On the L^p-theory of elliptic operators in divergence form

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Preface

In his seminal work [12], Auscher investigated the L^p-theory of second-order divergence form operators

$$L = -\operatorname{div}(A\nabla\cdot) \quad \text{on } \mathbb{R}^d.$$

Here, A is a matrix-valued function with bounded, measurable, and complex entries that satisfies the *ellipticity* condition

$$\Delta_2(A) = \lambda(A) \coloneqq \operatorname*{essinf}_{x\in\mathbb{R}^d} \min_{|\xi|=1} \operatorname{Re}(A(x)\xi\cdot\overline{\xi}) > 0.$$

Using formal integration by parts, the operator L is realized as an m-accretive operator in $\mathrm{L}^2(\mathbb{R}^d)$ via the sesquilinear form

$$a\colon V\times V\to\mathbb{C}, \quad a(u,v) = \int_{\mathbb{R}^d} A\nabla u\cdot\overline{\nabla v}\,\mathrm{d}x,$$

where the form domain V is the Sobolev space $\mathrm{W}^{1,2}(\mathbb{R}^d)$. We denote by $(\mathrm{e}^{-tL})_{t>0}$ the contraction semigroup associated with L.

Auscher observed that *'four critical numbers rule the L^p behavior'* (see [12, p. 14]). More precisely, the main interest lies in determining the maximal intervals for which the heat semigroup $(\mathrm{e}^{-t^2L})_{t>0}$ and its gradient family $(t\nabla\,\mathrm{e}^{-t^2L})_{t>0}$ extrapolate to uniformly bounded operator families in $\mathrm{L}^p(\mathbb{R}^d)$. To this end, he defined the four critical numbers $p_\pm(L)$ and $q_\pm(L)$ as the left and right endpoints of the intervals:

$$\mathcal{J}(L) \coloneqq \Big\{p\in[1,\infty]\colon (\mathrm{e}^{-t^2L})_{t>0} \text{ is } \mathrm{L}^p\text{-bounded}\Big\},$$
$$\mathcal{N}(L) \coloneqq \Big\{p\in[1,\infty]\colon (t\nabla\,\mathrm{e}^{-t^2L})_{t>0} \text{ is } \mathrm{L}^p\text{-bounded}\Big\}.$$

The study of these intervals is intimately related to L^p-bounds for the L-adapted Riesz transform $\nabla L^{-1/2}$, the existence of a bounded H^∞-calculus for L in the Lebesgue and Sobolev scales, and L^p-estimates for vertical and conical square functions. These relations reveal deep connections between the theory of elliptic boundary value problems in the upper half space $\mathbb{R}^{1+d}_+$ and the underlying harmonic analysis of L. For this reason, the critical numbers serve as a *leitmotif*.

The outline is as follows. First, we provide an overview of the most important concepts: functional calculus, $\mathrm{L}^p-\mathrm{L}^q$-estimates, p-ellipticity, and relevant results from geometric measure theory. Second, we delve into the first topic and establish *explicit improvements* for the critical numbers depending on ellipticity and dimension. Third, we examine the relation between the critical numbers, local elliptic regularity of second-order equations, and elliptic boundary value problems with a *non-block* structure. Fourth, we explore the extended Lebesgue scale when L is realized on an open set $O \subseteq \mathbb{R}^d$ under *mixed boundary conditions*. This ultimately leads to the study of Gaussian estimates and real-variable methods in the spirit of De Giorgi–Nash–Moser (exponents 'beyond ∞'), as well as (geometric) Hardy spaces (exponents 'below 1'). Last but not least, we analyze the L^p-functional calculus when L models so-called *dynamical boundary conditions* in a general geometric framework.

Let us provide a brief overview of each topic. For a more detailed exposition, we refer to the corresponding chapters and the references therein.

Explicit improvements for the critical numbers

Throughout this chapter, we assume $d \geq 3$. Since the critical numbers play a key role in the study of elliptic boundary value problems, elliptic regularity theory, and the harmonic analysis of L, it is of independent interest to derive explicit improvements for them. The equality $p_-(L) = q_-(L)$ reduces the degree of freedom by one. Moreover, the duality relation $p_-(L) = p_+(L^*)'$ allows us to focus on lower bounds for $p_+(L)$ and $q_+(L)$. These two quantities are related by $2d/(d-2) = 2^* < q_+(L)^* \leq p_+(L)$. Furthermore, it is shown in [113] that, for any $\varepsilon > 0$, one can construct a divergence form operator L such that $p_+(L) \leq 2^* + \varepsilon$. In particular, $q_+(L)$ is typically not much larger than 2, provided no additional structural information is available. The construction is based on Frehse's irregular solution to complex, scalar-valued elliptic equations [93]. An inspection of the coefficient matrix entails that the

ellipticity constant degenerates as ε tends to 0. Consequently, we expect that $p_+(L) \geq 2^* + c$, where $c > 0$ depends on ellipticity and dimension. Results of this type have been established by Egert [80] and independently by ter–Elst, Haller, Rehberg, and Tolksdorf in [85]. Both approaches rely on the concept of *p-ellipticity*, which originates from the work of Cialdea and Maz'ya [61]:

$$\Delta_p(A) \coloneqq \operatorname*{essinf}_{x\in\mathbb{R}^d} \min_{|\xi|=1} \operatorname{Re}\left(A(x)\xi \cdot \overline{\left(\xi + (1-{}^2\!/_p)\overline{\xi}\right)}\right) > 0.$$

This algebraic property of A bridges the class of complex elliptic matrices and real-valued ones. The precise notion was introduced by Carbonaro and Dragičević in the context of bilinear embeddings and generalized convexity of power functions [52] and independently by Dindoš and Pipher in the study of elliptic boundary value problems [69]. This concept plays a fundamental role throughout this thesis and should be kept in mind.

We choose a different path and measure ellipticity via a scale invariant distance to the identity matrix: They persist as long as

$$\mathrm{d}(A) \coloneqq \min_{t>0} \|\mathbf{1}_{\mathbb{C}^d} - tA\|_{\mathrm{L}^\infty(\mathbb{R}^d;\mathbb{C}^{d\times d})} < 1.$$

This perspective originates from Koshelev's work [121] on local *qualitative* elliptic regularity theory. Based on a result by Auscher and Tchamitchian [29] (which we shall revisit later), we reinterpret their local results *quantitatively* as a global one and prove a perturbation theorem for Gaussian estimates of the heat kernel associated with the semigroup $(\mathrm{e}^{-tL})_{t>0}$:

$$\mathrm{d}(A) < \delta(d) \coloneqq \left(1 + \frac{(d-1)^2}{d-2}\right)^{-\frac{1}{2}}.$$

This furnishes a criterion ensuring that $p_+(L) = \infty$. A classical counterexample of De Giorgi [103] for real elliptic systems demonstrates that our result is sharp in the class of elliptic matrices.

With this result in hand, our idea is to embed A into an analytic family $\{A_z\}_z$ of elliptic matrices, defined on the closed strip $\{z \in \mathbb{C} \colon 0 \leq \operatorname{Re}(z) \leq 1\}$, and then to invoke Stein-interpolation. For $\operatorname{Re}(z) = 0$, we use only the information that $p_+(L_z) \geq 2^*$, while for $\operatorname{Re}(z) = 1$, we adjust the operator family such that $\mathrm{d}(A_z) < \delta(d)$. Notably, both results extend to elliptic systems and to bounded domains with C^1-boundary, where L is subject to Dirichlet boundary conditions.

The second main contribution is a d-independent improvement for $q_+(L)$, which will later be used in the context of a priori estimates for solutions of

elliptic boundary value problems, see Chapter 3. The argument relies on an equivalent characterization of $q_+(L)$ from [12] and d-independent upper bounds for the L^p-norm of the classical Riesz transform $\nabla(-\Delta)^{-1/2}$, established by Volberg and Dragičević in [76].

These results have been published in a joint article with Egert [46].

Next, we develop equivalent characterizations of the critical numbers in virtue of local elliptic regularity of second-order equations and L^p-estimates for the associated first-order theory.

Meyers meets Dirac

The Meyers exponent $m_+(L)$ associated to the elliptic operator L is defined as the largest exponent $p > 2$ for which the gradient of any weak solution $Lu = 0$ in a ball $B \subseteq \mathbb{R}^d$ is p-integrable in $\frac{1}{2}B$, with quantitative L^2-norm control. Phrased differently, the gradient of locally L-harmonic functions fulfills a weak reverse Hölder estimate with exponent p. Therefore, $m_+(L)$ serves as a measure for local elliptic regularity. Using ideas from Shen [145] and an equivalent characterization of $q_+(L)$, we establish the identity

$$m_+(L) = q_+(L).$$

As a second step, we bridge the second-order theory of the critical numbers to the first-order theory of *perturbed Dirac operators*. Let A be a t-independent elliptic copy of A in $1 + d$ dimensions. More precisely, we consider an elliptic matrix-valued function

$$\mathsf{A} = \begin{bmatrix} a & b \\ c & A \end{bmatrix} : \mathbb{R}^d \to \mathcal{L}(\mathbb{C} \times \mathbb{C}^d) \tag{0.1}$$

with bounded and measurable entries. From this, we define the bounded multiplication operator

$$\mathsf{B} \coloneqq \widehat{\mathsf{A}} = \underline{\mathsf{A}}\overline{\mathsf{A}}^{-1} = \begin{bmatrix} 1 & 0 \\ c & A \end{bmatrix} \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}^{-1}.$$

The first-order differential operator

$$\mathsf{D} \coloneqq \begin{bmatrix} 0 & \operatorname{div}_x \\ -\nabla_x & 0 \end{bmatrix}$$

acts in the distributional sense. The DB-formalism plays a fundamental role in the study of elliptic boundary value problems, which we explore further in Chapter 4. Notably, the operator DB admits a robust L^2-theory. The exponents of interest are those $p \in (1,\infty)$ in the maximal open interval around $p = 2$, where the L^p-realization of DB is bisectorial. Within this range, a bounded functional calculus and a DB-adapted kernel range splitting are available, see [27]. Auscher and Stahlhut introduced the notation $I(\mathsf{DB})$ for this interval, with endpoints denoted by $p_\pm(\mathsf{DB})$. This interval is important to extend the range of exponents for which a priori estimates for solutions of elliptic boundary value problems are available, see Chapter 4.

The main result of this chapter establishes the identity

$$I(\mathsf{DB}) = (q_+(\boldsymbol{L}^*)', q_+(\boldsymbol{L})) = (m_+(\boldsymbol{L}^*)', m_+(\boldsymbol{L})),$$

where $\boldsymbol{L} = -\operatorname{div}_{t,x}(\mathsf{A}\nabla_{t,x}\cdot)$ is the homogeneous 'parent' of $L = -\operatorname{div}_x(A\nabla_x\cdot)$ in $1+d$ dimensions. While the second equality follows directly from $q_+(L) = m_+(L)$, the first requires a more intricate argument.

By duality, it suffices to argue for $p_+(\mathsf{DB})$. To outline the proof, we apply the Fourier transform in the time variable, relating the homogeneous gradient $\nabla_{t,x}$ to the inhomogeneous, τ-dependent gradient

$$S_\tau \coloneqq [\mathrm{i}\tau, \nabla_x]^\top.$$

With this in mind, the homogeneous operator $\boldsymbol{L}$ in $1+d$ dimensions can be associated with the inhomogeneous family $(\mathsf{L}_\tau)_{\tau\in\mathbb{R}} = (S_\tau^*\mathsf{A}S_\tau)_{\tau\in\mathbb{R}}$ in one fewer dimension. For this one parameter family, we define a Meyers exponent $m_+((\mathsf{L}_\tau)_\tau)$ and the critical number $q_+((\mathsf{L}_\tau)_\tau)$, both uniform in τ. The proof of $p_+(\mathsf{DB}) = q_+(\boldsymbol{L})$ follows from the following chain of equalities:

$$p_+(\mathsf{DB}) = q_+((\mathsf{L}_\tau)_\tau) = m_+((\mathsf{L}_\tau)_\tau) = q_+(\boldsymbol{L}) = m_+(\boldsymbol{L}).$$

The first equality follows from an algebraic relation between the resolvent of DB and the second-order Hodge operators $S_\tau\mathsf{L}_\tau^{-1}S_\tau^*$, which we further exploit in Chapter 4. The second equality follows from refined proofs of the identity $q_+(L) = m_+(L)$. Last but not least, to lift L^p-estimates for $(\mathsf{L}_\tau)_{\tau\in\mathbb{R}}$ in d dimensions to $\boldsymbol{L}$ in $1+d$ dimensions, we interpret $\mathrm{L}^2(\mathbb{R}^{1+d}) = \mathrm{L}^2(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d))$ as a vector-valued L^2-space. This perspective enables us to apply a celebrated theorem of Weis [161] for vector-valued multipliers.

These results are based on [18], in collaboration with Auscher and Egert.

To conclude Chapter 3, we provide an alternative characterization of $p_+(L)$. Specifically, we show that $p_+(L)$ is the largest exponent $p > 2$ for which

a uniform, quantitative weak reverse Hölder estimate holds for local weak solutions to $(L+s)u=0$, for all $s \geq 0$.

The next chapter shifts focus to the first-order Dirac operator and its applications to elliptic boundary value problems.

Elliptic boundary value problems with non-block structure

In this chapter, we investigate the range of exponents $p \in (1_*, \infty)$ for which a priori estimates for solutions of the elliptic boundary value problem

$$\text{(BVP)} \qquad \begin{cases} -\operatorname{div}_{t,x}(\mathsf{A}(x)\nabla_{t,x}U(t,x)) &= 0 \qquad (t>0, x \in \mathbb{R}^d), \\ U(0,\cdot) &= f \in \mathrm{H}^p(\mathbb{R}^d), \end{cases}$$

are accessible. Here, $\mathrm{H}^p(\mathbb{R}^d)$ is the real Hardy space for $p \leq 1$ and $\mathrm{H}^p(\mathbb{R}^d) = \mathrm{L}^p(\mathbb{R}^d)$ for $p > 1$. Different to the previous chapter, A, as defined in (0.1), does not satisfy a pointwise ellipticity condition, but rather the inequality

$$\operatorname{Re}\left(\int_{\mathbb{R}^d} \mathsf{A}\begin{bmatrix} f \\ \nabla_x u \end{bmatrix} \cdot \overline{\begin{bmatrix} f \\ \nabla_x u \end{bmatrix}}\, \mathrm{d}x\right) \geq \lambda(\|f\|_2^2 + \|\nabla_x u\|_2^2)$$

for some $\lambda > 0$, each $f \in \mathrm{L}^2(\mathbb{R}^d)$ and all $u \in \dot{\mathrm{W}}^{1,2}(\mathbb{R}^d)$. If A has *block-structure*, meaning that b and c vanish, then (BVP) simplifies significantly to $\partial_t^2 U - a^{-1}LU = 0$, which admits a *Poisson extension* solution ansatz:

$$U(t,x) = (\mathrm{e}^{-t\sqrt{a^{-1}L}}f)(x).$$

Using this *second-order approach*, Auscher and Egert unified in [20] the theory of (BVP) in the block-structure case and established its deep connection to the critical numbers of L.

However, in the non-block case, the problem becomes significantly more sophisticated. The foundation for an extensive L^2-theory of (BVP) was laid in the breakthrough work of Axelsson, Keith, and McIntosh [32]. Building on their results, Auscher, Axelsson, and McIntosh invented in a series of papers [14–16] the so-called *first-order approach*. The key idea is a one-to-one correspondence between weak solutions $U = U(t,x)$ of (BVP) and their

conormal gradient $F = \overline{\mathsf{A}}\nabla_{t,x}U$. Crucially, F solves the *first-order* (evolution) equation of Cauchy–Riemann type:

$$\text{(FO)} \qquad F'(t) + \mathsf{DB}F(t) = 0 \qquad (t > 0).$$

The main result from [32], namely that DB admits a bounded functional calculus on $\overline{\mathrm{ran}(\mathsf{D})}$, allows to generate a semigroup solution to (FO), which ultimately yields a priori estimates for solutions of (BVP) when $p = 2$.

The key tool for extending results beyond $p = 2$ is the study of DB-*adapted Hardy spaces* $\mathbb{H}^p_{\mathsf{DB}}$. These function spaces are tailored to the operator such that its functional calculus behaves optimally. The ultimate goal is to identify $\mathbb{H}^p_{\mathsf{DB}}$ with the classical function space $\mathbb{H}^p_{\mathsf{D}} = \mathrm{H}^p(\mathbb{R}^d)^{1+d} \cap \overline{\mathrm{ran}(\mathsf{D})}$. This leads to the identification interval

$$\mathcal{H}(\mathsf{DB}) = \Big\{p \in (1_*, \infty) \colon \mathbb{H}^p_{\mathsf{DB}} = \mathbb{H}^p_{\mathsf{D}}\Big\},$$

where $h_\pm(\mathsf{DB})$ denote its left and right endpoints. In the block-structure case, it was shown in [20] that $h_-(\mathsf{DB}) = p_-(L)$ and $h_+(\mathsf{DB}) = q_+(L)$. This already indicates a fundamental dichotomy between $h_-(\mathsf{DB})$ and $h_+(\mathsf{DB})$. On the one hand, we cannot expect $h_+(\mathsf{DB})$ to be much larger than 2. On the other hand, it is shown in [28] that $h_-(\mathsf{DB}) \leq p_-(\mathsf{DB})_* < 2_* = {}^{2d}\!/_{(d+2)}$. Moreover, when A is real, it holds $h_-(\mathsf{DB}) < 1$.

Our first main result establishes that, for all $M \in \mathbb{N}$, it holds

$$h_-(\mathsf{DB}) = \inf\Big\{p \in (1_*, 2] \colon ((1 + \mathrm{i}t\mathsf{DB})^{-M})_{t>0} \text{ is } \mathbb{H}^p_{\mathsf{D}}\text{-bounded}\Big\}.$$

This provides a quantitative criterion for determining $h_-(\mathsf{DB})$ based on resolvent bounds. The proof builds on abstract interpolation principles and a refined version of an extrapolation argument for conical square functions, see [20, 28, 94].

Our second contribution leverages the algebraic relation between the resolvent of DB and the inhomogeneous Hodge operators $S_\tau \mathsf{L}_\tau^{-1} S_\tau^*$ from Chapter 3. This rephrases the $\mathbb{H}^p_{\mathsf{D}}$-boundedness of $((1 + \mathrm{i}t\mathsf{DB})^{-1})_{t>0}$, directly connecting $h_-(\mathsf{DB})$ with the critical numbers. Using this, we 'interpolate' between the general estimate $h_-(\mathsf{DB}) < 2_*$ and the improved result $h_-(\mathsf{DB}) < 1$ when A is real. More precisely, if c is *divergence-free* in the distributional sense and A is *p-elliptic* for some $p \in (2, \infty)$, we establish the refined bound

$$h_-(\mathsf{DB}) \leq p_\circ := ({}^{p2^*}\!/_2)'.$$

Building on molecular Hardy space theory, we provide a new, shorter proof of the fact that $h_-(\mathsf{DB}) < 1$ when A is real.

The characterization of $h_{-}(\mathsf{DB})$ in this chapter is unpublished work by the author, while the remaining results are joint work with Auscher and Egert.

The next topics bring other exciting areas into play, namely geometric measure theory, real-variable methods for partial differential equations and real Hardy spaces on open sets.

Gaussian estimates vs. elliptic regularity on open sets

In this chapter, we study regularity properties of (local) L-harmonic functions and their connection to (global) pointwise Gaussian estimates and the Hölder continuity of the heat kernel of $(\mathrm{e}^{-tL})_{t>0}$. Let $d \geq 2$. We consider L on an open set $O \subseteq \mathbb{R}^d$ and incorporate mixed boundary conditions into L via the form domain: let $V = \mathrm{W}^{1,2}_D(O)$ be the $\mathrm{W}^{1,2}(O)$-closure of test functions whose support avoids the *Dirichlet part* D. Here, $D \subseteq \partial O$ is closed and we call $N = \partial O \setminus D$ the *Neumann part*.

The primary goal is to establish the equivalence of the following three *quantitative* properties:

(D) Growth control of the Dirichlet energy of L- and L^*-harmonic functions.

(G) Pointwise Gaussian estimates and Hölder continuity of the heat kernel of $(\mathrm{e}^{-tL})_{t>0}$.

(H) Local Hölder continuity of L- and L^*-harmonic functions, with quantitative L^2-control.

In light of property (G), these conditions can be interpreted as expressing that $p_+(L)$ extends 'beyond $p = \infty$'.

A key observation is that a localization technique allows us to derive (H) from (G) without imposing geometric constraints on O. However, obtaining the global property (G) from local conditions requires additional geometric assumptions. Specifically, we introduce two complementary conditions:

(Fat) O^c is locally 2-fat away from N.

(LU) O is locally uniform near N.

Assumption (Fat) imposes a lower bound on the relative 2-capacity of O^c away from N at small scales. Near N, it replaces O^c with $D(\subseteq O^c)$, strengthening the requirement as points approach N.

On the other hand, condition (LU) serves as a local, quantitative connectedness condition near N. It also ensures that 'pure Neumann' connected components maintain a uniformly positive diameter. This notion originates from Jones' definition of uniform domains [117] and was introduced by Bechtel, Egert, and Haller in [40].

The combination of (Fat) and (LU) yields the key tool for the proof of the main result: a weak 2-Poincaré inequality for boundary balls that correctly scales in the radius and respects the boundary conditions. We denote this key condition as (P).

The proof of (P) relies on two main ingredients: the extension operator constructed by Bechtel, Brown, Haller, and Tolksdorf in [39], along with Maz'ya's capacitary Poincaré inequality [122]. Notably, we show that under the sole assumption of (LU), condition (Fat) is equivalent to (P). This suggests that our geometric framework is nearly optimal.

Our equivalence theorem builds upon prior work by Auscher [11] and Auscher and Tchamitchian [29] on $\mathbb{R}^d$, as well as their work for pure Dirichlet/Neumann boundary conditions on special Lipschitz domains [30]. Our contributions include the treatment of mixed Dirichlet/Neumann boundary conditions, a unified and streamlined approach, and an extension of admissible geometries. In particular, we do not only recover the known results for pure Dirichlet and Neumann boundary conditions.

We then investigate whether any of the properties (D), (G), or (H) hold in general. To approach this question systematically, we adopt an axiomatic framework and establish the following key results:

In dimension $d \geq 3$, property (H) always holds when A is real. This follows from a combination of De Giorgi's classical regularity results [67, 97] and the non-linear techniques of DiBenedetto [72, 73]. A crucial step in our argument leverages the fact that (Fat) is an open ended condition, as established by Lewis [128] and Mikkonen [134]. Notably, we use a non-linear technique for a linear operator, which is surprising in itself.

When $d = 2$, we prove that every elliptic operator L satisfies (G). The proof is based on an extrapolation technique 'à la Šneĭberg'. To achieve this, we slightly strengthen (Fat) by requiring that D is a $(d-1)$-*set*.

These results appeared in [45], a joint work with Ciani and Egert.

Building on the geometric and operator theoretic setting developed in this

chapter, we now turn to an important application: the study of Hardy spaces.

Hardy spaces adapted to elliptic operators on open sets

The real Hardy spaces $\mathrm{H}^p(\mathbb{R}^d)$ play a central role in modern harmonic analysis, providing a natural extension of the Lebesgue scale. In particular, the H^p-boundedness of $(\mathrm{e}^{-tL})_{t>0}$ for some $p \in (0,1)$ expresses that $p_-(L)$ is 'below $p = 1$'. Auscher and Egert observed in [20] that the condition $p_-(L) < 1$ is equivalent to L satisfying property (G).

From now on, we focus on the case $p = 1$. The Hardy space $\mathrm{H}^1(\mathbb{R}^d)$ serves as the natural substitute for $\mathrm{L}^1(\mathbb{R}^d)$ in various contexts, including singular integral operators, elliptic boundary value problems, and interpolation theory. Many characterizations of $\mathrm{H}^1(\mathbb{R}^d)$ have been established, such as atomic, maximal, and square function characterizations. They already foreshadow that $\mathrm{H}^1(\mathbb{R}^d)$ bridges the harmonic analysis of second-order elliptic operators (prototypically the Laplacian) and the underlying geometry.

The goal of this chapter is to strengthen this connection by providing a geometric description of the L-adapted (square function) Hardy space $\mathrm{H}^1_L(O)$. Specifically, assuming that L satisfies a variant of property (G), we characterize $\mathrm{H}^1_L(O)$ in terms of D-adapted atoms as the building blocks:

$$\mathrm{H}^1_L(O) = \mathrm{H}^1_D(O).$$

To complement the geometric framework introduced in the previous chapter, we append two assumptions to (Fat) and (LU):

(UITC) O is uniformly interior thick.

(UP) D is uniformly porous.

The first condition ensures the uniform comparability $|O \cap B| \simeq |B|$ for all balls centered in O with radii up to the diameter of O. In particular, O is a space of homogeneous type. The second condition guarantees the existence of corkscrew points near D.

Our main theorem is based on earlier results by Auscher and Russ [26], Coifman and Weiss [63], and Miyachi [136]. As in the previous chapter, our contributions include the treatment of mixed boundary conditions, unified proofs, and an expanded class of admissible geometries. We further refine this

theory by studying the Laplacian in more detail. Indeed, leveraging results from Bui, Duong and Ly [48], as well as from Hofmann, Lu, Mitrea, Mitrea and Yan [112], we show that the square and maximal function characterizations of $\mathrm{H}^1_{-\Delta_D}(O)$ all coincide with the geometric (atomic) Hardy space $\mathrm{H}^1_D(O)$.

This chapter is based on joint work with Bechtel [38].

Leaving behind the realm of Gaussian estimates and Hardy spaces but retaining the setting of mixed boundary conditions and non-smooth geometries, we now turn to the final part of this thesis.

Bounded functional calculus for dynamical boundary conditions

Dynamical boundary conditions naturally arise in various scientific contexts [70, 71, 154], governing the time evolution of a physical state on a lower dimensional surface $\Sigma \subseteq \overline{O}$, say of Hausdorff dimension $\ell \in (d-2, d)$. To build intuition, let us consider the following parabolic problem as an example:

$$\begin{aligned}
\partial_t u - \operatorname{div}(A\nabla u) &= f|_{O\setminus\Sigma} && \text{in } (0,T)\times(O\setminus\Sigma),\\
\partial_t u + \nu\cdot A\nabla u &= f|_{\Sigma\cap\partial O} && \text{on } (0,T)\times(\Sigma\cap\partial O),\\
\partial_t u + \nu_{\Sigma\cap O}\cdot A\nabla u &= f|_{\Sigma\cap O} && \text{on } (0,T)\times(\Sigma\cap O),\\
u &= 0 && \text{on } (0,T)\times D,\\
\nu\cdot A\nabla u &= 0 && \text{on } (0,T)\times\partial O\setminus(D\cup\Sigma),\\
u(0) &= u_0 && \text{in } O\cup\Sigma.
\end{aligned}$$

Here, $\nu_{\Sigma\cap O}$ represents a jump in the outer unit normal ν. Keeping in mind that Σ is a Lebesgue null-set, we interpret these equations as an abstract Cauchy problem

$$\begin{aligned}
u'(t) + \mathscr{L}u(t) &= f(t) \qquad (t\in(0,T]),\\
u(0) &= u_0,
\end{aligned}$$

in the Hilbert space

$$\mathbb{L}^2 = \mathrm{L}^2(O\cup\Sigma, \mathrm{d}x\oplus\mathrm{d}\mathcal{H}^\ell) \cong \mathrm{L}^2(O,\mathrm{d}x)\oplus\mathrm{L}^2(\Sigma,\mathrm{d}\mathcal{H}^\ell).$$

Unlike in the 'static' setting, the operator $\mathscr{L}$ is associated to a pair (a, j). As before, the form a, defined on $V = \mathrm{W}^{1,2}_D(O)$, accounts for the mixed boundary conditions, while the *identification operator* $j\colon V\to\mathbb{L}^2, j(u) = u|_O\oplus\operatorname{Tr}(u)$

captures the dynamics on Σ. To define this framework rigorously, we require a bounded trace operator $\mathrm{Tr}\colon V \to \mathrm{L}^2(\Sigma, \mathrm{d}\mathcal{H}^\ell)$ satisfying the natural identity $\mathrm{Tr}(u) = u|_\Sigma$ for $u \in V \cap \mathrm{C}(\overline{O})$. Remarkably, we show that merely assuming (LU) and that Σ is an *upper ℓ-set* for some $\ell \in (d-2, d)$ suffices. To reach this goal, we opt for an axiomatic approach that maximizes flexibility while isolating the key issues: pointwise existence of a trace (capacity theory) and boundedness of the trace operator (geometric measure theory). Our main tools include the extension operator from [39] and classical trace results of Jonsson and Wallin [118].

Beyond these geometric considerations, we investigate whether $\mathscr{L}$ admits a bounded functional calculus in $\mathbb{L}^p$. This property is crucial, as it guarantees maximal regularity for the abstract Cauchy problem. While this is known for $p = 2$ due to the m-accretivity of $\mathscr{L}$ [9], results for $p \neq 2$ remain scarce. So far, only the case of real coefficients has been studied, using a positivity criterion [84].

We close this gap by employing the *heat flow method*, a powerful tool most recently popularized by Carbonaro and Dragičević [52, 53, 55, 56]. This approach allows us to establish a bilinear estimate for the gradient of the heat semigroup associated to $\mathscr{L}$, provided that A is p-elliptic.

As a first step, we show that $(\mathrm{e}^{-t\mathscr{L}})_{t>0}$ extends to a bounded analytic semigroup in $\mathbb{L}^p$, thereby generalizing Nittka's invariance criterion for closed and convex sets [137]. Next, we address the role of j. A crucial challenge in applying the heat flow method is verifying that the identification operator j commutes with 'V-preserving' non-linear maps. This step plays a fundamental role in ensuring that the method can be applied in our setting. With this bilinear estimate at our disposal, we follow a standard approach to deduce weak quadratic estimates for $\mathscr{L}$. By a classical result of Cowling, Doust, McIntosh, and Yagi [64], this immediately implies that $\mathscr{L}$ has a bounded H^∞-calculus in $\mathbb{L}^p$.

We complement this chapter by studying regularity results for dynamical elliptic problems in a variational setup. Our approach stems from mapping properties of the form a and a 'transference principle' for $\mathscr{L}^{-1}$. This allows us to incorporate deep results from the static theory, such as heat kernel bounds (see Chapter 5) or Kato square root estimates [40]. These results are particularly valuable in applications to semilinear parabolic problems, see, e.g., [86].

This chapter is based on [47], a joint work with Egert and Rehberg.

Accepted papers

The articles [45] and [46] have been published.

Preprints

The articles [18], [38], and [47] have been submitted for publication.

Unpublished work

Chapter 4 contains unpublished material. The results in Section 4.3 were obtained by the author of this thesis, while Sections 4.4 and 4.5 present joint work with Auscher and Egert.

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Zusammenfassung in Deutscher Sprache

Diese Arbeit befasst sich mit der L^p-Theorie elliptischer Operatoren in Divergenzform $L = -\operatorname{div}(A\nabla\cdot)$. Hierbei ist A eine elliptische, Matrix-wertige Funktion mit beschränkten, messbaren und komplexen Koeffizienten. Das Ziel dieser Thesis ist es, diese Theorie in verschiedener Art und Weise zu bereichern. Wir zeigen neue Resultate auf dem Ganzraum $\mathbb{R}^d$, verallgemeinern Aussagen im Kontext nicht glatter Geometrien, wenn L *gemischte Randbedingungen* dekodiert, und beweisen neue Resultate, falls L *dynamische Randbedingungen* modelliert.

Im ersten Kapitel bieten wir eine Übersicht über alle relevanten technischen Werkzeuge dieser Arbeit: Funktionenräume, Kapazitätstheorie, geometrische Maßtheorie, $\mathrm{L}^p - \mathrm{L}^q$-Abschätzungen für Operatorfamilien, der H^∞-Funktionalkalkül und die wichtigsten bekannten Resultate bezüglich der L^p-Theorie von L.

Im zweiten Kapitel beschäftigen wir uns mit expliziten Verbesserungen der *kritischen Zahlen* $p_\pm(L)$ und $q_\pm(L)$ in Abhängigkeit von Elliptizität und Raumdimension. Die kritischen Zahlen charakterisieren die Exponenten $p \in [1,\infty]$ für welche die Wärmeleitungshalbgruppe $(\mathrm{e}^{-t^2L})_{t>0}$ und die assoziierte Gradientenfamilie $(t\nabla\,\mathrm{e}^{-t^2L})_{t>0}$ gleichmäßig L^p-beschränkt sind. In diesem Zusammenhang beweisen wir auch ein optimales Störungsresultat für die Eigenschaft, dass der Wärmeleitungskern von $(\mathrm{e}^{-t^2L})_{t>0}$ durch eine Hölder stetige Funktion gegeben ist, die außerdem punktweise Gaußabschätzungen erfüllt.

Im dritten und vierten Kapitel studieren wir *a priori Abschätzungen* für Lösungen elliptischer Randwertprobleme der Form $\boldsymbol{L}U = -\operatorname{div}_{t,x}\mathsf{A}(x)\nabla_{t,x}U = 0$ im oberen Halbraum $\mathbb{R}^{1+d}_+$. Dazu benutzen wir den ersten Ordnungs Ansatz

und reformulieren die Gleichung $\boldsymbol{L}U = 0$ als Evolutionsgleichung $F'(t) + \mathsf{DB}F(t) = 0$ für den konormalen Gradienten von U. Hierbei ist D ein Differentialoperator erster Ordnung, auch *Dirac Operator* genannt, und $\mathsf{B} = \widehat{\mathsf{A}}$ ist ein beschränkter Multiplikationsoperator, der durch eine algebraische Transformation aus A entsteht. Das Problem a priori Abschätzungen für Lösungen zu $\boldsymbol{L}U = 0$ zu finden, reduziert sich dann darauf, DB-adaptierte Hardy-Räume mit klassischen Funktionenräumen in der Lebesgue- und Hardy-Raum Skala zu identifizieren.

In Kapitel drei finden wir äquivalente Charakterisierungen des *Auscher–Stahlhut-Intervalls* $I(\mathsf{DB})$ mittels des Operators $\boldsymbol{L}$ von zweiter Ordnung. Hierbei ist $I(\mathsf{DB})$ das maximale offene Intervall um $p = 2$, in dem DB in L^p bisektoriell ist. Dieses Intervall ist von Bedeutung, um explizite Verbesserungen für das Identifizierungsintervall $\mathcal{H}(\mathsf{DB})$ der DB-adaptierten Hardy-Räume zu finden. Wir beschreiben $I(\mathsf{DB})$ auf zwei Arten: Einerseits ist $I(\mathsf{DB})$ das maximale offene Intervall um $p = 2$, in welchem der Lax–Milgram Operator $\boldsymbol{L}\colon \dot{\mathrm{W}}^{1,p}(\mathbb{R}^{1+d}) \to \dot{\mathrm{W}}^{-1,p}(\mathbb{R}^{1+d})$ ein Isomorphismus ist. Andererseits können wir $I(\mathsf{DB})$ durch das größte $p > 2$ charakterisieren, sodass der Gradient lokal $\boldsymbol{L}$-harmonischer Funktionen eine umgekehrte schwache Hölder-Abschätzung mit Exponent p erfüllt.

Im vierten Kapitel charakterisieren wir den linken Endpunkt von $\mathcal{H}(\mathsf{DB})$, welchen wir mit $h_-(\mathsf{DB})$ bezeichnen, mittels der Beschränktheit von DB-Resolventen. Danach beweisen wir unter der Annahme, dass A obere Dreiecksgestalt hat und A p-elliptisch ist, die Abschätzung $h_-(\mathsf{DB}) \leq (pd/d-2)'$. Diese Verbesserung interpoliert zwischen der allgemeinen Abschätzung $h_-(\mathsf{DB}) \leq 2d/(d+2)$ und $h_-(\mathsf{DB}) < 1$, wenn A reell ist.

Für den Rest der Arbeit verlassen wir den Ganzraum-Fall und konzentrieren uns auf die Situation, dass L gemischte Dirichlet/Neumann Randbedingungen auf einer irregulären offenen Menge $O \subseteq \mathbb{R}^d$ modelliert.

Im fünften Kapitel beschäftigen wir uns erneut mit der Eigenschaft, dass der Kern der Halbgruppe $(\mathrm{e}^{-t^2L})_{t>0}$ durch eine Hölder stetige Funktion gegeben ist, die punktweise Gaußabschätzungen erfüllt (Eigenschaft (G)). Das erste Hauptresultat ist, Eigenschaft (G) durch quantitative Regularitätseigenschaften lokal L-harmonischer Funktionen auszudrücken. Unsere geometrischen Annahmen sind hierbei so allgemein wie möglich gewählt. Im zweiten Abschnitt dieses Kapitels beweisen wir dann unter denselben geometrischen Annahmen, dass eine der lokalen Eigenschaften (und damit auch die andere) stets erfüllt ist, falls A reelle Einträge besitzt.

Das Hauptresultat des sechsten Kapitels ist eine geometrische Charakteri-

sierung des L-adaptierten Hardy-Raums $\mathrm{H}^1_L(O)$. Dazu führen wir einen atomaren Hardy-Raum $\mathrm{H}^1_D(O)$ ein, der die Dirichlet/Neumann Randbedingungen respektiert und zeigen die Gleichheit $\mathrm{H}^1_L(O) = \mathrm{H}^1_D(O)$ unter minimalen geometrischen Annahmen und der Zusatzbedingung, dass L eine Variante von Eigenschaft (G) erfüllt. Es folgen Anwendungen für den Laplace Operator und weitere äquivalente Charakterisierungen von $\mathrm{H}^1_L(O)$ mittels Maximalfunktionen.

Zu guter Letzt untersuchen wir L im Kontext dynamischer Randbedingungen. Wir zeigen unter minimalen geometrischen Anforderungen und der algebraischen Annahme der p-Elliptizität an A, dass L einen beschränkten H^∞-Funktionalkalkül in einem hybriden L^p-Raum besitzt. Das Resultat impliziert maximale parabolische Regularität. Im letzten Abschnitt des Kapitels studieren wir Regularitätsaussagen von semilinearen parabolischen Gleichungen im Kontext dynamischer Randbedingungen.

CHAPTER 1

Function spaces, geometric and analytical tools

1.1 Function spaces

In this section, we introduce all relevant function spaces.

1.1.1 Function spaces on $\mathbb{R}^d$

Let us begin with the classical smoothness spaces.

Definition 1.1.1. For $p \in [1, \infty)$, we define the **homogeneous Sobolev space**

$$\dot{\mathrm{W}}^{1,p} := \Big\{ u \in \mathrm{L}^p_{\mathrm{loc}} \colon \|u\|_{\dot{\mathrm{W}}^{1,p}} := \|\nabla u\|_p < \infty \Big\} / \mathbb{C},$$

and denote its anti-dual space by

$$\dot{\mathrm{W}}^{-1,p'} := (\dot{\mathrm{W}}^{1,p})^*.$$

Definition 1.1.2. The space $\mathcal{Z}$ consists of all Schwartz functions f with $(\partial^\alpha \widehat{f})(0) = 0$ for all $\alpha \in \mathbb{N}_0^n$. We denote by $\mathcal{Z}'$ its topological dual space.

Lemma 1.1.3 (Universal approximation technique, [157, Sec. 5.1.5])**.** *For every $f \in \mathcal{Z}'$ there is a sequence $(f_n)_n \subseteq \mathcal{Z}$ such that $f_n \to f$ in each L^p and $\dot{\mathrm{W}}^{1,p}$ which f belongs to.*

Next, we introduce the real Hardy spaces H^p and we refer to [152, Chap. III] for more.

Definition 1.1.4. Let $p \in (1_*, 1]$, $a\colon \mathbb{R}^d \to \mathbb{C}^m$ be a measurable function and $B = B(x,r) \subseteq \mathbb{R}^d$ be a ball. We say that a is an **$\mathbf{L^2}$-atom for $\mathbf{H^p}$ associated with $\boldsymbol{B}$** if

$$\operatorname{supp}(a) \subseteq B, \quad \|a\|_2 \leq |B|^{\frac{1}{2}-\frac{1}{p}} \quad \& \quad \int_{\mathbb{R}^d} a \,\mathrm{d}x = 0.$$

We refer to these conditions as **localization**, **size** and **cancellation condition**.

Definition 1.1.5. Let $p \in (1_*, 1]$. The real **Hardy space $\mathbf{H^p}$** consists of those $f \in \mathcal{S}'$ for which there is a sequence $(\lambda_n)_n \in \ell^p$ and L^2-atoms a_n for H^p such that the **atomic decomposition** $f = \sum_n \lambda_n a_n$ holds in $\mathcal{S}'$. We endow this space with the (quasi-)norm

$$\|f\|_{\mathrm{H}^p} \coloneqq \inf \|(\lambda_n)_n\|_{\ell^p},$$

where the infimum is taken over all atomic decompositions of f.

Remark 1.1.6. For $p = 1$, the space H^1 can be viewed as a subspace of L^1 with continuous inclusion. In particular, every element of H^1 is a proper function.

1.1.2 Function spaces adapted to mixed boundary conditions

The aim of this subsection is to introduce all relevant function spaces that involve boundary conditions. To this end, we work with

Assumption 1.1.7. *Let $d \geq 1$, $O \subseteq \mathbb{R}^d$ be open and $\Xi \subseteq \mathbb{R}^d$ be non-empty. The **Dirichlet part** $D \subseteq \partial O \cap \overline{\Xi}$ is closed. We call its complement $N \coloneqq \partial O \setminus D$ the **Neumann part**.*

Definition 1.1.8. Let $\mathrm{C}_{\mathrm{c}}^\infty(O)$ be the space of smooth and compactly supported functions on O. We set

$$\mathrm{C}_D^\infty(\Xi) \coloneqq \{\varphi|_\Xi \colon \varphi \in \mathrm{C}_{\mathrm{c}}^\infty(\mathbb{R}^d \setminus D)\}.$$

Definition 1.1.9. For $p \in [1, \infty)$, we define $\mathrm{W}_D^{1,p}(O)$ as the closure of $\mathrm{C}_D^\infty(O)$ with respect to the $\mathrm{W}^{1,p}(O)$-norm

$$\|\cdot\|_{1,p} \coloneqq (\|\cdot\|_p^p + \|\nabla \cdot\|_p^p)^{\frac{1}{p}}.$$

For $p = 2$, we put

$$V \coloneqq \mathrm{W}_D^{1,2}(O),$$

and call V the **form domain**.

Remark 1.1.10. In general, $\mathrm{W}^{1,2}_{\varnothing}(O)$ is a proper subspace of $\mathrm{W}^{1,2}(O)$ and models so-called *good Neumann boundary conditions*. However, if we have a bounded Sobolev extension operator $\mathcal{E}\colon \mathrm{W}^{1,2}(O) \to \mathrm{W}^{1,2}(\mathbb{R}^d)$ at our disposal, then equality holds, because $\mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^d)$ is dense in $\mathrm{W}^{1,2}(\mathbb{R}^d)$.

The Sobolev spaces $\mathrm{W}^{1,p}_D(O;\mathbb{R})$ are closed under truncation in the following sense.

Lemma 1.1.11 ([87, Lem. 2.2 (a)])**.** *Let $p \in (1,\infty)$, $u \in \mathrm{W}^{1,p}_D(O;\mathbb{R})$ and $k \geq 0$. Then $(u-k)^+$ and $u \wedge k$ are contained in $\mathrm{W}^{1,p}_D(O;\mathbb{R})$.*

Next, we introduce a terminology that turns out to be useful in this thesis.

Definition 1.1.12. Let $d \geq 3$. We say that V has the

- **inhomogeneous embedding property** if there is $C_E > 0$ such that

$$\|u\|_{2^*} \leq C_E \|u\|_{1,2} \qquad (u \in V).$$

- **homogeneous embedding property** if there is $C_E > 0$ such that

$$\|u\|_{2^*} \leq C_E \|\nabla u\|_2 \qquad (u \in V).$$

Finally, we need Hölder spaces adapted to boundary conditions.

Definition 1.1.13. Let $\mu \in (0,1]$. The

- **homogeneous Hölder space $\dot{\mathrm{C}}^{\mu}_{D}(\Xi)$** consists of all functions $u\colon \Xi \to \mathbb{C}$ whose continuous extension to $\overline{\Xi}$ vanishes in D and for which the seminorm

$$[u]^{(\mu)}_{\Xi} \coloneqq \sup_{x,y\in\Xi, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\mu} \quad \text{is finite.}$$

- **Hölder space $\mathrm{C}^{\mu}_{D}(\Xi)$** consists of all functions $u\colon \Xi \to \mathbb{C}$ whose continuous extension to $\overline{\Xi}$ vanishes in D and for which the norm

$$\|u\|_{\mathrm{C}^\mu(\Xi)} \coloneqq \|u\|_{\mathrm{L}^\infty(\Xi)} + [u]^{(\mu)}_{\Xi} \quad \text{is finite.}$$

If $D = \varnothing$, we also write $\dot{\mathrm{C}}^{\mu}_{\varnothing}(\Xi) = \dot{\mathrm{C}}^{\mu}(\Xi)$ and $\mathrm{C}^{\mu}_{\varnothing}(\Xi) = \mathrm{C}^{\mu}(\Xi)$.

1.1.3 Morrey and Campanato spaces

The L^2-adapted Morrey and Campanato spaces play a key role in the theory of elliptic regularity. We refer to [50] or [96, Chap. 3] for more information.

Definition 1.1.14. Let $\kappa \in [0,d)$, $x \in \overline{\Xi}$ and $r > 0$. We define the **local Morrey space**

$$\mathcal{M}^{\kappa}_{x,r}(\Xi) := \Big\{ u \in \mathrm{L}^2(\Xi) \colon \|u\|_{\mathcal{M}^{\kappa}_{x,r}(\Xi)} := \sup_{\rho\in(0,r]} \rho^{-\frac{\kappa}{2}} \|u\|_{\mathrm{L}^2(\Xi(x,\rho))} < \infty \Big\}.$$

In addition, the **global Morrey space** consists of all $u \in \mathrm{L}^2(\Xi)$ for which the norm

$$\|u\|_{\mathcal{M}^{\kappa}(\Xi)} := \sup_{x\in\overline{\Xi}, r<\operatorname{diam}(\Xi)} \|u\|_{\mathcal{M}^{\kappa}_{x,r}(\Xi)} \quad \text{is finite.}$$

Definition 1.1.15. Let $\kappa \in [0,d+2)$, $x \in \overline{\Xi}$ and $r > 0$. We define the **local Campanato space**

$$\mathcal{L}^{\kappa}_{x,r}(\Xi) := \Big\{ u \in \mathrm{L}^2(\Xi) \colon [u]_{\mathcal{L}^{\kappa}_{x,r}(\Xi)} := \sup_{\rho\in(0,r]} \rho^{-\frac{\kappa}{2}} \|u - (u)_{\Xi(x,\rho)}\|_{\mathrm{L}^2(\Xi(x,\rho))} < \infty \Big\}.$$

In addition, the **global Campanato space** consists of all $u \in \mathrm{L}^2(\Xi)$ for which the seminorm

$$[u]_{\mathcal{L}^{\kappa}(\Xi)} := \sup_{x\in\overline{\Xi}, r<\operatorname{diam}(\Xi)} [u]_{\mathcal{L}^{\kappa}_{x,r}(\Xi)} \quad \text{is finite.}$$

The following results furnish the main properties that we use in this thesis.

Lemma 1.1.16 ([87, Lem. 3.1])**.** *Let $\gamma \in (0,1)$, $r \in (0,1]$, $\kappa \in [0,d+2)$ and $x, y \in \overline{\Xi}$ with $|\Xi(x,\rho)| \wedge |\Xi(y,\rho)| \geq \gamma |B(x,\rho)|$ for all $\rho \in (0,r]$. The following properties hold true with implicit constants depending only on $[d,\gamma,\kappa]$:*

(i) *If $\kappa < d$, then $\mathcal{M}^{\kappa}_{x,r}(\Xi) \cong \mathcal{L}^{\kappa}_{x,r}(\Xi)$ with estimate*

$$[u]_{\mathcal{L}^{\kappa}_{x,r}(\Xi)} \leq \|u\|_{\mathcal{M}^{\kappa}_{x,r}(\Xi)} \lesssim r^{-\frac{d}{2}} \|u\|_{\mathrm{L}^2(\Xi(x,r))} + [u]_{\mathcal{L}^{\kappa}_{x,r}(\Xi)} \qquad (u \in \mathcal{M}^{\kappa}_{x,r}(\Xi)).$$

(ii) *If $\kappa > d$ and $u \in \mathcal{L}^{\kappa}_{x,r}(\Xi)$, then $u(x) := \lim_{\rho\searrow 0} (u)_{\Xi(x,\rho)}$ exists and*

$$|u(x)| \lesssim r^{\frac{\kappa-d}{2}} [u]_{\mathcal{L}^{\kappa}_{x,r}(\Xi)} + r^{-d} \|u\|_{\mathrm{L}^1(\Xi(x,r))}.$$

(iii) *If* $\kappa > d$, $|x-y| \le r/2$ *and* $u \in \mathcal{L}^\kappa_{x,r}(\Xi) \cap \mathcal{L}^\kappa_{y,r}(\Xi)$, *then*

$$|u(x)-u(y)| \lesssim ([u]_{\mathcal{L}^\kappa_{x,r}(\Xi)} + [u]_{\mathcal{L}^\kappa_{y,r}(\Xi)})|x-y|^{\frac{\kappa-d}{2}}.$$

In particular, if $x \neq y$, *then we have*

$$\frac{|u(x)-u(y)|}{|x-y|^{(\kappa-d)/2}} \lesssim [u]_{\mathcal{L}^\kappa_{x,r}(\Xi)} + [u]_{\mathcal{L}^\kappa_{y,r}(\Xi)} + r^{-\frac{\kappa-d}{2}-d}(\|u\|_{\mathrm{L}^1(\Xi(x,r))} + \|u\|_{\mathrm{L}^1(\Xi(y,r))}).$$

1.2 Capacity theory

In this section, we introduce the notion of relative or variational capacities. This concept serves as an important tool in Chapters 5, 6 and 7. We refer to [2] for general background on capacities. Throughout this section, we assume that $d \ge 2$.

1.2.1 Relative capacities

Definition 1.2.1. Let $p \in (1,d]$, $U \subseteq \mathbb{R}^d$ be open and $K \subseteq U$ be compact. We define the ***p*-capacity of the condenser (K,U)** as

$$\operatorname{cap}_p(K;U) \coloneqq \inf\left\{\|\nabla u\|^p_{\mathrm{L}^p(U)} \colon u \in \mathrm{C}^\infty_{\mathrm{c}}(U;\mathbb{R}) \text{ with } u \ge 1 \text{ pointwise on } K\right\}.$$

Definition 1.2.2. Let $F \subseteq \mathbb{R}^d$ be closed, $\widehat{F} \subseteq F$ and $p \in (1,d]$. We call F **locally *p*-fat in $\widehat{F}$** if there is $C > 0$ such that

$$Cr^{d-p} \le \operatorname{cap}_p(\overline{B(x,r)} \cap F; B(x,2r)) \qquad (x \in \widehat{F}, r \in (0,1]). \tag{1.1}$$

If $\widehat{F} = F$, then we say that F **is locally *p*-fat**.

For the reader's convenience, we include some results related to capacities that we could not find in the literature.

Lemma 1.2.3. *We can make the following replacements in* (1.1).

(i) *For all* $\kappa > 1$, *we can exchange* $B(x,2r)$ *by* $B(x,\kappa r)$.

(ii) *For each* $r_0 > 0$, *we can replace* $r \in (0,1]$ *by* $r \in (0,r_0]$.

(iii) *We can replace the requirement* $u \ge 1$ *on* $\overline{B(x,r)} \cap F$ *by* $u = 1$ *in an open neighborhood of* $\overline{B(x,r)} \cap F$ *and* $0 \le u \le 1$ *everywhere.*

Proof of (i). We set $B = B(x, r)$. The interesting direction is when fatness is formulated with κB and we want to switch to $2B$. We pick $u \in \mathrm{C}_{\mathrm{c}}^{\infty}(2B)$ such that $u \geq 1$ on $\overline{B} \cap F$. Now, we pick $\eta \in \mathrm{C}_{\mathrm{c}}^{\infty}(\kappa B)$ with values in $[0, 1]$, $\eta = 1$ on $\overline{B}$ and $\|\nabla\eta\|_{\mathrm{L}^\infty} \lesssim_\kappa r^{-1}$. Poincaré's inequality yields

$$\mathrm{cap}_p(\overline{B} \cap F; \kappa B) \leq \|\nabla(u\eta)\|_{\mathrm{L}^p(\kappa B)}^p \lesssim_{\kappa,d,p} \|\nabla u\|_{\mathrm{L}^p(2B)}^p,$$

and thus

$$\mathrm{cap}_p(\overline{B} \cap F; \kappa B) \lesssim \mathrm{cap}_p(\overline{B} \cap F; 2B).$$

Proof of (ii). This follows from (i) and the monotonicity of capacities in the first argument.

Proof of (iii). For the interesting direction we pick $u \in \mathrm{C}_{\mathrm{c}}^{\infty}(2B; \mathbb{R})$ with $u \geq 1$ on $\overline{B} \cap F$. We fix $\varepsilon \in (0, 1)$ and put $v_\varepsilon \coloneqq ((1-\varepsilon)^{-1}u \wedge 1) \vee 0$. Then v_ε is continuous with

$$v_\varepsilon = \begin{cases} 1 & (u \geq 1-\varepsilon), \\ (1-\varepsilon)^{-1}u & (0 \leq u \leq 1-\varepsilon), \\ 0 & (u \leq 0). \end{cases}$$

In particular, as $\overline{B} \cap F$ is compact and u is continuous, there is some $\delta > 0$ such that $v_\varepsilon = 1$ on $(\overline{B} \cap F)_\delta$. We set $v_n \coloneqq \eta_n * v_\varepsilon$ for all $n \in \mathbb{N}$, where $\eta_n(x) = n^d\eta(nx)$ is a standard mollifier. Then v_n is smooth, $[0, 1]$-valued and there is some $N \in \mathbb{N}$ with $v_n = 1$ on $(\overline{B} \cap F)_{\delta/2}$ and v_n has compact support in $2B$ for all $n \geq N$. Next, using Young's inequality for convolutions and $\nabla v_\varepsilon = (1-\varepsilon)^{-1}\mathbf{1}_{[0\leq u\leq 1-\varepsilon]}\nabla u$, we get

$$\|\nabla v_n\|_{\mathrm{L}^p(2B)}^p \leq \|\nabla v_\varepsilon\|_{\mathrm{L}^p(2B)}^p \leq (1-\varepsilon)^{-p}\|\nabla u\|_{\mathrm{L}^p(2B)}^p.$$

Hence, we derive

$$\inf_w \|\nabla w\|_{\mathrm{L}^p(2B)}^p \leq (1-\varepsilon)^{-p}\|\nabla u\|_{\mathrm{L}^p(2B)}^p,$$

where the infimum is taken over all $[0, 1]$-valued $w \in \mathrm{C}_{\mathrm{c}}^{\infty}(2B)$ that are 1 in an open neighborhood of $\overline{B} \cap F$. We conclude by sending $\varepsilon \to 0$. □

1.2.2 Riesz capacities

Lewis' celebrated result on the self-improvement property of p-fatness [128] will be crucial in Section 5.6. However, the notion of Riesz capacities is used in the main result. We use the opportunity to include a proof that gives rise

to an equivalent notion of p-fatness when $p \in (1,d)$. Let us mention that this does not cover Lewis' result in the case $p = 2 = d$ and we refer to [134] for a proof using variational capacities.

We denote by $I_1(x) \coloneqq |x|^{1-d}$ the Riesz potential of order 1. Given $f \in \mathrm{L}^p$ with $f \geq 0$ a.e., the convolution

$$(I_1 * f)(x) = \int_{\mathbb{R}^d} |x-y|^{1-d} f(y)\,\mathrm{d}y$$

is defined for all $x \in \mathbb{R}^d$ as a real number in $[0,\infty]$. Using classical results for the Riesz transform [150, Chap. III.1], we obtain for all non-negative $f \in \mathrm{C}_\mathrm{c}^\infty$ that

$$\|\nabla(I_1 * f)\|_{\mathrm{L}^p} \lesssim_{d,p} \|f\|_{\mathrm{L}^p}. \tag{1.2}$$

Definition 1.2.4. Let $p \in (1,d)$ and $F \subseteq \mathbb{R}^d$ be closed. We define the **Riesz capacity of F** by

$$R_{1,p}(F) \coloneqq \inf\Big\{\|f\|_{\mathrm{L}^p}^p \colon f \in \mathrm{L}^p \text{ with } f \geq 0 \text{ a.e. and } I_1 * f \geq 1 \text{ on } F\Big\}.$$

Let us explicitly prove that the Riesz and variational capacity give rise to the same notion of p-fatness when $p \in (1,d)$.

Lemma 1.2.5. *Let $p \in (1,d)$ and $F \subseteq \mathbb{R}^d$ be closed. Then F is locally p-fat if and only if there is $C > 0$ such that*

$$Cr^{d-p} \leq R_{1,p}(\overline{B(x,r)} \cap F) \qquad (x \in F, r \in (0,1]). \tag{1.3}$$

Proof. First, we assume that F is locally p-fat with lower bound $C > 0$ and abbreviate $B \coloneqq B(x,r)$. We fix $f \in \mathrm{L}^p$ with $f \geq 0$ a.e. and $u \coloneqq I_1 * f \geq 1$ on $\overline{B} \cap F$. For $\varepsilon \in (0,1)$ we set $f_\varepsilon \coloneqq (1-\varepsilon)^{-1} f$. We define for $n \in \mathbb{N}$ the non-negative functions $f_n \coloneqq (\mathbf{1}_{B(0,n)} f_\varepsilon) * \eta_n \in \mathrm{C}_\mathrm{c}^\infty$, where $\eta_n(x) = n^d \eta(nx)$ is again a standard mollifier. We put $u_n \coloneqq I_1 * f_n$ and claim that there is some $N \in \mathbb{N}$ such that $u_n \geq 1$ on $\overline{B} \cap F$ for all $n \geq N$.

Indeed, let us assume that this is not the case. Then there is a strictly increasing sequence $(n_j)_j \subseteq \mathbb{N}$ and a sequence $(x_j)_j \subseteq \overline{B} \cap F$ such that $u_{n_j}(x_j) < 1$ for all $j \in \mathbb{N}$. Since $\overline{B} \cap F$ is compact, there is some $x \in \overline{B} \cap F$ such that $x_j \to x$ along a subsequence. Using Fatou's lemma and passing to a further subsequence to guarantee $f_{n_j} \to f_\varepsilon$ a.e., we get $(I_1 * f_\varepsilon)(x) \leq 1$. Thus, $u(x) \leq 1 - \varepsilon < 1$, which is a contradiction to our assumption.

From now on, we let $n \geq N$.

If $C\frac{r^{d/p}}{4} \leq \|u_n\|_{\mathrm{L}^p(2B)}$, then we obtain by Hölder's inequality and a Sobolev embedding that

$$r^{\frac{d}{p}-1} \lesssim r^{-1}\|u_n\|_{\mathrm{L}^p(2B)} \lesssim \|u_n\|_{\mathrm{L}^{p^*}} \lesssim \|\nabla u_n\|_{\mathrm{L}^p}.$$

Now, we consider the case $\|u_n\|_{\mathrm{L}^p(2B)} < C\frac{r^{d/p}}{4}$. We let $\varphi \in \mathrm{C}_{\mathrm{c}}^\infty(2B)$ be $[0,1]$-valued with $\varphi = 1$ on $\overline{B}$ and $\|\nabla\varphi\|_{\mathrm{L}^\infty} \leq 2/r$. Using that F is locally p-fat, we estimate

$$\begin{aligned} Cr^{\frac{d}{p}-1} &\leq \|\nabla(u_n\varphi)\|_{\mathrm{L}^p(2B)} \\ &\leq \|\nabla u_n\|_{\mathrm{L}^p(2B)} + 2r^{-1}\|u_n\|_{\mathrm{L}^p(2B)} \\ &\leq \|\nabla u_n\|_{\mathrm{L}^p} + \frac{C}{2}r^{\frac{d}{p}-1} \end{aligned}$$

and subtract the second summand on the right to obtain the same lower bound for $\|\nabla u_n\|_{\mathrm{L}^p}$ as before.

Using the very definition of u_n and (1.2), the upshot is that in either case we have

$$r^{d-p} \lesssim \|\nabla u_n\|_{\mathrm{L}^p}^p \lesssim \|f_n\|_{\mathrm{L}^p}^p \leq (1-\varepsilon)^{-p}\|f\|_{\mathrm{L}^p}^p.$$

Thus, (1.3) follows as $\varepsilon \to 0$.

Now, we suppose that (1.3) holds true. To prove that F is locally p-fat, we pick $u \in \mathrm{C}_{\mathrm{c}}^\infty(2B;\mathbb{R})$ with $u \geq 1$ on $\overline{B} \cap F$. By [150, p. 125] we find $f \in \mathrm{L}^p$ such that $u = I_1 * f$ and $\|f\|_{\mathrm{L}^p} \lesssim \|\nabla u\|_{\mathrm{L}^p}$. In particular, $I_1 * f^+ \geq 1$ on $\overline{B} \cap F$. Hence, (1.3) yields

$$r^{d-p} \lesssim \|f^+\|_{\mathrm{L}^p}^p \leq \|f\|_{\mathrm{L}^p}^p \lesssim \|\nabla u\|_{\mathrm{L}^p}^p = \|\nabla u\|_{\mathrm{L}^p(2B)}^p. \qquad \square$$

1.3 Geometric measure theory

In this Section we introduce all relevant notions from geometric measure theory. We refer to [163] for background information.

Definition 1.3.1. Let $s \in [0,\infty)$. We define the

- ***s*-dimensional Hausdorff measure $\mathcal{H}^s$** as

$$\begin{aligned} \mathcal{H}^s(\Xi) &:= \lim_{\delta\to 0} \mathcal{H}_\delta^s(\Xi) \\ &:= \lim_{\delta\to 0} \inf\left\{\sum_n r_n^s \colon \Xi \subseteq \bigcup_n B(x_n,r_n), x_n \in \Xi, r_n \leq \delta\right\}. \end{aligned}$$

- **s-dimensional Hausdorff content $\mathcal{H}^s_\infty$** as

$$\mathcal{H}^s_\infty(\Xi) := \inf\left\{\sum_n r_n^s \colon \Xi \subseteq \bigcup_n B(x_n, r_n), x_n \in \Xi\right\}.$$

Definition 1.3.2. Let $s \in (0, d]$ and $\Xi' \subseteq \Xi$. We say that Ξ is an

- **upper s-set in Ξ'** if there is $C > 0$ such that

$$\mathcal{H}^s(\Xi(x, r)) \le C r^s \qquad (x \in \Xi', r \in (0, 1]).$$

- **s-set in Ξ'** if there are $C, c > 0$ such that

$$c r^s \le \mathcal{H}^s(\Xi(x, r)) \le C r^s \qquad (x \in \Xi', r \in (0, 1]).$$

If $\Xi' = \Xi$, we say that Ξ is an **(upper) s-set**. Furthermore, if the condition $r \in (0, 1]$ is replaced by $r \in (0, \operatorname{diam}(\Xi))$, then we speak of **uniformly (upper) s-sets**.

The following result compares p-fatness with the s-set property. We frequently use this fact in Chapter 5.

Lemma 1.3.3. *Let $d \ge 2$, $\delta > 0$, $p \in (1, d]$ and $s \in (d - p, d]$. Let $F \subseteq \mathbb{R}^d$ be closed and $\widehat{F} \subseteq \mathbb{R}^d$. If F satisfies*

$$\mathcal{H}^s(F(x, r)) \simeq r^s \qquad (x \in F \cap \widehat{F}_{2\delta}, r \in (0, \delta]),$$

then F is locally p-fat in $F \cap \widehat{F}_\delta$.

Proof. In view of [131, Thm. 3.1] applied with $h(r) = r^s$, it suffices to prove the lower bound

$$r^s \lesssim \mathcal{H}^s_\infty(F(x, r)) \qquad (x \in F \cap \widehat{F}_\delta, r \in (0, \delta]). \tag{1.4}$$

We adapt the argument in [17, Lem. 6.6] to our needs.

Let us fix $x \in F \cap \widehat{F}_\delta$, $r \in (0, \delta]$ and a covering $\{B_n\}_n = \{B(x_n, r_n)\}_n$ of $F(x, r)$ with open balls centered in $F(x, r)$. The assumption yields

$$r^s \lesssim \mathcal{H}^s(F(x, r)) = \mathcal{H}^s\Big(F(x, r) \cap \bigcup_n B_n\Big) \le \sum_n \mathcal{H}^s(F(x, r) \cap B_n).$$

If $r_n > \delta$, then we use the other part of the assumption in the form

$$\mathcal{H}^s(F(x, r) \cap B_n) \le \mathcal{H}^s(F(x, r)) \lesssim r^s \le r_n^s.$$

If $r_n \leq \delta$, then we note that $F(x,r) \cap B_n \subseteq F(x_n, r_n)$ and $x_n \in F \cap \widehat{F}_{2\delta}$ in order to deduce the same bound $\mathcal{H}^s(F(x,r) \cap B_n) \lesssim r_n^s$. In total, we obtain

$$r^s \lesssim \sum_n r_n^s.$$

By definition, $\mathcal{H}^s_\infty(F(x,r))$ is the infimum over all expressions as on the right-hand side and (1.4) follows. □

1.4 $\mathrm{L}^p - \mathrm{L}^q$-estimates

The goal of this section in to provide a brief summary of $\mathrm{L}^p - \mathrm{L}^q$ mapping principles for operator families that are initially defined on L^2. We refer to [12, 20, 37, 79] for detailed proofs and further background information.

Throughout this section, we assume that $\Xi \subseteq \mathbb{R}^d$ is measurable, $m_1, m_2, m_3 \in \mathbb{N}$ and $U \subseteq \mathbb{C} \setminus \{0\}$. In addition, we suppose that $\mathcal{T} = (T(z))_{z \in U}$ and $\mathcal{S} = (S(z))_{z \in U}$ are families of bounded linear operators from $\mathrm{L}^2(\Xi)^{m_1}$ into $\mathrm{L}^2(\Xi)^{m_2}$ and $\mathrm{L}^2(\Xi)^{m_2}$ into $\mathrm{L}^2(\Xi)^{m_3}$, respectively.

Definition 1.4.1. Let $1 \leq p \leq q \leq \infty$ and $\mu \in (0,1]$. We say that $\mathcal{T}$ is

(i) **$\mathrm{L}^p - \mathrm{L}^q$-bounded** if there is $C > 0$ such that

$$\|T(z)f\|_q \leq C|z|^{\frac{d}{q} - \frac{d}{p}} \|f\|_p \qquad (z \in U, f \in (\mathrm{L}^p \cap \mathrm{L}^2)(\Xi)^{m_1}).$$

If $p = q$, then we say that $\mathcal{T}$ is **L^p-bounded**.

(ii) **$\mathrm{L}^p - \dot{\mathrm{C}}^\mu$-bounded** if there is $C > 0$ such that

$$[T(z)f]^{(\mu)}_\Xi \leq C|z|^{-\mu - \frac{d}{p}} \|f\|_p \qquad (z \in U, f \in (\mathrm{L}^p \cap \mathrm{L}^2)(\Xi)^{m_1}).$$

The next definition provides a notion of averaged kernel bounds.

Definition 1.4.2. Let $1 \leq p \leq q \leq \infty$. We say that $\mathcal{T}$ satisfies **$\mathrm{L}^p - \mathrm{L}^q$ off-diagonal estimates** if there are $C, c > 0$ such that

$$\|\mathbf{1}_F T(z) \mathbf{1}_E f\|_q \leq C|z|^{\frac{d}{q} - \frac{d}{p}} \, \mathrm{e}^{-c\frac{\mathrm{d}(E,F)}{|z|}} \, \|\mathbf{1}_E f\|_p \tag{1.5}$$

for all $z \in U$, $f \in (\mathrm{L}^p \cap \mathrm{L}^2)(\Xi)^{m_1}$ and measurable sets $E, F \subseteq \Xi$. If $p = q$, then we say that $\mathcal{T}$ satisfies **L^p off-diagonal estimates**.

Convention 1.4.3. *If $\mathcal{T} = (\mathrm{e}^{-t^2\mathsf{L}})_{t>0}$, then we replace the exponential term $\mathrm{e}^{-c\frac{\mathrm{d}(E,F)}{t}}$ by the Gaussian $\mathrm{e}^{-c\left(\frac{\mathrm{d}(E,F)}{t}\right)^2}$ on the right-hand side of* (1.5).

The main tools that we frequently use in this thesis are the following to lemmas.

Lemma 1.4.4 ([20, Lem. 4.3, 4.6])**.** *Let $1 \leq p \leq q \leq r \leq \infty$.*

(i) *$\mathcal{T}$ is $\mathrm{L}^p - \mathrm{L}^q$-bounded if and only if $\mathcal{T}^* = (T(z)^*)_{z\in U}$ is $\mathrm{L}^{q'} - \mathrm{L}^{p'}$-bounded.*

(ii) *If $\mathcal{T}$ is $\mathrm{L}^p - \mathrm{L}^q$-bounded and $\mathcal{S}$ is $\mathrm{L}^q - \mathrm{L}^r$-bounded, then $(S(z)T(z))_{z\in U}$ is $\mathrm{L}^p - \mathrm{L}^r$-bounded.*

The same assertions hold true when we replace $\mathrm{L}^p - \mathrm{L}^q$-boundedness by $\mathrm{L}^p - \mathrm{L}^q$ off-diagonal estimates.

Lemma 1.4.5 ([20, Lem. 4.7, Lem. 4.14])**.** *Let $1 \leq p \leq q \leq \infty$ with $(p,q) \neq (2,2)$, assume that $\mathcal{T}$ is $\mathrm{L}^p - \mathrm{L}^q$-bounded with constant $C_{p,q} > 0$ and satisfies L^2 off-diagonal estimates with implicit constants $C, c > 0$.*

(i) *If $p \geq 2$, then $\mathcal{T}$ satisfies $\mathrm{L}^r - \mathrm{L}^s$ off-diagonal estimates for all $2 \leq r \leq s < q$.*

(ii) *If $p < 2 < q$, then $\mathcal{T}$ satisfies $\mathrm{L}^r - \mathrm{L}^s$ off-diagonal estimates for all $p < r \leq s < q$.*

(iii) *If $q \leq 2$, then $\mathcal{T}$ satisfies $\mathrm{L}^r - \mathrm{L}^s$ off-diagonal estimates for all $p < r \leq s \leq 2$.*

Implicit constants depend only on $d, p, q, C_{p,q}, C, c$.

Proof. We only treat the first item. The other ones can be deduced analogously. We use the results in [20, Chap. 4]. Lemma 4.14 asserts that $\mathcal{T}$ satisfies $\mathrm{L}^{[2,p]_\theta} - \mathrm{L}^{[2,q]_\theta}$ off-diagonal estimates for all $\theta \in [0,1)$. Remark 4.8 implies that $\mathcal{T}$ satisfies L^t off-diagonal estimates for all $t \in [2,q)$. Finally, we interpolate the L^t off-diagonal estimates with the $\mathrm{L}^p - \mathrm{L}^q$-boundedness (Lemma 4.14) to deduce the claim. □

1.5 H^∞-calculus

Throughout this section, we let X be a reflexive Banach space and T a **sectorial operator** in X of angle $\omega \in [0,\pi)$: it holds $\sigma(T) \subseteq \overline{\mathrm{S}_\omega}$, where

$$\mathrm{S}_\omega \coloneqq \{z \in \mathbb{C} \colon |\arg(z)| < \omega\} \quad \text{if } \omega > 0 \text{ and} \quad \mathrm{S}_0 \coloneqq (0,\infty),$$

and for each $\mu \in (\omega,\pi)$ there is $C > 0$ such that

$$\|z(z-T)^{-1}\| \leq C \qquad (z \in \mathbb{C} \setminus \overline{\mathrm{S}_\mu}).$$

We follow the book [105] to define the functional calculus of T and start with some basic properties.

Proposition 1.5.1 ([105, Prop. 2.1.1 & Rem. 2.1.2])**.** *The following assertions hold true.*

(i) *T is densely defined.*

(ii) *T induces the topological kernel-range splitting*

$$X = \ker(T) \oplus \overline{\operatorname{ran}(T)}$$

with corresponding projection $\mathcal{P} \colon X \to \overline{\operatorname{ran}(T)}$.

(iii) *The **injective part** $T\mathcal{P} = T \cap (\overline{\operatorname{ran}(T)} \times \overline{\operatorname{ran}(T)})$ is an injective sectorial operator of angle $\omega(T)$ in $\overline{\operatorname{ran}(T)}$.*

Let us fix

$$\omega < \nu < \mu < \pi.$$

Definition 1.5.2. The **Dunford–Riesz class $\mathbf{H_0^\infty(S_\mu)}$** consists of all holomorphic functions $f \colon \mathrm{S}_\mu \to \mathbb{C}$ such that $|f(z)| \leq C(|z|^s \wedge |z|^{-s})$ for some $C, s > 0$ and all $z \in \mathrm{S}_\mu$.

Definition 1.5.3. The **extended Dunford–Riesz class $\boldsymbol{\mathcal{E}(\mathrm{S}_\mu)}$** consists of all holomorphic functions $g \colon \mathrm{S}_\mu \to \mathbb{C}$ for which there are $a, b \in \mathbb{C}$ and $f \in \mathrm{H}_0^\infty(\mathrm{S}_\mu)$ with $g = f + a(1+\boldsymbol{z})^{-1} + b$.

Following [105, Sec. 2.3], we define

$$g(T) \coloneqq \frac{1}{2\pi \mathrm{i}} \int_{\partial \mathrm{S}_\nu} f(z)(z-T)^{-1} \,\mathrm{d}z + a(1+T)^{-1} + b \in \mathcal{L}(X).$$

This gives rise to an algebra homomorphism

$$(g \mapsto g(T))\colon \mathcal{E}(\mathrm{S}_\mu) \to \mathcal{L}(X),$$

which is consistent with $\mathcal{P}$ in the sense that

$$g(T)\mathcal{P} = \mathcal{P}g(T) = g(T\mathcal{P}) \qquad (g \in \mathcal{E}(\mathrm{S}_\mu)).$$

For $T\mathcal{P}$ we can extend this definition to the class $\mathrm{H}^\infty(\mathrm{S}_\mu)$ of **bounded holomorphic functions** on S_μ by **regularization**: we let $e \coloneqq \boldsymbol{z}(1+\boldsymbol{z})^{-2}$. The operator $e(T\mathcal{P}) = T\mathcal{P}(1+T\mathcal{P})^{-2}$ is injective and $ef \in \mathcal{E}(\mathrm{S}_\mu)$ for all $f \in \mathrm{H}^\infty(\mathrm{S}_\mu)$. We define

$$f(T\mathcal{P}) \coloneqq e(T\mathcal{P})^{-1}(ef)(T\mathcal{P})$$

as a closed operator in $\overline{\operatorname{ran}(T)}$.

Definition 1.5.4. We say that T admits a **bounded H^∞-calculus** of angle $\mu \in [\omega, \pi)$ if for all $\nu \in (\mu, \pi)$ there is some $C > 0$ such that

$$\|f(T\mathcal{P})\| \leq C\|f\|_{\mathrm{L}^\infty(\mathrm{S}_\nu)} \qquad (f \in \mathrm{H}^\infty(\mathrm{S}_\nu)).$$

Remark 1.5.5. Owing to the convergence lemma [105, Sec. 5.1], T admits a bounded H^∞-calculus of angle $\mu \in [\omega, \pi)$ if and only if for all $\nu \in (\mu, \pi)$ there is some $C > 0$ such that

$$\|f(T)\| \leq C\|f\|_{\mathrm{L}^\infty(\mathrm{S}_\nu)} \qquad (f \in \mathrm{H}^\infty_0(\mathrm{S}_\nu)).$$

Remark 1.5.6. The same constructions can be made for a **bisectorial operator** T: there is $\omega \in (0, \pi/2)$ such that $\sigma(T) \subseteq \overline{\dot{\mathrm{S}}_\omega}$, where

$$\dot{\mathrm{S}}_\omega \coloneqq \mathrm{S}_\omega \cup -\mathrm{S}_\omega,$$

and for each $\mu \in (\omega, \pi/2)$ there is $C > 0$ such that

$$\|z(z-T)^{-1}\| \leq C \qquad (z \in \mathbb{C} \setminus \overline{\dot{\mathrm{S}}_\mu}).$$

We refer to [78, Chap. 3] for a comprehensive exposition.

1.6 L^2-theory of elliptic operators in divergence form

In this section, we define a proper $\mathrm{L}^2(O)$-notion of divergence form operators with bounded, measurable and complex coefficients.

Assumption 1.6.1. *Let $O \subseteq \mathbb{R}^d$ be open. The function*

$$\mathsf{A} = \begin{bmatrix} a & b \\ c & A \end{bmatrix} : O \to \mathcal{L}(\mathbb{C}^m \times (\mathbb{C}^m)^d)$$

has bounded and measurable coefficients. We set

$$\Lambda(\mathsf{A}) \coloneqq \|\mathsf{A}\|_{\mathrm{L}^\infty(O;\mathcal{L}(\mathbb{C}^m \times (\mathbb{C}^m)^d))}.$$

We associate to A the sesquilinear form

$$\mathsf{a}(u,v) = \int_O \mathsf{A} \begin{bmatrix} u \\ \nabla u \end{bmatrix} \cdot \overline{\begin{bmatrix} v \\ \nabla v \end{bmatrix}} \,\mathrm{d}x \qquad (u, v \in V).$$

There are three types of ellipticity conditions that we use in this work.

Definition 1.6.2. We say that A is

(i) **elliptic** if

$$\lambda(\mathsf{A}) \coloneqq \operatorname*{essinf}_{x \in O} \min_{|\xi|=1} \operatorname{Re}(\mathsf{A}(x)\xi \cdot \overline{\xi}) > 0.$$

(ii) **Gårding elliptic** if there is some $\lambda > 0$ such that

$$\operatorname{Re} \mathsf{a}(u,u) \geq \lambda \|\nabla u\|_2^2 \qquad (u \in V).$$

(iii) **strongly Gårding elliptic** if there is some $\lambda > 0$ such that

$$\operatorname{Re} \mathsf{a}(u,u) \geq \lambda(\|u\|_2^2 + \|\nabla u\|_2^2) \qquad (u \in V).$$

In the case of (ii) or (iii), we also say that a satisfies a (strong) Gårding inequality.

Definition 1.6.3. Let A be Gårding elliptic. We define

$$\mathsf{L}u = -\operatorname{div}(A\nabla u + cu) + b\nabla u + au$$

in $\mathrm{L}^2(O)^m$ as an unbounded linear operator with domain

$$\operatorname{dom}(\mathsf{L}) \coloneqq \Big\{ u \in V \colon \exists\, \mathsf{L}u \in \mathrm{L}^2(O)^m \,\forall\, v \in V \colon (\mathsf{L}u \,|\, v) = \mathsf{a}(u,v) \Big\}.$$

We encode boundary conditions into L via the form domain $V = \mathrm{W}^{1,2}_D(O)^m$.

The following notation allows us to treat the homogeneous and inhomogeneous theory simultaneously.

Convention 1.6.4. *If a, b, c vanish, then we write $a = \mathsf{a}$ and $L = \mathsf{L}$.*

From now on we make the following

Assumption 1.6.5. *We assume that A is strongly Gårding elliptic. We replace this assumption by Gårding ellipticity of A if a, b and c vanish.*

Since

$$|\operatorname{Im} \mathsf{a}(u,u)| \leq \Lambda(\mathsf{A})\|u\|_{1,2}^2 \leq \frac{\Lambda(\mathsf{A})}{\lambda(\mathsf{A})} \operatorname{Re} \mathsf{a}(u,u) \qquad (u \in V),$$

it follows that $\mathsf{a}(u,u)$ belongs for all $u \in V$ to the closed sector $\overline{\mathrm{S}_\omega}$, where $\tan(\omega) \coloneqq {}^{\Lambda(\mathsf{A})}/_{\lambda(\mathsf{A})}$. We write

$$\omega(\mathsf{A}) \coloneqq \inf\Big\{\omega \in [0, \tfrac{\pi}{2}) \colon \mathsf{a}(u,u) \in \overline{\mathrm{S}_\omega} \text{ for all } u \in V\Big\}$$

for the **accretivity angle** of A. By Kato's form method [119, Chap. 6], Definition 1.6.3 gives rise to an **m-$\omega(\mathsf{A})$-accretive** realization of L in $\mathrm{L}^2(O)^m$:

$$\sigma(\mathsf{L}) \subseteq \overline{\mathrm{S}_{\omega(\mathsf{A})}} \quad \& \quad \|(z-\mathsf{L})^{-1}\| \leq \mathrm{d}(z, \overline{\mathrm{S}_{\omega(\mathsf{A})}})^{-1} \qquad (z \in \mathbb{C} \setminus \overline{\mathrm{S}_{\omega(\mathsf{A})}}).$$

The Lumer–Phillips theorem implies that $-\mathsf{L}$ generates a strongly continuous contraction semigroup $(\mathrm{e}^{-t\mathsf{L}})_{t\geq 0}$ of angle ${}^{\pi}/_{2} - \omega(\mathsf{A})$ in $\mathrm{L}^2(O)^m$. The starting point for a rich L^p-theory of L are the following results.

Proposition 1.6.6 (Crouzeix–Delyon, [105, Thm. 7.1.16])**.** *The operator L admits a bounded H^∞-calculus in $\mathrm{L}^2(O)^m$ of angle $\omega(\mathsf{A})$ with universal estimate*

$$\|f(\mathsf{L})\| \leq 4\|f\|_{\mathrm{L}^\infty(\mathrm{S}_\omega)} \qquad (\omega \in (\omega(\mathsf{A}), \pi), f \in \mathrm{H}^\infty(\mathrm{S}_\omega)).$$

Proposition 1.6.7 ([37, Prop. 3.2], [79, Prop. 4.2])**.** *Let $\psi \in [0, {}^{\omega(\mathsf{A})}/_{2} - {}^{\pi}/_{4})$. The operator families $(\mathrm{e}^{-z^2\mathsf{L}})_{z\in\mathrm{S}_\psi}$, $(z\nabla\, \mathrm{e}^{-z^2\mathsf{L}})_{z\in\mathrm{S}_\psi}$ and $(z^2\mathsf{L}\, \mathrm{e}^{-z^2\mathsf{L}})_{z\in\mathrm{S}_\psi}$ satisfy L^2 off-diagonal estimates with implicit constants depending only on $\lambda(\mathsf{A}), \Lambda(\mathsf{A})$ and ψ.*

1.7 L^p-theory of elliptic operators in divergence form

Based on [12, 20, 37, 79], we introduce all relevant tools to investigate the L^p-theory of L. For the rest of this section we work under

Assumption 1.7.1. *Throughout this section, we work with Assumption 1.6.5 and suppose that V has the inhomogeneous embedding property, see Definition 1.1.12. Moreover, if $\mathsf{L} = L$ we use the stronger homogeneous embedding property.*

1.7.1 The critical numbers

For the purpose of this thesis, we work for the rest of this subsection on $O = \mathbb{R}^d$. This allows us to avoid several case distinctions and simplifies the arguments. We refer to [37, 79] for all necessary modifications. Let us mention that the homogeneous Sobolev space $\dot{\mathrm{W}}^{1,p}$ has to be replaced by its inhomogeneous version $\mathrm{W}^{1,p}$ when passing from L to L.

The main objects of interest are the following two sets:

$$\mathcal{J}(L) \coloneqq \Big\{p \in [1,\infty] \colon (\mathrm{e}^{-t^2 L})_{t>0} \text{ is } \mathrm{L}^p\text{-bounded}\Big\},$$
$$\mathcal{N}(L) \coloneqq \Big\{p \in [1,\infty] \colon (t\nabla\, \mathrm{e}^{-t^2 L})_{t>0} \text{ is } \mathrm{L}^p\text{-bounded}\Big\}.$$

In view of Proposition 1.6.7 both contain $p = 2$ and complex interpolation ensures that both are intervals. We denote the left and right endpoints of $\mathcal{J}(L)$ and $\mathcal{N}(L)$ by $p_\pm(L)$ and $q_\pm(L)$, respectively. Let us collect some basic facts, see [12, Chap. 3] and [20, Chap. 6].

Proposition 1.7.2. *The following assertions hold true.*

(i) *We have $p_-(L) = q_-(L)$ and $p_+(L) \geq q_+(L)^*$.*

(ii) *In dimension $d = 1$,*

$$\mathcal{J}(L) = \mathcal{N}(L) = [1,\infty].$$

(iii) *In dimension $d \geq 2$, there is $\varepsilon > 0$ depending on dimensions and ellipticity such that*

$$p_-(L) \leq 2_* - \varepsilon \quad \& \quad q_+(L) \geq 2 + \varepsilon.$$

The first main result that emphasizes the importance of $\mathcal{J}(L)$ is

Theorem 1.7.3 ([12, Thm. 5.1])**.** *Let $p \in (p_-(L), p_+(L))$. Then L admits a bounded H^∞-calculus in L^p of angle ω_L: for all $\mu \in (\omega_L, \pi)$ there is $C > 0$ such that*

$$\|f(L)u\|_p \leq C\|f\|_{\mathrm{L}^\infty(\mathrm{S}_\mu)}\|u\|_p \qquad (f \in \mathrm{H}^\infty_0(\mathrm{S}_\mu), u \in \mathrm{L}^p \cap \mathrm{L}^2).$$

In order to state the second main result, we need the resolution of Kato's conjecture, see [21, 31, 40, 81] for more.

Theorem 1.7.4. *We have $\mathrm{dom}(\sqrt{L}) = \mathrm{W}^{1,2}$ with homogeneous estimate*

$$\|\sqrt{L}u\|_2 \simeq \|\nabla u\|_2 \qquad (u \in \mathrm{W}^{1,2}).$$

Since $\mathrm{W}^{1,2}$ is dense in $\dot{\mathrm{W}}^{1,2}$ and $\sqrt{L}$ has dense range, it follows from the previous result that $\sqrt{L}$ extends to an isomorphism from $\dot{\mathrm{W}}^{1,2}$ onto L^2. We denote this **square root isomorphism** by the same symbol. In particular, we can define the second-order **Riesz transform**

$$\nabla L^{-\frac{1}{2}} \colon \mathrm{L}^2 \to \mathrm{L}^2,$$

which enjoys the two-sided estimate

$$\|\nabla L^{-\frac{1}{2}} f\|_2 \simeq \|f\|_2 \qquad (f \in \mathrm{L}^2).$$

One main observation is that $\mathcal{N}(L)$ consists of exactly those $p \in [1, \infty]$ for which the square root extends to a p-isomorphism. In particular, the L^p-boundedness of the operator family $(t\nabla \mathrm{e}^{-t^2 L})_{t>0}$ can be reduced to L^p-estimates of one single operator and vice versa.

Theorem 1.7.5 ([20, Thm. 11.1])**.** *We have $p \in (q_-(L), q_+(L))$ if and only if the square root $\sqrt{L} \colon \dot{\mathrm{W}}^{1,p} \cap \mathrm{W}^{1,2} \to \mathrm{L}^p \cap \mathrm{L}^2$ extends to an isomorphism from $\dot{\mathrm{W}}^{1,p}$ onto L^p and its inverse agrees with $L^{-1/2}$ on $\mathrm{L}^p \cap \mathrm{L}^2$.*

The next result characterizes the range for which $p \in (1, \infty)$ the Lax–Milgram isomorphism

$$\mathcal{L} \coloneqq -\operatorname{div} A\nabla \colon \dot{\mathrm{W}}^{1,2} \to \dot{\mathrm{W}}^{-1,2}$$

extends to a p-isomorphism with compatible inverse.

Theorem 1.7.6 ([20, Lem. 13.4, Thm. 13.12])**.** *Let $p \in (1, \infty)$. Then $p \in (q_+(L^*)', q_+(L))$ if and only if $\mathcal{L}$ extends by density from $\dot{\mathrm{W}}^{1,p} \cap \dot{\mathrm{W}}^{1,2}$ to an isomorphism $\mathcal{L}_p \colon \dot{\mathrm{W}}^{1,p} \to \dot{\mathrm{W}}^{-1,p}$ whose inverse agrees with $\mathcal{L}^{-1}$ on $\dot{\mathrm{W}}^{-1,p} \cap \dot{\mathrm{W}}^{-1,2}$.*

1.7.2 p-ellipticity of A

Let $O \subseteq \mathbb{R}^d$ be open and $A\colon O \to \mathcal{L}(\mathbb{C}^d)$ be elliptic. The notion of p-ellipticity refers to an algebraic condition on the coefficients that originates from Cialdea–Maz'ya [61].

Definition 1.7.7. Let $p \in (1,\infty)$. We call A **p-elliptic** if

$$\Delta_p(A) \coloneqq \operatorname*{essinf}_{x\in O} \min_{|\xi|=1} \operatorname{Re}\Big(A(x)\xi \cdot \overline{(\xi + (1-{}^2\!/_p)\bar{\xi})}\Big) > 0. \tag{1.6}$$

This notion was introduced by Carbonaro–Dragičević [52] and independently by Dindoš–Pipher [69]. It bridges between the class of elliptic complex matrices ($p=2$) and the class of real matrices ($p=\infty$). An important fact is that every elliptic A is p-elliptic in a range of p's that depends on $\lambda(A), \Lambda(A)$, see [80, Cor. 3]. The following elementary properties will be useful throughout this thesis.

Lemma 1.7.8 ([52, Sec. 1.2 & Sec. 5.3].)**.** *The following hold true.*

(i) $\Delta_2(A) = \lambda(A)$.

(ii) $\Delta_p(A^*) \geq \Delta_p(A)({}^p\!/_{p'} \wedge {}^p\!/_{p'})$.

(iii) $\Delta_p(A) = \Delta_{p'}(A)$.

(iv) *A is real-valued if and only if $\Delta_p(A) > 0$ for all $p \in (1,\infty)$.*

(v) *The map $p \mapsto \Delta_p(A)$ is decreasing and Lipschitz continuous in $[2,\infty)$.*

(vi) *The map $\theta \mapsto \Delta_p(\mathrm{e}^{\mathrm{i}\theta} A)$ is continuous in $(-{}^\pi\!/_2, {}^\pi\!/_2)$.*

A key observation is that imposing a p-ellipticity assumption on A delivers a lower bound for $p_+(L)$ (and an upper bound for $p_-(L)$). This has been proven in [80, Thm. 1] and independently in [85, Thm. 3.1] via a different method. For the purpose of uniform estimates for a family of second-order operators, see Section 4.4, we take care of implicit constants and define

$$\langle p \rangle \coloneqq [p, d, \lambda(\mathsf{A}), \Lambda(\mathsf{A}), \Delta_p(A), \Delta_\bullet(A), C_E].$$

The key observation is that $\langle p \rangle$ is invariant under the transformation $A \mapsto A(t\cdot)$ for all $t \in \mathbb{R}^*$. We set

$$p^\circ \coloneqq \frac{p}{2}2^* \quad \& \quad p_\circ \coloneqq (p^\circ)'.$$

Theorem 1.7.9 ([80, Prop. 18])**.** *Let $d \geq 3$, $p \in (2,\infty)$ and A be p-elliptic. Then $(\mathrm{e}^{-t^2 L})_{t>0}$ is L^q-bounded for all $q \in [p_\circ, p^\circ]$ with bound depending only on $\langle p \rangle$.*

Let us extend this result to the inhomogeneous elliptic operator L. We start with an $\mathrm{L}^p - \mathrm{L}^{p^\circ}$-estimate and bootstrap L^p-estimates for the semigroup afterwards.

Lemma 1.7.10. *Let $d \geq 3$, $p \in (2,\infty)$ and A be p-elliptic. Then there is some $\varepsilon = \varepsilon(p, \Delta_\bullet(A)) \in (0,1)$ and $C > 0$ depending only on $\Lambda(\mathsf{A})$, $\Delta_p(A)$, p and C_E such that for all $q \in [2, p+\varepsilon)$ and $u \in \mathrm{L}^q(O) \cap \mathrm{dom}(\mathsf{L})$ with $\mathsf{L}u \in \mathrm{L}^q(O)$ it holds*

$$\|u\|_{q^\circ} \leq C(\|\mathsf{L}u\|_q^{\frac{1}{q}} \|u\|_q^{1-\frac{1}{q}} + \|u\|_q).$$

Proof. By Lemma 1.7.8 (v) we find $\varepsilon = \varepsilon(p, \Delta_\bullet(A)) \in (0,1)$ with $\Delta_{p+\varepsilon}(A) \geq \Delta_p(A)/2$. We pick $q \in [2, p+\varepsilon)$, fix $n \in \mathbb{N}$ and define the truncated functions

$$v \coloneqq (|u|^{\frac{q}{2}-1} \wedge n)u \quad \& \quad w \coloneqq (|u|^{q-2} \wedge n^2)u.$$

We also abbreviate

$$\chi \coloneqq \mathbf{1}_{\{|u|^{q-2} \geq n^2\}} \quad \& \quad \chi_c \coloneqq \mathbf{1}_{\{|u|^{q-2} < n^2\}}.$$

First, [80, Prop. 11] yields the estimate

$$\begin{aligned}
\frac{\Delta_p(A)}{p+1} \|\nabla v\|_2^2 &\leq \frac{2\Delta_q(A)}{q} \|\nabla v\|_2^2 \\
&\leq \mathrm{Re}(A\nabla u \,|\, \nabla w) \\
&= \mathrm{Re}(\mathsf{L}u \,|\, w) - \mathrm{Re}(au \,|\, w) - \mathrm{Re}(\nabla u \,|\, b^* w) - \mathrm{Re}(cu \,|\, \nabla w) \\
&=: \mathrm{(I)} + \mathrm{(II)} + \mathrm{(III)} + \mathrm{(IV)}.
\end{aligned}$$

Estimate for (I) and (II). Hölder's inequality yields

$$\begin{aligned}
|\mathrm{(I)}| + |\mathrm{(II)}| &\leq \|\mathsf{L}u\|_q \|w\|_{q'} + \|a\|_\infty \|u\|_q \|w\|_{q'} \\
&\leq \|\mathsf{L}u\|_q \|u\|_q^{q-1} + \|a\|_\infty \|u\|_q^q.
\end{aligned}$$

Estimate for (III). This term is more involved. First, we write

$$\begin{aligned}
(\nabla u \,|\, b^* w) &= (\nabla u \,|\, b^*(\chi n^2 u + \chi_c |u|^{q-2} u)) \\
&= ((n\chi + \chi_c |u|^{\frac{q}{2}-1}) \nabla u \,|\, b^* v).
\end{aligned}$$

At this point, we use [80, p. 714, Equ. (10)] joint with the last equality on the same page to continue with

$$= (\nabla v - \chi_c(1-\tfrac{2}{q})\operatorname{sgn}(v)\nabla|v|\,|\,b^*v).$$

Since $\nabla|v| = \operatorname{Re}(\overline{\operatorname{sgn}(v)}\nabla v)$ by [140, Prop. 4.4] we get by Hölder's and Young's inequality that

$$|(\mathrm{III})| \le 2\|b\|_\infty\|\nabla v\|_2\|v\|_2 \le \delta\|\nabla v\|_2^2 + \frac{\|b\|_\infty^2}{\delta}\|v\|_2^2 \le \delta\|\nabla v\|_2^2 + \frac{\|b\|_\infty^2}{\delta}\|u\|_q^q.$$

Estimate for (IV). We start by proving the auxiliary estimate

$$|\nabla w| \le 2|u|^{\frac{q}{2}-1}|\nabla v|. \tag{1.7}$$

If $u \neq 0$, then we get by [80, Lem. 10] that

$$\begin{aligned}|\nabla w| &\le n\chi|\nabla v| + \chi_c|u|^{\frac{q}{2}-1}(2-\tfrac{2}{q})|\nabla v| \\ &\le \chi|u|^{\frac{q}{2}-1}|\nabla v| + \chi_c|u|^{\frac{q}{2}-1}(2-\tfrac{2}{q})|\nabla v| \\ &\le 2|u|^{\frac{q}{2}-1}|\nabla v|.\end{aligned}$$

On the other hand, if $u = 0$, then $\nabla w = 0$ by [80, p. 714, Equ. (11)] and (1.7) follows immediately. Now, we use Hölder's inequality, (1.7) and Young's inequality to obtain

$$|(\mathrm{IV})| \le \|c\|_\infty\|u\|_q\|\nabla w\|_{q'} \le 2\|c\|_\infty\|u\|_q^{\frac{q}{2}}\|\nabla v\|_2 \le \delta\|\nabla v\|_2^2 + \frac{\|c\|_\infty^2}{\delta}\|u\|_q^q.$$

The result of the last four estimates is

$$\frac{\Delta_p(A)}{p+1}\|\nabla v\|_2^2 \le \|\mathsf{L}u\|_q\|u\|_q^{q-1} + 2\delta\|\nabla v\|_2^2 + \Big(\|a\|_\infty + \frac{\|b\|_\infty^2 + \|c\|_\infty^2}{\delta}\Big)\|u\|_q^q.$$

We choose $\delta := {\Delta_p(A)}/{4(p+1)}$ to absorb the second term on the right and get

$$\frac{\Delta_p(A)}{2(p+1)}\|\nabla v\|_2^2 \le \|\mathsf{L}u\|_q\|u\|_q^{q-1} + \Big(\|a\|_\infty + 4(p+1)\frac{\|b\|_\infty^2 + \|c\|_\infty^2}{\Delta_p(A)}\Big)\|u\|_q^q.$$

Hence, we find $C \ge 1$, depending only on $\Lambda(\mathsf{A})$, $\Delta_p(A)$ and p, with

$$\|v\|_2^2 + \|\nabla v\|_2^2 \le C(\|\mathsf{L}u\|_q\|u\|_q^{q-1} + \|u\|_q^q).$$

Finally, we make use of the inhomogeneous embedding property on the left-hand side, take q-th roots on both sides and apply monotone convergence to conclude. □

Theorem 1.7.11. *Let $d \geq 3$, $p \in (2,\infty)$ and A be p-elliptic. Then $(\mathrm{e}^{-t^2\mathsf{L}})_{t>0}$ is L^q-bounded for all $q \in [p_\circ, p^\circ]$ with bound depending only on $\langle p\rangle$.*

Proof. We pick $\varepsilon \in (0,1)$ as in Lemma 1.7.10, $q \in [2, p+\varepsilon)$ and assume that $\mathcal{T} \coloneqq (\mathrm{e}^{-t^2\mathsf{L}})_{t>0}$ is L^q-bounded with implicit constants depending only on $\langle p\rangle$. Let us mention that this is the case for $q = 2$. We pick $\delta \in (0, \varepsilon \wedge {}^4\!/_{(d-2)})$, which also implies $q^\circ - \delta > q$. Now, we show that $\mathcal{T}$ is L^s-bounded for all $s \in [2, q^\circ - \delta]$ with implicit constants depending only on $\langle p\rangle$. According to the L^q-boundedness of $\mathcal{T}$ and Lemma 1.7.10, we have for all $t > 0$ and $f \in (\mathrm{L}^q \cap \mathrm{L}^2)(O)$ the estimate

$$\begin{aligned}\|\,\mathrm{e}^{-t^2\mathsf{L}} f\|_{q^\circ} &\lesssim \|\mathsf{L}\,\mathrm{e}^{-t^2\mathsf{L}} f\|_q^{\frac{1}{q}} \|\,\mathrm{e}^{-t^2\mathsf{L}} f\|_q^{1-\frac{1}{q}} + \|\,\mathrm{e}^{-t^2\mathsf{L}} f\|_q \\ &\lesssim \|\mathsf{L}\,\mathrm{e}^{-t^2\mathsf{L}} f\|_q^{\frac{1}{q}} \|f\|_q^{1-\frac{1}{q}} + \|f\|_q.\end{aligned}$$

Next, we infer by Theorem 1.7.3 that the angle of holomorphy of the semigroup is p-independent. Hence, we have with $r = t\sin(\arctan({}^{\Lambda(\mathsf{A})}\!/_{\lambda(\mathsf{A})}))$ the representation

$$-\mathsf{L}\,\mathrm{e}^{-t\mathsf{L}} f = \frac{1}{2\pi\mathrm{i}} \int_{\partial B(t,r)} \frac{\mathrm{e}^{-z\mathsf{L}} f}{(z-t)^2}\,\mathrm{d}z$$

and we obtain for all $c \in (0,1)$, $t > 0$ and $f \in (\mathrm{L}^q \cap \mathrm{L}^2)(O)$ the estimate

$$\|\,\mathrm{e}^{-t^2\mathsf{L}} f\|_{q^\circ} \lesssim (t^{-\frac{2}{q}} + 1)\|f\|_q \lesssim c^{-\frac{1}{p+1}} t^{-\frac{2}{q}}\,\mathrm{e}^{ct^2}\,\|f\|_q.$$

Lemma 1.4.5 implies that $(c^{\frac{1}{p+1}}\,\mathrm{e}^{-t^2(c+\mathsf{L})})_{t>0}$ is L^s-bounded for all $s \in [2, q^\circ)$ with implicit constants depending only on $\langle p\rangle$. Now, complex interpolation with the L^2-bound

$$\|\,\mathrm{e}^{-t^2\mathsf{L}} f\|_2 \leq \mathrm{e}^{-\frac{\lambda}{2}t^2}\,\|f\|_2$$

and choosing c sufficiently small depending on $\langle p\rangle$ and δ, implies that $\mathcal{T}$ is L^s-bounded for all $s \in [2, q^\circ - \delta]$ and the implicit constant depends only on $\langle p\rangle$ and δ.

