space is already contained in $\tilde{Y}$. Thus it is enough to show equality of global sections of $\mathcal{E}$ when passing from $U_{\varpi\leq t\neq 0} \dot{\times} \operatorname{Spa}(\mathbb{Z}_p)$ to $(U_{\varpi\leq t\neq 0} \dot{\times} \operatorname{Spa}(\mathbb{Z}_p)) - V(p,\varpi)$. Now the former is not affinoid, but rather it has a cover by rational open affinoid subsets[22]

$$\operatorname{Spa}(C_n, C_n^+) := (U_{\varpi\leq t\neq 0} \dot{\times} \operatorname{Spa}\mathbb{Z}_p)_{p^n\leq [t]\neq 0},$$

so we need to show that the restriction map

$$\Gamma(\operatorname{Spa}(C_n, C_n^+), \mathcal{E}) \to \Gamma(\operatorname{Spa}(C_n, C_n^+) - V(p, [\varpi]), \mathcal{E})$$

is a bijection for each n. For readability, let's fix n, write $q = p^n$ and omit n from the index. Now as taking Witt vectors commute with completions, one computes

$$C^+ = B^+[\frac{[\varpi]}{[t]}, \frac{q}{[t]}]^{\wedge t}, \qquad C = B^+\left\langle \frac{[\varpi]}{[t]}, \frac{q}{[t]} \right\rangle = C^+[t^{-1}]$$

for

$$B^+ := W(A^+[[t^{1/p^\infty}]]).$$

As $[t]$ is invertible in C one has $V(q,[\varpi]) = V(q/[t], [\varpi]/[t])$. Setting $a = q/[t]$ and $b = \varpi/[t]$, this places us in the situation of Proposition 4.27. Indeed, (C, C^+) is a Tate-Huber pair, both $q/[t]$ and $[\varpi]/[t]$ are non-zero divisors and as (t,a,b) is clearly a regular sequence in B^+, Lemma 4.26 below implies that $(q/[t], \varpi/[t])$ is a regular sequence in $C^+/([t])$. $\square$

Lemma 4.26. *Let B^+ be topological ring carrying the $(t, a_1, \ldots, a_n)$-adic topology for $(t, a_1, \ldots, a_n)$ a regular sequence in B^+. Then $(\frac{a_1}{t}, \ldots, \frac{a_n}{t})$ is a regular sequence in*

$$B\left\langle \frac{a_1}{t}, \ldots, \frac{a_n}{t} \right\rangle^+ /(t)$$

Proof. One has

$$\begin{aligned} B\left\langle \frac{a_1}{t}, \ldots, \frac{a_n}{t} \right\rangle^+ &= (B^+[X_1, \ldots, X_n]/I)^{\wedge(t,a_1,\ldots,a_n)} \\ &= (B^+[X_1 \ldots, X_n]/I)^{\wedge t} \end{aligned}$$

[22]One might want to denote $\operatorname{Spa}(C_n, C_n^+)$ by $\mathcal{Y}_{[0,n]}(U_{\varpi\leq t\neq 0})$, but we are not doing this here as it might cause confusion between ϖ (the pseudouniformizer in A) and t (the pseudouniformizer in the ring of global sections of $U_{\varpi\leq t\neq 0}$).

for

$$I = (tX_1 - a_1, \dots, tX_n - a_n),$$

with the two completions agreeing because in the localized ring t divides the a_i. Taking the quotient ring of this by t, one gets

$$(B^+/(t, a_1, \dots a_n))[X_1, \dots, X_n],$$

in which $(X_1, \dots, X_n) = (\frac{a_1}{t}, \dots, \frac{a_n}{t})$ is clearly a regular sequence. □

Proposition 4.27. *Let (C, C^+) be a sheafy Tate-Huber pair with pseudouniformizer t and let $a, b \in C^+$ be non-zero divisors such that (a, b) form a regular sequence in $C^+/(t)$ (in the classical sense, e.g. [Sta20, 0AUH]). Then for any vector bundle $\mathcal{E}$ on $Z := \operatorname{Spa}(C, C^+)$ one has*

$$\Gamma(Z, \mathcal{E}) = \Gamma(Z - V(a, b), \mathcal{E}).$$

Proof. First, by a standard affineness-argument, it is enough to show the statement for $\mathcal{E} = \mathcal{O}_Z$. Indeed, Z is affinoid and analytic, so $M \mapsto \big(U \mapsto (H^0(U, \mathcal{O}_Z) \otimes_C M)\big)$ defines an equivalence between finite projective C-modules and vector bundles on Z. Now if $(U_i)_i$ is an affinoid open cover of $\tilde{Z} := Z - V(a, b)$ we can compute

$$\begin{aligned}
H^0(\tilde{Z}, \mathcal{E}) &= \ker\left(\prod_i H^0(U_i, \mathcal{E}) \to \prod_{i,j} H^0(U_i \cap U_j, \mathcal{E})\right) \\
&= \ker\left(\prod_i H^0(U_i, \mathcal{O}_Z) \otimes_C M \to \prod_{i,j} H^0(U_i \cap U_j, \mathcal{O}_Z) \otimes_C M\right) \\
&= \ker\left(\prod_i H^0(U_i, \mathcal{O}_Z) \to \prod_{i,j} H^0(U_i \cap U_j, \mathcal{O}_Z)\right) \otimes_C M \\
&= H^0(\tilde{Z}, \mathcal{O}_Z) \otimes_C M,
\end{aligned}$$

where the third equality uses finite projectiveness of M to commute the tensor product with (possibly infinite) products and kernels.

Now cover $\tilde{Z}$ by the rational subsets[23]

$$U_n := \tilde{Z}_{t^n \le \{a, t^n\} \neq 0} = \tilde{Z}_{t^n \le a \neq 0} \qquad V_n := \tilde{Z}_{t^n \le \{b, t^n\} \neq 0} = \tilde{Z}_{t^n \le b \neq 0}$$

which have local sections

$$\Gamma(U_n, \mathcal{O}_Z) = C\left\langle \frac{t^n}{a} \right\rangle, \qquad \Gamma(V_n, \mathcal{O}_Z) = C\left\langle \frac{t^n}{b} \right\rangle.$$

[23]note that $\tilde{Z}_{a \le b} \neq 0$ is not a rational subset as $bC_0 \subset C_0$ is not open.

