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Nonconvex Quadratically-Constrained Feasibility Problems: An Inside-Ellipsoids Outside-Sphere Model

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Abstract

This paper proposes a new approach for solving Quadratically Constrained Feasibility Problems (QCFPs). We introduce an isomorphic mapping (one-to-one and onto correspondence), which equivalently converts the QCFP to an optimization problem called the Inside-Ellipsoids Outside-Sphere Problem (IEOSP). This mapping preserves the convexity of convex constraints, but it converts all non-convex constraints to convex ones. The QCFP is a feasibility problem with non-convex constraints, while the IEOSP is an optimization problem with a convex feasible region and a non-convex objective function. It is shown that the global optimal solution of IEOSP is a feasible solution of the QCFP. Comparing the structures of QCFP and the proposed IEOSP, the second model only has one extra variable compared to the original QCFP because it employs one slack variable for the mapping. Thus, the problem dimension approximately remains unchanged. Due to the convexity of all constraints in IEOSP, it has a well-defined feasible region. Therefore, it can be solved much easier than the original QCFP. This paper proposes a solution algorithm for IEOSP that iteratively solves a convex optimization problem. The algorithm is mathematically shown to reach either a feasible solution of the QCFP or a local solution of the IEOSP. To illustrate our theoretical developments, a comprehensive numerical experiment is performed, and 500 different QCFPs are studied. All these numerical experiments confirm the promising performance and applicability of our theoretical developments in the current paper.

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1 Introduction

1.1 Motivation

The Quadratically Constrained Feasibility Problem (QCFP) arises in a wide range of areas involving combinatorial optimization, economic equilibria, and mixed linear integer programming [15]. Elegance in the formulation and wide ranges of practical applications have made these problems an active area of research [5, 7, 20].

The non-convex QCFPs are generally NP-hard [20] due to the existence of non-convex quadratic constraints, and hence they are difficult to solve. To cope with this issue, various ideas have been developed to solve these problems efficiently. The solution algorithms for solving QCFPs can be broadly categorized as convex-relaxation, convex-conservative, branch-and-bound, and Inside-Ellipsoids Outside-Sphere (IEOS)-based methods. We review these solution methods in the next subsection, and we present our theoretical contributions. It must be emphasized that some of these methods are specially developed for Quadratically Constrained Quadratic Problems (QCQPs) that can be intuitively exploited for QCFPs, as well.

1.2 Branch-and-Bound Methods

The branch-and-bound methods have been widely used for solving QCQPs and QCFPs [3, 10, 12, 17, 21]. These methods generally consider a box for the decision variables and iteratively shrink the box volume in order to find global or feasible solutions. Various types of search strategies, including depth-first and breadth-first, and branching strategies, such as most-strong and most-infeasible, can be exploited to develop methods with better convergence performances. Branch-and-bound methods have been developed to find the exact solution of QCQPs or exactly determine whether QCFPs are infeasible.

Paper [3] uses an outer approximation and linear relaxed problem as its bounding step when it explores the search tree. Paper [21] presents a symbolical branch-and-bound method in which linear subproblems and simplicial partitions are utilized to improve its computational efficiency. Recently, paper [13] presents a suitable branch-and-bound algorithm for exactly solving QCQPs. It uses the property of quadratic constraints in order to construct a linear, relaxed version of the problem, which is employed in a branch-and-bound algorithm without adding either new variables or constraints. Through solving a series of consecutive linear problems, the algorithm is capable of reaching the global solution of the QCQP. It has been reported that the numerical performance of this method is more efficient than their benchmark methods for solving QCQPs.

The tightness of the relaxation and the computational efficiency in finding the lower bound are principal factors that may exert an adverse influence on the applicability and efficiency of the branch-and-bound methods [26]. Different relaxation techniques for QCQPs are also used in branch-and-bound methods to enhance their computational efficiency. The computational burden is the biggest demerit of these relaxation techniques since they have to search all points of the given primary search box in the worst case. Added to this, the number of required searches increases extremely whenever the number of variables increases. These issues widely limit the applicability and performance of the relaxation techniques.

1.3 Convex Relaxation Methods

These methods attempt to convert the QCQP into a convex problem, which relaxes the main problem. Recently, relaxation methods have increasingly attracted great attention among researchers due to their efficiencies. Briefly, branch-and-bound methods majorly suffer from their computational cost, while the convex relaxation methods result in a better performance from this perspective.

These methods first reformulate the problem by an equivalent problem consisting of rank-one plus some convex constraints. Then, the rank-one constraint is relaxed through exploiting various techniques [6] to obtain the Semi-Definite Programming (SDP) form of the problem. Although these methods are basically proposed for QCQPs, they can be used to solve QCFPs by omitting the cost function from the optimization problem. Paper [6] reviewed and compared these SDP techniques in view of computational burden and feasibility performances.

One SDP relaxation technique is to omit the rank-one constraint and only solve the SDP constraints that are known as the Shor SDP relaxation method [6]. Another way to reach an SDP relaxation is through deriving the Lagrangian dual problem of the Shor SDP relaxed problem [6]. Since the Shor relaxed and its dual problems are convex, the dual problem can be exploited to find a lower bound for the optimal value of the original QCQP problem.

Sherali and Adams [23] originally introduced the notion of reformulation–linearization-technique (RLT) in order to solve general nonlinear problems. The RLT uses the lower and upper bounds of decision variables to extract a tighter linear relaxation of the nonlinear problem. Afterward, Anstreicher [4] exploited this RLT concept in the SDP reformulation of the QCQP to convexify the problem by replacing the original non-convex constraints with their relaxed versions. This technique leads to the SDP-RLT relaxation method. In this method, extra constraints are added to the Shor relaxed problem, which obviously tightens the feasible region of the Shor relaxed problem in order to reduce the relaxation bound [4].

Sturm and Zhang [24] combined RLT and SDP reformulations along with linear constraints of QCQP to formulate Second Order Cone (SOC) constraints. They proposed a new set of convex constraints for the QCQP optimization, which is well-known by the SOC-RLT method. It is apparent that SOC-RLT is tighter than Shor SDP and SDP-RLT relaxations since more constraints are involved in SOC-RLT [24].

In SOC-RLT, the convex quadratic constraints of QCQP are exactly reformulated as SOC constraints, and the product of SOC and linear constraints are linearized.

The idea of SOC-RLT is further extended to other special cases of QCQPs. Paper [9] explains the extension of this idea in the mixed integer QCQPs in order to obtain a better relaxation in comparison with SDP-RLT relaxation. Also, it has been proven that the SDP-RLT relaxation leads to a zero relaxation gap in an extended trust region problem with a quadratic function and a unit sphere and linear constraints [11].

Study [16] recently proposed the idea of a Generalized SOC-RLT (GSRT) relaxation method whose constraints are derived through combining the idea of SOC-RLT and positive and negative definite decompositions of quadratic terms. Indeed, the GSRT combines their previous ideas in order to develop valid inequalities through linearizing the product of linear and non-convex constraints. Although GSRT results in a better feasibility performance, its computational burden can be considered as its main drawback.

It should be noted that these methods finally reach a convex relaxation version of QCQP whose optimal objective function value does not exceed that of QCQP. In QCFPs, the feasible region of the relaxed problem includes the feasible region of their corresponding QCFP; therefore, these methods may find a point outside the feasible region of the QCFP.

1.4 Convex Conservative Methods

Convexification of quadratic constraints is the other methodology for solving QCQPs and QCFPs. These methods strive for the convexification of the non-convex constraints in order to reach a convex conservative problem as presented in papers [14, 18, 22]. In these methods, the non-convex quadratic constraints of the problem are substituted by convex quadratic constraints whose feasible regions are included in their corresponding non-convex constraints. Therefore, they can be categorized as conservative methods. Firstly, McCormick [18] uses the lower and upper bounds of decision variables to extract valid linear constraints for the polynomial problems. Obviously, this idea can be directly exploited in the QCQPs and QCFPs without any change due to the polynomial nature of these problems. This idea is further investigated in other similar works in order to reach less conservative constraints than the linear constraints of McCormick.

Saeki [22] exploited the idea of convex-concave decomposition for quadratic constraints in Bilinear Matrix Inequalities (BMIs) that is also applicable to QCFPs and QCQPs. In this idea, the non-convex constraints are convexified by conservative constraints, which are theoretically valid for the problem. This idea is denoted by the convex-concave method in the rest of [22]. The other method is noted by PN-convex-concave developed by [14], in which the matrices of quadratic constraints are decomposed to positive and negative sub-matrices and the convex-concave decomposition idea is only applied to the negative part whose dimension is smaller than the number of variables.

The idea of convexification is deeply investigated in the other related studies which focus on specific types of QCQPs [8, 19, 25]. Conservatism is the most important demerit of these methods because it majorly restricts the original feasible region. The feasible region of the conservative problem is the largest convex sub-region of the original feasible region of the QCFPs in the best case; this is while the feasible region of the original problem is inherently non-convex.

1.5 IEOS-Based Methods

The idea of IEOS has recently been developed and exploited for solving QCFPs through our previous studies [1] and [2]. In the methodology of IEOS, mapping is used to convert the QCFP to an Inside-Ellipsoids Outside-Sphere Problem (IEOSP), which is a quadratic optimization problem with only one non-convex constraint. This conversion does not change the dimension of the problem significantly since only one variable is added to the problem. This means that the original QCFP is equivalently transformed into IEOSP at almost the same complexity level. This is the most important advantage of the IEOS idea, as it enables us to reduce the non-convexity of the original QCFP to another problem with the least possible non-convexity.

The IEOS method has been introduced by paper [2] and applied to the power-flow problem in power-system engineering [1]. It has been numerically shown that the proposed IEOS algorithm is capable of solving large-scale power-flow feasibility problems.

1.6 Paper Contribution

This paper utilizes the IEOS idea to reduce the impact of QCPF non-convexity. We significantly modify the previous idea given in [2] to improve the performances of the final algorithm. Inspired by our previous study in [2], we equivalently convert the QCFP into IEOSP, which is another quadratic optimization problem with convex constraints and the non-convex quadratic objective function. Then, we define the acceptable region of the IEOSP as the intersection of its feasible region with the outer space of the unit sphere. Our proposed mapping has been proven to be one-to-one and onto over the feasible region of QCFP, which in turn means the feasible region of the QCFP and acceptable region of IEOSP have isomorphism. This isomorphic mapping implies that the QCFP and IEOSP models are equivalent and enables us to solve the IEOSP equivalently instead of the original hard QCFP.

An algorithm is presented for solving the IEOSP in order to reach an acceptable point, which is an isomorphism of the feasible region of the original QCFP. Our algorithm iteratively solves a convexified version of the IEOSP in which the non-convex objective function of the IEOSP is linearized over the previously obtained optimal solution. Then, it is theoretically proved that the sequence of optimal solutions tends to be the optimum solution of the IEOSP. The proposed algorithm is potentially able to find the global solution of the IEOSP. It is proved that there are some initial points that lead us to the global optimum. To be sure about it, the algorithm should be repeatedly called with different initial points to increase the chance of convergence to the global solution.