Now, we can iterate the above argument and choose $\delta < {}^{\varepsilon d}\!/_{(d-2)}$ (so that $(p+\varepsilon)^\circ - \delta > p^\circ$) to conclude that $\mathcal{T}$ is L^s-bounded for all $s \in [2, p^\circ]$ with implicit constant depending only on $\langle p\rangle$.

Finally, we repeat the argument with L^* instead of L. Since L^* is of the same type as L, see also Lemma 1.7.8 (ii), we conclude by duality (see Lemma 1.4.4 (i)) that $\mathcal{T}$ is L^s-bounded for all $s \in [p_\circ, p^\circ]$. □

Corollary 1.7.12. *Let $d \geq 3$, $p \in (2,\infty)$ and A be p-elliptic. Then the gradient family $(t\nabla\,\mathrm{e}^{-t^2\mathsf{L}})_{t>0}$ is L^q-bounded for all $q \in [p_\circ, 2]$ with bound depending only on $\langle p\rangle$.*

Proof. Throughout this proof, implicit constants depend only on $\langle p \rangle$. By Lemma 1.7.8 (v), there is $\varepsilon = \varepsilon(p, \Delta_\bullet(A)) \in (0,1)$ such that $\Delta_{p+2\varepsilon}(A) \geq {}^{\Delta_p(A)}/_2$. Theorem 1.7.11 joint with Lemma 1.4.5 implies that $(\mathrm{e}^{-t^2\mathsf{L}})_{t>0}$ is $\mathrm{L}^{(p+\varepsilon)_\circ} - \mathrm{L}^2$-bounded. Thus, by the L^2-boundedness of $(t\nabla \mathrm{e}^{-t^2 L})_{t>0}$, the semigroup law and composition, we conclude that $(t\nabla \mathrm{e}^{-t^2\mathsf{L}})_{t>0}$ is $\mathrm{L}^{(p+\varepsilon)_\circ} - \mathrm{L}^2$-bounded. Lemma 1.4.5 completes the proof. □

CHAPTER 2

Explicit improvements for the critical numbers

In dimension $d \geq 3$, we consider elliptic systems on $\mathbb{R}^d$ of $m \geq 1$ equations in divergence form

$$(Lu)^\alpha = -(\mathrm{div}(A\nabla u))^\alpha = -\sum_{i,j=1}^{d}\sum_{\beta=1}^{m} \partial_i A_{i,j}^{\alpha,\beta}\partial_j u^\beta \qquad (\alpha = 1,\dots,m)$$

with bounded, measurable and complex coefficients, see Definition 1.6.2. Surprisingly little is known about explicit L^p-estimates for $(\mathrm{e}^{-t^2L})_{t>0}$ when no further regularity on the coefficients is imposed. For systems with minimally smooth coefficients we refer e.g. to [75]. More precisely, we consider the interval $\mathcal{J}(L)$ from Subsection 1.7.1 and its left and right endpoints $p_\pm(L)$. All of our results will be stable under taking adjoints. Since $p_-(L) = (p_+(L^*))'$, we shall concentrate on the upper endpoint $p_+(L)$. It is known that $p_+(L) > 2^*$ and that the improvement $p_+(L) - 2^*$ can be arbitrarily small even when $m = 1$ [113, Sec. 2.2]. What seems to be missing though, are explicit lower bounds for $p_+(L)$ in terms of the data of L, such as ellipticity constants and dimensions, in particular when the improvement is expected to be large or even covers $p_+(L) = \infty$. Indeed, all results for systems (that we are aware of) are perturbative from the general L^2-theory and provide small, non-quantifiable improvements [11, 12, 155]. In contrast, we proceed by interpolation from the L^∞-theory for special systems described further below. Our results are new also for elliptic equations ($m = 1$).

The number $p_+(L)$ is related to the optimal ranges of various L^p-estimates for L, such as Riesz transforms, boundary value problems and functional calculus, see the introduction of [20] for a comprehensive account on the literature. Thus, improving lower bounds for $p_+(L)$, as we shall do here, leads to automatic improvements in all these topics.

All of our results are perturbative from the diagonal Laplacian system corresponding to $A = \mathbf{1}_{(\mathbb{C}^m)^d} \coloneqq (\delta_{\alpha,\beta}\delta_{i,j})_{i,j}^{\alpha,\beta}$ but not necessarily on a small scale. This is in the nature of things, because every elliptic A is an L^∞-perturbation of $\mathbf{1}_{(\mathbb{C}^m)^d}$ of size smaller than 1 up to normalization:

$$\mathrm{d}(A) \coloneqq \min_{t\geq 0} \|\mathbf{1}_{(\mathbb{C}^m)^d} - tA\|_{\mathrm{L}^\infty(\mathbb{R}^d;\mathcal{L}((\mathbb{C}^m)^d))} < 1. \tag{2.1}$$

The 'distance' $\mathrm{d}(A)$ is a well-known measure of ellipticity [121]. It can be bounded from above and below in terms of the usual ellipticity constants and when $A = A^*$, there is an easy formula (Lemma 2.1.1). The dimensional constant

$$\delta(d) \coloneqq \left(1 + \frac{(d-2)^2}{d-1}\right)^{-\frac{1}{2}}$$

will play an important role in this paper. Our main result is as follows.

Theorem 2.0.1. *The following three statements hold.*

(i) *If* $\mathrm{d}(A) \geq \delta(d)$*, then (with* $2^*/0 \coloneqq \infty$*)*

$$p_+(L) \geq \frac{2^*}{1 - \frac{\ln(\mathrm{d}(A))}{\ln(\delta(d))}}.$$

(ii) *If* $\mathrm{d}(A) < \delta(d)$*, then* $p_+(L) = \infty$*.*

(iii) *The result in (ii) is optimal in the sense that for each* $\varepsilon > 0$ *there is some* A_ε *with* $\mathrm{d}(A_\varepsilon) \leq \delta(d) + \varepsilon$ *and* $p_+(L_\varepsilon) < \infty$*.*

The dimensional constants in Theorem 2.0.1 are quite large in small dimensions and we collect some values in Figure 1.

Part (ii) is proven in Section 2.2 by combining results of Koshelev [121], see also [127], with a characterization of Gaussian estimates and Hölder regularity of the kernel associated with e^{-t^2L} due to Auscher–Tchamitchian [29]. In fact, in Theorem 2.2.3 we shall not only prove that $\mathrm{d}(A) < \delta(d)$ implies $p_+(L) = \infty$, but that e^{-t^2L} has a Hölder regular integral kernel with Gaussian decay. Since

d	$\delta(d)$	$-\ln(\delta(d))^{-1}$	$\varrho(A)$
3	0.8165	4.9326	0.1010
4	0.6547	2.3604	0.2087
5	0.5547	1.6968	0.2864
6	0.4880	1.3936	0.3441

Figure 1: Approximate constants in Theorem 2.0.1 in small dimensions. The third column contains the ellipticity ratio $\varrho(A) = {}^{\lambda(A)}/_{\Lambda(A)} \in (0,1]$ that is sufficient for having $\mathrm{d}(A) = \delta(d)$ in the special case $A = A^*$, see Lemma 2.1.1.

$\delta(d) > {}^{1}/\sqrt{d}$, this disproves the conjecture in [29, Chap. 1, Sec. 1.4.6] that the best possible perturbation result would be $\mathrm{d}(A) \in O(d^{-1})$. The optimality statement in (iii) is almost classical, see Proposition 2.2.5.

Part (i) is proved in Section 2.3. The idea is to rewrite $\mathrm{d}(A) < 1$ as $A = \tau(\mathbf{1}_{(\mathbb{C}^m)^d} - B)$, where $\|B\|_\infty = \mathrm{d}(A)$ and $\tau > 0$. We embed A as A_1 into an analytic family of elliptic matrices given by

$$A_z \coloneqq \tau(\mathbf{1}_{(\mathbb{C}^m)^d} - zB),$$

where $r \leq |z| \leq R$ with $0 < r < 1 < R$. Then, in the spirit of Stein interpolation, we estimate $\mathrm{e}^{-t^2 L} = \mathrm{e}^{-t^2 L_1}$ by using the generic information $p_+(L_z) \geq 2^*$ on the outer circle $|z| = R$ and $p_+(L_z) = \infty$ on the inner circle $|z| = r$ provided r is small. This gives a lower bound for $p_+(L_1)$ that becomes the larger, the closer $z = 1$ is to the inner circle and the farther away it is from the outer one. Thus, the best bound is achieved when r, R are the largest possible and the optimal choice for r comes from (ii). We believe that this simple analytic perturbation argument is of independent interest and has multiple applications to other types of L^p-estimates for divergence form operators.

Writing Theorem 2.0.1 (i) as ${}^{1}/_{2^*} - {}^{1}/_{p_+(L)} \geq {}^{\ln(\mathrm{d}(A))}/_{2^* \ln(\delta(d))} =: \varepsilon(d, \mathrm{d}(A))$, we see that $\varepsilon(d, \mathrm{d}(A)) \to 0$ as $d \to \infty$. Inspired by Stein's result [151] on dimensionless bounds for the Riesz transform, we ask whether an improvement can be given independently of d. To this end, it will be advantageous to consider $\mathcal{N}(L)$ instead of $\mathcal{J}(L)$ and its left and right endpoints $q_\pm(L)$,

see Subsection 1.7.1. Since $q_-(L) = p_-(L)$ and $p_+(L) \geq q_+(L)^*$ by Proposition 1.7.2, it follows that the improvement $q_+(L) - 2$ can be arbitrarily small. In the next result, proven in Section 2.5, we improve $q_+(L)$ in terms of $\mathrm{d}(A)$ alone. Writing the conclusion as $1/2 - 1/q_+(L) \geq \varepsilon(\mathrm{d}(A))$ gives the dimensionless improvement $1/2^* - 1/p_+(L) \geq \varepsilon(\mathrm{d}(A))$.

Theorem 2.0.2. *It holds*

$$q_+(L) \geq \begin{cases} \frac{2}{1+\frac{\sigma-1}{\sigma^2}\ln(\mathrm{d}(A))} & \text{if} \quad \frac{1}{4(\sigma-1)^2} \leq \mathrm{d}(A), \\ \frac{1}{2\sqrt{\mathrm{d}(A)}} + 1 & \text{if} \quad \mathrm{d}(A) \leq \frac{1}{4(\sigma-1)^2}, \end{cases}$$

where $\sigma \approx 5.69061$ *is the unique real solution to*

$$\ln(2\sigma - 2) = \frac{\sigma(\sigma-2)}{2(\sigma-1)}.$$

For curiosity, let us mention that the first bound in Theorem 2.0.2 produces a larger improvement for $p_+(L)$ compared to Theorem 2.0.1 (i) in dimension $d \geq 922100$.

It would be interesting to know to what extent our results can be extended to more general domains and boundary conditions. In case of Theorem 2.0.1 we provide an extension to bounded C^1-domains with Dirichlet boundary conditions in Section 2.4.

In view of the results in Chapter 3, it is worth mentioning that both improvements provide information about local elliptic regularity.

2.1 Elliptic systems

We reader may recall the *distance function* $\mathrm{d}(A)$ from (2.1). By compactness, the minimum is attained in some $t^* \geq 0$. Let us verify that $t^* > 0$ and that $\mathrm{d}(A)$ is an appropriate quantity to measure ellipticity. To this end, we introduce the **ellipticity quotient** of A:

$$\varrho(A) \coloneqq \frac{\lambda(A)}{\Lambda(A)}.$$

Lemma 2.1.1. *It holds*

$$\frac{1-\varrho(A)}{1+\varrho(A)} \leq \mathrm{d}(A) \leq \sqrt{1-\varrho(A)^2}.$$

Furthermore, if $A = A^*$*, then the first inequality becomes an equality.*

Proof. We have for all $t \geq 0$, each normalized $\xi \in (\mathbb{C}^m)^d$ and almost every $x \in \mathbb{R}^d$ that

$$|\xi - tA(x)\xi|^2 = 1 - 2t \operatorname{Re}(A(x)\xi \,|\, \xi) + t^2 |A(x)\xi|^2 \leq 1 - 2t\lambda(A) + t^2 \Lambda(A)^2.$$

We choose $t \coloneqq {}^{\lambda(A)}\!/_{\Lambda(A)^2}$ to get the upper bound for $\mathrm{d}(A)$.

Now, we fix $t^* > 0$ such that $\|\mathbf{1}_{(\mathbb{C}^m)^d} - t^* A\|_\infty = \mathrm{d}(A)$. Then

$$\operatorname{Re}(t^* A(x)\xi \,|\, \xi) = 1 - \operatorname{Re}((\mathbf{1}_{(\mathbb{C}^m)^d} - t^* A(x))\xi \,|\, \xi) \geq 1 - \mathrm{d}(A).$$

Hence, $t^*\lambda(A) \geq 1 - \mathrm{d}(A) > 0$ and by the triangle inequality $t^*\Lambda(A) \leq 1 + \mathrm{d}(A)$. Rearranging gives the lower bound for $\mathrm{d}(A)$.

For the second claim we take $t \coloneqq {}^{2}\!/_{(\Lambda(A)+\lambda(A))}$. Then $\mathbf{1}_{(\mathbb{C}^m)^d} - tA(x)$ is a self-adjoint matrix for almost every $x \in \mathbb{R}^d$ with eigenvalues contained in

$$[1 - t\Lambda(A), 1 - t\lambda(A)] = \Big[-\tfrac{\Lambda(A)-\lambda(A)}{\Lambda(A)+\lambda(A)}, \tfrac{\Lambda(A)-\lambda(A)}{\Lambda(A)+\lambda(A)} \Big] = \Big[-\tfrac{1-\varrho(A)}{1+\varrho(A)}, \tfrac{1-\varrho(A)}{1+\varrho(A)} \Big],$$

and thus

$$\mathrm{d}(A) \leq \tfrac{1-\varrho(A)}{1+\varrho(A)}$$

by the spectral radius formula. □

The next smoothing of the coefficients lemma will be important in Section 2.2 and Section 2.4 to absorb terms, which are a priori not finite for non-smooth coefficients. We include the simple proof for convenience. To this end, we let $\eta \in \mathrm{C}_\mathrm{c}^\infty(B(0,1))$ be non-negative with $\int_{\mathbb{R}^d} \eta \,\mathrm{d}x = 1$ and put $\eta_n(x) \coloneqq n^d \eta(nx)$ for $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$. We define the smoothed coefficients $A_n \coloneqq A * \eta_n$.

Lemma 2.1.2. *Let $O \subseteq \mathbb{R}^d$ be open and bounded, $u \in \mathrm{W}^{1,2}(O)^m$ be a weak solution to $Lu = 0$ in O and for all $n \in \mathbb{N}$ let $u_n \in \mathrm{W}^{1,2}(O)^m$ be the unique weak solution to*

$$-\operatorname{div}(A_n \nabla u_n) = 0 \quad \text{in } O \quad \& \quad u - u_n \in \mathrm{W}_0^{1,2}(O)^m.$$

Then the following assertions are satisfied.

(i) *For all $n \in \mathbb{N}$ it holds $\lambda(A_n) \geq \lambda(A)$, $\Lambda(A_n) \leq \Lambda(A)$, $\mathrm{d}(A_n) \leq \mathrm{d}(A)$ and u_n is smooth in O.*

(ii) *Along a subsequence $u_n \to u$ in $\mathrm{L}^2(O)^m$ and pointwise a.e. on O.*

Proof of (i). As A_n is smooth, so is u_n by elliptic regularity theory, see e.g. [88, Sec. 6.3.1, Thm. 3] adapted to systems. The rest of (i) follows from

$$A_n(x)\xi \cdot \overline{\zeta} = \int_{\mathbb{R}^d} \eta_n(y) A(x-y)\xi \cdot \overline{\zeta}\,\mathrm{d}y \qquad (\xi, \zeta \in (\mathbb{C}^m)^d).$$

For instance, to prove the estimate $\mathrm{d}(A_n) \leq \mathrm{d}(A)$, we let $t > 0$ be such that $\|\mathbf{1}_{(\mathbb{C}^m)^d} - tA\|_\infty = \mathrm{d}(A)$ and use $\int_{\mathbb{R}^d} \eta_n(y)\,\mathrm{d}y = 1$ twice in order to get for all $\xi \in (\mathbb{C}^m)^d$ that

$$\begin{aligned} |(\mathbf{1}_{(\mathbb{C}^m)^d} - tA_n(x))\xi| &= \left| \int_{\mathbb{R}^d} \eta_n(y)(\xi - tA(x-y)\xi)\,\mathrm{d}y \right| \\ &\leq \mathrm{d}(A) \int_{\mathbb{R}^d} \eta_n(y)\,\mathrm{d}y \\ &= \mathrm{d}(A). \end{aligned}$$

Proof of (ii). Let us show in a first step that $(v_n)_n := (u_n - u)_n \subseteq \mathrm{W}^{1,2}_0(O)^m$ is bounded. Indeed, since

$$-\operatorname{div}(A_n \nabla v_n) = \operatorname{div}(A_n \nabla u) \quad \text{in } O,$$

this follows from the Lax–Milgram lemma and (i). Thus, we can find a subsequence $(v_k)_k$, and some $v \in \mathrm{W}^{1,2}_0(O)^m$ such that $v_k \to v$ weakly in $\mathrm{W}^{1,2}_0(O)^m$. By compactness, we can additionally assume $v_k \to v$ strongly in $\mathrm{L}^2(O)^m$ and a.e. on O. We put $w := v + u$. In particular, $u_k \to w$ in $\mathrm{L}^2(O)^m$ and a.e. on O, and $\nabla u_k \to \nabla w$ weakly in $\mathrm{L}^2(O)^{dm}$. We claim that $w = u$. To this end, we fix some $\varphi \in \mathrm{C}^\infty_\mathrm{c}(O)^m$. Then

$$\begin{aligned} 0 &= \int_O A_k \nabla u_k \cdot \overline{\nabla \varphi}\,\mathrm{d}x \\ &= \int_O \nabla u_k \cdot \overline{A_k^* \nabla \varphi}\,\mathrm{d}x \longrightarrow \int_O \nabla w \cdot \overline{A^* \nabla \varphi}\,\mathrm{d}x = \int_O A \nabla w \cdot \overline{\nabla \varphi}\,\mathrm{d}x, \end{aligned}$$

using also strong L^2-convergence $A_k^* \nabla \varphi \to A^* \nabla \varphi$, which follows from dominated convergence. This proves that w solves

$$Lw = 0 \quad \text{in } O \quad \& \quad w - u \in \mathrm{W}^{1,2}_0(O)^m,$$

hence $w = u$. □

Remark 2.1.3. If A would be only defined on O, then we can extend it to $\mathbb{R}^d$ without changing the 'distance': Indeed, we let $t^* > 0$ be such that $\|\mathbf{1}_{(\mathbb{C}^m)^d} - t^* A\|_{\mathrm{L}^\infty(O)} = \mathrm{d}(A)$ and extend A to $\mathbb{R}^d$ by $(t^*)^{-1}\mathbf{1}_{(\mathbb{C}^m)^d}$. Hence, we can always assume that A is defined on $\mathbb{R}^d$.

2.2 New thoughts on old results of Koshelev

In a series of articles, culminating in the monograph [121], Koshelev studied qualitative (Hölder) regularity of weak solutions to elliptic systems. In this section we explain how they lead us to an optimal perturbation result for Gaussian estimates for heat semigroups, when reinterpreted appropriately as quantitative statements.

Definition 2.2.1. Let $O \subseteq \mathbb{R}^d$ be open. We call a function $u \in \mathrm{W}^{1,2}(O)^m$ ***L*-harmonic in** O, if we have for all $\varphi \in \mathrm{C}_c^\infty(O)^m$ that

$$\int_O A\nabla u \cdot \overline{\nabla \varphi}\,\mathrm{d}x = 0.$$

The appropriate setting to study regularity of L-harmonic functions turns out to be the following **weighted Morrey spaces** $\mathbf{H_\alpha(O)^m}$, $\alpha \in \mathbb{R}$, which are defined as the spaces of all $u \in \mathrm{W}^{1,2}(O)^m$ modulo $\mathbb{C}^m$ for which the norm[1]

$$\|u\|_{\mathrm{H}_\alpha(O)} \coloneqq \sup_{x_0 \in O} \|u\|_{\mathrm{H}_{\alpha,x_0}(O)} \coloneqq \sup_{x_0 \in O} \left(\int_O |\nabla u|^2 |x - x_0|^{-\alpha}\,\mathrm{d}x \right)^{\frac{1}{2}},$$

is finite. For $\alpha > d-2$ sufficiently close to $d-2$ and $\varepsilon > 0$ small enough we have

$$c(\alpha, d, \varepsilon) \coloneqq \left(1 + \frac{\alpha(d-2)}{d-1} + \varepsilon\right)^{\frac{1}{2}} \left(1 - \frac{\alpha(\alpha - (d-2))}{2(d-1)} - \varepsilon\right)^{-1} > 0. \tag{2.2}$$

This quantity will play an important role. In fact, $c(\alpha, d, \varepsilon) \to \delta(d)^{-1}$ in the limit as $\alpha \to d-2$ and $\varepsilon \to 0$. From now on we shall assume $\mathrm{d}(A) < \delta(d)$.

We begin by looking at L-harmonic functions on the unit ball B. Let us fix $t > 0$. Guided by the perturbation principle in Lemma 2.1.1, it begins with writing the equation $Lu = 0$ in B in the weak sense as

$$-\Delta u = -\operatorname{div}(F) \quad \text{with} \quad F \coloneqq (\mathbf{1}_{(\mathbb{C}^m)^d} - tA)\nabla u.$$

Due to technical reasons we replace A by A_n and u by u_n as defined in Lemma 2.1.2 and call the term on the right-hand side $-\operatorname{div}(F_n)$. In addition, we choose $t > 0$ such that $\mathrm{d}(A_n) = \|\mathbf{1}_{(\mathbb{C}^m)^d} - tA_n\|_\infty$.

[1] Let us mention that Koshelev defines $\mathrm{H}_{\alpha,x}(O)$ as $\mathrm{H}_{-\alpha,x}(O)$. We use this definition in order to be consistent with the notation for Morrey spaces.

Temporarily, we fix $x_0 \in \frac{1}{4}B$. In order to derive optimal Morrey estimates for the solutions u_n, Koshelev considers two variational integrals

$$X'(u_n, v_n) \coloneqq \int_{\frac{1}{4}B} \nabla u_n \cdot \nabla v_n \,\mathrm{d}x,$$
$$Y'_{-\alpha}(v_n) \coloneqq \int_{\frac{1}{4}B} |\nabla v_n|^2 |x - x_0|^\alpha \,\mathrm{d}x,$$

where $\alpha > d - 2$ is as above and v_n is an ingeniously chosen test function for the equation on $\frac{1}{4}B$ that they constructs from u_n using spherical harmonics [121, Equ. (2.3.2)]. The precise formula for v_n is not needed here – it suffices to use the estimates below "off-the-shelf". In fact, this specific v_n dates back to Giaquinta and Nečas [98]. Koshelev goes on by proving in [121, Cor. 2.3.1] the bounds

$$\begin{aligned} X'(u_n, v_n) &\geq \left(1 - \frac{\alpha(\alpha - (d-2))}{2(d-1)} - \varepsilon\right) \|u_n\|^2_{\mathrm{H}_{\alpha,x_0}(\frac{1}{4}B)} \\ &\quad - C(\alpha, d, \varepsilon) \|\nabla u_n\|^2_{\mathrm{L}^2(\frac{1}{4}B)}, \\ Y'_{-\alpha}(v_n) &\leq \left(1 + \frac{(d-2)\alpha}{d-1} + \varepsilon\right) \|u_n\|^2_{\mathrm{H}_{\alpha,x_0}(\frac{1}{4}B)} + C(\alpha, d, \varepsilon) \|\nabla u_n\|^2_{\mathrm{L}^2(\frac{1}{4}B)}. \end{aligned}$$

Since v_n is a test function for the equation for u_n in $\frac{1}{4}B$, we have

$$X'(u_n, v_n) = \int_{\frac{1}{4}B} (\mathbf{1}_{(\mathbb{C}^m)^d} - tA_n) \nabla u_n \cdot \nabla v_n \,\mathrm{d}x$$

and the Cauchy–Schwarz inequality along with the bound $\mathrm{d}(A_n) \leq \mathrm{d}(A)$ in Lemma 2.1.2 (i) yields

$$|X'(u_n, v_n)| \leq \mathrm{d}(A) \|u_n\|_{\mathrm{H}_{\alpha,x_0}(\frac{1}{4}B)} Y'_{-\alpha}(v_n)^{\frac{1}{2}}.$$

Combining the previous three estimates and recalling the definition of $c(\alpha, d, \varepsilon)$ leads to

$$\begin{aligned} \|u_n\|^2_{\mathrm{H}_{\alpha,x_0}(\frac{1}{4}B)} &\leq \mathrm{d}(A) \|u_n\|_{\mathrm{H}_{\alpha,x_0}(\frac{1}{4}B)} \Big(c(\alpha, d, \varepsilon) \|u_n\|_{\mathrm{H}_{\alpha,x_0}(\frac{1}{4}B)} \\ &\qquad + C(\alpha, d, \varepsilon) \|\nabla u_n\|_{\mathrm{L}^2(\frac{1}{4}B)} \Big) + C(\alpha, d, \varepsilon) \|\nabla u_n\|^2_{\mathrm{L}^2(\frac{1}{4}B)}, \end{aligned}$$

where $C(\alpha, d, \varepsilon)$ varies from line to line. The smoothing of the coefficients guarantees that the first summand on the right-hand side is finite and this is

the very reason why we have to include this argument. By Young's inequality, it follows that

$$\|u_n\|^2_{\mathrm{H}_{\alpha,x_0}(\frac{1}{4}B)} \leq (c(\alpha,d,\varepsilon)+\varepsilon)\,\mathrm{d}(A)\|u_n\|^2_{\mathrm{H}_{\alpha,x_0}(\frac{1}{4}B)} + C(\alpha,d,\varepsilon)\|\nabla u_n\|^2_{\mathrm{L}^2(\frac{1}{4}B)}.$$

Since $\mathrm{d}(A) < \delta(d)$, we can fix ε small and α close to $d-2$ depending only on dimension and $\mathrm{d}(A)$ such that the first term on the right can be absorbed. This is the key point and the result is

$$\|u_n\|_{\mathrm{H}_{\alpha,x_0}(\frac{1}{4}B)} \lesssim_{d,\mathrm{d}(A)} \|\nabla u_n\|_{\mathrm{L}^2(\frac{1}{4}B)}.$$

After taking the supremum over all $x_0 \in \frac{1}{4}B$ we arrive at

$$\|u_n\|_{\mathrm{H}_{\alpha}(\frac{1}{4}B)} \lesssim_{d,\mathrm{d}(A)} \|\nabla u_n\|_{\mathrm{L}^2(\frac{1}{4}B)}.$$

Then, Koshelev proves in [121, Thm. 2.1.1] that the left-hand side controls the Hölder seminorm of order $\mu = {}^{(\alpha-d+2)}\!/_2$ on $\frac{1}{4}B$. Applying Caccioppoli's inequality on the right-hand side eventually leads to

$$[u_n]^{(\mu)}_{\frac{1}{4}B} \lesssim_{d,\mathrm{d}(A)} \|u_n\|_{\mathrm{L}^2(B)}.$$

Finally, we invoke Lemma 2.1.2 (ii) in order to deduce

$$[u]^{(\mu)}_{\frac{1}{4}B} \leq \limsup_{n\to\infty} [u_n]^{(\mu)}_{\frac{1}{4}B} \lesssim_{d,\mathrm{d}(A)} \|u\|_{\mathrm{L}^2(B)}.$$

This estimate holds for any L-harmonic function on the unit ball. Since we have

$$\mathrm{d}(A) = \mathrm{d}(A^*) \quad \& \quad \mathrm{d}(A) = \mathrm{d}(A(x_0 + r\,\cdot))$$

for each $x_0 \in \mathbb{R}^d$ and $r > 0$, a scaling argument shows that the outcome of revisiting Koshelev's results is the following

Proposition 2.2.2. *Suppose* $\mathrm{d}(A) < \delta(d)$. *There are* $\mu \in (0,1]$ *and* $C > 0$, *both depending only on* d *and* $\mathrm{d}(A)$, *such that we have for all balls* $B = B(x,r) \subseteq \mathbb{R}^d$ *and every* L- *or* L^*-*harmonic* u *in* B *that*

$$r^{\mu}[u]^{(\mu)}_{\frac{1}{4}B} \leq Cr^{-\frac{d}{2}}\|u\|_{\mathrm{L}^2(B)}.$$

This quantitative Hölder estimate appeared much later in a different context. Namely, Auscher and Tchamitchian [11, 29] called it property (H) and proved that it implies that e^{-t^2L} has a kernel with pointwise Gaussian bounds. If we combine their Theorem 10 in [29, Chap. 1, Sec. 1.4.1] with Proposition 2.2.2 above, then we obtain the following result for Gaussian estimates. It can be seen as a perturbation result from the Laplacian.

Theorem 2.2.3. *Suppose* $\mathrm{d}(A) < \delta(d)$. *The kernel of* $(\mathrm{e}^{-t^2 L})_{t>0}$ *is represented by a Hölder regular function* $(K_t)_{t>0}$, *which admits pointwise Gaussian estimates: there are* $C, c > 0$ *and* $\mu \in (0, 1]$ *such that*

$$|K_t(x,y)| \leq Ct^{-d}\,\mathrm{e}^{-c\left(\frac{|x-y|}{t}\right)^2},$$
$$|K_t(x,y) - K_t(x',y')| \leq Ct^{-\mu-d}(|x-x'| + |y-y'|)^{\mu}$$

for all $x, x', y, y' \in \mathbb{R}^d$ *and* $t > 0$. *The constants* C, c *and* μ *depend only on* $d, \mathrm{d}(A), \lambda(A)$ *and* $\Lambda(A)$.

From Young's inequality for convolutions, we obtain

Corollary 2.2.4. *If* $\mathrm{d}(A) < \delta(d)$, *then* $p_+(L) = \infty$.

We shall see next that the "radius" $r = \delta(d)$ is optimal for the conclusion in Corollary 2.2.4 and hence also for the one in Theorem 2.2.3. Again this is implicit in Koshelev's work and relies on a counterexample due to De Giorgi.

Let us fix $c > 0$ and $C \geq {}^{(c^2+1)}\!/_{(d-2)c}$. We define for $x \in \mathbb{R}^d \setminus \{0\}$ the elliptic system with coefficients

$$(A_{\mathrm{DG}}(x))_{i,j}^{\alpha,\beta} := \left(c\delta_{ij} + C\frac{x_i x_j}{|x|^2}\right)\left(c\delta_{\alpha\beta} + C\frac{x_\alpha x_\beta}{|x|^2}\right) \qquad (i,j,\alpha,\beta = 1,\dots,d).$$

Then $u(x) := {}^{x}\!/_{|x|^b}$ with

$$b = \frac{d}{2} - \left(\frac{d^2}{4} - \frac{d(d-1)cC + (d-1)C^2}{1 + (c+C)^2}\right)^{\frac{1}{2}} \in [1, {}^{d}\!/_{2})$$

is a weak solution to $-\operatorname{div}(A_{\mathrm{DG}}\nabla u) = 0$ in $B(0,1)$, see [121, Sec. 2.5] or [103]. Let us mention that $b = 1$ if and only if $C = {}^{(c^2+1)}\!/_{(d-2)c}$. Koshelev continues in [121, Sec. 2.5] by showing that for this choice of C they can pick $c = c(d) > 0$ such that $\mathrm{d}(A_{\mathrm{DG}}) = \delta(d)$. Since $\mathrm{d}(A_{\mathrm{DG}})$ depends continuously on C and c, we can pick these parameters for any given $\varepsilon > 0$ in such a way that

$$b > 1 \quad \& \quad \mathrm{d}(A_{\mathrm{DG}}) < \delta(d) + \varepsilon. \tag{2.3}$$

Now we use a localization argument from [29, Chap. 1, Sec. 1.3] to prove

Proposition 2.2.5. *For any* $\varepsilon > 0$ *there is* A *such that* $\mathrm{d}(A) < \delta(d) + \varepsilon$ *and* $p_+(L) < \infty$.

Proof. We pick $A = A_{\mathrm{DG}}$ as in (2.3) and set $u(x) := {}^{x}/_{|x|^b}$. Let us fix some $\phi \in \mathrm{C}_{\mathrm{c}}^{\infty}$ with $\mathbf{1}_{B(0,1/2)} \le \phi \le \mathbf{1}_{B(0,1)}$. Then $v := \phi u \in \mathrm{dom}(L)$ can be seen as follows: using the equation for u, we deduce

$$Lv = -\operatorname{div}(A(\nabla\phi \otimes u)) - \operatorname{div}(\phi A \nabla u) = -\operatorname{div}(A(\nabla\phi \otimes u)) - \nabla\phi \odot (A\nabla u).$$

The only thing that matters is that $\nabla\phi$ vanishes near the origin and hence $Lv \in (\mathrm{C}_{\mathrm{c}}^{\infty})^m$. If we had $p_+(L) = \infty$, then according to [12, Prop. 5.3] we would get $v \in (\mathrm{L}^q)^m$ for every $q \in (2^*, \infty)$. However, $|v(x)| = |x|^{1-b}$ in a neighborhood of 0 can not belong to $(\mathrm{L}^q)^m$ for $q \ge {}^{d}/_{(b-1)}$. □

2.3 The interpolation argument

We come to the proof of Theorem 2.0.1 for the case $\mathrm{d}(A) \ge \delta(d)$. We will use basic properties of semigroups and vector-valued holomorphic functions. For further background we refer to [5].

We begin with a Stein-type interpolation principle tailored to our needs. In particular, we take care of implicit constants. For $\delta > 0$ we write

$$\mathrm{S}^{\delta} := \{z \in \mathbb{C} : -\delta < \mathrm{Re}(z) < 1 + \delta\} \quad \& \quad \mathrm{S} := \mathrm{S}^0.$$

Proposition 2.3.1. *Let $\delta > 0$ and suppose that $\{A_z\}_{z \in \mathrm{S}^{\delta}} \subseteq (\mathrm{L}^{\infty})^{dm \times dm}$ are elliptic such that:*

(i) *There are $0 < \lambda \le \Lambda$ with $\lambda(A_z) \ge \lambda$ and $\Lambda(A_z) \le \Lambda$ for all $z \in \overline{\mathrm{S}}$.*

(ii) *We have $c := \sup_{t \in \mathbb{R}} \mathrm{d}(A_{\mathrm{i}t}) < \delta(d)$.*

(iii) *For all $u, v \in (\mathrm{W}^{1,2})^m$ the map $z \mapsto a_{A_z}(u, v)$ is holomorphic in S^{δ}.*

Then for $\theta \in [0, 1]$ we have $p_+(L^{A_\theta}) \ge {}^{2^*}/_{\theta}$.

Proof. By (i) and (ii) implicit constants in this proof depend only on d, c, λ and Λ. We fix $t > 0$, $\theta \in [0, 1]$ and define $\Phi(z) := \mathrm{e}^{-t^2 L^{A_z}}$ for $z \in \mathrm{S}^{\delta}$. As an $\mathcal{L}((\mathrm{L}^2)^m)$-valued map, Φ is bounded by 1, holomorphic on S^{δ} and in particular continuous on $\overline{\mathrm{S}}$. This follows from (iii), see [160]. Now, we estimate Φ on the boundary of S. We fix $f \in (\mathrm{L}^2)^m$ and write $f_t^{A_z} = \mathrm{e}^{-t^2 L^{A_z}} f$.

(A) Estimate on the left boundary. We fix $z \in \overline{\mathrm{S}}$ with $\mathrm{Re}(z) = 0$. Due to (ii) and Theorem 2.2.3 we have Gaussian estimates for the kernel of $(\mathrm{e}^{-t^2 L^{A_z}})_{t>0}$ at our disposal. Young's inequality for convolutions yields

$$\|f_t^{A_z}\|_{\infty} \lesssim t^{-\frac{d}{2}} \|f\|_2.$$

(B) Estimate on the right boundary. We fix $z \in \overline{\mathrm{S}}$ with $\mathrm{Re}(z) = 1$. By holomorphy of the semigroup it follows that $f_t^{A_z} \in (\mathrm{W}^{1,2})^m$. Thus, by a Sobolev embedding, (i) and Hölder's inequality, we get

$$\|f_t^{A_z}\|_{2^*}^2 \lesssim \|\nabla f_t^{A_z}\|_2^2 \lesssim |a_{A_z}(f_t^{A_z}, f_t^{A_z})| = |(L^{A_z} f_t^{A_z} \mid f_t^{A_z})| \leq \|L^{A_z} f_t^{A_z}\|_2 \|f_t^{A_z}\|_2.$$

The semigroup is contractive on $(\mathrm{L}^2)^m$ in the sector S_φ, where $\tan(\varphi) = \Lambda/\lambda$. In particular, $\|f_t^{A_z}\|_2 \leq \|f\|_2$. With $r = t\sin(\varphi)$, Cauchy's formula for the complex derivative gives

$$\|L^{A_z} \mathrm{e}^{-tL^{A_z}} f\|_2 = \Big\| \frac{1}{2\pi \mathrm{i}} \int_{\partial B(t,r)} \frac{\mathrm{e}^{-wL^{A_z}} f}{(w-t)^2} \, \mathrm{d}w \Big\|_2 \lesssim t^{-1} \|f\|_2.$$

Altogether, we obtain

$$\|f_t^{A_z}\|_{2^*} \lesssim t^{-1} \|f\|_2.$$

Combining (A) and (B), Stein's interpolation theorem [149, Thm. 1] implies that

$$\|f_t^{A_\theta}\|_{\frac{2^*}{\theta}} \lesssim t^{\frac{d}{2^*/\theta} - \frac{d}{2}} \|f\|_2.$$

Hence, $(\mathrm{e}^{-t^2 L^{A_\theta}})_{t>0}$ is $\mathrm{L}^2 - \mathrm{L}^{2^*/\theta}$-bounded and Lemma 1.4.5 yields the claim. □

Proof of Theorem 2.0.1 (i). We let A be elliptic with $\mathrm{d}(A) \in [\delta(d), 1)$ and choose $t^* > 0$ with $\|\mathbf{1}_{(\mathbb{C}^m)^d} - t^* A\|_\infty = \mathrm{d}(A)$. We abbreviate $B \coloneqq \mathbf{1}_{(\mathbb{C}^m)^d} - t^* A$, which means $A = (t^*)^{-1}(\mathbf{1}_{(\mathbb{C}^m)^d} - B)$. As sketched in the introduction of this chapter, we perturb B by multiplication with complex numbers from a suitable annulus. However, it will be convenient to parameterize these numbers via an entire function F.

Let us pick $0 < r < 1 < R$ and $\delta > 0$ to be chosen. We embed A into the analytic family

$$A_z \coloneqq (t^*)^{-1}(\mathbf{1}_{(\mathbb{C}^m)^d} - F(z)B), \quad \text{where} \quad F(z) \coloneqq r^{1-z} R^z = r\, \mathrm{e}^{z \ln(R/r)}.$$

We notice that F is entire with $|F(z)| = r(R/r)^{\mathrm{Re}(z)}$ and maps $\overline{\mathrm{S}}$ onto the annulus $\{z \in \mathbb{C} : r \leq |z| \leq R\}$. It follows

$$\begin{aligned}
\sup_{z \in \mathrm{S}^\delta} \|A_z\|_\infty &\leq (t^*)^{-1}(1 + r(R/r)^{1+\delta}\, \mathrm{d}(A)), \\
\sup_{t \in \mathbb{R}} \mathrm{d}(A_{\mathrm{i}t}) &\leq \sup_{t \in \mathbb{R}} \|F(\mathrm{i}t)B\|_\infty = r\, \mathrm{d}(A), \\
\sup_{z \in \mathrm{S}^\delta} \mathrm{d}(A_z) &\leq r(R/r)^{1+\delta}\, \mathrm{d}(A).
\end{aligned}$$

We fix $\varepsilon \in (0, 1 - \mathrm{d}(A))$. In view of Proposition 2.3.1 we choose

$$r \coloneqq \frac{\delta(d)}{\mathrm{d}(A) + \varepsilon} \quad \& \quad R \coloneqq \frac{1}{\mathrm{d}(A) + \varepsilon}$$

and pick δ sufficiently small in order to guarantee that

$$\sup_{t \in \mathbb{R}} \mathrm{d}(A_{\mathrm{i}t}) < \delta(d) \quad \& \quad \sup_{z \in \mathrm{S}^\delta} \mathrm{d}(A_z) = \frac{\mathrm{d}(A)}{\mathrm{d}(A) + \varepsilon} \delta(d)^{-\delta} < 1.$$

Now we pick $\theta \in (0, 1)$ such that $1 = r^{1-\theta} R^\theta$. Then $L = L^{A_\theta}$ and Proposition 2.3.1 implies that $p_+(L) \geq {}^{2^*}\!/_\theta$. Finally, we notice that

$$\theta = 1 - \frac{\ln(R)}{\ln({}^R\!/_r)} \longrightarrow 1 - \frac{\ln(\mathrm{d}(A))}{\ln(\delta(d))} \qquad (\varepsilon \to 0). \qquad \square$$

Remark 2.3.2. In Proposition 2.3.1 we assume that $z \mapsto a_{A_z}(u, v)$ is holomorphic in a larger strip for convenience to get continuity of $z \mapsto \mathrm{e}^{-t^2 L^{A_z}}$ up to $\overline{\mathrm{S}}$. If this holds true for any other reason, it is enough to suppose that $z \mapsto a_{A_z}(u, v)$ is holomorphic in S.

Remark 2.3.3. The proof of Theorem 2.0.1 reveals that the same results hold for divergence form operators with form domain V on open sets $O \subseteq \mathbb{R}^d$ if the following conditions are satisfied:

- Theorem 2.2.3 holds true with implicit constants depending only on geometry, ellipticity and dimension.
- We have the homogeneous embedding property, see Definition 1.1.12. This was used in (B) above.

2.4 Extension to bounded C^1-domains

Let us extend Theorem 2.0.1 and Theorem 2.2.3 to bounded C^1-domains with Dirichlet boundary conditions. The divergence form operator L with elliptic $A \in \mathrm{L}^\infty(\Omega; \mathcal{L}((\mathbb{C}^m)^d))$ is now realized in $\mathrm{L}^2(\Omega)^m$ with form domain $V = \mathrm{W}^{1,2}_0(\Omega)^m$. We fix our geometric setup.

Assumption 2.4.1. *Throughout this section $\Omega \subseteq \mathbb{R}^d$, $d \geq 3$, is a bounded domain with C^1-boundary. This means that there is some $M > 0$ such that for each $x_0 \in \partial\Omega$ there is an open neighborhood U of x_0 and a C^1-diffeomorphism $\phi \colon U \to B(0,1)$, $\phi(x) = (x', \psi(x') - x_d)$ such that $\phi(U \cap \Omega) = B(0,1) \cap \mathbb{R}^d_+$ and $\|D\psi\|_\infty \leq M$.*

We can ensure that M is arbitrarily small by choosing the neighborhoods small enough. This is exactly the reason, why we assume that the boundary is C^1 and not just Lipschitz.

Theorem 2.4.2. *In the setting above suppose that* $\mathrm{d}(A) < \delta(d)$. *Then the kernel of* $(\mathrm{e}^{-t^2 L})_{t>0}$ *is represented by a measurable function* $(K_t)_{t>0}$ *for which there are* $C, c > 0$ *and* $\mu \in (0,1]$ *such that*

$$|K_t(x,y)| \leq Ct^{-d}\,\mathrm{e}^{-c\left(\frac{|x-y|}{t}\right)^2},$$
$$|K_t(x,y) - K_t(x',y')| \leq Ct^{-\mu-d}(|x-x'| + |y-y'|)^\mu$$

for all $x, x', y, y' \in \Omega$ *and* $t > 0$. *The constants* C, c *and* μ *depend only on* d, $\mathrm{d}(A)$, $\lambda(A)$, $\Lambda(A)$ *and geometry.*

Proof. The proof is very similar to the one of Theorem 2.2.3. We use again elliptic estimates for L-harmonic functions due to Koshelev, this time also in the half space after localization and transformation. The kernel estimates will then follow from [30, Thm. 12] provided we can check what they call property (D)[2] for L and L^*. By the easy argument in [29, p. 37] it suffices to show a property similar to Proposition 2.2.2 and formulated as follows:

There are $C > 0$ and $\gamma \in (0,1)$ such that for all $x_0 \in \overline{\Omega}$, $r \in (0, \rho_0]$ and $u \in \mathrm{W}^{1,2}_0(\Omega)^m$ with $Lu = 0$ or $L^*u = 0$ in $\Omega(x_0, r)$ it holds

$$r^\mu [u]^{(\mu)}_{\Omega(x_0,\gamma r)} \leq Cr^{-\frac{d}{2}} \|u\|_{\mathrm{L}^2(\Omega(x_0,r))}. \tag{2.4}$$

Here, $\rho_0 \in (0,1]$ is chosen small as explained in [30, p. 20].

Since $\mathrm{d}(A) = \mathrm{d}(A^*)$, we stick to the case of L-harmonic functions. When $x_0 \in \Omega$ and $B(x_0, r) \subseteq \Omega$, this estimate has already been obtained in Section 2.2 with $\gamma = 1/4$. By a case distinction (whether or not $B(x_0, r/2)$ intersects $\partial\Omega$) it suffices to treat in addition the case $x_0 \in \partial\Omega$.

So, we let $r \in (0, \rho_0]$, $u \in \mathrm{W}^{1,2}_0(\Omega)^m$ with $Lu = 0$ in $\Omega(x_0, r)$ and pick $\rho \simeq_M r$ small enough such that $\phi^{-1}(B(0,\rho)) \subseteq B(x_0, r)$. We put $B_+ \coloneqq (\mathbb{R}^d_+)(0,1)$ and write $u_{\phi,\rho} \coloneqq (u \circ \phi^{-1})(\rho\,\cdot)$. A change of coordinates implies that $u_{\phi,\rho} \in \mathrm{W}^{1,2}(B_+)^m$ is a weak solution of

$$-\operatorname{div}\Big((D\phi)_{\phi,\rho} A_{\phi,\rho} (D\phi)^\top_{\phi,\rho} \nabla(u_{\phi,\rho})\Big) = 0 \quad \text{in } B_+$$

[2] We stress that the bounded C^1-domain Ω falls into class (II) in [30]. In this case, and since we consider Dirichlet boundary conditions, the properties (D) and $(\mathrm{D}_{\mathrm{loc}})$ in [30] coincide by definition.

that vanishes on $\partial\mathbb{R}^d_+ \cap \partial B_+$.

Next, we use a smoothing procedure as in Section 2.2. We let A_n and u_n be defined as in Lemma 2.1.2 with A replaced by $(D\phi)_{\phi,\rho}A_{\phi,\rho}(D\phi)^\top_{\phi,\rho}$ (see also Remark 2.1.3). Then the same lemma assures that $u_n \to u_{\phi,\rho}$ in $\mathrm{L}^2(B_+)^m$ and a.e. along a subsequence. In addition, we have $u_n \in \mathrm{C}^\infty(\overline{\frac{3}{4}B_+})^m$ by elliptic regularity [88, Sec. 6.3.1, Thm. 5]. As in Section 2.2, we write $-\operatorname{div}(A_n\nabla u_n) = 0$ as

$$-\Delta u_n = -\operatorname{div}(F_n) \quad \text{with} \quad F_n \coloneqq (\mathbf{1}_{(\mathbb{C}^m)^d} - tA_n)\nabla u_n$$

and $t > 0$ is chosen such that $\|\mathbf{1}_{(\mathbb{C}^m)^d} - tA_n\|_\infty = \mathrm{d}(A_n)$. We note that

$$\|\mathbf{1}_{(\mathbb{C}^m)^d} - tA_n\|_\infty \le \mathrm{d}((D\phi)_{\phi,\rho}A_{\phi,\rho}(D\phi)^\top_{\phi,\rho}) \le M^2 + (1+M)^2\mathrm{d}(A), \tag{2.5}$$

where the first inequality is due to Lemma 2.1.2 (i) and the second one follows by definition of ϕ. Koshelev proves in [121, (2.4.13)] for each $x_0 \in \frac{1}{4}B_+$ that

$$\begin{aligned}\|u_n\|_{\mathrm{H}_{\alpha,x_0}(\frac{1}{4}B_+)} &\le c(\alpha,d,\varepsilon)\|F_n|x-x_0|^{-\frac{\alpha}{2}}\|_{\mathrm{L}^2(\frac{1}{4}B_+)} \\ &\quad + C(\alpha,d,\varepsilon)\Big(\|\nabla u_n\|_{\mathrm{L}^2(\frac{1}{4}B_+)} + \|F_n\|_{\mathrm{L}^2(\frac{1}{4}B_+)}\Big),\end{aligned}$$

provided that the right-hand side is finite, and u_n and f_n are sufficiently smooth, which is the case thanks to our smoothing procedure. Here, $c(\alpha,d,\varepsilon)$ is as in (2.2), where $\alpha > d-2$ and $\varepsilon > 0$ are chosen such that $c(\alpha,d,\varepsilon)$ is positive and finite. By definition of F_n and (2.5), we derive the estimate

$$\begin{aligned}\|u_n\|_{\mathrm{H}_{\alpha,x_0}(\frac{1}{4}B_+)} &\le \Big((1+M)^2\mathrm{d}(A) + M^2\Big)\Big(c(\alpha,d,\varepsilon)+\varepsilon\Big)\|u_n\|_{\mathrm{H}_{\alpha,x_0}(\frac{1}{4}B_+)} \\ &\quad + C(\alpha,d,\varepsilon,M)\|\nabla u_n\|_{\mathrm{L}^2(\frac{1}{4}B_+)}.\end{aligned}$$

As $\mathrm{d}(A) < \delta(d)$, we can pick $\varepsilon > 0$, $\alpha > d-2$ and $M > 0$ depending only on d and $\mathrm{d}(A)$ such that the first term on the right can be absorbed in order to obtain

$$\|u_n\|_{\mathrm{H}_\alpha(\frac{1}{4}B_+)} \lesssim_{d,\mathrm{d}(A)} \|\nabla u_n\|_{\mathrm{L}^2(\frac{1}{4}B_+)}.$$

As in Section 2.2 we deduce with the choice $\mu \coloneqq {}^{(\alpha-d+2)}\!/_2$ that

$$[u_{\phi,\rho}]^{(\mu)}_{\frac{1}{4}B_+} \lesssim_{d,\mathrm{d}(A)} \|u_{\phi,\rho}\|_{\mathrm{L}^2(B_+)}.$$

Transforming back gives (2.4) for some $\gamma = \gamma(M) \in (0,1)$. □

At this point we are in the same situation as on $\mathbb{R}^d$ and we can derive

Corollary 2.4.3. *In the setup of this section the following assertions hold true.*

(i) *If* $\mathrm{d}(A) > \delta(d)$, *then*

$$p_+(L) \geq \frac{2^*}{1 - \frac{\ln(\mathrm{d}(A))}{\ln(\delta(d))}}.$$

(ii) *If* $\mathrm{d}(A) \leq \delta(d)$, *then* $p_+(L) = \infty$.

(iii) *Part (ii) is sharp in the sense that for all bounded* C^1*-domains* $\Omega \subseteq \mathbb{R}^d$ *and for each* $\varepsilon > 0$ *there is some* A_ε *with* $\mathrm{d}(A_\varepsilon) \leq \delta(d) + \varepsilon$ *and* $p_+(L_\varepsilon) < \infty$.

Proof. The estimates for $p_+(L)$ follow as before, see also Remark 2.3.3. As for the sharpness of the radius $\mathrm{d}(A) = \delta(d)$ we can, after translation, assume $0 \in \Omega$. We take the same coefficients $A = A_{\mathrm{DG}}$ as in the proof of Proposition 2.2.5 and localize u to a ball contained in Ω. As before, this produces some $v \in \mathrm{dom}(L)$ with $v \notin \mathrm{L}^q(\Omega)^m$ for q large but $Lv \in \mathrm{C}_{\mathrm{c}}^\infty(\Omega)^m$. Arriving at a contradiction with $p_+(L) = \infty$ requires a different (and in fact simpler) argument compared to the case $\Omega = \mathbb{R}^d$.

By ellipticity and Poincaré's inequality, there is some $\theta_2 > 0$ such that $L - \theta_2$ is still m-accretive. Hence, L is invertible in $\mathrm{L}^2(\Omega)^m$ and the semigroup enjoys the exponential bound $\| \mathrm{e}^{-tL} f \|_2 \leq \mathrm{e}^{-\theta_2 t} \| f \|_2$ for all $t > 0$ and $f \in \mathrm{L}^2(\Omega)^m$. By interpolation with the uniform bound on $\mathrm{L}^p(\Omega)^m$ for some $p > q$, we get $\| \mathrm{e}^{-tL} f \|_q \lesssim \mathrm{e}^{-\theta_q t} \| f \|_q$ with some $\theta_q > 0$. But then the formula

$$L^{-1} f = \int_0^\infty \mathrm{e}^{-tL} f \, \mathrm{d}t,$$

valid in $(\mathrm{L}^q \cap \mathrm{L}^2)(\Omega)^m$ by the exponential estimate, implies that L^{-1} maps $(\mathrm{L}^q \cap \mathrm{L}^2)(\Omega)^m$ into itself, in contradiction with the properties of v. □

2.5 Dimensionless improvements

Here, we prove Theorem 2.0.2. Given $p \in (1, \infty)$, we consider $\mathcal{L}_p \coloneqq -\operatorname{div} A \nabla$ as a bounded operator from $(\dot{\mathrm{W}}^{1,p})^m$ into $(\dot{\mathrm{W}}^{-1,p})^m$. We denote by $c(p)$ the operator norm of the Riesz transform $R \coloneqq \nabla(-\Delta)^{-1/2}$, which we define via a Fourier multiplication operator with symbol $-\mathrm{i}\frac{\xi}{|\xi|} \otimes \mathbf{1}_{\mathbb{C}^m}$ from $(\mathrm{L}^p)^m$ to $(\mathrm{L}^p)^{dm}$. By Plancherel's theorem we have $c(2) = 1$.

Proposition 2.5.1. *Let* $p > 2$. *Then* $c(p) < 1/\sqrt{\mathrm{d}(A)}$ *implies* $q_+(L) \geq p$.

Proof. By Theorem 1.7.6 it suffices to prove for $p > 2$ with $c(p) < 1/\sqrt{\mathrm{d}(A)}$ that $\mathcal{L}_p$ is invertible such that $(\mathcal{L}_p)^{-1} = (\mathcal{L}_2)^{-1}$ on $(\dot{\mathrm{W}}^{-1,p} \cap \dot{\mathrm{W}}^{-1,2})^m$.

We borrow an idea from [12, Lem. 3.4]. Let us fix $t^* > 0$ such that $\mathrm{d}(A) = \|\mathbf{1}_{(\mathbb{C}^m)^d} - t^*A\|_\infty$ and put $B \coloneqq \mathbf{1}_{(\mathbb{C}^m)^d} - t^*A$. Then we can factorize

$$\begin{aligned} -\operatorname{div} A\nabla &= (t^*)^{-1}(-\Delta) + (t^*)^{-1} \operatorname{div} B\nabla \\ &= (t^*)^{-1}(-\Delta)^{\frac{1}{2}} \left(\mathbf{1} + R^*BR\right)(-\Delta)^{\frac{1}{2}}. \end{aligned}$$

Since

$$\|R^*BRf\|_p \leq c(p)^2 \mathrm{d}(A) \|f\|_p$$

for $f \in (\mathrm{L}^p \cap \mathrm{L}^2)^m$, it suffices that $c(p) < 1/\sqrt{\mathrm{d}(A)}$ to invert

$$(\mathbf{1} + R^*BR)^{-1} = \sum_{n=0}^{\infty} (-R^*BR)^n,$$

in $(\mathrm{L}^p)^m$. The inverse is compatible since the same Neumann series converges also in $(\mathrm{L}^2)^m$ owing to $c(2) = 1$. Finally, since $(-\Delta)^{1/2} \colon (\dot{\mathrm{W}}^{s,p})^m \to (\dot{\mathrm{W}}^{s-1,p})^m$ is an isomorphism for $s = 0, 1$ and all $p \in (1, \infty)$, it follows that $\mathcal{L}_p$ is invertible with compatible inverse, too. □

The constant $c(p)$ has a long history and that $c(p)$ is controlled from above by a dimensionless constant goes back to Stein [151]. Its exact value remains unknown to date. The best known estimates can be used to determine an improvement for $q_+(L) - 2$ explicitly. Dragičević and Volberg have shown in [76, Cor. 0.2] that

$$c(p) \leq 2(p-1) \qquad (p \geq 2). \tag{2.6}$$

Let us mention that their short argument applies word-by-word to $\mathbb{C}^m$-valued functions. However, this does not give $c(2) = 1$, suggesting that their bound can be improved by interpolation for $p > 2$ not too far away from 2. In fact, this is the case we are most interested in and we include the proof of the following elementary lemma. A similar argument for the Ahlfors–Beurling transform is found in [34].

Lemma 2.5.2. *If $\sigma \approx 5.69061$ is the unique real solution to*

$$\ln(2\sigma - 2) = \frac{\sigma(\sigma - 2)}{2(\sigma - 1)},$$

and $p \in [2, \sigma]$, then

$$c(p) \leq \left(\mathrm{e}^{\frac{\sigma^2}{\sigma - 1}}\right)^{\frac{1}{2} - \frac{1}{p}} \leq 2(p-1). \tag{2.7}$$

Proof. We fix $q \geq p$. Riesz–Thorin interpolation and the fact that $c(2) = 1$ yields

$$c(p) \leq c(q)^\theta \quad \text{with} \quad \frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{q}.$$

We insert this value of θ and use (2.6) to get

$$c(p) \leq \left((2(q-1))^{\frac{2q}{q-2}}\right)^{\frac{1}{2}-\frac{1}{p}} = \left(\mathrm{e}^{\frac{2q}{q-2}\ln(2q-2)}\right)^{\frac{1}{2}-\frac{1}{p}}.$$

Since $1/2 - 1/p > 0$, we have to minimize the expression $\frac{2q}{q-2}\ln(2q-2)$ over $q \geq p$. A straightforward calculation shows that the minimum over $q \geq 2$ is attained in $q = \sigma$. Since $p \leq \sigma$, this must also be the global minimum over $q \geq p$ and the first estimate in (2.7) follows. The second estimate in (2.7) follows simply because $q = p$ cannot give a better bound. □

Proof of Theorem 2.0.2. We define

$$\Phi\colon [2,\infty) \to [1,\infty), \quad \Phi(p) \coloneqq \begin{cases} \left(\mathrm{e}^{\frac{\sigma^2}{\sigma-1}}\right)^{\frac{1}{2}-\frac{1}{p}} & (p \leq \sigma), \\ 2(p-1) & (p > \sigma), \end{cases}$$

and notice that Φ is continuous, bijective and $c(p) \leq \Phi(p)$ for all $p \geq 2$, see (2.6) and (2.7). Hence, Proposition 2.5.1 yields

$$q_+(L) \geq \sup\left\{p > 2 : \Phi(p) < 1/\sqrt{\mathrm{d}(A)}\right\} = \Phi^{-1}\left(1/\sqrt{\mathrm{d}(A)}\right).$$

Inverting Φ leads to

$$q_+(L) \geq \begin{cases} \frac{2}{1+\frac{\sigma-1}{\sigma^2}\ln(\mathrm{d}(A))} & \text{if} \quad \frac{1}{4(\sigma-1)^2} \leq \mathrm{d}(A), \\ \frac{1}{2\sqrt{\mathrm{d}(A)}} + 1 & \text{if} \quad \mathrm{d}(A) \leq \frac{1}{4(\sigma-1)^2}, \end{cases}$$

which proves the theorem. □

CHAPTER 3

Meyers meets Dirac

Let $d \geq 1$. In this chapter, we investigate the elliptic operator

$$\boldsymbol{L}U = -\operatorname{div}_{t,x} \mathsf{A} \nabla_{t,x} U, \tag{3.1}$$

where the t-independent coefficient function

$$\mathsf{A} = \begin{bmatrix} a & b \\ c & A \end{bmatrix} : \mathbb{R}^d \to \mathcal{L}(\mathbb{C} \times \mathbb{C}^d)$$

has bounded, measurable entries and is elliptic in the sense of Definition 1.6.2. Historically, such operators arise as pullbacks of the Laplacian from the domain above a Lipschitz graph. Since the pioneering work of Dahlberg [66], harmonic analysts have been particularly interested in a priori estimates, uniqueness, and solvability of boundary value problems for the equation $\boldsymbol{L}U = 0$ in the upper half-space $\mathbb{R}^{1+d}_+$ with prescribed data of Dirichlet- and Neumann-type in L^p-spaces.

When the coefficients of A are real, such questions can be efficiently studied through the properties of elliptic measure and layer potentials [110, 111]. These tools are unavailable for complex coefficients since they rely on the maximum principle and pointwise estimates of weak solutions in the style of the celebrated DeGiorgi–Nash–Moser theorem.

In [15], Auscher, Axelsson, and McIntosh introduced the so-called *first-order approach*. They observed that the equation $\boldsymbol{L}U = 0$ can be reformulated as

an evolution equation for its *conormal gradient* $F = \overline{\mathsf{A}}\nabla_{t,x}U$:

$$F'(t) + \mathsf{DB}F(t) = 0 \qquad (t > 0). \tag{3.2}$$

Here, D is a *first-order* differential operator (Dirac operator), and B is a bounded multiplication operator that corresponds to A via a self-inverse algebraic transformation, see Section 3.3. The problem of generating solutions to (3.2) that yield appropriate estimates for (3.1) then reduces to the following questions:

- What is the correct space of initial data h for which the semigroup ansatz $F(t) = \mathrm{e}^{-t\mathsf{DB}}\, h$ solves (3.2)?
- Does the semigroup solution admit additional properties such as boundedness and analyticity?

It turns out that the right functional analytic tool to answer these questions is the theory of DB-adapted function spaces [4,28], which we thoroughly examine in Chapter 4. In a nutshell, these function spaces are tailored to DB in such a way that it admits a bounded H^∞-calculus. The main task then lies in determining the range of exponents for which these spaces coincide with classical function spaces in the Hardy, Lebesgue, and Sobolev scale. From now on, we restrict our attention to the Hardy–Lebesgue scale and denote the identification interval by $\mathcal{H}(\mathsf{DB})$, with $h_\pm(\mathsf{DB})$ as its left and right endpoints.

The starting point is the well-developed L^2-theory for DB. In the breakthrough paper [32], it was shown that DB is a bisectorial operator in $\mathrm{L}^2(\mathbb{R}^d)^{1+d}$ and admits a bounded H^∞-calculus. In particular, $2 \in \mathcal{H}(\mathsf{DB})$. This observation paved the wave for further research in this direction, culminating in a series of papers [4, 22, 24, 25, 27, 28].

In the light of [27, 28], the main object of interest for extrapolating the L^2-theory of DB—including, for example, a bounded H^∞-calculus in L^p and a topological kernel-range splitting—is the *Auscher–Stahlhut interval* $I(\mathsf{DB})$. This is the maximal open interval around $p = 2$ for which DB is L^p-bisectorial. We denote its left and right endpoints by $p_\pm(\mathsf{DB})$. By the results in [28, Chap. 5], the intervals $I(\mathsf{DB})$ and $\mathcal{H}(\mathsf{DB})$ are related via

$$h_-(\mathsf{DB}) \le p_-(\mathsf{DB})_* \quad \& \quad p_+(\mathsf{DB}) \le h_+(\mathsf{DB}).$$

Since no competing method exists to reproduce the aforementioned results, the question remains whether the interval $I(\mathsf{DB})$ is merely an artifact of

the first-order approach or whether it can also be characterized through the second-order operator $\boldsymbol{L}$.

The main goal of this chapter is to provide an affirmative answer to this question.

Let us begin with the block case, where b and c vanish. In this case, the equation $\boldsymbol{L}U = 0$ reduces to $\partial_t^2 U + a^{-1} \operatorname{div}_x A \nabla_x U = 0$. This allows for a solution ansatz via the *Poisson extension* $U(t,x) = (\mathrm{e}^{-t\sqrt{a^{-1}L_0}} f)(x)$, where $L_0 = -\operatorname{div}_x A \nabla_x$. Using this *second-order approach*, a priori estimates for solutions of (3.1) have been extensively studied in [20]. In particular, it has been shown that $I(\mathsf{DB})$ coincides with the maximal open interval around $p = 2$ in which the Lax–Milgram operator $L_0 \colon \dot{\mathrm{W}}^{1,p}(\mathbb{R}^d) \to \dot{\mathrm{W}}^{-1,p}(\mathbb{R}^d)$ is an isomorphism. We denote this interval by $\mathcal{P}(L_0)$ and its left and right endpoints by $\mathcal{P}_\pm(L_0)$. Our goal is to extend this identity to the non-block case by showing that $I(\mathsf{DB}) = \mathcal{P}(\boldsymbol{L})$.

To achieve this, we take a different perspective and interpret the global estimate for $\boldsymbol{L}$ in the homogeneous Sobolev scale in terms of local elliptic regularity of $\boldsymbol{L}$-harmonic functions. The *Meyers exponent* $m_+(\boldsymbol{L})$ of $\boldsymbol{L}$ is defined as the largest $p > 2$ for which the gradient of any weak solution $\boldsymbol{L}U = 0$ in an axis-parallel cube $Q \subseteq \mathbb{R}^{1+d}$ is p-integrable in $\frac{1}{2}Q$ with quantitative L^2-norm control, see [133]. Phrased differently, the gradient of locally $\boldsymbol{L}$-harmonic functions satisfies a weak reverse Hölder estimate with exponent p. In this sense, $m_+(\boldsymbol{L})$ serves as a measure for local elliptic regularity. Our main result is

Theorem 3.0.1. *It holds*

$$I(\mathsf{DB}) = (m_+(\boldsymbol{L}^*)', m_+(\boldsymbol{L})) = \mathcal{P}(\boldsymbol{L}) = (q_+(\boldsymbol{L}^*)', q_+(\boldsymbol{L})).$$

Moreover, if $d = 1$, then all intervals coincide with $(1, \infty)$.

In view of Theorem 2.0.2, we have explicit improvements for $q_+(\boldsymbol{L})$, which depend solely on $\lambda(\mathsf{A})$ and $\Lambda(\mathsf{A})$. In particular, we can record

Corollary 3.0.2. *Let $d \geq 2$. Then there is some $\varepsilon > 0$ depending only on $\lambda(\mathsf{A})$ and $\Lambda(\mathsf{A})$ such that*

$$(2 - \varepsilon, 2 + \varepsilon) \subseteq I(\mathsf{DB}).$$

Let us now explain the proof of Theorem 3.0.1. Since the final equality is a well-known result from [20, Thm. 13.12], we focus on establishing the first and second equality.

We begin with a simple observation: Applying the Fourier transform in the time variable to the homogeneous gradient $\nabla_{t,x}$ creates a family $S_\tau \coloneqq [\mathrm{i}\tau, \nabla_x]^\top$ of τ-dependent inhomogenous gradients in $\mathbb{R}^d$. Therefore, the homogeneous operator $\boldsymbol{L} = -\operatorname{div}_{t,x} \mathsf{A} \nabla_{t,x}$ can be associated with the family $(\mathsf{L}_\tau)_{\tau\in\mathbb{R}} = (S_\tau^* \mathsf{A} S_\tau)_{\tau\in\mathbb{R}}$ of inhomogeneous second-order operators but one dimension lower. For this family, we can define a uniform Meyers exponent $m_+((\mathsf{L}_\tau)_\tau)$ and the maximal open interval $\mathcal{P}((\mathsf{L}_\tau)_\tau)$ around $p = 2$ in which the operators $\mathsf{L}_\tau \colon \mathrm{W}^{1,p}_\tau(\mathbb{R}^d) \to \mathrm{W}^{-1,p}_\tau(\mathbb{R}^d)$ are isomorphisms uniformly in τ. Here, $\mathrm{W}^{1,p}_\tau(\mathbb{R}^d)$ is a τ-adapted Sobolev space, which we introduce in Section 3.1. The proof of Theorem 3.0.1 follows from the following chain of equalities:

$$I(\mathsf{DB}) = \mathcal{P}((\mathsf{L}_\tau)_\tau) = (m_+((\mathsf{L}_\tau^*)_\tau)', m_+((\mathsf{L}_\tau)_\tau)) = (m_+(\boldsymbol{L}^*)', m_+(\boldsymbol{L})) = \mathcal{P}(\boldsymbol{L}).$$

To derive the first equality, we link the L^p-boundedness of the inhomogeneous *Hodge projector* $S_\tau \mathsf{L}_\tau^{-1} S_\tau^*$ to the compatible invertibilty of $\mathsf{L}_\tau \colon \mathrm{W}^{1,p}_\tau(\mathbb{R}^d) \to \mathrm{W}^{-1,p}_\tau(\mathbb{R}^d)$. Our argument generalizes and unifies the approach given in [20, Chap. 13] for the homogeneous case $\tau = 0$, see Section 3.2.

Next, we exploit an algebraic relation between the resolvent of DB and the second-order Hodge operators $S_\tau \mathsf{L}_\tau^{-1} S_\tau^*$, which was first observed in [28], see Section 3.3. Since $\mathcal{P}_-(\boldsymbol{L}) = \mathcal{P}_+(\boldsymbol{L}^*)'$, we focus on the upper endpoint from now on.

The second and last equality follow from a refined argument of Shen [145], see also [116]. Indeed, we derive the inequality $m_+(\mathsf{L}_\tau) \le \mathcal{P}_+(\mathsf{L}_\tau)$ by a particular application of Shen's extrapolation theorem, while the reverse bound follows from localization and bootstrapping local elliptic regularity, see Section 3.4.

We are left with the third equality $m_+((\mathsf{L}_\tau)_\tau) = m_+(\boldsymbol{L})$: On the one hand, the inequality "$\ge$" turns out to be a simple consequence of specifying U in the equation $\boldsymbol{L}U = 0$ as $U(t,x) = \mathrm{e}^{\mathrm{i}\tau t} u(x)$ for L_τ-harmonic functions u, see Section 3.5. On the other hand, the reverse estimate requires more effort and its proof is the content of Section 3.6.

We use the well-known characterization $q_+(\boldsymbol{L}) = \mathcal{P}_+(\boldsymbol{L})$ from [20, Chap. 13]. To lift L^p-estimates for resolvents of $(\mathsf{L}_\tau)_{\tau\in\mathbb{R}\setminus\{0\}}$ in d dimensions to estimates for $\boldsymbol{L}$ but one dimension higher, we view $\mathrm{L}^p(\mathbb{R}^{1+d}) = \mathrm{L}^p(\mathbb{R}; \mathrm{L}^p(\mathbb{R}^d))$ as a vector-valued L^p-space. Using Weis' multiplier theorem [161] and an equivalent characterization of $q_+(\boldsymbol{L})$ via higher-order resolvents [20, Lem. 6.5], we reduce the problem to proving square function estimates for $(sS_\tau(1 + s^2\mathsf{L}_\tau)^{-N})_{\tau\in\mathbb{R}\setminus\{0\}}$ in $\mathrm{L}^p(\mathbb{R}^d)$ for sufficiently large $N \in \mathbb{N}$.

To this end, we develop the necessary L^2-theory and bootstrap $\mathrm{L}^p - \mathrm{L}^q$-estimates for specific operator families associated to L_τ up to the exponent $m_+((\mathsf{L}_\tau)_\tau)$. Once in place, we combine a pointwise domination technique by the Hardy–Littlewood maximal operator with the vector-valued Fefferman–Stein inequality (see [41] for a similar argument) to conclude.

This being said, Theorem 3.0.1 follows from

Theorem 3.0.3. *It holds*

$$I(\mathsf{DB}) = \mathcal{P}((\mathsf{L}_\tau)_\tau) = (m_+((\mathsf{L}_\tau^*)_\tau)', m_+((\mathsf{L}_\tau)_\tau)) = (m_+(\boldsymbol{L}^*)', m_+(\boldsymbol{L})) = \mathcal{P}(\boldsymbol{L}).$$

Let us mention that the proof for $d = 1$ requires a different (and much simpler) argumentation at some points that we elaborate in Section 3.7. Figure 2 comprises the structure of the proof of our main result.

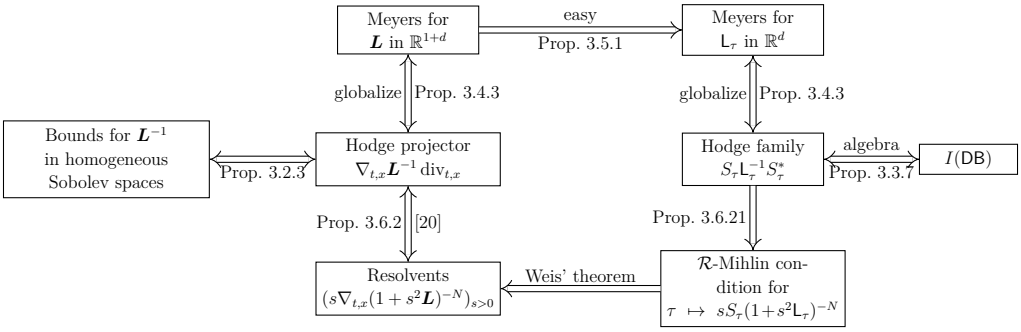

Figure 2: Roadmap to Theorem 3.0.3. Equivalent characterizations of the Auscher–Stahlhut interval $I(\mathsf{DB})$.

Last but not least, we complement this chapter by providing an equivalent characterization of $p_+(\mathsf{L})$ using quantitative weak reverse Hölder estimates for local weak solutions of L-harmonic functions. Here, L is the full divergence form operator introduced in Section 1.6.

Notation. We abbreviate $\mathbb{R}^* \coloneqq \mathbb{R} \setminus \{0\}$. Furthermore, we identify an operator $T\colon X \to Y$ with its graph $T \subseteq X \times Y$. We write $T|_Z \coloneqq T \cap (Z \times Z)$ for the part of T in Z. In particular, it holds

$$\operatorname{dom}(T|_Z) = \{u \in \operatorname{dom}(T) \cap Z\colon Tu \in Z\}.$$

For $Z = \mathrm{L}^p$, we write $T_p \coloneqq T|_{\mathrm{L}^p}$ and $\operatorname{dom}_p(T) \coloneqq \operatorname{dom}(T_p)$. We use the same notation for the range $\operatorname{ran}(T)$ and kernel $\ker(T)$.

3.1 L^2-theory of (τ-dependent) elliptic operators

In this section, we define the L^2-realization of

$$\mathsf{L}_\tau \coloneqq S_\tau^* \mathsf{A} S_\tau \colon \mathrm{W}^{1,2}(\mathbb{R}^d) \to \mathrm{W}^{-1,2}(\mathbb{R}^d),$$

where

$$S_\tau \coloneqq [\mathrm{i}\tau, \nabla_x]^\top \quad \& \quad S_\tau^* = [-\mathrm{i}\tau, -\operatorname{div}_x].$$

We construct this realization via Kato's form method [119, Chap. 6]. To this end, we introduce a family of τ-dependent function spaces.

Definition 3.1.1. For $\tau \in \mathbb{R}^*$ and $p \in (1,\infty)$, we define $\mathrm{W}^{1,p}_\tau(\mathbb{R}^d)$ as $\mathrm{W}^{1,p}(\mathbb{R}^d)$ endowed with the norm

$$\|u\|_{\mathrm{W}^{1,p}_\tau} \coloneqq (\|\tau u\|_p^p + \|\nabla_x u\|_p^p)^{\frac{1}{p}}.$$

We define the dual space $\mathrm{W}^{-1,p}_\tau(\mathbb{R}^d) \coloneqq \mathrm{W}^{1,p'}_\tau(\mathbb{R}^d)^*$.

Remark 3.1.2. For consistency in proofs, the space $\mathrm{W}^{1,p}_\tau(\mathbb{R}^d)$ should be understood as $\dot{\mathrm{W}}^{1,p}(\mathbb{R}^d)$ when $\tau = 0$, see Definition 1.1.1.

For $\tau \in \mathbb{R}$, the (boundedly invertible) operator L_τ acts in the distributional sense and is associated with the form

$$\mathsf{a}_\tau(u,v) \coloneqq \int_{\mathbb{R}^d} \mathsf{A} S_\tau u \cdot \overline{S_\tau v}\,\mathrm{d}x \qquad (u, v \in \mathrm{W}^{1,2}(\mathbb{R}^d)).$$

With a slight abuse of notation, we denote its part in $\mathrm{L}^2(\mathbb{R}^d)$ by the same symbol. Its L^2-domain is

$$\operatorname{dom}_2(\mathsf{L}_\tau) = \Big\{ u \in \mathrm{L}^2(\mathbb{R}^d) \colon \exists\, \mathsf{L}_\tau u \in \mathrm{L}^2 \,\forall\, v \in \mathrm{W}^{1,2} : (L_\tau u \,|\, v) = a_\tau(u,v) \Big\}.$$

By ellipticity of A, we have for all $u \in \mathrm{W}^{1,2}_\tau(\mathbb{R}^d)$ that

$$|\mathsf{a}_\tau(u,u)| \le \Lambda(\mathsf{A}) \|u\|^2_{\mathrm{W}^{1,2}_\tau} \quad \& \quad \operatorname{Re} \mathsf{a}_\tau(u,u) \ge \lambda(\mathsf{A}) \|u\|^2_{\mathrm{W}^{1,2}_\tau}.$$

This implies that L_τ is m-accretive in $\mathrm{L}^2(\mathbb{R}^d)$, ensuring that $-\mathsf{L}_\tau$ generates a holomorphic C_0-semigroup of contractions in $\mathrm{L}^2(\mathbb{R}^d)$.

3.2 Inhomogeneous Hodge theory

In this section, we investigate the extrapolation property of the Lax–Milgram isomorphism $\mathsf{L}_\tau\colon \mathrm{W}^{1,2}_\tau \to \mathrm{W}^{-1,2}_\tau$ in the $\mathrm{W}^{1,p}_\tau - \mathrm{W}^{-1,p}_\tau$ scale. To this end, we introduce the following

Definition 3.2.1. We define $\mathcal{P}(\mathsf{L}_\tau)$ as the set of all $p \in (1,\infty)$ for which there is $C > 0$ such that for all $u \in \mathrm{W}^{-1,p}_\tau \cap \mathrm{W}^{-1,2}_\tau$ it holds

$$\|\mathsf{L}_\tau^{-1} u\|_{\mathrm{W}^{1,p}_\tau} \leq C \|u\|_{\mathrm{W}^{-1,p}_\tau}.$$

We denote by $\mathcal{P}_\pm(\mathsf{L}_\tau)$ the lower and upper endpoints of $\mathcal{P}(\mathsf{L}_\tau)$.

We also write $p \in \mathcal{P}((\mathsf{L}_\tau)_\tau)$ if $p \in \mathcal{P}(\mathsf{L}_\tau)$ for all $\tau \in \mathbb{R}^*$ and the implicit constants do not depend on τ. We denote the left and right endpoints of this set by $\mathcal{P}_\pm((\mathsf{L}_\tau)_\tau)$.

Remark 3.2.2. The Lax–Milgram lemma implies that $\mathcal{P}(\mathsf{L}_\tau)$ contains $p = 2$. Using the same argument as in the proof of Lemma 3.3.6, we conclude that $\mathcal{P}(\mathsf{L}_\tau)$ is the maximal open interval around $p = 2$ for which $\mathsf{L}_\tau\colon \mathrm{W}^{1,p}_\tau \to \mathrm{W}^{-1,p}_\tau$ is an isomorphism such that its inverse is compatible with the inverse on $\mathrm{W}^{-1,2}_\tau$.

The main goal of this section is to characterize $\mathcal{P}(\mathsf{L}_\tau)$ via the L^p-boundedness of the Hodge projector

$$S_\tau \mathsf{L}_\tau^{-1} S_\tau^* = \begin{bmatrix} \tau^2 \mathsf{L}_\tau^{-1} & -\mathrm{i}\tau \mathsf{L}_\tau^{-1} \operatorname{div}_x \\ -\mathrm{i}\tau \nabla_x \mathsf{L}_\tau^{-1} & -\nabla_x \mathsf{L}_\tau^{-1} \operatorname{div}_x \end{bmatrix}.$$

To be more precise, we demonstrate

Proposition 3.2.3. *For all $\tau \in \mathbb{R}$, it holds*

$$\mathcal{P}(\mathsf{L}_\tau) = \Big\{ p \in (1,\infty)\colon S_\tau \mathsf{L}_\tau^{-1} S_\tau^* \text{ is } \mathrm{L}^p\text{-bounded} \Big\}.$$

In addition, we have

$$\mathcal{P}((\mathsf{L}_\tau)_\tau) = \Big\{ p \in (1,\infty)\colon (S_\tau \mathsf{L}_\tau^{-1} S_\tau^*)_{\tau \in \mathbb{R}^*} \text{ is } \mathrm{L}^p\text{-bounded} \Big\}.$$

In order to prove this result, we generalize the homogeneous theory in [20, Chap. 13]. We start by investigating the projection

$$\mathbb{P}_\tau \coloneqq S_\tau (\tau^2 - \Delta_x)^{-1} S_\tau^*,$$

which is initially defined on $\mathcal{Z}'$, see Definition 1.1.2.

Lemma 3.2.4. *Let $\tau \in \mathbb{R}$ and $p \in (1,\infty)$. Then $\mathbb{P}_\tau$ restricts to an L^p-bounded projection and the operator norm is independent of τ.*

Proof. Clearly, $\mathbb{P}_\tau$ satisfies the identity $\mathbb{P}_\tau^2 = \mathbb{P}_\tau$ on $\mathcal{Z}'$. The symbol of $\mathbb{P}_\tau$ is given by

$$m_\tau(\xi) = |\boldsymbol{\xi}|^{-2}(\boldsymbol{\xi} \otimes \boldsymbol{\xi}) \qquad (\boldsymbol{\xi} = (\tau,\xi) \in \mathbb{R}^{1+d} \setminus \{0\}).$$

As m is homogeneous of degree 0, that is $m(s\boldsymbol{\xi}) = m(\boldsymbol{\xi})$ for all $s > 0$ and $\boldsymbol{\xi} \in \mathbb{R}^{1+d} \setminus \{0\}$, it follows that for all $\alpha \in \mathbb{N}_0^{1+d}$ there is $C = C(\alpha, d) > 0$ such that

$$|\partial^\alpha m(\boldsymbol{\xi})| \leq C|\boldsymbol{\xi}|^{-|\alpha|} \qquad (\boldsymbol{\xi} \in \mathbb{R}^{1+d} \setminus \{0\}).$$

Thus, by restricting to $\alpha \in \mathbb{N}_0^d$ and noting that $|\xi| \leq |\boldsymbol{\xi}|$, we obtain

$$|\partial_\xi^\alpha m_\tau(\xi)| \leq C|\xi|^{-|\alpha|} \qquad (\tau \in \mathbb{R}, \xi \in \mathbb{R}^d \setminus \{0\}).$$

As a consequence, Mihlin's theorem [102, Thm. 6.2.7] yields that $\mathbb{P}_\tau$ restricts to a bounded projection on L^p and its operator norm is independent of τ. □

Lemma 3.2.5. *For all $\tau \in \mathbb{R}$ and $p \in (1,\infty)$, there is a topological splitting*

$$\mathrm{L}^p(\mathbb{R}^d)^{1+d} = \ker_p(S_\tau^*) \oplus \overline{\operatorname{ran}_p(S_\tau)} \tag{3.3}$$

and the projection onto $\overline{\operatorname{ran}_p(S_\tau)}$ is given by $\mathbb{P}_\tau$. In particular, norm estimates do not depend on τ.

Proof. According to Lemma 3.2.4, the L^p-bounded projection $\mathbb{P}_\tau$ induces the topological direct splitting

$$\mathrm{L}^p(\mathbb{R}^d)^{1+d} = \operatorname{ran}_p(1 - \mathbb{P}_\tau) \oplus \operatorname{ran}_p(\mathbb{P}_\tau).$$

By definition of $\mathbb{P}_\tau$, we have $\operatorname{ran}_p(1-\mathbb{P}_\tau) \subseteq \ker_p(S_\tau^*)$ and $\operatorname{ran}_p(\mathbb{P}_\tau) \subseteq \overline{\operatorname{ran}_p(S_\tau)}$. Thus, it suffices to prove that $\ker_p(S_\tau^*) \cap \overline{\operatorname{ran}_p(S_\tau)} = \{0\}$ to conclude. We pick $f = S_\tau g$ with $g \in \mathrm{W}_\tau^{1,p}$ in this intersection. Then $0 = S_\tau^* f = (\tau^2 - \Delta_x)g$ in $\mathcal{Z}'$ and hence $g = 0$, which also implies that $f = 0$. □

In view of the splitting in (3.3), we can record

Corollary 3.2.6. *Let $\tau \in \mathbb{R}$ and $p \in (1,\infty)$. Then the operators*

$$S_\tau \colon \mathrm{W}_\tau^{1,p} \cap \mathrm{W}_\tau^{1,2} \to \overline{\operatorname{ran}_p(S_\tau)} \cap \overline{\operatorname{ran}_2(S_\tau)},$$
$$S_\tau^* \colon \overline{\operatorname{ran}_p(S_\tau)} \cap \overline{\operatorname{ran}_2(S_\tau)} \to \mathrm{W}_\tau^{-1,p} \cap \mathrm{W}_\tau^{-1,2}$$

are bijective and bounded from above and below for the p-norms. All implicit constants do not depend on τ.

Proof. The assertion for S_τ is clear. Next, Lemma 3.2.5 implies injectivity and a two-sided bound for S_τ^*: for all $f \in \overline{\operatorname{ran}_p(S_\tau)} \cap \overline{\operatorname{ran}_2(S_\tau)}$, we have

$$\begin{aligned}\|S_\tau^* f\|_{\mathrm{W}_\tau^{-1,p}} &= \sup\Big\{|(f \,|\, S_\tau g)| \colon g \in \mathrm{W}_\tau^{1,p'} \cap \mathrm{W}_\tau^{1,2}, \|S_\tau g\|_{p'} \leq 1\Big\} \\ &\simeq \sup\Big\{|(f \,|\, h)| \colon h \in \mathrm{L}^{p'} \cap \mathrm{L}^2, \|h\|_{p'} \leq 1\Big\} \\ &= \|f\|_p.\end{aligned}$$

Finally, surjectivity holds because $S_\tau(\tau^2 - \Delta_x)^{-1}$ is an explicit right inverse for S_τ^*. □

Proof of Proposition 3.2.3. The second assertion follows once we have shown the first one. In view of Lemma 3.2.5, the operator $S_\tau \mathsf{L}_\tau^{-1} S_\tau^*$ is L^p-bounded if and only if

$$S_\tau \mathsf{L}_\tau^{-1} S_\tau^* \colon \overline{\operatorname{ran}_p(S_\tau)} \cap \overline{\operatorname{ran}_2(S_\tau)} \to \overline{\operatorname{ran}_p(S_\tau)} \cap \overline{\operatorname{ran}_2(S_\tau)}$$

is bounded for the p-norms. By Corollary 3.2.6, the latter is equivalent to saying that

$$\mathsf{L}_\tau^{-1} \colon \mathrm{W}_\tau^{-1,p} \cap \mathrm{W}_\tau^{-1,2} \to \mathrm{W}_\tau^{1,p} \cap \mathrm{W}_\tau^{1,2}$$

is bounded for the p-norms. This means exactly that $p \in \mathcal{P}(\mathsf{L}_\tau)$. □

3.3 The first-order Dirac operator DB

Armed with the inhomogeneous Hodge theory, we can understand L^p-resolvent estimates of first-order Dirac operators in terms of L^p-boundedness of second-order operators. Let us begin with the precise definitions.

Definition 3.3.1. We define

$$\underline{\mathsf{A}} \coloneqq \begin{bmatrix} 1 & 0 \\ c & A \end{bmatrix}, \quad \overline{\mathsf{A}} \coloneqq \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \quad \& \quad \mathsf{B} \coloneqq \underline{\mathsf{A}}\overline{\mathsf{A}}^{-1}.$$

Remark 3.3.2. The ellipticity assumption on A implies that A is boundedly invertible in $\mathrm{L}^\infty(\mathbb{R}^d; \mathcal{L}(\mathbb{C}^{1+d}))$. Hence, the same holds true for B. In particular, B is an isomorphism on any L^p space.

Definition 3.3.3. We define the **Dirac operator**

$$\mathsf{D} \coloneqq \begin{bmatrix} 0 & \operatorname{div}_x \\ -\nabla_x & 0 \end{bmatrix} \colon \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)^{1+d} \to \mathcal{D}'(\mathbb{R}^d)^{1+d}, \quad \langle Du \,|\, \varphi \rangle \coloneqq \int_{\mathbb{R}^d} u \cdot D\varphi \,\mathrm{d}x.$$

For $p \in (1,\infty)$, we denote by

$$\mathsf{D}_p \coloneqq \mathsf{D} \cap (\mathrm{L}^p \times \mathrm{L}^p)$$

its part in $\mathrm{L}^p(\mathbb{R}^d)^{1+d}$. Similarly, we define the L^p-realization of the **perturbed Dirac operator**

$$\mathsf{DB}\colon \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)^{1+d} \to \mathcal{D}'(\mathbb{R}^d)^{1+d}, \quad \langle \mathsf{DB}u \,|\, \varphi\rangle \coloneqq \int_{\mathbb{R}^d} \mathsf{B}u \cdot \mathsf{D}\varphi \,\mathrm{d}x,$$

as

$$(\mathsf{DB})_p \coloneqq \mathsf{DB} \cap (\mathrm{L}^p \times \mathrm{L}^p) = \mathsf{D}_p\mathsf{B}.$$

We drop the subscript p when $p = 2$.

Remark 3.3.4. Let us collect important facts about these L^p-realizations.

(i) For all $p, q \in (1,\infty)$, they are consistent in the sense that

$$(\mathsf{DB})_p \cap (\mathsf{DB})_q = \mathsf{DB} \cap ((\mathrm{L}^p \cap \mathrm{L}^q) \times (\mathrm{L}^p \cap \mathrm{L}^q)).$$

(ii) For all $p \in (1,\infty)$, the operator $(\mathsf{DB})_p = \mathsf{D}_p\mathsf{B}$ is closed. Indeed, D_p is closed and B is bounded on L^p.

(iii) Invertibility of B in L^∞ entails that B satisfies p-lower bounds on $\mathrm{ran}_p(\mathsf{D})$ for all $p \in (1,\infty)$, that is $\|\mathsf{BD}u\|_p \gtrsim \|\mathsf{D}u\|_p$ for all $u \in \mathrm{dom}_p(\mathsf{D})$.

It is shown in [32, Prop. 2.5] that DB is a bisectorial operator in L^2, see also Remark 1.5.6. In particular, $((1 + \mathrm{i}t\mathsf{DB})^{-1})_{t\in\mathbb{R}^*}$ is L^2-bounded. We now determine for which $p \in (1,\infty)$ the resolvent family remains L^p-bounded.

Definition 3.3.5. We define the **Auscher–Stahlhut interval** $I(\mathsf{DB})$ as the maximal open interval of exponents $p \in (1,\infty)$ around $p = 2$ for which $(\mathsf{DB})_p$ is bisectorial in L^p. We denote the lower and upper endpoints of $I(\mathsf{DB})$ by $p_\pm(\mathsf{DB})$.

Let us provide equivalent characterizations of this interval.

Lemma 3.3.6. *Let $p \in (1,\infty)$. Then the following assertions are equivalent.*

(i) *The operator family $((1 + \mathrm{i}t\mathsf{DB})^{-1})_{t\in\mathbb{R}^*}$ is L^p-bounded.*

(ii) *It holds $p \in I(\mathsf{DB})$.*

(iii) *It holds $p \in I(\mathsf{DB})$ and the resolvent of $(\mathsf{DB})_p$ is compatible with the resolvent of $(\mathsf{DB})_2$.*

Proof. Only the implication from (iii) to (i) is obvious.

(i) $\Longrightarrow$ (ii). For all $t \in \mathbb{R}^*$, we denote by $R_p(t)$ the unique continuous extension of $(1 + \mathrm{i}t\mathsf{DB})^{-1}$ to L^p. Clearly, we have $\|R_p(t)f\|_p \lesssim \|f\|_p$ for all $f \in \mathrm{L}^p$. Let us show that $1 + \mathrm{i}t(\mathsf{DB})_p$ is bijective and $(1 + \mathrm{i}t(\mathsf{DB})_p)^{-1} = R_p(t)$, which ultimately yields the bisectoriality of $(\mathsf{DB})_p$. Notably, observing that condition (i) holds true for $p = 2$ and interpolates with respect to p, and using [28, Thm. 3.6], this already implies that $p \in I(\mathsf{DB})$.

To prove that $1 + \mathrm{i}t(\mathsf{DB})_p$ is surjective, we take $f \in \mathrm{L}^p$ and show that $R_p(t)f \in \mathrm{dom}_p(\mathsf{DB})$ with $(1 + \mathrm{i}t(\mathsf{DB})_p)R_p(t)f = f$. To achieve this, we take a sequence $(f_n)_n \subseteq \mathrm{L}^p \cap \mathrm{L}^2$ with $f_n \to f$ in L^p. Then $u_n \coloneqq (1 + \mathrm{i}t\mathsf{DB})^{-1}f_n \to R_p(t)f$ in L^p and $(u_n)_n \subseteq \mathrm{L}^p \cap \mathrm{dom}_2(\mathsf{DB})$ with $\mathrm{i}t\mathsf{DB}u_n = f_n - u_n \in \mathrm{L}^p$. We conclude that $(u_n)_n \subseteq \mathrm{dom}_p(\mathsf{DB})$ with $(1 + \mathrm{i}t(\mathsf{DB})_p)u_n = f_n \to f$ in L^p. Since $1 + \mathrm{i}t(\mathsf{DB})_p$ is closed by Remark 3.3.4 (ii), we deduce that $R_p(t)f \in \mathrm{dom}_p(\mathsf{DB})$ along with $(1 + \mathrm{i}t(\mathsf{DB})_p)R_p(t)f = f$.

To establish injectivity of $1 + \mathrm{i}t(\mathsf{DB})_p$, we assume that $u \in \mathrm{dom}_p(\mathsf{DB})$ satisfies $(1 + \mathrm{i}t(\mathsf{DB})_p)u = 0$. To show that $u = 0$, we take $\varphi \in \mathrm{C}_\mathrm{c}^\infty$ and calculate

$$\begin{aligned} 0 &= (R_p 0 \,|\, \varphi) \\ &= ((1 + \mathrm{i}t\mathsf{DB})^{-1}(u + \mathrm{i}t(\mathsf{DB})_p u) \,|\, \varphi). \end{aligned}$$

As $(1 + \mathrm{i}t\mathsf{DB})^{-1}$ is L^p-bounded, we deduce by duality that $(1 - \mathrm{i}t\mathsf{B}^*\mathsf{D})^{-1}$ is $\mathrm{L}^{p'}$-bounded. It follows that $v \coloneqq (1 - \mathrm{i}t\mathsf{B}^*\mathsf{D})^{-1}\varphi \in \mathrm{L}^{p'} \cap \mathrm{dom}_2(\mathsf{D})$ with $-\mathrm{i}t\mathsf{B}^*\mathsf{D}v = \varphi - v \in \mathrm{L}^{p'}$. Thus, we have $v \in \mathrm{dom}_{p'}(\mathsf{D}) \cap \mathrm{dom}_2(\mathsf{D})$. Now, we approximate v by $v_n \coloneqq \eta_n * (\psi_n v)$, where $\eta_n(x) \coloneqq n^d\eta(nx)$ is a standard mollifier and $\psi_n(x) \coloneqq \psi(x/n)$ is a bump function with $\mathbf{1}_{B(0,1)} \leq \psi \leq \mathbf{1}_{B(0,2)}$. The crucial point is that $v_n \in \mathrm{C}_\mathrm{c}^\infty$ with $v_n \to v$ in $\mathrm{dom}_{p'}(D)$. Consequently, we get

$$\begin{aligned} 0 &= (u + \mathrm{i}t(\mathsf{DB})_p u \,|\, v) \\ &= \lim_{n\to\infty} (u + \mathrm{i}t(\mathsf{DB})_p u \,|\, v_n) \\ &= \lim_{n\to\infty} (u \,|\, (1 - \mathrm{i}t\mathsf{B}^*\mathsf{D})v_n) \\ &= (u \,|\, (1 - \mathrm{i}t\mathsf{B}^*\mathsf{D})v) \\ &= (u \,|\, \varphi). \end{aligned}$$

We conclude that $u = 0$.