Now by the sheaf condition for Z, we have to show that any

$$f \in \Gamma(U_1 \cap V_1, \mathcal{O}_Z) = C\left\langle \frac{t}{a}, \frac{t}{b} \right\rangle$$

that lies in the image of each of the $\Gamma(U_n, \mathcal{O}_Z)$ and $\Gamma(V_n, \mathcal{O}_Z)$ has a unique preimage in C, or in symbols

$$\bigcap_{n=1}^{\infty} \left(C\left\langle \frac{t^n}{a} \right\rangle \cap C\left\langle \frac{t^n}{b} \right\rangle \right) = C,$$

with "$\cap$" from now on being a shorthand for the equalizer along the natural maps to $C\left\langle \frac{t}{a}, \frac{t}{b} \right\rangle$ and equality being interpreted as "isomorphic as rings under $C\left\langle \frac{t}{a}, \frac{t}{b} \right\rangle$". We proceed in several reduction steps:

1. It would be enough to show that

$$C\left\langle \frac{t}{a} \right\rangle^+ \cap C\left\langle \frac{t}{b} \right\rangle^+ = C$$

as all the maps $\Gamma(U_n, \mathcal{O}_Z) \to \Gamma(U_1 \cap V_1, \mathcal{O}_Z)$ factor over $\Gamma(U_1, \mathcal{O}_Z)$ injectively, and similarly for the V_n (by the assumption that a and b are non-zero divisors). This would amount to showing that

$$\Gamma(U_1 \cup V_1, \mathcal{O}_Z) = C,$$

which somewhat surprisingly already holds true.[24]

2. Furthermore, it suffices to show that

$$C\left\langle \frac{t}{a} \right\rangle^+ \cap C\left\langle \frac{t}{b} \right\rangle^+ = C^+.$$

Indeed, suppose we would find some

$$f \in C\left\langle \frac{t}{a} \right\rangle \cap C\left\langle \frac{t}{b} \right\rangle = C\left\langle \frac{t}{a} \right\rangle^+ [(ta)^{-1}] \cap C\left\langle \frac{t}{b} \right\rangle^+ [(tb)^{-1}].$$

not contained in $C = C^+[t^{-1}]$. Then multiplication with $(tab)^n$ for n sufficiently large would produce an element in

$$C\left\langle \frac{t}{a} \right\rangle^+ \cap C\left\langle \frac{t}{b} \right\rangle^+$$

not contained in C^+.

[24] The relevant picture here is the following: We want to exclude the subset $V(a, b)$ which should be thought of as a codimension two subset from Z, but due to the particular nature of adic spaces we cannot do this on the nose, but rather we need to approximate $Z - V(a)$ by "tubes" of the form $Z - U_n$, and similarly for $V(b)$ and $Z - V_n$. Now, surprisingly, just $U_1 \cup V_1$ is big enough to produce the same global sections as all of Z, which, when trying to draw a picture of the situation, would be missing a lot more than a codimension 2 subset.

3. The intersection we are looking at is a limit, so it commutes with the limit involved in the definition of the t-adic completion, so we can write

$$C\left\langle \frac{t}{a}\right\rangle^+ \cap C\left\langle \frac{t}{b}\right\rangle^+ = \lim_{n,m}\left(C^+[\frac{t}{a}]/t^n \cap C^+[\frac{t}{b}]/t^m\right).$$

Now using that we are only intersecting two (hence finitely many) rings we can even let the power of t grow uniformly and thus it is enough to show that

$$C^+[\frac{t}{a}]/t^n \cap C^+[\frac{t}{b}]/t^n = C^+/t^n$$

(with the intersection now taken inside $C^+[\frac{t}{a}, \frac{t}{b}]/t^n$) for all large enough n.

4. Now $(C^+[\frac{t}{a}])/t^n \subset (C^+[a^{-1}])/t^n = (C^+/t^n)[a^{-1}]$ and similarly for b, so it suffices to show that
$$(C^+/t^n)[a^{-1}] \cap (C^+/t^n)[b^{-1}] = C^+/t^n,$$
with the intersection now taken inside $(C^+/t^n)[(ab)^{-1}]$.

Finally, this last statement follows by a standard algebraic geometry argument from the condition of (a,b) being a regular sequence inside C^+/t^n (Recall that (t, a, b) is a regular sequence if and only if (t^n, a, b) is). Indeed, assume $f = g/a^l = h/b^k \in (C^+/t^n)[a^{-1}] \cap (C^+/t^n)[b^{-1}]$ and assume further $l \neq 0$ and $a \nmid g$. Then $gb^k = ha^k \in C^+/t^n$ gets sent to 0 inside $C^+/(t^n, a)$ by our assumption that $l \neq 0$, and because b is regular inside $C^+/(t^n, a)$ we must also have $g = 0 \in C^+/(t^n, a)$, so a divides g in C^+/t^n, contradicting our assumption. □

Proof. (of Theorem 4.18)
By our Lemma 4.21 and the descent results Proposition 2.13 and Proposition 3.9, we can immediately assume that $R^+ = A^+$ is a product of points in the sense of the Lemma 4.21, equipped with the ϖ-adic topology.

Just for this proof, consider the category "$\mathcal{Y}_{[0,\infty]}$-bundles on $\mathrm{Spd}(A, A^+)$" consisting of triplets of the form $(\mathcal{E}_{(0,\infty]}, \mathcal{E}_{[0,\infty)}, \alpha)$ where

1. $\mathcal{E}_{(0,\infty]}$ is a rational $\mathcal{Y}$-bundle on $\mathrm{Spd}(A^+)$,
2. $\mathcal{E}_{[0,\infty)}$ is an integral $\mathcal{Y}$-bundle on $\mathrm{Spd}(A, A^+)$,
3. α is an isomorphism of rational $\mathcal{Y}$-bundles on $\mathrm{Spd}(A, A^+)$ between the relevant restrictions of $\mathcal{E}_{(0,\infty]}$ and $\mathcal{E}_{[0,\infty)}$.

This is a fibre product of the three different categories of $\mathcal{Y}$-bundles in the obvious way and was already seen implicitly in Construction 4.24. I have only made it explicit here since I think it makes it much more transparent what's going on.

There is a commutative diagram of categories

$$\begin{array}{ccc} \text{vector bundles on } \mathrm{Spec}(A_{\mathrm{inf}}(A^+)) & \longrightarrow & \text{integral } \mathcal{Y}\text{-bundles on } \mathrm{Spd}(A^+) \\ \downarrow & & \downarrow \\ \text{vector bundles on } \mathcal{Y}_{[0,\infty]}(A, A^+) & \longrightarrow & \text{“}\mathcal{Y}_{[0,\infty]}\text{-bundles on } \mathrm{Spd}(A, A^+)\text{”}. \end{array}$$

The bottom arrow is an equivalence of categories by analytic gluing, Proposition 4.15 and Proposition 3.12 (3) – this was already used in Construction 4.24. The functor in Construction 4.24 is the composition of the right functor with the inverse of the bottom one and was shown to be an equivalence in Proposition 4.25. The left arrow is an equivalence by the “in particular” part of Lemma 4.23. Putting these together, we see that the top functor is an equivalence of categories.