Various aspects of our proposed IEOS-based algorithm are verified in this paper. The performance and computational efficiency of the proposed algorithm are compared with the relaxation and convexification methods using 500 randomly generated QCFPs. The proposed methodology in this paper majorly differs from our previous paper [2] as follows:

1. In [2], it has been proven that the proposed algorithm converges to a KKT point of the IEOSP. However, this paper modifies the algorithm such that it finally reaches the local solution of the IEOSP, whose global solution is a feasible solution to the original QCFP.
2. In [2], the final solution of the algorithm, which may not be the feasible solution of QCFP, is allowed to violate both convex and non-convex constraints of QCFP. To cope with this issue, we modify the IEOS mapping such that the feasible solutions of IEOSP satisfy convex constraints and just violate the non-convex ones.
3. This paper proves that the feasible region of QCFP and acceptable region of IEOSP have isomorphism.

The isomorphism of the feasible region of QCFP and the acceptable region of IEOSP are proved by Theorems 1 and 2. Then, the algorithm is presented to solve the IEOSP efficiently instead of QCFP. The convergence and relaxation analyses of the proposed algorithm are discussed in Theorems 3, 4, and 5. Theorem 3 proves the convergence of the proposed algorithm to a feasible solution of IEOSP. Then, Theorem 4 shows this limit point is a local solution of the IEOSP, as well. Theorem 5 investigates the relaxation of the consecutive solutions of the proposed algorithm, which mathematically measures how much these solutions violate the non-convex constraints of the original QCFP.

1.7 Paper Structure

This paper is organized as follows: In Sect. 2, QCFPs are introduced. In Sect. 3, the QCFP is transformed into the IEOSP through three subsections. Section 4 presents the proposed algorithm for solving IEOSP and its related results, such as convergence and relaxation analyses. Section 5 consists of subsections that numerically assess various aspects of the proposed algorithm by simulation studies. Finally, Sect. 6 concludes the paper.

2 Problem Description

Consider the QCFP

$$\textbf{QCFP}$$

$$Find\ x \in \mathbb{R}^n \tag{1a}$$

$$Such\ that:$$

$$Ax \leq b, \tag{1b}$$

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} P_i & p_i \\ p_i^T & \mathfrak{p}_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} + 1 \leq 0, \quad \forall i \in \{1, \dots, m_1\} \tag{1c}$$

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} Q_i & q_i \\ q_i^T & \mathfrak{q}_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} + 1 \leq 0, \quad \forall i \in \{1, \dots, m_2\} \tag{1d}$$

where $x \in \mathbb{R}^n$ is the vector of decision variables, $\{P_i\}_{i=1}^{m_1} \subset \mathbb{S}^{n\times n}$, and $\{Q_i\}_{i=1}^{m_2} \subset \mathbb{S}^{n\times n}$ are real symmetric matrices, $\{p_i\}_{i=1}^{m_1} \subset \mathbb{R}^n$ and $\{q_i\}_{i=1}^{m_2} \subset \mathbb{R}^n$ are real vectors, and $\{\mathfrak{p}_i\}_{i=1}^{m_1} \subset \mathbb{R}$ and $\{\mathfrak{q}_i\}_{i=1}^{m_2} \subset \mathbb{R}$ are real numbers. Also, $A \in \mathbb{R}^{l\times n}$ and $b \in \mathbb{R}^l$ are real matrices that play the role of linear constraints in (1). In QCFP (1), constraints (1c) and (1d) are supposed to be quadratic such that they cannot be exactly reformulated as linear constraints. It means there is no way to equivalently rewrite these constraints by a set of linear constraints. These quadratic constraints are precisely divided into two sets of convex and non-convex constraints in which constraints (1c) are convex and (1d) are non-convex. Splitting the constraints in these two categories can be useful in problems, where there are only a few non-convex constraints. The convex quadratic constraints need no change and can be directly inserted in the convex optimization problems, while the non-convex constraints need to be modified, mapped, or relaxed to be applicable in the final convex optimization problem.

Assume the feasible region of QCFP (1) is denoted by notation $\mathcal{F} \subset \mathbb{R}^n$. It should be noted that region $\mathcal{F}$ is supposed to be non-convex in this paper. This is mainly because the convex version of the problem can be simply solved by the existing convex solvers. Furthermore, the feasible region is assumed not to contain the origin, which means $0 \notin \mathcal{F}$, because it can be considered as a trivial feasible solution. In the following, we will discuss an important property of the QCFP (1), which is frequently addressed in the next theoretical results of this paper.

Lemma 2.1 *If QCFP (1) is feasible, then the following inequalities are satisfied for all $i \in \{1, \dots, m_1\}$:*

$$P_i \succeq 0, \tag{2a}$$

$$\mathfrak{p}_i + 1 < p_i^T V_i S_i^{-1} V_i^T p_i, \tag{2b}$$

where $S_i \in \mathbb{R}^{n_i}$ is a diagonal matrix containing positive eigenvalues of matrix P_i on its diagonal, n_i is the number of positive eigenvaluesof P_i, and $V_i \in \mathbb{R}^{n\times n_i}$ contains eigenvectors of P_i on its columns (i.e. $P_i = V_i S_i V_i^T$ and $V_i^T V_i = I_{n_i}$).

The proof of Lemma 2.1 is given in Appendix A.1.

Remark 2.1 Although this paper mainly focuses on the feasibility version of quadratically constrained problems, the proposed methodology is also applicable to the optimization version. This claim is completely explained in part A.2 in the Appendix section.

3 Converting QCFP to IEOSP

This section converts the QCFP (1) into an equivalent optimization problem which has a simpler formulation. The simpler version is noted by IEOSP, which is equivalent to the QCFP from the mathematical perspective, but it does not contain any non-convex constraints. This section mainly focuses on this idea and consists of three subsections, namely, the presentation of the IEOSP model, the equivalency of the QCFP and IEOSP models, and the properties of the proposed IEOSP model.

The first subsection mathematically defines the IEOSP and indicates its relation to QCFP (1). The second subsection proves an important relationship between the feasible regions of the QCFP and the IEOSP, which will be further utilized to prove the equivalency of these problems. The third subsection explicitly mentions some important properties and advantages of the proposed IEOSP model compared to the original QCFP.

3.1 Presentation of the IEOSP

Based on the QCFP (1), we define the optimization problem (3) known as the IEOSP:

$$\underset{\hat{x}\in\mathbb{R}^{n+1}}{\text{Maximize}} \quad \hat{x}^T\hat{x} \tag{3a}$$

Subject to

$$\hat{A}\hat{x} \leq 0, \tag{3b}$$

$$\hat{x}^T\hat{P}_i\hat{x} \leq 0, \qquad \forall i \in \{1,\ldots,m_1\} \tag{3c}$$

$$\hat{x}^T\hat{Q}_i\hat{x} \leq 1, \qquad \forall i \in \{1,\ldots,m_2\} \tag{3d}$$

where $\hat{A} \in \mathbb{R}^{l\times(n+1)}$, $\left\{\hat{P}_i\right\}_{i=1}^{m_1} \subset \mathbb{R}^{(n+1)\times(n+1)}$, and $\left\{\hat{Q}_i\right\}_{i=1}^{m} \subset \mathbb{R}^{(n+1)\times(n+1)}$ are defined as:

$$\hat{A} = \begin{bmatrix}\sqrt{\gamma-1}A & -\sqrt{\gamma}b\end{bmatrix}, \tag{4}$$

$$\hat{P}_i = \begin{bmatrix}\sqrt{\gamma-1}I_n & 0\\ 0 & \sqrt{\gamma}\end{bmatrix}^T \begin{bmatrix}P_i & p_i\\ p_i^T & \mathfrak{p}_i+1\end{bmatrix} \begin{bmatrix}\sqrt{\gamma-1}I_n & 0\\ 0 & \sqrt{\gamma}\end{bmatrix}, \qquad \forall i \in \{1,\ldots,m_1\} \tag{5}$$

$$\hat{Q}_i = \begin{bmatrix}\frac{1}{\sqrt{\gamma}}I_n & 0\\ 0 & \frac{1}{\sqrt{\gamma-1}}\end{bmatrix}^T \left(\begin{bmatrix}Q_i & q_i\\ q_i^T & \mathfrak{q}_i\end{bmatrix} + \gamma I_{n+1}\right) \begin{bmatrix}\frac{1}{\sqrt{\gamma}}I_n & 0\\ 0 & \frac{1}{\sqrt{\gamma-1}}\end{bmatrix}, \qquad \forall i \in \{1,\ldots,m_2\} \tag{6}$$

$$\gamma = \max\left(1+\varepsilon, \max_{i\in\{1,\ldots,m_2\}} \bar{\lambda}\left(-\begin{bmatrix}Q_i & q_i\\ q_i^T & \mathfrak{q}_i\end{bmatrix}\right)\right). \tag{7}$$

In (7), operator $\bar{\lambda}(\bullet)$ stands for the maximum eigenvalue, I_{n+1} is the $(n+1)^{th}$ identity matrix, and $\varepsilon > 0$ is a small positive number.

One of the most important aims of this paper is to show that the feasible region of QCFP (1) (denoted by $\mathcal{F}$) and the acceptable region of IEOSP (3) have isomorphism. It provides us with an opportunity to solve the IEOSP equivalently instead of the QCFP.

It should be noted that all constraints of IEOSP (3) are convex, which is one of the most important properties of this problem. This claim is theoretically proved through the next lemma.

Lemma 3.1 *All constraints of IEOSP (3) are convex.*

Proof Clearly, equation (7) results

$$\gamma I_{n+1} \succeq - \begin{bmatrix} Q_i & q_i \\ q_i^T & \mathfrak{q}_i \end{bmatrix} \Rightarrow \begin{bmatrix} Q_i & q_i \\ q_i^T & \mathfrak{q}_i \end{bmatrix} + \gamma I_{n+1} \succeq 0, \qquad \forall i \in \{1, \ldots, m_2\}. \tag{8}$$

It can be simply concluded that all constraints in (3d) are convex according to relation (8). Constraint (3c) can be equivalently rewritten as (9) and (10) for an arbitrary $i \in \{1, \ldots, m_1\}$ considering $P_i = V_i S_i V_i^T$ is singular value decomposition of matrix P_i:

$$(\gamma - 1)\, \hat{x}_b^T P_i \hat{x}_b + 2\sqrt{\gamma - 1}\sqrt{\gamma}\hat{x}_b^T p_i \hat{x}_e + \gamma\,(\mathfrak{p}_i + 1)\,\hat{x}_e^2 \leq 0, \tag{9}$$

$$\begin{aligned} &(\sqrt{\gamma - 1} S_i^{\frac{1}{2}} V_i^T \hat{x}_b + \sqrt{\gamma}\hat{x}_e S_i^{-\frac{1}{2}} V_i^T p_i)^2 + 2\sqrt{\gamma}\sqrt{\gamma - 1}\hat{x}_b^T \left(I_n - V_i V_i^T\right) p_i \hat{x}_e \\ &+ \gamma \left(\mathfrak{p}_i + 1 - p_i^T V_i S_i^{-1} V_i^T p_i\right) \hat{x}_e^2 \leq 0, \end{aligned} \tag{10}$$

where $\hat{x}_b = \left[\hat{x}_1 \ldots \hat{x}_n\right]^T$ and $\hat{x}_e = \hat{x}_{n+1}$.