In summary, $1+\mathrm{i}t(\mathsf{DB})_p$ is bijective and we have $R_p(t)f = (1+\mathrm{i}t(\mathsf{DB})_p)^{-1}f$ for all $f \in \mathrm{L}^p$.

(ii) $\Longrightarrow$ (iii). Let us denote by $M \subseteq I(\mathsf{DB})$ the set of exponents for which compatibility with the L^2-resolvent of DB holds. As $2 \in M$, M is non-empty. In addition, Šneĭberg's theorem [78, 146] implies that M is open in $I(\mathsf{DB})$. In order to see that M is also closed in $I(\mathsf{DB})$, we take a sequence $(p_n)_n \subseteq M$ with $p_n \to p \in I(\mathsf{DB})$. Again Šneĭberg's theorem yields that $(1+\mathrm{i}t(\mathsf{DB})_p)^{-1}f = (1+\mathrm{i}t(\mathsf{DB})_{p_n})^{-1}f = (1+\mathrm{i}t\mathsf{DB})^{-1}f$ for all $f \in \mathrm{L}^p \cap \mathrm{L}^{p_n} \cap \mathrm{L}^2$ and large enough $n \in \mathbb{N}$. Since $\mathrm{L}^p \cap \mathrm{L}^{p_n} \cap \mathrm{L}^2$ is dense in $\mathrm{L}^p \cap \mathrm{L}^2$, we conclude that $p \in M$. Therefore, $M = I(\mathsf{DB})$. □

To gain further insight into the interval $I(\mathsf{DB})$, we now characterize it in terms of the Hodge projectors $S_\tau \mathsf{L}_\tau^{-1} S_\tau^*$.

Proposition 3.3.7. *Let $p \in (1,\infty)$. Then $((1+\mathrm{i}t\mathsf{DB})^{-1})_{t\in\mathbb{R}^*}$ is L^p-bounded if and only if $(S_\tau \mathsf{L}_\tau^{-1} S_\tau^*)_{\tau\in\mathbb{R}^*}$ is L^p-bounded. In particular, it holds $I(\mathsf{DB}) = \mathcal{P}((\mathsf{L}_\tau)_\tau)$.*

Proof. The algebraic key identity is given by [28, Lem. 13.4]: for all $\tau \in \mathbb{R}^*$ and $f = [f_\perp, f_\parallel]^\top \in \mathrm{L}^p \cap \mathrm{L}^2$ it holds

$$(1-\mathrm{i}\tau^{-1}\mathsf{DB})^{-1}f = \overline{\mathsf{A}} S_\tau \mathsf{L}_\tau^{-1} S_\tau^* N f + \begin{bmatrix} bf_\parallel \\ f_\parallel \end{bmatrix}, \quad \text{where } N = \begin{bmatrix} 1 & -b \\ 0 & -A \end{bmatrix}.$$

Hence, the estimate

$$\|(1+\mathrm{i}t\mathsf{DB})^{-1}f\|_p \lesssim \|f\|_p \qquad (t \in \mathbb{R}^*, f \in \mathrm{L}^p \cap \mathrm{L}^2)$$

is valid if and only if

$$\|S_\tau \mathsf{L}_\tau^{-1} S_\tau^* N f\|_p \lesssim \|f\|_p \qquad (\tau \in \mathbb{R}^*, f \in \mathrm{L}^p \cap \mathrm{L}^2).$$

As A is elliptic, it follows that N is invertible on L^p. Thus, the latter estimate is equivalent to the L^p-boundedness of the Hodge projectors $(S_\tau \mathsf{L}_\tau^{-1} S_\tau^*)_{\tau\in\mathbb{R}^*}$. The last assertion follows from Proposition 3.2.3 and Lemma 3.3.6. □

3.4 The Meyers exponent of elliptic operators

In this part, we identify the exponent $\mathcal{P}_+(\mathsf{L}_\tau)$ by means of local elliptic regularity. We write

$$Q(x,r) := \{y \in \mathbb{R}^d \colon |x-y|_\infty < r\} \qquad (x \in \mathbb{R}^d, r > 0).$$

Definition 3.4.1. Let $Q = Q(x,r) \subseteq \mathbb{R}^d$ and $u \in \mathrm{W}^{1,2}_{\mathrm{loc}}(Q)$. We say that u is **L_τ-harmonic in Q** and write $\boldsymbol{u \in \mathcal{H}_{\mathsf{L}_\tau,Q} = \mathcal{H}_{\mathsf{L}_\tau,x,r}}$, if $\mathsf{L}_\tau u = 0$ in Q, that is

$$\int_Q \mathsf{A} S_\tau u \cdot \overline{S_\tau \varphi} \,\mathrm{d}x = 0 \qquad (\varphi \in \mathrm{C}^\infty_{\mathrm{c}}(Q)).$$

Definition 3.4.2. We define the **Meyers exponent** $m_+(\mathsf{L}_\tau)$ of L_τ as the supremum of all $p \in [2,\infty)$ for which the following holds true: There is $C > 0$ such that for all open axis-parallel cubes $Q \subseteq \mathbb{R}^d$ and every $u \in \mathcal{H}_{\mathsf{L}_\tau,2Q}$ it holds

$$\left(\fint_Q |S_\tau u|^p \,\mathrm{d}x \right)^{\frac{1}{p}} \leq C \left(\fint_{2Q} |S_\tau u|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}.$$

We also write $m_+((\mathsf{L}_\tau)_\tau) \coloneqq \inf_{\tau \in \mathbb{R}^*} m_+(\mathsf{L}_\tau)$ with the additional assumption that the implicit constants are independent of τ.

The main goal of this section is to prove

Proposition 3.4.3. *Let $d \geq 2$. For all $\tau \in \mathbb{R}$ it holds $m_+(\mathsf{L}_\tau) = \mathcal{P}_+(\mathsf{L}_\tau)$ and additionally we have $m_+((\mathsf{L}_\tau)_\tau) = \mathcal{P}_+((\mathsf{L}_\tau)_\tau)$.*

To this end, we rely on the equivalent characterization of $\mathcal{P}_+(\mathsf{L}_\tau)$ from Proposition 3.2.3. The proof will be a refinement of the argument in [145].

Proof. The proof will show that the implicit constants governing the Meyers exponent remain independent of τ precisely when those in the Hodge theory do. Hence, the second assertion is an immediate consequence of the first one.

$\boldsymbol{m_+(\mathsf{L}_\tau) \leq \mathcal{P}_+(\mathsf{L}_\tau)}$. We pick $p \in (2, m_+(\mathsf{L}_\tau))$, an open axis-parallel cube $Q \subseteq \mathbb{R}^d$ and $f = [f_\perp, f_\parallel]^\top \in \mathrm{C}^\infty_{\mathrm{c}}$ with $f|_{4Q} = 0$. We set $u_\tau \coloneqq \mathsf{L}_\tau^{-1} S_\tau^* f$ and observe that $\mathsf{L}_\tau u_\tau = S_\tau^* f = 0$ in $2Q$. Hence, the choice of p gives

$$\left(\fint_Q |S_\tau u_\tau|^p \,\mathrm{d}x \right)^{\frac{1}{p}} \lesssim \left(\fint_{2Q} |S_\tau u_\tau|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}.$$

As $S_\tau \mathsf{L}_\tau^{-1} S_\tau^*$ is L^2-bounded with norm at most $\lambda(\mathsf{A})^{-1}$, Shen's extrapolation theorem [145, Thm. 3.1] implies that $S_\tau \mathsf{L}_\tau^{-1} S_\tau^*$ is L^q-bounded for all $q \in (2,p)$. Thus, $p \leq \mathcal{P}_+(\mathsf{L}_\tau)$ by Proposition 3.2.3.

$\boldsymbol{\mathcal{P}_+(\mathsf{L}_\tau) \leq m_+(\mathsf{L}_\tau)}$. We let $p \in (2, \mathcal{P}_+(\mathsf{L}_\tau))$, take $Q = Q(x,r) \subseteq \mathbb{R}^d$ and $u \in \mathcal{H}_{\mathsf{L}_\tau,2Q}$. In addition, we pick $\gamma \in (1,2)$ and $\varphi \in \mathrm{C}^\infty_{\mathrm{c}}$ with $\mathbf{1}_Q \leq \varphi \leq \mathbf{1}_{\gamma Q}$ and $\|\nabla_x \varphi\|_\infty \lesssim r^{-1}$. Then $v \coloneqq \varphi(u - \kappa) \in \mathrm{W}^{1,2}$ with $\kappa \coloneqq (u)_{\gamma Q}$ is a global weak solution to

$$\mathsf{L}_\tau v = f_1 + \tau f_2 - \mathrm{div}_x(F),$$

where

$$f_1 + \tau f_2 \coloneqq [-A\nabla_x u \cdot \nabla_x \varphi - \mathrm{i}\tau\kappa c \cdot \nabla_x \varphi] - \tau[\mathrm{i}(u-\kappa)(b+c^T)\nabla_x\varphi + \tau\kappa a\varphi],$$
$$F \coloneqq (u-\kappa)d\nabla_x\varphi - \mathrm{i}\tau\kappa\varphi c.$$

The inhomogeneous scaling in τ necessitates a different approach compared to the homogeneous setting. Specifically, we use the assumption $2 < p < \mathcal{P}_+(\mathsf{L}_\tau)$ along with the embeddings $\mathrm{L}^p \subseteq |\tau|\mathrm{W}_\tau^{-1,p}$ and $\mathrm{L}^{p_*} \subseteq \mathrm{W}_\tau^{-1,p}$. Notably, the constraints $d \geq 2$ and $p > 2$ imply that $p_* > 1$. We obtain

$$\begin{aligned}\|v\|_{\mathrm{W}_\tau^{1,p}} &\lesssim \|f_1\|_{\mathrm{W}_\tau^{-1,p}} + |\tau|\|f_2\|_{\mathrm{W}_\tau^{-1,p}} + \|\operatorname{div}_x F\|_{\dot{\mathrm{W}}_\tau^{-1,p}} \\ &\lesssim \|f_1\|_{p_*} + \|f_2\|_p + \|F\|_p \\ &\lesssim |\tau||\kappa||Q|^{\frac{1}{p}} + r^{-1}\|u-\kappa\|_{\mathrm{L}^p(\gamma Q)} + r^{-1}\|\nabla_x u\|_{\mathrm{L}^{p_*}(\gamma Q)}.\end{aligned}$$

Next, we use $|\tau\kappa| \leq (⨍_{\gamma Q} |\tau u|^{p_*})^{1/p_*}$ and apply Poincaré's inequality to the second term to get

$$\lesssim |Q|^{\frac{1}{p}} \left(⨍_{\gamma Q} |\tau u|^{p_*}\,\mathrm{d}x\right)^{\frac{1}{p_*}} + r^{-1}\|\nabla_x u\|_{\mathrm{L}^{p_*}(\gamma Q)}.$$

In total, we have shown that

$$\begin{aligned}\|S_\tau u\|_{\mathrm{L}^p(Q)} &\leq |\tau\kappa||Q|^{\frac{1}{p}} + \|S_\tau v\|_p \\ &\lesssim |Q|^{\frac{1}{p}} \left(⨍_{\gamma Q} |\tau u|^{p_*}\,\mathrm{d}x\right)^{\frac{1}{p_*}} + r^{-1}\|\nabla_x u\|_{\mathrm{L}^{p_*}(\gamma Q)},\end{aligned}$$

and dividing both sides by $|Q|^{1/p}$ reveals the estimate

$$\left(⨍_Q |S_\tau u|^p\,\mathrm{d}x\right)^{\frac{1}{p}} \lesssim \left(⨍_{\gamma Q} |S_\tau u|^{p_*}\,\mathrm{d}x\right)^{\frac{1}{p_*}}.$$

If $p \in [2, 2^*]$, then we are done by applying Hölder's inequality on the right-hand side. For $p > 2^*$, we define an iterative sequence $(p_*^{[j]})_j$ via $p_*^{[1]} \coloneqq p_*$ and $p_*^{[j]} \coloneqq (p_*^{[j-1]})_*$ for $j \geq 2$. We select the smallest integer $j_0 \in \mathbb{N}$ such that $p_*^{[j_0]} \in [2, 2^*]$ and repeat the above argument $(j_0 - 1)$-times, adjusting φ and γ accordingly. □

Having characterized $I(\mathsf{DB})$ via families of second-order operators in $\mathbb{R}^d$:

$$I(\mathsf{DB}) = \mathcal{P}((\mathsf{L}_\tau)_\tau) = (m_+((\mathsf{L}_\tau^*)_\tau)', m_+((\mathsf{L}_\tau)_\tau)),$$

we now seek to express it in terms of a single higher-dimensional operator. This leads us to consider

$$\boldsymbol{L} \coloneqq -\operatorname{div}_{t,x} \mathsf{A}\nabla_{t,x} \colon \dot{\mathrm{W}}^{1,2}(\mathbb{R}^{1+d}) \to \dot{\mathrm{W}}^{-1,2}(\mathbb{R}^{1+d}),$$

the homogeneous counterpart of $L_0 = -\operatorname{div}_x A\nabla_x$ but one dimension higher. As before, we denote its part in $\mathrm{L}^2(\mathbb{R}^{1+d})$ by the same symbol. Let us record an immediate consequence of Proposition 3.4.3.

Corollary 3.4.4. *It holds* $\mathcal{P}_+(\boldsymbol{L}) = m_+(\boldsymbol{L})$.

The next two sections are devoted to establish the identity

$$m_+(\boldsymbol{L}) = m_+((\mathsf{L}_\tau)_\tau).$$

Passing from $\boldsymbol{L}$ to $(\mathsf{L}_\tau)_\tau$ follows naturally by choosing an appropriate function U satisfying $\boldsymbol{L}U = 0$. However, the converse direction is more delicate, which we thoroughly discuss in Section 3.6.

3.5 From L to $(\mathsf{L}_\tau)_\tau$

Proposition 3.5.1. *It holds* $m_+(\boldsymbol{L}) \le m_+((\mathsf{L}_\tau)_\tau)$.

Proof. We fix $p \in (2, m_+(\boldsymbol{L}))$, $\tau \in \mathbb{R}$, $Q = Q(x,r) \subseteq \mathbb{R}^d$ and $u \in \mathcal{H}_{\mathsf{L}_\tau,2Q}$. Then $U(t,x) \coloneqq \mathrm{e}^{\mathrm{i}t\tau}\, u(x) \in \mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbb{R} \times 2Q)$ solves $\boldsymbol{L}U = 0$ in $(0,4r) \times 2Q$. The assumption yields a constant $C > 0$ independent of τ such that

$$\left(\fint_0^{2r} \fint_Q |\nabla_{t,x} U|^p \,\mathrm{d}x \,\mathrm{d}t \right)^{\frac{1}{p}} \le C \left(\fint_0^{4r} \fint_{2Q} |\nabla_{t,x} U|^2 \,\mathrm{d}x \,\mathrm{d}t \right)^{\frac{1}{2}}.$$

As $|\nabla_{t,x} U| = |S_\tau u|$, this estimate is equivalent to

$$\left(\fint_Q |S_\tau u|^p \,\mathrm{d}x \right)^{\frac{1}{p}} \le C \left(\fint_{2Q} |S_\tau u|^2 \,\mathrm{d}x \right)^{\frac{1}{2}},$$

and hence $p \le m_+((\mathsf{L}_\tau)_\tau)$. □

3.6 From $(\mathsf{L}_\tau)_\tau$ to L

In view of the identities $m_+(\boldsymbol{L}) = \mathcal{P}_+(\boldsymbol{L})$ and $m_+((\mathsf{L}_\tau)_\tau) = \mathcal{P}_+((\mathsf{L}_\tau)_\tau)$ from Section 3.4, the main goal of this section is to prove

Proposition 3.6.1. *Let $d \geq 2$. We have $\mathcal{P}_+((\mathsf{L}_\tau)_\tau) \leq \mathcal{P}_+(\boldsymbol{L})$.*

To this end, we take advantage of a characterization of $\mathcal{P}_+(\boldsymbol{L})$ via resolvents.

Proposition 3.6.2 ([20, Thm. 13.12])**.** *It holds*

$$\mathcal{P}_+(\boldsymbol{L}) = q_+(\boldsymbol{L}).$$

Therefore, we aim to prove that the gradient family $(s\nabla_{t,x}(1+s^2\boldsymbol{L})^{-1})_{s>0}$ is L^p-bounded, provided that $2 < p < \mathcal{P}_+((\mathsf{L}_\tau)_\tau)$. Our main observation to bridge the homogeneous theory in $\mathbb{R}^{1+d}$ with the inhomogeneous one in $\mathbb{R}^d$ is to reinterpret $s\nabla_{t,x}(1+s^2\boldsymbol{L})^{-1}$ as an operator in $\mathrm{L}^2(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d))$ with an $\mathrm{L}^2(\mathbb{R}^d)$-valued symbol. To this end, we define the one-dimensional Fourier transform as

$$(\mathcal{F}_t f)(\tau) \coloneqq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t)\, \mathrm{e}^{-\mathrm{i}t\cdot\tau}\, \mathrm{d}t \qquad (f \in \mathcal{S}(\mathbb{R}), \tau \in \mathbb{R}).$$

Lemma 3.6.3. *Let $s > 0$. Then $(1+s^2\boldsymbol{L})^{-1}$ is an $\mathcal{L}(\mathrm{L}^2(\mathbb{R}^d))$-valued Fourier multiplier with symbol*

$$\tau \mapsto (1+s^2\mathsf{L}_\tau)^{-1},$$

that is

$$(1+s^2\boldsymbol{L})^{-1} = \mathcal{F}_t^{-1}(1+s^2\mathsf{L}_\tau)^{-1}\mathcal{F}_t.$$

Proof. We let $f \in \mathrm{L}^2(\mathbb{R}^{1+d})$ and set $g \coloneqq (1+s^2\boldsymbol{L})^{-1}f \in \mathrm{dom}_2(\boldsymbol{L})$. In particular, we have $g \in \mathrm{L}^2(\mathbb{R};\mathrm{W}^{1,2}(\mathbb{R}^d)) \cap \mathrm{W}^{1,2}(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d))$. Then

$$\begin{aligned} f &= g + s^2\boldsymbol{L}g \\ &= g + s^2(-\operatorname{div}_x(d\nabla_x g) - \operatorname{div}_x(c\partial_t g) - \partial_t(b\nabla_x g) - a\partial_t^2 g). \end{aligned}$$

We take $h \in \mathcal{S}(\mathbb{R}^{1+d}) = \mathcal{S}(\mathbb{R};\mathcal{S}(\mathbb{R}^d))$ and calculate by means of Fubini's theorem

$$(f \,|\, h) = (g \,|\, h) + s^2\Big((d\nabla_x g \,|\, \nabla_x h) - (cg \,|\, \partial_t\nabla_x h) + (b\nabla_x g \,|\, \partial_t h) - (ag \,|\, \partial_t^2 h)\Big).$$

Now, we invoke Plancherel's theorem in the time variable to get

$$\begin{aligned} (\mathcal{F}_t f \,|\, \mathcal{F}_t h) &= (\mathcal{F}_t g \,|\, \mathcal{F}_t h) \\ &\quad + s^2\Big((d\nabla_x \mathcal{F}_t g \,|\, \nabla_x \mathcal{F}_t h) + (\mathrm{i}\tau c\mathcal{F}_t g \,|\, \nabla_x \mathcal{F}_t h) \\ &\quad - (\mathrm{i}\tau b\nabla_x \mathcal{F}_t g \,|\, \mathcal{F}_t h) + (a\tau^2 \mathcal{F}_t g \,|\, \mathcal{F}_t h)\Big) \\ &= ((1 + s^2(-\operatorname{div}_x d\nabla_x - \mathrm{i}\tau \operatorname{div}_x c - \mathrm{i}\tau b\nabla_x + a\tau^2))\mathcal{F}_t g \,|\, \mathcal{F}_t h) \\ &= ((1+s^2\mathsf{L}_\tau)\mathcal{F}_t g \,|\, \mathcal{F}_t h). \end{aligned}$$

From this calculation, we conclude that $(\mathcal{F}_t g)(\tau,\cdot) \in \mathrm{dom}_2(\mathsf{L}_\tau)$ for all $\tau \in \mathbb{R}$, satisfying

$$(\mathcal{F}_t f)(\tau,\cdot) = (1 + s^2 \mathsf{L}_\tau)(\mathcal{F}_t g)(\tau,\cdot).$$

Hence, we obtain

$$(1 + s^2 \boldsymbol{L})^{-1} f = g = \mathcal{F}_t^{-1}(1 + s^2 \mathsf{L}_\tau)^{-1} \mathcal{F}_t f. \qquad \square$$

A sufficient condition to prove an L^p-extrapolation result for vector-valued multipliers is furnished by the celebrated theorem of Weis [161]. To state this result, we need

Definition 3.6.4. Let $p \in [1,\infty)$ and $U \subseteq \mathbb{C} \setminus \{0\}$. We say that $(T(z))_{z\in U} \subseteq \mathcal{L}(\mathrm{L}^2)$ satisfies **square function estimates on $\mathbf{L}^p$**, if there is $C > 0$ such that

$$\Big\| \Big(\sum_{j\le n} |T(z_j) f_j|^2 \Big)^{\frac{1}{2}} \Big\|_p \le C \Big\| \Big(\sum_{j\le n} |f_j|^2 \Big)^{\frac{1}{2}} \Big\|_p$$

for all $n \in \mathbb{N}$, $z_1,\dots,z_n \in U$ and $f_1,\dots,f_n \in \mathrm{L}^p \cap \mathrm{L}^2$.

Remark 3.6.5. The latter definition means that (the L^p-extension of) the operator family $(T(z))_{z\in U}$ is $\mathcal{R}$-bounded in L^p, see [125, Rem. 2.9]. As we do not need this notion in depth, we decided to use this reformulation.

Theorem 3.6.6 (Weis, [161, Thm. 3.4])**.** *Assume that $p \in (1,\infty)$ and that $m \in \mathrm{C}^1(\mathbb{R}^*;\mathcal{L}(\mathrm{L}^p))$ is such that $(m(\tau))_{\tau\in\mathbb{R}^*}$ and $(\tau m'(\tau))_{\tau\in\mathbb{R}^*}$ satisfy square function estimates on L^p. Then, for all $q \in (1,\infty)$, the operator*

$$Tf = \mathcal{F}_t^{-1} m \mathcal{F}_t f \qquad (f \in \mathcal{S}(\mathbb{R};\mathrm{L}^p))$$

extends to a bounded operator from $\mathrm{L}^q(\mathbb{R};\mathrm{L}^p)$ into itself. The implicit constant depends on m only through the square function bound in L^p.

In view of Lemma 3.6.3 and Theorem 3.6.6, the proof of Proposition 3.6.1 boils down to checking smoothness and square function estimates in L^p for the symbol associated to $s\nabla_{t,x}(1 + s^2\boldsymbol{L})^{-1}$. Using a pointwise domination technique via the Hardy–Littlewood maximal operator and the vector-valued Fefferman–Stein inequality, the claim follows once we find $\mathcal{P}_-((\mathsf{L}_\tau)_\tau) < r < 2 < p < q < \mathcal{P}_+((\mathsf{L}_\tau)_\tau)$ such that this symbol satisfies $\mathrm{L}^r - \mathrm{L}^q$ off-diagonal estimates. We learned this argument from [41]. However, as the gain of integrability of one resolvent is limited, we decided to work with powers of resolvents. According to Lemma 3.6.3, we can record

Corollary 3.6.7. *Let $s > 0$ and $N \in \mathbb{N}$. The symbol of $s\nabla_{t,x}(1 + s^2\boldsymbol{L})^{-N}$ is given by*

$$m_s(\tau) = \begin{bmatrix} m_s(\tau)_\perp \\ m_s(\tau)_\parallel \end{bmatrix} = \begin{bmatrix} \mathrm{i}s\tau(1 + s^2\mathsf{L}_\tau)^{-N} \\ s\nabla_x(1 + s^2\mathsf{L}_\tau)^{-N} \end{bmatrix}. \tag{3.4}$$

In view of the outline above, the rest of this section is dedicated to establishing $\mathrm{L}^p - \mathrm{L}^q$ mapping properties of $m_s(\tau)$, where N will be specified later on. We begin by developing the L^2-theory of particular operator families associated to L_τ and pass afterwards to the study of $\mathrm{L}^p - \mathrm{L}^q$-estimates. The families of interest are the following.

Definition 3.6.8. Let $s > 0$ and $\tau \in \mathbb{R}^*$. We introduce the operator families

$$\begin{aligned}
{}^{\alpha}R_{(0,0)}(s,\tau) &\coloneqq (s\tau)^{\alpha}(1 + s^2\mathsf{L}_\tau)^{-1} \qquad (\alpha \in \{0,1,2\}),\\
{}^{\alpha}R_{(1,0)}(s,\tau) &\coloneqq (s\tau)^{\alpha}s\nabla_x(1 + s^2\mathsf{L}_\tau)^{-1} \qquad (\alpha \in \{0,1\}),\\
{}^{\alpha}R_{(0,1)}(s,\tau) &\coloneqq (s\tau)^{\alpha}(1 + s^2\mathsf{L}_\tau)^{-1}s\,\mathrm{div}_x \qquad (\alpha \in \{0,1\}),\\
{}^{\alpha}R_{(1,1)}(s,\tau) &\coloneqq (s\tau)^{\alpha}s\nabla_x(1 + s^2\mathsf{L}_\tau)^{-1}s\,\mathrm{div}_x \qquad (\alpha = 0).
\end{aligned}$$

For $\alpha = 0$, we omit this index.

3.6.1 L^2-theory for the auxiliary operators

Our first goal is to prove that the families ${}^{\alpha}R_{\beta}(\cdot,\tau)$ satisfy L^2 off-diagonal estimates with implicit constants depending only on $\lambda(\mathsf{A})$ and $\Lambda(\mathsf{A})$. To achieve this, we employ a scaling argument. Using the notation

$$(\delta_t f)(x) \coloneqq f(tx) \qquad (t \in \mathbb{R}^*, x \in \mathbb{R}^d),$$

the transformation rule yields

$$\begin{aligned}
\tau^{-2}\delta_{\tau^{-1}}\mathsf{L}_\tau\delta_\tau &= \delta_{\tau^{-1}}\begin{bmatrix} -\mathrm{i} & -\tau^{-1}\,\mathrm{div}_x \end{bmatrix}\mathsf{A}\begin{bmatrix} \mathrm{i} \\ \tau^{-1}\nabla_x \end{bmatrix}\delta_\tau\\
&= S_1^*\mathsf{A}(\tau^{-1}\cdot)S_1 =: \mathsf{L}_{\mathsf{A}(\tau^{-1}\cdot)}.
\end{aligned}$$

The ellipticity assumption on A implies that $\mathsf{L}_{\mathsf{A}(\tau^{-1}\cdot)}$ is a divergence form operator associated with a form that has $\lambda(\mathsf{A})$ as a lower bound and $\Lambda(\mathsf{A})$ as an upper bound. Thus, [37, Prop. 3.2] yields

Lemma 3.6.9. *There are $C, c > 0$ depending only on $\lambda(\mathsf{A})$ and $\Lambda(\mathsf{A})$ such that*

$$\| \mathrm{e}^{-s^2 \mathsf{L}_{\mathsf{A}(\tau\cdot)}} f\|_{\mathrm{L}^2(F)} + \|s\nabla_x \mathrm{e}^{-s^2 \mathsf{L}_{\mathsf{A}(\tau\cdot)}} f\|_{\mathrm{L}^2(F)} \leq C \mathrm{e}^{-c\frac{\mathrm{d}(E,F)^2}{s^2} - cs^2} \|f\|_{\mathrm{L}^2(E)}$$

for all $s > 0$, $\tau \in \mathbb{R}^*$, measurable sets $E, F \subseteq \mathbb{R}^d$ and $f \in \mathrm{L}^2(E)$.

By duality and composition, we can record

Corollary 3.6.10. *There are $C, c > 0$ depending only on $\lambda(\mathsf{A})$ and $\Lambda(\mathsf{A})$ such that*

$$\begin{aligned}\| \mathrm{e}^{-s^2 \mathsf{L}_{\mathsf{A}(\tau\cdot)}} s \operatorname{div}_x f\|_{\mathrm{L}^2(F)} + \|s\nabla_x \mathrm{e}^{-s^2 \mathsf{L}_{\mathsf{A}(\tau\cdot)}} s \operatorname{div}_x f\|_{\mathrm{L}^2(F)} \\ \leq C \mathrm{e}^{-c\frac{\mathrm{d}(E,F)^2}{s^2} - cs^2} \|f\|_{\mathrm{L}^2(E)}\end{aligned}$$

for all $s > 0$, $\tau \in \mathbb{R}^*$, measurable sets $E, F \subseteq \mathbb{R}^d$ and $f \in \mathrm{L}^2(E)$.

In a next step, we transform these estimates to resolvents. For technical reasons, we begin with the L^2-boundedness of $R_{(1,1)}(\cdot, \tau)$.

Lemma 3.6.11. *For all $s > 0$, $\tau \in \mathbb{R}^*$ and $f \in \mathrm{L}^2$, we have*

$$\|R_{(1,1)}(s, \tau) f\|_2 \leq \lambda(\mathsf{A})^{-1} \|f\|_2.$$

Proof. We fix $s > 0$, $\tau \in \mathbb{R}^*$ and $u \in \mathrm{W}^{1,2}_\tau$. By the Lax–Milgram lemma and the estimate

$$\lambda(\mathsf{A}) s^2 \|u\|^2_{\mathrm{W}^{1,2}_\tau} \leq \operatorname{Re}\langle (1 + s^2 \mathsf{L}_\tau) u \,|\, u\rangle \leq \|(1 + s^2 \mathsf{L}_\tau) u\|_{\mathrm{W}^{-1,2}_\tau} \|u\|_{\mathrm{W}^{1,2}_\tau},$$

we obtain that $1 + s^2 \mathsf{L}_\tau \colon \mathrm{W}^{1,2}_\tau \to \mathrm{W}^{-1,2}_\tau$ is boundedly invertible and its inverse has norm at most $\lambda(\mathsf{A})^{-1} s^{-2}$. Hence, we get for all $f \in \mathrm{L}^2$ that

$$\begin{aligned}\|s\nabla_x (1 + s^2 \mathsf{L}_\tau)^{-1} s \operatorname{div}_x f\|_2 &\leq s^2 \|(1 + s^2 \mathsf{L}_\tau)^{-1} \operatorname{div}_x f\|_{\mathrm{W}^{1,2}_\tau} \\ &\leq \lambda(\mathsf{A})^{-1} \|\operatorname{div}_x f\|_{\mathrm{W}^{-1,2}_\tau} \\ &\leq \lambda(\mathsf{A})^{-1} \|f\|_2.\end{aligned}$$

□

Lemma 3.6.12. *Let $\beta \in \{0, 1\}^2$, $\alpha \in \{0, \dots, 2 - |\beta|\}$ and $\tau \in \mathbb{R}^*$. Then the operator family ${}^\alpha R_\beta(\cdot, \tau)$ satisfies L^2 off-diagonal estimates with implicit constants depending only on $\lambda(\mathsf{A})$ and $\Lambda(\mathsf{A})$.*

Proof. We fix $s > 0$, $\tau \in \mathbb{R}^*$, measurable sets $E, F \subseteq \mathbb{R}^d$, $f \in \mathrm{L}^2(E)$ and normalize $\|f\|_{\mathrm{L}^2(E)} = 1$. In view of Lemma 3.6.11, we can additionally assume that $\mathrm{d}(E,F)/s \geq 1$ if $\beta = (1,1)$. Let us write

$$ {}^{\alpha}R_{\beta}^{\mathsf{A}(\tau^{-1}\cdot)}(s,1) $$

for the operator ${}^{\alpha}R_{\beta}(s,1)$ with A replaced by $\mathsf{A}(\tau^{-1}\cdot)$. By the Laplace-transform joint with Lemma 3.6.9 and Corollary 3.6.10, we get

$$\begin{aligned} \| {}^{\alpha}R_{\beta}^{\mathsf{A}(\tau^{-1}\cdot)}(s,1)f\|_{\mathrm{L}^2(F)} &\lesssim s^{\alpha} \int_0^{\infty} \mathrm{e}^{-t-cts^2-c\frac{\mathrm{d}(E,F)^2}{ts^2}} \, \frac{\mathrm{d}t}{t^{|\beta|/2}} \\ &= s^{\alpha} \int_0^{\infty} \mathrm{e}^{-\frac{t}{2}-cts^2-\left(\frac{t}{2}+c\frac{\mathrm{d}(E,F)^2}{2ts^2}\right)-c\frac{\mathrm{d}(E,F)^2}{2ts^2}} \, \frac{\mathrm{d}t}{t^{|\beta|/2}}. \end{aligned}$$

Applying the AM-GM inequality and using that $\mathrm{d}(E,F)/s \geq 1$ for $\beta = (1,1)$, we proceed as follows:

$$\begin{aligned} &\leq \mathrm{e}^{-c\frac{\mathrm{d}(E,F)}{s}} \, s^{\alpha} \int_0^{\infty} \mathrm{e}^{-\frac{1}{2}(1+cs^2)t} \, \mathrm{e}^{-c\frac{\mathrm{d}(E,F)^2}{ts^2}} \, t^{-\frac{|\beta|}{2}} \, \mathrm{d}t \\ &\leq \mathrm{e}^{-c\frac{\mathrm{d}(E,F)}{s}} \, s^{\alpha} \int_0^{\infty} \mathrm{e}^{-\frac{1}{2}(1+cs^2)t} (\mathbf{1}_{[|\beta|<2]} t^{-\frac{|\beta|}{2}} + \mathbf{1}_{[|\beta|=2]}) \, \mathrm{d}t \\ &\lesssim \mathrm{e}^{-c\frac{\mathrm{d}(E,F)}{s}} \, \frac{s^{\alpha}}{(1+cs^2)^{1-|\beta|/2}} \\ &\lesssim \mathrm{e}^{-c\frac{\mathrm{d}(E,F)}{s}} \, . \end{aligned}$$

Now, we finish the proof by scaling. By similarity, it holds

$$ \mathrm{sgn}(\tau)^{\alpha+|\beta|} \, {}^{\alpha}R_{\beta}^{\mathsf{A}(\tau^{-1}\cdot)}(s|\tau|,1) = \delta_{\tau^{-1}} \, {}^{\alpha}R_{\beta}(s,\tau)\delta_{\tau}. $$

Hence, the first part of the proof reveals the estimate

$$ \|\delta_{\tau^{-1}} \, {}^{\alpha}R_{\beta}(s,\tau)f\|_{\mathrm{L}^2(\tau F)} \leq C \, \mathrm{e}^{-c\frac{\mathrm{d}(\tau E,\tau F)}{s|\tau|}} \, \|\delta_{\tau^{-1}} f\|_{\mathrm{L}^2(\tau E)} $$

for all $s > 0$, $\tau \in \mathbb{R}^*$, measurable sets $E, F \subseteq \mathbb{R}^d$ and $f \in \mathrm{L}^2(E)$. Finally, we use $\mathrm{d}(\tau E, \tau F) = |\tau|\mathrm{d}(E,F)$ and the transformation $x \mapsto \tau x$ to conclude. □

3.6.2 L^p-theory for the auxiliary operators

We start with the following Sobolev embeddings related to the resolvent. The qualitative result is not new but once again τ-independence of constants matters.

Lemma 3.6.13. *If $1 \vee \mathcal{P}_-((\mathsf{L}_\tau)_\tau)_* < p < \mathcal{P}_+((\mathsf{L}_\tau)_\tau)^*$, then $(R_{(0,0)}(s,\tau))_{s>0}$ is L^p-bounded with implicit constant independent of $\tau \in \mathbb{R}^*$.*

Proof. We use a bootstrapping argument to increase p step-by-step.

Base case $2_* < p < 2^*$. We first consider exponents $p \geq 2$. By the Gagliardo–Nirenberg inequality and the L^2-estimates from Lemma 3.6.12, we find for all $s > 0$ and $f \in \mathrm{L}^2$ that

$$\begin{aligned}\|(1+s^2\mathsf{L}_\tau)^{-1}f\|_p &\lesssim s^{\frac{d}{p}-\frac{d}{2}}\|s\nabla_x(1+s^2\mathsf{L}_\tau)^{-1}f\|_2^{\frac{d}{2}-\frac{d}{p}}\|(1+s^2\mathsf{L}_\tau)^{-1}f\|_2^{1-\frac{d}{2}+\frac{d}{p}}\\ &\lesssim s^{\frac{d}{p}-\frac{d}{2}}\|f\|_2.\end{aligned}$$

This means that $(R_{(0,0)}(s,\tau))_{s>0}$ is $\mathrm{L}^2 - \mathrm{L}^p$-bounded. Since the restriction on p is open-ended, L^p-boundedness follows from Lemma 1.4.5.

So far, we have only used L^2-theory for L_τ. Hence, the same conclusion is valid for the adjoint L_τ^*, which is an operator in the same class as L_τ. By duality, $(R_{(0,0)}(s,\tau))_{s>0}$ is $\mathrm{L}^{p'} - \mathrm{L}^2$-bounded and $\mathrm{L}^{p'}$-bounded. This covers all exponents in $(2_*, 2)$.

Inductive case. The next case to consider is $2^* < p < (2^*)^* \wedge \mathcal{P}_+((\mathsf{L}_\tau)_\tau)^*$. (We do not have to consider $p = 2^*$ explicitly since all assumptions on p are open-ended). This scenario can only appear in dimension $d \geq 3$ since $2^* = \infty$ when $d \leq 2$. Since $(p_*)_* > 2_*$, we know from the base case that $(R_{(0,0)}(s,\tau))_{s>0}$ is $\mathrm{L}^{(p_*)_*}$-bounded. Given $f \in \mathrm{L}^{(p_*)_*} \cap \mathrm{L}^2$, Sobolev embeddings yields

$$\begin{aligned}\|(1+s^2\mathsf{L}_\tau)^{-1}f\|_p &\lesssim s^{-2}\|\nabla_x\mathsf{L}_\tau^{-1}(1-(1+s^2\mathsf{L}_\tau)^{-1})f\|_{p_*}\\ &\leq s^{-2}\|\mathsf{L}_\tau^{-1}(1-(1+s^2\mathsf{L}_\tau)^{-1})f\|_{\mathrm{W}_\tau^{1,p_*}}.\end{aligned}$$

Since $2 < p_* < \mathcal{P}_+((\mathsf{L}_\tau)_\tau)$, the L^{p_*}-boundedness of the Hodge projectors in its equivalent form in Proposition 3.2.3 and another Sobolev embedding lead to

$$\begin{aligned}&\lesssim s^{-2}\|(1-(1+s^2\mathsf{L}_\tau)^{-1})f\|_{\mathrm{W}_\tau^{-1,p_*}}\\ &\lesssim s^{-2}\|(1-(1+s^2\mathsf{L}_\tau)^{-1})f\|_{(p_*)_*}.\end{aligned}$$

This means that $(R_{(0,0)}(s,\tau))_{s>0}$ is $\mathrm{L}^{(p_*)_*} - \mathrm{L}^p$-bounded and Lemma 1.4.5 yields the desired L^p-boundedness as before. All implicit constants in this argument are independent of τ.

Iterating the procedure covers the full range $2^* < p < \mathcal{P}_+((\mathsf{L}_\tau)_\tau)^*$ in a finite number of steps and we conclude by duality. □

Our proof in the base case revealed an additional result that we record for later.

Corollary 3.6.14. *If $1 \vee 2_* < p < 2$, then $(R_{(0,0)}(s,\tau))_{s>0}$ is $\mathrm{L}^p - \mathrm{L}^2$-bounded with implicit constant independent of $\tau \in \mathbb{R}^*$.*

We turn to the other operator families.

Lemma 3.6.15. *Let $d \geq 2$, $p \in \mathcal{P}((\mathsf{L}_\tau)_\tau)$, $|\beta| \leq 1$ and $\alpha = 1$ when $\beta = (0,1)$. Then $({}^\alpha R_\beta(s,\tau))_{s>0}$ is L^p-bounded with implicit constant independent of $\tau \in \mathbb{R}^*$.*

Proof. For $R_{(0,0)}$, this is Lemma 3.6.13. The other families can be classified into three groups.

The family ${}^\alpha\boldsymbol{R}_{(0,0)}$ with $\boldsymbol{\alpha \in \{1,2\}}$. Since $|s\tau| \leq 1 + |s\tau|^2$, it suffices to treat ${}^2R_{(0,0)}$. However, we have

$$(s\tau)^2(1+s^2\mathsf{L}_\tau)^{-1} = \tau^2\mathsf{L}_\tau^{-1}(1-(1+s^2\mathsf{L}_\tau)^{-1}),$$

and the claim follows by combining Proposition 3.2.3 and Lemma 3.6.13.

The families ${}^1\boldsymbol{R}_{(1,0)}$ and ${}^1\boldsymbol{R}_{(0,1)}$. We write

$$\begin{aligned}(s\tau)s\nabla_x(1+s^2\mathsf{L}_\tau)^{-1} &= \tau\nabla_x\mathsf{L}_\tau^{-1}(1-(1+s^2\mathsf{L}_\tau)^{-1}),\\ (s\tau)(1+s^2\mathsf{L}_\tau)^{-1}s\operatorname{div}_x &= (1-(1+s^2\mathsf{L}_\tau)^{-1})\tau\mathsf{L}_\tau^{-1}\operatorname{div}_x\end{aligned}$$

and conclude once again by Proposition 3.2.3 and Lemma 3.6.13.

The family $\boldsymbol{R}_{(1,0)}$. Our assumption $d \geq 2$ enables us to fix $q \in \mathcal{P}((\mathsf{L}_\tau)_\tau) \cap (1^*, \infty)$. As before, we write

$$s\nabla_x(1+s^2\mathsf{L}_\tau)^{-1} = s^{-1}\nabla_x\mathsf{L}_\tau^{-1}(1-(1+s^2\mathsf{L}_\tau)^{-1}).$$

Proposition 3.2.3, the Sobolev embedding $\mathrm{L}^{q_*} \subseteq \mathrm{W}_\tau^{-1,q}$ and Lemma 3.6.13 yield for all $f \in \mathrm{L}^{q_*} \cap \mathrm{L}^2$ the bound

$$\begin{aligned}\|s\nabla_x(1+s^2\mathsf{L}_\tau)^{-1}f\|_q &\leq s^{-1}\|\mathsf{L}_\tau^{-1}(1-(1+s^2\mathsf{L}_\tau)^{-1})f\|_{\mathrm{W}_\tau^{1,q}}\\ &\lesssim s^{-1}\|(1-(1+s^2\mathsf{L}_\tau)^{-1})f\|_{\mathrm{W}_\tau^{-1,q}}\\ &\lesssim s^{-1}\|(1-(1+s^2\mathsf{L}_\tau)^{-1})f\|_{q_*}\\ &\lesssim s^{-1}\|f\|_{q_*}.\end{aligned}$$

This means that $(R_{(1,0)}(s,\tau))_{s>0}$ is $\mathrm{L}^{q_*} - \mathrm{L}^q$-bounded with τ-independent bound. Open-endedness in q and Lemma 1.4.5 yield the claim. □

3.6.3 L^p-theory for the symbol

In this section, we assemble estimates for the auxiliary functions in order to derive smoothness and $\mathrm{L}^p - \mathrm{L}^q$-type bounds for the symbol $m_s(\tau)$ from (3.4).

Lemma 3.6.16. *Let $s > 0$, $N \in \mathbb{N}$ and $p \in \mathcal{P}((\mathsf{L}_\tau)_\tau)$. Then the map $\tau \mapsto (1 + s^2\mathsf{L}_\tau)^{-N}$ is of class $\mathrm{C}^1(\mathbb{R}^*; \mathcal{L}(\mathrm{L}^p, \mathrm{W}^{1,p}))$ with derivative*

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(1 + s^2\mathsf{L}_\tau)^{-N} = s^2 \sum_{k=1}^{N} (1 + s^2\mathsf{L}_\tau)^{-k}(\mathrm{i}\,\mathrm{div}_x\, c + \mathrm{i}b\nabla_x - 2\tau a)(1 + s^2\mathsf{L}_\tau)^{-(N+1-k)}.$$

In particular, we have $m_s \in \mathrm{C}^1(\mathbb{R}^*; \mathcal{L}(\mathrm{L}^p))$.

Proof. We begin with the assertion about the resolvent. By the product rule and induction, it suffices to do the case $N = 1$. For $\sigma \neq \tau$, we have

$$\begin{aligned}
&(1 + s^2\mathsf{L}_\tau)^{-1} - (1 + s^2\mathsf{L}_\sigma)^{-1} \\
&= s^2(1 + s^2\mathsf{L}_\tau)^{-1}(\mathsf{L}_\sigma - \mathsf{L}_\tau)(1 + s^2\mathsf{L}_\sigma)^{-1} \\
&= s^2(1 + s^2\mathsf{L}_\tau)^{-1}(\mathrm{i}(\tau - \sigma)\,\mathrm{div}_x\, c + \mathrm{i}(\tau - \sigma)b\nabla_x - (\tau^2 - \sigma^2)a)(1 + s^2\mathsf{L}_\sigma)^{-1} \\
&= (\tau - \sigma)s^2(1 + s^2\mathsf{L}_\tau)^{-1}(\mathrm{i}\,\mathrm{div}_x\, c + \mathrm{i}b\nabla_x - (\tau + \sigma)a)(1 + s^2\mathsf{L}_\sigma)^{-1} \\
&=: (\tau - \sigma)r(\sigma, \tau).
\end{aligned}$$

Thanks to Lemma 3.6.15, the remainder $r(\sigma, \tau)$ is bounded in $\mathcal{L}(\mathrm{L}^p)$-norm, uniformly in σ and τ in compact subsets of $\mathbb{R}^*$. Hence, $\tau \mapsto (1 + s^2\mathsf{L}_\tau)^{-1}$ is continuous in $\mathbb{R}^*$ with values in $\mathcal{L}(\mathrm{L}^p)$.

The same type of argument can be used to prove continuity in $\mathbb{R}^*$ with values in $\mathcal{L}(\mathrm{L}^p)$ for $\tau \mapsto \nabla_x(1 + s^2\mathsf{L}_\tau)^{-1}$ and $\tau \mapsto (1 + s^2\mathsf{L}_\tau)^{-1}\,\mathrm{div}_x$. In this calculation, the operator $s^2\nabla_x(1 + s^2\mathsf{L}_\tau)^{-1}\,\mathrm{div}_x$ appears, which cannot be handled via Lemma 3.6.15. However, we can write

$$s^2\nabla_x(1 + s^2\mathsf{L}_\tau)^{-1}\,\mathrm{div}_x = \nabla_x\mathsf{L}_\tau^{-1}\,\mathrm{div}_x - \nabla_x\mathsf{L}_\tau^{-1}(1 + s^2\mathsf{L}_\tau)^{-1}\,\mathrm{div}_x,$$

as a composition of operators that either fall in the scope of Lemma 3.6.15 or are controlled through the Hodge projector.

Altogether, the remainder function $\tau \mapsto r(\sigma, \tau)$ is continuous with values in $\mathcal{L}(\mathrm{L}^p, \mathrm{W}^{1,p})$. Thus, $\sigma \mapsto (1 + s^2\mathsf{L}_\sigma)^{-1}$ is of class $\mathrm{C}^1(\mathbb{R}^*; \mathcal{L}(\mathrm{L}^p, \mathrm{W}^{1,p}))$ with derivative

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}(1 + s^2\mathsf{L}_\sigma)^{-1} = r(\sigma, \sigma).$$

Continuous differentiability of m_s follows immediately by the product rule since we have $m_s(\tau)_\perp = \mathrm{i}s\tau(1 + s^2\mathsf{L}_\tau)^{-N}$ and $m_s(\tau)_\parallel = s\nabla_x(1 + s^2\mathsf{L}_\tau)^{-N}$. □

We have reached the point in the argument, where we choose N large.

Lemma 3.6.17. *Given $\mathcal{P}_-((\mathsf{L}_\tau)_\tau) < r < 2 < q < \mathcal{P}_+((\mathsf{L}_\tau)_\tau)$, there exists $N \in \mathbb{N}$, divisible by 4, such that $((1+s^2\mathsf{L}_\tau)^{-N/4})_{s>0}$ is $\mathrm{L}^r - \mathrm{L}^q$-bounded with implicit constant independent of $\tau \in \mathbb{R}^*$.*

Proof. We consider the resolvent family $((1+s^2\mathsf{L}_\tau)^{-1})_{s>0}$ and the following boundedness properties with implicit constant independent of $\tau \in \mathbb{R}^*$. By Lemma 3.6.13, we have L^ϱ-bounds for all $\varrho \in (q, q^*)$ and by Corollary 3.6.14 we have $\mathrm{L}^\sigma - \mathrm{L}^2$-bounds for all $\sigma \in (1 \vee 2_*, 2)$. By interpolation, we get an $\mathrm{L}^\sigma - \mathrm{L}^q$ bound for some $\sigma \in (1 \vee 2_*, q)$. Again by Lemma 3.6.13 we also have the L^σ- and the L^q-bound. Now, the argument in [20, Lem. 4.4] yields the claim. □

The proof of the next result clarifies why this choice is appropriate for our purpose.

Lemma 3.6.18. *Let $\mathcal{P}_-((\mathsf{L}_\tau)_\tau) < r < 2 < q < \mathcal{P}_+((\mathsf{L}_\tau)_\tau)$ and N be as in Lemma 3.6.17. Then $(m_s(\tau))_{s>0}$ and $(\tau m_s'(\tau))_{s>0}$ are $\mathrm{L}^r - \mathrm{L}^q$-bounded with implicit constants independent of $\tau \in \mathbb{R}^*$.*

Proof. The bound for $m_s(\tau)$ follows by composition from Lemma 3.6.17 and the L^q-bounds for ${}^1R_{(0,0)}$ and $R_{(1,0)}$ in Lemma 3.6.15. Let us come to the estimates for $\tau m_s'(\tau)$.

The scalar component. We compute

$$\tau m_s'(\tau)_\perp = m_s(\tau)_\perp + \mathrm{i}s\tau^2 \frac{\mathrm{d}}{\mathrm{d}\tau}(1+s^2\mathsf{L}_\tau)^{-N}$$

and still need to handle the second term on the right. To this end, we write the formula in Lemma 3.6.16 in the following form:

$$\mathrm{i}s\tau^2 \frac{\mathrm{d}}{\mathrm{d}\tau}(1+s^2\mathsf{L}_\tau)^{-N} = \sum_{k=1}^{N} (1+s^2\mathsf{L}_\tau)^{-(k-1)} T(s,\tau) (1+s^2\mathsf{L}_\tau)^{-(N-k)}, \tag{3.5}$$

where

$$\begin{aligned} T(s,\tau) &:= \mathrm{i}s^3\tau^2(1+s^2\mathsf{L}_\tau)^{-1}(\mathrm{i}\,\mathrm{div}_x\, c + \mathrm{i}b\nabla_x - 2\tau a)(1+s^2\mathsf{L}_\tau)^{-1} \\ &= -\Big((s\tau)(1+s^2\mathsf{L}_\tau)^{-1} s\,\mathrm{div}_x\Big)\Big(c(s\tau)(1+s^2\mathsf{L}_\tau)^{-1}\Big) \\ &\quad - \Big((s\tau)^2(1+s^2\mathsf{L}_\tau)^{-1}\Big)\Big(bs\nabla_x(1+s^2\mathsf{L}_\tau)^{-1}\Big) \\ &\quad - 2\mathrm{i}\Big((s\tau)(1+s^2\mathsf{L}_\tau)^{-1}\Big)\Big(a(s\tau)^2(1+s^2\mathsf{L}_\tau)^{-1}\Big) \end{aligned} \tag{3.6}$$

is a composition of operators ${}^{\alpha}R_\beta(s,\tau)$ with $|\beta| \le 1$ and $\alpha = 1$ for $\beta = (0,1)$. Thus, Lemma 3.6.15 yields that $(T(s,\tau))_{s>0}$ is L^r- and L^q-bounded, independently of τ. The upshot is that for each summand in (3.5) the exponents sum up to $(k-1)+(N-k) = N-1 \ge N/2$ and hence one exponent is at least $N/4$. By the choice of N, the entire expression, as a family indexed in $s>0$, is $\mathrm{L}^r - \mathrm{L}^q$-bounded. This concludes the treatment of the scalar component.

The vectorial component. Since $\nabla_x \colon \mathrm{W}^{1,p} \to (\mathrm{L}^p)^d$ is bounded, we obtain from Lemma 3.6.16 and with $T(s,\tau)$ as in (3.6) that

$$\begin{aligned}
\tau m_s'(\tau)_\parallel &= s\tau\nabla_x \frac{\mathrm{d}}{\mathrm{d}\tau}(1+s^2\mathsf{L}_\tau)^{-N} \\
&= \sum_{k=1}^{N} s\nabla_x(1+s^2\mathsf{L}_\tau)^{-(k-1)}(\mathrm{i}s\tau)^{-1}T(s,\tau)(1+s^2\mathsf{L}_\tau)^{-(N-k)} \\
&= \Big(s\nabla_x(\mathrm{i}s\tau)^{-1}T(s,\tau)\Big)(1+s^2\mathsf{L}_\tau)^{-(N-1)} \\
&\quad + \sum_{k=2}^{N} \Big(s\nabla_x(1+s^2\mathsf{L}_\tau)^{-1}\Big)(1+s^2\mathsf{L}_\tau)^{-(k-2)}\Big((\mathrm{i}s\tau)^{-1}T(s,\tau)\Big) \\
&\qquad\qquad (1+s^2\mathsf{L}_\tau)^{-(N-k)}.
\end{aligned}$$

Let us first handle the terms in $k = 2,\dots,N$. It follows from (3.6) that $((\mathrm{i}s\tau)^{-1}T(s,\tau))_{s>0}$ is a composition of the same type as $(T(s,\tau))_{s>0}$. Hence, this family is L^r- and L^q-bounded, independently of τ. By Lemma 3.6.15, the same is true for $(s\nabla_x(1+s^2\mathsf{L}_\tau)^{-1})_{s>0}$ appearing on the left. The sum of the exponents still satisfies $(k-2)+(N-k) = N-2 \ge N/2$, so we obtain the required $\mathrm{L}^r - \mathrm{L}^q$-bound as before.

For the final term, we already know that $((1+s^2\mathsf{L}_\tau)^{-(N-1)})_{s>0}$ is $\mathrm{L}^r - \mathrm{L}^q$-bounded with implicit constants independent of $\tau \in \mathbb{R}^*$. Thus, it suffices to prove L^q-boundedness, uniformly in τ, for

$$\begin{aligned}
s\nabla_x(\mathrm{i}s\tau)^{-1}T(s,\tau) = {}& \mathrm{i}\Big(s\nabla_x(1+s^2\mathsf{L}_\tau)^{-1}s\,\mathrm{div}_x\Big)\Big(c(s\tau)(1+s^2\mathsf{L}_\tau)^{-1}\Big) \\
&+ \mathrm{i}\Big((s\tau)s\nabla_x(1+s^2\mathsf{L}_\tau)^{-1}\Big)\Big(bs\nabla_x(1+s^2\mathsf{L}_\tau)^{-1}\Big) \\
&- 2\Big(s\nabla_x(1+s^2\mathsf{L}_\tau)^{-1}\Big)\Big(a(s\tau)^2(1+s^2\mathsf{L}_\tau)^{-1}\Big).
\end{aligned} \tag{3.7}$$

The second and third term on the right are L^q-bounded by Lemma 3.6.15 but this lemma does not cover the family $R_{(1,1)}$ that appears in the first term.

However, writing

$$\begin{aligned} s\nabla_x(1+s^2\mathsf{L}_\tau)^{-1}s\,\mathrm{div}_x &= \nabla_x\mathsf{L}_\tau^{-1}s^2\mathsf{L}_\tau(1+s^2\mathsf{L}_\tau)^{-1}\,\mathrm{div}_x \\ &= \nabla_x\mathsf{L}_\tau^{-1}\,\mathrm{div}_x - s\nabla_x(1+s^2\mathsf{L}_\tau)^{-1}\tau\mathsf{L}_\tau^{-1}\,\mathrm{div}_x(s\tau)^{-1}, \end{aligned}$$

we see that the first of the three terms on the right in (3.7) can also be decomposed into

$$\begin{aligned} &\mathrm{i}\Big(\nabla_x\mathsf{L}_\tau^{-1}\,\mathrm{div}_x\Big)\Big(c(s\tau)(1+s^2\mathsf{L}_\tau)^{-1}\Big) \\ &\qquad - \mathrm{i}\Big(s\nabla_x(1+s^2\mathsf{L}_\tau)^{-1}\Big)\Big(\tau\mathsf{L}_\tau^{-1}\,\mathrm{div}_x\Big)\Big(c(1+s^2\mathsf{L}_\tau)^{-1}\Big). \end{aligned}$$

Now, the L^q-bound follows from Proposition 3.2.3 and Lemma 3.6.15. □

The latter proof also demonstrates that $m_s(\tau)$ and $\tau m_s'(\tau)$ can be written as a sum and composition of the families ${}^{\alpha}R_{\beta}(s,\tau)$. Hence, Lemma 3.6.12 implies that $(m_s(\tau))_{s>0}$ and $(\tau m_s'(\tau))_{s>0}$ satisfy L^2 off-diagonal estimates with implicit constants independent of τ and Lemma 1.4.5 furnishes

Corollary 3.6.19. *For all $\mathcal{P}_-((\mathsf{L}_\tau)_\tau) < r < 2 < q < \mathcal{P}_+((\mathsf{L}_\tau)_\tau)$, there is some $N \in \mathbb{N}$ such that for every $\tau \in \mathbb{R}^*$ the families $(m_s(\tau))_{s>0}$ and $(\tau m_s'(\tau))_{s>0}$ satisfy $\mathrm{L}^r - \mathrm{L}^q$ off-diagonal estimates with constants independent of τ.*

Before we come to the proof of Proposition 3.6.1, we need a pointwise domination of L^q-averages by the Hardy–Littlewood maximal operator $\mathcal{M}$. Here, Corollary 3.6.19 enables us to apply [41, Lem. 5.3].

Lemma 3.6.20. *Let $\mathcal{P}_-((\mathsf{L}_\tau)_\tau) < r < 2 < q < \mathcal{P}_+((\mathsf{L}_\tau)_\tau)$. Then there is $C > 0$ such that*

$$\left(\fint_{B(x,s)} |m_s(\tau)f|^q + |\tau m_s'(\tau)f|^q\,\mathrm{d}x\right)^{\frac{1}{q}} \leq C\mathcal{M}(|f|^r)(x)^{\frac{1}{r}}$$

for all $s > 0$, $\tau \in \mathbb{R}^*$, $x \in \mathbb{R}^d$ and $f \in \mathrm{L}^r \cap \mathrm{L}^2$.

Proposition 3.6.21. *Let $2 < p < \mathcal{P}_+((\mathsf{L}_\tau)_\tau)$. Then there is some $N \in \mathbb{N}$ such that for all $s > 0$ the families $(m_s(\tau))_{\tau\in\mathbb{R}^*}$ and $(\tau m_s'(\tau))_{\tau\in\mathbb{R}^*}$ satisfy square function estimates in L^p with bounds independent of s.*

Proof. We only prove the square function estimates for $(m_s(\tau))_{\tau\in\mathbb{R}^*}$. The argument for $(\tau m_s'(\tau))_{\tau\in\mathbb{R}^*}$ is the same. We fix $\mathcal{P}_-((\mathsf{L}_\tau)_\tau) < r < 2 < p < q <$

$\mathcal{P}_+((\mathsf{L}_\tau)_\tau)$ and choose $N \in \mathbb{N}$ as in Corollary 3.6.19. We pick finitely many $f_j \in \mathrm{L}^p \cap \mathrm{L}^2$ and $\tau_j \in \mathbb{R}^*$. By [125, Prop. 8.13], we have

$$\Big\| \Big(\sum_j |m_s(\tau_j) f_j|^2 \Big)^{\frac{1}{2}} \Big\|_p \lesssim \Big\| \Big(\sum_j | \Big(\fint_{B(\cdot,s)} |m_s(\tau_j) f_j|^q \, \mathrm{d}x \Big)^{\frac{1}{q}} |^2 \Big)^{\frac{1}{2}} \Big\|_p .$$

Now, we apply Lemma 3.6.20 to get

$$\begin{aligned} &\lesssim \Big\| \Big(\sum_j |\mathcal{M}(|f_j|^r)|^{\frac{2}{r}} \Big)^{\frac{1}{2}} \Big\|_p \\ &= \Big\| \Big(\sum_j |\mathcal{M}(|f_j|^r)|^{\frac{2}{r}} \Big)^{\frac{r}{2}} \Big\|_{\frac{p}{r}}^{\frac{1}{r}} . \end{aligned}$$

Hence, as $2/r \wedge p/r > 1$, the Fefferman–Stein inequality [92, Thm. 1] implies

$$\begin{aligned} &\lesssim \Big\| \Big(\sum_j |f_j|^2 \Big)^{\frac{r}{2}} \Big\|_{\frac{p}{r}}^{\frac{1}{r}} \\ &= \Big\| \Big(\sum_j |f_j|^2 \Big)^{\frac{1}{2}} \Big\|_p . \end{aligned}$$

□

Remark 3.6.22. The previous proof is a variant of [41, Prop. 5.8]. The main difference lies in the application of [125, Prop. 8.13] with radii $s > 0$ instead of $(\tau_j)_j$ for this allows us to use the off-diagonal decay in $s > 0$.

Finally, we are able to give the

Proof of Proposition 3.6.1. We fix $2 < p < \mathcal{P}_+((\mathsf{L}_\tau)_\tau)$ and $N \in \mathbb{N}$ as in Proposition 3.6.21. By Lemma 3.6.16, it follows that $m_s \in \mathrm{C}^1(\mathbb{R}^*; \mathcal{L}(\mathrm{L}^p))$ for all $s > 0$. Next, we combine Proposition 3.6.21 with Theorem 3.6.6 to conclude that $(s\nabla_{t,x}(1 + s^2 \boldsymbol{L})^{-N})_{s>0}$ is L^p-bounded. Hence, [20, Lem. 6.5] yields that $(s\nabla_{t,x}(1 + s^2 \boldsymbol{L})^{-1})_{s>0}$ is L^p-bounded and we infer by Proposition 3.6.2 that $p \leq \mathcal{P}_+(\boldsymbol{L})$. Consequently, $\mathcal{P}_+((\mathsf{L}_\tau)_\tau) \leq \mathcal{P}_+(\boldsymbol{L})$. □

Let us finally collect all results that we have established so far in order to provide the

Proof of Theorem 3.0.3. First, we assume that $d \geq 2$. The first equality has been shown in Proposition 3.3.7, while the second and last equality follow from Proposition 3.4.3 and the duality relation $\mathcal{P}_-(\mathsf{L}_\tau) = \mathcal{P}_+(\mathsf{L}_\tau^*)'$. Last but not least, the third one is a consequence of Proposition 3.5.1 and Proposition 3.6.1.

If $d = 1$, then [28, Prop. 3.11] implies $I(\mathsf{DB}) = (1, \infty)$. Since we have used the restriction $d \geq 2$ only in the proofs of the inequalities $\mathcal{P}_+((\mathsf{L}_\tau)_\tau) \leq m_+((\mathsf{L}_\tau)_\tau)$ and $\mathcal{P}_+((\mathsf{L}_\tau)_\tau) \leq \mathcal{P}_+(\boldsymbol{L})$ (see Proposition 3.4.3 and Lemma 3.6.15), $\mathcal{P}_+((\mathsf{L}_\tau)_\tau) = p_+(\mathsf{DB}) = \infty$ by Proposition 3.3.7 and

$$\mathcal{P}_+(\boldsymbol{L}) = m_+(\boldsymbol{L}) \leq m_+((\mathsf{L}_\tau)_\tau),$$

we are left to prove that $m_+(\boldsymbol{L}) = \infty$. We present a new and concise argument in Proposition 3.7.1, see also [29, App. B] for an alternative proof. □

3.7 The Meyers exponent in dimension $d = 1$

Proposition 3.7.1. *Let $d = 1$. There is $C > 0$ that only depends on $\lambda(\mathsf{A})$ and $\Lambda(\mathsf{A})$, such that for all axis-parallel cubes $Q \subseteq \mathbb{R}^2$ and every $U \in \mathcal{H}_{\boldsymbol{L},2Q}$ we have*

$$\|\nabla_{t,x} U\|_{\mathrm{L}^\infty(Q)} \leq C \left(\fint_{2Q} |\nabla_{t,x} U|^2 \,\mathrm{d}(t,x) \right)^{\frac{1}{2}}.$$

Proof. For any fixed $x_0 \in \mathbb{R}$ and $r > 0$, the transformed coefficients $\mathsf{A}(x_0 + r\cdot)$ are of the same class as A with the same ellipticity bounds. Thus, it suffices to treat the case $Q = Q(0,1)$. We split vectors $f \in \mathbb{C}^{1+1}$ as $f = [f_\perp, f_\parallel]^\top$ and, as in Section 3.3, we write

$$\underline{\mathsf{A}} = \begin{bmatrix} 1 & 0 \\ c & A \end{bmatrix}, \quad \text{so that} \quad \underline{\mathsf{A}}^{-1} = \begin{bmatrix} 1 & 0 \\ -A^{-1}c & A^{-1} \end{bmatrix}.$$

Since A is elliptic, it follows that A is invertible with $\|A^{-1}\|_\infty \leq \lambda^{-1}(\mathsf{A})$. We introduce $V \coloneqq \underline{\mathsf{A}} \nabla_{t,x} U$ and aim for the bound

$$\|V\|_{\mathrm{L}^\infty(Q)} \leq C \|\nabla_{t,x} U\|_{\mathrm{L}^2(2Q)}. \tag{3.8}$$

This yields the claim because $|Q| = 1$ and we have the pointwise comparability $|V| \simeq |\nabla_{t,x} U|$ with implicit constants depending only on $\lambda(\mathsf{A})$ and $\Lambda(\mathsf{A})$.

The function U is qualitatively smooth in t since A is t-independent (see [29, App. B, Lem. 1]). Within $2Q$, we compute

$$\nabla_{t,x} V_\parallel = \nabla_{t,x} (\mathsf{A} \nabla_{t,x} U)_\parallel = \begin{bmatrix} \partial_t (\mathsf{A} \nabla_{t,x} U)_\parallel \\ \partial_x (\mathsf{A} \nabla_{t,x} U)_\parallel \end{bmatrix} = \begin{bmatrix} (\mathsf{A} \nabla_{t,x} \partial_t U)_\parallel \\ -(\mathsf{A} \nabla_{t,x} \partial_t U)_\perp \end{bmatrix},$$

where we have used the equation $\boldsymbol{L}U = 0$ in the final step. We also have $\nabla_{t,x} V_\perp = \nabla_{t,x}\partial_t U$, which altogether leads to the pointwise control $|\nabla_{t,x} V| \lesssim |\nabla_{t,x}\partial_t U|$. Since we are working in dimension $1 + d = 2$, Sobolev embeddings yield for any $p > 2$ a bound

$$\begin{aligned}\|V\|_{\mathrm{L}^\infty(Q)} &\lesssim \|V\|_{\mathrm{L}^p(Q)} + \|\nabla_{t,x} V\|_{\mathrm{L}^p(Q)} \\ &\lesssim \|\nabla_{t,x} U\|_{\mathrm{L}^p(Q)} + \|\nabla_{t,x}\partial_t U\|_{\mathrm{L}^p(Q)}.\end{aligned}$$

We pick $p > 2$ such that we have the classical Meyers estimate [133] for $\boldsymbol{L}$-harmonic functions at our disposal. Since $\partial_t U$ is $\boldsymbol{L}$-harmonic by t-independence of the coefficients, Meyers estimate applies to U and $\partial_t U$, allowing us to continue by

$$\lesssim \|\nabla_{t,x} U\|_{\mathrm{L}^2(\frac{3}{2}Q)} + \|\nabla_{t,x}\partial_t U\|_{\mathrm{L}^2(\frac{3}{2}Q)}.$$

Now, (3.8) follows from the Caccioppoli inequality for $\partial_t U$. □

3.8 Remarks on the block case

We conclude this part of the chapter with a discussion of the block case. In this setting, we consider the operators

$$\boldsymbol{L} = -\operatorname{div}_x A\nabla_x - a\partial_t^2 \quad \& \quad \mathsf{L}_\tau = -\operatorname{div}_x A\nabla_x + \tau^2 a.$$

As established in [20, Chap. 15], we have $q_+(a^{-1}L_0) = q_+(L_0) = p_+(\mathsf{DB})$ and hence $q_+(L_0) = q_+(\boldsymbol{L})$. In other words, the critical number for the gradient family of the resolvent is invariant under lifting the homogeneous operator one dimension higher via the (one-dimensional) Laplacian. It is possible to prove this equality directly.

3.9 A characterization of $p_+(\mathsf{L})$ via local elliptic regularity

In this last section of this chapter, we highlight the importance of the critical number $p_+(\mathsf{L})$ in local regularity results for elliptic equations. Here, L was introduced in Definition 1.6.3. In order to simplify the exposition and to keep the focus on the critical numbers, we work on $O = \mathbb{R}^d$ and refer to Chapter 5 for broader geometric considerations and mixed boundary conditions. At this point, the reader should replace L_τ by L in Definition 3.4.1. Our main result reads as follows.

Theorem 3.9.1. *The exponent $p_+(\mathsf{L})$ is the supremum over all $p > 2$ for which there is $C > 0$ such that for all cubes $Q(x,r) \subseteq \mathbb{R}^d$, $s \geq 0$ and $u \in \mathcal{H}_{\mathsf{L}+s,x,2r}$ it holds*

$$\|u\|_{\mathrm{L}^p(Q)} \leq C r^{\frac{d}{p}-\frac{d}{2}} \|u\|_{\mathrm{L}^2(2Q)}.$$

For the proof, we require an inhomogeneous version of Caccioppoli's inequality.

Lemma 3.9.2 (Caccioppoli)**.** *There is $C > 0$, depending only on d, $\lambda(\mathsf{A})$ and $\|b\|_\infty \vee \|c\|_\infty \vee \|A\|_\infty$, such that for all cubes $Q(x,r) \subseteq \mathbb{R}^d$, every $u \in \mathcal{H}_{\mathsf{L},x,r}$ and $c \in (0,1)$ it holds*

$$\|u\|_{\mathrm{L}^2(cQ)} + \|\nabla u\|_{\mathrm{L}^2(cQ)} \leq \frac{C}{(1-c)r} \|u\|_{\mathrm{L}^2(Q)}.$$

In the homogeneous setting, the same estimate holds true without the first term on the left-hand side.

Proof. We focus on the inhomogeneous case, as the homogeneous one is even simpler. We pick $\varphi \in \mathrm{C}_\mathrm{c}^\infty$ such that $\mathbf{1}_{cQ} \leq \varphi \leq \mathbf{1}_Q$ and $\|\nabla\varphi\|_\infty \leq {}^{C}/_{(1-c)r}$ for some dimensional constant $C > 0$. Then $\varphi^2 u \in \mathrm{W}_0^{1,2}(Q)^m$ is an admissible test function for the equation $\mathsf{L}u = 0$ in Q and hence

$$\begin{aligned}
0 &= \int_{\mathbb{R}^d} A\nabla u \cdot \overline{\nabla(\varphi^2 u)} + cu \cdot \overline{\nabla(\varphi^2 u)} + b\nabla u \cdot \varphi^2 \overline{u} + a|\varphi u|^2 \\
&= \mathsf{a}(\varphi u, \varphi u) \\
&\quad + \int_{\mathbb{R}^d} \varphi cu \cdot (\nabla\varphi \otimes \overline{u}) - \varphi b(\nabla\varphi \otimes u) \cdot \overline{u} + A(\nabla\varphi \otimes u) \cdot (\nabla\varphi \otimes \overline{u}) \\
&\qquad - A(\nabla\varphi \otimes u) \cdot \overline{\nabla(\varphi u)} - A\nabla(\varphi u) \cdot (\nabla\varphi \otimes \overline{u}).
\end{aligned}$$

Thus, using the strong Gårding ellipticity of a, the boundedness of A and Young's inequality, we obtain the estimate

$$\begin{aligned}
&\lambda(\mathsf{A}) \int_{\mathbb{R}^d} |\varphi u|^2 + |\nabla(\varphi u)|^2 \\
&\leq C \int_{\mathbb{R}^d} 2|\nabla\varphi \otimes u||\varphi u| + 2|\nabla\varphi \otimes u||\nabla(\varphi u)| + |\nabla\varphi \otimes u|^2 \\
&\leq \frac{\lambda(\mathsf{A})}{2} \int_{\mathbb{R}^d} \left(|\varphi u|^2 + |\nabla(\varphi u)|^2\right) + \left(\tfrac{4C^2}{\lambda(\mathsf{A})} + C\right) \int_{\mathbb{R}^d} |\nabla\varphi \otimes u|^2,
\end{aligned}$$

where $C \coloneqq \|b\|_\infty \vee \|c\|_\infty \vee \|A\|_\infty$. Finally, we absorb the first term on the right-hand side and use the properties of φ to obtain

$$\begin{aligned}\int_{cQ} |u|^2 + |\nabla u|^2 &\le \int_{\mathbb{R}^d} |\varphi u|^2 + |\nabla(\varphi u)|^2 \\ &\le \left(\tfrac{2C}{\lambda(\mathsf{A})} + \tfrac{8C^2}{\lambda(\mathsf{A})^2}\right) \int_{\mathbb{R}^d} |\nabla\varphi \otimes u|^2 \\ &\le \frac{C}{(1-c)^2 r^2} \int_Q |u|^2. \qquad \square\end{aligned}$$

Now we are able to start the proof of Theorem 3.9.1. In order to derive a weak reverse Hölder estimate from the L^p-boundedness of the semigroup, we adapt the argument in [29, Chap. 1]. We shall come back to that in Section 5.4 when we study Gaussian estimates on open sets.

Proposition 3.9.3. *If $p \in [2, p_+(\mathsf{L}))$, then there is $C > 0$ such that for all cubes $Q(x,r) \subseteq \mathbb{R}^d$, $s \ge 0$ and $u \in \mathcal{H}_{\mathsf{L}+s,x,2r}$ it holds*

$$r^{-\frac{d}{p}} \|u\|_{\mathrm{L}^p(Q)} \le C r^{-\frac{d}{2}} \|u\|_{\mathrm{L}^2(2Q)}.$$

Proof. We prove the result for the inhomogeneous operator and highlight the necessary modifications for the homogeneous case. We let $s \ge 0$, $Q(x,r) \subseteq \mathbb{R}^d$ and $u \in \mathcal{H}_{\mathsf{L}+s,x,2r}$. We pick $\varphi \in \mathrm{C}_\mathrm{c}^\infty(Q)$ with $\|\varphi\|_{p'} = 1$ and $\chi \in \mathrm{C}_\mathrm{c}^\infty$ with $\mathbf{1}_{\frac{3}{2}Q} \le \chi \le \mathbf{1}_{\frac{7}{4}Q}$ and $\|\nabla\chi\|_\infty \lesssim r^{-1}$. It suffices to prove that

$$|(u \,|\, \varphi)| = |(\chi u \,|\, \varphi)| \lesssim r^{\frac{d}{p}-\frac{d}{2}} \|u\|_{\mathrm{L}^2(2Q)}. \tag{3.9}$$

Using the notation $f_t = \mathrm{e}^{-t^2\mathsf{L}} f$ and $f_t^* = \mathrm{e}^{-t^2\mathsf{L}^*} f$, the fundamental theorem of calculus and Hölder's inequality give

$$\begin{aligned}|(\chi u \,|\, \varphi)| &= \Big| \mathrm{e}^{-sr^2} (\chi u \,|\, \varphi_r^*) + 2 \int_0^r (\chi u \,|\, (\mathsf{L}^* + s)\varphi_t^*) t\, \mathrm{e}^{-st^2}\, \mathrm{d}t \Big| \\ &\le |(\chi u \,|\, \varphi_r^*)| + 2 \int_0^r |(\chi u \,|\, (\mathsf{L}^* + s)\varphi_t^*)| t \,\mathrm{d}t \\ &\le \|u\|_{\mathrm{L}^2(2Q)} \|\varphi_r^*\|_2 + 2 \int_0^r |(\chi u \,|\, (\mathsf{L}^* + s)\varphi_t^*)| t \,\mathrm{d}t.\end{aligned}$$

Since $p \in [2, p_+(\mathsf{L}))$, duality and Lemma 1.4.5 imply that $(\mathrm{e}^{-t^2\mathsf{L}^*})_{t>0}$ is $\mathrm{L}^{p'}-\mathrm{L}^2$-bounded. Hence, we can continue the latter estimate by

$$\lesssim r^{\frac{d}{p}-\frac{d}{2}} \|u\|_{\mathrm{L}^2(2Q)} + \int_0^r |(\chi u \,|\, (\mathsf{L}^* + s)\varphi_t^*)| t \,\mathrm{d}t.$$

Let us focus on the integrand in the second term. We write

$$\begin{aligned}&|(\chi u \,|\, (\mathsf{L}^* + s)\varphi_t^*)|\\ &= |(\nabla(\chi u) \,|\, A^*\nabla\varphi_t^*) + (\nabla(\chi u) \,|\, b^*\varphi_t^*) + (\chi u \,|\, c^*\nabla\varphi_t^*) + (\chi u \,|\, (a^* + s)\varphi_t^*)|\\ &= |(A\nabla(\chi u) \,|\, \nabla\varphi_t^*) + (cu \,|\, \chi\nabla\varphi_t^*) + (b\nabla(\chi u) \,|\, \varphi_t^*) + ((a + s)u \,|\, \chi\varphi_t^*)|\end{aligned}$$

and use that $u \in \mathcal{H}_{\mathsf{L}+s,2Q}$ along with $\chi\varphi_t^* \in \mathrm{W}_0^{1,2}(2Q)^m$ to get

$$\begin{aligned}&= |(A(\nabla\chi \otimes u) \,|\, \nabla\varphi_t^*) - (A\nabla u \,|\, \nabla\chi \otimes \varphi_t^*) - (cu \,|\, \nabla\chi \otimes \varphi_t^*)\\ &\qquad + (b(\nabla\chi \otimes u) \,|\, \varphi_t^*)|.\end{aligned}$$

Thus, we can estimate

$$\begin{aligned}&t|(\chi u \,|\, (\mathsf{L}^* + s)\varphi_t^*)|\\ &\lesssim r^{-1}\|u\|_{\mathrm{L}^2(2Q)}\|t\nabla\varphi_t^*\|_{\mathrm{L}^2((\frac{3}{2}Q)^c)} + tr^{-1}\|u\|_{\mathrm{W}^{1,2}(\frac{7}{4}Q)}\|\varphi_t^*\|_{\mathrm{L}^2((\frac{3}{2}Q)^c)},\end{aligned}$$

where the implicit constant is independent of s. At this point, we stress that the $\mathrm{W}^{1,2}$-norm of u can be replaced by the L^2-norm of ∇u in the homogeneous case. By duality and Lemma 1.4.5, it follows that $(\mathrm{e}^{-t^2\mathsf{L}^*})_{t>0}$ satisfies $\mathrm{L}^{p'} - \mathrm{L}^2$ off-diagonal estimates. As $(t\nabla\,\mathrm{e}^{-t^2\mathsf{L}^*})_{t>0}$ satisfies L^2 off-diagonal estimates, we conclude by composition that this operator family has $\mathrm{L}^{p'} - \mathrm{L}^2$ off-diagonal estimates as well. Since φ has support in Q, we obtain

$$\lesssim r^{-1}t^{\frac{d}{p}-\frac{d}{2}}\,\mathrm{e}^{-c\left(\frac{r}{t}\right)^2}\,\|u\|_{\mathrm{L}^2(2Q)} + r^{-1}t^{\frac{d}{p}-\frac{d}{2}+1}\,\mathrm{e}^{-c\left(\frac{r}{t}\right)^2}\,\|u\|_{\mathrm{W}^{1,2}(\frac{7}{4}Q)}.$$

Applying the inhomogeneous version of Caccioppoli's inequality to the second term leads to the final bound

$$\lesssim r^{-1}t^{\frac{d}{p}-\frac{d}{2}}\,\mathrm{e}^{-c\left(\frac{r}{t}\right)^2}(1 + t/r)\|u\|_{\mathrm{L}^2(2Q)}.$$

In total, we get

$$\begin{aligned}\int_0^r |(\chi u \,|\, (\mathsf{L}^* + s)\varphi_t^*)|t\,\mathrm{d}t &\lesssim r^{-1}\|u\|_{\mathrm{L}^2(2Q)}\int_0^r t^{\frac{d}{p}-\frac{d}{2}}\,\mathrm{e}^{-c\left(\frac{r}{t}\right)^2}(1 + t/r)\,\mathrm{d}t\\ &= r^{\frac{d}{p}-\frac{d}{2}}\|u\|_{\mathrm{L}^2(2Q)}\int_1^\infty t^{\frac{d}{2}-\frac{d}{p}-2}\,\mathrm{e}^{-ct^2}(1 + t^{-1})\,\mathrm{d}t\\ &\lesssim r^{\frac{d}{p}-\frac{d}{2}}\|u\|_{\mathrm{L}^2(2Q)}.\end{aligned}$$

Hence, (3.9) follows. □

The converse direction is an immediate consequence of Shen's extrapolation theorem [145, Thm. 3.1], compare with Proposition 3.4.3.

Proposition 3.9.4. *Let $p \in (2,\infty)$. If there is $C > 0$ such that we have for all cubes $Q(x,r) \subseteq \mathbb{R}^d$, $s \geq 0$ and $u \in \mathcal{H}_{\mathsf{L}+s,x,2r}$ that*

$$\|u\|_{\mathrm{L}^p(Q)} \leq C r^{\frac{d}{p}-\frac{d}{2}} \|u\|_{\mathrm{L}^2(2Q)},$$

then $p_+(\mathsf{L}) \geq p$.

Proof. We fix $s > 0$, $Q \subseteq \mathbb{R}^d$ and $f \in \mathrm{C}_{\mathrm{c}}^\infty$ with $f|_{4Q} = 0$. Then $u \coloneqq s(s+\mathsf{L})^{-1} f \in \mathrm{W}^{1,2}(2Q)$ solves $(\mathsf{L}+s)u = 0$ in $2Q$ and the assumption yields the estimate

$$\left(\fint_Q |s(s+\mathsf{L})^{-1} f|^p \,\mathrm{d}x \right)^{\frac{1}{p}} \leq C \left(\fint_{2Q} |s(s+\mathsf{L})^{-1} f|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}.$$

As $(s(s+\mathsf{L})^{-1})_{s>0}$ is L^2-bounded, Shen's extrapolation theorem [145, Thm. 3.1] implies that $(s(s+\mathsf{L})^{-1})_{s>0}$ is L^q-bounded for all $q \in (2,p)$. We conclude that $p_+(\mathsf{L}) \geq p$. □

CHAPTER 4

Elliptic boundary value problems with non-block structure

This chapter aims to further develop the theory of elliptic boundary value problems in the upper half space using the DB-formalism. In particular, it serves as a complementary part to the previous chapter. Our main object of interest is the elliptic problem

$$\boldsymbol{L}U = -\operatorname{div}_{t,x} \mathsf{A}(x)\nabla_{t,x}U = 0 \qquad (\text{in } \mathbb{R}^{1+d}_+).$$

Unlike in Chapter 3, we impose the following weaker accretivity assumption on A: there is $\lambda > 0$ such that

$$\operatorname{Re}\left(\int_{\mathbb{R}^d} \mathsf{A}\begin{bmatrix} f \\ \nabla u \end{bmatrix} \cdot \overline{\begin{bmatrix} f \\ \nabla u \end{bmatrix}}\, \mathrm{d}x\right) \geq \lambda(\|f\|_2^2 + \|\nabla u\|_2^2) \tag{4.1}$$

for each $f \in \mathrm{L}^2(\mathbb{R}^d)$ and all $u \in \dot{\mathrm{W}}^{1,2}(\mathbb{R}^d)$. Identifying the boundary of $\mathbb{R}^{1+d}_+$ with $\mathbb{R}^d$, we formally impose the boundary condition $U(0,\cdot) \in \mathrm{X}(\mathbb{R}^d)$, where $\mathrm{X}(\mathbb{R}^d)$ is for example a Hardy, Lebesgue or Sobolev space. In the classical theory of elliptic boundary value problems, this condition is usually understood in terms of the existence of the limit $\lim_{n\to\infty} U(t_n, x_n)$ (in a suitable topology) for sequences $((t_n, x_n))_n \subseteq \mathbb{R}^{1+d}_+$ approaching the boundary in a *non-tangential* manner. For the Laplacian $\boldsymbol{L} = -\Delta_{t,x}$, it is classical (see, e.g., [120]) that the appropriate boundary spaces ensuring well-posedness are the real Hardy spaces $\mathrm{H}^p(\mathbb{R}^d)$ for $p \leq 1$ and the Lebesgue spaces $\mathrm{L}^p(\mathbb{R}^d)$ for $p > 1$. In particular, equivalent $-\Delta_{t,x}$-adapted characterizations of Hardy and Lebesgue

spaces are available, which motivates the notation $\mathrm{H}^p_{-\Delta_{t,x}}(\mathbb{R}^d) = \mathrm{H}^p(\mathbb{R}^d)$. For further details, we refer to [20] and the references therein.

However, when $\boldsymbol{L}$ differs from the Laplace operator, the study of elliptic boundary value problems is significantly more involved. In a series of papers [14–16], Auscher, Axelsson, and McIntosh developed the so-called *first-order approach*. The key observation is that the elliptic equation for U can be rewritten as a first-order evolution equation for its conormal gradient $F = \overline{\mathsf{A}}\nabla_{t,x}U$:

$$F'(t) + \mathsf{DB}F(t) = 0 \qquad (t > 0).$$

To generate a semigroup solution to this equation, a bounded functional calculus for DB is required. In the L^2-setting, this deep result was established by Axelsson, Keith, and McIntosh [32], whereas for $p \neq 2$, analogous results remain rare. The theory of DB-*adapted* function spaces $\mathbb{H}^p_{\mathsf{DB}}$ provides the appropriate framework, as these spaces are constructed to ensure that DB admits optimal functional analytic properties, notably a bounded functional calculus. A key consequence is that a priori estimates for solutions to $\boldsymbol{L}U = 0$ with boundary data in the H^p-scale (where $\mathrm{H}^p = \mathrm{L}^p$ for $p > 1$) follow from the identification of DB-adapted function spaces with classical Hardy and Lebesgue spaces. Moreover, the conormal gradient of every reasonable solution to $\boldsymbol{L}U = 0$ can be represented via a DB-type semigroup, see [28, Chap. 12] and the references therein. Thus, this chapter primarily investigates the identification interval

$$\mathcal{H}(\mathsf{DB}) \coloneqq \Big\{p \in (1_*, \infty) \colon \mathbb{H}^p_{\mathsf{DB}} = \mathrm{H}^p \cap \overline{\operatorname{ran}(\mathsf{D})}\Big\},$$

where we adopt the

Convention 4.0.1. *For $p > 1$, we identify H^p with L^p.*

We write $h_\pm(\mathsf{DB})$ for the left and right endpoints of $\mathcal{H}(\mathsf{DB})$. The remainder of this chapter focuses on the lower endpoint.

Notably, the inequality $h_-(\mathsf{DB}) \leq p_-(\mathsf{DB})_*$ was established in [28, Thm. 5.1], where $p_-(\mathsf{DB})$ is the left endpoint of the Auscher–Stahlhut interval from Definition 3.3.5. In particular, we have $h_-(\mathsf{DB}) = 1_*$ for $d = 1$, so we restrict our analysis to the case $d \geq 2$. The first main theorem explores this exponent in greater depth, see Section 4.3.

Theorem 4.0.2. *Let $d \geq 2$ and $M \in \mathbb{N}$. Then*

$$h_-(\mathsf{DB}) = \inf\Big\{p \in (1_*, 2] \colon ((1 + \mathrm{i}t\mathsf{DB})^{-M})_{t>0} \text{ is } \mathrm{H}^p \cap \overline{\operatorname{ran}(\mathsf{D})}\text{-bounded}\Big\}.$$

This theorem implies that DB admits a bounded H^∞-calculus on $\mathrm{L}^p \cap \overline{\mathrm{ran}(\mathsf{D})}$ provided that one resolvent power is uniformly bounded on this space. To prove the key inequality

$$h_-(\mathsf{DB}) \leq \inf\Big\{p \in (1_*, 2] \colon ((1 + \mathrm{i}t\mathsf{DB})^{-M})_{t>0} \text{ is } \mathrm{H}^p \cap \overline{\mathrm{ran}(\mathsf{D})}\text{-bounded}\Big\},$$

we proceed as follows. First, assuming that $((1+\mathrm{i}t\mathsf{DB})^{-M})_{t>0}$ is $\mathrm{H}^p \cap \overline{\mathrm{ran}(\mathsf{D})}$-bounded for some $p \in (1_*, 2)$, we invoke abstract interpolation methods to find some $K \geq M$ such that for all $q \in (p, 2]$, the operator family $((1+\mathrm{i}t\mathsf{DB})^{-K})_{t>0}$ maps $\mathrm{H}^q \cap \overline{\mathrm{ran}(\mathsf{D})}$ boundedly into L^2 with additional off-diagonal decay. In a second step, we utilize these mapping properties to adapt an extrapolation technique for conical square functions, previously used in [20, 28, 94].

In light of Chapter 2, the second main result provides explicit upper bounds for $h_-(\mathsf{DB})$ in terms of ellipticity and dimension, see Section 4.4.

Theorem 4.0.3. *Let $d \geq 3$, $m = 1$, $p \in (2, \infty)$, A be p-elliptic and $\mathrm{div}(c) = 0$ in the distributional sense. Then*

$$h_-(\mathsf{DB}) \leq p_\circ = \Big(\frac{p}{2}2^*\Big)'.$$

In the spirit of the p-ellipticity condition on A, this estimate interpolates between the general bound $h_-(\mathsf{DB}) \leq 2_*$ and the improvement $h_-(\mathsf{DB}) < 1$ when A is real [28]. Building on Theorem 4.0.2, we establish this explicit estimate for $h_-(\mathsf{DB})$ using second-order methods. The key ingredient is the algebraic relation between the DB-resolvent and the inhomogeneous Hodge operators, which we previously exploited in Section 3.3. A natural question remains open:

Conjecture 4.0.4. *The divergence-free condition on c in Theorem 4.0.3 is not necessary and can be removed.*

To resolve this question, it seems plausible that yet a deeper algebraic understanding of the DB-formalism is required.

Finally, in Section 4.5, we revisit the known result that $h_-(\mathsf{DB}) < 1$ when A is real-valued and present a new, more concise proof. Our approach relies on deriving a molecular estimate for the DB-resolvent acting on D-adapted atoms. By Theorem 4.0.2, this suffices to conclude.

4.1 L^2-theory of Dirac operators

In this section, we briefly discuss the L^2-theory of the Dirac operator DB under assumption (4.1). We begin by considering the unperturbed case.

4.1.1 The Dirac operator D

The L^2-realization of the Dirac operator D (see Section 3.3) is self-adjoint, leading to an orthogonal decomposition

$$\mathrm{L}^2(\mathbb{R}^d;(\mathbb{C}^m)^{1+d}) = \mathrm{L}^2 = \ker(\mathsf{D}) \oplus \overline{\operatorname{ran}(\mathsf{D})}$$

with corresponding orthogonal projection

$$\mathbb{P}_{\mathsf{D}} \colon \mathrm{L}^2 \to \overline{\operatorname{ran}(\mathsf{D})}.$$

By [20, Sec. 8.5], the operator $\mathbb{P}_{\mathsf{D}}$ coincides with $(-\Delta)^{-1}\mathsf{D}^2$ on $\mathscr{Z}'$, which can be seen as a Fourier multiplier operator with symbol

$$m_{\mathsf{D}}(\xi) \coloneqq \begin{bmatrix} \mathbf{1}_{\mathbb{C}^m} & 0 \\ 0 & \frac{\xi\otimes\xi}{|\xi|^2} \otimes \mathbf{1}_{\mathbb{C}^m} \end{bmatrix}.$$

Since m_{D} is homogeneous of degree 0, a generalized version of Mihlin's theorem [157, Thm. 5.2.2] implies that the restriction of $\mathbb{P}_{\mathsf{D}}$ to $\mathrm{H}^p \cap \mathrm{L}^2$ extends to a bounded operator on H^p for all $p \in (1_*, \infty)$. We denote this extension again by $\mathbb{P}_{\mathsf{D}}$. Moreover, the structure of m_{D} implies that $\mathbb{P}_{\mathsf{D}}$ admits the representation

$$\mathbb{P}_{\mathsf{D}} = \begin{bmatrix} 1 & 0 \\ 0 & \mathbb{P}_{\mathrm{curl}} \end{bmatrix}, \tag{4.2}$$

where $\mathbb{P}_{\mathrm{curl}} = \nabla\Delta^{-1}\operatorname{div}$ projects onto the space of curl-free vector fields.

4.1.2 The perturbed Dirac operator DB

In view of (4.2), the condition (4.1) states precisely that A is strictly accretive on $\overline{\operatorname{ran}(\mathsf{D})}$:

$$\operatorname{Re}(\mathsf{A}\mathsf{D}F \,|\, \mathsf{D}F) \geq \lambda \|\mathsf{D}F\|_2^2 \qquad (F \in \operatorname{dom}(\mathsf{D})).$$

Hence, A is strongly Gårding elliptic, a is elliptic and A is Gårding elliptic with $\lambda(\mathsf{A}) \wedge \lambda(A) \wedge \lambda(a) \geq \lambda$, see Definition 1.6.2. In particular, the bounded multiplication operator $\mathsf{B} = \underline{\mathsf{A}}\overline{\mathsf{A}}^{-1}$ (see Definition 3.3.1) is also strictly accretive on $\overline{\operatorname{ran}(\mathsf{D})}$, satisfying

$$\operatorname{Re}(\mathsf{B}\mathsf{D}F \,|\, \mathsf{D}F) \geq \lambda \|\overline{\mathsf{A}}\|_\infty^{-2} \|\mathsf{D}F\|_2^2 \qquad (F \in \operatorname{dom}(\mathsf{D})),$$

as shown in [78, Lem. 6.1.8]. This estimate is already sufficient to establish a well-behaved L^2-theory. Indeed, combining [20, Prop. 3.11] with [32, Prop. 2.5 & Thm. 2.7], we obtain

Theorem 4.1.1. *The operator* DB *is bisectorial in* L^2 *and has a bounded* H^∞*-calculus on* $\overline{\operatorname{ran}(\mathsf{DB})} = \overline{\operatorname{ran}(\mathsf{D})}$*. In addition,* $((1 + \mathrm{i}t\mathsf{DB})^{-1})_{t\in\mathbb{R}^*}$ *satisfies* L^2 *off-diagonal estimates.*

4.2 Operator-adapted spaces

In this section, we introduce relevant notions related to operator-adapted function spaces. We let $T \in \{\mathsf{D}, \mathsf{DB}\}$. For $x \in \mathbb{R}^d$, we define the *cone with vertex* x as

$$\Gamma(x) \coloneqq \{(t, y) \in \mathbb{R}^{1+d}_+ : |x - y| < t\}.$$

Given a measurable function $F\colon \mathbb{R}^{1+d}_+ \to \mathbb{C}^m$, we define its square function as

$$S(F)(x) \coloneqq \left(\iint_{\Gamma(x)} |F(t, y)|^2 \, \frac{\mathrm{d}t \, \mathrm{d}y}{t^{1+d}}\right)^{\frac{1}{2}} \qquad (x \in \mathbb{R}^d).$$

To define T-adapted spaces, we use an auxiliary function that is holomorphic on a suitable bisector and exhibits controlled growth at both the origin and infinity.

Definition 4.2.1. Let $\sigma, \tau \in \mathbb{R}$ and $\mu \in [0, \pi/2)$. The class $\Psi^\tau_\sigma(\dot{\mathrm{S}}_\mu)$ consists of all holomorphic functions $\psi\colon \dot{\mathrm{S}}_\mu \to \mathbb{C}$ satisfying $|\psi(z)| \leq C(|z|^\sigma \wedge |z|^{-\tau})$ for some $C > 0$ and all $z \in \dot{\mathrm{S}}_\mu$. In addition, we write $\psi \in \Psi^\infty_\infty(\dot{\mathrm{S}}_\mu)$ if for all $\sigma > 0$ there is $C > 0$ with $|\psi(z)| \leq C(|z|^\sigma \wedge |z|^{-\sigma})$ for all $z \in \dot{\mathrm{S}}_\mu$.

Definition 4.2.2. Let $p \in (1_*, \infty)$, $\mu \in (\omega_T, \pi/2)$, $\sigma \wedge \tau > |d/p - d/2|$ and $\psi \in \Psi^\tau_\sigma(\dot{\mathrm{S}}_\mu)$ be *non-degenerate*, meaning that ψ does not vanish on S_μ or $-\mathrm{S}_\mu$. We define the **pre-Hardy space of integrability p adapted to T** as

$$\mathbb{H}^p_T \coloneqq \left\{h \in \overline{\operatorname{ran}(\mathsf{D})}\colon S_{\psi,T}(h) \coloneqq S(\psi(tT)h) \in \mathrm{L}^p\right\}.$$

We endow this space with the (quasi-)norm

$$\|h\|_{\mathbb{H}^p_T} \coloneqq \|S_{\psi,T}(h)\|_p.$$

Remark 4.2.3. By [4, Prop. 4.4], the constraint on ψ ensures that $\mathbb{H}^p_T$ is independent of the choice of ψ.

The first key result is that $\mathbb{H}^p_{\mathsf{D}}$ can be characterized in terms of $\mathbb{P}_{\mathsf{D}}$ and classical function spaces. To state this result, we also need a notion of D-adapted atoms, compare with Definition 1.1.4.

Definition 4.2.4. Let $p \in (1_*, 1]$ and $B = B(x,r) \subseteq \mathbb{R}^d$. We say that $a \in (\mathrm{L}^2)^{1+d}$ is an $\mathbb{H}^p_{\mathsf{D}}$**-atom associated with** B, if $\operatorname{supp}(a) \subseteq B$, $\int_B a \,\mathrm{d}x = 0$ and there is $b \in \operatorname{dom}(\mathsf{D})$ with $\mathsf{D}b = a$ and

$$r^{-1}\|b\|_2 + \|a\|_2 \leq r^{\frac{d}{2}-\frac{d}{p}}.$$

The **atomic D-adapted Hardy space** $\mathbb{H}^p_{\mathsf{D},\mathrm{at}}$ consists of those $h \in \overline{\operatorname{ran}(\mathsf{D})}$, for which there is a sequence of $\mathbb{H}^p_{\mathsf{D}}$-atoms $(a_n)_n$ and a sequence $(\lambda_n)_n \in \ell^p$ such that

$$h = \sum_n \lambda_n a_n$$

converges in L^2. We endow this space with the (quasi-)norm

$$\|h\|_{\mathbb{H}^p_{\mathsf{D},\mathrm{at}}} = \inf \|(\lambda_n)_n\|_{\ell^p},$$

where the infimum is taken over all D-adapted atomic decompositions of h.