This leaves us with exactness, i.e., that a sequence of $\mathcal{O}_{\Delta}^{\mathrm{perf}}$-bundles on A^+ is exact if and only if the associated sequence of $\mathcal{Y}$-bundles is. By the exactness results of the first chapters, we can assume without loss of generality that our ring A^+ still has the form as in Lemma 4.21. In this case by Corollary 2.10, exactness of $\mathcal{O}_{\Delta}^{\mathrm{perf}}$-bundles is the same as exactness of finite projective $A_{\mathrm{inf}}(A^+)$-modules, which already implies the “only if” direction of the statement. For the if direction, assume our sequence of $\mathcal{O}_{\Delta}^{\mathrm{perf}}$-modules is exact when viewed as $\mathcal{Y}$-bundles, i.e., there is a cover by a representable $\mathrm{Spa}(B, B^+) \to \mathrm{Spd}(A_{\mathrm{inf}}(A^+))$ such that the associated sequence of vector bundles on $\mathcal{Y}_{[0,\infty)}(B, B^+)$ is exact. We need to show that already the sequence of $A_{\mathrm{inf}}(A^+)$-modules is exact. This is very similar to the proof of Definition + Proposition 3.14: The map of underlying topological spaces of

$$\mathcal{Y}_{[0,\infty)}(B, B^+) = \mathrm{Spa}(\mathbb{Z}_p) \dot{\times} \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(\mathbb{Z}_p) \dot{\times} \mathrm{Spa}(A^+) = \mathrm{Spa}(A_{\mathrm{inf}}(A^+))$$

is surjective, so the exactness follows as $\mathrm{Spa}(A_{\mathrm{inf}}(A^+))$ is reduced. □

4.5 Equivalence of Frobenius structures

Construction 4.28. The functor in Construction 4.17 extends to a functor

$$\text{perfect-prismatic } F\text{-crystals on } R^+ \to \mathrm{GL}_n\text{-shtukas with fixed leg on } \mathrm{Spd}(R^+).$$

Indeed, the map

$$\mathcal{Y}_{[0,\infty)}(B, B^+) \to \mathrm{Spa}(A_{\mathrm{inf}}(B^+))$$

from the previous construction commutes with Frobenii and sends the distinguished element $d \in A_{\mathrm{inf}}(B^+)$ to a distinguished element ξ in the global sections of $\mathcal{Y}_{[0,\infty)}(B, B^+)$, so we can pull back our Frobenius-linear isomorphism.

We want to show that this is still an equivalence. In fact, let's state it directly for general groups $\mathcal{G}$:

Theorem 4.29. *Let $\mathcal{G}$ be any affine algebraic group over $\mathbb{Z}_p$ and let R^+ be a complete topological ring carrying the Π-adic topology for some $\Pi \in R^+$ dividing p. Then the functor defined above induces an equivalence of groupoids*

$$\text{perfect-prismatic } \mathcal{G}\text{-crystals on } R^+ \to \mathcal{G}\text{-shtukas with fixed leg on } \mathrm{Spd}(R^+).$$

Proof. As before, it suffices to show what we have an exact tensor equivalence for $\mathcal{G} = \mathrm{GL}_n$. We have already showed the exact tensor equivalence of underlying bundles in Theorem 4.18, so it suffices to show that we can descend the Frobenius linear morphisms. By our simultaneous v- and arc-covers from Lemma 4.21 in conjunction with the descent results Proposition 2.21 and Proposition 3.22, it suffices to show this for rings A^+ as in Lemma 4.21. In this case, our $\mathcal{O}_{\mathbb{\Delta}}^{\mathrm{perf}}$-bundle $\mathcal{E}$ is known to be a free $A_{\mathrm{inf}}(A^+)$-module, and so are all the $\mathcal{E}|_{(B,B^+)}$ for maps from affinoid perfectoids $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A^+)$, and we can even equip them with compatible bases (by pulling back the one on $A_{\mathrm{inf}}(A^+)$), so our shtuka structure is given by a compatible (with pullbacks) family of invertible matrices

$$M_{(B,B^+)} \in \mathrm{GL}_n(\Gamma(\mathcal{Y}_{[0,\infty)}(B, B^+), \mathcal{O}_{\mathcal{Y}_{[0,\infty)}(B,B^+)})[1/\xi]),$$

indexed by maps from not necessarily characteristic p affinoid perfectoids $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A^+)$ (see Remark 3.21 for why we allow mixed characteristic perfectoids).

Thus what is left to show is that (1) every compatible family of $M_{(B,B^+)}$ as above descends to a single matrix M_{A^+} with coefficients in $A_{\mathrm{inf}}(A^+)[1/d]$ and (2) that the latter is invertible if the former is. As we can cover $\mathrm{Spd}(A^+)$ by a single $\mathrm{Spa}(B_1, B_1^+)$ by Lemma 3.3, we can multiply our sections with a single finite power of ξ to get a matrix with coefficients in

$$\Gamma(\mathcal{Y}_{[0,\infty)}(B_1, B_1^+), \mathcal{O}_{\mathcal{Y}_{[0,\infty)}(B_1,B_1^+)}).$$

Using compatibility with pullbacks (for both the $M_{(B,B^+)}$ and $\xi = \xi_{(B,B^+)}$), multiplication with the same power of ξ for any (B, B^+) produces sections

$$\tilde{M}_{(B,B^+)} = \xi^n M_{(B,B^+)} \in M_{n\times n}(\Gamma(\mathcal{Y}_{[0,\infty)}(B, B^+), \mathcal{O}_{\mathcal{Y}_{[0,\infty)}(B,B^+)})).$$

But this is just a morphism of (free) $\mathcal{Y}$-bundles over $\mathrm{Spd}(A^+)$, so by the equivalence of categories Theorem 4.18, it descends to a morphism of $A_{\mathrm{inf}}(A^+)$-modules, again given in our basis as

$$\tilde{M}_{A^+} \in M_{n\times n}(A_{\mathrm{inf}}(A^+)),$$

which we can now divide in $A_{\mathrm{inf}}(A^+)[1/d]$ by d^n to get a morphism $M_{A^+} = \tilde{M}_{A^+}/d^n$ which descends the $M_{(B,B^+)}$. This shows (1). What is left to show is (2), i.e., that this descended morphism is an isomorphism in $A_{\mathrm{inf}}(A^+)[1/d]$ if the family of morphisms over $\Gamma(\mathcal{Y}_{[0,\infty)}(B, B^+), \mathcal{O}_{\mathcal{Y}_{[0,\infty)}(B,B^+)})[1/\xi]$ for varying (B, B^+) was. For this, it is enough to show that the map from $A_{\mathrm{inf}}(A^+)[1/d]$ to the compatible families of global sections of $\mathcal{Y}_{[0,\infty)}(B, B^+)[1/\xi]$ is injective, as this means that the inverse morphism $M^{-1}_{(B,B^+)}$ at the shtuka level (which we can of course descend just as well) is also an inverse of maps of $A_{\mathrm{inf}}[1/d]$-modules. But indeed, already the map

$$A_{\mathrm{inf}}(A^+) \to \Gamma(\mathcal{Y}_{[0,\infty)}(A, A^+), \mathcal{O}_{\mathcal{Y}_{[0,\infty)}(A,A^+)})$$

for $A = A^+[1/\varpi]$ is injective (remember that ϖ is a non-zero divisor), so the one to compatible families indexed over all (B, B^+) certainly is. $\square$

5 Relation to prismatic cohomology

5.1 Perfect-prismatic F-crystals arising from prismatic cohomology

Let R^+ be a p-complete ring. We want to associate to any proper smooth formal scheme over R^+ with finite projective prismatic cohomology group in degree n a perfect-prismatic F-crystal over R^+.