Note that, Lemma 2.1 proves that $\mathfrak{p}_i + 1 < p_i^T V_i S_i^{-1} V_i^T p_i$ which enables us to rewrite inequality (10) as follows:

$$\begin{bmatrix} -\hat{x}_e I_n & \sqrt{\gamma - 1} S_i^{\frac{1}{2}} V_i^T \hat{x}_b + \sqrt{\gamma}\hat{x}_e S_i^{-\frac{1}{2}} V_i^T p_i \\ * & 2\sqrt{\gamma}\sqrt{\gamma - 1}\hat{x}_b^T \left(I_n - V_i V_i^T\right) p_i - \gamma \left(p_i^T V_i S_i^{-1} V_i^T p_i - \mathfrak{p}_i - 1\right) \hat{x}_e \end{bmatrix} \prec 0. \tag{11}$$

Above, notation $*$ stands for $\left(\sqrt{\gamma - 1} S_i^{\frac{1}{2}} V_i^T \hat{x}_b + \sqrt{\gamma}\hat{x}_e S_i^{-\frac{1}{2}} V_i^T p_i\right)^T$.

Based on (11), constraints (3c) are convex since they can be exactly represented by convex constraints. □

It should be noted that the feasible region of QCFP (1) and IEOSP (3) are related to each other in the sense that is indicated in the next subsection. For this purpose, we define the concepts of the acceptable point and the acceptable region for the proposed IEOSP as below:

Definition 3.1 Solution $\hat{x} \in \mathbb{R}^{n+1}$ is noted to be an acceptable point for IEOSP (3) if it is a feasible point of IEOSP (3) such that $\|\hat{x}\| \leq 1$ and $\max\limits_{i \in \{1,\ldots,m_2\}} \hat{x}^T \hat{Q}_i \hat{x} = 1$. The acceptable region $\mathcal{A} \subset \mathbb{R}^{n+1}$ is defined as a region that includes all acceptable points of IEOSP (3). Furthermore, notation $\hat{\mathcal{F}}$ stands for the feasible region of IEOSP (3).

3.2 The Isomorphism of QCFP and IEOSP

This subsection proves the feasible region of QCFP (1) denoted by $\mathcal{F}$ and acceptable region of IEOSP (3) denoted by $\mathcal{A}$ have isomorphism through Theorems 3.1 and 3.2. Theorem 3.1 presents mapping $\mathcal{T}$ that maps region $\mathcal{F}$ into region $\mathcal{A}$ (i.e. $\mathcal{T}:\mathcal{F} \to \mathcal{A}$) and proves $\mathcal{T}$ is invertible. Then, Theorem 3.2 proves mapping $\mathcal{T}$ is also onto which means $\mathcal{T}(\mathcal{F}) \subset \mathcal{A}$ and $\mathcal{T}^{-1}(\mathcal{A}) \subset \mathcal{F}$. Therefore, mapping $\mathcal{T}$ demonstrates regions $\mathcal{F}$ and $\mathcal{A}$ have isomorphism that can be easily converted to each other. This section finally proves the equivalency of QCFP and IEOSP.

Theorem 3.1 *Mapping $\mathcal{T} : \mathcal{F} \to \mathcal{A}$ defined as*

$$\forall x \in \mathcal{F}: \quad \mathcal{T}(x) = \frac{1}{\sqrt{\gamma \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} x \\ 1 \end{bmatrix} + \max_{i \in \{1,\dots,m_2\}} \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} Q_i & q_i \\ q_i^T & \mathfrak{q}_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}}} \begin{bmatrix} \sqrt{\gamma} x \\ \sqrt{\gamma - 1} \end{bmatrix}, \tag{12}$$

is definable and invertible.

Proof First, we prove the definability of mapping $\mathcal{T}$ over $\mathcal{F}$ that suffices to show term $\gamma \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} x \\ 1 \end{bmatrix} + \max_{i \in \{1,\dots,m_2\}} \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} Q_i & q_i \\ q_i^T & \mathfrak{q}_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$ is always greater than 0 for all $x \in \mathcal{F}$. For this purpose, consider the following derivations:

$$\begin{aligned} &\gamma \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} x \\ 1 \end{bmatrix} + \max_{i \in \{1,\dots,m_2\}} \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} Q_i & q_i \\ q_i^T & \mathfrak{q}_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \\ &\max_{i \in \{1,\dots,m_2\}} \begin{bmatrix} x \\ 1 \end{bmatrix}^T \left(\begin{bmatrix} Q_i & q_i \\ q_i^T & \mathfrak{q}_i \end{bmatrix} + \gamma I_{n+1} \right) \begin{bmatrix} x \\ 1 \end{bmatrix}. \end{aligned} \tag{13}$$

Inequality (13) proves the definability of mapping $\mathcal{T}$ over $\mathcal{F}$ because $\begin{bmatrix} Q_i & q_i \\ q_i^T & \mathfrak{q}_i \end{bmatrix} + \gamma I_{n+1} \succeq 0, \ \forall i \in \{1, \dots, m_2\}$.

Second, we prove mapping $\mathcal{T}$ is invertible that is denoted by $\mathcal{T}^{-1} : \mathcal{A} \to \mathcal{F}$ and obtained

$$\forall \hat{x} \in \mathcal{A}: \quad \mathcal{T}^{-1}(\hat{x}) = \frac{\sqrt{\gamma - 1}}{\sqrt{\gamma} \hat{x}_{n+1}} \begin{bmatrix} I_n & 0 \end{bmatrix} \hat{x}. \tag{14}$$

Therefore, it is needed to show $\mathcal{T}\left(\mathcal{T}^{-1}(\hat{x})\right) = x$ for all $\hat{x} \in \mathcal{A}$ that is followed by equation (15), considering $\hat{x}$ is arbitrary selected from $\mathcal{A}$:

$$\mathcal{T}\left(\mathcal{T}^{-1}\left(\hat{x}\right)\right)=\frac{\begin{bmatrix}\frac{\sqrt{\gamma}\sqrt{\gamma-1}}{\sqrt{\gamma}\hat{x}_{n+1}}\left[\,I_n\ 0\,\right]\hat{x}\\ \sqrt{\gamma-1}\end{bmatrix}}{\sqrt{\gamma\begin{bmatrix}\frac{\sqrt{\gamma-1}}{\sqrt{\gamma}\hat{x}_{n+1}}\left[\,I_n\ 0\,\right]\hat{x}\\ 1\end{bmatrix}^T\begin{bmatrix}\frac{\sqrt{\gamma-1}}{\sqrt{\gamma}\hat{x}_{n+1}}\left[\,I_n\ 0\,\right]\hat{x}\\ 1\end{bmatrix}+M_X}},$$
$$M_X=\max_{i\in\{1,\ldots,m_2\}}\begin{bmatrix}\frac{\sqrt{\gamma-1}}{\sqrt{\gamma}\hat{x}_{n+1}}\left[\,I_n\ 0\,\right]\hat{x}\\ 1\end{bmatrix}^T\begin{bmatrix}Q_i & q_i\\ q_i^T & \mathfrak{q}_i\end{bmatrix}\begin{bmatrix}\frac{\sqrt{\gamma-1}}{\sqrt{\gamma}\hat{x}_{n+1}}\left[\,I_n\ 0\,\right]\hat{x}\\ 1\end{bmatrix}. \tag{15}$$

Simplifying (15) results in

$$\mathcal{T}\left(\mathcal{T}^{-1}\left(\hat{x}\right)\right)=\frac{\hat{x}}{\sqrt{\gamma\begin{bmatrix}\frac{1}{\sqrt{\gamma}}\left[\,I_n\ 0\,\right]\hat{x}\\ \frac{1}{\sqrt{\gamma-1}}\hat{x}_{n+1}\end{bmatrix}^T\begin{bmatrix}\frac{1}{\sqrt{\gamma}}\left[\,I_n\ 0\,\right]\hat{x}\\ \frac{1}{\sqrt{\gamma-1}}\hat{x}_{n+1}\end{bmatrix}+N_X}},$$
$$N_X=\max_{i\in\{1,\ldots,m_2\}}\begin{bmatrix}\frac{1}{\sqrt{\gamma}}\left[\,I_n\ 0\,\right]\hat{x}\\ \frac{1}{\sqrt{\gamma-1}}\hat{x}_{n+1}\end{bmatrix}^T\begin{bmatrix}Q_i & q_i\\ q_i^T & \mathfrak{q}_i\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{\gamma}}\left[\,I_n\ 0\,\right]\hat{x}\\ \frac{1}{\sqrt{\gamma-1}}\hat{x}_{n+1}\end{bmatrix}, \tag{16}$$

$$\mathcal{T}\left(\mathcal{T}^{-1}\left(\hat{x}\right)\right)=\frac{\hat{x}}{\sqrt{\max\limits_{i\in\{1,\ldots,m_2\}}\hat{x}^T\begin{bmatrix}\frac{1}{\sqrt{\gamma}}I_n & 0\\ 0 & \frac{1}{\sqrt{\gamma-1}}\end{bmatrix}\left(\begin{bmatrix}Q_i & q_i\\ q_i^T & \mathfrak{q}_i\end{bmatrix}+\gamma I_{n+1}\right)\begin{bmatrix}\frac{1}{\sqrt{\gamma}}I_n & 0\\ 0 & \frac{1}{\sqrt{\gamma-1}}\end{bmatrix}\hat{x}}}$$
$$=\frac{\hat{x}}{\sqrt{\max\ \max\limits_{i\in\{1,\ldots,m_2\}}\hat{x}^T\hat{Q}_i\hat{x}}}. \tag{17}$$

Since $\hat{x}$ is arbitrary selected from $\mathcal{A}$, it satisfies $\max \hat{x}^T\hat{Q}_i\hat{x}_l\, imitsi \in \{1,\ldots,m_2\}=1$ based on Definition 3.1. This fact intuitively implies $\mathcal{T}\left(\mathcal{T}^{-1}\left(\hat{x}\right)\right)=\hat{x}$ due to (17). □

Theorem 3.2 *The feasible region of QCFP (1) ($\mathcal{F}$) and the acceptable region of IEOSP (3) ($\mathcal{A}$) have isomorphism through $\mathcal{T}:\mathcal{F}\rightarrow\mathcal{A}$ defined in (12).*

Proof Mapping $\mathcal{T}$ is proved to be definable and invertible by Theorem 3.1.

