Supersymmetry of Scattering Amplitudes and Green Functions in Perturbation Theory

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Dipl.-Phys. Jürgen Reuter

aus Frankfurt am Main

Referent: Prof. Dr. P. Manakos
Korreferent: Prof. Dr. N. Grewe

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Cerro de la Estrella

Aquí los antiguos recibían al fuego
Aquí el fuego creaba al mundo
Al mediodía las piedras se abren como frutos
El agua abre los párpados
La luz resbala por la piel del día
Gota inmensa donde el tiempo se refleja y se sacia [...]

Octavio Paz, *"Libertad Bajo Palabra"*

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Chapter 1

Introduction and Motivation

Although it is a theory of the utmost accuracy and success, the Standard Model (SM) of elementary particle physics cannot describe Nature up to arbitrarily high energy scales and therefore is not the last answer on our way in uncovering Nature's secrets. Today we look upon the SM as merely an effective field theory which is described by a local, causal quantum field theory up to an energy scale yet unknown, but assumed to lie at about 10^{15} GeV. Though all experimental data available today are in perfect agreement with the description of Nature by the Standard Model, there are some loose ends in the framework of the SM from which we mention just one, the so called *naturalness* or *hierarchy* problem. If the breaking of the electroweak gauge symmetry is provided by an elementary scalar getting a vacuum expectation value, the mass of that scalar, the Higgs boson, should be of the order of the electroweak breaking scale. Typically, the radiative corrections to the mass square of a scalar are proportional to the square of the energy scale at which its quantum field theory is embedded in a more fundamental theory, candidates for which being the Planck scale, a GUT or a string scale of the order given above or higher. This is not the case for fermions which receive only logarithmic corrections. An immense fine tuning for the bare mass of the Higgs scalar at the scale of the more fundamental theory is therefore necessary to cancel the quadratic contributions from the renormalization group flow. If we did not have these cancellations, the "natural" mass square of the Higgs scalar at the electroweak breaking scale would be of the order of the square of the high scale; this is called the naturalness problem. The hierarchy problem means the sheer existence of the vast differences between the two energy scales.

A possible solution of the naturalness problem serves as the strongest motivation for supersymmetry. Supersymmetry is a symmetry which interchanges bosons and fermions and could therefore naturally explain the existence of light scalars. In the supersymmetric limit each fermion loop contributing to the quadratically divergent Higgs self-energy is accompanied by a scalar loop with the opposite sign. Furthermore the coupling constants are required to be equal by supersymmetry, hence the quadratic divergence cancels out and only the logarithmic survives. As a second motivation we may mention gauge coupling unification which is compatible with current data only in supersymmetric extensions of the Standard Model but not in the SM itself. Hence, in spite of technicolour models – theories where the Higgs is a composite object – and models with extra dimensions (whether "large" or not) as competitors, supersymmetric extensions of the SM are the most widely accepted of the hypothesized models beyond the Standard Model.

After the first supersymmetric models had been established in the early 1970s

[1], phenomenology started and supersymmetric extensions of the SM have been constructed, e.g. see the reviews given in [2], [3]. The simplest of these extensions is called the *Minimal Supersymmetric Standard Model* (MSSM), where the predicate “minimal” stands for minimal field content: Each SM field is embedded into a superfield where the SM fermions are accompanied by scalars, the gauge bosons by fermions, called gauginos, and the Higgs bosons also by fermionic superpartners. Moreover, the constraint of being supersymmetric forces the existence of at least two Higgs superfields, one with hypercharge $+1$ and one with hypercharge -1, to give mass to the up- as well as down-type fermions; the appearance of two Higgs doublets is necessary also to avoid anomalies.

Therefore the prediction of supersymmetry is the existence of superpartners for all yet known SM particles. Since they are constrained by SUSY to have the same masses as the SM particles but have not been observed yet, supersymmetry has to be broken. Until today the mechanism of supersymmetry breaking is unknown, so we parameterize our ignorance by the most general explicit breaking of supersymmetry, the so called *soft breaking terms*. They are motivated by the fact that SUSY has to be broken by a whatsoever mechanism at a high scale, producing these explicit breaking terms by the renormalization group evolution of all relevant operators compatible with all symmetries. Though SUSY is a very simple concept and an enormously powerful symmetry, in addition to the huge number of particles, these soft breaking terms make the MSSM tremendously complicated as all particles which are by their quantum numbers allowed to mix really do mix. Also the pure number of free parameters in the MSSM becomes one order of magnitude higher as in the SM, namely 124 [4], or even, in a more general version, 178 [3], [5].

Another issue is the incredible number of vertices considering all Feynman rules of the MSSM (cf. tables B.3, B.4, B.5, B.6) and the sometimes very complex structure of the coupling constants, [6], [7], and [5]. There are some simplifying assumptions for the structure of the coefficients of the soft breaking terms (e.g. *flavour alignment* or *universality*) which are motivated by supergravity embeddings of the MSSM, but need not be fulfilled. One can steer a middle course as a compromise for the model: as general as possible, but as simple as necessary. We choose coefficients which are diagonal in generation space (actually, the generation mixings must be very small not to contradict the experimental thresholds for violation of the separate lepton numbers L_e, L_μ, L_τ) but the diagonal elements need not be equal in contrast to the prejudice given by the universality constraint. The number of vertices in tables B.3, B.4, B.5 and B.6 has been estimated under this assumption, but even as this is not the most complex of the “minimal” MSSMs, it has a discouraging number of more than four thousand vertices.

Today’s generation of running and planned colliders (Tevatron, LHC, and TESLA) will bring the decision which way Nature has chosen for electroweak symmetry breaking (cf. e.g. [8]). But even if a Higgs boson is detected at one of the world’s huge colliders in the next years, it will not be easy to determine whether it is a “standard”, a minimal supersymmetric, a next to minimal supersymmetric one [9], [10], [11], or something else. For this, extensive knowledge about the alternatives to the SM must be available, and besides the ubiquitous radiative corrections (within the SM, the MSSM and other models), it is indispensable to calculate tree level processes with up to eight particles in the final state, as in highly energetic processes ($10^2 - 10^3$ GeV for the colliders above) the final states are very complex. (The interest in eight final particles comes from the desire to study $WW \to WW$ scattering, the inclusion of the $WWWW$-vertex in eight-fermion production processes, production of $t\bar{t}$-pairs and their decays as well as the production of superpartners and SUSY cascade decays.) Of course, such calculations with $10^4 - 10^8$ participating Feynman diagrams have to be done automatically by matrix element generators like *O'Mega* [12]. Alternative models to the SM have therefore to be incorporated into such matrix element generators as the SM was. The goal for the next years will be to compare possibly found experimental deviations from the SM predictions with the theoretical results from alternative models like the MSSM.

As it soon becomes clear, the work is not done by simply writing a model file for the MSSM to incorporate it in an matrix element generator like *O'Mega*. Since the complexity of the model grows immensely from the SM to the MSSM (compare tables B.1-B.2 with tables B.3-B.6) it is inevitable to check the consistency of such models like the MSSM. This is necessary for making sure that all parameters (masses, coupling constants, widths, etc.) are compatible with each other, to debug computer programs (model files, numerical function library, etc.), and not to forget, to have the numerical stability under control. Symmetry principles which have always been strong concepts in physical theories provide such tests for consistency checks here. The MSSM like the SM has its $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge symmetry as a powerful tool for those checks; what is often used is the independence of all physical results from the gauge parameter ξ in general R_ξ gauges. Our aim is to make use of the Ward, or better, the Slavnov-Taylor identities of the gauge symmetry [13], [14], [15], [16]. Both kinds of identities originate from the quantum generalization of the symmetry principle of the classical field theory, the first expressing current conservation and being only valuable in the case of global symmetries, the latter stemming from the BRST symmetry left over after gauge fixing.

In supersymmetric field theories we can, of course, use supersymmetry as the underlying symmetry, and there, as long as we are not concerned with local supersymmetry (supergravity), we are able to employ Ward identities. As we will see for supersymmetric gauge theories it is indispensable – even at tree level – to use the Slavnov-Taylor identities. The stringency of the consistency checks is also a drawback: the relations mentioned as vehicles for those tests are quite complicated and involve a number of sophisticated techniques. As a first and fundamental step, extensive knowledge about

Process	# *O'Mega* fusions	# Propagators	# Diagrams
$e^+e^- \to \tilde\chi_1^0\tilde\chi_2^0$	24	8	8
$e^+e^- \to \tilde e_1^+\tilde e_1^-$	27	9	9
$e^+e^- \to \tilde u_1\tilde u_1^*\tilde u_1\tilde u_1^*$	346	41	660
$e^+e^- \to e^+e^-\tilde\chi_1^0\tilde\chi_2^0$	610	60	1,552
$e^+e^- \to \tilde\chi_1^0\tilde\chi_2^0\tilde\chi_3^0\tilde\chi_4^0$	782	66	2,208
$e^+e^- \to \tilde e_1^+\tilde e_1^-\tilde u_1\tilde u_1^*\tilde u_1\tilde u_1^*$	4,002	153	141,486
$e^+e^- \to e^+e^-\mu^+\mu^-\tilde\chi_1^0\tilde\chi_2^0$	4,389	172	239,518
$e^+e^- \to e^+e^-\tilde\chi_1^0\tilde\chi_2^0\tilde\chi_3^0\tilde\chi_4^0$	11,870	280	1,056,810
$e^+e^- \to \tilde\chi_1^0\tilde\chi_1^0\tilde\chi_2^0\tilde\chi_2^0\tilde\chi_3^0\tilde\chi_4^0$	17,075	322	2,191,845
$e^+e^- \to e^+e^-\mu^+\mu^-u\bar u\tilde\chi_1^0\tilde\chi_2^0$	23,272	434	50,285,616
$e^+e^- \to \tilde\chi_1^0\tilde\chi_1^0\tilde\chi_2^0\tilde\chi_2^0\tilde\chi_3^0\tilde\chi_3^0\tilde\chi_4^0\tilde\chi_4^0$	273,950	1,370	470,267,024

Table 1.1: *Juxtaposition of the number of Feynman diagrams and of* O'Mega *fusions for some MSSM processes at a linear collider. By fusions we mean the fundamental calculational steps for constructing the amplitudes in* O'Mega.

how Ward- and Slavnov-Taylor identities for supersymmetric (gauge) field theories work *analytically* (in perturbation theory) has to be gained to use such identities for numerical checks. This will be the concern for the major part of this thesis; first of all, the investigation of the applicability (on-shell and/or off-shell, what kind of method for which model) of the several kinds of methods to be presented here, furthermore – and even more important – to understand the way the cancellations happen in these identities. The latter point is inevitable in deciding which expressions to use in numerical checks: expressions adjusted to the technical nature of cancellations are likely to be numerically more stable than those which are not. A third and last issue then is to transfer these analytical expressions to the matrix element generator and perform numerical checks. Since it is not possible to produce reliable theoretical predictions for future experiments without having powerful consistency checks at hand, and since such consistency checks cannot be under (numerical) control without a deeper understanding of how they work analytically, the original intention of this work has changed: from a purely phenomenological issue at the beginning – to implement realistic supersymmetric models as alternatives to the Standard Model into the matrix element generator *O'Mega* – to a more theoretical one – to develop stringent tests as consistency checks for these models and to understand their fine points in detail. We hope to have convinced the reader that the latter is the *sine qua non* for the first. Thus the main part of this thesis is concerned with analytical perturbative calculations of three different kinds of identities within several models, to our knowledge never been done before. Let us briefly summarize the content of this thesis.

1.1 Structure and Content

After a short introduction to supersymmetry transformations, the main text is divided into four parts, the first showing a method to gain on-shell Ward identities for supersymmetric field theories originally invented in the late 1970s by Grisaru, Pendleton and van Nieuwenhuizen but as far as we know this method has never been used diagrammatically. We investigate that kind of Supersymmetric Ward Identities (SWI) for the Wess-Zumino model and a more complex toy model to uncover some new effects. As this formalism relies on the annihilation of the vacuum by the supercharge, it does not work for spontaneously broken supersymmetry. We provide an example within the framework of the O'Raifeartaigh model.

The second part is concerned with SWI constructed from Green functions with one current insertion and contracted with the momentum brought into the Green function by the current. At tree level these identities are fulfilled on-shell and off-shell. For the latter the SWI are more complicated due to the contributions of several "contact terms" and provide more stringent tests than the on-shell identities. Examples are calculated for the Wess-Zumino model, the toy model from part one and for the O'Raifeartaigh model, as the supersymmetric current is still conserved for spontaneously broken SUSY. It will be shown that this method does not work for supersymmetric gauge theories. The explanation of this phenomenon then blends over to the next part.

There we introduce the BRST formalism for supersymmetric theories where supersymmetry as a global symmetry is quantized with the help of constant ghosts, [17], [18]. In order not to cloud the intricacies by a huge amount of fields and diagrams, we construct the simplest possible supersymmetric Abelian toy model. We summarize the BRST transformations with inclusion of supersymmetry and translations and show several examples of supersymmetric Slavnov-Taylor identities in that toy model and also in supersymmetric QCD.

In the last part we discuss the problems concerned with the implementation of supersymmetric models and the consistency checks mentioned above. Connected with supersymmetric field theories is the appearence of Majorana fermions – real fermions – which are their own antiparticles. The solution of how to let the matrix element generator evaluate the signs coming from Fermi statistics without expanding the Feynman diagrams is presented based on ideas in [19]. Furthermore it is presented there how one- and two-point vertices arising together with the BRST formalism can be handled within *O'Mega*, though their topologies are not compatible with the way the amplitudes are built by *O'Mega*. It is demonstrated that Slavnov-Taylor identities for gauge symmetries and supersymmetry can be done within the same framework. Finally we will give an outlook of what remains to be done in that field, possible generalizations and improvements.

Chapter 2

SUSY Transformations

2.1 Classical transformations

First of all, we want to summarize the supersymmetry transformations of classical fields; as a general reference for the basics of supersymmetry we mention the book of Julius Wess and Jonathan Bagger, *Supersymmetry and Supergravity* [1]. By contraction with a fermionic (i.e. Grassmann odd) spinor transformation parameter we make the supercharges bosonic

$$Q(\xi) \equiv \xi Q + \bar{\xi}\bar{Q} \tag{2.1}$$

The component fields of a chiral multiplet, the scalar field ϕ, the Weyl-spinor field ψ and the scalar auxiliary field F with dimension two undergo the following transformations generated by the supercharge $Q(\xi)$ (cf. the appendix as well)

$$\begin{aligned} \delta_\xi \phi &= \sqrt{2}\xi\psi \\ \delta_\xi \psi &= -\mathrm{i}\sqrt{2}\sigma^\mu \bar{\xi}\partial_\mu \phi + \sqrt{2}\xi F \\ \delta_\xi F &= -\mathrm{i}\sqrt{2}\bar{\xi}\bar{\sigma}^\mu \partial_\mu \psi \end{aligned} \tag{2.2}$$

Compared to the book of Wess/Bagger the relative signs in the last two transformations have their origin in the different convention for the metric used by Wess/Bagger. This causes differences in the definition of the 4-vector of the Pauli matrices.

Because $Q(\xi)$ is real (Hermitean as a generator for quantum fields), the transformation properties of a field imply the properties of the complex conjugated field. One simply has to define:

$$(\delta_\xi \Psi)^* = \delta_\xi \Psi^* \quad , \tag{2.3}$$

This is the natural choice for a real generator. The relation will still be fulfilled in the quantized calculus.

Better suited for our aim – application of SUSY transformations in a phenomenological particle physics context – will be a formulation of the transformation rules with bispinors. Therefore we reformulate the transformations given above in this formalism. We also split the lowest and the highest components of the superfields into their scalar and pseudoscalar parts, called "chiral". This will prove useful later.

$$\phi = \frac{1}{\sqrt{2}}(A + \mathrm{i}B), \qquad F = \frac{1}{\sqrt{2}}(\mathcal{F} - \mathrm{i}\mathcal{G}). \tag{2.4}$$

The resulting transformations are:

$$\boxed{\begin{aligned}&\delta_\xi A = \bar{\xi}\psi,\\ &\delta_\xi B = \mathrm{i}\bar{\xi}\gamma^5\psi,\\ &\delta_\xi \psi = -\mathrm{i}\partial\!\!\!/\left(A + \mathrm{i}\gamma^5 B\right)\xi + \left(\mathcal{F} + \mathrm{i}\gamma^5\mathcal{G}\right)\xi,\\ &\delta_\xi \mathcal{F} = -\mathrm{i}\,\bar{\xi}\partial\!\!\!/\psi,\\ &\delta_\xi \mathcal{G} = -\bar{\xi}\partial\!\!\!/\gamma^5\psi.\end{aligned}} \tag{2.5}$$

In this list all spinors are understood as bispinors. For the translation of the "fundamental" component fields to the "chiral" fields we refer to section 2.3.

2.2 SUSY transformations in Hilbert space

The following discussion should prevent the confusion with factors i and signs when talking about SUSY transformations on the classical level and in the context of quantum field theory. Classically we review the results of the last section:

$$\begin{aligned}\delta_\xi\phi \equiv \left(\xi Q + \bar{\xi}\bar{Q}\right), \qquad \left(\xi Q + \bar{\xi}\bar{Q}\right)^* = \xi Q + \bar{\xi}\bar{Q}\\ \delta_\xi\phi^* = \left(\delta_\xi\phi\right)^* = \left(\xi Q + \bar{\xi}\bar{Q}\right)\phi^* \quad ,\end{aligned} \tag{2.6}$$

wherein ϕ could be a field of any geometrical character and any Grassmann parity.

In the quantum theory the transformation is represented by a unitary operator, which is created by exponentiation of the supercharge – now a Hermitean generator – multiplied with i:

$$[\mathrm{i}Q(\xi), \phi] = \delta_\xi\phi \tag{2.7}$$

Again ϕ is a field (operator) of arbitrary geometrical character and Grassmann parity. Moreover, $\delta_\xi\phi$ is the transformation of the classical fields incorporated into Hilbert space, i.e. the classical term, in which the fields have been replaced by operators acting in the Hilbert space. For the Hermitean adjoint one finds:

$$\begin{aligned}[\mathrm{i}Q(\xi), \phi]^\dagger &= \left[\mathrm{i}Q(\xi), \phi^\dagger\right]\\ &= \left(\delta_\xi\phi\right)^\dagger\\ &\Longrightarrow \boxed{\left[\mathrm{i}Q(\xi), \phi^\dagger\right] = \left(\delta_\xi\phi\right)^\dagger = \delta_\xi\phi^\dagger}\end{aligned} \tag{2.8}$$

There is no subtlety in dealing with fermionic fields here because the rule for reversing the order of Grassmann odd parameters classically is translated to the rule for reversing the order of field operators when Hermitean adjoined – no matter whether they are fermionic or bosonic. But one still has to take into account that Grassmann odd classical parameters like ξ and fermionic field operators have to be reversed in order when Hermitean adjoined.

Finally there is a simple rule for the embedding of the classical transformations into the quantum theory: Replace left multiplication with $Q(\xi)$ by application of the commutator with $\mathrm{i}Q(\xi)$.

2.3 General problems with auxiliary fields in supersymmetric field theories

As we will see, there is a possibility to implement SUSY Ward identities for theories with exact supersymmetry and an S-matrix invariant under SUSY transformations, by examining the transformation properties of the creation and annihilation operators of *in* and *out* states. For the extraction of the relations between amplitudes provided by supersymmetry, (in this ansatz) asymptotic fields (cf., for example, Kugo, [13]) have to be taken into account. The only important parts of the asymptotic fields are the one-particle poles, so we only have to keep those terms in the equations of motion of the auxiliary fields F and D which stem from the bilinear parts of the superpotential.

For example in the Wess-Zumino model we have:

$$\begin{aligned} F &= -m\phi^* - \frac{1}{2}\lambda(\phi^*)^2 \\ &= -\frac{m}{\sqrt{2}}\left(A - \mathrm{i}B\right) - \frac{\lambda}{4}\left(A - \mathrm{i}B\right)^2 \\ &\stackrel{!}{=} \frac{1}{\sqrt{2}}\left(\mathcal{F} - \mathrm{i}\mathcal{G}\right) \end{aligned} \tag{2.9}$$

Out of this we obtain the equations of motion for the auxiliary fields:

$$\boxed{\begin{aligned} \mathcal{F} &= -mA - \frac{\lambda}{2\sqrt{2}}\left(A^2 - B^2\right) \\ \mathcal{G} &= -mB - \frac{\lambda}{\sqrt{2}}AB \end{aligned}} \tag{2.10}$$

Off-shell there is no distinction possible between fields and auxiliary fields. The auxiliary fields are necessary to preserve the lemma stating that the number of bosonic and fermionic degrees of freedom has to be equal. For physical processes (with fields on the mass shell) one has to insert the equations of motion for the auxiliary fields. For the derivation of the S-matrix via the LSZ reduction formula all one-particle poles have to be accounted for. This implies further that in the equations of motion only the one-particle poles have to be kept. In the MSSM these poles exclusively appear in the mass terms (soft SUSY breaking terms) and the bilinear Higgs term, the latter also generating masses.

2.4 SUSY transformations of quantum fields

Finally, we are able to write down the SUSY transformations in Hilbert space for the chiral superfield:

$$\boxed{\begin{aligned} [\mathrm{i}Q(\xi), A] &= \overline{\xi}\psi, \\ [\mathrm{i}Q(\xi), B] &= \mathrm{i}\overline{\xi}\gamma^5\psi, \\ [\mathrm{i}Q(\xi), \psi] &= -\mathrm{i}\not{\partial}\left(A + \mathrm{i}\gamma^5 B\right)\xi + \left(\mathcal{F} + \mathrm{i}\gamma^5\mathcal{G}\right)\xi, \\ [\mathrm{i}Q(\xi), \mathcal{F}] &= -\mathrm{i}\overline{\xi}\not{\partial}\psi, \\ [\mathrm{i}Q(\xi), \mathcal{G}] &= -\overline{\xi}\not{\partial}\gamma^5\psi \end{aligned}} \tag{2.11}$$

Taking into account only the one-particle poles, e.g. in the Wess-Zumino model, yields:

$$\boxed{\begin{aligned}&[\mathrm{i}Q(\xi), A] = \overline{\xi}\psi,\\ &[\mathrm{i}Q(\xi), B] = \mathrm{i}\overline{\xi}\gamma^5\psi,\\ &[\mathrm{i}Q(\xi), \psi] = -\left(\mathrm{i}\partial\!\!\!/ + m\right)\left(A + \mathrm{i}\gamma^5 B\right)\xi\end{aligned}} \tag{2.12}$$

Part I

SUSY Ward identities for asymptotic states

Chapter 3

SUSY Ward Identities [SWI] for asymptotic fields

3.1 Consequences of SUSY for S-matrix elements

In supersymmetric field theories supersymmetry is a symmetry of the theory, meaning that the S-operator commutes with the supercharges: $[Q, S] = 0$. Later on we will see that in supersymmetric gauge theories the gauge fixing required for quantization breaks supersymmetry, with the result that the supercharge no longer commutes with the S-operator on the complete Hilbert space but only with the S-operator on the cohomology of the supercharge [18]. The S-operator maps the Hilbert space basis of asymptotic *in* states onto the one of the asymptotic *out* states. Therefore we immediately conclude that the *in* and *out* creation and annihilation operators have the same algebra, i.e. commutation relations with the supercharge Q. Remember that we are dealing at the moment with exact supersymmetry, so the vacuum is invariant under SUSY transformations and must be annihilated by the supercharge:

$$Q\,|0\rangle = 0. \tag{3.1}$$

At this point we mention some common grounds and some differences of supersymmetry and BRST symmetry. Both have in common that they are fermionic generators of global symmetries of the theory (we do not treat supergravity and local supersymmetry here) so there are some similarities between them. BRST transformations leave many more states of Hilbert space invariant (namely all physical states) than supersymmetry under which only the vacuum (and perhaps soliton solutions) are invariant. So for constructing relations between amplitudes of different processes we are (in case of supersymmetry) left with on-shell relations between S-matrix amplitudes whereas in BRST identities different off-shell Green functions can be compared. Later on we will bring SUSY and BRST together and derive the most general identities for supersymmetric gauge theories.

For the derivation of SWIs the following relation is the basic ingredient to start with:

$$\boxed{\begin{aligned} 0 &= \left\langle 0 \middle| \left[Q, a_1^{\mathrm{out}} \dots a_n^{\mathrm{out}} a_1^{\dagger\,\mathrm{in}} \dots a_m^{\dagger\,\mathrm{in}} \right] \middle| 0 \right\rangle \\ &= \sum_i \langle 0 | a_1^{\mathrm{out}} \dots \left[Q, a_i^{\mathrm{out}} \right] \dots | 0 \rangle + \sum_j \left\langle 0 \middle| a_1^{\mathrm{out}} \dots \left[Q, a_j^{\dagger\,\mathrm{in}} \right] \dots \middle| 0 \right\rangle \end{aligned}} \tag{3.2}$$

It follows, of course, from the invariance of the vacuum under SUSY transformations. So starting with a string of creation operators differing in spin by half a unit from the spin of the annihilation operators we get a sum of amplitudes for different processes where all incoming and outgoing particles are SUSY transformed successively. The creation and annihilation operators needed in the SWI of that kind have to be extracted from the field operators. An explanation for the way this is done will be given in the next section.

3.2 Projecting out creation and annihilation operators

In this section we only summarize the inverse Fourier transformations by which the creation and annihilation operators of excitations of a scalar or fermionic quantum field can be projected out with, following these prescriptions:

$$\boxed{\begin{aligned} a(k) &= \mathrm{i} \int d^3\vec{x}\; e^{\mathrm{i}kx} \overleftrightarrow{\partial}_t \phi(x) \\ b(k,\sigma) &= \int d^3\vec{x}\; \overline{u}(k,\sigma)\, \gamma^0 \psi(x) e^{\mathrm{i}kx} \\ d^\dagger(k,\sigma) &= \int d^3\vec{x}\; \overline{v}(k,\sigma)\, \gamma^0 \psi(x) e^{-\mathrm{i}kx} \end{aligned}} \tag{3.3}$$

In the first line we made use of the famous abbreviation:

$$\left(a \overleftrightarrow{\partial}_\mu b\right) \equiv a(\partial_\mu b) - (\partial_\mu a) b.$$

In the case of Majorana spinor fields, which are important in supersymmetric field theories, the last two equations are identical. The verification of (3.3) can be found in appendix C.1.

3.3 Transformations of creation and annihilation operators

As was discussed in the first section of this chapter for the derivation of the SWIs we need the SUSY transformation properties of the creation and annihilation operators. To derive them we go back to the so called “chiral” fields, ϕ and ϕ^*, which are now called ϕ_- and ϕ_+. We write down their definitions again:

$$\boxed{\phi_\pm \equiv \frac{1}{\sqrt{2}}\left(A \mp \mathrm{i}B\right)} \tag{3.4}$$

At this point, there is a difference in the choice of sign compared to the work of Grisaru, Pendleton and van Nieuwenhuizen [20].

Now we are – by the use of the SUSY transformations of the quantum fields and projecting the creation and annihilation operators out of the field operators – able to get the SUSY transformations of the ladder operators. First we discuss the transformations of creation and annihilation operators of the "chiral" scalar fields $\phi_\pm$, for which the notation $a^{(\dagger)}(k,\sigma), \sigma \equiv \pm$ is:

$$\begin{aligned}[Q(\xi), a(k,\sigma)] &= \mathrm{i}\int d^3\vec{x}\, e^{\mathrm{i}kx} \overleftrightarrow{\partial}_0 [Q(\xi), \phi_\sigma(x)] \\ &= -\frac{1}{\sqrt{2}}\int d^3\vec{x}\, e^{\mathrm{i}kx} \overleftrightarrow{\partial}_0 \left(\overline{\xi}(1+\sigma\gamma^5)\psi\right) \\ &= \frac{\mathrm{i}}{\sqrt{2}}\overline{\xi}(1+\sigma\gamma^5)\sum_\tau b(k,\tau)u(k,\tau) \end{aligned} \tag{3.5}$$

We find the transformation law

$$\boxed{[Q(\xi), a(k,\sigma)] = \frac{\mathrm{i}}{\sqrt{2}}\overline{\xi}(1+\sigma\gamma^5)\sum_\tau b(k,\tau)u(k,\tau)} \tag{3.6}$$

Consider a massless theory, where the spinors $u(k,\tau)$ und $v(k,\tau)$ are eigenstates of the matrix γ^5. We end up with the concise result:

$$[Q(\xi), a(k,\sigma)] = \sqrt{2}\mathrm{i}\,\overline{\xi}u(k,\sigma)\, b(k,\sigma) \tag{3.7}$$

Now we derive the transformation properties for the fermionic annihilation operators:

$$\begin{aligned}[Q(\xi), b(k,\sigma)] &= \int d^3\vec{x}\, \overline{u}(k,\sigma)\gamma^0 [Q(\xi), \psi(x)]\, e^{\mathrm{i}kx} \\ &= -\mathrm{i}\overline{u}(k,\sigma)\left(a_A(k) + \mathrm{i}\gamma^5 a_B(k)\right)\xi, \end{aligned} \tag{3.8}$$

where we have used the spinor $\overline{u}$'s equation of motion:

$$\overline{u}(p,\sigma)\,(\not{p} - m) = 0. \tag{3.9}$$

When using the chiral fields instead of the scalar and pseudoscalar ones, it follows:

$$\boxed{[Q(\xi), b(k,\sigma)] = -\frac{\mathrm{i}}{\sqrt{2}}\sum_\tau \left(\overline{u}(k,\sigma)\left(1-\tau\gamma^5\right)\xi\right) a(k,\tau)} \tag{3.10}$$

In the massless case the bispinor is again an eigenstate of the chiral projectors, so we find:

$$[Q(\xi), b(k,\sigma)] = -\sqrt{2}\mathrm{i}\,\overline{u}(k,\sigma)\xi a(k,\sigma). \tag{3.11}$$

We will derive the latter result in a more general context following the discussion of Grisaru and Pendleton [20] in section 3.5.

3.4 Anticommutativity, Grassmann numbers and Generators

There is a subtlety which may easily be overlooked, but without it, it is not possible to derive the SUSY transformations of the asymptotic creation operators.

For the quantization of field theories including fermions, Grassmann fields are being used, i.e. spinor fields whose components are Grassmann odd. This is necessary to fulfill the demands of the fermions having Fermi-Dirac statistics. Consider SUSY transformations which contain Grassmann odd constant spinors (as ξ above). Those parameters must *anti*commute with the Fermi fields. Consequently, spinor products normally being skew become symmetric between Fermi fields or between a Fermi field and such a Grassmann odd parameter (There are two signs when interchanging the two spinors in the product, one which causes the skewness of the product, namely the contraction direction of the spinor indices, but also another one from anticommuting the Grassmann numbers (cf. the appendix and [5])). In quantizing such a theory, the anticommutativity must be maintained when going from the classical Fermi fields to the field operators. Because – with the exception of the creation and annihilation operators (about which one could be tempted to assume that they only are responsible for the anticommutativity of fermions on Hilbert space) – there are only commuting terms in the field operators, we have to deduce that the creation and annihilation operators for fermions remain Grassmann odd with respect to "classical" Grassmann numbers. This means

$$\{\xi, b(k,\sigma)\} = \left\{\xi, b^\dagger(k,\sigma)\right\} = \{\xi, d(k,\sigma)\} = \left\{\xi, d^\dagger(k,\sigma)\right\} = 0, \tag{3.12}$$

which has noteworthy technical consequences.

What happens after taking the Hermitean adjoint of an equation like (3.6)? The left hand side yields:

$$\Big([Q(\xi), a(k,\sigma)]\Big)^\dagger = -\left[Q(\xi), a^\dagger(k,\sigma)\right] \tag{3.13}$$

Again we used the Hermiticity of $Q(\xi)$:

$$Q(\xi)^\dagger = \left(\xi Q + \bar{\xi}\bar{Q}\right)^\dagger = \bar{\xi}\bar{Q} + \xi Q = Q(\xi) \tag{3.14}$$

On the right hand side of (3.6) it has been taken into account that a Hermitean adjoint for operators includes complex conjugation of ordinary numbers and Grassmann numbers. The order of Grassmann numbers has to be reversed in complex conjugation:

$$(g_1 g_2 \dots g_n)^* = g_n^* \dots g_2^* g_1^* \qquad g_i \text{ Grassmann odd} \tag{3.15}$$

One therefore gets:

$$\begin{aligned}\left(\frac{\mathrm{i}}{\sqrt{2}}\bar{\xi}\left(1+\sigma\gamma^5\right)\sum_\tau u(k,\tau)b(k,\tau)\right)^\dagger &= -\frac{\mathrm{i}}{\sqrt{2}}\sum_\tau b^\dagger(k,\tau)u^\dagger(k,\tau)\left(1+\sigma\gamma^5\right)\gamma^0\xi \\ &= -\frac{\mathrm{i}}{\sqrt{2}}\sum_\tau b^\dagger(k,\tau)\overline{u}(k,\tau)\left(1-\sigma\gamma^5\right)\xi \\ &= +\frac{\mathrm{i}}{\sqrt{2}}\sum_\tau \overline{u}(k,\tau)\left(1-\sigma\gamma^5\right)\xi\, b^\dagger(k,\tau)\end{aligned} \tag{3.16}$$

In the last line we used (3.12). This finally produces the relation:

$$\boxed{\left[Q(\xi), a^\dagger(k,\sigma)\right] = -\frac{\mathrm{i}}{\sqrt{2}}\sum_\tau \overline{u}(k,\tau)\left(1-\sigma\gamma^5\right)\xi\, b^\dagger(k,\tau)} \quad . \tag{3.17}$$

Altogether there are three signs: One due to the Hermitean adjoint of the commutator, one by complex conjugation of the explicit factor i and a third one due to the anticommutativity of Fermi field operators and Grassmann numbers.

Another important difficulty about signs, related to the anticommutativity of Fermi field operators and Grassmann numbers, will be discussed in chapter 5.

3.5 General derivation of the transformations

When translating the identities of that kind introduced in the first section of this chapter into the graphical language of Feynman diagrams, we discover several subtleties concerning signs (a trade mark of supersymmetry), which seem to be confusing at the first sight. We discuss these specialties using an example with two incoming and two outgoing particles. Here we have two *in* creation operators and two *out* annihilation operators. With the abbreviation $c_\sigma(k_i)$ instead of $c(k_i, \sigma)$ for $c \equiv a, b$ this SWI reads:

$$\begin{aligned}
0 = &\left\langle 0\middle|\left[Q(\xi), a_-^{\text{out}}(k_3) b_+^{\text{out}}(k_4) a_-^{\dagger\,\text{in}}(k_1) a_-^{\dagger\,\text{in}}(k_2)\right]\middle|0\right\rangle \\
= &\left\langle 0\middle|\left[Q(\xi), a_-^{\text{out}}(k_3)\right] b_+^{\text{out}}(k_4) a_-^{\dagger\,\text{in}}(k_1) a_-^{\dagger\,\text{in}}(k_2)\middle|0\right\rangle \\
&+ \left\langle 0\middle| a_-^{\text{out}}(k_3) \left[Q(\xi), b_+^{\text{out}}(k_4)\right] a_-^{\dagger\,\text{in}}(k_1) a_-^{\dagger\,\text{in}}(k_2)\middle|0\right\rangle \\
&+ \left\langle 0\middle| a_-^{\text{out}}(k_3) b_+^{\text{out}}(k_4) \left[Q(\xi), a_-^{\dagger\,\text{in}}(k_1)\right] a_-^{\dagger\,\text{in}}(k_2)\middle|0\right\rangle \\
&+ \left\langle 0\middle| a_-^{\text{out}}(k_3) b_+^{\text{out}}(k_4) a_-^{\dagger\,\text{in}}(k_1) \left[Q(\xi), a_-^{\dagger\,\text{in}}(k_2)\right]\middle|0\right\rangle
\end{aligned} \tag{3.18}$$

With the help of the relations (3.6), (3.10) and (3.17) this can be transformed into:

$$\begin{aligned}
0 = &\frac{\mathrm{i}}{\sqrt{2}} \sum_\sigma \left\langle 0\middle| \left(\overline{\xi}\mathcal{P}_L u(k_3,\sigma)\right) b_\sigma^{\text{out}}(k_3) b_+^{\text{out}}(k_4) a_-^{\dagger\,\text{in}}(k_1) a_-^{\dagger\,\text{in}}(k_2)\middle|0\right\rangle \\
&- \frac{\mathrm{i}}{\sqrt{2}} \left\langle 0\middle| a_-^{\text{out}}(k_3) \left(\overline{u}(k_4,+)\mathcal{P}_L\xi\right) a_+^{\text{out}}(k_4) a_-^{\dagger\,\text{in}}(k_1) a_-^{\dagger\,\text{in}}(k_2)\middle|0\right\rangle \\
&- \frac{\mathrm{i}}{\sqrt{2}} \left\langle 0\middle| a_-^{\text{out}}(k_3) \left(\overline{u}(k_4,+)\mathcal{P}_R\xi\right) a_-^{\text{out}}(k_4) a_-^{\dagger\,\text{in}}(k_1) a_-^{\dagger\,\text{in}}(k_2)\middle|0\right\rangle \\
&- \frac{\mathrm{i}}{\sqrt{2}} \sum_\sigma \left\langle 0\middle| a_-^{\text{out}}(k_3) b_+^{\text{out}}(k_4) \left(\overline{u}(k_1,\sigma)\mathcal{P}_R\xi\right) b_\sigma^{\dagger\,\text{in}}(k_1) a_-^{\dagger\,\text{in}}(k_2)\middle|0\right\rangle \\
&- \frac{\mathrm{i}}{\sqrt{2}} \sum_\sigma \left\langle 0\middle| a_-^{\text{out}}(k_3) b_+^{\text{out}}(k_4) a_-^{\dagger\,\text{in}}(k_1) \left(\overline{u}(k_2,\sigma)\mathcal{P}_R\xi\right) b_\sigma^{\dagger\,\text{in}}(k_2)\middle|0\right\rangle
\end{aligned} \tag{3.19}$$

The sum in (3.10) has been split up so that there are five terms now. To separate the spinor bilinears produced by the SUSY transformations from the S-matrix elements, we bring all these factors to the utmost left. Be aware of picking up a sign in the last two lines by anticommuting the Grassmann odd spinor bilinear and the fermionic annihilator. One ends up with

$$\begin{aligned}
0 = &\frac{\mathrm{i}}{\sqrt{2}} \sum_\sigma \left(\overline{\xi}\mathcal{P}_L u(k_3,\sigma)\right) \left\langle 0\middle| b_\sigma^{\text{out}}(k_3) b_+^{\text{out}}(k_4) a_-^{\dagger\,\text{in}}(k_1) a_-^{\dagger\,\text{in}}(k_2)\middle|0\right\rangle \\
&- \frac{\mathrm{i}}{\sqrt{2}} \left(\overline{u}(k_4,+)\mathcal{P}_L\xi\right) \left\langle 0\middle| a_-^{\text{out}}(k_3) a_+^{\text{out}}(k_4) a_-^{\dagger\,\text{in}}(k_1) a_-^{\dagger\,\text{in}}(k_2)\middle|0\right\rangle \\
&- \frac{\mathrm{i}}{\sqrt{2}} \left(\overline{u}(k_4,+)\mathcal{P}_R\xi\right) \left\langle 0\middle| a_-^{\text{out}}(k_3) a_-^{\text{out}}(k_4) a_-^{\dagger\,\text{in}}(k_1) a_-^{\dagger\,\text{in}}(k_2)\middle|0\right\rangle \\
&+ \frac{\mathrm{i}}{\sqrt{2}} \sum_\sigma \left(\overline{u}(k_1,\sigma)\mathcal{P}_R\xi\right) \left\langle 0\middle| a_-^{\text{out}}(k_3) b_+^{\text{out}}(k_4) b_\sigma^{\dagger\,\text{in}}(k_1) a_-^{\dagger\,\text{in}}(k_2)\middle|0\right\rangle \\
&+ \frac{\mathrm{i}}{\sqrt{2}} \sum_\sigma \left(\overline{u}(k_2,\sigma)\mathcal{P}_R\xi\right) \left\langle 0\middle| a_-^{\text{out}}(k_3) b_+^{\text{out}}(k_4) a_-^{\dagger\,\text{in}}(k_1) b_\sigma^{\dagger\,\text{in}}(k_2)\middle|0\right\rangle
\end{aligned} \tag{3.20}$$

There is yet another source for producing signs, but it can only arise in the context of Dirac fermions – i.e. charged fermions. Anticommutation of fermionic annihilators and/or creators due to the Wick theorem is the origin of these additional sign factors; we will go into the details in chapter 5, which deals with a model in which Dirac fermions appear.

Now we want to revisit part of a general derivation of the SWIs in the formalism originally written down by M.T. Grisaru and H.N. Pendleton used to derive helicity selection rules in gravitino–graviton scattering [21]. Because the supercharges commute with the momentum operator and change the particles' spin by half a unit, we can derive the following relations for the *in* annihilators of particles with spin j and chirality σ in a supersymmetric theory:

$$\begin{aligned} [Q(\xi), a_j(k,\sigma)] &= \Delta_j(\xi,k,\sigma)\cdot a_{j-\frac{1}{2}}(k,\sigma), \\ \left[Q(\xi), a_{j-\frac{1}{2}}(k,\sigma)\right] &= \Delta_{j-\frac{1}{2}}(\xi,k,\sigma)\cdot a_j(k,\sigma). \end{aligned} \tag{3.21}$$

The momentum operator has the form:

$$P^\mu = \sum_\sigma \int d^3\vec{p}\,p^\mu \left(a_j^\dagger(p,\sigma)a_j(p,\sigma) + a^\dagger_{j-\frac{1}{2}}(p,\sigma)a_{j-\frac{1}{2}}(p,\sigma)\right). \tag{3.22}$$

From the fact that the supercharge and the momentum operator commute, an equation for the two unknown functions Δ_j, $\Delta_{j-\frac{1}{2}}$ on the right hand side can be deduced

$$\begin{aligned} [Q(\xi), P^\mu] = &\sum_\sigma \int d^3\vec{p}\,p^\mu \Big(a_j^\dagger(p,\sigma)\,[Q(\xi), a_j(p,\sigma)] + \left[Q(\xi), a_j^\dagger(p,\sigma)\right] a_j(p,\sigma) \\ &+ a^\dagger_{j-\frac{1}{2}}(p,\sigma)\left[Q(\xi), a_{j-\frac{1}{2}}(p,\sigma)\right] + \left[Q(\xi), a^\dagger_{j-\frac{1}{2}}(p,\sigma)\right] a_{j-\frac{1}{2}}(p,\sigma)\Big) \\ = &\sum_\sigma \int d^3\vec{p}\,p^\mu \Big(a_j^\dagger(p,\sigma)a_{j-\frac{1}{2}}(p,\sigma)\left(\Delta_j(\xi,p,\sigma) - \Delta^*_{j-\frac{1}{2}}(\xi,p,\sigma)\right) \\ &+ a^\dagger_{j-\frac{1}{2}}(p,\sigma)a_j(p,\sigma)\left(\Delta_{j-\frac{1}{2}}(\xi,p,\sigma) - \Delta_j^*(\xi,p,\sigma)\right)\Big) \\ \stackrel{!}{=} &\, 0 \\ \Longrightarrow &\, \Delta_{j-\frac{1}{2}}(\xi,p,\sigma) \;=\; \Delta_j^*(\xi,p,\sigma) \end{aligned} \tag{3.23}$$

Defining $\Delta_j \equiv \Delta$ (3.21) reads

$$\boxed{\begin{aligned} [Q(\xi), a_j(k,\sigma)] &= \Delta(\xi,k,\sigma)\cdot a_{j-\frac{1}{2}}(k,\sigma), \\ \left[Q(\xi), a_{j-\frac{1}{2}}(k,\sigma)\right] &= \Delta^*(\xi,k,\sigma)\cdot a_j(k,\sigma) \end{aligned}}\;, \tag{3.24}$$

to be compared with (3.7) and (3.11).

More relations can be gained from the Jacobi identity:

$$[[Q(\xi), Q(\zeta)], a_j(k,\sigma)] + [[Q(\zeta), a_j(k,\sigma)], Q(\xi)] + [[a_j(k,\sigma), Q(\xi)], Q(\zeta)] = 0 \tag{3.25}$$

This implies the equation:

$$\Delta(\zeta,k,\sigma)\cdot\Delta^*(\xi,k,\sigma) - \Delta(\xi,k,\sigma)\cdot\Delta^*(\zeta,k,\sigma) = 2\bar{\xi}\not{k}\zeta \quad . \tag{3.26}$$

As is shown in [21], the explicit form of these functions can be found in the context of special models. In the last section we derived them directly by projecting out the annihilators from the field operators. In a general model this procedure can become arbitrarily complicated, especially if one has a nondiagonal metric on the space of states or if unphysical modes are involved.

Chapter 4

The Wess-Zumino Model

We want to test the SUSY Ward identities of the kind derived in the last chapter for the Wess-Zumino (WZ) model. This is the simplest supersymmetric field theoretic model with just one superfield but the most general renormalizable superpotential. For details about the model, the particle content and the Feynman rules see appendix E.1.

4.1 SWI for the WZ model

We can use the formula (3.2) derived in the last chapter to check SWI in the WZ model. The starting point – similar to the derivation of the Slavnov-Taylor identities – is a string of field operators with half integer spin, which only by application of the symmetry generator (here the supercharge), becomes a physically possible (in particular non-vanishing) amplitude. First, we have to translate the formulae from the previous chapter to the physical fields of the WZ model – by this we mean the real and imaginary part of the complex scalar field ϕ or the scalar and pseudoscalar part, respectively.

To get the transformation properties of annihilators and creators of the real part A of the complex scalar field ϕ one has to set the term proportional to γ^5 in equation (3.6) equal to zero and to multiply the result by $\sqrt{2}$. For the imaginary part B one has to set the term proportional to unity equal to zero, to set σ equal to one and multiply the result by a factor $\sqrt{2}\mathrm{i}$. This results in:

$$[Q(\xi), a_A(k)] = \mathrm{i}\sum_{\sigma}\overline{\xi}u(k,\sigma)b(k,\sigma) \tag{4.1}$$

$$[Q(\xi), a_B(k)] = -\sum_{\sigma}\overline{\xi}\gamma^5 u(k,\sigma)b(k,\sigma) \quad . \tag{4.2}$$

For the transformation law of the fermion annihilator it suffices to use (3.8),

$$[Q(\xi), b(k,\sigma)] = -\mathrm{i}\overline{u}(k,\sigma)\Big(a_A(k) + \mathrm{i}\gamma^5 a_B(k)\Big)\xi \quad . \tag{4.3}$$

As an example, we take a transformation of a product of an *in* creation operator for one A and one B field, and *out* annihilators for an A field and a Majorana fermion of positive helicity. We therefore write relation (3.2) in the form

$$
\begin{aligned}
0 \overset{!}{=}\ & \left\langle 0 \middle| \left[Q(\xi), a_A^{\text{out}}(k_3) b^{\text{out}}(k_4,+) a_A^{\dagger\,\text{in}}(k_1) a_B^{\dagger\,\text{in}}(k_2) \right] \middle| 0 \right\rangle \\
=\ & \quad \left\langle 0 \middle| a_A^{\text{out}}(k_3) b^{\text{out}}(k_4,+) a_A^{\dagger\,\text{in}}(k_1) \left[Q(\xi), a_B^{\dagger\,\text{in}}(k_2) \right] \middle| 0 \right\rangle \\
& + \left\langle 0 \middle| a_A^{\text{out}}(k_3) b^{\text{out}}(k_4,+) \left[Q(\xi), a_A^{\dagger\,\text{in}}(k_1) \right] a_B^{\dagger\,\text{in}}(k_2) \middle| 0 \right\rangle \\
& + \left\langle 0 \middle| a_A^{\text{out}}(k_3) \left[Q(\xi), b^{\text{out}}(k_4,+) \right] a_A^{\dagger\,\text{in}}(k_1) a_B^{\dagger\,\text{in}}(k_2) \middle| 0 \right\rangle \\
& + \left\langle 0 \middle| \left[Q(\xi), a_A^{\text{out}}(k_3) \right] b^{\text{out}}(k_4,+) a_A^{\dagger\,\text{in}}(k_1) a_B^{\dagger\,\text{in}}(k_2) \middle| 0 \right\rangle
\end{aligned} \tag{4.4}
$$

This seems to relate the amplitudes of four different physical processes. But as the transformation of a fermionic annihilator produces a linear combination of annihilators for the scalar and pseudoscalar fields, A and B, respectively, we get indeed five different processes (here we adopt the convention that processes only differing by the helicity of a fermion are counted as *one* process).

$$
\begin{aligned}
0 \overset{!}{=}\ & -\sum_\sigma \overline{u}(k_2,\sigma)\gamma^5\xi \left\langle 0 \middle| a_A^{\text{out}}(k_3) b^{\text{out}}(k_4,+) a_A^{\dagger\,\text{in}}(k_1) b^{\dagger\,\text{in}}(k_2,\sigma) \middle| 0 \right\rangle \\
& + \mathrm{i} \sum_\sigma \overline{u}(k_1,\sigma)\xi \left\langle 0 \middle| a_A^{\text{out}}(k_3) b^{\text{out}}(k_4,+) b^{\dagger\,\text{in}}(k_1,\sigma) a_B^{\dagger\,\text{in}}(k_2) \middle| 0 \right\rangle \\
& - \mathrm{i}\, \overline{u}(k_4,+)\xi \left\langle 0 \middle| a_A^{\text{out}}(k_3) a_A^{\text{out}}(k_4) a_A^{\dagger\,\text{in}}(k_1) a_B^{\dagger\,\text{in}}(k_2) \middle| 0 \right\rangle \\
& + \overline{u}(k_4,+)\gamma^5\xi \left\langle 0 \middle| a_A^{\text{out}}(k_3) a_B^{\text{out}}(k_4) a_A^{\dagger\,\text{in}}(k_1) a_B^{\dagger\,\text{in}}(k_2) \middle| 0 \right\rangle \\
& + \mathrm{i} \sum_\sigma \overline{\xi} u(k_3,\sigma) \left\langle 0 \middle| b^{\text{out}}(k_3,\sigma) b^{\text{out}}(k_4,+) a_A^{\dagger\,\text{in}}(k_1) a_B^{\dagger\,\text{in}}(k_2) \middle| 0 \right\rangle
\end{aligned} \tag{4.5}
$$

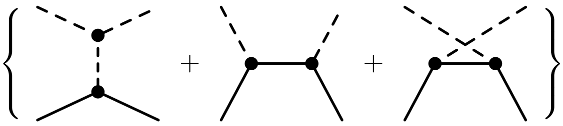
Note the double sign arising in the last two lines – as explained in section 3.1 – coming from a relative sign between the transformation properties of a creation and an annihilation operator and one from equation (3.17). With the help of the relation for S-matrix elements and amplitudes, which e.g. can be read off from [22], p. 105,

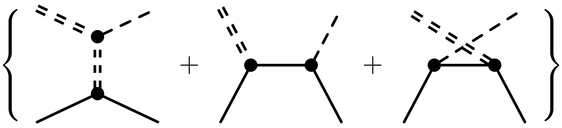
$$
\begin{aligned}
& \langle q_1 \dots q_n | S | p_1 \dots p_m \rangle_{\text{conn.}} = \\
& \qquad \mathrm{i}\mathcal{M}(p_1,\dots,p_m \longrightarrow q_1,\dots,q_n)\,(2\pi)^4 \delta^4\left(\sum_{i=1}^m p_i - \sum_{j=1}^n q_j \right) \quad ,
\end{aligned} \tag{4.6}
$$

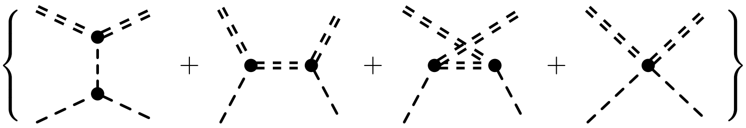
equation (4.5) can immediately be transferred into Feynman diagrams (omitting the overall factor i and also the delta function for global momentum conservation):

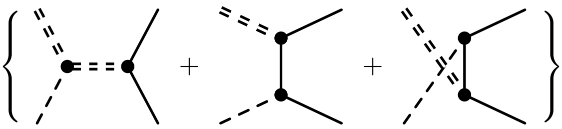
$$
\begin{aligned}
0 \overset{!}{=}\ & -\sum_\sigma \overline{u}(k_2,\sigma)\gamma^5\xi \cdot \mathcal{M}(A(k_1)\Psi(k_2,\sigma) \longrightarrow A(k_3)\Psi(k_4,+)) \\
& + \mathrm{i}\sum_\sigma \overline{u}(k_1,\sigma)\xi \cdot \mathcal{M}(\Psi(k_1,\sigma)B(k_2) \longrightarrow A(k_3)\Psi(k_4,+)) \\
& - \mathrm{i}\overline{u}(k_4,+)\xi \cdot \mathcal{M}(A(k_1)B(k_2) \longrightarrow A(k_3)A(k_4)) \\
& + \overline{u}(k_4,+)\gamma^5\xi \cdot \mathcal{M}(A(k_1)B(k_2) \longrightarrow A(k_3)B(k_4)) \\
& + \mathrm{i}\sum_\sigma \overline{\xi}u(k_3,\sigma) \cdot \mathcal{M}(A(k_1)B(k_2) \longrightarrow \Psi(k_3,\sigma)\Psi(k_4,+)) \quad .
\end{aligned} \tag{4.7}
$$

Diagrammatically we can write down the following expression:

$0 = -\sum_{\sigma} \overline{u}(k_2,\sigma)\gamma^5\xi \cdot$ 

$+\,\mathrm{i}\sum_{\sigma} \overline{u}(k_1,\sigma)\xi \cdot$  $+\ 0$

$+\,\overline{u}(k_4,+)\gamma^5\xi \cdot$ 

$+\,\mathrm{i}\sum_{\sigma} \overline{\xi}u(k_3,\sigma) \cdot$ 

(4.8)

For the calculation of the amplitudes it is useful to introduce the Mandelstam variables,

$$s = (k_1 + k_2)^2 = (k_3 + k_4)^2 \ , \tag{4.9}$$
$$t = (k_3 - k_1)^2 = (k_4 - k_2)^2 \ , \tag{4.10}$$
$$u = (k_4 - k_1)^2 = (k_3 - k_2)^2 \ . \tag{4.11}$$

The explicit analytical expressions for diagrams in which only scalar (or pseudoscalar) particles are involved are easily found and work in the same manner as in ϕ^4 theory or the Standard Model. For the diagrams with Majorana fermions the Feynman rules for general fermions worked out by Denner et al. [19] are needed.