Proposition 4.2.5 ([28, Thm. 4.16 & Lem. 5.12])**.** *Let $p \in (1_*, \infty)$. Then*

$$\mathbb{H}^p_{\mathsf{D}} = \mathbb{P}_{\mathsf{D}}(\mathrm{H}^p \cap \mathrm{L}^2) = \mathrm{H}^p \cap \overline{\operatorname{ran}(\mathsf{D})}.$$

Moreover, if $p \leq 1$, then we have

$$\mathbb{H}^p_{\mathsf{D}} = \mathbb{H}^p_{\mathsf{D},\mathrm{at}},$$

and, if $p > 1$, then the following assertions are equivalent:

(i) *It holds $h \in \mathbb{H}^p_{\mathsf{D}}$.*

(ii) *There exists $F \in \dot{\mathrm{W}}^{1,p} \cap \dot{\mathrm{W}}^{1,2}$ such that $h = \mathsf{D}F$ and $\|h\|_p \simeq \|\nabla F\|_p$, as well as $\|h\|_2 \simeq \|\nabla F\|_2$.*

(iii) *There are $f \in \mathrm{L}^p \cap \mathrm{L}^2$ and $g \in \dot{\mathrm{W}}^{1,p} \cap \dot{\mathrm{W}}^{1,2}$ such that $h = [f, \nabla g]^\top$.*

4.3 Equivalent characterization of $h_-(\mathsf{DB})$

In this section, we prove Theorem 4.0.2. To establish this theorem, we introduce an appropriate notion of D-adapted $\mathrm{H}^p - \mathrm{H}^q$-estimates.

Definition 4.3.1. Let $1_* < p \leq q < \infty$, $T \in \{\mathsf{D}, \mathsf{DB}\}$, $\mu \in (\omega_T, \pi/2)$, $\psi \in \mathrm{H}^\infty(\dot{\mathrm{S}}_\mu)$ and $U \subseteq \mathbb{R}^*$. We say that $(\psi(tT))_{t\in U}$ is $\mathbb{H}^p_{\mathsf{D}} - \mathbb{H}^q_{\mathsf{D}}$**-bounded** if there is $C > 0$ such that

$$\|\psi(tT)h\|_{\mathrm{H}^q} \leq C|t|^{\frac{d}{q}-\frac{d}{p}}\|h\|_{\mathrm{H}^p} \qquad (t \in U, h \in \mathbb{H}^p_{\mathsf{D}}).$$

For $p = q$, this is simply referred to as $\mathbb{H}^p_{\mathsf{D}}$**-boundedness**.

With this notion at hand, Theorem 4.0.2 will follow from

Theorem 4.3.2. *Let $d \geq 2$ and $p \in (1_*, 2]$. Then the following assertions hold true.*

(i) *If $\mathbb{H}^p_{\mathsf{DB}} = \mathbb{H}^p_{\mathsf{D}}$, then $((1 + \mathrm{i}t\mathsf{DB})^{-M})_{t\in\mathbb{R}^*}$ is $\mathbb{H}^p_{\mathsf{D}}$-bounded for all $M \in \mathbb{N}$.*

(ii) *If $((1 + \mathrm{i}t\mathsf{DB})^{-M})_{t>0}$ is $\mathbb{H}^p_{\mathsf{D}}$-bounded for some $M \in \mathbb{N}$, then for each $q \in (p, 2)$, there exists $K \geq M$ such that $((1 + \mathrm{i}t\mathsf{DB})^{-K})_{t>0}$ is $\mathbb{H}^q_{\mathsf{D}} - \mathbb{H}^2_{\mathsf{D}}$-bounded.*

(iii) *If $((1 + \mathrm{i}t\mathsf{DB})^{-K})_{t>0}$ is $\mathbb{H}^p_{\mathsf{D}} - \mathbb{H}^2_{\mathsf{D}}$-bounded for some $K \in \mathbb{N}$, then $\mathbb{H}^q_{\mathsf{DB}} = \mathbb{H}^q_{\mathsf{D}}$ for all $q \in (p, 2]$.*

Proof of (i). The first claim follows from the bounded H^∞-calculus of $\mathsf{DB}\mathbb{P}_{\mathsf{D}}$ on $\mathbb{H}^p_{\mathsf{DB}}$, see [4, Thm. 4.14].

Proof of (ii). To prove the second assertion, we start by showing that $((1 + \mathrm{i}t\mathsf{DB})^{-M})_{t>0}$ is $\mathbb{H}^{2_*}_{\mathsf{D}} - \mathbb{H}^2_{\mathsf{D}}$-bounded. We fix $t > 0$, $h \in \mathbb{H}^{2_*}_{\mathsf{D}}$ and treat the cases $d \geq 3$ and $d = 2$ separately.

(1) $d \geq 3$. According to Proposition 4.2.5, we find $F \in \dot{\mathrm{W}}^{1,2_*} \cap \dot{\mathrm{W}}^{1,2} \subseteq \mathrm{dom}(\mathsf{D})$ with $h = \mathsf{D}F$ and $\|h\|_{2_*} \simeq \|\nabla F\|_{2_*}$. Using the intertwining relation of DB [20, Lem. 3.7] along with the strict accretivity of B on $\mathrm{ran}(\mathsf{D})$ and the L^2-boundedness of $((1 + \mathrm{i}t\mathsf{BD})^{-1})_{t\in\mathbb{R}^*}$ (see [32, Prop. 2.5]), we get

$$\begin{aligned}\|(1 + \mathrm{i}t\mathsf{DB})^{-M}h\|_2 &= \|\mathsf{D}(1 + \mathrm{i}t\mathsf{BD})^{-M}F\|_2\\ &\simeq t^{-1}\|t\mathsf{BD}(1 + \mathrm{i}t\mathsf{BD})^{-M}F\|_2\\ &\lesssim t^{-1}\|F\|_2\\ &\lesssim t^{-1}\|\nabla F\|_{2_*}\\ &\simeq t^{-1}\|h\|_{2_*}.\end{aligned}$$

(2) $d = 2$. In this case, we have $2_* = 1$. By Proposition 4.2.5, we find a sequence of $\mathbb{H}^1_{\mathsf{D}}$-atoms $(a_n)_n$ and a sequence $(\lambda_n)_n \in \ell^1$ such that $h = \sum_n \lambda_n a_n$ converges in L^2 and $\|(\lambda_n)_n\|_{\ell^1} \leq 2\|h\|_{\mathrm{H}^1}$. By the very definition of an $\mathbb{H}^1_{\mathsf{D}}$-atom, we can write $a_n = \mathsf{D}b_n$ for some $b_n \in \mathrm{dom}(\mathsf{D})$. Since $d = 2$ and $p = 1$, we have $\|b_n\|_2 \leq 1$ for all n. Using again the intertwining relation along with the strict accretivity of B on $\mathrm{ran}(\mathsf{D})$ and the L^2-boundedness of

$((1+\mathrm{i}t\mathsf{BD})^{-1})_{t\in\mathbb{R}^*}$, we obtain the estimate

$$\begin{aligned}\|(1+\mathrm{i}t\mathsf{DB})^{-M}h\|_2 &\le \sum_n |\lambda_n|\|(1+\mathrm{i}t\mathsf{DB})^{-M}a_n\|_2\\ &= t^{-1}\sum_n |\lambda_n|\|t\mathsf{D}(1+\mathrm{i}t\mathsf{BD})^{-M}b_n\|_2\\ &\simeq t^{-1}\sum_n |\lambda_n|\|t\mathsf{BD}(1+\mathrm{i}t\mathsf{BD})^{-M}b_n\|_2\\ &\lesssim t^{-1}\sum_n |\lambda_n|\|b_n\|_2\\ &\le t^{-1}\|(\lambda_n)_n\|_{\ell^1}\\ &\le 2t^{-1}\|h\|_{\mathrm{H}^1}.\end{aligned}$$

Next, since the spaces $\{\mathbb{P}_{\mathsf{D}}\mathrm{H}^p\}_{p\in(1_*,\infty)}$ form a complex interpolation scale, see [35, Cor. 1.1.3] and [65], we can apply the argument in [20, Lem. 4.4] with $\varrho = 2_*$. As a result, for each $q\in(p,2)$ there is some $K\ge M$ such that $((1+\mathrm{i}t\mathsf{DB})^{-K})_{t>0}$ is $\mathbb{H}^q_{\mathsf{D}}-\mathbb{H}^2_{\mathsf{D}}$-bounded.

Proof of (iii). Our approach is based on a square function extrapolation technique from [20, 28, 94]. The main difference to earlier approaches is that we avoid DB-adapted Hodge decompositions and Sobolev embeddings but instead invoke $\mathbb{H}^p_{\mathsf{D}}-\mathbb{H}^2_{\mathsf{D}}$-estimates for higher-order resolvents of DB. We fix $q\in(p,2)$. Since $\mathbb{H}^q_{\mathsf{DB}}\subseteq\mathbb{H}^q_{\mathsf{D}}$ by [28, Prop. 4.17], it remains to prove that $\mathbb{H}^q_{\mathsf{D}}\subseteq\mathbb{H}^q_{\mathsf{DB}}$. We let $M\in\mathbb{N}$ with $M>|{}^d\!/_p-{}^d\!/_2|\vee K$ and define

$$\psi := (\mathrm{i}\boldsymbol{z})^M(1+\mathrm{i}\boldsymbol{z})^{-2M}\in\Psi^M_M.$$

By Remark 4.2.3, ψ is admissible for defining $\mathbb{H}^s_{\mathsf{DB}}$ for any $s\in[p,2]$ and our assumption yields that the family $(\psi(t\mathsf{DB}))_{t>0}$ is $\mathbb{H}^p_{\mathsf{D}}-\mathbb{H}^2_{\mathsf{D}}$-bounded. We investigate the cases $p\ge1$ and $p<1$ individually.

(1) $p\ge1$. Proposition 4.2.5 implies that for any $h\in\mathbb{H}^q_{\mathsf{D}}$ there is $F\in\dot{\mathrm{W}}^{1,q}\cap\dot{\mathrm{W}}^{1,2}$ with $h=\mathsf{D}F$ and $\|h\|_{\mathbb{H}^q_{\mathsf{D}}}\simeq\|\nabla F\|_q$. Hence, we have to prove that

$$\|S_{\psi,\mathsf{DB}}(\mathsf{D}F)\|_q\lesssim\|\nabla F\|_q. \tag{4.3}$$

Given $r\in(p,q)$, we claim that it is enough to show the weak type bound

$$|\{S_{\psi,\mathsf{DB}}(\mathsf{D}F)>\alpha\}|\lesssim\alpha^{-r}\|\nabla F\|_r^r\qquad(\alpha>0,F\in\dot{\mathrm{W}}^{1,r}\cap\dot{\mathrm{W}}^{1,2}). \tag{4.4}$$

Indeed, we consider the positive sublinear operator

$$T\colon\mathcal{Z}\to\mathrm{L}^2,\quad TF := S_{\psi,\mathsf{DB}}(\mathsf{D}(-\Delta)^{-\frac12}F),$$

recall from Lemma 1.1.3 that $\mathcal{Z}$ is dense in L^s for every $s \in (1,\infty)$ and notice that $(-\Delta)^{-1/2}$ is invertible on $\mathcal{Z}$ with $\|(-\Delta)^{1/2}F\|_r \simeq \|\nabla F\|_r$. The square function estimates of DB in L^2 (see Theorem 4.1.1) guarantee that T is of strong type $(2,2)$. Moreover, (4.4) means that T is also of weak type (r,r). Hence, the Marcinkiewicz interpolation theorem reveals that T is of strong type (q,q), so (4.3) holds true for all $F \in \mathcal{Z}$. By the universal approximation property of $\mathcal{Z}$, (4.3) extends to all $F \in \dot{\mathrm{W}}^{1,q} \cap \dot{\mathrm{W}}^{1,2}$, see [20, Prop. 9.12] for this argument.

To prove (4.4), we apply the Calderón–Zygmund decomposition for Sobolev functions to F at height α, as given in [12, Lem. 4.12]. For a correction of an inaccuracy in the original proof, see [13], and for a careful verification of additional properties, see [20, Lem. 9.10]. More precisely, there exists a countable collection of cubes $(Q_j)_{j\in J}$ and measurable functions g and $(b_j)_{j\in J}$ satisfying

$$F = g + \sum_{j\in J} b_j =: g + b$$

pointwise a.e. in $\mathbb{R}^d$ and:

(i) $\|\nabla g\|_\infty \lesssim \alpha$.

(ii) For all $j \in J$, we have $b_j \in \mathrm{W}^{1,r}_0(Q_j) \cap \mathrm{W}^{1,2}_0(Q_j)$ with $\|\nabla b_j\|_r \lesssim \alpha |Q_j|^{1/r}$, and $\sum_{j\in J} b_j$ converges unconditionally in $\dot{\mathrm{W}}^{1,r} \cap \dot{\mathrm{W}}^{1,2}$.

(iii) $\sum_{j\in J} |Q_j| \lesssim \alpha^{-r} \|\nabla F\|_r^r$.

(iv) $\sum_{j\in J} \mathbf{1}_{Q_j} \lesssim 1$.

(v) $\| \sum_{j\in J} \nabla b_j\|_r + \|\nabla g\|_r \lesssim \|\nabla F\|_r$.

All implicit constants depend only on r, m and d. We start with the estimate

$$|\{S_{\psi,\mathsf{DB}}(\mathsf{D}F) > \alpha\}| \le |\{S_{\psi,\mathsf{DB}}(\mathsf{D}g) > \tfrac{\alpha}{2}\}| + |\{S_{\psi,\mathsf{DB}}(\mathsf{D}b) > \tfrac{\alpha}{2}\}|.$$

Estimate for the good part. We apply Markov's inequality, use square function estimates of DB in L^2, and interpolate (i) and (v) to get

$$|\{S_{\psi,\mathsf{DB}}(\mathsf{D}g) > \tfrac{\alpha}{2}\}| \lesssim \alpha^{-2} \|S_{\psi,\mathsf{DB}}(\mathsf{D}g)\|_2^2 \lesssim \alpha^{-2} \|\nabla g\|_2^2 \lesssim \alpha^{-r} \|\nabla F\|_r^r.$$

Estimate for the bad part. For $n \in \mathbb{N}$, we put $J_n := J \cap \{1,\dots,n\}$. Since b converges in $\dot{\mathrm{W}}^{1,2}$ by (ii) and $\mathsf{D}\colon \dot{\mathrm{W}}^{1,2} \to \mathrm{L}^2$ is bounded, we have

$$\mathsf{D}b = \sum_{j\in J} \mathsf{D}b_j = \sum_{j\in J_n} \mathsf{D}b_j + \sum_{j\in J_n^c} \mathsf{D}b_j =: (\mathsf{D}b)^{(n)} + (\mathsf{D}b)_{(n)}$$

with convergence in L^2. Markov's inequality joint with the square function estimates in L^2 entails

$$\begin{aligned}|\{S_{\psi,\mathsf{DB}}(\mathsf{D}b) > \tfrac{\alpha}{2}\}| &\le |\{S_{\psi,\mathsf{DB}}((\mathsf{D}b)^{(n)}) > \tfrac{\alpha}{4}\}| + |\{S_{\psi,\mathsf{DB}}((\mathsf{D}b)_{(n)}) > \tfrac{\alpha}{4}\}| \\ &\le |\{S_{\psi,\mathsf{DB}}((\mathsf{D}b)^{(n)}) > \tfrac{\alpha}{4}\}| + 16\alpha^{-2}\|S_{\psi,\mathsf{DB}}((\mathsf{D}b)_{(n)})\|_2^2 \\ &\lesssim |\{S_{\psi,\mathsf{DB}}((\mathsf{D}b)^{(n)}) > \tfrac{\alpha}{4}\}| + \alpha^{-2}\|(\mathsf{D}b)_{(n)}\|_2^2.\end{aligned}$$

Since $\sum_{j\in J}\mathsf{D}b_j$ converges in L^2, the second term vanishes as $n\to\infty$. It therefore suffices to show that

$$|\{S_{\psi,\mathsf{DB}}((\mathsf{D}b)^{(n)}) > \tfrac{\alpha}{4}\}| \lesssim \alpha^{-r}\|\nabla F\|_r^r$$

with an implicit constant independent of n. To this end, we introduce a regularizing function

$$\varphi(z) \coloneqq (1-(1+\mathrm{i}z)^{-K})^M = \sum_{m=0}^{M}\binom{M}{m}(-1)^m(1+\mathrm{i}z)^{-Km}.$$

Indeed, the function $1-\varphi$ serves as an approximation of the identity. Next, we write $\ell_j \coloneqq \ell(Q_j)$, decompose

$$\begin{aligned}(\mathsf{D}b)^{(n)} &= \sum_{j\in J_n}(1-\varphi(\ell_j\mathsf{DB}))\mathsf{D}b_j + \sum_{j\in J_n}\varphi(\ell_j\mathsf{DB})\mathsf{D}b_j \\ &=: (\mathsf{D}b)_1^{(n)} + (\mathsf{D}b)_2^{(n)},\end{aligned}$$

and estimate

$$|\{S_{\psi,\mathsf{DB}}((\mathsf{D}b)^{(n)}) > \tfrac{\alpha}{4}\}| \le |\{S_{\psi,\mathsf{DB}}((\mathsf{D}b)_1^{(n)}) > \tfrac{\alpha}{8}\}| + |\{S_{\psi,\mathsf{DB}}((\mathsf{D}b)_2^{(n)}) > \tfrac{\alpha}{8}\}|.$$

It remains to control each term by a generic multiple of $\alpha^{-r}\|\nabla F\|_r^r$.

Estimate for the regularized bad part. We use Markov's inequality and the square function estimates in L^2 in order to get

$$|\{S_{\psi,\mathsf{DB}}((\mathsf{D}b)_1^{(n)}) > \tfrac{\alpha}{8}\}| \lesssim \alpha^{-2}\|S_{\psi,\mathsf{DB}}((\mathsf{D}b)_1^{(n)})\|_2^2 \lesssim \alpha^{-2}\|(\mathsf{D}b)_1^{(n)}\|_2^2.$$

To estimate $\|(\mathsf{D}b)_1^{(n)}\|_2$, we fix $h\in\mathrm{L}^2$ with $\|h\|_2\le 1$ and obtain

$$\begin{aligned}|((\mathsf{D}b)_1^{(n)}\,|\,h)| &\le \sum_{j\in J_n}\sum_{k\ge 0}|((1-\varphi(\ell_j\mathsf{DB}))\mathsf{D}b_j\,|\,\mathbf{1}_{C_k(Q_j)}h)| \\ &\le \sum_{j\in J_n}\sum_{k\ge 0}\|(1-\varphi(\ell_j\mathsf{DB}))\mathsf{D}b_j\|_{\mathrm{L}^2(C_k(Q_j))}\|\mathbf{1}_{C_k(Q_j)}h\|_2.\end{aligned}$$

Since $1-\varphi(\ell_j\mathsf{DB})$ is a linear combination of resolvents of DB with at least order K, our assumption implies that this operator is $\mathbb{H}^p_{\mathsf{D}}-\mathbb{H}^2_{\mathsf{D}}$-bounded. In addition, this operator satisfies L^2 off-diagonal estimates. Now, since $b_j\in \mathrm{W}^{1,r}_0(Q_j)$ and the spaces $(\mathrm{W}^{1,s}_0(Q_j))_{s\in(1,\infty)}$ interpolate by the complex method with implicit constants independent of Q_j (see [94, Sec. 2.3] for a similar argument), we obtain with the help of (ii) that

$$\|(1-\varphi(\ell_j\mathsf{DB}))\mathsf{D}b_j\|_{\mathrm{L}^2(C_k(Q_j))}\lesssim |Q_j|^{\frac12-\frac1r}\,\mathrm{e}^{-c2^k}\,\|\nabla b_j\|_r\lesssim \alpha|Q_j|^{\frac12}\,\mathrm{e}^{-c2^k}\,.$$

As for the second factor, we use the maximal operator to find

$$\|\mathbf{1}_{C_k(Q_j)}h\|_2\lesssim |2^kQ_j|^{\frac12}\inf_{x\in Q_j}(\mathcal{M}(|h|^2))^{\frac12}(x)\le 2^{\frac{kd}{2}}|Q_j|^{-\frac12}\int_{Q_j}(\mathcal{M}(|h|^2))^{\frac12}\,\mathrm{d}x.$$

By (iv), we obtain

$$\begin{aligned}|((\mathsf{D}b)^{(n)}_1\,|\,h)|&\lesssim \alpha\sum_{j\in J}\sum_{k\ge0}2^{\frac{kd}{2}}\,\mathrm{e}^{-c2^k}\int_{Q_j}(\mathcal{M}(|h|^2))^{\frac12}\,\mathrm{d}x\\&\lesssim\alpha\int_{\bigcup_{j\in J}Q_j}(\mathcal{M}(|h|^2))^{\frac12}\,\mathrm{d}x.\end{aligned}$$

At this point, we use Kolmogorov's lemma and the normalization of h in L^2 to bound further

$$\alpha^{-2}|((\mathsf{D}b)^{(n)}_1\,|\,h)|^2\lesssim\Big|\bigcup_{j\in J}Q_j\Big|\||h|^2\|_1\le\sum_{j\in J}|Q_j|.$$

This estimate and (iii) imply that

$$|\{S_{\psi,\mathsf{DB}}((\mathsf{D}b)^{(n)}_1)>\tfrac{\alpha}{8}\}|\lesssim\alpha^{-2}\|(\mathsf{D}b)^{(n)}_1\|_2^2\lesssim\alpha^{-r}\|\nabla F\|_r^r.$$

Estimate for the non-regular bad part. Next, we estimate

$$|\{S_{\psi,\mathsf{DB}}((\mathsf{D}b)^{(n)}_2)>\tfrac{\alpha}{8}\}|.$$

In view of (iii), it suffices to bound the measure of

$$E\coloneqq\Big\{S_{\psi,\mathsf{DB}}((\mathsf{D}b)^{(n)}_2)>\tfrac{\alpha}{8}\Big\}\setminus\bigcup_{j\in J}4Q_j.$$

We apply Markov's inequality and use Fubini's theorem to get

$$\begin{aligned}|E|&\lesssim\alpha^{-2}\int_E|S_{\psi,\mathsf{DB}}((\mathsf{D}b)^{(n)}_2)(x)|^2\,\mathrm{d}x\\&\le\alpha^{-2}\iint_{\mathbb{R}^{1+d}_+}|\sum_{j\in J_n}(\varphi(\ell_j\mathsf{DB})\psi(t\mathsf{DB})\mathsf{D}b_j)(y)|^2\frac{|B(y,t)\setminus\bigcup_{j\in J}4Q_j|}{t^d}\,\frac{\mathrm{d}t\,\mathrm{d}y}{t}.\end{aligned}$$

Since $B(y,t) \setminus \bigcup_{j\in J} 4Q_j$ is empty provided that $y \in 2Q_j$ and $t \in (0,\ell_j]$ for some $j \in J$, we can split the integrand into a local and a global part

$$f_{\mathrm{loc}}(t,y) \coloneqq \sum_{j\in J_n} \mathbf{1}_{2Q_j}(y)\mathbf{1}_{(\ell_j,\infty)}(t)\,(\varphi(\ell_j\mathsf{DB})\psi(t\mathsf{DB})\mathsf{D}b_j)\,(y),$$
$$f_{\mathrm{glob}}(t,y) \coloneqq \sum_{j\in J_n} \mathbf{1}_{(2Q_j)^c}(y)\,(\varphi(\ell_j\mathsf{DB})\psi(t\mathsf{DB})\mathsf{D}b_j)\,(y).$$

So far, we have shown that

$$|E| \lesssim \alpha^{-2} \iint_{\mathbb{R}^{1+d}_+} |f_{\mathrm{loc}}(t,y)|^2 + |f_{\mathrm{glob}}(t,y)|^2 \,\frac{\mathrm{d}t\,\mathrm{d}y}{t}$$

and it remains to bound the integral on the right-hand side by a generic multiple of $\alpha^{2-r}\|\nabla F\|_r^r$.

Bound for the local part. We pick a normalized $h \in \mathrm{L}^2(\mathbb{R}^{1+d}_+, \frac{\mathrm{d}t\,\mathrm{d}x}{t})$ and denote the corresponding inner product by $(\cdot\,|\,\cdot)$. We abbreviate

$$I_j \coloneqq \left(\int_{\ell_j}^\infty \int_{2Q_j} |\varphi(\ell_j\mathsf{DB})\psi(t\mathsf{DB})\mathsf{D}b_j(y)|^2 \,\frac{\mathrm{d}y\,\mathrm{d}t}{t}\right)^{\frac12}$$

and

$$H(y) \coloneqq \left(\int_0^\infty |h(t,y)|^2 \,\frac{\mathrm{d}t}{t}\right)^{\frac12}. \tag{4.5}$$

The Cauchy–Schwarz inequality implies

$$\begin{aligned}
|(f_{\mathrm{loc}}\,|\,h)| &\le \sum_{j\in J} I_j \left(\int_{2Q_j}\int_{\ell_j}^\infty |h(t,y)|^2 \,\frac{\mathrm{d}t\,\mathrm{d}y}{t}\right)^{\frac12} \\
&\le \sum_{j\in J} I_j \left(\int_{2Q_j} |H(y)|^2\,\mathrm{d}y\right)^{\frac12} \\
&\le \sum_{j\in J} I_j |2Q_j|^{\frac12} \inf_{x\in Q_j} (\mathcal{M}(H^2)(x))^{\frac12}.
\end{aligned}$$

To estimate I_j, we use the L^2-boundedness of $\varphi(\ell_j\mathsf{DB})$ along with the $\mathbb{H}^r_{\mathsf{D}}-\mathbb{H}^2_{\mathsf{D}}$-boundedness of $(\psi(t\mathsf{DB}))_{t>0}$, and (ii) to get

$$\|\varphi(\ell_j\mathsf{DB})\psi(t\mathsf{DB})\mathsf{D}b_j\|_2 \lesssim t^{\frac{d}{2}-\frac{d}{r}}\|\nabla b_j\|_r \lesssim \alpha t^{\frac{d}{2}-\frac{d}{r}}|Q_j|^{\frac1r}. \tag{4.6}$$

Since $r < 2$, we infer

$$I_j \lesssim \alpha|Q_j|^{\frac1r}\left(\int_{\ell_j}^\infty t^{d-\frac{2d}{r}}\,\frac{\mathrm{d}t}{t}\right)^{\frac12} \simeq \alpha|Q_j|^{\frac12},$$

and hence

$$|(f_{\mathrm{loc}}\,|\,h)| \lesssim \alpha \sum_{j\in J} |Q_j| \inf_{x\in Q_j} (\mathcal{M}(H^2)(x))^{\frac{1}{2}} \leq \alpha \int_{\bigcup_{j\in J} Q_j} \mathcal{M}(H^2)^{\frac{1}{2}}\,\mathrm{d}x.$$

We use again Kolmogorov's inequality, the normalization of H and (iii) to find

$$|(f_{\mathrm{loc}}\,|\,h)|^2 \lesssim \alpha^2 \Big|\bigcup_{j\in J} Q_j\Big| \|H^2\|_1 \leq \alpha^2 \sum_{j\in J} |Q_j| \lesssim \alpha^{2-r} \|\nabla F\|_r^r. \tag{4.7}$$

This shows the estimate

$$\iint_{\mathbb{R}^{1+d}_+} |f_{\mathrm{loc}}(t,y)|^2 \,\frac{\mathrm{d}t\,\mathrm{d}y}{t} \lesssim \alpha^{2-r} \|\nabla F\|_r^r.$$

Bound for the global part. We use the same duality argument as for the local part. We decompose $(2Q_j)^c = \bigcup_{k\geq 1} C_k(Q_j)$, define H as in (4.5) and put

$$I_{j,k} \coloneqq \left(\int_0^\infty \int_{C_k(Q_j)} |\varphi(\ell_j \mathsf{DB})\psi(t\mathsf{DB})\mathsf{D}b_j(y)|^2 \,\frac{\mathrm{d}y\,\mathrm{d}t}{t} \right)^{\frac{1}{2}}.$$

We obtain the estimate

$$|(f_{\mathrm{glob}}\,|\,h)| \leq \sum_{j\in J} \sum_{k\geq 1} I_{j,k} |2^{k+1}Q_j|^{\frac{1}{2}} \inf_{x\in Q_j} (\mathcal{M}(H^2)(x))^{\frac{1}{2}}. \tag{4.8}$$

At this stage, we apply the argument in [20, Lem. 4.16] with $p = q = 2$, $\sigma = \tau = M$, $E = Q_j$, $F = C_k(Q_j)$ and first-order scaling: the first item yields the bound

$$\|\varphi(\ell_j \mathsf{DB})\psi(t\mathsf{DB})\mathsf{D}b_j\|_{\mathrm{L}^2(C_k(Q_j))} \lesssim 2^{-kM} \,(t/\ell_j)^M \,\|\nabla b_j\|_2,$$

while item (ii) entails for $\ell_j \leq t$ the improved estimate

$$\|\varphi(\ell_j \mathsf{DB})\psi(t\mathsf{DB})\mathsf{D}b_j\|_{\mathrm{L}^2(C_k(Q_j))} \lesssim 2^{-kM} \|\nabla b_j\|_2.$$

Thus, we obtain

$$\|\varphi(\ell_j \mathsf{DB})\psi(t\mathsf{DB})\mathsf{D}b_j\|_{\mathrm{L}^2(C_k(Q_j))} \lesssim 2^{-kM} \Big((t/\ell_j)^M \wedge 1 \Big) \|\nabla b_j\|_2.$$

Now, we fix $\theta \in (0,1)$ such that $r = [p,2]_\theta$ and interpolate the latter bound with (4.6) (with p instead of r and arbitrary $\eta \in \mathrm{C}_{\mathrm{c}}^\infty(Q_j)$ instead of b_j) to find

$$\|\varphi(\ell_j \mathsf{DB})\psi(t\mathsf{DB})\mathsf{D}b_j\|_{\mathrm{L}^2(C_k(Q_j))} \lesssim t^{\frac{d}{2}-\frac{d}{r}} 2^{-k\theta M} \Big((t/\ell_j)^{\theta M} \wedge 1 \Big) \|\nabla b_j\|_r.$$

We split the time integral in $I_{j,k}$ at $t = \ell_j$ and use the last estimate to derive

$$I_{j,k} \lesssim 2^{-k\theta M} \left(\ell_j^{-\theta M} \left(\int_0^{\ell_j} t^{d-\frac{2d}{r}+2M\theta} \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}} + \left(\int_{\ell_j}^{\infty} t^{d-\frac{2d}{r}} \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}} \right) \|\nabla b_j\|_r.$$

We pick $M > {}^{d}\!/_{r\theta} - {}^{d}\!/_{2\theta}$ to ensure that the first integral is finite and obtain by (ii) that

$$I_{j,k} \lesssim \alpha 2^{-k\theta M} |Q_j|^{\frac{1}{2}}.$$

Inserting this bound into (4.8), choosing $M > {}^{d}\!/_{2\theta}$ and using (iv), we infer

$$\begin{aligned} |(f_{\mathrm{glob}} \,|\, h)| &\lesssim \alpha \sum_{j\in J} \sum_{k\geq 1} 2^{\frac{kd}{2}} 2^{-k\theta M} |Q_j| \inf_{x\in Q_j} (\mathcal{M}(H^2)(x))^{\frac{1}{2}} \\ &\lesssim \alpha \int_{\bigcup_{j\in J} Q_j} (\mathcal{M}(H^2))^{\frac{1}{2}} \,\mathrm{d}x. \end{aligned}$$

From now on we proceed as in (4.7) for the local part.

(2) $p < 1$. By complex interpolation (see [4, Thm. 4.28]), it suffices to consider $q \in (p, 1)$. In addition, according to the atomic decomposition of $\mathbb{H}^q_{\mathsf{D}}$ (see Proposition 4.2.5) and the argument in the proof of [20, Thm. 8.17], it is sufficient to show for every $\mathbb{H}^q_{\mathsf{D}}$-atom a that

$$\|S_{\psi,\mathsf{DB}}(a)\|_q \lesssim 1. \tag{4.9}$$

We pick such an atom a with associated ball $B = B(x, r)$. Our argument is similar to the proof of [20, Lem. 8.21].

Local estimate. By Hölder's inequality along with the square function estimates of DB in L^2 and the size estimate for a, we have

$$\|S_{\psi,\mathsf{DB}}(a)\|_{\mathrm{L}^q(16B)} \leq |16B|^{\frac{1}{q}-\frac{1}{2}} \|S_{\psi,\mathsf{DB}}(a)\|_2 \lesssim |B|^{\frac{1}{q}-\frac{1}{2}} \|a\|_2 \lesssim 1.$$

Global estimate. We decompose $(16B)^c = \bigcup_{j\geq 4} C_j(B)$ and estimate

$$\|S_{\psi,\mathsf{DB}}(a)\|_{\mathrm{L}^q(C_j(B))} \lesssim (2^j r)^{\frac{d}{q}-\frac{d}{2}} \|S_{\psi,\mathsf{DB}}(a)\|_{\mathrm{L}^2(C_j(B))}.$$

Thus, if we can find $C, \alpha > 0$ such that

$$I_j \coloneqq \|S_{\psi,\mathsf{DB}}(a)\|_{\mathrm{L}^2(C_j(B))} \leq C 2^{-\alpha j} (2^j r)^{\frac{d}{2}-\frac{d}{q}} \qquad (j \geq 4),$$

then (4.9) follows after summation in j. To achieve this, we start by invoking Fubini's theorem to write

$$I_j = \left(\int_0^{\infty} \int_{\mathbb{R}^d} |(\psi(t\mathsf{DB})a)(y)|^2 \frac{|B(y,t) \cap C_j(B)|}{t^d} \,\mathrm{d}y \,\frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}}.$$

We introduce the notation $D_j(B) := 2^{j+2}B \setminus 2^{j-1}B$ and notice that the set $B(y,t) \cap C_j(B)$ is empty, provided that $y \in D_j(B)^c$ and $t \leq \tau := 2^{\theta(j-1)}r$, where $\theta \in (0,1)$ will be chosen later. This observation entails the estimate

$$\lesssim \left(\int_0^\tau \|\psi(t\mathsf{DB})a\|^2_{\mathrm{L}^2(D_j(B))} \, \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}} + \left(\int_\tau^\infty \|\psi(t\mathsf{DB})a\|_2^2 \, \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}}.$$

For the first term, we use the L^2 off-diagonal estimates of $(\psi(t\mathsf{DB}))_{t>0}$ (with $N \in \mathbb{N}$ to be chosen at our disposal), while for the second term we apply the $\mathbb{H}^p_{\mathsf{D}} - \mathbb{H}^2_{\mathsf{D}}$-boundedness. This gives

$$\begin{aligned} &\lesssim (2^j r)^{-N} \left(\int_0^\tau t^{2N} \, \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}} \|a\|_2 + \left(\int_\tau^\infty t^{d-\frac{2d}{p}} \, \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}} \|a\|_{\mathrm{H}^p} \\ &\simeq (2^j r)^{-N} \tau^N \|a\|_2 + \tau^{\frac{d}{2}-\frac{d}{p}} \|a\|_{\mathrm{H}^p}. \end{aligned}$$

Since a is an L^2-atom for H^q, we can continue with

$$\begin{aligned} &\lesssim (2^j r)^{-N} (2^{\theta(j-1)} r)^N r^{\frac{d}{2}-\frac{d}{q}} + (2^{\theta(j-1)} r)^{\frac{d}{2}-\frac{d}{p}} r^{\frac{d}{p}-\frac{d}{q}} \\ &\lesssim \left(2^{-j(N(1-\theta)+\frac{d}{2}-\frac{d}{q})} + 2^{-j(\theta(\frac{d}{p}-\frac{d}{2})-(\frac{d}{q}-\frac{d}{2}))} \right) (2^j r)^{\frac{d}{2}-\frac{d}{q}}. \end{aligned}$$

Now, we pick $\theta = \theta(p,q) \in (0,1)$ such that $\theta(d/p - d/2) - (d/q - d/2) > 0$ and then choose $N = N(d,p,q) \in \mathbb{N}$ such that $N(1-\theta) + d/2 - d/q > 0$ to obtain for some $\alpha = \alpha(d,p,q)$ that

$$\lesssim 2^{-\alpha j} (2^j r)^{\frac{d}{2}-\frac{d}{q}}. \qquad \square$$

4.4 The improvement $h_-(\mathsf{DB}) \leq p_\circ$

This section is devoted to the proof of Theorem 4.0.3. In the previous section, we observed that $h_-(\mathsf{DB})$ can be characterized via resolvent bounds of DB. As we have already seen in Section 3.3, the algebraic structure of the DB-resolvent provides the link to the second-order Hodge operators $S_\tau \mathsf{L}_\tau^{-1} S_\tau^*$, see Section 3.1 for the notation. The proof of Proposition 3.3.7 further implies the following

Proposition 4.4.1. *Let $p \in (1,2]$. Then $((1+\mathrm{i}t\mathsf{DB})^{-1})_{t\in\mathbb{R}^*}$ is $\mathbb{H}^p_{\mathsf{D}}$-bounded if and only if*

$$\|S_\tau \mathsf{L}_\tau^{-1} \tau f\|_p \lesssim \|f\|_p \qquad (\tau \in \mathbb{R}^*, f \in \mathrm{L}^p \cap \mathrm{L}^2) \tag{4.10}$$

and

$$\|S_\tau \mathsf{L}_\tau^{-1} \operatorname{div} A \nabla h\|_p \lesssim \|\nabla h\|_p \qquad (\tau \in \mathbb{R}^*, h \in \dot{\mathrm{W}}^{1,p} \cap \mathrm{W}^{1,2}). \tag{4.11}$$

Proof of Theorem 4.0.3. By Theorem 4.0.2 and Proposition 4.4.1, it suffices to show (4.10) and (4.11) for $p_\circ$. To this end, we set $\mathsf{L}_\mathsf{A} \coloneqq S_1^* \mathsf{A} S_1$. The transformation formula yields for all $\tau \in \mathbb{R}^*$ the identity

$$S_\tau \mathsf{L}_\tau^{-1} = \tau^{-1} \delta_\tau S_1 \mathsf{L}_{\mathsf{A}(\tau^{-1}\cdot)}^{-1} \delta_{\tau^{-1}}.$$

Hence, (4.10) and (4.11) are equivalent to

$$\|\mathsf{L}_{\mathsf{A}(t\cdot)}^{-1} f\|_{1,p_\circ} \lesssim \|f\|_{p_\circ} \qquad (t \in \mathbb{R}^*, f \in \mathrm{L}^{p_\circ} \cap \mathrm{L}^2) \tag{4.12}$$

and

$$\|\mathsf{L}_{\mathsf{A}(t\cdot)}^{-1} \operatorname{div} A(t\cdot)\nabla h\|_{1,p_\circ} \lesssim \|\nabla h\|_{p_\circ} \qquad (t \in \mathbb{R}^*, h \in \dot{\mathrm{W}}^{1,p_\circ} \cap \mathrm{W}^{1,2}). \tag{4.13}$$

Next, we notice that $A(t\cdot)$ is p-elliptic with $\Delta_p(A(t\cdot)) = \Delta_p(A)$, $\mathsf{A}(t\cdot)$ satisfies (4.1) with lower bound $\lambda > 0$ and $\Lambda(\mathsf{A}(t\cdot)) = \Lambda(\mathsf{A})$. Thus, $(\mathsf{L}_{\mathsf{A}(t\cdot)})_{t\in\mathbb{R}^*}$ forms a family of invertible elliptic operators within the same class. Applying Lemma 1.7.8 (v) and Theorem 1.7.11, we find some $\varepsilon = \varepsilon(p, \Delta_\bullet(A)) > 0$ such that $(\mathrm{e}^{-s^2 \mathsf{L}_{\mathsf{A}(t\cdot)}})_{s>0}$ is $\mathrm{L}^{p_\circ - \varepsilon}$-bounded with bound independent of t. Hence, interpolation with the L^2-boundedness of $(\mathrm{e}^{-s^2(\mathsf{L}_{\mathsf{A}(t\cdot)} - \lambda/2)})_{s>0}$ furnishes some $\theta = \theta(p, d, \lambda, \Delta_\bullet(A)) > 0$ such that $(\mathrm{e}^{-s^2(\mathsf{L}_{\mathsf{A}(t\cdot)} - \theta)})_{s>0}$ is $\mathrm{L}^{p_\circ}$-bounded. Using Corollary 1.7.12, the same argument applies to the gradient family. Hence, the formula

$$\mathsf{L}_{\mathsf{A}(t\cdot)}^{-1} f = \int_0^\infty \mathrm{e}^{-s\mathsf{L}_{\mathsf{A}(t\cdot)}} f \, \mathrm{d}s \qquad (t \in \mathbb{R}^*, f \in \mathrm{L}^{p_\circ} \cap \mathrm{L}^2)$$

is valid in $\mathrm{W}^{1,p_\circ}$ by the exponential decay and implies (4.12).

To show (4.13), we write $L_{a(t\cdot)}$ for the L^2-realization of the perturbed operator $-a(t\cdot)^{-1} \operatorname{div} A(t\cdot)\nabla$. We observe that $L_{a(t\cdot)}$ is again a sectorial operator in L^2 that admits the same critical numbers (defined via resolvents) as $L_{A(t\cdot)} = -\operatorname{div} A(t\cdot)\nabla$, see [20, Chap. 6] for all details. For the rest of this proof, we drop the dependence on t and ensure that all estimates are t-independent. First, we combine Theorems 1.7.3 and 1.7.5 with Theorem 1.7.9 to obtain the following facts:

(i) L_a has a bounded H^∞-calculus on $\mathrm{L}^{p_\circ}$.

(ii) The square root $\sqrt{L_a}\colon \dot{\mathrm{W}}^{1,p_\circ} \cap \mathrm{W}^{1,2} \to \mathrm{L}^{p_\circ} \cap \operatorname{ran}(\sqrt{L_a})$ is bijective and extends to a $p_\circ$-isomorphism whose inverse agrees with $L_a^{-1/2}$ on $\mathrm{L}^{p_\circ} \cap \operatorname{ran}(\sqrt{L_a})$.

The implicit constants are t-independent, see [20, Chap. 6–12]. By (ii), it follows that (4.13) is equivalent to the estimate

$$\|\mathsf{L}_\mathsf{A}^{-1} \operatorname{div} A\nabla L_a^{-1/2} f\|_{1,p_\circ} \lesssim \|f\|_{p_\circ} \qquad (f \in \mathrm{L}^{p_\circ} \cap \operatorname{ran}(\sqrt{L_a})).$$

Next, [20, Lem. 7.2] asserts that $\mathrm{L}^{p_\circ} \cap \operatorname{dom}(L_a) \cap \operatorname{ran}(\sqrt{L_a})$ is dense in $\mathrm{L}^{p_\circ} \cap \mathrm{L}^2$. For f in this intersection, we have

$$-\operatorname{div} A\nabla L_a^{-1/2} f = aL_aL_a^{-1/2} f = a\sqrt{L_a} f = (a+L)(1+L_a)^{-1}\sqrt{L_a} f.$$

Hence, by density, it suffices to show

$$\|\mathsf{L}_\mathsf{A}^{-1}(a+L)(1+L_a)^{-1}\sqrt{L_a} f\|_{1,p_\circ} \lesssim \|f\|_{p_\circ} \qquad (f \in \mathrm{L}^{p_\circ} \cap \operatorname{dom}(L_a) \cap \operatorname{ran}(\sqrt{L_a})).$$

Indeed, if $(f_n)_n \subseteq \mathrm{L}^{p_\circ} \cap \operatorname{dom}(L_a) \cap \operatorname{ran}(\sqrt{L_a})$ converges to $f \in \mathrm{L}^{p_\circ} \cap \operatorname{ran}(\sqrt{L_a})$ in $\mathrm{L}^{p_\circ} \cap \mathrm{L}^2$, then $(g_n)_n := (\operatorname{div} A\nabla L_a^{-1/2} f_n)_n$ converges to $g := \operatorname{div} A\nabla L_a^{-1/2} f$ in $\mathrm{W}^{-1,2}$. Thus, $(\mathsf{L}_\mathsf{A}^{-1} g_n)_n$ converges to $\mathsf{L}_\mathsf{A}^{-1} g$ in $\mathrm{W}^{1,2}$. In summary, the estimate $\|\mathsf{L}_\mathsf{A}^{-1} g\|_{1,p_\circ} \lesssim \|f\|_{p_\circ}$ follows by Fatou's lemma and the $\mathrm{L}^{p_\circ}$-convergence $f_n \to f$.

At this point, we use that $\operatorname{div}(c) = 0$ to write

$$\mathsf{L}_\mathsf{A}^{-1}(a+L) = 1 + \mathrm{i}\mathsf{L}_\mathsf{A}^{-1}(b\nabla + \operatorname{div} c) = 1 + \mathrm{i}\mathsf{L}_\mathsf{A}^{-1}(b\nabla + c\cdot\nabla).$$

Since $\mathsf{L}_\mathsf{A}^{-1} \colon \mathrm{L}^{p_\circ} \cap \mathrm{L}^2 \to \mathrm{W}^{1,p_\circ} \cap \mathrm{W}^{1,2}$ is bounded for the $p_\circ$-norms, it amounts to show that

$$\|(1+L_a)^{-1}\sqrt{L_a} f\|_{1,p_\circ} \lesssim \|f\|_{p_\circ} \qquad (f \in \mathrm{L}^{p_\circ} \cap \operatorname{dom}(L_a) \cap \operatorname{ran}(\sqrt{L_a})).$$

To this end, we use (i) and (ii) from above to get

$$\begin{aligned}\|(1+L_a)^{-1}\sqrt{L_a} f\|_{1,p_\circ} &\simeq \|(1+L_a)^{-1}\sqrt{L_a} f\|_{p_\circ} + \|\nabla(1+L_a)^{-1}\sqrt{L_a} f\|_{p_\circ} \\ &\lesssim \|f\|_{p_\circ} + \|L_a(1+L_a)^{-1} f\|_{p_\circ} \\ &\lesssim \|f\|_{p_\circ}.\end{aligned}$$

□

4.5 The identification $h_-(\mathsf{DB}) = 1$ for real A

In this section, we provide an alternative proof of [28, Cor. 13.3].

Theorem 4.5.1. *If $m = 1$ and A is real-valued, then $h_-(\mathsf{DB}) < 1$.*

Our strategy is as follows. First, by Theorem 4.0.2, we reduce the problem to showing the $\mathbb{H}^p_{\mathsf{D}}$-boundedness of higher-order resolvents of DB for some $p \in (1_*, 1)$. Next, we prove this boundedness by deriving a 'molecular' estimate when the resolvent is applied to D-adapted atoms. To obtain this estimate, we invoke the algebraic relation to the second-order theory furnished by Proposition 4.4.1. Finally, we bootstrap local elliptic regularity in Morrey and Campanato spaces. At this stage, we use the assumption that A is real, which yields a local growth property for the classical Dirichlet energy of L^*-harmonic functions, called *property (D) for* L^*.

There are $\mu_0 \in (0,1]$ and $C > 0$ such that we have for all balls $B(x,R) \subseteq \mathbb{R}^d$, $u \in \mathcal{H}_{L^*,x,R}$ and $r \in (0,R]$ that

$$\int_{B(x,r)} |\nabla u|^2 \,\mathrm{d}x \leq C \left(\frac{r}{R}\right)^{d-2+2\mu_0} \int_{B(x,R)} |\nabla u|^2 \,\mathrm{d}x.$$

Here, μ_0 and C depend only on $d, \lambda(A), \Lambda(A)$, see Section 5.6. In particular, L^* has property (D) if and only if the scaled operator $L^*_t = -\operatorname{div} A^*(t\cdot)\nabla$ does so for all $t > 0$ with the same implicit constants.

We will revisit this in detail in Chapter 5 and this section serves as a primer. For the rest of this section, we advice having copies of [20] and [28] on hand.

4.5.1 Bootstrapping elliptic regularity in Morrey and Campanato spaces

The reader may recall the definitions of Morrey and Campanato spaces from Subsection 1.1.3. We introduce the operator

$$T \coloneqq \begin{bmatrix} 1 & 0 \\ 0 & \mathrm{i}\nabla \end{bmatrix} : \mathrm{L}^2 \times \dot{\mathrm{W}}^{1,2} \to \mathrm{L}^2 \times (\mathrm{L}^2)^d.$$

Using the notation $\mathsf{L} = S_{-1}^* \mathsf{A} S_{-1}$ (see Section 3.1), we set

$$\mathcal{R} \coloneqq \begin{bmatrix} (a + \mathrm{i}b\nabla)\mathsf{L}^{-1} & -a + (a + \mathrm{i}b\nabla)\mathsf{L}^{-1}(a + \mathrm{i}\operatorname{div} c) \\ \mathsf{L}^{-1} & \mathsf{L}^{-1}(a + \mathrm{i}\operatorname{div} c) \end{bmatrix},$$

viewed as a bounded operator on $\mathrm{L}^2 \times \mathrm{L}^2$. Its adjoint is given by

$$\mathcal{R}^* = \begin{bmatrix} (\mathsf{L}^*)^{-1}(a^* + \mathrm{i}\operatorname{div} b^*) & (\mathsf{L}^*)^{-1} \\ -a^* + (a^* + \mathrm{i}c^*\nabla)(\mathsf{L}^*)^{-1}(a^* + \mathrm{i}\operatorname{div} b^*) & (a^* + \mathrm{i}c^*\nabla)(\mathsf{L}^*)^{-1} \end{bmatrix}. \tag{4.14}$$

As shown in [28, Chap. 13.1], we have

$$(1 + \mathrm{i}\mathsf{DB})^{-1} T = T\mathcal{R} \quad \text{on } \mathrm{L}^2 \times \dot{\mathrm{W}}^{1,2},$$

and hence

$$(1 + \mathrm{i}\mathsf{DB})^{-M} T = T\mathcal{R}^M \qquad (M \in \mathbb{N}).$$

Remark 4.5.2. For the rest of this section, implicit constants depend on A and A only through $\lambda(\mathsf{A}), \Lambda(\mathsf{A})$ and $\lambda(A), \Lambda(A)$. Especially, all results remain valid if we replace B with $\mathsf{B}(t\cdot)$ for any $t > 0$, with the same implicit constants.

Lemma 4.5.3 ([28, Lem. 13.6])**.** *Let $\mu \in (0, \mu_0)$. There is $M \in \mathbb{N}$ such that*

$$(\mathcal{R}^*)^{M-1} \colon \mathrm{L}^2 \times \mathrm{L}^2 \to \mathcal{L}^{d+2\mu} \times \mathcal{M}^{d-2+2\mu} \quad \textit{is bounded.}$$

4.5.2 Molecular decay

The key reduction argument is contained in the next

Lemma 4.5.4. *Let $p \in (d/(d+\mu_0), 1]$. If there are $C, \varepsilon > 0$ and $M \in \mathbb{N}$ such that*

$$\|(1 + \mathrm{i}\mathsf{DB})^{-M} a\|_{\mathrm{L}^2(C_j(B))} \leq C(2^j r)^{\frac{d}{2} - \frac{d}{p}} 2^{-\varepsilon j} \tag{4.15}$$

holds true for all balls $B = B(x, r) \subseteq \mathbb{R}^d$, $\mathbb{H}^p_{\mathsf{D}}$-atoms a associated with B and $j \geq 0$, then

$$h_-(\mathsf{DB}) \leq \frac{d}{d + \mu_0}.$$

Proof. The conservation property for DB [20, Prop. 5.3] implies that the function $(1 + \mathrm{i}\mathsf{DB})^{-M} a$ has vanishing integral. Applying [20, Lem. 4.10] together with the identification $\mathbb{H}^p_{\mathsf{D}} = \mathrm{H}^p \cap \overline{\mathrm{ran}(\mathsf{D})}$ furnished by Proposition 4.2.5, we obtain some $C' > 0$ depending only on d, C, ε such that

$$\|(1 + \mathrm{i}\mathsf{DB})^{-M} a\|_{\mathrm{H}^p} \leq C'.$$

Now, we pick $f \in \mathbb{H}^p_{\mathsf{D}}$. By Proposition 4.2.5, there is a sequence of $\mathbb{H}^p_{\mathsf{D}}$-atoms $(a_n)_n$ and a sequence $(\lambda_n)_n \in \ell^p$ such that $f = \sum_n \lambda_n a_n$ converges in L^2 and $\|(\lambda_n)_n\|_{\ell^p} \leq 2\|f\|_{\mathrm{H}^p}$. Following the argument in [20, p. 97], we obtain

$$\|(1 + \mathrm{i}\mathsf{DB})^{-M} f\|_{\mathrm{H}^p} \leq 2C' \|f\|_{\mathrm{H}^p}.$$

By Remark 4.5.2, the same bound holds with $\mathsf{B}(t\cdot)$ instead of B for all $t > 0$. A scaling argument delivers the bound

$$\begin{aligned}\|(1+\mathrm{i}t\mathsf{DB})^{-M} f\|_{\mathrm{H}^p} &= t^{\frac{d}{p}} \|(1+\mathrm{i}\mathsf{DB}(t\cdot))^{-M} \delta_t f\|_{\mathrm{H}^p} \\ &\leq 2C' t^{\frac{d}{p}} \|\delta_t f\|_{\mathrm{H}^p} \\ &= 2C' \|f\|_{\mathrm{H}^p}.\end{aligned}$$

Finally, we use Theorem 4.3.2 (iii) to conclude. □

Proof of Theorem 4.5.1. We fix $p \in (d/(d+\mu_0), 1)$ and choose $\mu \in (0, \mu_0)$ such that $d/(d+\mu) < p$. By Lemma 4.5.4, it suffices to prove (4.15). To this end, we let a be an $\mathbb{H}^p_{\mathsf{D}}$-atom associated with $B = B(x, r)$ and pick $M \in \mathbb{N}$ as in Lemma 4.5.3. We set $\mathsf{R} := (1+\mathrm{i}\mathsf{DB})^{-1}$. The L^2 off-diagonal estimates for R^M joint with the localization and size condition for a yield

$$\|\mathsf{R}^M a\|_{\mathrm{L}^2(C_j(B))} \lesssim \mathrm{e}^{-c(2^j-1)r} \, \|a\|_2 \leq r^{\frac{d}{2}-\frac{d}{p}} \, \mathrm{e}^{-c(2^j-1)r} \qquad (j \geq 0).$$

This shows the estimate for $j = 0$. Now, we let $j \in \mathbb{N}$ and $\varphi \in \mathrm{L}^2(C_j(B))$. As a is an $\mathbb{H}^p_{\mathsf{D}}$-atom, we find $b \in \mathrm{dom}(\mathsf{D})$ such that $a = \mathsf{D}b = [f, \mathrm{i}\nabla h]^\top = T[f,h]^\top$. We set $U := (\mathcal{R}^*)^M T^* \varphi$ (with $\mathcal{R}^*$ as in (4.14)) and calculate

$$\begin{aligned}(\mathsf{R}^M a \,|\, \varphi) &= (\mathsf{R}^M T[f,h]^\top \,|\, \varphi) \\ &= (T\mathcal{R}^M [f,h]^\top \,|\, \varphi) \\ &= ([f,h]^\top \,|\, (\mathcal{R}^*)^M T^* \varphi) \\ &= (f \,|\, U_\perp) + (h \,|\, U_\parallel).\end{aligned}$$

Since a is an $\mathbb{H}^p_{\mathsf{D}}$-atom, it follows that f has vanishing integral and we have the size estimates $\|f\|_2 \leq r^{d/2-d/p}$ and $\|h\|_2 \leq r^{1+d/2-d/p}$. This delivers

$$\begin{aligned}|(\mathsf{R}^M a \,|\, \varphi)| &\leq \|f\|_2 \|U_\perp - (U_\perp)_B\|_{\mathrm{L}^2(B)} + \|h\|_2 \|U_\parallel\|_{\mathrm{L}^2(B)} \\ &\leq r^{\frac{d}{2}-\frac{d}{p}} \|U_\perp - (U_\perp)_B\|_{\mathrm{L}^2(B)} + r^{1+\frac{d}{2}-\frac{d}{p}} \|U_\parallel\|_{\mathrm{L}^2(B)}.\end{aligned}$$

Since $\mathcal{R}^* T^* \colon (\mathrm{L}^2)^{1+d} \to \mathrm{L}^2 \times \mathrm{L}^2$ is bounded, composition and Lemma 4.5.3 imply the boundedness of $(\mathcal{R}^*)^M T^* \colon (\mathrm{L}^2)^{1+d} \to \mathcal{L}^{d+2\mu} \times \mathcal{M}^{d-2+2\mu}$. Hence, we can continue the latter estimate by

$$\lesssim r^{d+\mu-\frac{d}{p}} \|\varphi\|_2.$$

Notably, $d + \mu - d/p > 0$ by our choice of μ. Altogether, we have shown

$$\|\mathsf{R}^M a\|_{\mathrm{L}^2(C_j(B))} \lesssim r^{\frac{d}{2}-\frac{d}{p}} \, \mathrm{e}^{-c2^j r} \quad \& \quad \|\mathsf{R}^M a\|_{\mathrm{L}^2(C_j(B))} \lesssim r^{d+\mu-\frac{d}{p}} \qquad (j \in \mathbb{N}).$$

Finally, we pick $\theta = \theta(d, p, \mu) \in (0, 1)$ such that

$$\varepsilon \coloneqq (1-\theta)\left(\frac{d}{2} - \frac{d}{p}\right) + \theta\left(d + \mu - \frac{d}{p}\right) > 0$$

and interpolate the previous two estimates to find

$$\|\mathsf{R}^M a\|_{\mathrm{L}^2(C_j(B))} \lesssim r^{\varepsilon}\, \mathrm{e}^{-c2^j r} = (2^j r)^{\frac{d}{p} - \frac{d}{2} + \varepsilon}\, \mathrm{e}^{-c2^j r} (2^j r)^{\frac{d}{2} - \frac{d}{p}} 2^{-\varepsilon j} \lesssim (2^j r)^{\frac{d}{2} - \frac{d}{p}} 2^{-\varepsilon j}.$$

This completes the proof. □

CHAPTER 5

Gaussian estimates vs. elliptic regularity on open sets

Let $m \geq 1$, $d \geq 2$, $O \subseteq \mathbb{R}^d$ be open, A be Gårding elliptic and assume that L is subject to mixed boundary conditions with form domain $V = \mathrm{W}^{1,2}_D(O)^m$, see Section 1.6. To give a first idea of our results, let us consider the simplest case of pure Dirichlet boundary conditions, that is $D = \partial O$. We show that the following three properties are equivalent:

(D) De Giorgi estimates for L and L^*-harmonic functions: there exist $C > 0$ and $\mu \in (0, 1]$ such that for all $x \in \overline{O}$, $R \in (0, 1]$ and $u \in \mathrm{W}^{1,2}_0(O)^m$ that are L or L^*-harmonic in $O(x, R)$ it holds

$$\int_{O(x,r)} |\nabla u|^2 \, \mathrm{d}y \leq C \left(\frac{r}{R}\right)^{d-2+2\mu} \int_{O(x,R)} |\nabla u|^2 \, \mathrm{d}y \qquad (r \in (0, R]).$$

(G) Hölder continuity and pointwise Gaussian estimates for the kernel of the semigroup $(\mathrm{e}^{-tL})_{t\geq 0}$.

(H) Local Hölder regularity of L and L^*-harmonic functions with L^2-norm control: there exist $C > 0$ and $\mu \in (0, 1]$ such that for all $x \in \overline{O}$, $r \in (0, 1]$ and $u \in \mathrm{W}^{1,2}_0(O)^m$ that are L or L^*-harmonic in $O(x, r)$ it holds

$$\|u\|_{\mathrm{L}^\infty(O(x,\frac{r}{2}))} + r^\mu [u]^{(\mu)}_{O(x,\frac{r}{2})} \leq C r^{-\frac{d}{2}} \|u\|_{\mathrm{L}^2(O(x,r))}.$$

Our main interest lies in property (G) and we consider the other two properties as a means of getting there. This opens the door for developing for the

first time harmonic analysis, and in particular a theory of geometric Hardy spaces for L as in [26, 48, 77, 147] on open sets far beyond Lipschitz domains. This link is explored in Chapter 6. Let us stress that positivity methods via the Beurling–Deny criterion as in [7] are not suitable for getting (G) — this approach can give the pointwise Gaussian bound, but it misses the Hölder continuity of the kernel that is key to treating the semigroup via methods from singular integral operators.

The geometric framework

In order to prove this equivalence, we only assume that O^c is locally 2-fat. It is shown in [124, Thm. 3.3], see also Proposition 5.1.12, that this is equivalent to the following weak Poincaré inequality at the boundary: there are $c_0, r_0 > 0$ and $c_1 \geq 1$ such that

$$\|u\|_{\mathrm{L}^2(O(x,r))} \leq c_0 r \|\nabla u\|_{\mathrm{L}^2(O(x,c_1 r))} \tag{P_D}$$

for all $x \in \partial O$, $r \in (0, r_0]$ and $u \in \mathrm{W}_0^{1,2}(O)^m$. This condition seems indispensable, for example to control the growth of the Dirichlet energy in the derivation of (D) from (H), making it reasonable to conjecture that our geometric assumptions are in the realm of the best possible.

Now, we pass to the case of pure Neumann boundary conditions for L. The Poincaré inequality that we need (for the same reason as above) is that for the same balls as before and $u \in \mathrm{W}^{1,2}(O)^m$ we have

$$\|u - (u)_{O(x,r)}\|_{\mathrm{L}^2(O(x,r))} \leq c_0 r \|\nabla u\|_{\mathrm{L}^2(O(x,c_1 r))}. \tag{P_N}$$

However, in the Neumann case we need different geometric properties of O, because, roughly speaking, extending functions in the form domain $\mathrm{W}^{1,2}(O)^m$ by 0 is not meaningful anymore. The most general and appropriate geometric framework that we are aware of is the class of locally uniform domains with a positive radius condition. This should be thought of as a quantitative connectedness condition of O, see [39, 117].

Our methods are flexible enough to treat mixed Dirichlet/Neumann boundary conditions with hardly any additional effort. Then (D), (G) and (H) are understood in this context (see Section 5.2).

Remarkably, and in contrast to several earlier results [87, 148], we can work without an interface condition between D and N, which is where typically the main difficulties lie. Our geometric assumptions "interpolate" between the two extremal cases. Namely, we need:

(Fat) O^c is locally 2-fat away from N,

(LU) O is locally uniform near N,

see Section 5.1.1 for precise definitions. In this setting we are able to unite (P_D) and (P_N) to the single property

$$\text{(P)} \qquad \|u - \mathbf{1}_{[x\in N]} \cdot (u)_{O(x,r)}\|_{\mathrm{L}^2(O(x,r))} \leq c_0 r \|\nabla u\|_{\mathrm{L}^2(O(x,c_1 r))},$$

for all balls as before and $u \in V$.

We introduce all relevant geometric concepts in Section 5.1. The rest of the chapter is divided into three parts.

The equivalence of (D), (G) and (H)

In Sections 5.2 to 5.5 we prove the equivalence of (D), (G) and (H) for mixed boundary conditions. This is the main result and illustrated in Figure 3.

Theorem 5.0.1. *Let $m \geq 1$, $d \geq 2$, $O \subseteq \mathbb{R}^d$ be locally uniform near N, O^c locally 2-fat away from N and $\mu_0 \in (0, 1]$. Then the following assertions are equivalent.*

(i) *L and L^* have property $\mathrm{D}(\mu)$ for all $\mu \in (0, \mu_0)$.*

(ii) *L has property $\mathrm{G}(\mu)$ for all $\mu \in (0, \mu_0)$.*

(iii) *L and L^* have property $\mathrm{H}(\mu)$ for all $\mu \in (0, \mu_0)$.*

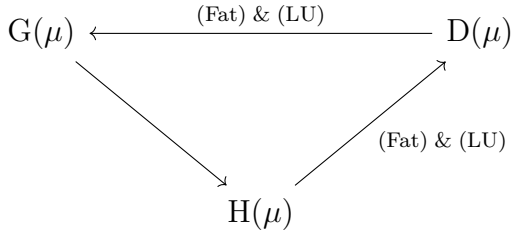

Figure 3: The geometric assumptions needed in Theorem 5.0.1. More precisely, $\mathrm{D}(\mu_0)$ implies $\mathrm{G}(\mu)$, $\mathrm{G}(\mu_0)$ implies $\mathrm{H}(\mu)$ for all $\mu \in (0, \mu_0)$ and $\mathrm{D}(\mu_0)$ is equivalent to $\mathrm{H}(\mu_0)$.

Theorem 5.0.1 originates from results of Auscher and Tchamitchian on $O = \mathbb{R}^d$, see [11, Chap. 4] and [29, Chap. 1]. They have been extended to special

Lipschitz domains with pure Dirichlet or Neumann boundary conditions using localization techniques [30]. We highlight that we do not only recover the known statements of [30], but we generalize the geometric setting to a far larger class of admissible geometries.

The proof of Theorem 5.0.1 is inspired by the monograph [29] and the work of ter Elst and Rehberg [87], who studied property (G) for real-valued coefficients, when O has a weakly Lipschitz boundary around the Neumann part, satisfies an exterior thickness condition around the Dirichlet part and an interface condition in between. Concerning the geometric setup, the present work also extends their result, see the discussion in Subsection 5.1.2.

It remains the question whether some operator L that satisfies one or equivalently all three of these properties exists. This leads us to the second part.

Real-valued coefficients

In Sections 5.6 and 5.7 we study real-valued A and show

Theorem 5.0.2. *Let $m = 1$, $d \geq 2$, $O \subseteq \mathbb{R}^d$ be locally uniform near N, O^c locally 2-fat away from N and let A be real-valued. Then L has property* (H).

To this end, we combine De Giorgi's classical approach [67, 97] with a method of DiBenedetto for non-linear operators of p-growth [72, 73]. The simple underlying idea is that a Poincaré inequality, with a lower exponent than the 2-growth of the operator, implies an estimate for the growth of the level sets as in the case of the isoperimetric inequality. Here, we use the deep fact that p-fatness is an open ended condition in p [128, 134]. Furthermore, we prove local boundedness and Hölder continuity up to the boundary for functions lying in a wider function class than the mere solutions to the equation (Definition 5.6.2). In fact, we can also be slightly more general on the geometric side by replacing (Fat) and (LU) by a p-adapted version of (P) for some $p \in (1, 2)$ joint with the embedding $V \subseteq \mathrm{L}^{2^*}(O)$ for $d \geq 3$ or an interpolation type inequality when $d = 2$.

Theorem 5.0.2 is already known in the pure Dirichlet case provided that O satisfies an exterior thickness condition [123, Chap. II, App. C & D]. In several papers [59, 108, 109, 129, 148] the Hölder regularity of solutions to non-homogeneous elliptic problems was studied for the case of mixed boundary conditions. However, all aforementioned papers use either Lipschitz coordinate charts around the Neumann part, stronger assumptions on D and an interface condition between D and N, or their geometric setup is almost impossible to check [148]. We refer to [87] for further references and applications.

Perturbations and $d = 2$

Following an argument in [11], we prove in Section 5.7 (in the setup of Theorem 5.0.1) that property (D), and thus also (G) and (H), are stable under small complex perturbations of the coefficients. Property (G) has been obtained in [85] for small perturbations of real-valued A via a different method. Moreover, when $d = 2$, only a slightly stronger geometric assumption on D — the $(d-1)$-set property — is needed to deduce that every elliptic system L has property (G) and hence also (D) and (H). This is the content of Section 5.8.

5.1 The geometric setup

5.1.1 Assumptions (Fat) and (LU)

We introduce the geometric setup and explain its consequences. The reader may recall the notion of relative capacities from Section 1.2.

Definition 5.1.1 (Assumption $(\mathrm{Fat})_p$)**.** Let $p \in (1, d]$. We say that $\boldsymbol{O^c}$ **is locally** $\boldsymbol{p}$**-fat away from** $\boldsymbol{N}$, if there is some $\delta > 0$ such that:

(i) D is locally p-fat in $D \cap N_\delta$,

(ii) O^c is locally p-fat in D.

For $p = 2$, we write (Fat) instead of $(\mathrm{Fat})_2$ to mean that O^c is locally 2-fat away from N.

This terminology is motivated by the fact that (Fat) is a fatness assumption on $O^c(\supseteq D)$ but as points get closer to N (point x instead of y in Figure 4), the lower bound on the capacity already has to come from the complementary boundary part $D(\subseteq O^c)$. In Section 5.6 we will need a self-improvement property of $(\mathrm{Fat})_p$ with respect to p in the spirit of Lewis' result [128]. To this end, the equivalent formulation of $(\mathrm{Fat})_p$ below will be useful. In the rest of this chapter, we shall exclusively work with (Fat) and set $p = 2$.

Given $\delta > 0$, we let $\Sigma \subseteq \mathbb{R}^d$ be a grid of closed, axis-parallel cubes of diameter $\delta/8$ and define

$$N_\delta^\Sigma := \Big(\bigcup \{Q \in \Sigma \colon\ Q \cap \overline{N_\delta} \neq \varnothing\}\Big)^\circ .$$

The set N_δ^Σ is a regularized version of N_δ such that $N_\delta \subseteq N_\delta^\Sigma \subseteq N_{9\delta/8}$. In particular, as a union of cubes of the same size, $(N_\delta^\Sigma)^c$ is locally p-fat for

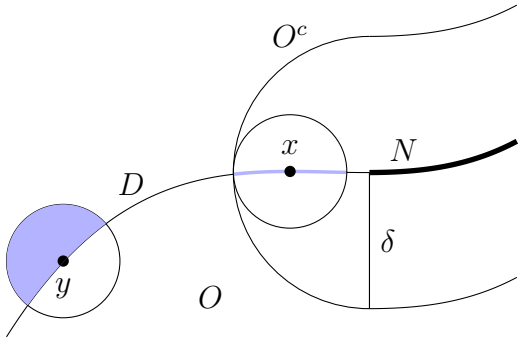

Figure 4: The parts of the balls around x and y for which we require a lower bound on the 2-capacity are in blue. The full picture of assumption (Fat) is obtained by letting x, y and the size of the balls vary.

any $p \in (1, d]$ by Poincaré's inequality, see also Subsection 5.1.2, which is not necessarily true for $(N_\delta)^c$.

Lemma 5.1.2. *Let $p \in (1, d]$. The following assertions are equivalent.*

(i) *$D \cup (O^c \setminus N_\delta^\Sigma)$ is locally p-fat for some $\delta > 0$.*

(ii) *O^c is locally p-fat away from N.*

In addition, if one of the conditions holds true with $\delta > 0$, then the other one holds true with $\delta/2$.

Proof. (ii) $\Longrightarrow$ (i): We fix $\delta > 0$ as in Definition 5.1.1 and show that $U := D \cup (O^c \setminus N_{\delta/2}^\Sigma)$ is locally p-fat. To this end, we let $x \in U$, $r \in (0, \delta/4]$ and make the following case distinction:

(1) $\boldsymbol{\overline{B(x, r/2)} \cap (D \cap N_\delta) \neq \varnothing}$. We pick $z \in \overline{B(x, r/2)} \cap (D \cap N_\delta)$ and get $\overline{B(z, r/2)} \cap D \subseteq \overline{B(x, r)} \cap U$. Hence, the local p-fatness of D in $D \cap N_\delta$ yields the claim.

(2) $\boldsymbol{\overline{B(x, r/2)} \cap (D \cap N_\delta) = \varnothing}$. In this case we have $x \in O^c \setminus N_{\delta/2}^\Sigma$ and we consider two subcases.

(2.1) $\boldsymbol{\overline{B(x, r/2)} \cap D \neq \varnothing}$. We pick $z \in \overline{B(x, r/2)} \cap (D \setminus N_\delta)$ and since $r \leq \delta/4$ we get $\overline{B(z, r/2)} \cap O^c \subseteq \overline{B(x, r)} \cap U$. We conclude from the local p-fatness of O^c in D.

(2.2) $\boldsymbol{\overline{B(x, r/2)} \cap D = \varnothing}$. It follows that $\overline{B(x, r/2)} \subseteq O^c$. Thus, we have $\overline{B(x, r/2)} \cap (N_{\delta/2}^\Sigma)^c \subseteq \overline{B(x, r)} \cap U$ and deduce the claim from the local p-fatness of $(N_{\delta/2}^\Sigma)^c$.

(i) $\Longrightarrow$ (ii): We set $U := D \cup (O^c \setminus N_\delta^\Sigma)$ and show that D is locally p-fat in $D \cap N_{\delta/2}$ and that O^c is locally p-fat in D. The second assertion follows, since $U \subseteq O^c$ and $D \subseteq U$. For the first assertion, we let $x \in D \cap N_{\delta/2}$ and $r \in (0, \delta/4]$. Then $\overline{B(x,r)} \cap D = \overline{B(x,r)} \cap U$ and we conclude again from the local p-fatness of U. □

We will see in Proposition 5.1.9 that (Fat) is substantial for having a boundary Poincaré inequality on V without average. There are essentially two ways to get this inequality: either the extension of u to the whole ball B vanishes on a set that has measure comparable to B or u vanishes on a portion of D that is "nice enough" in this capacitary sense. The fatness assumption treats both cases simultaneously.

While (Fat) describes O away from N in our main result, we use the following quantitative connectedness condition near N, see also Figure 5.

Definition 5.1.3 (Assumption (LU))**.** Let $\varepsilon \in (0, 1]$ and $\delta \in (0, \infty]$. We call $\boldsymbol{O}$ **locally an $(\boldsymbol{\varepsilon}, \boldsymbol{\delta})$-domain near $\boldsymbol{N}$**, if the following properties hold:

(i) All points $x, y \in O \cap N_\delta$ with $0 < |x - y| < \delta$ can be joined in O by an **$\boldsymbol{\varepsilon}$-cigar with respect to $\boldsymbol{\partial O \cap N_\delta}$**, that is to say, a rectifiable curve $\gamma \subseteq O$ of length $\ell(\gamma) \leq |x-y|/\varepsilon$ such that we have for all $z \in \operatorname{tr}(\gamma)$ that

$$\operatorname{d}(z, \partial O \cap N_\delta) \geq \frac{\varepsilon |z - x||z - y|}{|x - y|}. \tag{5.1}$$

(ii) O has **positive radius near $\boldsymbol{N}$**: there is $C > 0$ such that all connected components O' of O with $\partial O' \cap N \neq \varnothing$ satisfy $\operatorname{diam}(O') \geq C\delta$.

If the values C, ε, δ need not be specified, then O is called **locally uniform near $\boldsymbol{N}$**.

Definition 5.1.4. Let $C > 0$. We say that

(i) C depends on the **geometry** if C depends only on dimensions and the parameters in the definitions of (Fat) and (LU).

(ii) C depends on **ellipticity** if C depends on $\lambda = \lambda(A)$ and $\Lambda = \Lambda(A)$.

Assumption (LU) has been introduced in [40], to which we refer for a detailed discussion. It is slightly stronger than the related condition in [39], see [40, Prop. 2.5 (ii)].

Let us observe that (Fat) and (LU) become weaker as δ decreases. Hence, we will work from now on under

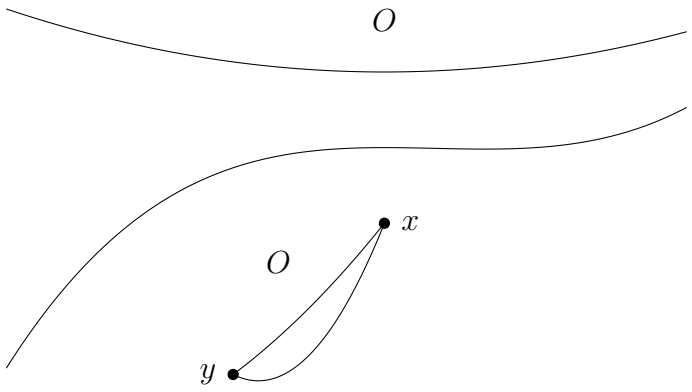

Figure 5: An illustration of an ε-cigar between x and y. In view of (5.1), the cigar is contained in O. In order to understand the nature of the cigar shape, we use the length condition $\ell(\gamma) \leq {}^{|x-y|}/_{\varepsilon}$ to obtain from (5.1) that

$$\frac{\varepsilon}{2}(|z-x| \wedge |z-y|) \leq \frac{\varepsilon|z-x||z-y|}{|x-y|} \leq |z-x| \wedge |z-y|.$$

Hence, (5.1) means essentially that every point z on $\mathrm{tr}(\gamma)$ keeps distance to $\partial O \cap N_\delta$ that is at least the minimum of its distance to x and y. We refer to [104, 159] for more information.

Assumption 5.1.5. *From now on we assume that $\delta \in (0,1]$ with the same choice of δ in* (Fat) *and* (LU).

Now, we draw important consequences from (Fat) and (LU). The first one is the existence of an extension operator that will be useful at several places of this thesis.

Definition 5.1.6. An **extension operator** $\mathcal{E}$ is a linear map from $\mathrm{L}^1_{\mathrm{loc}}(O)^m$ into the space of $\mathbb{C}^m$-valued measurable functions on $\mathbb{R}^d$ such that $(\mathcal{E}f)|_O = f$ for all $f \in \mathrm{L}^1_{\mathrm{loc}}(O)^m$.

Theorem 5.1.7 ([39, Thm. 10.2], ($\mathcal{E}$))**.** *Assume* (LU). *There are $K \in [1,\infty)$, $A \in (0, 1/2]$ and an extension operator $\mathcal{E}$ such that for all $p \in [1,\infty)$ we have:*

- *$\mathcal{E}$ restricts to a bounded operator from $\mathrm{W}^{1,p}_D(O)^m$ to $\mathrm{W}^{1,p}_D(\mathbb{R}^d)^m$.*
- *$\mathcal{E}$ is **local and homogeneous**, that is*

$$\|\nabla^\ell \mathcal{E}u\|_{\mathrm{L}^p(B(x,r))} \lesssim \|\nabla^\ell u\|_{\mathrm{L}^p(O(x,Kr))} \tag{5.2}$$

holds true for all $x \in \partial O$, $r \in (0, A\delta]$, $u \in \mathrm{W}^{1,p}_D(O)^m$ *and* $\ell \in \{0,1\}$.

The implicit constant depends only on the parameters in (LU), m, d *and* p.

The second key implication of (LU) is that O has no exterior cusps near N.

Proposition 5.1.8 ([40, Prop. 2.9])**.** *Assume* (LU). *We have an* ***interior corkscrew condition*** *for* O *near* N*:*

$$(\mathrm{ICC}_{N_\delta}) \qquad \exists\, \alpha > 0\ \forall\, x \in \overline{O \cap N_{\frac{\delta}{2}}}, r \in (0,1]\ \exists\, z \in O\colon \qquad B(z, \alpha r) \subseteq O(x,r).$$

The last one is a weak Poincaré inequality with correct scaling. In the formulation, $\mathcal{E}, A, \delta$ and K are as in Theorem 5.1.7. In the proof, we frequently use the fact that

$$\inf_{c\in\mathbb{C}} \|u-c\|_{\mathrm{L}^p(E)} \leq \|u-(u)_E\|_{\mathrm{L}^p(E)} \leq 2 \inf_{c\in\mathbb{C}} \|u-c\|_{\mathrm{L}^p(E)}$$

whenever $p \in [1,\infty)$, $E \subseteq \mathbb{R}^d$ has positive and finite measure, and $u \in \mathrm{L}^p(E)^m$.

Proposition 5.1.9 (Weak Poincaré inequality)**.** *Let* $p \in (1,d]$ *and let us assume* $(\mathrm{Fat})_p$ *and* (LU). *There is* $c_0 > 0$ *depending on the geometry and* p *such that*

$$(\mathrm{P})_p \qquad \|u - \mathbf{1}_{[\mathrm{d}_D(x)>r]} \cdot (u)_{O(x,r)}\|_{\mathrm{L}^p(O(x,r))} \leq c_0 r \|\nabla u\|_{\mathrm{L}^p(O(x,3Kr))}$$

for all $x \in \partial O$, $r \in (0, {}^{A\delta}/_{2}]$ *and* $u \in \mathrm{W}^{1,p}_D(O)^m$.

Proof. By density, we can assume $u \in \mathrm{C}^\infty_D(O)^m$. First, we note that if $\overline{B(x,r)} \subseteq O$, then $\mathrm{d}_D(x) > r$ and $O(x,r) = B(x,r)$, and $(\mathrm{P})_p$ follows from the standard Poincaré inequality (with subtraction of the average) on the ball $B(x,r)$. Hence, we assume from now on that $\overline{B(x,r)} \cap \partial O \neq \varnothing$. We distinguish two cases.

(1) $\boldsymbol{\mathrm{d}_D(x) \leq r}$**.** In this case there exists $x_D \in \overline{B(x,r)} \cap D$.

(1.1) $\boldsymbol{x_D \in D \cap N_\delta}$**.** We estimate

$$\begin{aligned}\|u\|_{\mathrm{L}^p(O(x,r))} &\leq \|u\|_{\mathrm{L}^p(O(x_D,2r))} \\ &\leq \|\mathcal{E}u\|_{\mathrm{L}^p(B(x_D,2r))}.\end{aligned}$$

Since $u \in \mathrm{C}_D^\infty(O)^m$ and $\mathcal{E}$ is local and homogeneous, we have for all $y \in D$ and sufficiently small $r > 0$ that $\|\mathcal{E}u\|_{\mathrm{L}^p(B(y,r))} \lesssim \|u\|_{\mathrm{L}^p(O(y,Kr))} = 0$. From this we conclude that $\mathcal{E}u$ vanishes almost everywhere on an open neighborhood of D. Hence, in view of $(\mathrm{Fat})_p$, we can apply Mazya's Poincaré inequality [122, Lem. 3.1] and (5.2) to continue by

$$\begin{aligned}&\lesssim r\|\nabla\mathcal{E}u\|_{\mathrm{L}^p(B(x_D,2r))}\\&\lesssim r\|\nabla u\|_{\mathrm{L}^p(O(x_D,2Kr))}\\&\leq r\|\nabla u\|_{\mathrm{L}^p(O(x,3Kr))}.\end{aligned}$$

(1.2) $\boldsymbol{x_D \in D \setminus N_\delta}$**.** Since $2r \leq \delta/2$, we know that $u_0 \in \mathrm{W}^{1,p}(B(x_D,2r))^m$, and of course u_0 vanishes on O^c. Hence, the same argument as in case (1.1) applies with u_0 replacing $\mathcal{E}u$.

(2) $\boldsymbol{r < \mathrm{d}_D(x)}$**.** In this case there exists $x_N \in \overline{B(x,r)} \cap N$. Using the standard Poincaré inequality on $B(x_N, 2r)$ and (5.2), we get the desired estimate

$$\begin{aligned}\|u-(u)_{O(x,r)}\|_{\mathrm{L}^p(O(x,r))} &\leq 2\|u-(\mathcal{E}u)_{B(x_N,2r)}\|_{\mathrm{L}^p(O(x,r))}\\&\leq 2\|\mathcal{E}u-(\mathcal{E}u)_{B(x_N,2r)}\|_{\mathrm{L}^p(B(x_N,2r))}\\&\lesssim r\|\nabla\mathcal{E}u\|_{\mathrm{L}^p(B(x_N,2r))}\\&\lesssim r\|\nabla u\|_{\mathrm{L}^p(O(x_N,2Kr))}\\&\leq r\|\nabla u\|_{\mathrm{L}^p(O(x,3Kr))}.\end{aligned}$$

□

As with $(\mathrm{Fat})_p$ we simply write (P) instead of $(\mathrm{P})_2$. From now on we set $p = 2$ and use the fixed constants

$$c_1 \coloneqq 3K \geq 3 \tag{c_1}$$

and

$$r_0 \coloneqq (A \wedge C)\frac{\delta}{2}. \tag{r_0}$$

Incorporating $C > 0$ from Definition 5.1.3 in the definition of the radius r_0 will be useful at later occurrences.

Remark 5.1.10. As we have seen in the last proof, working with weak Poincaré inequalities bears the advantage that the ball can be centered at the boundary. Because of this, $(\mathrm{P})_p$ could equivalently be required with $x \in \partial O$ instead of $x \in \overline{O}$. Moreover, we get the Poincaré inequality with average

$$\|u-(u)_{O(x,r)}\|_{\mathrm{L}^p(O(x,r))} \lesssim r\|\nabla u\|_{\mathrm{L}^p(O(x,c_1r))},$$

for all $x \in \overline{O}$, $r \in (0, r_0]$ and $u \in \mathrm{W}_D^{1,p}(O)^m$.

5.1.2 Comparison of the geometric setup

Now, we provide a short comparison of our chosen geometry with the one in [70, 87]. We believe that it is instructive to see how their assumptions are built into our general framework.

To show (G) for real-valued A, the following geometric setup is used, compare with [87, Thm. 7.5]:

(I) **Uniform Lipschitz charts around N:** There is $K \geq 1$ such that for all $x \in \overline{N}$ there is an open neighborhood U_x of x and a bi-Lipschitz map $\Phi_x \colon U_x \to B(0,1)$ with bi-Lipschitz constant at most K and the properties $\Phi_x(x) = 0$ and $\Phi_x(U_x \cap O) = (\mathbb{R}^d_+)(0,1)$.

(II) **O is exterior thick in D:** There is $C > 0$ such that

$$Cr^d \leq |(O^c)(x,r)| \qquad (x \in D, r \in (0,1]).$$

(III) **Interface condition between D and N:** There are $C, c > 0$ with

$$Cr^{d-1} \leq \mathcal{H}^{d-1}(((\mathbb{R}^{d-1} \times \{0\}) \cap [\mathrm{d}_{\Phi_x(N \cap U_x)}(\cdot) > cr])(y,r)),$$

for all $x \in D \cap \overline{N}$, $y \in \Phi_x(D \cap \overline{N} \cap U_x)$ and $r \in (0,1]$.

Lemma 5.1.11. *If O, D and N satisfy* (I), (II) *and* (III), *then they also fulfill* (Fat) *and* (LU).

Proof. It is classical that (I) implies (LU), see [40, p. 9] and references therein. The full details have been written out in [78, Lem. 2.2.20].

To see that (II) implies that O^c is locally 2-fat in D, we let $x \in D$, $r \in (0,1]$ and $u \in \mathrm{C}^\infty_\mathrm{c}(B(x,2r))$ with $u = 1$ on $\overline{B(x,r)} \cap O^c$. Then (II) joint with Poincaré's inequality yields

$$r^{d-2} \lesssim r^{-2}|(O^c)(x,r)| \leq r^{-2}\|u\|^2_{\mathrm{L}^2(B(x,2r))} \lesssim \|\nabla u\|^2_{\mathrm{L}^2(B(x,2r))},$$

and hence

$$r^{d-2} \lesssim \mathrm{cap}_2(\overline{B(x,r)} \cap O^c; B(x,2r)).$$

Finally, let us explain why D is locally 2-fat in $D \cap N_\delta$ for some $\delta > 0$. In fact, (I), (III), and [87, Lem. 5.4] show that there is some $\delta \in (0,1]$ such that D is a $(d-1)$-set in $D \cap N_{2\delta}$, compare also with [87, p. 304]. It seems to be folklore that this implies the local 2-fatness of D in $D \cap N_\delta$. For convenience, we included the details in Lemma 1.3.3. Altogether, we have concluded (Fat) from (I), (II) and (III). □

To construct explicitly a set O that fulfills (LU) and (Fat), but not the geometric setup from above, we consider

$$\mathbb{R}^2 \setminus \{(x,y) \in \mathbb{R}^2 \colon x \geq 0 \,\&\, 0 \leq y \leq x^2\}$$

and add a part of the von Koch snowflake, see Figure 6. We put Neumann boundary conditions on the "fractal part" coming from the snowflake and Dirichlet boundary conditions on its complement. Let us sketch that O is an admissible example.

(i) As $|(O^c)(0,r)| \leq \int_0^r x^2 \,\mathrm{d}x = r^3/3$ for small enough $r > 0$, it follows that O is not exterior thick at the origin.

(ii) The boundary of the von Koch snowflake is not rectifiable, so there are no Lipschitz coordinate charts around N.

(iii) By inspection, D is a $(d-1)$-set. Hence, Lemma 1.3.3 reveals that D is locally 2-fat (in itself). As $D \subseteq O^c$, also (Fat) is satisfied.

(iv) By [104, Prop. 6.30] the von Koch snowflake is an (ε, ∞)-domain. Hence, one can verify that O is an (ε, ∞)-domain as well.

We close this section by showing that (under the background assumption (LU)) having (Fat) is just as good as having the abstract assumption (P). Let us recall that (LU) is void in the case of Dirichlet boundary conditions.

Proposition 5.1.12. *Let $m \geq 1$, $d \geq 2$ and assume* (LU). *Then* (Fat) *is equivalent to* (P).

Proof. That (Fat) and (LU) imply (P) has been shown in Proposition 5.1.9.

For the converse statement, we borrow ideas from [124, Thm. 3.3]. To show (Fat), we fix $x \in D$ and $r \in (0, r_0]$. We consider two cases.

(1) $\boldsymbol{x \in D \setminus N_\delta}$. We pick $u \in \mathrm{C}_\mathrm{c}^\infty(B(x,2r))$ with $u = 1$ on $\overline{B(x,r)} \cap O^c$ and assume first that

$$\frac{1}{4}|B(x, r/2c_1)| \leq \|u\|^2_{\mathrm{L}^2(B(x,r/2c_1))}.$$

Then Poincaré's inequality applied on $B(x,2r)$ implies

$$r^{d-2} \lesssim \|\nabla u\|^2_{\mathrm{L}^2(B(x,2r))}.$$

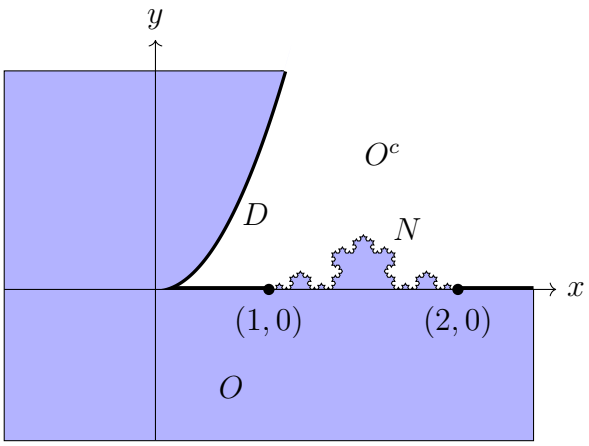

Figure 6: A geometric constellation in $\mathbb{R}^2$ that satisfies (Fat) and (LU), but not the geometric setup introduced in this subsection. Here, N can be constructed by the following algorithm: divide the line segment between $(1,0)$ and $(2,0)$ into three parts of equal length, remove the middle one, and build an equilateral triangle over this segment. Then apply this procedure to each of the four remaining segments and iterate.

Now, we assume the converse estimate. Then

$$\begin{aligned}\frac{1}{2}|B(x, r/2c_1)| &\leq \|1-u\|^2_{\mathrm{L}^2(B(x,r/2c_1))} + \|u\|^2_{\mathrm{L}^2(B(x,r/2c_1))} \\ &\leq \|1-u\|^2_{\mathrm{L}^2(B(x,r/2c_1))} + \frac{1}{4}|B(x, r/2c_1)|\end{aligned}$$

and hence

$$r^d \lesssim \|1-u\|^2_{\mathrm{L}^2(B(x,r/2c_1))}.$$

We choose $\varphi \in \mathrm{C}_\mathrm{c}^\infty(B(x,r))$ with $\varphi = 1$ on $B(x, r/2)$, put $v \coloneqq \varphi(1-u)$ and note that $v \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^d)$ with $v = 0$ on D. Hence, $v \in V$ by [2, Thm. 9.1.3] and (P) yields

$$r^d \lesssim \int_{B(x,r/2c_1)} |1-u|^2 = \int_{O(x,r/2c_1)} |v|^2 \lesssim r^2 \int_{O(x,r/2)} |\nabla v|^2 = r^2 \int_{B(x,r/2)} |\nabla u|^2.$$

This shows that O^c is locally 2-fat in $D \setminus N_\delta$.

(2) $\boldsymbol{x \in D \cap N_\delta}$. To prove that D is locally 2-fat in $D \cap N_\delta$, we systematically replace O^c by D and $B(x, r/2c_1)$ by $O(x, r/2c_1)$ in (1) and apply the same argument. The key points are that we now have $v \in \mathrm{W}^{1,2}_0(\mathbb{R}^d \setminus D)$ and hence $v|_O \in V$, and $|O(x, r/2c_1)| \simeq r^d$ due to $(\mathrm{ICC}_{N_\delta})$. □

5.2 Properties (D), (G) and (H)

The property that we are mostly interested in is the Gaussian estimate for the kernel of the semigroup $(\mathrm{e}^{-tL})_{t\geq 0}$. Let us introduce this property in detail:

Definition 5.2.1. Let $\mu \in (0,1]$. We say that L has **property G(μ)** if the following holds:

(G1) For any $t > 0$ there is a measurable function $K_t \colon O \times O \to \mathbb{C}$ such that

$$(\mathrm{e}^{-t^2 L} f)(x) = \int_O K_t(x,y) f(y)\,\mathrm{d}y$$

for all $f \in \mathrm{L}^2(O)^m$ and almost every $x \in O$.

(G2) There are $C, c, \omega > 0$ such that

$$|K_t(x,y)| \leq C\,\mathrm{e}^{\omega t^2}\, t^{-d}\,\mathrm{e}^{-c\left(\frac{|x-y|}{t}\right)^2} \qquad (t > 0).$$

(G3) There are $C, \omega > 0$ such that

$$|K_t(x,y) - K_t(x',y')| \leq C\,\mathrm{e}^{\omega t^2}\, t^{-\mu-d}(|x-x'| + |y-y'|)^\mu$$

for all $x, x', y, y' \in O$ and $t > 0$.

Furthermore, we say that L has **property (G)** if L has property G(μ) for some $\mu \in (0,1]$.

Remark 5.2.2. The following facts will be useful:

(i) Property G(μ) is stable under taking adjoints since the kernel of the adjoint semigroup is given by $K_t^*(x,y) = \overline{K_t(y,x)}$.

(ii) Logarithmic convex combinations of (G2) and (G3) yield for all $\nu \in (0,\mu)$ constants $C, c, \omega > 0$ such that

$$\begin{aligned}&|K_t(x,y+h) - K_t(x,y)|\\ &\leq C\,\mathrm{e}^{\omega t^2}\, t^{-d}\left(\frac{|h|}{t}\right)^\nu \left(\mathbf{1}_{[0,2|h|)}(|x-y|) + \mathbf{1}_{[2|h|,\infty)}(|x-y|)\,\mathrm{e}^{-c\left(\frac{|x-y|}{t}\right)^2}\right),\end{aligned}$$

for each $x, y \in O$, $h \in \mathbb{R}^d$ with $y + h \in O$ and $t > 0$. A similar estimate holds true in the x-variable.

The (eventually equivalent) properties (D) and (H) quantify the regularity of weak solutions in subsets of O. For Dirichlet boundary conditions they have appeared in the introduction, but their adaptation to general boundary conditions requires some care. Following [87], we do that by looking at solutions to $-\operatorname{div} A\nabla u = 0$ in $O(x,r)$ that are compatible with the "global" boundary conditions (Dirichlet on D, Neumann on N). More precisely, we use test functions with Dirichlet boundary conditions only on $\partial O(x,r) \setminus N(x,r)$:

$$V_{x,r} := \mathrm{W}^{1,2}_{\partial O(x,r)\setminus N(x,r)}(O(x,r))^m.$$

The precise variational formulation is as follows:

Definition 5.2.3. Let $x \in \overline{O}$ and $r > 0$.

(i) Let $u \in \mathrm{W}^{1,2}(O(x,r))^m$, $f \in \mathrm{L}^2(O(x,r))^m$ and $F \in \mathrm{L}^2(O(x,r))^{dm}$. We write $L_D u = f - \operatorname{div} F$ in $O(x,r)$ if

$$\int_{O(x,r)} A\nabla u \cdot \overline{\nabla\varphi} = \int_{O(x,r)} f\overline{\varphi} + F \cdot \overline{\nabla\varphi} \qquad (\varphi \in V_{x,r}).$$

In addition, we write $\boldsymbol{u \in \mathcal{H}_{L,x,r}}$ if $L_D u = 0$ in $O(x,r)$.

(ii) Let $u \in V$, $f \in \mathrm{L}^2(O)^m$ and $F \in \mathrm{L}^2(O)^{dm}$. We write $L_D u = f - \operatorname{div} F$ in O if

$$\int_{O} A\nabla u \cdot \overline{\nabla\varphi} = \int_{O} f\overline{\varphi} + F \cdot \overline{\nabla\varphi} \qquad (\varphi \in V).$$

In addition, we write $\boldsymbol{u \in \mathcal{H}_L}$ if $L_D u = 0$ in O.

This definition of $L_D u = f - \operatorname{div} F$ is natural for our purpose, because $V_{x,r}$ is canonically embedded into V: If $u \in V$ satisfies a global equation $L_D u = f - \operatorname{div} F$, then also $L_D u = f - \operatorname{div} F$ in all local sets $O(x,r)$ due to part (i) of the following lemma applied with $U = B(x,r)$.

Lemma 5.2.4. *Let $U \subseteq \mathbb{R}^d$ be open and $p \in [1,\infty)$.*

(i) *The set $N \cap U$ is open in $\partial(O \cap U)$ and the 0-extension*

$$\mathcal{E}_0 \colon \mathrm{W}^{1,p}_{\partial(O\cap U)\setminus(N\cap U)}(O\cap U)^m \to \mathrm{W}^{1,p}_D(O)^m \quad \textit{is isometric.}$$

(ii) *If $\psi \in \mathrm{C}^\infty_{\mathrm{c}}(U)$, then multiplication by ψ maps the space $\mathrm{W}^{1,p}_D(O)^m$ boundedly into $\mathrm{W}^{1,p}_{\partial(O\cap U)\setminus(N\cap U)}(O\cap U)^m$.*

Proof. For (i) we refer to [87, Lem. 6.3]. For (ii) we pick $u \in \mathrm{W}^{1,p}_D(O)^m$. By definition this means that there is a sequence $(\varphi_n)_n \subseteq \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^d \setminus D)^m$ with $\varphi_n|_O \to u$ in $\mathrm{W}^{1,p}(O)^m$. Then $\psi\varphi_n \in \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^d)^m$ and to see that $\operatorname{supp}(\psi\varphi_n) \cap [\partial(O \cap U) \setminus (N \cap U)] = \varnothing$, we notice that $\operatorname{supp}(\psi\varphi_n) \subseteq U \setminus D$ and

$$(U \setminus D) \cap (\partial(O \cap U) \setminus (N \cap U)) \subseteq (\partial O \cap U) \setminus (D \cup (N \cap U)) = \varnothing.$$

We have shown that $\psi\varphi_n \in \mathrm{C}^\infty_{\partial(O\cap U)\setminus(N\cap U)}(\mathbb{R}^d)^m$ and the claim follows by passing to the limit in n. □

Now, we introduce the local properties (D) and (H).