Therefore, it suffices to prove $\mathcal{T}^{-1}\left(\hat{x}\right)\subset\mathcal{F}$ for all $\hat{x}\in\mathcal{A}$ and $\mathcal{T}\left(x\right)\subset\mathcal{A}$ for $x\in\mathcal{F}$ that means the mapping and its inverse are onto mappings.

Assume $\hat{x}\in\mathcal{A}$ and define $x=\mathcal{T}^{-1}\left(\hat{x}\right)$. Using these assumptions, we have:

$$Ax-b=\frac{1}{\sqrt{\gamma}\hat{x}_{n+1}}\left(\sqrt{\gamma-1}A\left[\,I_n\ 0\,\right]\hat{x}-\sqrt{\gamma}b\hat{x}_{n+1}\right)=\frac{1}{\sqrt{\gamma}\hat{x}_{n+1}}\hat{A}\hat{x}\leq 0, \tag{18}$$

$$\begin{bmatrix}x\\ 1\end{bmatrix}^T\begin{bmatrix}P_i & p_i\\ p_i^T & \mathfrak{p}_i\end{bmatrix}\begin{bmatrix}x\\ 1\end{bmatrix}+1=$$
$$\frac{\gamma-1}{\hat{x}_{n+1}^2}\hat{x}^T\begin{bmatrix}\frac{1}{\sqrt{\gamma}}I_n & 0\\ 0 & \frac{1}{\sqrt{\gamma-1}}\end{bmatrix}\begin{bmatrix}P_i & p_i\\ p_i^T & \mathfrak{p}_i\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{\gamma}}I_n & 0\\ 0 & \frac{1}{\sqrt{\gamma-1}}\end{bmatrix}\hat{x}+1 \tag{19}$$
$$=\frac{\hat{x}^T\hat{P}_i\hat{x}}{\gamma\hat{x}_{n+1}^2}\leq 0,\quad \forall i\in\{1,\ldots,m_1\}$$

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} Q_i & q_i \\ q_i^T & \mathfrak{q}_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} + 1 = \frac{(\gamma-1)\hat{x}^T \begin{bmatrix} \frac{1}{\sqrt{\gamma}} I_n & 0 \\ 0 & \frac{1}{\sqrt{\gamma-1}} \end{bmatrix} \begin{bmatrix} Q_i & q_i \\ q_i^T & \mathfrak{q}_i \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{\gamma}} I_n & 0 \\ 0 & \frac{1}{\sqrt{\gamma-1}} \end{bmatrix} \hat{x} + \hat{x}_{n+1}^2}{\hat{x}_{n+1}^2} = \frac{(\gamma-1)\left(\hat{x}^T \hat{Q}_i \hat{x} - \hat{x}^T \hat{x}\right)}{\hat{x}_{n+1}^2} \leq 0, \quad \forall i \in \{1, \ldots, m_2\} \tag{20}$$

According to (18)–(20), point x belongs to $\mathcal{F}$. Now, assume $x \in \mathcal{F}$ and let $\hat{x} = \mathcal{T}(x)$ that concludes the following derivations ($j = \underset{i \in \{1,\ldots,m_2\}}{\operatorname{argmax}} \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} Q_i & q_i \\ q_i^T & \mathfrak{q}_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$):

$$\hat{A}\hat{x} = \frac{\sqrt{\gamma}\sqrt{\gamma-1}}{\sqrt{\gamma \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} x \\ 1 \end{bmatrix} + \max_{i \in \{1,\ldots,m_2\}} \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} Q_i & q_i \\ q_i^T & \mathfrak{q}_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}}} (Ax - b) \leq 0, \tag{21}$$

$$\begin{aligned} \hat{x}^T \hat{P}_i \hat{x} &= \hat{x}^T \begin{bmatrix} \sqrt{\gamma-1} I_n & 0 \\ 0 & \sqrt{\gamma} \end{bmatrix}^T \begin{bmatrix} P_i & p_i \\ p_i^T & \mathfrak{p}_i + 1 \end{bmatrix} \begin{bmatrix} \sqrt{\gamma-1} I_n & 0 \\ 0 & \sqrt{\gamma} \end{bmatrix} \hat{x} \\ &= \frac{\sqrt{\gamma}\sqrt{\gamma-1}}{\gamma \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} x \\ 1 \end{bmatrix} + \max_{i \in \{1,\ldots,m_2\}} \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} Q_i & q_i \\ q_i^T & \mathfrak{q}_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}} \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} P_i & p_i \\ p_i^T & \mathfrak{p}_i + 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \leq 0, \\ &\quad \forall i \in \{1, \ldots, m_1\} \end{aligned} \tag{22}$$

$$\hat{x}^T \hat{Q}_i \hat{x} = \frac{\begin{bmatrix} x \\ 1 \end{bmatrix}^T \left(\begin{bmatrix} Q_i & q_i \\ q_i^T & \mathfrak{q}_i \end{bmatrix} + \gamma I_{n+1} \right) \begin{bmatrix} x \\ 1 \end{bmatrix}}{\gamma \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} x \\ 1 \end{bmatrix} + \max_{i \in \{1,\ldots,m_2\}} \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} Q_i & q_i \\ q_i^T & \mathfrak{q}_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}} \leq 1, \quad \forall i \in \{1, \ldots, m_2\} - \{j\} \tag{23}$$

$$\hat{x}^T \hat{Q}_j \hat{x} = \frac{\begin{bmatrix} x \\ 1 \end{bmatrix}^T \left(\begin{bmatrix} Q_j & q_j \\ q_j^T & \mathfrak{q}_j \end{bmatrix} + \gamma I_{n+1} \right) \begin{bmatrix} x \\ 1 \end{bmatrix}}{\gamma \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} x \\ 1 \end{bmatrix} + \max_{i \in \{1,\ldots,m_2\}} \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} Q_i & q_i \\ q_i^T & \mathfrak{q}_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}} = 1, \tag{24}$$

$$\hat{x}^T \hat{x} = \frac{\gamma x^T x + \gamma - 1}{\gamma \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} x \\ 1 \end{bmatrix} + \max_{i \in \{1,\ldots,m_2\}} \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} Q_i & q_i \\ q_i^T & \mathfrak{q}_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}} \geq 1. \tag{25}$$

Based on (21)–(25), point $\hat{x}$ belongs to $\mathcal{A}$. □

Remark 3.1 Theorem 3.2 states that the feasible region of QCFP (1) can be equivalently mapped into the acceptable region of IEOSP (3) through mapping $\mathcal{T}$. This theorem suggests finding an acceptable point of IEOSP (3) that equals the solution of QCFP (1).

Remark 3.2 Let $\hat{x}$ be a feasible solution of IEOSP (3) that is not acceptable (i.e. $\hat{A}\hat{x} \le 0$, $\max_{R \in \left\{\hat{P}_i\right\}_{i=1}^{m_1}} \hat{x}^T R \hat{x} \le 0$, and $\max_{R \in \left\{\hat{Q}_i\right\}_{i=1}^{m_2}} \hat{x}^T R \hat{x} < 1$). Then, we can find another point $\frac{\hat{x}}{\sqrt{\max_{R \in \{\hat{Q}_i\}_{i=1}^{m_2}} \hat{x}^T R \hat{x}}}$ belonging to acceptable region $\mathcal{A}$. Therefore, finding the acceptable point of IECQP reversibly equals finding a feasible solution for this model.

Based on Theorems 3.1 and 3.2 and Remarks 3.1 and 3.2, it is desired to develop an algorithm for solving IEOSP in order to find its optimum solutions. Notice that the algorithm should only find a feasible solution for the proposed IEOSP that lies outside the unit sphere.

3.3 Properties of the Proposed IEOSP

The IEOSP (3) has a series of interesting mathematical properties which benefit the numerical solver. These properties are listed below:

1. The IEOSP does not have any non-convex constraint, and all its constraints are convex. Notice that the IEOSP is not a convex optimization problem due to its non-convex quadratic objective function. Although IEOSP is non-convex, all its constraints are convex, which helps us solve this model more effectively as compared to the original QCFP (1).
2. The QCFP (1) and IEOPS (3) have approximately the same number of free variables. The QCFP employs n free variables, while the IEOSP has $n+1$ free variables. Therefore, both problems have the same computational burden once they are solved by numerical algorithms.
3. The other important property of the IEOSP (3) is the geometry of its feasible region, which is the intersection of some half-spaces with several ellipsoids whose centers are at the origin.
4. It can be simply shown that the origin belongs to the feasible region of the IEOSP (3). This fact enables us to find an initial feasible solution for the IEOSP in the local neighborhood of the origin.
5. Finally, the main advantage of the IEOSP model is its ability to be effectively solved by numerical algorithms such as the proposed algorithm that will be presented in Sect. 4.

4 Proposed Solution Algorithm for IEOSP

This section presents an effective algorithm for solving our proposed IEOSP (3) that can be equivalently used to solve QCFP (1) as described in Sect. 3. This section consists of three subsections, namely, presentation of the solution algorithm, convergence analysis of the proposed algorithm, and relaxation analysis of the proposed algorithm. The first subsection presents the proposed algorithm and its related explanatory notes, and the second and third ones present theoretical results about the algorithm and its convergence properties.

4.1 Presentation of the Solution Algorithms

Assume $\left\{\hat{x}^{(k)}\right\}_{k=0}^{\infty}$ is a set of feasible points that are iteratively obtained through solving convex problem (26)

$$\max_{\hat{x}\in\mathbb{R}^{n+1}} \bar{x}^T\hat{x} \tag{26a}$$

Subject to :

$$\hat{A}\hat{x} \leq 0, \tag{26b}$$

$$\hat{x}^T\hat{P}_i\hat{x} \leq 0, \qquad \forall i \in \{1,\ldots,m_1\} \tag{26c}$$

$$\hat{x}^T\hat{Q}_i\hat{x} \leq 1, \qquad \forall i \in \{1,\ldots,m_2\} \tag{26d}$$

which convexifies the IEOSP (3) considering $\bar{x}$ is a given feasible solution of IEOSP (3). Point $\bar{x}$ is consecutively replaced by the solution obtained in the previous iteration of the algorithm. In fact, point $\hat{x}^{(k+\mathbf{1})}$ will be the optimal solution of convex problem (26) considering $\bar{x} = \hat{x}^{(k)}$. Our proposed solution algorithm exploits the consecutive optimal solutions $\left\{\hat{x}^{(k)}\right\}_{k=0}^{\infty}$ to reach a solution for the IEOSP (3). The steps of the algorithm are briefly given below:

The proposed solution algorithm for IEOSP

1. Set $k = 1$ and consider $\hat{x}^{(\mathbf{0})}$ as an initial feasible solution.
2. Solve convex problem (26) to find $\hat{x}^{(k)}$ considering $\bar{x} = \hat{x}^{(k-\mathbf{1})}$.
3. If $\|\hat{x}^{(k)}\| \geq 1$, return point $\hat{x}^{(k)}$ as an acceptable point and terminate.
4. If $\|\hat{x}^{(k)} - \hat{x}^{(k-\mathbf{1})}\| \leq \delta$, return $\hat{x}^{(k)}$ as a convergence point and terminate
5. Otherwise, set $k = k + 1$ and go to Step 2.