There are two ways in which one can generalize this construction: First, one can consider usual (i.e., non-perfect) prismatic F-crystals as well; secondly, working with perfect complexes instead of finite projective modules, one can consider derived prismatic F-crystals. Just for this subsection, we will consider both of these generalizations, but only for quasiregular-semiperfectoid (qrsp) base rings R^+. A generalization to quasi-syntomic rings[25] is straightforward, but requires some more definitions and ∞-categorical methods, see e.g. [AB23]. As the prismatic site over a qrsp ring R^+ admits an initial object $\mathbb{\Delta}_{R^+}$, we can avoid these issues:

Definition 5.1. 1. Let R^+ be a qrsp ring. A *derived prismatic F-module* on R^+ is a perfect complex $\mathcal{E}$ over the ring $\mathbb{\Delta}_{R^+}$ together with an isomorphism in the derived category

$$\Phi\colon \varphi^*\mathcal{E}[1/d] \to \mathcal{E}[1/d],$$

where $(\mathbb{\Delta}_{R^+}, (d))$ is the initial object of the prismatic site over R^+.

2. A derived prismatic F-module $(\mathcal{E}, \Phi)$ is called *of geometric type* if:
   - The underlying perfect complex $\mathcal{E}$ is concentrated in non-negative cohomological degrees.
   - It is effective, i.e. Φ extends to a necessarily unique map $\varphi^*\mathcal{E} \to \mathcal{E}$.
   - The induced map $H^i(\varphi^*\mathcal{E}) \to H^i(\mathcal{E})$[26] on the i'th cohomology group has its kernel- and cokernel killed by multiplication with d^i for all $i < 0$.

We will need a little reformulation:

Lemma 5.2. *Let A be a ring and $d \in A$ some element. Let $\Phi\colon P \to Q$ be a map between perfect complexes on A sitting in non-negative degrees and assume that for $n \geq 0$ there are maps $V^n\colon \tau_{\leq n}((d^n) \otimes_A Q) \to \tau_{\leq n}(P)$ such that the compositions*

$$V^n \circ \tau_{\leq n}(\Phi)\colon \tau_{\leq n}((d^n) \otimes_A P) \to \tau_{\leq n} P$$

[25] It seems like generalizing non-perfect prismatic crystals to even more general bounded p-adic rings is possible, but requires the methods of Bhatt-Lurie. Also, in a fully derived setting like Bhatt-Lurie, there is no good notion a $\mathcal{G}$-structure, while on the other hand for perfect-prismatic F-crystals generalizing to arbitrary (not even bounded) p-complete base rings with extra structure is easy. On a fundamental level this ability to work with non-derived objects throughout can be explained by flatness results like Lemma 2.11.

[26] Note that in order to write the left side as $\varphi^* H^i(\mathcal{E})$, as is necessary to obtain a non-derived prismatic F-crystal, we need flatness of φ. This is the case notably if R^+ is perfectoid so that φ is an isomorphism.

and
$$\tau_{\leq n}(\Phi)\circ V^n\colon \tau_{\leq n}((d^n)\otimes_A Q)\to \tau_{\leq n}Q$$
are both given by the natural inclusions. Then $H^i(\Phi)\colon H^i(P)\to H^i(Q)$ has its kernel- and cokernel killed by multiplication with d^i. In particular, the induced map $P[1/d]\to Q[1/d]$ is an isomorphism (in the derived category).

Proof. Note first that Φ and its restriction $\tau_{\leq n}(\Phi)\colon \tau_{\leq n}P\to\tau_{\leq n}Q$ both induce the same map on the n'th cohomology group. Thus it suffices to show that for any n, $H^n(\tau_{\leq n}\Phi)\colon H^n(P)\to H^n(Q)$ has kernel and cokernel killed by d^n. Now
$$\ker(H^n(\tau_{\leq n}\Phi))\subseteq\ker(H^n(V_n\circ\tau_{\leq n}\Phi))$$
and
$$\operatorname{coker}(H^n(\tau_{\leq n}\Phi\circ V_n))\twoheadrightarrow\operatorname{coker}(H^n(\tau_{\leq n}\Phi)),$$
so if every element in the kernel and cokernel of the compositions $H^n(\tau_{\leq n}\Phi\circ V^n)$ and $H^n(V^n\circ\tau_{\leq n}\Phi)$ are killed by d^n, then the same is true for $H^n(\tau_{\leq n}\Phi)$ itself. But this is clearly the case by assumption. □

Lemma 5.3. *Let R^+ be a qrsp ring. We can functorially associate to every smooth proper formal scheme X over R^+ a derived prismatic F-module of geometric type over R^+, given by its prismatic cohomology perfect complex.*

Proof. Let $A:=\mathbb{\Delta}_{R^+}$ be the initial object of the prismatic site over R^+. Then by smoothness and properness, the prismatic cohomology $\mathbb{\Delta}_{X/A}$ is a perfect complex and, by [BS21, Corollary 15.5], carries a φ-linear isomorphism with isogeny properties as described in the previous lemma. □

Now for the global statement; as mentioned before, we will from now on only talk about perfect-prismatic and non-derived objects.

Proposition 5.4. *Now let R^+ be any p-complete ring and let $\mathcal{C}$ be a class of smooth proper formal schemes X over R^+ having the property that, for some fixed n and any perfect prism $(A,(d))\in(R^+)^{\mathrm{perf}}_{\mathbb{\Delta}}$, the n'th prismatic cohomology group of $X\times_{\mathrm{Spf}(R^+)}\mathrm{Spf}(A/d)$ over $(A,(d))$ is finite projective and satisfies base change in the perfect prism $(A,(d))$. Then we can functorially associate to any $X\in\mathcal{C}$ a perfect-prismatic F-crystal of geometric type over R^+.*

Note that the base change condition is always met if for all $i>n$ also the i'th prismatic cohomology is finite projective. Indeed, we always have base change for the cohomology perfect complex and the finite projectiveness ensures that the derived tensor product is equal to the normal one for the i'th cohomology group.