The terms in braces yield the following analytical expressions, in the first line of (4.8)

$$-\frac{\mathrm{i}\lambda^2}{2}\,\overline{u}(k_4,+)\left(\frac{3m}{t-m^2} + \frac{\not{k}_1 + \not{k}_2 + m}{s-m^2} + \frac{\not{k}_2 - \not{k}_3 + m}{u-m^2}\right)u(k_2,\sigma) \quad , \tag{4.12}$$

in the second line

$$-\frac{\lambda^2}{2}\,\overline{u}(k_4,+)\left(\frac{m\gamma^5}{u-m^2} + \frac{(\not{k}_1 + \not{k}_2 + m)\gamma^5}{s-m^2} + \frac{\gamma^5(\not{k}_1 - \not{k}_3 + m)}{t-m^2}\right)u(k_1,\sigma) \quad . \tag{4.13}$$

The diagrams in the third line add up to

$$-\frac{\mathrm{i}\lambda^2}{2}\left(\frac{3m^2}{t-m^2} + \frac{m^2}{s-m^2} + \frac{m^2}{u-m^2} + 1\right) \quad , \tag{4.14}$$

and finally in the last line of (4.8):

$$-\frac{\lambda^2}{2}\,\overline{u}(k_4,+)\left(\frac{m\gamma^5}{s-m^2} + \frac{\gamma^5(\not{k}_4 - \not{k}_2 + m)}{t-m^2} + \frac{(\not{k}_4 - \not{k}_1 + m)\gamma^5}{u-m^2}\right)v(k_3,\sigma) \quad . \tag{4.15}$$

It proves to be more convenient for further simplification – remember that we still have to multiply the prefactors from equation (4.8) – to modify the analytical expression for the diagrams in the last line. To apply the spin summation formula

$$\sum_\sigma u(p,\sigma)\overline{u}(p,\sigma) = \not{p} + m \tag{4.16}$$

we reverse the calculational direction of the Majorana fermion line for the last process. How this works is explained in detail in [19]. The result looks like

$$+\frac{\lambda^2}{2}\,\overline{u}(k_3,\sigma)\left(\frac{m\gamma^5}{s-m^2} + \frac{(\not{k}_3 - \not{k}_1 + m)\gamma^5}{t-m^2} + \frac{\gamma^5(\not{k}_3 - \not{k}_2 + m)}{u-m^2}\right) v(k_4,+) \quad , \tag{4.17}$$

with the change in sign coming from the antisymmetry of the charge conjugation matrix. There are no additional signs from the vertices because all couplings are scalar or pseudoscalar (cf. again [19]). It is important to keep track of the momenta's signs in the fermion propagators.

Equation (4.8) now has the form:

$$\begin{aligned}
&0 \stackrel{!}{=} \\
&\frac{\mathrm{i}\lambda^2}{2}\,\overline{u}(k_4,+)\left(\frac{3m}{t-m^2} + \frac{\not{k}_1 + \not{k}_2 + m}{s-m^2} + \frac{\not{k}_2 - \not{k}_3 + m}{u-m^2}\right)(\not{k}_2 + m)\gamma^5\xi \\
&-\frac{\mathrm{i}\lambda^2}{2}\overline{u}(k_4,+)\left(\frac{m\gamma^5}{u-m^2} + \frac{(\not{k}_1 + \not{k}_2 + m)\gamma^5}{s-m^2} + \frac{\gamma^5(\not{k}_1 - \not{k}_3 + m)}{t-m^2}\right)(\not{k}_1 + m)\xi \\
&-\frac{\mathrm{i}\lambda^2}{2}\,\overline{u}(k_4,+)\left(\frac{3m^2}{t-m^2} + \frac{m^2}{s-m^2} + \frac{m^2}{u-m^2} + 1\right)\gamma^5\xi \\
&+\frac{\mathrm{i}\lambda^2}{2}\overline{\xi}(\not{k}_3 + m)\left(\frac{m\gamma^5}{s-m^2} + \frac{(\not{k}_3 - \not{k}_1 + m)\gamma^5}{t-m^2} + \frac{\gamma^5(\not{k}_3 - \not{k}_2 + m)}{u-m^2}\right) v(k_4,+)
\end{aligned} \tag{4.18}$$

We divide everything by the common factor $\frac{\mathrm{i}\lambda^2}{2}$. To achieve the same structure for all four contributions we reverse the fermion line in the last process a second time to arrive at

$$\begin{aligned}
&0 \stackrel{!}{=} \\
&\overline{u}(k_4,+)\left(\frac{3m}{t-m^2} + \frac{\not{k}_1 + \not{k}_2 + m}{s-m^2} + \frac{\not{k}_2 - \not{k}_3 + m}{u-m^2}\right)(\not{k}_2 + m)\gamma^5\xi \\
&-\overline{u}(k_4,+)\left(\frac{m\gamma^5}{u-m^2} + \frac{(\not{k}_1 + \not{k}_2 + m)\gamma^5}{s-m^2} + \frac{\gamma^5(\not{k}_1 - \not{k}_3 + m)}{t-m^2}\right)(\not{k}_1 + m)\xi \\
&-\overline{u}(k_4,+)\left(\frac{3m^2}{t-m^2} + \frac{m^2}{s-m^2} + \frac{m^2}{u-m^2} + 1\right)\gamma^5\xi \\
&-\overline{u}(k_4,+)\left(\frac{m\gamma^5}{s-m^2} + \frac{\gamma^5(\not{k}_4 - \not{k}_2 + m)}{t-m^2} + \frac{(\not{k}_4 - \not{k}_1 + m)\gamma^5}{u-m^2}\right)(-\not{k}_3 + m)\xi
\end{aligned} \tag{4.19}$$

The terms proportional to m^2 in the first and third row cancel and we are left with:

$$\begin{aligned}
0 \stackrel{!}{=} \overline{u}(k_4,+)\Bigg[&\frac{3m\not{k}_2}{t-m^2} + \frac{(\not{k}_1 + \not{k}_2)(\not{k}_2 + m) + m\not{k}_2}{s-m^2} + \frac{(\not{k}_2 - \not{k}_3)(\not{k}_2 + m) + m\not{k}_2}{u-m^2} \\
&+ \frac{m(\not{k}_1 - m)}{u-m^2} + \frac{(\not{k}_1 + \not{k}_2 + m)(\not{k}_1 - m)}{s-m^2}
\end{aligned}$$

$$
- \frac{(\not k_1 - \not k_3 - m)(\not k_1 - m)}{t - m^2} - 1 - \frac{m(\not k_3 + m)}{s - m^2}
+ \frac{(\not k_4 - \not k_2 - m)(\not k_3 + m)}{t - m^2} - \frac{(\not k_4 - \not k_1 + m)(\not k_3 + m)}{u - m^2} \Bigg] \gamma^5 \xi \tag{4.20}
$$

Considering the terms proportional to $(t - m^2)^{-1}$ and applying the Dirac equation,

$$
\overline{u}(k_4, +)\,(\not k_4 - m) = 0 \quad , \tag{4.21}
$$

and momentum conservation

$$
k_1 + k_2 = k_3 + k_4 \quad , \tag{4.22}
$$

one gets

$$
\begin{aligned}
& (t - m^2)^{-1} \Big[3m\not k_2 + \not k_2(\not k_1 - m) - \not k_2(\not k_3 + m) \Big] \\
&= (t - m^2)^{-1} \Big[m\not k_2 + \not k_2(\not k_1 - \not k_3) \Big] \;=\; (t - m^2)^{-1} \Big[\not k_4 \not k_2 + \not k_2 \not k_4 - m^2 \Big] \\
&= (t - m^2)^{-1} \Big[2(k_2 k_4) - m^2 \Big] \;=\; (t - m^2)^{-1}(-t + m^2) \\
&= \, -1
\end{aligned} \tag{4.23}
$$

The terms proportional to $(s - m^2)^{-1}$ add up to

$$
\begin{aligned}
& (s - m^2)^{-1} \Big[\not k_1 \not k_2 + m^2 + m(\not k_1 + \not k_2) + m\not k_2 + \not k_2 \not k_1 - m\not k_2 - m\not k_3 - m^2 \Big] \\
&= (s - m^2)^{-1} \Big[\not k_1 \not k_2 + \not k_2 \not k_1 + m(\not k_1 + \not k_2 - \not k_3) \Big] \\
&= (s - m^2)^{-1} \Big[2(k_1 k_2) + m^2 \Big] \;=\; (s - m^2)^{-1}(s - m^2) \\
&= \, +1,
\end{aligned} \tag{4.24}
$$

while the remaining u terms yield:

$$
\begin{aligned}
& (u - m^2)^{-1} \Big[m\not k_2 + m^2 - m\not k_3 - \not k_3 \not k_2 + m\not k_2 + m\not k_1 - m^2 - \not k_2 \not k_3 - m\not k_2 \Big] \\
&= (u - m^2)^{-1} \Big[-\not k_2 \not k_3 - \not k_3 \not k_2 + m(\not k_1 + \not k_2 - \not k_3) \Big] \\
&= (u - m^2)^{-1} \Big[-2(k_2 k_3) + m^2 \Big] \;=\; (u - m^2)^{-1}(u - m^2) \\
&= \, +1 \qquad .
\end{aligned} \tag{4.25}
$$

So finally all terms add up to zero and the SWI is fulfilled.

4.2 Jacobi identities for the WZ model

An important possibility to test the consistency of the SWI themselves is to check whether the Jacobi identities for the appearing operators, i.e. the supercharge and the annihilation and creation operators for the particles, are valid.

In the sequel we frequently will use the properties of Grassmann odd bilinears under the exchange of the two spinors. These can e.g. be found in [5] (cf. also appendix A.3):

$$
\overline{\eta}\Gamma\xi = \left\{ \begin{array}{ll} +\overline{\xi}\Gamma\eta & \text{für } \;\Gamma = 1, \gamma^5, \gamma^5\gamma^\mu \\ -\overline{\xi}\Gamma\eta & \text{für } \;\Gamma = \gamma^\mu, [\gamma^\mu, \gamma^\nu] \end{array} \right. \tag{4.26}
$$

There is no complication in proving the Jacobi identities for the scalar annihilation operators:

$$-\Big[[Q(\xi),Q(\eta)]\,,a_A(k)\Big] \overset{!}{=} \Big[[Q(\eta),a_A(k)]\,,Q(\xi)\Big] + \Big[[a_A(k),Q(\xi)]\,,Q(\eta)\Big] \tag{4.27}$$

For the left hand side we have

$$\text{LHS (4.27)} \;=\; -\left[2\overline{\xi}\not{P}\eta, a_A(k)\right] = +2\left(\overline{\xi}\not{k}\eta\right) a_A(k) \quad .$$

The right hand side results in

$$\begin{aligned}
\text{RHS (4.27)} \;&=\; -\,\mathrm{i}\sum_\sigma \overline{\eta}u(k,\sigma)\,[Q(\xi),b(k,\sigma)] - (\xi\leftrightarrow\eta)\\
&=\; -\sum_\sigma \overline{\eta}u(k,\sigma)\,\overline{u}(k,\sigma)\Big(a_A(k)+\mathrm{i}\gamma^5 a_B(k)\Big)\xi - (\xi\leftrightarrow\eta)\\
&=\; -\,\overline{\eta}\,(\not{k}+m)\,\Big(a_A(k)+\mathrm{i}\gamma^5 a_B(k)\Big)\xi - (\xi\leftrightarrow\eta)\\
&=\; -\,(\overline{\eta}\not{k}\xi)a_A(k) + (\overline{\xi}\not{k}\eta)a_A(k) \;=\; 2(\overline{\xi}\not{k}\eta)a_A(k) \qquad \surd
\end{aligned}$$

The calculation for the annihilator of the pseudoscalar particle B is analogous, the only difference being the appearance of γ^5, which lets the parts containing a_A vanish and those with a_B remain.

$$-\Big[[Q(\xi),Q(\eta)]\,,a_B(k)\Big] = \Big[[Q(\eta),a_B(k)]\,,Q(\xi)\Big] + \Big[[a_B(k),Q(\xi)]\,,Q(\eta)\Big] \tag{4.28}$$

$$\text{LHS (4.28)} \;=\; -\left[2\overline{\xi}\not{P}\eta, a_B(k)\right] = +2\left(\overline{\xi}\not{k}\eta\right) a_B(k)$$

$$\begin{aligned}
\text{RHS (4.28)} \;&=\; +\sum_\sigma \overline{\eta}\gamma^5 u(k,\sigma)\,[Q(\xi),b(k,\sigma)] - (\xi\leftrightarrow\eta)\\
&=\; -\,\mathrm{i}\sum_\sigma \overline{\eta}\gamma^5 u(k,\sigma)\,\overline{u}(k,\sigma)\Big(a_A(k)+\mathrm{i}\gamma^5 a_B(k)\Big)\xi - (\xi\leftrightarrow\eta)\\
&=\; -\,\mathrm{i}\overline{\eta}\gamma^5\,(\not{k}+m)\,\Big(a_A(k)+\mathrm{i}\gamma^5 a_B(k)\Big)\xi - (\xi\leftrightarrow\eta)\\
&=\; -\,(\overline{\eta}\not{k}\xi)a_B(k) + (\overline{\xi}\not{k}\eta)a_B(k) \;=\; 2(\overline{\xi}\not{k}\eta)a_B(k) \qquad \surd
\end{aligned}$$

A more complicated task is the calculation of the Jacobi identity for the fermion annihilators. We are forced to use the Fierz transformations, the Gordon identity and all other formulae for spinors needed before. First of all the Jacobi identity has, of course, the same form as usual:

$$-\Big[[Q(\xi),Q(\eta)]\,,b(k,\sigma)\Big] \overset{!}{=} \Big[[Q(\eta),b(k,\sigma)]\,,Q(\xi)\Big] + \Big[[b(k,\sigma),Q(\xi)]\,,Q(\eta)\Big] \tag{4.29}$$

For the momentum operator on the left hand side one has to insert only the part of the particle number operators of the fermions, which yields

$$\text{LHS (4.29)} \;=\; -\left[2\overline{\xi}\not{P}\eta, b(k,\sigma)\right] = +2\left(\overline{\xi}\not{k}\eta\right) b(k,\sigma) \quad .$$

The right hand side can be manipulated in the following way:

$$\begin{aligned}
\text{RHS (4.29)} \;&=\; +\,\mathrm{i}\overline{u}(k,\sigma)\Big([Q(\xi),a_A(k)] + \mathrm{i}\gamma^5\,[Q(\xi),a_B(k)]\Big)\eta - (\xi\leftrightarrow\eta)\\
&=\; -\sum_\tau (\overline{u}(k,\sigma)\eta)(\overline{\xi}u(k,\tau))b(k,\tau)\\
&\quad\; +\sum_\tau (\overline{u}(k,\sigma)\gamma^5\eta)(\overline{\xi}\gamma^5 u(k,\tau))b(k,\tau) - (\xi\leftrightarrow\eta)
\end{aligned}$$

To calculate these products of spinor bilinears we have to use the Fierz identities to be found in appendix C.2 as well as e.g. in [23]. For arbitrary commuting spinors $\lambda_i, i = 1, \ldots, 4$ we therefore introduce these abbreviations:

$$\begin{aligned}
s(4,2;3,1) &= (\overline{\lambda}_4\lambda_2)\,(\overline{\lambda}_3\lambda_1)\\
v(4,2;3,1) &= (\overline{\lambda}_4\gamma^\mu\lambda_2)\,(\overline{\lambda}_3\gamma_\mu\lambda_1)\\
t(4,2;3,1) &= \frac{1}{2}(\overline{\lambda}_4\sigma^{\mu\nu}\lambda_2)\,(\overline{\lambda}_3\sigma_{\mu\nu}\lambda_1)\\
a(4,2;3,1) &= (\overline{\lambda}_4\gamma^5\gamma^\mu\lambda_2)\,(\overline{\lambda}_3\gamma_\mu\gamma^5\lambda_1)\\
p(4,2;3,1) &= (\overline{\lambda}_4\gamma^5\lambda_2)\,(\overline{\lambda}_3\gamma^5\lambda_1)
\end{aligned} \tag{4.30}$$

The scalar and pseudoscalar combinations (take care of the sign which has to be accounted for in case of spinors 2 and 3 being Grassmann odd!) give us the following relations:

$$s(4,2;3,1) = -\frac{1}{4}\Big(s(4,1;3,2) + v(4,1;3,2) + t(4,1;3,2) + a(4,1;3,2) + p(4,1;3,2)\Big) \tag{4.31}$$

$$p(4,2;3,1) = -\frac{1}{4}\Big(s(4,1;3,2) - v(4,1;3,2) + t(4,1;3,2) - a(4,1;3,2) + p(4,1;3,2)\Big) \tag{4.32}$$

Due to equation (4.26) the scalar, the pseudoscalar and the pseudovector are symmetric under interchange of the two Grassmann odd spinors, hence after subtracting the "exchange" term ($\xi \leftrightarrow \eta$) these contributions vanish. The scalar and pseudoscalar combination appear on the right hand side of equation (4.29) with different signs, so the tensorial part of the equation cancels. Only the vector contribution remains four times (scalar/pseudoscalar and a factor two by adding the "exchange" term), so we have

$$\text{RHS (4.29)} \;=\; +\sum_\tau \left(\overline{u}(k,\sigma)\gamma^\mu u(k,\tau)\right)\left(\overline{\xi}\gamma_\mu\eta\right) b(k,\tau) \tag{4.33}$$

Finally the Gordon identity (cf. e.g. [23], eq. (2.54))

$$\overline{u}(p,\sigma)\gamma^\mu u(p',\tau) = \frac{1}{2m}\overline{u}(p,\sigma)\Big((p+p')^\mu + \mathrm{i}\sigma^{\mu\nu}(p-p')_\nu\Big)u(p',\tau) \tag{4.34}$$

for identical momenta $p = p' \equiv k$ is used, that is why the second term vanishes. With the normalization of the Dirac spinors

$$\overline{u}(k,\sigma)u(k,\tau) = 2m\,\delta_{\sigma\tau} \tag{4.35}$$

the polarization sum over τ collapses and we end up with the desired result

$$\text{RHS (4.29)} \;=\; +2\,\left(\overline{\xi}\not{k}\eta\right) b(k,\sigma)\,. \qquad \surd \tag{4.36}$$

Chapter 5

A toy model

5.1 General remarks

To study the effects stemming from mixings of component fields from different superfields – independent of the difficulty of spontaneous breakdown of supersymmetry as in the O'Raifeartaigh model – we consider another toy model. It consists of two superfields, a mass term and a trilinear coupling. Like for the WZ model we summarize details about the model and the derivation of the Feynman rules in appendix E.2.

5.2 SUSY transformations of Dirac spinors

The main difference between this toy model and the WZ model is the problem of diagonalizing the mass terms which arise by the existence of more than one (at least two as here) superfields. By fusing a left- and a righthanded Weyl spinor from different superfields (not connected through Hermitean adjoint) a Dirac bispinor has been constructed. Moreover there is the problem of "clashing arrows" in Feynman diagrams, i.e. vertices with apparently incompatible directions of the fermion lines. More accurately this means the appearance of two fermions or two antifermions attached to a vertex in such models. This may happen if quadratic terms of superfields, whose fermionic components are combined into Dirac spinors, appear in the trilinear part of the superpotential. Another possibility is within the kinetic terms of the vector superfields in the Lagrangean density of supersymmetric gauge theories if their fermionic components are combined into Dirac fermions together with the Weyl components of chiral matter superfields, as is the case for the charginos in the MSSM.

First of all we want to derive the SUSY transformations of the scalar annihilators, in analogy to the calculations in chapter 3. The mode expansions of the *charged* scalar fields – the scalar component fields of the second superfield – are as follows

$$\begin{aligned}\phi(x) &= \int \frac{d^3\vec{p}}{(2\pi)^3 2E}\left(a_-(p)e^{-\mathrm{i}px} + a_+^\dagger(p)e^{+\mathrm{i}px}\right)\\ \phi^*(x) &= \int \frac{d^3\vec{p}}{(2\pi)^3 2E}\left(a_+(p)e^{-\mathrm{i}px} + a_-^\dagger(p)e^{+\mathrm{i}px}\right)\end{aligned} \tag{5.1}$$

Analogously, the projection onto the two different annihilators results in

$$a_-(k) = \mathrm{i}\int d^3\vec{x}\, e^{\mathrm{i}kx} \overleftrightarrow{\partial}_t \phi(x), \qquad a_+(k) = \mathrm{i}\int d^3\vec{x} \;\overleftrightarrow{\partial}_t \phi^*(x) \tag{5.2}$$

This enables us to write down the transformation laws of the annihilators.

$$\begin{aligned}[Q(\xi), a_+(k)] &= \mathrm{i} \int d^3\vec{x}\, e^{\mathrm{i}kx} \overleftrightarrow{\partial}_t \,[Q(\xi), \phi^*(x)] \\ &= -\sqrt{2} \int d^3\vec{x}\, e^{\mathrm{i}kx} \overleftrightarrow{\partial}_t \left(\bar{\xi}\mathcal{P}_R \chi_2(x)\right)\end{aligned}$$

Here and in the sequel χ_1 and χ_2 are the Majorana bispinors which could be built of the fermionic component fields of the first and the second superfield,

$$\chi_1 = \begin{pmatrix}\psi_1 \\ \bar{\psi}_1\end{pmatrix}, \qquad \chi_2 = \begin{pmatrix}\psi_2 \\ \bar{\psi}_2\end{pmatrix}. \tag{5.3}$$

With the definition of the Dirac field (E.17) we are able to express the righthanded Majorana field in terms of the Dirac field:

$$\Psi = \begin{pmatrix}\psi_1 \\ \bar{\psi}_2\end{pmatrix} = \mathcal{P}_L\chi_1 + \mathcal{P}_R\chi_2 \quad \Longrightarrow \quad \mathcal{P}_R\chi_2 = \mathcal{P}_R\Psi \tag{5.4}$$

Inserting this in the above equation and performing a calculation in the same manner as in chapter 3 one finally gets the relation

$$\boxed{[Q(\xi), a_+(k)] = \mathrm{i}\sqrt{2}\sum_\sigma \left(\bar{\xi}\mathcal{P}_R u(k,\sigma)\right) b(k,\sigma)} \tag{5.5}$$

Trying to proceed analogously for the annihilator $a_-(k)$ reveals a problem,

$$\begin{aligned}[Q(\xi), a_-(k)] &= \mathrm{i} \int d^3\vec{x}\, e^{\mathrm{i}kx} \overleftrightarrow{\partial}_t \,[Q(\xi), \phi(x)] \\ &= -\sqrt{2} \int d^3\vec{x}\, e^{\mathrm{i}kx} \overleftrightarrow{\partial}_t \left(\bar{\xi}\mathcal{P}_L \chi_2(x)\right) \quad ,\end{aligned}$$

which consists of an impossibility – at first look – to express the lefthanded Majorana field built of the spinor components of the second superfield in terms of the components of the Dirac field. The solution is to pass over to the charge conjugated Dirac field,

$$\Psi^c \equiv \mathcal{C}\overline{\Psi}^T = \begin{pmatrix}\psi_2 \\ \bar{\psi}_1\end{pmatrix} \quad \Longrightarrow \quad \mathcal{P}_L\chi_2 = \mathcal{P}_L\Psi^c \quad , \tag{5.6}$$

with the charge conjugation matrix $\mathcal{C}$. Remembering the mode expansion of the charge conjugated field operator,

$$\Psi^c(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 2E} \sum_\sigma \Big(u(p,\sigma)d(p,\sigma)e^{-\mathrm{i}px} + v(p,\sigma)b^\dagger(p,\sigma)e^{\mathrm{i}px}\Big), \tag{5.7}$$

the result for the SUSY transformation of the antifermion annihilator is found:

$$\boxed{[Q(\xi), a_-(k)] = \mathrm{i}\sqrt{2}\sum_\sigma \left(\bar{\xi}\mathcal{P}_L u(k,\sigma)\right) d(k,\sigma)} \tag{5.8}$$

How to project the annihilation operators out of the scalar component fields is well known by now:

$$\begin{aligned} a_A(k) &= \mathrm{i} \int d^3\vec{x}\, e^{\mathrm{i}kx} \overleftrightarrow{\partial}_t \, A(x) \\ a_B(k) &= \mathrm{i} \int d^3\vec{x}\, e^{\mathrm{i}kx} \overleftrightarrow{\partial}_t \, B(x).\end{aligned} \tag{5.9}$$

The derivation of the transformation laws is at first identical to those of the annihilators $a_+(k)$ and $a_-(k)$:

$$\begin{aligned}[Q(\xi), a_A(k)] &= \mathrm{i} \int d^3\vec{x}\, e^{\mathrm{i}kx} \overleftrightarrow{\partial}_t\, [Q(\xi), A(x)] \\ &= -\int d^3\vec{x}\, e^{\mathrm{i}kx} \overleftrightarrow{\partial}_t\, \left(\bar{\xi}\chi_1\right)\end{aligned}$$

$$\begin{aligned}[Q(\xi), a_B(k)] &= \mathrm{i} \int d^3\vec{x}\, e^{\mathrm{i}kx} \overleftrightarrow{\partial}_t\, [Q(\xi), B(x)] \\ &= -\mathrm{i} \int d^3\vec{x}\, e^{\mathrm{i}kx} \overleftrightarrow{\partial}_t\, \left(\bar{\xi}\gamma^5\chi_1\right)\end{aligned}$$

The difference to the scalar fields of the second superfield is, that now the whole Majorana spinor fields and not only the left- or righthanded parts are present. In consequence, the Dirac spinor field and its charge conjugate both appear in the transformation laws for $a_A(k)$ and $a_B(k)$ according to the expansion

$$\chi_1 = \begin{pmatrix}\psi_1\\ \bar{\psi}_1\end{pmatrix} = \begin{pmatrix}\psi_1\\ 0\end{pmatrix} + \begin{pmatrix}0\\ \bar{\psi}_1\end{pmatrix} = \mathcal{P}_L\Psi + \mathcal{P}_R\Psi^c \tag{5.10}$$

After inserting the above we arrive at the final form of the transformation laws for $a_A(k)$ and $a_B(k)$, which yield linear combinations of the Dirac fermion's particle and antiparticle annihilation operators:

$$\boxed{\begin{aligned}[Q(\xi), a_A(k)] &= \mathrm{i}\sum_\sigma\Big(\left(\bar{\xi}\mathcal{P}_L u(k,\sigma)\right) b(k,\sigma) + \left(\bar{\xi}\mathcal{P}_R u(k,\sigma)\right) d(k,\sigma)\Big)\\ [Q(\xi), a_B(k)] &= \sum_\sigma\Big(\left(\bar{\xi}\mathcal{P}_L u(k,\sigma)\right) b(k,\sigma) - \left(\bar{\xi}\mathcal{P}_R u(k,\sigma)\right) d(k,\sigma)\Big)\end{aligned}} \tag{5.11}$$

Remark: If the annihilators $b(k,\sigma)$ and $d(k,\sigma)$ are identical we have a real, i.e. a Majorana fermion and the equations (5.11) are reduced to the relations (4.1) and (4.2). For the chiral scalar fields ϕ and ϕ^* the same is true if we identify b and d and form the linear combinations $(\sqrt{2})^{-1}(\phi+\phi^*)$ and $(\mathrm{i}\sqrt{2})^{-1}(\phi-\phi^*)$, respectively. Hence the generalization of the Wess-Zumino model for Dirac fermions is consistent.

Deriving the SUSY transformations of the fermionic annihilators is more complicated. We must be aware of the fact that the Dirac bispinor field is composed from the Weyl spinor field ψ_1 as its lefthanded component and from the Weyl spinor field $\bar{\psi}_2$ as its righthanded component. Only these two chiral fields appear (we did not construct a Majorana bispinor field of the component fields $\psi_{1/2}$ and $\bar{\psi}_{1/2}$ from the first chiral superfield or from the second superfield, respectively) which means that here we only have to consider the transformations of the components of the leftchiral superfield $\hat{\Phi}_1$ and the rightchiral superfield $\hat{\Phi}_2^\dagger$ and not of their Hermitean adjoints. Everything is consistent and chirality is conserved. Going back to the roots, the transformation laws are:

$$\begin{aligned}[Q(\xi), \mathcal{P}_L\chi_1] = \mathcal{P}_L\,[Q(\xi), \chi_1] &= -\mathrm{i}\mathcal{P}_L(\mathrm{i}\partial\!\!\!/)\left(A+\mathrm{i}\gamma^5 B\right)\xi + \mathrm{i}\sqrt{2}\,F_1\mathcal{P}_L\xi\\ &\to -\mathrm{i}(\mathrm{i}\partial\!\!\!/)\mathcal{P}_R\left(A+\mathrm{i}\gamma^5 B\right)\xi - \mathrm{i}m\sqrt{2}\phi^*\mathcal{P}_L\xi\\ &= -\mathrm{i}(\mathrm{i}\partial\!\!\!/)\,(A+\mathrm{i}B)\,\mathcal{P}_R\xi - \mathrm{i}m\sqrt{2}\phi^*\mathcal{P}_L\xi\end{aligned} \tag{5.12}$$

In the second line we inserted the equation of motion for the auxiliary field F_1 and took the one-particle pole for the asymptotic fields of the theory. By the same method one gets for the righthanded fermion field of the second superfield

$$\begin{aligned}[Q(\xi),\mathcal{P}_R\chi_2] = \mathcal{P}_R\,[Q(\xi),\chi_2] =& -\mathrm{i}\mathcal{P}_R(\mathrm{i}\partial\!\!\!/)\sqrt{2}\,(\mathcal{P}_R\phi+\mathcal{P}_L\phi^*)\,\xi+\mathrm{i}\sqrt{2}\,F_2^*\mathcal{P}_R\xi\\ \to& -\mathrm{i}(\mathrm{i}\partial\!\!\!/)\sqrt{2}\mathcal{P}_L\,(\mathcal{P}_R\phi+\mathcal{P}_L\phi^*)\,\xi-\mathrm{i}m\,(A+\mathrm{i}B)\,\mathcal{P}_R\xi\\ =& -\mathrm{i}(\mathrm{i}\partial\!\!\!/)\sqrt{2}\phi^*\mathcal{P}_L\xi-\mathrm{i}m\,(A+\mathrm{i}B)\,\mathcal{P}_R\xi \quad .\end{aligned} \tag{5.13}$$

Here, for inserting the poles of the asymptotic fields into the equation of motion for the auxiliary field F_2, it is important to note that for the SUSY transformation of the lefthanded Weyl spinor field the auxiliary field is multiplied with the lefthanded Grassmann spinor ξ, whereas for the transformation of the righthanded Weyl spinor field we have the complex conjugated auxiliary field multiplied by the righthanded Grassmann spinor $\bar{\xi}$ (cf. chapter 1, and [5], [3], [1]).

Combining the two transformation laws (5.12) and (5.13) one reaches

$$\begin{aligned}[Q(\xi),\Psi] &= \mathcal{P}_L\,[Q(\xi),\chi_1]+\mathcal{P}_R\,[Q(\xi),\chi_2]\\ &= -\mathrm{i}(\mathrm{i}\partial\!\!\!/+m)\,(A+\mathrm{i}B)\,\mathcal{P}_R\xi-\mathrm{i}(\mathrm{i}\partial\!\!\!/+m)\sqrt{2}\,\phi^*\,\mathcal{P}_L\xi\end{aligned} \tag{5.14}$$

With the help of the equations (3.3) from chapter 3 we are able to deduce the SUSY transformations of the asymptotic annihilation operators (and as a by-product also those for the creation operators). The calculations are analogous to those in (3.8) so that the positive-frequency part (the one with the annihilators) remains.

$$\begin{aligned}[Q(\xi),b(k,\sigma)] &= \int d^3\vec{x}\,\overline{u}(k,\sigma)\gamma^0 e^{\mathrm{i}kx}\,[Q(\xi),\Psi(x)]\\ &= -\mathrm{i}\int d^3\vec{x}\,\overline{u}(k,\sigma)\gamma^0 e^{\mathrm{i}kx}(\mathrm{i}\partial\!\!\!/+m)\,(A+\mathrm{i}B)\,\mathcal{P}_R\xi\\ &\quad -\mathrm{i}\int d^3\vec{x}\,\overline{u}(k,\sigma)\gamma^0 e^{\mathrm{i}kx}(\mathrm{i}\partial\!\!\!/+m)\sqrt{2}\,\phi^*\,\mathcal{P}_L\xi\end{aligned}$$

This implies:

$$\boxed{[Q(\xi),b(k,\sigma)] = -\mathrm{i}\overline{u}(k,\sigma)\Big(a_A(k)\mathcal{P}_R+\mathrm{i}a_B(k)\mathcal{P}_R+\sqrt{2}a_+(k)\mathcal{P}_L\Big)\xi} \tag{5.15}$$

Finally, we reconsider in detail the calculation for the antifermion creator on which originally is projected, wherein we use the notation $k=(E,\vec{k})$ und $\tilde{k}=(E,-\vec{k})$:

$$\begin{aligned}\left[Q(\xi),d^\dagger(k,\sigma)\right] &= \int d^3\vec{x}\,\overline{v}(k,\sigma)\gamma^0 e^{-\mathrm{i}kx}\,[Q(\xi),\Psi(x)]\\ &= -\mathrm{i}\int d^3\vec{x}\,\overline{v}(k,\sigma)\gamma^0 e^{-\mathrm{i}kx}(\mathrm{i}\partial\!\!\!/+m)\,(A+\mathrm{i}B)\,\mathcal{P}_R\xi\\ &\quad -\mathrm{i}\int d^3\vec{x}\,\overline{v}(k,\sigma)\gamma^0 e^{-\mathrm{i}kx}(\mathrm{i}\partial\!\!\!/+m)\sqrt{2}\,\phi^*\,\mathcal{P}_L\xi\\ &= -\mathrm{i}\int \frac{d^3\vec{x}\,d^3\vec{p}}{(2\pi)^3 2E}\overline{v}(k,\sigma)\gamma^0\Big((p\!\!\!/+m)\,(a_A(p)+\mathrm{i}a_B(p))\,e^{-\mathrm{i}(k+p)x}\\ &\qquad -(p\!\!\!/-m)\,\Big(a_A^\dagger(p)+\mathrm{i}a_B^\dagger(p)\Big)\,e^{\mathrm{i}(p-k)x}\Big)\mathcal{P}_R\xi\end{aligned}$$

$$
\begin{aligned}
&- \mathrm{i}\sqrt{2}\int \frac{d^3\vec{x}\, d^3\vec{p}}{(2\pi)^3 2E}\overline{v}(k,\sigma)\gamma^0\Big((\not{p}+m)\,a_+(p)e^{-\mathrm{i}(k+p)x} \\
&\qquad\qquad - (\not{p}-m)\,a_-^\dagger(p)e^{\mathrm{i}(k-p)x}\Big)\mathcal{P}_L\xi \\
= &- \frac{\mathrm{i}}{2E}\overline{v}(k,\sigma)\gamma^0\Big((\tilde{\not{k}}+m)\,\big(a_A(\tilde{k})+\mathrm{i}a_B(\tilde{k})\big) \\
&\qquad\qquad - (\not{k}-m)\,\big(a_A^\dagger(k)+\mathrm{i}a_B^\dagger(k)\big)\Big)\mathcal{P}_R\xi \\
&- \frac{\sqrt{2}\mathrm{i}}{2E}\overline{v}(k,\sigma)\gamma^0\Big((\tilde{\not{k}}+m)\,a_+(\tilde{k}) - (\not{k}-m)\,a_-^\dagger(k)\Big)\mathcal{P}_L\xi \\
= &- \frac{\mathrm{i}}{2E}\overline{v}(k,\sigma)\Big((\not{k}+m)\,\gamma^0\,\big(a_A(\tilde{k})+\mathrm{i}a_B(\tilde{k})\big) \\
&\qquad\qquad + \left(-2E\gamma^0+\not{k}+m\right)\gamma^0\,\big(a_A^\dagger(k)+\mathrm{i}a_B^\dagger(k)\big)\Big)\mathcal{P}_R\xi \\
&- \frac{\sqrt{2}\mathrm{i}}{2E}\overline{v}(k,\sigma)\Big((\not{k}+m)\,\gamma^0 a_+(\tilde{k}) + \left(-2E\gamma^0+\not{k}+m\right)\gamma^0 a_-^\dagger(k)\Big)\mathcal{P}_L\xi \\
= &+\mathrm{i}\overline{v}(k,\sigma)\Big(a_A^\dagger(k)\mathcal{P}_R + \mathrm{i}a_B^\dagger(k)\mathcal{P}_R + \sqrt{2}a_-^\dagger(k)\mathcal{P}_L\Big)\xi
\end{aligned}
$$

In the last line we used the Dirac equation in the form $\overline{v}(k,\sigma)\,(\not{k}+m) = 0$. Complex conjugation changes this result into

$$[Q(\xi), d(k,\sigma)] = \; +\mathrm{i}\overline{\xi}\Big(a_A(k)\mathcal{P}_L - \mathrm{i}a_B(k)\mathcal{P}_L + \sqrt{2}a_-(k)\mathcal{P}_R\Big)v(k,\sigma).$$

"Reversing the calculational direction of the fermion line" with respect to the Feynman rules [19] (this way of speaking originates from changing the calculational directions of fermion lines in diagrams and refers to the property of fermion bilinears summarized in appendix A.3) gives rise to the final result:

$$\boxed{[Q(\xi), d(k,\sigma)] = -\mathrm{i}\overline{u}(k,\sigma)\Big(a_A(k)\mathcal{P}_L - \mathrm{i}a_B(k)\mathcal{P}_L + \sqrt{2}a_-(k)\mathcal{P}_R\Big)\xi} \tag{5.16}$$

5.3 A cross-check: Jacobi identities

The Jacobi identities for this toy model are mostly in complete analogy to the Jacobi identities for the WZ model, but there are some fine points which have to be handled carefully. So we show the calculations in detail here.

The Jacobi identity has the standard structure:

$$-\Big[[Q(\xi), Q(\eta)]\,, a_A(k)\Big] = \Big[[Q(\eta), a_A(k)]\,, Q(\xi)\Big] + \Big[[a_A(k), Q(\xi)]\,, Q(\eta)\Big] \tag{5.17}$$

Up to now it is well known how to manipulate the left hand side

$$\text{LHS (5.17)} = +2\left(\overline{\xi}\not{k}\eta\right)\, a_A(k) \tag{5.18}$$

There are more steps to take on the right hand side compared to the case of the WZ model and they are a little bit more complex, too,

$$\text{RHS (5.17)} = \;\; \mathrm{i}\sum_\sigma \left(\overline{\eta}\mathcal{P}_L u(k,\sigma)\right)[b(k,\sigma), Q(\xi)]$$

$$
\begin{aligned}
&+\mathrm{i}\sum_\sigma\left(\overline{\eta}\mathcal{P}_R u(k,\sigma)\right)\left[d(k,\sigma),Q(\xi)\right]-(\xi\leftrightarrow\eta)\\
=\;&-\overline{\eta}\mathcal{P}_L\left(\not{k}+m\right)\Big(a_A(k)\mathcal{P}_R+\mathrm{i}a_B(k)\mathcal{P}_R+\sqrt{2}\mathcal{P}_L a_+(k)\Big)\xi\\
&-\overline{\eta}\mathcal{P}_R\left(\not{k}+m\right)\Big(a_A(k)\mathcal{P}_L-\mathrm{i}a_B(k)\mathcal{P}_L+\sqrt{2}\mathcal{P}_R a_-(k)\Big)\xi-(\xi\leftrightarrow\eta)\\
=\;&-\left(\overline{\eta}\mathcal{P}_L\not{k}\xi\right)a_A(k)-\mathrm{i}\left(\overline{\eta}\mathcal{P}_L\not{k}\xi\right)a_B(k)-\sqrt{2}\,m\left(\overline{\eta}\mathcal{P}_L\xi\right)a_+(k)\\
&-\left(\overline{\eta}\mathcal{P}_R\not{k}\xi\right)a_A(k)+\mathrm{i}\left(\overline{\eta}\mathcal{P}_R\not{k}\xi\right)a_B(k)-\sqrt{2}\,m\left(\overline{\eta}\mathcal{P}_R\xi\right)a_+(k)-(\xi\leftrightarrow\eta)\\
=\;&+2\left(\overline{\xi}\not{k}\eta\right)a_A(k)\quad\surd
\end{aligned}
$$

In the second equation we used the polarization sum for the Dirac spinors $u(k,\sigma)$, in the third equation the anticommutativity of γ^5 with the other gamma matrices and finally, in the fourth equation, we made use of the identity (4.26), which, after subtracting the term $(\xi\leftrightarrow\eta)$, forces the scalar and pseudoscalar parts to vanish so that only the vector contribution with the annihilator $a(k,\sigma)$ remains.

The calculation for the annihilation operator of the pseudoscalar particle, $a_B(k)$, is almost completely analogous.

What about the annihilators of the chiral scalar fields, i.e. the component fields from the second supermultiplet? The difference lies only in the commutator of the supercharge with the annihilator now producing either the fermion or the antifermion annihilator. In particular,

$$
-\Big[\left[Q(\xi),Q(\eta)\right],a_+(k)\Big]=\Big[\left[Q(\eta),a_+(k)\right],Q(\xi)\Big]+\Big[\left[a_+(k),Q(\xi)\right],Q(\eta)\Big],\tag{5.19}
$$

$$
-\Big[\left[Q(\xi),Q(\eta)\right],a_-(k)\Big]=\Big[\left[Q(\eta),a_-(k)\right],Q(\xi)\Big]+\Big[\left[a_-(k),Q(\xi)\right],Q(\eta)\Big].\tag{5.20}
$$

The left hand sides look as usual,

$$
\text{LHS (5.19)}=+2\left(\overline{\xi}\not{k}\eta\right)a_+(k),\qquad \text{LHS (5.20)}=+2\left(\overline{\xi}\not{k}\eta\right)a_-(k)\quad.
$$

No problems show up for the right hand sides:

$$
\begin{aligned}
\text{RHS (5.19)}=\;&\mathrm{i}\sum_\sigma\left(\overline{\eta}\mathcal{P}_R u(k,\sigma)\right)\left[b(k,\sigma),Q(\xi)\right]-(\xi\leftrightarrow\eta)\\
=\;&-\sqrt{2}\overline{\eta}\mathcal{P}_R\left(\not{k}+m\right)\Big(a_A(k)\mathcal{P}_R+\mathrm{i}a_B(k)\mathcal{P}_R+\sqrt{2}a_+(k)\mathcal{P}_L\Big)\xi-(\xi\leftrightarrow\eta)\\
=\;&-\sqrt{2}\,m\left(\overline{\eta}\mathcal{P}_R\xi\right)a_A(k)-\sqrt{2}\,\mathrm{i}m\left(\overline{\eta}\mathcal{P}_R\xi\right)a_B(k)\\
&-2\left(\overline{\eta}\mathcal{P}_R\not{k}\xi\right)a_+(k)-(\xi\leftrightarrow\eta)\\
=\;&+2\left(\overline{\xi}\not{k}\eta\right)a_+(k)\quad\surd\qquad,
\end{aligned}
$$

$$
\begin{aligned}
\text{RHS (5.20)}=\;&\mathrm{i}\sum_\sigma\left(\overline{\eta}\mathcal{P}_L u(k,\sigma)\right)\left[d(k,\sigma),Q(\xi)\right]-(\xi\leftrightarrow\eta)\\
=\;&-\sqrt{2}\overline{\eta}\mathcal{P}_L\left(\not{k}+m\right)\Big(a_A(k)\mathcal{P}_L-\mathrm{i}a_B(k)\mathcal{P}_L+\sqrt{2}a_-(k)\mathcal{P}_R\Big)\xi-(\xi\leftrightarrow\eta)\\
=\;&-\sqrt{2}\,m\left(\overline{\eta}\mathcal{P}_L\xi\right)a_A(k)+\sqrt{2}\,\mathrm{i}m\left(\overline{\eta}\mathcal{P}_L\xi\right)a_B(k)\\
&-2\left(\overline{\eta}\mathcal{P}_L\not{k}\xi\right)a_-(k)-(\xi\leftrightarrow\eta)\\
=\;&+2\left(\overline{\xi}\not{k}\eta\right)a_-(k)\quad\surd\qquad,
\end{aligned}
$$

where the last line again follows from (4.26).

There is nothing new about the Jacobi identities of the fermion annihilation operators, but for the sake of completeness, we list the calculations here, too. Again we have the standard structure

$$-\Big[\left[Q(\xi),Q(\eta)\right],b(k,\sigma)\Big] = \Big[\left[Q(\eta),b(k,\sigma)\right],Q(\xi)\Big] + \Big[\left[b(k,\sigma),Q(\xi)\right],Q(\eta)\Big] \tag{5.21}$$

$$-\Big[\left[Q(\xi),Q(\eta)\right],d(k,\sigma)\Big] = \Big[\left[Q(\eta),d(k,\sigma)\right],Q(\xi)\Big] + \Big[\left[d(k,\sigma),Q(\xi)\right],Q(\eta)\Big] \tag{5.22}$$

The left hand sides are:

$$\text{LHS (5.21)} = +2\left(\overline{\xi}\not{k}\eta\right) b(k,\sigma) \quad ,$$

$$\text{LHS (5.22)} = +2\left(\overline{\xi}\not{k}\eta\right) d(k,\sigma) \quad .$$

For the right hand side we find

$$\begin{aligned}
\text{RHS (5.21)} = & -\mathrm{i}\overline{u}(k,\sigma)\Big(\left[a_A(k),Q(\xi)\right]\mathcal{P}_R + \mathrm{i}\left[a_B(k),Q(\xi)\right]\mathcal{P}_R \\
& + \sqrt{2}\left[a_+(k),Q(\xi)\right]\mathcal{P}_L\Big) - (\xi\leftrightarrow\eta) \\
= & -\sum_\tau \left(\overline{u}(k,\sigma)\mathcal{P}_R\eta\right)\Big(\overline{\xi}\mathcal{P}_L u(k,\tau)b(k,\tau) + \overline{\xi}\mathcal{P}_R u(k,\tau)d(k,\tau)\Big) \\
& -\sum_\tau \left(\overline{u}(k,\sigma)\mathcal{P}_R\eta\right)\Big(\overline{\xi}\mathcal{P}_L u(k,\tau)b(k,\tau) - \overline{\xi}\mathcal{P}_R u(k,\tau)d(k,\tau)\Big) \\
& -2\sum_\tau \left(\overline{u}(k,\sigma)\mathcal{P}_L\eta\right)\left(\overline{\xi}\mathcal{P}_R u(k,\tau)\right) b(k,\tau) \quad - \quad (\xi\leftrightarrow\eta)
\end{aligned}$$

Obviously the contributions of the antifermion annihilators cancel out. In this calculation, by multiplying out the chiral spinor bilinears, one gets the same scalar and pseudoscalar terms as for the Jacobi identity for the fermion annihilator of the WZ model (4.29), so we can use that earlier result.

$$\begin{aligned}
\text{RHS (5.21)} = & -2\sum_\tau \Big(\left(\overline{u}(k,\sigma)\mathcal{P}_R\eta\right)\left(\overline{\xi}\mathcal{P}_L u(k,\tau)\right) \\
& + \left(\overline{u}(k,\sigma)\mathcal{P}_L\eta\right)\left(\overline{\xi}\mathcal{P}_R u(k,\tau)\right)\Big) b(k,\tau) - (\xi\leftrightarrow\eta) \\
= & -\sum_\tau \left(\overline{u}(k,\sigma)\eta\right)\left(\overline{\xi}u(k,\tau)\right) b(k,\tau) \\
& + \sum_\tau \left(\overline{u}(k,\sigma)\gamma^5\eta\right)\left(\overline{\xi}\gamma^5 u(k,\tau)\right) b(k,\tau) - (\xi\leftrightarrow\eta) \\
= & +2\left(\overline{\xi}\not{k}\eta\right) b(k,\sigma) \quad \surd
\end{aligned}$$

The calculation for $d(k,\sigma)$ is analogous.

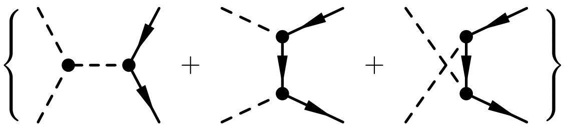
5.4 Wick theorem and plenty of signs

Another point of utmost importance appears whenever charged fermions come into play: We have to take care of relative signs between amplitudes belonging to different processes in the same SWI. This is due to the Wick theorem, with the signs stemming from disentangling the contractions of the interaction operators of Yukawa type $\overline{\Psi}\Psi\phi$. To illuminate this further, we want to show an example considering the SWI:

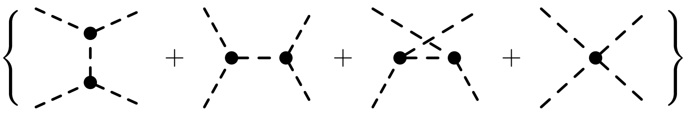
$$0 = \left\langle 0 \middle| \left[Q(\xi), a_A^{\text{out}}(k_3) d^{\text{out}}(k_4,+) a_A^{\text{in}\,\dagger}(k_1) a_A^{\text{in}\,\dagger}(k_2) \right] \middle| 0 \right\rangle \tag{5.23}$$

This produces a relation between the following processes of the diagrammatical form (for the vertices and propagators see appendix E.2):

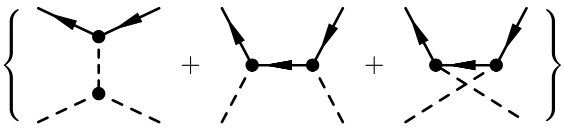
$$0 = \boxed{(-1)\cdot}\,\mathrm{i} \sum_{\sigma} \left(\bar{\xi} \mathcal{P}_L u(k_3,\sigma) \right) \cdot$$



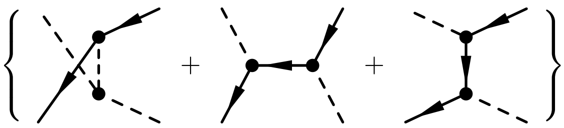
$$- \mathrm{i}\overline{u}(k_4,+) \mathcal{P}_L \xi \cdot$$



$$+ \boxed{(-1)\cdot}\,\mathrm{i} \sum_{\sigma} \left(\overline{u}(k_1,\sigma) \mathcal{P}_L \xi \right) \cdot$$



$$+ \boxed{(-1)\cdot}\,\mathrm{i} \sum_{\sigma} \left(\overline{u}(k_2,\sigma) \mathcal{P}_L \xi \right) \cdot$$



(5.24)

Here we have omitted several processes giving vanishing contributions, $AA \to AB$, $AA \to A\phi^{(*)}$, $AA \to \overline{\Psi}\,\overline{\Psi}$ and $A\Psi \to A\overline{\Psi}$. At first glance, the signs in boxes might seem totally arbitrary, but can be verified by the Wick theorem. Before proving this statement we show that without these signs the SWI would indeed not be valid.

The calculation for the SWI is principally analogous to similar calculations in chapter 4 done within the WZ model. Thus we may omit the details here. No difficulties arise as we can switch directly from analytical Feynman rules to diagrams. We use the polarization sum of Dirac spinors and the change of sign, but not of chirality, when "reversing" a fermion line, [19],

$$-\mathrm{i} \left(\overline{u}(k_4,+) \mathcal{P}_L \xi \right) = +\mathrm{i} \left(\bar{\xi} \mathcal{P}_L v(k_4,+) \right) \qquad . \tag{5.25}$$

The first process $A(k_1)A(k_2) \to \Psi(k_3,\sigma)\overline{\Psi}(k_4,+)$ yields, after multiplication with its prefactor and performing the polarization sum,

$$- 2g^2\, \bar{\xi} \mathcal{P}_L \left(\frac{3m\,(\not{k}_3 + m)}{s - m^2} + \frac{(\not{k}_3 + m)\,(\not{k}_3 - \not{k}_2 + m)}{t - m^2} \right.$$
$$\left. + \frac{(\not{k}_3 + m)\,(\not{k}_3 - \not{k}_1 + m)}{u - m^2} \right) v(k_4,+) \qquad . \tag{5.26}$$

For the purely scalar process $A(k_1)A(k_2) \to A(k_3)A(k_4)$ we have to "reverse a fermion line" (i.e. the spinorial prefactor) as mentioned above

$$+6g^2\,\bar{\xi}\mathcal{P}_L\left(\frac{3m^2}{s-m^2}+\frac{3m^2}{t-m^2}+\frac{3m^2}{u-m^2}+1\right)v(k_4,+) \qquad . \tag{5.27}$$

The scattering $A(k_1)\overline{\Psi}(k_2,\sigma) \to A(k_3)\overline{\Psi}(k_4,+)$ of the scalar particle and the antifermion sums up to give the amplitude as follows:

$$\begin{aligned} -2g^2\,\bar{\xi}\mathcal{P}_L\Bigg(\frac{-3m\,(\not{k}_2-m)}{u-m^2}&+\frac{(\not{k}_2-m)\,(\not{k}_1+\not{k}_2-m)}{s-m^2} \\ &+\frac{(\not{k}_2-m)\,(\not{k}_2-\not{k}_3-m)}{t-m^2}\Bigg)v(k_4,+) \end{aligned} \tag{5.28}$$

With the help of the substitutions $k_1 \leftrightarrow k_2$ and $t \leftrightarrow u$ we get the amplitude for the remaining fourth process $\overline{\Psi}(k_1,\sigma)A(k_2) \to A(k_3)\overline{\Psi}(k_4,+)$.

Summing up the amplitudes of these four processes with the appropriate prefactors gives zero. The calculation is totally identical to the corresponding one done in the WZ model. Now it is obvious that the three added signs are necessary for the SWI to be fulfilled. But where do they come from?