Some explanatory notes about the steps of the algorithm should be included. Step 1 simply initializes the iteration number k and begins with a feasible point $\hat{x}^{(\mathbf{0})}$ (notice that, $\hat{x}^{(\mathbf{0})}$ should be a feasible point of IEOSP (3) that can be simply obtained). As stated earlier, the first feasible point can be easily obtained in the local neighborhood of the origin. Step 2 solves convex optimization problem (26) to reach the unique optimal solution $\hat{x}^{(k)}$ at the k^{th} iteration. If $\hat{x}^{(k)}$ is outside the unit sphere or $\|\hat{x}^{(k)} - \hat{x}^{(k-\mathbf{1})}\|$ is smaller than a convergence threshold denoted by δ, then the algorithm terminates at Steps 3 and 4. It is mainly because it finds an acceptable point (in Step 3) or it converges to a feasible point (in Step 4). Step 5 updates the iteration number and repeats the procedure from Step 2.

4.2 Convergence Analysis of the Proposed Algorithm

Our proposed solution algorithm has some significant properties, which are discussed in this section. In the sequel, Theorem 4.1 proves the convergence of the algorithm to a feasible solution of the IEOSP (3).

Theorem 4.1 *The proposed algorithm will asymptotically converge to a feasible solution of IEOSP (3) for each arbitrarily selected point $\hat{x}^{(\mathbf{0})}$ which is feasible for IEOSP (3).*

Proof To prove this theorem, it suffices to show sequence $\{\hat{x}^{(k)}\}_{k=0}^{\infty}$ certainly converges to a feasible point of the model based on the algorithm's steps.

If the algorithm finds a feasible point that is located outside the unit sphere, it returns the point as the acceptable point and terminates. Hence, we assume $\|\hat{x}^{(k)}\|$ is smaller than 1 at all iterations k in the rest of the proof to prove the opposite side.

According to (26b)–(26d), it is clear that $\hat{x}^{(k)}$ belongs to the feasible region of the IEOSP (3). Using this fact and optimality of point $\hat{x}^{(k+\mathbf{1})}$ in problem (26) with $\bar{x} = \hat{x}^{(k)}$, we have:

$$(\hat{x}^{(k)})^T \hat{x}^{(k+\mathbf{1})} \geq \|\hat{x}^{(k)}\|^2. \tag{27}$$

Now, three cases about the optimal solutions $\hat{x}^{(k)}$ and $\hat{x}^{(k+\mathbf{1})}$ can be considered that are separately discussed in the following.

Case 1. Assume $\hat{x}^{(k+\mathbf{1})} = \hat{x}^{(k)}$. In this case, point $\hat{x}^{(k+\mathbf{2})}$ will be the optimal solution of problem (26) considering $\bar{x} = \hat{x}^{(k+\mathbf{1})} = \hat{x}^{(k)}$ and $\hat{x}^{(k+\mathbf{1})}$ is the optimal solution of problem (26) for $\bar{x} = \hat{x}^{(k)}$ that apparently implies $\hat{x}^{(k+\mathbf{2})} = \hat{x}^{(k)}$. Through this iterative procedure, it can be simply concluded that $\hat{x}^{(k)}$ is a limit point for the mentioned sequence that proves the statement of the theorem.

Case 2. Assume $\hat{x}^{(k+\mathbf{1})} \neq \hat{x}^{(k)}$ are parallel vectors that means $\hat{x}^{(k+\mathbf{1})} = \beta\hat{x}^{(k)}$ in which $\beta > 1$ due to $\hat{x}^{(k+\mathbf{1})} \neq \hat{x}^{(k)}$ and (27). This fact clearly implies that $\|\hat{x}^{(k)}\| < \|\hat{x}^{(k+1)}\|$ because $\beta > 1$.

Case 3. Assume $x^{(k+1)} \neq x^{(k)}$ are not parallel that implies $(\hat{x}^{(k)})^T \hat{x}^{(k+\mathbf{1})} < \|\hat{x}^{(k)}\| \, \|\hat{x}^{(k+1)}\|$.

Notice that the feasible region of IEOSP (3) is composed of the intersection of some ellipsoids with the same centers (at the origin), which in turn implies the feasible region is a compact space. The compactness of the feasibility region obviously contradicts the divergence of the sequence $\{\hat{x}^{(k)}\}_{k=0}^{\infty}$ considering inequality (27). In fact, cases 2 and 3 cannot always occur for all $k \in \{0, \ldots, \infty\}$ because $\hat{x}^{(k)}$ must stay inside a compact feasible region. □

Finally, Theorem 4.2 proves that the algorithm definitely converges to a local solution of the IEOSP (3) for each arbitrarily selected feasible point $\hat{x}^{(\mathbf{0})}$.

Theorem 4.2 *Assume the proposed algorithm converges to* $\hat{x}$. *If* $\hat{x}$ *is the unique optimal solution of problem (26) for* $\bar{x} = \hat{x}$ *and* $\|\hat{x}\| < 1$, *then* $\hat{x}$ *is a local solution of IEOSP (3).*

Proof Theorem 4.1 proves the convergence of sequence $\{\hat{x}^{(k)}\}_{k=0}^{\infty}$ that can be denoted by $\hat{x}$ in this proof. Due to the statement of the theorem, it suffices to prove the theorem for case $\|\hat{x}\| < 1$.

Suppose $\hat{x}$ is not the local solution of the IEOSP (3) which implies the existence of a unit vector $\hat{u} \in \mathbb{R}^{n+1}$ and a sufficiently small positive number $\varepsilon > 0$ holding next relations owing to the convexity of all constraints of IEOSP (3):

$$\forall t \in [0, \varepsilon] : \quad \hat{x} + t\hat{u} \in \hat{\mathcal{F}}, \tag{28}$$

$$\frac{d}{dt}(\|\hat{x} + t\hat{u}\|^2)|_{t=0} \geq 0 \Rightarrow \hat{x}^T \hat{u} \geq 0. \tag{29}$$

Using (28) and (29), we will have:

$$\forall t \in [0, \varepsilon]: \quad \hat{x} + t\hat{u} \in \hat{\mathcal{F}} \wedge \hat{x}^T \left(\hat{x} + t\hat{u}\right) \geq \hat{x}^T \hat{x}. \tag{30}$$

According to (30), point $\hat{x} + t\hat{u}$ for a given $t \in [0, \varepsilon]$ meets all constraints of problem (26) which contradicts the unique optimality of $\hat{x}$. This fact concludes $\hat{x}$ is the local solution of IEOSP (3). The proof is complete. □

Remark 4.1 Theorem 4.2 proves that the proposed algorithm reaches an acceptable point or asymptotically tends to a local solution of the IEOSP (3).

Corollary 4.1 *There exists a potential initial point such that the algorithm tends to an acceptable point of IEOSP (3) once it starts from this point.*

Proof Notice that the global solution $\hat{x}^*$ is also a local solution to the problem. If the algorithm starts at the global solution $\hat{x}^{(\mathbf{0})} = \hat{x}^*$, it stays at this point forever, which also belongs to the acceptable region $\mathcal{A}$. □

It is suggested to repeat the algorithm for different initial points that theoretically increases the chance of tending to an acceptable point of the IEOSP (3). This claim has been further investigated later in Sect. 5, numerical experiments.

4.3 Relaxation Analysis of the Proposed Algorithm

This subsection focuses on the intermediate optimal solutions $\left\{\hat{x}^{(k)}\right\}_{k=0}^{\infty}$ in order to investigate how much their inverse points $\left\{x^{(k)} = \mathcal{T}^{-1}\left(\hat{x}^{(k)}\right)\right\}_{k=0}^{\infty}$ violate constraints of QCFP (1). To follow this aim, Theorem 4.3 will prove that each point $x^{(k)}$ which is not a feasible solution of QCFP (1) meets all convex constraints (1b) and (1c) and only violates the non-convex constraints (1d) with a known upper bound.

Theorem 4.3 *Assume $\hat{x}^{(k)}$ for a given positive integer number, k is an intermediate optimal solution which is obtained by the proposed algorithm, and its inverse point $x^{(k)} = \mathcal{T}^{-1}\left(\hat{x}^{(k)}\right)$ is not a feasible solution of QCFP (1) (i.e. $(\hat{x}^{(k)})^T \hat{x}^{(k)} < 1$). Then, $x^{(k)}$ satisfies all convex constraints of QCFP (1) which are (1b) and (1c) and violates the non-convex constraints (1d) with the following upper bound for all $i \in \{1, \ldots, m_2\}$:*

$$\begin{bmatrix} x^{(k)} \\ 1 \end{bmatrix}^T \begin{bmatrix} Q_i & q_i \\ q_i^T & \mathfrak{q}_i \end{bmatrix} \begin{bmatrix} x^{(k)} \\ 1 \end{bmatrix} + 1 \leq \frac{\gamma - 1}{(\hat{x}_{n+1}^{(k)})^2} \left(1 - (\hat{x}^{(k)})^T \hat{x}^{(k)}\right). \tag{31}$$