Proof. We associate to any $(A,(d))\in(R)^{\mathrm{perf}}_{\mathbb{\Delta}}$ the A-module
$$H^n(R\Gamma_{\mathbb{\Delta}}(X\times_{\mathrm{Spf}(R^+)}\mathrm{Spf}(A/d))).$$
By Lemma 5.3, this will have the appropriate Frobenius structure, and compatibility with base change implies that this construction satisfies the crystal condition. □

5.2 Orthogonal/symplectic structu re on middle cohomology group

We want to use the Tannaka formalism and Poincaré duality to establish some GO_n resp. GSp_n structure on middle prismatic cohomology groups. To avoid subtleties concerning skew symmetric/alternating bilinear forms, we will assume $p \neq 2$ for this subsection. Let S be a suitable base ring (in our case $\mathbb{Z}_p$) and fix a non-degenerate symmetric bilinear form b_s (resp. non-degenerate alternating bilinear form b_a) on S^n. The group of orthogonal (resp. symplectic) similitudes GO_n (resp. GSp_n) is defined by

$$R \mapsto \{A \in M_{n\times n}(R) | \exists a \in R^\times : \forall v \in R^n : b_*(Av, Av) = ab_*(v, v)\}$$

for $b_* = b_s$ (resp. $b_* = b_a$). Under the Tannaka formalism, a GO_n-bundle on a locally ringed site over $\mathbb{Z}_p$ corresponds to a rank n locally free sheaf $\mathcal{E}$ together with a line bundle $\mathcal{L}$ and a non-degenerate symmetric bilinear map $\mathcal{E} \otimes \mathcal{E} \to \mathcal{L}$, and similarly for GSp_n. By keeping track of the extra structure, we get a similar description for perfect-prismatic GO_n resp. GSp_n-crystals, with line bundles being replaced for $\otimes$-invertible objects, which are exactly the perfect-prismatic F-crystals where the underlying locally free sheaf has rank 1.

Theorem 5.5. *(Prismatic Poincaré duality, [Tan22, Theorem 6.3][27]) Let (A, I) be a perfect prism corresponding to an integral perfectoid ring A/I and let $X \to \mathrm{Spf}(A/I)$ a smooth proper formal scheme of dimension d. Then there is an isomorphism of derived prismatic F-modules (in the sense of the previous subsection)*

$$R\Gamma_{\Delta}(X/A)^\vee \cong R\Gamma_{\Delta}(X/A)[2d]\{d\}.$$

For our purposes, we need to transform this in two ways: Firstly, we want to exchange the map of complexes for a map of cohomology groups, and secondly, we want that map to be symmetric/alternating:

Construction 5.6. In the formulation of [Tan22, Theorem 6.3], the first map of

$$A \to R\Gamma_{\Delta}((X \times_{\mathrm{Spf}(A)} X)/A)\{d\}[2d] \to (R\Gamma_{\Delta}(X/A) \otimes^L_A (R\Gamma_{\Delta}(X/A))\{d\}[2d]$$

comes from the cycle class of the diagonal $X \to X \times_{\mathrm{Spf}(A)} X$. Basically by definition, the diagonal is compatible with the isomorphism exchanging the two copies of X, and thus the same is true for the composition above as well as its dual map. Writing $D^\bullet = R\Gamma_{\Delta}(X/A)$ for readability, we thus get a commutative diagram

$$\begin{tikzcd}
D^\bullet \otimes D^\bullet \arrow[d, "\cong"] \arrow[dr] & \\
& A[-2d]\{-d\} \\
D^\bullet \otimes D^\bullet \arrow[ur] &
\end{tikzcd}$$

[27] In fact, Tangs result is significantly stronger, working with prismatic F-gauges, which are a version of filtered derived prismatic F-crystals, and arbitrary base prisms instead of perfect ones

with the vertical isomorphism swapping the two copies of $D^\bullet$. Let's take a look at passage to cohomology groups. We will have to assume that all cohomology groups of $D^\bullet$ are finite projective, which means that we can write $D^\bullet$ as

$$\cdots \to 0 \to H^0(D^\bullet) \to \cdots \to H^{2d}(D^\bullet) \to 0 \to \dots$$

with all differentials zero. The dual of this complex $(D^\bullet)^\vee$ can be readily computed as

$$\cdots \to 0 \to H^{2d}(D^\bullet)^\vee \to \cdots \to H^0(D^\bullet)^\vee \to 0 \to \dots,$$

with the last non-zero entry sitting in degree zero. Hence the isomorphism

$$(D^\bullet)^\vee \cong D^\bullet[2d]\{d\}$$

from Tang's theorem induces an isomorphism on cohomology groups

$$H^k(D^\bullet) \cong H^{2d-k}(D^\bullet)^\vee\{d\},$$

or in other words a non-degenerate bilinear form represented by a map

$$H^k(D^\bullet) \otimes_A H^{2d-k}(D^\bullet) \to A\{d\}.$$

Now the swapping isomorphism in the diagram above induces a swapping isomorphism on cohomology groups

$$H^d(D^\bullet) \otimes_A H^d(D^\bullet) \xrightarrow{\sim} H^d(D^\bullet) \otimes_A H^d(D^\bullet)$$

given by $(-1)^d\,\mathrm{id}$. Thus the commutativity of the diagram above implies that the bilinear form

$$H^d(D^\bullet) \otimes_A H^d(D^\bullet) \to A\{d\}.$$

is symmetric resp. alternating, depending on parity of d.

After applying the Tannaka formalism and globalizing, we have obtained the following:

Theorem 5.7. *Let R^+ be any p-adically complete ring and $\mathcal{C}$ be a class of smooth proper formal schemes as in Proposition 5.4 such that* all *prismatic cohomology groups are finite projective, with the additional assumption that each $X \in \mathcal{C}$ has constant dimension d. Then one can functorially associate to any $X \in \mathcal{C}$ a perfect-prismatic* GO_n*- (if d is even) resp.* GSp_n*- (if d is odd) crystal given by*

$$H^d_{\Delta}(X/A)$$

together with the tensor-invertible object $A\{d\}$ and the symmetric/alternating map

$$H^d_{\Delta}(X/A) \otimes_A H^d_{\Delta}(X/A) \to A\{d\}.$$

5.3 Examples

In [AB23] it was shown that (the formal completions of) abelian varieties have finite projective prismatic cohomology groups. Inspired by their argument, we want to show the same in two other cases:

Proposition 5.8. *Let (A, I) be a bounded prism and let $X \to \operatorname{Spec}(A/I)$ be the formal completion of a smooth proper scheme that is either*

1. *a K3-surface*

2. *a complete intersection of projective space with even dimension.*

Then the prismatic cohomology groups $H^i_{\Delta}(X/A)$ are finite projective of the same rank as the deRham-cohomology groups and commute with base change in A.

The condition of even dimension in the second case is probably unnecessary. A hint in this direction is that from [Del73, Exposé XI, Théorème 1.5, Théorème 1.6] we know that deRham and ℓ-adic cohomology in case of odd dimension are finite projective resp. lisse.