Take a look at the first process as an S-matrix element:

$$\Big\langle 0\Big|b^{\text{out}}(k_3,\sigma)d^{\text{out}}(k_4,+)a_A^{\text{in}\,\dagger}(k_1)a_A^{\text{in}\,\dagger}(k_2)\Big|0\Big\rangle \tag{5.29}$$

When examining the three diagrams in the first line of (5.24), the following expression arises, where we suppress the momentum and spin arguments as well as the *in* and *out* labels,

$$\langle 0|b\;d\;(\overline{\Psi}\Psi A)\,(AAA)a^\dagger a^\dagger|0\rangle = (-1)\cdot\langle 0|b(\overline{\Psi}\Psi dA)\,(AAA)a^\dagger a^\dagger|0\rangle$$

To disentangle the contraction lines we had to anticommute the fermion annihilation operators. We used the conventional notations for contractions

$$\begin{aligned} A(x)A(y) &= \int\frac{d^4p}{(2\pi)^4}e^{-\mathrm{i}p(x-y)}\frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\epsilon} \\ \langle 0|\,d(p,\sigma)\Psi &= v(p,\sigma) \\ A\;a^\dagger\,|0\rangle &= 1 \\ \dots\quad&\;\dots \end{aligned}$$

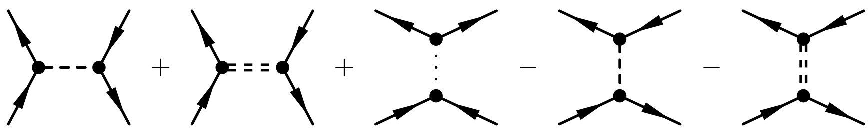
Using them, we can correctly convert the Feynman rules into analytical expressions. By means of this anticommutation, a sign emerges. One is easily convinced that the SWI with a fermion in the final state instead of an antifermion does not need this anticommutation. Due to the reversed order of the two fermion annihilation operators, no such sign arises in that case. After a short calculation we find that the two other diagrams contributing to the process considered above pick up signs by the same mechanism, whereas this would not be the case if the two diagrams contained the fermion instead of the antifermion annihilator in the S-matrix element.

The structure of these signs can be understood with the help of [19], on top of page 4. From there we can read off the sign of an S-matrix element to be $(-1)^{P+L+V}$, where L is the number of closed fermion loops, P is the parity of the permutation of asymptotic annihilators and creators after having disentangled the fermion lines, and V is the number of incoming and outgoing antifermions. We do only deal with tree level diagrams here, so the number of loops always is zero and no sign is produced by them. The signs stemming from the permutation of the ladder operators are the same as those between the different contributions from s- and t-channel in Bhabha scattering. We had already taken them into account for the WZ model. While we had only Majorana fermions there and could have contracted the field operators in an arbitrary way with the ladder operators for external particles, the fact that we now have to handle Dirac fermions and the sign problem connected with the existence of antifermions discussed in [19] is a new topic arising within our toy model. The signs in boxes in (5.24) are due to this effect.

Because we need some additional techniques for calculating an SWI for $(2 \to 2)$-processes, we show a detailed calculation here, starting with three fermions. In that case vertices with "clashing arrows" will appear. This is examplified with

$$0 \stackrel{!}{=} \left\langle 0 \right| \left[Q(\xi), a_A^{\text{out}}(k_3) b^{\text{out}}(k_4,+) b^{\text{in}\,\dagger}(k_1,+) d^{\text{in}\,\dagger}(k_2,-) \right] \left| 0 \right\rangle \quad . \tag{5.30}$$

For the first process, $\Psi(k_1,+)\overline{\Psi}(k_2,-) \to \overline{\Psi}(k_3,\sigma)\Psi(k_4,+)$, five diagrams contribute,

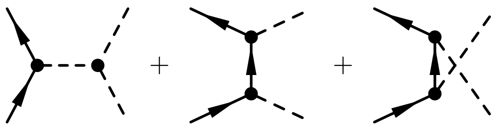

(5.31)

The relative sign of the third diagram (containing the "clashing arrows") has to be determined carefully from the Wick theorem and depends on the "position of the fermion lines relative to each other". More signs possibly arise here, depending on the calculational directions of the fermion lines as explained in [19]; this can happen, if it is necessary to anticommute the two fermion field operators in the interaction terms. Nevertheless this is compensated (cf. again [19]) by additional signs produced at the gamma matrices attached to the vertices, giving the same result. For the last two diagrams the relative signs, too, stem from the Wick theorem and can be understood as belonging to exchange diagrams in the same manner as for Bhabha scattering. The positive sign of the third diagram can be seen as belonging to a u-channel, as the u-channel diagram has a relative sign with respect to the t-channel diagrams but not to the s-channel diagrams as in quantum electrodynamics (Of course, without Feynman number violating vertices it is not possible to have s-, t- *and* u-channel diagrams there). But the global sign (which is indispensable for comparison with the other processes contributing to the SWI) is only calculable with the Wick theorem. For more complicated processes it is inevitable to use the Wick theorem to get the correct signs. Fortunately, as will be discussed later, it is possible to do this in a way compatible with the *O'Mega* factorization procedure.

The five diagrams give, together with all signs and after summing over the spin σ:

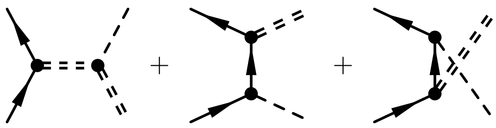
$$
\begin{aligned}
\text{1. process, (5.31)} \; = 4g^2 \Bigg\{ & \frac{1}{s-m^2}\Big(m\,(\overline{v}(k_2,-)\mathcal{P}_L u(k_1,+))\,(\overline{u}(k_4,+)\mathcal{P}_R\xi) \\
& \quad - (\overline{v}(k_2,-)\mathcal{P}_R u(k_1,+))\,(\overline{u}(k_4,+)\not{k}_3\mathcal{P}_R\xi)\Big) \\
& + \frac{1}{t-m^2}\Big((\overline{u}(k_4,+)\mathcal{P}_R u(k_1,+))\,(\overline{v}(k_2,-)\not{k}_3\mathcal{P}_R\xi) \\
& \quad - m\,(\overline{u}(k_4,+)\mathcal{P}_L u(k_1,+))\,(\overline{v}(k_2,-)\mathcal{P}_R\xi)\Big) \\
& - \frac{1}{u-m^2}\,(\overline{u}(k_4,+)\mathcal{P}_R u(k_2,-))\,(\overline{v}(k_1,+)\not{k}_3\mathcal{P}_R\xi)\Bigg\}
\end{aligned}
$$

The second process is decomposed into the two separate parts $\Psi(k_1,+)\overline{\Psi}(k_2,-) \to A(k_3)A(k_4)$,



(5.32)

as well as $\Psi(k_1,+)\overline{\Psi}(k_2,-) \to A(k_3)B(k_4)$:



(5.33)

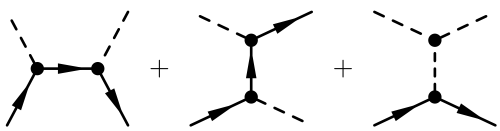
It is not difficult to derive the analytical expressions. For (5.32) we get

$$
-2g^2\,\overline{v}(k_2,-)\left(\frac{3m}{s-m^2} + \frac{\not{k}_1-\not{k}_3+m}{u-m^2} + \frac{\not{k}_1-\not{k}_4+m}{t-m^2}\right)u(k_1,+)\;\;(\overline{u}(k_4,+)\mathcal{P}_R\xi)\,,
$$

and for (5.33):

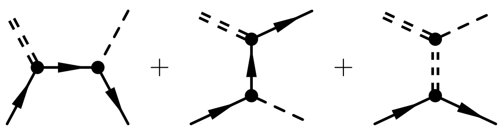
$$
-2g^2\,\overline{v}(k_2,-)\left(\frac{m}{s-m^2} - \frac{\not{k}_1-\not{k}_3-m}{u-m^2} + \frac{\not{k}_1-\not{k}_4+m}{t-m^2}\right)\gamma^5 u(k_1,+)\;\;(\overline{u}(k_4,+)\mathcal{P}_R\xi)
$$

SUSY transforming the antifermion in the initial state again gives rise to two different processes, $\Psi(k_1,+)A(k_2) \to A(k_3)\Psi(k_4,+)$,



(5.34)

and $\Psi(k_1,+)B(k_2) \to A(k_3)\Psi(k_4,+)$:



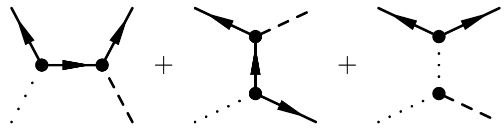
(5.35)

The corresponding terms are:

$$
2g^2\,\overline{u}(k_4,+)\left(\frac{3m}{t-m^2} + \frac{\not{k}_1+\not{k}_2+m}{s-m^2} + \frac{\not{k}_1-\not{k}_3+m}{u-m^2}\right)u(k_1,+)\;\;(\overline{v}(k_2,-)\mathcal{P}_R\xi)\,,
$$

$$2g^2\,\overline{u}(k_4,+)\left(\frac{m}{t-m^2}-\frac{\not{k}_1-\not{k}_3-m}{u-m^2}+\frac{\not{k}_1+\not{k}_2+m}{s-m^2}\right)\gamma^5 u(k_1,+)\;\left(\overline{v}(k_2,-)\mathcal{P}_R\xi\right).$$

There still remains to perform the SUSY transformation of the fermion in the initial state with the diagrams of the process $\phi^*(k_1)\overline{\Psi}(k_2,-)\to A(k_3)\Psi(k_4,+)$:



$$+ \qquad + \tag{5.36}$$

The relative (and, again, the global sign) of the second diagram results from the Wick theorem (here we only show the fermion contractions explicitly):

$$(-1)^2\cdot\langle 0|a\,b\left(\overline{\Psi}\Psi^c\phi^*\right)\left(\overline{\Psi^c}\Psi^c A\right)a^\dagger\,d^\dagger|0\rangle \tag{5.37}$$

The trick in this calculation is to disentangle the contractions by rewriting the second interaction operator,

$$\overline{\Psi}\Psi A\equiv\overline{\Psi}\Gamma\Psi A=\left(\overline{\Psi}\Gamma\Psi A\right)^T=(-1)\cdot\Psi^T\mathcal{C}^{-1}\left(\mathcal{C}\Gamma\mathcal{C}^{-1}\right)\Psi^c A\equiv(-1)^2\cdot\overline{\Psi^c}\Psi^c A \quad ,$$

because in this model only scalar, pseudoscalar or chiral scalar couplings appear that are invariant (i.e. their gamma matrices) under the charge conjugation transformation. One of the additional signs is due to the anticommutation of the Fermi field operators when transposing, the other stems from the relations

$$\mathcal{C}\overline{\Psi}^T=\Psi^c,\qquad \Psi^T\mathcal{C}^{-1}=-\overline{\Psi^c}\quad . \tag{5.38}$$

The sum of the last three diagrams results in:

$$\begin{aligned}-4g^2\cdot\overline{u}(k_4,+)\Bigg(&\frac{\not{k}_1+\not{k}_2+m}{s-m^2}\mathcal{P}_R-\mathcal{P}_R\frac{\not{k}_1-\not{k}_4-m}{t-m^2}\\&+\frac{2m}{u-m^2}\mathcal{P}_R\Bigg)u(k_2,-)\left(\overline{v}(k_1,+)\mathcal{P}_R\xi\right)\quad .\end{aligned}$$

Now we sum up the contributions of the several processes of this SWI separately for each of the reaction channels. A common prefactor $2g^2$ is suppressed in the following.

$$\begin{aligned}\textbf{s-channel}\propto\;&2m\big(\overline{v}(k_2,-)\mathcal{P}_L u(k_1,+)\big)\big(\overline{u}(k_4,+)\mathcal{P}_R\xi\big)\\&-2\big(\overline{v}(k_2,-)\mathcal{P}_R u(k_1,+)\big)\big(\overline{u}(k_4,+)\not{k}_3\mathcal{P}_R\xi\big)\\&-3m\big(\overline{v}(k_2,-)u(k_1,+)\big)\big(\overline{u}(k_4,+)\mathcal{P}_R\xi\big)\\&-m\big(\overline{v}(k_2,-)\gamma^5 u(k_1,+)\big)\big(\overline{u}(k_4,+)\mathcal{P}_R\xi\big)\\&+\big(\overline{u}(k_4,+)(\not{k}_1+\not{k}_2+m)u(k_1,+)\big)\big(\overline{v}(k_2,-)\mathcal{P}_R\xi\big)\\&+\big(\overline{u}(k_4,+)(\not{k}_1+\not{k}_2+m)\gamma^5 u(k_1,+)\big)\big(\overline{v}(k_2,-)\mathcal{P}_R\xi\big)\\&-2\big(\overline{u}(k_4,+)(\not{k}_1+\not{k}_2+m)\mathcal{P}_R u(k_2,-)\big)\big(\overline{v}(k_1,+)\mathcal{P}_R\xi\big)\end{aligned}\tag{5.39}$$

$$\begin{aligned}\textbf{t-channel}\propto\;&2\big(\overline{u}(k_4,+)\mathcal{P}_R u(k_1,+)\big)\big(\overline{v}(k_2,-)\not{k}_3\mathcal{P}_R\xi\big)\\&-2m\big(\overline{u}(k_4,+)\mathcal{P}_L u(k_1,+)\big)\big(\overline{v}(k_2,-)\mathcal{P}_R\xi\big)\\&-\big(\overline{v}(k_2,-)(\not{k}_1-\not{k}_4+m)u(k_1,+)\big)\big(\overline{u}(k_4,+)\mathcal{P}_R\xi\big)\\&-\big(\overline{v}(k_2,-)(\not{k}_1-\not{k}_4+m)\gamma^5 u(k_1,+)\big)\big(\overline{u}(k_4,+)\mathcal{P}_R\xi\big)\\&+3m\big(\overline{u}(k_4,+)u(k_1,+)\big)\big(\overline{v}(k_2,-)\mathcal{P}_R\xi\big)\\&+m\big(\overline{u}(k_4,+)\gamma^5 u(k_1,+)\big)\big(\overline{v}(k_2,-)\mathcal{P}_R\xi\big)\\&+2\big(\overline{u}(k_4,+)\mathcal{P}_R(\not{k}_1-\not{k}_4-m)u(k_2,-)\big)\big(\overline{v}(k_1,+)\mathcal{P}_R\xi\big)\end{aligned}\tag{5.40}$$

$$
\begin{aligned}
\textbf{u-channel} \propto\; & -2\big(\overline{u}(k_4,+)\mathcal{P}_R u(k_2,-)\big)\big(\overline{v}(k_1,+)\not{k}_3\mathcal{P}_R\xi\big)\\
& -\big(\overline{v}(k_2,-)(\not{k}_1-\not{k}_3+m)u(k_1,+)\big)\big(\overline{u}(k_4,+)\mathcal{P}_R\xi\big)\\
& +\big(\overline{v}(k_2,-)(\not{k}_1-\not{k}_3-m)\gamma^5 u(k_1,+)\big)\big(\overline{u}(k_4,+)\mathcal{P}_R\xi\big)\\
& +\big(\overline{u}(k_4,+)(\not{k}_1-\not{k}_3+m)u(k_1,+)\big)\big(\overline{v}(k_2,-)\mathcal{P}_R\xi\big)\\
& -\big(\overline{u}(k_4,+)(\not{k}_1-\not{k}_3-m)\gamma^5 u(k_1,+)\big)\big(\overline{v}(k_2,-)\mathcal{P}_R\xi\big)\\
& -4m\big(\overline{u}(k_4,+)\mathcal{P}_R u(k_2,-)\big)\big(\overline{v}(k_1,+)\mathcal{P}_R\xi\big)
\end{aligned}
\tag{5.41}
$$

The first, third and fourth line of (5.39) can be combined to give

$$
-4m\big(\overline{v}_2\mathcal{P}_R u_1\big)\big(\overline{u}_4\mathcal{P}_R\xi\big)
$$

(in the sequel we abbreviate $u(k_1,+)$ by u_1 etc.). Adding the second line from equation (5.39), we arrive at

$$
-2\big(\overline{v}_2\mathcal{P}_R u_1\big)\big(\overline{u}_4\,(\not{k}_3+2m)\,\mathcal{P}_R\xi\big) \quad . \tag{5.42}
$$

Adding the fifth and sixth line of (5.39) yields

$$
2\big(\overline{v}_2\mathcal{P}_R\xi\big)\big(\overline{u}_4\,(\not{k}_1+\not{k}_2+m)\,\mathcal{P}_R u_1\big) \quad . \tag{5.43}
$$

Applying the Fierz identities, we bring this expression and also the term of the last line in (5.39) into the form of (5.42). In the following calculation we use the notation $k_{12} \equiv k_1 + k_2$. The brackets indicate our fundamental spinors in spinor products of the Fierz identities. In contrast to the Fierz identities used for checking the Jacobi identities, there is no additional sign in here as there is only one anticommuting spinor.

$$
\begin{aligned}
2\big([\overline{u}_4(\not{k}_{12}+m)]\,\mathcal{P}_R u_1\big)\big(\overline{v}_2\,[\mathcal{P}_R\xi]\big) = & +\frac{1}{2}\big(\overline{u}_4(\not{k}_{12}+m)\mathcal{P}_R\xi\big)\big(\overline{v}_2\mathcal{P}_R u_1\big)\\
& +\frac{1}{2}\big(\overline{u}_4(\not{k}_{12}+m)\gamma^5\mathcal{P}_R\xi\big)\big(\overline{v}_2\gamma^5\mathcal{P}_R u_1\big)\\
& +\frac{1}{2}\big(\overline{u}_4(\not{k}_{12}+m)\gamma^\mu\mathcal{P}_R\xi\big)\big(\overline{v}_2\gamma_\mu\mathcal{P}_R u_1\big)\\
& +\frac{1}{2}\big(\overline{u}_4(\not{k}_{12}+m)\gamma^5\gamma^\mu\mathcal{P}_R\xi\big)\big(\overline{v}_2\gamma_\mu\gamma^5\mathcal{P}_R u_1\big)\\
& +\frac{1}{4}\big(\overline{u}_4(\not{k}_{12}+m)\sigma^{\mu\nu}\mathcal{P}_R\xi\big)\big(\overline{v}_2\sigma_{\mu\nu}\mathcal{P}_R u_1\big)
\end{aligned}
\tag{5.44}
$$

By Fierzing, the last line of (5.39) can be written as

$$
\begin{aligned}
-2\big([\overline{u}_4(\not{k}_{12}+m)]\,\mathcal{P}_R u_2\big)\big(\overline{v}_1\,[\mathcal{P}_R\xi]\big) = & -\frac{1}{2}\big(\overline{u}_4(\not{k}_{12}+m)\mathcal{P}_R\xi\big)\big(\overline{v}_1\mathcal{P}_R u_2\big)\\
& -\frac{1}{2}\big(\overline{u}_4(\not{k}_{12}+m)\gamma^5\mathcal{P}_R\xi\big)\big(\overline{v}_1\gamma^5\mathcal{P}_R u_2\big)\\
& -\frac{1}{2}\big(\overline{u}_4(\not{k}_{12}+m)\gamma^\mu\mathcal{P}_R\xi\big)\big(\overline{v}_1\gamma_\mu\mathcal{P}_R u_2\big)\\
& -\frac{1}{2}\big(\overline{u}_4(\not{k}_{12}+m)\gamma^5\gamma^\mu\mathcal{P}_R\xi\big)\big(\overline{v}_1\gamma_\mu\gamma^5\mathcal{P}_R u_2\big)\\
& -\frac{1}{4}\big(\overline{u}_4(\not{k}_{12}+m)\sigma^{\mu\nu}\mathcal{P}_R\xi\big)\big(\overline{v}_1\sigma_{\mu\nu}\mathcal{P}_R u_2\big) \quad .
\end{aligned}
\tag{5.45}
$$

To give the expressions a common structure we again use the rules of [19] to "turn round" the second term in parentheses on the right hand side of (5.45):

$$
\begin{aligned}
(5.45) \;=\; &+\frac{1}{2}\big(\overline{u}_4(\not{k}_{12}+m)\mathcal{P}_R\xi\big)\big(\overline{v}_2\mathcal{P}_R u_1\big)\\
&+\frac{1}{2}\big(\overline{u}_4(\not{k}_{12}+m)\gamma^5\mathcal{P}_R\xi\big)\big(\overline{v}_2\gamma^5\mathcal{P}_R u_1\big)\\
&-\frac{1}{2}\big(\overline{u}_4(\not{k}_{12}+m)\gamma^\mu\mathcal{P}_R\xi\big)\big(\overline{v}_2\gamma_\mu\mathcal{P}_L u_1\big)\\
&+\frac{1}{2}\big(\overline{u}_4(\not{k}_{12}+m)\gamma^5\gamma^\mu\mathcal{P}_R\xi\big)\big(\overline{v}_2\gamma_\mu\gamma^5\mathcal{P}_L u_1\big)\\
&-\frac{1}{4}\big(\overline{u}_4(\not{k}_{12}+m)\sigma^{\mu\nu}\mathcal{P}_R\xi\big)\big(\overline{v}_2\sigma_{\mu\nu}\mathcal{P}_R u_1\big)
\end{aligned}
\tag{5.46}
$$

When adding (5.44) and (5.46) the tensor part vanishes. Absorbing the γ^5 matrices into the chiral projectors the vector contributions in (5.44) and (5.46) cancel the terms containing the axial vector, while the scalar and pseudoscalar contributions can be combined to give:

$$2\big(\overline{u}_4(\not{k}_{12}+m)\mathcal{P}_R\xi\big)\big(\overline{v}_2\mathcal{P}_R u_1\big) \quad . \tag{5.47}$$

Summing up (5.42) and (5.47) yields the following result for the whole s-channel contributions

$$2\big(\overline{u}_4(\not{k}_1+\not{k}_2-\not{k}_3-m)\mathcal{P}_R\xi\big)\big(\overline{v}_2\mathcal{P}_R u_1\big) = 2\big(\overline{u}_4(\not{k}_4-m)\mathcal{P}_R\xi\big)\big(\overline{v}_2\mathcal{P}_R u_1\big) = 0 \quad . \tag{5.48}$$

In the analytical expression for the t-channel diagrams (5.40), combining the first two as well as the fifth and the sixth line gives

$$2\big(\overline{v}_2(\not{k}_3+2m)\mathcal{P}_R\xi\big)\big(\overline{u}_4\mathcal{P}_R u_1\big) \quad . \tag{5.49}$$

On the other hand, the third and fourth line yield

$$-2\big(\overline{v}_2(\not{k}_1-\not{k}_4+m)\mathcal{P}_R u_1\big)\big(\overline{u}_4\mathcal{P}_R\xi\big) \quad . \tag{5.50}$$

To perform the calculation in a more effective way, we manipulate the last line in (5.40), in particular we "turn round" the first term in parentheses,

$$+2\big(\overline{v}_2(\not{k}_1-\not{k}_4+m)\mathcal{P}_R v_4\big)\big(\overline{v}_1\mathcal{P}_R\xi\big) \quad . \tag{5.51}$$

It also has to be Fierz transformed, together with (5.50), to get the same spinor structure as (5.49). Again we use the notation $k_{14} \equiv k_1 - k_4$, the brackets distinguishing the spinors used as the fundamental ones in the Fierz identities. From (5.50) we obtain

$$
\begin{aligned}
-2\big([\overline{v}_2(\not{k}_{14}+m)]\,\mathcal{P}_R u_1\big)\big(\overline{u}_4\,[\mathcal{P}_R\xi]\big) \;=\; &-\frac{1}{2}\big(\overline{v}_2(\not{k}_{14}+m)\mathcal{P}_R\xi\big)\big(\overline{u}_4\mathcal{P}_R u_1\big)\\
&-\frac{1}{2}\big(\overline{v}_2(\not{k}_{14}+m)\gamma^5\mathcal{P}_R\xi\big)\big(\overline{u}_4\gamma^5\mathcal{P}_R u_1\big)\\
&-\frac{1}{2}\big(\overline{v}_2(\not{k}_{14}+m)\gamma^\mu\mathcal{P}_R\xi\big)\big(\overline{u}_4\gamma_\mu\mathcal{P}_R u_1\big)\\
&-\frac{1}{2}\big(\overline{v}_2(\not{k}_{14}+m)\gamma^5\gamma^\mu\mathcal{P}_R\xi\big)\big(\overline{u}_4\gamma_\mu\gamma^5\mathcal{P}_R u_1\big)\\
&-\frac{1}{4}\big(\overline{v}_2(\not{k}_{14}+m)\sigma^{\mu\nu}\mathcal{P}_R\xi\big)\big(\overline{u}_4\sigma_{\mu\nu}\mathcal{P}_R u_1\big)
\end{aligned}
\tag{5.52}
$$

For the Fierz transformation of (5.51) we "turn round" the product containing the spinors $\overline{v}_1$ and v_4, getting the spinors $\overline{u}_4$ and u_1.

$$\begin{aligned}2\big([\overline{v}_2(\not{k}_{14}+m)]\,\mathcal{P}_R v_4\big)\big(\overline{v}_1\,[\mathcal{P}_R\xi]\big) = &-\frac{1}{2}\big(\overline{v}_2(\not{k}_{14}+m)\mathcal{P}_R\xi\big)\big(\overline{u}_4\mathcal{P}_R u_1\big)\\ &-\frac{1}{2}\big(\overline{v}_2(\not{k}_{14}+m)\gamma^5\mathcal{P}_R\xi\big)\big(\overline{u}_4\gamma^5\mathcal{P}_R u_1\big)\\ &+\frac{1}{2}\big(\overline{v}_2(\not{k}_{14}+m)\gamma^\mu\mathcal{P}_R\xi\big)\big(\overline{u}_4\gamma_\mu\mathcal{P}_L u_1\big)\\ &-\frac{1}{2}\big(\overline{v}_2(\not{k}_{14}+m)\gamma^5\gamma^\mu\mathcal{P}_R\xi\big)\big(\overline{u}_4\gamma_\mu\gamma^5\mathcal{P}_L u_1\big)\\ &+\frac{1}{4}\big(\overline{v}_2(\not{k}_{14}+m)\sigma^{\mu\nu}\mathcal{P}_R\xi\big)\big(\overline{u}_4\sigma_{\mu\nu}\mathcal{P}_R u_1\big)\end{aligned} \tag{5.53}$$

As was the case for the s-channel, the tensor contributions to (5.52) and (5.53) cancel out, while in each equation the vector part again cancels the axial vector. The scalar and pseudoscalar parts from both Fierz transformations give

$$+2\big(\overline{v}_2(\not{k}_4-\not{k}_1-m)\mathcal{P}_R\xi\big)\big(\overline{u}_4\mathcal{P}_R u_1\big)\quad , \tag{5.54}$$

so finally the result for the t-channel is written as:

$$2\big(\overline{v}_2(\not{k}_3+\not{k}_4-\not{k}_1+m)\mathcal{P}_R\xi\big)\big(\overline{u}_4\mathcal{P}_R u_1\big) = 2\big(\overline{v}_2(\not{k}_2+m)\mathcal{P}_R\xi\big)\big(\overline{u}_4\mathcal{P}_R u_1\big) = 0 \tag{5.55}$$

The same calculation goes through for the u-channel, transferring (5.41):

$$\begin{aligned}(5.41) = &-2\big(\overline{v}_1(\not{k}_3+2m)\mathcal{P}_R\xi\big)\big(\overline{u}_4\mathcal{P}_R u_2\big)\\ &+2\big(\overline{v}_1(\not{k}_2-\not{k}_4+m)\mathcal{P}_R u_2\big)\big(\overline{u}_4\mathcal{P}_R\xi\big)\\ &-2\big(\overline{v}_1(\not{k}_2-\not{k}_4+m)\mathcal{P}_R v_4\big)\big(\overline{v}_2\mathcal{P}_R\xi\big)\end{aligned} \tag{5.56}$$

The Fierz transformations of the last two lines (again we "invert" the products containing $\overline{v}_2$ and v_4 in the third line and abbreviate k_2-k_4 by k_{24}) are:

$$\begin{aligned}2\big([\overline{v}_1(\not{k}_{24}+m)]\,\mathcal{P}_R u_2\big)\big(\overline{u}_4\,[\mathcal{P}_R\xi]\big) = &\ \frac{1}{2}\big(\overline{v}_1(\not{k}_{24}+m)\mathcal{P}_R\xi\big)\big(\overline{u}_4\mathcal{P}_R u_2\big)\\ &+\frac{1}{2}\big(\overline{v}_1(\not{k}_{24}+m)\gamma^5\mathcal{P}_R\xi\big)\big(\overline{u}_4\gamma^5\mathcal{P}_R u_2\big)\\ &+\frac{1}{2}\big(\overline{v}_1(\not{k}_{24}+m)\gamma^\mu\mathcal{P}_R\xi\big)\big(\overline{u}_4\gamma_\mu\mathcal{P}_R u_2\big)\\ &+\frac{1}{2}\big(\overline{v}_1(\not{k}_{24}+m)\gamma^5\gamma^\mu\mathcal{P}_R\xi\big)\big(\overline{u}_4\gamma_\mu\gamma^5\mathcal{P}_R u_2\big)\\ &+\frac{1}{4}\big(\overline{v}_1(\not{k}_{24}+m)\sigma^{\mu\nu}\mathcal{P}_R\xi\big)\big(\overline{u}_4\sigma_{\mu\nu}\mathcal{P}_R u_2\big)\end{aligned} \tag{5.57}$$

$$\begin{aligned}-2\big([\overline{v}_1(\not{k}_{24}+m)]\,\mathcal{P}_R v_4\big)\big(\overline{v}_2\,[\mathcal{P}_R\xi]\big) = &\ \frac{1}{2}\big(\overline{v}_1(\not{k}_{24}+m)\mathcal{P}_R\xi\big)\big(\overline{u}_4\mathcal{P}_R u_2\big)\\ &+\frac{1}{2}\big(\overline{v}_1(\not{k}_{24}+m)\gamma^5\mathcal{P}_R\xi\big)\big(\overline{u}_4\gamma^5\mathcal{P}_R u_2\big)\\ &-\frac{1}{2}\big(\overline{v}_1(\not{k}_{24}+m)\gamma^\mu\mathcal{P}_R\xi\big)\big(\overline{v}_2\gamma_\mu\mathcal{P}_R v_4\big)\\ &-\frac{1}{2}\big(\overline{v}_1(\not{k}_{24}+m)\gamma^5\gamma^\mu\mathcal{P}_R\xi\big)\big(\overline{v}_2\gamma_\mu\gamma^5\mathcal{P}_R v_4\big)\\ &-\frac{1}{4}\big(\overline{v}_1(\not{k}_{24}+m)\sigma^{\mu\nu}\mathcal{P}_R\xi\big)\big(\overline{u}_4\sigma_{\mu\nu}\mathcal{P}_R u_2\big)\end{aligned} \tag{5.58}$$

The vector contributions as well as the axial vector parts vanish separately for each process as in the s- and t-channels, while the tensor contributions of (5.57) and (5.58) cancel each other. The scalar and pseudoscalar contributions are equal und sum up to

$$2\big(\overline{v}_1(\not{k}_{24}+m)\mathcal{P}_R\xi\big)\big(\overline{u}_4\mathcal{P}_R u_2\big). \tag{5.59}$$

Therefore the result of (5.56) is

$$2\big(\overline{v}_1(\not{k}_2-\not{k}_3-\not{k}_4-m)\mathcal{P}_R\xi\big)\big(\overline{u}_4\mathcal{P}_R u_2\big)=-2\big(\overline{v}_1(\not{k}_1+m)\mathcal{P}_R\xi\big)\big(\overline{u}_4\mathcal{P}_R u_2\big)=0 \quad . \tag{5.60}$$

So finally we can see that s-, t- and u-channel diagrams vanish separately and we find the SWIs of $(2\to 2)$ processes containing two as well as four fermions to be fulfilled.

Chapter 6

The O'Raifeartaigh model

6.1 Spontaneous breaking of Supersymmetry

The simplest model in which supersymmetry is spontaneously broken is the O'Raifeartaigh model. To be more precise it is a whole class of models (cf. [3]), the particular O'Raifeartaigh model being only a special case. The particle content, some special remarks and the Feynman rules of the O'Raifeartaigh model (from hereon referred to as the OR model) are collected in the appendix. As was proven by O'Raifeartaigh, at least three chiral superfields are needed to make spontaneous supersymmetry breaking possible.

This model offers the opportunity to examine what happens to the SWI in the case of spontaneous breaking. Of course, the derivation of identity (3.2) breaks down together with our symmetry since the vacuum is no longer left invariant by the action of the supercharge. But we want to show an example of an SWI, in the sense, that we calculate a SWI as if (3.2) were still valid and take a look at the terms violating the SWI. The latter should turn out to be proportional to the parameters of SUSY breaking.

6.2 Preliminaries to the O'Raifeartaigh model

For the OR model as a spontaneously broken supersymmetric model the relation

$$Q\,|0\rangle = 0 \tag{6.1}$$

is no longer fulfilled, but this had to be postulated to be able to derive the SWI. This section will show what happens to the SWI if we were to assume (6.1) to be valid anyhow.

There is a higher number of particles in the OR model than in previously considered models. We gratefully make use of this fact as the number of participating diagrams in an SWI shrinks enormously with a growing variety of external particles. Unfortunately this advantage is partly lost since up to three different scalar particles appear as a result of the SUSY transformations of fermionic annihilation and creation operators.

With the experience from last chapter's toy model we can immediately write down the transformation laws of the annihilators (and therefore also for the creators).

First of all we want to introduce a common notation for all particles: The annihilators of the scalars are denoted by a_A, a_B, $a_\pm^\phi$ and $a_\pm^\Phi$, the Majorana fermion's annihilator by c, while the annihilators for the Dirac fermion are denoted by b and d as usual. The creators are the Hermitean adjoints, of course.

As for the toy model, the fermionic partner of the scalar field which is split into real and imaginary parts, gives the lefthanded component of a Dirac fermion so we can directly take over the result (5.11):

$$\boxed{\begin{aligned}[Q(\xi), a_A(k)] &= \mathrm{i}\sum_\sigma \Big(\big(\bar\xi \mathcal{P}_L u(k,\sigma)\big)\, b(k,\sigma) + \big(\bar\xi \mathcal{P}_R u(k,\sigma)\big)\, d(k,\sigma) \Big) \\ [Q(\xi), a_B(k)] &= \sum_\sigma \Big(\big(\bar\xi \mathcal{P}_L u(k,\sigma)\big)\, b(k,\sigma) - \big(\bar\xi \mathcal{P}_R u(k,\sigma)\big)\, d(k,\sigma) \Big)\end{aligned}} \tag{6.2}$$

The fermionic partner for the complex scalar field from the third superfield and its Hermitean adjoint are the righthanded component of that Dirac spinor. Consequently we can maintain (5.5) and (5.8),

$$\boxed{\left[Q(\xi), a_+^\Phi(k)\right] = \mathrm{i}\sqrt{2}\sum_\sigma \Big(\bar\xi \mathcal{P}_R u(k,\sigma)\Big)\, b(k,\sigma)} \quad , \tag{6.3}$$

$$\boxed{\left[Q(\xi), a_-^\Phi(k)\right] = \mathrm{i}\sqrt{2}\sum_\sigma \Big(\bar\xi \mathcal{P}_L u(k,\sigma)\Big)\, d(k,\sigma)} \quad . \tag{6.4}$$

In the case of the scalar field ϕ – the scalar component of the first superfield and superpartner of the Goldstino – we just have to set the two annihilators b and d equal to the Majorana annihilator c:

$$\boxed{\left[Q(\xi), a_+^\phi(k)\right] = \mathrm{i}\sqrt{2}\sum_\sigma \Big(\bar\xi \mathcal{P}_R u(k,\sigma)\Big)\, c(k,\sigma)} \quad , \tag{6.5}$$

$$\boxed{\left[Q(\xi), a_-^\phi(k)\right] = \mathrm{i}\sqrt{2}\sum_\sigma \Big(\bar\xi \mathcal{P}_L u(k,\sigma)\Big)\, c(k,\sigma)} \quad . \tag{6.6}$$

The transformations of the Dirac annihilators are analogous to (5.15) and (5.16), respectively:

$$\boxed{[Q(\xi), b(k,\sigma)] = -\mathrm{i}\overline{u}(k,\sigma)\Big(a_A(k)\mathcal{P}_R + \mathrm{i}a_B(k)\mathcal{P}_R + \sqrt{2}a_+^\Phi(k)\mathcal{P}_L\Big)\xi} \quad , \tag{6.7}$$

$$\boxed{[Q(\xi), d(k,\sigma)] = -\mathrm{i}\overline{u}(k,\sigma)\Big(a_A(k)\mathcal{P}_L - \mathrm{i}a_B(k)\mathcal{P}_L + \sqrt{2}a_-^\Phi(k)\mathcal{P}_R\Big)\xi} \quad . \tag{6.8}$$

For the first superfield we use equation (3.10) und get

$$\boxed{[Q(\xi), c(k,\sigma)] = -\mathrm{i}\sqrt{2}\,\overline{u}(k,\sigma)\Big(a_-^\phi \mathcal{P}_R + a_+^\phi \mathcal{P}_L\Big)\xi} \tag{6.9}$$

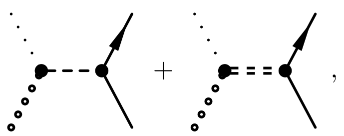
6.3 Example for an SWI in the OR model

As for the WZ model before, we want to construct an example for an SWI. Again we start with a string of fields in which the spin of initial and final states differ by half a unit. As mentioned above, we want to make use of the greater variety of particles available in this model.

Our choice for an example is the following:

$$\begin{aligned} 0 \neq & \left\langle 0 \middle| \left[Q(\xi), a_A(k_3) c(k_4,+) a_-^{\Phi\,\dagger}(k_1) a_+^{\phi\,\dagger}(k_2) \right] \middle| 0 \right\rangle \\ = & \quad \mathrm{i} \sum_\sigma \left(\bar{\xi} \mathcal{P}_L u(k_3,\sigma) \right) \cdot \mathcal{M}\left(\Phi(k_1) \phi^*(k_2) \to \Psi(k_3,\sigma) \chi(k_4,+) \right) \\ & + \mathrm{i} \sum_\sigma \left(\bar{\xi} \mathcal{P}_R u(k_3,\sigma) \right) \cdot \mathcal{M}\left(\Phi(k_1) \phi^*(k_2) \to \overline{\Psi}(k_3,\sigma) \chi(k_4,+) \right) \\ & + \mathrm{i}\sqrt{2} \sum_\sigma \left(\overline{u}(k_1,\sigma) \mathcal{P}_L \xi \right) \cdot \mathcal{M}\left(\overline{\Psi}(k_1,\sigma) \phi^*(k_2) \to A(k_3) \chi(k_4,+) \right) \end{aligned} \tag{6.10}$$

The processes resulting from the SUSY transformations of the Majorana fermion in the final state and the massless boson in the initial state do not contribute. For the transformation of the remaining particles we write down only the nonvanishing terms. The first process with two diagrams

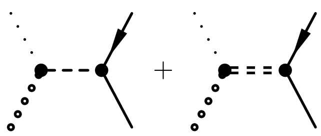


$$, \tag{6.11}$$

produces, after multiplication with the appropriate prefactor, the analytical expression

$$2g^2 m \cdot \left(\bar{\xi} \mathcal{P}_L \not{k}_3 v(k_4,+) \right) \cdot \left(\frac{1}{s - m^2 + 2\lambda g} - \frac{1}{s - m^2 - 2\lambda g} \right) \quad . \tag{6.12}$$

The second process is analogous:

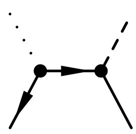


(6.13)

The result is

$$-2g^2 m \cdot \left(\bar{\xi} \mathcal{P}_R \not{k}_3 v(k_4,+) \right) \cdot \left(\frac{1}{s - m^2 + 2\lambda g} + \frac{1}{s - m^2 - 2\lambda g} \right) . \tag{6.14}$$

There exists just one diagram for the third process,



$$. \tag{6.15}$$

Here we have to keep an eye on the signs again, while having to apply the Wick theorem. The resulting amplitude is

$$+4g^2 m \cdot \left(\bar{\xi} \mathcal{P}_R (\not{k}_1 + \not{k}_2) v(k_4,+) \right) \cdot \frac{1}{s - m^2} = +4g^2 m \cdot \left(\bar{\xi} \mathcal{P}_R \not{k}_3 v(k_4,+) \right) \cdot \frac{1}{s - m^2} \tag{6.16}$$

Implicitly we used momentum conservation and the Dirac equation for $v(k_4)$, which simply is $\not{k}_4 v(k_4) = 0$ for the Majorana fermion being the Goldstino.

When choosing special transformation spinors ξ, we see that the righthanded and lefthanded part of the identity must be fulfilled separately. As is immediately seen the SWI is violated as we expected from the beginning. Inspecting the limit $\lambda \to 0$ shows that the contribution containing the lefthanded chiral projector vanishes and the parts with the righthanded chiral projectors cancel each other. This is understandable by remembering that the parameter λ controls the spontaneous symmetry breaking of the OR model as it produces the mass splitting between the particles of the second and the third superfield, which came from diagonalizing the mass terms.

This violation of the SWI of the type derived in [20] and [21] stems from the non-invariance of the vacuum under SUSY transformations in spontaneously broken SUSY theories. It can be avoided by using a formalism based on the concept of a conserved Noether current for the supersymmetry; this will be shown in the next part.

Part II

SUSY Ward identities via the current

Chapter 7

The supersymmetric current and SWI

There are some inherent problems in the method of calculating SWIs the way presented in the last part: It does not work for spontaneously broken supersymmetry and is also only applicable for on-shell identities. To develop stringent tests for supersymmetric field theories, it will prove useful to consider off-shell identities as well, as much more of the underlying physics is involved in such relations. In this part we will first present how SWI can be implemented when using the current of the supersymmetry and then show examples for the Wess-Zumino model. To verify that this method is also valid for spontaneously broken supersymmetry, we extend our calculations to the O'Raifeartaigh model. Afterwards we turn to the combination of (global) supersymmetry and gauge symmetries when examining currents in supersymmetric Yang-Mills theories. This is important because realistic models should, of course, incorporate at least the gauge symmetries of the Standard Model.

7.1 Ward identities – current vs. external states

In this section we describe the connection between the SWI in the formalism derived in [20] and [21] and similar relations which can be obtained with the help of supersymmetric current conservation. The name "supersymmetric" current is a bit misleading as this current is not invariant under SUSY transformations. In fact, the current mentioned here is closely related to a spinor component of a real superfield provided with an additional vector index, called the supercurrent (cf. [3], [24]). The scalar component of the supercurrent is the current of R symmetry, while the vector component is given by the energy-momentum tensor. The supersymmetric current has the Lorentz transformation properties of a vectorspinor. In a local version of supersymmetry – supergravity – the corresponding gauge field is the gravitino.

To derive this kind of SWI we write down a time-ordered product of a string of field operators (appearing in the supersymmetric model under consideration) with the operator insertion of the supersymmetric current,

$$\langle 0|\mathrm{T}\left[\mathcal{J}^{\mu}(x)\mathcal{O}_1(y_1)\mathcal{O}_2(y_2)\ldots\mathcal{O}_n(y_n)\right]|0\rangle \tag{7.1}$$

Taking the derivative of this expression with respect to x^μ (we use the abbreviation $\partial^x_\mu \equiv \partial/\partial x^\mu$), we get:

$$\begin{aligned}
&\mathrm{i}\partial^x_\mu \langle 0|\mathrm{T}\left[\mathcal{J}^\mu(x)\mathcal{O}_1(y_1)\ldots\mathcal{O}_n(y_n)\right]|0\rangle \\
&\quad = \sum_{i=1}^{n}\chi_i \left\langle 0\middle|\mathrm{T}\left[\mathcal{O}_1\ldots\mathcal{O}_{i-1}\left[\mathrm{i}\mathcal{J}^0(x),\mathcal{O}_i(y_i)\right]_{P_i}\delta(x^0-y_i^0)\,\mathcal{O}_{i+1}\ldots\mathcal{O}_n\right]\middle|0\right\rangle \\
&\qquad\qquad + \mathrm{i}\left\langle 0\middle|\mathrm{T}\left[\partial^x_\mu\mathcal{J}^\mu(x)\mathcal{O}_1(y_1)\ldots\mathcal{O}_n(y_n)\right]\middle|0\right\rangle
\end{aligned} \tag{7.2}$$

Here χ_i has the meaning of a sign prefactor

$$\chi_i \equiv (-1)^{\sum_{j=1}^{i-1}P_j}, \tag{7.3}$$

which arises by anticommuting the Grassmann odd current with Fermi field operators. P is the Grassmann parity of the fields, 1 for fermions and 0 for bosons. In the same manner we have introduced the graded commutator

$$[A,B]_{P=1} \equiv \{A,B\} \text{ for fermions,} \quad [A,B]_{P=0} \equiv [A,B] \text{ otherwise} \tag{7.4}$$

as an anticommutator in the case of two fermionic operators and a commutator in all other cases.

The last term in (7.2), which is created by applying the derivative to the current, vanishes due to current conservation. The terms with the graded commutators arise when acting with the time derivative on the step functions in the time ordered product. We make use of the fact that the equal time commutator (or anticommutator in the case of a fermionic operator) of the zero component of the current with an operator (for instance, the field operator of the fundamental fields of the theory) equals the symmetry transformation (in our case the SUSY transformation) of the considered field:

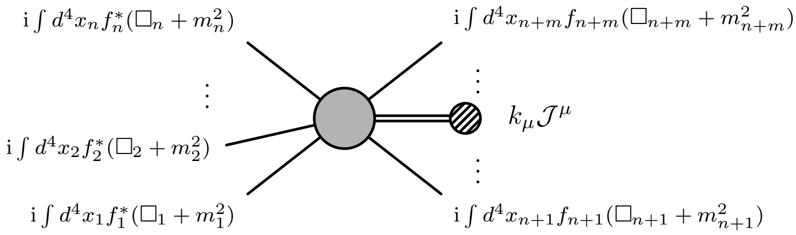
$$\left[\mathrm{i}\bar{\xi}\mathcal{J}^0(x),\mathcal{O}(y)\right]\delta(x^0-y^0) = \delta_\xi\mathcal{O}(y)\cdot\delta^4(x-y) \tag{7.5}$$

With the help of this relation we can rewrite the right hand side of (7.2). Furthermore we switch to momentum space and replace the spacetime derivative acting on the left hand side of equation (7.2) by the momentum k_μ which flows into the Green function through the current operator insertion (so $-k^\mu = \sum_i p_i^\mu$ is the sum over the incoming momenta of all other external legs).

$$\begin{aligned}
&k_\mu \mathrm{F.T.}\left\langle 0\middle|\mathrm{T}\left[\bar{\xi}\mathcal{J}^\mu(x)\mathcal{O}_1(y_1)\ldots\mathcal{O}_n(y_n)\right]\middle|0\right\rangle \\
&\qquad = \sum_{i=1}^{n}\mathrm{F.T.}\,\langle 0|\mathrm{T}\left[\mathcal{O}_1\ldots\mathcal{O}_{i-1}\left(\delta_\xi\mathcal{O}_i(y_i)\right)\mathcal{O}_{i+1}\ldots\mathcal{O}_n\right]|0\rangle\cdot\delta^4(x-y_i)
\end{aligned} \tag{7.6}$$

In (7.6) the supersymmetric current has been multiplied by the SUSY transformation parameter ξ and hence became a bosonic operator. There are two consequences: we could forget about the sign prefactor which was part of (7.2) and all graded commutators became commutators. In (7.5) and (7.6) we used the usual notation for the SUSY transformations of the fields (with transformation parameter ξ).

At tree level the identity (7.2) is valid for linearly as well as nonlinearly realized symmetries both for on-shell and off-shell processes (cf. for instance the path integral derivation of the Ward identities in [25]). In the case of nonlinearly realized symmetries, not only higher than quadratic terms will appear in the current operator but also composite operators in the transformations of the fields. To put the identity (7.2) on the mass shell we have to apply the LSZ reduction formula [13], [15], [22] to all external legs except the current itself, which remains unamputated:



We used the abbreviation $f_i \equiv e^{-\mathrm{i}k_i x_i}/\sqrt{(2\pi)^3 2k_i^0}$. For simplicity we denoted only the amputation procedure for bosons. The big grey blob stands for the process under consideration (i.e. the interaction operators needed to connect the external fields in (7.2)), while the smaller blob will become our standard convention for a current insertion. On shell, all the so called contact terms on the right hand side of equation (7.2) vanish. This is seen by inspection of the amputation procedure for those Green functions with the transformed fields: Let the external particle corresponding to the ith field have momentum p_i on the left hand side, then on the right hand side the particle corresponding to the transformed field has its momentum increased by the momentum influx through the current $p_i + k$. For the sake of simplicity, we show an example involving only scalar fields:

$$D_F(p_i)^{-1} \cdot D_F(p_i + k) = \frac{p_i^2 - m_i^2}{(p_i + k)^2 - m_i^2} \tag{7.7}$$

These two propagator factors do not cancel like all other propagators of external particles do, hence when setting the external momenta $p_j, j = 1, 2, \ldots$ on the mass shell, this yields zero for every term on the right hand side.

Another interesting phenomenon happens for spontaneously broken symmetries, where a field gets a vacuum expectation value and is therefore shifted by a constant. A term linear in the field appears in the current, or more precisely, a term proportional to the derivative of the Goldstone boson. This contributes tadpole-like diagrams which, if resummed, shift the appropriate poles of the fields according to the mass splitting from the spontaneous symmetry breaking. Since coupling constants and vacuum expectation values are combined to yield masses of particles, there is a mixing of different orders in perturbation theory contributing to the Ward identity. For supersymmetric field theories the corresponding term in the current is given by a gamma matrix times the derivative of the Goldstino field. We will study this in detail in the O'Raifeartaigh model below.

7.2 Simplest example – Wess-Zumino model

Like any continous symmetry in a field theory, supersymmetry possesses a conserved current whose charge is the generator of the symmetry transformation. Supersymmetry is no symmetry of the Lagrangean density but only of the action. It transforms the Lagrangean density into a total derivative which vanishes upon integration over space-time. The following discussion is similar to that in [3]. If we assume that the change of the Lagrangean density under a SUSY transformation takes on the form

$$\delta_\xi \mathcal{L} = \bar{\xi} \partial_\mu K^\mu, \tag{7.8}$$

we can calculate the structure of K^μ. We want to derive the supersymmetric current for the WZ model as this is the simplest supersymmetric model. In the Lagrangean density only the D-term of the kinetic part and the F-terms from the superpotential appear. The SUSY transformation of a D-term of an arbitrary superfield is given by [3], [5] [1]

$$\delta_\xi D = \bar{\xi}\gamma^5 \partial\!\!\!/ \lambda \quad . \tag{7.9}$$

Here λ is a spinor being the θ^3 coefficient in a superspace expansion of a general superfield. We conclude, that for the kinetic part of the WZ model Lagrangean density as a product of a right- and a lefthanded chiral superfield, $\hat{\Phi}^\dagger\hat{\Phi}$,

$$\delta_\xi \mathcal{L}_{\rm kin} = \delta_\xi \left[\frac{1}{2}\hat{\Phi}^\dagger\hat{\Phi}\right]_D = \bar{\xi}\gamma^5 \partial\!\!\!/ \frac{1}{2}\left[\hat{\Phi}^\dagger\hat{\Phi}\right]_\lambda \quad . \tag{7.10}$$

The appropriate λ can be read off from equation (26.2.24) in [3] or, in our conventions, from equation (5.116) in [5], by taking into consideration that the general superfield $\hat{\Phi}_1$ there is to be set to the right chiral superfield $\hat{\Phi}^\dagger$ and the second superfield $\hat{\Phi}_2$ to the Hermitean adjoint left chiral superfield $\hat{\Phi}$. This enables us to make the following replacements (of course, the SUSY transformation can be done by brute force in a component language but the superfield formalism is much more elegant) [2]:

$$\begin{array}{ll}
\lambda_1 \equiv 0 & \lambda_2 \equiv 0 \\
V_1^\mu \equiv -\mathrm{i}\partial^\mu\phi^* & V_2^\mu \equiv \mathrm{i}\partial^\mu\phi \\
C_1 \equiv \phi^* & \omega_1 \equiv \mathrm{i}\sqrt{2}\Psi_R \\
C_2 \equiv \phi & \omega_2 \equiv -\mathrm{i}\sqrt{2}\Psi_L \\
N_1 \equiv F^* & M_1 \equiv -\mathrm{i}F^* \\
N_2 \equiv F & M_2 \equiv \mathrm{i}F
\end{array} \tag{7.11}$$

The result is

$$K^\mu_{\rm kin} = \frac{1}{\sqrt{2}}\gamma^\mu\Big((\partial\!\!\!/\phi)\,\Psi_R + (\partial\!\!\!/\phi^*)\,\Psi_L - \mathrm{i}F\Psi_R - \mathrm{i}F^*\Psi_L\Big) \quad . \tag{7.12}$$

From the transformation of the superpotential's F-terms we write down the relation

$$\delta_\xi \mathcal{L}_{\rm pot} = \delta_\xi\left[\frac{m}{2}\hat{\Phi}^2 + \frac{\lambda}{3!}\hat{\Phi}^3\right]_F + \text{h.c.} = -\mathrm{i}\sqrt{2}\,\bar{\xi}\mathcal{P}_L\partial\!\!\!/\left[\frac{m}{2}\hat{\Phi}^2 + \frac{\lambda}{3!}\hat{\Phi}^3\right]_\psi + \text{h.c.} \tag{7.13}$$

The contribution from the potential becomes

$$K^\mu_{\rm pot} = -\mathrm{i}\sqrt{2}\gamma^\mu\Big(m\Psi_L\phi + m\Psi_R\phi^* + \frac{1}{2}\lambda\Psi_L\phi^2 + \frac{1}{2}\lambda\Psi_R(\phi^*)^2\Big) \tag{7.14}$$

(NB: Herein λ is the coupling constant of the WZ model, not a spinor component of a superfield.) So altogether we get for this contribution to the supersymmetric current

$$K^\mu = \frac{1}{\sqrt{2}}\gamma^\mu\Big((\partial\!\!\!/\phi)\,\Psi_R + (\partial\!\!\!/\phi^*)\,\Psi_L - \mathrm{i}F\Psi_R - \mathrm{i}F^*\Psi_L - 2m\mathrm{i}\Psi_L\phi$$

[1] The relative factor of i between both references comes from the different conventions concerning the metric and hence the gamma matrices.