Proof Since $\hat{x}^{(k)}$ is a feasible solution of IEOSP (3) and $x^{(k)} = \mathcal{T}^{-1}\left(\hat{x}^{(k)}\right)$, the following expressions yield:

$$Ax^{(k)} - b = \frac{1}{\sqrt{\gamma}\hat{x}_{n+1}^{(k)}} \left(\sqrt{\gamma - 1} A \left[\, I_n \;\; 0 \,\right] \hat{x}^{(k)} - \sqrt{\gamma} b \hat{x}_{n+1}^{(k)}\right) = \frac{1}{\sqrt{\gamma}\hat{x}_{n+1}^{(k)}} \hat{A}\hat{x}^{(k)} \leq 0, \tag{32}$$

$$\begin{bmatrix} x^{(k)} \\ 1 \end{bmatrix}^T \begin{bmatrix} P_i & p_i \\ p_i^T & \mathfrak{p}_i \end{bmatrix} \begin{bmatrix} x^{(k)} \\ 1 \end{bmatrix} + 1 =$$
$$\frac{\gamma - 1}{(\hat{x}_{n+1}^{(k)})^2} (\hat{x}^{(k)})^T \begin{bmatrix} \frac{1}{\sqrt{\gamma}} I_n & 0 \\ 0 & \frac{1}{\sqrt{\gamma - 1}} \end{bmatrix} \begin{bmatrix} P_i & p_i \\ p_i^T & \mathfrak{p}_i \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{\gamma}} I_n & 0 \\ 0 & \frac{1}{\sqrt{\gamma - 1}} \end{bmatrix} \hat{x}^{(k)} + 1 = \tag{33}$$
$$\frac{(\hat{x}^{(k)})^T \hat{P}_i \hat{x}^{(k)}}{\gamma (\hat{x}_{n+1}^{(k)})^2} \leq 0, \quad \forall i \in \{1, \ldots, m_1\}$$

$$\begin{bmatrix} x^{(k)} \\ 1 \end{bmatrix}^T \begin{bmatrix} Q_i & q_i \\ q_i^T & \mathfrak{q}_i \end{bmatrix} \begin{bmatrix} x^{(k)} \\ 1 \end{bmatrix} + 1 =$$
$$\frac{(\gamma - 1)\, (\hat{x}^{(k)})^T \begin{bmatrix} \frac{1}{\sqrt{\gamma}} I_n & 0 \\ 0 & \frac{1}{\sqrt{\gamma - 1}} \end{bmatrix} \begin{bmatrix} Q_i & q_i \\ q_i^T & \mathfrak{q}_i \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{\gamma}} I_n & 0 \\ 0 & \frac{1}{\sqrt{\gamma - 1}} \end{bmatrix} \hat{x}^{(k)} + (\hat{x}_{n+1}^{(k)})^2}{(\hat{x}_{n+1}^{(k)})^2} = \tag{34}$$
$$\frac{\gamma - 1}{(\hat{x}_{n+1}^{(k)})^2} \left((\hat{x}^{(k)})^T \hat{Q}_i \hat{x}^{(k)} - (\hat{x}^{(k)})^T \hat{x}^{(k)} \right) \leq 0, \quad \forall i \in \{1, \ldots, m_2\}.$$

Based on (31) and (32), point $x^{(k)}$ meets all convex constraints of QCFP (1). Also, expression (34) proves Eq. (31) in the statement of the theorem, which completes the proof. □

Therefore, the proposed algorithm reaches a solution that only relaxes the non-convex constraints of the original QCFP (1) with the approved upper bound.

5 Numerical Experiments

This section provides some examples to show how the proposed algorithm works and compares its properties with the existing methods. The examples are given in separate subsections to focus on their own aims.

All numerical examples are programmed in MATLAB as software (convex problems are solved by SDPT3 and Yalmip) and implemented in a dual-core (2.3GH each) CPU. The proposed method is compared to three groups of existing methods. In all comparison examples, two performances are considered, namely the mean computational time and the number of feasible QCFPs. The mean computational time refers to the meantime that lasts for each batch by each method. The number of feasible QCFPs stands for the number of feasible solutions finally obtained by each method.

5.1 An Illustrative Two-Dimensional QCFP

A two-dimensional QCFP has been considered in this subsection to present the algorithm and its related concepts in detail. The problem is about finding a point inside a set of two-dimensional convex or non-convex quadratic constraints that depend on an unknown parameter. For a better understanding, Fig. 1 schematically presents our illustrative QCFP in this subsection:

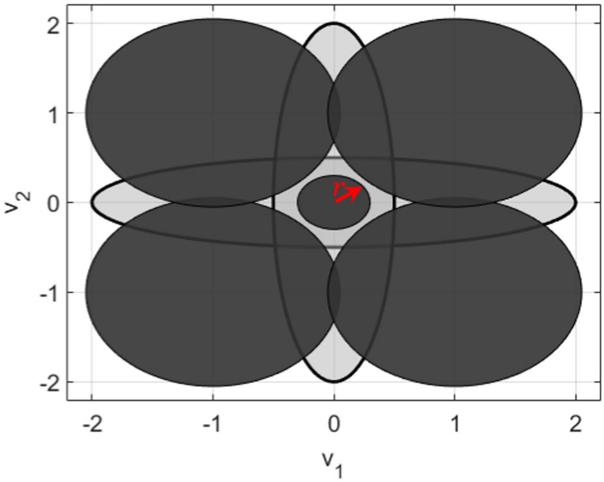

Fig. 1 The schematic presentation of the parametric QCFP, it is desired to find a point inside the light gray ellipsoids and outside the dark spheres (r is the radius of the center sphere)

In this problem, a point must be found inside the light gray ellipsoids and outside the dark spheres. Notice that parameter r indicates the radius of the center sphere, which can vary. In fact, the QCFP is parametric since its feasible region depends on the defined parameter r. This parametric QCFP, $P(r)$, is mathematically defined below:

Problem $P(r)$

$$\text{Find } x \in \mathbb{R}^2 \text{ such that:} \tag{35a}$$

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \le 0, \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} 0.25 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \le 0, \tag{35b}$$

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -0.8975 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \le 0, \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & -0.8975 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \le 0, \tag{35c}$$

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & -1 & -0.8975 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \le 0, \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ -1 & -1 & -0.8975 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \le 0, \tag{35d}$$

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & r^2 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \le 0. \tag{35e}$$

The feasible region of QCFP (35) for the specific value $r = 0.3$ is shown by Fig. 2.

As can be seen from Fig. 2, the feasible region of $P(0.3)$ is not a convex space. In the following, we numerically assess the smallest value of r_f in which QCFP $P\left(r_f\right)$

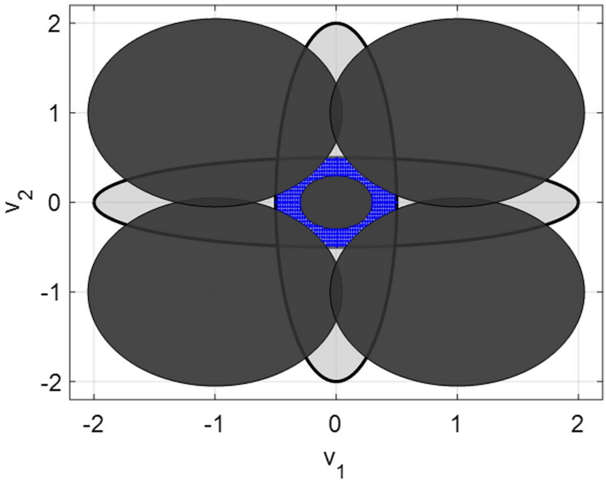

Fig. 2 The feasible region of the QCFP (35) for r = 0.3 (the blue region)

does not include any feasible solution, which means its corresponding feasible region becomes empty. The numerical results reveal that $r_f = 0.5$ that is graphically approved by Fig. 3.

In the following, the QCFP (35) is converted to the IEOSP (3). Then, the proposed algorithm is applied to the obtained IEOSP and finally reaches the local solution of the IEOSP based on Theorem 4.2. Figure 4 shows the trajectory of the proposed algorithm for various initial points and their related local solutions considering $r = 0.3$ (Note that the algorithm solves IEOSP, and the obtained solutions at the iterations are mapped into the space of the QCFP (35) based on inverse mapping $\mathcal{T}^{-1}$).

Figure 4 shows the convergence of some sequences of $\left\{\mathcal{T}^{-1}\left(x^{(k)}\right)\right\}_{k=0}^{\infty}$ defined in Sect. 3 that are obtained by the algorithm. This figure shows the local solutions of the IEOSP, which are also located inside the feasible region of QCFP with $r = 0.3$, $P\,(0.3)$.

In the following, the above procedure has been repeated for $r = 0.5$, and the obtained trajectories are shown in Fig. 5. The algorithm reaches the following local solutions whose inverse mapped points are feasible solutions of QCFP with $r = 0.3$, $P\,(0.5)$, which are $\left\{\begin{bmatrix}0.5\\0\end{bmatrix}, \begin{bmatrix}-0.5\\0\end{bmatrix}, \begin{bmatrix}0\\0.5\end{bmatrix}, \begin{bmatrix}0\\-0.5\end{bmatrix}\right\}$.

5.2 Comparison of the Proposed Algorithm with Relaxation Methods

This subsection provides a comparative example that evaluates the performance of our proposed algorithm with four relaxation methods. The relaxation methods are noted by SDP, SDP-RLT, SOC-RLT, and GSRT that are respectively adopted from previous studies [16], [5], [24], and [15].

These relaxation methods are generally developed for non-convex QCQPs. To fairly compare the methods, our feasibility problem QCFP (1) has been equivalently

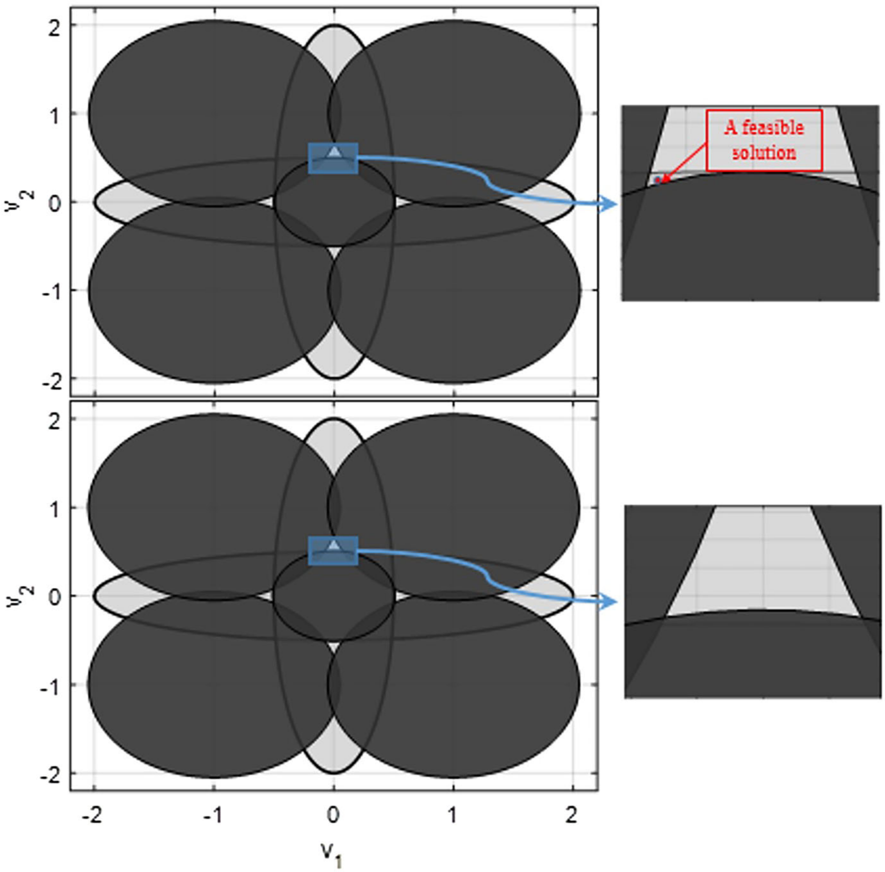

Fig. 3 The feasible regions of QCFP $P\,(0.5)$ and $P(0.51)$ shown at top and bottom

rewritten as the following non-convex QCQP:

$$\underset{x\in\mathbb{R}^n,\gamma\in\mathbb{R}}{\text{Minimize}}\ \gamma \tag{36a}$$