Proof. In both cases, the conjugate prismatic spectral sequence

$$E_2^{i,j} = H^i(X, \Omega^j_{X/(A/I)})\{-j\} \Rightarrow H^{i+j}(X, \overline{\Delta}_{X/A})$$

degenerates on the second page. Indeed, the Hodge numbers $h^{i,j} := H^i(X, \Omega_j)$ of both a K3-Surface and a complete intersection of projective space are 0 whenever $i \neq j \neq \dim(X) - i$, so that the Hodge diamond will simply be a cross. For K3-surfaces, this is e.g. [Lie23, Proposition 2.4], while for complete intersections, this is [Del73, Exposé XI, Théorème 1.5 (iii)a]. In both cases (for $\dim X$ even) all differentials have either target or source 0. So the spectral sequence degenerates, which shows that the reduced prismatic cohomology $\overline{\Delta}_{X/A}$ has finite projective cohomology groups and commutes with base change.

Now for the full prismatic cohomology groups: Since the question is local, we can assume our prism is oriented with $I = (d)$. The transition map

$$\mathcal{O}_{\Delta}/d^{n+1} \to \mathcal{O}_{\Delta}/d^n$$

is surjective, so

$$\mathcal{O}_{\Delta} = \lim_n \mathcal{O}_{\Delta}/d^n = R\lim_n \mathcal{O}_{\Delta}/d^n.$$

Since at least every second cohomology group is 0, we can use the long exact cohomology sequence associated to the SES

$$0 \to \mathcal{O}_{\Delta}/d^n \xrightarrow{\cdot d} \mathcal{O}_{\Delta}/d^{n+1} \to \mathcal{O}_{\Delta}/d \to 0$$

to inductively show that $R^k\Gamma((X/A)_{\Delta}, \mathcal{O}_{\Delta}/d^n)$ is still a finite projective $A/(d^n)$-module and that the transition maps are still surjective, so for all k

$$\lim_n H^k(R\Gamma((X/A)_{\Delta}, \mathcal{O}_{\Delta}/d^n)) = R\lim_n H^k(R\Gamma((X/A)_{\Delta}, \mathcal{O}_{\Delta}/d^n)).$$

is also finite projective. This implies that $R\Gamma((X/A)_{\Delta}, \mathcal{O}_{\Delta}/d^n)$ can be written as the shifted direct sum of it cohomology groups, and as finite sums commute with (derived) limits, we can perform the limit in the equation above before taking cohomology groups, so that

$$H^k\left(R\lim_n R\Gamma((X/A)_{\Delta}, \mathcal{O}_{\Delta}/d^n)\right)$$

is finite projective. Finite projectiveness and hence commutation with base change for the full prismatic cohomology groups now follow from the fact that $R\lim$'s commute with $R\Gamma$'s, so

$$R\lim_n R\Gamma((X/A)_{\Delta}, \mathcal{O}_{\Delta}/d^n) = R\Gamma((X/A)_{\Delta}, R\lim_n \mathcal{O}_{\Delta}/d^n).$$

The equality of rank follows directly from the deRham-comparison theorem. □

Combining Proposition 5.8 and Proposition 5.4 with Theorem 5.7 we get the following:

Corollary 5.9. *Let A be a p-complete ring. Let $\mathbf{K3}(A)$, $\mathbf{CI}^d(A)$ and $\mathbf{CI}^d_n(A)$ be, respectively: The category of K3-surfaces over A, the category of smooth dimension d schemes that can be realized as complete intersections of some projective space over A, and the subcategory thereof containing schemes with middle deRham-cohomology group having rank n. For any of these, denote by $(\cdot)^{\cong}$ the underlying groupoid. Then one has functors:*

$$\begin{aligned}
\mathbf{K3}(A) &\to \textit{Perfect-prismatic F-crystals on } A\\
\mathbf{CI}^{2d}(A) &\to \textit{Perfect-prismatic F-crystals on } A\\
\mathbf{K3}(A)^{\cong} &\to \textit{Perfect-prismatic } \mathrm{GO}_{22}\textit{-crystals on } A\\
\mathbf{CI}^{2d}_n(A)^{\cong} &\to \textit{Perfect-prismatic } \mathrm{GO}_{n}\textit{-crystals on } A
\end{aligned}$$

5.4 The Newton- and Hodge stratification

We want to finish with some constructions and speculations concerning different stratifications arising from perfect prismatic F-crystals. For this chapter we assume $\mathcal{G}$ to be connected and split[28] reductive and fix a Borel and a Torus $T \subset B \subset \mathcal{G}$.

Plugging the perfect-prismatic F-crystal associated to a formal scheme $X \to S$ into (the easy direction of) the equivalence Theorem 4.18 we obtain a family of shtukas on S. One can then study the Newton- and Hodge stratification of such a thing using the olivine spectrum of Gleason, constructed in [Gle22].

Construction 5.10. We have two maps from the stack of shtukas to simpler objects:

$$\begin{tikzcd}
 & \mathrm{Sht}_{\mathcal{G}} \arrow[dl] \arrow[dr] & \\
\mathrm{Bun}_{\mathcal{G}} & & \mathrm{Hk}_{\mathcal{G}}^{B^+_{dR}}
\end{tikzcd}$$

[28]being split is probably unnecessary, but it simplifies notation

- On the left, we have a map to the moduli stack of vector bundles on the Fargues-Fontaine curve: For a shtuka $(\mathcal{E}, \Phi)$ on some perfectoid (B, B^+), we consider the φ^{-1}-module $\mathcal{E}|_{\mathcal{Y}_{[r,\infty)}(B)}$ for sufficiently large r, and use the Frobenius spreading out trick from e.g. [SW20] to obtain a φ-bundle on all of $\mathcal{Y}_{(0,\infty)}(B)$.

- On the right, we map to the B_{dR^+} Hecke stack, defined as the étale sheafification of
$$(R, R^+) \mapsto \mathcal{G}(B_{dR}^+(R))\backslash\mathcal{G}(B_{dR}(R))/\mathcal{G}(B_{dR}^+(R)),$$
with the map $\mathrm{Sht}_{\mathcal{G}} \to \mathrm{Hk}_{\mathcal{G}}^{B_{dR}^+}$ defined by sending a shtuka to the modification of vector bundles $\varphi^*\mathcal{E}[1/\xi] \to \mathcal{E}[1/\xi]$ while forgetting that one is a the Frobenius pullback of the other.

Both of these objects carry a natural stratification:

- $\mathrm{Bun}_{\mathcal{G}}$ has a stratification by the Newton- and Kottwitz points, indexed by $X_*(T)^+_{\mathbb{Q}}$ and $\pi_1(\mathcal{G})$, respectively, see e.g. [FS21, Chapter III] for details.