Definition 5.2.5. Let $\mu \in (0,1]$. We say that L has **property D(μ)** if there is $c_{\mathrm{D}(\mu)} > 0$ such that for all $x \in \overline{O}$, $R \in (0,1]$ and $u \in V \cap \mathcal{H}_{L,x,R}$ we have

$$\int_{O(x,r)} |\nabla u|^2 \leq c_{\mathrm{D}(\mu)} \left(\frac{r}{R}\right)^{d-2+2\mu} \int_{O(x,R)} |\nabla u|^2 \qquad (r \in (0,R]).$$

In addition, we say that L has **property (D)** if L has property D(μ) for some $\mu \in (0,1]$.

Remark 5.2.6. The interest in property (D) lies in radii that are not comparable by absolute constants. Otherwise the estimate holds by monotonicity of the integral with any choice of μ. In particular, we can replace the condition $0 < r \leq R \leq 1$ by the more flexible condition $0 < cr \leq R \leq R_0$ for any fixed $R_0 > 0$ and $c > 1$.

Definition 5.2.7. Let $\mu \in (0,1]$. We say that L has **property H(μ)** if there is $c_{\mathrm{H}(\mu)} > 0$ such that for all $x \in \overline{O}$, $r \in (0,1]$ and $u \in V \cap \mathcal{H}_{L,x,r}$ we have that u has a continuous representative in $O(x, r/2)$ that satisfies

$$r^\mu [u]^{(\mu)}_{O(x,\frac{r}{2})} \leq c_{\mathrm{H}(\mu)} r^{-\frac{d}{2}} \|u\|_{\mathrm{L}^2(O(x,r))}. \tag{5.3}$$

In addition, we say that L has **property (H)** if L has property H(μ) for some $\mu \in (0,1]$.

Let us mention that this definition is different from the one given in the introduction, but with some work it turns out to be equivalent as we will see in Lemma 5.2.9 below. In the setting of (H) the function u extends continuously to $\overline{O(x,r/2)}$ and the Dirichlet condition gets a pointwise meaning:

Lemma 5.2.8. *Assume* (Fat) *and* (LU). *Let $x \in \overline{O}$, $r > 0$ and $u \in V$. If u has a continuous representative in $\overline{O(x,r)}$, then $u(y) = 0$ for all $y \in \overline{D(x,r)}$.*

Proof. By continuity, and since y and r are arbitrary, it suffices to prove $u(y) = 0$ when $y \in D$. We can assume that u is real-valued, since otherwise we consider real and imaginary parts separately. For the sake of a contradiction, we suppose that $|u(y)| \neq 0$. By continuity, we pick $\rho \in (0, r_0 \wedge r]$ and $c > 0$ such that $|u| \geq c$ on $O(y, \rho)$. Repeated application of the truncation property in Lemma 1.1.11 gives $|u| \wedge c \in V$. But on $O(y, \rho)$ this function is the constant c and the Poincaré inequality in Proposition 5.1.9 yields the contradiction $c = 0$. □

A further consequence of our geometric setup is that property (H) yields a posteriori local boundedness of L-harmonic functions.

Lemma 5.2.9. *Assume* (Fat) *and* (LU) *and let L have property* (H)*. Then there is $C > 0$ such that for all $x \in \overline{O}$, $r \in (0, 1]$ and $u \in V \cap \mathcal{H}_{L,x,r}$ we have that u has a continuous representative in $O(x, r/2)$ that satisfies*

$$\|u\|_{\mathrm{L}^\infty(O(x,\frac{r}{2}))} \leq C r^{-\frac{d}{2}} \|u\|_{\mathrm{L}^2(O(x,r))}.$$

Proof. We distinguish the following cases:

(1) $x \in N_{\delta/2}$ or $B(x, r/4) \subseteq O$. By H(μ) we have for all $y, z \in O(x, r/2)$ that

$$|u(y)| \leq r^\mu [u]^{(\mu)}_{O(x,\frac{r}{2})} + |u(z)| \lesssim r^{-\frac{d}{2}} \|u\|_{\mathrm{L}^2(O(x,r))} + |u(z)|.$$

Now, we average with respect to z on $O(x, r/2)$ and use $|O(x, r/2)| \simeq r^d$ (either by (ICC_{N_δ}) or $B(x, r/4) \subseteq O$) to infer

$$|u(y)| \lesssim (r^{-\frac{d}{2}} + |O(x, r/2)|^{-\frac{1}{2}}) \|u\|_{\mathrm{L}^2(O(x,r))} \lesssim r^{-\frac{d}{2}} \|u\|_{\mathrm{L}^2(O(x,r))}.$$

(2) $x \in (N_{\delta/2})^c$ and $(\partial O)(x, r/4) \neq \varnothing$. We consider two subcases.

(2.1) $D(x, r/4) \neq \varnothing$. We pick $y \in D(x, r/4)$. Lemma 5.2.8 implies $u(y) = 0$ and we get for all $z \in O(x, r/2)$ that

$$|u(z)| = |u(z) - u(y)| \leq r^\mu [u]^{(\mu)}_{O(x,\frac{r}{2})} \lesssim r^{-\frac{d}{2}} \|u\|_{\mathrm{L}^2(O(x,r))}.$$

(2.2) $D(x, r/4) = \varnothing$. Pick $w \in N(x, r/4)$. Then $O(w, r/4) \subseteq O(x, r/2)$ and we have for all $y \in O(x, r/2)$ and $z \in O(w, r/4)$ that

$$|u(y)| \leq r^\mu [u]^{(\mu)}_{O(x,\frac{r}{2})} + |u(z)|.$$

Now, we average with respect to z on $O(w, r/4)$ and conclude as in the first case. Here, we have used (ICC_{N_δ}) at $w \in N$. □

An inspection of the previous proof reveals the following

Corollary 5.2.10. *Assume* (Fat) *and* (LU) *and let* $\mu \in (0,1]$. *Then there is* $C > 0$ *depending only on* d *and the constants in* (LU) *such that*

$$\|u\|_{\mathrm{L}^\infty(O(x,r))} \leq 2r^\mu [u]^{(\mu)}_{O(x,r)} + Cr^{-d}\|u\|_{\mathrm{L}^1(O(x,r))}$$

for all $x \in \overline{O}$, $r \in (0,1]$ *and* $u \in V \cap \mathrm{C}^\mu(O(x,r))^m$.

Remark 5.2.11. Using Lemma 5.2.9, the following modifications can be made in Definition 5.2.7 and Lemma 5.2.9.

(i) It is possible to replace the condition $r \in (0,1]$ by $r \in (0,R]$ for any $R > 0$. Indeed, it suffices to consider $R > 1$ and $r \in (1,R)$. The L^∞-part is clear, as it is a pointwise estimate for all $y \in O(x, r/2)$. To bound the Hölder seminorm, we pick $y, z \in O(x, r/2)$. If $|y - z| \geq 1/8$, then we can use the L^∞-bound. If $|y - z| < 1/8$, then we can apply the estimate in $O(x, 1/4)$.

(ii) By the same type of argument, the radius $r/2$ on the left-hand side of (5.3) can be replaced by γr for any $\gamma \in (0,1)$.

Next, we discuss the solvability of the local problem $L_D u = f - \operatorname{div} F$ in $O(x,R)$ with an a priori bound that has the correct scaling in R. To see that this does not come for free, we consider a simple counterexample.

We fix $r \in (0,1]$ and put $O_r := B(0,r) \cup B(4e_1, 1)$ as the union of two disjoint balls. We impose Neumann boundary conditions on $\partial B(0,r)$ and Dirichlet boundary conditions on $\partial B(4e_1, 1)$. Now we take some $f \in \mathrm{L}^2(O_r)$ that is not average free over $B(0,r)$ and set $R = 2$ to see that the local problem $L_D u = f$ in $O_r(0,R) = B(0,r)$ cannot have a solution. Indeed, we can take the constant 1-function as a test function.

However, changing the radius from $R = 2$ to $\rho = r/2$ yields a pure Dirichlet problem $L_D u = f$ in $O_r(0,\rho) = B(0,\rho)$, which admits a unique solution by the Lax–Milgram lemma. The correct scaling in ρ in the a priori estimate comes from the classical Poincaré inequality on balls. Now, the key observation is that our geometric setup does not allow that r shrinks to 0 (see Definition 5.1.3). This ensures that the ratio ρ/R is bounded from below. We will need this fact in Section 5.3.

Getting the correct scaling in R can be more difficult. Here, the geometry has to ensure that the local Dirichlet part $\partial O(x,R) \setminus N(x,R)$ is large enough in a suitable sense.

The key point in the next lemma is the following: even when we cannot solve every local problem with an a priori bound that has the correct scaling in R, we can use our geometric setup to do it for some smaller radius ρ still comparable to R.

Lemma 5.2.12. *Assume* (Fat) *and* (LU). *Let* $x \in \overline{O}$, $r \in (0, r_0]$, $f \in \mathrm{L}^2(O(x,r))^m$ *and* $F \in \mathrm{L}^2(O(x,r))^{md}$. *There is some* $\rho \in [r/4, r]$ *such that the problem*

$$L_D v = f - \operatorname{div} F \quad \text{in } O(x,\rho)$$

has a unique weak solution $v \in V_{x,\rho}$ *that satisfies*

$$\lambda \|\nabla v\|_{\mathrm{L}^2(O(x,\rho))} \le C\rho \|f\|_{\mathrm{L}^2(O(x,\rho))} + \|F\|_{\mathrm{L}^2(O(x,\rho))}, \tag{5.4}$$

where $C > 0$ *depends only on the geometry.*

Proof. The main issue, as seen from the example above, is the coercivity of the form on $V_{x,\rho}$. If $\overline{B(x,r)} \subseteq O$, then $O(x,r) = B(x,r)$ and we have Poincaré's inequality on $V_{x,r} = \mathrm{W}_0^{1,2}(B(x,r))^m$ at our disposal. Hence, the result for $\rho = r$ follows from the Lax–Milgram lemma. We assume from now on that $\overline{B(x,r)} \cap \partial O \neq \varnothing$.

If $\mathrm{d}_D(x) \le r$, then Lemma 5.2.4 (i) joint with (P) implies

$$\|\varphi\|_{\mathrm{L}^2(O(x,r))} \lesssim r \|\nabla \varphi_0\|_{\mathrm{L}^2(O(x,c_1 r))} = r \|\nabla \varphi\|_{\mathrm{L}^2(O(x,r))} \qquad (\varphi \in V_{x,r})$$

and we conclude as in the first case.

Finally, we come to the most technical part $\mathrm{d}_N(x) \le r < \mathrm{d}_D(x)$. Here, we need to consider two subcases:

(1) $\boldsymbol{\partial B(x,r) \cap O = \varnothing}$**.** Since $x \in \overline{O}$, this means that O splits into two components $O = O_{\mathrm{loc}} \cup (O \setminus O_{\mathrm{loc}})$, where O_{loc} is open and contained in $B(x,r)$. As $r < \mathrm{d}_D(x)$, the boundary of all connected components of O_{loc} intersects ∂O in N. Hence, O has a connected component with diameter less than $2r \le C\delta$ that intersects N in contradiction with (LU). Thus, this case can never occur.

(2) $\boldsymbol{\partial B(x,r) \cap O \neq \varnothing}$**.** As for $\mathrm{d}_D(x) \le r$, it suffices to find some $\rho \in [r/4, r]$ with

$$\|\varphi\|_{\mathrm{L}^2(O(x,\rho))} \lesssim \rho \|\nabla \varphi\|_{\mathrm{L}^2(O(x,\rho))} \qquad (\varphi \in V_{x,\rho}). \tag{5.5}$$

The argument in (1) implies that there is some $y \in \partial B(x, r/2) \cap O \subseteq N_{\delta/2} \cap O$. Now, we find a radius ρ comparable to r such that $\partial O(x,\rho)$ carries a large portion of Dirichlet boundary conditions not necessarily coming from D. For this, we find a ball $B(z, \alpha\rho/4)$ that lies inside O with center $z \in \partial B(x,\rho)$.

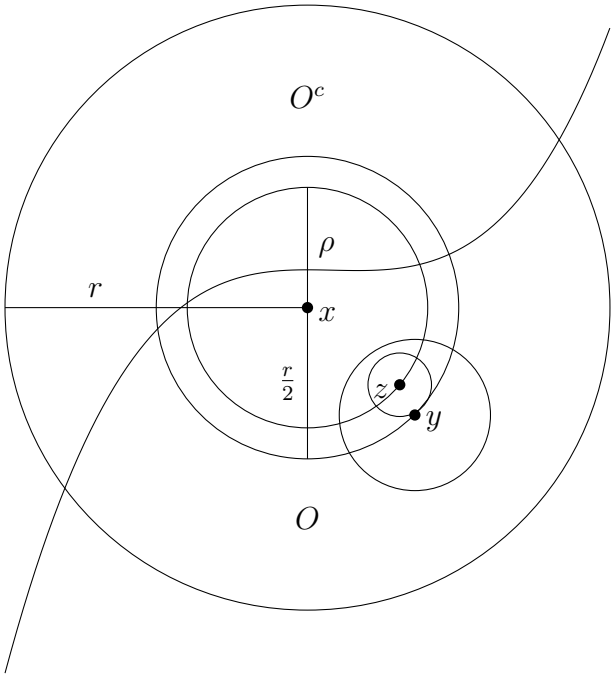

Figure 7: Geometric configuration, where $\partial O(x,\rho) \setminus N(x,\rho)$ is large enough.

By $(\mathrm{ICC}_{N_\delta})$, there is some z with $B(z, {}^{\alpha r}/_4) \subseteq O(y, {}^{r}/_4)$. We set $\rho := |z-x|$ so that $\rho \in [{}^{r}/_4, {}^{3r}/_4]$. Then $z \in \partial B(x,\rho)$ and $B(z, {}^{\alpha\rho}/_4) \subseteq O$, which is exactly what we need (see Figure 7).

Now, we pick $\varphi \in \mathrm{C}^\infty_{\partial O(x,\rho)\setminus N(x,\rho)}(O(x,\rho))^m$. By Lemma 5.2.4 (i), the 0-extension of φ belongs to V and hence we get

$$\|\varphi\|_{\mathrm{L}^2(O(x,\rho))} \leq \|\mathcal{E}\varphi_0\|_{\mathrm{L}^2(B(x,\rho))}.$$

Since $\mathcal{E}\varphi_0 \in \mathrm{W}^{1,2}(B(x,\rho))^m$ vanishes on $\partial B(x,\rho) \cap B(z, {}^{\alpha\rho}/_4)$, we have

$$\lesssim \rho \|\nabla \mathcal{E}\varphi_0\|_{\mathrm{L}^2(B(x,\rho))}.$$

Indeed, if $x = 0$, $\rho = 1$ and z is the north pole of $B(x,\rho)$, then this Poincaré inequality follows by compactness. The general case follows from scaling and a rigid motion. Finally, we use that $\mathcal{E}$ is local and homogeneous in order to derive

$$\begin{aligned} &\lesssim \rho \|\nabla \varphi_0\|_{\mathrm{L}^2(O(x,c_1\rho))} \\ &= \rho \|\nabla \varphi\|_{\mathrm{L}^2(O(x,\rho))}. \end{aligned}$$

This proves (5.5) and completes the proof. □

We will show Theorem 5.0.1 by proving the implications (i) $\Longrightarrow$ (ii), (ii) $\Longrightarrow$ (iii), and (iii) $\Longrightarrow$ (i) in this order as in [29]. This is the content of the following three sections.

5.3 From (D) to (G)

In this section, we prove the implication (i) $\Longrightarrow$ (ii) of Theorem 5.0.1. Throughout the entire section, we make the geometric assumptions (Fat) and (LU), see Definitions 5.1.1 and 5.1.3, and our goal is thus to show

Theorem 5.3.1. *Let L and L^* have property $\mathrm{D}(\mu_0)$ and fix $\mu \in (0, \mu_0)$. Then L has property $\mathrm{G}(\mu)$ with implicit constants depending on the geometry, ellipticity, μ, μ_0 and $c_{\mathrm{D}(\mu_0)}$.*

Let us fix $t > 0$ and $f \in \mathrm{L}^2(O)^m$. The key idea is to consider the elliptic problem

$$L_D \, \mathrm{e}^{-t^2 L} f = L \, \mathrm{e}^{-t^2 L} f = g \quad \text{in } O.$$

Property $\mathrm{D}(\mu_0)$ allows us to bootstrap regularity of $\nabla f_t = \nabla \, \mathrm{e}^{-t^2 L} f$ in Morrey and Campanato spaces, see Subsection 1.1.3 for the definition of these spaces. This enables us to find for all $\mu \in (0, \mu_0)$ some $\omega > 0$ such that $(\mathrm{e}^{-t^2(L+\omega)})_{t>0}$ and $(\mathrm{e}^{-t^2(L^*+\omega)})_{t>0}$ are $\mathrm{L}^2 - \dot{\mathrm{C}}^\mu$-bounded. Once in place, existence of the kernel with estimates follow from the Dunford–Pettis theorem, see Theorem 5.3.9 and Lemma 5.3.10. Compared to [29, 87], we replace Davies' perturbation method by the usage of off-diagonal estimates. This provides a much shorter and streamlined argument, since it does not produce lower order perturbations for the divergence form operator.

We begin with gradient estimates for global weak solutions in Morrey spaces. Later, we apply these estimates iteratively to ∇f_t.

Convention 5.3.2. *We drop the dependence on O when using Morrey and Campanato (semi)norms whenever the context is clear.*

Proposition 5.3.3. *Let $\mu \in (0, 1]$ and assume that L has property $\mathrm{D}(\mu)$. Let $\kappa \in [0, d)$, $\sigma \in (0, 2]$ with $\kappa + \sigma < d - 2 + 2\mu$, $x \in \overline{O}$, $R_0 \in (0, r_0]$ and $f \in \mathcal{M}^\kappa_{x,R_0}(O)^m$. Then we have for all $u \in V$ with $L_D u = f$ in $O(x, R_0)$ and $\varepsilon \in (0, 1]$ the estimate*

$$\|\nabla u\|_{\mathcal{M}^{\kappa+\sigma}_{x,R_0}} \lesssim R_0^{1-\frac{\sigma}{2}} \varepsilon^{2-\sigma} \|f\|_{\mathcal{M}^\kappa_{x,R_0}} + R_0^{-\frac{\kappa+\sigma}{2}} \varepsilon^{-(\kappa+\sigma)} \|\nabla u\|_{\mathrm{L}^2(O(x,R_0))},$$

where the implicit constant depends only on λ, $c_{\mathrm{D}(\mu)}$, μ, $\kappa + \sigma$ and geometry.

Before we start the proof, we recall the classical Campanato lemma.

Lemma 5.3.4 ([96, Chap. 3, Lem. 2.1])**.** *Let $\phi\colon \mathbb{R} \to [0,\infty)$ be increasing, and let $C_1, C_2, R_0, \varepsilon \geq 0$ as well as $0 \leq \beta < \gamma$. If we have*

$$\phi(r) \leq C_1\left[\left(\frac{r}{R}\right)^{\gamma} + \varepsilon\right]\phi(R) + C_2 R^{\beta} \qquad (0 < r \leq R \leq R_0),$$

then there exist $C, \varepsilon_0 > 0$ depending only on C_1, γ and β such that if $\varepsilon < \varepsilon_0$, then

$$\phi(r) \leq C\left[\left(\frac{r}{R}\right)^{\beta}\phi(R) + C_2 r^{\beta}\right] \qquad (0 < r \leq R \leq R_0).$$

Proof of Proposition 5.3.3. We let $0 < r \leq R \leq R_0\varepsilon^2$ and define the function

$$\phi(r) \coloneqq \|\nabla u\|_{\mathrm{L}^2(O(x,r))}.$$

By Lemma 5.2.12 there are $\rho \in [{}^{R}\!/_{4}, R]$ and $v \in V_{x,\rho}$ such that $L_D v = f$ in $O(x,\rho)$, and v satisfies the a priori bound (5.4). This implies

$$\lambda\|\nabla v\|_{\mathrm{L}^2(O(x,\rho))} \leq CR^{\frac{\kappa+2}{2}}\|f\|_{\mathcal{M}^{\kappa}_{x,R}} \leq C(R_0\varepsilon^2)^{\frac{2-\sigma}{2}}\|f\|_{\mathcal{M}^{\kappa}_{x,R_0}} R^{\frac{\kappa+\sigma}{2}}, \tag{5.6}$$

where $C > 0$ depends only on the geometry. By Lemma 5.2.4 (i) we know that $v_0 \in V$. Hence, $w \coloneqq u - v_0 \in V \cap \mathcal{H}_{L,x,\rho}$ and, provided that $r \leq \rho$, we can use property $\mathrm{D}(\mu)$ to get

$$\begin{aligned}
\phi(r) &\leq \|\nabla w\|_{\mathrm{L}^2(O(x,r))} + \|\nabla v\|_{\mathrm{L}^2(O(x,r))} \\
&\leq c^{1/2}_{\mathrm{D}(\mu)}\left(\frac{r}{\rho}\right)^{\frac{d}{2}-1+\mu} \|\nabla w\|_{\mathrm{L}^2(O(x,\rho))} + \|\nabla v\|_{\mathrm{L}^2(O(x,r))} \\
&\leq c^{1/2}_{\mathrm{D}(\mu)}\left(\frac{r}{\rho}\right)^{\frac{d}{2}-1+\mu} \phi(\rho) + (c^{1/2}_{\mathrm{D}(\mu)} + 1)\|\nabla v\|_{\mathrm{L}^2(O(x,\rho))}. \qquad (5.7)
\end{aligned}$$

Inserting (5.6) into (5.7) and using ${}^{R}\!/_{4} \leq \rho \leq R$ delivers

$$\phi(r) \leq 2^d c^{1/2}_{\mathrm{D}(\mu)}\left(\frac{r}{R}\right)^{\frac{d}{2}-1+\mu}\phi(R) + \frac{C(c^{1/2}_{\mathrm{D}(\mu)}+1)}{\lambda}(R_0\varepsilon^2)^{1-\frac{\sigma}{2}}\|f\|_{\mathcal{M}^{\kappa}_{x,R_0}} R^{\frac{\kappa+\sigma}{2}}.$$

If $r \geq \rho$, then ${}^{r}\!/_{R} \geq {}^{1}\!/_{4}$ and the same estimate follows by monotonicity of ϕ, even without the second summand. Lemma 5.3.4 improves this bound to

$$\phi(r) \lesssim \left(\frac{r}{R}\right)^{\frac{\kappa+\sigma}{2}}\phi(R) + (R_0\varepsilon^2)^{1-\frac{\sigma}{2}}\|f\|_{\mathcal{M}^{\kappa}_{x,R_0}} r^{\frac{\kappa+\sigma}{2}},$$

with an implicit constant depending only on λ, $c_{\mathrm{D}(\mu)}$, μ, $\kappa+\sigma$ and geometry. Hence, we get for all r and R as above that

$$r^{-\frac{\kappa+\sigma}{2}}\phi(r) \lesssim R^{-\frac{\kappa+\sigma}{2}}\|\nabla u\|_{\mathrm{L}^2(O(x,R_0))} + (R_0\varepsilon^2)^{1-\frac{\sigma}{2}}\|f\|_{\mathcal{M}^{\kappa}_{x,R_0}}.$$

In particular, if we pick $R \coloneqq R_0\varepsilon^2$, then we get for $0 < r \leq R_0\varepsilon^2$ the estimate

$$r^{-\frac{\kappa+\sigma}{2}}\phi(r) \lesssim R_0^{-\frac{\kappa+\sigma}{2}}\varepsilon^{-(\kappa+\sigma)}\|\nabla u\|_{\mathrm{L}^2(O(x,R_0))} + R_0^{1-\frac{\sigma}{2}}\varepsilon^{2-\sigma}\|f\|_{\mathcal{M}^{\kappa}_{x,R_0}}.$$

As before, this estimate remains valid for $R_0\varepsilon^2 < r \leq R_0$ by monotonicity of ϕ. Taking the supremum in $r \in (0, R_0]$ yields the claim. □

Lemma 5.3.5. *Let $\kappa \in [0,d)$ and $\sigma \in (0,2]$. There is $C > 0$ depending only on κ and the geometry such that*

$$[u]_{\mathcal{L}^{\kappa+\sigma}_{x,r}} \leq Cr^{1-\frac{\sigma}{2}}\varepsilon^{2-\sigma}\|\nabla u\|_{\mathcal{M}^{\kappa}_{x,c_1r}} + r^{-\frac{\kappa+\sigma}{2}}\varepsilon^{-(\kappa+\sigma)}\|u\|_{\mathrm{L}^2(O(x,r))}$$

for all $x \in \overline{O}$, $r \in (0, r_0]$, $\varepsilon \in (0,1]$ and $u \in V$. Moreover, if $x \notin N_{\delta/2}$, then the same estimate holds for u_0 on $B(x,r)$.

Proof. If $\rho \in (0, \varepsilon^2 r]$, then we have by Remark 5.1.10 that

$$\begin{aligned}\rho^{-\frac{\kappa+\sigma}{2}}\|u - (u)_{O(x,\rho)}\|_{\mathrm{L}^2(O(x,\rho))} &\leq c_0\rho^{1-\frac{\sigma}{2}}\rho^{-\frac{\kappa}{2}}\|\nabla u\|_{\mathrm{L}^2(O(x,c_1\rho))}\\ &\leq c_0c_1^{\frac{\kappa}{2}}r^{1-\frac{\sigma}{2}}\varepsilon^{2-\sigma}\|\nabla u\|_{\mathcal{M}^{\kappa}_{x,c_1r}}.\end{aligned}$$

In the other case, we get

$$\begin{aligned}\rho^{-\frac{\kappa+\sigma}{2}}\|u - (u)_{O(x,\rho)}\|_{\mathrm{L}^2(O(x,\rho))} &\leq \rho^{-\frac{\kappa+\sigma}{2}}\|u\|_{\mathrm{L}^2(O(x,\rho))}\\ &\leq r^{-\frac{\kappa+\sigma}{2}}\varepsilon^{-(\kappa+\sigma)}\|u\|_{\mathrm{L}^2(O(x,r))}.\end{aligned}$$

Finally, if $x \notin N_{\delta/2}$, then $u_0 \in \mathrm{W}^{1,2}(B(x,r))^m$ since we have $r_0 < \delta/2$ by definition. Hence, we can use the standard Poincaré inequality on balls in the first case. □

Now, we increase the regularity of ∇f_t for $f \in \mathrm{L}^2(O)^m$ in Morrey spaces up to the critical exponent $d-2+2\mu$ and eventually pass to an estimate for the Campanato seminorm of f_t with exponent $d+2\mu$.

Convention 5.3.6. *For the rest of this section, implicit constants depend only on the geometry, ellipticity and the premises in the statement.*

Lemma 5.3.7. *Let L have property $\mathrm{D}(\mu_0)$ and let $\mu \in (0, \mu_0)$. There are $C, \omega > 0$ and $\gamma \in (0, 1]$ such that*

$$\mathbf{1}_{[x\in N_{\delta/2}]}[f_t]_{\mathcal{L}^{d+2\mu}_{x,\gamma r_0}(O)} + \mathbf{1}_{[x\notin N_{\delta/2}]}[(f_t)_0]_{\mathcal{L}^{d+2\mu}_{x,\gamma r_0}(\mathbb{R}^d)} \le C\,\mathrm{e}^{\omega t^2}\, t^{-\mu-\frac{d}{2}}\|f\|_2 \tag{5.8}$$

for all $x \in \overline{O}$, $t > 0$ and $f \in \mathrm{L}^2(O)^m$.

Proof. We treat only the case $x \in N_{\delta/2}$. The other one can be discussed analogously, replacing (P) by the standard Poincaré inequality on balls.

For $\kappa \in [0, d-2+2\mu_0)$ and $r \in (0, r_0]$ we consider the following statement:

***P*(κ, r).** There are $C, \omega > 0$ such that

$$\|f_t\|_{\mathcal{M}^\kappa_{x,r}} + \|t\nabla f_t\|_{\mathcal{M}^\kappa_{x,r}} \le C\,\mathrm{e}^{\omega t^2}\, t^{-\frac{\kappa}{2}}\|f\|_2 \qquad (t > 0, f \in \mathrm{L}^2(O)^m).$$

Here, $P(0, r_0)$ holds true by the L^2-theory of L. We fix $\sigma \in (0, 2]$ with $\kappa+\sigma < d-2+2\mu_0$ and claim that

$$P(\kappa, r_0) \Longrightarrow P(\kappa+\sigma, {}^{r_0}\!/_{c_1}).$$

This will yield (5.8) with $\gamma := {}^1\!/_{c_1^{m_0+2}}$, where m_0 is the largest integer with $2m_0 < d-2+2\mu$, by iterating (m_0+1)-times and a final application of the Poincaré inequality in Remark 5.1.10.

Let us assume that $P(\kappa, r_0)$ is valid. We define $\varepsilon := \sqrt{t}\,\mathrm{e}^{-t^2} \in (0, 1]$ and prove $P(\kappa+\sigma, {}^{r_0}\!/_{c_1})$ in two steps. We abbreviate $r_1 := {}^{r_0}\!/_{c_1}$.

Non-gradient bound. Since $x \in N_{\delta/2}$ we have $(\mathrm{ICC}_{N_\delta})$ at hand. We apply Lemma 1.1.16 (i) and then Lemma 5.3.5 to get

$$\begin{aligned}\|f_t\|_{\mathcal{M}^{\kappa+\sigma}_{x,r_1}} &\lesssim [f_t]_{\mathcal{L}^{\kappa+\sigma}_{x,r_1}} + \|f_t\|_2 \\ &\lesssim (\sqrt{t}\,\mathrm{e}^{-t^2})^{2-\sigma}\|\nabla f_t\|_{\mathcal{M}^\kappa_{x,r_0}} + (\sqrt{t}\,\mathrm{e}^{-t^2})^{-(\kappa+\sigma)}\|f_t\|_2.\end{aligned}$$

Using $P(\kappa, r_0)$ and the L^2-contractivity of the semigroup gives us

$$\begin{aligned}&\lesssim t^{-\frac{\kappa+\sigma}{2}}\,\mathrm{e}^{\omega t^2}\,\|f\|_2 + t^{-\frac{\kappa+\sigma}{2}}\,\mathrm{e}^{dt^2}\,\|f\|_2 \\ &\lesssim t^{-\frac{\kappa+\sigma}{2}}\,\mathrm{e}^{(\omega\vee d)t^2}\,\|f\|_2.\end{aligned}$$

Gradient bound. Proposition 5.3.3 with right-hand side $Lf_t = \mathrm{e}^{-\frac{t^2}{2}L}\,Lf_{t/\sqrt{2}}$ reveals the bound

$$\begin{aligned}\|t\nabla f_t\|_{\mathcal{M}^{\kappa+\sigma}_{x,r_1}} &\lesssim t(\sqrt{t}\,\mathrm{e}^{-t^2})^{2-\sigma}\|\,\mathrm{e}^{-\frac{t^2}{2}L}\,Lf_{\frac{t}{\sqrt{2}}}\|_{\mathcal{M}^\kappa_{x,r_1}} + (\sqrt{t}\,\mathrm{e}^{-t^2})^{-(\kappa+\sigma)}\|t\nabla f_t\|_2 \\ &\lesssim t^{-\frac{\sigma}{2}}\|\,\mathrm{e}^{-\frac{t^2}{2}L}(t^2Lf_{\frac{t}{\sqrt{2}}})\|_{\mathcal{M}^\kappa_{x,r_1}} + t^{-\frac{\kappa+\sigma}{2}}\,\mathrm{e}^{dt^2}\,\|t\nabla f_t\|_2.\end{aligned}$$

Next, we use $P(\kappa, r_0)$ joint with the bound $\|t^2 L f_{t/\sqrt{2}}\|_2 \lesssim \|f\|_2$ for analytic semigroups for the first summand and the L^2-estimate $\|t\nabla f_t\|_2 \lesssim \|f\|_2$ for the second one to deduce

$$\lesssim t^{-\frac{\kappa+\sigma}{2}} \, \mathrm{e}^{(\frac{\omega}{2} \vee d)t^2} \, \|f\|_2. \qquad \square$$

Finally, we can prove

Proposition 5.3.8. *Assume that L has property $\mathrm{D}(\mu_0)$ and fix $\mu \in (0, \mu_0)$. Then there is $\omega > 0$ such that $(\mathrm{e}^{-t^2(L+\omega)})_{t>0}$ is $\mathrm{L}^2 - \dot{\mathrm{C}}^\mu$-bounded.*

Proof. We fix $x, y \in O$ with $x \neq y$, $t > 0$ and $f \in \mathrm{L}^2(O)^m$. It suffices to assume that $x, y \in O \cap N_{\delta/2}$. Indeed, if $x \in O \setminus N_{\delta/2}$ or $y \in O \setminus N_{\delta/2}$, then we replace f_t and O by $(f_t)_0$ and $\mathbb{R}^d$. This being said, we fix γ as in Lemma 5.3.7, apply Lemma 1.1.16 (iii) with $r = \gamma r_0$ and then Hölder's inequality to get

$$\frac{|f_t(x) - f_t(y)|}{|x-y|^\mu} \lesssim [f_t]_{\mathcal{L}^{d+2\mu}_{x,\gamma r_0}} + [f_t]_{\mathcal{L}^{d+2\mu}_{y,\gamma r_0}} + \|f_t\|_2.$$

Now, we invoke Lemma 5.3.7 joint with the estimates $\|f_t\|_2 \leq \|f\|_2$ and $t^{\mu+d/2} \lesssim \mathrm{e}^{\omega t^2}$ in order to find

$$\lesssim \mathrm{e}^{\omega t^2} \, t^{-\mu-\frac{d}{2}} \|f\|_2. \qquad \square$$

We come to the main result, Theorem 5.3.1. For this we need a criterion to decide when a linear operator is given by a measurable kernel, known as the Dunford–Pettis theorem.

Theorem 5.3.9 ([6, Thm. 1.3])**.** *Let $p \in [1, \infty)$. The map*

$$\mathrm{L}^\infty(O; \mathrm{L}^{p'}(O)^m)^m \to \mathcal{L}(\mathrm{L}^p(O)^m, \mathrm{L}^\infty(O)^m), \quad K \mapsto \left(f \mapsto \int_O K(\cdot, y) f(y) \, \mathrm{d}y \right)$$

is an isometric isomorphism.

Armed with this result, we are going to use the $\mathrm{L}^2 - \dot{\mathrm{C}}^\mu$-estimate as follows:

Lemma 5.3.10. *Consider the following two statements:*

(i) *There are $p \in (p_-(L) \wedge p_-(L^*), 2]$, $\mu \in (0, 1]$ and $\omega > 0$ such that $(\mathrm{e}^{-t^2(L+\omega)})_{t>0}$ and $(\mathrm{e}^{-t^2(L^*+\omega)})_{t>0}$ are $\mathrm{L}^p - \dot{\mathrm{C}}^\mu$-bounded.*

(ii) *L has property $\mathrm{G}(\mu)$.*

Then (i) *implies* (ii) *and conversely,* (ii) *implies* (i) *for every* $p \in [1,2]$ *and any* $\nu \in (0,\mu)$ *in place of* μ*. Furthermore, both statements imply that* $(\mathrm{e}^{-t^2(L+\omega)})_{t>0}$ *and* $(\mathrm{e}^{-t^2(L+\omega)})_{t>0}$ *satisfy* $\mathrm{L}^p - \mathrm{L}^q$ *off-diagonal estimates for some* $\omega > 0$ *and all* $1 \le p \le q \le \infty$*.*

The result is not particularly deep, but we believe that the precise formulation and the flexibility coming from the exponent p will also be useful for other applications.

Proof. (i) $\Longrightarrow$ (ii): We set $\mathcal{T}_\omega = (\mathrm{e}^{-t^2(L+\omega)})_{t>0}$ and $\mathcal{T} = \mathcal{T}_0$. Let us prove in a first step that $\mathcal{T}_\omega$ and $\mathcal{T}_\omega^*$ satisfy $\mathrm{L}^p - \mathrm{L}^\infty$ off-diagonal estimates for a different choice of ω. Since L and L^* are of the same type it suffices to argue for L. We let $E, F \subseteq O$ be measurable, $x \in F$, $t > 0$ and $f \in (\mathrm{L}^p \cap \mathrm{L}^2)(E)^m$. Let us also fix $c' > 0$ to be chosen. We set

$$r := \gamma r_0 t\, \mathrm{e}^{-t^2}\, \mathrm{e}^{-c'\left(\frac{\mathrm{d}(E,F)}{t}\right)^2} \le \gamma r_0.$$

Corollary 5.2.10 joint with Hölder's inequality implies that

$$|f_t(x)| \lesssim r^\mu [f_t]_O^{(\mu)} + r^{-\frac{d}{p}} \|f_t\|_{\mathrm{L}^p(O(x,r))}.$$

Next, we use the $\mathrm{L}^p - \dot{\mathrm{C}}^\mu$-estimate for $\mathcal{T}_\omega$ and the definition of r to find

$$\begin{aligned}
&\lesssim \mathrm{e}^{\omega t^2} (r/t)^\mu t^{-\frac{d}{p}} \|f\|_{\mathrm{L}^p(E)} + r^{-\frac{d}{p}} \|f_t\|_{\mathrm{L}^p(O(x,r))} \\
&\lesssim \mathrm{e}^{\omega t^2}\, t^{-\frac{d}{p}}\, \mathrm{e}^{-\mu c'\left(\frac{\mathrm{d}(E,F)}{t}\right)^2} \|f\|_{\mathrm{L}^p(E)} + \mathrm{e}^{\frac{d}{p}t^2}\, t^{-\frac{d}{p}}\, \mathrm{e}^{\frac{d}{p}c'\left(\frac{\mathrm{d}(E,F)}{t}\right)^2} \|f_t\|_{\mathrm{L}^p(O(x,r))}.
\end{aligned}$$

Since $p \in (p_-(L), 2]$, Lemma 1.4.5 implies that $\mathcal{T}$ satisfies L^p off-diagonal estimates. It follows that there is $c'' > 0$ such that

$$\lesssim \mathrm{e}^{(\omega \vee \frac{d}{p})t^2}\, t^{-\frac{d}{p}} \Big(\mathrm{e}^{-\mu c'\left(\frac{\mathrm{d}(E,F)}{t}\right)^2} + \mathrm{e}^{\frac{d}{p}c'\left(\frac{\mathrm{d}(E,F)}{t}\right)^2 - c''\left(\frac{\mathrm{d}(E,O(x,r))}{t}\right)^2} \Big) \|f\|_{\mathrm{L}^p(E)}.$$

In particular, for $\mathrm{d}(E,F) \le 2t$ the claim follows. Now, we let $\mathrm{d}(E,F) > 2t \ge 2r$. Then $\mathrm{d}(E, O(x,r)) \ge \mathrm{d}(E,F)/2$ and therefore

$$\|f_t\|_{\mathrm{L}^\infty(F)} \lesssim \mathrm{e}^{(\omega \vee \frac{d}{p})t^2}\, t^{-\frac{d}{p}}\, \mathrm{e}^{-c\left(\frac{\mathrm{d}(E,F)}{t}\right)^2} \|f\|_{\mathrm{L}^p(E)}$$

for some $c > 0$ provided that $c' < pc''/4d$.

With this result at our disposal, Lemma 1.4.5 implies that $\mathcal{T}_\omega$ and $\mathcal{T}_\omega^*$ satisfy $\mathrm{L}^2 - \mathrm{L}^\infty$ off-diagonal estimates and are $\mathrm{L}^2 - \dot{\mathrm{C}}^\mu$-bounded. By duality and composition, see Lemma 1.4.4, it follows that $\mathcal{T}_{2\omega}$ and $\mathcal{T}_{2\omega}^*$ satisfy $\mathrm{L}^1 - \mathrm{L}^\infty$ off-diagonal estimates and are $\mathrm{L}^1 - \dot{\mathrm{C}}^\mu$-bounded. Hence, Theorem 5.3.9 and

Remark 5.2.2 (i) imply that e^{-t^2L} can be represented by integration against a μ-Hölder continuous function K_t and there are $C, c > 0$ such that

$$|K_t(x,y)| \le C\,\mathrm{e}^{2\omega t^2}\,t^{-d}\,\mathrm{e}^{-c\left(\frac{|x-y|}{t}\right)^2},$$
$$|K_t(x,y) - K_t(x',y')| \le C\,\mathrm{e}^{2\omega t^2}\,t^{-\mu-d}(|x-x'| + |y-y'|)^\mu,$$

for all $x, x', y, y' \in O$ and $t > 0$.

(ii) $\Longrightarrow$ (i): By Remark 5.2.2 (i) it suffices to prove that $\mathcal{T}_\omega$ is $\mathrm{L}^p - \dot{\mathrm{C}}^\mu$-bounded. We let $x \in O$, $h \in \mathbb{R}^d \setminus \{0\}$ with $x + h \in O$, $t > 0$ and $f \in (\mathrm{L}^p \cap \mathrm{L}^2)(O)^m$. Hölder's inequality gives

$$\begin{aligned}|f_t(x+h) - f_t(x)| &\le \int_O |K_t(x+h,y) - K_t(x,y)||f(y)|\,\mathrm{d}y \\ &\le \left(\int_O |K_t(x+h,y) - K_t(x,y)|^{p'}\,\mathrm{d}y\right)^{\frac{1}{p'}} \|f\|_p.\end{aligned}$$

Let us fix $\nu \in (0,\mu)$. By Remark 5.2.2 (ii) we can continue with

$$\lesssim \mathrm{e}^{\omega t^2}\,t^{-\nu-d}|h|^\nu \cdot \mathrm{I} \cdot \|f\|_p,$$

where

$$\mathrm{I} := \left(\int_{\mathbb{R}^d} (\mathbf{1}_{[0,2|h|)}(|x-y|) + \mathbf{1}_{[2|h|,\infty)}(|x-y|)\,\mathrm{e}^{-cp'\left(\frac{|x-y|}{t}\right)^2})\,\mathrm{d}y\right)^{\frac{1}{p'}}.$$

We calculate I using polar coordinates and get

$$\begin{aligned}\mathrm{I} &\simeq \left(\int_0^{2|h|} r^d\,\frac{\mathrm{d}r}{r} + \int_{2|h|}^\infty \mathrm{e}^{-cp'\left(\frac{r}{t}\right)^2} r^d\,\frac{\mathrm{d}r}{r}\right)^{\frac{1}{p'}} \\ &\lesssim |h|^{\frac{d}{p'}} + t^{\frac{d}{p'}}.\end{aligned}$$

In total, we infer

$$|f_t(x+h) - f_t(x)| \lesssim \mathrm{e}^{\omega t^2}\,t^{-\nu-\frac{d}{p}}|h|^\nu\Big(1 + (|h|/t)^{\frac{d}{p'}}\Big)\|f\|_p.$$

Hence, if $|h| \le t$, then we are done. For $|h| > t$, we invoke the triangle inequality, (G1) and Hölder's inequality to deduce

$$\begin{aligned}|f_t(x+h) - f_t(x)| &\le |f_t(x+h)| + |f_t(x)| \\ &\le 2C\,\mathrm{e}^{\omega t^2}\,t^{-d}\left(\int_{\mathbb{R}^d} \mathrm{e}^{-cp'\left(\frac{|x-y|}{t}\right)^2}\,\mathrm{d}y\right)^{\frac{1}{p'}} \|f\|_p \\ &\lesssim \mathrm{e}^{\omega t^2}\,t^{-\frac{d}{p}}\|f\|_p \\ &\le \mathrm{e}^{\omega t^2}\,t^{-\nu-\frac{d}{p}}|h|^\nu\|f\|_p.\end{aligned}$$

Let us come to the proof of the last assertion. We have already seen in the proof from (i) to (ii) that (i) yields $\mathrm{L}^1 - \mathrm{L}^\infty$ off-diagonal estimates for $\mathcal{T}_\omega$ and $\mathcal{T}_\omega^*$. Thus, the claim follows from (the proof of) Lemma 1.4.5. □

Proof of Theorem 5.3.1. We apply Proposition 5.3.8 to L and L^* and eventually use Lemma 5.3.10 with $p = 2$. □

5.4 From (G) to (H)

We highlight that this implication does not depend on the geometry at all, which is a fundamental difference compared to the other parts of the equivalence. Heuristically, this is due to the fact that (G) is a global property and (D) and (H) are local ones.

Theorem 5.4.1. *Assume that L has property $\mathrm{G}(\mu_0)$ and let $\mu \in (0, \mu_0)$. Then L and L^* have property $\mathrm{H}(\mu)$.*

The idea of the proof dates back to [29]. However, using the equivalent formulation of $\mathrm{G}(\mu)$ from Lemma 5.3.10, we provide a shorter argument even when $O = \mathbb{R}^d$. Before we start with the proof, let us recall Caccioppoli's inequality for mixed boundary conditions. The proof is identical to the standard argument in e.g. [20, Lem. 16.6] since the test function classes $V_{x,r}$ are invariant under multiplication with functions in $\mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^d)$.

Lemma 5.4.2 (Caccioppoli inequality)**.** *There is $C > 0$ depending only on ellipticity such that for all $x \in \overline{O}$, $r > 0$, $c \in (0,1)$ and $u \in V \cap \mathcal{H}_{L,x,r}$ it holds*

$$(1-c)r\|\nabla u\|_{\mathrm{L}^2(O(x,cr))} \leq C\|u - \mathbf{1}_{[\mathrm{d}_D(x)>r]} \cdot (u)_{O(x,r)}\|_{\mathrm{L}^2(O(x,r))}.$$

Proof of Theorem 5.4.1. We are going to use $\mathrm{G}(\mu_0)$ through Lemma 5.3.10. Since $\mathrm{G}(\mu_0)$ is stable under taking adjoints, it suffices to show $\mathrm{H}(\mu)$ for L.

Let us fix $x \in \overline{O}$, $r \in (0,1]$ and $u \in V \cap \mathcal{H}_{L,x,r}$. We let $y \in O(x, r/2)$ and $h \in \mathbb{R}^d$ be such that $y + h \in O(x, r/2)$. Finally, we pick $\varepsilon > 0$ with $B(y,\varepsilon) \cup B(y+h,\varepsilon) \subseteq O(x, r/2)$ and define $\tau_h f \coloneqq f(\cdot + h)$. We abbreviate $B \coloneqq B(x,r)$.

First, we claim that there are $C, c > 0$ depending on the constants in $\mathrm{G}(\mu_0)$, ellipticity, μ and μ_0 such that

$$\begin{aligned}\|\mathbf{1}_{(\frac{7}{8}B)^c}\,\mathrm{e}^{-t^2L^*}(\tau_{-h}-1)f\|_2 + \|\mathbf{1}_{(\frac{7}{8}B)^c}t\nabla\,\mathrm{e}^{-t^2L^*}(\tau_{-h}-1)f\|_2 \\ \leq C|h|^\mu t^{-\mu-\frac{d}{2}}\,\mathrm{e}^{-c\left(\frac{r}{t}\right)^2}\|f\|_{\mathrm{L}^1(B(y,\varepsilon))}\end{aligned} \tag{5.9}$$

for all $t \in (0,1]$ and $f \in (\mathrm{L}^1 \cap \mathrm{L}^2)(B(y,\varepsilon))^m$.

To see the claim, we fix $\nu \in (\mu, \mu_0)$. Property $\mathrm{G}(\mu_0)$ implies that $(\mathrm{e}^{-t^2L})_{t\in(0,1]}$ is $\mathrm{L}^2 - \dot{\mathrm{C}}^\nu$-bounded, so that

$$\|(\tau_h - 1)\,\mathrm{e}^{-t^2L} f\|_{\mathrm{L}^\infty(B(y,\varepsilon))} \lesssim |h|^\nu t^{-\nu-\frac{d}{2}} \|f\|_2 \qquad (t \in (0,1], f \in \mathrm{L}^2(O)^m).$$

By duality, we get

$$\|\,\mathrm{e}^{-t^2L^*}(\tau_{-h} - 1)f\|_2 \lesssim |h|^\nu t^{-\nu-\frac{d}{2}} \|f\|_{\mathrm{L}^1(B(y,\varepsilon))} \tag{5.10}$$

for all $t \in (0,1]$ and $f \in (\mathrm{L}^1 \cap \mathrm{L}^2)(B(y,\varepsilon))^m$. By the semigroup law and the L^2-boundedness of $(t\nabla\,\mathrm{e}^{-t^2L^*})_{t>0}$ the same estimate holds true when we replace $\mathrm{e}^{-t^2L^*}$ by $t\nabla\,\mathrm{e}^{-t^2L^*}$ on the left-hand side.

Now, (5.9) follows from interpolation between (5.10) and the $\mathrm{L}^1 - \mathrm{L}^2$ off-diagonal estimates of $(\mathrm{e}^{-t^2L^*})_{t>0}$ and $(t\nabla\,\mathrm{e}^{-t^2L^*})_{t>0}$ with interpolation parameter $\theta = \mu/\nu$.

With the claim at hand, we prove $\mathrm{H}(\mu)$. By duality, it suffices to show for all $\varphi \in \mathrm{C}_\mathrm{c}^\infty(B(y,\varepsilon))^m$ normalized to $\|\varphi\|_1 = 1$ that

$$|((\tau_h - 1)u \,|\, \varphi)| = |(u \,|\, (\tau_{-h} - 1)\varphi)| \lesssim r^{-\mu-\frac{d}{2}} \|u\|_{\mathrm{L}^2(O(x,r))} |h|^\mu,$$

with an implicit constant not depending on ε.

We normalize $\|u\|_{\mathrm{L}^2(O(x,r))} = 1$, abbreviate $\psi := (\tau_{-h} - 1)\varphi$ and pick $\chi \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^d)$ with $\mathbf{1}_{\frac{7}{8}B} \le \chi \le \mathbf{1}_{\frac{8}{9}B}$ and $\|\nabla\chi\|_\infty \lesssim r^{-1}$. The fundamental theorem of calculus and Hölder's inequality deliver

$$\begin{aligned} |(u \,|\, \psi)| &= \Big|(\chi u \,|\, \psi_r^*) + 2\int_0^r (\chi u \,|\, L^*\psi_t^*) t \,\mathrm{d}t\Big| \\ &\le \|\psi_r^*\|_2 + 2\int_0^r |(\chi u \,|\, L^*\psi_t^*)|\, t \,\mathrm{d}t. \end{aligned}$$

Now, we invoke (5.10) to continue with

$$\begin{aligned} &\lesssim r^{-\mu-\frac{d}{2}}|h|^\mu + \int_0^r |(\chi u \,|\, L^*\psi_t^*)|\, t \,\mathrm{d}t \\ &= r^{-\mu-\frac{d}{2}}|h|^\mu + \int_0^r |(A\nabla(\chi u) \,|\, \nabla\psi_t^*)|\, t \,\mathrm{d}t \\ &=: r^{-\mu-\frac{d}{2}}|h|^\mu + (\mathrm{I}). \end{aligned}$$

Next, Lemma 5.2.4 (ii) implies that $\chi\psi_t^* \in V_{x,r}$. Hence, as $u \in \mathcal{H}_{L,x,r}$, we get

$$\begin{aligned} (A\nabla(\chi u) \,|\, \nabla\psi_t^*) &= (A(\nabla\chi \otimes u) \,|\, \nabla\psi_t^*) + (A\nabla u \,|\, \chi\nabla\psi_t^*) \\ &= (A(\nabla\chi \otimes u) \,|\, \nabla\psi_t^*) - (A\nabla u \,|\, \nabla\chi \otimes \psi_t^*). \end{aligned}$$

Thus, using the properties of χ and Hölder's inequality, we obtain

$$\begin{aligned}(\mathrm{I}) &\leq \int_0^r |(A(\nabla\chi\otimes u)\,|\,t\nabla\psi_t^*)|\,\mathrm{d}t + \int_0^r |(A\nabla u\,|\,\nabla\chi\otimes\psi_t^*)|\,t\,\mathrm{d}t\\ &\lesssim r^{-1}\int_0^r \|\mathbf{1}_{(\frac{7}{8}B)^c}t\nabla\psi_t^*\|_2\,\mathrm{d}t + r^{-1}\|\nabla u\|_{\mathrm{L}^2(O(x,\frac{8}{9}r))}\int_0^r \|\mathbf{1}_{(\frac{7}{8}B)^c}\psi_t^*\|_2 t\,\mathrm{d}t.\end{aligned}$$

At this point we invoke (5.9) joint with Caccioppoli's inequality to get

$$\begin{aligned}&\lesssim r^{-1}|h|^\mu\int_0^r \mathrm{e}^{-c\left(\frac{r}{t}\right)^2} t^{-\mu-\frac{d}{2}}\,\mathrm{d}t + r^{-2}|h|^\mu\int_0^r \mathrm{e}^{-c\left(\frac{r}{t}\right)^2} t^{-\mu-\frac{d}{2}+1}\,\mathrm{d}t\\ &= r^{-1}|h|^\mu\int_0^r \mathrm{e}^{-c\left(\frac{r}{t}\right)^2} t^{-\mu-\frac{d}{2}}(1+{}^t\!/_r)\,\mathrm{d}t\\ &= r^{-\mu-\frac{d}{2}}|h|^\mu\int_1^\infty \mathrm{e}^{-ct^2}\,t^{\mu+\frac{d}{2}-2}(1+t^{-1})\,\mathrm{d}t\\ &\simeq r^{-\mu-\frac{d}{2}}|h|^\mu.\end{aligned}$$

□

5.5 From (H) to (D)

In this section, we close the circle of implications by proving

Theorem 5.5.1. *Let* (Fat) *and* (LU) *be satisfied and let* $\mu\in(0,1]$. *If* L *has property* $\mathrm{H}(\mu)$, *then* L *has property* $\mathrm{D}(\mu)$.

Proof. By Remark 5.2.6 it suffices to control the growth of the Dirichlet integral when $x\in\overline{O}$, $0<4c_1r\leq R\leq r_0$ and $u\in V\cap\mathcal{H}_{L,x,R}$.

(1) $\boldsymbol{{}^R\!/_{c_1} < \mathbf{d}_{\boldsymbol{D}}(\boldsymbol{x})}$**.** We fix $\varphi\in\mathrm{C}_D^\infty(\mathbb{R}^d)$ with $\varphi=1$ on $\overline{B(x,{}^R\!/_{c_1})}$. It follows that $v\coloneqq\varphi(u-(u)_{O(x,R/c_1)})\in V\cap\mathcal{H}_{L,x,R/c_1}$ and $\mathrm{H}(\mu)$ implies that v has a continuous representative in $O(x,{}^R\!/_{2c_1})\supseteq O(x,2r)$. We start with the Caccioppoli inequality

$$\begin{aligned}\int_{O(x,r)}|\nabla u|^2 &= \int_{O(x,r)}|\nabla v|^2\\ &\lesssim r^{-2}\int_{O(x,2r)}|v-v(x)|^2.\end{aligned}$$

Now, we can bring H(μ) into play and then apply (P) in order to get

$$\begin{aligned}
&\lesssim r^{-2}R^{-d-2\mu}\left(\int_{O(x,2r)}|x-y|^{2\mu}\,\mathrm{d}y\right)\int_{O(x,R/c_1)}|v|^2\\
&= r^{-2}R^{-d-2\mu}\left(\int_{O(x,2r)}|x-y|^{2\mu}\,\mathrm{d}y\right)\int_{O(x,R/c_1)}|u-(u)_{O(x,R/c_1)}|^2\\
&\lesssim \left(\frac{r}{R}\right)^{d-2+2\mu}\int_{O(x,R)}|\nabla u|^2.
\end{aligned}$$

(2) $\mathbf{d_D(x) \le R/c_1}$. We consider two subcases.

(2.1) $\mathbf{d_D(x) \le 2r}$. We pick $x_D \in D$ with $\mathrm{d}_D(x) = |x - x_D|$. The most important observation is that (the continuous representative of) u vanishes in x_D by Lemma 5.2.8. The Caccioppoli inequality yields

$$\int_{O(x,r)}|\nabla u|^2 \lesssim r^{-2}\int_{O(x,2r)}|u|^2.$$

By property H(μ) on $O(x, R/2c_1) \supseteq O(x, 2r)$ and (P), we obtain

$$\begin{aligned}
&\lesssim r^{-2}\int_{O(x,2r)}|u(y)-u(x)|^2\,\mathrm{d}y + r^{d-2}|u(x)-u(x_D)|^2\\
&\lesssim R^{-d-2\mu}(r^{d-2+2\mu}+r^{d-2}\mathrm{d}_D(x)^{2\mu})\int_{O(x,R/c_1)}|u|^2\\
&\lesssim \left(\frac{r}{R}\right)^{d-2+2\mu}\int_{O(x,R)}|\nabla u|^2.
\end{aligned}$$

(2.2) $\mathbf{2r < d_D(x)}$. In this case we can replace u by $u - u(x)$ when we apply the Caccioppoli inequality in case (2.1). Then x_D is not needed and we conclude by the same chain of estimates. □

Remark 5.5.2. An inspection of the proof above reveals that the result is even local: if L has property H(μ) in $x \in \overline{O}$, then L has property D(μ) in x.

Let us complete the first part of this chapter with a bonus.

Theorem 5.5.3. *Let* (Fat) *and* (LU) *be satisfied and let* $\mu \in (0, 1]$. *If* L *has property* D(μ), *then* L *has property* H(μ).

Proof. Let us fix $x \in \overline{O}$, $r \in (0, r_0]$ and $u \in V \cap \mathcal{H}_{L,x,r}$. Since $u \in V \cap \mathcal{H}_{L,y,r/4}$ for all $y \in O(x, r/2)$, property D(μ) implies

$$\sup_{\rho \le r/4}\rho^{-\frac{d+2\mu}{2}}\rho\|\nabla u\|_{\mathrm{L}^2(O(y,\rho))} \lesssim r^{-\mu-\frac{d}{2}}r\|\nabla u\|_{\mathrm{L}^2(O(y,r/4))}.$$

Applying Caccioppoli's inequality on the right and (P) on the left delivers

$$[u]_{\mathcal{L}^{d+2\mu}_{y,r/4c_1}} = \sup_{\rho\le r/4c_1} \rho^{-\frac{d+2\mu}{2}} \|u-(u)_{O(y,\rho)}\|_{\mathrm{L}^2(O(y,\rho))} \lesssim r^{-\mu-\frac{d}{2}} \|u\|_{\mathrm{L}^2(O(y,r/2))} \le r^{-\mu-\frac{d}{2}} \|u\|_{\mathrm{L}^2(O(x,r))}.$$

Hence, Lemma 1.1.16 (iii) reveals that u has a μ-Hölder continuous representative in $O(x, r/16c_1)$ with estimate

$$r^{\mu}[u]^{(\mu)}_{O(x,r/16c_1)} \lesssim r^{-\frac{d}{2}} \|u\|_{\mathrm{L}^2(O(x,r))}.$$

Remark 5.2.11 implies that L has property $\mathrm{H}(\mu)$. □

5.6 Property (H) for divergence form operators with real coefficients

Our goal in this section is to show Theorem 5.0.2. As mentioned in the introduction, we can go one step further and relax (Fat) and (LU) to the following axiomatic framework, where $p \in (1,2)$.

(E) **Embedding property for V.** There are $\theta \in [0,1)$ and $C_E > 0$ such that with $1/q := 1/2 - (1-\theta)/d$ it holds

$$\|u\|_q \le C_E \|u\|_{1,2}^{1-\theta} \|u\|_2^{\theta} \qquad (u \in V). \tag{5.11}$$

$(\mathrm{P})_p$ **Weak p-Poincaré inequality.** There are $c_0, r_0 > 0$ and $c_1 \ge 1$ with

$$\|u - \mathbf{1}_{[\mathrm{d}_D(x)>r]} \cdot (u)_{O(x,r)}\|_{\mathrm{L}^p(O(x,r))} \le c_0 r \|\nabla u\|_{\mathrm{L}^p(O(x,c_1 r))}$$

for all $x \in \overline{O}$, $r \in (0, r_0]$ and $u \in \mathrm{W}^{1,p}_D(O)$.

The implicit constants in $(\mathrm{P})_p$ might be different from the ones chosen in and after Proposition 5.1.9 but they serve the exact same purpose. It is only that in this section we postulate $(\mathrm{P})_p$ instead of deriving it from geometric assumptions.

Let us explain why these properties follow from our concrete geometric assumptions in the previous sections.

Lemma 5.6.1. *Assumption* (LU) *implies* (E) *and assumptions* (Fat) *and* (LU) *imply* $(\mathrm{P})_p$ *for some* $p \in (1,2)$. *In particular,* (E) *holds true in the pure Dirichlet case on any open set* O.

Proof. Let us first show that (LU) implies (E). For $d \geq 3$, (E) with $\theta = 0$ follows from $(\mathcal{E})$ and the embedding $\mathrm{W}^{1,2} \subseteq \mathrm{L}^{2^*}$. By [138, Lec. 2, Thm.] we can use for $d = 2$ all $\theta \in (0,1)$, since we have the interpolation inequality

$$\|u\|_{\mathrm{L}^q} \lesssim \|\nabla u\|_{\mathrm{L}^2}^{1-\frac{2}{q}} \|u\|_{\mathrm{L}^2}^{\frac{2}{q}} \qquad (u \in \mathrm{W}^{1,2}).$$

To show $(\mathrm{P})_p$, we borrow a deep result from capacity theory:

> Let $d \geq 2$ and F be closed. If F is locally 2-fat, then F is locally p-fat for some $p \in (1,2)$.

This is called 'self-improvement of p-fatness', a phenomenon that is attributed to Lewis [128, Thm. 1] but for a slightly different version of capacities. With our definition, the proof can be found in [134, Thm. 8.2]. At least when $d \geq 3$, one can rely directly on Lewis' result because the two versions of local 2-fatness coincide, see Subsection 1.2.2.

We apply this result to the auxiliary set $D \cup (O^c \setminus N_\delta^\Sigma)$ in Lemma 5.1.2 to see that (Fat) self-improves to $(\mathrm{Fat})_p$ for some $p \in (1,2)$. Hence, $(\mathrm{P})_p$ follows from Proposition 5.1.9. □

To prove Theorem 5.0.2, we show in a first step that any L-harmonic function u is locally bounded. For this we adapt the proof of [97, Prop. 8.15] to derive a decay condition on the super-level sets of u. Here, we only need the embedding (E).
In a second step, we follow a classical approach [67, 97] and use the local boundedness of u to obtain estimates for its oscillation, which eventually results in local Hölder continuity. At this point, $(\mathrm{P})_p$ is crucial, as it allows us to get a *quantitative* decay in the super-level sets of u. This uses ideas from non-linear methods [72, 73].

5.6.1 Local boundedness of functions in the De Giorgi class

The central point of this subsection is that the local boundedness of a function is a consequence of lying in a function class rather than solving an equation. However, the latter is needed to obtain uniform control on the implicit constants, which is essential for Theorem 5.0.2.

Definition 5.6.2. Let $x \in \overline{O}$ and $R > 0$. We define $\mathrm{DG}_{D,x,R}(O)$ as the set of all $u \in \mathrm{W}_D^{1,2}(O;\mathbb{R})$ for which there is $C > 0$ such that for all $r \in (0,R)$ and $k \in [0,\infty)$ it holds

$$\int_{O(x,r)} |\nabla (u \mp k)^\pm|^2 \leq \frac{C}{(R-r)^2} \int_{O(x,R)} |(u \mp k)^\pm|^2. \tag{5.12}$$

In addition, if $R < \mathrm{d}_D(x)$, then we require both estimates for all $k \in \mathbb{R}$.

Remark 5.6.3. Let us mention that $u \in \mathrm{DG}_{D,x,R}(O)$ if and only if $-u \in \mathrm{DG}_{D,x,R}(O)$ due to $(-u-k)^+ = (u+k)^-$ and $(-u+k)^- = (u-k)^+$.

To state the main result of this section, we define

$$\delta \coloneqq \frac{d}{1-\theta} \quad \& \quad \beta \coloneqq \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2}{\delta}} > 1, \tag{5.13}$$

where $\theta \in [0,1)$ is chosen as in (5.11).

Theorem 5.6.4. *Assume* (E). *Let $x \in \overline{O}$, $R_0 > 0$ and $u \in \mathrm{DG}_{D,x,R_0}(O)$. There is $C > 0$ such that we have for all $k \geq 0$ and $R \in (0, R_0]$ the estimate*

$$\operatorname*{esssup}_{O(x,\frac{R}{2})} u^+ \leq k + C \left(R^{-d} \int_{O(x,R)} |(u-k)^+|^2 \right)^{\frac{1}{2}} \left(R^{-d} |\{u > k\}(x,R)| \right)^{\frac{\beta-1}{2}}.$$

In addition, if $R < \mathrm{d}_D(x)$, then we can allow for all $k \in \mathbb{R}$ in the estimate. Furthermore, if A is real-valued and $u \in \mathrm{W}^{1,2}_D(O;\mathbb{R}) \cap \mathcal{H}_{L,x,R}$, then C depends only on d, ellipticity, R_0 and (E).

Before we come to the proof of Theorem 5.6.4, let us show that $\mathrm{DG}_{D,x,R}(O)$ is the natural energy class associated to the equation $L_D u = 0$ in $O(x,R)$.

Lemma 5.6.5. *Let A be real-valued, $x \in \overline{O}$, $R > 0$ and $u \in \mathrm{W}^{1,2}_D(O;\mathbb{R}) \cap \mathcal{H}_{L,x,R}$. Then $u \in \mathrm{DG}_{D,x,R}(O)$ with an implicit constant depending only on ellipticity.*

Proof. Owing to Remark 5.6.3 we only have to show the estimate for $(u-k)^+$. First, we assume that $\mathrm{d}_D(x) \leq R$ and $k \geq 0$.

We let $r \in (0,R)$ and $\varphi \in \mathrm{C}^\infty_{\mathrm{c}}(B(x,R))$ be $[0,1]$-valued with $\varphi = 1$ on $B(x,r)$ and $\|\nabla\varphi\|_{\mathrm{L}^\infty} \leq 2/(R-r)$.

We put $w \coloneqq (u-k)^+\varphi$ and $v \coloneqq (u-k)^+\varphi^2$. They are contained in V and test functions for the equation for u, see Lemma 1.1.11 and Lemma 5.2.4 (ii).

By the product rule and ellipticity, we have

$$\begin{aligned}\frac{\lambda}{2}\int_O |\nabla(u-k)^+|^2\varphi^2 &\le \lambda\int_O |\nabla w|^2 + |(u-k)^+\nabla\varphi|^2\\ &\le \int_O A\nabla w\cdot\nabla w + \lambda|(u-k)^+\nabla\varphi|^2\\ &= \int_O A\nabla u\cdot\varphi\nabla w + A(u-k)^+\nabla\varphi\cdot\nabla w\\ &\quad + \lambda|(u-k)^+\nabla\varphi|^2\\ &= \int_O -A\nabla u\cdot w\nabla\varphi + A(u-k)^+\nabla\varphi\cdot\nabla w\\ &\quad + \lambda|(u-k)^+\nabla\varphi|^2,\end{aligned}$$

where we have used the equation for u with test function v. By definition of w we can continue by

$$\begin{aligned}&\le \int_O -A\varphi\nabla(u-k)^+\cdot(u-k)^+\nabla\varphi\\ &\quad + A(u-k)^+\nabla\varphi\cdot\varphi\nabla(u-k)^+\\ &\quad + A(u-k)^+\nabla\varphi\cdot(u-k)^+\nabla\varphi\\ &\quad + \lambda|(u-k)^+\nabla\varphi|^2.\end{aligned}$$

At this point, we can use the boundedness of A and Young's inequality to absorb all terms with $\varphi\nabla(u-k)^+$ on the right. We are left with

$$\int_O |\nabla(u-k)^+|^2\varphi^2 \le C\int_O |(u-k)^+\nabla\varphi|^2,$$

where $C>0$ depends on ellipticity, and (5.12) follows by the choice of φ.

Finally, if $R < \mathrm{d}_D(x)$, then $v, w \in V$ for all $k\in\mathbb{R}$ and the same argument applies. □

Proof of Theorem 5.6.4. We begin with the case $\mathrm{d}_D(x)\le R$.

We fix $r\in(0,R)$, $k\ge 0$ and let $\eta\in \mathrm{C}_{\mathrm{c}}^\infty(B(x,{}^{(r+R)}/_2))$ be $[0,1]$-valued with $\eta=1$ on $B(x,r)$ and $\|\nabla\eta\|_{\mathrm{L}^\infty}\le {}^4/_{(R-r)}$. Then $\eta(u-k)^+\in V$ by Lemmas 1.1.11 and 5.2.4 (ii). We deduce from (5.12) that

$$\begin{aligned}\|\nabla(\eta(u-k)^+)\|_{\mathrm{L}^2(O(x,\frac{r+R}{2}))} &\le \|(u-k)^+\nabla\eta\|_{\mathrm{L}^2(O(x,\frac{r+R}{2}))}\\ &\quad + \|\eta\nabla(u-k)^+\|_{\mathrm{L}^2(O(x,\frac{r+R}{2}))}\\ &\lesssim \frac{1}{R-r}\|(u-k)^+\|_{\mathrm{L}^2(O(x,R))}.\end{aligned}\tag{5.14}$$

By Lemma 5.6.5 the implicit constant depends only on ellipticity in the case that $L_D u = 0$ in $O(x, R)$. We abbreviate

$$A_{k,R} \coloneqq \{u > k\}(x, R).$$

Using (E) and (5.14), we get

$$\begin{aligned}\|u-k\|_{\mathrm{L}^q(A_{k,r})} &\leq \|\eta(u-k)^+\|_q \\ &\lesssim \|\eta(u-k)^+\|^{1-\theta}_{\mathrm{W}^{1,2}(O(x,\frac{r+R}{2}))}\|\eta(u-k)^+\|^{\theta}_{\mathrm{L}^2(O(x,\frac{r+R}{2}))} \\ &\lesssim \left(\frac{1+R_0}{R-r}\right)^{1-\theta}\|u-k\|_{\mathrm{L}^2(A_{k,R})}.\end{aligned}$$

Hölder's inequality yields

$$\begin{aligned}\|u-k\|_{\mathrm{L}^2(A_{k,r})} &\leq |A_{k,r}|^{\frac{1-\theta}{d}}\|u-k\|_{\mathrm{L}^q(A_{k,r})} \\ &\lesssim \frac{|A_{k,R}|^{\frac{1-\theta}{d}}}{(R-r)^{1-\theta}}\|u-k\|_{\mathrm{L}^2(A_{k,R})}.\end{aligned} \tag{5.15}$$

Let us recall the definitions of β and δ from (5.13) and define

$$\Phi(k, r) \coloneqq \|u-k\|^{\delta\beta}_{\mathrm{L}^2(A_{k,r})}|A_{k,r}|.$$

We raise (5.15) to the $\delta\beta$-th power and multiply by $|A_{k,r}|$ to get for all $h < k$ the estimate

$$\begin{aligned}\Phi(k,r) &\lesssim \frac{1}{(R-r)^{d\beta}}|A_{k,R}|^{\beta}|A_{k,r}|\|u-k\|^{\delta\beta}_{\mathrm{L}^2(A_{k,R})} \\ &\lesssim \frac{1}{(R-r)^{d\beta}}|A_{h,R}|^{\beta}|A_{k,r}|\|u-h\|^{\delta\beta}_{\mathrm{L}^2(A_{h,R})} \\ &\leq \frac{1}{(R-r)^{d\beta}}\frac{1}{(k-h)^2}|A_{h,R}|^{\beta}\|u-h\|^{2+\delta\beta}_{\mathrm{L}^2(A_{h,R})}.\end{aligned}$$

Using that β is the positive solution to $\beta^2 - \beta - 2/\delta = 0$, we have proven so far that there is some $c > 0$ such that we have for all $0 < r < R$, $k \geq 0$ and $h < k$ the estimate

$$\Phi(k,r) \leq \frac{c}{(R-r)^{d\beta}}\frac{1}{(k-h)^2}\Phi(h,R)^{\beta}.$$

At this point, we set up an iteration scheme to conclude. To this end, we fix $\zeta > 0$, set $k_n \coloneqq k + \zeta - \zeta/2^n$ and $r_n \coloneqq R/2 + R/2^{n+1}$, so that

$$\Phi(k_{n+1}, r_{n+1}) \leq c2^{d\beta}\frac{2^{(n+1)(d\beta+2)}}{R^{d\beta}\zeta^2}\Phi(k_n, r_n)^{\beta}.$$

We set $\mu := {}^{(d\beta+2)}/_{(\beta-1)} > 0$ and put $\psi_n := 2^{\mu n}\Phi(k_n, r_n)$. Then

$$\psi_{n+1} \leq c\frac{2^{\beta(\mu+d)}}{R^{d\beta}\zeta^2}\psi_n^{\beta}.$$

We choose $\zeta > 0$ such that

$$\psi_0^{1-\beta} = c\frac{2^{\beta(\mu+d)}}{R^{d\beta}\zeta^2},$$

that is,

$$\begin{aligned}\zeta &= c^{\frac{1}{2}}2^{\frac{\beta(\mu+d)}{2}}R^{-\frac{d\beta}{2}}\|u-k\|_{\mathrm{L}^2(A_{k,R})}|A_{k,R}|^{\frac{\beta-1}{2}}\\ &= c'R^{-\frac{d}{2}}\|u-k\|_{\mathrm{L}^2(A_{k,R})}(R^{-d}|A_{k,R}|)^{\frac{\beta-1}{2}}.\end{aligned}$$

Our choice of ζ and induction yields $\psi_n \leq \psi_0$ for all $n \in \mathbb{N}_0$. This eventually implies

$$\Phi(k+\zeta, {}^{R}/_{2}) \leq \limsup_{n\to\infty}\Phi(k_n, r_n) = \limsup_{n\to\infty} 2^{-\mu n}\psi_n = 0.$$

Thus, $\Phi(k+\zeta, {}^{R}/_{2}) = 0$ and hence $|A_{k+\zeta,{}^{R}/_{2}}| = 0$ or $u = k+\zeta$ on $A_{k+\zeta,{}^{R}/_{2}}$. In both cases we conclude that

$$\operatorname*{esssup}_{O(x,\frac{R}{2})} u^+ \leq k+\zeta,$$

as claimed.

The restriction $k \geq 0$ was only used in order to apply (5.12) and to guarantee that $\eta(u-k)^+ \in V$. But when $R < \mathrm{d}_D(x)$, this is true for all $k \in \mathbb{R}$ simply by the support of η and the same argument applies. □

5.6.2 Property (H) for L

So far, we have worked under assumption (E) alone. Now, we will add $(\mathrm{P})_p$ in order to upgrade local boundedness to local Hölder continuity.

Theorem 5.6.6. *Assume* (E) *and* $(\mathrm{P})_p$. *Let* $x \in \overline{O}$, $R \in (0, r_0]$ *and* $u \in \mathrm{DG}_{D,x,R}(O)$. *Then* u *is locally Hölder continuous in* $O(x, {}^{R}/_{4})$ *with*

$$R^{\mu}[u]^{(\mu)}_{O(x,\frac{R}{4})} \lesssim R^{-\frac{d}{2}}\|u\|_{\mathrm{L}^2(O(x,R))}.$$

The implicit constant and μ *depend only on* d, $(\mathrm{P})_p$, (E) *and the constant in* (5.12).

This result implies Theorem 5.0.2.

Proof of Theorem 5.0.2 from Theorem 5.6.6. Let us fix $x \in \overline{O}$, $r \in (0,1]$ and $u \in V \cap \mathcal{H}_{L,x,r}$. By Remark 5.2.11 we can assume $r \leq r_0$ and it suffices to prove (5.3) with $O(x, r/4)$ on the left-hand side.

Since A is real-valued, $\mathrm{Re}(u), \mathrm{Im}(u) \in \mathrm{W}^{1,2}_D(O;\mathbb{R})$ solve the same equation as u. According to Lemma 5.6.5 they belong to $\mathrm{DG}_{D,x,r}(O)$. As the geometric assumptions (Fat) and (LU) imply (E) and $(\mathrm{P})_p$ by Lemma 5.6.1, we can apply Theorem 5.6.6 to $\mathrm{Re}(u)$ and $\mathrm{Im}(u)$ to complete the proof. □

In order to prove Theorem 5.6.6, we still need two short lemmas. The slight asymmetry in the scaling of the radius between (i) and (ii) is unavoidable and the reader might want to think of $c_1 = 1$ on a first reading.

Lemma 5.6.7. *Assume* $(\mathrm{P})_p$, *let* $x \in \overline{O}$, $r \in (0, r_0]$ *and* $u \in \mathrm{W}^{1,2}_D(O;\mathbb{R})$.

(i) *If* $r < \mathrm{d}_D(x)$, *then it holds for all* $h < k$ *that*

$$(k-h)^p |\{u \geq k\}(x, r/c_1)| \leq c_0^p r^p \left(\frac{|O(x, r/c_1)|}{|\{u \leq h\}(x, r/c_1)|} \right)^p \int_{\{h \leq u \leq k\}(x,r)} |\nabla u|^p. \tag{5.16}$$

(ii) *If* $\mathrm{d}_D(x) \leq r$, *then it holds for all* $0 \leq h < k$ *that*

$$(k-h)^p |\{u \geq k\}(x, r)| \leq c_0^p r^p \int_{\{h \leq u \leq k\}(x, c_1 r)} |\nabla u|^p. \tag{5.17}$$

Proof of (i). We pick $\varphi \in \mathrm{C}^\infty_D(\mathbb{R}^d)$ with $\varphi = 1$ on $B(x,r)$, set $w := \varphi((u-h)^+ - (u-k)^+)$ and estimate

$$(w)_{O(x, r/c_1)} \leq \frac{(k-h)|\{u > h\}(x, r/c_1)|}{|O(x, r/c_1)|} \leq k - h.$$

As $w = k - h$ on $\{u \geq k\}(x, r/c_1)$, we can use this bound and $(\mathrm{P})_p$ to get

$$\begin{aligned}
&(k-h)^p \left(1 - \frac{|\{u > h\}(x, r/c_1)|}{|O(x, r/c_1)|}\right)^p |\{u \geq k\}(x, r/c_1)| \\
&= \int_{\{u \geq k\}(x, r/c_1)} \left| k - h - \frac{(k-h)|\{u > h\}(x, r/c_1)|}{|O(x, r/c_1)|} \right|^p \\
&\leq \int_{O(x, r/c_1)} |w - (w)_{O(x, r/c_1)}|^p \\
&\leq c_0^p r^p \int_{O(x,r)} |\nabla w|^p.
\end{aligned}$$

Since $\nabla w = \mathbf{1}_{\{h \le u \le k\}} \nabla u$ in $O(x,r)$, we are done.

Proof of (ii). Lemma 1.1.11 yields $w \coloneqq (u-h)^+ - (u-k)^+ \in V$ and $(\mathrm{P})_p$ implies

$$(k-h)^p |\{u \ge k\}(x,r)| = \int_{\{u \ge k\}(x,r)} |w|^p \le c_0^p r^p \int_{O(x,c_1 r)} |\nabla w|^p,$$

and we conclude as before. □

Definition 5.6.8. Let $u \colon O \to \mathbb{R}$ be a measurable function, $x \in \overline{O}$ and $r > 0$. We define

(i) $M_{x,u}(r) \coloneqq \operatorname*{esssup}_{O(x,r)} u.$

(ii) $m_{x,u}(r) \coloneqq \operatorname*{essinf}_{O(x,r)} u.$

(iii) $\operatorname{osc}_{x,u}(r) \coloneqq M_{x,u}(r) - m_{x,u}(r).$

Here, the third expression is only defined when at least one of the summands is finite or both are infinite with a different sign.

Lemma 5.6.9 (Shrinking Lemma)**.** *Assume* (E) *and* $(\mathrm{P})_p$*. Let* $x \in \overline{O}$*,* $r \in (0, 4c_1 r_0]$*,* $u \in \mathrm{DG}_{D,x,r}(O)$ *with* $\operatorname{osc}_{x,u}(r/2) > 0$ *and define* $\gamma(p) \coloneqq 1 - p/2 \in (0,1)$*.*

(i) *If* $r/4c_1 < \mathrm{d}_D(x)$ *and*

$$\left|\left\{u > M_{x,u}(r/2) - \tfrac{1}{2}\operatorname{osc}_{x,u}(r/2)\right\}(x, r/4c_1^2)\right| \le \tfrac{1}{2}|O(x, r/4c_1^2)|,$$

then there is some $c > 0$ *depending only on* $(\mathrm{P})_p$ *and the implicit constant in* (5.12) *for* u *such that for each* $n \in \mathbb{N}$ *the super-level sets of* u *shrink by the law*

$$\left|\left\{u \ge M_{x,u}(r/2) - 2^{-(n+1)}\operatorname{osc}_{x,u}(r/2)\right\}(x, r/4c_1^2)\right| \le c|O(x, r/4c_1)| \cdot n^{-\gamma(p)}.$$

(ii) *If* $\mathrm{d}_D(x) \le r/4c_1$ *and*

$$M_{x,u}(r/2) - \tfrac{1}{2}\operatorname{osc}_{x,u}(r/2) \ge 0,$$

then there is some $c > 0$ *depending only on* $(\mathrm{P})_p$ *and the implicit constant in* (5.12) *for* u *such that for each* $n \in \mathbb{N}$ *the super-level sets of* u *shrink by the law*

$$\left|\left\{u \ge M_{x,u}(r/2) - 2^{-(n+1)}\operatorname{osc}_{x,u}(r/2)\right\}(x, r/4c_1)\right| \le c|O(x, r/4)| \cdot n^{-\gamma(p)}.$$

Proof. Theorem 5.6.4 implies $\operatorname{osc}_{x,u}(r/2) < \infty$. We define for $i = 0, \dots, n$ the numbers

$$\begin{aligned} k_i &:= k_i(u) := M_{x,u}(r/2) - 2^{-(i+1)}\operatorname{osc}_{x,u}(r/2), \\ A_{i,r} &:= \{u \geq k_i\}(x,r). \end{aligned} \tag{5.18}$$

Proof of (i). Using that $A_{i,r}$ is decreasing in i joint with the assumption

$$|\{u \leq k_0\}(x, r/4c_1^2)| \geq \frac{1}{2}|O(x, r/4c_1^2)|,$$

we derive from (5.16) that

$$|k_{i+1} - k_i|^p |A_{i+1,r/4c_1^2}| \lesssim r^p \int_{\{k_i \leq u \leq k_{i+1}\}(x,r/4c_1)} |\nabla u|^p. \tag{5.19}$$

Now, we use Hölder's inequality and $u \in \mathrm{DG}_{D,x,r}(O)$ to infer

$$\begin{aligned} |k_{i+1} - k_i|^p |A_{i+1,r/4c_1^2}| &\lesssim r^p \left(\int_{O(x,r/4c_1)} |\nabla (u-k_i)^+|^2 \right)^{\frac{p}{2}} |A_{i,r/4c_1} \setminus A_{i+1,r/4c_1}|^{1-\frac{p}{2}} \\ &\lesssim \left(\int_{O(x,r/2)} |(u-k_i)^+|^2 \right)^{\frac{p}{2}} |A_{i,r/4c_1} \setminus A_{i+1,r/4c_1}|^{1-\frac{p}{2}}. \end{aligned}$$

Now, we use that $(u-k_i)^+ \leq 2^{-(i+1)}\operatorname{osc}_{x,u}(r/2) = 2(k_{i+1} - k_i)$ on $O(x, r/2)$ and that $|O(x,r)| \lesssim r_0^d$ to conclude the bound

$$|k_{i+1} - k_i|^p |A_{i+1,r/4c_1^2}| \lesssim |k_{i+1} - k_i|^p |A_{i,r/4c_1} \setminus A_{i+1,r/4c_1}|^{\gamma(p)}.$$

Finally, we cancel the term $|k_{i+1} - k_i|^p$, raise both sides to the $1/\gamma(p)$-th power, sum from $i = 1$ to n and bound the left from below by $n|A_{n+1,r/4c_1^2}|^{1/\gamma(p)}$ in order to get

$$n|A_{n+1,r/4c_1^2}|^{\gamma(p)^{-1}} \lesssim |O(x, r/4c_1)|.$$

Proof of (ii). We perform the same proof as in (i), using (5.17) instead of (5.16) in (5.19). Here, we also use the assumption $k_0(u) \geq 0$. □

Finally, we come to the

Proof of Theorem 5.6.6. We can assume that $\operatorname{osc}_{x,u}(R/2) > 0$, since otherwise u is equal to a constant almost everywhere and there is nothing to prove. For $\operatorname{osc}_{x,u}(R/2)$ we divide the proof into two parts.

(1) $R/4c_1 < \mathrm{d}_D(x)$. We define $k_n = k_n(u)$ as in (5.18) with $r = R$ and write

$$O(x, R/4c_1^2) \supseteq \{u > k_0(u)\}(x, R/4c_1^2) \cup \{-u > k_0(-u)\}(x, R/4c_1^2).$$

At least one of the disjoint sets on the right has measure at most $\frac{1}{2}|O(x, R/4c_1^2)|$, say the first one, because otherwise we work with $-u$. Theorem 5.6.4 implies that

$$M_{x,u}(R/8c_1^2) \leq k_n + c(M_{x,u}(R/2) - k_n) \left(\frac{|\{u \geq k_n\}(x, R/4c_1^2)|}{R^d} \right)^{\frac{\beta-1}{2}},$$

with β as in (5.13). Since $|\{u > k_0(u)\}(x, R/4c_1^2)| \leq \frac{1}{2}|O(x, R/4c_1^2)|$, Lemma 5.6.9 (i) yields that there is some $n \in \mathbb{N}$ depending only on (E), $(\mathrm{P})_p$ and the implicit constant in (5.12) for u such that

$$c \left(\frac{|\{u \geq k_n\}(x, R/4c_1^2)|}{R^d} \right)^{\frac{\beta-1}{2}} < \frac{1}{2}. \tag{5.20}$$

This implies that

$$M_{x,u}(R/8c_1^2) \leq M_{x,u}(R/2) - 2^{-(n+2)} \mathrm{osc}_{x,u}(R/2).$$

We set $\sigma := 1 - 2^{-(n+2)}$, subtract $m_{x,u}(R/8c_1^2)$ and use $m_{x,u}(R/2) \leq m_{x,u}(R/8c_1^2)$ to obtain

$$\mathrm{osc}_{x,u}(R/8c_1^2) \leq \sigma \mathrm{osc}_{x,u}(R/2).$$

Now, we let $0 < r \leq \rho \leq R/2$ and fix $k \in \mathbb{N}$ with $r \in ((4c_1^2)^{-k}\rho, (4c_1^2)^{-k+1}\rho]$. The latter estimate delivers with $\mu := -\ln(\sigma)/\ln(4c_1^2) \in (0,1)$ that

$$\mathrm{osc}_{x,u}(r) \leq \mathrm{osc}_{x,u}((4c_1^2)^{-k+1}\rho) \leq \sigma^{k-1} \mathrm{osc}_{x,u}(\rho) \leq \sigma^{-1} \left(\frac{r}{\rho} \right)^{\mu} \mathrm{osc}_{x,u}(\rho), \tag{5.21}$$

where we have used that $\sigma^k = ((4c_1^2)^{-k})^{\mu} \leq (r/\rho)^{\mu}$.

Finally, we fix $y, z \in O(x, R/4)$. If $|y - z| > R/8$, then Theorem 5.6.4 with $k = 0$ yields

$$\frac{|u(y) - u(z)|}{|y - z|^{\mu}} \leq \frac{2M_{x,|u|}(R/4)}{(R/8)^{\mu}} \lesssim R^{-\mu - \frac{d}{2}} \|u\|_{\mathrm{L}^2(O(x,R))}.$$

If $|y - z| \leq R/8$, then $O(y, R/8) \subseteq O(x, R/2)$ and (5.21) gives

$$\begin{aligned} |u(y) - u(z)| &\leq \mathrm{osc}_{y,u}(|y - z|) \\ &\lesssim \left(\frac{|y - z|}{R/8} \right)^{\mu} \mathrm{osc}_{y,u}(R/8) \\ &\lesssim \left(\frac{|y - z|}{R} \right)^{\mu} 2M_{x,|u|}(R/2). \end{aligned}$$

We conclude by Theorem 5.6.4 as before.

(2) $\mathbf{d_D(x) \leq R/4c_1}$. Since $k_0(-u) = -k_0(u)$ we can assume that $k_0(u) \geq 0$. Now, we argue as before, using Lemma 5.6.9 (ii) instead of (i) in (5.20) and consequently replacing c_1^2 by c_1 everywhere. □

5.7 Property (D) for complex perturbations

Property (D) is stable under small complex perturbations. This observation is due to Auscher when $O = \mathbb{R}^d$ [11, Thm. 4.4] and one of the main reasons to study Gaussian bounds through property (D). Indeed, the former is much harder to perturb.

Theorem 5.7.1. *Let $m \geq 1$, $d \geq 2$, assume* (Fat) *and* (LU) *and let $\mu \in (0, 1]$. Let $A_0 \in \mathrm{L}^\infty(O; \mathcal{L}((\mathbb{C}^m)^d))$ be Gårding elliptic with ellipticity constant $\lambda > 0$ and suppose that $L_0 \coloneqq -\operatorname{div}(A_0\nabla\cdot)$ has property $\mathrm{D}(\mu)$. Then for all $\nu \in (0, \mu)$ there is some $\varepsilon > 0$, depending only on d, λ, $c_{\mathrm{D}(\mu)}$, μ and ν, such that if $\|A - A_0\|_\infty < \varepsilon$, then L has property $\mathrm{D}(\nu)$.*

In view of Theorems 5.0.1 and 5.0.2 we record

Corollary 5.7.2. *Let $m = 1$. Under the geometric assumptions* (Fat) *and* (LU) *there is some $\varepsilon > 0$ depending only on geometry and ellipticity such that if $\|\operatorname{Im}(A)\|_\infty < \varepsilon$, then L has properties* (D), (G) *and* (H).

Proof. We adapt the argument in [11, Thm. 4.4]. To this end, we let $x \in \overline{O}$, $0 < r \leq R/4 \leq r_0/4$, $u \in V \cap \mathcal{H}_{L,x,R}$ and define the function

$$\phi(r) \coloneqq \|\nabla u\|_{\mathrm{L}^2(O(x,r))}.$$

By Lemma 5.2.12 we find $\rho \in [R/4, R]$ and $v \in V_{x,\rho}$ such that

$$L_{0,D}v = -\operatorname{div}(A_0\nabla u) = -\operatorname{div}((A_0 - A)\nabla u) \quad \text{in } O(x, \rho),$$

with estimate

$$\lambda\|\nabla v\|_{\mathrm{L}^2(O(x,\rho))} \leq \|(A - A_0)\nabla u\|_{\mathrm{L}^2(O(x,\rho))} \leq \|A - A_0\|_\infty \phi(R). \tag{5.22}$$

Lemma 5.2.4 (i) implies that $v_0 \in V$. Hence, $w \coloneqq u - v_0 \in V \cap \mathcal{H}_{L_0,x,\rho}$ and

since L_0 has property $\mathrm{D}(\mu)$, we get

$$\begin{aligned}\phi(r) &\leq \|\nabla w\|_{\mathrm{L}^2(O(x,r))} + \|\nabla v\|_{\mathrm{L}^2(O(x,r))} \\ &\leq c_{\mathrm{D}(\mu)}^{1/2} \left(\frac{r}{\rho}\right)^{\frac{d}{2}-1+\mu} \|\nabla w\|_{\mathrm{L}^2(O(x,\rho))} + \|\nabla v\|_{\mathrm{L}^2(O(x,r))} \\ &\leq 2^d c_{\mathrm{D}(\mu)}^{1/2} \left(\frac{r}{R}\right)^{\frac{d}{2}-1+\mu} \phi(R) + (c_{\mathrm{D}(\mu)}^{1/2} + 1)\|\nabla v\|_{\mathrm{L}^2(O(x,\rho))}.\end{aligned}$$

Now, we insert (5.22) in order to find

$$\leq 2^d c_{\mathrm{D}(\mu)}^{1/2} \left(\left(\frac{r}{R}\right)^{\frac{d}{2}-1+\mu} + \frac{\|A - A_0\|_\infty}{2^d \lambda}\left(1 + c_{\mathrm{D}(\mu)}^{-1/2}\right)\right) \phi(R).$$

Finally, Lemma 5.3.4 yields for all $\nu \in (0, \mu)$ some $C', \varepsilon_0 > 0$ depending only on d, $c_{\mathrm{D}(\mu)}$, μ and ν such that if $\|A - A_0\|_\infty < {2^d \lambda \varepsilon_0}/{(1 + c_{\mathrm{D}(\mu)}^{-1/2})}$, then

$$\phi(r) \leq C' \left(\frac{r}{R}\right)^{\frac{d}{2}-1+\nu} \phi(R) \qquad (r \in (0, R]). \qquad \square$$

5.8 Property (G) in dimension $d = 2$

In this section, we prove that in dimension $d = 2$ *every* elliptic system L has property (G). On $O = \mathbb{R}^d$, this is due to [23]. In doing so, we need to assume that D is a $(d-1)$-set. By Lemma 1.3.3 this implies that D is locally 2-fat. Hence, as $D \subseteq O^c$, also (Fat) is satisfied. We need this enhanced version of (Fat) to apply an extrapolation result from [37].

Theorem 5.8.1. *Let $m \geq 1$, $d = 2$, assume* (LU) *and that D is a $(d-1)$-set. Then L has property* (G) *with implicit constants depending on geometry and ellipticity.*

Proof. We are going to use the following two properties of the operator L from [37]. First, the extrapolation result from [37, Prop. 7.1] with $p = 2$ furnishes some $q \in (2, \infty)$ such that the restriction of the Lax–Milgram isomorphism $1 + \mathcal{L}\colon V \to V^*$ to $\mathrm{W}_D^{1,q}(O)^m \cap V$ extends to an isomorphism from $\mathrm{W}_D^{1,q}(O)^m$ to $\mathrm{W}_D^{-1,q}(O)^m$ with inverse that coincides with $(1 + \mathcal{L})^{-1}$ on $\mathrm{W}_D^{-1,q}(O)^m \cap V^*$. Second, [37, Cor. 3.5 & Prop. 3.6] implies the estimate

$$\|t(1 + L)\,\mathrm{e}^{-t(1+L)}\, f\|_{q_*} \lesssim \|f\|_{q_*} \qquad (t > 0, f \in (\mathrm{L}^{q_*} \cap \mathrm{L}^2)(O)^m).$$

Now, we can start the actual proof. We set $\mu \coloneqq 1 - {}^{d}\!/_{q} \in (0,1)$. Due to $(\mathcal{E})$ we have the Sobolev embeddings $\mathrm{W}^{1,q}_D(O)^m \subseteq \mathrm{C}^\mu(O)^m$ and $\mathrm{W}^{1,q'}_D(O)^m \subseteq \mathrm{L}^{(q_*)'}(O)^m$. By duality, the second embedding entails $\mathrm{L}^{q_*}(O)^m \subseteq \mathrm{W}^{-1,q}_D(O)^m$. Using these embeddings and the fact that $(1+\mathcal{L})^{-1}$ maps $\mathrm{W}^{-1,q}_D(O)^m \cap V^*$ boundedly into $\mathrm{W}^{1,q}_D(O)^m$ for the q-norms, we conclude that

$$[(1+\mathcal{L})^{-1} f]^{(\mu)}_O \lesssim \|f\|_{q_*} \qquad (f \in (\mathrm{L}^{q_*} \cap \mathrm{L}^2)(O)^m).$$

We take $t > 0$, $f \in (\mathrm{L}^{q_*} \cap \mathrm{L}^2)(O)^m$ and use that $(1+\mathcal{L})|_{\mathrm{L}^2(O)^m} = 1 + L$ to get

$$\begin{aligned}
[\mathrm{e}^{-t^2 L} f]^{(\mu)}_O &= [(1+\mathcal{L})^{-1}(1+L)\,\mathrm{e}^{t^2}\,\mathrm{e}^{-t^2(1+L)} f]^{(\mu)}_O \\
&\lesssim \mathrm{e}^{t^2}\,\|(1+L)\,\mathrm{e}^{-t^2(1+L)} f\|_{q_*} \\
&\lesssim \mathrm{e}^{t^2}\, t^{-\mu-\frac{d}{q_*}} \|f\|_{q_*}.
\end{aligned}$$

Hence, $(\mathrm{e}^{-t^2(L+1)})_{t>0}$ is $\mathrm{L}^{q_*} - \dot{\mathrm{C}}^\mu$-bounded. Replacing L by L^*, we obtain the same result for $(\mathrm{e}^{-t^2(L^*+1)})_{t>0}$. Since $d = 2$, it follows from [79, Thm. 1.6] that $p_-(L) = p_-(L^*) = 1$. We conclude by Lemma 5.3.10. □

CHAPTER 6

Hardy spaces adapted to elliptic operators on open sets

6.1 Introduction

Let us start with the classical Hardy space $\mathrm{H}^1(\mathbb{R}^d)$. There are many equivalent descriptions for it, including atomic, maximal and square function characterizations. All these perspectives are valuable and prove their merit in various applications. For instance, both the maximal and square function characterizations are essential tools in the treatment of harmonic boundary value problems on the half-space: the maximal characterization allows a better understanding in which sense boundary values can actually be attained, whereas the square function characterization gives rise to solutions of boundary value problems using techniques from evolution equations. The atomic characterization makes cancellation properties of the Hardy space visible and accessible in the treatment of general singular integrals.

The space $\mathrm{H}^1(\mathbb{R}^d)$ can be seen as a space adapted to the negative Laplacian $-\Delta$ on the Euclidean space $\mathbb{R}^d$. Consequently, if one is interested in studying boundary value problems with respect to other elliptic operators than the Laplacian [20], or maybe even boundary value problems in cylindrical domains [19], it seems logical to study Hardy spaces that are adapted to a different elliptic operator than the Laplacian.

Let $d \geq 2$, $O \subseteq \mathbb{R}^d$ be open and $A \in \mathrm{L}^\infty(O; \mathcal{L}(\mathbb{C}^d))$ be Gårding elliptic. We consider the elliptic operator $L = -\operatorname{div}(A\nabla\cdot)$ subject to mixed boundary

conditions (see Definition 1.6.3). For the moment, the reader can keep a bounded domain subject to pure Dirichlet or Neumann boundary conditions in mind. Nevertheless, we mention already here that the interplay between different boundary conditions and components of O leads to interesting effects that we are going to discuss in a minute. For the precise geometric setup, we refer to Section 6.2. We consider the operator-adapted Hardy space $\mathrm{H}^1_L(O)$ that consists of all $f \in \mathrm{L}^1(O)$ such that the square function

$$(S_L f)(x) = \Big(\int_0^\infty \fint_{O(x,t)} |t\partial_t \, \mathrm{e}^{-t\sqrt{L}} \, f(y)|^2 \, \mathrm{d}y \, \frac{\mathrm{d}t}{t} \Big)^{\frac{1}{2}} \qquad (x \in O) \tag{6.1}$$

is again in $\mathrm{L}^1(O)$. We are going to omit the underlying set O in the notation of function spaces from now on.