Subject to:

$$Ax \leq b + \gamma e, \tag{36b}$$

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T Q_i \begin{bmatrix} v \\ 1 \end{bmatrix} \leq \gamma, \quad \forall i \in \{1, \ldots, m\}. \tag{36c}$$

It can be easily proved that QCFP (1) is feasible if and only if the QCQP (36) has a non-positive optimal value; hence, we can solve either (36) or (1). At this stage, 200 different examples of the QCFP (1) with different numbers of variables, linear, and quadratic constraints are generated, and our proposed IEOSP solution algorithm along with the above-mentioned relaxation methods are individually applied to each of these 200 different examples. These QCFP examples are grouped into twenty batches

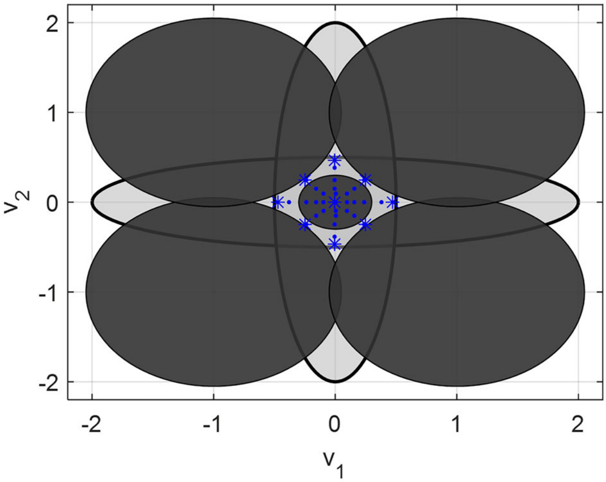

Fig. 4 The trajectory of consecutive solutions obtained by the proposed algorithm applied to P (0.3); 'o' stands for an intermediate solution (obtained by solving the problem (26)) and '*' stands for a local solution

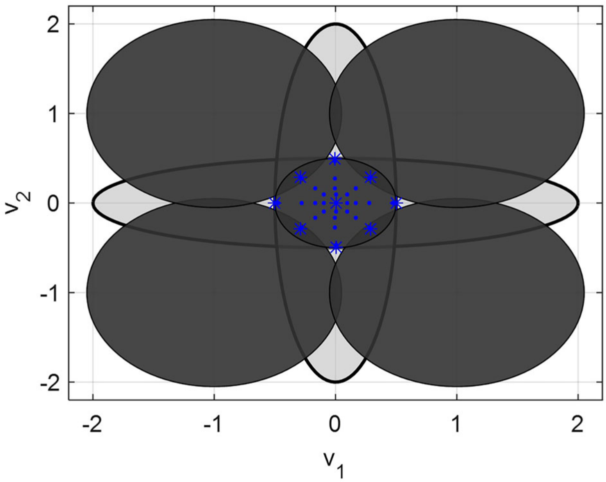

Fig. 5 The negative of optimal values of examined QCFP in which $n = 10$

of problems with the same problem size. In fact, each batch contains 10 QCFPs with the same number of variables and constraints (i.e., parameters n, m, and l are the same for each of these 10 QCFP batches, note that l is supposed to be equal to m).

The number of feasible cases and mean computational time are evaluated in each batch. Table 1 and Figs. 6 and 7 give information about the results. The detailed results are available in https://figshare.com/s/bd64bcbc9a031acf018e. Also, the source codes of the methods are available via the same link.

According to the above results, the following points should be highlighted:

Table 1 The number of feasible QCFPs solved by our proposed algorithm and benchmark relaxation methods SDP, RLT, SOC-RLT, and GSRT for each 10-QCFP batch; the mean computational times of these methods (in second) are given in parentheses

	Our proposed algorithm	SDP relaxation	SDP-RLT relaxation	SOC-RLT relaxation	GSRT relaxation
n = 3, m = 2	10 (0.90)	4(1.54)	4 (1.51)	4 (1.48)	5 (2.32)
n = 3, m = 4	10 (0.56)	3 (0.86)	4 (0.83)	4 (0.93)	4 (2.24)
n = 3, m = 6	10 (0.91)	3 (0.69)	3 (0.76)	3 (0.76)	3 (3.29)
n = 3, m = 8	10 (1.10)	2 (0.54)	2 (0.62)	2 (0.62)	2 (3.96)
n = 3,m = 10	10 (1.42)	5 (0.50)	5 (0.65)	5 (0.65)	5 (5.76)
n = 5, m = 2	10 (0.50)	1 (0.81)	4 (0.81)	4 (0.80)	7 (1.29)
n = 5, m = 4	10 (0.71)	0 (0.85)	0 (0.88)	0 (0.87)	3 (2.00)
n = 5, m = 6	10 (0.84)	1 (0.85)	1 (0.89)	1 (0.90)	1 (3.19)
n = 5, m = 8	10 (1.90)	0 (0.70)	0 (0.80)	0 (0.80)	0 (4.52)
n = 5,m = 10	10 (1.72)	2 (0.63)	2 (0.80)	2 (0.80)	2 (6.38)
n = 8, m = 2	10 (0.48)	1 (0.83)	2 (0.85)	2 (0.85)	4 (1.30)
n = 8, m = 4	10 (0.58)	0 (0.85)	0(0.87)	0 (0.88)	0 (2.28)
n = 8, m = 6	10 (1.39)	0 (0.87)	0 (0.93)	0 (0.93)	0 (3.80)
n = 8, m = 8	10 (0.85)	0 (0.88)	0 (0.98)	0 (0.98)	0 (5.98)
n = 8,m = 10	10 (0.75)	0(0.85)	0 (1.02)	0 (1.03)	0 (8.45)
n = 10, m = 2	10 (0.42)	3 (0.84)	3 (0.85)	3 (0.85)	4 (1.56)
n = 10, m = 4	10 (0.89)	0 (0.87)	0 (0.89)	0 (0.90)	3 (2.61)
n = 10, m = 6	10 (0.87)	0 (0.89)	0 (0.95)	0 (0.94)	2 (4.51)
n = 10, m = 8	10 (0.81)	0 (0.90)	0 (1.02)	0 (1.02)	0 (7.12)
n = 10,m = 10	10 (2.16)	0 (0.88)	0 (1.07)	0 (1.08)	0 (10.73)

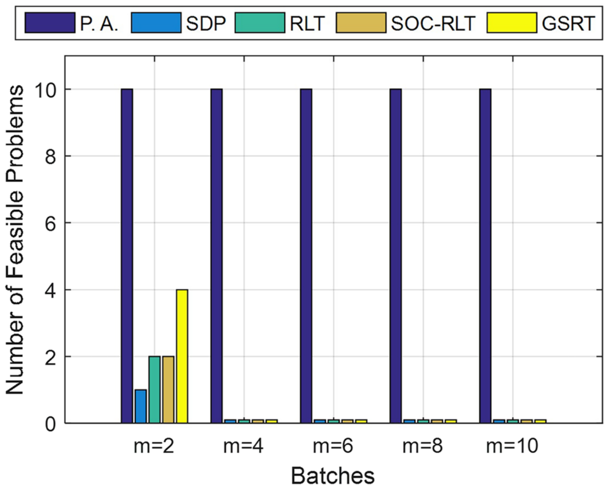

Fig. 6 The number of feasible QCFP instances from Table 1 for batches in which $n = 8$, P. A. stands for our Proposed Algorithm

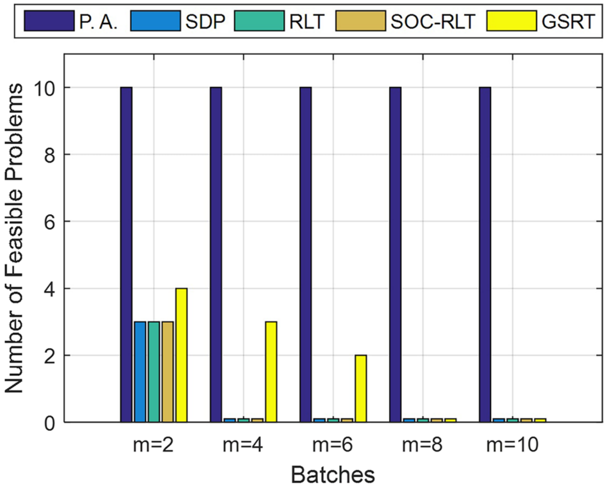

Fig. 7 The number of feasible QCFP instances from Table 1 for batches in which $n = 10$, P. A. stands for our Proposed Algorithm

1. Based on Table 1, our proposed algorithm solves all randomly generated QCFPs in all batches; however, the studied relaxation methods cannot successfully find feasible solutions for several QCFP cases.
2. The relaxation methods theoretically search a relaxed region, including the feasible region of the original QCFP. According to Figs. 6 and 7, the relaxation methods work better for QCFPs with a small number of variables and constraints since the QCFP relaxed and original feasibility regions are close enough for these small cases.
3. Based on these figures, the relaxation methods cannot find even one feasible solution for batches in which $n \geq 8$ and $m \geq 8$. Besides, the feasibility performance of our proposed algorithm does not depend on the sizes of the QCFP cases examined here.

5.3 Comparison of the Proposed Algorithm with Conservative Methods

This subsection compares our proposed algorithm with two convex conservative methods. The conservative methods convexify the non-convex problems, which theoretically means they try to find the largest convex sub-region of the non-convex problem. Exactly the same as in the previous subsection, 200 QCFPs are randomly generated (note that these 200 QCFPs differ from those generated in the previous subsection), and they are grouped into twenty batches of QCFPs with an equivalent number of variables and constraints. Then, our proposed algorithm and the two convexification methods, noted by convex-concave and PN-convex-concave methods that respectively adopted from previous studies [22] and [14], are implemented for each generated QCFP of each batch. The number of feasible QCFPs and mean computational times of these methods are computed and given in Table 2.