[2] In the appendix a detailed derivation for the supersymmetric current in supersymmetric Yang-Mills theories can be found.

$$- 2mi\Psi_R\phi^* - i\lambda\Psi_L\phi^2 - i\lambda\Psi_R(\phi^*)^2\Big) \quad (7.15)$$

Inserting the definitions of the fields A, B, $\mathcal{F}$ and $\mathcal{G}$ yields

$$\begin{aligned} K^\mu = &\frac{1}{2}\gamma^\mu(\partial\!\!\!/ A)\Psi - \frac{\mathrm{i}}{2}\gamma^\mu\gamma^5(\partial\!\!\!/ B)\Psi - \frac{\mathrm{i}}{2}\gamma^\mu\mathcal{F}\Psi - \frac{1}{2}\gamma^\mu\gamma^5\mathcal{G}\Psi \\ &- \mathrm{i}m\gamma^\mu A\Psi - m\gamma^\mu\gamma^5 B\Psi - \frac{\mathrm{i}\lambda}{2\sqrt{2}}\gamma^\mu\left(A^2 - B^2\right)\Psi - \frac{\lambda}{\sqrt{2}}\gamma^\mu\gamma^5 AB\Psi \end{aligned} \quad (7.16)$$

The so called Noether part of the supersymmetric current (by which the current is given in the case of an invariant Lagrangean density) reads

$$\sum_{\text{all fields}} \frac{\partial_R\mathcal{L}}{\partial(\partial_\mu\Phi)}\delta_\xi\Phi = -\bar{\xi}N^\mu \quad . \quad (7.17)$$

In the WZ models these derivatives are

$$\frac{\partial_R\mathcal{L}}{\partial(\partial_\mu A)} = \partial^\mu A, \qquad \frac{\partial_R\mathcal{L}}{\partial(\partial_\mu B)} = \partial^\mu B, \qquad \frac{\partial_R\mathcal{L}}{\partial(\partial_\mu\Psi)} = \frac{\mathrm{i}}{2}\overline{\Psi}\gamma^\mu, \quad (7.18)$$

while the SUSY transformations of the several fields are stated in (2.5). The Noether part therefore is

$$N^\mu = -\left(\partial^\mu A\right)\Psi - \mathrm{i}\left(\partial^\mu B\right)\gamma^5\Psi - \frac{1}{2}\left[\partial\!\!\!/\left(A - \mathrm{i}\gamma^5 B\right)\right]\gamma^\mu\Psi + \frac{\mathrm{i}}{2}\left(\mathcal{F} + \mathrm{i}\gamma^5\mathcal{G}\right)\gamma^\mu\Psi \quad (7.19)$$

Adding the two parts (7.16) and (7.17) results in the supersymmetric current for the WZ model

$$\boxed{\begin{aligned} \mathcal{J}^\mu &= K^\mu + N^\mu \\ &= \mathrm{i}\left((\mathrm{i}\partial\!\!\!/ - m)A\right)\gamma^\mu\Psi + \left((\mathrm{i}\partial\!\!\!/ + m)B\right)\gamma^5\gamma^\mu\Psi \\ &\quad - \frac{\mathrm{i}\lambda}{2\sqrt{2}}\gamma^\mu\left(A^2 - B^2\right)\Psi - \frac{\lambda}{\sqrt{2}}\gamma^\mu\gamma^5 AB\Psi \end{aligned}} \quad (7.20)$$

Now we can check – even if it is a little bit cumbersome – the current conservation explicitly.

$$\begin{aligned} \partial_\mu\mathcal{J}^\mu = &-(\Box A)\Psi - \mathrm{i}mA(\partial\!\!\!/\Psi) - \underline{(\partial\!\!\!/ A)(\partial\!\!\!/\Psi)} - \underline{\mathrm{i}m(\partial\!\!\!/ A)\Psi} - \mathrm{i}(\Box B)\gamma^5\Psi + mB\gamma^5(\partial\!\!\!/\Psi) \\ &+ \underline{\mathrm{i}(\partial\!\!\!/ B)\gamma^5(\partial\!\!\!/\Psi)} - \underline{m(\partial\!\!\!/ B)\gamma^5\Psi} - \frac{\mathrm{i}\lambda}{2\sqrt{2}}\left(A^2 - B^2\right)\partial\!\!\!/\Psi - \underline{\frac{\mathrm{i}\lambda}{\sqrt{2}}(\partial\!\!\!/ A)A\Psi} \\ &+ \underline{\frac{\mathrm{i}\lambda}{\sqrt{2}}(\partial\!\!\!/ B)B\Psi} + \frac{\lambda}{\sqrt{2}}\gamma^5 AB\partial\!\!\!/\Psi + \underline{\frac{\lambda}{\sqrt{2}}\gamma^5(\partial\!\!\!/ A)B\Psi} + \underline{\frac{\lambda}{\sqrt{2}}\gamma^5 A(\partial\!\!\!/ B)\Psi} \\ = &\frac{\lambda}{2\sqrt{2}}(\overline{\Psi}\Psi)\Psi - m\mathcal{F}\Psi - \frac{\lambda}{\sqrt{2}}A\mathcal{F}\Psi - \frac{\lambda}{\sqrt{2}}B\mathcal{G}\Psi + \frac{\lambda}{2\sqrt{2}}(\overline{\Psi}\gamma^5\Psi)\gamma^5\Psi - \mathrm{i}m\mathcal{G}\gamma^5\Psi \\ &+ \frac{\mathrm{i}\lambda}{\sqrt{2}}B\mathcal{F}\gamma^5\Psi - \frac{\mathrm{i}\lambda}{\sqrt{2}}A\mathcal{G}\gamma^5\Psi + \mathrm{i}\mathcal{F}(\partial\!\!\!/\Psi) - \mathcal{G}\gamma^5(\partial\!\!\!/\Psi) \end{aligned}$$

The underlined terms cancel due to the equation of motion of the Majorana field Ψ. In the second equality the first eight terms stem from the equations of motion for the scalar fields A and B, while the last two come from inserting the equations of motion for the spinor field into the terms not underlined. The terms linear in $\mathcal{F}$ and $\mathcal{G}$ can be combined to give the equations of motion for the Majorana field and we are left with the trilinear fermion terms. Noting that third powers of Grassmann odd two component spinors $(\psi\psi)\psi$ vanish, the calculation

$$\begin{aligned} \left(\overline{\Psi}\Psi\right)\Psi &= (\psi\psi + \bar{\psi}\bar{\psi}) \cdot \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} = \begin{pmatrix} (\bar{\psi}\bar{\psi})\psi \\ (\psi\psi)\bar{\psi} \end{pmatrix} \\ \left(\overline{\Psi}\gamma^5\Psi\right)\gamma^5\Psi &= (-\psi\psi + \bar{\psi}\bar{\psi}) \cdot \begin{pmatrix} -\psi \\ \bar{\psi} \end{pmatrix} = \begin{pmatrix} -(\bar{\psi}\bar{\psi})\psi \\ -(\psi\psi)\bar{\psi} \end{pmatrix}, \end{aligned} \tag{7.21}$$

shows the cancellation of the trilinear fermion terms. This finishes the proof of the desired current conservation:

$$\boxed{\partial_\mu \mathcal{J}^\mu = 0} \tag{7.22}$$

The current for a general model with an arbitrary number of superfields and the proof for its conservation can be found in appendix D.1.

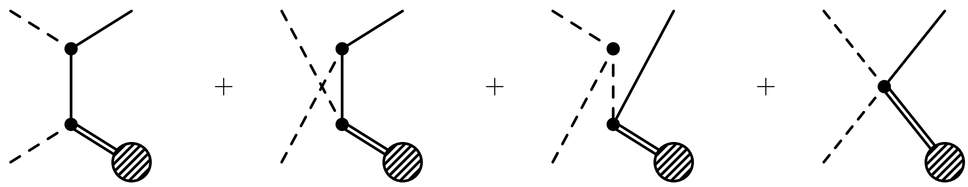
Chapter 8

SWI via the current

8.1 Starting point: WZ model

In this section we want to calculate supersymmetric Ward identities (SWI) for the WZ model obtained with the help of the current as constructed in the previous chapter. The current for the WZ model is given by (7.20). We will show an example for an on-shell identity with three external particles (SWI with two external particles are just given by the propagators of the theory in the contact terms and are rather trivial) as well as for an off-shell SWI with the same number of external particles.

For the on-shell example, where the contact terms are absent, we choose a $(2 \to 1)$ process with two incoming scalar particles A, one outgoing fermion Ψ and a current insertion, to which (in lowest order perturbation theory) four different diagrams contribute:



The momenta of the incoming As are denoted by k_1 and k_2 while the outgoing Majorana fermion's momentum is k'. The analytical expressions for the four diagrams are (from right to left):

$$(1) \qquad -\frac{\mathrm{i}\lambda}{\sqrt{2}}\gamma^\mu v(k'), \tag{8.1a}$$

$$(2) \qquad +\frac{3\mathrm{i}m\lambda}{\sqrt{2}\,(s-m^2)}\,(\not{k}_1+\not{k}_2-m)\,\gamma^\mu v(k'), \tag{8.1b}$$

$$(3) \qquad +\frac{\mathrm{i}\lambda}{\sqrt{2}\,(t-m^2)}\,(\not{k}_1-m)\,\gamma^\mu\,(\not{k}_2-\not{k}'+m)\,v(k'), \tag{8.1c}$$

$$(4) \qquad +\frac{\mathrm{i}\lambda}{\sqrt{2}\,(u-m^2)}\,(\not{k}_2-m)\,\gamma^\mu\,(\not{k}_1-\not{k}'+m)\,v(k'). \tag{8.1d}$$

For this problem the Mandelstam variables are

$$s \equiv (k_1+k_2)^2, \qquad t \equiv (k_2-k')^2, \qquad u \equiv (k_1-k')^2.$$

The verification of the SWI only needs the use of the Dirac equation $(\not{k}' + m)\, v(k') = 0$ and the relation $\not{k}\not{k} = k^2$. Applying the 4-gradient to the above matrix element produces the following sum that can be easily confirmed to be zero:

$$\begin{aligned}
\partial_\mu \langle \Psi | \mathcal{J}^\mu | AA \rangle &= \frac{\lambda}{\sqrt{2}} \Bigg[(\not{k}_1 + \not{k}_2 + m) - \frac{3m}{s - m^2} (\not{k}_1 + \not{k}_2 - m)\, (\not{k}' - \not{k}_1 - \not{k}_2) \\
&\qquad - \frac{(\not{k}_1 - m)\, (\not{k}' - \not{k}_2 - \not{k}_1)\, (\not{k}' - \not{k}_2 - m)}{t - m^2} \\
&\qquad - \frac{(\not{k}_2 - m)\, (\not{k}' - \not{k}_1 - \not{k}_2)\, (\not{k}' - \not{k}_1 - m)}{u - m^2} \Bigg] v(k') \\
&= \frac{\lambda}{\sqrt{2}} \Bigg[(\not{k}_1 + \not{k}_2 + m) - 3m - \frac{(\not{k}_1 - m)\, (t + m\not{k}_1)}{t - m^2} \\
&\qquad - \frac{(\not{k}_1 - m)\, (\not{k}_1 + m)\, (\not{k}_2 - \not{k}')}{t - m^2} - \frac{(\not{k}_2 - m)\, (u + m\not{k}_2)}{u - m^2} \\
&\qquad - \frac{(\not{k}_2 - m)\, (\not{k}_2 + m)\, (\not{k}_1 - \not{k}')}{u - m^2} \Bigg] v(k') \\
&= \frac{\lambda}{\sqrt{2}} \Big[m + \not{k}_1 + \not{k}_2 - 3m - \not{k}_2 + m - \not{k}_1 + m \Big] v(k') \;\; = \;\; 0 \quad \surd
\end{aligned} \tag{8.2}$$

Concerning (nonlinear) transformations, on-shell only the one-particle pole contributes. But for off-shell Ward identities the nonlinear terms give nonvanishing contributions in contact terms. The correct method to handle that difficulty is to define local operator insertions for every nonlinear term appearing in the transformations.

As an example for an off-shell identity we take the insertion of an A, a B and a Ψ field as the left hand side in (7.6)

$$\begin{aligned}
&\text{F.T.} \left\langle 0 \middle| \mathrm{T} \overline{\mathcal{J}_\mu}(y) \xi A(x_1) B(x_2) \Psi(x_3) \middle| 0 \right\rangle = \\
&\qquad \frac{\mathrm{i}}{p_1^2 - m^2} \frac{\mathrm{i}}{p_2^2 - m^2} \frac{-\mathrm{i}}{\not{p}_3 + m} \left(\text{F.T.} \left\langle 0 \middle| \mathrm{T} \overline{\mathcal{J}_\mu}(y) A(x_1) B(x_2) \Psi(x_3) \middle| 0 \right\rangle_{\text{amp.}} \right) \xi,
\end{aligned} \tag{8.3}$$

where F.T. stands for the Fourier transform. Compared to the on-shell identity we just changed one scalar into a pseudoscalar. As this is an off-shell identity we need not to distinguish incoming and outgoing particles. The nonvanishing contributions to the contact terms for this SWI are:

$$\text{F.T.} \left\langle 0 \middle| \mathrm{T} \bar{\xi} \Psi(x_1) B(x_2) \Psi(x_3) \middle| 0 \right\rangle = \frac{-\lambda}{\sqrt{2}} \frac{\mathrm{i}}{p_2^2 - m^2} \frac{-\mathrm{i}}{\not{p}_3 + m} \gamma^5 \frac{\mathrm{i}}{\not{p}_1 + \not{k} - m} \xi \tag{8.4a}$$

$$\text{F.T.} \left\langle 0 \middle| \mathrm{T} A(x_1) (\mathrm{i} \bar{\xi} \gamma^5 \Psi(x_2)) \Psi(x_3) \middle| 0 \right\rangle = \frac{\lambda}{\sqrt{2}} \frac{\mathrm{i}}{p_1^2 - m^2} \frac{-\mathrm{i}}{\not{p}_3 + m} \frac{\mathrm{i}}{\not{p}_2 + \not{k} - m} \gamma^5 \xi \tag{8.4b}$$

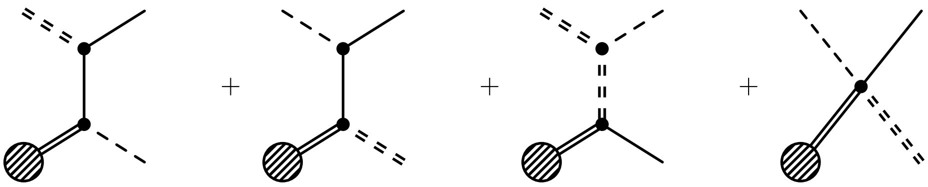
$$\begin{aligned}
&\text{F.T.} \left\langle 0 \middle| \mathrm{T} A(x_1) B(x_2) (-\mathrm{i})\, (\mathrm{i} \not{\partial}_{x_3} + m)\, B(x_3) \gamma^5 \xi \middle| 0 \right\rangle = \\
&\qquad \frac{-m\lambda}{\sqrt{2}} \frac{\mathrm{i}}{p_1^2 - m^2} \frac{\mathrm{i}}{p_2^2 - m^2} \frac{\mathrm{i}}{(p_3 + k)^2 - m^2} (-\not{p}_3 - \not{k} + m)\, \gamma^5 \xi
\end{aligned} \tag{8.4c}$$

$$\frac{-\mathrm{i}\lambda}{\sqrt{2}} \text{F.T.} \left\langle 0 \middle| \mathrm{T} A(x_1) B(x_2) (AB)(x_3) \gamma^5 \xi \middle| 0 \right\rangle = \frac{-\mathrm{i}\lambda}{\sqrt{2}} \frac{\mathrm{i}}{p_1^2 - m^2} \frac{\mathrm{i}}{p_2^2 - m^2} \gamma^5 \xi \tag{8.4d}$$

To evaluate the 4-point function with the current insertion we rewrite the current $\bar{\xi}\mathcal{J}_\mu$ as $\overline{\mathcal{J}_\mu}\xi$, which is identical due to the Majorana properties of the current and the transformation parameter:

$$\begin{aligned}\bar{\xi}\mathcal{J}_\mu = \bar{\xi}\Big\{ & \mathrm{i}\left((\mathrm{i}\overleftrightarrow{\partial\!\!\!/} - m)A\right)\gamma_\mu\Psi + \left((\mathrm{i}\overleftrightarrow{\partial\!\!\!/} + m)B\right)\gamma^5\gamma_\mu\Psi \\ & - \frac{\mathrm{i}\lambda}{2\sqrt{2}}\gamma_\mu\left(A^2 - B^2\right)\Psi - \frac{\lambda}{\sqrt{2}}\gamma_\mu\gamma^5 AB\Psi\Big\} \\ = \Big\{ & \overline{\Psi}\gamma_\mu \mathrm{i}\left(\mathrm{i}\overleftrightarrow{\partial\!\!\!/} + m\right)A - \overline{\Psi}\gamma_\mu\left(\mathrm{i}\overleftrightarrow{\partial\!\!\!/} + m\right)B\gamma^5 \\ & + \frac{\mathrm{i}\lambda}{2\sqrt{2}}\overline{\Psi}\gamma_\mu\left(A^2 - B^2\right) - \frac{\lambda}{\sqrt{2}}\overline{\Psi}\gamma_\mu\gamma^5 AB\Big\}\xi \end{aligned} \tag{8.5}$$

This brings the propagator of the (matter) fermion to the left. Again there are four diagrams for the Green function with current insertion:



(8.6)

For the sign of the fermion propagator one has to take care of the momentum flow.

$$\begin{aligned}\mathrm{F.T.}\,\langle 0|\mathrm{T}\overline{\mathcal{J}_\mu}(y)A(x_1)B(x_2)\Psi(x_3)|0\rangle_{\mathrm{amp.}}\,\xi = & -\frac{\mathrm{i}\lambda}{\sqrt{2}}\gamma^5\frac{\mathrm{i}}{p\!\!\!/_1 + k\!\!\!/ - m}\gamma_\mu\left(p\!\!\!/_1 + m\right)\xi \\ & + \frac{\mathrm{i}\lambda}{\sqrt{2}}\frac{\mathrm{i}}{p\!\!\!/_2 + k\!\!\!/ - m}\gamma_\mu\left(p\!\!\!/_2 + m\right)\gamma^5\xi - \frac{\lambda}{\sqrt{2}}\gamma_\mu\gamma^5\xi \\ & - \frac{\mathrm{i}m\lambda}{\sqrt{2}}\frac{\mathrm{i}}{(p_3 + k)^2 - m^2}\gamma_\mu\left(p\!\!\!/_3 + k\!\!\!/ - m\right)\gamma^5\xi \end{aligned} \tag{8.7}$$

Dotting the momentum $k_\mu = -(p_1 + p_2 + p_3)_\mu$ into this expression yields

$$\frac{\mathrm{i}}{p\!\!\!/_3 + m}\frac{1}{(p_1^2 - m^2)(p_2^2 - m^2)}\,k^\mu\mathrm{F.T.}\,\langle 0|\mathrm{T}\overline{\mathcal{J}_\mu}(y)A(x_1)B(x_2)\Psi(x_3)|0\rangle_{\mathrm{amp.}}\,\xi$$

$$\begin{aligned} = & \frac{\mathrm{i}\lambda}{\sqrt{2}}\frac{1}{p\!\!\!/_3 + m}\frac{1}{(p_1^2 - m^2)(p_2^2 - m^2)}\cdot\Big\{\left(p\!\!\!/_1 + p\!\!\!/_2 + p\!\!\!/_3\right) \\ & - \frac{1}{p\!\!\!/_2 + p\!\!\!/_3 - m}\left(p\!\!\!/_1 + p\!\!\!/_2 + p\!\!\!/_3\right)\left(p\!\!\!/_1 - m\right) \\ & - \frac{1}{p\!\!\!/_1 + p\!\!\!/_3 + m}\left(p\!\!\!/_1 + p\!\!\!/_2 + p\!\!\!/_3\right)\left(p\!\!\!/_2 + m\right) \\ & + m\frac{1}{(p_1 + p_2)^2 - m^2}\left(p\!\!\!/_1 + p\!\!\!/_2 + p\!\!\!/_3\right)\left(p\!\!\!/_1 + p\!\!\!/_2 + m\right)\Big\}\gamma^5\xi \\ = & \frac{\mathrm{i}\lambda}{\sqrt{2}}\frac{1}{p\!\!\!/_3 + m}\frac{1}{(p_1^2 - m^2)(p_2^2 - m^2)}\left(p\!\!\!/_1 + p\!\!\!/_2 + p\!\!\!/_3\right)\gamma^5\xi \\ & - \frac{\mathrm{i}\lambda}{\sqrt{2}}\frac{1}{p\!\!\!/_3 + m}\frac{1}{(p_1^2 - m^2)(p_2^2 - m^2)}\left(p\!\!\!/_1 - m\right)\gamma^5\xi \end{aligned}$$

$$
\begin{aligned}
&-\frac{\mathrm{i}\lambda}{\sqrt{2}}\frac{1}{\not{p}_3+m}\frac{1}{p_2^2-m^2}\frac{1}{\not{p}_2+\not{p}_3-m}\gamma^5\xi \\
&-\frac{\mathrm{i}\lambda}{\sqrt{2}}\frac{1}{\not{p}_3+m}\frac{1}{(p_1^2-m^2)(p_2^2-m^2)}(\not{p}_2+m)\gamma^5\xi \\
&-\frac{\mathrm{i}\lambda}{\sqrt{2}}\frac{1}{\not{p}_3+m}\frac{1}{p_1^2-m^2}\frac{1}{\not{p}_1+\not{p}_3+m}\gamma^5\xi \\
&+\frac{\mathrm{i}m\lambda}{\sqrt{2}}\frac{1}{(p_1^2-m^2)(p_2^2-m^2)}\frac{1}{(p_1+p_2)^2-m^2}(\not{p}_1+\not{p}_2+m)\gamma^5\xi \\
&+\frac{\mathrm{i}m\lambda}{\sqrt{2}}\frac{1}{\not{p}_3+m}\frac{1}{(p_1^2-m^2)(p_2^2-m^2)}\gamma^5\xi
\end{aligned} \tag{8.8}
$$

The third, fifth and sixth term equal the ones from the linearly transformed fields of the r.h.s.:

$$
\begin{aligned}
-\frac{\mathrm{i}\lambda}{\sqrt{2}}\Bigg\{&\frac{1}{p_2^2-m^2}\frac{1}{\not{p}_3+m}\frac{1}{\not{p}_2+\not{p}_3-m}+\frac{1}{p_1^2-m^2}\frac{1}{\not{p}_3+m}\frac{1}{\not{p}_1+\not{p}_3+m} \\
&-m\frac{1}{p_1^2-m^2}\frac{1}{p_2^2-m^2}\frac{1}{(p_1+p_2)^2-m^2}(\not{p}_1+\not{p}_2+m)\Bigg\}\gamma^5\xi
\end{aligned} \tag{8.9}
$$

The remaining terms add up to:

$$
\begin{aligned}
-\frac{\mathrm{i}\lambda}{\sqrt{2}}&\frac{1}{(p_1^2-m^2)(p_2^2-m^2)}\frac{1}{\not{p}_3+m}\Big\{-\not{p}_1-\not{p}_2-\not{p}_3+\not{p}_1-m+\not{p}_2+m-m\Big\}\gamma^5\xi \\
&=\frac{\mathrm{i}\lambda}{\sqrt{2}}\frac{1}{(p_1^2-m^2)(p_2^2-m^2)}\gamma^5\xi
\end{aligned} \tag{8.10}
$$

This equals the single term coming from the local operator insertion, so that the Ward identity is indeed fulfilled.

8.2 Currents and SWI in the O'Raifeartaigh model

Taking the general formula (D.8) derived in appendix D.1 we can derive the supersymmetric current for the O'Raifeartaigh model (short: OR model). From the superpotential in which the superfields have been substituted by their scalar components

$$
f(\phi_1,\phi_2,\phi_3)=\lambda\phi_1+m\phi_2\phi_3+g\phi_1\phi_2^2 \tag{8.11}
$$

we can read off the derivatives with respect to the scalar fields (there is no difference whether we take the mixings of the fields into account first and take the derivatives afterwards or vice versa):

$$
\frac{\partial f(\phi_1,\phi_2,\phi_3)}{\partial\phi_1}=\lambda+g\phi_2^2 \quad = \lambda+\frac{g}{2}\left(A^2-B^2+2\mathrm{i}AB\right) \tag{8.12}
$$

$$
\frac{\partial f(\phi_1,\phi_2,\phi_3)}{\partial\phi_2}=m\phi_3+2g\phi_1\phi_2 = m\Phi+\sqrt{2}g\,(A+\mathrm{i}B) \tag{8.13}
$$

$$
\frac{\partial f(\phi_1,\phi_2,\phi_3)}{\partial\phi_3}=m\phi_2 \quad = \frac{m}{\sqrt{2}}(A+\mathrm{i}B) \tag{8.14}
$$

After inserting these derivatives and sorting the terms we get

$$\begin{aligned}\mathcal{J}^\mu = & -\sqrt{2}(\partial\!\!\!/\phi)\gamma^\mu\mathcal{P}_R\chi - \sqrt{2}(\partial\!\!\!/\phi^*)\gamma^\mu\mathcal{P}_L\chi - \sqrt{2}\mathrm{i}\lambda\gamma^\mu\chi + \mathrm{i}\mathcal{P}_L\left((\mathrm{i}\partial\!\!\!/ - m)A\right)\gamma^\mu\Psi \\ & + \mathrm{i}\mathcal{P}_R\left((\mathrm{i}\partial\!\!\!/ - m)A\right)\gamma^\mu\Psi^c + \mathcal{P}_L\left((\mathrm{i}\partial\!\!\!/ - m)B\right)\gamma^\mu\Psi - \mathcal{P}_R\left((\mathrm{i}\partial\!\!\!/ - m)B\right)\gamma^\mu\Psi^c \\ & + \mathrm{i}\sqrt{2}\mathcal{P}_R\left((\mathrm{i}\partial\!\!\!/ - m)\Phi\right)\gamma^\mu\Psi + \mathrm{i}\sqrt{2}\mathcal{P}_L\left((\mathrm{i}\partial\!\!\!/ - m)\Phi^*\right)\gamma^\mu\Psi^c \\ & - \frac{\mathrm{i}g}{\sqrt{2}}\left(A^2 - B^2\right)\gamma^\mu\chi - \sqrt{2}gAB\gamma^\mu\gamma^5\chi - 2\mathrm{i}g\gamma^\mu A\phi\mathcal{P}_L\Psi - 2\mathrm{i}g\gamma^\mu A\phi^*\mathcal{P}_R\Psi^c \\ & + 2g\gamma^\mu B\phi\mathcal{P}_L\Psi - 2g\gamma^\mu B\phi^*\mathcal{P}_R\Psi^c\end{aligned} \tag{8.15}$$

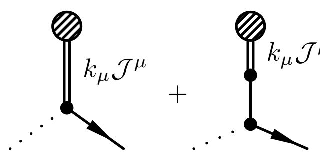
Let us start with a rather trivial example, which relates 2- and 3-point functions in lowest order perturbation theory. We consider the SWI

$$\begin{aligned}& k_\mu\,\mathrm{F.T.}\left\langle 0\middle|\mathrm{T}\left[\left(\bar{\xi}\mathcal{J}^\mu\right)A(x_1)\Psi(x_2)\right]\middle|0\right\rangle \\ \stackrel{!}{=}\quad & \mathrm{F.T.}\left\langle 0\middle|\mathrm{T}\left[\Psi(x_2)\left(\overline{\Psi}(x_1)\mathcal{P}_R\xi\right)\right]\middle|0\right\rangle\delta^4(x-x_1) \\ & \qquad + \mathrm{F.T.}\left\langle 0\middle|\mathrm{T}\left[A(x_1)\left(-\mathrm{i}\partial\!\!\!/ - m\right)A(x_2)\mathcal{P}_R\xi\right]\middle|0\right\rangle\delta^4(x-x_2) + \mathcal{O}(g)\end{aligned} \tag{8.16}$$

We have only kept those of the contact terms giving nonvanishing contributions. The right hand side will be calculated first; we adopt the convention that all momenta be incoming. The right hand side is

$$\mathrm{RHS}\,(8.16) = \frac{\mathrm{i}(-k\!\!\!/_2 + m)}{k_2^2 - m^2}\mathcal{P}_R\xi - \frac{\mathrm{i}(k\!\!\!/_1 + m)}{k_1^2 - m^2 - 2\lambda g}\mathcal{P}_R\xi \;+\; \mathcal{O}(g) \tag{8.17}$$

As mentioned earlier, for the calculation of the left hand side care has to be taken about possible higher orders in perturbation theory which may contribute to this SWI. In these diagrams the linear part of the current will be coupled to the external particles via the Goldstino, wherein the coupling constant combined with the parameter for the spontaneous symmetry breaking λ is responsible for the mass splitting between the participating particles A and Ψ. This will prove important – as we will see soon – for constructing the propagators with the correct poles. The pole of the Goldstino at zero mass always cancels out of those diagrams against the momentum influx from the current. Diagrammatically the left hand side looks like ($k = k_1 + k_2$):

LHS (8.16) =  (8.18)

The analytical expression for the left hand side (8.16) is

$$\begin{aligned}\mathrm{LHS}\,(8.16) = \;& \mathrm{i}\partial^x_\mu\left\langle 0\middle|\mathrm{T}\left[\left(\bar{\xi}\mathcal{J}^\mu(x)\right)A(x_1)\Psi(x_2)\right]\middle|0\right\rangle_{(0)} \\ & + \mathrm{i}\partial^x_\mu\cdot\mathrm{i}\int d^4y\left\langle 0\middle|\mathrm{T}\left[\left(\bar{\xi}\mathcal{J}^\mu(x)\right)A(x_1)\Psi(x_2)\mathcal{L}_{\mathrm{int}}\right]\middle|0\right\rangle + \ldots \\ = \;& \mathrm{i}\partial^x_\mu\left\langle 0\middle|\mathrm{T}\left[A(x_1)\Psi(x_2)\mathrm{i}\Big(\overline{\Psi}(x)\gamma^\mu\left[(\mathrm{i}\partial\!\!\!/ + m)A(x)\right]\mathcal{P}_R\xi\Big)\right]\middle|0\right\rangle \\ & - \partial^x_\mu\int d^4y\left\langle 0\middle|\mathrm{T}\left[\left(\sqrt{2}\mathrm{i}\lambda\bar{\xi}\gamma^\mu\chi(x)\right)\left(\sqrt{2}g\overline{\Psi}(y)\mathcal{P}_R\chi(y)\right)A(x_1)\Psi(x_2)\right]\middle|0\right\rangle \\ & + \text{higher orders}\end{aligned}$$

$$
\begin{aligned}
&\stackrel{\mathrm{FT}}{\longrightarrow} \frac{\mathrm{i}(-\not{k}_2 + m)}{k_2^2 - m^2} \, (\not{k}_1 + \not{k}_2) \, \frac{\not{k}_1 + m}{k_1^2 - m^2 - 2\lambda g} \mathcal{P}_R \xi \\
&\quad - \frac{\mathrm{i}(-\not{k}_2 + m)}{k_2^2 - m^2} \frac{2\lambda g}{k_1^2 - m^2 - 2\lambda g} \mathcal{P}_R \xi \; + \; \mathcal{O}(g) \\
&= \frac{\mathrm{i}(-\not{k}_2 + m)}{k_2^2 - m^2} \mathcal{P}_R \xi - \frac{\mathrm{i}(\not{k}_1 + m)}{k_1^2 - m^2 - 2\lambda g} \mathcal{P}_R \xi \; + \; \mathcal{O}(g) \quad = \quad \mathrm{RHS}\,(8.16) \quad \checkmark
\end{aligned}
\tag{8.19}
$$

The SWI is fulfilled. Amputating the external legs (except for the current) by means of the LSZ reduction formula produces the on-shell identity which is thence automatically fulfilled as well.

Chapter 9

Gauge theories and Supersymmetry

In gauge theories there appears a new phenomenon not met in the previous chapters: the participation of (massless or massive) vector bosons connected to the concept of gauge symmetry and gauge transformations. These are indispensable ingredients for a realistic field theoretic model describing elementary particle phenomenology. The gauge principle, i.e. the covariance of the fields under local phase transformations, must in a supersymmetric field theory be incorporated in a SUSY covariant manner. As shown in [3] and [5] the kinetic terms with minimal coupling can be written down in a SUSY-covariant form by introducing a vector superfield $\hat{V}$ (this is a real superfield with $\hat{V}^\dagger = \hat{V}$), and making the replacement

$$S_{\text{kin}} = \int d^4x \, \frac{1}{2} \left[\hat{\Phi}^\dagger \hat{\Phi}\right]_D \longrightarrow \int d^4x \, \frac{1}{2} \left[\hat{\Phi}^\dagger e^{\pm c\hat{V}} \Phi\right]_D . \tag{9.1}$$

Therein c is a normalization constant depending on the normalization of the algebra of the gauge symmetry which is as changing from author to author as the choice of sign. The sign of c is related to the sign in the gauge-covariant derivative,

$$D_\mu = \partial_\mu \pm \mathrm{i}g \sum_a T^a A^a_\mu. \tag{9.2}$$

The kinetic term for the gauge fields is produced with the help of spinor superfields, chiral superfields equipped with an additional spinor index. They are established by triply applying the super-covariant derivative $\mathcal{D}$ to the vector superfield

$$\hat{W}(x,\theta) = -\frac{1}{4} \left(\overline{\mathcal{D}}\mathcal{D}\right) \mathcal{D} \, \hat{V}(x,\theta). \tag{9.3}$$

Then the kinetic part of the gauge fields can be expressed as

$$S_{\text{gauge}} = \frac{1}{2} \int d^4x \, \mathrm{Re}\Big[\sum_a \overline{W^a_R} W^a_L\Big]. \tag{9.4}$$

There is a high redundancy in the superfield formulation of supersymmetric gauge theories. The new superfield $\hat{V}$ there contains a huge amount of unphysical degrees of freedom. But we can get rid of them. The kinetic part (and the superpotential as well) are not only invariant under SUSY and gauge transformations but also under so called extended gauge transformations. These are gauge transformations where the gauge parameter (usually a scalar spacetime dependent parameter) is replaced by a complete superfield $\hat{\Lambda}(x,\theta)$. We can use these transformations to gauge away the superfluous degrees of freedom, three scalar and one spinor component field so that only the gauge field, the gaugino and a scalar field with canonical dimension two remain. This is called the Wess-Zumino gauge. After having fixed the above mentioned components, only the ordinary gauge transformations survive from the extended gauge transformations.

The Lagrangean density of the matter fields with minimal coupling therefore has the structure:

$$\begin{aligned}\mathcal{L}_{\mathrm{mat}} = (D_\mu\phi)^\dagger\,(D^\mu\phi) + \frac{\mathrm{i}}{2}\left(\overline{\Psi}\not{D}\Psi\right) + F^\dagger F - \sqrt{2}g\overline{\lambda^a}\phi^\dagger T^a\Psi_L \\ - \sqrt{2}g\overline{\Psi_L}T^a\phi\lambda^a + g\phi^\dagger T^a\phi D^a + \mathcal{W}(\phi,\Psi,F)\end{aligned} \tag{9.5}$$

Here $\mathcal{W}(\phi,\Psi,F)$ stands for the superpotential parts of the matter Lagrangean density which are globally and locally invariant under the gauge symmetry group. It does not contain any derivatives of the fields.

The kinetic terms of the gauge fields and gauginos are:

$$\mathcal{L}_{\mathrm{gauge}} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + \frac{\mathrm{i}}{2}\overline{\lambda^a}\,(\not{D}\lambda)^a + \frac{1}{2}D^aD^a \tag{9.6}$$

Since it is consistent with the gauge symmetry, we may add a *Fayet-Iliopoulos* term

$$\mathcal{L}_{\mathrm{FI}} = \zeta^a D^a \qquad \text{with} \qquad f^a_{bc}\zeta^a = 0. \tag{9.7}$$

The last condition is necessary in the non-Abelian case for this term to transform into a total derivative under SUSY. It forces the gauge field part in the covariant derivative of the gauginos produced when SUSY-transforming the auxiliary field to vanish.

9.1 The de Wit–Freedman transformations

The Wess-Zumino supergauge fixing procedure destroys invariance of the Lagrangean density under SUSY transformations as well as under extended gauge transformations. When performing a SUSY transformation the states gauged away in the WZ gauge are populated again with the effect that the Lagrangean density is no longer WZ gauged. This can be remedied by performing another extended gauge transformation to newly reach WZ gauge. From last section's discussion this is understandable from the fact that SUSY and gauge transformations are not completely orthogonal to each other. The several transformations and their relations are displayed below:

$$\begin{tikzcd} \mathcal{L} \arrow[r, "\mathcal{T}_{\mathrm{SUSY}}"] \arrow[d, "\mathcal{T}_{\mathrm{ext.g.}}"'] & \mathcal{L} \arrow[d, "\mathcal{T}_{\mathrm{ext.g.}}"] \\ \mathcal{L}^{WZ} \arrow[ur, "\mathcal{T}_{\mathrm{SUSY}}"] \arrow[r, "\mathcal{T}_{\mathrm{deW-F}}"'] & \mathcal{L}^{WZ} \end{tikzcd}$$

When performing a SUSY transformation and an extended gauge transformation afterwards (for the details cf. [3]) this results in a combined transformation called *de Wit–Freedman transformation* which leaves the Lagrangean density in WZ gauge invariant [26]. In de Wit–Freedman transformations the spacetime derivatives are replaced by gauge covariant derivatives; furthermore there are some additional terms. So de Wit–Freedman transformations are the gauge-covariant version of the SUSY transformations. For supersymmetric Yang–Mills theories they are (we put a tilde on them to distinguish them from the ordinary supersymmetry transformations; for more details see appendix A.5):

$$\boxed{\begin{aligned}
\tilde{\delta}_\xi \phi &= \sqrt{2}\left(\bar{\xi}\Psi_L\right), \\
\tilde{\delta}_\xi \psi &= -\mathrm{i}\sqrt{2}\gamma^\mu\left((D_\mu\phi)\mathcal{P}_R + (D_\mu\phi)^\dagger\mathcal{P}_L\right)\xi + \sqrt{2}(F\mathcal{P}_L + F^\dagger\mathcal{P}_R)\xi, \\
\tilde{\delta}_\xi F &= -\mathrm{i}\sqrt{2}\left(\bar{\xi}\not{D}\Psi_L\right) + 2g\bar{\xi}T^a\phi\lambda_R^a, \\
\tilde{\delta}_\xi A_\mu^a &= -\left(\bar{\xi}\gamma_\mu\gamma_5\lambda^a\right), \\
\tilde{\delta}_\xi \lambda^a &= -\frac{\mathrm{i}}{2}F_{\mu\nu}^a\gamma^\mu\gamma^\nu\gamma^5\xi + D^a\xi, \\
\tilde{\delta}_\xi D^a &= -\mathrm{i}\bar{\xi}\left(\not{D}\lambda\right)^a.
\end{aligned}} \tag{9.8}$$

9.2 The current in supersymmetric Yang–Mills theories

Because it is a complicated and lengthy topic we postpone the detailed derivation of the supersymmetric current for supersymmetric Yang–Mills theories (SYM) to the appendix, D.2. We simply state the result for the SUSY current in a supersymmetric Yang-Mills theory

$$\boxed{\begin{aligned}
\mathcal{J}^\mu = &-\sqrt{2}\gamma^\nu\gamma^\mu(D_\nu\phi)^T\Psi_R - \sqrt{2}\gamma^\nu\gamma^\mu(D_\nu\phi)^\dagger\Psi_L - \mathrm{i}\gamma^\mu\zeta^a\lambda^a \\
&+ \frac{1}{2}\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^5F_{\alpha\beta}^a\lambda^a - \mathrm{i}g\gamma^\mu\left(\phi^\dagger\vec{T}\phi\right)\cdot\lambda \\
&- \mathrm{i}\sqrt{2}\gamma^\mu\left(\frac{\partial f(\phi)}{\partial\phi}\right)^T\Psi_L - \mathrm{i}\sqrt{2}\gamma^\mu\left(\frac{\partial f(\phi)}{\partial\phi}\right)^\dagger\Psi_R
\end{aligned}} \quad . \tag{9.9}$$

It is conserved,

$$\boxed{\partial_\mu\mathcal{J}^\mu = 0} \quad , \tag{9.10}$$

as will also be proven in the appendix, D.3.

9.3 Comparison of the currents – physical interpretation

The use of the de Wit–Freedman transformation is not mandatory [3]. It is also possible to use the "ordinary" SUSY transformations to calculate the current. We do want to show now that the current in SYM theories remains the same when using SUSY instead of de Wit–Freedman transformations in its derivation. The difference between both transformations shows up in the auxiliary fields F, $F^\dagger$ and D^a, as well as in the matter fermions. The two Noether parts – calculated on the one hand with the dWF transformation, on the other hand with the ordinary SUSY transformation – differ by a term

$$\bar{\xi}\mathcal{J}^\mu_{\text{dWF}} - \bar{\xi}\mathcal{J}^\mu_{\text{ord.}} = -\frac{\mathrm{i}g}{\sqrt{2}}\left(\bar{\xi}\gamma^\nu\gamma^\mu\phi^T\vec{T}^T\vec{A}_\nu\Psi_R\right) + \frac{\mathrm{i}g}{\sqrt{2}}\left(\bar{\xi}\gamma^\nu\gamma^\mu\phi^\dagger\vec{T}\vec{A}_\nu\Psi_L\right). \tag{9.11}$$

We now list all terms produced by a dWF transformation of the Lagrangean density in addition to the SUSY transformation:

$$\begin{aligned} F^\dagger F \longrightarrow & -2gF^\dagger\left(\bar{\xi}\vec{T}\phi\cdot\vec{\lambda}_R\right) - 2g\left(\bar{\xi}\phi^\dagger\vec{T}F\cdot\vec{\lambda}_L\right) + \sqrt{2}g\left(\bar{\xi}F^\dagger\vec{T}\vec{\not{A}}\Psi_L\right) \\ & -\sqrt{2}g\left(\bar{\xi}F^T\vec{T}^T\vec{\not{A}}\Psi_R\right) \end{aligned} \tag{9.12}$$

$$\begin{aligned} \frac{\mathrm{i}}{2}\left(\overline{\Psi}\not{D}\Psi\right) \longrightarrow & \frac{\mathrm{i}g}{\sqrt{2}}\left(\bar{\xi}\gamma^\mu\gamma^\nu(D_\nu\vec{T}^T\vec{A}_\mu\phi^T)\Psi_R\right) - \frac{\mathrm{i}g}{\sqrt{2}}\left(\bar{\xi}\gamma^\mu\gamma^\nu(D_\nu\vec{T}\vec{A}_\mu\phi)^\dagger\Psi_L\right) \\ & + \frac{\mathrm{i}g}{\sqrt{2}}\left(\bar{\xi}\gamma^\mu(\vec{T}\vec{A}_\mu\phi^\dagger)\not{D}\Psi_L\right) - \frac{\mathrm{i}g}{\sqrt{2}}\left(\bar{\xi}\gamma^\mu(\vec{T}^T\vec{A}_\mu\phi^T)\not{D}\Psi_R\right) \end{aligned} \tag{9.13}$$

$$-\sqrt{2}g\left(\vec{\bar{\lambda}}\cdot\phi^\dagger\vec{T}\Psi_L\right) \longrightarrow 2g^2\left(\bar{\xi}\phi^\dagger\vec{T}(\vec{T}\vec{\not{A}}\phi)\cdot\vec{\lambda}_L\right) \tag{9.14}$$

$$-\sqrt{2}g\left(\overline{\Psi_L}\vec{T}\phi\cdot\vec{\lambda}\right) \longrightarrow -2g^2\left(\bar{\xi}\gamma^\mu(\vec{T}\vec{A}_\mu\phi^\dagger)\vec{T}\phi\cdot\vec{\lambda}_R\right) \tag{9.15}$$

$$g\left(\phi^\dagger\vec{T}\phi\right)\cdot D \longrightarrow \mathrm{i}g^2\left(\phi^\dagger T^a\phi\right)\left(\bar{\xi}f^a_{\ bc}\not{A}^b\lambda^c\right) \tag{9.16}$$

$$\mathcal{L}_{\text{gauge}} \longrightarrow \mathrm{i}D^a\left(\bar{\xi}f^a_{\ bc}\not{A}^b\lambda^c\right) \tag{9.17}$$

From these additional terms as many as possible are eliminated. The first two terms out of (9.12) vanish by the condition (D.59). From the last two equations (9.16) and (9.17), the contributions cancel due to the equation of motion for the auxiliary field D^a. Next we multiply the terms with the covariant derivatives of the fermions in (9.13) by a factor two and subtract them once. In the doubled expressions we insert the equation of motion for the fermions; with the help of the two identities derived from (D.59) in the paragraph below that equation we see, that the remaining terms from (9.12) cancel as well as the gaugino contributions (9.14) and (9.15). We are left with

$$\begin{aligned} & \frac{\mathrm{i}g}{\sqrt{2}}\left(\bar{\xi}\gamma^\mu\gamma^\nu(D_\nu\vec{T}^T\vec{A}_\mu\phi^T)\Psi_R\right) - \frac{\mathrm{i}g}{\sqrt{2}}\left(\bar{\xi}\gamma^\mu\gamma^\nu(D_\nu\vec{T}\vec{A}_\mu\phi)^\dagger\Psi_L\right) \\ & -\frac{\mathrm{i}g}{\sqrt{2}}\left(\bar{\xi}\gamma^\mu(\vec{T}\vec{A}_\mu\phi^\dagger)\not{D}\Psi_L\right) + \frac{\mathrm{i}g}{\sqrt{2}}\left(\bar{\xi}\gamma^\mu(\vec{T}^T\vec{A}_\mu\phi^T)\not{D}\Psi_R\right). \end{aligned} \tag{9.18}$$

All terms containing two gauge fields cancel each other so that the remaining term is the following derivative:

$$\partial_\nu\left[\frac{\mathrm{i}g}{\sqrt{2}}\left(\bar{\xi}\gamma^\mu\gamma^\nu(\vec{T}^T\vec{A}_\mu\phi^T)\Psi_R\right) - \frac{\mathrm{i}g}{\sqrt{2}}\left(\bar{\xi}\gamma^\mu\gamma^\nu(\vec{T}\vec{A}_\mu\phi^\dagger)\Psi_L\right)\right]. \tag{9.19}$$

This cancels exactly the contribution to the current from the Noether part, (9.11), and both currents are equal.

The fact that the two currents are identical can be interpreted physically in the following way: The supersymmetric current of SYM may be derived in the superfield formalism independent from any choice of supergauge. Since the supercurrent (the superfield containing the supersymmetric current and the energy-momentum tensor) is a scalar with respect to (extended) gauge transformations, the supersymmetric current remains the same when expressed in different supergauges.

9.4 SWI in an Abelian toy model

To test our supersymmetric Ward identities (SWI) for a supersymmetric gauge theory, we choose the simplest possible example, a model with one matter superfield and a $U(1)$ gauge symmetry. This is not SQED – the supersymmetric extension of QED, since there is only a single superfield. Gauge invariance then forces the superpotential to vanish, so that all particles of our model are massless. Furthermore the matter fermion is of Majorana type and the gauge field must couple to it as an axial vector, as this is the only possibility for a gauge field to have a nonvanishing coupling to a Majorana fermion. The whole gauge superfield must then be axial as well and the model bears an anomaly, the supersymmetric extension of the axial vector anomaly. But as long as we are only concerned with tree level processes we do not have to care about anomalies – they will only become important for higher order calculations. The details of our Abelian toy model can again be found in appendix E.4. Here we just quote the Lagrangean density and the current (to avoid confusion with the scalar particle A we denote the gauge boson by G_μ in this model):

$$\boxed{\begin{aligned}\mathcal{L} = \frac{1}{2}(\partial_\mu A)(\partial^\mu A) + \frac{1}{2}(\partial_\mu B)(\partial^\mu B) + \frac{\mathrm{i}}{2}\overline{\Psi}\partial\!\!\!/\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\mathrm{i}}{2}\overline{\lambda}\partial\!\!\!/\lambda\\ +eG_\mu\left(B\partial^\mu A - A\partial^\mu B\right) + \frac{e^2}{2}G_\mu G^\mu\left(A^2+B^2\right) - e\left(\overline{\Psi}\lambda\right)A\\ -\mathrm{i}e\left(\overline{\Psi}\gamma^5\lambda\right)B - \frac{e}{2}\overline{\Psi}G\!\!\!/\gamma^5\Psi - \frac{e^2}{8}\left(A^4+B^4+2A^2B^2\right)\end{aligned}} \tag{9.20}$$

$$\boxed{\begin{aligned}\mathcal{J}^\mu = -(\partial\!\!\!/A)\gamma^\mu\Psi - \mathrm{i}(\partial\!\!\!/B)\gamma^\mu\gamma^5\Psi + \mathrm{i}eAG\!\!\!/\gamma^\mu\gamma^5\Psi - eBG\!\!\!/\gamma^\mu\Psi\\ +\frac{1}{2}[\gamma^\alpha,\gamma^\beta]\gamma^\mu\gamma^5(\partial_\alpha G_\beta)\lambda - \frac{\mathrm{i}e}{2}\left(A^2+B^2\right)\gamma^\mu\lambda\end{aligned}} \tag{9.21}$$

We first show a simple example for an on-shell Ward identity:

$$\begin{aligned}J_\mu(p_1,p_2) &= \text{F.T.}\,\langle 0|\mathcal{J}_\mu(x)|A(p_1)\Psi(p_2)\rangle\\ &= \text{F.T.}\,\langle 0|\mathcal{J}_\mu(x)|A(p_1)\Psi(p_2)\rangle_{(0)} = -p\!\!\!/_1\gamma_\mu u(p_2)\end{aligned} \tag{9.22}$$

$$(p_1+p_2)^\mu J_\mu(p_1,p_2) = -p\!\!\!/_1\left(p\!\!\!/_1+p\!\!\!/_2\right)u(p_2) = 0 \tag{9.23}$$

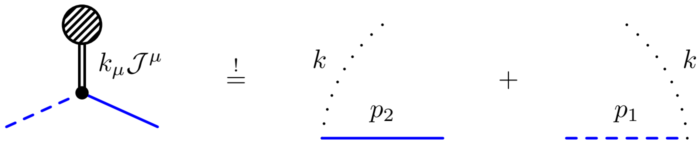
The second term in parentheses vanishes due to the Dirac equation while p_1^2 is zero (all particles are massless in this model by gauge invariance).

We now discuss several examples of Ward identites calculated off-shell. By amputation in the sense of LSZ reduction we can then easily get back to the on-shell identities. Let us first – as a warm-up – examine an SWI with two fields beneath the current insertion written down in the form (7.6)

$$\begin{aligned}k^\mu\text{F.T.}\left\langle 0\middle|\mathrm{T}\,\overline{\xi}\mathcal{J}_\mu(y)A(x_1)\Psi(x_2)\middle|0\right\rangle &\stackrel{!}{=} \text{F.T.}\,\langle 0|\mathrm{T}\left(\delta_\xi A(x_1)\right)\Psi(x_2)|0\rangle\,\delta^4(x_1-y)\\ &+ \text{F.T.}\,\langle 0|\mathrm{T}A(x_1)\left(\delta_\xi\Psi(x_2)\right)|0\rangle\,\delta^4(x_2-y)\end{aligned}$$

$$
\begin{aligned}
&= \text{F.T.}\left\langle 0\middle|\mathrm{T}\Psi(x_2)\overline{\Psi}(x_1)\xi\middle|0\right\rangle \delta^4(x_1-y) \\
&\qquad + \text{F.T.}\left\langle 0|\mathrm{T}A(x_1)(-\mathrm{i}\not{\partial}A(x_2))\xi|0\right\rangle \delta^4(x_2-y) \qquad (9.24)
\end{aligned}
$$

Graphically we denote the momentum influx by a dotted line. Then we have the following relation ($k+p_1+p_2=0$ and all momenta incoming)

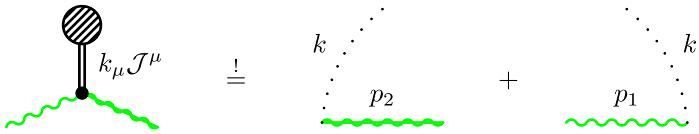


(9.25)

$$
-k^\mu \frac{\mathrm{i}}{p_1^2}\frac{-\mathrm{i}}{\not{p}_2}\gamma_\mu(-\mathrm{i}\not{p}_1)\xi = \frac{-\mathrm{i}}{p_1^2}\frac{1}{\not{p}_2}\left(\not{p}_1+\not{p}_2\right)\not{p}_1\xi \overset{!}{=} \left(\frac{-\mathrm{i}}{\not{p}_2}+\frac{-\mathrm{i}\not{p}_1}{p_1^2}\right)\xi \qquad (9.26)
$$

The SWI is fulfilled.

Another SWI for a 2-point Green function will be calculated now to show a new effect.