- $\mathrm{Hk}_{\mathcal{G}}^{B_{dR}^+}$ inherits the stratification by Schubert cells through the quotient map from the Beilinson-Drinfeld affine Grassmanian
$$\mathrm{Gr}_{\mathcal{G},\mathrm{Spd}(\mathbb{Z}_p)} \to \mathrm{Hk}_{\mathcal{G}}^{B_{dR}^+}.$$
The stratification is indexed by dominant cocharacters $X_*(T)^+$, see [SW20, Section 20.3], and descends from the Grassmanian to the Hecke stack e.g. by looking at the generic- and special fibres separately.

We will refer to those as the Newton- resp. Hodge stratification.

Construction 5.11. Let R^+ be a p-complete ring. Then every perfect-prismatic $\mathcal{G}$-crystal $\mathcal{E}$ on R^+ (e.g. arising from a suitable smooth proper formal scheme over $\mathrm{Spf}(R^+)$) yields a Newton- and Hodge stratification on $|\mathrm{Spd}(R^+)|$ and thus on Gleason's olivine spectrum $\mathrm{Spo}(R^+)$. We will construct this for the Newton stratification, the Hodge stratification works similarly. Cover $\mathrm{Spd}(R^+)$ by some affinoid perfectoid $\mathrm{Spa}(B, B^+)$ as in Lemma 3.3. Then pullback of $\mathcal{E}$, viewed as a family of shtukas, to $\mathrm{Spa}(B, B^+)$, determines a morphism
$$\mathrm{Spa}(B, B^+) \to \mathrm{Sht}_{\mathcal{G}} \to \mathrm{Bun}_{\mathcal{G}}$$
and thus a stratification on $\mathrm{Spa}(B, B^+)$. Now the fact that the stratification comes via pullback from $\mathrm{Bun}_{\mathcal{G}}$ forces the Strata to be compatible with pullbacks themselves, which imply that all the preimages of a point $x \in \mathrm{Spd}(R^+)$ lie in the same stratum, and we attach x to that particular stratum. Now $|\mathrm{Spa}(B, B^+)| \to |\mathrm{Spd}(R^+)|$ is a quotient map (basically by definition of the topology on $|\mathrm{Spd}(R^+)|$), and thus the newly defined strata on $\mathrm{Spd}(R^+)$ satisfy the correct openess/closeness properties. Finally, via $|\mathrm{Spd}(R^+)| = \mathrm{Spo}(R^+)$, we get a stratification on the latter.

If $G = \mathrm{GL}_n$, we can associate to the Newton- and the Hodge strata a Newton- and Hodge polygon in the usual way. Even for general G, we can still talk about the relative position of the Newton- and Hodge "polygon" (assumed to be concave), even though there are no polygons in the literal sense. See [RR96] for details.

Proposition 5.12. *([CS17, Lemma 3.5.4]) The Newton polygon lies on or below the negative of the Hodge polygon.*

Remark 5.13. Some words about sign conventions:

1. In loc. cit. and some of the classical literature, one reads that the Newton polygon lies on or above the Hodge polygon. This is because they use the convention of *convex* polygons, while ours are concave. Our convention is in line with using the upper triangular matrices as standard Borel of GL_n and is compatible with the relation $\preceq$, which e.g. in loc. cit. points in the opposite direction of what one would expect in terms of polygons.

2. having to put a minus sign to the Hodge polygon is a result of the convention for slopes flipping when passing from isocrystals to vector bundles on the Fargues-Fontaine curve, while the convention for cocharacters have stayed the same (i.e., in line with what is used for isocrystals).

3. Newton polygon lying above the Hodge polygon is however independent from the global sign convention, i.e., if we were to flip the convention for cocharacters and Bun_G simultaneously, the relation would stay the same. This is because after e.g. inverting a cocharacter, one has to pick a new dominant representative — or, in terms of polygons, after negating, one has to reorder the slopes to make sure the polygon stays concave.

The Hodge polygon can vary in families:

Example 5.14. Let $R^+ = \mathbb{F}_p[[t^{1/p^\infty}]]$ and

$$\Phi\colon (R^+)^2 \to (R^+)^2, v \mapsto \begin{pmatrix} p^2 & tp \\ tp & 1 \end{pmatrix}.$$

This evidently has Hodge polygon $(0,2)$ in $t=0$, but Hodge polygon $(1,1)$ in $t \neq 0$.

However, I make the following speculation:

Remark 5.15. For perfect-prismatic F-crystals arising as the cohomology group of a smooth proper scheme such that the Hodge-to-prismatic spectral sequence degenerates plus possibly extra assumptions, I expect some prismatic version of Mazur's theorem to hold, i.e.: that the Hodge polygon can be read-off from the Hodge numbers and is locally constant in families.

We also have a second Newton stratification coming from the special fibre. Most of what follows has been explained to me by Ian Gleason, so I credit the insights to him (Though any possible mistakes and misconceptions in what follows are all mine).

Remark 5.16. (meromorphicity and isocrystals) One key difference between isocrystals corresponding to (families of) shtukas with leg at $p = 0$ on the one hand and vector bundles on the Fargues-Fontaine curve (classified by morphisms to $\mathrm{Bun}_{\mathcal{G}}$) on the other is that the former have some meromorphicity condition. While perfect-prismatic F-crystals on mixed characteristic rings, thus shtukas with legs outside $p = 0$, are also meromorphic around the leg, this meromorphicity gets destroyed when applying the Frobenius spreading out. Thus there is no obvious way to produce an isocrystal on $R^{+\flat}$ from a perfect prismatic F-crystal on a mixed characteristic perfectoid R^+.

If our ring R^+ has characteristic p however, there is no Frobenius spreading out necessary. Indeed, in this case (effective) perfect-prismatic F-crystals are the same thing as Frobenius crystals[29] and the map to $\mathrm{Bun}_{\mathcal{G}}$ factors through the moduli space of isocrystals. We can then use tubular neighbourhoods to pull this back to mixed characteristic rings:

Construction 5.17. Let R^+ be a p-complete ring, which we can assume to be integral perfectoid for this construction, and let $\mathcal{E}$ be a perfect prismatic $\mathcal{G}$-crystal on R^+. Consider the reduced special fibre $R^{+,\mathrm{red}} = (R^+/p)^{\mathrm{perf}}$. Base changing $\mathcal{E}$ along the map $R^+ \to R^{+,\mathrm{red}}$ yields a $\mathcal{G}$-bundle with Frobenius action on $W(R^{+,\mathrm{red}})$ which is the same thing as a $\mathcal{G}$-isocrystal. By classical results ([RR96]) we get a Newton- and Kottwitz stratification[30] on $\mathrm{Spec}(R^{+,\mathrm{red}})$. In [Gle21] one constructs a continuous specialization map

$$\mathrm{Spo}(R^+) \to \mathrm{Spec}(R^{+,\mathrm{red}}),$$

and pulling back along this map we have arrived at what we will call the *schematic Newton stratification.*

Note that in geometric situations, e.g. when considering the perfect-prismatic F-crystal associated to some smooth proper formal scheme $X \to \mathrm{Spf}(R^+)$, this schematic Newton polygon depends only on the crystalline cohomology of the special fibre $X \times_{\mathrm{Spf}(R^+)} \mathrm{Spf}(R^{+,\mathrm{red}})$. So also in this geometric sense, it can be viewed as a mixed characteristic shadow of the special fibre.