In the application to boundary value problems, adapted Hardy spaces describe the space of admissible boundary data. Therefore, it is desirable to have a description of H^1_L which is independent of the concrete operator L. Atomic spaces can provide such a description. Examples of atomic Hardy spaces on open sets include the space $\mathrm{H}^1_{\mathrm{CW}}$ studied by Coifman–Weiss [63] and $\mathrm{H}^1_{\mathrm{Mi}}$ of Miyachi [136]. It turns out, see [26, 48], that the space $\mathrm{H}^1_{\mathrm{CW}}$ corresponds to pure Neumann and $\mathrm{H}^1_{\mathrm{Mi}}$ to pure Dirichlet boundary conditions. In other words, under suitable assumptions, $\mathrm{H}^1_{\mathrm{CW}}$ coincides with H^1_L if L is subject to pure Neumann boundary conditions and $\mathrm{H}^1_{\mathrm{Mi}}$ coincides with H^1_L if L is subject to pure Dirichlet boundary conditions. We are going to review Hardy spaces on open sets in more depth in Section 6.3. To the best of our knowledge, if the matrix A is not self-adjoint then only the results from Auscher and Russ [26] relate square function based spaces with (L-independent) atomic spaces. Their main geometric assumption is that O is connected with Lipschitz boundary. The purpose of this paper is a generalization of [26] in the following two directions:

(i) Can we work with a rough and unconnected open set O far below the Lipschitz class?

(ii) Can we replace pure Dirichlet and Neumann boundary conditions by mixed boundary conditions?

Let us start with an example that highlights that some geometric conditions are necessary.

Example 6.1.1. Consider the punctured plane $O = \mathbb{R}^2 \setminus \{0\}$ with boundary $\partial O = \{0\}$. Then $\mathrm{W}^{1,2}_0(O) = \mathrm{W}^{1,2}(\mathbb{R}^2)$. Hence, the negative Dirichlet

Laplacian $-\Delta_{\partial O}$ on O coincides with the usual negative Laplacian $-\Delta$ on $\mathbb{R}^2$. Consequently, $\mathrm{H}^1_{-\Delta_{\partial O}} = \mathrm{H}^1(\mathbb{R}^2)$ by [26]. But $\mathrm{H}^1_{\mathrm{Mi}}$ contains functions that are not mean value free, therefore $\mathrm{H}^1_{-\Delta_{\partial O}} \neq \mathrm{H}^1_{\mathrm{Mi}}$. This show that a fatness condition for ∂O cannot be avoided.

Pushing this example further, if $D \subseteq \mathbb{R}^d$ is of Hausdorff dimension less than $d-2$, then an application of Netrusov's theorem [2, Ch. 10] gives a counterexample to $\mathrm{H}^1_{-\Delta_{\partial O}} \neq \mathrm{H}^1_{\mathrm{Mi}}$ for $O = \mathbb{R}^d \setminus D$. On the other hand, if the boundary has dimension larger than $d-2$, our theory gives an answer to the affirmative.

As a first gentle step into mixed boundary conditions, we consider an open and bounded set $O = O_1 \cup O_2$ that consists of two separated connected components. Then we let L be an operator on O subject to pure Dirichlet boundary conditions on O_1, and pure Neumann boundary conditions on O_2. A natural candidate for a D-adapted atomic Hardy space $\mathrm{H}^1_D(O)$ is

$$\mathrm{H}^1_D(O) = \mathrm{H}^1_{\mathrm{Mi}}(O_1) \oplus \mathrm{H}^1_{\mathrm{CW}}(O_2). \tag{6.2}$$

Taking a look at the square function (6.1), such a decoupling is not to be expected as the square function always integrates over both components for t large enough. Nevertheless, (6.2) is the correct guess and with its help we eventually even find

$$\mathrm{H}^1_D(O) = \mathrm{H}^1_L(O) = \mathrm{H}^1_{L_1}(O_1) \oplus \mathrm{H}^1_{L_2}(O_2),$$

where L_1 and L_2 are restrictions of the elliptic operator L to the components O_1 and O_2. Going one step further, we want to study the situation where $D \subseteq \partial O$ is a non-trivial subset of the boundary of an, say connected, open set O. We have indicated in Example 6.1.1 that Dirichlet boundary conditions can lead to a lack of cancellation for atoms near the boundary in the sense that they are not mean value free anymore, in contrast to the case of classical atoms in the sense of Coifman–Weiss. The correct definition for an atomic space H^1_D therefore has to "interpolate" the definitions for $\mathrm{H}^1_{\mathrm{CW}}$ and $\mathrm{H}^1_{\mathrm{Mi}}$ in a non-trivial way. We comment in Remark 6.3.8 on the upcoming challenges in this quest. Eventually, we come up with a suitable notion of H^1_D-atoms in Definition 6.4.3. The abstract main result of this chapter is the following theorem. The exact assumptions on geometry and the elliptic operator will be introduced in Section 6.2 and we give a precise formulation of the result in Theorem 6.5.6.

Theorem 6.1.2. *Let $O \subseteq \mathbb{R}^d$ be open and $D \subseteq \partial O$ be closed. Assume that O is uniformly interior thick and locally uniform near $N = \partial O \setminus D$, that D is porous and that O^c is locally 2-fat away from N. Suppose that L satisfies Hölder-type Gaussian estimates. Then*

$$\mathrm{H}^1_L = \mathrm{H}^1_D.$$

We give some sufficient conditions for our geometric framework.

Example 6.1.3. Uniform interior thickness ensures that O is at least a space of homogeneous type. Being a locally uniform domain near N is a generalization of Jones' notion of an (ε, δ)-domain [117]. All open sets whose boundary is represented by a sufficiently regular graph are admissible, but also fractals like the von Koch snowflake, see Figure 8. The condition for D means that the Dirichlet part is not of the same dimension as O and is essentially always fulfilled. It is in particular the case when D is Ahlfors–David regular, in which case also the local 2-fatness of O^c away from N follows. Also exterior thick sets (see Proposition 6.3.10) satisfy the fatness condition.

Our definition also unifies the cases of pure Dirichlet and Neumann boundary conditions, see Proposition 6.4.7 for the precise assumptions.

Theorem 6.1.4. *Under minimal geometric conditions it holds*

$$\mathrm{H}^1_{\varnothing} = \mathrm{H}^1_{\mathrm{CW}} \qquad \& \qquad \mathrm{H}^1_{\partial O} = \mathrm{H}^1_{\mathrm{Mi}}.$$

An essential tool to prove Theorem 6.5.6 is a full duality theory for H^1_D. We are going to introduce D-adapted spaces VMO_D and BMO_D. A key feature of our approach is the fact that the BMO-seminorm is defined by bounded means and not bounded mean oscillation near the Dirichlet boundary part. The following theorem recaptures the duality theory for H^1_D. Precise versions are given in Theorems 6.4.12 and 6.4.15.

Theorem 6.1.5. *Under minimal geometric assumptions it holds*

$$(\mathrm{VMO}_D)^* = \mathrm{H}^1_D \qquad \& \qquad (\mathrm{H}^1_D)^* = \mathrm{BMO}_D.$$

We have seen in Example 6.1.3 that our geometric framework comprises many relevant situations. So far, we have not discussed the Hölder-type Gaussian estimates (see Definition 6.2.10) that are imposed on L in our main result. In Lipschitz domains they were pioneered in [30] and for mixed boundary conditions they were discussed in Chapter 5 of this thesis. As a combination of

our main result, the Gaussian estimates from Chapter 5 and the recent article [48] in a self-adjoint setting, we give a full picture (including maximal characterizations) for $-\Delta$ subject to mixed boundary conditions on a bounded open set. The following theorem contains the special case where O is connected and D is non-trivial. The precise formulation in the general case can be found in Theorem 6.6.5. The special cases of pure Dirichlet or Neumann boundary conditions are presented in Corollaries 6.6.7 and 6.6.8.

Theorem 6.1.6. *Let $O \subseteq \mathbb{R}^d$ be a bounded domain that is interior thick and let $D \subseteq \partial O$ be non-trivial, closed and porous. Suppose that O is locally uniform near N and that O^c is locally 2-fat away from N. Then*

$$\{f \in \mathrm{L}^1 \colon S_{-\Delta_D}(f) \in \mathrm{L}^1\} = \mathrm{H}^1_D = \mathrm{H}^1_{-\Delta_D,\max} = \mathrm{H}^1_{-\Delta_D,\mathrm{rad}},$$

where $-\Delta_D$ is the negative Laplacian subject to mixed boundary conditions. The spaces $\mathrm{H}^1_{-\Delta_D,\max}$ and $\mathrm{H}^1_{-\Delta_D,\mathrm{rad}}$ are non-tangential and radial maximal spaces.

In the preceding result we can always replace the Laplacian by an elliptic operator with a real and self-adjoint coefficient matrix. In dimension $d = 2$ the first identity stays true if all restrictions but ellipticity on the coefficients are dropped [45, Thm. 10.1].

Let us emphasize that all assumptions of the last theorem as well as the space H^1_D appearing in its conclusion are purely geometrical.

Notation specific in this chapter

We abbreviate $\mathrm{X} \coloneqq \mathrm{X}(O)$ and $\mathrm{d}(O) \coloneqq \operatorname{diam}(O)$.

6.2 Geometric framework and the elliptic operator

6.2.1 The geometric setup

We start with a list of geometric concepts.

Assumption 6.2.1. *Throughout this chapter, we make use of the following assumptions.*

(UITC) *O is **uniformly interior thick**: There is $c > 0$ such that for all $x \in O$ and $r \in (0, \mathrm{d}(O))$ it holds $|O(x,r)| \geq cr^d$.*

(Fat) O^c *is* ***locally 2-fat away from*** $\boldsymbol{N}$*, see Definition 5.1.1.*

(LU) O *is* ***locally uniform near*** $\boldsymbol{N}$*, see Definition 5.1.3.*

(UP) D *is* ***uniformly porous****: There is* $\kappa \in (0,1]$ *such that for all* $x \in D$ *and* $r \in (0, \mathrm{d}(O))$ *there is some* $y \in \mathbb{R}^d$ *with* $B(y, \kappa r) \subseteq B(x,r) \setminus \partial O$.

Remark 6.2.2. There is a common variation of (UITC) in which only radii $r \in (0,1]$ are considered. We refer to it as (ITC). It turns out that the constraint $r \in (0,1]$ in (ITC) can be changed to $r \in (0, r_0]$ for any $r_0 > 0$ up to a change of the implied constant c. Therefore, (UITC) implies (ITC). The converse is false in general, but holds if O is bounded. We are going to write (ITC) instead of (UITC) whenever we want to make explicit that O is bounded. Let us observe that due to (UITC), O is bounded if and only if O has finite measure.

We work with a concept of mean value free and locally constant functions that is adapted to D. This terminology and notation is non-standard, but turns out to be very natural and convenient for our setting.

Definition 6.2.3. We enumerate the bounded connected components O' of O that satisfy $\partial O' \cap D = \varnothing$ by $\{O_m\}_m$ and write I for the countable index set. Let $p \in [1, \infty)$.

(i) The **space** $\mathbf{L^p_0}$ consists of $f \in \mathrm{L}^p$ such that $\int_{O_m} f \,\mathrm{d}x = 0$ for every $m \in I$.

(ii) We say that $f \in \mathrm{L}^p$ is **locally constant** if f restricted to O_m is constant for every $m \in I$ and is zero otherwise.

Remark 6.2.4. (i) In view of (LU) we can identify the space of locally constant L^p-functions with $\ell^p(I)$.

(ii) In the situation of Definition 6.2.3 we put

$$O_N \coloneqq \bigcup_{m \in I} O_m \quad \& \quad O_D \coloneqq O \setminus O_N.$$

We elaborate on the preceding definition by introducing a notion of "components".

Definition 6.2.5. The elements of $\Sigma \coloneqq \{O_D, O_1, \ldots, O_m\}$ are called **components** of O. Let us mention that we have the disjoint decomposition $O = \bigcup_{O' \in \Sigma} O'$ by construction.

Remark 6.2.6. Some remarks are in order.

(i) We stress that components need not be connected components. This is only ensured for the pure Neumann components O_m and may fail for the component O_D. The components reflect a decoupling of the elliptic operator L that stems from the presence of boundary conditions. We will see in Proposition 6.2.9 why it is natural to have such a decoupling. In this sense, locally constant functions in the sense of Definition 6.2.3 are functions that are constant on decoupled parts of the operator L and that respect the Dirichlet boundary conditions in D.

(ii) Let us consider for instance the case where O_D consists of two connected components, both equipped with Dirichlet boundary conditions. Large parts of our theory work if we consider the two connected components of O_D as components of O. The notable exception is Lemma 6.2.7 below. It is of course possible that O_D satisfies (ITC) whereas this is false for all its connected components. To make our theory robust for such generalizations, we often write $D \cap \partial O'$, even though this intersection is always D or $\varnothing$ with the choice of components from Definition 6.2.5.

In the next result we derive some geometric properties from assumption (LU) that will be beneficial for the rest of this chapter.

Lemma 6.2.7. *Assume* (LU)*, let $\delta > 0$ be as in Definition 5.1.3 and let O_i be a connected component of O with $\partial O_i \cap D = \varnothing$. Then the following assertions hold true.*

(i) *It holds $\mathrm{d}(O_i, O \setminus O_i) \geq \delta$. In particular, the components of O have at least distance δ to each other.*

(ii) *If O_i is unbounded, then O_i has infinite measure.*

(iii) *If O is bounded and satisfies* (ITC)*, then each component satisfies* (ITC) *as well.*

(iv) *If O is bounded, then I is finite.*

Proof of (i). Let us suppose to the contrary that $\mathrm{d}(O_i, O \setminus O_i) < \delta$. Then there are $x \in O_i \cap N_\delta$ and $y \in (O \setminus O_i) \cap N_\delta$ with $|x - y| < \delta$ and (LU) yields a path $\gamma \subseteq O$ that connects x with y. As O_i is connected we conclude that $y \in O_i$, which cannot be the case.

Proof of (ii). Since O_i is unbounded, we can find $(x_n)_n \subseteq O_i$ such that the balls $B(x_n, \delta)$ are pairwise disjoint. By item (i) and Proposition 5.1.8 we find $C > 0$ such that

$$|O_i| \geq \sum_n |O_i(x_n, \delta)| = \sum_n |O(x_n, \delta)| \geq C \sum_n \delta^d = \infty.$$

Proof of (iii). Since O is bounded, it suffices to check (ITC) for balls with radius $r \in (0, \delta]$. We let O' be a component of O. Balls centered in O' cannot hit other components of O according to item (i). Therefore, (ITC) can be transferred from O to O'.

Proof of (iv). Since O is bounded, so is $O_{\delta/2}$ and according to the first assertion the sets $(O_m)_{\delta/2}$ for $m \in I$ are pairwise disjoint. Since each $(O_m)_{\delta/2}$ contains a ball of radius $\delta/2$, we find some $C > 0$ with $|(O_m)_{\delta/2}| \geq C$ for all $m \in I$. It follows

$$C|I| \leq \sum_m |(O_m)_{\frac{\delta}{2}}| \leq |O_{\frac{\delta}{2}}|,$$

which yields that I is finite. □

Lemma 6.2.8. *Assume* (UP). *Then there are $c > 0$ and $\eta \in (0, 1)$ such that*

$$|\{x \in O(x_0, r) : \mathrm{d}_D(x) < s\}| \leq c s^\eta r^{d-\eta}$$

holds true for all $x_0 \in O$, $r \in (0, \mathrm{d}(O))$ and $s > 0$.

Proof. If $r \leq s$, then we can take $c = |B(0,1)|$ and any $\eta > 0$. Now, we assume that $s \leq r$. Since $y \mapsto x^y$ is decreasing on $(0, \infty)$ for $x \in (0, 1]$, we can ensure that $\eta \in (0, 1)$ as soon as we have found some $\eta > 0$. Using that D is uniformly porous, we can mimic the argument in [35, Lem. 5.4.2 & Lem. A.1.8] to prove the claim. □

6.2.2 Characterization of $\ker(L)$

In general, the elliptic operator $L = -\operatorname{div}(A\nabla\cdot)$ (see Definition 1.6.3) is not injective. We characterize its kernel in the following proposition. A similar observation was indicated in [54, p. 5]. We write

$$\ker(\nabla_D) \coloneqq \{u \in \mathrm{W}^{1,2}_D : \nabla u = 0\}.$$

Proposition 6.2.9. *The following assertions hold true:*

(i) *It holds* $\ker(L) = \ker(\nabla_D)$ *and* $\mathrm{L}^2 = \overline{\operatorname{ran}(L)} \oplus \ker(\nabla_D)$ *as an orthogonal decomposition.*

(ii) *If* (Fat) *and* (LU) *are satisfied, then* $\ker(\nabla_D) = \ell^2(I)$ *and* $\overline{\operatorname{ran}(L)} = \mathrm{L}^2_0$. *In particular,* $\mathrm{L}^2 = \mathrm{L}^2_0 \oplus \ell^2(I)$ *as an orthogonal decomposition.*

If $D = \partial O$*, then* L *is injective and the second statement holds true without the geometric assumptions.*

Proof of (i). We have for all $u \in \operatorname{dom}(L)$ and $v \in \mathrm{W}^{1,2}_D$ that

$$(Lu \,|\, v) = \int_O A\nabla u \cdot \overline{\nabla v}\,\mathrm{d}x \quad \& \quad \operatorname{Re}(Lu \,|\, u) \geq \lambda \|\nabla u\|_2^2.$$

Hence, $\ker(L) = \ker(\nabla_D)$. Since the same holds true for L^* replaced by L, we have $\overline{\operatorname{ran}(L)} = \ker(L^*)^\perp = \ker(\nabla_D)^\perp$ and the orthogonal splitting $\mathrm{L}^2 = \overline{\operatorname{ran}(L)} \oplus \ker(\nabla_D)$ follows.

Proof of (ii). First, we notice that $u \in \ker(\nabla_D)$ if and only if $u \in \mathrm{W}^{1,2}_D$ is constant on each connected component of O. If $D = \partial O$, then we can extend any $u \in \ker(\nabla_D) \subseteq \mathrm{W}^{1,2}_0$ by 0 to $u_0 \in \mathrm{W}^{1,2}(\mathbb{R}^d)$ and obtain $\nabla u_0 = (\nabla u)_0 = 0$. It follows $u_0 = 0$ and hence $u = u_0|_O = 0$. Now, we assume that D is not the full boundary. In this case we have $\ell^2(I) \subseteq \ker(\nabla_D)$ by Lemma 6.2.7 (i). Conversely, if $u \in \ker(\nabla_D)$, then u is constant on each bounded connected Neumann component ($\partial O_i \cap D = \varnothing$) and vanishes by Lemma 6.2.7 (ii) on each unbounded connected Neumann component. We are left to show that u vanishes on each connected component O_i of O with $\partial O_i \cap D \neq \varnothing$. To this end, we pick any $x \in \partial O_i \cap D$. According to Proposition 5.1.12, there are $c, r > 0$ depending on the geometry such that

$$\int_{O_i(x,r)} |u|^2\,\mathrm{d}x \lesssim \int_{O(x,cr)} |\nabla u|^2\,\mathrm{d}x = 0.$$

Therefore, $u = 0$ on O_D and hence $u \in \ell^2(I)$. Finally, we notice that

$$\overline{\operatorname{ran}(L)} = \ker(\nabla_D)^\perp = \ell^2(I)^\perp = \mathrm{L}^2_0. \qquad \square$$

6.2.3 Property (G_O)

We continue with the precise notion of kernel bounds that we employ to prove our main result, Theorem 6.5.6, see also Section 5.2 for the related global property (G).

Definition 6.2.10. We say that L has **property (G_O)** if the following holds:

(G1) For any $t > 0$ there is a measurable function $K_t \colon O \times O \to \mathbb{C}$ such that

$$(\mathrm{e}^{-t^2 L} f)(x) = \int_O K_t(x,y) f(y) \,\mathrm{d}y \qquad (f \in \mathrm{L}^2, \text{ a.e. } x \in O).$$

(G2) There are $C, c > 0$ such that for each $t \in (0, \mathrm{d}(O))$ and $x, y \in O$ it holds

$$|K_t(x,y)| \leq C t^{-d} \,\mathrm{e}^{-c\left(\frac{|x-y|}{t}\right)^2}.$$

(G3) There are $C > 0$ and $\mu \in (0,1]$ such that for all $t \in (0, \mathrm{d}(O))$ and every $x, x', y, y' \in O$ it holds

$$|K_t(x,y) - K_t(x',y')| \leq C t^{-d-\mu} (|x-x'| + |y-y'|)^\mu.$$

Remark 6.2.11. The difference to Definition 5.2.1 is that we require (G2) and (G3) with $\omega = 0$ and for all $t \in (0, \mathrm{d}(O))$ instead of some $\omega > 0$ and every $t > 0$.

Keeping Remark 5.2.2 (i) and (ii) (with obvious modifications) in mind, we add

Remark 6.2.12. For any $x, y \in O$ the function $z \mapsto K_z(x,y)$ defines an analytic map in the open sector $\mathrm{S}_{\pi/4 - \omega_L/2}$, where $\omega_L \in [0, \pi/2)$ is the angle of sectoriality of L. Then a straightforward calculation shows that $\frac{1}{2} t \partial_t K_t$ is the kernel associated with $-t^2 L \,\mathrm{e}^{-t^2 L}$ and satisfies the same kernel bounds as K_t owing to Cauchy's theorem.

The following lemma relates (G_O) with $\ker(\nabla_D)$.

Lemma 6.2.13. *Assume that O is unbounded and that there is some elliptic operator L that satisfies (G_O). Then $\ker(\nabla_D)$ is trivial.*

Proof. Let us fix $u \in \ker(\nabla_D) = \ker(L)$. Then $\mathrm{e}^{-tL} u = u$ for all $t > 0$ by the functional calculus of L. Therefore, as $\mathrm{d}(O) = \infty$, (G_O) joint with Young's convolution inequality gives

$$\|u\|_\infty = \|\,\mathrm{e}^{-tL} u\|_\infty \lesssim t^{-\frac{d}{4}} \|u\|_2 \longrightarrow 0 \qquad (t \to \infty).$$

We conclude that $u = 0$ as claimed. □

Surprisingly, (G_O) also imposes the following crucial geometrical constraint in the case of mixed boundary conditions.

Corollary 6.2.14. *Assume* (Fat) *and* (LU) *and suppose that there is some elliptic operator L that satisfies* (G_O)*. If O is unbounded, then $O_N = \varnothing$.*

Proof. According to Proposition 6.2.9 (ii) joint with Lemma 6.2.13, we have $\ell^2(I) = \ker(\nabla_D) = \{0\}$, which yields the claim. □

The following fact was also stated in [26]. We present a proof based on the previous two results. Another more direct proof could be given with the conservation property.

Corollary 6.2.15. *Assume that $\mathrm{W}_D^{1,2} = \mathrm{W}_\varnothing^{1,2}$ and that there is some elliptic operator L that satisfies* (G2) *and* (G3) *in* (G_O) *for all $t > 0$. Then O is unbounded.*

Proof. Let us assume to the contrary that O is bounded. Then every constant function belongs to $\ker(\nabla_D)$, but on the other hand Lemma 6.2.13 yields $\ker(\nabla_D) = \{0\}$. □

As L is m-accretive, L admits an m-accretive square root $\sqrt{L}$ which generates the *Poisson semigroup*

$$(P_t)_{t\geq 0} \coloneqq (\mathrm{e}^{-t\sqrt{L}})_{t\geq 0}.$$

To prove polynomial decay and Hölder continuity of the Poisson kernel, we use a subordination technique that we cite for the reader's convenience.

Lemma 6.2.16 ([90, p. 169])**.** *For all $t > 0$ and $f \in \mathrm{L}^2$, it holds*

$$\mathrm{e}^{-t\sqrt{L}} f = \frac{2}{\sqrt{\pi}} \int_0^\infty \mathrm{e}^{-s^2}\, \mathrm{e}^{-\frac{t^2}{4s^2}L} f \,\mathrm{d}s.$$

With this representation at our disposal, we can record

Lemma 6.2.17. *Assume that L satisfies* (G_O)*. Then, for all $t \in (0, \mathrm{d}(O))$, the kernel p_t of the Poisson semigroup is given by*

$$p_t(x,y) = \frac{2}{\sqrt{\pi}} \int_0^\infty \mathrm{e}^{-s^2}\, K_{\frac{t}{2s}}(x,y) \,\mathrm{d}s. \tag{6.3}$$

In addition, there is $C > 0$ such that

$$|p_t(x,y)| \leq C t^{-d} \left(1 + \frac{|x-y|}{t}\right)^{-(d+1)}, \tag{6.4}$$

and

$$|p_t(x,y) - p_t(x',y')| \leq C t^{-d-\mu} (|x-x'| + |y-y'|)^\mu \tag{6.5}$$

for all $t \in (0, \mathrm{d}(O))$ and $x, x', y, y' \in O$.

Remark 6.2.18. Similar observations as in Remarks 5.2.2 and 6.2.12 for the heat semigroup can be made for the Poisson semigroup, too.

Lemma 6.2.19. *Assume* (G_O)*, let* $t \in (0, \mathrm{d}(O))$ *and* $\varphi \in \{\sqrt{z}\,\mathrm{e}^{-\sqrt{z}}, z\,\mathrm{e}^{-z}\}$*. Then there is a continuous kernel* $q_t \colon O \times O \to \mathbb{C}$ *such that for all* $p \in [1, \infty]$ *the operator*

$$Q_t f := \int_O q_t(\cdot, y) f(y)\,\mathrm{d}y$$

is bounded on L^p *and agrees with* $\varphi(t^2 L)$ *on* $\mathrm{L}^p \cap \mathrm{L}^2$*.*

Proof. We start with $\varphi(z) = z\,\mathrm{e}^{-z}$. Then $\varphi(t^2 L) = t^2 L\,\mathrm{e}^{-t^2 L}$ and its kernel is given by $-\frac{1}{2} t \partial_t K_t$, which satisfies the bound (G2) from Definition 6.2.10, see Remark 6.2.12. Hence, the L^p-bound for Q_t follows from Young's inequality and consistency follows by construction, compare also with (G1).

The argument for $\varphi(z) = \sqrt{z}\,\mathrm{e}^{-\sqrt{z}}$ is similar. Indeed, the kernel bounds can be obtained in the same way, see Lemma 6.2.17 and Remark 6.2.18. Let us mention that q_t satisfies only the Poisson bound (6.4) in this case, which is nevertheless sufficient to derive L^p-boundedness. □

Lemma 6.2.20. *Assume* (Fat)*,* (LU)*,* (G_O) *and let* $\varphi \in \{\sqrt{z}\,\mathrm{e}^{-\sqrt{z}}, z\,\mathrm{e}^{-z}\}$*. Then* $q_t(x, y) = 0$ *for all* $t \in (0, \mathrm{d}(O))$*,* $x \in D$ *and* $y \in O$*.*

Proof. As before, we consider first $\varphi(z) = z\,\mathrm{e}^{-z}$ and fix $f \in \mathrm{L}^2$. On the one hand, Lemma 5.3.10 yields $\varphi(t^2 L) f \in \dot{\mathrm{C}}^\mu$. On the other hand, $\varphi(t^2 L) f \in \mathrm{dom}(L) \subseteq \mathrm{W}^{1,2}_D$. Hence, $\varphi(t^2 L) f \in \dot{\mathrm{C}}^\mu \cap \mathrm{W}^{1,2}_D$ and Lemma 5.2.8 implies that

$$\int_O q_t(x, y) f(y)\,\mathrm{d}y = \varphi(t^2 L) f(x) = 0 \qquad (x \in D).$$

Since $q_t(x, \cdot)$ is continuous we infer $q_t(x, y) = 0$ for all $y \in O$.

Now if $\varphi(z) = \sqrt{z}\,\mathrm{e}^{-\sqrt{z}}$, then the claim follows from the first case in conjunction with (6.3) and Remark 6.2.18. □

The conclusions of Lemmas 6.2.19 and 6.2.20 are the central properties of the operators $\varphi(t^2 L)$. Thus, we can usually ignore the precise choice of φ, which we manifest in the following

Definition 6.2.21. In both cases $\varphi(z) = \sqrt{z}\,\mathrm{e}^{-\sqrt{z}}$ or $\varphi(z) = z\,\mathrm{e}^{-z}$ we simply write q_t for the kernel of the operator $Q_t = \varphi(t^2 L)$.

6.3 Review of Hardy spaces on open sets

There appear different notions of Hardy spaces on an open set O in the literature. On the one hand, several atomic descriptions exist, notably those by Coifman–Weiss [63] and Miyachi [136]. On the other hand, there are also "global" constructions in which $\mathrm{H}^1(\mathbb{R}^d)$ is the point of departure to define new spaces as subspaces or quotient spaces [58].

It turns out that all these spaces can be sorted into two categories: spaces that are associated with an operator subject to either pure Dirichlet or pure Neumann boundary conditions.

To find a reasonable definition for the case of mixed boundary conditions, it is essential to gain a deeper understanding of these constructions. Furthermore, our general geometric framework allows us to sharpen results on the pure Dirichlet and Neumann cases found in the literature.

6.3.1 Atomic spaces on an open set

All atomic H^1-spaces consist of functions $f \in \mathrm{L}^1$ that can be represented by an **atomic representation** $f = \sum_j \lambda_j a_j$, where $(\lambda_j)_j \in \ell^1$ and the a_j are "atoms". The decisive question is how to define an atom. We begin with the space $\mathrm{H}^1_{\mathrm{CW}}$ introduced by Coifman and Weiss [63].

Definition 6.3.1. A function $a\colon O \to \mathbb{C}$ is called **Coifman–Weiss atom** if there is a ball $B \subseteq \mathbb{R}^d$ centered in O such that

(i) $\operatorname{supp}(a) \subseteq \overline{O} \cap B$ (localization condition),

(ii) $\|a\|_\infty \leq |O \cap B|^{-1}$ (size condition) and

(iii) $\int_O a \,\mathrm{d}x = 0$ (cancellation condition).

The **Coifmann–Weiss Hardy space $\mathbf{H^1_{CW}}$** consists of those $f \in \mathrm{L}^1$ for which there are Coifmann–Weiss atoms $(a_j)_j$ and a sequence $(\lambda_j)_j \in \ell^1$ such that $f = \sum_j \lambda_j a_j$ holds pointwise almost everywhere. We endow this space with the norm

$$\|f\|_{\mathrm{H}^1_{\mathrm{CW}}} = \inf \|(\lambda_j)_j\|_{\ell^1},$$

where the infimum is taken over all atomic decompositions of f.

Moreover, if O has finite measure, we require that constant functions also belong to $\mathrm{H}^1_{\mathrm{CW}}$. In other words, $\mathrm{H}^1_{\mathrm{CW}}$ has to be replaced by $\mathrm{H}^1_{\mathrm{CW}} \oplus \mathbb{C}$.

Remark 6.3.2. For brevity, we also call a an atom if the size condition is only satisfied up to multiplication with an absolute constant.

Remark 6.3.3. For $O = \mathbb{R}^d$, we recover $\mathrm{H}^1_{\mathrm{CW}}(\mathbb{R}^d) = \mathrm{H}^1(\mathbb{R}^d)$, where $\mathrm{H}^1(\mathbb{R}^d)$ is the classical Hardy space from Definition 1.1.5.

For the space $\mathrm{H}^1_{\mathrm{Mi}}$ by Miyachi [136], the notion of an atom is relaxed.

Definition 6.3.4. A measurable function $a\colon O \to \mathbb{C}$ is called **Miyachi atom** if there is a ball $B \subseteq \mathbb{R}^d$ centered in O such that $\operatorname{supp}(a) \subseteq \overline{O} \cap B$ and one of the following two conditions is satisfied.

(i) There is a classical L^∞-atom A for $\mathrm{H}^1(\mathbb{R}^d)$ associated with B such that $A|_O = a$.

(ii) It holds $2B \subseteq O$ but $5B \cap O^c \neq \varnothing$ and $\|a\|_\infty \leq |B|^{-1}$.

The **Miyachi Hardy space $\mathbf{H^1_{Mi}}$** consists of those $f \in \mathrm{L}^1$ for which there are Miyachi atoms $(a_j)_j$ and a sequence $(\lambda_j)_j \in \ell^1$ such that $f = \sum_j \lambda_j a_j$ holds pointwise almost everywhere. We endow this space with the norm

$$\|f\|_{\mathrm{H}^1_{\mathrm{Mi}}} = \inf \|(\lambda_j)_j\|_{\ell^1},$$

where the infimum is taken over all atomic decompositions of f.

Remark 6.3.5. In view of Definition 6.3.4 (i), all Coifman–Weiss atoms are also Miyachi atoms under (UITC).

There appears another variation of Miyachi atoms in the literature, employed for instance in [48, Def. 4.3] and implicitly used in [26].

Definition 6.3.6. A function $a\colon O \to \mathbb{C}$ associated with a ball $B \subseteq \mathbb{R}^d$ is called **Miyachi atom** if $2B \subseteq O$, $\operatorname{supp}(a) \subseteq B$ and $\|a\|_\infty \leq |B|^{-1}$, and $\int_O a \,\mathrm{d}x = 0$ if $4B \subseteq O$.

Let us demonstrate that both spaces coincide.

Proposition 6.3.7. *The Miyachi space in the sense of Definition 6.3.4 coincides with the one in the sense of Definition 6.3.6.*

Proof. Up to a change of implied constants, it is negligible that one definition is with respect to $4B$ and the other one with respect to $5B$. In particular, the Miyachi space in the sense of Definition 6.3.6 embeds into the space in the sense of Definition 6.3.4.

For the converse inclusion, we let a be a Miyachi atom in the sense of Definition 6.3.4 associated with a ball B. If $5B \subseteq O$, then a is of type (i) and therefore an atom in the sense of Definition 6.3.6. Now, we assume that $5B \cap O^c \neq \varnothing$.

If $2B \subseteq O$, then a is clearly an atom in the sense of Definition 6.3.6. Hence, we are left with the case that $2B \cap O^c \neq \varnothing$. Then $a = A|_O$ is of type (i) and we decompose a into a sum of atoms without cancellation. We start with the trivial covering

$$B \subseteq \bigcup_{x \in B} B(x, {}^{\mathrm{d}_{\partial O}(x)}/_{10})$$

and invoke Vitali's covering lemma to get a countable subcollection of balls $\{B_j\}_j \coloneqq \{B(x_j, {}^{\mathrm{d}_{\partial O}(x_j)}/_2)\}_j$ with $B \subseteq \bigcup_j B_j$ and $\frac{1}{5}B_i \cap \frac{1}{5}B_j = \varnothing$ for $i \neq j$. By construction, they satisfy $2B_j \subseteq O$ as well as $4B_j \cap \partial O \neq \varnothing$. Now, we write

$$a = \sum_j \lambda_j a_j \quad \text{with} \quad \lambda_j \coloneqq \frac{|B_j|}{|B|} \quad \text{and} \quad a_j \coloneqq \lambda_j^{-1} \frac{\mathbf{1}_{B_j} a}{\sum_i \mathbf{1}_{B_i}}.$$

Then we have $\operatorname{supp}(a_j) \subseteq B_j$ and $\|a_j\|_\infty \leq |B_j|^{-1}$. Hence, every a_j is an atom without cancellation and thus

$$\|a\|_{\mathrm{H}^1_{\mathrm{Mi}}} \leq \sum_j \lambda_j = |B|^{-1} \sum_j |B_j| = \frac{5^d}{|B|} \sum_j |\tfrac{1}{5}B_j| = \frac{5^d}{|B|} \Big| \bigcup_j \tfrac{1}{5}B_j \Big|.$$

Finally, since $\mathrm{d}_{\partial O}(x_j) \leq r(B) + \mathrm{d}_{\partial O}(c(B)) \leq 3r(B)$ for every j, we conclude that $\bigcup_j \frac{1}{5}B_j \subseteq 2B$ and hence $\|a\|_{\mathrm{H}^1_{\mathrm{Mi}}} \leq 10^d$. □

The following remark explains why neither of the two definitions is right away suitable to define spaces adapted to mixed boundary conditions.

Remark 6.3.8. The Miyachi space, in the sense of Definition 6.3.6, has a priori a better localization of atoms. This is needed in the identification of L-adapted spaces, as becomes apparent from the references given in the proof. A naive approach to treating mixed boundary conditions might be to simply replace the distance function $\mathrm{d}_{\partial O}(\cdot)$ by $\mathrm{d}_D(\cdot)$ in the proof of the previous proposition. Unfortunately, this does not work. The reason is that balls defined with respect to ∂O are either entirely contained within O or entirely outside of O. Thus, balls that are completely outside of O can be "thrown away". However, this property no longer holds in the case of mixed boundary conditions.

The definition that we are going to give in Section 6.4 is a third alternative way to define the Miyachi space $\mathrm{H}^1_{\mathrm{Mi}}$ as becomes apparent from Proposition 6.4.7.

6.3.2 Constructions based on the Euclidean Hardy space

Now, we reformulate the intrinsic definitions of Hardy spaces via atoms in terms of a global perspective. The first construction is by means of restriction of functions in the classical Hardy space $\mathrm{H}^1(\mathbb{R}^d)$.

Definition 6.3.9. The **restriction Hardy space $\mathbf{H}^1_r$** is defined by

$$\mathrm{H}^1_r := \Big\{ f \colon f = F|_O,\ F \in \mathrm{H}^1(\mathbb{R}^d) \Big\}, \qquad \|f\|_{\mathrm{H}^1_r} := \inf \|F\|_{\mathrm{H}^1(\mathbb{R}^d)},$$

where the infimum is taken over all $F \in \mathrm{H}^1(\mathbb{R}^d)$ with $f = F|_O$.

Under suitable geometric assumptions, H^1_r corresponds to the Miyachi Hardy space. For instance, using that O is uniform in a certain sense, $\mathrm{H}^1_r = \mathrm{H}^1_{\mathrm{Mi}}$ follows from [136, Sec. 6]. Another approach uses thickness of O^c. The main argument was already employed in [58, p. 304], even though the formulation was less general.

Proposition 6.3.10. *Assume that O is uniformly exterior thick, that is to say, there is $c > 0$ such that for any ball B centered in ∂O it holds $|O^c \cap B| \geq c|B|$. Then $\mathrm{H}^1_r = \mathrm{H}^1_{\mathrm{Mi}}$.*

Proof. The inclusion $\mathrm{H}^1_r \subseteq \mathrm{H}^1_{\mathrm{Mi}}$ is explained in [58, Proof of Thm. 2.7] without imposing any geometric assumptions. We mention that this inclusion reuses the argument presented in the proof of Proposition 6.3.7.

For the converse inclusion, we only have to consider Miyachi atoms a supported in a ball B such that $2B \subseteq O$ but $5B \cap O^c \neq \varnothing$, because the other atoms are by definition restrictions of atoms for $\mathrm{H}^1(\mathbb{R}^d)$. We take $x \in 5B \cap \partial O$, set $r := r(B)$ and define $A := a - c\mathbf{1}_{O^c(x,r)}$ with

$$c := \frac{1}{|O^c(x,r)|} \int_B a \,\mathrm{d}x.$$

Then $A|_O = a$, $\operatorname{supp}(A) \subseteq 6B$ and $\int_{6B} A \,\mathrm{d}x = 0$. In addition, the size condition for a and the uniform exterior thickness condition for O entails

$$\|A\|_\infty \leq \|a\|_\infty + |c| \leq |B|^{-1} + \frac{|B|}{|O^c(x,r)|}\|a\|_\infty \lesssim |6B|^{-1},$$

which shows the size condition for A. Hence, a is a restriction of an L^∞-atom for $\mathrm{H}^1(\mathbb{R}^d)$. □

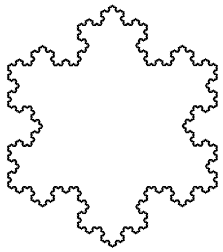

Figure 8: Sketch of a von Koch snowflake.

The previous proposition already appeared on Lipschitz domains as a side-product in [26]. The following example shows that we can go even below the class of Lipschitz regular domains.

Example 6.3.11. Let $\bigstar \subseteq \mathbb{R}^2$ denote the interior of the von Koch snowflake depicted in Figure 8. Then it holds

$$\mathrm{H}^1_r(\bigstar) = \mathrm{H}^1_{\mathrm{Mi}}(\bigstar).$$

Indeed, the assumptions of Proposition 6.3.10 are satisfied, see [135, p. 349].

The second construction is based on extension by zero. We write $\mathcal{E}_0$ for the zero extension operator.

Definition 6.3.12. The **zero extension Hardy space $\mathbf{H^1_z}$** is defined by

$$\mathrm{H}^1_z := \Big\{ f \in \mathrm{L}^1(O) \colon \mathcal{E}_0 f \in \mathrm{H}^1(\mathbb{R}^d) \Big\}, \qquad \|f\|_{\mathrm{H}^1_z} := \|\mathcal{E}_0 f\|_{\mathrm{H}^1(\mathbb{R}^d)}.$$

Let us prove the following fact.

Lemma 6.3.13. *Assume* (UITC) *and that O has infinite measure. Then it holds* $\mathrm{H}^1_{\mathrm{CW}} \subseteq \mathrm{H}^1_z$.

Proof. First, we prove a uniform bound for $\mathrm{H}^1_{\mathrm{CW}}$-atoms a associated with a ball B. Since O has infinite measure, we know that a is non-constant. In addition, we have $\operatorname{supp}(\mathcal{E}_0 a) \subseteq B$, $\int_B \mathcal{E}_0 a \,\mathrm{d}x = \int_{O\cap B} a \,\mathrm{d}x = 0$ and (UITC) implies

$$\|\mathcal{E}_0 a\|_\infty = \|a\|_\infty \leq |O \cap B|^{-1} \lesssim |B|^{-1}.$$

Hence, $\mathcal{E}_0 a$ is an L^∞-atom for $\mathrm{H}^1(\mathbb{R}^d)$ and thus,

$$\|a\|_{\mathrm{H}^1_z} = \|\mathcal{E}_0 a\|_{\mathrm{H}^1(\mathbb{R}^d)} \lesssim 1.$$

Now, we take $f \in \mathrm{H}^1_{\mathrm{CW}}$ with atomic decomposition $f = \sum_j \lambda_j a_j$ as well as $\|(\lambda_j)_j\|_{\ell^1} \simeq \|f\|_{\mathrm{H}^1_{\mathrm{CW}}}$. Then we have $\mathcal{E}_0 f = \sum_j \lambda_j (\mathcal{E}_0 a_j)$ and hence

$$\|f\|_{\mathrm{H}^1_z} \leq \sum_j |\lambda_j| \|a_j\|_{\mathrm{H}^1_z} \lesssim \|(\lambda_j)_j\|_{\ell^1} \simeq \|f\|_{\mathrm{H}^1_{\mathrm{CW}}}. \qquad \square$$

The converse inclusion is less trivial. Using operator-adapted spaces, $\mathrm{H}^1_z = \mathrm{H}^1_{\mathrm{CW}}$ was shown in [26]. We are going to sharpen this result in our less restrictive geometric framework.

Theorem 6.3.14. *Suppose that $O \subseteq \mathbb{R}^d$ is open, unbounded, $D = \varnothing$, that* (UITC) *and* (LU) *are satisfied, and that some elliptic operator L has property* (G_O). *Then it holds $\mathrm{H}^1_z = \mathrm{H}^1_{\mathrm{CW}}$.*

We postpone the proof to Subsection 6.5.5.

6.3.3 Dirichlet vs Neumann Hardy spaces

We write $-\Delta_{\partial O}$ and $-\Delta_{\varnothing}$ for the Dirichlet and Neumann Laplacians on O. If O is a Lipschitz domain, then the following classification was derived in [26]:

Dirichlet	Neumann
$\mathrm{H}^1_{-\Delta_{\partial O}}$	$\mathrm{H}^1_{-\Delta_{\varnothing}}$
$\mathrm{H}^1_{\mathrm{Mi}}$	$\mathrm{H}^1_{\mathrm{CW}}$
H^1_r	H^1_z

Our main result, Theorem 6.5.6, can be interpreted as adding a third column that corresponds to mixed boundary conditions to this classification. The third line, a global characterization, has yet to remain empty in the case of mixed boundary conditions, but gives rise to subsequent research.

6.4 Duality theory for D-adapted spaces

We introduce atomic Hardy spaces H^1_D subject to a cancellation condition in D and provide duality relations with corresponding spaces VMO_D and BMO_D.

6.4.1 Philosophy behind our atoms

So far we have seen that classical atoms in the sense of Coifman–Weiss correspond to Neumann boundary conditions and atoms in the sense of Miyachi to Dirichlet boundary conditions. While classical atoms have a cancellation condition up to the boundary, for Miyachi atoms this is not the case anymore. Instead, the boundary conditions are supposed to compensate for a lack of cancellation of the atoms. Indeed, Lemma 6.2.20 applied to the pure Dirichlet case gives that the kernel of the operator $\varphi(t^2L)$ vanishes in ∂O. These operators will be the starting point to define operator-adapted spaces in Section 6.5 and we will see, for instance in the proof of the central Lemma 6.5.7, that the cancellation of atoms can often be replaced by a vanishing kernel in a straightforward way.

In the realm of mixed boundary conditions, Lemma 6.2.20 shows that the kernel of $\varphi(t^2L)$ vanishes in D. Consequently, atoms for a space H^1_D should have the cancellation condition provided they are not close to D. We have already discussed in Remark 6.3.8 that it is not sufficient to simply replace ∂O in the definition of a Miyachi atom by D. Instead, we provide another variation of Miyachi's definition that works in the situation of mixed boundary conditions and at the same time allows to treat the pure Dirichlet and Neumann cases in a unified way.

6.4.2 Precise definitions

Definition 6.4.1. If B is a ball centered in a component O' as in Definition 6.2.3, then we write

$$B_O = O' \cap B.$$

Definition 6.4.2. Any ball $B \subseteq \mathbb{R}^d$ centered in O is called **usual**. If B is centered in O' for $O' \in \Sigma$ and $2B \cap (D \cap \partial O') \neq \varnothing$, then we say that B is **near D**.

Definition 6.4.3. Let B be a usual ball centered in $O' \in \Sigma$ and $a\colon O \to \mathbb{C}$ be measurable. We call a an H^1_D-atom if it satisfies one of the following conditions.

(i) We call a a **usual $\mathbf{H^1_D}$-atom**, if

$$\operatorname{supp}(a) \subseteq \overline{O'} \cap B, \quad \|a\|_2 \leq |B_O|^{-\frac{1}{2}} \quad \& \quad \int_O a \,\mathrm{d}x = 0.$$

(ii) We call a an **$\mathbf{H^1_D}$-atom near $\boldsymbol{D}$**, if B is also near D,

$$\operatorname{supp}(a) \subseteq \overline{O'} \cap B \quad \& \quad \|a\|_2 \leq |B_O|^{-\frac{1}{2}}.$$

Definition 6.4.4. The **atomic Hardy space $\mathbf{H^1_D}$** consists of all functions $f\colon O \to \mathbb{C}$ for which there is a sequence $(\lambda_j)_j \in \ell^1$ and a sequence of H^1_D-atoms $(a_j)_j$ such that $f = \sum_j \lambda_j a_j$ pointwise almost everywhere. We equip this space with the norm

$$\|f\|_{\mathrm{H}^1_D} \coloneqq \inf \|(\lambda_j)_j\|_{\ell^1},$$

where the infimum is taken over all atomic decompositions of f.

In addition, we denote by $\mathbf{H^1_{D,a}}$ the subspace of all **finite linear combinations of $\mathbf{H^1_D}$-atoms**. This space is by construction dense in H^1_D.

Remark 6.4.5. Let us mention that every atomic decomposition also converges in L^1 and $\|f\|_1 \leq \|f\|_{\mathrm{H}^1_D}$ for all $f \in \mathrm{H}^1_D$. This means that the inclusion map $\mathrm{H}^1_D \subseteq \mathrm{L}^1$ is a contraction.

Remark 6.4.6. Let us emphasize that we impose a weaker L^2-size condition on H^1_D-atoms. This choice appears more natural in our context, as we are working with an operator defined on L^2, and it also simplifies the arguments involved in the $\mathrm{H}^1_D - \mathrm{BMO}_D$ duality. Nevertheless, under (UITC), (Fat), (LU) and (G_O) (see also Lemma 6.2.7 and Corollary 6.2.14), the proof of [63, Thm. A, p. 626–631] shows that H^1_D can equivalently be defined using atoms that satisfy an L^∞-size condition. The only substantive difference is that a_0 qualifies as an H^1_D-atom near D as long as b is near D, see [63, p. 628–629] for details.

The following proposition shows that Definition 6.4.3 indeed extends and unifies the pure Dirichlet and Neumann cases in our geometric setting.

Proposition 6.4.7. *Suppose* (UITC), (Fat), (LU) *and that there exists an operator L satisfying* (G_O).

(i) *In the Neumann case $D = \varnothing$, it holds* $\mathrm{H}^1_\varnothing \oplus \ell^1(I) = \mathrm{H}^1_{\mathrm{CW}}$.

(ii) *In the Dirichlet case $D = \partial O$, we recover* $\mathrm{H}^1_{\partial O} = \mathrm{H}^1_{\mathrm{Mi}}$.

Proof. According to Remark 6.4.6, we can assume that H^1_D-atoms are defined with an L^∞-size condition.

Proof of (i). The main difference between $\mathrm{H}^1_{\varnothing}$ and $\mathrm{H}^1_{\mathrm{CW}}$ is the concept of components in the definition of $\mathrm{H}^1_{\varnothing}$. We distinguish the following cases.

(1) O is unbounded. In this case, O has infinite measure by Remark 6.2.2, $I = \varnothing$ by Corollary 6.2.14 and all $\mathrm{H}^1_{\varnothing}$-atoms are usual and coincide with the Coifman–Weiss atoms. Therefore, $\mathrm{H}^1_{\varnothing} \oplus \ell^1(I) = \mathrm{H}^1_{\varnothing} = \mathrm{H}^1_{\mathrm{CW}}$.

(2) O is bounded. Then constants are added to $\mathrm{H}^1_{\mathrm{CW}}$ and $I \neq \varnothing$ is finite by Lemma 6.2.7 (iv). Throughout this proof, all implicit constants depend only on the geometry.

$\mathrm{H}^1_{\varnothing} \oplus \ell^1(I) \subseteq \mathrm{H}^1_{\mathrm{CW}}$. We take $f = g + \sum_m c_m \mathbf{1}_{O_m} = \sum_j \lambda_j a_j + \sum_m c_m \mathbf{1}_{O_m} \in \mathrm{H}^1_{\varnothing} \oplus \ell^1(I)$ such that $\|g\|_{\mathrm{H}^1_{\varnothing}} \simeq \|(\lambda_j)_j\|_{\ell^1}$ and every a_j is an $\mathrm{H}^1_{\varnothing}$-atom associated with a ball $B(x_j, r_j)$. Using Lemma 6.2.7 and a case distinction whether $r_j \leq \delta$ or not (with $\delta > 0$ as in (LU)), we conclude that every a_j is also a constant multiple of a Coifman–Weiss atom. Hence, we get

$$\|g\|_{\mathrm{H}^1_{\mathrm{CW}}} \leq \sum_j |\lambda_j| \|a_j\|_{\mathrm{H}^1_{\mathrm{CW}}} \lesssim \|(\lambda_j)_j\|_{\ell^1} \lesssim \|g\|_{\mathrm{H}^1_{\varnothing}}.$$

Next, we write $\sum_m c_m \mathbf{1}_{O_m} = h + c$, where $c \coloneqq |O|^{-1} \sum_m c_m |O_m|$. Then we have

$$\int_O h \,\mathrm{d}x = 0, \quad \|h\|_\infty \lesssim |O|^{-1} \sum_m |c_m| \quad \& \quad |c| \lesssim \sum_m |c_m|.$$

In particular, $\|(c_m)_m\|^{-1}_{\ell^1(I)} h$ is a constant multiple of a Coifman–Weiss atom associated with a ball of radius $\mathrm{d}(O)$. In total, we obtain

$$\begin{aligned} \|f\|_{\mathrm{H}^1_{\mathrm{CW}}} &= \|g + h + c\|_{\mathrm{H}^1_{\mathrm{CW}}} \leq \|g\|_{\mathrm{H}^1_{\mathrm{CW}}} + \|h\|_{\mathrm{H}^1_{\mathrm{CW}}} + |c| \\ &\lesssim \|g\|_{\mathrm{H}^1_{\varnothing}} + \sum_m |c_m| = \|f\|_{\mathrm{H}^1_{\varnothing} \oplus \ell^1(I)}. \end{aligned}$$

$\mathrm{H}^1_{\mathrm{CW}} \subseteq \mathrm{H}^1_{\varnothing} \oplus \ell^1(I)$. Now, we take $f = g + c$ with $g = \sum_j \lambda_j a_j$, $c \in \mathbb{C}$ and $\|f\|_{\mathrm{H}^1_{\mathrm{CW}}} \simeq \|(\lambda_j)_j\|_{\ell^1} + |c|$. Here, every a_j is a Coifman–Weiss atom associated with a ball $B(x_j, r_j)$. Again Lemma 6.2.7 implies that a_j is an $\mathrm{H}^1_{\varnothing}$-atom provided that $r_j \leq \delta$. On the other hand, if $r_j > \delta$, then we decompose

$$a_j = \sum_m \mathbf{1}_{O_m}(a_j - (a_j)_{O_m}) + \sum_m \mathbf{1}_{O_m}(a_j)_{O_m} =: \sum_m a_{j,m} + \sum_m \mathbf{1}_{O_m} c_{j,m}.$$

Clearly, every $a_{j,m}$ is now a constant multiple of an $\mathrm{H}^1_{\varnothing}$-atom associated with a ball centered in O_m of radius $\mathrm{d}(O)$. With these observations in hand, we write

$$f = \left(\sum_{j : r_j \leq \delta} \lambda_j a_j + \sum_{j : r_j > \delta} \sum_m \lambda_j a_{j,m} \right) + \sum_m \mathbf{1}_{O_m} \Big(c + \sum_{j : r_j > \delta} \lambda_j c_{j,m} \Big).$$

In the light of Lemma 6.2.7, we infer from this decomposition the estimate

$$\|f\|_{\mathrm{H}^1_{\varnothing}\oplus\ell^1(I)} \lesssim \|(\lambda_j)_j\|_{\ell^1} + |c| \simeq \|f\|_{\mathrm{H}^1_{\mathrm{CW}}}.$$

Proof of (ii). First, we observe that we could equivalently define $\mathrm{H}^1_{\partial O}$-atoms near ∂O by the condition $5B \cap \partial O \neq \varnothing$. We use this throughout this proof. Let us recall from Definition 6.3.4 that the space $\mathrm{H}^1_{\mathrm{Mi}}$ consists of two types of atoms. Either $a = A|_O$ is the restriction of an $\mathrm{H}^1(\mathbb{R}^d)$-atom associated with a ball B centered in O or a is associated with a ball B that satisfies $2B \subseteq O$, $5B \cap O^c \neq \varnothing$ and a satisfies the localization and size conditions.

$\mathbf{H^1_{Mi} \subseteq H^1_{\partial O}}$**.** First, we let a be of type (i). Then either $2B \subseteq O$, in which case a is a usual atom for $\mathrm{H}^1_{\partial O}$, or $2B \cap \partial O \neq \varnothing$, in which case a is an $\mathrm{H}^1_{\partial O}$-atom near ∂O. Next, we let a be an atom of type (ii). Then a is automatically an $\mathrm{H}^1_{\partial O}$-atom near ∂O.

$\mathbf{H^1_{\partial O} \subseteq H^1_{Mi}}$**.** We let a be an $\mathrm{H}^1_{\partial O}$-atom associated with B. If a is a usual $\mathrm{H}^1_{\partial O}$-atom, then we extend a from B_O to B by zero and write A for this function on B. By (UITC), A is an $\mathrm{H}^1(\mathbb{R}^d)$-atom, so $a = A|_O$ is of type (i). Otherwise, if a is an atom near ∂O, we employ the localization argument used in the proof of Proposition 6.3.7 to conclude. □

The space $\mathrm{L}^2_{\mathrm{loc}}$ in the following definition consists of all measurable functions that are square integrable on any bounded subset of O.

Definition 6.4.8. The **space of bounded mean oscillation $\mathbf{BMO_D}$** consists of all $f \in \mathrm{L}^2_{\mathrm{loc}}$ for which the (semi-)norm

$$\|f\|_{\mathrm{BMO}_D} \coloneqq \sup_{B \text{ usual}} \left(\fint_{B_O} |f - (f)_{B_O}|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \vee \sup_{B \text{ near } D} \left(\fint_{B_O} |f|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}$$

is finite.

The definition of BMO_D already indicates the natural $\mathrm{H}^1_D - \mathrm{BMO}$-duality. The key observation is the following

Lemma 6.4.9. *It holds*

$$\left| \int_O fg \,\mathrm{d}x \right| \leq \|f\|_{\mathrm{H}^1_D} \|g\|_{\mathrm{BMO}_D} \qquad (f \in \mathrm{H}^1_D, g \in \mathrm{L}^\infty). \tag{6.6}$$

Proof. We let $f = \sum_j \lambda_j a_j$ be an atomic decomposition of $f \in \mathrm{H}^1_D$. Since $g \in \mathrm{L}^\infty$ and the series converges in L^1 we obtain

$$\int_O fg \,\mathrm{d}x = \sum_j \lambda_j \int_{(B_j)_O} a_j g \,\mathrm{d}x.$$

If a_j is near D, then we directly apply Hölder's inequality on $\int_{(B_j)_O} a_j g \,\mathrm{d}x$. In the other case we exploit the moment condition of a_j on $(B_j)_O$, which allows us to subtract $(g)_{(B_j)_O}$ from g. Afterwards, we apply Hölder's inequality. Hence, using the size condition for the atoms and the very definition of the BMO_D-norm, we get

$$\left| \int_O f g \,\mathrm{d}x \right| \leq \|(\lambda_j)_j\|_{\ell^1} \|g\|_{\mathrm{BMO}_D}.$$

We conclude by taking the infimum over all atomic decompositions of f. □

Remark 6.4.10. The localization property of H^1_D-atoms and its counterpart in the BMO_D-norm give rise to the topological splittings

$$\mathrm{H}^1_D = \bigoplus_{O' \in \Sigma} \mathrm{H}^1_{D \cap \partial O'}(O') \quad \& \quad \mathrm{BMO}_D = \bigoplus_{O' \in \Sigma} \mathrm{BMO}_{D \cap \partial O'}(O').$$

As a consequence of the preceding remark we are going to impose the following assumption in order to ease the arguments.

Assumption 6.4.11. *We suppose that O satisfies* (UITC) *and consists of only one component.*

It follows from Remark 6.4.10 that all results from this section stay true if O consists of multiple components that all satisfy (UITC). According to Lemma 6.2.7 and Corollary 6.2.14, this is a consequence of (G_O) in conjunction with (Fat) and (LU), so we can safely use all results developed in this section in the setting of Theorem 6.5.6.

6.4.3 $\mathrm{H}^1_D - \mathrm{BMO}_D$-duality

In this part we extend the classical $\mathrm{H}^1 - \mathrm{BMO}$-duality of Fefferman [91]. This comprises also the pure Neumann case established by Coifman and Weiss [63] and we stress that no geometric assumptions are used.

Theorem 6.4.12. *If O satisfies* (UITC) *and consists of only one component, then the space BMO_D is the dual space of H^1_D in the following sense:*

(i) *Each $g \in \mathrm{BMO}_D$ extends to a bounded linear functional on H^1_D via $\ell_g(f) = \int_O g f \,\mathrm{d}x$ for each $f \in \mathrm{H}^1_{D,a}$ and $\|\ell_g\|_{(\mathrm{H}^1_D)^*} \leq c\|g\|_{\mathrm{BMO}_D}$ for some generic constant c.*

(ii) *For any* $\ell \in (\mathrm{H}^1_D)^*$ *there is a unique* $g \in \mathrm{BMO}_D$ *such that* $\ell = \ell_g$ *and* $\|g\|_{\mathrm{BMO}_D} \leq c\|\ell\|_{(\mathrm{H}^1_D)^*}$ *for some generic constant* c.

Proof of (i). With (6.6) at our disposal, the argument in [63, p. 632] applies verbatim.

Proof of (ii). We take $\ell \in (\mathrm{H}^1_D)^*$. If D is empty, then the argument is classical, see [63, p. 632–634]. Otherwise, we have to put in some extra effort to improve the classical estimate on balls near D. That being said, we pick a reference ball B' near D, which is possible in the current case. By definition, B' also qualifies as a usual ball.

Following the classical proof in the case $D = \varnothing$ (see [152, Chap. IV, Sec. 1.2]), there exists for any usual ball B a mean value free function $F_B \in \mathrm{L}^2(B_O)$ such that $\|F_B\|_2 \leq |B_O|^{1/2}\|\ell\|_{(\mathrm{H}^1_D)^*}$. We set $f_B \coloneqq F_B - (F_B)_{B'_O}$. As in the classical case, the function $f(x) \coloneqq f_B(x)$, whenever $x \in B_O$ is a usual ball, is well-defined. We claim that there is a constant c depending on ℓ and B' such that $g \coloneqq f + c \in \mathrm{BMO}_D$ and $\ell_g = \ell$.

First, it follows readily that

$$\left(⨍_{B_O} |f + (F_B)_{B'_O}|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} = \left(⨍_{B_O} |F_B|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \leq \|\ell\|_{(\mathrm{H}^1_D)^*}.$$

Therefore, the BMO_D-norm of f is under control on usual balls. The same is true if we add a constant to f later on.

Second, to improve the bound on balls $\widetilde{B}$ near D we construct an auxiliary function. For any such $\widetilde{B}$ there is a unique function $h_{\widetilde{B}} \in \mathrm{L}^2(\widetilde{B}_O)$ that represents ℓ restricted to $\mathrm{L}^2(\widetilde{B}_O)$ and fulfills the bound $\|h_{\widetilde{B}}\|_2 \leq |\widetilde{B}_O|^{1/2}\|\ell\|_{(\mathrm{H}^1_D)^*}$. Now we can define a function h by $h(x) = h_{\widetilde{B}}(x)$ whenever $x \in \widetilde{B}_O$ is a ball near D. The difference to the classical construction is that $h_{\widetilde{B}}$ is unique without the necessity to perform a normalization with respect to a reference ball. Testing $h - f$ with the mean value free functions of $\mathrm{L}^2(\widetilde{B}_O)$ shows that $h - f$ is constant on $\widetilde{B}_O$. Since f is mean value free over the reference ball B'_O, averaging $h - f$ over B'_O shows that this constant is $(h)_{B'_O}$ independent of the ball $\widetilde{B}$. Exhausting O by such balls $\widetilde{B}$ yields $h = f + (h)_{B'_O}$. Therefore,

$$\left(⨍_{\widetilde{B}_O} |f + (h)_{B'_O}|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} = \left(⨍_{\widetilde{B}_O} |h|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \leq \|\ell\|_{(\mathrm{H}^1_D)^*}.$$

Thus, with $c = (h)_{B'_O}$ we have $g = f + c \in \mathrm{BMO}_D$ with $\|g\|_{\mathrm{BMO}_D} \leq c\|\ell\|_{(\mathrm{H}^1_D)^*}$ as claimed. □

6.4.4 Predual of the atomic Hardy space

The main goal of this subsection is to determine a predual of H^1_D. To this end, we introduce a D-adapted version of the space of vanishing mean oscillation in the spirit of Coifman and Weiss.

Definition 6.4.13. The **space of vanishing mean oscillation $\mathbf{VMO_D}$** is the closure of $\mathrm{C_c}$ with respect to $\|\cdot\|_{\mathrm{BMO}_D}$.

Let us mention that VMO_D is separable as $\mathrm{C_c}$ contains a $\|\cdot\|_\infty$-dense countable subset.

Remark 6.4.14. There is some flexibility in the definition of VMO_D. Indeed, it is possible to take the closure of a suitable space of regular functions that vanish only in D. We will not pursue this direction in this thesis.

The following theorem is the main result of this section.

Theorem 6.4.15. *Under* (UITC) *the dual space of* VMO_D *is* H^1_D *up to equivalent norms.*

This result naturally extends [63, Thm. 4.1] and we will need it in Section 6.5.4 to show the inclusion $\mathrm{H}^1_L \subseteq \mathrm{H}^1_D$. For its proof, we adapt the arguments in [63]. We are going to use a trick to circumvent the *a priori* norming property of $\mathrm{C_c}$ for H^1_D used in [63], which we learned from [164, p. 85].

The inclusion $\mathbf{H^1_D \subseteq (VMO_D)^*}$. This one is rather easy to see. Indeed, the proof of Lemma 6.4.9 reveals that this inclusion map is a contraction. Moreover, injectivity follows from the fact that $\mathrm{C_c}$ separates the points of $\mathrm{H}^1_D \subseteq \mathrm{L}^1$.

The inclusion $\mathbf{(VMO_D)^* \subseteq H^1_D}$. By the closed graph theorem, it suffices to show the inclusion in the set theoretic sense. The proof consists of two parts. The first one is a purely functional analytic reduction argument and the second one is a particular D-adapted modification of the argument in [63].

We write B_X for the closed unit ball in a Banach space X.

Step 1. We pick $\ell \in (\mathrm{VMO}_D)^*$ and assume without loss of generality that $\ell \neq 0$. After normalization we can assume that $\|\ell\|_{(\mathrm{VMO}_D)^*} \leq 1$. We let $\widetilde{\ell} \in (\mathrm{BMO}_D)^*$ be a Hahn–Banach extension of ℓ. By Theorem 6.4.12 there is a universal constant $c > 0$ such that $\widetilde{\ell} \in cB_{(\mathrm{H}^1_D)^{**}}$. Now, Goldstine's theorem [162, Thm. 8.3.17] implies that $cB_{\mathrm{H}^1_D}$ is w^*-dense in $cB_{(\mathrm{H}^1_D)^{**}}$. Using this along with the separability of VMO_D, we find a sequence $(f_n)_n \subseteq cB_{\mathrm{H}^1_D}$ such that $f_n \to \ell$ in the w^*-topology of $(\mathrm{VMO}_D)^*$.

Step 2. The upshot of the previous discussion is that the proof of the inclusion $(\mathrm{VMO}_D)^* \subseteq \mathrm{H}^1_D$ boils down to proving that any bounded sequence in H^1_D admits a subsequence which converges to some $f \in \mathrm{H}^1_D$ in the w^*-topology of $(\mathrm{VMO}_D)^*$.

The first auxiliary result is in the spirit of [63, Lem. 4.4]. When $D = \varnothing$, our proof gives a technically less involved version of their result in the Euclidean setting. For non-trivial D an additional argument is needed.

Lemma 6.4.16. *Under Assumption 6.4.11 any $f \in \mathrm{H}^1_D$ admits an atomic decomposition*

$$f = \sum_{(k,i)\in\mathbb{Z}\times\mathbb{Z}^d} \lambda_{k,i} a_{k,i} \quad \textit{with} \quad \|f\|_{\mathrm{H}^1_D} \simeq \|(\lambda_{k,i})_{k,i}\|_{\ell^1}.$$

In addition, it holds:

(i) *If $a_{k,i}$ is associated with the ball $B_{k,i}$, then $r(B_{k,i}) \simeq 2^k$.*

(ii) *For fixed $k \in \mathbb{N}$, only finitely many balls of $\{B_{k,i}\}_{i\in\mathbb{Z}^d}$ can intersect the same compact set in $\overline{O}$.*

Proof. We take an atomic decomposition $f = \sum_j \lambda_j a_j$ with $\lambda_j \in (0,\infty)$ satisfying $\sum_j \lambda_j \le 2\|f\|_{\mathrm{H}^1_D}$ and let a_j be associated with $B_j := B(x_j, r_j)$. For each $j \in \mathbb{N}$ there is a unique $k \in \mathbb{Z}$ such that $2^{k-1} < r_j \le 2^k$. Now we let $Q_{k,i}$ be the unique cube in the standard half open dyadic grid of cubes with side length 2^{k+1} that contains x_j. We express this relation by the notation $j \equiv (k,i) \in \mathbb{Z}\times\mathbb{Z}^d$.

We put $J_{k,i} := \{j\colon j \equiv (k,i)\}$ and let $j \in J_{k,i}$. We denote by $B_{k,i}$ a ball centered in $Q_{k,i} \cap O$ of smallest radius such that $2Q_{k,i}$ is contained. By construction, its radius is comparable to 2^k, so (i) is satisfied. Also, since for fixed $k \in \mathbb{Z}$ cubes from the dyadic grid $\{Q_{k,i}\}_{i\in\mathbb{Z}^d}$ are disjoint, a counting argument yields (ii).

To define a new atomic decomposition of f, we put

$$\lambda_{k,i} := \sum_{j\in J_{k,i}} \lambda_j \quad \& \quad a_{k,i} := \frac{\sum_{j\in J_{k,i}} \lambda_j a_j}{\lambda_{k,i}}.$$

We check that $a_{k,i}$ is an H^1_D-atom. First, since by construction $B_j \subseteq 2Q_{k,i} \subseteq B_{k,i}$ for $j \in J_{k,i}$, $a_{k,i}$ is supported in $\overline{O} \cap B_{k,i}$. Second, for $j \in J_{k,i}$ it follows

from $r(B_j) \simeq 2^k \simeq r(B_{k,i})$ and (UITC) that $|(B_j)_O| \geq c^{-2}|(B_{k,i})_O|$ for an absolute constant $c > 0$. Hence,

$$\|a_{k,i}\|_2 \leq (\lambda_{k,i})^{-1} \sum_{j \in J_{k,i}} |\lambda_j| \|a_j\|_2 \leq (\lambda_{k,i})^{-1} \sum_{j \in J_{k,i}} |\lambda_j| |(B_j)_O|^{-\frac{1}{2}} \leq c|(B_{k,i})_O|^{-\frac{1}{2}}.$$

Third, if $B_{k,i}$ is not near D, then every B_j for $j \in J_{k,i}$ is not near D. Hence, since $J_{k,i}$ is finite, $a_{k,i}$ has mean value zero. In summary, $a_{k,i}$ is a constant multiple of an H^1_D-atom and we deduce the atomic decomposition

$$f = \sum_{k,i} (c\lambda_{k,i})(c^{-1}a_{k,i}) \quad \text{with} \quad \sum_{k,i} c\lambda_{k,i} \leq 2c\|f\|_{\mathrm{H}^1_D}. \qquad \square$$

The second key lemma is an extension of [63, Lem. 4.2].

Lemma 6.4.17. *Under Assumption 6.4.11 any bounded sequence $(f_n)_n \subseteq \mathrm{H}^1_D$ has a subsequence which converges to some $f \in \mathrm{H}^1_D$ in the w^*-topology of $(\mathrm{VMO}_D)^*$. Moreover, it holds $\|f\|_{\mathrm{H}^1_D} \lesssim \sup_n \|f_n\|_{\mathrm{H}^1_D}$.*

Proof. After normalization we can assume that $\sup_n \|f_n\|_{\mathrm{H}^1_D} = 1$. Then Lemma 6.4.16 furnishes for all $n \in \mathbb{N}$ a decomposition $f_n = \sum_{k,i} \lambda_{k,i}(n) a_{k,i}(n)$ with $\lambda_{k,i}(n) \in (0, \infty)$ and $\sup_n \sum_{k,i} \lambda_{k,i}(n) \lesssim 1$. After passing to an appropriate subsequence, the Bolzano–Weierstrass theorem joint with a diagonalization argument allows us to assume that $\lambda_{k,i} \coloneqq \lim_{n\to\infty} \lambda_{k,i}(n)$ exists for all (k,i) and $\sum_{k,i} \lambda_{k,i} \lesssim 1$, see also [63, Lem. 4.3]. Since $(a_{k,i}(n))_n \subseteq \mathrm{L}^2((B_{k,i})_O)$ is bounded, there is some $a_{k,i} \in \mathrm{L}^2((B_{k,i})_O)$ such that $a_{k,i}(n) \to a_{k,i}$ weakly in $\mathrm{L}^2((B_{k,i})_O)$ along a subsequence. A diagonalization argument ensures that we can choose the same subsequence for all (k,i). The weak convergence implies that $a_{k,i}$ inherits the size and (potentially) moment conditions from the atoms $(a_{k,i}(n))_n$. Thus, each $a_{k,i}$ is an H^1_D-atom associated with $B_{k,i}$. We define the candidate

$$f \coloneqq \sum_{k,i} \lambda_{k,i} a_{k,i} \in \mathrm{H}^1_D.$$

By construction, $\|f\|_{\mathrm{H}^1_D} \lesssim \sup_n \|f_n\|_{\mathrm{H}^1_D}$. It remains to show that f is the appropriate limit of $(f_n)_n$. By a 2ε-argument it suffices to show $\langle f_n \,|\, \phi \rangle \to \langle f \,|\, \phi \rangle$ for all $\phi \in \mathrm{C_c}$. We let $K \in \mathbb{N}$ to be chosen and split

$$\begin{aligned} \langle f_n \,|\, \phi \rangle &= \Big(\sum_{k<-K} + \sum_{|k|\leq K} + \sum_{k>K} \Big) \sum_i \lambda_{k,i}(n) \int_{(B_{k,i})_O} a_{k,i}(n)\phi \,\mathrm{d}x \\ &=: \mathrm{(I)} + \mathrm{(II)} + \mathrm{(III)}. \end{aligned}$$

Compared to the argument in [63], the major change lies in the treatment of the small cubes (I) as we have to consider atoms near D. Let us fix $\varepsilon > 0$. We claim that K can be chosen independently of (k, i) such that

$$|(\mathrm{I})| + |(\mathrm{III})| \lesssim \varepsilon.$$

The same argument will also entail a similar estimate for the decomposition of f.

Estimate for (I). We denote by $x_{k,i}$ the center of $B_{k,i}$. Since ϕ is uniformly continuous, there is some $K \in \mathbb{N}$ such that $|\phi(x)-\phi(y)| \leq \varepsilon$ for all $x, y \in 2B_{k,i}$, where $i \in \mathbb{Z}^d$ and $k < -K$. If $B_{k,i}$ is usual, then $a_{k,i}(n)$ is average free over $(B_{k,i})_O$. Hence, with the size condition for atoms, we infer

$$\int_{(B_{k,i})_O} |a_{k,i}(n)||\phi - \phi(x_{k,i})| \,\mathrm{d}x \lesssim \varepsilon.$$

Now, we let $B_{k,i}$ be near D. Since ϕ vanishes in D, we can apply uniform continuity with $y \in D \cap 2B_{k,i}$ to give $|\phi(x)| \leq \varepsilon$ for all $x \in B_{k,i}$, $i \in \mathbb{Z}^d$ and $k < -K$. Consequently,

$$\int_{(B_{k,i})_O} |a_{k,i}(n)||\phi| \,\mathrm{d}x \lesssim \varepsilon.$$

Combining the last two estimates, we obtain

$$|(\mathrm{I})| \lesssim \varepsilon \sum_{k<-K} \sum_i \lambda_{k,i}(n) \lesssim \varepsilon.$$

Estimate for (III). By the size condition for atoms joint with (UITC) and Lemma 6.4.16 (i), we get

$$(\mathrm{III}) \leq \|\phi\|_2 \sum_{k>K} \sum_i \lambda_{k,i}(n) |(B_{k,i})_O|^{-\frac{1}{2}} \lesssim |B_{K,i}|^{-\frac{1}{2}} \|\phi\|_2 \sum_{k,i} \lambda_{k,i}(n) \lesssim 2^{-K\frac{d}{2}} \|\phi\|_2.$$

The right-hand side can be made smaller than ε provided that K is large enough.

Next, by Lemma 6.4.16 (ii) and the compact support of ϕ, we get

$$(\mathrm{II}) \longrightarrow \int_O \sum_{|k|\leq K,i} \lambda_{k,i} a_{k,i} \phi \,\mathrm{d}x \qquad (n \to \infty).$$

Combining the estimates for (I) and (III) with the convergence for (II) we finally obtain

$$\limsup_{n\to\infty} |\langle f - f_n \,|\, \phi \rangle| \lesssim \varepsilon. \qquad \square$$

6.5 Identification of operator-adapted and atomic Hardy spaces

We introduce the operator-adapted spaces H^1_L and state and prove our main result, Theorem 6.5.6. The proof heavily relies on the duality theory for atomic spaces developed in the last section. Throughout, we assume (G_O), (UITC), (Fat) and (LU). Also (UP) is used, but only once. We will make its usage explicit.

6.5.1 Relevant definitions

Definition 6.5.1. Let $x \in O$. For a measurable function $f\colon (0,\infty)\times O \to \mathbb{C}$, we define its **local square function** by

$$(S^{\mathrm{loc}} f)(x) \coloneqq \left(\int_0^{\mathrm{d}(O)} \fint_{O(x,t)} |f(t,y)|^2 \,\frac{\mathrm{d}y\,\mathrm{d}t}{t} \right)^{\frac{1}{2}}.$$

For either $\varphi(z) = \sqrt{z}\,\mathrm{e}^{-\sqrt{z}}$ or $\varphi(z) = z\,\mathrm{e}^{-z}$ and $f \in \mathrm{L}^1$, we set

$$S^{\mathrm{loc}}_{\varphi,L}(f)(x) \coloneqq S^{\mathrm{loc}}(\varphi(\boldsymbol{t}^2 L)f)(x).$$

Analogously, S and S^{glob} are defined by the same expression as S^{loc} but with $(0,\mathrm{d}(O))$ replaced by $(0,\infty)$ or $(\mathrm{d}(O),\infty)$ in the t-integral, respectively. These square functions lead to the L-adapted square functions $S_{\varphi,L}$ and $S^{\mathrm{glob}}_{\varphi,L}$.

Remark 6.5.2. Due to (UITC), the local square function can equivalently be expressed via

$$(S^{\mathrm{loc}} f)(x) = \left(\int_0^{\mathrm{d}(O)} \int_{O(x,t)} |f(t,y)|^2 \,\frac{\mathrm{d}y\,\mathrm{d}t}{t^{1+d}} \right)^{\frac{1}{2}} \qquad (x \in O).$$

This is not possible for S^{glob}.

As already hinted at in (6.1), we have the following

Definition 6.5.3. The **operator-adapted Hardy space $\mathbf{H^1_L}$** consists of all $f \in \mathrm{L}^1_0$ for which $S^{\mathrm{loc}}_{\varphi,L}(f) \in \mathrm{L}^1$, where $\varphi \in \{\sqrt{\boldsymbol{z}}\,\mathrm{e}^{-\sqrt{\boldsymbol{z}}}, \boldsymbol{z}\,\mathrm{e}^{-\boldsymbol{z}}\}$. We endow H^1_L with the norm

$$\|f\|_{\mathrm{H}^1_L} \coloneqq \|S^{\mathrm{loc}}_{\varphi,L}(f)\|_1 + \mathbf{1}_{[\mathrm{d}(O)<\infty]}\|f\|_1.$$

Remark 6.5.4 (dependence on φ)**.** At this stage, the definition of H^1_L using $\varphi \in \{\sqrt{z}\,\mathrm{e}^{-\sqrt{z}}, z\,\mathrm{e}^{-z}\}$ is a slight abuse of notation. We will of course treat both of them in all proofs. As a consequence of Theorem 6.5.6 it will turn out that the definition of H^1_L is indeed independent of the choice of φ up to equivalent norms.

Remark 6.5.5. Assume (Fat) and (LU).

(i) Let us suppose that $f \in \mathrm{H}^1_L \cap \mathrm{L}^2$. Then in particular $f \in \mathrm{L}^2_0$. According to Proposition 6.2.9 this gives $f \in \overline{\mathrm{ran}(L)}$. A practical consequence is that if $\psi \in \mathrm{H}^\infty$ on a suitable sector, then $\psi(L)f \in \overline{\mathrm{ran}(L)} = \mathrm{L}^2_0$.

(ii) If L satisfies (G_O), then $(P_t)_{t\in(0,\mathrm{d}(O))}$ and $(Q_t)_{t\in(0,\mathrm{d}(O))}$ map L^1_0 into $\mathrm{L}^1 \cap \overline{\mathrm{ran}(L)}$. Indeed, let us fix $f \in \mathrm{L}^1_0$ and $t \in (0, \mathrm{d}(O))$. We demonstrate the argument only for P_t. Then $P_t f \in \mathrm{L}^1 \cap \mathrm{L}^2$ by property (G_O) and we are left to show that $P_t f \in \overline{\mathrm{ran}(L)}$. To this end, we invoke Proposition 6.2.9. We take $u = \sum_m c_m \mathbf{1}_{O_m} \in \ker(L^*) \cong \ell^2(I)$ and calculate

$$(P_t f \,|\, u) = (f \,|\, \mathrm{e}^{-t\sqrt{L^*}}\, u) = (f \,|\, u) = \sum_m \overline{c_m} \int_{O_m} f \,\mathrm{d}x = 0.$$

It follows $P_t f \in \ker(L^*)^\perp = \overline{\mathrm{ran}(L)}$.

(iii) If L satisfies (G_O), then for each $f \in \mathrm{L}^1$ the sequence $(P_{1/n} f)_n$ converges to f in L^1 and the sequence $(Q_{1/n} f)_n$ converges to 0 in L^1. Indeed, for $f \in \mathrm{L}^1 \cap \mathrm{L}^2$ with compact support, say in a ball $B = B(x, r)$, it follows by the strong continuity in L^2 and the kernel estimates that

$$\begin{aligned} \|P_{1/n} f - f\|_1 &\le \|P_{1/n} f - f\|_2 |2B|^{\frac{1}{2}} + \|P_{1/n} f\|_{\mathrm{L}^1(O\setminus 2B)} \\ &\lesssim \|P_{1/n} f - f\|_2 |2B|^{\frac{1}{2}} + n^d (1 + rn)^{-(d+1)} \|f\|_1 \end{aligned}$$

tends to 0 as $n \to \infty$. Since $(P_{1/n})_n \subseteq \mathcal{L}(\mathrm{L}^1)$ is uniformly bounded, the desired convergence follows by a 2ε-argument. The assertion for $(Q_{1/n} f)_n$ can be deduced by the same reasoning.

6.5.2 Precise result and strategy of the proof

Finally, we are able to state the main result of this chapter.

Theorem 6.5.6. *Let $O \subseteq \mathbb{R}^d$ be open and $D \subseteq \partial O$ be closed. Suppose the geometric conditions* (UITC), (Fat), (LU) *and* (UP)*. Moreover, suppose that L satisfies (G_O). Then*

$$\mathrm{H}^1_L(O) = \mathrm{H}^1_D(O).$$

To prove it, we show both inclusions separately. The proof for $\mathrm{H}^1_L \subseteq \mathrm{H}^1_D$ builds on the ideas in [26, Sec. 3.4]. For the converse inclusion, we choose a different route: the aforementioned work takes a detour through a maximal function characterization, which relies on the Lipschitz property of the domain. Taking a direct approach is one of the reasons why we can drastically lower the geometric requirements in our main result. In the self-adjoint case, we obtain nevertheless a maximal function characterization in Section 6.6. The strategy for the individual inclusions is as follows.

$\mathbf{H^1_D \subseteq H^1_L}$. Taking Remark 6.4.5 into account, this inclusion can be reinterpreted as proving boundedness of

$$S^{\mathrm{loc}}_{\varphi,L} \colon \mathrm{H}^1_D \to \mathrm{L}^1.$$

Using the atomic decomposition for H^1_D, it turns out that this follows from the uniform control on H^1_D-atoms

$$\|S^{\mathrm{loc}}_{\varphi,L}(a)\|_1 \lesssim 1.$$

Let us assume that a is associated with B. To prove the above bound, we split the L^1-integral into a local and a global part, $O \cap 4B$ and $O \setminus 4B$. The local part can be purely treated by the L^2-theory of the operator. To control the global part, we split the t-integral in $(S^{\mathrm{loc}}_{\varphi,L}(a))(x)$ into several parts that let us exploit decay properties of the kernel stemming from (G_O) combined with either the cancellation condition of a or the vanishing of the kernel of $\varphi(t^2L)$ in D (obtained in Lemma 6.2.20), depending on whether the atom is usual or near D.

$\mathbf{H^1_L \subseteq H^1_D}$. We explain the case of unbounded O first. The main tool to show this inclusion is Theorem 6.4.15. To make it applicable, we reproduce $f \in \mathrm{H}^1_L$ in the functional calculus for L. After combining the reproducing formula with a duality estimate in tent spaces, it remains to prove that the (local) Carleson functional

$$C^{\mathrm{loc}}(\varphi(\boldsymbol{t}^2L^*)\,\cdot\,) \colon \mathrm{VMO}_D \to \mathrm{L}^\infty,$$

that we are going to introduce in (6.18), is bounded. To this aim, we fix a ball B centered in O and decompose the test function $\phi \in \mathrm{C_c}$ as

$$\phi = (\phi)_{B_O} + (\phi - (\phi)_{B_O})\mathbf{1}_{2B_O} + (\phi - (\phi)_{B_O})\mathbf{1}_{O\setminus 2B_O}.$$

As in the proof of $\mathrm{H}^1_D \subseteq \mathrm{H}^1_L$, the local part $C^{\mathrm{loc}}(\varphi(\boldsymbol{t}^2L^*)(\phi - (\phi)_{B_O})\mathbf{1}_{2B_O})$ can be controlled by the L^2-theory of L, while the decay properties of the kernel

q_t provide good control of the global part $C^{\mathrm{loc}}(\varphi(\boldsymbol{t}^2L^*)(\phi-(\phi)_{B_O})\mathbf{1}_{O\setminus 2B_O})$. The main innovation lies in controlling $C^{\mathrm{loc}}(\varphi(\boldsymbol{t}^2L^*)(\phi)_{B_O})$. There are three ingredients.

First, we derive in Lemma 6.5.8 a D-adapted conservation property for the operator $\varphi(t^2L)$.

Second, uniform porosity of D is used to estimate the size of boundary strips close to D. Notably, this is the first and last time in this thesis that we need property (UP).

Third, our definition of H^1_D-atoms lets us treat certain large balls as boundary terms of a telescopic sum. At this point, we stress that this change of definition also leads to a simplified argument in the case of pure Dirichlet boundary conditions compared to [26, Lem. 15].

If O is bounded, more technical challenges appear, notably a reduction to small scales in the reproducing formula and a localization of B to multiple components. In each component, a different constant is subtracted in the decomposition of ϕ.

The rest of this section is devoted to filling in the details.

6.5.3 The inclusion $\mathbf{H^1_D \subseteq H^1_L}$

For this part we need (UITC), (Fat), (LU) and (G_O). The following lemma is central. We recall the kernel q_t from Definition 6.2.21.

Lemma 6.5.7. *Assume* (Fat), (LU) *and* (G_O) *with exponent* $\mu \in (0,1]$. *Let* $\theta \in [0,1]$ *and* a *be an* H^1_D*-atom associated with* $B = B(x,r)$. *Then it holds*

$$\left|\int_{B_O} q_t(y,z)a(z)\,\mathrm{d}z\right| \lesssim t^{-d}\left(\frac{r}{t}\right)^{\theta\mu}\left(1+\frac{\mathrm{d}_{B_O}(y)}{t}\right)^{-(d+1)(1-\theta)}$$

for all $t \in (0,\mathrm{d}(O))$ *and* $y \in O$.

Proof. By the kernel estimate for q_t and the size condition of a, we obtain

$$\begin{aligned}\left|\int_{B_O} q_t(y,z)a(z)\,\mathrm{d}z\right| &\lesssim t^{-d}\left(1+\frac{\mathrm{d}_{B_O}(y)}{t}\right)^{-(d+1)}\|a\|_1 \\ &\leq t^{-d}\left(1+\frac{\mathrm{d}_{B}(y)}{t}\right)^{-(d+1)}.\end{aligned} \tag{6.7}$$

Next, if a is usual, then a satisfies the moment condition. Hence, the version of (6.5) for q_t delivers

$$\begin{aligned}\left|\int_{B_O} q_t(y,z)a(z)\,\mathrm{d}z\right| &= \left|\int_{B_O} (q_t(y,z)-q_t(y,x))a(z)\,\mathrm{d}z\right| \\ &\lesssim t^{-d}\left(\frac{r}{t}\right)^{\mu}\|a\|_1 \\ &\leq t^{-d}\left(\frac{r}{t}\right)^{\mu}. \end{aligned} \tag{6.8}$$

Otherwise, if a is near D, then in particular B is near D. We fix $x_D \in 2B \cap D$. Lemma 6.2.20 applied to L^* gives $q_t(y, x_D) = 0$. Then a similar calculation as before in which $q_t(y, x_D)$ takes the role of $q_t(y, x)$ gives (6.8) in this situation.

We conclude by taking logarithmic convex combinations between the estimates (6.7) and (6.8). □

Proof of the inclusion $\mathbf{H^1_D \subseteq H^1_L}$. Since $\mathrm{H}^1_D \subseteq \mathrm{L}^1$ is a contraction, it suffices to prove that

$$\|S^{\mathrm{loc}}_{\varphi,L}(f)\|_1 \lesssim \|f\|_{\mathrm{H}^1_D}.$$

We tacitly use Remark 6.5.2 in this proof. Let us assume for the moment that there is some $C > 0$ such that

$$\|S^{\mathrm{loc}}_{\varphi,L}(a)\|_1 \leq C \tag{6.9}$$

holds true for any H^1_D-atom a. Now, we take $f \in \mathrm{H}^1_D$ and fix an atomic decomposition $f = \sum_j \lambda_j a_j$ with $\|(\lambda_j)_j\|_{\ell^1} \leq 2\|f\|_{\mathrm{H}^1_D}$. As $f = \sum_j \lambda_j a_j$ converges in L^1, property (G_O) implies for all fixed $t \in (0, \mathrm{d}(O))$ that

$$Q_t f = \sum_j \lambda_j \varphi(t^2 L) a_j$$

converges in L^2. This yields for every $x \in O$ the estimate

$$\|Q_t f\|^2_{\mathrm{L}^2(O(x,t))} \leq \Big(\sum_j |\lambda_j| \|\varphi(t^2L)a_j\|_{\mathrm{L}^2(O(x,t))}\Big)^2,$$

which (together with monotone convergence) entails that

$$\begin{aligned} S^{\mathrm{loc}}_{\varphi,L}(f)(x) &= \left(\int_0^{\mathrm{d}(O)} \|Q_t f\|^2_{\mathrm{L}^2(O(x,t))}\,\frac{\mathrm{d}t}{t^{1+d}}\right)^{\frac{1}{2}} \\ &\leq \lim_{n\to\infty}\left(\int_0^{\mathrm{d}(O)} \Big(\sum_{j=1}^n |\lambda_j| \|\varphi(t^2L)a_j\|_{\mathrm{L}^2(O(x,t))}\Big)^2 \frac{\mathrm{d}t}{t^{1+d}}\right)^{\frac{1}{2}} \\ &\leq \sum_j |\lambda_j| S^{\mathrm{loc}}_{\varphi,L}(a_j)(x). \end{aligned}$$

Next, we take L^1-norms on both sides, apply monotone convergence and use (6.9) to get

$$\|S^{\mathrm{loc}}_{\varphi,L}(f)\|_1 \leq C \sum_j |\lambda_j| \leq 2C\|f\|_{\mathrm{H}^1_D}.$$

Proof of (6.9)**.** We let a be an H^1_D-atom associated with $B = B(x_0, r)$. The proof is divided into two parts.

Local part. By Hölder's inequality and the L^2-boundedness of $S_{\varphi,L}$, we get

$$\|S^{\mathrm{loc}}_{\varphi,L}(a)\|_{\mathrm{L}^1(O(x_0,4r))} \leq |O(x_0,4r)|^{\frac{1}{2}} \|S^{\mathrm{loc}}_{\varphi,L}(a)\|_2 \lesssim |O(x_0,4r)|^{\frac{1}{2}} \|a\|_2.$$

Now, (UITC) for O and all its components along with the size condition of a provide the uniform bound

$$\|S^{\mathrm{loc}}_{\varphi,L}(a)\|_{\mathrm{L}^1(O(x_0,4r))} \lesssim 1.$$

Global part. We fix $x \in O \setminus 4B$ and note that $r < {\mathrm{d}_B(x)}/{2}$. To find upper bounds for

$$S^{\mathrm{loc}}_{\varphi,L}(a)(x) = \left(\int_0^{\mathrm{d}(O)} \int_{O(x,t)} \left| \int_B q_t(z,y) a(z) \, \mathrm{d}z \right|^2 \frac{\mathrm{d}y \, \mathrm{d}t}{t^{1+d}} \right)^{\frac{1}{2}},$$

we split the time integral into the three parts

$$0 < t \leq r, \quad r < t \leq {\mathrm{d}_B(x)}/{2} \quad \& \quad {\mathrm{d}_B(x)}/{2} \leq t < \mathrm{d}(O).$$

We denote the corresponding integrals by (I), (II) and (III), respectively. Hence,

$$S^{\mathrm{loc}}_{\varphi,L}(a)(x) \leq \mathrm{(I)} + \mathrm{(II)} + \mathrm{(III)}.$$

Before we start estimating the right-hand side, we observe that if $t \leq {\mathrm{d}_B(x)}/{2}$ and $y \in O(x,t)$, then

$$\mathrm{d}_B(y) \geq \mathrm{d}_B(x) - |x-y| \geq \mathrm{d}_B(x) - t \geq \frac{\mathrm{d}_B(x)}{2}. \tag{6.10}$$

Estimate for (I). Lemma 6.5.7 with $\theta = 0$ joint with (6.10) gives

$$\mathrm{(I)} \lesssim \left(\int_0^r \int_{O(x,t)} \left(t^{-d} \mathrm{d}_B(x)^{-(d+1)} t^{d+1} \right)^2 \frac{\mathrm{d}y \, \mathrm{d}t}{t^{1+d}} \right)^{\frac{1}{2}} \lesssim r \, \mathrm{d}_B(x)^{-(d+1)}.$$

Estimate for (II). With $\mu \in (0,1]$ as in (G_O), we fix $\theta \in (1/(d+1+\mu), 1/(d+1))$. This gives $d + \theta\mu > (d+1)(1-\theta) > d$. We use this θ in Lemma 6.5.7 along with (6.10) to obtain

$$\begin{aligned}
\text{(II)} &\lesssim \left(\int_r^{\frac{\mathrm{d}_B(x)}{2}} \int_{O(x,t)} \left(t^{-d} \left(\frac{r}{t} \right)^{\mu\theta} t^{(d+1)(1-\theta)} \mathrm{d}_B(x)^{-(d+1)(1-\theta)} \right)^2 \frac{\mathrm{d}y\,\mathrm{d}t}{t^{1+d}} \right)^{\frac{1}{2}} \\
&\lesssim r^{\mu\theta} \mathrm{d}_B(x)^{-(d+1)(1-\theta)} \left(\int_r^{\frac{\mathrm{d}_B(x)}{2}} t^{-2(d+\mu\theta-(d+1)(1-\theta))} \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}} \\
&\lesssim r^{(d+1)(1-\theta)-d} \mathrm{d}_B(x)^{-(d+1)(1-\theta)}.
\end{aligned}$$

Estimate for (III). Lemma 6.5.7 with $\theta = 1$ gives

$$\begin{aligned}
\text{(III)} &\lesssim \left(\int_{\frac{\mathrm{d}_B(x)}{2}}^{\infty} \int_{O(x,t)} \left(t^{-d} \left(\frac{r}{t} \right)^{\mu} \right)^2 \frac{\mathrm{d}y\,\mathrm{d}t}{t^{1+d}} \right)^{\frac{1}{2}} \\
&\lesssim r^{\mu} \left(\int_{\frac{\mathrm{d}_B(x)}{2}}^{\infty} t^{-2(d+\mu)} \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}} \\
&\simeq r^{\mu} \mathrm{d}_B(x)^{-(d+\mu)}.
\end{aligned}$$

We denote $\gamma_1 \coloneqq 1$, $\gamma_2 \coloneqq (d+1)(1-\theta) - d$, $\gamma_3 \coloneqq \mu$ and recall that $\gamma_2 > 0$. Combining all three bounds and integrating over $O \setminus 4B$ yields

$$\|S^{\mathrm{loc}}_{\varphi,L}(a)\|_{\mathrm{L}^1(O\setminus 4B)} \lesssim \sum_{i=1}^{3} r^{\gamma_i} \int_{O\setminus 4B} \mathrm{d}_B(x)^{-(d+\gamma_i)}\,\mathrm{d}x \lesssim \sum_{i=1}^{3} r^{\gamma_i} \int_{4r}^{\infty} s^{-\gamma_i} \frac{\mathrm{d}s}{s} \simeq 1.$$

This completes the proof. □

6.5.4 The inclusion $\mathbf{H}^1_L \subseteq \mathbf{H}^1_D$

In this part we explicitly make use of (UITC), (UP) and (G_O), even though the other assumptions from Theorem 6.5.6 are implicitly used in some geometric reductions. We start with the D-adapted conservation property that we alluded to in Subsection 6.5.2.

Lemma 6.5.8 (D-adapted conservation property)**.** *Assume* (LU), (G_O) *and let O' be a component of O. It holds*

$$\left| \int_{O'} q_t(x,y)\,\mathrm{d}x \right| \lesssim \left(1 + \frac{\mathrm{d}_{D\cap\partial O'}(y)}{t} \right)^{-1} \qquad (t \in (0, \mathrm{d}(O)), y \in O'). \tag{6.11}$$

If $D \cap \partial O' = \varnothing$ it holds $|\int_{O'} q_t(x,y)\,\mathrm{d}x| = 0$. With the convention $\mathrm{d}_\varnothing(y) = \infty$, this is consistent with the formula given above.

Proof. The case $D \cap \partial O' = \varnothing$ follows from the classical conservation property in the case of pure Neumann boundary conditions, see [35, Prop. 6.5.6]. Hence, we assume that $\partial O'$ hits D. We fix $y \in O'$ and set $r \coloneqq \mathrm{d}_{D\cap\partial O'}(y)$, as well as $B \coloneqq B(y,r)$.

(1) The case $\boldsymbol{\varphi(z) = z\mathrm{e}^{-z}}$. We show in a first step that there are $C, c > 0$ such that

$$\left|\int_{O'} t\partial_t K_t(x,y)\,\mathrm{d}x\right| \leq C\,\mathrm{e}^{-c\left(\frac{r}{t}\right)^2} \qquad (t \in (0, \mathrm{d}(O))). \tag{6.12}$$

In particular, this implies (6.11) when $\varphi(z) = z\,\mathrm{e}^{-z}$. We let $f \in \mathrm{L}^1 \cap \mathrm{L}^2$ with $\operatorname{supp}(f) \subseteq B_O(y, r/8)$ and $\|f\|_1 = 1$. It suffices to prove that there are $C, c > 0$ such that

$$\left|\int_{O'} t^2 L\,\mathrm{e}^{-t^2L} f\,\mathrm{d}x\right| \leq C\,\mathrm{e}^{-c\left(\frac{r}{t}\right)^2} \qquad (t \in (0, \mathrm{d}(O))). \tag{6.13}$$

Indeed, assuming (6.13) and taking into account that $\frac{1}{2}t\partial_t K_t$ is the kernel associated with $-t^2L\,\mathrm{e}^{-t^2L}$ (see Remark 6.2.12), we get

$$\left|\int_{O'}\int_{B(y,\frac{r}{8})} t\partial_t K_t(x,z) f(z)\,\mathrm{d}z\,\mathrm{d}x\right| \leq C\,\mathrm{e}^{-c\left(\frac{r}{t}\right)^2}. \tag{6.14}$$

Now, we pick a non-negative $g \in \mathrm{C}_\mathrm{c}^\infty(B(y,1))$ with $\int_{\mathbb{R}^d} g\,\mathrm{d}x = 1$, we put $g_n(x) \coloneqq n^d g(nx)$ for all $n \in \mathbb{N}$ and choose $N \in \mathbb{N}$ such that $1/N \leq r/8$ and $B(y, 2/N) \subseteq O'$. For $n \geq N$, we calculate

$$\begin{aligned}\left|\int_{O'}\int_{B(y,\frac{1}{n})} t\partial_t K_t(x,z) g_n(z)\,\mathrm{d}z\,\mathrm{d}x - \int_{O'} t\partial_t K_t(x,y)\,\mathrm{d}x\right| \\ \leq \int_{O'}\int_{B(y,\frac{1}{n})} |t\partial_t K_t(x,z) - t\partial_t K_t(x,y)| g_n(z)\,\mathrm{d}z\,\mathrm{d}x.\end{aligned} \tag{6.15}$$

By construction, (6.14) can be applied with $f = g_n$ for all $n \geq N$. Therefore, (6.12) follows if the right-hand side of (6.15) goes to 0 as $n \to \infty$. To this end, we split its x-integral into the two parts $B(y, 2/n)$ and $O' \setminus B(y, 2/n)$. We denote the corresponding integrals by I_loc and I_glob, respectively.