(9.27)

$$
\begin{aligned}
&\text{F.T.}\left\langle 0|\mathrm{T}\left(\delta_\xi G_\nu(x_1)\right)\lambda(x_2)|0\right\rangle \delta^4(x_1-y) \\
&\qquad + \text{F.T.}\left\langle 0|\mathrm{T}G_\nu(x_1)\left(\delta_\xi\lambda(x_2)\right)|0\right\rangle \delta^4(x_2-y) \\
=&\ -\text{F.T.}\left\langle 0\middle|\mathrm{T}\lambda(x_2)\overline{\lambda}(x_1)\gamma_\nu\gamma^5\xi\middle|0\right\rangle \delta^4(x_1-y) \\
&\qquad -\frac{\mathrm{i}}{2}\text{F.T.}\left\langle 0\middle|\mathrm{T}G_\nu(x_1)(\partial_\alpha^{x_2}G_\beta(x_2))[\gamma^\alpha,\gamma^\beta]\gamma^5\xi\middle|0\right\rangle \delta^4(x_2-y) \\
\overset{!}{=}&\ k^\mu\text{F.T.}\left\langle 0\middle|\mathrm{T}\overline{\xi}\mathcal{J}_\mu(y)G_\nu(x_1)\lambda(x_2)\middle|0\right\rangle \\
=&\ \frac{1}{2}k^\mu\text{F.T.}\left\langle 0\middle|\mathrm{T}\lambda(x_2)\overline{\lambda}(y)\gamma^5\gamma_\mu[\gamma^\alpha,\gamma^\beta](\partial_\alpha^y G_\beta(y))G_\nu(x_1)\xi\middle|0\right\rangle \quad , \qquad (9.28)
\end{aligned}
$$

i.e.

$$
\begin{aligned}
&\frac{\mathrm{i}}{\not{p}_2}\gamma_\nu\gamma^5\xi - \frac{1}{2}\frac{\mathrm{i}}{p_1^2}[-\not{p}_1,\gamma_\nu]\gamma^5\xi \\
&\qquad \overset{!}{=} \frac{1}{2}(-1)(p_1^\mu+p_2^\mu)\frac{-\mathrm{i}}{\not{p}_2}\gamma^5\gamma_\mu[\gamma^\alpha,\gamma^\beta](-\mathrm{i}p_{1,\alpha})\frac{-\mathrm{i}\eta_{\beta\nu}}{p_1^2}\xi \qquad (9.29)
\end{aligned}
$$

Multiplication by a factor i and simplification yields the result

$$
\begin{aligned}
&\frac{-1}{\not{p}_2}\gamma_\nu\gamma^5\xi - \frac{1}{2}\frac{1}{p_1^2}[\not{p}_1,\gamma_\nu]\gamma^5\xi \\
&\quad \overset{!}{=} \frac{1}{2}\frac{1}{\not{p}_2}\gamma^5(\not{p}_1+\not{p}_2)[\not{p}_1,\gamma_\nu]\frac{1}{p_1^2}\xi = -\frac{1}{2}\frac{1}{p_1^2}[\not{p}_1,\gamma_\nu]\gamma^5\xi - \frac{1}{2}\gamma^5\frac{1}{\not{p}_2}\not{p}_1[\not{p}_1,\gamma_\nu]\frac{1}{p_1^2}\xi \\
&\quad = -\frac{1}{2}\frac{1}{p_1^2}[\not{p}_1,\gamma_\nu]\gamma^5\xi - \frac{1}{\not{p}_2}\gamma_\nu\gamma^5\xi + \frac{1}{\not{p}_2}\frac{\not{p}_1}{p_1^2}p_{1,\nu}\gamma^5\xi \qquad (9.30)
\end{aligned}
$$

Astonishingly there is an additional term on the side of the current so that *this SWI is not fulfilled off-shell.* But we notice that this additional (the third) term is proportional to the momentum of the gauge boson. Using LSZ reduction (multiplying with the inverse propagator and with the polarization vector) and setting the particles on the mass shell yields zero by gauge boson transversality. Hence, *on-shell* the SWI *is* valid.

Now we attack a more complicated example, a 3-point function. In formula (7.6) we choose – together with the current insertion – the fields $A(x_1)$, $G_\nu(x_2)$ and $\Psi(x_3)$. We get four nonvanishing contributions from the SUSY transformations of these fields.

$$\text{F.T.}\left\langle 0\middle|\mathrm{T}\bar{\xi}\Psi(x_1)G_\nu(x_2)\Psi(x_3)\middle|0\right\rangle = +\mathrm{i}e\frac{(-\mathrm{i})}{p_2^2}\frac{\mathrm{i}}{\not{p}_3}\gamma_\nu\gamma^5\frac{\mathrm{i}}{\not{p}_1+\not{k}}\xi \tag{9.31a}$$

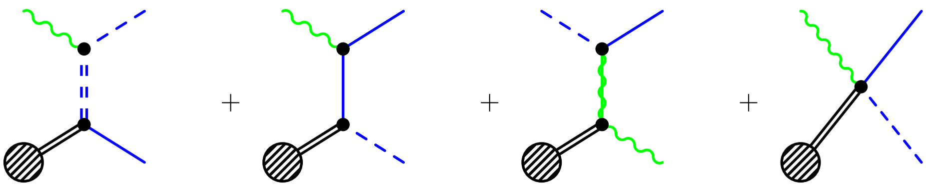
$$\text{F.T.}\left\langle 0\middle|\mathrm{T}A(x_1)(-\bar{\xi}\gamma_\nu\gamma^5\lambda(x_2))\Psi(x_3)\middle|0\right\rangle = -\mathrm{i}e\frac{\mathrm{i}}{p_1^2}\frac{\mathrm{i}}{\not{p}_3}\frac{\mathrm{i}}{\not{p}_2+\not{k}}\gamma_\nu\gamma^5\xi \tag{9.31b}$$

$$\begin{aligned}\text{F.T.}&\left\langle 0\middle|\mathrm{T}A(x_1)G_\nu(x_2)(-\gamma^5\not{\partial}B(x_3)\xi)\middle|0\right\rangle = \\ &+\mathrm{i}e\frac{\mathrm{i}}{p_1^2}\frac{-\mathrm{i}}{p_2^2}(p_{3,\nu}-p_{1,\nu}+k_\nu)\frac{\mathrm{i}}{(p_3+k)^2}\gamma^5(\not{p}_3-\not{k})\xi\end{aligned} \tag{9.31c}$$

$$\text{F.T.}\left\langle 0\middle|\mathrm{T}A(x_1)G_\nu(x_2)(-\gamma^\mu(G_\mu A)(x_3)\gamma^5\xi)\middle|0\right\rangle = \frac{\mathrm{i}}{p_1^2}\frac{-\mathrm{i}\eta_{\mu\nu}}{p_2^2}(-e)\gamma^\mu\gamma^5\xi \tag{9.31d}$$

The last term includes a composite operator insertion coming from the nonlinearities in the current of the supersymmetry due to the de Wit-Freedman transformation (others stem from the elimination of the auxiliary fields). For the first two processes one has to take care of the sign of the rightmost fermion propagator whose calculational direction is opposite to the momentum flow.

Now we have to calculate the 4-point function with current insertion; there are four contributing diagrams (again we use the trick to rewrite $\bar{\xi}\mathcal{J}_\mu$ as $\overline{\mathcal{J}_\mu}\xi$, which brings the propagator of the matter fermion to the rightmost position in the fermion line):



(9.32)

$$\begin{aligned}\text{F.T.}&\left\langle 0\middle|\mathrm{T}\overline{\mathcal{J}_\mu}(y)\xi A(x_1)G_\nu(x_2)\Psi(x_3)\middle|0\right\rangle = \\ &\frac{\mathrm{i}}{p_1^2}\frac{-\mathrm{i}}{p_2^2}\frac{-\mathrm{i}}{\not{p}_3}\left(\text{F.T.}\left\langle 0\middle|\mathrm{T}\overline{\mathcal{J}_\mu}(y)A(x_1)G_\nu(x_2)\Psi(x_3)\middle|0\right\rangle_{\text{amp.}}\right)\xi\end{aligned} \tag{9.33}$$

Note the momentum flow for the sign of the fermion propagator.

$$\begin{aligned}\text{F.T.}&\left\langle 0\middle|\mathrm{T}\overline{\mathcal{J}_\mu}(y)A(x_1)G_\nu(x_2)\Psi(x_3)\middle|0\right\rangle_{\text{amp.}}\xi = -\mathrm{i}e\gamma_\mu\gamma^5\gamma_\nu\xi \\ &-\mathrm{i}e\frac{\mathrm{i}}{\not{p}_2+\not{k}}\left(-\frac{1}{2}\right)(-\mathrm{i}p_{2,\alpha})\gamma_\mu\gamma^5[\gamma^\alpha,\gamma_\nu]\xi+\mathrm{i}e\gamma_\nu\gamma^5\frac{\mathrm{i}}{\not{p}_1+\not{k}}\gamma_\mu(-\mathrm{i}\not{p}_1)\xi \\ &+\frac{\mathrm{i}}{(p_3+k)^2}e\,(p_{1,\nu}-p_{3,\nu}-k_\nu)\,\mathrm{i}\gamma_\mu\gamma^5\mathrm{i}\,(\not{p}_3+\not{k})\,\xi\end{aligned} \tag{9.34}$$

with $k_\mu = -(p_1+p_2+p_3)_\mu$

$$\begin{aligned}
&\frac{-\mathrm{i}}{\not p_3}\frac{1}{p_1^2p_2^2}\,k^\mu \mathrm{F.T.}\,\big\langle 0\big|\mathrm{T}\overline{\mathcal{J}_\mu}(y)A(x_1)G_\nu(x_2)\Psi(x_3)\big|0\big\rangle_{\mathrm{amp.}}\,\xi \\
=&-\frac{\mathrm{i}}{\not p_3}\frac{1}{p_1^2p_2^2}\Big\{+\mathrm{i}e\,(\not p_1+\not p_2+\not p_3)\,\gamma^5\gamma_\nu+\frac{\mathrm{i}e}{2}\frac{1}{\not p_1+\not p_3}\,(\not p_1+\not p_2+\not p_3)\,\gamma^5[\not p_2,\gamma_\nu] \\
&\quad-\mathrm{i}e\,(p_{1,\nu}-p_{3,\nu}+k_\nu)\,\frac{1}{(p_3-k)^2}\,(\not p_1+\not p_2+\not p_3)\,\gamma^5\,(\not p_1+\not p_2) \\
&\quad+\mathrm{i}e\gamma_\nu\gamma^5\frac{1}{\not p_2+\not p_3}\,(\not p_1+\not p_2+\not p_3)\,\not p_1\Big\}\xi \\
=&-\frac{e}{p_1^2p_2^2}\gamma_\nu\gamma^5\xi-\frac{e}{p_1^2p_2^2}\frac{1}{\not p_3}\,(\not p_1+\not p_2)\,\gamma_\nu\gamma^5\xi \\
&+\frac{e}{2}\frac{1}{p_1^2p_2^2}\frac{1}{\not p_3}\gamma^5[\not p_2,\gamma_\nu]\xi+\frac{e}{p_1^2}\frac{1}{\not p_3}\frac{1}{\not p_1+\not p_3}\gamma_\nu\gamma^5\xi-\frac{e}{p_1^2p_2^2}\frac{1}{\not p_3}\frac{1}{\not p_1+\not p_3}\not p_2 p_{2,\nu}\gamma^5\xi \\
&+\frac{e}{p_1^2p_2^2}\frac{1}{\not p_3}\,(2p_{1,\nu}+p_{2,\nu})\,\gamma^5\xi-\frac{e}{p_1^2p_2^2}\,(2p_{1,\nu}+p_{2,\nu})\,\frac{1}{(p_1+p_2)^2}\gamma^5\,(\not p_1+\not p_2)\,\xi \\
&+e\frac{1}{p_1^2p_2^2}\frac{1}{\not p_3}\gamma_\nu\gamma^5\not p_1\xi+e\frac{1}{p_2^2}\frac{1}{\not p_3}\gamma_\nu\gamma^5\frac{1}{\not p_2+\not p_3}\xi
\end{aligned}\tag{9.35}$$

The first term from the first line, each of the second terms from the second and third line as well as the second term from the last line yield the sum of the four contact terms with the SUSY transformed fields, given by:

$$\begin{aligned}
&e\frac{1}{p_2^2}\frac{1}{\not p_3}\gamma_\nu\gamma^5\frac{1}{\not p_2+\not p_3}\xi+e\frac{1}{p_1^2}\frac{1}{\not p_3}\frac{1}{\not p_1+\not p_3}\gamma_\nu\gamma^5\xi-\frac{e}{p_1^2p_2^2}\gamma_\nu\gamma^5\xi \\
&\qquad-e\frac{1}{p_1^2}\frac{1}{p_2^2}\,(2p_{1,\nu}+p_{2,\nu})\,\frac{1}{(p_1+p_2)^2}\gamma^5(\not p_1+\not p_2)\xi
\end{aligned}\tag{9.36}$$

The remaining terms sum up to

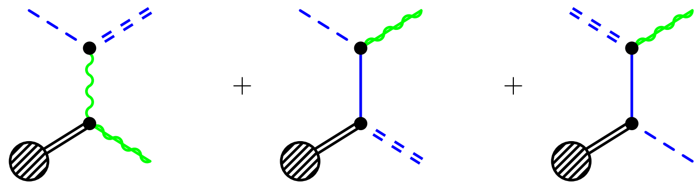
$$-\frac{e}{p_1^2p_2^2}\frac{1}{\not p_3}\frac{1}{\not p_1+\not p_3}\not p_2 p_{2,\nu}\gamma^5\xi \tag{9.37}$$

Again the term violating the off-shell validity of the SWI is proportional to the momentum of the gauge boson as observed for the second of our 2-point function examples $\langle 0|\mathrm{T}\,[[Q(\xi),G_\mu(x_1)\lambda(x_2)]]|0\rangle$, so that it will not survive LSZ reduction.

One is tempted to say that it is the external gauge boson's "fault" but this allegation is contradicted by another example not containing any gauge boson,

$$\begin{aligned}
&k^\mu \mathrm{F.T.}\,\big\langle 0\big|\mathrm{T}\,\big[\bar\xi\mathcal{J}_\mu A(x_1)B(x_2)\lambda(x_3)\big]\big|0\big\rangle \\
&\quad\stackrel{?}{=}\mathrm{F.T.}\,\big\langle 0\big|\mathrm{T}\,\big[(\bar\xi\Psi(x_1))B(x_2)\lambda(x_3)\big]\big|0\big\rangle\,\delta^4(x_1-y) \\
&\qquad+\mathrm{F.T.}\,\big\langle 0\big|\mathrm{T}\,\big[A(x_1)(\mathrm{i}\bar\xi\gamma^5\Psi(x_2))\lambda(x_3)\big]\big|0\big\rangle\,\delta^4(x_2-y) \\
&\qquad-\frac{\mathrm{i}}{2}\mathrm{F.T.}\,\big\langle 0\big|\mathrm{T}\,\big[A(x_1)B(x_2)\partial_\alpha G_\beta(x_3)[\gamma^\alpha,\gamma^\beta]\gamma^5\xi\big]\big|0\big\rangle\,\delta^4(x_3-y)
\end{aligned}\tag{9.38}$$

There are three diagrams contributing at tree level to the Green function with current insertion:



(9.39)

All momenta are understood as incoming (k_μ is the momentum influx through the current). With the relation

$$[(\not p_1 + \not p_2), (\not p_1 - \not p_2)] = -2[\not p_1, \not p_2] \tag{9.40}$$

we can calculate the three diagrams (for simplicity we multiply the relation by a factor +i)

$$\begin{aligned}
&-\mathrm{i}(p_1+p_2+p_3)^\mu \left(-\frac{\mathrm{i}e}{p_1^2 p_2^2 \not p_3}\right)\left\{\frac{1}{(p_1+p_2)^2}\gamma_\mu[\not p_1, \not p_2] - \frac{1}{\not p_1 + \not p_3}\gamma_\mu \not p_2 + \frac{1}{\not p_2 + \not p_3}\gamma_\mu \not p_1\right\}\gamma^5\xi \\
&\quad = -\frac{e}{p_1^2 p_2^2 (p_1+p_2)^2}[\not p_1, \not p_2]\gamma^5\xi - \frac{e}{p_1^2 p_2^2 (p_1+p_2)^2 \not p_3}(\not p_1 + \not p_2)[\not p_1, \not p_2]\gamma^5\xi \\
&\qquad + \frac{e}{p_1^2 p_2^2 \not p_3}\not p_2 \gamma^5\xi + \frac{e}{p_1^2 \not p_3}\frac{1}{\not p_1 + \not p_3}\gamma^5\xi - \frac{e}{p_1^2 p_2^2 \not p_3}\not p_1 \gamma^5\xi - \frac{e}{p_2^2 \not p_3}\frac{1}{\not p_2 + \not p_3}\gamma^5\xi
\end{aligned}$$

The first part of the identity

$$(\not p_1 + \not p_2)[\not p_1, \not p_2] = -(p_1+p_2)^2(\not p_1 - \not p_2) + (\not p_1 + \not p_2)\left(p_1^2 - p_2^2\right), \tag{9.41}$$

inserted into the second term of the analytical expression for the current Green function cancels its third and fifth term. It remains:

$$\begin{aligned}
&\mathrm{i}k^\mu\, \mathrm{F.T.}\, \langle 0|\mathrm{T}\left[\mathcal{J}_\mu A(x_1)B(x_2)\lambda(x_3)\right]|0\rangle = -\frac{e}{p_1^2 p_2^2 (p_1+p_2)^2}[\not p_1, \not p_2]\gamma^5\xi \\
&\quad - \frac{e}{p_1^2 p_2^2 (p_1+p_2)^2 \not p_3}(\not p_1 + \not p_2)(p_1^2 - p_2^2)\gamma^5\xi + \frac{e}{p_1^2 \not p_3}\frac{1}{\not p_1 + \not p_3}\gamma^5\xi - \frac{e}{p_2^2 \not p_3}\frac{1}{\not p_2 + \not p_3}\gamma^5\xi
\end{aligned} \tag{9.42}$$

The three contact terms yield (again multiplied by +i)

$$-\frac{e}{p_2^2 \not p_3}\frac{1}{\not p_2 + \not p_3}\gamma^5\xi + \frac{e}{p_1^2 \not p_3}\frac{1}{\not p_1 + \not p_3}\gamma^5\xi - \frac{e}{p_1^2 p_2^2 (p_1+p_2)^2}[\not p_1, \not p_2]\gamma^5\xi. \tag{9.43}$$

We see that the difference of the left and right hand sides of the SWI is an additional term from the Green function with current insertion, the second one on the right hand side of (9.42), proportional to the difference of the squared momenta of the scalar and pseudoscalar particles which vanishes on-shell but need not to do so off-shell.

As a final statement we can say that everything is fine when treating SWI calculated with the current as on-shell identities. But there are two obstacles: Ward-Takahashi identities for on-shell amplitudes do not provide tests as stringent as do off-shell amplitudes since they only check the current's couplings to external lines. Hence it would be more satisfying to have also off-shell checks for the consistency at hand. That is the practical point; but more disturbing is the theoretical aspect: Why are the SWI not fulfilled off-shell? The deeper cause is that supersymmetry is no longer a symmetry of the S-matrix for SUSY gauge theories. As will be explained in the next part, the gauge-fixing procedure required for the quantization of gauge theories is not compatible with SUSY, since it breaks the invariance of the action under supersymmetry. This restricts the validity of the SWI built with the current from the whole Hilbert space to its physical part. From this it is clear, that in the case of supersymmetric gauge theories the SWI presented in this part are only fulfilled for physical on-shell states. The next part will bring a way to get rid of that obstacle.

Part III

SUSY Slavnov-Taylor identities

The problem of supersymmetric gauge theories that Ward identities of the supersymmetry are not valid off-shell, arises from the fact that the supercharge does no longer commute with the S-operator, or in other words, it is no longer constant in time, cf. [18]. As derived therein, the difference of the action of the SUSY charge operator on the space of asymptotic *in* and asymptotic *out* states is the BRST transformation of the derivative of the effective action with respect to the ghost of the supersymmetry:

$$Q_{\text{out}} - Q_{\text{in}} = \mathrm{i}\left[Q_{\text{BRST}}, \frac{\partial \Gamma_{\text{eff}}}{\partial \epsilon}\right] \tag{III.1}$$

This can be rewritten, in the language of [24], as a commutator of the SUSY charge with the S-operator

$$[Q_{\text{in}}, S] = -\mathrm{i}\left[Q_{\text{BRST}}, \frac{\partial \Gamma_{\text{eff}}}{\partial \overline{\epsilon}} \circ S\right] \quad , \tag{III.2}$$

where the symbol $\circ$ means that the derivative of the effective action has to be understood as an operator insertion on the right hand side. The right hand side vanishes between physical states, so the SUSY charge – if not conserved on Hilbert space – is a conserved symmetry operator on the cohomology of the BRST charge, the physical Hilbert space.

There are some remarks in order: Following the pioneering idea of Peter L. White [17] we enlarge the BRST formalism not only to include supersymmetry transformations but also translations. As long as supergravity is not considered, supersymmetry remains a global symmetry and hence the ghosts of supersymmetry being *commuting* spinors are constants, as well as the translation ghosts. This allows a filtration, a power series expansion of functionals and also of Slavnov-Taylor identities with respect to the constant ghosts. Since they are constant, we were able to take an ordinary derivative in (III.1) instead of a functional one. Using the BRST formalism, the nonlinear representation of supersymmetry causes no problems for renormalization any longer, but we are not interested in that topic here. The crucial point being responsible for the nonconservation of the SUSY charge, eq. (III.1), is that gauge fixing, necessary for constructing a well-defined perturbation theory for quantized gauge theories, does not take place in a SUSY invariant manner and hence breaks supersymmetry – fortunately only in the unphysical sector of Hilbert space. Another noteworthy matter is the fact that the enlarged BRST algebra initially closes only on-shell, which can be remedied by the Batalin-Vilkovisky formalism [16], introducing quadratic terms for the sources of the non-linear parts of the BRST transformations (those for the fermions), sometimes called antifields. However, in lowest order perturbation theory we do not have to take care of that subtlety.

The important thing for us is that with the generalization of the BRST formalism we have a possibility at hand to calculate off-shell identities for supersymmetry – the Slavnov-Taylor identities (STI) of the generalized BRST algebra in lowest order perturbation theory. The prize to pay for this is the proliferation of several kinds of ghosts and BRST vertices. The details of this formalism and its application to supersymmetric gauge theories will be the content of this part. Therein we obtain strong insights into the structure of supersymmetric STI. In particular, we will see that in the Abelian case the Faddeev-Popov ghosts do no longer decouple from matter when considering SUSY STI. An example for the non-Abelian case reveals the details of the cancellations between the gauge and the SUSY parts of the BRST transformations and shows that almost all ingredients for a non-Abelian model are necessary to fulfill the STI there.

Chapter 10

BRST formalism and SUSY transformations

10.1 Definitions of the ghosts

In this section we try to solve the problem of the ghosts' properties concerning reality and statistics. We take account of the existence of not only the "gauge ghosts" (Faddeev-Popov ghosts) but also of ghosts for translations and SUSY transformations. This is necessary to achieve closure of the algebra, as well as nilpotency for the BRST charge. We will discuss this in detail in the sequel.

Let us consider a pure gauge transformation with real gauge (transformation) parameter $\theta^{a\,*} = \theta^a$, with a being the index of the gauge group. The ghost of a gauge symmetry can be derived by splitting a Grassmann odd, constant parameter λ from the gauge parameter; from the parameter being, of course, itself spacetime dependent, remains a Grassmann odd spacetime dependent field, the *(Faddeev-Popov) ghost*. As it is not consistent to choose ghost and antighost as Hermitean adjoints of each other the obvious alternative is to consider both as real, i.e. Hermitean fields. The ghost is an anticommuting scalar field which (as unphysical degree of freedom) violates the spin-statistics theorem; this would be the case for all ghosts. From the Hermiticity of the ghost it follows that the parameter λ is imaginary:

$$\mathbb{R} \ni \theta^a = \lambda c^a \qquad (\lambda c^a)^* = c^a \lambda^* = -\lambda^* c^a \;\Rightarrow\; \boxed{\lambda^* = -\lambda} \tag{10.1}$$

We have to proceed in the same manner for the SUSY transformation parameter (at first, we use the two component notation) and establish SUSY ghosts: transformation parameter $\xi^\alpha, \bar{\xi}_{\dot{\alpha}} \longrightarrow$ SUSY ghosts $\epsilon^\alpha, \bar{\epsilon}_{\dot{\alpha}}$. If we define

$$\xi^\alpha = \lambda \epsilon^\alpha, \tag{10.2}$$

use the property $(\xi^\alpha)^* = \bar{\xi}^{\dot{\alpha}}$ and require the same relation to hold for the SUSY ghosts $(\epsilon^\alpha)^* = \bar{\epsilon}^{\dot{\alpha}}$ (this is necessary for consistency as there is an identical relation between the corresponding generators Q^α and $\bar{Q}^{\dot{\alpha}}$ of their SUSY transformations), we get the following identity:

$$(\xi^\alpha)^* = (\lambda \epsilon^\alpha)^* = \lambda^* (\epsilon^\alpha)^* = -\lambda \bar{\epsilon}^{\dot{\alpha}} \stackrel{!}{=} \bar{\xi}^{\dot{\alpha}}, \tag{10.3}$$

so altogether

$$\xi^\alpha = \lambda \epsilon^\alpha, \qquad \bar{\epsilon}_{\dot{\alpha}} = -\lambda \bar{\epsilon}_{\dot{\alpha}}. \tag{10.4}$$

Introducing the bispinor notation now,

$$\xi \equiv \begin{pmatrix} \xi_\alpha \\ \bar{\xi}^{\dot{\alpha}} \end{pmatrix}, \qquad \epsilon \equiv \begin{pmatrix} \epsilon_\alpha \\ \bar{\epsilon}^{\dot{\alpha}} \end{pmatrix}, \tag{10.5}$$

we get the final result:

$$\boxed{\xi = -\lambda \gamma^5 \epsilon} \tag{10.6}$$

For deriving an analogous relation for the translation ghosts we note, that in general an infinitesimal translation of a function has the form

$$\delta_a f(x) = a^\mu \partial_\mu f(x). \tag{10.7}$$

[18] and [27] adopt the following conventions

$$a^\mu = \mathrm{i}\lambda\omega^\mu \tag{10.8}$$

for the connection between transformation parameter and translation ghost. The translation (of course only as a global symmetry here) is a bosonic symmetry like (ordinary) gauge symmetry, so the translation ghost ω^μ is a Grassmann odd vector. From the reality of the transformation parameter a^μ we conclude with the help of

$$\mathbb{R}^4 \ni a^\mu \;\Rightarrow\; (\mathrm{i}\lambda\omega^\mu)^* = -\mathrm{i}\omega^{\mu\,*}\lambda^* = +\mathrm{i}\lambda^*\omega^{\mu\,*} = -\mathrm{i}\lambda\omega^{\mu\,*} \stackrel{!}{=} \mathrm{i}\lambda\omega^\mu \tag{10.9}$$

$$\Longrightarrow \boxed{\omega^{\mu\,*} = -\omega^\mu} \tag{10.10}$$

We summarize the properties of all ghosts in the following table (d_s is the unspecified dimension of the BRST charge).

Ghost	Dim.	Grassmann P.	Charge	Ghost Number
c	s	1	0	+1
$\bar{c}$	$2 - d_s$	1	0	−1
ϵ	$d_s - \frac{1}{2}$	0	0	+1
ω^μ	$d_s - 1$	1	0	+1

10.2 BRST symmetry in our Abelian toy model

10.2.1 The model

To illustrate the BRST formalism for supersymmetric gauge theories in detail, we use the Abelian toy model invented in the last part in the context of the supersymmetric current. The Lagrangean density, the field content, the propagators, vertices, equations of motion, as well as the SUSY transformations can be found in the appendix.

10.2.2 BRST transformations

With the above given definitions for the ghosts and the summary of transformations from the appendix we can immediately write down the BRST transformations for our Abelian toy model

$$
\begin{aligned}
sA(x) =&\ -ec(x)B(x) - \bar{\epsilon}\gamma^5\Psi(x) - \mathrm{i}\omega^\nu\partial_\nu A(x) && (10.11a)\\
sB(x) =&\ +ec(x)A(x) - \mathrm{i}\bar{\epsilon}\Psi(x) - \mathrm{i}\omega^\nu\partial_\nu B(x) && (10.11b)\\
s\Psi(x) =&\ -\mathrm{i}ec(x)\gamma^5\Psi(x) + \mathrm{i}(\partial\!\!\!/ - \mathrm{i}eG\!\!\!/\gamma^5)(A(x)\gamma^5 + \mathrm{i}B(x))\epsilon - \mathrm{i}\omega^\nu\partial_\nu\Psi(x) && (10.11c)\\
s\overline{\Psi}(x) =&\ \mathrm{i}e\overline{\Psi}(x)\gamma^5 c(x) - \mathrm{i}\bar{\epsilon}(\partial\!\!\!/ + \mathrm{i}e\gamma^5 G\!\!\!/)(A(x)\gamma^5 - \mathrm{i}B(x)) - \mathrm{i}\omega^\nu\partial_\nu\overline{\Psi}(x) && (10.11d)\\
s\lambda(x) =&\ \frac{\mathrm{i}}{2}F_{\alpha\beta}(x)\gamma^\alpha\gamma^\beta\epsilon + \frac{e}{2}\left(A^2(x) + B^2(x)\right)\gamma^5\epsilon - \mathrm{i}\omega^\nu\partial_\nu\lambda(x) && (10.11e)\\
s\overline{\lambda}(x) =&\ -\frac{\mathrm{i}}{2}\bar{\epsilon}F_{\alpha\beta}(x)\gamma^\alpha\gamma^\beta + \frac{e}{2}\bar{\epsilon}\gamma^5\left(A^2(x) + B^2(x)\right) - \mathrm{i}\omega^\nu\partial_\nu\overline{\lambda}(x) && (10.11f)\\
sG_\mu(x) =&\ \partial_\mu c(x) - \bar{\epsilon}\gamma_\mu\lambda(x) - \mathrm{i}\omega^\nu\partial_\nu G_\mu(x) && (10.11g)\\
sc(x) =&\ \mathrm{i}\bar{\epsilon}\gamma^\mu\epsilon G_\mu(x) - \mathrm{i}\omega^\nu\partial_\nu c(x) && (10.11h)\\
s\bar{c}(x) =&\ \mathrm{i}\tilde{B}(x) - \mathrm{i}\omega^\nu\partial_\nu\bar{c}(x) && (10.11i)\\
s\tilde{B}(x) =&\ \bar{\epsilon}\gamma^\mu\epsilon\partial_\mu\bar{c}(x) - \mathrm{i}\omega^\nu\partial_\nu\tilde{B}(x) && (10.11j)\\
s\epsilon =&\ 0 && (10.11k)\\
s\omega^\mu =&\ \bar{\epsilon}\gamma^\mu\epsilon && (10.11l)
\end{aligned}
$$

We have denoted the Nakanishi-Lautrup field by $\tilde{B}$ to avoid confusion with the pseudoscalar field. To derive the identities for adjoint fields we use that for bosonic fields B and for fermionic fields F the relations

$$sB^\dagger = (sB)^\dagger, \qquad sF^\dagger = -(sF)^\dagger. \tag{10.12}$$

hold. In the case of the adjoint spinors we have to take care of the commutation properties of the several fields. The first part of (10.11) – if present – stems from the gauge transformation, the second from the SUSY transformation, and the last obviously from translation. The somewhat strange looking and unmotivated transformations of the several ghosts are necessary for the closure of the algebra (cf. [17], [18]) and can be understood from examination of the super-Poincaré algebra.

It is not hard to check that the BRST transformation is nilpotent except for the transformation of the fermion fields, where the square of the BRST operator gives the equation of motion for them:

$$
\begin{aligned}
&s^2A = s^2B = s^2G_\mu = s^2c = s^2\bar{c} = s^2\tilde{B} = s^2\epsilon = s^2\omega_\mu = 0,\\
&s^2\Psi = -\frac{1}{2}(\bar{\epsilon}\gamma^\mu\epsilon)\gamma_\mu\frac{\delta\Gamma}{\delta\overline{\overline{\Psi}}}, \qquad s^2\lambda = -\frac{1}{4}(\bar{\epsilon}\gamma^\mu\epsilon)\gamma_\mu\frac{\delta\Gamma}{\delta\overline{\overline{\lambda}}}
\end{aligned} \tag{10.13}
$$

For the derivation of the identity for the matter fermion one has to multiply use the Fierz identities. As the calculations are a little bit intricate, we show one example for the matter fermion:

$$
\begin{aligned}
s^2\Psi =&\ -\mathrm{i}e(sc)\gamma^5\Psi + \mathrm{i}ec\gamma^5(s\Psi) + \mathrm{i}(\partial\!\!\!/ - \mathrm{i}eG\!\!\!/\gamma^5)\left((sA)\gamma^5 + \mathrm{i}(sB)\right)\epsilon\\
&+ e\gamma^\mu(sG_\mu)\gamma^5\left(A\gamma^5 + \mathrm{i}B\right)\epsilon - \mathrm{i}(s\omega^\nu)\partial_\nu\Psi + \mathrm{i}\omega^\nu\partial_\nu(s\Psi)\\
=&\ e(\bar{\epsilon}\gamma^\mu\epsilon)G_\mu\gamma^5\Psi - \underline{e\omega^\nu(\partial_\nu c)\gamma^5\Psi} + \underline{\underline{e^2c^2\Psi}} - \underline{ec\gamma^5(\partial\!\!\!/ - \mathrm{i}eG\!\!\!/\gamma^5)\left(A\gamma^5 + \mathrm{i}B\right)\epsilon}
\end{aligned}
$$

$$
\begin{aligned}
&+ \underline{ec\gamma^5\omega^\nu\partial_\nu\Psi} + \underline{\mathrm{i}(\not\partial - \mathrm{i}e\not{G}\gamma^5)\mathrm{i}ec\gamma^5\left(A\gamma^5 + \mathrm{i}B\right)\epsilon} \\
&+ \underline{\mathrm{i}(\not\partial - \mathrm{i}e\not{G}\gamma^5)(-\mathrm{i})\omega^\nu\partial_\nu\left(A\gamma^5 + \mathrm{i}B\right)\epsilon} + \mathrm{i}(\not\partial - \mathrm{i}e\not{G}\gamma^5)\left[(\bar\epsilon\Psi) - (\bar\epsilon\gamma^5\Psi)\gamma^5\right]\epsilon \\
&+ \underline{e(\not\partial c)\gamma^5\left(A\gamma^5 + \mathrm{i}B\right)\epsilon} - e\gamma^\mu(\bar\epsilon\gamma_\mu\lambda)\gamma^5\left(A\gamma^5 + \mathrm{i}B\right)\epsilon \\
&- \underline{\mathrm{i}e\omega^\nu(\partial_\nu\not{G})\gamma^5\left(A\gamma^5 + \mathrm{i}B\right)\epsilon} - \mathrm{i}(\bar\epsilon\gamma^\nu\epsilon)\partial_\nu\Psi + \underline{e\omega^\nu\partial_\nu(c\gamma^5\Psi)} \\
&- \underline{\omega^\nu\partial_\nu(\not\partial - \mathrm{i}e\not{G}\gamma^5)\left(A\gamma^5 + \mathrm{i}B\right)\epsilon} + \underline{\underline{\omega^\mu\omega^\nu\partial_\mu\partial_\nu\Psi}} + \underline{\mathrm{i}^2 e(\not\partial c)\gamma^5\left(A\gamma^5 + \mathrm{i}B\right)\epsilon}
\end{aligned}
\tag{10.14}
$$

Underlined terms cancel each other, while doubly underlined ones vanish identically. Sorting and ordering the remaining terms we are left with:

$$
\begin{aligned}
s^2\Psi = \; & -\mathrm{i}(\bar\epsilon\gamma^\mu\epsilon)\left(\partial_\mu\Psi + \mathrm{i}eG_\mu\gamma^5\Psi\right) + \mathrm{i}(\not\partial - \mathrm{i}e\not{G}\gamma^5)\left[(\bar\epsilon\Psi) - (\bar\epsilon\gamma^5\Psi)\gamma^5\right]\epsilon \\
& - e\gamma^\mu(\bar\epsilon\gamma_\mu\lambda)\gamma^5\left(A\gamma^5 + \mathrm{i}B\right)\epsilon
\end{aligned}
\tag{10.15}
$$

We use the following Fierz identities

$$
(\bar\epsilon\Psi)\epsilon = \frac{1}{8}(\bar\epsilon\sigma_{\mu\nu}\epsilon)\sigma^{\mu\nu}\Psi + \frac{1}{4}(\bar\epsilon\gamma^\mu\epsilon)\gamma_\mu\Psi \tag{10.16a}
$$

$$
(\bar\epsilon\gamma^5\Psi)\gamma^5\Psi = \frac{1}{8}(\bar\epsilon\sigma_{\mu\nu}\epsilon)\sigma^{\mu\nu}\Psi - \frac{1}{4}(\bar\epsilon\gamma^\mu\epsilon)\gamma_\mu\Psi. \tag{10.16b}
$$

For *commuting* Majorana spinors the scalar, pseudoscalar and pseudovectorial combinations vanish due to the symmetry properties of Majorana bilinears $\bar\epsilon\Gamma\epsilon$. We get for the second term of (10.15):

$$
\begin{aligned}
&\mathrm{i}(\not\partial - \mathrm{i}e\not{G}\gamma^5)\left[(\bar\epsilon\Psi) - (\bar\epsilon\gamma^5\Psi)\gamma^5\right]\epsilon \\
&\qquad = -\frac{1}{2}(\bar\epsilon\gamma^\mu\epsilon)\mathrm{i}\gamma_\mu(\not\partial + \mathrm{i}e\not{G}\gamma^5)\Psi + \mathrm{i}(\bar\epsilon\gamma^\mu\epsilon)\left(\partial_\mu + \mathrm{i}eG_\mu\gamma^5\Psi\right)
\end{aligned}
\tag{10.17}
$$

Here we have used the Dirac algebra; the second term cancels the first from (10.15). When Fierzing the last term, only the vector combination is nonvanishing and we can directly write down:

$$
-e\gamma^\mu(\bar\epsilon\gamma_\mu\lambda)\gamma^5\left(A\gamma^5 + \mathrm{i}B\right)\epsilon = \frac{1}{2}(\bar\epsilon\gamma^\mu\epsilon)e\gamma_\mu\left(A - \mathrm{i}B\gamma^5\right)\lambda \tag{10.18}
$$

Summing up all terms gives the desired result.

10.3 Gauge fixing and kinetic ghost term

For the quantization of a gauge theory we have to carry out a gauge fixing in the usual way.

$$
S_{\mathrm{GF+FP}} = -\mathrm{i}\int d^4x\; s(\bar c F) = -\mathrm{i}\int d^4x\left[(s\bar c)F - \bar c(sF)\right] \tag{10.19}
$$

F is the gauge fixing function:

$$
F = \partial_\mu G^\mu + \frac{\xi}{2}\tilde{B}. \tag{10.20}
$$

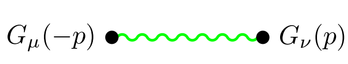
By ξ we denote the gauge parameter, not to be confused with the (anticommuting) SUSY transformation parameter. After applying the BRST transformation the terms containing the translation ghost ω^μ cancel out since the gauge fixing function is translation invariant. It remains:

$$\boxed{S_{\mathrm{GF+FP}} = \int d^4x \left\{ \tilde{B}\partial_\mu G^\mu + \frac{\xi}{2}\tilde{B}^2 + \mathrm{i}\bar{c}\Box c - \mathrm{i}\bar{c}(\bar{\epsilon}\partial\!\!\!/\lambda) + \mathrm{i}\frac{\xi}{2}\bar{c}(\bar{\epsilon}\gamma^\mu\epsilon)\partial_\mu\bar{c} \right\}} \tag{10.21}$$

Integrating out the Nakanishi-Lautrup field gives:

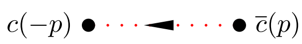
$$S_{\mathrm{GF+FP}} = \int d^4x \left\{ -\frac{1}{2\xi}(\partial_\mu G^\mu)^2 + \mathrm{i}\bar{c}\Box c - \mathrm{i}\bar{c}(\bar{\epsilon}\partial\!\!\!/\lambda) + \mathrm{i}\frac{\xi}{2}\bar{c}(\bar{\epsilon}\gamma^\mu\epsilon)\partial_\mu\bar{c} \right\} \tag{10.22}$$

The contributions with the derivative of the gauge field yield, together with the terms from the kinetic part, the gauge boson propagator in R_ξ gauge:



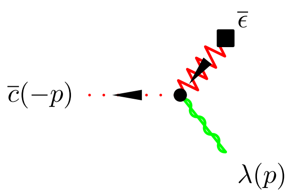
$$= \frac{-\mathrm{i}}{p^2 + \mathrm{i}\epsilon}\left(\eta_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right) \tag{10.23}$$

Furthermore we get the ghost propagator:

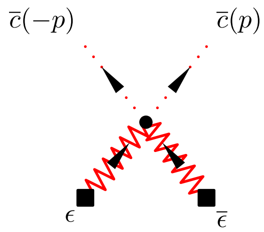


$$= \frac{-1}{p^2 + \mathrm{i}\epsilon} \tag{10.24}$$

But there also arise two new vertices containing the gauge antighosts as well as the SUSY ghosts. Albeit being an Abelian model the ghosts do not decouple from the (matter) fields. To say it sloppily, since supersymmetry and gauge symmetry are not commuting subalgebras in the de Wit–Freedman description, the model becomes formally non-Abelian. The two vertices are



$$= -\mathrm{i}p\!\!\!/ \tag{10.25a}$$



$$= \xi p\!\!\!/ \tag{10.25b}$$

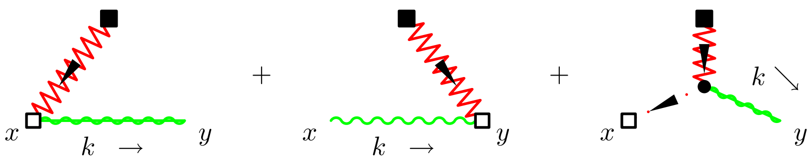
For the four-ghost vertex there is a symmetry factor two for the gauge antighost, but no symmetry factor for the SUSY ghost. This is because the SUSY ghost is a constant, so what we do is merely a Taylor expansion in the power of the SUSY ghosts where the factorials cancel the symmetry factors. Only the gauge ghosts are propagating fields, all other ghosts are simply constant insertions. The black box in the Feynman diagrams should indicate that for the constance of the ghost the line ends there.

10.4 Slavnov-Taylor identities in the Abelian toy model

In this section we want to review the Green functions from the last section of the second part in the light of the BRST formalism and the Slavnov-Taylor identities. There is no one-to-one correspondence between the Ward identities of the last part and the Slavnov-Taylor identities; but one can replace the current insertion from the SWI by the BRST charge acting on the same string of fields as in the SWI. So for the first example where the SWI had not been valid off-shell we write down the STI [1]

$$\langle 0|\mathrm{T}\left[\{Q_{\mathrm{BRST}}, G_\nu(x_1)\lambda(x_2)\}\right]|0\rangle = 0 \quad \text{wegen} \quad Q_{\mathrm{BRST}}\,|0\rangle = 0. \tag{10.26}$$

There are three (!) contributing diagrams:



(10.27)

For the BRST vertices we use the notations of [13]. The first diagram yields

$$-\langle 0|\mathrm{T}\left[(\bar\epsilon\gamma_\nu\lambda(x))\lambda(y)\right]|0\rangle = +\left\langle 0\middle|\mathrm{T}\left[\lambda(y)(\bar\lambda(x)\gamma_\nu\epsilon)\right]\middle|0\right\rangle = \frac{\mathrm{i}}{\not k}\gamma_\nu\epsilon. \tag{10.28}$$

Several signs have to be accounted for: $(\bar\epsilon\gamma_\nu\lambda) = +(\bar\lambda\gamma_\nu\epsilon)$, as ϵ is commuting; for the same reason: $(\bar\lambda\gamma_\nu\epsilon)\lambda = -\lambda(\bar\lambda\gamma_\nu\epsilon)$.

From the second diagram we get

$$\frac{\mathrm{i}}{2}\left\langle 0\middle|\mathrm{T}\left[G_\nu(x)F_{\alpha\beta}(y)\gamma^\alpha\gamma^\beta\epsilon\right]\middle|0\right\rangle = \frac{\mathrm{i}}{2}\frac{-\mathrm{i}\eta_{\nu\beta}}{k^2}(-\mathrm{i}k_\alpha)[\gamma^\alpha,\gamma^\beta]\epsilon = -\frac{\mathrm{i}}{2}\frac{1}{k^2}[\not k,\gamma_\nu]\epsilon. \tag{10.29}$$

After insertion of an interaction operator the third diagram contributes

$$\begin{aligned}\langle 0|\mathrm{T}\left[\partial^x_\nu c(x)\lambda(y)\right]|0\rangle &= -\left\langle 0\middle|\mathrm{T}\left[\partial^x_\nu c(x)\bar c(y)\lambda(y)(\bar\lambda(z)\overleftarrow{\not\partial}_z\,\epsilon)\right]\middle|0\right\rangle \\ &= -\frac{-1}{k^2}(\mathrm{i}k_\nu)\frac{\mathrm{i}}{\not k}(\mathrm{i}\not k)\epsilon = -\frac{\mathrm{i}k_\nu}{k^2}.\end{aligned} \tag{10.30}$$

Adding the three terms yields

$$\frac{\mathrm{i}}{\not k}\gamma_\nu\epsilon - \frac{\mathrm{i}}{2}\frac{1}{k^2}[\not k,\gamma_\nu]\epsilon - \frac{\mathrm{i}k_\nu}{k^2} = \frac{\mathrm{i}}{\not k}\gamma_\nu\epsilon - \frac{\mathrm{i}}{\not k}\gamma_\nu\epsilon + \frac{\mathrm{i}k_\nu}{k^2} - \frac{\mathrm{i}k_\nu}{k^2} = 0. \tag{10.31}$$

The STI is in contrast to the SWI fulfilled *off-shell* (and therefore automatically on-shell). Note the crucial importance of the term where the two ghosts couple to the gaugino.

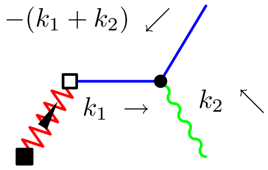
We now turn to the more complex examples. At first,

$$0 = \langle 0|\mathrm{T}\left[\{Q_{\mathrm{BRST}}, A(x_1)G_\nu(x_2)\Psi(x_3)\}\right]|0\rangle$$

[1] All calculations in this chapter are done in the Feynman gauge, $\xi \equiv 1$, but as is easily seen all calculations go through analogously for a general R_ξ gauge and are therefore independent from the choice of gauge.

$$\begin{aligned}
={}& - e\,\langle 0|\mathrm{T}\,[c(x_1)B(x_1)G_\nu(x_2)\Psi(x_3)]\,|0\rangle + \left\langle 0\middle|\mathrm{T}\left[(\overline{\Psi}(x_1)\gamma^5\epsilon)G_\nu(x_2)\Psi(x_3)\right]\middle|0\right\rangle \\
&+ \langle 0|\mathrm{T}\,[A(x_1)(\partial_\nu c(x_2))\Psi(x_3)]\,|0\rangle - \left\langle 0\middle|\mathrm{T}\left[A(x_1)(\overline{\lambda}(x_2)\gamma_\nu\epsilon)\Psi(x_3)\right]\middle|0\right\rangle \\
&- \mathrm{i}e\left\langle 0\middle|\mathrm{T}\left[A(x_1)G_\nu(x_2)c(x_3)\gamma^5\Psi(x_3)\right]\middle|0\right\rangle \\
&+ \mathrm{i}\left\langle 0\middle|\mathrm{T}\left[A(x_1)G_\nu(x_2)\left(\not\partial - \mathrm{i}e\not G(x_3)\gamma^5\right)\left(A(x_3)\gamma^5 + \mathrm{i}B(x_3)\right)\epsilon\right]\middle|0\right\rangle
\end{aligned} \tag{10.32}$$

The first and the penultimate Green function do not contribute. Graphically the second Green function in (10.32) yields

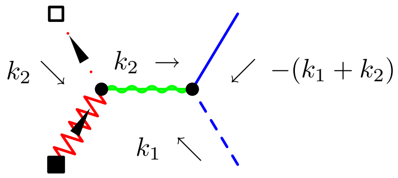


(10.33)

and analytically

$$-\frac{-\mathrm{i}\eta_{\nu\beta}}{k_2^2}\frac{\mathrm{i}}{\not k_1+\not k_2}\mathrm{i}e\gamma^5\gamma^\beta\frac{\mathrm{i}}{\not k_1}\gamma^5\epsilon = \frac{e}{k_2^2}\frac{1}{\not k_1+\not k_2}\gamma_\nu\frac{1}{\not k_1}\epsilon \tag{10.34}$$

As was the case for the 2-point function, a term from the third Green function in (10.32) does exist with an additional interaction vertex here, too:

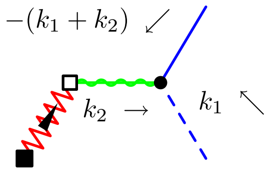


(10.35)

It yields

$$-(\mathrm{i}k_{2,\nu})\frac{\mathrm{i}}{k_1^2}\frac{\mathrm{i}}{\not k_1+\not k_2}(-\mathrm{i}e)\frac{\mathrm{i}}{\not k_2}\mathrm{i}\not k_2\frac{-1}{k_2^2}\epsilon = \frac{ek_{2,\nu}}{k_1^2k_2^2}\frac{1}{\not k_1+\not k_2}\epsilon \tag{10.36}$$

The fourth Green function with the diagram



(10.37)

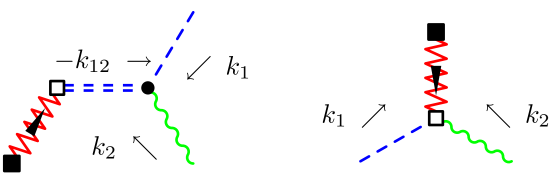
gives the result

$$\frac{\mathrm{i}}{k_1^2}\frac{\mathrm{i}}{\not k_1+\not k_2}(-\mathrm{i}e)\frac{\mathrm{i}}{\not k_2}\gamma_\nu\epsilon = -\frac{e}{k_1^2}\frac{1}{\not k_1+\not k_2}\frac{1}{\not k_2}\gamma_\nu\epsilon \tag{10.38}$$

From the last Green function there are two contributions,

$$-\langle 0|\mathrm{T}\,[A(x_1)G_\nu(x_2)\not\partial B(x_3)]\,|0\rangle + e\left\langle 0\middle|\mathrm{T}\left[A(x_1)G_\nu(x_2)\gamma^\lambda(G_\lambda A)(x_3)\epsilon\right]\middle|0\right\rangle \tag{10.39}$$

with the two diagrams ($k_{12} \equiv k_1 + k_2$)



. (10.40)

The right diagram yields the analytical expression

$$e\frac{\mathrm{i}}{k_1^2}\frac{-\mathrm{i}\eta_{\nu\lambda}}{k_2^2}\gamma^\lambda\epsilon = \frac{e}{k_1^2k_2^2}\gamma_\nu\epsilon \quad , \tag{10.41}$$

while the left diagram gives the result

$$\begin{aligned} -\frac{\mathrm{i}}{k_1^2}\frac{-\mathrm{i}\eta_{\nu\beta}}{k_2^2}\frac{\mathrm{i}}{(k_1+k_2)^2}e(k_1+(k_1+k_2))^\beta(-\mathrm{i})(\not{k}_1+\not{k}_2)\epsilon \\ = \frac{-e}{k_1^2k_2^2(k_1+k_2)^2}(2k_1+k_2)_\nu(\not{k}_1+\not{k}_2)\epsilon \end{aligned} \tag{10.42}$$

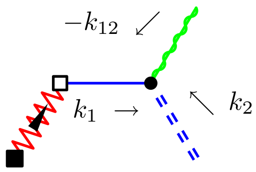
The sum of all four terms vanishes:

$$\begin{aligned} &\frac{e}{k_2^2}\frac{1}{\not{k}_1+\not{k}_2}\gamma_\nu\frac{1}{\not{k}_1}\epsilon+\frac{ek_{2,\nu}}{k_1^2k_2^2}\frac{1}{\not{k}_1+\not{k}_2}\epsilon-\frac{e}{k_1^2}\frac{1}{\not{k}_1+\not{k}_2}\frac{1}{\not{k}_2}\gamma_\nu\epsilon+\frac{e}{k_1^2k_2^2}\gamma_\nu\epsilon \\ &\qquad -\frac{e}{k_1^2k_2^2(k_1+k_2)^2}(2k_1+k_2)_\nu(\not{k}_1+\not{k}_2)\epsilon \\ &= \frac{e}{k_1^2k_2^2(\not{k}_1+\not{k}_2)}\left\{\gamma_\nu\not{k}_1+k_{2,\nu}-\not{k}_2\gamma_\nu+(\not{k}_1+\not{k}_2)\gamma_\nu-(2k_1+k_2)_\nu\right\}\epsilon=0 \end{aligned} \tag{10.43}$$

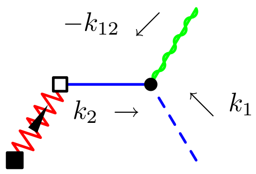
So this STI is fulfilled. Let us check the other example from the earlier part of the text as well.

$$\begin{aligned} 0 =\ &\langle 0|\mathrm{T}\left[\{Q_{\mathrm{BRST}},A(x_1)B(x_2)\lambda(x_3)\}\right]|0\rangle \\ =\ &-e\,\langle 0|\mathrm{T}\left[c(x_1)B(x_1)B(x_2)\lambda(x_3)\right]|0\rangle+\left\langle 0\middle|\mathrm{T}\left[(\overline{\Psi}(x_1)\gamma^5\epsilon)B(x_2)\lambda(x_3)\right]\middle|0\right\rangle \\ &+e\,\langle 0|\mathrm{T}\left[A(x_1)c(x_2)A(x_2)\Psi(x_3)\right]|0\rangle+\mathrm{i}\left\langle 0\middle|\mathrm{T}\left[A(x_1)(\overline{\Psi}(x_2)\epsilon)\lambda(x_3)\right]\middle|0\right\rangle \\ &+\frac{\mathrm{i}}{2}e\left\langle 0\middle|\mathrm{T}\left[A(x_1)B(x_2)\partial_\alpha G_\beta(x_3)[\gamma^\alpha,\gamma^\beta]\epsilon\right]\middle|0\right\rangle \\ &+\frac{e}{2}\left\langle 0\middle|\mathrm{T}\left[A(x_1)B(x_2)\left(A^2(x_3)+B^2(x_3)\right)\gamma^5\epsilon\right]\middle|0\right\rangle \end{aligned} \tag{10.44}$$

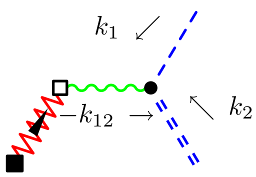
The only vanishing term is the last one with trilinear terms from the BRST transformations. The nonvanishing contributions are ($k_{12}=k_1+k_2$):



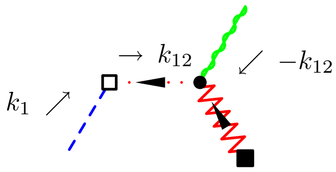
$$= -\frac{\mathrm{i}e}{k_2^2}\frac{1}{\not{k}_1+\not{k}_2}\frac{1}{\not{k}_1}\epsilon \tag{10.45}$$



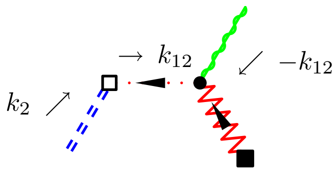
$$= \frac{\mathrm{i}e}{k_1^2}\frac{1}{\not{k}_1+\not{k}_2}\frac{1}{\not{k}_2}\epsilon \tag{10.46}$$



$$= \frac{\mathrm{i}e}{2k_1^2k_2^2(k_1+k_2)^2}[\not{k}_1+\not{k}_2,\not{k}_1-\not{k}_2]\epsilon \tag{10.47}$$



$$= -\frac{\mathrm{i}e}{k_1^2(k_1+k_2)^2}\epsilon \tag{10.48}$$



$$= \frac{\mathrm{i}e}{k_2^2(k_1+k_2)^2}\epsilon \tag{10.49}$$

Adding up the five contributions yields:

$$\frac{\mathrm{i}e}{k_1^2k_2^2(k_1+k_2)^2}\left\{-(\not{k}_1+\not{k}_2)\not{k}_1+(\not{k}_1+\not{k}_2)\not{k}_2-[\not{k}_1,\not{k}_2]-k_2^2+k_1^2\right\}\epsilon = 0 \tag{10.50}$$

This STI is valid, too.