Table 2 The number of feasible QCFPs solved by our proposed algorithm and convexification methods convex-concave and PN-convex-concave for each 10-QCFP batch; the mean computational times of these methods (in second) are given in parentheses

	Our proposed algorithm	Convex-concave convexification	PN-convex-concave convexification
n = 3, m = 2	10 (0.75)	4 (1.15)	10 (0.53)
n = 3, m = 4	10 (1.41)	5 (0.49)	6 (0.54)
n = 3, m = 6	10 (0.87)	2 (0.52)	7 (0.51)
n = 3, m = 8	10 (1.75)	2 (0.58)	5 (0.52)
n = 3,m = 10	10 (2.35)	2 (0.59)	2 (0.55)
n = 5, m = 2	10 (0.65)	8 (0.40)	9 (0.40)
n = 5, m = 4	10 (0.72)	3 (0.52)	8 (0.45)
n = 5, m = 6	10 (0.83)	0 (0.67)	8 (0.49)
n = 5, m = 8	10 (1.32)	0 (1.04)	3 (0.57)
n = 5,m = 10	10 (1.20)	0 (0.66)	3 (0.56)
n = 8, m = 2	10 (0.65)	6 (0.50)	10 (0.37)
n = 8, m = 4	10 (0.85)	5 (0.96)	10 (0.45)
n = 8, m = 6	10 (1.02)	0 (0.68)	5 (0.50)
n = 8, m = 8	10 (1.41)	1 (0.72)	6 (0.48)
n = 8,m = 10	10 (1.90)	0 (0.72)	3 (0.56)
n = 10, m = 2	10 (0.64)	8 (0.47)	10 (0.37)
n = 10, m = 4	10 (0.68)	4 (0.95)	10 (0.44)
n = 10, m = 6	10 (0.91)	1 (0.66)	7 (0.51)
n = 10, m = 8	10 (1.35)	0 (0.72)	8 (0.54)
n = 10,m = 10	10 (1.75)	0 (0.75)	2 (0.58)

In the following, Fig. 8 shows the number of problem instances solved by our proposed IEOS-based algorithm and PN-convex-concave method in all batches.

According to the above results, the following points should be highlighted:

1. Based on Table 2, our proposed algorithm solves all generated QCFP cases in all batches; however, convexification methods are able to solve a few QCFPs.
2. According to Fig. 8, the feasibility performance of the PN-convex-concave convexification method depends on the number of constraints. This is completely consistent with the nature of this method since it is only able to search the largest convex region inside the feasible region of the QCFP.
3. The results of our proposed algorithm are not numerically affected by the size of the problem. This is an interesting feature of our approach, which makes it suitable for solving large QCFPs with numerous variables and constraints.

5.4 Comparison of the Proposed Algorithm with Branch-and-Bound Methods

This subsection compares the proposed algorithm with branch-and-bound methods. Exactly the same as the previous subsections, 100 QCFPs are randomly generated

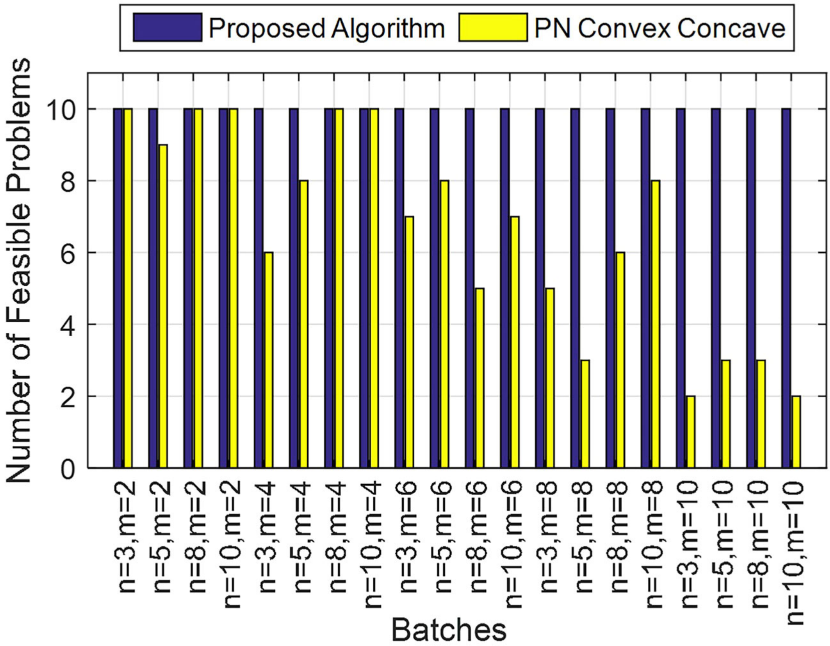

Fig. 8 The number of problem instances solved by our proposed algorithm and PN-convex-concave convexification for all batches

Table 3 The number of feasible QCFPs solved by the proposed algorithm and Baron for each 10-QCFP batch; the mean computational times of these methods (in second) are given in

	Proposed algorithm	Baron
n = 3, m = 1	10 (0.44)	7 (359.98)
n = 3, m = 2	10 (0.65)	6 (460.32)
n = 3, m = 3	10 (2.56)	4 (52.72)
n = 3, m = 4	10 (1.42)	3 (107.85)
n = 3, m = 5	10 (1.62)	4 (1.69)
n = 4, m = 1	10 (0.34)	4 (356.54)
n = 4, m = 2	10 (0.45)	5 (460.23)
n = 4, m = 3	10 (1.80)	4 (457.95)
n = 4, m = 4	10 (0.76)	5 (102.89)
n = 4, m = 5	10 (1.33)	2 (23.93)

(notice that these 100 QCFPs differ from those generated in the former subsections) and are grouped into twenty batches of QCFPs with the same number of variables and constraints. Then, our proposed algorithm and Baron solver are applied to these generated problems, where Baron inherently uses a branch-and-bound algorithm for solving QCFPs. The number of feasible QCFPs and mean computational times of Baron are calculated and given in Table 3:

The computational times of the Baron solver are much higher than the ones for our proposed algorithm. We could solve all ten examples in each batch using our proposed algorithm, but Baron solver has not achieved this. This promising performance of our proposed algorithm is mainly due to interesting mathematical properties, which are discussed comprehensively in the current paper.

6 Conclusions

In the current paper, we propose a new approach for solving general non-convex QCFPs. We introduced a mapping that converts the QCFP to an equivalent IEOSP. The equivalent IEOSP has a series of interesting mathematical properties that are theoretically discussed in our paper. The IEOSP has a convex and well-defined feasibility region, which makes it much easier to solve than the original QCFP. Then, an IEOS-based solution algorithm is developed to solve these IEOSPs. An illustrative example with 500 different QCFPs is numerically examined in our paper. All these numerical results confirm the promising performance of our solution algorithm. This work can be fruitfully expanded by applying our theoretical findings to solve real-life applications.

Appendix

Proof of Lemma 2.1

Proof It can be directly concluded that matrices $\{P_i\}_{i=1}^{m_1}$ are positive semi-definite since constraints (1c) are asssumed to be convex. In QCFP (1), constraints (1c) are supposed to be naturally quadratic, so they cannot be equivalently rewritten as linear constraints. It implies that matrices $\left\{\begin{bmatrix} P_i & p_i \\ p_i^T & \mathfrak{p}_i + 1 \end{bmatrix}\right\}_{i=1}^{m_1}$ are not positive semi-definite. It is because the positive semi-definiteness of matrix $\begin{bmatrix} P_i & p_i \\ p_i^T & \mathfrak{p}_i + 1 \end{bmatrix}$ for a given $i \in \{1, \ldots, m_1\}$ and constraint (1c) conclude that $\begin{bmatrix} P_i & p_i \\ p_i^T & \mathfrak{p}_i + 1 \end{bmatrix}\begin{bmatrix} x \\ 1 \end{bmatrix} = 0$ for all feasible points of QCFP (1) like $x \in \mathcal{F}$. This means the i^{th} convex constraint of (1c) can be exactly reformulated as linear equalities, which contradicts the quadratic nature of these constraints.

Now, assume $p_i^T V_i S_i^{-1} V_i^T p_i \leq \mathfrak{p}_i + 1$ for a given $i \in \{1, \ldots, m_1\}$, then, the following relations in (37) and (38) will be obtained based on Schur lemma:

$$P_i \geq 0 \wedge p_i^T V_i S_i^{-1} V_i^T p_i \leq \mathfrak{p}_i + 1 \Rightarrow \begin{bmatrix} S_i & V_i^T p_i \\ p_i^T V_i & \mathfrak{p}_i + 1 \end{bmatrix} \geq 0, \tag{37}$$

$$\begin{bmatrix} S_i & V_i^T p_i \\ p_i^T V_i & \mathfrak{p}_i + 1 \end{bmatrix} = \begin{bmatrix} V_i & 0 \\ 0 & 1 \end{bmatrix}^T \begin{bmatrix} P_i & p_i \\ p_i^T & \mathfrak{p}_i + 1 \end{bmatrix} \begin{bmatrix} V_i & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} P_i & p_i \\ p_i^T & \mathfrak{p}_i + 1 \end{bmatrix} \geq 0. \tag{38}$$

Equation (38) contradicts the former fact that matrix $\begin{bmatrix} P_i & p_i \\ p_i^T & \mathfrak{p}_i + 1 \end{bmatrix}$ is not positive semi-definite. Thus, $\mathfrak{p}_i + 1 < p_i^T V_i S_i^{-1} V_i^T p_i$. □

Converting a QCQP to the Equivalent QCFP

Consider the following general optimization form of QCQP:

$$\underset{x\in\mathbb{R}^n}{\text{Minimize}} \begin{bmatrix} x \\ 1 \end{bmatrix}^T Q_0 \begin{bmatrix} x \\ 1 \end{bmatrix} \tag{39a}$$

Subject to:

$$Ax \leq b, \tag{39b}$$

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T Q_i \begin{bmatrix} x \\ 1 \end{bmatrix} \leq 0, \quad \forall i \in \{1, \dots, m\} \tag{39c}$$

where $x \in \mathbb{R}^n$ is the vector of decision variables, $A \in \mathbb{R}^{l\times n}$, $b \in \mathbb{R}^l$, and set $\{Q_i\}_{i=0}^m \subset \mathbb{R}^{(n+1)\times(n+1)}$ includes symmetric matrices that generally are neither positive-definite nor negative-definite. Now, assume γ is a given number that plays the role of optimal value in problem (39). Using this parameter, QCQP (39) can be equivalently rewritten as follows:

$$\underset{x\in\mathbb{R}^n,\gamma\in\mathbb{R}}{\text{Find}} \; x \in \mathbb{R}^n \tag{40a}$$

Such that:

$$Ax \leq b, \tag{40b}$$

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T Q_i \begin{bmatrix} x \\ 1 \end{bmatrix} \leq 0, \quad \forall i \in \{1, \dots, m\} \tag{40c}$$

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T Q_0 \begin{bmatrix} x \\ 1 \end{bmatrix} \leq \gamma. \tag{40d}$$

The feasibility problem (40) can be used to determine the optimal value of QCQP (39). In (40), parameter γ is supposed to be a given number, and the proposed algorithm tries to solve its equivalent IEOS model in order to find an acceptable point. If it reaches an acceptable point, the optimal value of the main QCQP (39) will certainly be less than γ. Therefore, a line search technique can be exploited to iteratively solve QCFP (40) for various values of γ to determine the optimal value of the original QCQP (39).

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Data Availability The data of the constructed test instances can be found in https://figshare.com/s/bd64bcbc9a031acf018e and is publicly available.

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