Estimate for $\mathbf{I_{loc}}$. We use the Hölder estimate (G3) for $t\partial_t K_t$ to get

$$\begin{aligned}\mathrm{I}_\mathrm{loc} &\lesssim t^{-d-\mu}\int_{B(y,\frac{2}{n})}\int_{B(y,\frac{1}{n})} |y-z|^\mu g_n(z)\,\mathrm{d}z\,\mathrm{d}x \\ &\lesssim t^{-d-\mu} n^{-\mu-d} \longrightarrow 0 \qquad (n \to \infty).\end{aligned}$$

Estimate for $\mathrm{I}_{\mathrm{glob}}$. We let $\nu \in (0,\mu)$ and $x \in O' \setminus B(y, 2/n)$. Consequently, $|y-z| \leq 1/n \leq |x-y|/2$ for all $z \in B(y,1/n)$ and $x \in O' \setminus B(y,2/n)$. Remark 5.2.2 (ii) delivers for all such x and z the estimate

$$|t\partial_t K_t(x,y) - t\partial_t K_t(x,z)| \lesssim t^{-d-\nu}|y-z|^\nu \, \mathrm{e}^{-c\left(\frac{|x-y|}{t}\right)^2} \lesssim_t \frac{1}{n^\nu}\, \mathrm{e}^{-c\left(\frac{|x-y|}{t}\right)^2}.$$

Hence, we obtain that

$$\begin{aligned}\mathrm{I}_{\mathrm{glob}} &\lesssim_t n^{-\nu-d} \int_{O'\setminus B(y,\frac{2}{n})} \mathrm{e}^{-c\left(\frac{|x-y|}{t}\right)^2}\,\mathrm{d}x \\ &\lesssim n^{-\nu-d} \int_0^\infty \mathrm{e}^{-c\left(\frac{r}{t}\right)^2} r^d \,\frac{\mathrm{d}r}{r} \longrightarrow 0 \qquad (n\to\infty).\end{aligned}$$

Therefore, it only remains to show (6.13). To this end, we choose $\psi \in \mathrm{C}_{\mathrm{c}}^\infty(\mathbb{R}^d)$ such that $\mathbf{1}_{B(y,r/4)} \leq \psi \leq \mathbf{1}_{B(y,r/2)}$ and $\|\nabla\psi\|_\infty \lesssim r^{-1}$. We decompose

$$\begin{aligned}\left|\int_{O'} t^2 L\,\mathrm{e}^{-t^2L} f \,\mathrm{d}x\right| &\leq |(t^2 L\,\mathrm{e}^{-t^2L} f \,|\, \mathbf{1}_{O'}\psi)| + |(t^2 L\,\mathrm{e}^{-t^2L} f \,|\, (\mathbf{1}_{O'} - \mathbf{1}_{O'}\psi))| \\ &=: \mathrm{(I)} + \mathrm{(II)}.\end{aligned}$$

Estimate for (I). Since $O' = O_D$, we conclude by Lemma 6.2.7 (i) and our choice of ψ that $\mathbf{1}_{O'}\psi \in \mathrm{W}_D^{1,2}$ and $\nabla(\mathbf{1}_{O'}\psi) = \mathbf{1}_{O'}\nabla\psi$. Hence, we can bound

$$\begin{aligned}|(t^2 L\,\mathrm{e}^{-t^2L} f \,|\, \mathbf{1}_{O'}\psi)| &= t^2 |(A\nabla\,\mathrm{e}^{-t^2L} f \,|\, \mathbf{1}_{O'}\nabla\psi)| \\ &\lesssim t^2 \|\mathbf{1}_{O'\setminus\frac{1}{4}B}\nabla\,\mathrm{e}^{-t^2L} f\|_2 \|\nabla\psi\|_2 \\ &\lesssim r^{\frac{d}{2}-1} t \|\mathbf{1}_{O'\setminus\frac{1}{4}B} t\nabla\,\mathrm{e}^{-t^2L} f\|_2.\end{aligned}$$

A particular consequence of (G_O) are $\mathrm{L}^1 - \mathrm{L}^2$ off-diagonal estimates for the family $(\mathrm{e}^{-t^2L})_{t\in(0,\mathrm{d}(O))}$. By composition, also $(t\nabla\,\mathrm{e}^{-t^2L})_{t\in(0,\mathrm{d}(O))}$ satisfies L^1–L^2 off-diagonal estimates and hence

$$|(t^2 L\,\mathrm{e}^{-t^2L} f \,|\, \mathbf{1}_{O'}\psi)| \lesssim \left(\frac{r}{t}\right)^{\frac{d}{2}-1} \mathrm{e}^{-c\left(\frac{r}{t}\right)^2} \lesssim \mathrm{e}^{-c\left(\frac{r}{t}\right)^2}.$$

Estimate for (II). Since (G_O) implies that $(t^2 L\,\mathrm{e}^{-t^2L})_{t\in(0,\mathrm{d}(O))}$ satisfies L^1 off-diagonal estimates, it follows

$$|(t^2 L\,\mathrm{e}^{-t^2L} f \,|\, \mathbf{1}_{O'}(1-\psi))| \leq \|\mathbf{1}_{O'\setminus B(y,r/4)} t^2 L\,\mathrm{e}^{-t^2L} f\|_1 \lesssim \mathrm{e}^{-c\left(\frac{r}{t}\right)^2}.$$

(2) The case $\varphi(z) = \sqrt{z}\mathrm{e}^{-\sqrt{z}}$. We differentiate the subordination formula (6.3) to get

$$\begin{aligned} t\partial_t p_t(x,y) &= \frac{2}{\sqrt{\pi}} \int_0^\infty \mathrm{e}^{-s^2} \tfrac{t}{2s} (\partial_t K)_{\frac{t}{2s}}(x,y)\,\mathrm{d}s \\ &= \frac{2}{\sqrt{\pi}} \int_0^{t/2\mathrm{d}(O)} \mathrm{e}^{-s^2} \tfrac{t}{2s} (\partial_t K)_{\frac{t}{2s}}(x,y)\,\mathrm{d}s \\ &\quad + \frac{2}{\sqrt{\pi}} \int_{t/2\mathrm{d}(O)}^\infty \mathrm{e}^{-s^2} \tfrac{t}{2s} (\partial_t K)_{\frac{t}{2s}}(x,y)\,\mathrm{d}s. \end{aligned}$$

Now, we use Fubini's theorem, [26, Lem. A.1] for the first integral and (6.12) for the second one in order to deduce

$$\begin{aligned} \left| \int_{O'} t\partial_t p_t(x,y)\,\mathrm{d}x \right| &\lesssim \int_0^{t/2\mathrm{d}(O)} \mathrm{e}^{-s^2}\,\mathrm{d}s + \int_0^\infty \mathrm{e}^{-(1+c\left(\frac{r}{t}\right)^2)s^2}\,\mathrm{d}s \\ &\lesssim t\mathbf{1}_{[\mathrm{d}(O)<\infty]} + \left(1+\frac{r}{t}\right)^{-1} \\ &\lesssim \left(1+\frac{r}{t}\right)^{-1}, \end{aligned}$$

where the implicit constant depends on $\mathrm{d}(O)$ if O is bounded. □

Proof of the inclusion $\mathbf{H^1_L \subseteq H^1_D}$. We start with an approximation argument. We take $f \in \mathrm{H}^1_L \subseteq \mathrm{L}^1_0$ and denote $f_n \coloneqq P_{1/n} f$. Then $f_n \in \mathrm{L}^1 \cap \overline{\mathrm{ran}(L)}$ by Remark 6.5.5 (ii). We fix $0 < \varepsilon < b < \mathrm{d}(O)$ and obtain from an integration by parts

$$P_\varepsilon f_n + Q_{2\varepsilon} f_n = P_b f_n + Q_{2b} f_n + 4\int_\varepsilon^b Q_t^2 f_n \,\frac{\mathrm{d}t}{t}.$$

By Remark 6.5.5 (iii), we can take the L^1-limit $n \to \infty$ and afterwards the L^1-limit $\varepsilon \to 0$ to get

$$f = (P_b f + Q_{2b} f) + 4\int_0^b Q_t^2 f \,\frac{\mathrm{d}t}{t} \eqqcolon f_1 + f_2.$$

Now, we claim that

$$|(f_2 \,|\, \phi)| = 4\left| \int_0^b (Q_t f \,|\, Q_t^* \phi)\,\frac{\mathrm{d}t}{t} \right| \lesssim \|S^{\mathrm{loc}}_{\varphi,L}(f)\|_1 \|\phi\|_{\mathrm{BMO}_D} \qquad (\phi \in \mathrm{C_c}), \tag{6.16}$$

where the implicit constant is independent of b. Assuming (6.16) for the moment, we can conclude as follows.

(1) O is bounded. In this case, we send $b \to \mathrm{d}(O)$ and deduce from (6.16) that $f_2 \in (\mathrm{VMO}_D)^*$. Now, Theorem 6.4.15 yields $\|f_2\|_{\mathrm{H}^1_D} \lesssim \|S^{\mathrm{loc}}_{\varphi,L}(f)\|_1$. Next, we show that f_1 is a finite linear combination of H^1_D-atoms. We write $f_1 = \sum_{O' \in \Sigma} \mathbf{1}_{O'} f_1$. First, we deduce with (G_O) the bound

$$\|f_1\|_2 = \|P_{\mathrm{d}(O)} f + Q_{2\mathrm{d}(O)} f\|_2 \lesssim \mathrm{d}(O)^{-\frac{d}{2}} \|f\|_1. \tag{6.17}$$

As $O' = O' \cap B(x, \mathrm{d}(O))$ for any $x \in O'$ and keeping in mind that all components satisfy (ITC) by Lemma 6.2.7, we get $\mathrm{d}(O)^d \simeq |B_O(x, \mathrm{d}(O))|$. It follows that $\mathbf{1}_{O'} f_1$ satisfies the localization and size conditions of an H^1_D-atom associated with the ball $B(x, \mathrm{d}(O))$. Moreover, $P_{2\mathrm{d}(O)} + Q_{2\mathrm{d}(O)}$ maps L^1_0 into $\overline{\mathrm{ran}(L)} = \mathrm{L}^2_0$ by Remark 6.5.5 (ii). Therefore, in the case that $O' = O_m$ is a Neumann component, the cancellation condition is fulfilled as well. Hence, we infer $f_1 \in \mathrm{H}^1_D$ and $\|f_1\|_{\mathrm{H}^1_D} \lesssim \|f\|_1$.

(2) O is unbounded. Arguing as in (6.17), we deduce the estimate $|(f_1 \,|\, \phi)| \lesssim b^{-\frac{d}{2}} \|f\|_1 \|\phi\|_2$, which vanishes as $b \to \infty$. Since (6.16) holds for all $b > 0$, we infer $|(f \,|\, \phi)| \lesssim \|S^{\mathrm{loc}}_{\varphi,L}(f)\|_1 \|\phi\|_{\mathrm{BMO}_D}$. This allows to conclude using Theorem 6.4.15 right away.

It only remains to prove (6.16). To do so, we rely on a duality estimate in tent spaces. This duality estimate involves the so-called **local Carleson functional** defined for a measurable function $g\colon (0, \mathrm{d}(O)) \times O \to \mathbb{C}$ by

$$C^{\mathrm{loc}}(g)(x) \coloneqq \sup_{B \ni x} \left(\int_0^{r(B)} \fint_{O \cap B} |g(t,y)|^2 \,\mathrm{d}y \,\frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}} \qquad (x \in O). \tag{6.18}$$

Here, the supremum is taken over all Euclidean balls centered in O that contain x and are of radius $r(B) < \mathrm{d}(O)$. The duality estimate reads then as follows.

Lemma 6.5.9 ([3, Lem. 3.16 & Cor. 3.17])**.** *Assume* (UITC)*. Then there is $C > 0$ such that we have for all measurable functions $f, g\colon (0, \mathrm{d}(O)] \times O \to \mathbb{C}$ satisfying $S^{\mathrm{loc}}(f) \in \mathrm{L}^1$ and $C^{\mathrm{loc}}(g) \in \mathrm{L}^\infty$ and for all $b \in (0, \mathrm{d}(O)]$ the estimate*

$$\left| \int_0^b \int_O f(t,x) g(t,x) \,\mathrm{d}x \,\frac{\mathrm{d}t}{t} \right| \leq C \|S^{\mathrm{loc}}(f)\|_1 \|C^{\mathrm{loc}}(g)\|_\infty.$$

Now, we let $b \in (0, \mathrm{d}(O)]$ and $\phi \in \mathrm{C_c}$. For convenience, we assume that ϕ is normalized in BMO_D. We invoke Lemma 6.5.9 to infer

$$\left| \int_0^b (Q_t f \,|\, Q_t^* \phi) \,\frac{\mathrm{d}t}{t} \right| \lesssim \|S^{\mathrm{loc}}_{\varphi,L}(f)\|_1 \|C^{\mathrm{loc}}(Q_t^* \phi)\|_\infty.$$

To bound $\|C^{\mathrm{loc}}(Q_{\boldsymbol{t}}^*\phi)\|_\infty$, we fix $x \in O$ and take a ball B centered in $x_B \in O$ that contains x with $r = r(B) < \mathrm{d}(O)$. At this point, we recall from Corollary 6.2.14 that O_N is empty provided that O is unbounded and, in the case that O is bounded, from Lemma 6.2.7 (iii) that O and all components satisfy (ITC). We pick $O'' \in \Sigma$ with $x_B \in O''$. For each component $O' \in \Sigma$ with $O' \neq O''$ and $O' \cap B \neq \varnothing$, we fix some $z \in O' \cap B$. It follows

$$O \cap B \subseteq B_O \cup \Big(\bigcup_{O'} B_O(z, 2r)\Big),$$

where the union is taken over all such components O'. We write

$$\phi = \sum_{O' \in \Sigma} \phi \mathbf{1}_{O'} =: \sum_{O' \in \Sigma} \phi_{O'}.$$

We fix one component O' and put either $\widetilde{B} = B$ if $O' = O''$ or $\widetilde{B} = B(z, 2r)$, where $z \in O' \cap B$ from the above decomposition. We split

$$\phi_{O'} = (\phi)_{\widetilde{B}_O} \mathbf{1}_{O'} + (\phi - (\phi)_{\widetilde{B}_O}) \mathbf{1}_{2\widetilde{B}_O} + (\phi - (\phi)_{\widetilde{B}_O}) \mathbf{1}_{O' \setminus 2\widetilde{B}}. \tag{6.19}$$

First, we focus on the case that $O' = O_D$ and explain the necessary changes for the pure Neumann components $O' = O_m$ afterwards. Let us write $\phi_i \coloneqq \phi_{i,O'}$ for the three terms on the right-hand side of (6.19). It suffices to bound

$$\sum_{i=1}^{3} \int_0^r \fint_{O \cap B} |(Q_t^* \phi_i)(y)|^2 \, \frac{\mathrm{d}y \, \mathrm{d}t}{t} =: (\mathrm{I}) + (\mathrm{II})_{\mathrm{loc}} + (\mathrm{II})_{\mathrm{glob}} \lesssim 1.$$

Estimate for $(\mathrm{II})_{\mathrm{loc}}$. McIntosh's theorem and (UITC) yield

$$(\mathrm{II})_{\mathrm{loc}} \lesssim |O \cap B|^{-1} \|\phi_2\|_2^2 \simeq |2\widetilde{B}_O|^{-1} \|\phi_2\|_2^2 = \fint_{2\widetilde{B}_O} |\phi - (\phi)_{\widetilde{B}_O}|^2 \, \mathrm{d}x \lesssim 1.$$

Estimate for $(\mathrm{II})_{\mathrm{glob}}$. We write

$$\phi_3 = \sum_{j \geq 1} \mathbf{1}_{O' \cap C_j(\widetilde{B})} (\phi - (\phi)_{\widetilde{B}_O}) =: \sum_{j \geq 1} \phi_3^{(j)}$$

and invoke (G_O) as well as (UITC) to get

$$\begin{aligned}
(\mathrm{II})_{\mathrm{glob}} &= \int_0^r \fint_{O\cap B} |(Q_t^*\phi_3)(y)|^2 \,\mathrm{d}y \,\frac{\mathrm{d}t}{t} \\
&\le \int_0^r \fint_{O\cap B} \Big(\sum_{j\ge 1} |(Q_t^*\phi_3^{(j)})(y)|\Big)^2 \,\mathrm{d}y \,\frac{\mathrm{d}t}{t} \\
&\lesssim \int_0^r \fint_{O\cap B} \Big(\sum_{j\ge 1} t^{-d}\big(2^j r/t\big)^{-(d+1)} \|\phi_3^{(j)}\|_1\Big)^2 \,\mathrm{d}y \,\frac{\mathrm{d}t}{t} \\
&= \int_0^r t\,\mathrm{d}t \cdot \Big(\sum_{j\ge 1} (2^j r)^{-(d+1)} \|\phi_3^{(j)}\|_1\Big)^2 \\
&\simeq r^2 \Big(\sum_{j\ge 1} (2^j r)^{-(d+1)} \|\phi_3^{(j)}\|_1\Big)^2 \\
&\lesssim \Big(\sum_{j\ge 1} 2^{-j} \fint_{2^{j+1}\widetilde{B}_O} |\phi(z) - (\phi)_{\widetilde{B}_O}| \,\mathrm{d}z\Big)^2,
\end{aligned}$$

where we have used in the second estimate that $|z-y| \simeq 2^j r$ for all $y \in O\cap B$ and $z \in O' \cap C_j(\widetilde{B})$. Next, using (UITC), we estimate

$$\begin{aligned}
&\fint_{2^{j+1}\widetilde{B}_O} |\phi(z) - (\phi)_{\widetilde{B}_O}| \,\mathrm{d}z \\
&\le \fint_{2^{j+1}\widetilde{B}_O} |\phi(z) - (\phi)_{2^{j+1}\widetilde{B}_O}| \,\mathrm{d}z + \sum_{k=0}^{j} |(\phi)_{2^{k+1}\widetilde{B}_O} - (\phi)_{2^k\widetilde{B}_O}| \\
&\lesssim 1 + \sum_{k=0}^{j} \fint_{2^k\widetilde{B}_O} |\phi(z) - (\phi)_{2^{k+1}\widetilde{B}_O}| \,\mathrm{d}z \\
&\lesssim 1 + \sum_{k=0}^{j} \fint_{2^{k+1}\widetilde{B}_O} |\phi(z) - (\phi)_{2^{k+1}\widetilde{B}_O}| \,\mathrm{d}z \\
&\lesssim j.
\end{aligned}$$

In total, we get $(\mathrm{II})_{\mathrm{glob}} \lesssim (\sum_{j\ge 1} j 2^{-j})^2 \lesssim 1$.

Estimate for (I). In view of Lemma 6.5.8 applied with O' we have

$$(\mathrm{I}) \lesssim \left(\int_0^r \fint_{O\cap B} \left(1 + \frac{\mathrm{d}_{D\cap\partial O'}(y)}{t}\right)^{-2} \mathrm{d}y \,\frac{\mathrm{d}t}{t}\right) |(\phi)_{\widetilde{B}_O}|^2.$$

We write $x_{\widetilde{B}}$ for the center of $\widetilde{B}$. To bound the integral on the right, we distinguish two cases.

(1) $\mathbf{d_{D\cap\partial O'}(x_{\widetilde{B}}) < 2r}$. By assumption $\widetilde{B}$ is near D. We set

$$h(y) := \left(1 + \frac{\mathrm{d}_{D\cap\partial O'}(y)}{t}\right)^{-2} \qquad (y \in \widetilde{B}),$$

and note that for all $\alpha \in (0,1)$ it holds

$$h(y) > \alpha \quad \Longleftrightarrow \quad \mathrm{d}_{D\cap\partial O'}(y) < t(\alpha^{-\frac{1}{2}} - 1) =: s.$$

The layer cake formula joint with the substitution $s = t(\alpha^{-1/2} - 1)$, that is $\mathrm{d}s = -\frac{t}{2}\alpha^{-\frac{3}{2}}\,\mathrm{d}\alpha$, gives

$$\begin{aligned}\int_0^r \int_{O\cap B} \left(1 + \frac{\mathrm{d}_{D\cap\partial O'}(y)}{t}\right)^{-2} \mathrm{d}y\,\frac{\mathrm{d}t}{t} \\ &= \int_0^r \int_0^1 |\{y \in O\cap B : h(y) > \alpha\}|\,\mathrm{d}\alpha\,\frac{\mathrm{d}t}{t} \\ &= 2\int_0^r \int_0^\infty |\{y \in O\cap B : \mathrm{d}_{D\cap\partial O'}(y) < s\}| \left(1 + \frac{s}{t}\right)^{-3} \mathrm{d}s\,\frac{\mathrm{d}t}{t^2}.\end{aligned}$$

At this point we use assumption (UP). Thus, Lemma 6.2.8 furnishes some $\eta \in (0,1)$ such that

$$|\{y \in O\cap B : \mathrm{d}_{D\cap\partial O'}(y) < s\}| \lesssim s^\eta r^{d-\eta}.$$

Hence, using the substitution $s \mapsto ts$, the fact that $\eta < 2$ and (UITC) we get the upper estimate

$$\begin{aligned}\int_0^r \fint_{O\cap B} \left(1 + \frac{\mathrm{d}_{D\cap\partial O'}(y)}{t}\right)^{-2} \mathrm{d}y\,\frac{\mathrm{d}t}{t} &\lesssim r^{-\eta}\int_0^r \left(\int_0^\infty s^\eta \left(1+\frac{s}{t}\right)^{-3} \mathrm{d}s\right)\frac{\mathrm{d}t}{t^2} \\ &\simeq r^{-\eta}\int_0^r t^\eta\,\frac{\mathrm{d}t}{t} \\ &\simeq 1.\end{aligned}$$

In summary, we have (I) $\lesssim |(\phi)_{\widetilde{B}_O}|^2$. But since $\widetilde{B}$ is near D, $|(\phi)_{\widetilde{B}_O}| \le 1$ by Hölder's inequality and the very definition of the BMO_D-norm. Therefore, (I) $\lesssim 1$ and we are done with this case. Let us stress that the localization of ϕ to the components of O is used in a crucial way here, for the BMO_D-norm only gives control on $\widetilde{B}_O = O' \cap \widetilde{B}$ and not on the full $O \cap B$.

(2) $2r \le \mathrm{d}_{D\cap\partial O'}(x_{\widetilde{B}})$. In the current case it holds $\mathrm{d}_{D\cap\partial O'}(x_{\widetilde{B}}) \simeq \mathrm{d}_{D\cap\partial O'}(y)$ for all $y \in \widetilde{B}$ and hence

$$\begin{aligned}\int_0^r \fint_{O\cap B} \left(1 + \frac{\mathrm{d}_{D\cap\partial O'}(y)}{t}\right)^{-2} \mathrm{d}y\,\frac{\mathrm{d}t}{t} &\lesssim \mathrm{d}_{D\cap\partial O'}(x_{\widetilde{B}})^{-2}\int_0^r t^2\,\frac{\mathrm{d}t}{t} \\ &\simeq \left(\frac{r}{\mathrm{d}_{D\cap\partial O'}(x_{\widetilde{B}})}\right)^2.\end{aligned}$$

Now we treat $|(\phi)_{\widetilde{B}_O}|$. To this end, we pick the smallest $k \geq 1$ with $2^{k+1}\widetilde{B} \cap (D \cap \partial O') \neq \varnothing$. Such k exists since $O' \neq O_m$ by assumption and it follows that $2^k\widetilde{B}$ is a ball near D. Using that $2^k r \simeq \mathrm{d}_{D\cap\partial O'}(x_{\widetilde{B}})$, we get by a telescope sum and (UITC) that

$$|(\phi)_{\widetilde{B}_O}| \leq \sum_{j=0}^{k-1} |(\phi)_{2^j\widetilde{B}_O} - (\phi)_{2^{j+1}\widetilde{B}_O}| + |(\phi)_{2^k\widetilde{B}_O}| \lesssim k \simeq \ln\left(\frac{\mathrm{d}_{D\cap\partial O'}(x)}{r}\right).$$

Since $\sup_{\lambda\geq 1} {}^{\ln(\lambda)}\!/\!_{\lambda}$ is finite, we derive $(\mathrm{I}) \lesssim 1$ from the last two estimates as desired.

Finally, if $O' = O_m$, then the terms $(\mathrm{II})_{\mathrm{loc}}$ and $(\mathrm{II})_{\mathrm{glob}}$ are handled in the same way. The treatment of (I) is even easier: this term vanishes right away in virtue of the conservation property on O_m, see Lemma 6.5.8. This completes the proof. □

Let us show *a posteriori* that H^1_L can be equipped with several equivalent norms. In the sequel, we are going to use this fact freely.

Proposition 6.5.10. *Assume* (UITC), (Fat), (LU) *and* (UP), *and that* L *satisfies* (G_O). *Then* H^1_L *can be equipped with the equivalent norms*

$$\|f\|_{\mathrm{H}^1_D} \simeq \|S_{\varphi,L}(f)\|_1 \simeq \|S_{\varphi,L}(f)\|_1 + \|f\|_1 \simeq \|S^{\mathrm{loc}}_{\varphi,L}(f)\|_1 + \|f\|_1.$$

Proof. We argue only for the Poisson case for the other one can be treated similarly. Let us begin with the case that O is unbounded. Then Theorem 6.5.6 yields the first norm equivalence. As the inclusion $\mathrm{H}^1_D \subseteq \mathrm{L}^1$ is a contraction, the second equivalence follows and we are done, because $S^{\mathrm{loc}}_{\varphi,L} = S_{\varphi,L}$.

Now, we assume that O is bounded. We start with the last equivalence. It suffices to prove $\|S^{\mathrm{glob}}_{\varphi,L}(f)\|_1 \lesssim \|f\|_1$. To this end, we use Hölder's inequality, the averaging trick, the substitution $t \mapsto t + {}^{\mathrm{d}(O)}\!/\!_2$, the square function estimate in L^2 and (G_O) to obtain

$$\begin{aligned}
\|S^{\mathrm{glob}}_{\varphi,L}(f)\|_1 &\leq |O|^{\frac{1}{2}} \|S^{\mathrm{glob}}_{\varphi,L}(f)\|_2 \\
&= |O|^{\frac{1}{2}} \left(\int_{\mathrm{d}(O)}^{\infty} \int_O |\sqrt{L}\,\mathrm{e}^{-(t-\frac{\mathrm{d}(O)}{2})\sqrt{L}}\, P_{\frac{\mathrm{d}(O)}{2}} f(y)|^2 \,\mathrm{d}y\,\mathrm{d}t\right)^{\frac{1}{2}} \\
&= |O|^{\frac{1}{2}} \left(\int_{\frac{\mathrm{d}(O)}{2}}^{\infty} \int_O |\sqrt{L}\,\mathrm{e}^{-t\sqrt{L}}\, P_{\frac{\mathrm{d}(O)}{2}} f(y)|^2 \,\mathrm{d}y\,\mathrm{d}t\right)^{\frac{1}{2}} \\
&\leq |O|^{\frac{1}{2}} \|S_{\varphi,L}(P_{\frac{\mathrm{d}(O)}{2}} f)\|_2 \\
&\lesssim |O|^{\frac{1}{2}} \|P_{\frac{\mathrm{d}(O)}{2}} f\|_2 \\
&\lesssim \|f\|_1.
\end{aligned}$$

Next, Theorem 6.5.6 joint with the first part of this proof reveals

$$\|f\|_{\mathrm{H}^1_D} \simeq \|S^{\mathrm{loc}}_{\varphi,L}(f)\|_1 + \|f\|_1 \simeq \|S_{\varphi,L}(f)\|_1 + \|f\|_1.$$

In order to show that the L^1-norm can be dropped, we invoke the molecular theory that we will discuss in Section 6.6. Since any $f \in \mathrm{H}^1_{D,a}$ belongs to $\mathrm{L}^2_0 = \overline{\mathrm{ran}(L)}$ (see Proposition 6.2.9) it follows from Lemma 6.6.18 that

$$\|f\|_{\mathrm{H}^1_D} \simeq \|S^{\mathrm{loc}}_{\varphi,L}(f)\|_1 + \|f\|_1 \simeq \|S_{\varphi,L}(f)\|_1 + \|f\|_1 \simeq \|S_{\varphi,L}(f)\|_1.$$

Since $\mathrm{H}^1_{D,a}$ is by definition dense in $\mathrm{H}^1_D = \mathrm{H}^1_L$ and the first as well as the second norm equivalence holds for all $f \in \mathrm{H}^1_L$, the claim follows by density. □

6.5.5 The equality $\mathrm{H}^1_z = \mathrm{H}^1_{\mathrm{CW}}$

Let us close this section by giving the proof of Theorem 6.3.14. On recalling the definition of H^1_z from Definition 6.3.12, we can record

Corollary 6.5.11. *If O is unbounded, $D = \varnothing$ and L satisfies (G_O), then $\mathrm{H}^1_z \subseteq \mathrm{H}^1_L$.*

Proof. Using a trick from [26, Sec. 3.2], the conclusion follows from the same calculation as in the proof of the inclusion $\mathrm{H}^1_D \subseteq \mathrm{H}^1_L$ in Subsection 6.5.3. The argument is as follows. The kernel q_t on O can be extended to a kernel $\widetilde{q}_t$ on $\mathbb{R}^d$ still satisfying (6.4) and (6.5). We need that O is unbounded since otherwise the kernel bounds for $\widetilde{q}$ are only valid for finite times t and not up to $\mathrm{diam}(\mathbb{R}^d) = \infty$. Now we take $f \in \mathrm{H}^1_z$. Its zero extension F from O to $\mathbb{R}^d$ is in $\mathrm{H}^1(\mathbb{R}^d)$. It is classical that F admits an atomic decomposition $F = \sum_j \lambda_j a_j$ into usual $\mathrm{H}^1_\varnothing(\mathbb{R}^d)$-atoms. We observe that

$$\int_O q_t(z,y) f(z)\,\mathrm{d}z = \int_{\mathbb{R}^d} \widetilde{q}_t(z,y) F(z)\,\mathrm{d}z$$

by the support property of f. Therefore, all arguments from the proof of the inclusion $\mathrm{H}^1_D \subseteq \mathrm{H}^1_L$ in Subsection 6.5.3 apply verbatim when the operator $\varphi(t^2L)$ is replaced by the operator on $\mathbb{R}^d$ associated with the kernel $\widetilde{q}_t$. □

Proof of Theorem 6.3.14. By Corollary 6.5.11, Theorem 6.5.6 and Proposition 6.4.7, it holds $\mathrm{H}^1_z \subseteq \mathrm{H}^1_L = \mathrm{H}^1_\varnothing = \mathrm{H}^1_{\mathrm{CW}}$. The converse inclusion follows from Lemma 6.3.13. □

6.6 Maximal characterization for hermitian coefficients

Throughout this section, we assume that L satisfies (G_O) and is self-adjoint. The goal is to enrich the result derived in Theorem 6.5.6 by a maximal function characterization. This extends the results given in [48, Sec. 4] on the pure Dirichlet and Neumann cases to mixed boundary conditions. For the negative Laplacian subject to mixed boundary conditions on a bounded domain, our very mild geometric assumptions let us recover the full picture of classical Hardy space theory in this way, see Theorem 6.6.5.

Definition 6.6.1. For $f \in \mathrm{L}^2$ we define the **non-tangential maximal function** adapted to L by

$$f_L^*(x) \coloneqq \sup_{t>0} \sup_{y \in O(x,t)} |\,\mathrm{e}^{-tL} f(y)| \qquad (x \in O),$$

and the **radial maximal function** adapted to L by

$$f_L^+(x) \coloneqq \sup_{t>0} |\,\mathrm{e}^{-tL} f(x)| \qquad (x \in O).$$

Remark 6.6.2. As a consequence of (G_O), the pointwise evaluation of $\mathrm{e}^{-tL} f$ is well-defined.

This enables us to define L-adapted Hardy spaces using a maximal function characterization.

Definition 6.6.3. The **non-tangential** and **radial maximal pre-Hardy spaces** $\mathbb{H}^1_{L,\mathrm{max}}$ and $\mathbb{H}^1_{L,\mathrm{rad}}$ are defined as

$$\mathbb{H}^1_{L,\mathrm{max}} \coloneqq \left\{ f \in \mathrm{L}^2 \colon f_L^* \in \mathrm{L}^1 \right\}, \qquad \|f\|_{\mathbb{H}^1_{L,\mathrm{max}}} \coloneqq \|f_L^*\|_1,$$

and

$$\mathbb{H}^1_{L,\mathrm{rad}} \coloneqq \left\{ f \in \mathrm{L}^2 \colon f_L^+ \in \mathrm{L}^1 \right\}, \qquad \|f\|_{\mathbb{H}^1_{L,\mathrm{rad}}} \coloneqq \|f_L^+\|_1.$$

The **non-tangential** and **radial maximal Hardy spaces** $\mathbf{H}^1_{L,\mathrm{max}}$ and $\mathbf{H}^1_{L,\mathrm{rad}}$ are defined as the completions of $\mathbb{H}^1_{L,\mathrm{max}}$ and $\mathbb{H}^1_{L,\mathrm{rad}}$, respectively.

The abstract main result of this section is the following.

Theorem 6.6.4. *Assume that L is self-adjoint and satisfies* (G_O). *Suppose that the geometric requirements* (UITC), (Fat), (LU) *and* (UP) *are fulfilled. Then it holds*

$$\mathrm{H}^1_{L,\max} = \mathrm{H}^1_{L,\mathrm{rad}} = \mathrm{H}^1_L \oplus \ell^1(I) = \mathrm{H}^1_D \oplus \ell^1(I).$$

The framework developed in Chapter 5 lets us conclude a powerful application for the negative Laplacian $-\Delta_D$ on a bounded open set.

Theorem 6.6.5. *Let $O \subseteq \mathbb{R}^d$ be open and bounded, and assume* (ITC), (Fat), (LU) *and* (UP). *Then we have the identities*

$$\{f \in \mathrm{L}^1 \colon S_{-\Delta_D}(f) \in \mathrm{L}^1\} = \mathrm{H}^1_D \oplus \ell^1(I) = \mathrm{H}^1_{-\Delta_D,\max} = \mathrm{H}^1_{-\Delta_D,\mathrm{rad}},$$

where the space on the left-hand side is equipped with the norm

$$\|f\| \coloneqq \|f\|_1 + \|S_{-\Delta_D}(f)\|_1.$$

Remark 6.6.6. (i) We wish to emphasize that both the space $\mathrm{H}^1_D \oplus \ell^1(I)$ and the assumptions of Theorem 6.6.5 are purely geometrical and independent of any operator.

(ii) Theorem 6.6.5 is formulated for the negative Laplacian for convenience, but it is true for any elliptic operator with real and self-adjoint coefficient matrix.

(iii) We assume that O is bounded in Theorem 6.6.5 to ensure that $-\Delta_D$ has property (G_O), see Theorems 5.0.1 and 5.0.2 and compare with Remark 6.2.11.

We summarize as corollaries the pure Dirichlet and Neumann cases. This takes Proposition 6.4.7 into account.

Corollary 6.6.7 (pure Dirichlet case)**.** *Let $O \subseteq \mathbb{R}^d$ be open and bounded, and assume* (ITC), (Fat), *and that ∂O is porous. Then we have the identities*

$$\{f \in \mathrm{L}^1 \colon S_{-\Delta_{\partial O}}(f) \in \mathrm{L}^1\} = \mathrm{H}^1_{\mathrm{Mi}} = \mathrm{H}^1_{-\Delta_{\partial O},\max} = \mathrm{H}^1_{-\Delta_{\partial O},\mathrm{rad}},$$

where the space on the left-hand side carries the norm $\|f\| \coloneqq \|S_{-\Delta_{\partial O}}(f)\|_1$.

Corollary 6.6.8 (pure Neumann case)**.** *Let $O \subseteq \mathbb{R}^d$ be open and bounded, and assume* (LU). *Then we have the identities*

$$\{f \in \mathrm{L}^1 \colon S_{-\Delta_\varnothing}(f) \in \mathrm{L}^1\} = \mathrm{H}^1_{\mathrm{CW}} = \mathrm{H}^1_{-\Delta_\varnothing,\max} = \mathrm{H}^1_{-\Delta_\varnothing,\mathrm{rad}},$$

where the space on the left-hand side is equipped with the norm

$$\|f\| \coloneqq \|f\|_1 + \|S_{-\Delta_\varnothing}(f)\|_1.$$

6.6.1 Bridging the theories using pre-adapted spaces

We introduce a pre-adapted version $\mathbb{H}^1_L$ of our space H^1_L. Such spaces can be defined and studied assuming only that the operator L has a decent L^2 theory, which is notably weaker than our requirement (G_O).

For the rest of this section, we suppose at least that L is self-adjoint and that the set O satisfies (UITC). Then O is a space of homogeneous type and $(\mathrm{e}^{-t^2L})_{t>0}$ satisfies L^2 off-diagonal estimates. Hence, our constellation falls into the scope of [112].

Definition 6.6.9. The **pre-adapted Hardy space** $\mathbb{H}^1_L$ is defined by

$$\mathbb{H}^1_L \coloneqq \Big\{ f \in \overline{\operatorname{ran}(L)} \colon S_L(f) \in \mathrm{L}^1 \Big\}, \qquad \|f\|_{\mathbb{H}^1_L} \coloneqq \|S_L(f)\|_1.$$

Remark 6.6.10. The square function $S_L(f)$ can either denote the heat or Poisson version. The respective spaces in [112] are denoted by $\mathbb{H}^1_{L,S_h}$ and $\mathbb{H}^1_{L,S_P}$. It turns out that they coincide [112, Thm. 2.5], so we do not bother to distinguish them and simply write $\mathbb{H}^1_L$ and S_L.

Remark 6.6.11. In virtue of Proposition 6.5.10, the spaces $\mathbb{H}^1_L$ and H^1_L are precisely the functions f such that $S_L(f) \in \mathrm{L}^1$ equipped with the same norm, where in the first case f is quantified over $\overline{\operatorname{ran}(L)} = \mathrm{L}^2_0$ and in the second case over L^1_0.

In a nutshell, the line of reasoning to obtain Theorem 6.6.4 is as follows.

(i) The space H^1_L is complete and its intersection with $\overline{\operatorname{ran}(L)}$ is a dense subspace. It follows that H^1_L is a completion of $\mathbb{H}^1_L$.

(ii) We show that $\mathbb{H}^1_L \oplus \ell^1(I) = \mathbb{H}^1_{L,\max} = \mathbb{H}^1_{L,\mathrm{rad}}$. Therefore, taking completions gives

$$\mathrm{H}^1_L \oplus \ell^1(I) = \mathrm{H}^1_{L,\max} = \mathrm{H}^1_{L,\mathrm{rad}}.$$

The next proposition takes care of the first item.

Proposition 6.6.12. *Assume that L satisfies* (G_O) *and that O fulfills the geometric assumptions* (UITC), (Fat), (LU) *and* (UP). *Then the space H^1_L is complete and contains $\mathbb{H}^1_L$ as a dense subspace. Phrased differently, H^1_L is a completion of the pre-adapted space $\mathbb{H}^1_L$.*

Proof. Many functional analytic properties of H^1_L are not evident from its very definition. Instead, we use the atomic viewpoint developed in Theorem 6.5.6.

Indeed, H^1_D is the dual space of VMO_D by Theorem 6.4.15 and therefore complete. In addition, $\mathrm{H}^1_{D,a}$, the space of finite linear combinations of H^1_D-atoms, is dense in $\mathrm{H}^1_D = \mathrm{H}^1_L$. The second claim follows if we can show the chain of inclusions

$$\mathrm{H}^1_{D,a} \subseteq \mathbb{H}^1_L \subseteq \mathrm{H}^1_L.$$

The second inclusion follows readily from Remark 6.6.11 and Lemma 6.6.18. For the first inclusion we note that atoms are by definition elements of $\mathrm{L}^2_0 = \overline{\mathrm{ran}(L)}$ (see Proposition 6.2.9) and satisfy $S_L(a) \in \mathrm{L}^1$ by Theorem 6.5.6. □

Remark 6.6.13. This observation could have been drawn already in [26] but was not explicitly stated.

In both [112] and [48] the authors derive an L-adapted atomic decomposition, in the first case coming from $\mathbb{H}^1_L$, in the second case coming from $\mathbb{H}^1_{L,\max}$. In [112, Def. 2.1 & 2.2], they give the following

Definition 6.6.14. Let $M \in \mathbb{N}$. A function $a \in \mathrm{L}^2$ is called a **$(1, 2, M)$-atom** associated to L if there exists $b \in \mathrm{dom}(L^M)$ and a ball B of radius $r > 0$ centered in O such that $a = L^M b$ and for all $k = 0, \dots, M$ it holds

$$\mathrm{supp}(L^k b) \subseteq O \cap B \quad \& \quad \|(r^2 L)^k b\|_2 \leq r^{2M} |O \cap B|^{-\frac{1}{2}}.$$

The atomic pre-Hardy space $\mathbb{H}^1_{L,\mathrm{at},M}$ consists of all $f \in \mathrm{L}^2$ that admit a $(1, 2, M)$-atomic representation $f = \sum_j \lambda_j a_j$, where $(\lambda_j)_j \in \ell^1$, each a_j is a $(1, 2, M)$-atom, and the sum converges in L^2. The space $\mathbb{H}^1_{L,\mathrm{at},M}$ is equipped with the quotient norm

$$\|f\|_{\mathbb{H}^1_{L,\mathrm{at},M}} \coloneqq \inf \|(\lambda_j)_j\|_{\ell^1},$$

where the infimum is taken over all $(1, 2, M)$-atomic decompositions $\sum_j \lambda_j a_j$ of f.

Remark 6.6.15. We remark that $\mathbb{H}^1_{L,\mathrm{at},M} \subseteq \overline{\mathrm{ran}(L)}$, in accordance with the definition of $\mathbb{H}^1_L$.

Remark 6.6.16 (comparison with [48])**.** Definition 6.6.14 coincides with the definition given in [48, Def. 1.1 & 1.2] up to two remarks.

(i) The definition in [48] is formulated with $(1, \infty, M)$-atoms, in which the size condition is formulated in L^∞. However, the results of [48] are true for $(1, q, M)$-atoms if $q \in (1, \infty]$. Indeed, having a $(1, \infty, M)$-atomic representation is stronger than having a $(1, q, M)$-atomic representation and the inclusion $\mathbb{H}^1_{L,\mathrm{at},M} \subseteq \mathbb{H}^1_{L,\max}$ was already formulated for more general q.

(ii) If O has finite measure, then $\mathbb{H}^1_{L,\mathrm{at},M}$ is replaced[1] by $\mathbb{H}^1_{L,\mathrm{at},M} \oplus \ker(L)$ in [48]. In our case, $\ker(L) = \ell^1(I)$. We will have to take this into account by a case distinction later on.

The following is one of the main results of [112], see [112, Prop. 4.16].

Proposition 6.6.17 (L-atomic characterization of $\mathbb{H}^1_L$)**.** *Assume* (G_O) *and that L is self-adjoint. For $M > d/4$ it holds*

$$\mathbb{H}^1_{L,\mathrm{at},M} = \mathbb{H}^1_L.$$

The inclusion $\mathbb{H}^1_{L,\mathrm{at},M} \subseteq \mathbb{H}^1_L$ is the simpler one and similar to the arguments in Section 6.5.3. For the converse inclusion, the authors first derive a so-called molecular decomposition. Molecules are a generalization of $(1, 2, M)$-atoms in which the function a is not supported in $O \cap B$ anymore, but has rapid decay over annuli around B. This does not yet require self-adjointness of L but a bounded H^∞-calculus on $\overline{\mathrm{ran}(L)}$ is sufficient. This is always the case for L by the Crouzeix–Delyon theorem, see Theorem 1.6.6. Later, they decompose L-molecules into L-atoms. Only this latter part relies heavily on self-adjointness.

We record a lemma that was already mentioned and used in Section 6.5.

Lemma 6.6.18. *We have the estimate*

$$\|f\|_1 \lesssim \|S_L(f)\|_1 \qquad (f \in \mathbb{H}^1_L).$$

In other words, $\mathbb{H}^1_L \subseteq \mathrm{L}^1$ with continuous inclusion.

The argument is well-known to the experts and has been mentioned in [112, p. 75].

[1] In the published version, $\ker(L)$ is replaced by $\mathbb{C}$. This is true in their applications but has to be changed in the abstract formulation.

Proof. We take $k = M$ in the definition of a $(1, 2, M)$-atom associated to L to find $\|a\|_2 \leq |O \cap B|^{-1/2}$ for all such atoms a. Thus, $\mathbb{H}^1_{L,\mathrm{at},M} \subseteq \mathrm{L}^1$ follows with the same argument as for traditional atomic Hardy spaces. The argument using a molecular decomposition works the same up to an additional summation over the annuli. Therefore, the result is true without assuming that L is self-adjoint. □

To conclude the second claim from the outline we use [48, Cor. 1.5]. If O has finite measure, this is their main result [48, Thm. 1.4]. The case of infinite measure was already known in the literature, see their references. Here, we recall also the second part from Remark 6.6.16.

Proposition 6.6.19 (maximal characterization)**.** *Assume that L is self-adjoint and satisfies* (G_O) *as well as the geometric assumptions* (UITC), (Fat), (LU) *and* (UP) *and let $M \in \mathbb{N}$. We have the identities*

$$\mathbb{H}^1_{L,\mathrm{at},M} \oplus \ell^1(I) = \mathbb{H}^1_{L,\max} = \mathbb{H}^1_{L,\mathrm{rad}}.$$

Let us record the following corollary, which is the second claim from the outline.

Corollary 6.6.20. *It holds*

$$\mathbb{H}^1_L \oplus \ell^1(I) = \mathbb{H}^1_{L,\max} = \mathbb{H}^1_{L,\mathrm{rad}}.$$

From Proposition 6.6.12 and Corollary 6.6.20 we conclude Theorem 6.6.4.

Proof of Theorem 6.6.4. We perform the argument sketched in the outline. Corollary 6.6.20 yields for instance

$$\mathbb{H}^1_L \oplus \ell^1(I) = \mathbb{H}^1_{L,\max}. \tag{6.20}$$

We take the completion of both sides of (6.20). Then, according to Proposition 6.6.12, H^1_L is a completion of $\mathbb{H}^1_L$. Since we can take the completion componentwise, $\mathrm{H}^1_L \oplus \ell^1(I)$ is a completion of the left-hand side of (6.20). But, by definition, $\mathrm{H}^1_{L,\max}$ is a completion of the right-hand side of (6.20), so $\mathrm{H}^1_L \oplus \ell^1(I) = \mathrm{H}^1_{L,\max}$ follows. The argument for $\mathrm{H}^1_{L,\mathrm{rad}}$ is the same. □

6.6.2 Application to the negative Laplacian

We apply Theorem 6.6.4 to $-\Delta_D$ defined on a bounded and open set O satisfying our geometric assumptions (ITC), (Fat), (LU) and (UP) to give the

Proof of Theorem 6.6.5. We appeal to Theorem 6.6.4. The geometric requirements are all fulfilled by assumption and $-\Delta_D$ is self-adjoint. To invoke the theorem, it remains to check (G_O). Since the coefficients of $-\Delta_D$ are real and O is bounded, this follows from Theorems 5.0.1 and 5.0.2. Therefore, Theorem 6.6.4 can be applied and yields all identities except

$$\{f \in \mathrm{L}^1 \colon S_{-\Delta_D}(f) \in \mathrm{L}^1\} = \mathrm{H}^1_{-\Delta_D} \oplus \ell^1(I).$$

Since $S^{\mathrm{loc}}_{-\Delta_D}(c) = 0$ for all $c \in \ell^1(I)$ by the conservation property on O_N, see Lemma 6.5.8, it follows that the mapping

$$f \mapsto \left(f - \sum_m \mathbf{1}_{O_m}(f)_{O_m} \right) \oplus ((f)_{O_m})_m$$

is an isomorphism between both spaces. Here, we also used Proposition 6.5.10 and that I is finite by Lemma 6.2.7 (iv). □

CHAPTER 7

Bounded functional calculus for divergence form operators with dynamical boundary conditions

This chapter is dedicated to elliptic operators in divergence form along with their associated parabolic problems, which are commonly encountered in science. These scenarios frequently pose challenges due to discontinuous coefficient functions and singular objects on the right-hand side, which reside on sets with Hausdorff dimension smaller than the spatial dimension. It is widely recognized, e.g. in the theory of electricity (see the monograph of the Nobel prize winner I. Tamm [154, Chap. 1.4]) that (spatial) jumps in the coefficient function are intimately connected to the presence of surface densities on the right-hand side.

The parabolic equation

First, let us give a formal description of the linear parabolic problem with dynamical boundary conditions that we have in mind. An excellent exposition for the derivation of such equations in sciences can be found in [100]. In dimension $d \geq 2$ we let $O \subseteq \mathbb{R}^d$ be open, $D \subseteq \partial O$, $\Sigma \subseteq \overline{O} \setminus D$, $T \in (0, \infty]$

and consider the system

$$
\begin{aligned}
\partial_t u - \operatorname{div}(A\nabla u) &= f|_{O\setminus\Sigma} && \text{in } (0,T)\times(O\setminus\Sigma),\\
\partial_t u + \nu\cdot A\nabla u &= f|_{\Sigma\cap\partial O} && \text{on } (0,T)\times(\Sigma\cap\partial O),\\
\partial_t u + \nu_{\Sigma\cap O}\cdot A\nabla u &= f|_{\Sigma\cap O} && \text{on } (0,T)\times(\Sigma\cap O),\\
u &= 0 && \text{on } (0,T)\times D,\\
\nu\cdot A\nabla u &= 0 && \text{on } (0,T)\times\partial O\setminus(D\cup\Sigma),\\
u(0) &= u_0 && \text{in } O\cup\Sigma.
\end{aligned}
\tag{7.1}
$$

Here, $A\in \mathrm{L}^\infty(O;\mathcal{L}(\mathbb{C}^d))$ is elliptic in the sense of Definition 1.6.2, the vector ν is the outer unit normal, $\nu_{\Sigma\cap O}$ denotes a 'jump' in the normal derivative on $\Sigma\cap O$, and f, u_0 are functions defined on $O\cup\Sigma$. Hence, the underlying set for the dynamics is $O\cup\Sigma$, where the 'volume' O is equipped with the d-dimensional Lebesgue measure, but the 'surface' Σ is a Lebesgue null set and carries a different Radon measure m. In science, Σ would typically be a Lipschitz surface with $(d-1)$-dimensional Hausdorff measure, but our mathematical treatment allows it to be as wild as the von Koch snowflake with its natural measure of fractal dimension, or of Hausdorff co-dimension larger than 1.

The variational setting

Following [10, 71, 84, 137], we model (7.1) as an abstract Cauchy problem

$$
\begin{aligned}
u'(t) + \mathscr{L}^A u(t) &= f(t) \qquad (t\in(0,T)),\\
u(0) &= u_0,
\end{aligned}
$$

in the natural L^2-space $\mathbb{L}^2 := \mathrm{L}^2(O\cup\Sigma, \mathrm{d}x\oplus\mathrm{d}m)$ that takes the two dynamical parts into account. This can efficiently be done by an application of the form method due to Arendt and ter Elst [9], see Section 7.2 for details. In this construction, $\mathscr{L}^A$ is associated with the usual sesquilinear form for divergence form operators,

$$
a\colon V\times V\to\mathbb{C},\quad a(u,v)=\int_O A\nabla u\cdot\overline{\nabla v}\,\mathrm{d}x,
\tag{7.2}
$$

where $V=\mathrm{W}^{1,2}_D(O)$ models the homogeneous Dirichlet condition on D in the fourth line of (7.1). However, V is not considered as a subspace of $\mathrm{L}^2(O)$ but of $\mathbb{L}^2$ through the identification operator $j(u)=u|_O\oplus\operatorname{Tr}(u)$ in order to account for the dynamics on Σ, compare with [84, Sec. 1]. To this end, we need minimal geometric assumptions that we describe next.

Assumption 7.0.1. *Throughout the entire chapter, $d \geq 2$ and:*

(i) *The set $O \subseteq \mathbb{R}^d$ is open and non-empty.*

(ii) *The* Dirichlet part *$D \subseteq \partial O$ is closed and possibly empty.*

(iii) *The* dynamical part *$\Sigma \subseteq \overline{O} \setminus D$ is a non-empty Borel set with Lebesgue measure zero that is equipped with a Radon measure m.*

(iv) *The* trace *to Σ, defined for $u \in V \cap \mathrm{C}(\overline{O})$ by pointwise restriction*

$$\mathrm{Tr}(u) \coloneqq u|_{\Sigma},$$

extends by density to a bounded linear operator from V into $\mathrm{L}^2(\Sigma, \mathrm{d}m)$.

Remark 7.0.2. We denote the extension of Tr in Assumption 7.0.1 (iv) by the same symbol. As usual, a Radon measure is a Borel measure that is finite on (relative) compact sets, outer regular on Borel sets, and inner regular on (relative) open sets.

Assumption 7.0.1 (iv) is always satisfied when m is the zero measure but in applications m is not free to choose — it belongs to the geometric data of the problem just as O, D and Σ. Proving the boundedness of the trace onto geometrically intricate objects with their 'natural' measure m can be a challenging task and in Section 7.1 we propose a unified approach that covers a variety of examples, including fractal sets Σ. For the rest of the paper it is conceptually simpler to stick to the general assumption above.

Main results

Abstract theory of sectorial forms tells us that $\mathscr{L}^A$ generates a holomorphic contraction semigroup in $\mathbb{L}^2$. Thus, $\mathscr{L}^A$ has a bounded H^∞-calculus in $\mathbb{L}^2$ and notably exhibits maximal parabolic regularity, serving as a powerful tool for handling non-linear versions of (7.1) via fixed-point methods, and even stochastic maximal regularity [158].

The primary focus of this work is to elaborate whether these properties extend to $\mathbb{L}^p \coloneqq \mathrm{L}^p(O \cup \Sigma, \mathrm{d}x \oplus \mathrm{d}m)$ for $p \neq 2$. For operators with *real coefficients* this problem and related ones are extensively investigated in the full Lebesgue scale, see e.g. [71, 84, 101, 114]. Their techniques do not carry over to the complex case, which, to the best of our knowledge, has been an open problem. We settle the question in our main result by using the notion of p-ellipticity, see Subsection 1.7.2:

Theorem 7.0.3. *Let $p \in (1,\infty)$ and suppose that A is p-elliptic. The semigroup $(\mathrm{e}^{-t\mathscr{L}^A})_{t\geq 0}$ extends to an analytic contraction semigroup on $\mathbb{L}^p$ of angle $\theta_p \in (0,\pi/2)$ defined in (7.3) and its generator $\mathscr{L}_p^A$ has a bounded H^∞-calculus of angle $\pi/2-\theta_p$. In particular, $\mathscr{L}_p^A$ has maximal parabolic regularity.*

Every uniformly strongly elliptic A is p-elliptic in a range of p's that depends on $\lambda(A),\Lambda(A)$ [80, Cor. 3]. Therefore, Theorem 7.0.3 always applies in an A-dependent range of p's. The angle above is

$$\theta_p := \sup\Big\{\theta\in[0,\pi/2)\colon\ \mathrm{e}^{\pm\mathrm{i}\theta}\,A \text{ is } p\text{-elliptic}\Big\} \tag{7.3}$$

and we address the problem of finding lower bounds for θ_p in terms of the 'data' of A in Section 7.7.

Compared to many L^p-extrapolation results related to elliptic operators [12,37,43,44,46,80,85,145], there is no clear dimensional scaling on the spatial domain of our operator due to the presence of Σ. This seems to forbid any use of (generalized) kernel estimates on objects related to $\mathscr{L}^A$. In proving Theorem 7.0.3, we had to watch out for methods that predominantly work on the level of the sesquilinear form a in (7.2) and not on the associated operator $\mathscr{L}^A$, because the former one does not involve Σ. We found a suitable approach in the non-linear heat flow technique of Carbonaro and Dragičević [51–53, 55, 56]. Largely inspired by their results, we first prove a *bilinear embedding* in Section 7.4:

Theorem 7.0.4. *Let $p\in(1,\infty)$ and A,B be p-elliptic. Then there is a constant $C>0$ that depends only on p, $\lambda(A,B)$, $\Lambda(A,B)$ and $\Delta_p(A,B)$ such that*

$$\int_0^\infty\int_O \Big|\nabla(\mathrm{e}^{-t\mathscr{L}^A}f)|_O\Big|\cdot\Big|\nabla(\mathrm{e}^{-t\mathscr{L}^B}g)|_O\Big|\,\mathrm{d}x\,\mathrm{d}t \leq C\|f\|_p\|g\|_{p'}$$

for all $f,g\in\mathbb{L}^p\cap\mathbb{L}^{p'}$.

Theorem 7.0.4 implies Theorem 7.0.3 by standard quadratic estimates due to Cowling, Doust, McIntosh and Yagi [64], see Section 7.5. However, this isn't the whole story, as the boundedness of the semigroup in $\mathbb{L}^p$ is needed in the proof of Theorem 7.0.4. We will address this prerequisite beforehand in Section 7.3 by providing a generalization of Nittka's invariance criterion [137] tailored to our needs.

In both, the bilinear embedding and Nittka's criterion, the main novelty is the presence of the identification operator j. It poses new technical challenges

that we resolve in this work. We make essential use of the particular choice of j, or more precisely, that it is injective and that j as well as j^{-1} commute with certain non-linear maps.

Discussion of the L^p-setting

Let us provide further motivation for considering the elliptic/parabolic operators in the particular spaces $\mathbb{L}^p$.

In the realm of mathematical semiconductor modeling, a widely studied model is the Van Roosbroeck system [130, 132, 144]. This system comprises a set of non-linear drift-diffusion equations with surface charge densities on the right-hand sides. Mathematically, they are understood as measures concentrated on surfaces. In an advanced writing of the system [95], dynamical boundary conditions emerge on parts of the boundary. Solving the system numerically is a nontrivial task and the only known successful algorithm, due to Gummel and Scharfetter, bases on two essentials. First, if the system is studied in some function space, then its dual has to contain indicator functions of subsets, such as boxes or tetrahedra, as 'test functions'. This is true in $\mathbb{L}^p$ and false in the more general distribution spaces used e.g. in [132]. Second, once having tested the system with indicator functions on a partition of subdomains, a point balance across all subdomains needs to be established. Usually, this conversion involves transforming local volume integrals into surface integrals utilizing Gauß' theorem. While this method is effective when the flux's divergence is a measure, it fails when it is only a distribution.

Further applications to non-linear problems and elliptic regularity

Already when dealing with semilinear parabolic equations, a standard approach to handling right-hand sides R that depend nonlinearly on the solution, is based on the fact that R is (locally) Lipschitz continuous with respect to the solution. This holds when the solution space is equipped with the topology of an interpolation space between the underlying Banach space and the domain of the elliptic operator [141, Chap. 6]. For parabolic equations on O it typically suffices to know that domains of fractional powers of the linear elliptic operator embed into L^∞ in order to catch the nonlinearities [86] but in the case of the $\mathbb{L}^p$ spaces involving two different measures, a pointwise uniform control is needed.

We address this issue in Section 7.6, where for large p we establish embeddings of fractional power domains of $\mathscr{L}_p^A$ into spaces of Hölder continuous functions on $O \cup \Sigma$ in various settings. These embeddings generalize results in [70, 71, 84, 107, 115] for parabolic equations on O and open the door for proving Hölder regularity for solutions of the parabolic equations simultaneously in space and time, which can be very useful in applications, see e.g. [42, 57] and references therein. The proof relies on delicate yet known mapping properties of the sesquilinear form (7.2), which are independent of the dynamical boundary conditions, and a 'transference formula' for the inverse of $\mathscr{L}^A$, which might be of independent interest.

7.1 Geometry

This section contains prerequisites on function spaces and a unified treatment of Sobolev traces, leading to a variety of geometric configurations that match with our background assumption (Assumption 7.0.1), see also Subsection 1.1.2.

By uniform continuity, all functions in $\mathrm{C}_D^\infty(O)$ extend continuously to $\overline{O}$. Hence, $V \cap \mathrm{C}(\overline{O})$ is dense in V and Assumption 7.0.1 (iv) makes sense. Let us stress that we use $\mathrm{W}_\varnothing^{1,2}(O)$ when the Dirichlet part is empty, which should be thought of as a regularized version of the usual Sobolev space $\mathrm{W}^{1,2}(O)$ that contains enough functions with a well-defined trace, compare with [140, Chap. 4] and [8, 10, 143].

7.1.1 The identification map j

To define dynamical boundary conditions rigorously via the form method, we use an embedding of V into an L^2-space on $O \cup \Sigma$.

Definition 7.1.1. Let $p \in [1, \infty]$. We denote by $\mathrm{L}^p(O) = \mathrm{L}^p(O, \mathrm{d}x)$ the usual Lebesgue space of p-integrable functions, put $\mathrm{d}\mu = \mathrm{d}x \oplus \mathrm{d}m$ and write

$$\mathbb{L}^p \coloneqq \mathrm{L}^p(O \cup \Sigma, \mathrm{d}\mu) \coloneqq \mathrm{L}^p(O) \oplus \mathrm{L}^p(\Sigma).$$

We refer to this space as the L^p-space over the **hybrid measure space** $(O \cup \Sigma, \mathrm{d}\mu)$. The **identification operator** is given by

$$j \colon V \to \mathbb{L}^2, \quad j(u) \coloneqq u|_O \oplus \mathrm{Tr}(u).$$

The terminology 'hybrid' highlights the fact that parts of such functions 'live' on the volume $(O, \mathrm{d}x)$, whereas another part 'lives' on the typically lower dimensional set $(\Sigma, \mathrm{d}m)$. We could have also used $O \setminus \Sigma$ for the volume part in order to have 'restrictions' to two disjoint sets. However, there's no confusion here as Σ is assumed to be a Lebesgue null set according to Assumption 7.0.1 (iii).

Lemma 7.1.2. *The space* $\mathrm{C}_D^\infty(\Sigma)$ *is dense in* $\mathrm{L}^2(\Sigma)$*. In particular,* $j(V)$ *is dense in* $\mathbb{L}^2$*.*

Proof. Density of $\mathrm{C}_D^\infty(\Sigma)$ in $\mathrm{L}^2(\Sigma)$ follows by combining three density results. First, by dominated convergence, $\mathrm{L}^2(\Sigma)$-functions with support in a compact set $\Sigma' \subseteq \Sigma$ with $\mathrm{d}(\Sigma', D) > 0$ are dense in $\mathrm{L}^2(\Sigma)$. Second, Σ' carries the Radon measure $m|_{\Sigma'}$, so $\mathrm{C}(\Sigma')$ is dense in $\mathrm{L}^2(\Sigma', \mathrm{d}m)$, see for instance [163, Thm. 19.38]. Third, Σ' keeps a positive distance to D, so the restrictions $\mathrm{C}_D^\infty(\Sigma')$ form a unital $*$-algebra that separates the points of $\mathrm{C}(\Sigma')$ and the Stone–Weierstrass theorem yields that $\mathrm{C}_D^\infty(\Sigma')$ is $\|\cdot\|_\infty$-dense in $\mathrm{C}(\Sigma')$.

In order to see that the above implies that j has dense range in $\mathbb{L}^2$, we suppose that $u \in j(V)^\perp$. Then, we have

$$0 = (u|_O \,|\, v|_O) + (u|_\Sigma \,|\, v|_\Sigma) \qquad (v \in j(V) \subseteq \mathbb{L}^2).$$

The second term on the right vanishes if $v \in \mathrm{C}_\mathrm{c}^\infty(O)$ vanishes on $\Sigma \cap O$. Since Σ is a Lebesgue null set, such functions are dense in $\mathrm{L}^2(O)$ and we conclude that $u|_O = 0$. Now, we can use the density result from the first part to conclude $u|_\Sigma = 0$ and thus $u = 0$. □

For the $\mathbb{L}^p$-theory, we shall need to commute j and its inverse with certain non-linear maps. While this is clear for j^{-1}, because applying j^{-1} means restricting functions in $j(V)$ from $O \cup \Sigma$ to O, the argument for j is more involved.

Lemma 7.1.3. *Let* $k, l \in \mathbb{N}$*,* $\Phi \colon \mathbb{C}^k \to \mathbb{C}^l$ *be Lipschitz continuous with* $\Phi(0) = 0$ *and let* $U \coloneqq (u^i)_{i=1}^k \in V^k$*. Then* $\Phi(U) \in V^l$ *and*

$$(j(\Phi(U)^1), \dots, j(\Phi(U)^l)) = \Phi(j(u^1), \dots, j(u^k)).$$

Proof. We follow the argument in [80, Lem. 4] and pick a sequence $(U_n)_n = ((u_n^i)_{i=1}^k)_n \subseteq \mathrm{C}_D^\infty(O)^k$ such that $U_n \to U$ in V^k as $n \to \infty$. Since Φ is Lipschitz continuous with $\Phi(0) = 0$, it follows that $(\Phi(U_n))_n \subseteq V^l$ is bounded. Hence, we find some $v \in V^l$ such that $\Phi(U_n) \to v$ weakly in V^l along a

subsequence. On the other hand, $\Phi(U_n) \to \Phi(U)$ in $\mathrm{L}^2(O)^l$ and we conclude that $\Phi(U) = v \in V^l$. This proves the first assertion.

To show the second one, we use the first part and that $j\colon V \to \mathbb{L}^2$ is continuous to conclude that $(j(\Phi(U_n)^1), \dots, j(\Phi(U_n)^l)) \to (j(\Phi(U)^1), \dots, j(\Phi(U)^l))$ weakly in $(\mathbb{L}^2)^l$ along a subsequence. Since all $\Phi(U_n)^i$ are continuous on $\overline{O}$, the map Tr acts as an honest pointwise restriction and we can commute

$$(j(\Phi(U_n)^1), \dots, j(\Phi(U_n)^l)) = \Phi(j(u_n^1), \dots, j(u_n^k)).$$

The right-hand side tends to $\Phi(j(u^1), \dots, j(u^k))$ in $\mathbb{L}^2$ and the proof is complete. □

7.1.2 Lebesgue points and traces

For a globally defined function $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ we say that $x \in \mathbb{R}^d$ is a **Lebesgue point** of f if there exists $z \in \mathbb{C}$ such that

$$\lim_{r \to 0} \fint_{B(x,r)} |f(y) - z| \, \mathrm{d}y = 0.$$

In this case, we also have

$$z = \lim_{r \to 0} \fint_{B(x,r)} f(y) \, \mathrm{d}y.$$

Lebesgue points allow us to assign pointwise values to equivalence classes of functions, sometimes called 'precise' or 'refined' representative.

Definition 7.1.4. Let $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $F \subseteq \mathbb{R}^d$. The **global trace** of f to F is defined as

$$\mathrm{Tr}_{\mathrm{glob},F}(f)(x) \coloneqq \lim_{r \to 0} \fint_{B(x,r)} f(y) \, \mathrm{d}y,$$

for all $x \in F$ for which the limit exists.

If the set F happens to be a subset of $\overline{O}$, we can also define a trace for functions defined only on O as follows.

Definition 7.1.5. Let f be integrable on bounded subsets of O and $F \subseteq \overline{O}$. The **interior trace** of f to F is defined as

$$\mathrm{Tr}_{\mathrm{int},F}(f)(x) \coloneqq \lim_{r \to 0} \fint_{O(x,r)} f(y) \, \mathrm{d}y,$$

for all $x \in F$ for which the limit exists.

Whether global and interior traces exist in a suitable sense has been investigated extensively [10, 82, 118, 143, 153]. Here, our focus lies on making 'soft' assumptions on V that are commonly used in heat kernel theory on domains [140, Sec. 6.3] rather than relying on geometric measure theory. The reader should be reminiscent of the embedding property (E) from (5.11).

Definition 7.1.6. We say that V has the **extension property** if there is a bounded extension operator $\mathcal{E}\colon V \to \mathrm{W}^{1,2}(\mathbb{R}^d)$, which should satisfy the additional L^2-bound $\|\mathcal{E}u\|_2 \leq C_E\|u\|_2$ for some $C_E > 0$ and all $u \in V$ if we work in dimension $d = 2$.

Remark 7.1.7. In Section 7.1.4 we come back to the extension property in concrete settings. The extension property implies (E). Indeed, if $d \geq 3$, then we can take $q = {}^{2d}/_{(d-2)}$ and $\theta = 0$, and use the commutative diagram

$$\begin{array}{ccc} \mathrm{W}^{1,2}(\mathbb{R}^d) & \xrightarrow{\ \subseteq\ } & \mathrm{L}^q(\mathbb{R}^d) \\ {\scriptstyle\mathcal{E}}\uparrow & & \downarrow{\scriptstyle|_O} \\ V & \xrightarrow{\ \subseteq\ } & \mathrm{L}^q(O) \end{array} \tag{7.4}$$

where the first line is the Sobolev embedding on $\mathbb{R}^d$. If $d = 2$, then we can take any $\theta \in (0, 1)$ and transfer the respective Gagliardo–Nirenberg inequality from $\mathbb{R}^d$ to O in the same manner. This is where the additional L^2-bound for $\mathcal{E}$ is needed.

Assumption (E) ensures that O is thick enough in points away from D. Indeed, the next result is a generalization of [106, Thm. 1], see also [36]. Unlike the original result, it also applies in the critical case $d = 2$ without any connectivity assumption on O, compare with the proof of [106, Lem. 13].

Proposition 7.1.8. *Assume* (E). *Then there is $C > 0$ depending only on d, θ and C_E such that*

$$Cr^d \leq |O(x, r)|$$

for all $x \in \overline{O} \setminus D$ and $r \in (0, 1 \wedge \mathrm{d}_D(x))$.

Proof. As in [106], we consider a 'halving radius' $\widehat{r} \in (0, r)$ such that

$$|O(x, \widehat{r})| = \frac{1}{2}|O(x, r)|.$$

We fix $\varphi \in \mathrm{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^d)$ such that $\mathbf{1}_{B(x,\widehat{r})} \leq \varphi \leq \mathbf{1}_{B(x,r)}$ and $\|\nabla\varphi\|_\infty \leq {}^{c}\!/_{(r-\widehat{r})}$ with a dimensional constant $c \geq 1$. Then $\varphi|_O \in V$ with estimates

$$\begin{aligned}\|\varphi\|_2 &\leq |O(x,r)|^{\frac{1}{2}},\\ \|\varphi\|_{1,2} &\leq 2c(r-\widehat{r})^{-1}|O(x,r)|^{\frac{1}{2}},\\ \|\varphi\|_q &\geq \left(\frac{1}{2}|O(x,r)|\right)^{\frac{1}{2}-\frac{(1-\theta)}{d}},\end{aligned}$$

where for the Sobolev norm we have used that $r-\widehat{r} \leq 1$. We plug this bound into the embedding property (E) in order to obtain

$$r-\widehat{r} \leq 2c(2^{1/q}C_E)^{\frac{1}{(1-\theta)}}|O(x,r)|^{\frac{1}{d}}.$$

Now, we iterate: $r_1 \coloneqq r$ and $r_{n+1} \coloneqq \widehat{r_n}$. Since $|O(x,r_n)| = 2^{-n}|O(x,r)|$ tends to 0 as $n \to \infty$, we find $r_n \to 0$. A telescoping series yields the claim

$$\begin{aligned}r &= \sum_{n=1}^{\infty}(r_n - r_{n+1})\\ &\leq 2c(2^{1/q}C_E)^{\frac{1}{(1-\theta)}}\sum_{n=1}^{\infty}|O(x,r_n)|^{\frac{1}{d}}\\ &= 2c(2^{1/q}C_E)^{\frac{1}{(1-\theta)}}\left(\sum_{n=1}^{\infty}2^{-\frac{n}{d}}\right)|O(x,r)|^{\frac{1}{d}}\\ &= \frac{2c(2^{1/q}C_E)^{\frac{1}{(1-\theta)}}}{2^{1/d}-1}|O(x,r)|^{\frac{1}{d}}.\end{aligned}$$

□

With the previous result at hand, we can prove that our various notions of traces coincide on Σ.

Corollary 7.1.9. *Suppose* (E)*, let f be integrable on bounded subsets of O and let $u \in V \cap \mathrm{C}(\overline{O})$.*

(i) *If $\mathcal{E}f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ is any extension of f, then at every Lebesgue point $x \in \Sigma$ of $\mathcal{E}f$ we have*

$$\mathrm{Tr}_{\mathrm{glob},\Sigma}(\mathcal{E}f)(x) = \mathrm{Tr}_{\mathrm{int},\Sigma}(f)(x).$$

(ii) *For every $x \in \Sigma$ we have*

$$\mathrm{Tr}(u)(x) = \mathrm{Tr}_{\mathrm{int},\Sigma}(u)(x).$$

Proof of (i). We set $z := \mathrm{Tr}_{\mathrm{glob},\Sigma}(\mathcal{E}f)(x)$. For $r \in (0, 1 \wedge \mathrm{d}_D(x))$ we obtain from Proposition 7.1.8 that

$$\left| \fint_{O(x,r)} f \,\mathrm{d}y - z \right| \leq \fint_{O(x,r)} |f(y) - z| \,\mathrm{d}y \leq \frac{|B(0,1)|}{C} \fint_{B(x,r)} |(\mathcal{E}f)(y) - z| \,\mathrm{d}y.$$

The right-hand side converges to 0 as $r \to 0$ since x is a Lebesgue point of $\mathcal{E}f$. Hence, $\mathrm{Tr}_{\mathrm{int},\Sigma}(f)(x)$ exists and equals z.

Proof of (ii). This follows since u is continuous at x. □

7.1.3 Continuity of the trace

In order to get a continuous trace map into $\mathrm{L}^2(\Sigma)$ as required in Assumption 7.0.1, we need to guarantee that Sobolev functions have sufficiently many Lebesgue points on Σ and that the so-obtained trace is controlled in norm. For this part only, we switch to a more concrete setup inspired by Jonsson and Wallin [118].

We work with the Hausdorff measure $\mathcal{H}^\ell$ of dimension $\ell \in (d-2, d)$ on $\mathbb{R}^d$, see Section 1.3. In particular, sets with finite $\mathcal{H}^\ell$-measure are Lebesgue null, and if $F \subseteq \mathbb{R}^d$ is a Borel set with $\mathcal{H}^\ell(F) < \infty$, then the restriction of $\mathcal{H}^\ell$ to F is a Radon measure. The restriction on the dimension stems from a fundamental result in potential theory [2, Thm. 6.2.1 & Thm. 5.1.13]: every $u \in \mathrm{W}^{1,2}(\mathbb{R}^d)$ has Lebesgue points $\mathcal{H}^\ell$-almost everywhere.

If Σ is an upper ℓ-set, then we take $m = \mathcal{H}^\ell|_\Sigma$ as the measure on Σ. Here is the main result of the section:

Theorem 7.1.10. *Assume that V has the extension property (with extension operator $\mathcal{E}$) and that Σ is an upper ℓ-set for some $\ell \in (d-2, d)$. Then*

$$\mathrm{Tr}_{\mathrm{glob},\Sigma}(\mathcal{E}u) = \mathrm{Tr}_{\mathrm{int},\Sigma}(u) = \mathrm{Tr}(u) \qquad (u \in V)$$

and all three linear operators are bounded from V into $\mathrm{L}^2(\Sigma)$. Furthermore, if Σ is bounded, then they are compact.

Proof. Since $\mathcal{E}u$ has Lebesgue points $\mathcal{H}^\ell$-a.e. on $\mathbb{R}^d$, we conclude from Corollary 7.1.9 (i) that $\mathrm{Tr}_{\mathrm{glob},\Sigma}(\mathcal{E}u) = \mathrm{Tr}_{\mathrm{int},\Sigma}(u)$ holds $\mathcal{H}^\ell$-a.e. on $\mathbb{R}^d$. Since $\ell > d-2$, we obtain from [118, Chap. VI, Thm. 1 & Rem. 1] that $\mathrm{Tr}_{\mathrm{glob},\Sigma} \colon \mathrm{W}^{1,2}(\mathbb{R}^d) \to \mathrm{L}^2(\Sigma)$ is bounded. Hence, also $\mathrm{Tr}_{\mathrm{glob},\Sigma} \circ \mathcal{E} \colon V \to \mathrm{L}^2(\Sigma)$ is bounded. If in addition $u \in \mathrm{C}(\overline{O})$, then by Corollary 7.1.9 (ii) we have $\mathrm{Tr}(u) = \mathrm{Tr}_{\mathrm{int},\Sigma}(u)$ and we have already seen that the interior trace is bounded and everywhere defined on V. Thus, $\mathrm{Tr}_{\mathrm{int},\Sigma}$ is the continuous extension of Tr to V.

It remains to prove that $\mathrm{Tr}_{\mathrm{glob},\Sigma} \circ \mathcal{E}$ is compact if Σ is bounded. To this end, we fix an open ball $B \supseteq \overline{\Sigma}$ and a function $\eta \in \mathrm{C}_{\mathrm{c}}^{\infty}(B)$ with $\eta = 1$ on $\overline{\Sigma}$, so that $\mathrm{Tr}_{\Sigma} \circ \mathcal{E} = \mathrm{Tr}_{\Sigma} \circ (\eta \mathcal{E})$. The key point is that [118, Chap. VI, Thm. 1] even gives continuity of $\mathrm{Tr}_{\mathrm{glob},\Sigma} \colon \mathrm{B}^{2,2}_{1-\varepsilon}(\mathbb{R}^d) \to \mathrm{L}^2(\Sigma)$ on Besov spaces for sufficiently small $\varepsilon > 0$. Similar to (7.4), $\mathrm{Tr}_{\Sigma} \circ \mathcal{E}$ factorizes through the embedding $\mathrm{W}^{1,2}_{\partial B}(B) \subseteq \mathrm{B}^{2,2}_{1-\varepsilon}(\mathbb{R}^d)$, which is compact [33, Cor. 2.96]. □

Corollary 7.1.11. *In the setting of Theorem 7.1.10 suppose that O is bounded. Then $j \colon V \to \mathbb{L}^2$ is compact.*

Proof. We already know that $\mathrm{Tr} \colon V \to \mathrm{L}^2(\Sigma)$ is compact. It remains to see that the inclusion $V \subseteq \mathrm{L}^2(O)$ is compact. But this follows by taking an open ball $B \supseteq \overline{O}$ and a function $\eta \in \mathrm{C}_{\mathrm{c}}^{\infty}(B)$ with $\eta = 1$ on $\overline{O}$ and factorizing the inclusion through the same compact embedding as before. □

7.1.4 Concrete geometric setups

We already know from Theorem 5.1.7 that the extension property for V holds if O is locally uniform near N. Theorem 7.1.10 implies the following concrete version of Assumption 7.0.1. In Figure 9 we illustrate a geometric configuration that goes far beyond the Lipschitz class.

Corollary 7.1.12. *Suppose that D is closed, Σ is an upper ℓ-set for some $\ell \in (d-2, d)$ and O is locally uniform near N. Then Assumption 7.0.1 is satisfied.*

We close this section by extrapolating Theorem 7.1.10 to an admissible range of $\mathbb{L}^p$-spaces in the geometric setting from above. This will be important for regularity theory, see Section 7.6.

Theorem 7.1.13. *Let $\ell \in (d-2, d)$, assume that O is locally uniform near N, Σ is an upper ℓ-set and let $\mathcal{E}$ be the extension operator from Theorem 5.1.7. Then the following hold true for all $p \in ((d-\ell) \vee 1, d)$ and $q \in [p, {}^{\ell p}/_{(d-p)})$:*

(i) *We have*

$$\mathrm{Tr}_{\mathrm{glob},\Sigma}(\mathcal{E}u) = \mathrm{Tr}_{\mathrm{int},\Sigma}(u) = \mathrm{Tr}(u) \qquad (u \in \mathrm{W}^{1,p}_D(O) \cap V)$$

and all three operators admit bounded extensions from $\mathrm{W}^{1,p}_D(O)$ into $\mathrm{L}^q(\Sigma)$.

(ii) *If Σ is bounded, then the operators in (i) are compact.*

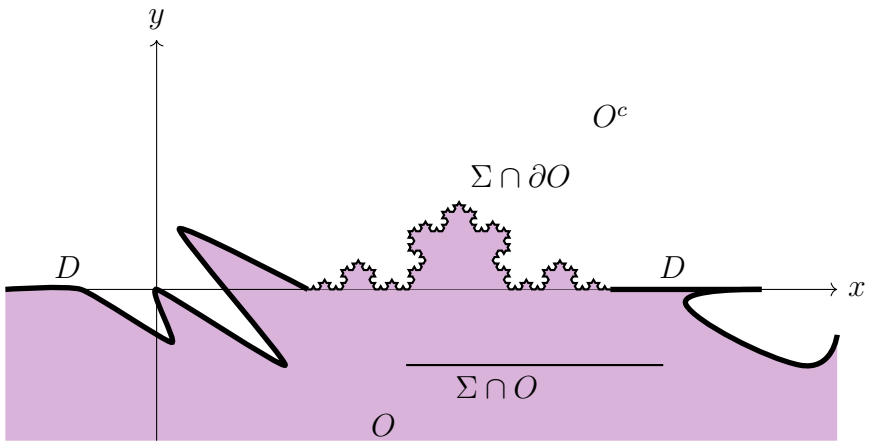

Figure 9: A geometric constellation in $\mathbb{R}^2$ that matches with Corollary 7.1.12 and hence satisfies Assumption 7.0.1. The set $\Sigma \cap \partial O$ is a part of the von Koch snowflake, which is an upper ℓ-set for $\ell = \log_3(4) > 1$, see [89, Sec. 2.3], and we take $m = \mathcal{H}^\ell$ on this dynamical boundary part. A proof of local uniformity of O near N can be found in [104, Prop. 6.30]. The proof of Theorem 7.1.10 shows that Σ could be the union of multiple disjoint upper ℓ-sets with different values of ℓ and we can add a jump condition over the line segment $\Sigma \cap O$, which is a 1-set.

.

(iii) *If O is bounded, then j has a compact extension from $\mathrm{W}^{1,p}_D(O)$ into $\mathbb{L}^q$.*

We follow the proof of Theorem 7.1.10 and explain all necessary changes.

Proof of (i). We set $s \coloneqq 1 - (d/p - d/q)$, that is $1 - d/p = s - d/q$. Our choice of p and q implies that $s \in ((d-\ell)/q, 1]$. By Theorem 5.1.7, [33, Prop. 2.71] and [118, Chap. VI, Thm. 1 & Rem. 1], we have the chain of continuous operators

$$\mathrm{W}^{1,p}_D(O) \xrightarrow{\mathcal{E}} \mathrm{W}^{1,p}(\mathbb{R}^d) \subseteq \mathrm{B}^{q,q}_s(\mathbb{R}^d) \xrightarrow{\mathrm{Tr}_{\mathrm{glob},\Sigma}} \mathrm{L}^q(\Sigma).$$

Thus, $\mathrm{Tr}_{\mathrm{glob},\Sigma} \circ \mathcal{E} \colon \mathrm{W}^{1,p}_D(O) \to \mathrm{L}^q(\Sigma)$ is bounded.

Proof of (ii). We only need to observe that [118, Chap. VI, Thm. 1] also gives continuity of $\mathrm{Tr}_{\mathrm{glob},\Sigma} \colon \mathrm{B}^{q,q}_{s-\varepsilon}(\mathbb{R}^d) \to \mathrm{L}^q(\Sigma)$ for sufficiently small $\varepsilon > 0$. Using [33, Cor. 2.96], we conclude that $\mathrm{Tr}_\Sigma \circ \mathcal{E}$ factorizes through the compact embedding $\mathrm{W}^{1,p}_{\partial B}(B) \subseteq \mathrm{B}^{q,q}_{s-\varepsilon}(\mathbb{R}^d)$.

Proof of (iii). This is the same argument as in the proof of Corollary 7.1.11 with $(V, \mathrm{L}^2(O))$ replaced by $(\mathrm{W}^{1,p}_D(O), \mathrm{L}^q(O))$. □

7.2 L^2-theory

This section contains all relevant definitions and proofs for $p = 2$.

7.2.1 The extended form method

We let A be elliptic, see Definition 1.6.2 and consider the bounded sesquilinear form

$$a\colon V \times V \to \mathbb{C}, \quad a(u,v) = \int_O A\nabla u \cdot \overline{\nabla v}\,\mathrm{d}x.$$

Ellipticity implies that a is **j-elliptic** in the sense of [9], because

$$\operatorname{Re} a(u,u) + \lambda\|j(u)\|_2^2 \geq \lambda\|u\|_V^2 \qquad (u \in V).$$

Moreover, a is **sectorial** and

$$0 \leq \omega(A) \coloneqq \operatorname*{esssup}_{x\in O} \sup_{|\xi|=1} |\arg(A(x)\xi \cdot \overline{\xi})| \leq \arccos\left(\tfrac{\lambda(A)}{\Lambda(A)}\right) < \tfrac{\pi}{2},$$

see also [83]. The associated operator in $\mathbb{L}^2$ is defined as follows. We say $u \in \operatorname{dom}(\mathscr{L}^A)$ if and only if there are $\mathscr{L}^A u \in \mathbb{L}^2$ and $w \in V$ such that $j(w) = u$ and

$$(\mathscr{L}^A u \,|\, j(v)) = a(w,v) \qquad (v \in V).$$

We notice that $\mathscr{L}^A u$ is unique since j has dense range by Lemma 7.1.2. Abstract theory of sectorial forms [9, Thm. 2.1 (ii)] implies that $-\mathscr{L}^A$ generates a strongly continuous semigroup $(\mathrm{e}^{-t\mathscr{L}^A})_{t\geq 0}$ that extends to an analytic contraction semigroup of angle $\pi/2 - \omega(A)$. Hence, $\mathrm{e}^{-z\mathscr{L}^A}$ is defined for $z \in \mathrm{S}_\mu$ with $\mu \in [0, \pi/2 - \omega(A))$. Let us also remark that $\mathscr{L}^A$ admits compact resolvents if $j\colon V \to \mathbb{L}^2$ is compact [9, Lem. 2.7]. This happens in the setting of Theorem 7.1.10 when O is bounded, compare with Corollary 7.1.11.

7.2.2 A formula for $\mathscr{L}^A$

The 'Lax–Milgram' operator associated with the form a is defined as

$$\mathcal{L}\colon V \to V^*, \quad \langle \mathcal{L}u \,|\, v\rangle = a(u,v) \qquad (u,v \in V).$$

Just as a, this operator is independent of the boundary dynamics and it is natural to ask for a formula relating $\mathscr{L}^A$ and $\mathcal{L}$. If $\mathcal{L}$ is invertible, then j and its adjoint provide the link between $\mathcal{L}^{-1}$ and $(\mathscr{L}^A)^{-1}$ as in the following lemma. It has nothing to do with the concrete choices of a, V and j, and is valid in the general j-elliptic framework of [9].

Lemma 7.2.1. *Suppose that $\mathcal{L}$ is invertible. Then, so is $\mathscr{L}^A$ and*

$$(\mathscr{L}^A)^{-1} = j\mathcal{L}^{-1}j^*.$$

Proof. To prove that $\mathscr{L}^A$ is injective, we let $u \in \mathrm{dom}(\mathscr{L}^A)$ with $\mathscr{L}^A u = 0$. Then $u = j(w)$ for some $w \in V$ and

$$0 = a(w, v) = \langle \mathcal{L}w \,|\, v\rangle \qquad (v \in V).$$

Hence, $\mathcal{L}w = 0$ and as $\mathcal{L}$ is injective, we conclude that $w = 0$. Consequently, $u = j(w) = 0$.

As for surjectivity, we take any $f \in \mathbb{L}^2$. Since $\mathcal{L}$ is surjective, we find $w \in V$ such that $\mathcal{L}w = j^*(f)$, that is

$$(f \,|\, j(v)) = \langle j^*(f) \,|\, v\rangle = \langle \mathcal{L}w \,|\, v\rangle = a(w, v) \qquad (v \in V).$$

This means that $j(w) \in \mathrm{dom}(\mathscr{L}^A)$ with $\mathscr{L}^A j(w) = f$. In total, $\mathscr{L}^A$ is invertible with inverse $(\mathscr{L}^A)^{-1} f = j(w) = j\mathcal{L}^{-1}j^* f$. □

Remark 7.2.2. By standard form theory, $\mathcal{L}$ is invertible if a is strongly Gårding elliptic or, equivalently, if V admits the global Poincaré inequality $\|u\|_2 \leq C\|\nabla u\|_2$ for some $C > 0$ and all $u \in V$. If O is a bounded domain and V has the extension property, then Poincaré's inequality always holds unless $V = \mathrm{W}^{1,2}_{\varnothing}(O)$ models good Neumann boundary conditions [80, Lem. 6].

Lemma 7.2.1 allows us to transfer mapping properties from the 'non-dynamical' operator $\mathcal{L}^{-1}$ to the 'dynamical operator' $(\mathscr{L}^A)^{-1}$. We shall give a striking application in Section 7.6. Let us stress that for the functional calculus of $\mathscr{L}^A$ no such simple transference can be used (and hence the results in this paper are non-trivial). Indeed, already for resolvents we do not have the formula

$$(t + \mathscr{L}^A)^{-1} = j(t + \mathcal{L})^{-1}j^* \qquad (t > 0),$$

because the form corresponding to the left-hand side is $a(\cdot,\cdot) + t(j(\cdot) \,|\, j(\cdot))_{\mathbb{L}^2}$ and not just $a(\cdot,\cdot) + t(\cdot \,|\, \cdot)_{\mathrm{L}^2(O)}$.

7.2.3 The bilinear embedding for $p = 2$

We find it instructive to give an elementary proof of Theorem 7.0.4 in the case $p = 2$ first. We fix $f, g \in \mathbb{L}^2$ and for $t > 0$ we abbreviate

$$f_t^A := \mathrm{e}^{-t\mathscr{L}^A} f \quad \& \quad g_t^B := \mathrm{e}^{-t\mathscr{L}^B} g.$$

We consider the power function

$$Q(\zeta,\eta) \coloneqq |\zeta|^2 + |\eta|^2 \qquad (\zeta,\eta \in \mathbb{C})$$

and define the corresponding heat flow as

$$\mathscr{E}(t) \coloneqq \int_{O\cup\Sigma} Q(f_t^A, g_t^B)\,\mathrm{d}\mu = \|f_t^A\|_2^2 + \|g_t^B\|_2^2.$$

Since $(\mathrm{e}^{-t\mathscr{L}^A})_{t\geq 0}$ and $(\mathrm{e}^{-t\mathscr{L}^B})_{t\geq 0}$ are bounded analytic C_0-semigroups in $\mathbb{L}^2$, we obtain $\mathscr{E} \in \mathrm{C}[0,\infty)\cap\mathrm{C}^1(0,\infty)$ and

$$-\mathscr{E}'(t) = 2\,\mathrm{Re}\int_{O\cup\Sigma} \mathscr{L}^A f_t^A \cdot \overline{f_t^A} + \mathscr{L}^B g_t^B \cdot \overline{g_t^B}\,\mathrm{d}\mu.$$

Analyticity of the semigroups also entails that $f_t^A \in \mathrm{dom}(\mathscr{L}^A)$ and $g_t^B \in \mathrm{dom}(\mathscr{L}^B)$. In particular, $f_t^A, g_t^B \in j(V)$ with $f_t^A = j(f_t^A|_O)$ and $g_t^B = j(g_t^B|_O)$. Hence, ellipticity and the elementary inequality $2XY \leq X^2 + Y^2$ yield the lower bound

$$\begin{aligned}
-\mathscr{E}'(t) &= 2\,\mathrm{Re}\int_O A\nabla(f_t^A|_O)\cdot\overline{\nabla(f_t^A|_O)} + B\nabla(g_t^B|_O)\cdot\overline{\nabla(g_t^B|_O)}\,\mathrm{d}x \\
&\geq 2\lambda(A,B)\int_O \big|\nabla(f_t^A|_O)\big|^2 + \big|\nabla(g_t^B|_O)\big|^2\,\mathrm{d}x \\
&\geq 4\lambda(A,B)\int_O \big|\nabla(f_t^A|_O)\big|\cdot\big|\nabla(g_t^B|_O)\big|\,\mathrm{d}x.
\end{aligned}$$

Integration in $t\in[0,T]$ leads us to the estimate

$$4\lambda(A,B)\int_0^T\int_O \big|\nabla(f_t^A|_O)\big|\cdot\big|\nabla(g_t^B|_O)\big|\,\mathrm{d}x\,\mathrm{d}t \leq \mathscr{E}(0) - \mathscr{E}(T) \leq \mathscr{E}(0).$$

Sending $T\to\infty$, we obtain

$$4\lambda(A,B)\int_0^\infty\int_O \big|\nabla(f_t^A|_O)\big|\cdot\big|\nabla(g_t^B|_O)\big|\,\mathrm{d}x\,\mathrm{d}t \leq \mathscr{E}(0) = \|f\|_2^2 + \|g\|_2^2.$$

Finally, we replace the pair (f,g) by $(sf, s^{-1}g)$ for $s>0$ and optimize in s to deduce the bilinear estimate

$$4\lambda(A,B)\int_0^\infty\int_O \big|\nabla(f_t^A|_O)\big|\cdot\big|\nabla(g_t^B|_O)\big|\,\mathrm{d}x\,\mathrm{d}t \leq 2\|f\|_2\|g\|_2.$$

The main obstacle in generalizing this argument to $p\neq 2$ lies in finding the correct p-adapted version of Q and getting p-adapted estimates of the semigroups from above and below. Here, the p-ellipticity assumption on A and B plays a key role.

7.3 p-ellipticity and L^p-contractivity of the semigroup

Let us recall that A is p-elliptic if (1.6) holds true. The elementary properties summarized in Lemma 1.7.8 will be useful.

7.3.1 Nittka's invariance criterion for j-elliptic forms

We denote by P_q the projection from $\mathbb{L}^2$ onto the closed and convex set

$$B_q := \{u \in \mathbb{L}^q \cap \mathbb{L}^2 \colon \|u\|_q \leq 1\}.$$

The following invariance criterion of closed and convex sets is due to Ouhabaz [139]. It continuous to hold in the j-elliptic setting [9, Prop. 2.9 (i), (ii)], see also [137, Rem. 4.2]. For our injective j, it takes the following, simpler form.

Proposition 7.3.1. *Let $q \in [1, \infty]$. The following assertions are equivalent.*

(i) *The semigroup $(\mathrm{e}^{-t\mathscr{L}^A})_{t\geq 0}$ is $\mathbb{L}^q$-contractive, that is*

$$\|\mathrm{e}^{-t\mathscr{L}^A} f\|_q \leq \|f\|_q \qquad (t \geq 0, f \in \mathbb{L}^q \cap \mathbb{L}^2).$$

(ii) *The set $j(V)$ is invariant under P_q and $\operatorname{Re} a(j^{-1}(P_q u), j^{-1}(u - P_q u)) \geq 0$ for all $u \in j(V)$.*

First, we use Proposition 7.3.1 to get the invariance of $j(V)$ under P_q for all $q \in [1, \infty]$ by complex interpolation and the fact that $(\mathrm{e}^{-t\mathscr{L}^A})_{t\geq 0}$ is $\mathbb{L}^\infty$-contractive when $A = \mathbf{1}$ is the identity. To this end, we need the precise form of j.

Lemma 7.3.2. *The semigroup $(\mathrm{e}^{-t\mathscr{L}^{\mathbf{1}}})_{t\geq 0}$ is $\mathbb{L}^\infty$-contractive. In particular, $j(V)$ is invariant under P_q for all $q \in [1, \infty]$.*

Proof. As explained above, we only need the L^∞-contractivity. The projection from $\mathbb{L}^2$ onto B_∞ is given by

$$P_\infty u = (|u| \wedge 1) \operatorname{sgn}(u) =: \Phi(u),$$

where $\operatorname{sgn}(z) := {}^{z}\!/_{|z|}\mathbf{1}_{\{|z|\neq 0\}}$ and $\Phi(z) := (|z| \wedge 1) \operatorname{sgn}(z)$ is Lipschitz continuous with $\Phi(0) = 0$. Hence, Lemma 7.1.3 asserts that $j(V)$ is invariant under P_∞. Let us fix $u \in j(V)$. Since j^{-1} is the pointwise restriction to O, we deduce

$$j^{-1}(P_\infty u) = \Phi(u|_O) \quad \& \quad j^{-1}(u - P_\infty u) = u|_O - \Phi(u|_O).$$

We set $w \coloneqq u|_O$. Since $A = \mathbf{1}$, we have

$$\operatorname{Re} a(j^{-1}(P_\infty u), j^{-1}(u - P_\infty u)) = \operatorname{Re} \int_O \nabla\Phi(w) \cdot \overline{(\nabla w - \nabla\Phi(w))}\,\mathrm{d}x.$$

By [140, Prop. 4.11], the weak gradient of $\Phi(w)$ is given by

$$\begin{aligned}\nabla\Phi(w) &= \mathbf{1}_{\{|w|>1\}} \mathrm{i}\operatorname{sgn}(w) \frac{\operatorname{Im}(\operatorname{sgn}(\overline{w})\nabla w)}{|w|} + \mathbf{1}_{\{|w|\leq 1\}}\nabla w \\ &\eqqcolon \mathbf{1}_{\{|w|>1\}}\mathrm{i}\Psi(w) + \mathbf{1}_{\{|w|\leq 1\}}\nabla w.\end{aligned}$$

Inserting this identity, we obtain

$$\begin{aligned}\operatorname{Re} a(j^{-1}(P_\infty u), j^{-1}(u - P_\infty u)) &= \operatorname{Re} \int_{\{|w|>1\}} \mathrm{i}\Psi(w) \cdot \overline{(\nabla w - \mathrm{i}\Psi(w))}\,\mathrm{d}x \\ &= \int_{\{|w|>1\}} \frac{|\operatorname{Im}(\operatorname{sgn}(\overline{w})\nabla w)|^2}{|w|} - |\Psi(w)|^2\,\mathrm{d}x \\ &= \int_{\{|w|>1\}} |\Psi(w)|^2|w| - |\Psi(w)|^2\,\mathrm{d}x \geq 0.\end{aligned}$$

Proposition 7.3.1 yields the claim. □

One of Nittka's contributions in [137] is an equivalent formulation of Proposition 7.3.1 (ii) that is more practical in applications. He observed the following fact.