From this examples it can be seen that the formalism of Slavnov-Taylor identities with the inclusion of the constant SUSY ghosts does indeed work for supersymmetric gauge theories. Let us give some comments about that: We only calculated the lowest order of the STI in the gauge coupling constant *and* the SUSY ghost (first order) *and* the translation ghost (zeroth order). This kind of identities can easily be extended to higher orders in the number of the constant ghosts due to the filtration property of functionals and Hilbert space, but *not in this model* to higher orders in the gauge coupling since that special model carries an anomaly and the symmetry is spoiled when going to the one-loop level or beyond.

Chapter 11

Non-Abelian gauge theories: $SU(N)$

The generalization of the ideas and formalisms for supersymmetric gauge theories and BRST quantization developed up to now to the non-Abelian case is the topic of this chapter. For the sake of simplicity we restrict ourselves to the special unitary groups $SU(N)$; within these gauge groups the structure of the vertices of four scalar particles is very simple. For more general gauge groups cf. [28].

We consider an SQCD-like model with two (matter) superfields $\hat{\Phi}_+$ and $\hat{\Phi}_-$ and their Hermitean adjoints. In this model the fermion becomes a Dirac fermion by combining the left-handed components of the $+$-superfield with those of the conjugated $-$-superfield. Quadratic superpotential terms are allowed here, cubic are still forbidden by gauge invariance. The Abelian limit of this model will give SQED, the supersymmetric extension of quantum electrodynamics.

The main part of the model's details concerning Lagrangean density, equations of motion, propagators, Feynman rules, etc. can be found in appendix E.5. After diagonalizing the mass terms of the fermions, the model contains two charged scalar fields ϕ_+ and ϕ_- living in the fundamental representation and its complex conjugate, respectively, as well as a charged fermion Ψ in the fundamental representation. Moreover there is the gauge boson A_μ, the gaugino λ, the ghost c and the antighost $\bar{c}$, each in the adjoint representation.

The BRST transformations of these fields are:

$$s\phi_{+,i}(x) = -\mathrm{i}gc^a(x)T^a_{ij}\phi_{+,j}(x) + \sqrt{2}\,(\bar{\epsilon}\mathcal{P}_L\Psi_i(x)) - \mathrm{i}\omega^\nu\partial_\nu\phi_{+,i}(x) \tag{11.1a}$$

$$s\phi^\dagger_{+,i}(x) = \mathrm{i}gc^a(x)\phi^\dagger_{+,j}(x)T^a_{ji} + \sqrt{2}\left(\overline{\Psi}_i(x)\mathcal{P}_R\epsilon\right) - \mathrm{i}\omega^\nu\partial_\nu\phi^\dagger_{+,i}(x) \tag{11.1b}$$

$$s\phi_{-,i}(x) = \mathrm{i}gc^a(x)\phi_{-,j}(x)T^a_{ji} - \sqrt{2}\left(\overline{\Psi}_i(x)\mathcal{P}_L\epsilon\right) - \mathrm{i}\omega^\nu\partial_\nu\phi_{-,i}(x) \tag{11.1c}$$

$$s\phi^\dagger_{-,i}(x) = -\mathrm{i}gc^a(x)T^a_{ij}\phi^\dagger_{-,j}(x) - \sqrt{2}\,(\bar{\epsilon}\mathcal{P}_R\Psi_i(x)) - \mathrm{i}\omega^\nu\partial_\nu\phi^\dagger_{-,i}(x) \tag{11.1d}$$

$$\begin{aligned} s\Psi_i(x) = {} & -\mathrm{i}gc^a(x)T^a_{ij}\Psi_j(x) + \sqrt{2}\Big[(\mathrm{i}\partial\!\!\!/ + m)\phi_{+,i}(x)\mathcal{P}_R + (\mathrm{i}\partial\!\!\!/ - m)\phi^\dagger_{-,i}(x)\mathcal{P}_L \\ & + gA\!\!\!/^a(x)T^a_{ij}\left(\phi_{+,j}(x)\mathcal{P}_R + \phi^\dagger_{-,j}(x)\mathcal{P}_L\right)\Big]\epsilon - \mathrm{i}\omega^\nu\partial_\nu\Psi_i(x) \end{aligned} \tag{11.1e}$$

$$\begin{aligned} s\overline{\Psi}_i(x) = {} & \mathrm{i}gc^a(x)\overline{\Psi}_j(x)T^a_{ji} + \sqrt{2}\bar{\epsilon}\Big[\mathcal{P}_L(\mathrm{i}\partial\!\!\!/ - m)\phi^\dagger_{+,i}(x) + \mathcal{P}_R(\mathrm{i}\partial\!\!\!/ - m)\phi_{-,i}(x) \\ & - g\left(\phi^\dagger_{+,j}(x)\mathcal{P}_L + \phi_{-,j}(x)\mathcal{P}_R\right)T^a_{ji}A\!\!\!/^a(x)\Big] - \mathrm{i}\omega^\nu\partial_\nu\overline{\Psi}_i(x) \end{aligned} \tag{11.1f}$$

$$\begin{aligned}
sA^a_\mu(x) =&\; (D_\mu c(x))^a - \bar\epsilon\gamma_\mu\lambda^a(x) - \mathrm{i}\omega^\nu\partial_\nu A^a_\mu(x) && (11.1g)\\
s\lambda^a(x) =&\; gf_{abc}c^b(x)\lambda^c(x) + \frac{\mathrm{i}}{2}F^a_{\alpha\beta}(x)\gamma^\alpha\gamma^\beta\epsilon + g\left(\phi^\dagger_+(x)T^a\phi_+(x)\right)\gamma^5\epsilon \\
&- g\left(\phi_-(x)T^a\phi^\dagger_-(x)\right)\gamma^5\epsilon - \mathrm{i}\omega^\nu\partial_\nu\lambda^a(x) && (11.1h)\\
s\overline{\lambda^a}(x) =&\; gf_{abc}c^b(x)\overline{\lambda^c}(x) - \frac{\mathrm{i}}{2}\bar\epsilon\gamma^\alpha\gamma^\beta F^a_{\alpha\beta}(x) + g\bar\epsilon\gamma^5\left(\phi^\dagger_+(x)T^a\phi_+(x)\right) \\
&- g\bar\epsilon\gamma^5\left(\phi_-(x)T^a\phi^\dagger_-(x)\right) - \mathrm{i}\omega^\nu\partial_\nu\overline{\lambda^a}(x) && (11.1i)\\
sc^a(x) =&\; -\frac{g}{2}f_{abc}c^b(x)c^c(x) + \mathrm{i}(\bar\epsilon\gamma^\mu\epsilon)A_\mu(x) - \mathrm{i}\omega^\nu\partial_\nu c^a(x) && (11.1j)\\
s\bar c^a(x) =&\; \mathrm{i}B^a(x) - \mathrm{i}\omega^\nu\partial_\nu\bar c^a(x) && (11.1k)\\
sB^a(x) =&\; (\bar\epsilon\gamma^\mu\epsilon)\partial_\mu\bar c^a(x) - \mathrm{i}\omega^\nu\partial_\nu B^a(x) && (11.1l)\\
s\epsilon =&\; 0 && (11.1m)\\
s\omega^\mu =&\; (\bar\epsilon\gamma^\mu\epsilon) && (11.1n)
\end{aligned}$$

The gauge fixing is in complete analogy to the Abelian case

$$S_{\mathrm{GF+FP}} = -\mathrm{i}\int d^4x\; s(\bar c^a F^a) = -\mathrm{i}\int d^4x\,[(s\bar c^a)F^a - \bar c^a(sF^a)] \tag{11.2}$$

with the gauge fixing function

$$F^a = \partial^\mu A^a_\mu + \frac{\xi}{2}B^a. \tag{11.3}$$

ξ is the gauge parameter. For the translational invariance of the gauge fixing term no translation ghosts ω^μ appear, too, and we get:

$$\boxed{\begin{aligned} S_{\mathrm{GF+FP}} = \int d^4x\,\Big\{& B^a\partial^\mu A^a_\mu + \frac{\xi}{2}B^aB^a + \mathrm{i}\bar c^a\partial_\mu(D^\mu c)^a \\ &- \mathrm{i}\bar c^a(\bar\epsilon\not{\partial}\lambda^a) + \mathrm{i}\frac{\xi}{2}\bar c^a(\bar\epsilon\gamma^\mu\epsilon)\partial_\mu\bar c^a\Big\}\end{aligned}} \tag{11.4}$$

Integrating the Nakanishi-Lautrup field out yields:

$$\begin{aligned} S_{\mathrm{GF+FP}} = \int d^4x\,\Big\{& -\frac{1}{2\xi}(\partial^\mu A^a_\mu)(\partial^\nu A^a_\nu) + \mathrm{i}\bar c^a\partial_\mu(D^\mu c)^a \\ &- \mathrm{i}\bar c^a(\bar\epsilon\not{\partial}\lambda^a) + \mathrm{i}\frac{\xi}{2}\bar c^a(\bar\epsilon\gamma^\mu\epsilon)\partial_\mu\bar c^a\Big\}\end{aligned} \tag{11.5}$$

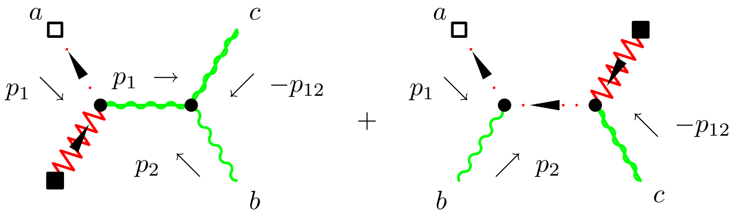
11.1 An example for an STI in SQCD

We just want to show one example for a Slavnov-Taylor identity in supersymmetric quantum chromodynamics, SQCD:

$$\begin{aligned}
0 \overset{!}{=}&\; \left\langle 0\middle|\mathrm{T}\left[\left\{Q_{\mathrm{BRST}}, A^a_\mu(x_1)A^b_\nu(x_2)\lambda^c(x_3)\right\}\right]\middle|0\right\rangle \\
=&\; \left\langle 0\middle|\mathrm{T}\left[(D_\mu c)^a(x_1)A^b_\nu(x_2)\lambda^c(x_3)\right]\middle|0\right\rangle - \left\langle 0\middle|\mathrm{T}\left[\left(\bar\epsilon\gamma_\mu\lambda^a(x_1)\right)A^b_\nu(x_2)\lambda^c(x_3)\right]\middle|0\right\rangle \\
&+ \left\langle 0\middle|\mathrm{T}\left[A^a_\mu(x_1)(D_\nu c)^b(x_2)\lambda^c(x_3)\right]\middle|0\right\rangle - \left\langle 0\middle|\mathrm{T}\left[A^a_\mu(x_1)\left(\bar\epsilon\gamma_\nu\lambda^b(x_2)\right)\lambda^c(x_3)\right]\middle|0\right\rangle
\end{aligned}$$

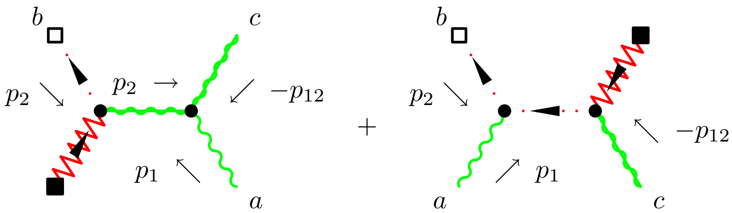
$$+\frac{\mathrm{i}}{2}\left\langle 0\middle|\mathrm{T}\left[A^a_\mu(x_1)A^b_\nu(x_2)\partial_\lambda A^c_\kappa(x_3)[\gamma^\lambda,\gamma^\kappa]\epsilon\right]\middle|0\right\rangle$$
$$+\frac{\mathrm{i}g}{4}\left\langle 0\middle|\mathrm{T}\left[A^a_\mu(x_1)A^b_\nu(x_2)\left(A^e_\lambda A^f_\kappa\right)(x_3)[\gamma^\lambda,\gamma^\kappa]f^{cef}\epsilon\right]\middle|0\right\rangle \tag{11.6}$$

(In the sequel p_{12} means p_1+p_2.)



$$= -\mathrm{F.T.}\left\langle 0\middle|\mathrm{T}\left[\partial_\mu c^a(x_1)\bar{c}^d(z)\lambda^c(x_3)\left(\overline{\lambda^d}(z)\overleftarrow{\partial\!\!\!/}\,\epsilon\right)A^b_\nu(x_2)\right]\middle|0\right\rangle$$

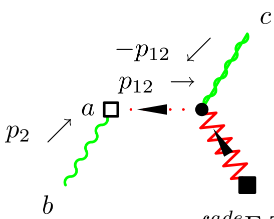
$$-\frac{-1}{p_1^2}\frac{-\mathrm{i}}{p_2^2}\frac{\mathrm{i}}{p\!\!\!/_1+p\!\!\!/_2}g\gamma_\nu f^{abc}\frac{\mathrm{i}}{p\!\!\!/_1}(\mathrm{i}p\!\!\!/_1)\mathrm{i}p_{1,\mu}\epsilon = \frac{-\mathrm{i}gf^{abc}}{p_1^2p_2^2(p_1+p_2)^2}\left(p\!\!\!/_1+p\!\!\!/_2\right)\gamma_\nu p_{1,\mu}\epsilon$$
$$-\frac{\mathrm{i}}{p\!\!\!/_1+p\!\!\!/_2}\mathrm{i}\left(p\!\!\!/_1+p\!\!\!/_2\right)\epsilon\frac{-1}{(p_1+p_2)^2}.$$
$$(-\mathrm{i}gf^{abc})p_{1,\nu}\frac{-\mathrm{i}}{p_2^2}\frac{-1}{p_1^2}(\mathrm{i}p_{1,\mu}) = \frac{-\mathrm{i}gf^{abc}}{p_1^2p_2^2(p_1+p_2)^2}p_{1,\mu}p_{1,\nu}\epsilon \tag{11.7}$$



$$= -\mathrm{F.T.}\left\langle 0\middle|\mathrm{T}\left[\partial_\nu c^b(x_2)\bar{c}^d(z)\lambda^c(x_3)\left(\overline{\lambda^d}(z)\overleftarrow{\partial\!\!\!/}\,\epsilon\right)A^a_\mu(x_1)\right]\middle|0\right\rangle$$

This is just the earlier result with the replacements $(a\leftrightarrow b)$, $(\mu\leftrightarrow\nu)$, $(p_1\leftrightarrow p_2)$:

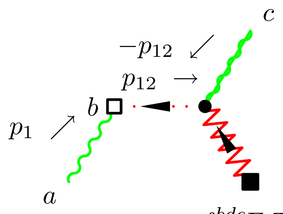
$$\frac{\mathrm{i}gf^{abc}}{p_1^2p_2^2(p_1+p_2)^2}\left(p\!\!\!/_1+p\!\!\!/_2\right)\gamma_\mu p_{2,\nu}\epsilon+\frac{\mathrm{i}gf^{abc}}{p_1^2p_2^2(p_1+p_2)^2}p_{2,\mu}p_{2,\nu}\epsilon \tag{11.8}$$



$$= -gf^{ade}\mathrm{F.T.}\left\langle 0\middle|\mathrm{T}\left[\left(A^d_\mu c^e\right)(x_1)\bar{c}^f(z)\lambda^c(x_3)\left(\overline{\lambda^f}(z)\overleftarrow{\partial\!\!\!/}\,\epsilon\right)A^b_\nu(x_2)\right]\middle|0\right\rangle$$

This gives the analytical expression:

$$-gf^{abc}\frac{-\mathrm{i}}{p_2^2}\eta_{\mu\nu}\frac{-1}{(p_1+p_2)^2}\frac{\mathrm{i}}{p\!\!\!/_1+p\!\!\!/_2}\mathrm{i}\left(p\!\!\!/_1+p\!\!\!/_2\right)\epsilon = \frac{\mathrm{i}gf^{abc}}{p_1^2p_2^2(p_1+p_2)^2}p_1^2\eta_{\mu\nu}\epsilon \tag{11.9}$$

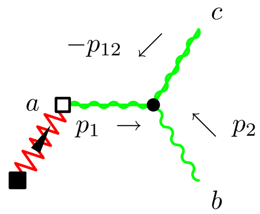


$$= -gf^{bde}\text{F.T.}\left\langle 0\middle|\mathrm{T}\left[\left(A^d_\nu c^e\right)(x_2)\bar{c}^f(z)\lambda^c(x_3)\left(\overline{\lambda^f}(z)\overleftarrow{\not\partial}\,\epsilon\right)A^a_\mu(x_1)\right]\middle|0\right\rangle$$

Again we can use the replacements made above:

$$\frac{-\mathrm{i}gf^{abc}}{p_1^2p_2^2(p_1+p_2)^2}p_2^2\eta_{\mu\nu}\epsilon \tag{11.10}$$

All of the expressions calculated up to now came from the gauge part of the BRST transformations, now we discuss the SUSY part.



$$= \text{F.T.}\left\langle 0\middle|\mathrm{T}\left[A^b_\nu(x_2)\lambda^c(x_3)\left(\overline{\lambda^a}(x_1)\gamma_\mu\epsilon\right)\right]\middle|0\right\rangle$$

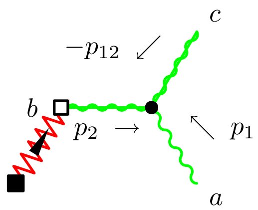
We have made use of the two following identities:

$$(\bar{\epsilon}\gamma_\mu\lambda) = +\left(\overline{\lambda}\gamma_\mu\epsilon\right), \qquad \left(\overline{\lambda_1}\gamma_\mu\epsilon\right)\lambda_2 = -\lambda_2\left(\overline{\lambda_1}\gamma_\mu\epsilon\right). \tag{11.11}$$

The last diagram yields the analytical expression:

$$\frac{\mathrm{i}}{\not p_1+\not p_2}\frac{-\mathrm{i}}{(p_1+p_2)^2}g\gamma_\nu f^{abc}\frac{\mathrm{i}}{\not p_1}\gamma_\mu\epsilon = \frac{\mathrm{i}gf^{abc}}{p_1^2p_2^2(p_1+p_2)^2}\left(\not p_1+\not p_2\right)\gamma_\nu\not p_1\gamma_\mu\epsilon \tag{11.12}$$

The terms for

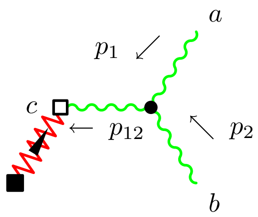


$$= \text{F.T.}\left\langle 0\middle|\mathrm{T}\left[A^a_\mu(x_1)\lambda^c(x_3)\left(\overline{\lambda^b}(x_2)\gamma_\nu\epsilon\right)\right]\middle|0\right\rangle$$

are again available by the replacements $(a\leftrightarrow b)$, $(\mu\leftrightarrow\nu)$, $(p_1\leftrightarrow p_2)$:

$$\frac{-\mathrm{i}gf^{abc}}{p_1^2p_2^2(p_1+p_2)^2}\left(\not p_1+\not p_2\right)\gamma_\mu\not p_2\gamma_\nu\epsilon \tag{11.13}$$

The next step is the transformation of the gluino:

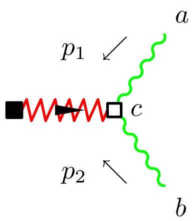


$$= \frac{\mathrm{i}}{2}\text{F.T.}\left\langle 0\middle|\mathrm{T}\left[A^a_\mu(x_1)A^b_\nu(x_2)\partial_\lambda A^c_\kappa(x_3)[\gamma^\lambda,\gamma^\kappa]\epsilon\right]\middle|0\right\rangle$$

From this diagram we get

$$\frac{\mathrm{i}}{2}\frac{-\mathrm{i}}{p_1^2}\frac{-\mathrm{i}}{p_2^2}\frac{-\mathrm{i}}{(p_1+p_2)^2}gf^{abc}(-\mathrm{i})(p_1+p_2)_\lambda[\gamma^\lambda,\gamma^\kappa]\cdot$$
$$\Big[\eta_{\mu\nu}\left(p_1-p_2\right)_\kappa+\eta_{\nu\kappa}\left(2p_2+p_1\right)_\mu+\eta_{\mu\kappa}\left(-2p_1-p_2\right)_\nu\Big]\epsilon$$
$$=\frac{\mathrm{i}gf^{abc}}{p_1^2p_2^2(p_1+p_2)^2}\frac{1}{2}\Big[\eta_{\mu\nu}[\not p_1+\not p_2,\not p_1-\not p_2]+\left(2p_2+p_1\right)_\mu[\not p_1+\not p_2,\gamma_\nu]$$
$$-\left(2p_1+p_2\right)_\nu[\not p_1+\not p_2,\gamma_\mu]\Big]\epsilon \quad (11.14)$$

The final contribution to the STI comes from



$$=\frac{\mathrm{i}gf^{cde}}{4}\mathrm{F.T.}\left\langle 0\middle|\mathrm{T}\left[A_\mu^a(x_1)A_\nu^b(x_2)\left(A_\lambda^dA_\kappa^e\right)(x_3)[\gamma^\lambda,\gamma^\kappa]\epsilon\right]\middle|0\right\rangle$$

It yields (with a symmetry factor two)

$$\frac{\mathrm{i}}{2}gf^{abc}\frac{-\mathrm{i}}{p_1^2}\frac{-\mathrm{i}}{p_2^2}[\gamma_\mu,\gamma_\nu]\epsilon=\frac{-\mathrm{i}gf^{abc}}{p_1^2p_2^2(p_1+p_2)^2}\frac{1}{2}[\gamma_\mu,\gamma_\nu]\left(p_1+p_2\right)^2\epsilon \quad (11.15)$$

Adding up all the contributions (11.7)-(11.15) should give zero; in the following calculation we forget about the common prefactor $\mathrm{i}gf^{abc}/(p_1^2p_2^2(p_1+p_2)^2)$. First we collect all terms containing a factor $\not p_1+\not p_2$. For the terms of the contribution with the three-gauge boson vertex we use

$$\frac{1}{2}[\not p_1+\not p_2,\not a]=(\not p_1+\not p_2)\not a-(p_1+p_2)\cdot a \quad , \quad (11.16)$$

while for the last contributing term we use $(p_1+p_2)^2=(\not p_1+\not p_2)(\not p_1+\not p_2)$. We get

$$(\not p_1+\not p_2)\Big\{-\gamma_\nu p_{1,\mu}-\left(2p_1+p_2\right)_\nu\gamma_\mu+\gamma_\mu p_{2,\nu}+\gamma_\nu\not p_1\gamma_\mu-\gamma_\mu\not p_2\gamma_\nu+\eta_{\mu\nu}\left(\not p_1-\not p_2\right)$$
$$+\left(2p_2+p_1\right)_\mu\gamma_\nu-\frac{1}{2}\left(\not p_1+\not p_2\right)[\gamma_\mu,\gamma_\nu]\Big\}$$
$$=(\not p_1+\not p_2)\Big\{\eta_{\mu\nu}\left(\not p_1-\not p_2\right)-\frac{1}{2}\left(\not p_1+\not p_2\right)[\gamma_\mu,\gamma_\nu]+\not p_2\gamma_\mu\gamma_\nu-\not p_1\gamma_\nu\gamma_\mu\Big\}$$
$$=\frac{1}{2}\left(\not p_1+\not p_2\right)\left(\not p_1-\not p_2\right)\Big[2\eta_{\mu\nu}-\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu\Big]=0. \quad (11.17)$$

In the second equation the Dirac algebra was used for the fourth and fifth term to cancel out all terms with only one gamma matrix (besides the prefactor).

The terms that do not contain any gamma matrices yield:

$$\left(2p_1+p_2\right)_\nu\left(p_1+p_2\right)_\mu-\eta_{\mu\nu}\left(p_1+p_2\right)\cdot\left(p_1-p_2\right)+\eta_{\mu\nu}\left(p_1^2-p_2^2\right)$$

$$- (2p_2 + p_1)_\mu (p_1 + p_2)_\nu - p_{1,\mu} p_{1,\nu} + p_{2,\mu} p_{2,\nu} = 0. \quad (11.18)$$

So everything cancels, and the STI is fulfilled. Note that this STI contains almost all elements of the non-Abelian character of the theory, the coupling of the gluon to the ghosts and to the gluinos, and also the three-gluon vertex. This identity would be trivial in SQED.

11.2 BRST formalism for spontaneously broken supersymmetry

For spontaneously broken supersymmetry the BRST formalism brings nothing new. Since supersymmetry is a global symmetry, the Goldstino always is a physical particle. Only in a supergravity theory with a super-Higgs mechanism the Goldstino is eaten up by the gravitino to make it massive and add the missing two $J_z = \pm\frac{1}{2}$ polarizations. But we do not want to discuss the BRST formalism for supergravity in this thesis.

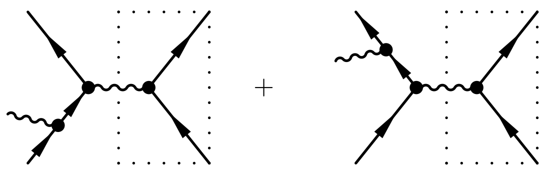
Part IV

Implementation, Summary and Outlook

Chapter 12

Implementation in *O'Mega*

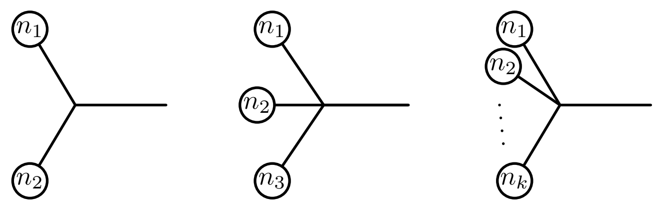
The *"**O**ptimizing **M**atrix **E**lement **G**ener**a**tor" O'Mega* invented by Thorsten Ohl [12], [29], [30] provides the most efficient way to calculate tree level amplitudes available today. This is achieved by building up amplitudes recursively by fusions of one-particle off-shell wave functions where the redundacies of the diagrammatic representation are removed by a procedure called *common subexpression elimination.* That is most easily illustrated by an example, $e^+e^- \to \mu^+\mu^-$ with one additional bremsstrahlung quantum:



When we consider the initial state radiation then the muon part (in the dotted box) remains the same and needs only to be calculated once. If the result of the one calculation is kept in memory and inserted when needed instead of being calculated again, a lot of computation time is saved. That is the way *O'Mega* works and how it is able to reduce the factorial growth of the number of Feynman diagrams with the number of external particles to an exponential, cf. table 1.1. This chapter will be rather formal since we do not have the space to go into the details of *O'Mega* here, they can be found in the commented source code of the program [30].

12.1 BRST vertices

As was briefly mentioned in the foregoing section, *O'Mega* constructs amplitudes by fusing sets of subamplitudes built up recursively from one-particle off-shell wave functions (1POWs) fused out of partitions of the external particles. The fusion works with k-ary topologies ($k \geq 2$), so that a new 1POW is fused out of two, three or more 1POWs depending on subsets of the external momenta:



Binary topology Ternary topology k-ary topology

The first two cases are needed for the Standard Model's 3- and 4-point vertices (or other models') while higher topologies are useful for introducing vertices of a degree higher than four to parameterize irrelevant operator insertions from an effective theory. But it is *not* possible for *O'Mega* to handle 2-point vertices like self energy insertions, current insertions for operator product expansions, or also BRST vertices, since every 1POW has to be uniquely labelled by the external momenta on which it depends. So for *O'Mega* it would be impossibile to distinguish between a 1POW and one with just an additional 2-point vertex.

The solution to that problem is not difficult: When expressing the theory with the help of functionals and for deriving Slavnov-Taylor identities in a functional language, then in the effective action one has to introduce external sources K for each BRST transformation [13], [16],

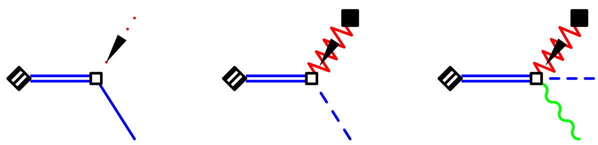
$$\mathrm{i}\int d^4x \sum_{\text{all }\Phi} K_\Phi \cdot (s\Phi) \tag{12.1}$$

sometimes called *antifields*. If we add those sources to the particle content of an *O'Mega* model file then we are able to generate Slavnov-Taylor identities in a manner similar to using functional derivatives,

$$\langle 0|\mathrm{T}\left[(s\Phi)\phi_1\dots\phi_n\right]|0\rangle = \left\langle 0\middle|\mathrm{T}\left[\phi_1\dots\phi_n\frac{\delta}{\delta K_\Phi}e^{\mathrm{i}\Gamma(\phi_i,K_i)}\right]\middle|0\right\rangle\Bigg|_{K_j=0} \quad .$$

We also add the BRST vertices in the form (12.1) to our model file and generate them in an amplitude by using the source as an external particle. There are always two possibilities to define particles (or better: fields) in *O'Mega*, as propagating, or as being "only insertion". In the first case they can appear as virtual particles in inner lines while in the latter they are forced to serve exclusively as external particles. The source K_Φ, which always has the same Lorentz structure as the transformed field Φ, has to be defined as nonpropagating, since it just generates a single operator insertion (that corresponds to setting the sources to zero above). Graphically, we use again the notation of [13] for the BRST vertex, a small square. For the source we just double the line of the transformed field and put a diamond at the end to demonstrate its nature as a pure insertion (in analogy to the constant SUSY ghosts). To make this more pictorial we give an example for the BRST transformation of the matter fermion of our Abelian toy model from the last part (without the translation ghost, and for simplicity we also omit the diagrams with the pseudoscalar field B)

$$\overline{K_\Psi}\,(s\Psi) = -\mathrm{i}ec\,\overline{K_\Psi}\gamma^5\Psi + \mathrm{i}\overline{K_\Psi}(\partial\!\!\!/ - \mathrm{i}e\,C\!\!\!/\gamma^5)(A\gamma^5 + \mathrm{i}B)\epsilon$$

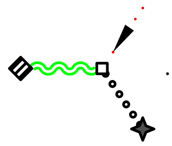
$\Longleftrightarrow$ 

From these diagrams we can immediately see the advantages of this construction: Slavnov-Taylor identities (STI) for the gauge symmetries as well as for supersymmetry can be done in the same formalism, one just has to tell which ghost should be produced, c or ϵ. Without the BRST transformation source the last of the diagrams written down above would have been an ordinary 3-point vertex and could have been handled in the usual manner in *O'Mega*, but this vertex arises due to the nonlinearities of supersymmetry (additional to the nonlinear structure of BRST) while most of the BRST vertices are of the type of the leftmost diagrams, and furthermore, the ability to have a unified formalism for managing all kinds of STI is worth the effort.

The only objection could lie in the BRST transformation of the gauge boson into the derivative of the ghost

$$sA_\mu = \partial_\mu c + \ldots,$$

which – even when coupled to the antifield – is still only a 2-point vertex. There we must use a trick: we establish a dummy field, another nonpropagating local operator which couples to the BRST source of the photon and to the ghost, which we denote by a tetragram here:



This means that for generating an STI – where we have to replace each of the fields $\phi_i, i = 1, \ldots n$ in $\langle 0|\mathrm{T}\left[\{Q_{\mathrm{BRST}}, \phi_1\phi_2 \ldots \phi_n(\text{ghost})\}\right]|0\rangle$ successively by the source of its BRST transformation – each gauge boson must be replaced not only by the source of its BRST transformation but also by the product of the source and that local operator. By the choice of "ghost" we specify whether we want to study a SUSY or a gauge STI. (Since the BRST charge has ghost number +1 we must have one ghost in our string of fields above to get a nontrivial result.) The problem with the one-point vertex can be avoided by using physical polarization vectors (cf. the table at the end of that section) for the external states of the BRST transformed vector bosons, i.e. transversal states but still off-shell. Since the BRST vertex with the derivative of the ghost produces the momentum of that gauge boson, the contribution from that term vanishes. In that formalism the gauge boson can be handled as a matter particle in the adjoint representation of the gauge group. Of course, a more stringent test is done by taking the STIs component-wise, where the term $\partial_\mu c^a$ has been taken accounted for.

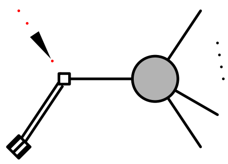
There is yet another problem which is not so difficult to resolve with our BRST source construction: *O'Mega* produces code for calculating S-matrix elements, i.e. for amputated Green functions, whereas we want to study STI, relations between off-shell Green functions usually *not* amputated. But we can consider all Green functions participating in our STIs as being amputated while still remaining off the mass shell. So all external legs of the Green functions have been multiplied by their inverse propagators to arrive at a matrix element-like object, something that can be generated by *O'Mega*.

For gauge theories one of the fields in the STI must be an antighost which combines with the ghost from the BRST transformation to give a ghost propagator. Dividing by that ghost propagator cancels the ghost propagator in all of the Green functions including the gauge boson propagator coming from the scalar mode of the gauge boson produced by BRST transforming the antighost. This is clear for exact gauge symmetries, whereas for the spontaneously broken case we present a short proof (we omit the propagator of the Goldstone boson ϕ since it always has the same pole as the ghost, $p^2 - \xi m^2$):

$$(-\mathrm{i})s\bar{c}^a = \frac{1}{\xi}\partial^\mu A^a_\mu + m\phi^a \quad \text{produces the propagator with contracted momentum:}$$

$$\frac{1}{\xi}\left(\mathrm{i}p^{\mu}\right)\frac{\mathrm{i}}{p^2-m^2}\left(-\eta_{\mu\nu}+\frac{(1-\xi)p_\mu p_\nu}{p^2-\xi m^2}\right)=\frac{-1}{\xi}\frac{p_\nu}{p^2-m^2}\left(-1+\frac{(1-\xi)p^2}{p^2-\xi m^2}\right)$$
$$=\frac{1}{\xi}\frac{p_\nu}{p^2-m^2}\frac{\xi(p^2-m^2)}{p^2-\xi m^2} \qquad (12.2)$$

The BRST transformed "physical fields" (matter fields and gauge bosons) become internal lines, but so they are in *O'Mega* due to our construction using the sources,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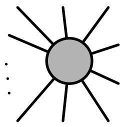
and their propagators are produced automatically in *O'Mega*. When multiplying with the inverse propagators of all external particles there is a mismatch according to the formula (for simplicity we assume to have a gauge, not a supersymmetry here and all particles to be scalars; the generalization is straightforward):

$$\prod_{i=1}^{n}(-\mathrm{i})\left(p_i^2-m_i^2\right)\mathrm{F.T.}\,\langle 0|\mathrm{T}\left[\{Q_{\mathrm{BRST}},\phi_1\ldots\phi_n\}\right]|0\rangle$$
$$=\sum_{j=1}^{n}\frac{p_j^2-m_j^2}{(p_j+k)^2-\tilde{m}_j^2}\cdot\left[\quad\right]'_{\text{amputated}}, \qquad (12.3)$$



because the transformed field now has an internal propagator with a momentum shifted by the ghost's momentum and therefore does not cancel the inverse propagator (the prime indicates that this internal propagator has been extracted from the diagram). Besides, in spontaneously broken symmetries there is the possibility that field and transformed field have different masses, e.g. when connecting electron and neutrino by an $SU(2)_L$-BRST transformation or the several scalar and fermionic fields in the O'Raifeartaigh model. This is indicated by a tilde placed over the mass symbol. In supersymmetric theories the ghost is constant and brings no momentum into the Green functions, but obviously the propagators of field and transformed field cannot cancel since they belong to a fermion and a boson or vice versa. Thus, generally, one inverse propagator survives which has to be taken into account. In our formalism this is easily done by absorbing it into the wavefunction of the BRST source. Consequently, we associate the following expressions as wavefunctions to our BRST sources[1] (gauge boson sources in unitarity gauge):

Particle	Wavefunction	Particle	Wavefunction
K_ϕ, incoming	$-\mathrm{i}(p^2-m^2)$	K_Ψ, incoming	$-\mathrm{i}(\not p-m)u(p)$
K_ϕ, outgoing	$-\mathrm{i}(p^2-m^2)$	$K_{\overline{\Psi}}$, outgoing	$\mathrm{i}(\not p+m)v(p)$
K_{A_μ}, incoming	$\mathrm{i}(p^2-m^2)\epsilon_\mu(p)$	K_Ψ, outgoing	$-\mathrm{i}\overline{u}(p)(\not p-m)$
K_{A_μ}, outgoing	$\mathrm{i}(p^2-m^2)\epsilon^*_\mu(p)$	$K_{\overline{\Psi}}$, incoming	$\mathrm{i}\overline{v}(p)(\not p+m)$

(12.4)

[1] This is based on an idea of Christian Schwinn. ϵ, u and v could be the ordinary wavefunctions for external states which were off-shell in that context, or for the most general tests unit four-vectors and unit four-spinors.

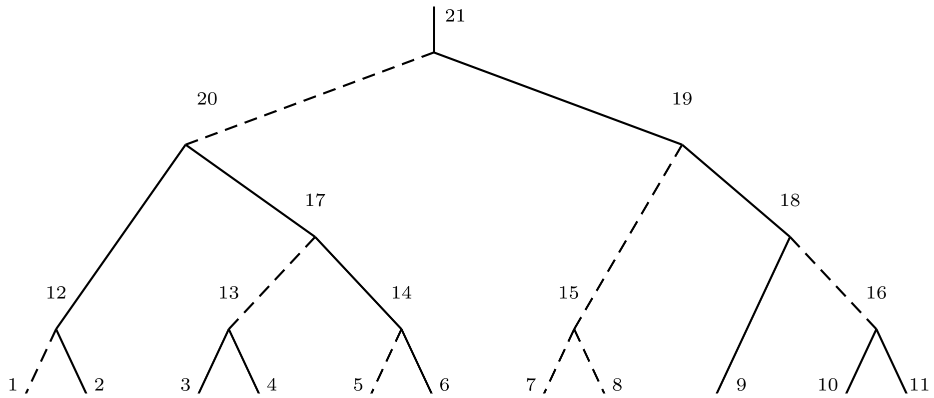


Figure 12.1: *Example for Fermi statistics sign evaluation. Fermions are denoted by plain lines, bosons by dashed ones.*

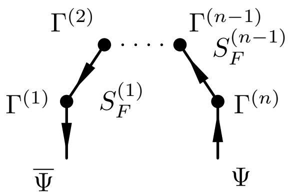
For Majorana fermions there are some minor changes; we will come back to that point after having discussed another important topic – the handling of Fermi statistics with the inclusion of real fermions within the framework of *O'Mega*.

12.2 Fermi Statistics - Evaluation of Signs

Fermions are anticommuting objects – whenever their order is changed in Green functions or different diagrams contributing to a Green function they produce sign factors. These signs can be read off from the Wick theorem. They arise as relative signs (e.g. for Bhabha scattering between s- and t-channel) between whole diagrams, but to avoid the explicit construction of all diagrams was the strongest motivation for *O'Mega*. How can we cope with Fermi statistics within its framework? The solution was found by Thorsten Ohl [30]: When fusing two (or more) one-particle off-shell wavefunctions (1POWs), then the sign of the newly produced 1POW is calculated and divided by the signs of the subamplitudes out of which it was combined. Thus, for every fusion only the sign relative to the subamplitudes is kept in memory, which makes evaluating the signs compatible with the factorization procedure of *O'Mega*. What remains is to answer the question how to evaluate those signs from Fermi statistics for a subamplitude.

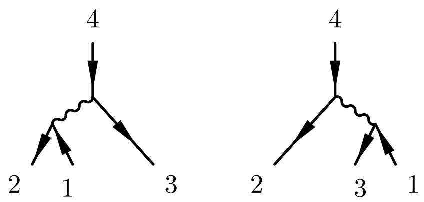
Each subamplitude depends on a subset of the external particles, part of which are fermions while the others are not. The solution for calculating the sign factors lies in a bookkeeping concept for fermion lines: We "follow the fermion lines" through the graphs. At each fusion we examine whether a fermion line is continued or, together with another fermion line, fused into a boson line while keeping in mind all "closed" fermion lines appearing in the fused 1POWs. The best way to explain this is an example which is provided by figure 12.1. There the numbers 1-11 denote external particles on which the 1POW labelled by 21 depends; this 1POW is a fermion. (It would be more consistent to label each line by the external particles on which it depends, e.g. $(1, 2)$ for 12, $(3, 4)$ for 13, $(7, 8, 9, 10, 11)$ for 19 etc.) So, the fermion 21 is part of an open fermion line beginning (or ending) with the external particle 9 while its 1POW contains the closed fermion lines $(2 \leftrightarrow 6)$, $(3 \leftrightarrow 4)$ and $(10 \leftrightarrow 11)$. And what about the direction of the lines?

This is the point where the difference arises between the way Ohl calculates the sign for theories with Dirac fermions only and our way for theories with Dirac as well as Majorana fermions. The Feynman rules for Majorana fermions and their implementation in *O'Mega* rely upon the ideas in [19]. When a theory is endowed only with Dirac fermions and is not supersymmetric, then always a well defined direction can be asserted to each fermion line – which is based on the fact that we can always distinguish between incoming particle and outgoing antiparticle or vice versa. Out of this concept of directed fermion lines Ohl further specified, whether a fermionic 1POW is a fermion or an antifermion (it would be better to say, a spinor or a conjugated spinor, respectively) – in the picture of figure 12.1 whether the arrow at the line points upward or downward, respectively. Closed fermion lines then consist of pairs of numbers indicating the conjugated spinor of the external particle at which the line begins, and the spinor of the external particle at which the line ends. This is actually an abuse of language, since the arrows point from the "barred" to the "unbarred" spinor, but when writing down an analytical expression we start with the barred spinor since we write from left to right:



$$\equiv \overline{\Psi}\Gamma^{(1)}S_F^{(1)}\Gamma^{(2)}\ldots\Gamma^{(n-1)}S_F^{(n-1)}\Gamma^{(n)}\Psi$$

In Ohl's ansatz the sign from Fermi statistics is calculated by collecting the closed fermion lines (running from the conjugated spinor to the spinor) and then comparing them with a reference order of the external fermions and antifermions. The number of transpositions needed to bring the collected pairs of external fermionic particles into that reference order gives the relative sign between the two different orders. As an example we consider Bhabha scattering (we denote the incoming electron and positron by 1 and 2, respectively, the outgoing electron and positron by 3 and 4, respectively):



The left diagram is the s-channel, the right one the t-channel. In our formalism we get $(21)(34) \rightarrow \{2,1,3,4\}$ for the left diagram, and $(31)(24) \rightarrow \{3,1,2,4\}$ for the right one[2]. If we take $\{1,2,3,4\}$ as our reference order then the s-channel gets a relative sign, while the t-channel not; as the global sign does not matter, the relative sign between the two channels is produced[3].

[2] Normally the fusions run up only to 1POWS depending on nearly half of the external momenta. In our case, each possible pairing of the fermions would produce a photon, and in a final step, two off-shell photon wavefunctions are fused to yield the whole amplitude. Each of those photons would contain one closed fermion line, but as pairs of numbers $(\_,\_)$ commute with each other (4 transpositions) the order of the pairs plays no role.

[3] As discussed above, *O'Mega* evaluates the sign(s) subamplitude-wise, but for human beings the principle is better understandable when thinking in diagrams.

We therefore have the following "fusion rules" for solely Dirac fermions (again the dashed line is an unspecified boson), by which we mean the collecting of the fermion lines to evaluate the sign from the number of transpositions needed to bring the pairs of fermions into the reference order:

$\phi = \overline{\Psi}_b \Gamma \Psi_a$ $\quad$ $\phi = \overline{\Psi}_a \Gamma \Psi_b$ $\quad$ $\Psi'_a = \phi \Gamma \Psi_a$ $\quad$ $\overline{\Psi}'_a = \phi \overline{\Psi}_a \Gamma$

a, l_a $\quad$ b, l_b $\quad$ a, l_a $\quad$ b, l_b $\quad$ a, l_a $\quad$ $\_, l_b$ $\quad$ a, l_a $\quad$ $\_, l_b$

$\boxed{\_\,, \{b,a\} \cup l_a \cup l_b}$ $\quad$ $\boxed{\_\,, \{a,b\} \cup l_a \cup l_b}$ $\quad$ $\boxed{a\,, l_a \cup l_b}$ $\quad$ $\boxed{a\,, l_a \cup l_b}$

(12.5)

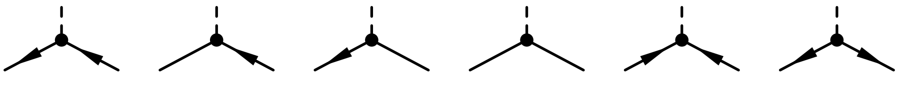
Therein l_a and l_b are lists of antifermion–fermion pairs of external particles belonging to lines already closed, contained in the 1POWs which take part in that fusion. a and b are fermion labels for the left and right leg in the fusion. In the two left fusions a boson is produced so there is no "open" fermion index as for the rightmost fusions where a fermion (or antifermion) is produced again. What *O'Mega* calculates numerically is shown above the diagrams. For the last two diagrams there also exists a mirror version omitted here, with the fermion leg on the right.

The expressions in *O'Mega*'s Dirac fermion version are produced numerically by a final closure of the fermion line where – via the bilinear product $\overline{\Psi}_1 \Gamma \Psi_2$ – a conjugated spinor is multiplied with a spinor to give a non-spinorial expression. For external particles we assign a spinor to incoming fermions (u) and to outgoing antifermions (v), and use conjugated spinors for outgoing fermions ($\overline{u}$) and incoming antifermions ($\overline{v}$). Each conjugated spinor is continued through the subamplitudes representing classes of subdiagrams by right multiplication with the vertex and propagator factors which consist of linear combinations of gamma matrices. In contrast, spinors are left multiplied by the corresponding factors when following the line, respectively:

$$\overline{\Psi}' = \overline{\Psi}\Gamma \;\text{ or }\; \overline{\Psi}'' = \overline{\Psi} S_F \qquad\qquad \Psi' = \Gamma\Psi \;\text{ or }\; \Psi'' = S_F \Psi \tag{12.6}$$

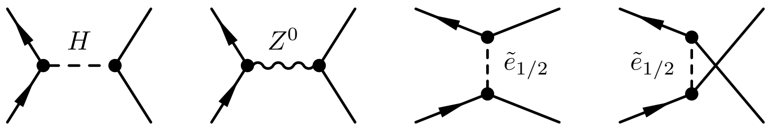
The bilinear product closing the fermion line could either produce a new bosonic off-shell wavefunction or means the last keystone in constructing a part of the whole amplitude. For the latter case it is indeed a scalar product, e.g. in the process $e^+e^- \to \gamma\gamma$, where the positron wavefunction made out of the incoming positron and one outgoing photon is fused with the electron wavefunction constructed from the incoming electron and the other photon.

In supersymmetric theories there are two difficulties which make the implementation by Thorsten Ohl, described in the last paragraphs inapplicable: the existence of Majorana fermions, i.e. real fermions, to whose lines no specific direction can be assigned since they do not transport a conserved quantum number, and the possibility of "clashing arrows" already seen in our toy model in the first part. So there are now six types of vertices involving fermions (the dashed line indicates a boson of whatsoever type):



(12.7)

Since there no longer is a well-defined direction along the fermion lines (there may be parts of lines having no direction at all due to propagating Majorana fermions, or the line's direction changes by one of the rightmost vertices) the method presented above is no longer feasible. One possibility is to artificially assign a direction to a Majorana line and define different vertices for each case in which the arrows direct to or from the vertex [2]. This would give the correct analytical expressions for the Feynman diagrams, but the signs between different diagrams contributing to the same amplitude, e.g. for the production of two neutralinos at a linear collider (H is a shortcut for the three neutral Higgs bosons, H^0, h^0, A^0),



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have to be derived by the Wick theorem, which is no good solution for implementing such rules in a computer program. Instead we follow the Feynman rules invented by Ansgar Denner et al. [19]. As definite directions along the fermion lines are only partially available they give up the concept of fermion number conservation and use *fermion conservation* as an alternative, which is simply the statement that fermion lines must still run through the diagrams without being interrupted. [19] describe in detail how the contractions of the fermionic field operators can be disentangled with the help of that concept. The ingredients of Denner's rules are the "ordinary" vertices, propagators and external states as well as their charge conjugated versions:

$$\mathcal{C}\overline{v}^T(p,\sigma) = u(p,\sigma) \tag{12.8}$$

$$\mathcal{C}\overline{u}^T(p,\sigma) = v(p,\sigma) \tag{12.9}$$

$$S'_F \equiv \mathcal{C}S_F^T(p)\mathcal{C}^{-1} = S_F(-p) = \frac{\mathrm{i}}{-\not{p} - m} \tag{12.10}$$

$$\Gamma' \equiv \mathcal{C}\Gamma^T\mathcal{C}^{-1} = \begin{cases} +\Gamma & \text{for } \Gamma \equiv \mathbb{I}, \gamma^5\gamma^\mu, \gamma^\mu \\ -\Gamma & \text{for } \Gamma \equiv \gamma^\mu, \sigma^{\mu\nu} \end{cases} , \tag{12.11}$$

where $\mathcal{C}$ is the charge conjugation matrix. (The basics of Denner's idea for disentangling the contractions are to replace an interaction operator bilinear in the fermionic field operators by its transpose, which should leave everything unchanged as the bilinear is a number (or an array of numbers), $\overline{\Psi}\Gamma\Psi = \left(\overline{\Psi}\Gamma\Psi\right)^T$:

$$\ldots\ \overline{\Psi}\Gamma\Psi\ \ldots \quad \longrightarrow \quad (-1)\cdot\ \ldots\ \Psi^T\,\mathcal{C}^{-1}\left(\mathcal{C}\Gamma^T\mathcal{C}^{-1}\right)\mathcal{C}\ \overline{\Psi}^T \ldots \tag{12.12}$$

The sign comes from anticommuting the field operators. The contractions could have been got tangled up in that way, since all four possible contractions between field operators and conjugated field operators are allowed for Majorana fermions, and also due to the appearance of explicitly charged conjugated fermions as in the chargino–lepton–slepton vertex. For Dirac fermions this happens only for the contractions of the field operators with the asymptotic creation and annihilation operators for incoming or outgoing antifermions, which have produced the global signs mentioned in the first part of the text, and for closed fermion loops, of course. On the level of the analytical expressions for the fermion lines within Feynman diagrams the relation (12.12) for complete lines reads ($w \in \{u,v\}$, $w^c \in \{u^c = v, v^c = u\}$ with $w^c \equiv \mathcal{C}\overline{w}^T$)

$$\overline{w}_1\Gamma^{(1)}S_F^{(1)}\Gamma^{(2)}\ldots\Gamma^{(n)}w_2 = (-1)\cdot\overline{w_2^c}\,\Gamma'^{(n)}\ldots S_F'^{(1)}\Gamma'^{(1)}w_1^c, \tag{12.13}$$

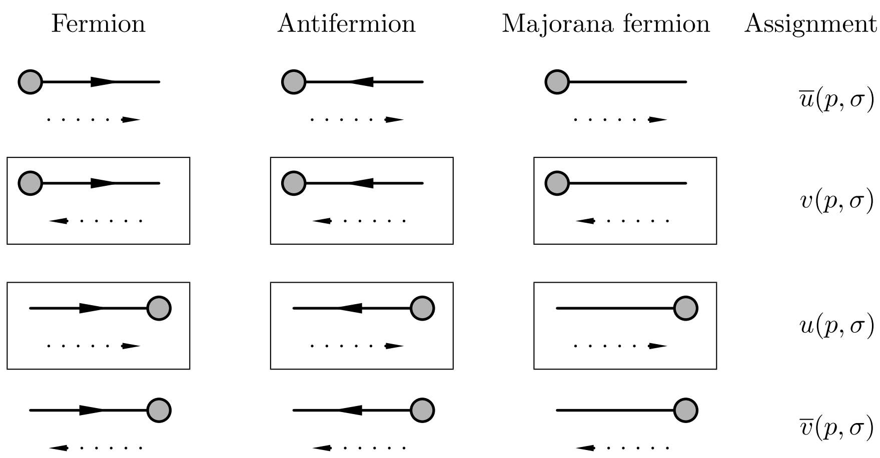
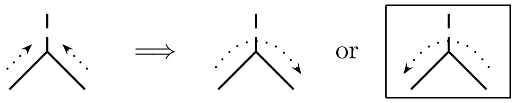


Figure 12.2: *Denner's rules for external fermionic particles depending upon the direction along which the line is calculated. Only those in boxes are used for the implementation.*

where the sign now does not come from fermion anticommutation but from the antisymmetry of the charge conjugation matrix: $w^T = \overline{w^c}\,\mathcal{C}^T = -\overline{w^c}\,\mathcal{C}$. This was also the basis for the "reversing of fermion lines" for the asymptotic STI in part one. Both ways of calculating a fermion line produce the same result as the sign in (12.13) is cancelled by the one from fermion anticommutation in (12.12). The key ingredient of Denner's rules is to write down the Feynman diagram(s), to choose a calculational direction for each fermion line (i.e. to decide where to start and where to end at a given fermion line) and to write down the primed expressions when the calculational direction is opposite to the arrows of a Dirac fermion. Disentanglement of the contractions is thereby achieved automatically, and there is no need to explicitly use the Wick theorem. The relative sign of different diagrams can be evaluated by the method of permutations with respect to a reference order where the pairs now are of the form (*endpoint*, *startpoint*) instead of (*conjugated spinor*, *spinor*). For details cf. [19].)