Remark 5.18. In [KL15] and for characteristic p rings, the Newton polygon produced via passage to $\mathrm{Bun}_{\mathcal{G}}$ is called the *special Newton polygon*, while the one we just produced through isocrystals is called the *generic Newton polygon.* We have the following familiar phenomena:

1. After accounting for the sign flip the two stratifications have reversed topologies. More precisely, using the sign convention of Bun_G, the Newton stratification via Bun_G is upper-semicontinuous while the schematic Newton stratification is lower semicontinuous.

[29] at least on perfect rings, which form a basis of the arc topology

[30] For isocrystals, and thus for the schematic Newton stratification, we will use the convention that $\Phi\colon W(R^+)[1/p] \to W(R^+)[1/p], a \mapsto p\varphi(a)$ has slope 1. Thus, passage to vector bundles on the Fargues Fontaine curve reverses slopes

2. If R^+ itself is characteristic p, the special Newton polygon lies on or above the negative of generic one. This is [KL15, Lemma 7.4.3] and it is independent of which sign convention we use, cf. Remark 5.13 (3).

There is probably much more to be said about those two Newton polygons. For example, one might suspect them to coincide for a suitable collection of "classical" points of $\mathrm{Spo}(R^+)$.

References

[AB23] Johannes Anschütz and Arthur-César Le Bras. "Prismatic Dieudonné theory". In: *Forum of Mathematic, Pi* 11 (2023).

[Ans18] Johannes Anschütz. "Extending torsors on the punctured Spec(A_{inf})". In: *Journal für die reine und angewandte Mathematik (Crelles Journal)* 2022 (Apr. 2018). DOI: 10.1515/crelle-2021-0077.

[Ans23] Johannes Anschütz. "G-bundles on the absolute Fargues-Fontaine curve". In: *Acta Aritmetica* 207 (2023).

[BC22] Alexis Bouthier and Kestutis Cesnavicius. *Torsors on loop groups and the Hitchin fibration*. 2022. arXiv: 1908.07480 [math.AG].

[BMS19] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. *Integral p-adic Hodge theory*. 2019. arXiv: 1602.03148 [math.AG].

[BS21] Bhargav Bhatt and Peter Scholze. *Prisms and Prismatic Cohomology*. 2021. arXiv: 1905.08229 [math.AG].

[BS23] Bhargav Bhatt and Peter Scholze. "Prismatic F-crystals and crystalline Galois representations". In: *Cambridge Journal of Mathematics* 11 (2023).

[CS17] Ana Caraiani and Peter Scholze. "On the generic part of the cohomology of compact unitary Shimura varieties". In: *Annals of Mathematics* 186.3 (2017), pp. 649–766.

[ČS21] Kestutis Česnavicius and Peter Scholze. *Purity for flat cohomology*. 2021. arXiv: 1912.10932 [math.AG].

[Del73] Pierre Deligne. *Groupes de Monodromie en Géometétrie Algébrique*. 1973.

[FF18] Laurent Fargues and Jean Marc Fontaine. "Courbes et fibrés vectoriels en théorie de Hodge p-adique". In: *Astérisque* 406 (2018), pp. 1–382.

[FS21] Laurent Fargues and Peter Scholze. *Geometrization of the local Langlands correspondence*. 2021. arXiv: 2102.13459 [math.RT].

[GI23] Ian Gleason and Alexander Ivanov. *Meromorphic vector bundles on the Fargue-Fontaine curve*. in preparation. 2023.

[Gle21] Ian Gleason. "Specialization maps for Scholze's category of diamonds (Ph. D. thesis)". In: (2021). URL: https://ianandreigf.github.io/Website/tesis.pdf.

[Gle22] Ian Gleason. *Specialization maps for Scholze's category of diamonds*. 2022. arXiv: 2012.05483 [math.AG].

[Heu22] Ben Heuer. "Line bundles on rigid spaces in the v-topology". In: *Forum of Mathematics* 10 (2022).

[HK20] David Hansen and Kiran S. Kedlaya. *Sheafiness Criteria For Huber Rings*. 2020. URL: https://kskedlaya.org/papers/criteria.pdf.

[Ito21] Kazuhiro Ito. *p-complete arc-descent for perfect complexes over integral perfectoid rings.* 2021. DOI: 10.48550/ARXIV.2112.02443. URL: https://arxiv.org/abs/2112.02443.

[Ked] Kiran Kedlaya. *Kedlaya's website, "The arc topology and friends.* URL: https://kskedlaya.org/prismatic/sec_arc-topology.html.

[KL15] Kiran S. Kedlaya and Ruochuan Liu. *Relative p-adic Hodge theory: Foundations.* 2015. arXiv: 1301.0792 [math.NT].

[Lie23] Christian Liedtke. *Lectures on Supersingular K3 Surfaces and the Crystalline Torelli Theorem.* 2023. arXiv: 1403.2538 [math.AG].

[PR21] Georgios Pappas and Michael Rapoport. *p-adic shtukas and the theory of global and local Shimura varieties.* 2021. DOI: 10.48550/ARXIV.2106.08270. URL: https://arxiv.org/abs/2106.08270.

[RR96] Michael Rapoport and Melanie Richartz. "On the classification and specialization of F-isocrystals with additional structure". In: *Composition Mathematica* 103.2 (1996), pp. 153–181.

[Sch17] Peter Scholze. "Étale Cohomology of Diamonds". In: *SSRN Electronic Journal* (2017).

[Sen93] Shankar Sen. "An infinite dimensional Hodge-Tate theory". en. In: *Bulletin de la Société Mathématique de France* 121.1 (1993), pp. 13–34. DOI: 10.24033/bsmf.2199. URL: http://www.numdam.org/articles/10.24033/bsmf.2199/.

[Sta20] The Stacks Project Authors. *Stacks Project.* https://stacks.math.columbia.edu. 2020.

[SW20] Peter Scholze and Jared Weinstein. *Berkeley Lectures on p-adic Geometry.* Princeton University Press, 2020.

[Tan22] Longke Tang. *Syntomic cycle classes and prismatic Poincaré duality.* 2022. DOI: 10.48550/ARXIV.2210.14279. URL: https://arxiv.org/abs/2210.14279.

[Wed19] Torsten Wedhorn. *Adic Spaces.* 2019. arXiv: 1910.05934 [math.AG].