Lemma 7.3.3 ([137, Prop. 2.4 & Lem. 3.1])**.** *Let $q \in [2,\infty)$ and $f \in \mathbb{L}^2$. There are unique $u \in B_q$ and $t \geq 0$ such that $f = u + t|u|^{q-2}u$. Moreover, $u = P_q f$.*

Armed with this result, we obtain the following version of Nittka's invariance criterion [137, Thm. 4.1] for our specific setting of V and j.

Proposition 7.3.4. *Let $q \in [2,\infty)$. The following assertions are equivalent.*

(i) *The semigroup $(\mathrm{e}^{-t\mathscr{L}^A})_{t\geq 0}$ is $\mathbb{L}^q$-contractive.*

(ii) *We have $\operatorname{Re} a(u|_O, (|u|^{q-2}u)|_O) \geq 0$ for all $u \in j(V)$ with $|u|^{q-2}u \in j(V)$.*

Proof. First, we assume that (ii) is valid. To show part (ii) of Proposition 7.3.1, we fix $f \in j(V)$. Lemma 7.3.2 yields $u \coloneqq P_q f \in j(V)$. By

Lemma 7.3.3 we find $t \geq 0$ such that $f = u + t|u|^{q-2}u$. Hence, $f - u = t|u|^{q-2}u \in j(V)$ and

$$\operatorname{Re} a(j^{-1}(P_q f), j^{-1}(f - P_q f)) = t \operatorname{Re} a(u|_O, (|u|^{q-2}u)|_O) \geq 0.$$

Now, we assume that Proposition 7.3.1 (ii) holds true and pick $u \in j(V)$ with $|u|^{q-2}u \in j(V)$. We note that $u \in \mathbb{L}^q$. If $u = 0$ there is nothing to prove. So we assume that $u \neq 0$ and put $\alpha := \|u\|_q^{-1} > 0$. Then $f := \alpha u + |u|^{q-2}u \in j(V)$ and Lemma 7.3.3 implies that $P_q f = \alpha u$. Consequently,

$$\operatorname{Re} a(u|_O, (|u|^{q-2}u)|_O) = \alpha^{-1} \operatorname{Re} a(j^{-1}(P_q f), j^{-1}(f - P_q f)) \geq 0. \qquad \square$$

Corollary 7.3.5. *Let $p \in (1, \infty)$ and A be p-elliptic. Then $(\mathrm{e}^{-t\mathscr{L}^A})_{t\geq 0}$ is $\mathbb{L}^p$-contractive.*

Proof. By duality and items (ii) and (iii) of Lemma 1.7.8 we can assume that $p \geq 2$. In view of Proposition 7.3.4 we have to show for all $u \in j(V)$ with $|u|^{p-2}u \in j(V)$ that

$$\operatorname{Re} a(w, |w|^{p-2}w) \geq 0,$$

where $w := u|_O$. Since $w, |w|^{p-2}w \in V$, this is the crucial estimate that follows from p-ellipticity of A, see for instance [80, Cor. 12]. $\square$

7.3.2 Generators in $\mathbb{L}^p$

Let us prove the first assertion of Theorem 7.0.3, namely that $(\mathrm{e}^{-t\mathscr{L}^A})_{t\geq 0}$ extrapolates to an analytic C_0-semigroup of contractions in $\mathbb{L}^p$.

Proof of Theorem 7.0.3, part 1. First, Lemma 1.7.8 (vi) yields $\theta_p > 0$ and part (v) of the same lemma implies $\pi/2 - \theta_p \geq \pi/2 - \theta_2 \geq \omega(A)$. Let us fix $\theta \in (-\theta_p, \theta_p)$. Since $\mathrm{e}^{-t\,\mathrm{e}^{\mathrm{i}\theta}\,\mathscr{L}^A} = \mathrm{e}^{-t\mathscr{L}^{\mathrm{e}^{\mathrm{i}\theta}A}}$ for all $t \geq 0$, Corollary 7.3.5 entails

$$\|\,\mathrm{e}^{-z\mathscr{L}^A} f\|_p \leq \|f\|_p \qquad (z \in \mathrm{S}_{\theta_p} \cup \{0\}, f \in \mathbb{L}^p \cap \mathbb{L}^2).$$

By density, each $\mathrm{e}^{-z\mathscr{L}^A}$ extends to a contractive linear operator $T_p(z)$ on $\mathbb{L}^p$. In view of Lemma 1.7.8 (v), the interval of exponents p, in which this conclusion holds, is open. Hence, we conclude by [140, Prop. 3.12] that $T_p := (T_p(z))_{z\in \mathrm{S}_{\theta_p}\cup\{0\}}$ defines an analytic C_0-semigroup of contractions in $\mathbb{L}^p$. $\square$

The above result gives rise to a proper notion of an $\mathbb{L}^p$-realization of $\mathscr{L}^A$.

Definition 7.3.6. Let $-\mathscr{L}_p^A$ be the generator of the semigroup T_p. We call $\mathscr{L}_p^A$ the **$\mathbb{L}^p$-realization** of $\mathscr{L}^A$.

Letting $\mathscr{L}^A|_{\mathbb{L}^p}$ be the part of $\mathscr{L}^A$ in $\mathbb{L}^p$, we have the following, more concrete description for the $\mathbb{L}^p$-realization.

Lemma 7.3.7 ([74, Lem. 3.1])**.** *Let $p \in (1,\infty)$ and A be p-elliptic. The operator $\mathscr{L}_p^A$ is the closure of $\mathscr{L}^A|_{\mathbb{L}^p}$ in $\mathbb{L}^p$. In particular, if $p \in (2,\infty)$ and μ is bounded, then $\mathscr{L}_p^A = \mathscr{L}^A|_{\mathbb{L}^p}$.*

Clearly, $\mathscr{L}_p^A$ is sectorial of angle at most $\pi/2 - \theta_p$ in $\mathbb{L}^p$. For convenience of the reader, we collect elementary properties of $\mathscr{L}_p^A$.

Lemma 7.3.8. *Let $p \in (1,\infty)$ and A be p-elliptic.*

(i) *We have $(z - \mathscr{L}_p^A)^{-1} f = (z - \mathscr{L}^A)^{-1} f$ for all $z \in \mathbb{C} \setminus \overline{\mathrm{S}_{\pi/2-\theta_p}}$ and $f \in \mathbb{L}^p \cap \mathbb{L}^2$.*

(ii) *We have $\mathscr{L}_p^A u = \mathscr{L}^A u$ for all $u \in \mathrm{dom}(\mathscr{L}_p^A) \cap \mathrm{dom}(\mathscr{L}^A)$.*

(iii) *If $\mathscr{L}^A$ is injective, then so is $\mathscr{L}_p^A$.*

(iv) *If a is strongly Gårding elliptic, then $\mathscr{L}_p^A$ is invertible and (i) holds for $z = 0$.*

Proof. Parts (i) and (ii) are already in [74, Lem. 3.1]. Let us prove (iii). For any $f \in \mathbb{L}^p \cap \mathbb{L}^2$ we have $\lim_{t\to\infty} T_p(t) f = \lim_{t\to\infty} T_2(t) f = 0$ strongly in $\mathbb{L}^2$. Since T_p is uniformly bounded on $\mathbb{L}^p$ and $\mathbb{L}^p \cap \mathbb{L}^2$ is dense in $\mathbb{L}^p$, the same convergence holds weakly in $\mathbb{L}^p$ for every $f \in \mathbb{L}^p$. Now, if $f \in \ker(\mathscr{L}_p^A)$, then $T_p(t) f = f$ for all $t > 0$ and $f = 0$ follows.

To prove (iv), we note that the strong Gårding estimate for a implies that T_2 is exponentially stable. Since A is p-elliptic, we can combine Lemma 1.7.8 (v) and part 1 of Theorem 7.0.3 to find $q \in (1,\infty)$ such that p lies between 2 and q, and T_q is contractive. By complex interpolation, T_p is exponentially stable. Hence, $\mathscr{L}_p^A$ is invertible and consistency of the inverse on $\mathbb{L}^p \cap \mathbb{L}^2$ follows from consistency of the semigroups by taking the Laplace transform at $\lambda = 0$. □

7.4 Bilinear embedding

In this section, we prove the bilinear embedding, Theorem 7.0.4, for $p \neq 2$. By symmetry of the assumptions we suppose from now on that $p > 2$ and that A is p-elliptic. We fix $t > 0$, $f, g \in \mathbb{L}^p \cap \mathbb{L}^{p'}$ and let $Q\colon \mathbb{C} \times \mathbb{C} \to [0,\infty)$

be a continuously (real) differentiable function. We are interested in the monotonicity behaviour of the adapted heat flow

$$\mathscr{E}(t) \coloneqq \int_{O\cup\Sigma} Q(f_t^A, g_t^B)\,\mathrm{d}\mu.$$

The fundamental theorem of calculus yields

$$-\int_0^\infty \mathscr{E}'(t)\,\mathrm{d}t \le \mathscr{E}(0) = \int_{O\cup\Sigma} Q(f,g)\,\mathrm{d}\mu,$$

and as in Subsection 7.2.3 we need an upper bound for Q and a uniform lower bound for $-\mathscr{E}'(t)$. The difficulty is to find the correct Q. As in [52, 53], we use the ingenious Nazarov–Treil Bellmann function and we refer to these references for historical background.

Definition 7.4.1. Let $\delta \in (0,1]$ and $\zeta, \eta \in \mathbb{C}$. The **Nazarov–Treil Bellmann function** is defined as

$$Q(\zeta,\eta) \coloneqq |\zeta|^p + |\eta|^{p'} + \delta \begin{cases} |\zeta|^2|\eta|^{2-p'}, & (|\zeta|^p \le |\eta|^{p'}), \\ \frac{2}{p}|\zeta|^p + (1-\frac{2}{p})|\eta|^{p'}, & (|\zeta|^p \ge |\eta|^{p'}). \end{cases}$$

Here, δ is a degree of freedom that will be chosen small at a later point. We note that

$$Q(\zeta,\eta) \le 2(|\zeta|^p + |\eta|^{p'}) \qquad (\zeta,\eta\in\mathbb{C}), \tag{7.5}$$

and for $p = 2$ we recover (up to a multiplicative constant) the energy from Subsection 7.2.3. We write

$$\partial_z = \frac{1}{2}(\partial_x - \mathrm{i}\partial_y)$$

for the (Wirtinger) derivative with respect to the complex variable $z = x + \mathrm{i}y$. Since for any $q \in (1,\infty)$ the power function $z \mapsto |z|^q$ is continuously differentiable with derivative

$$\partial_z|z|^q = \partial_z(z\cdot\overline{z})^{\frac{q}{2}} = \frac{q}{2}(z\cdot\overline{z})^{\frac{q}{2}-1}\partial_z(z\cdot\overline{z}) = \frac{q}{2}|z|^{q-2}\overline{z},$$

the very definition of Q implies that $Q \in \mathrm{C}^1(\mathbb{C}^2)$. As a matter of fact, we need to re-tell some of the results found by Carbonaro and Dragičević in [52, 53]. To simplify the exposition, we shall follow their outline in [55].

The first part of Theorem 7.0.3 implies that $(\mathrm{e}^{-t\mathscr{L}^A})_{t\geq 0}$ and $(\mathrm{e}^{-t\mathscr{L}^B})_{t\geq 0}$ are bounded analytic semigroups in $\mathbb{L}^p$ and $\mathbb{L}^{p'}$, respectively. Hence, $\mathscr{E} \in \mathrm{C}[0,\infty) \cap \mathrm{C}^1(0,\infty)$ with derivative

$$\begin{aligned} -\mathscr{E}'(t) &= 2\,\mathrm{Re}\int_{O\cup\Sigma} \mathscr{L}^A f_t^A \cdot (\partial_\zeta Q)(f_t^A, g_t^B) + \mathscr{L}^B g_t^B \cdot (\partial_\eta Q)(f_t^A, g_t^B)\,\mathrm{d}\mu \\ &= 2\,\mathrm{Re}\int_{O\cup\Sigma} \mathscr{L}^A j(f_t^A|_O) \cdot (\partial_\zeta Q)(j(f_t^A|_O), j(g_t^B|_O)) \\ &\qquad + \mathscr{L}^B j(g_t^B|_O) \cdot (\partial_\eta Q)(j(f_t^A|_O), j(g_t^B|_O))\,\mathrm{d}\mu, \end{aligned} \tag{7.6}$$

see [53, App. C, Prop. C1]. We would like to use the definition of $\mathscr{L}^A$ and $\mathscr{L}^B$ to get rid of the hybrid measure space, but this is not immediately possible, because $(\partial_\zeta Q, \partial_\eta Q)$ lacks in regularity and, among other things, we cannot use Lemma 7.1.3 to commute $(\partial_\zeta Q, \partial_\eta Q)$ with j. A similar issue has also occurred in [53] when transforming the heat flow method from $\mathbb{R}^d$ [52] to arbitrary open sets. It has been resolved by approximating Q with smooth functions $R_{n,\nu}$ and we will see that the approximations also fit into the setting of Lemma 7.1.3. To this end, we need:

- a radial function $\varphi \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^4)$ with $0 \leq \varphi \leq \mathbf{1}_{B(0,1)}$ and $\int_{\mathbb{R}^4} \varphi\,\mathrm{d}x = 1$,
- a radial function $\psi \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^4)$ with $\mathbf{1}_{B(0,3)} \leq \psi \leq \mathbf{1}_{B(0,4)}$.

For $n \in \mathbb{N}$, $\nu \in (0,1]$ and $\omega \in \mathbb{R}^4$ we set

$$\varphi_\nu(\omega) \coloneqq \nu^{-4}\varphi(\omega/\nu) \quad \& \quad \psi_n(\omega) \coloneqq \psi(\omega/n).$$

We identify $\mathbb{C}^2$ explicitly with $\mathbb{R}^4$ via

$$\mathcal{W}_2 \colon \mathbb{C}^2 \to \mathbb{R}^4, \quad \mathcal{W}_2(z_1, z_2) \coloneqq (\mathrm{Re}(z_1), \mathrm{Im}(z_1), \mathrm{Re}(z_2), \mathrm{Im}(z_2))$$

and define the convolution of Q with φ_ν at $z \in \mathbb{C}^2$ by

$$(Q * \varphi_\nu)(z) \coloneqq \int_{\mathbb{R}^4} Q(z - \mathcal{W}_2^{-1}(\omega))\varphi_\nu(\omega)\,\mathrm{d}\omega.$$

Definition 7.4.2. Let $n \in \mathbb{N}$ and $\nu \in (0,1]$. We define

$$R_{n,\nu} \coloneqq \psi_n(Q * \varphi_\nu) + P_{n,\nu}.$$

Here, $P_{n,\nu}$ is a cleverly chosen correction term that makes the generalized Hessian of $R_{n,\nu}$ (A,B)-convex, see Definition 7.4.4 below. Its precise form is not needed and we refer to [53, Sec. 5.2]. Let us collect useful properties of $R_{n,\nu}$. We write $D = (\partial_\zeta, \partial_\eta)$ and denote by $D^2 R_{n,\nu}$ the Hessian of $R_{n,\nu}$ in the variables ζ and η.

Lemma 7.4.3 ([53, Lem. 14 & Thm. 16])**.** *Let $n \in \mathbb{N}$ and $\nu \in (0,1]$. Then $R_{n,\nu} \in \mathrm{C}^\infty(\mathbb{C}^2)$ has the following additional properties:*

(Q1) We have $D^2 R_{n,\nu} \in \mathrm{L}^\infty(\mathbb{C}^2; \mathbb{C}^{2\times 2})$.

(Q2) We have $(DR_{n,\nu})(0) = 0$.

(Q3) We have

$$\lim_{n\to\infty} ((DR_{n,\nu})(z), (D^2 R_{n,\nu})(z)) = ((D(Q * \varphi_\nu))(z), (D^2(Q * \varphi_\nu))(z))$$

for all $z \in \mathbb{C}^2$ and

$$\lim_{\nu\to 0} \lim_{n\to\infty} ((DR_{n,\nu})(z), (D^2 R_{n,\nu})(z)) = ((DQ)(z), (D^2 Q)(z))$$

for almost every $z \in \mathbb{C}^2$.

(Q4) We have $|(DR_{n,\nu})(z)| + |D(Q * \varphi_\nu)(z)| \leq C(|z|^{p-1} + |z|^{p'-1})$ for some $C = C(\delta, p) > 0$ and all $z \in \mathbb{C}^2$.

Let us continue in (7.6). Using (Q3), (Q4) and the boundedness of the semigroup on $\mathbb{L}^p$ and $\mathbb{L}^{p'}$ (part 1 of Theorem 7.0.3), we can apply dominated convergence twice and get

$$-\frac{1}{2}\mathscr{E}'(t) = \lim_{\nu\to 0} \lim_{n\to\infty} \operatorname{Re}\bigg(\int_{O\cup\Sigma} \mathscr{L}^A j(f_t^A|_O) \cdot (\partial_\zeta R_{n,\nu})(j(f_t^A|_O), j(g_t^B|_O)) + \mathscr{L}^B j(g_t^B|_O) \cdot (\partial_\eta R_{n,\nu})(j(f_t^A|_O), j(g_t^B|_O))\,\mathrm{d}\mu \bigg).$$

Owing to (Q1) and (Q2), $DR_{n,\nu}$ falls into the scope of Lemma 7.1.3. Thus, we can commute j as desired and obtain

$$-\frac{1}{2}\mathscr{E}'(t) = \lim_{\nu\to 0} \lim_{n\to\infty} \int_O \operatorname{Re}\Big(A\nabla(f_t^A|_O) \cdot \nabla(\partial_\zeta R_{n,\nu})(f_t^A|_O, g_t^B|_O)\Big) + \operatorname{Re}\Big(B\nabla(g_t^B|_O) \cdot \nabla(\partial_\eta R_{n,\nu})(f_t^A|_O, g_t^B|_O)\Big)\,\mathrm{d}x. \tag{7.7}$$

At this point, j has disappeared and we are exactly in the setting of [53].

Let us focus on the first term in the integral. By the chain rule, applying ∇ to $\partial_\zeta R_{n,\nu}$ at the point $(z_1, z_2) := (f_t^A|_O, g_t^B|_O) \in \mathbb{C}^2$ produces second-order derivatives of $R_{n,\nu}$ that are multiplied with the gradient pair

$(X_1, X_2) \coloneqq (\nabla(f_t^A|_O), \nabla(g_t^B|_O)) \in (\mathbb{C}^d)^2$ and we obtain $(\partial_\zeta^2 R_{n,\nu})(z_1, z_2)X_1 + (\partial_\eta \partial_\zeta R_{n,\nu})(z_1, z_2)X_2$. Thus, the first term in the integral is

$$\operatorname{Re}\Big(AX_1 \cdot (\partial_\zeta^2 R_{n,\nu})(z_1, z_2)X_1 + AX_1 \cdot (\partial_\eta \partial_\zeta R_{n,\nu})(z_1, z_2)X_2\Big)$$

and we have a similar formula for the second one. Their sum is the following object for $F = R_{n,\nu}$.

Definition 7.4.4. Let $F\colon \mathbb{C}^2 \to \mathbb{R}$ be twice (real) differentiable, $x \in O$, $z_1, z_2 \in \mathbb{C}$ and $X_1, X_2 \in \mathbb{C}^d$. We define the **generalized Hessian** of F with respect to A and B as

$$H_F^{(A(x),B(x))}[(z_1, z_2); (X_1, X_2)] \coloneqq \operatorname{Re}\left(\begin{bmatrix} A(x)X_1 \\ B(x)X_2 \end{bmatrix} \cdot (D^2F)(z_1, z_2) \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\right),$$

where we identify each entry of the Hessian $(D^2F)(z_1, z_2) \in \mathbb{C}^{2\times 2}$ with a multiplication operator in $\mathbb{C}^d$.

In this terminology, (7.7) becomes

$$-\frac{1}{2}\mathscr{E}'(t) = \lim_{\nu\to 0}\lim_{n\to\infty}\int_O H_{R_{n,\nu}}^{(A,B)}\big[(f_t^A|_O, g_t^B|_O); (\nabla(f_t^A|_O), \nabla(g_t^B|_O))\big]\,\mathrm{d}x. \tag{7.8}$$

The beautiful insight of Carbonaro and Dragičević in [52] was to establish the link between p-ellipticity of A and B and pointwise lower bounds for certain generalized Hessians, which they call **(A, B)-convexity**.

Proposition 7.4.5 ([52, Cor. 5.5] & [53, Thm. 16])**.** *Let $\nu \in (0, 1]$. There is some $\delta \in (0, 1)$, depending only on $\lambda(A, B)$, $\Lambda(A, B)$ and $\Delta_p(A, B)$, such that for almost every $x \in O$, all $z_1, z_2 \in \mathbb{C}$ and every $X_1, X_2 \in \mathbb{C}^d$ we have:*

(i) *The generalized Hessian of $R_{n,\nu}$ is non-negative.*

(ii) *The generalized Hessian of $Q * \varphi_\nu$ satisfies the explicit lower bound*

$$H_{Q*\varphi_\nu}^{(A(x),B(x))}[(z_1, z_2); (X_1, X_2)] \geq \frac{\Delta_p(A, B)}{5} \cdot \frac{\lambda(A, B)}{\Lambda(A, B)}|X_1||X_2|.$$

Since the generalized Hessian of $R_{n,\nu}$ is non-negative almost everywhere, we can use (Q3) and Fatou's lemma in (7.8) in order to obtain

$$\begin{aligned} -\frac{1}{2}\mathscr{E}'(t) &\geq \liminf_{\nu\to 0}\int_O H_{Q*\varphi_\nu}^{(A,B)}\big[(f_t^A|_O, g_t^B|_O); (\nabla(f_t^A|_O), \nabla(g_t^B|_O))\big]\,\mathrm{d}x \\ &\geq \frac{\Delta_p(A, B)}{5} \cdot \frac{\lambda(A, B)}{\Lambda(A, B)}\int_O |\nabla(f_t^A|_O)||\nabla(g_t^B|_O)|\,\mathrm{d}x. \end{aligned}$$

Now, integration with respect to $t \in [0,T]$ gives

$$\frac{\Delta_p(A,B)}{5} \cdot \frac{\lambda(A,B)}{\Lambda(A,B)} \int_0^T \int_O \left|\nabla(f_t^A|_O)\right| \cdot \left|\nabla(g_t^B|_O)\right| \mathrm{d}x\,\mathrm{d}t \le \mathscr{E}(0) - \mathscr{E}(T) \le \mathscr{E}(0)$$

and from (7.5) and letting $T \to \infty$ we obtain

$$\frac{\Delta_p(A,B)}{5} \cdot \frac{\lambda(A,B)}{\Lambda(A,B)} \int_0^\infty \int_O \left|\nabla(f_t^A|_O)\right| \cdot \left|\nabla(g_t^B|_O)\right| \mathrm{d}x\,\mathrm{d}t \le 2\Big(\|f\|_p^p + \|g\|_{p'}^{p'}\Big).$$

Finally, we replace (f,g) by $(sf, s^{-1}g)$ for $s>0$ and optimize in s to conclude the proof of Theorem 7.0.4.

7.5 Bounded H$^\infty$-calculus in $\mathbb{L}^p$ and maximal regularity

In this section, we complete the proof of Theorem 7.0.3. We refer to [68, 105] and Section 1.5 for background on functional calculus and maximal parabolic regularity.

To prove the second assertion of Theorem 7.0.3, we follow [53] and first prove the following **weak quadratic estimates** in the spirit of [64]. Since $\mathbb{L}^p \cap \mathbb{L}^{p'} \subseteq \mathbb{L}^2$, we can avoid $\mathbb{L}^p$-realizations at this stage.

Proposition 7.5.1. *Let $p \in (1,\infty)$, A be p-elliptic and $\theta \in (-\theta_p, \theta_p)$. Then there exists $C>0$ such that we have*

$$\int_0^\infty \left|\Big(t\,\mathrm{e}^{\mathrm{i}\theta}\,\mathscr{L}^A\,\mathrm{e}^{-t\,\mathrm{e}^{\mathrm{i}\theta}\,\mathscr{L}^A} f \,\Big|\, g\Big)\right| \frac{\mathrm{d}t}{t} \le C\|f\|_p\|g\|_{p'} \qquad (f,g \in \mathbb{L}^p \cap \mathbb{L}^{p'}).$$

Proof. We abbreviate $A(\theta) \coloneqq \mathrm{e}^{\mathrm{i}\theta} A$. Thanks to Lemma 1.7.8 (ii), we can apply Theorem 7.0.4 to the pair $(A(\theta), A(\theta)^*)$ and get

$$\begin{aligned}
&\int_0^\infty \left|\Big(t\mathscr{L}^{A(\theta)}\,\mathrm{e}^{-2t\mathscr{L}^{A(\theta)}} f \,\Big|\, g\Big)\right| \frac{\mathrm{d}t}{t} \\
&\qquad = \int_0^\infty \left|\Big(\mathscr{L}^{A(\theta)} f_t^{A(\theta)} \,\Big|\, g_t^{A(\theta)^*}\Big)\right| \mathrm{d}t \\
&\qquad = \int_0^\infty \left|\int_O A(\theta)\nabla(f_t^{A(\theta)})|_O \cdot \overline{\nabla(g_t^{A(\theta)^*})|_O}\,\mathrm{d}x\right| \mathrm{d}t \\
&\qquad \le \Lambda(A) \int_0^\infty \int_O \left|\nabla(f_t^{A(\theta)})|_O\right| \cdot \left|\nabla(g_t^{A(\theta)^*})|_O\right| \mathrm{d}x\,\mathrm{d}t \\
&\qquad \le C\Lambda(A)\|f\|_p\|g\|_{p'}.
\end{aligned}$$

□

Proof of Theorem 7.0.3, part 2. We fix $\mu \in (\pi/2 - \theta_p, \pi)$. A classical theorem on quadratic estimates [64, Thm. 4.6 & Ex. 4.8] states that the estimate in Proposition 7.5.1 implies that there is $C > 0$ such that

$$\|\varphi(\mathscr{L}^A)f\|_p \leq C\|\varphi\|_{\mathrm{L}^\infty(\mathrm{S}_\mu)}\|f\|_p \qquad (\varphi \in \mathrm{H}^\infty_0(\mathrm{S}_\mu), f \in \mathbb{L}^p \cap \mathbb{L}^{p'}).$$

Indeed, we apply the results in [64] to the dual pair $(\mathbb{L}^p \cap \mathbb{L}^2, \mathbb{L}^{p'} \cap \mathbb{L}^2)$. They require that $\mathscr{L}^A$ has dense range only for the convergence lemma [64, p.69]. If $\varphi \in \mathrm{H}^\infty_0(\mathrm{S}_\mu)$, then the latter holds without this extra assumption. Since $\varphi(\mathscr{L}^A_p)f = \varphi(\mathscr{L}^A)f$ for all $f \in \mathbb{L}^p \cap \mathbb{L}^2$ by Lemma 7.3.8 (i), we can conclude by density. □

Finally, we establish maximal regularity for $\mathscr{L}^A_p$.

Definition 7.5.2. Let $T \in (0, \infty)$. We say that $\mathscr{L}^A_p$ has **maximal parabolic regularity** if for some $q \in (1, \infty)$ (or equivalently all $q \in (1, \infty)$) and all $f \in \mathrm{L}^q(0, T; \mathbb{L}^p)$ the mild solution

$$u\colon [0, T) \to \mathbb{L}^p, \quad u(t) = \int_0^t \mathrm{e}^{-(t-s)\mathscr{L}^A_p} f(s)\,\mathrm{d}s$$

to the abstract Cauchy problem

$$\begin{aligned} u'(t) + \mathscr{L}^A_p u(t) &= f(t) \qquad (t \in (0, T)), \\ u(0) &= 0, \end{aligned}$$

is (Fréchet-)differentiable a.e., takes its values in $\mathrm{dom}(\mathscr{L}^A_p)$ a.e. and $u', \mathscr{L}^A_p u$ belong to $\mathrm{L}^q(0, T; \mathbb{L}^p)$.

Proof of Theorem 7.0.3, part 3. This is just the Dore–Venni theorem [105, Cor. 9.3.12] applied to the invertible operator $1 + \mathscr{L}^A_p$. Indeed, the property of maximal regularity is invariant under shifting the operator and $1 + \mathscr{L}^A_p$ has a bounded H^∞-calculus of angle $\pi/2 - \theta_p$. Therefore, $1 + \mathscr{L}^A_p$ has bounded imaginary powers [105, p. 88]. □

7.6 Elliptic regularity for dynamical boundary conditions

In this section, we illustrate how to prove Hölder estimates for elements in the domain of (fractional powers of) $\mathscr{L}^A_p$ for large p. In order to avoid further technicalities, we work in the concrete geometric setting of Section 7.1.4 and more specifically assume the following.

Assumption 7.6.1. *Throughout this section, we suppose that*

(i) *O is a bounded domain and $V \neq \mathrm{W}^{1,2}_{\varnothing}(O)$.*

(ii) *O^c is locally 2-fat away from N, O is locally uniform near N and there is $\ell \in (d-2, d)$ such that Σ is an upper ℓ-set.*

Boundedness of O implies that $\mathrm{W}^{-1,q}_D(O) \subseteq V^*$ and $\mathbb{L}^p \subseteq \mathbb{L}^2$ for all $p, q \in [2, \infty)$. By Remark 7.2.2, the Lax–Milgram operator $\mathcal{L}\colon V \to V^*$ is invertible. Hence, Lemma 7.2.1 implies that also $\mathscr{L}^A$ is invertible. Moreover, Theorem 7.1.13 yields that $j^*\colon \mathbb{L}^p \to \mathrm{W}^{-1,q}_D(O)$ is bounded provided that

$$q \in [2, ((d-\ell) \vee 1)') \quad \& \quad p \in ((\tfrac{\ell q'}{d-q'})', q] \cap [2, q]. \tag{7.9}$$

Our first result concerns global Hölder regularity to a variational problem with dynamical boundary conditions.

Theorem 7.6.2. *Let $\mu \in (0, 1]$ and p, q as in (7.9). If $\mathcal{L}^{-1}\colon \mathrm{W}^{-1,q}_D(O) \to \mathrm{C}^\mu(O)$ is bounded, then for every $f \in \mathbb{L}^p$ the unique solution u to the variational problem $\mathscr{L}^A u = f$ belongs to $\mathrm{C}^\mu_D(O)$.*

Proof. The assumption joint with Lemma 7.2.1 and Theorem 7.1.13 entail that

$$u \coloneqq (\mathscr{L}^A)^{-1} f = j\mathcal{L}^{-1} j^* f \in \mathrm{C}^\mu(O) \cap \mathrm{dom}(\mathscr{L}^A).$$

Thanks to Lemma 5.2.8 we can conclude that the continuous extension of u to $\overline{O}$ vanishes in D. □

Remark 7.6.3. The assumption on $\mathcal{L}^{-1}$ in Theorem 7.6.2 is known in a variety of settings, compare with [17,37,79,108]. Interestingly, these references rely on results from harmonic analysis, such as Hölderian Gaussian estimates (see Chapter 5) and the solution of the Kato problem in the 'non-dynamical setting' [40]. We do not know whether these results themselves also hold in our 'dynamical setting'.

Theorem 7.6.2 is the starting point for investigating operator theoretic regularity, that is, embeddings of domains of $\mathbb{L}^p$-realizations into Hölder spaces. The following result is new even for small complex perturbations of real coefficients, so for p-elliptic A with very large p.

Corollary 7.6.4. *Let $\mu \in (0, 1]$, p, q as in (7.9) and let A be p-elliptic. If $\mathcal{L}^{-1}\colon \mathrm{W}^{-1,q}_D(O) \to \mathrm{C}^\mu(O)$ is bounded, then $\mathrm{dom}(\mathscr{L}^A_p) \subseteq \mathrm{C}^\mu_D(O)$ with continuous inclusion.*

Proof. Since O is bounded, $\mathscr{L}_p^A$ is the part of $\mathscr{L}^A$ in $\mathbb{L}^p$, compare with Lemma 7.3.7. Hence, if $u \in \operatorname{dom}(\mathscr{L}_p^A)$, then $u = (\mathscr{L}^A)^{-1} f = j\mathcal{L}^{-1} j^* f$ with $f := \mathscr{L}_p^A u \in \mathbb{L}^p$. The claim follows from Theorem 7.6.2 and continuity of the respective linear operators. □

As pointed out in the introduction, applications to semilinear equations [71] require similar embedding results already for domains of fractional powers $(\mathscr{L}_p^A)^\sigma$ with $\sigma \in (0,1)$. Fractional powers are defined by functional calculus, see e.g. [105, Chap. 3], and all relevant knowledge appears in the proof below. The same goes with interpolation theory and we refer to [156] for background information.

Theorem 7.6.5. *Let $\mu \in (0,1]$, p, q as in (7.9) and A be p-elliptic. If $\mathcal{L}^{-1} \colon \mathrm{W}_D^{-1,q}(O) \to \mathrm{C}^\mu(O)$ is bounded, then*

$$\operatorname{dom}((\mathscr{L}_p^A)^\sigma) \subseteq \mathrm{C}_D^{\kappa\mu}(O)$$

for all $\kappa \in (0,1)$ and $\sigma \in (\frac{1}{2} \vee ((1-\kappa)\frac{d}{p\mu+d} + \kappa), 1]$.

For the proof, we modify an argument in [108, Sec. 3]. We also need a new interpolation inequality and a compatibility property for domains of fractional powers, the proofs of which we postpone until the end of the section.

Lemma 7.6.6. *Let $p \in [1,\infty]$, $\mu \in (0,1]$ and $\theta := {}^{d}\!/_{(p\mu+d)}$. Then there is $C > 0$ depending on d, p and geometry such that*

$$\|u\|_{\mathrm{L}^\infty(O)} \le C \|u\|_{\mathrm{L}^p(O)}^{1-\theta} \|u\|_{\mathrm{C}^\mu(O)}^{\theta} \qquad (u \in \mathrm{L}^p(O) \cap \mathrm{C}_D^\mu(O)).$$

In addition, we have for all $\mu \in (0,1]$ and $\theta \in [0,1]$ that

$$\|u\|_{\mathrm{C}^{\theta\mu}(O)} \le 3 \|u\|_{\mathrm{L}^\infty(O)}^{1-\theta} \|u\|_{\mathrm{C}^\mu(O)}^{\theta} \qquad (u \in \mathrm{C}^\mu(O)).$$

Lemma 7.6.7. *If $\sigma > {}^1\!/_2$, then $\operatorname{dom}((\mathscr{L}^A)^\sigma) \subseteq j(V)$.*

Proof of Theorem 7.6.5. We obtain from [156, Thm. 1.15.2] that

$$\operatorname{dom}((\mathscr{L}_p^A)^\sigma) \subseteq (\mathbb{L}^p, \operatorname{dom}(\mathscr{L}_p^A))_{\sigma,\infty}.$$

Since $\sigma > (1-\kappa)\frac{d}{p\mu+d} + \kappa$, [156, Thm. 1.3.3 (b) & (e)] entails

$$\subseteq (\mathbb{L}^p, \operatorname{dom}(\mathscr{L}_p^A))_{(1-\kappa)\frac{d}{p\mu+d}+\kappa,1}.$$

By the reiteration theorem [156, Thm. 1.10.2] the latter coincides with

$$= ((\mathbb{L}^p, \mathrm{dom}(\mathscr{L}_p^A))_{\frac{d}{p\mu+d},1}, \mathrm{dom}(\mathscr{L}_p^A))_{\kappa,1}.$$

Now, we apply the bounded operator $|_O$ on both sides and use Corollary 7.6.4 in order to get

$$\begin{aligned}\mathrm{dom}((\mathscr{L}_p^A)^\sigma)|_O &\subseteq ((\mathrm{L}^p(O), \mathrm{dom}(\mathscr{L}_p^A)|_O)_{\frac{d}{p\mu+d},1}, \mathrm{dom}(\mathscr{L}_p^A)|_O)_{\kappa,1}\\ &\subseteq ((\mathrm{L}^p(O), \mathrm{C}_D^\mu(O))_{\frac{d}{p\mu+d},1}, \mathrm{C}_D^\mu(O))_{\kappa,1}.\end{aligned}$$

In the language of [156, Lem. 1.10.1 (a)], the first statement of Lemma 7.6.6 means that $\mathrm{L}^\infty(O)$ is of class $J(d/(p\mu+d))$ between $\mathrm{L}^p(O)$ and $\mathrm{C}_D^\mu(O)$ and hence contains $(\mathrm{L}^p(O), \mathrm{C}_D^\mu(O))_{d/(p\mu+d),1}$ with continuous inclusion. A similar reformulation applies to the second statement of the lemma. We deduce

$$\mathrm{dom}((\mathscr{L}_p^A)^\sigma)|_O \subseteq (\mathrm{L}^\infty(O), \mathrm{C}_D^\mu(O))_{\kappa,1} \subseteq \mathrm{C}_D^{\kappa\mu}(O).$$

Now, we let $u \in \mathrm{dom}((\mathscr{L}_p^A)^\sigma)$. We already know that $u|_O \in \mathrm{C}_D^{\kappa\mu}(O)$. To prove that the same is true for u itself, we let $\widetilde{u} \in \mathrm{C}_D^{\kappa\mu}(O)$ be the unique continuous extension of $u|_O$. We have $\mathrm{dom}((\mathscr{L}_p^A)^\sigma) \subseteq \mathrm{dom}((\mathscr{L}^A)^\sigma)$, see [105, Prop. 2.6.5 b)], and therefore Lemma 7.6.7 yields $u \in j(V)$. We conclude that $u|_\Sigma = \mathrm{Tr}(u|_O) = \widetilde{u}|_\Sigma$ and hence $u = \widetilde{u} \in \mathrm{C}_D^{\kappa\mu}(O)$. □

Proof of Lemma 7.6.6. The second assertion is [108, Lem. 3.5]. Let us turn to the first part. To this end, we fix $\delta \in (0,1]$ as in the definition of (LU). If $\|u\|_{\mathrm{L}^p(O)} \geq (\frac{\delta}{2})^{d/\theta p}\|u\|_{\mathrm{C}^\mu(O)}$, then we simply estimate

$$\|u\|_{\mathrm{L}^\infty(O)} = \|u\|_{\mathrm{L}^\infty(O)}^{1-\theta}\|u\|_{\mathrm{L}^\infty(O)}^{\theta} \leq (\tfrac{2}{\delta})^{\frac{d(1-\theta)}{\theta p}}\|u\|_{\mathrm{L}^p(O)}^{1-\theta}\|u\|_{\mathrm{C}^\mu(O)}^{\theta}.$$

Hence, we assume from now on that $\|u\|_{\mathrm{L}^p(O)} < (\frac{\delta}{2})^{d/\theta p}\|u\|_{\mathrm{C}^\mu(O)}$ and set

$$r := (\|u\|_{\mathrm{L}^p(O)}\|u\|_{\mathrm{C}^\mu(O)}^{-1})^{\frac{\theta p}{d}} \in (0, \tfrac{\delta}{2}).$$

We fix $x \in O$ and consider two cases.

(1) $\overline{B(x,r)} \cap D \neq \varnothing$. In this case there is $x_D \in \overline{B(x,r)} \cap D$ and since u vanishes in D we get

$$|u(x)| = |u(x) - u(x_D)| \leq r^\mu\|u\|_{\mathrm{C}^\mu(O)} \leq \|u\|_{\mathrm{L}^p(O)}^{1-\theta}\|u\|_{\mathrm{C}^\mu(O)}^{\theta}.$$

(2) $\overline{B(x,r)} \cap D = \varnothing$. Then $\overline{B(x,r)} \subseteq O$ or $\overline{B(x,r)} \cap N \neq \varnothing$ and we obtain for all $y \in O(x,r)$ that

$$|u(x)| \leq |u(x) - u(y)| + |u(y)| \leq r^\mu\|u\|_{\mathrm{C}^\mu(O)} + |u(y)|.$$

We average the latter with respect to $y \in O(x,r)$ and use Hölder's inequality, which leads us to

$$|u(x)| \leq r^{\mu}\|u\|_{\mathrm{C}^{\mu}(O)} + |O(x,r)|^{-\frac{1}{p}}\|u\|_{\mathrm{L}^{p}(O)}.$$

If $\overline{B(x,r)} \subseteq O$, then $|O(x,r)| = |B(0,1)|r^d$. In the other case, $\mathrm{d}_N(x) \leq r < (\delta/2 \wedge 1)$. It follows from Proposition 5.1.8 that there is $C > 0$ depending on d and geometry such that $|O(x,r)| \geq Cr^d$. We conclude that

$$\|u\|_{\mathrm{L}^{\infty}(O)} \leq (1 + C^{-\frac{1}{p}})\|u\|_{\mathrm{L}^{p}(O)}^{1-\theta}\|u\|_{\mathrm{C}^{\mu}(O)}^{\theta}. \qquad \square$$

Proof of Lemma 7.6.7. By [105, Prop. 3.1.9 a)] we have $\mathrm{dom}((\mathscr{L}^A)^{\sigma}) = \mathrm{dom}((1+\mathscr{L}^A)^{\sigma})$. Since $1 + \mathscr{L}^A$ is invertible, so are its fractional powers [105, Prop. 3.1.1 e)]. We let $u \in \mathrm{dom}((1+\mathscr{L}^A)^{\sigma})$ and put $f \coloneqq (1+\mathscr{L}^A)^{\sigma}u$. We have the following formula [105, Cor. 3.3.6]:

$$u = (1+\mathscr{L}^A)^{-\sigma}f = \frac{1}{\Gamma(\sigma)}\int_0^{\infty} t^{\sigma}\,\mathrm{e}^{-t}\,f_t^A\,\frac{\mathrm{d}t}{t} = \frac{1}{\Gamma(\sigma)}\int_0^{\infty} t^{\sigma}\,\mathrm{e}^{-t}\,j((f_t^A)|_O)\,\frac{\mathrm{d}t}{t},$$

where in the last step we have used that $f_t^A \in \mathrm{dom}(\mathscr{L}^A) \subseteq j(V)$. Since $j\colon V \to \mathbb{L}^2$ is bounded, the assertion follows once we have checked that

$$\int_0^{\infty} t^{\sigma}\,\mathrm{e}^{-t}(f_t^A)|_O\,\frac{\mathrm{d}t}{t}$$

converges absolutely in V. Due to the exponential decay and the assumption $\sigma > 1/2$, it suffices to show that there is $C > 0$ such that $\|f_t^A|_O\|_V \leq Ct^{-1/2}$ for all $t > 0$. In order to see this, we use the strong Gårding estimate for a and the Cauchy–Schwarz inequality to get

$$C\|f_t^A|_O\|_V^2 \leq \mathrm{Re}\,a(f_t^A|_O, f_t^A|_O) = \mathrm{Re}(\mathscr{L}^A f_t^A \mid f_t^A) \leq \|\mathscr{L}^A f_t^A\|_2\|f_t^A\|_2.$$

Since $(\mathrm{e}^{-t\mathscr{L}^A})_{t\geq 0}$ is a bounded analytic semigroup in $\mathbb{L}^2$, the right-hand side is bounded from above by $Ct^{-1}\|f\|_2^2$. This completes the proof. $\square$

7.7 Explicit angle

We close this chapter by elaborating a quantitative lower bound for θ_p, depending on p and the data $\Lambda(A)$, $\lambda(A)$ and $\omega(A)$. Under additional symmetry assumptions on A, results of this type already exist, see Remark 7.7.3. For $p \in (1,\infty)$ we set

$$\sigma_p \coloneqq \frac{|p-2|}{2\sqrt{p-1}}$$

and note that $\sigma_p = \sigma_{p'}$. We will use the fact that (quantitative) smallness of $\|\operatorname{Im}(A)\|_\infty$ implies p-ellipticity of A.

Lemma 7.7.1 ([85, Lem. 2.16])**.** *Let $p \in (2,\infty)$ and assume that*

$$\sigma_p \|\operatorname{Im}(A)\|_\infty < \lambda(A).$$

Then A is p-elliptic.

Hence, our goal is to find the largest possible θ such that $\mathrm{e}^{\pm\mathrm{i}\theta} A$ satisfies this smallness assumption. The method cannot be optimal, because the same sesquilinear form a may be represented by a variety of matrices A with different size of $\|\operatorname{Im}(A)\|_\infty$, see [61, Ex. 1]. On the other hand, it does not require any symmetry for A.

Proposition 7.7.2. *Let $p \in (1,\infty)$ and assume that $\sigma_p \|\operatorname{Im}(A)\|_\infty < \lambda(A)$. Then*

$$\tan(\theta_p) \geq \frac{\lambda(A) - \sigma_p \|\operatorname{Im}(A)\|_\infty}{\lambda(A)\tan(\omega(A)) + \sigma_p \|\operatorname{Re}(A)\|_\infty}.$$

Proof. By symmetry and Lemma 1.7.8 (iii) we can assume $p > 2$. We fix $\theta \in (0, \pi/2 - \omega(A))$. We need to prove $\Delta_p(\mathrm{e}^{\pm\mathrm{i}\theta} A) > 0$ provided that

$$\tan(\theta) < \frac{\lambda(A) - \sigma_p \|\operatorname{Im}(A)\|_\infty}{\lambda(A)\tan(\omega(A)) + \sigma_p \|\operatorname{Re}(A)\|_\infty}. \tag{7.10}$$

Lemma 7.7.1 says that $\Delta_p(\mathrm{e}^{\mathrm{i}\theta} A) > 0$ if

$$\sigma_p \|\operatorname{Im}(\mathrm{e}^{\mathrm{i}\theta} A)\|_\infty < \lambda(\mathrm{e}^{\mathrm{i}\theta} A).$$

We have the rough estimate

$$\begin{aligned} \|\operatorname{Im}(\mathrm{e}^{\mathrm{i}\theta} A)\| &= \|\sin(\theta)\operatorname{Re}(A) + \cos(\theta)\operatorname{Im}(A)\|_\infty \\ &\leq \sin(\theta)\|\operatorname{Re}(A)\|_\infty + \cos(\theta)\|\operatorname{Im}(A)\|_\infty \end{aligned}$$

and, by rotating the set $\{z \in \mathbb{C} \colon |\arg(z)| \leq \omega(A) \,\&\, \operatorname{Re}(z) \geq \lambda\}$ and finding the point with smallest real part, we get

$$\lambda(\mathrm{e}^{\mathrm{i}\theta} A) \geq \lambda(A)\frac{\cos(\omega(A)+\theta)}{\cos(\omega(A))} = \lambda(A)\Big(\cos(\theta) - \sin(\theta)\tan(\omega(A))\Big).$$

So, a sufficient criterion for $\Delta_p(\mathrm{e}^{\mathrm{i}\theta} A) > 0$ is

$$\sigma_p\Big(\sin(\theta)\|\operatorname{Re}(A)\|_\infty + \cos(\theta)\|\operatorname{Im}(A)\|_\infty\Big) < \lambda(A)\Big(\cos(\theta) - \sin(\theta)\tan(\omega(A))\Big).$$

We divide both sides by $\cos(\theta) > 0$ and rearrange terms to get

$$\tan(\theta)\Big(\lambda(A)\tan(\omega(A)) + \sigma_p \|\operatorname{Re}(A)\|_\infty\Big) < \lambda(A) - \sigma_p \|\operatorname{Im}(A)\|_\infty,$$

which is equivalent to (7.10).

Since the right-hand side in (7.10) stays the same when replacing A by A^*, the restriction on θ in (7.10) also implies $\Delta_p(\mathrm{e}^{\mathrm{i}\theta} A^*) > 0$ and hence $\Delta_p(\mathrm{e}^{-\mathrm{i}\theta} A) > 0$, see Lemma 1.7.8 (ii). This completes the proof. □

Remark 7.7.3. The following results are known from the literature.

(i) For real-valued A, [60, Thm. 3.4] gives an upper bound for the angle of the numerical range of $\mathscr{L}_p^A$. This leads to the better bound

$$\tan(\theta_p) \geq \frac{1}{\sqrt{\sigma_p^2 + \frac{p^2}{4(p-1)}\tan(\omega(A))^2}}.$$

The proof uses the fact that the coefficients are real and there is no simple adaptation in the complex case.

(ii) Cialdea and Maz'ya characterize in [62, Thm. 9] the optimal angle of the L^p-dissipativity under Dirichlet boundary conditions by an implicit formula. For symmetric A (that is $A = A^t$ in their terminology) the expression simplifies, see [62, Thm. 1].

(iii) If $\operatorname{Im}(A)$ is symmetric and $|1 - 2/p| < \cos(\omega(A))$, Do found in [74, Thm. 1.1] a lower bound for θ_p in terms of $\omega(A)$ and p:

$$\begin{aligned} \theta_p &\geq \arccos(|1 - 2/p|) - \omega(A) && (\operatorname{Re}(A) \text{ symmetric}), \\ \tan(\theta_p) &\geq \frac{1 - \tan(\omega(A))\sigma_p}{\frac{p-\sqrt{p-1}}{\sqrt{p-1}}\tan(\omega(A)) + \sigma_p} && (\operatorname{Re}(A) \text{ not symmetric}). \end{aligned}$$

This bound does not recover the bound in (i) when A is real.

Bibliography

[1] D. R. Adams. *A note on Riesz potentials*. Duke Math. J. **42** (1975), no. 4, 765–778.

[2] D. R. Adams and L. I. Hedberg. Function spaces and potential theory, Grundlehren der Mathematischen Wissenschaften, vol. 314, Springer, Berlin, 1995.

[3] A. Amenta. *Tent spaces over metric measure spaces under doubling and related assumptions*. Operator theory in harmonic and non-commutative analysis. 23rd international workshop in operator theory and its applications, IWOTA, Sydney, Australia, July 16–20, 2012, Cham: Birkhäuser/Springer (2014), 1–29.

[4] A. Amenta and P. Auscher. Elliptic boundary Value Problems with Fractional Regularity Data, CRM Monograph Series, vol. 37, AMS, Providence, 2018.

[5] W. Arendt, C. J. K. Batty, M. Hieber and F. Neubrander. Vector-Valued Laplace transforms and Cauchy problems, Monographs in Mathematics, vol. 94, Birkhäuser, Basel, 2011.

[6] W. Arendt and A. V. Bukhvalov. *Integral representations of resolvents and semigroups*. Forum Math. **6** (1994), no. 1, 111–135.

[7] W. Arendt and A. F. M. ter Elst. *Gaussian estimates for second order elliptic operators with boundary conditions*. J. Oper. Theory **38** (1997), no. 1, 87–130.

[8] W. Arendt and A. F. M. ter Elst. *The Dirichlet-to-Neumann operator on rough domains.* J. Differential Equations, **251** (2011), 2100–2124.

[9] W. Arendt and A. F. M. ter Elst. *Sectorial forms and degenerate differential operators.* J. Operator Theory, **67** (2012), no. 1, 33–72.

[10] W. Arendt and M. Sauter. *The Wentzell Laplacian via forms and the approximative trace.* Discrete and Continuous Dynamical Systems - S, **17** (2024), no. 5–6, 1750–1765.

[11] P. Auscher. *Regularity theorems and heat kernel for elliptic operators.* J. Lond. Math. Soc., II. Ser. **54** (1996), no. 2, 284–296.

[12] P. Auscher. *On necessary and sufficient conditions for L^p-estimates of Riesz transforms associated to elliptic operators on $\mathbb{R}^n$ and related estimates.* Mem. Amer. Math. Soc. **186** (2007), no. 871.

[13] P. Auscher. *On the Calderón-Zygmund lemma for Sobolev functions.* Preprint (2008), `https://hal.science/hal-00335122`.

[14] P. Auscher and A. Axelsson. *Weighted maximal regularity estimates and solvability of non-smooth elliptic systems. I* Invent. Math., **184** (2011), no. 1, 47–115.

[15] P. Auscher, A. Axelsson and A. McIntosh. On a quadratic estimate related to the Kato conjecture and boundary value problems. Harmonic analysis and partial differential equations. 8th international conference, El Escorial, Madrid, Spain, June 16–20, 2008, Providence, RI: (AMS), 2010, 105–129.

[16] P. Auscher, A. Axelsson and A. McIntosh. *Solvability of elliptic systems with square integrable boundary data.* Ark. Mat., **48** (2010), no. 2, 253–287.

[17] P. Auscher, N. Badr, R. Haller–Dintelmann and J. Rehberg. *The square root problem for second-order, divergence form operators with mixed boundary conditions on L^p.* J. Evol. Equ., **15** (2015), no. 1, 165–208.

[18] P. Auscher, T. Böhnlein and M. Egert. *Meyers exponent rules the first-order approach to second-order elliptic boundary value problems.* Preprint (2025), `https://arxiv.org/abs/2504.00650`.

[19] P. Auscher and M. Egert. *Mixed boundary value problems on cylindrical domains.* Adv. Differential Eq. **22** (2017), no. 1-2, 101–168.

[20] P. Auscher and M. Egert. Boundary value problems and Hardy

spaces for elliptic systems with block structure, Progress in Mathematics, vol. 346, Birkhäuser, Cham, 2023.

[21] P. Auscher, S. Hofmann, M. Lacey, A. McIntosh and P. Tchamitchian. *The solution of the Kato square root problem for second order elliptic operators on* $\mathbb{R}^n$. Ann. Math. (2), **156** (2002), no. 2, 633–654.

[22] P. Auscher, A. McIntosh and M. Mourgoglou. *On* L^2 *solvability of BVPs for elliptic systems.* J. Fourier Anal. Appl., **19** (2013), no. 3, 478–494.

[23] P. Auscher, A. McIntosh and P. Tchamitchian. *Heat Kernels of Second Order Complex Elliptic Operators and Applications.* J. Funct. Anal., **152** (1998), no. 1, 22–73.

[24] P. Auscher and M. Mourgoglou. *Boundary layers, Rellich estimates and extrapolation of solvability for elliptic systems.* Proc. Lond. Math. Soc., **(3) 109** (2014), no. 2, 446–482.

[25] P. Auscher and M. Mourgoglou. *Representation and uniqueness for boundary value elliptic problems via first order systems.* Rev. Mat. Iberoam., **35** (2019), no. 1, 241–315.

[26] P. Auscher and E. Russ. *Hardy spaces and divergence operators on strongly Lipschitz domains of* $\mathbb{R}^n$. J. Funct. Anal., **201** (2003), no. 1, 148–184.

[27] P. Auscher and S. Stahlhut. *Remarks on Functional Calculus for Perturbed First-order Dirac Operators.* Oper. Theory: Adv. Appl. **240** (2014), 31–43.

[28] P. Auscher and S. Stahlhut. *Functional calculus for first order systems of Dirac type and boundary value problems.* Mém. Soc. Math. Fr., Nouv. Sér. **144** (2016), 1–127, 157–164.

[29] P. Auscher and P. Tchamitchian. Square root problem for divergence operators and related topics, Astérisque, vol. 249, Société Mathématique de France, Paris, 1998.

[30] P. Auscher and P. Tchamitchian. Gaussian estimates for second order elliptic divergence operators on Lipschitz and C^1 domains, Evolution equations and their applications in physical and life sciences (Bad Herrenalb, 1998), Lecture Notes in Pure and Appl. Math., vol. 215, Dekker, New York, 2001, 15–32.

[31] A. Axelsson , S. Keith and A. McIntosh. *The Kato square root problem for mixed boundary value problems*. J. Lond. Math. Soc., II. Ser., **74** (2006), no. 1, 113–130.

[32] A. Axelsson , S. Keith and A. McIntosh. *Quadratic estimates and functional calculi of perturbed Dirac operators*. Invent. Math., **163** (2006), no. 3, 455–497.

[33] H. Bahouri , J.–Y. Chemin and R. Danchin. Fourier Analysis and Nonlinear Partial Differential Equations, Grundlehren der mathematischen Wissenschaft, vol. 343, Berlin: Springer, 2011.

[34] R. Bañuelos and P. Janakiraman. *L^p-bounds for the Beurling-Ahlfors transform*. Trans. Amer. Math. Soc. **360** (2008), 3603–3612.

[35] S. Bechtel. On mixed boundary conditions, function spaces, and Kato's square root problem, PhD Thesis, TU Darmstadt, 2021.

[36] S. Bechtel. *The extension problem for fractional Sobolev spaces with a partial vanishing trace condition*. Arch. Math., **117** (2021), no. 1, 79–85.

[37] S. Bechtel. L^p*-estimates for the square root of elliptic systems with mixed boundary conditions* II. J. Differential Equations, **379** (2024), 104–124.

[38] S. Bechtel and T. Böhnlein. *Hardy spaces adapted to elliptic operators on open sets*. Preprint (2023), `https://arxiv.org/abs/2311.13316`.

[39] S. Bechtel, R. M. Brown, R. Haller–Dintelmann and P. Tolksdorf. *Extendability of functions with partially vanishing trace*. Preprint (2021), `https://arxiv.org/abs/1910.06009v2`.

[40] S. Bechtel, M. Egert and R. Haller–Dintelmann. *The Kato square root problem on locally uniform domains*. Adv. Math., **375** (2020), id/no. 107410, 37.

[41] S. Bechtel and E. M. Ouhabaz. *Off-diagonal bounds for the Dirichlet–to–Neumann operator on Lipschitz domains*. J. Math. Anal. Appl., **530** (2024), no. 2, 18.

[42] L. M. Betz. *Second-order sufficient optimality conditions for optimal control of nonsmooth, semilinear parabolic equations*. SIAM J. Control Optim., **57** (2019), no. 6, 4033–4062.

[43] S. Blunck and P. Kunstmann. *Calderón–Zygmund theory for non-*

integral operators and the H^∞-calculus. Rev. Mat. Iberoamericana, **19** (2003), no. 3, 919–942.

[44] S. Blunck and P. Kunstmann. *Weak type (p,p) estimates for Riesz transforms*. Math. Z., **247** (2004), no. 1, 137–148.

[45] T. Böhnlein, S. Ciani and M. Egert. *Gaussian estimates vs. elliptic regularity on open sets*. Math. Ann., (2024), https://doi.org/10.1007/s00208-024-02939-0.

[46] T. Böhnlein and M. Egert. *Explicit improvements for L^p-estimates related to elliptic systems*. Bull. Lon. Math. Soc., **56** (2024), no. 3, 914–930.

[47] T. Böhnlein, M. Egert and J. Rehberg. *Bounded functional calculus for divergence form operators with dynamical boundary conditions*. Preprint (2024), https://arxiv.org/abs/2406.09583.

[48] T. A. Bui, X. T. Duong and F. K. Ly. *Maximal function characterizations for Hardy spaces on spaces of homogeneous type with finite measure and applications*. J. Funct. Anal., **278** (2020), no. 8, 55.

[49] X. Cabré and F. Charro. *The optimal exponent in the embedding into the Lebesgue spaces for functions with gradient in the Morrey space*. Adv. Math., **380** (2021), id/no. 107592, 36.

[50] S. Campanato. *Proprietà di hölderianità di alcune classi di funzioni*. Ann. Sc. Norm. Super. Pisa, **17** (1963), no. 1–2, 175–188.

[51] A. Carbonaro and O. Dragičević. *Bellmann function and linear dimension-free estimates in a theorem of Bakry*. J. Funct. Anal., **265** (2013), no. 7, 1085–1104.

[52] A. Carbonaro and O. Dragičević. *Convexity of power functions and bilinear embedding for divergence-form operators with complex coefficients*. J. Eur. Math. Soc. (JEMS), **22** (2020), no. 10, 3175–3221.

[53] A. Carbonaro and O. Dragičević. *Bilinear embedding for divergence-form operators with complex coefficients on irregular domains*. Calc. Var. Partial Differ. Equ., **59** (2020), no. 3, id/no. 104, 36.

[54] A. Carbonaro and O. Dragičević. *On semigroup maximal operators associated with divergence-form operators with complex coefficients*. Preprint (2022), https://arxiv.org/abs/2207.11045v2.

[55] A. Carbonaro and O. Dragičević. *Bilinear embedding for*

Schrödinger-type operators with complex coefficients. Preprint (2023), https://arxiv.org/abs/1908.00143v2.

[56] A. Carbonaro, O. Dragičević, V. Kovač and K. Škreb. *Trilinear embedding for divergence-form operators with complex coefficients*. Adv. Math., **431** (2023), id/no. 109239, 72.

[57] E. Casas and J. Yong. *Optimal control of a parabolic equation with memory*. Control Optim. Calc. Var., **29** (2023), no. 23, 16.

[58] D-C. Chang, S. Krantz, and E. Stein. *H^p Theory on a Smooth Domain in $\mathbb{R}^N$ and Elliptic Boundary Value Problems*. Journal of Functional Analysis **114** (1993), 286–347.

[59] M. Chicco and M. Venturino. *Hölder regularity for solutions of mixed boundary value problems containing boundary terms*. Boll. Unione Mat. Ital., Sez. B, Artic. Ric. Mat. (8), **9** (2006), no. 2, 267–281.

[60] R. Chill, H. Meinlschmidt and J. Rehberg. *On the numerical range of second-order elliptic operators with mixed boundary conditions in L^p*. J. Evol. Equ., **21** (2021), no. 3, 3267–3288.

[61] A. Cialdea and V. Maz'ya. *Criterion for the L^p-dissipativity of second order differential operators with complex coefficients*. J. Math. Pures Appl., **84** (2005), no. 8, 1067–1100.

[62] A. Cialdea and V. Maz'ya. *Criteria for the L^p-dissipativity of systems of second order differential equations*. Ric. Mat., **55** (2006), no. 2, 233–265.

[63] R. Coifman and G. Weiss. *Extensions of Hardy spaces and their use in analysis*. Bull. Amer. Math. Soc., **83** (1977), no. 4, 569–645.

[64] M. Cowling, I. Doust, A. McIntosh and A. Yagi. *Banach space operators with a bounded H^∞ functional calculus*. J. Aust. Math. Soc., Ser. A, **60** (1996), no. 1, 51–89.

[65] M. Cwikel and M. Milman. *Complex Interpolation of Some Quasi-Banach Spaces*. J. Funct. Anal., **65** (1986), no. 3, 339–347.

[66] B.E.J. Dahlberg. *Estimates of harmonic measure*. Arch. Rational Mech. Anal., **65** (1977), no. 3, 275–288.

[67] E. De Giorgi. *Sulla differenziabilità e l'analiticità delle estremale degli integrali multipli regolari (Differentiability and analyticity of the extremals of regular multiple integrals)*. Mem. Acc. Sci. Torino, Ser, **3** (1957), no. 3, 25–43.

[68] R. Denk, M. Hieber and J. Prüss. $\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type. Mem. Amer. Math. Soc. **166** (2003), no. 788.

[69] M. Dindoš and J. Pipher. *Regularity theory for solutions to second order elliptic operators with complex coefficients and the L^p Dirichlet problem*. Adv. Math., **341** (2019), 255–298.

[70] K. Disser, A. F. M. ter Elst and J. Rehberg. *Hölder estimates for parabolic operators on domains with rough boundary*. Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5), **17** (2017), no. 1, 65–79.

[71] K. Disser, M. Meyries and J. Rehberg. *A unified framework for parabolic equations with mixed boundary conditions and diffusion on interfaces*. J. Math. Anal. Appl., **430** (2015), no. 2, 1102–1123.

[72] E. DiBenedetto. *Harnack estimates in certain function classes*. Atti Sem. Mat. Fis. Univ. Modena, **37** (1989), no. 1, 173–182.

[73] E. DiBenedetto. Partial differential equations, ed. 2nd, Springer Science & Business Media, 2009.

[74] T. D. Do. *Sectoriality of degenerate elliptic operators via p-ellipticity*. Forum Math., **35** (2023), no. 1, 61–81.

[75] H. Dong and D. Kim. *Parabolic and elliptic systems with VMO coefficients*. Methods Appl. Anal. **16** (2009), no. 3, 365–388.

[76] O. Dragičević and A. Volberg. *Bellman functions and dimensionless estimates of Littlewood-Paley type*. J. Oper. Theory **56** (2006), no. 1, 167–198.

[77] T. X. Duong and L. Yan. *Duality of Hardy and BMO spaces asscociated with operators with heat kernel bounds*. J. Amer. Math. Soc., **18** (2005), no. 4, 943–973.

[78] M. Egert. On Kato's conjecture and mixed boundary conditions, PhD Thesis, Sierke Verlag, Göttingen, 2015. Available online: `https://www.mathematik.tu-darmstadt.de/media/analysis/personen_analysis/egert/Thesis.pdf`

[79] M. Egert. *L^p-estimates for the square root of elliptic systems with mixed boundary conditions*. J. Differential Equations, **265** (2018), no. 4, 1279–1323.

[80] M. Egert. *On p-elliptic divergence form operators and holomorphic semigroups*. J. Evol. Equ., **20** (2020), no. 3, 705–724.

[81] M. Egert, R. Haller–Dintelmann and P. Tolksdorf. *The Kato square root problem for mixed boundary conditions*. J. Funct. Anal., **267** (2014), no. 5, 1419–1461.

[82] M. Egert and P. Tolksdorf. *Characterizations of Sobolev functions that vanish on a part of the boundary*. Discrete Contin. Dyn. Syst. Ser. S 10 (2017), no. 4, 729–743.

[83] A. F. M. ter Elst, A. Linke and J. Rehberg. *On the numerical range of sectorial forms*. Pure Appl. Funct. Anal., **7** (2022), no. 5, 1929–1938.

[84] A. F. M. ter Elst, M. Meyries and J. Rehberg. *Parabolic equations with dynamical boundary conditions and source terms on interfaces*. Ann. Mat. Pura Appl. (4), **193** (2014), no. 5, 1295–1318.

[85] A. F. M. ter Elst and R. Haller–Dintelmann and J. Rehberg and P. Tolksdorf. *On the* L^p*-theory for second order elliptic operators in divergence form with complex coefficients*. J. Evol. Equ. **21** (2021), 3963–4003.

[86] A. F. M. ter Elst and J. Rehberg. L^∞*-estimates for divergence operators on bad domains*. Anal. Appl. (Singap.), **10** (2012), no. 2, 207–214.

[87] A. F. M. ter Elst and J. Rehberg. *Hölder estimates for second-order operators on domains with rough boundary*. Adv. Differ. Equ. **20** (2015), no. 3–4, 299–360.

[88] L. C. Evans. Partial Differential Equations, Graduate Studies in Mathematics, ed. 2nd, vol. 19, RI: American Mathematical Society, Providence, 2010.

[89] K. Falconer. Fractal Geometry: Mathematical Foundations and Applications, 2nd ed., John Wiley & Sons, Ltd, 2003.

[90] H. Fattorini. Second Order Linear Differential Equations in Banach Spaces, North-Holland Math. Stud., vol. 108, Elsevier, Amsterdam, 1985.

[91] C. Fefferman. *Characterizations of bounded mean oscillation*. Bull. Amer. Math. Soc. **77** (1971), 587–588.

[92] C. Fefferman and E. M. Stein. *Some maximal inequalities*. Am. J. Math., **93** (1971), no. 1, 107–115.

[93] J. Frehse. *An irregular complex valued solution to a scalar uniformly*

elliptic equation. Calc. Var. Partial Differ. Equ., **33** (2008), no. 3, 263–266.

[94] D. Frey, A. McIntosh and P. Portal. *Conical square function estimates and functional calculi for perturbed Hodge-Dirac operators in L^p*. J. Anal. Math. **134** (2018), no. 2, 399–453.

[95] H. Gajewski. *Analysis and numerics of carrier transport in semiconductors*. Mitt. Ges. Angew. Math. Mech., **16** (1993), no. 1, 35–57.

[96] M. Giaquinta. Multiple integrals in the calculus of variations and nonlinear elliptic systems, Ann. Math. Stud., vol. 105, Princeton University Press, Princeton, NJ, 1983.

[97] M. Giaquinta and L. Martinazzi. An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs, Appunti, Sc. Norm. Super. Pisa (N.S.), vol. 2, Edizioni della Normale, Pisa, 2005.

[98] M. Giaquinta and J. Nečas. *On the regularity of weak solutions to non linear elliptic systems of partial differential equations*. Journal für die reine und angewandte Mathematik **316** (1980), 140–159.

[99] D. Gilbarg and N. S. Trudinger. Elliptic partial differential equations of second order, Classics in Mathematics, Springer, Berlin, 2001.

[100] G. R. Goldstein. *Derivation and physical interpretation of general boundary conditions*. Adv. Differ. Equ., **11** (2006), no. 4, 457–480.

[101] G. R. Goldstein, J. Goldstein, D. Guidetti and S. Romanelli. *Maximal regularity, analytic semigroups, and dynamic and general Wentzell boundary conditions with a diffusion term on the boundary*. Ann. Mat. Pura Appl. (4), **199** (2020), no. 1, 127–146.

[102] L. Grafakos. Classical Fourier Analysis, ed. 3rd, Graduate Texts in Mathematics, vol. 250, Springer, New York, 2014.

[103] P. P. Guidugli. *De Giorgi's counterexample in elasticity*. Q. Appl. Math. **34** (1977), no. 1, 411–419.

[104] P. Gyrya and L. Saloff–Coste. Neumann and Dirichlet heat kernels in inner uniform domains, Astérisque, vol. 336, Société Mathématique de France, Paris, 2011.

[105] M. Haase. The Functional Calculus for Sectorial Operators. Operator Theory: Advances and Applications, vol. 169, Birkhäuser, Basel, 2006.

[106] P. Hajłasz, P. Koskela and H. Tuominen. *Sobolev embeddings,*

extensions and measure density condition. J. Funct. Anal., **254** (2008), no. 5, 1217–1234.

[107] R. Haller–Dintelmann, A. Jonsson, D. Knees and J. Rehberg. *Elliptic and parabolic regularity for second-order divergence operators with mixed boundary conditions*. Math. Methods Appl. Sci. **39** (2016), no. 17, 5007–5026.

[108] R. Haller, H. Meinlschmidt and J. Rehberg. *Hölder regularity for domains of fractional powers of elliptic operators with mixed boundary conditions*. Pure Appl. Funct. Anal. **9** (2024), no. 1, 169–194.

[109] R. Haller–Dintelmann, C. Meyer, J. Rehberg and A. Schiela. *Hölder continuity and optimal control for nonsmooth elliptic problems*. Appl. Math. Optim. **60** (2009), no. 3, 397–428.

[110] S. Hofmann, C. Kenig, S. Mayboroda and J. Pipher. *Square function/non-tangential maximal function estimates and the Dirichlet problem for non-symmetric elliptic operators*. J. Amer. Math. Soc., **28** (2015), no. 2, 483–529.

[111] S. Hofmann, C. Kenig, S. Mayboroda and J. Pipher. *The regularity problem for second order elliptic operators with complex-valued bounded measurable coefficients*. Math. Ann., **361** (2015), no. 3-4, 863–907.

[112] S. Hofmann, G. Lu, D. Mitrea, M. Mitrea and L. Yan. *Hardy spaces associated to non-negative self-adjoint operators satisfying Davies-Gaffney estimates*. Mem. Amer. Math. Soc. **214** (2011), no. 1007.

[113] S. Hofmann, S. Mayboroda and A. McIntosh. *Second order elliptic operators with complex bounded measurable coefficients in L^p, Sobolev and Hardy spaces*. Ann. Sci. Éc. Norm. Supér. (4) **44** (2011), no. 5, 723–800.

[114] D. Hömberg, K. Krumbiegel and J. Rehberg. *Optimal Control of a Parabolic Equation with Dynamic Boundary Condition*. Appl. Math. Optim., **67** (2013), no. 1, 3–31.

[115] F. Hoppe, H. Meinlschmidt and I. Neitzel. *Global-in-time solutions for quasilinear parabolic PDEs with mixed boundary conditions in the Bessel scale*. Preprint (2024), `https://arxiv.org/abs/2303.04659`

[116] R. Jiang and F. Lin. *Riesz transform on exterior Lipschitz domains and applications*. Adv. Math., **453** (2024), id/no. 109852, 48.

[117] P. W. Jones. *Quasiconformal mappings and extendability of functions in Sobolev spaces*. Acta Math. **147** (1981), 71–88.

[118] A. Jonsson and H. Wallin. Function Spaces on Subsets of $\mathbb{R}^n$, Mathematical Reports, vol. 2, Harwood Academic Publishers GmbH, 1984.

[119] T. Kato. Perturbation Theory for Linear Operators, Classics in Mathematics, vol. 132, Springer, Berlin, 1995.

[120] C. E. Kenig. Harmonic Analysis Techniques for Second Order Elliptic Boundary Value Problems, CBMS Regional Conference Series in Mathematics, vol. 83, Providence, RI: (AMS), 1994.

[121] A. Koshelev. Regularity problem for quasilinear elliptic and parabolic systems, Lecture Notes in Mathematics, vol. 1614, Springer-Verlag, Berlin, 1995.

[122] T. Kilpeläinen and P. Koskela. *Global integrability of the gradients of solutions to partial differential equations*. Nonlinear Anal., Theory Methods Appl. **23** (1994), no. 7, 899–909.

[123] D. Kinderlehrer and G. Stampacchia. An Introduction to Variational Inequalities and Their Applications, Classics in Applied Mathematics, vol. 31, PA: Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 2000.

[124] J. Kinnunen and R. Korte. Characterizations for the Hardy inequality, Around the research of Vladimir Maz'ya. I. Function spaces, Springer, Dordrecht; Tamara Rozhkovskaya Publisher, Novosibirsk, 2010.

[125] P. C. Kunstmann and L. Weis. Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus. Lecture Notes in Mathematics 1855, Springer, Berlin, 2004, 65–311.

[126] J. Lehrbäck. *Pointwise Hardy inequalities and uniformly fat sets*. Proc. Amer. Math. Soc. **136** (2008), no. 6, 2193–2200.

[127] S. Leonardi. Remarks on the regularity of solutions of elliptic systems, Applied nonlinear analysis. In honor of the 70th birthday of Professor Jindřich Nečas, 325–344, Kluwer Academic/Plenum Publishers, New York, 1999.

[128] J. L. Lewis. *Uniformly fat sets*. Trans. Amer. Math. Soc. **308** (1988), no. 1, 177–196.

[129] G. M. Liebermann. *Optimal Hölder regularity for mixed boundary value problems*. J. Math. Anal. Appl. **143** (1989), no. 2, 572–586.

[130] P. A. Markowich. The Stationary Semiconductor Device Equations, Computational Microelectronics, Springer-Verlag: Wien, 1986.

[131] O. Martio. *Capacity and measure densities*. Ann. Acad. Sci. Fenn., Ser. A I, Math. **4** (1979), 109–118.

[132] H. Meinlschmidt and J. Rehberg. *Extrapolated elliptic regularity and application to the van Roosbroeck system of semiconductor equations*. J. Differential Equations, **280** (2021), 375–404.

[133] N. Meyers. *An L^p-estimate for the gradient of solutions of second order elliptic divergence equations*. Ann. Sc. Norm. Super. Pisa, Sci. Fis. Mat., III., **17** (1963), no. 3, 189–206.

[134] P. Mikkonen. *On the Wolff potential and quasilinear elliptic equations involving measures*. Ann. Acad. Sci. Fenn., Math., Diss. **104** (1996).

[135] D. Mitrea, I. Mitrea, and M. Mitrea. Geometric Harmonic Analysis I. A Sharp Divergence Theorem with Nontangential Pointwise Traces. Developments in Mathematics, vol. 72, Springer, Cham, 2022.

[136] A. Miyachi. *H^p spaces over open subsets of R^n*. Stud. Math., **96** (1990), no. 3, 205–228.

[137] R. Nittka. *Projections onto convex sets and L^p-quasi-contractivity of semigroups*. Arch. Math., **98** (2012), no. 4, 341–353.

[138] L. Nirenberg. *On elliptic partial differential equations*. Ann. Sc. Norm. Super. Pisa, Sci. Fis. Mat., III. Ser. **13** (1959), 115–162.

[139] E. M. Ouhabaz. *Invariance of Closed Convex Sets and Domination Criteria for Semigroups*. Potential Ana., **5** (1996), no. 6, 611–625.

[140] E. M. Ouhabaz. Analysis of Heat Equations on Domains, London Mathematical Society Monographs Series, vol. 31, Princeton University Press, 2005.

[141] A. Pazy. Semigroups of linear operators and applications to partial differential equations, Applied Mathematical Sciences, vol. 44, Springer, Cham, 1983.

[142] J. Prüss and H. Sohr. *On Operators with Bounded Imaginary Powers in Banach Spaces*. Math. Z., **203** (1990), no. 3, 429–452.

[143] M. Sauter. *Uniqueness of the approximative trace*. Indiana Univ. Math. J., **69** (2020), no. 1, 171–204.

[144] S. Selberherr. Analysis and Simulations of Semiconductor Devices, Springer-Verlag: Wien, New York, 1984.

[145] Z. Shen. *Bounds of Riesz Transform on L^p Spaces for Second Order Elliptic Operators*. Ann. Inst. Fourier., **55** (2005), no. 1, 173–197.

[146] I. Šneĭberg. *Spectral properties of linear operators in interpolation families of Banach spaces*. Mat. Issled., **9** (1974), no. 2, 214–229, 254–255.

[147] L. Song and L. Yan. *Maximal function characterizations for Hardy spaces associated with nonnegative self-adjoint operators on spaces of homogeneous type*. J. Evol. Equ. **18** (2018), no. 1, 221–243.

[148] G. Stampacchia. *Problemi al contorno ellitici, con dati discontinui, dotati di soluzionie höderiane*. Ann. Mat. Pura. Appl., **51** (1960), 1–37.

[149] E. M. Stein. *Interpolation of linear operators*. Trans. Amer. Math. Soc. **83** (1956), 482–492.

[150] E. M. Stein. Singular Integrals and Differentiability Properties of Functions, Princeton University Press, New Jersey, 1970.

[151] E. M. Stein. *Some results in harmonic analysis in $\mathbb{R}^n$, for $n \to \infty$*. Bull. Amer. Math. Soc., New Ser. **9** (1983), 71–73.

[152] E. M. Stein. Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton University Press, New Jersey, 1993.

[153] D. Swanson and W. P. Ziemer. *Sobolev functions whose inner trace at the boundary is zero*. Ark. Mat., **37** (1999), no. 2, 373–380.

[154] I. E. Tamm. Fundamentals of the Theory of Electricity, Mir publishers Moscow, 1979.

[155] P. Tolksdorf. *$\mathcal{R}$-sectoriality of higher-order elliptic systems on general bounded domains*. J. Evol. Eq. **(**2018), no. 2, 323–349.

[156] H. Triebel. Interpolation theory, function spaces, differential operators, North-Holland Mathematical Library, vol. 18, Elsevier (North-Holland), Amsterdam, 1978.

[157] H. Triebel. Theory of function spaces. Modern Birkhäuser Classics, Birkhäuser/Springer-Basel AG, Basel, 2010.

[158] J. van Neerven, M. Veraar and L. Weis. *Stochastic maximal L^p-regularity*. Ann. Probab., **40** (2012), no. 2, 788–812.

[159] J. Väisälä. *Uniform domains*. Tôhoku Math. J., **40** (1988), no. 1, 101–118.

[160] H. Vogt and J. Voigt. *Holomorphic families of forms, operators and C_0-semigroups*. Monatsh. Math. **187** (2018), no. 2, 375–380.

[161] L. Weis. *Operator-valued Fourier multiplier theorems and maximal L_p-regularity*. Math. Ann. **319** (2001), no. 4, 735–758.

[162] D. Werner. Funktional-analysis, ed. 8th, Springer-Verlag, Berlin, 2018.

[163] J. Yeh. Real Analysis, Theory of Measure and Integration, ed. 2nd, World Scientific Publishing Co. Pte. Ldt., Hackensack, 2006.

[164] P. Zorin–Kranich. *Interpolation between Banach spaces and continuity of Radon-like integral transforms*. Diploma thesis (2011), `https://arxiv.org/pdf/1301.1025.pdf`.

List of notations

Capacity theory

$\mathrm{cap}_p(K;U)$	relative p-capacity of the condenser (K,U), where U is open and $K \subseteq U$ is compact
$R_{1,p}(F)$	$(1,p)$-Riesz capacity of a closed set F

Comparability

$x \lesssim y$	shorthand for $x \leq Cy$ for some generic constant $C \geq 0$.
$x \lesssim_a y$	shorthand for $x \leq C(a)y$
$x \simeq y$	shorthand for $x \lesssim y$ and $y \lesssim x$
$x \wedge y$	minimum between x and y
$x \vee y$	maximum between x and y

Critical numbers

$\mathcal{J}(L)$	interval of $p \in [1,\infty]$ for which $(\mathrm{e}^{-t^2L})_{t>0}$ is L^p-bounded
$p_\pm(L)$	left and right endpoints of $\mathcal{J}(L)$
$\mathcal{N}(L)$	interval of $p \in [1,\infty]$ for which $(t\nabla\,\mathrm{e}^{-t^2L})_{t>0}$ is L^p-bounded

$q_\pm(L)$	left and right endpoints of $\mathcal{N}(L)$
$\mathcal{P}(\mathsf{L}_\tau)$	interval of $p \in (1,\infty)$ for which $S_\tau \mathsf{L}_\tau^{-1} S_\tau^*$ is L^p-bounded
$\mathcal{P}_\pm(\mathsf{L}_\tau)$	left and right endpoints of $\mathcal{P}(\mathsf{L}_\tau)$
$\mathcal{P}((\mathsf{L}_\tau)_\tau)$	interval of $p \in (1,\infty)$ for which $S_\tau \mathsf{L}_\tau^{-1} S_\tau^*$ is L^p-bounded uniformly in $\tau \in \mathbb{R}^*$
$\mathcal{P}_\pm((\mathsf{L}_\tau)_\tau)$	left and right endpoints of $\mathcal{P}((\mathsf{L}_\tau)_\tau)$
$I(\mathsf{DB})$	interval of $p \in (1,\infty)$ around $p=2$ for which DB is L^p-bisectorial
$p_\pm(\mathsf{DB})$	left and right endpoints of $I(\mathsf{DB})$
$\mathcal{H}(\mathsf{DB})$	interval of $p \in (1_*,\infty)$ for which $\mathbb{H}^p_{\mathsf{DB}} = \mathbb{H}^p_{\mathsf{D}}$
$h_\pm(\mathsf{DB})$	left and right endpoints of $\mathcal{H}(\mathsf{DB})$
$m_+(\mathsf{L}_\tau)$	Meyers exponent of L_τ
$m_+((\mathsf{L}_\tau)_\tau)$	Meyers exponent of the family $(\mathsf{L}_\tau)_{\tau\in\mathbb{R}^*}$

Derivatives

∂^α	the derivative $\partial^{\alpha_1}\dots\partial^{\alpha_d}$ for $\alpha = (\alpha_1,\dots,\alpha_d) \in \mathbb{N}_0^d$
∂_t	time derivative
$\nabla = \nabla_x$	spatial gradient $\nabla_x u = (\partial_i u)_{i=1}^d$
$\nabla_{t,x}$	full gradient $[\partial_t, \nabla]^\top$
S_τ	inhomogeneous gradient $[\mathrm{i}\tau, \nabla_x]^\top$
$\mathrm{div} = \mathrm{div}_x$	spatial divergence $\mathrm{div}\, F = \sum_{i=1}^d \partial_i F_i$
$\mathrm{div}_{t,x}$	full divergence $\mathrm{div}_{t,x}\, F = \partial_t F_0 + \sum_{i=1}^d \partial_i F_i$

Dirac operators

D	the Dirac operator $\begin{bmatrix} 0 & \operatorname{div} \\ -\nabla & 0 \end{bmatrix}$
$\mathbb{P}_{\mathsf{D}}$	projection onto $\overline{\operatorname{ran}(\mathsf{D})}$
$\overline{\mathsf{A}}$	the matrix $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$
$\underline{\mathsf{A}}$	the matrix $\begin{bmatrix} 1 & 0 \\ c & A \end{bmatrix}$
B	the bounded multiplication operator $\widehat{\mathsf{A}} = \underline{\mathsf{A}}\overline{\mathsf{A}}^{-1}$
$f_\perp$	the scalar component of $f = [f_\perp, f_\parallel]^\top \colon \mathbb{R}^d \to \mathbb{C}^m \times (\mathbb{C}^m)^d$
$f_\parallel$	the vector component of $f = [f_\perp, f_\parallel]^\top \colon \mathbb{R}^d \to \mathbb{C}^m \times (\mathbb{C}^m)^d$

Elliptic operators

$\mathbf{1}_{\mathbb{C}^d}$	identity matrix in $\mathbb{C}^d$
A	$\mathcal{L}(\mathbb{C}^d)$-valued function with bounded and measurable entries
$a = a_A$	the form $a(u,v) = \int_O A\nabla u \cdot \overline{\nabla v}\,\mathrm{d}x$
$\mathcal{L}$	the Lax–Milgram operator $V \ni u \mapsto a(u,\cdot) \in V^*$
A	the coefficient function $\begin{bmatrix} a & b \\ c & A \end{bmatrix}$ with complex, bounded and measurable entries
$\mathsf{a} = \mathsf{a}_{\mathsf{A}}$	the form $\int_O (A\nabla u \cdot \overline{\nabla v} + cu \cdot \overline{\nabla v} + b\nabla u \cdot \overline{v} + au \cdot \overline{v})\,\mathrm{d}x$.
$\mathrm{d}(A)$	'distance' to the identity: $\min_{t\geq 0} \|\mathbf{1}_{\mathbb{C}^d} - tA\|_\infty$
$\lambda(A)$	ellipticity constant of A
$\Delta_p(A)$	p-ellipticity constant of A

$\Delta_p(A,B)$	shorthand for $\Delta_p(A) \wedge \Delta_p(B)$
$\Lambda(A)$	shorthand for $\|A\|_\infty$
$\Lambda(A,B)$	shorthand for $\Lambda(A) \vee \Lambda(B)$
$\varrho(A)$	the ellipticity ratio $\frac{\lambda(A)}{\Lambda(A)}$
$\omega(A)$	the angle $\inf\{\omega \in [0, \pi/2) \colon a(u,u) \in \overline{\mathrm{S}_\omega}$ for all $u \in V\}$, which has to be replaced by $\operatorname*{esssup}_{x\in O} \sup_{\|\xi\|=1} \|\arg(A(x)\xi \cdot \overline{\xi})\|$ if A is elliptic
L	the operator $Lu = -\operatorname{div}(A\nabla u)$
L	the operator $\mathsf{L}u = -\operatorname{div}(A\nabla u + cu) + b\nabla u + au$
L_τ	the operator $\mathsf{L}_\tau u = -\operatorname{div}(A\nabla u + \mathrm{i}\tau cu) - \mathrm{i}\tau b\nabla u + \tau^2 au$
$\mathbb{P}_\tau$	projection onto $\overline{\operatorname{ran}(S_\tau)}$ given by $S_\tau \mathsf{L}_\tau^{-1} S_\tau^*$
$\boldsymbol{L}$	the operator $\boldsymbol{L}U = -\operatorname{div}_{t,x}(\mathsf{A}\nabla_{t,x}U)$
$\mathscr{L} = \mathscr{L}^A$	the operator $\mathscr{L}u = -\operatorname{div}(A\nabla u)$ subject to dynamical boundary conditions
f_t, f_t^*	shorthand for $\mathrm{e}^{-t^2\mathsf{L}} f$ and $\mathrm{e}^{-t^2\mathsf{L}^*} f$
f_t^A	shorthand for $\mathrm{e}^{-t\mathscr{L}^A} f$ or $\mathrm{e}^{-t^2 L^A} f$
K_t	the kernel of $\mathrm{e}^{-t^2 L}$
p_t	the kernel of $\mathrm{e}^{-t\sqrt{L}}$
P_t	the Poisson semigroup $P_t f = \int_O p_t(\cdot, y) f(y) \,\mathrm{d}y$
q_t	the kernel of $t\sqrt{L}\,\mathrm{e}^{-t\sqrt{L}}$ or $t^2 L\,\mathrm{e}^{-t^2 L}$
Q_t	the operator $Q_t f = \int_O q_t(\cdot, y) f(y) \,\mathrm{d}y$
$u \in \mathcal{H}_{L,x,r}$	shorthand for $u \in \mathrm{W}^{1,2}(O(x,r))$ with $Lu = 0$ in $O(x,r)$
$\mathcal{H}_{L,B}$	shorthand for $\mathcal{H}_{L,x,r}$ when $B = B(x,r)$
$u \in \mathcal{H}_L$	shorthand for $u \in V$ and $Lu = 0$ in O

Euclidean geometry

$\mathbb{R}^{1+d}_+$	the upper half space $(0,\infty)\times\mathbb{R}^d$
$\Gamma(x)$	the cone $\{(t,y)\in\mathbb{R}^{1+d}_+\colon \lvert x-y\rvert<t\}$ with vertex x
$B=B(x,r)$	open ball with center $x=c(B)$ and radius $r=r(B)$
cB	the concentric ball $B(x,cr)$
$C_j(B)$	the annulus $2^{j+1}B\setminus 2^jB$ with $C_0(B)\coloneqq 2B$
$\Xi(x,r)$	the relative ball $\Xi\cap B(x,r)$
$\partial\Xi(x,r)$	short notation for $\partial(\Xi(x,r))$
$Q=Q(x,r)$	the open axis-parallel cube $\{y\in\mathbb{R}^d\colon \lvert x-y\rvert_\infty<r\}$
$\ell(Q)$	side length of the cube Q
$\mathrm{d}(E,F)$	Euclidean semi-distance $\inf\{\lvert x-y\rvert\colon x\in E,y\in F\}$
$\mathrm{d}_\Xi(x)$	abbreviation for $\mathrm{d}(\{x\},\Xi)$
$\mathrm{d}(\Xi)=\mathrm{diam}(\Xi)$	diameter of Ξ: $\sup\{\lvert x-y\rvert\colon x,y\in\Xi\}$
Ξ_δ	open δ-neighborhood of Ξ: $\{x\in\mathbb{R}^d\colon \mathrm{d}_\Xi(x)<\delta\}$

Exponents

p'	Hölder conjugate of $p\in[1,\infty]$: $\frac{1}{p'}=1-\frac{1}{p}$
p_*	lower Sobolev conjugate of $p\in(0,\infty]$: $\frac{1}{p_*}=\frac{1}{p}+\frac{1}{d}$
p^*	upper Sobolev conjugate of $p\in(0,d)$: $\frac{1}{p^*}=\frac{1}{p}-\frac{1}{d}$
$[p_0,p_1]_\theta$	interpolation index of $p_0,p_1\in(0,\infty]$: $\frac{1}{[p_0,p_1]_\theta}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$
p°	the exponent $\frac{p}{2}2^*$ for $p\geq 2$
$p_\circ$	the exponent $(p^\circ)'$

Function spaces

X^m	abbreviation for $\mathrm{X}(\mathbb{R}^d;\mathbb{C}^m)\cong\mathrm{X}(\mathbb{R}^d)^m$ with $\mathrm{X}^1=\mathrm{X}$
H^p	real Hardy space with $p\in(0,1]$

$\mathcal{Z}$	space of Schwartz functions f with $(\partial^\alpha \widehat{f})(0) = 0$ for all $\alpha \in \mathbb{N}_0^d$
$\dot{\mathrm{W}}^{1,p}$	homogeneous Sobolev space of $u \in \mathrm{L}^p_{\mathrm{loc}}$ with $\nabla u \in (\mathrm{L}^p)^d$
$\mathrm{W}^{1,p}(O)$	Sobolev space of $u \in \mathrm{L}^p(O)$ with $\nabla u \in \mathrm{L}^p(O)^d$
$\mathrm{C}^\infty_{\mathrm{c}}(O)$	smooth and in O compactly supported functions
$\mathrm{C}^\infty_D(\mathbb{R}^d)$	the space $\mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^d \setminus D)$
$\mathrm{C}^\infty_D(\Xi)$	the space $\mathrm{C}^\infty_D(\mathbb{R}^d)\vert_\Xi$
$\mathrm{W}^{1,p}_D(O)$	closure of $\mathrm{C}^\infty_D(O)$ in the $\mathrm{W}^{1,p}(O)$-norm, $p \in (1,\infty)$
V	the form domain $\mathrm{W}^{1,2}_D(O)$
$V_{x,r}$	the space $\mathrm{W}^{1,2}_{\partial O(x,r)\setminus N(x,r)}(O(x,r))^m$
$\mathrm{C}^\mu_D(O)$	space of μ-Hölder continuous functions on O that vanish in D
$\mathrm{H}_{\alpha,x_0}(O)$	L^2-adapted, weighted Morrey space with exponent $\alpha \in \mathbb{R}$ of $u \in \mathrm{W}^{1,2}(O)/\mathbb{C}$ with $\nabla u \in \mathrm{L}^2(O, \lvert x-x_0\rvert^{-\alpha}\,\mathrm{d}x)$
$\mathrm{H}_\alpha(O)$	L^2-adapted, global weighted Morrey space with exponent α
$\mathcal{L}^\kappa(\Xi)$	L^2-adapted, global Campanato space with exponent κ
$\mathcal{L}^\kappa_{x,r}(\Xi)$	L^2-adapted Campanato space localized to $\Xi(x,r)$ with exponent κ
$\mathcal{M}^\kappa(\Xi)$	L^2-adapted, global Morrey space with exponent κ
$\mathcal{M}^\kappa_{x,r}(\Xi)$	L^2-adapted Morrey space localized to $\Xi(x,r)$ with exponent κ
H^∞	space of bounded and holomorphic functions on some (bi)sector
H^∞_0	space of holomorphic functions ψ on some (bi)sector with decay $\lvert\psi(z)\rvert \lesssim \lvert z\rvert^s \wedge \lvert z\rvert^{-s}$ for some $s > 0$ and all z
Ψ^τ_σ	space of holomorphic functions on some bisector with $\lvert\psi(z)\rvert \lesssim \lvert z\rvert^\sigma \wedge \lvert z\rvert^{-\tau}$ for all z

$\mathbb{H}^p_T(O)$	pre-Hardy space adapted to T of $u \in \overline{\operatorname{ran}(T)}$ with $S_{\psi,T}(u) \in \mathrm{L}^p(O)$ for an admissible choice of ψ
$\mathbb{H}^p_{T,\mathrm{at}}$	T-adapted atomic Hardy space
$\mathrm{H}^1_L(O)$	L-adapted Hardy space as a subspace of $\mathrm{L}^1(O)$
$\mathrm{H}^1_D(O)$	D-adapted atomic Hardy space
$\mathrm{L}^p_0(O)$	space of $f \in \mathrm{L}^p(O)$ with $\int_{O_m} f \,\mathrm{d}x = 0$ for all $O_m \in O_N$
$\ell^p(I)$	identification with the space of $f \in \mathrm{L}^p(O)$ that vanish on O_D and are constant on each $O_m \in O_N$
$\mathrm{L}^p_{\mathrm{loc}}(O)$	space of measurable functions $f\colon O \to \mathbb{C}$ with $f \in \mathrm{L}^p(E)$ for every bounded set $E \subseteq O$
$\mathbb{L}^p$	the space $\mathrm{L}^p(O \cup \Sigma, \mathrm{d}x \oplus \mathrm{d}m)$

Geometric measure theory, mixed and dynamical boundary conditions

$\mathcal{H}^\ell(\Xi)$	ℓ-dimensional Hausdorff measure of Ξ
$\mathcal{H}^\ell_\infty(\Xi)$	ℓ-dimensional Hausdorff content of Ξ
D	the Dirichlet part $D \subseteq \partial O$
N	the Neumann part $N = \partial O \setminus D$
O_N	collection of all bounded connected components O_m of O with $\partial O_m \cap D = \varnothing$
O_D	the set $O \setminus O_N$
Σ	the set $\{O_D\} \cup \{O_m\}_m$ in the context of Hardy spaces on open sets or the dynamical boundary part $\Sigma \subseteq \overline{O} \setminus D$ in the context of dynamical boundary conditions

Operator theory

$\operatorname{dom}(T), \ker(T), \operatorname{ran}(T)$	domain, kernel and range of the operator T
$\sigma(T)$	spectrum of the operator T

ω_T	(bi)sectoriality angle of the (bi)sectorial operator T
$\mathcal{L}(X,Y)$	space of bounded linear operators from X to Y
$\mathcal{L}(X)$	short notation for $\mathcal{L}(X,X)$
X^*	topological anti-dual space of X
$\langle\cdot\,\vert\,\cdot\rangle$	duality pairing
$(\cdot\,\vert\,\cdot)_H$	inner product in the Hilbert space H

Sets

$\mathbb{N},\mathbb{R},\mathbb{C},\mathbb{N}_0$	natural, real and complex numbers as well as natural numbers with zero
$\mathbb{K}^*$	the set $\mathbb{K}\setminus\{0\}$ for $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$
Ξ	general subset of $\mathbb{R}^d$
Ξ°	interior of Ξ
$\overline{\Xi}$	closure of Ξ
$\partial\Xi$	boundary of Ξ
O	open subset of $\mathbb{R}^d$
S_μ	the open sector $\{z\in\mathbb{C}\colon \vert\arg(z)\vert<\mu\}$
$\dot{\mathrm{S}}_\mu$	the bisector $\mathrm{S}_\mu\cup-\mathrm{S}_\mu$

Specific functions and operators

sgn	the function $\operatorname{sgn}(z)=\frac{z}{\vert z\vert}$ for $z\neq0$ and $\operatorname{sgn}(0)=0$
$u\otimes v$	the tensor $(u\otimes v)_{i,j}=(u_iv_j)_{i,j}$
$u\odot v$	shorthand for $\sum_{i=1}^d u_iv_i$, where $u\in\mathbb{C}^d$ and $v\in(\mathbb{C}^m)^d$
δ_t	the dilation operator $\delta_t f=f(t\cdot)$
τ_y	the translation operator $\tau_y f=f(y+\cdot)$

$\mathbf{1}_{\Xi}(x) = \mathbf{1}_{[x\in\Xi]}$	indicator function of the set $\Xi \subseteq \mathbb{R}^d$
$(f)_E = \fint_E f \,\mathrm{d}x$	shorthand for $\frac{1}{\lvert E\rvert}\int_E f \,\mathrm{d}x$
$\mathcal{F}$	the Fourier transform
$\mathcal{F}_t$	the one-dimensional Fourier transform
$\mathcal{M}$	Hardy–Littlewood maximal operator
$\mathcal{E}$	extension operator
$\mathcal{E}_0$	the 0-extension operator $\mathcal{E}_0 f = f_0$
$S(f)(x)$	conical square function $\left(\int_0^\infty \fint_{O(x,t)} \lvert f(t,y)\rvert^2 \mathrm{d}y \,\frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}}$
$S^{\mathrm{loc}}(f)(x)$	local square function $\left(\int_0^{\mathrm{d}(O)} \fint_{O(x,t)} \lvert f(t,y)\rvert^2 \mathrm{d}y \,\frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}}$
$S^{\mathrm{glob}}(f)(x)$	global square function $\left(\int_{\mathrm{d}(O)}^\infty \fint_{O(x,t)} \lvert f(t,y)\rvert^2 \mathrm{d}y \,\frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}}$
$S_{\psi,T}(u)$	the function $S(\psi(\boldsymbol{t}T)u)$ if T is bisectorial and $S(\psi(\boldsymbol{t}^2T)u)$ if T is sectorial

Curriculum Vitæ

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