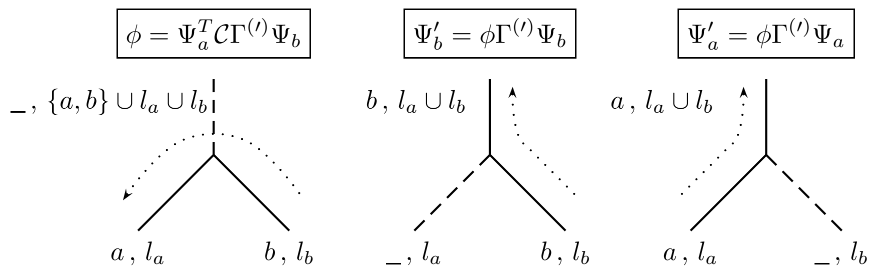
The biggest problem of the incorporation of Denner's rules is the factorization procedure used in *O'Mega*: We treat only subamplitudes by successively building up 1POWs, so we do not know where a beginning fermion line ends (or, where the beginning of an ending line is. If we knew this, we could choose a calculational direction and tell the program how to calculate the line numerically). At first we arbitrarily assume the calculational direction to point from the external fermion inwards into the (sub-)amplitude. Therefore we must assign a spinor instead of a conjugated spinor to each external fermion. According to Denner the assignments in table 12.2 for external fermionic particles have to be made, wherein the dotted line indicates the chosen calculational direction. As we use only spinors for the external particles in *O'Mega*, just the cases in boxes are relevant for us. Consequently every incoming fermion of whatsoever type is represented by a u spinor while to every outgoing fermion a v spinor is assigned. But this means that we can totally forget about conjugated spinors now, since the lines beginning with a spinor are continued by left multiplication with gamma matrices, by which a spinor is produced again. We simply have to take $\Psi_1^T\mathcal{C}\Gamma\Psi_2$ as a bilinear product instead of $\overline{\Psi}_1\Gamma\Psi_2$, and so conjugated spinors and right multiplication with gamma matrices are completely eliminated when dealing with fermions of mixed types. This choice for our bilinear product solves another problem: When fusing open ends of two fermion lines to a bosonic wavefunction (or for the final keystone) there is a mismatch in the calculational directions as for both legs they point from bottom to top:



(12.14)

As indicated in the figure there are now two equal alternatives, to reverse either the right or the left calculational direction. In *O'Mega* we have chosen the second alternative so that the calculational direction goes from the right to the left. The conjugated spinor $\overline{\Psi}$ is the same as $\Psi^{c\,T}\mathcal{C}$, but we take – as mentioned above – the product $\Psi^T\mathcal{C}\Gamma\Psi$ and not $\Psi^{c\,T}\mathcal{C}\Gamma\Psi$. The solution to that obstacle is the following: The part of the line coming from the left into the fusion has been calculated starting from the external particle against the calculational direction chosen to hold after the fusion of two fermion lines. Assume e.g. the left fermion line simply to be an external incoming fermion; due to the rules in table 12.2 *O'Mega* takes a u spinor for that particle but when *O'Mega* performs the fusion, it chooses the calculational direction of the whole line – now closed by the fusion – to go from the right leg of the fusion to the left. Hence, according to this direction, we actually would have had to take $\overline{v} = u^T\mathcal{C}$ as external wavefunction, so that performing the fusion like $\Psi^T_{\text{left}}\mathcal{C}\Gamma\Psi_{\text{right}}$ with $\Psi_{\text{left}} \equiv u$ is completely correct. The inclusion of propagators and vertex factors for the left leg will be discussed below.

The "fusion rules" for fermions of mixed types are:



(12.15)

As was discussed in the last but one paragraph, the pairings are now (*endpoint*, *startpoint*) instead of (*conjugated spinor*, *spinor*). One part of the Fermi statistics' sign is calculated as in the Dirac case from the number of transpositions needed to bring this collection of pairs into a reference order. In *O'Mega* the calculational direction always goes from the right leg of the fusion to the left, so the pair added to the list of closed fermion lines is $\{a, b\}$ each time, where a is the fermion index of the left leg and b that of the right one. When a line is continued as in the rightmost fusions in (12.15), then simply the gamma matrix from the vertex is multiplied from the left to the child spinor. The remaining part of the Fermi statistics' sign is produced from those gamma matrix vertex factors (and the propagators, cf. below), which also answers the question about the meaning of the primes at the Γ in (12.15): According to Denner's rules in (12.11) and the discussion in the text thereafter we must assign a primed vertex function whenever the calculational direction is opposite to the direction of an arrow at a vertex. As the calculational direction in *O'Mega* always points from right to left or from bottom to top, respectively, we must make the assignments shown in table 12.3 for the "fusion rules" at the vertices. Γ and Γ' refer to the property in (12.11) according to which in the left column always the ordinary vertex factor has to be taken with no additional

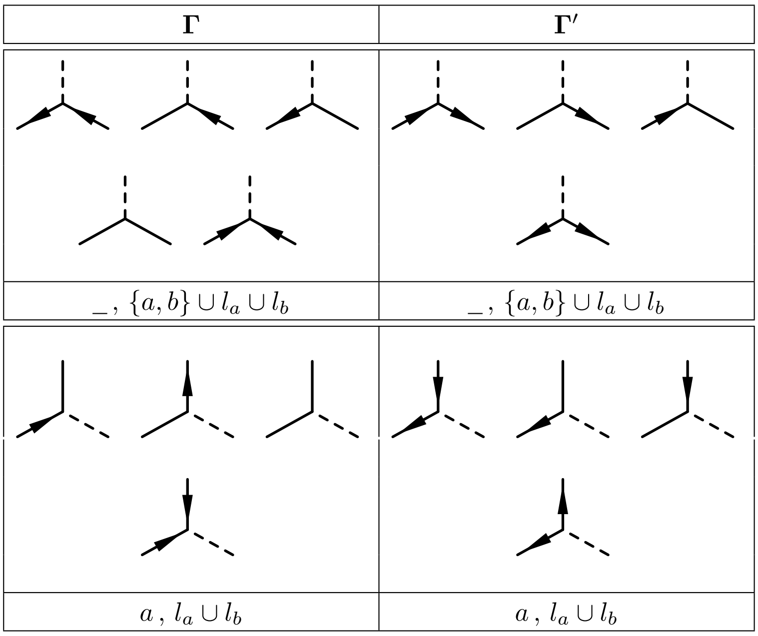
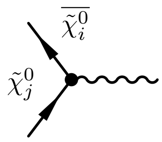


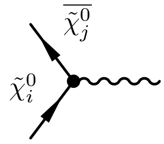
Figure 12.3: *"Fusion rules" for fermions of Dirac as well as Majorana type. For the fusions where a fermion is produced, there are also the mirror diagrams again. a and b are the fermion labels for the left fermion and the right fermion, respectively, and l_a and l_b are the corresponding closed fermion lines. For the conventions concerning the fusions with clashing arrows and with two Majorana fermions cf. the text.*

sign, while for the right column there is a sign if the vertex Γ represents a vectorial or tensorial coupling. Some remarks about the Majorana lines and the clashing arrows: In the case of the clashing arrows we must define the vertices in the *O'Mega* model files by $\overline{\Psi_1^c}\Gamma\Psi_2$ instead of $\overline{\Psi_1}\Gamma\Psi_2^c$; if we had defined them the other way round, the diagrams with clashing arrows would have to be exchanged between the two columns in table 12.3. There are no ambiguities for vertices with one Majorana and one Dirac line (e.g. the electron–selectron–neutralino vertex) because the Dirac fermion automatically gives a direction. For vertices with two Majorana fermions the case is more complex: If the two Majorana fermions are identical, the coupling has to be scalar, pseudoscalar or axial-vectorial, hence there is no problem with signs (otherwise that part of the interaction Lagrangean vanishes identically). Consider now the case where the Majorana fermions are different, e.g. in the vertex between different MSSM neutralinos and the Z boson. Here one has to decide whether to write the vertex as

$$\overline{\tilde{\chi}_i^0} \left(g_V + g_A\gamma^5\right) \not{Z} \tilde{\chi}_j^0 \qquad \text{or} \qquad \overline{\tilde{\chi}_j^0} \left(-g_V + g_A\gamma^5\right) \not{Z} \tilde{\chi}_i^0 \quad , \quad i \neq j \quad .$$

In the same way we have to choose which of the two possibilities should be included in the *O'Mega* model file. By taking one of the two versions, we implicitly introduce an arrow:



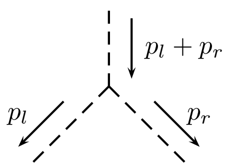


$$\boxed{\overline{\tilde{\chi}_i^0}\left(g_V + g_A\gamma^5\right)\not{Z}\tilde{\chi}_j^0} \qquad \boxed{\overline{\tilde{\chi}_j^0}\left(-g_V + g_A\gamma^5\right)\not{Z}\tilde{\chi}_i^0}$$

Consider the left of the possibilities above: Here the neutralino i becomes the particle with the arrow pointing away from the vertex whereas the neutralino j gets the arrow pointing to the vertex. For the right possibility the arrows are exchanged. That "pseudo-assigning of arrows" means that when contracting the field operators of that interaction vertex with external states or other interaction vertices, then we had to write down a conjugated spinor for the first neutralino and a spinor for the second in the left case, and vice versa for the right possibility.

In table 12.3 it is assumed that the vertex for two Majorana fermions is always written down in such a manner that in the case of fusioning two fermions the left one is the conjugated while in the case of the fermion line being continued, the fermion fused from the children is the conjugated. Henceforth no primed vertex factors have to be used. (In practice, there is a unique representation in *O'Mega* for such vertices, so when the fusion does not match that representation, then there *do* appear signs in front of the vertex factors. E.g. when we denote that neutralino neutral current by the left possiblity above, but the second neutralino appears as a left leg at a fusion and the first as a right leg, then the vector coupling constant has to be endowed with an extra minus [4].)

The last open point for the handling of real fermions in *O'Mega* is the question of the propagators. First of all, we must say a word about the momentum flow in *O'Mega*: It is always outgoing (pointing out of an amplitude), i.e. in those fusion diagrams the momentum flows always from top to bottom (for all vertices):

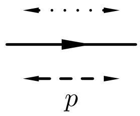
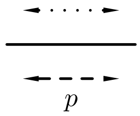


(12.16)

All momenta can be expressed as a sum of the external momenta, so each momentum can be uniquely labelled by a subset of the external particles. After a fusion has taken place, a fermionic wavefunction is multiplied by a propagator. Thus, every wavefunction appearing as a child (left or right leg) in a fusion is either a wavefunction of an external fermion or has already been multiplied with a propagator. An exception occurs if the wavefunction is the final keystone of a subamplitude; then to only one of the fermionic wavefunctions the propagator has to be assigned. Hence, a propagator is inserted in all fusion cases where the fermion line is not closed but runs through to the top. At first one could believe that more than one propagator type is needed when handling Dirac

[4] Do not get confused when reading [19]: There the abbreviaton $\overline{\chi}\Gamma\chi$ means $g^a_{ij}\overline{\chi}_i\Gamma^a\chi_j$ so when reversing the order of the field operators in that interaction vertex for two Majorana fermions yields $g^a_{ij}\overline{\chi}_j\Gamma'^a\chi_i$. If Γ'^a has a relative sign with respect to Γ^a then we also have $g^a_{ji} = -g^a_{ij}$ and in Denner's notation we get $\overline{\chi}\Gamma\chi$ again. But in *O'Mega* the coupling constant has a pre-defined value and we must get an additional sign when reversing the order of field operators for vectorial and tensorial couplings even for Majorana fermions.

and Majorana fermions, but fortunately this is not true. In connection with Denner's rules above we mentioned that, if the calculational direction is opposite to the arrow of the Dirac line, then we must use the primed propagator, i.e. the propagator with the negative momentum. Of course, also a momentum flow opposite to the arrow direction results in a minus sign for the momentum. Altogether the fermion propagator is of the form


$$= \frac{\mathrm{i}}{\xi \not{p} - m},$$

$$= \frac{\mathrm{i}}{\zeta \not{p} - m} \qquad (12.17)$$

where ξ and ζ are sign factors. ξ is $+1$ if the calculational direction and the momentum flow are both parallel or both antiparallel to the fermion's arrow, and -1 otherwise. For Majorana fermions ζ is $+1$ if calculational direction and momentum flow are parallel and -1 if they are antiparallel. The following table shows that this always leads to a negative sign for the propagator's momentum within *O'Mega*:

fermion type	fermion arrow	mom.	calc.	sign
Dirac fermion	$\uparrow$	$\downarrow$	$\uparrow$	negative
Dirac antifermion	$\downarrow$	$\downarrow$	$\uparrow$	negative
Majorana fermion	-	$\downarrow$	$\uparrow$	negative

So the universally used fermion propagator for all types of fermions – fermions, antifermions and Majorana fermions – is

$$\boxed{S_{F,O'Mega} = \frac{\mathrm{i}}{-\not{p} - m}} \qquad (12.18)$$

Now we are able to convince ourselves that everything is alright with the expressions for the left leg at the fusion. Let us assume that the spinor from the left child calculated by *O'Mega* up to the moment the fusion takes place, has the form

$$\Psi = S_F^{(1)} \Gamma^{(1)} S_F^{(2)} \ldots \Gamma^{(n)} w, \qquad w \in \{u, v\} \; . \qquad (12.19)$$

When the fusion happens, the expression $\Psi^T \mathcal{C}$ is made out of the left spinor, which after inserting (12.19) is equal to

$$\begin{aligned}
\Psi^T \mathcal{C} &= w^T {\Gamma^{(n)}}^T \ldots {S_F^{(2)}}^T {\Gamma^{(1)}}^T {S_F^{(1)}}^T \left(-\mathcal{C}^{-1}\right) \\
&= w^T \left(-\mathcal{C}\right) \mathcal{C} {\Gamma^{(n)}}^T \mathcal{C}^{-1} \ldots \mathcal{C} {S_F^{(2)}}^T \mathcal{C}^{-1} \mathcal{C} {\Gamma^{(1)}}^T \mathcal{C}^{-1} \mathcal{C} {S_F^{(1)}}^T \left(-\mathcal{C}^{-1}\right) \\
&= \overline{w^c}\, {\Gamma^{(n)}}' \ldots {S_F^{(2)}}' {\Gamma^{(1)}}' {S_F^{(1)}}' \qquad (12.20)
\end{aligned}$$

We have already explained that the assignment of the wavefunction for the external fermion is correct. Now we get the primed expressions for all vertex factors and propagators. Of course, that is what we really want, since, indeed, our calculational direction for the whole fermion line, after the fermion line has been closed (by the fusion), goes the opposite way as it was originally calculated by *O'Mega* for the left part of the line. There the calculational direction at first went from the external particle inwards, after the fusion it points outwards. When *O'Mega* calculated the left leg it took the primed expressions erroneously from the standpoint after the fusion, and as well had let other expressions unprimed erroneously. By performing (12.20) each vertex factor and propagator becomes primed so that the wrongly primed factors become unprimed again (the priming operation is involutory) while factors left originally unprimed become now correctly primed.

From the above discussion it is clear that for generating Slavnov-Taylor identities for theories including Majorana fermions within *O'Mega*, we must "reverse" the expressions for the conjugated spinors and assign the external wavefunction $-\mathrm{i}\,(\not{p}-m)\,u(p)$ to every incoming source of a BRST transformed fermion, while $\mathrm{i}\,(\not{p}+m)\,v(p)$ for every outgoing source, not depending on whether the transformed fermionic particle is a fermion, an antifermion or a Majorana fermion.

Of course, the formalism outlined above for theories including real fermions works as well for theories which contain exclusively Dirac fermions like the Standard Model. In that case the formalism has also been tested and shown to produce the same numerical results as with Ohl's ansatz. Finally, let us mention that the problems faced in section 5.4 concerning signs from external antifermions and from "clashing arrows" have been solved by the construction presented in this chapter: According to figure 12.2 the number of external v spinors is simply given by the number of all fermionic particles in the final state and thus is obvious, while the "clashing arrow"-obstacle has been remedied by our procedure for handling the vertices in figure 12.3.

12.3 Numerical checks

Numerical tests for Slavnov-Taylor identities (STI) have been made for gauge symmetries as well as for supersymmetry. Therein we investigated the ratio

$$R \equiv \frac{|\sum_i \mathcal{G}_i|}{\sum_i |\mathcal{G}_i|} \tag{12.21}$$

of the sum of Green functions contributing to the STI to the sum of their absolute values. In performing such tests for the Abelian toy model presented in section 10.2, we achieved ratios of better than 10^{-10}, so the STI are fulfilled numerically to a high level of accuracy. With their help, several errors in the model files and also in the numerical implementation of Majorana couplings in *O'Mega* have been discovered, showing these tests to produce nontrivial results.

Chapter 13

Summary and Outlook

Our original task was to implement the whole Minimal Supersymmetric Standard Model (MSSM) in the matrix element generator *O'Mega* and to further use the calculating power of that program to produce cross sections and decay rates as predictions for the coming generation of colliders, LHC and TESLA. This has be done by creating a new model file for *O'Mega*. A compromise has been made between the desire to be as general as possible and the constraint not to blow up the model to a complexity which is not only too difficult to handle within the framework of *O'Mega* but also contradicts current experimental knowledge (flavour-changing neutral currents, smallness of CP-violation). It became clear that the complexity of the model is still immense and that checking procedures had to be found to control the inner consistency of the theory and the program's numerical stability as well. Although gauge symmetries have been used as consistency checks ever since, supersymmetry, though being only global in supersymmetric field theories, is as powerful for this purpose as those – or perhaps even more.

The idea arose to develop a method to perform such consistency checks for supersymmetric field theories using supersymmetry as the vehicle. First, we picked up a formalism invented by Grisaru and Pendleton in the 1970s and calculated Ward identities for supersymmetry diagrammatically, out of which relations between *on-shell* S-matrix elements could be gained. This has been shown using the simplest supersymmetric field theoretic, the Wess-Zumino model, and then been extended to a more complex toy model to clarify questions concerning Fermi statistics and vertices with clashing arrows. But since this method relies on the annihilation of the ground state by the supercharge it is only applicable for theories with exact supersymmetry. The breakdown of the Ward identities has been demonstrated in the O'Raifeartaigh model. Generally, this formalism does not suit well enough for automatized tests (as well as for realistic models) but it enables some useful insights into the problems with fermions in supersymmetric theories.

Further progress was achieved by the investigation of the conserved current resulting from the supersymmetry of the action. Here Ward identities can be constructed by inserting the current operator into a Green function and then taking the derivative with respect to the spacetime argument of the current. Since the current is still conserved in the case of spontaneously broken symmetry, this method is applicable not only for exact supersymmetry. We derived in detail the supersymmetric current of general models including supersymmetric Yang-Mills theories. As the supersymmetric current is a spin-3/2 object to which the gravitino couples in gauged supersymmetry, the incorporation of that method in *O'Mega* provided the infrastructure for supergravity (propagating gravitinos, higher dimensional vertices with two bosons and two fermions, etc.). This may be the basis for one of the possible extensions of the work presented here. Calculations have been done for a simple toy model with a $U(1)$ gauge group, where examples are shown for the Ward identities constructed in that manner to be fulfilled only on-shell, but not off-shell. Hence the mentioned formalism can be used for on-shell tests in all models with global supersymmetry. Nevertheless we are also interested in off-shell tests since they are more stringent. The understanding of why Ward identities with current insertions in supersymmetric gauge theories are only fulfilled between physical on-shell states, opened the way to a more elegant formalism for consistency checks in part III.

Therein we introduced the BRST formalism for supersymmetric field theories based on the work of White and Sibold. A nilpotent BRST operator can only be found by including supersymmetry transformations and translations and using constant ghosts for them. The Slavnov-Taylor identities from this generalized BRST invariance is the desired consistency check working also off-shell. The deep-rooted reason that supersymmetry seems to be violated off-shell is that the supercharge for supersymmetric gauge theories does not commute with the S-operator for arbitrary states of the Hilbert space, but only for physical states from the cohomology of the BRST charge. Whenever we leave the mass shell, we have to include several additional diagrams containing the Faddeev-Popov ghosts, which shows that gauge symmetry and supersymmetry are inseparably entangled. These facts have been clarified in the context of algebraic renormalization of supersymmetric field theories by Sibold and co-workers. Here we presented analytic calculations in a diagrammatic language of Slavnov-Taylor identities in a pretty simple Abelian toy model and also for a general supersymmetric Yang-Mills theory. To our knowledge, this had not been done so far.

The Slavnov-Taylor identities (STI) have been used for numerical checks as well, where they justified their application by detecting several errors in the program libraries of *O'Mega*. Some of the difficulties and fine points concerning Majorana fermions, special fermion vertices in supersymmetric field theories and the inclusion of the STI in *O'Mega* have been listed for the sake of completeness in the last part; they maintain the connection to the amount of computational work not presented in this thesis. The MSSM, which served as a motivation for this work by its sheer complexity, is briefly reviewed in the appendix. All the physical fields with the abundance of mixing angles and phases have been included into the model file as has been mentioned above.

With this work the problem of how to test supersymmetry in scattering amplitudes and Green functions perturbatively within arbitrary models analytically and numerically has been solved. The mechanisms by which the cancellations in Ward and Slavnov-Taylor identities for supersymmetry happen have been understood in detail. Hence this thesis supplies the theoretical basis for testing supersymmetric models by means of these identities. All the infrastructure has been laid to perform these tests numerically; however this development has only been sketched in order not to go beyond the scope of this work. Notwithstanding the fact that this thesis – including the theoretical foundations and the implementation of these identities and providing a generally applicable checking tool – is an integral whole, we would like to give a brief outlook to further projects and additional ideas, which will be tackled in the future.

Among these points is a further debugging of the MSSM model file with the help of all existing symmetries: the gauge symmetry, supersymmetry, also Bose and Fermi symmetries. Since supersymmetry is explicitly broken in the MSSM, we will have to restore supersymmetry by a spurion formalism, where superpotential terms containing one or more new superfields are added. By spontaneous symmetry breaking these new superfields then generate the soft breaking terms of the MSSM. The price we have to pay for that restoring of supersymmetry are additional couplings to the component fields of the new superfields. After that we can start producing data for MSSM processes on a reliable basis. A further project is to enlarge the structure of *O'Mega* so that it can handle propagating fields violating the spin-statistics theorem with utmost generality. Strictly, this is only necessary when investigating loop processes or STI in supergravity but it would be satisfying to have a unified formalism managing all eventualities. The propagating Faddeev-Popov ghosts can be incorporated while evaluating their Fermi statistics signs separately from those of "physical" fermions. The further diagrammatical examination of supersymmetric Slavnov-Taylor identities, on the one hand with more than one SUSY ghost, on the other hand on an N-loop level, should finally be mentioned as a project.

Appendix A

Basics, notations and conventions

A.1 Basics

Metric:

$$\eta_{\mu\nu} = \text{diag}\,(1, -1, -1, -1) \tag{A.1}$$

Super-Poincaré algebra for $N = 1$, without central charges:

$$\boxed{\begin{aligned}
[P^\mu, P^\nu] &= 0 \\
[P^\mu, M^{\rho\sigma}] &= \mathrm{i}\,(\eta^{\mu\rho} P^\sigma - \eta^{\mu\sigma} P^\rho) \\
[M^{\mu\nu}, M^{\rho\sigma}] &= -\mathrm{i}(\eta^{\mu\rho} M^{\nu\sigma} - \eta^{\mu\sigma} M^{\nu\rho} - \eta^{\nu\rho} M^{\mu\sigma} + \eta^{\nu\sigma} M^{\mu\rho}) \\
\left\{Q, \overline{Q}\right\} &= 2\gamma^\mu P_\mu \\
[Q, M^{\mu\nu}] &= S^{\mu\nu} Q \quad \text{with} \quad S^{\mu\nu} = \frac{\mathrm{i}}{4}\,[\gamma^\mu, \gamma^\nu] \\
[Q, P^\mu] &= 0
\end{aligned}} \tag{A.2}$$

Generalized Jacobi identity for $\mathbb{Z}_2$-graded algebras:

$$(-1)^{\eta_C \eta_A}[[T_A, T_B\}, T_C\} + (-1)^{\eta_A \eta_B}[[T_B, T_C\}, T_A\} + (-1)^{\eta_B \eta_C}[[T_C, T_A\}, T_B\} = 0 \tag{A.3}$$

This implies the special cases where B is a bosonic and F a fermionic operator:

$$\begin{aligned}
[[B_A, B_B], B_C] + [[B_B, B_C], B_A] + [[B_C, B_A], B_B] &= 0 \\
[[F_A, B_B], B_C] + [[B_B, B_C], F_A] + [[B_C, F_A], B_B] &= 0 \\
\{[B_A, F_B], F_C\} + [\{F_B, F_C\}, B_A] - \{[F_C, B_A], F_B\} &= 0 \\
\{\{F_A, F_B\}, F_C\} + \{\{F_B, F_C\}, F_A\} + \{\{F_C, F_A\}, F_B\} &= 0
\end{aligned} \tag{A.4}$$

They can be proven by making all the generators bosonic, multiplying them with Grassmann numbers, writing down the ordinary Jacobi identity for bosonic operators and extracting all Grassmann numbers to the left (or to the right) while giving them the same order in all terms.

A.2 Superspace

Supersymmetric field theories are most easily represented on a $\mathbb{Z}_2$-graded vector space called superspace, containing the ordinary four-dimensional space-time and four Grassmann-odd coordinates,

$$(x^0, x^1, x^2, x^3, \theta^1, \theta^2, \theta^3, \theta^4) \equiv \begin{pmatrix} x^\mu \\ \theta \end{pmatrix} \quad . \tag{A.5}$$

with the bispinor (four-component spinor) θ. This is the special case of simple supersymmetry, for $N = n$-supersymmetry, $n \leq 4$ we have the space $\mathbb{R}^{(1,3)|2\cdot 2N}$. It is possible to combine the supercharges into a four-component Majorana spinor Q, [3]. The 14 generators of simple supersymmetry $(P^\mu, M^{\rho\sigma}, Q)$ generate superspace transformations – spacetime translations by P^μ, boosts and rotations by $M^{\rho\sigma}$ and translations of the Grassmann-odd coordinates by spinorial increments ξ, also combined into a Majorana spinor. They anticommute component-wise, and as these parameters are constant, they anticommute with the supercharges, too:

$$\{\xi, \xi\} = \{\xi, \bar{\xi}\} = \{\xi, Q\} = \{\bar{\xi}, Q\} = \left\{\xi, \overline{Q}\right\} = \left\{\bar{\xi}, \overline{Q}\right\} = 0. \tag{A.6}$$

Most general element of the Poincaré supergroup:

$$S(b, \omega, \xi) = \exp\left[\mathrm{i}\left(b_\mu P^\mu + \frac{1}{2}\omega_{\mu\nu} M^{\mu\nu} + \bar{\xi} Q\right)\right]. \tag{A.7}$$

Action of supercharge on superspace:

$$\boxed{Q = \frac{\partial}{\partial\bar{\theta}} - (\mathrm{i}\gamma^\mu\theta)\partial_\mu} \tag{A.8}$$

We check that Q fulfills the anticommutation relations

$$\{Q, Q\} = \left\{\overline{Q}, \overline{Q}\right\} = 0, \qquad \left\{Q, \overline{Q}\right\} = 2\mathrm{i}\partial\!\!\!/ \ . \tag{A.9}$$

The covariant derivatives with respect to the supergroup structure $\mathcal{D}$, called *superderivatives*, anticommute with the supercharges:

$$\{\mathcal{D}, Q\} = \left\{\mathcal{D}, \overline{Q}\right\} = \left\{\overline{\mathcal{D}}, Q\right\} = \left\{\overline{\mathcal{D}}, \overline{Q}\right\} = 0 \tag{A.10}$$

On superspace they can be given the representation

$$\boxed{\mathcal{D} = \frac{\partial}{\partial\bar{\theta}} + (\mathrm{i}\gamma^\mu\theta)\partial_\mu} \tag{A.11}$$

With each other they have the (anti-)commutation relations

$$\{\mathcal{D}, \mathcal{D}\} = \left\{\overline{\mathcal{D}}, \overline{\mathcal{D}}\right\} = 0, \qquad \left\{\mathcal{D}, \overline{\mathcal{D}}\right\} = -2\mathrm{i}\partial\!\!\!/ \ . \tag{A.12}$$

For constructing chiral superfields right- and left-handed versions of the superderivative are needed:

$$\boxed{\begin{aligned}\mathcal{D}_L &= \frac{1}{2}\left(1-\gamma^5\right)\mathcal{D} = \frac{\partial}{\partial\overline{\theta}_L} - (\mathrm{i}\gamma^\mu\theta_R)\partial_\mu \\ \mathcal{D}_R &= \frac{1}{2}\left(1+\gamma^5\right)\mathcal{D} = \frac{\partial}{\partial\overline{\theta}_R} - (\mathrm{i}\gamma^\mu\theta_L)\partial_\mu\end{aligned}} \tag{A.13}$$

For more details about Lie supergroups cf. [31].

A.3 Properties of Majorana spinors

Definition of a Majorana spinor

$$\overline{\Psi}_M \equiv \Psi_M^\dagger\gamma^0 = -\Psi_M^T\mathcal{C}, \tag{A.14}$$

where $\mathcal{C}$ is the antisymmetric charge conjugation matrix, usually chosen to be equal to $\mathcal{C} = \mathrm{i}\gamma^2\gamma^0$. In the sequel θ always means a Grassmann-odd spinor.

$$\begin{aligned}\overline{\theta}_1\Gamma\theta_2 &= \left(\overline{\theta}_1\Gamma\theta_2\right)^T = -(\theta_1^T\mathcal{C}\Gamma\theta_2)^T = -(\theta_2^T\Gamma^T\mathcal{C}\theta_1) \\ &= \overline{\theta}_2\mathcal{C}^{-1}\Gamma^T\mathcal{C}\theta_1\end{aligned} \tag{A.15}$$

Using the well-known relations about gamma matrices

$$\Gamma^T = \left\{\begin{array}{ll} +\mathcal{C}\Gamma\mathcal{C}^{-1} & \Gamma = \mathbb{I}, \gamma^5\gamma^\mu, \gamma^5 \\ -\mathcal{C}\Gamma\mathcal{C}^{-1} & \Gamma = \gamma^\mu, [\gamma^\mu,\gamma^\nu]\end{array}\right. \tag{A.16}$$

yields

$$\overline{\theta}_1\Gamma\theta_2 = \left\{\begin{array}{ll} +\,\overline{\theta}_2\Gamma\theta_1 & \Gamma = \mathbb{I}, \gamma^5\gamma^\mu, \gamma^5 \\ -\,\overline{\theta}_2\Gamma\theta_1 & \Gamma = \gamma^\mu, [\gamma^\mu,\gamma^\nu]\end{array}\right. \tag{A.17}$$

So the only possible bilinears with a single Grassmann-odd spinor are

$$\overline{\theta}\theta, \qquad \overline{\theta}\gamma^5\gamma^\mu\theta, \qquad \overline{\theta}\gamma^5\theta, \tag{A.18}$$

while the other combinations vanish identically:

$$\overline{\theta}\gamma^\mu\theta = \overline{\theta}\,[\gamma^\mu,\gamma^\nu]\,\theta = 0. \tag{A.19}$$

Note that for commuting spinors the signs in (A.17) are the other way round.

A.4 Superfields

The irreducible representations of the super-Poincaré algebra on superspace are called *superfields*. They are the basic ingredients of supersymmetric quantum field theories. We will denote them by a hat over the symbol. All superfields have expansions in the superspace coordinates which only run up to fourth order due to the latter's nilpotency. We do not go into the details here; especially we omit general superfields since they are not needed in the construction of supersymmetric field theories. We should only mention that products of superfields underlying some sort of constraints are general unconstrained superfields again; the highest component in the superspace expansion of general superfields is called D, being a scalar field of canonical dimension higher by two than the canonical dimension of the whole superfield. For details cf. [5], [3].

A superfield with the constraint

$$\mathcal{D}_R \hat{\Phi} = 0 \tag{A.20}$$

is called *left-chiral superfield*, while correspondingly a *right-chiral superfield*, distinguished by a bar, underlies the constraint

$$\mathcal{D}_L \hat{\bar{\Phi}} = 0 \ . \tag{A.21}$$

The chiral superfields have the superspace expansions

$$\boxed{\begin{aligned} \hat{\Phi}(x,\theta) &= \phi(x) + \sqrt{2}\left(\overline{\theta}\psi_L(x)\right) + \left(\overline{\theta}\left[\frac{1-\gamma^5}{2}\right]\theta\right)F(x) + \frac{\mathrm{i}}{2}\left(\overline{\theta}\gamma^5\gamma^\mu\theta\right)\partial_\mu\phi(x) \\ &\quad + \frac{\mathrm{i}}{\sqrt{2}}\left(\overline{\theta}\gamma^5\theta\right)\left(\overline{\theta}\partial\!\!\!/\psi_L(x)\right) + \frac{1}{8}\left(\overline{\theta}\gamma^5\theta\right)^2\Box\,\phi(x) \\ \hat{\bar{\Phi}}(x,\theta) &= \bar{\phi}(x) + \sqrt{2}\left(\overline{\theta}\psi_R(x)\right) + \left(\overline{\theta}\left[\frac{1+\gamma^5}{2}\right]\theta\right)\bar{F}(x) - \frac{\mathrm{i}}{2}\left(\overline{\theta}\gamma^5\gamma^\mu\theta\right)\partial_\mu\bar{\phi}(x) \\ &\quad - \frac{\mathrm{i}}{\sqrt{2}}\left(\overline{\theta}\gamma^5\theta\right)\left(\overline{\theta}\partial\!\!\!/\psi_R(x)\right) + \frac{1}{8}\left(\overline{\theta}\gamma^5\theta\right)^2\Box\,\bar{\phi}(x) \end{aligned}} \ . \tag{A.22}$$

Therein ϕ and $\bar{\phi}$ are complex scalar fields, ψ_L and ψ_R are left- and righthanded Weyl spinor fields, respectively, while F and $\bar{F}$ are again complex scalar fields of canonical dimension two if the dimension of $\phi, \bar{\phi}$ is one. From the expansions, one can see that the Hermitean adjoint of a left-chiral superfield is right-chiral and vice versa.

Products of left-chiral superfields are left-chiral superfields again. A function f consisting only of left-chiral superfields together with an identical contribution of right-chiral superfields with the complex conjugated prefactors, but neither containing superderivatives nor spacetime derivatives, is called *superpotential*. The F term (the highest component in the superspace expansion) of the product of two or three left-chiral superfields are:

$$\left[\hat{\Phi}_1\hat{\Phi}_2\right]_F = \left\{F_1\phi_2 + F_2\phi_1 - \left(\overline{\psi}_{L,1}\psi_{L,2}\right)\right\} \tag{A.23}$$

$$\begin{aligned} \left[\hat{\Phi}_1\hat{\Phi}_2\hat{\Phi}_3\right]_F = \Big\{ & F_1\phi_2\phi_3 + F_2\phi_3\phi_1 + F_3\phi_1\phi_2 - \left(\overline{\psi}_{L,1}\psi_{L,2}\right)\phi_3 \\ & - \left(\overline{\psi}_{L,2}\psi_{L,3}\right)\phi_1 - \left(\overline{\psi}_{L,3}\psi_{L,1}\right)\phi_2 \Big\} \end{aligned} \tag{A.24}$$

A superfield constrained by the reality condition

$$\hat{V}^\dagger = \hat{V} \tag{A.25}$$

is called a *vector superfield* since such superfields are the supersymmetric generalizations of the gauge boson fields:

$$\begin{aligned}\hat{V}^a(x,\theta) = \; & C^a(x) - \mathrm{i}\left(\overline{\theta}\gamma^5\omega^a(x)\right) - \frac{\mathrm{i}}{2}\left(\overline{\theta}\gamma^5\theta\right) M^a(x) - \frac{1}{2}\left(\overline{\theta}\theta\right) N^a(x) \\ & - \frac{1}{2}\left(\overline{\theta}\gamma^5\gamma^\mu\theta\right) V^a_\mu(x) - \mathrm{i}\left(\overline{\theta}\gamma^5\theta\right)\left(\overline{\theta}\left[\lambda^a(x) + \frac{\mathrm{i}}{2}\partial\!\!\!/\omega^a(x)\right]\right) \\ & - \frac{1}{4}\left(\overline{\theta}\gamma^5\theta\right)^2\left(D^a(x) - \frac{1}{2}\Box C^a(x)\right)\end{aligned} \tag{A.26}$$

As discussed in [3] there is an extended gauge symmetry in supersymmetric gauge theories with the gauge parameter replaced by a whole superfield. This freedom can be used to gauge away most of the components in (A.26). The remaining part of the extended gauge transformations, orthogonal to those used above, represents ordinary gauge invariance. Most famous is the *Wess-Zumino gauge*,

$$C^a(x) = \omega^a(x) = M^a(x) = N^a(x) = 0, \tag{A.27}$$

$$\boxed{\hat{V}^a(x,\theta) = \; - \frac{1}{2}\left(\overline{\theta}\gamma^5\gamma^\mu\theta\right) V^a_\mu(x) - \mathrm{i}\left(\overline{\theta}\gamma^5\theta\right)\left(\overline{\theta}\lambda^a(x)\right) - \frac{1}{4}\left(\overline{\theta}\gamma^5\theta\right)^2 D^a(x)} \quad , \tag{A.28}$$

in which all power series in the vector superfield breaks off after the quadratic term.

A.5 SUSY transformations of component fields

In the sequel we list the SUSY transformations for chiral and vector superfields: the "normal" ones, in the case of supersymmetric gauge theories the de Wit–Freedman transformations, where a mixing between the matter and the gauge superfields occurs. When inserting the equations of motion for the auxiliary fields, we can forget about the transformations of the auxiliary fields.

A.5.1 SUSY transformations for chiral superfields

SUSY transformation:

$$\begin{aligned}\delta_\xi\phi = \; & \sqrt{2}\left(\overline{\xi_R}\psi_L\right) \\ \delta_\xi\psi_L = \; & -\sqrt{2}\,\mathrm{i}(\partial\!\!\!/\phi)\xi_R + \sqrt{2}F\xi_L \\ \delta_\xi F = \; & -\sqrt{2}\,\mathrm{i}\left(\overline{\xi_L}\partial\!\!\!/\psi_L\right)\end{aligned} \tag{A.29}$$

De Wit-Freedman transformation:

$$\begin{aligned}\tilde{\delta}_\xi\phi = \; & \sqrt{2}\left(\overline{\xi_R}\psi_L\right) \\ \tilde{\delta}_\xi\psi_L = \; & -\sqrt{2}\,\mathrm{i}(D\!\!\!\!/\,\phi)\xi_R + \sqrt{2}F\xi_L \\ \tilde{\delta}_\xi F = \; & -\sqrt{2}\,\mathrm{i}\left(\overline{\xi_L}D\!\!\!\!/\,\psi_L\right) - 2\mathrm{i}T^a\phi\left(\overline{\xi_L}\lambda^a_R\right)\end{aligned} \tag{A.30}$$

Inserting the equations of motion for the auxiliary fields yields the "on-shell" de Wit-Freedman transformation:

$$\begin{aligned}\tilde{\delta}'_\xi\phi = \; & \sqrt{2}\left(\overline{\xi_R}\psi_L\right) \\ \tilde{\delta}'_\xi\psi_L = \; & -\sqrt{2}\,\mathrm{i}(D\!\!\!\!/\,\phi)\xi_R - \sqrt{2}\left(\frac{\partial f(\phi)}{\partial\phi}\right)^*\xi_L\end{aligned} \tag{A.31}$$

A.5.2 SUSY transformations for vector superfields

SUSY transformation:

$$\begin{aligned} \delta_\xi A^a_\mu &= -\left(\overline{\xi}\gamma_\mu\gamma^5\lambda^a\right) \\ \delta_\xi \lambda^a &= -\frac{\mathrm{i}}{2}[\gamma^\alpha,\gamma^\beta]\gamma^5\left(\partial_\alpha A^a_\beta - \partial_\beta A^a_\alpha\right) + D^a\xi \\ \delta_\xi D^a &= -\mathrm{i}\left(\overline{\xi}\partial\!\!\!/\lambda^a\right) \end{aligned} \tag{A.32}$$

De Wit-Freedman transformation:

$$\begin{aligned} \tilde{\delta}_\xi A^a_\mu &= -\left(\overline{\xi}\gamma_\mu\gamma^5\lambda^a\right) \\ \tilde{\delta}_\xi \lambda^a &= -\frac{\mathrm{i}}{2}[\gamma^\alpha,\gamma^\beta]\gamma^5 F^a_{\alpha\beta} + D^a\xi \\ \tilde{\delta}_\xi D^a &= -\mathrm{i}\left(\overline{\xi}\left(D\!\!\!\!/\lambda\right)^a\right) \end{aligned} \tag{A.33}$$

"On-shell" de Wit-Freedman transformation:

$$\begin{aligned} \tilde{\delta}'_\xi A^a_\mu &= -\left(\overline{\xi}\gamma_\mu\gamma^5\lambda^a\right) \\ \tilde{\delta}'_\xi \lambda^a &= -\frac{\mathrm{i}}{2}[\gamma^\alpha,\gamma^\beta]\gamma^5 F^a_{\alpha\beta} - e\left(\phi^\dagger T^a\phi\right)\xi \end{aligned} \tag{A.34}$$

A.6 Construction of supersymmetric field theories

Kinetic terms for matter fields (scalars and fermions):

$$\frac{1}{2}\left[\hat{\Phi}^\dagger\hat{\Phi}\right]_D = \partial_\mu\phi^\dagger\partial^\mu\phi + \frac{\mathrm{i}}{2}\overline{\psi_L}\partial\!\!\!/\psi_L - \frac{\mathrm{i}}{2}\left(\partial_\mu\overline{\psi_R}\right)\gamma^\mu\psi_R + |F|^2 \tag{A.35}$$

Kinetic terms for matter fields with minimal couplings to gauge boson fields from vector superfields in Wess-Zumino gauge:

$$\begin{aligned} \frac{1}{2}\left[\hat{\Phi}^\dagger\exp\left(-T^a\hat{V}^a\right)\hat{\Phi}\right]_D &= (D_\mu\phi)^\dagger(D^\mu\phi) \\ &\quad + \frac{\mathrm{i}}{2}\overline{\psi_L}D\!\!\!\!/\psi_L - \frac{\mathrm{i}}{2}\left(D_\mu\overline{\psi_R}\right)\gamma^\mu\psi_R + \sum_a|F^a|^2 \end{aligned} \tag{A.36}$$

From a vector superfield we construct a new superfield by acting triply with the superderivative:

$$\hat{W} = -\frac{1}{4}\left(\overline{\mathcal{D}}\mathcal{D}\right)\mathcal{D}\hat{V} \tag{A.37}$$

The complete superfield $\hat{W}$ has a spinor index and is therefore called a spinor superfield. Projecting with $\frac{1}{2}\left(1\pm\gamma^5\right)$ gives a right- and left-chiral superfield, respectively. By the following construction we get kinetic terms for the gauge boson and the gaugino as well as gauge boson–gaugino interaction terms:

$$\frac{1}{2}\mathrm{Re}\left[\overline{\hat{W}_R}\hat{W}_L\right]_F = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + \frac{\mathrm{i}}{2}\overline{\lambda^a}\left(D\!\!\!\!/\lambda\right)^a + \frac{1}{2}D^aD^a \tag{A.38}$$

Superpotentials (products of one, two or three chiral superfields) to construct scalar self-interactions and Yukawa couplings have already been discussed in section A.4.

Appendix B

Some details about the MSSM

All superfields of the MSSM:

Superfield	Bosons	Fermions	$U(1)_Y$	$SU(2)_L$	$SU(3)_C$
$\hat{V}^{U(1)_Y}$	B	$\tilde{B}$	0	0	1
$\hat{V}^{SU(2)_L}$	W^i	$\tilde{W}^i$	0	1	1
$\hat{V}^{SU(3)_C}$	G^i	$\tilde{G}^i$	0	0	8
$\hat{L}_1$	$(\tilde{\nu}_e, \tilde{e}_L^-)$	$(\nu_e, e^-)_L$	-1	$\frac{1}{2}$	1
$\hat{L}_2$	$(\tilde{\nu}_\mu, \tilde{\mu}_L^-)$	$(\nu_\mu, \mu^-)_L$	-1	$\frac{1}{2}$	1
$\hat{L}_3$	$(\tilde{\nu}_\tau, \tilde{\tau}_L^-)$	$(\nu_\tau, \tau^-)_L$	-1	$\frac{1}{2}$	1
$\hat{\bar{E}}_1$	$\tilde{e}_R^+$	e_L^+	2	0	1
$\hat{\bar{E}}_2$	$\tilde{\mu}_R^+$	μ_L^+	2	0	1
$\hat{\bar{E}}_3$	$\tilde{\tau}_R^+$	τ_L^+	2	0	1
$\hat{Q}_1$	$(\tilde{u}_L, \tilde{d}_L)$	$(u, d)_L$	$\frac{1}{3}$	$\frac{1}{2}$	3
$\hat{Q}_2$	$(\tilde{c}_L, \tilde{s}_L)$	$(c, s)_L$	$\frac{1}{3}$	$\frac{1}{2}$	3
$\hat{Q}_3$	$(\tilde{t}_L, \tilde{b}_L)$	$(t, b)_L$	$\frac{1}{3}$	$\frac{1}{2}$	3
$\hat{\bar{U}}_1$	$\tilde{u}_R^*$	u_L^c	$-\frac{4}{3}$	0	$\bar{3}$
$\hat{\bar{U}}_2$	$\tilde{c}_R^*$	c_L^c	$-\frac{4}{3}$	0	$\bar{3}$
$\hat{\bar{U}}_3$	$\tilde{t}_R^*$	t_L^c	$-\frac{4}{3}$	0	$\bar{3}$
$\hat{\bar{D}}_1$	$\tilde{d}_R^*$	d_L^c	$\frac{2}{3}$	0	$\bar{3}$
$\hat{\bar{D}}_2$	$\tilde{s}_R^*$	s_L^c	$\frac{2}{3}$	0	$\bar{3}$
$\hat{\bar{D}}_3$	$\tilde{b}_R^*$	b_L^c	$\frac{2}{3}$	0	$\bar{3}$
$\hat{H}_1$	(H_1^0, H_1^-)	$(\tilde{H}_1^0, \tilde{H}_1^-)_L$	-1	$\frac{1}{2}$	1
$\hat{H}_2$	(H_2^+, H_2^0)	$(\tilde{H}_2^+, \tilde{H}_2^0)_L$	1	$\frac{1}{2}$	1

Above we listed the superfields of the MSSM all of which are left-chiral superfields. Naturally, the total field content also includes their Hermitean conjugates which are right-chiral superfields. Since we only wanted to write down left-chiral superfields, all fermionic component fields are left-handed: Therefore those of the “barred” superfields are the left-handed parts of the antileptons and antiquarks. We omit right-handed neutrino fields here and so the neutrinos remain massless, but a generalization is obvious. A tilde on the component fields indicates a particle with negative R parity and hence a superpartner of a Standard Model field. The quantum numbers are given by the hypercharge, the third component of the weak isospin, while the number in the last column indicates whether the particle is a colour-singlet, triplet, antitriplet or octet.

MSSM - the Lagrangean density:

$$\boxed{\begin{aligned}
&\mathcal{L}_{\text{MSSM}} = \\
&\quad \sum_{i=1}^{3}\Big(\hat{Q}_i^\dagger \exp\Big[\hat{\mathcal{V}}\Big]\,\hat{Q}_i\Big)_D + \sum_{i=1}^{3}\Big(\hat{L}_i^\dagger \exp\Big[\hat{\mathcal{V}}\Big]\,\hat{L}_i\Big)_D \\
&+ \sum_{i=1}^{3}\Big(\hat{\bar{U}}_i^\dagger \exp\Big[\hat{\mathcal{V}}\Big]\,\hat{\bar{U}}_i\Big)_D + \sum_{i=1}^{3}\Big(\hat{\bar{D}}_i^\dagger \exp\Big[\hat{\mathcal{V}}\Big]\,\hat{\bar{D}}_i\Big)_D \\
&+ \sum_{i=1}^{3}\Big(\hat{\bar{E}}_i^\dagger \exp\Big[\hat{\mathcal{V}}\Big]\,\hat{\bar{E}}_i\Big)_D + \Big(\hat{H}_1^\dagger \exp\Big[\hat{\mathcal{V}}\Big]\,\hat{H}_1\Big)_D + \Big(\hat{H}_2^\dagger \exp\Big[\hat{\mathcal{V}}\Big]\,\hat{H}_2\Big)_D \\
&+ \frac{1}{2}\text{Re}\Big[\overline{\hat{W}_{R,a}^{SU(3)_C}}\,\hat{W}_{L,a}^{SU(3)_C}\Big]_F + \frac{1}{2}\text{Re}\Big[\overline{\hat{W}_{R,a}^{SU(2)_L}}\,\hat{W}_{L,a}^{SU(2)_L}\Big]_F \\
&+ \frac{1}{2}\text{Re}\Big[\overline{\hat{W}_{R}^{U(1)_Y}}\,\hat{W}_{L}^{U(1)_Y}\Big]_F - \frac{g_s^2\theta_{\text{QCD}}}{16\pi^2}\text{Im}\Big[\overline{\hat{W}_{R,a}^{SU(3)_C}}\,\hat{W}_{L,a}^{SU(3)_C}\Big]_F \\
&+ [\mathcal{W}]_F + \mathcal{L}_{SR}
\end{aligned}} \tag{B.1}$$

Summation over generation indices from 1 to 3 is explicitly shown while for gauge indices we use the summation convention. θ_{QCD} is the QCD vacuum angle, this term being the supersymmetric generalization of the term generating instanton solutions in QCD [16], [3]. The vector superfield of the Standard Model $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge group is

$$\hat{\mathcal{V}} = -\,\hat{V}_a^{U(1)_Y}\cdot\frac{Y}{2} - \sum_{a=1}^{3}\hat{V}_a^{SU(2)_L}\,\frac{\sigma^a}{2} - \sum_{a=1}^{8}\hat{V}_a^{SU(3)_C}\,\frac{\lambda^a}{2} \tag{B.2}$$

Furthermore $\mathcal{W}$ is the *superpotential*,

$$\begin{aligned}
\mathcal{W} = h_{kl}^E\left(\hat{L}_k^a\epsilon_{ab}\hat{H}_1^b\right)\hat{\bar{E}}_l + h_{kl}^D\left(\hat{Q}_k^a\epsilon_{ab}\hat{H}_1^b\right)\hat{\bar{D}}_l& \\
+ h_{kl}^U\left(\hat{Q}_k^a\epsilon_{ab}\hat{H}_2^b\right)\hat{\bar{U}}_l + \mu\left(\hat{H}_1^a\epsilon_{ab}\hat{H}_2^b\right) + \text{h.c.},&
\end{aligned} \tag{B.3}$$

and $\mathcal{L}_{SR}$ are the *superrenormalizable terms* parameterizing the unknown SUSY breaking mechanism:

$$\begin{aligned}
\mathcal{L}_{SR} = &-\sum_{ij}(M_{\tilde{Q}}^2)_{ij}(\tilde{Q}_i^\dagger\tilde{Q}_j) - \sum_{ij}(M_{\bar{U}}^2)_{ij}(\bar{U}_i^\dagger\bar{U}_j) - \sum_{ij}(M_{\bar{D}}^2)_{ij}(\bar{D}_i^\dagger\bar{D}_j) \\
&- \sum_{ij}(M_{\tilde{L}}^2)_{ij}(\tilde{L}_i^\dagger\tilde{L}_j) - \sum_{ij}(M_{\bar{E}}^2)_{ij}(\bar{E}_i^\dagger\bar{E}_j) - \Bigg\{\frac{1}{2}m_{\text{Gluino}}(\lambda_s\lambda_s) \\
&+ \frac{1}{2}m_{\text{Wino}}(\lambda\lambda) + \frac{1}{2}m_{\text{Bino}}(\lambda'\lambda') - \sum_{ij}A_{ij}^D h_{ij}^D(\tilde{Q}_i^T\epsilon H_1)\bar{D}_j \\
&- \sum_{ij}A_{ij}^E h_{ij}^E(\tilde{L}_i^T\epsilon H_1)\bar{E}_j - \sum_{ij}A_{ij}^U h_{ij}^U(\tilde{Q}_i^T\epsilon H_2)\bar{U}_j
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{ij} C^D_{ij} h^D_{ij} (\tilde{Q}^T_i H^*_2)\bar{D}_j + \sum_{ij} C^E_{ij} h^E_{ij} (\tilde{L}^T_i H^*_2)\bar{E}_j \\
&\left. +\sum_{ij} C^U_{ij} h^U_{ij} (\tilde{Q}^T_i H^*_1)\bar{U}_j + \frac{1}{2}\,(B\mu)\,(H_1^T \epsilon H_2) + \text{h.c.} \right\} \\
&+ m_1^2 |H_1^0|^2 + m_2^2 |H_2^0|^2 \qquad \text{(B.4)}
\end{aligned}
$$

In the last two formulae indices from the middle of the alphabet are used as generation labels while those from the beginning of the alphabet are gauge group indices. Note that in the soft breaking terms only components appear and not the whole superfields. h^E, h^D and h^U are arbitrary complex 3×3-matrices in the space of generations. There are more arbitrary complex 3×3-matrices in the soft supersymmetry breaking terms, A^E, A^U and A^D, and furthermore C^E, C^U and C^D, the latter not included in most reviews about the MSSM. $M_{\tilde{Q}/\bar{U}/\bar{D}/\tilde{L}/\bar{E}}$ are five Hermitean mass square matrices in generation space for the sparticles. The gaugino masses are allowed to be complex, as well as the Higgs potential parameters μ and $(B\mu)$, while the mass squares $m^2_{1/2}$ must be real.

VERTICES	#
Gauge-IA: $WW\gamma$, WWZ, ggg	3
Gauge-Lepton-IA: $\ell^+\ell^-\gamma$, $\ell^+\ell^-Z$, $\ell^+\nu W^-$, $\ell^-\bar{\nu}W^+$, $\nu\bar{\nu}Z$	$5\,G \to\ 15$
Gauge-Quark-IA: $q\bar{q}\gamma$, $q\bar{q}Z$, udW^-, $d\bar{u}W^+$, $q\bar{q}g$	$2\,G(G\ +\ 3) \to 36$
Higgs-IA: HHH	1
Higgs-Gauge-IA: HW^+W^-, HZZ	2
Higgs-Lepton-IA: $\ell^+\ell^-H$	$G \to\ 3$
Higgs-Quark-IA: $q\bar{q}H$	$2\,G \to 6$
Higgs-Gst.-IA: $H\phi^+\phi^-$, $H\phi\phi$	2
Gst.-Gauge-IA: $\phi^+\phi^-\gamma$, $\phi^+\phi^-Z$, $\phi^\pm\phi W^\mp$, $\phi^\pm HW^\mp$, ϕHZ, $\phi^\pm W^\mp Z$, $\phi^\pm W^\mp\gamma$	11
Gst.-Lepton-IA: $\ell^+\ell^-\phi$, $\ell^-\bar{\nu}\phi^+$, $\ell^+\nu\phi^-$	$3\,G \to 9$
Gst.-Quark-IA: $q\bar{q}\phi$, $ud\phi^-$, $d\bar{u}\phi^+$	$2\,G(G\!+\!1) \to 24$

Table B.1: **3-Vertices, SM:** $\big(2G^2 + 14G + 6\big) + \big(2G^2 + 5G + 13\big) \to 66 + 46$, *with* G *being the number of generations, set to* 3 *in the final step.*

VERTICES	#
Gauge-IA: $WW\gamma\gamma$, $WWZ\gamma$, $WWZZ$, $WWWW$, $gggg$	5
Higgs-IA: $HHHH$	1
Higgs-Gauge-IA: HHW^+W^-, $HHZZ$	2
Higgs-Gst.-IA: $HH\phi^+\phi^-$, $HH\phi\phi$, $\phi^+\phi^-\phi^+\phi^-$, $\phi^+\phi^-\phi\phi$, $\phi\phi\phi\phi$	5
Gst.-Gauge-IA: $\phi\phi W^+W^-$, $\phi\phi ZZ$, $\phi^+\phi^-WW$, $\phi^+\phi^-ZZ$, $\phi^+\phi^-Z\gamma$, $\phi^+\phi^-\gamma\gamma$, $H\phi^\pm W^\mp Z$, $H\phi^\pm W^\mp\gamma$, $\phi\phi^\pm W^\mp Z$, $\phi\phi^\pm W^\mp\gamma$	14

Table B.2: **4-Vertices, SM:** 8 *vertices and* 19 *additional Goldstone vertices.*

We now briefly summarize the mixings of the interaction eigenstates to the mass eigenstates. All particles with identical colour and electromagnetic quantum numbers are generally allowed to mix.

The two Higgs doublets are decomposed into the following mass eigenstates:

$$H_1 = \begin{pmatrix} \frac{1}{\sqrt{2}}\big(v_1 + H^0\cos\alpha - h^0\sin\alpha + \mathrm{i}A^0\sin\beta + \mathrm{i}\phi^0\cos\beta\big) \\ H^-\sin\beta + \phi^-\cos\beta \end{pmatrix}, \tag{B.5}$$

$$H_2 = \begin{pmatrix} H^+\cos\beta - \phi^+\sin\beta \\ \frac{1}{\sqrt{2}}\big(v_2 + H^0\sin\alpha + h^0\cos\alpha + \mathrm{i}A^0\cos\beta - \mathrm{i}\phi^0\sin\beta\big) \end{pmatrix} \tag{B.6}$$

There are now five physical Higgs particles, the scalars H^0 and h^0, the pseudoscalar A^0 and the charged Higgs' $H^\pm$, thus called in the case of a non-CP violating Higgs potential.

VERTICES	#
Gauge-IA: $WW\gamma$, WWZ, ggg	3
Gauge-Lepton-IA: $\ell^+\ell^-\gamma$, $\ell^+\ell^-Z$, $\ell^+\nu W^-$, $\ell^-\bar\nu W^+$, $\nu\bar\nu Z$	$5\,G \to\ 15$
Gauge-Quark-IA: $q\bar q\gamma$, $q\bar qZ$, $u\bar dW^-$, $d\bar uW^+$, $q\bar qg$	$2\,G(G\ +\ 3) \to 36$
Higgs-IA: H^+H^-H, H^+H^-h, HHH, HHh, Hhh, hhh, AAH, AAh	8
Higgs-Gauge-IA: $AW^\pm H^\mp$, HAZ, hAZ, W^+W^-H, W^+W^-h, $W^\pm H^\mp H$, $W^\mp H^\pm h$, ZZH, ZZh, $H^+H^-\gamma$, H^+H^-Z	14
Higgs-Lepton-IA: $H^+\bar\nu\ell^-$, $H^-\nu\ell^+$, $\ell^+\ell^-H$, $\ell^+\ell^-h$, $\ell^+\ell^-A$	$5\,G \to\ 15$
Higgs-Quark-IA: $q\bar qH$, $q\bar qh$, $q\bar qA$, $u\bar dH^-$, $d\bar uH^+$	$2\,G(G\ +\ 3) \to 36$
Higgs-Chargino-Neutralino-IA: $\tilde\chi\tilde\chi H$, $\tilde\chi\tilde\chi h$, $\tilde\chi\tilde\chi A$, $\tilde\chi^+\tilde\chi^-H$, $\tilde\chi^+\tilde\chi^-h$, $\tilde\chi^+\tilde\chi^-A$, $\tilde\chi^\pm\tilde\chi H^\mp$	52
Slepton-Gauge-IA: $\tilde\ell^+\tilde\ell^-\gamma$, $\tilde\ell^+\tilde\ell^-Z$, $\tilde\nu\tilde\nu^*Z$, $\tilde\ell^+\tilde\nu W^-$, $\tilde\ell^-\tilde\nu^*W^+$	$11\,G \to\ 33$
Squark-Gauge-IA: $\tilde q\tilde q^*\gamma$, $\tilde q\tilde q^*Z$, $\tilde d\tilde u^*W^+$, $\tilde u\tilde d^*W^-$, $\tilde q\tilde q^*g$	$8\,G(G\ +\ 2) \to 120$
Chargino-Neutralino-Gluino-Gauge-IA: $\tilde\chi\tilde\chi Z$, $\tilde\chi^+\tilde\chi^-Z$, $\tilde\chi^+\tilde\chi^-\gamma$, $\tilde\chi^+\tilde\chi W^-$, $\tilde\chi^-\tilde\chi W^+$, $\tilde g\tilde g g$	33
Other Chargino-Neutralino-Gluino-IA: $q\tilde g\tilde q^*$, $\bar q\tilde g\tilde q$, $\tilde\chi^+\ell^-\tilde\nu^*$, $\tilde\chi^-\ell^+\tilde\nu$, $\tilde\chi^+\bar\nu\tilde\ell^-$, $\tilde\chi^-\nu\tilde\ell^+$, $\tilde\chi^+\bar u\tilde d$, $\tilde\chi^-u\tilde d^*$, $\tilde\chi^+d\tilde u^*$, $\tilde\chi^-\bar d\tilde u$, $\tilde\chi\nu\tilde\nu^*$, $\tilde\chi\bar\nu\tilde\nu$, $\tilde\chi\ell^\mp\tilde\ell^\pm$, $\tilde\chi q\tilde q^*$, $\tilde\chi\bar q\tilde q$	$4\,G(4\,G\ +\ 19) \to 372$
Higgs-Slepton-IA: $H\tilde\nu\tilde\nu^*$, $h\tilde\nu\tilde\nu^*$, $H\tilde\ell^+\tilde\ell^-$, $h\tilde\ell^+\tilde\ell^-$, $A\tilde\ell^+\tilde\ell^-$, $H^+\tilde\ell^-\tilde\nu^*$, $H^-\tilde\ell^+\tilde\nu$	$18\,G \to\ 54$
Higgs-Squark-IA: $\tilde q\tilde q^*H$, $\tilde q\tilde q^*h$, $\tilde q\tilde q^*A$, $\tilde u\tilde d^*H^-$, $\tilde u^*\tilde dH^+$	$8\,G(3\,G\ +\ 1) \to\ 240$

Table B.3: **MSSM, 3-Vertices** $52\,G^2 + 151\,G + 110 \to 1031$, *with G being the number of generations, set to* 3 *in the final step.*

$\phi^{\pm}$ and ϕ^0 are the Goldstone bosons attached to the W and Z bosons, respectively. The vacuum expectation values are denoted by v_1 and v_2 while α and β are two real mixing angles.

Most important for the MSSM are the mixings of the charged Higgsinos (the SUSY partners of the Higgs') and the charged gauginos to mass eigenstates named *charginos*, as well as the neutral Higgsinos and neutral gauginos are linearly combined to states called *neutralinos*. For the charginos it is justified by the smallness of observed CP-violating effects to choose the imaginary parts of μ and m_{Wino} sufficiently small to define orthogonal instead of unitary mixing matrices

$$U = \begin{pmatrix} \cos\phi_- & \sin\phi_- \\ -\sin\phi_- & \cos\phi_- \end{pmatrix}, \tag{B.7}$$

VERTICES	#
Gauge-IA: $WW\gamma\gamma$, $WWZ\gamma$, $WWZZ$, $WWWW$, $gggg$	5
Higgs-IA: $H^+H^-H^+H^-$, H^+H^-HH, H^+H^-Hh, H^+H^-hh, H^+H^-AA, $HHHH$, $HHHh$, $HHhh$, $Hhhh$, $hhhh$, $HHAA$, $HhAA$, $hhAA$, $AAAA$	14
Higgs-Gauge-IA: $HHZZ$, $hhZZ$, $AAZZ$, H^+H^-ZZ, $H^+H^-Z\gamma$, $H^+H^-\gamma\gamma$, $H^{\pm}HW^{\mp}\gamma$, $H^{\pm}hW^{\mp}\gamma$, $H^{\pm}HW^{\mp}Z$, $H^{\pm}hW^{\mp}Z$, HHW^+W^-, hhW^+W^-, AAW^+W^-, $H^{\pm}AW^{\mp}\gamma, H^{\pm}AW^{\mp}Z$	21
Slepton-Gauge-IA: $\tilde{\ell}^+\tilde{\ell}^-\gamma\gamma$, $\tilde{\ell}^+\tilde{\ell}^-Z\gamma$, $\tilde{\ell}^+\tilde{\ell}^-ZZ$, $\tilde{\nu}\tilde{\nu}^*ZZ$, $\tilde{\ell}^+\tilde{\ell}^-W^+W^-$, $\tilde{\nu}\tilde{\nu}^*W^+W^-$, $\tilde{\ell}^-\tilde{\nu}^*W^+\gamma$, $\tilde{\ell}^+\tilde{\nu}W^-\gamma$, $\tilde{\ell}^-\tilde{\nu}^*W^+Z$, $\tilde{\ell}^+\tilde{\nu}W^-Z$	$24G \to\ 72$
Squark-Gauge-IA: $\tilde{q}\tilde{q}^*\gamma\gamma$, $\tilde{q}\tilde{q}^*Z\gamma$, $\tilde{q}\tilde{q}^*ZZ$, $\tilde{q}\tilde{q}^*W^+W^-$, $\tilde{u}\tilde{d}^*W^-\gamma$, $\tilde{u}^*\tilde{d}W^+\gamma$, $\tilde{u}\tilde{d}^*W^-Z$, $\tilde{u}^*\tilde{d}W^+Z$, $\tilde{q}\tilde{q}^*gg$, $\tilde{q}\tilde{q}^*g\gamma$, $\tilde{q}\tilde{q}^*gZ$, $\tilde{u}\tilde{d}^*gW^-$, $\tilde{u}^*\tilde{d}gW^+$	$4G(6G+11)$ $\to\ 348$
Slepton-Slepton-IA: $\tilde{\nu}\tilde{\nu}^*\tilde{\nu}\tilde{\nu}^*$, $\tilde{\nu}\tilde{\nu}^*\tilde{\ell}^-\tilde{\ell}^+$, $\tilde{\ell}^-\tilde{\ell}^+\tilde{\ell}^-\tilde{\ell}^+$	$\frac{25}{2}G^2+\frac{1}{2}G+1$ $\to\ 115$
Squark-Squark-IA: $\tilde{q}\tilde{q}^*\tilde{q}\tilde{q}^*$, $\tilde{u}\tilde{u}^*\tilde{d}\tilde{d}^*$	$2G(8G^3+8G$ $+1) \to\ 1446$
Slepton-Squark-IA: $\tilde{q}\tilde{q}^*\tilde{\nu}\tilde{\nu}^*$, $\tilde{q}\tilde{q}^*\tilde{\ell}^+\tilde{\ell}^-$, $\tilde{u}\tilde{d}^*\tilde{\ell}^-\tilde{\nu}^*$, $\tilde{u}^*\tilde{d}\tilde{\ell}^+\tilde{\nu}$	$8G^2(2G+5)$ $\to\ 792$
Higgs-Slepton-IA: $HH\tilde{\nu}\tilde{\nu}^*$, $Hh\tilde{\nu}\tilde{\nu}^*$, $hh\tilde{\nu}\tilde{\nu}^*$, $AA\tilde{\nu}\tilde{\nu}^*$, $H^+H^-\tilde{\nu}\tilde{\nu}^*$, $HH\tilde{\ell}^+\tilde{\ell}^-$, $Hh\tilde{\ell}^+\tilde{\ell}^-$, $hh\tilde{\ell}^+\tilde{\ell}^-$, $AA\tilde{\ell}^+\tilde{\ell}^-$, $H^+H^-\tilde{\ell}^+\tilde{\ell}^-$, $HH^+\tilde{\ell}^-\tilde{\nu}^*$, $hH^+\tilde{\ell}^-\tilde{\nu}^*$, $AH^+\tilde{\ell}^-\tilde{\nu}^*$, $HH^-\tilde{\ell}^+\tilde{\nu}$, $hH^-\tilde{\ell}^+\tilde{\nu}$, $AH^-\tilde{\ell}^+\tilde{\nu}$	$37G \to\ 111$
Higgs-Squark-IA: $\tilde{q}\tilde{q}^*HH$, $\tilde{q}\tilde{q}^*Hh$, $\tilde{q}\tilde{q}^*hh$, $\tilde{q}\tilde{q}^*AA$, $\tilde{q}\tilde{q}^*H^+H^-$, $\tilde{u}^*\tilde{d}H^+H$, $\tilde{u}^*\tilde{d}H^+h$, $\tilde{u}^*\tilde{d}H^+A$, $\tilde{u}\tilde{d}^*H^-H$, $\tilde{u}\tilde{d}^*H^-h$, $\tilde{u}\tilde{d}^*H^-A$	$8G(3G\ +\ 5)$ $\to 336$

Table B.4: **MSSM, 4-Vertices:** $16G^4+16G^3+\frac{233}{2}G^2+\frac{295}{2}G+41 \to 3260$

$$V = \begin{pmatrix} \cos\phi_+ & \sin\phi_+ \\ -\eta\sin\phi_+ & \eta\cos\phi_+ \end{pmatrix}, \qquad \eta = \text{sgn}\left[\mu\, m_{\text{Wino}} - m_W^2 \sin(2\beta)\right] \tag{B.8}$$

with the sign factor η guaranteeing that the chargino masses are positive. The mixing angles are

$$\tan(2\phi_+) = \frac{-2\sqrt{2}m_W\,(m_{\text{Wino}}\sin\beta + \mu\cos\beta)}{m_{\text{Wino}}^2 - \mu^2 + 2m_W^2\cos(2\beta)}, \tag{B.9}$$

$$\tan(2\phi_-) = \frac{-2\sqrt{2}m_W\,(m_{\text{Wino}}\cos\beta + \mu\sin\beta)}{m_{\text{Wino}}^2 - \mu^2 - 2m_W^2\cos(2\beta)}. \tag{B.10}$$

The charginos $\tilde{\chi}_i^{\pm}, i = 1, 2$ are then related to the charged Winos and Higgsinos as

$$\boxed{\begin{array}{ll} \tilde{W}_L^+ = V_{i1}^*\tilde{\chi}_{i,L}^+ & \overline{\tilde{W}_L^+} = \overline{\tilde{\chi}_{i,L}^+}V_{i1} \\ \tilde{W}_R^+ = U_{i1}\tilde{\chi}_{i,R}^+ & \overline{\tilde{W}_R^+} = \overline{\tilde{\chi}_{i,R}^+}U_{i1}^* \\ \tilde{H}_L^+ = V_{i2}^*\tilde{\chi}_{i,L}^+ & \overline{\tilde{H}_L^+} = \overline{\tilde{\chi}_{i,L}^+}V_{i2} \\ \tilde{H}_R^+ = U_{i2}\tilde{\chi}_{i,R}^+ & \overline{\tilde{H}_R^+} = \overline{\tilde{\chi}_{i,R}^+}U_{i2}^* \end{array}} \tag{B.11}$$

$$\boxed{\begin{array}{ll} \tilde{W}_L^- = U_{i1}^*\tilde{\chi}_{i,L}^- & \overline{\tilde{W}_L^-} = \overline{\tilde{\chi}_{i,L}^-}U_{i1} \\ \tilde{W}_R^- = V_{i1}\tilde{\chi}_{i,R}^- & \overline{\tilde{W}_R^-} = \overline{\tilde{\chi}_{i,R}^-}V_{i1}^* \\ \tilde{H}_L^- = U_{i2}^*\tilde{\chi}_{i,L}^- & \overline{\tilde{H}_L^-} = \overline{\tilde{\chi}_{i,L}^-}U_{i2} \\ \tilde{H}_R^- = V_{i2}\tilde{\chi}_{i,R}^- & \overline{\tilde{H}_R^-} = \overline{\tilde{\chi}_{i,R}^-}V_{i2}^* \end{array}} \tag{B.12}$$

For the neutralinos, we introduce the 4×4-matrices N used to diagonalize the mass matrix

$$Y^{0\prime} = \begin{pmatrix} m_{\text{Bino}} & 0 & m_Z\sin\theta_W\cos\beta & -m_Z\sin\theta_W\sin\beta \\ 0 & m_{\text{Wino}} & -m_Z\cos\theta_W\cos\beta & m_Z\cos\theta_W\sin\beta \\ m_Z\sin\theta_W\cos\beta & -m_Z\cos\theta_W\cos\beta & 0 & -\mu \\ -m_Z\sin\theta_W\sin\beta & m_Z\cos\theta_W\sin\beta & -\mu & 0 \end{pmatrix} \tag{B.13}$$

in the form $N^*Y^{0\prime}N^{-1} = N_D$; θ_W is the Weinberg angle of the electroweak theory. The neutralinos $\tilde{\chi}_i^0, i = 1, \ldots, 4$ are then defined as

$$\boxed{\begin{array}{ll} \tilde{B}_L = \eta_i^* N_{i1}^*\tilde{\chi}_{i,L}^0 & \overline{\tilde{B}_L} = \overline{\tilde{\chi}_{i,L}^0}\eta_i N_{i1} \\ \tilde{B}_R = \eta_i N_{i1}\tilde{\chi}_{i,R}^0 & \overline{\tilde{B}_R} = \overline{\tilde{\chi}_{i,R}^0}\eta_i^* N_{i1}^* \\ \tilde{W}_L^3 = \eta_i^* N_{i2}^*\tilde{\chi}_{i,L}^0 & \overline{\tilde{W}_L^3} = \overline{\tilde{\chi}_{i,L}^0}\eta_i N_{i2} \\ \tilde{W}_R^3 = \eta_i N_{i2}\tilde{\chi}_{i,R}^0 & \overline{\tilde{W}_R^3} = \overline{\tilde{\chi}_{i,R}^0}\eta_i^* N_{i2}^* \\ \tilde{H}_{1,L}^0 = \eta_i^* N_{i3}^*\tilde{\chi}_{i,L}^0 & \overline{\tilde{H}_{1,L}^0} = \overline{\tilde{\chi}_{i,L}^0}\eta_i N_{i3} \\ \tilde{H}_{1,R}^0 = \eta_i N_{i3}\tilde{\chi}_{i,R}^0 & \overline{\tilde{H}_{1,R}^0} = \overline{\tilde{\chi}_{i,R}^0}\eta_i^* N_{i3}^* \\ \tilde{H}_{2,L}^0 = \eta_i^* N_{i4}^*\tilde{\chi}_{i,L}^0 & \overline{\tilde{H}_{2,L}^0} = \overline{\tilde{\chi}_{i,L}^0}\eta_i N_{i4} \\ \tilde{H}_{2,R}^0 = \eta_i N_{i4}\tilde{\chi}_{i,R}^0 & \overline{\tilde{H}_{2,R}^0} = \overline{\tilde{\chi}_{i,R}^0}\eta_i^* N_{i4}^* \end{array}} \tag{B.14}$$

The η_i are phases to guarantee the positivity of the neutralino masses.

For each Standard Model fermion there are two complex scalar superpartners, which can also mix due to the soft breaking terms. Under (not too) special circumstances the transformations from the interaction to the mass eigenstates can be assumed to be orthogonal.

The matrices of the mass squares for the sfermions to be diagonalized will always be denoted in the form

$$M^2_{\tilde f} = \begin{pmatrix} m^2_{\tilde f_L} & m^2_{\tilde f_{L/R}} \\ \left(m^2_{\tilde f_{L/R}}\right)^* & m^2_{\tilde f_R} \end{pmatrix} \quad . \tag{B.15}$$

It is easy to perform the diagonalization leading to the mass square eigenvalues (usually taken as $m^2_{\tilde f_1} \leq m^2_{\tilde f_2}$):

$$m^2_{\tilde f_{1/2,i}} = \frac{1}{2}\left(m^2_{\tilde f_{L,i}} + m^2_{\tilde f_{R,i}}\right) \mp \frac{1}{2}\sqrt{\left(m^2_{\tilde f_{L,i}} - m^2_{\tilde f_{R,i}}\right)^2 + 4\left|m^2_{\tilde f_{L/R}}\right|^2} \quad , \tag{B.16}$$

while the mixing angle is the solution of

$$\tan\theta_{\tilde f_i} = \frac{2m^2_{\tilde f_{L/R,i}}}{m^2_{\tilde f_{L,i}} - m^2_{\tilde f_{R,i}}}. \tag{B.17}$$

The mixing is given by

$$\begin{aligned} \tilde f_{1,i} &= \tilde f_{L,i}\cos\theta_{\tilde f_i} + \tilde f_{R,i}\sin\theta_{\tilde f_i} \\ \tilde f_{2,i} &= -\tilde f_{L,i}\sin\theta_{\tilde f_i} + \tilde f_{R,i}\cos\theta_{\tilde f_i} \end{aligned} \quad . \tag{B.18}$$

Note that this is only an orthogonal transformation for real symmetric mass square matrices. If $m^2_{\tilde f_{L/R}}$ does have an imaginary part, then there are additional phases involved in (B.16) and (B.17), but they have to be drastically small in order not to contradict CP-violation observations.

We now list the mass square matrices for the up and down squarks as well as for the sleptons and sneutrinos.

VERTICES	#
Higgs-Gst.-IA: $HA\phi$, $hA\phi$, $H^\pm H\phi^\mp$, $H^\pm h\phi^\mp$, $H^\pm A\phi^\mp$, $H\phi\phi$, $h\phi\phi$, $H\phi^+\phi^-$, $h\phi^+\phi^-$	12
Higgs-Gst.-Gauge-IA: $Z\phi^+\phi^-$, $\gamma\phi^+\phi^-$, $W^\pm\phi^\mp\phi$, $ZH\phi$, $Zh\phi$, $W^\pm\phi^\mp H$, $W^\pm\phi^\mp h$, $W^\pm Z\phi^\mp$, $W^\pm\gamma\phi^\mp$	14
Gst.-Lepton-IA: $\ell^+\ell^-\phi$, $\ell^-\bar\nu\phi^+$, $\ell^+\nu\phi^-$	$3G \to 9$
Gst.-Quark-IA: $q\bar q\phi, u\bar d\phi^-$, $d\bar u\phi^+$	$2G(G+1) \to 24$
Gst.-C/N-ino-IA: $\tilde\chi\tilde\chi\phi$, $\tilde\chi^+\tilde\chi^-\phi$, $\tilde\chi^+\tilde\chi\phi^-$, $\tilde\chi^-\tilde\chi\phi^+$	30
Gst.-Slepton-IA: $\tilde\ell^+\tilde\ell^-\phi$, $\tilde\ell^-\tilde\nu^*\phi^+$, $\tilde\ell^+\tilde\nu\phi^-$	$8G \to 24$
Gst.-Squark-IA: $\tilde q\tilde q^*\phi$, $\tilde u\tilde d^*\phi^-$, $\tilde u^*\tilde d\phi^+$	$8G(G+1) \to 96$

Table B.5: **MSSM, 3-Goldstone-Vertices:** $10G^2 + 21G + 56 \to 209$

up squarks:

$$m^2_{\tilde u_{L,i}} = m^2_{\tilde Q_i} + m^2_{u_i} + m_Z^2 \cos(2\beta)\left(\frac{1}{2} - \frac{2}{3}\sin^2\theta_W\right) \tag{B.19a}$$

$$m^2_{\tilde u_{R,i}} = m^2_{\tilde U_i} + m^2_{u_i} + \frac{2}{3} m_Z^2 \cos(2\beta)\sin^2\theta_W \tag{B.19b}$$

$$m^2_{\tilde u_{L/R,i}} = m_{u_i}\cdot\left(A_i^U + (C_i^U + \mu^*)\cot\beta\right) \tag{B.19c}$$

down squarks:

$$m^2_{\tilde d_{L,i}} = m^2_{\tilde Q_i} + m^2_{d_i} - m_Z^2 \cos(2\beta)\left(\frac{1}{2} - \frac{1}{3}\sin^2\theta_W\right) \tag{B.21}$$

$$m^2_{\tilde d_{R,i}} = m^2_{\tilde D_i} + m^2_{d_i} - \frac{1}{3} m_Z^2 \cos(2\beta)\sin^2\theta_W \tag{B.22}$$

$$m^2_{\tilde d_{L/R,i}} = m_{d_i}\cdot\left(A_i^D - (C_i^D + \mu^*)\tan\beta\right) \tag{B.23}$$

sleptons:

$$m^2_{\tilde\ell_{L,i}} = m^2_{\tilde L_i} + m^2_{\ell_i} + \frac{1}{2} m_Z^2 \cos(2\beta)\sin^2\theta_W \tag{B.24a}$$

$$m^2_{\tilde\ell_{R,i}} = m^2_{\bar E_i} + m^2_{\ell_i} - m_Z^2 \cos(2\beta)\sin^2\theta_W \tag{B.24b}$$

$$m^2_{\tilde\ell_{L/R,i}} = m_{\ell_i}\cdot\left(A_i^E + (C_i^E + \mu^*)\tan\beta\right) \tag{B.24c}$$

sneutrinos:

VERTIZES	#
Higgs-Gst.-IA: $HHA\phi$, $HhA\phi$, $hhA\phi$, $AAA\phi$, $H^+H^-A\phi$, $H^\pm HH\phi^\mp$, $H^\pm Hh\phi^\mp$, $H^\pm hh\phi^\mp$, $H^\pm AA\phi^\mp$, $H^\pm HA\phi^\mp$, $H^\pm hA\phi^\mp$, $H^\pm H^+H^-\phi^\mp$, $HH\phi\phi$, $Hh\phi\phi$, $hh\phi\phi$, $AA\phi\phi$, $H^+H^-\phi\phi$, $H^\pm H\phi^\mp\phi$, $H^\pm h\phi^\mp\phi$, $H^\pm A\phi^\mp\phi$, $HH\phi^+\phi^-$, $Hh\phi^+\phi^-$, $hh\phi^+\phi^-$, $AA\phi^+\phi^-$, $H^+H^-\phi^+\phi^-$, $H^\pm H^\pm\phi^\mp\phi^\mp$, $A\phi\phi\phi$, $H^\pm\phi^\mp\phi\phi$, $A\phi^+\phi^-\phi$, $H^\pm\phi^\mp\phi^+\phi^-$, $\phi\phi\phi\phi$, $\phi^+\phi^-\phi\phi$, $\phi^+\phi^-\phi^+\phi^-$	46
Higgs-Gst.-Gauge-IA: $ZZ\phi\phi$, $ZZ\phi^+\phi^-$, $Z\gamma\phi^+\phi^-$, $\gamma\gamma\phi^+\phi^-$, $W^+W^-\phi^+\phi^-$, $W^+W^-\phi\phi$, $W^\pm Z\phi^\mp\phi$, $W^\pm\gamma\phi^\mp\phi$, $W^\pm Z\phi^\mp H$, $W^\pm\gamma\phi^\mp H$, $W^\pm Z\phi^\mp h$, $W^\pm\gamma\phi^\mp h$	18
Slepton-Gst.-IA: $\tilde\nu\tilde\nu^*A\phi$, $\tilde\nu\tilde\nu^*H^\pm\phi^\mp$, $\tilde\nu\tilde\nu^*\phi^+\phi^-$, $\tilde\nu\tilde\nu^*\phi\phi$, $\tilde\ell^+\tilde\ell^-A\phi$, $\tilde\ell^+\tilde\ell^-H^\pm\phi^\mp$, $\tilde\ell^+\tilde\nu h\phi^-$, $\tilde\ell^-\tilde\nu^*h\phi^+$, $\tilde\ell^+\tilde\nu A\phi^-$, $\tilde\ell^-\tilde\nu^*A\phi^+$, $\tilde\ell^+\tilde\nu\phi\phi^-$, $\tilde\ell^-\tilde\nu^*\phi\phi^+$, $\tilde\ell^+\tilde\ell^-\phi^+\phi^-$, $\tilde\ell^+\tilde\ell^-\phi\phi$, $\tilde\ell^+\tilde\nu H^-\phi$, $\tilde\ell^-\tilde\nu^*H^+\phi$, $\tilde\ell^+\tilde\nu H\phi^-$, $\tilde\ell^-\tilde\nu^*H\phi^+$	$45G$ $\to 135$
Squark-Gst.-IA: $\tilde q\tilde q^*A\phi$, $\tilde q\tilde q^*H^\pm\phi^\mp$, $\tilde q\tilde q^*\phi^+\phi^-$, $\tilde q\tilde q^*\phi\phi$, $\tilde u^*\tilde dH^+\phi$, $\tilde u\tilde d^*H^-\phi$, $\tilde u^*\tilde dH\phi^+$, $\tilde u\tilde d^*H\phi^-$, $\tilde u^*\tilde dh\phi^+$, $\tilde u\tilde d^*h\phi^-$, $\tilde u^*\tilde dA\phi^+$, $\tilde u\tilde d^*A\phi^-$, $\tilde u^*\tilde d\phi^+\phi$, $\tilde u\tilde d^*\phi^-\phi$	$40G\cdot$ $(G+1)$ $\to 480$

Table B.6: **MSSM, 4-Goldstone-Vertices:** $40G^2+85G+64\to 679$

As long as there are no right-handed neutrino fields there is no mixing between the left- and righthanded sneutrinos. Their mass square is

$$m_{L,i}^2 + \frac{1}{4} m_Z^2 \cos(2\beta) \sin^2 \theta_W \tag{B.25}$$

The discussion of the mass terms and the CKM mixing is more or less the same as in the Standard Model or the non-supersymmetric two Higgs-doublet model.

For more details about the MSSM cf. [7], [5].

Appendix C

Some technicalities

C.1 Proof of (3.3)

Here we want to prove the relations (3.3), which are nothing but the inverse Fourier transformations back from the Fourier modes to the field operators. The Fourier expansions of the field operators are

$$\begin{aligned}
\phi(x) &= \int \frac{d^3\vec{k}}{(2\pi)^3 2E} \left(a(k)e^{-\mathrm{i}kx} + a^\dagger(k)e^{+\mathrm{i}kx}\right), \\
\psi(x) &= \int \frac{d^3\vec{k}}{(2\pi)^3 2E} \sum_\sigma \left(b(k,\sigma)u(k,\sigma)e^{-\mathrm{i}kx} + d^\dagger(k,\sigma)v(k,\sigma)e^{+\mathrm{i}kx}\right), \\
\overline{\psi}(x) &= \int \frac{d^3\vec{k}}{(2\pi)^3 2E} \sum_\sigma \left(b^\dagger(k,\sigma)\overline{u}(k,\sigma)e^{+\mathrm{i}kx} + d(k,\sigma)\overline{v}(k,\sigma)e^{-\mathrm{i}kx}\right).
\end{aligned} \tag{C.1}$$

In the Majorana case the last two relations are equivalent and read:

$$\begin{aligned}
\psi(x) &= \int \frac{d^3\vec{k}}{(2\pi)^3 2E} \sum_\sigma \left(b(k,\sigma)u(k,\sigma)e^{-\mathrm{i}kx} + b^\dagger(k,\sigma)v(k,\sigma)e^{+\mathrm{i}kx}\right), \\
\overline{\psi}(x) &= \int \frac{d^3\vec{k}}{(2\pi)^3 2E} \sum_\sigma \left(b^\dagger(k,\sigma)\overline{u}(k,\sigma)e^{+\mathrm{i}kx} + b(k,\sigma)\overline{v}(k,\sigma)e^{-\mathrm{i}kx}\right)
\end{aligned} \tag{C.2}$$

The inverse relations (3.3) can be simply verified by inserting the Fourier expansions of the field operators:

$$\begin{aligned}
a(k) &\overset{!}{=} \mathrm{i}\int d^3\vec{x}\, e^{\mathrm{i}kx}\, \overleftrightarrow{\partial}_t \int \frac{d^3\vec{p}}{(2\pi)^3 2E}\left(a(p)e^{-\mathrm{i}px} + a^\dagger(p)e^{+\mathrm{i}px}\right) \\
&= \mathrm{i}\int d^3\vec{x} e^{\mathrm{i}kx} \int \frac{d^3\vec{p}}{(2\pi)^3 2E}\left(a(p)(-\mathrm{i}E - \mathrm{i}k^0)e^{-\mathrm{i}px} + a^\dagger(p)(+\mathrm{i}E - \mathrm{i}k^0)e^{+\mathrm{i}px}\right) \\
&= \int \frac{d^3\vec{p}}{2E}\left(a(p)(E+k^0)\delta^3(\vec{k}-\vec{p})\Big|_{k^0=E} + a^\dagger(p)(k^0-E)\delta^3(\vec{k}+\vec{p})\Big|_{k^0=E} e^{2\mathrm{i}Ex^0}\right) \\
&= a(k) \qquad \surd
\end{aligned} \tag{C.3}$$

In the fermionic case:

$$b(k,\sigma) \overset{!}{=} \int d^3\vec{x}\Bigg(\overline{u}(k,\sigma)\gamma^0 \int \frac{d^3\vec{p}}{(2\pi)^3 2E} \sum_\tau \big(b(p,\tau)u(p,\tau)e^{-\mathrm{i}px}$$

$$
\begin{aligned}
&\qquad\qquad\quad + d^\dagger(p,\tau)v(p,\tau)e^{+\mathrm{i}px}\Big)\Big)e^{\mathrm{i}kx} \\
&= \overline{u}(k,\sigma)\gamma^0 \int \frac{d^3\vec{p}}{2E}\sum_\tau \Big(b(p,\tau)u(p,\tau)\delta^3(\vec{p}-\vec{k})\Big|_{k^0=E} \\
&\qquad\qquad\quad + d^\dagger(p,\tau)v(p,\tau)\delta^3(\vec{k}+\vec{p})\Big|_{k^0=E} e^{2\mathrm{i}Ex^0}\Big) \\
&= \frac{1}{2E}\sum_\tau \Big(b(k,\tau)\overline{u}(\vec{k},\sigma)\gamma^0 u(\vec{k},\tau) + d^\dagger(k,\tau)\overline{u}(\vec{k},\sigma)\gamma^0 v(-\vec{k},\tau)\Big) \\
&= b(k,\sigma) \qquad \surd
\end{aligned} \tag{C.4}
$$

In the last step we used the identities

$$
\begin{aligned}
\overline{u}(\vec{k},\sigma)\gamma^0 u(\vec{k},\tau) = u^\dagger(\vec{k},\sigma)u(\vec{k},\tau) = 2E\,\delta_{\sigma\tau} \\
u^\dagger(\vec{k},\sigma)v(-\vec{k},\tau) = 0,
\end{aligned} \tag{C.5}
$$

which, for example, can be found in the book of Peskin/Schroeder [22] on p. 48.

C.2 Fierz identities

We briefly summarize the Fierz identities as they can be found in [23]. In the following, $\theta_i, i = 1, \ldots, 4$ are four anticommuting, i.e. Grassmann-odd four-component spinors like fermion field operators or superspace coordinates. The scalar, vectorial, tensorial, axialvectorial and pseudoscalar combination are

$$s(4,2\,;3,1) = \left(\overline{\theta}_4\theta_2\right)\left(\overline{\theta}_3\theta_1\right) \tag{C.6a}$$

$$v(4,2\,;3,1) = \left(\overline{\theta}_4\gamma^\mu\theta_2\right)\left(\overline{\theta}_3\gamma_\mu\theta_1\right) \tag{C.6b}$$

$$t(4,2\,;3,1) = \frac{1}{2}\left(\overline{\theta}_4\sigma^{\mu\nu}\theta_2\right)\left(\overline{\theta}_3\sigma_{\mu\nu}\theta_1\right) \tag{C.6c}$$

$$a(4,2\,;3,1) = \left(\overline{\theta}_4\gamma^5\gamma^\mu\theta_2\right)\left(\overline{\theta}_3\gamma_\mu\gamma^5\theta_1\right) \tag{C.6d}$$

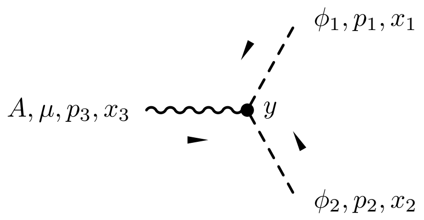
$$p(4,2\,;3,1) = \left(\overline{\theta}_4\gamma^5\theta_2\right)\left(\overline{\theta}_3\gamma^5\theta_1\right) \tag{C.6e}$$

Watch carefully the convention with respect to the axial vector adopted from [23]. The Fierz identities provide a possibility to rewrite the spinor products with combinations $(4,2\,;3,1)$ as $(4,1\,;3,2)$:

$$
\begin{pmatrix} s \\ v \\ t \\ a \\ p \end{pmatrix}(4,2\,;3,1) = -\frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & -2 & 0 & 2 & -4 \\ 6 & 0 & -2 & 0 & 6 \\ 4 & 2 & 0 & -2 & -4 \\ 1 & -1 & 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} s \\ v \\ t \\ a \\ p \end{pmatrix}(4,1\,;3,2) \tag{C.7}
$$

C.3 Derivation of couplings with momenta

In this short aside we want to get rid of the confusion with respect to the signs of momenta in 3-point vertices, e.g. arising in gauge theories by coupling two scalar fields to a gauge boson. The term under consideration is established by the trilinear terms in the kinetic parts in the Lagrangean density after having substituted the partial by the gauge covariant derivatives. By splitting the (necessarily complex) fields (charge!) in real and imaginary part we arrive at couplings with real fields. In the sequel we consider a 3-point vertex in which the scalar fields ϕ_1 and ϕ_2 possess the momenta p_1 and p_2 flowing into the vertex while the vector boson A_μ – not of interest in the following – has the incoming momentum p_3. The vertex looks like:



(C.8)

Its analytical form is:

$$\mathcal{L}_{\text{int}} = eA_\mu \left(\phi_1 \partial^\mu \phi_2 - \phi_2 \partial^\mu \phi_1\right) \tag{C.9}$$

All prefactors, numerical ones and also factors of i are understood to have been absorbed into the "coupling constant". We think about this vertex as being part of an n-point Green function, so that the fields of this interaction term are contracted with other field operators to give the propagators to be discussed below. This is shown here only with one term, the other is analogous; furthermore, as mentioned above, we ignore the vector field. It is not needed in the following discussion. To be more precise, we add to the derivatives the spacetime argument they act upon:

$$\phi_1(x_1)\phi_1(y)\partial_y^\mu \phi_2(y)\phi_2(x_2) = D_F(x_1 - y)\partial_y^\mu D_F(y - x_2) \tag{C.10}$$

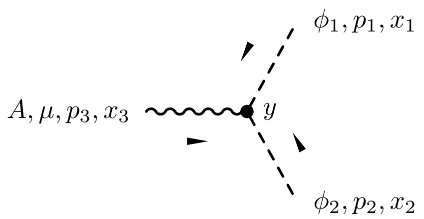
For the Feynman propagators in momentum space we have to perform the Fourier transformation from coordinate space with the momentum flowing from y to x_1, that is $-p_1$, and with the momentum flowing from x_2 to y, that is p_2. Note that due to

$$D_F(x - y) = \phi(x)\phi(y) = \langle 0|\mathrm{T}\left[\phi(x)\phi(y)\right]|0\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{i}e^{-\mathrm{i}p(x-y)}}{p^2 - m^2 + \mathrm{i}\epsilon} \tag{C.11}$$

the Fourier momentum flows from y to x according to the time ordering. This yields:

$$D_F(x_1 - y)\partial_y^\mu D_F(y - x_2) \stackrel{\text{F.T.}}{\longrightarrow} (-\mathrm{i}p_2^\mu)\text{F.T.}D_F(x_1 - y)D_F(y - x_2). \tag{C.12}$$

Finally, the analytical expression for the vertex (with the additional factor i stemming from the perturbation expansion) becomes:



$$= \quad e\left(p_2 - p_1\right)_\mu \tag{C.13}$$

This can be stated as the mnemonic:

$$\boxed{\mathrm{i}\partial^\mu \longrightarrow +(\text{incoming momentum})^\mu} \tag{C.14}$$

Appendix D

Details to the supersymmetric current

D.1 The current for a general model without gauge symmetry

In this section we want to briefly repeat the derivation of the supersymmetric current for a general model without gauge symmetries from [3] and prove its conservation not shown in the reference. The Lagrangean density of a general model given by a quantum field theory endowed with exact supersymmetry can be found in the equations (26.3.30), (26.4.7) and (26.7.7) in [3]:

$$\begin{aligned}\mathcal{L} = \sum_n \Big[(\partial_\mu \phi_n^*)(\partial^\mu \phi_n) + F_n^* F_n + \frac{\mathrm{i}}{2}\left(\overline{\psi_{n,L}}\,\partial\!\!\!/\,\psi_{n,L}\right) + \frac{\mathrm{i}}{2}\left(\overline{\psi_{n,R}}\,\partial\!\!\!/\,\psi_{n,R}\right)\Big] \\ - \frac{1}{2}\sum_{n,m} \frac{\partial^2 f(\phi)}{\partial\phi_n \partial\phi_m}\left(\overline{\psi_{n,R}}\psi_{m,L}\right) - \frac{1}{2}\sum_{n,m}\left(\frac{\partial^2 f(\phi)}{\partial\phi_n\partial\phi_m}\right)^*\left(\overline{\psi_{n,L}}\psi_{m,R}\right) \\ + \sum_n F_n \frac{\partial f(\phi)}{\partial\phi_n} + \sum_n F_n^* \left(\frac{\partial f(\phi)}{\partial\phi_n}\right)^*\end{aligned} \tag{D.1}$$

In this general model there are n different chiral superfields. f is an arbitrary function of these chiral superfields; when we impose renormalizability as a constraint, it is only allowed to be a polynom with degree three as an upper bound. The Noether part can be calculated by (7.17)

$$\begin{aligned}N^\mu = -\sum_n \Big[\sqrt{2}(\partial^\mu \phi_n^*)\psi_{n,L} + \sqrt{2}(\partial^\mu\phi_n)\psi_{n,R} + \frac{1}{\sqrt{2}}(\partial\!\!\!/\phi_n)\gamma^\mu\psi_{n,R} \\ + \frac{1}{\sqrt{2}}(\partial\!\!\!/\phi_n^*)\gamma^\mu\psi_{n,L} - \frac{\mathrm{i}}{\sqrt{2}}F_n\gamma^\mu\psi_{n,R} - \frac{\mathrm{i}}{\sqrt{2}}F_n^*\gamma^\mu\psi_{n,L}\Big]\end{aligned} \tag{D.2}$$

Next we derive the SUSY transformation of the general Lagrangean density in the same manner as for the WZ model, writing it in the form

$$S_{\text{general}} = \frac{1}{2}\sum_n \int d^4x \left[\hat{\Phi}_n^\dagger \hat{\Phi}_n\right]_D + \int d^4x \left[f(\hat{\Phi})\right]_F + \int d^4x \left[f(\hat{\Phi})\right]_F^*. \tag{D.3}$$

The kinetic part can be taken from (7.12),

$$K^{\mu}_{\text{kin}} = \frac{1}{\sqrt{2}}\gamma^{\mu}\sum_{n}\Big(\left(\partial\!\!\!/\phi_n\right)\psi_{n,R} + \left(\partial\!\!\!/\phi^*_n\right)\psi_{n,L} - \mathrm{i}F_n\psi_{n,R} - \mathrm{i}F^*_n\psi_{n,L}\Big) \quad . \tag{D.4}$$

For the contribution to the current from the potential we remember that

$$\delta_{\xi}\mathcal{L}_{\text{pot}} = \left(\delta_{\xi}\left[f(\hat{\Phi})\right]_F + \text{h.c.}\right) = \left(-\mathrm{i}\sqrt{2}\,\bar{\xi}\partial\!\!\!/\left[f(\hat{\Phi})\right]_{\psi_L} + \text{h.c.}\right) \tag{D.5}$$

for the SUSY transformation of the superpotential. When expanding the superpotential in a power series of the superfields,

$$f(\hat{\Phi}) = \sum_{k=0}^{\infty}\sum_{\substack{n_1,n_2,\ldots,n_k \\ n_1+n_2+\ldots+n_k=k}} f_{n_1 n_2\ldots n_k}\,\hat{\Phi}^{n_1}_{i_1}\cdot\hat{\Phi}^{n_2}_{i_2}\cdot\ldots\cdot\hat{\Phi}^{n_k}_{i_k} \qquad ,$$

with $i_j \in \{\text{appearing superfields}\}$, the spinor component of the superpotential can easily be read off:

$$\left[f(\hat{\Phi})\right]_{\psi_L} = \sum_n \left(\frac{\partial f(\phi)}{\partial\phi_n}\right)\psi_{n,L} \quad . \tag{D.6}$$

By the notation $f(\phi)$ we want to stress that the superfields as arguments of the function f have been replaced by their scalar components. This produces the potential part of the supersymmetric current

$$K^{\mu}_{\text{pot}} = -\mathrm{i}\sqrt{2}\sum_n \gamma^{\mu}\left[\left(\frac{\partial f(\phi)}{\partial\phi_n}\right)\psi_{n,L} + \left(\frac{\partial f(\phi)}{\partial\phi_n}\right)^*\psi_{n,R}\right]. \tag{D.7}$$

Finally the supersymmetric current for a general model (without gauge interactions which will be studied later on) is:

$$\boxed{\begin{aligned}\mathcal{J}^{\mu} = -\sqrt{2}\sum_n\Bigg[&(\partial\!\!\!/\phi_n)\gamma^{\mu}\psi_{n,R} + (\partial\!\!\!/\phi_n)^*\gamma^{\mu}\psi_{n,L} \\ &+\mathrm{i}\gamma^{\mu}\left(\frac{\partial f(\phi)}{\partial\phi_n}\right)\psi_{n,L} + \mathrm{i}\gamma^{\mu}\left(\frac{\partial f(\phi)}{\partial\phi_n}\right)^*\psi_{n,R}\Bigg]\end{aligned}} \tag{D.8}$$

We check the current conservation:

$$
\begin{aligned}
-\frac{1}{\sqrt{2}}\partial_\mu \mathcal{J}^\mu = \sum_n \Bigg[& (\Box\phi_n)\psi_{n,R} + (\Box\phi_n^*)\psi_{n,L} + \underline{(\partial\!\!\!/\phi_n)(\partial\!\!\!/\psi_{n,R})} + \underline{(\partial\!\!\!/\phi_n^*)(\partial\!\!\!/\psi_{n,L})} \\
& + \mathrm{i}\left(\frac{\partial f(\phi)}{\partial\phi_n}\right)\partial\!\!\!/\psi_{n,L} + \mathrm{i}\left(\frac{\partial f(\phi)}{\partial\phi_n}\right)^* \partial\!\!\!/\psi_{n,R}\Bigg] \\
& + \mathrm{i}\sum_{m,n}\Bigg[\underline{\left(\frac{\partial^2 f(\phi)}{\partial\phi_m\partial\phi_n}\right)(\partial\!\!\!/\phi_m)\psi_{n,L}} + \underline{\left(\frac{\partial^2 f(\phi)}{\partial\phi_m\partial\phi_n}\right)^*(\partial\!\!\!/\phi_m^*)\psi_{n,R}}\Bigg] \\
= \sum_n \Bigg[& -\frac{1}{2}\sum_{k,l}\left(\frac{\partial^3 f(\phi)}{\partial\phi_n\partial\phi_k\partial\phi_l}\right)\left(\overline{\psi_{k,R}}\psi_{l,L}\right)\psi_{n,L} \\
& -\frac{1}{2}\sum_{k,l}\left(\frac{\partial^3 f(\phi)}{\partial\phi_n\partial\phi_k\partial\phi_l}\right)^*\left(\overline{\psi_{k,L}}\psi_{l,R}\right)\psi_{n,R} \\
& + \underline{\underline{\sum_k F_k\left(\frac{\partial^2 f(\phi)}{\partial\phi_n\partial\phi_k}\right)\psi_{n,L}}} + \underline{\underline{\sum_k F_k^*\left(\frac{\partial^2\phi}{\partial\phi_n\partial\phi_k}\right)^*\psi_{n,R}}} \\
& - \underline{\underline{\mathrm{i}F_n(\partial\!\!\!/\psi_{n,L})}} - \underline{\underline{\mathrm{i}F_n^*(\partial\!\!\!/\psi_{n,R})}}\Bigg]
\end{aligned}
$$

In the first identity the underlined terms vanish due to the equations of motion for the fermions. For the second equality the equations of motion for the scalar particles were inserted yielding the leading four terms. The two rightmost terms are produced using the equation of motion for the auxiliary fields F_n and F_n^*. The doubly underlined terms cancel due to the fermions' equations of motion. There still remain the trilinear fermion terms. The three indices are summed over, so we can split the terms in three cyclic contributions (it will prove easier to use the 2-spinor formalism here):

$$
\begin{aligned}
& -\frac{1}{2}\sum_{n,k,l}\left(\frac{\partial^3 f(\phi)}{\partial\phi_n\partial\phi_k\partial\phi_l}\right)(\overline{\psi_{k,R}}\psi_{l,L})\psi_{n,L} + \text{h.c.} \\
& \quad = -\frac{1}{6}\sum_{n,k,l}\left(\frac{\partial^3 f(\phi)}{\partial\phi_n\partial\phi_k\partial\phi_l}\right)\Big((\psi_k\psi_l)\psi_n + (\psi_l\psi_n)\psi_k + (\psi_n\psi_k)\psi_l\Big) + \text{h.c.} = 0
\end{aligned} \tag{D.9}
$$

This vanishes due to the Schouten identity, cf. for instance [32], which is valid for Grassmann odd 2-spinors as well, for there is always an even number of transpositions in the cyclic sum.

D.2 Derivation of the SYM current

Here we present the detailed derivation of the SYM current omitted in the text.

We apply the de Wit–Freedman transformation to the matter Lagrangean density (9.5) and get:

$$
\begin{aligned}
\tilde{\delta}_\xi\Big[(D_\mu\phi)^\dagger(D^\mu\phi)\Big] = & \sqrt{2}\,(D_\mu\phi)^\dagger\left(\bar{\xi}D^\mu\Psi_L\right) + \mathrm{i}g\,(D_\mu\phi)^\dagger\left(\bar{\xi}\gamma^\mu\gamma^5\vec{T}\phi\cdot\vec{\lambda}\right) \\
& + \sqrt{2}\,(D_\mu\phi)^T\left(\bar{\xi}D^\mu\Psi_R\right) - \mathrm{i}g\phi^\dagger\left(\bar{\xi}\gamma_\mu\gamma^5\vec{T}\vec{\lambda}\right)D^\mu\phi
\end{aligned} \tag{D.10}
$$

$$\begin{aligned}
\tilde{\delta}_\xi\left[F^\dagger F\right] =& -\mathrm{i}\sqrt{2}F^\dagger\left(\bar{\xi}\not{D}\Psi_L\right) - \mathrm{i}\sqrt{2}F^T\left(\bar{\xi}\not{D}\Psi_R\right) + 2gF^\dagger\left(\bar{\xi}\vec{T}\phi\cdot\vec{\lambda}_R\right) \\
&+ 2g\phi^\dagger\left(\bar{\xi}\vec{T}F\cdot\vec{\lambda}_R\right) \qquad \text{(D.11)} \\
\tilde{\delta}_\xi\left[\frac{\mathrm{i}}{2}\overline{\Psi}\not{D}\Psi\right] =& -\frac{1}{\sqrt{2}}\left(\bar{\xi}\gamma^\mu\left(D_\mu\phi\right)^\dagger\not{D}\Psi_L\right) + \frac{\mathrm{i}}{\sqrt{2}}\left(\bar{\xi}F^\dagger\not{D}\Psi_L\right) \\
&+ \frac{1}{\sqrt{2}}\left(\bar{\xi}\gamma^\mu\gamma^\nu\left(D_\nu D_\mu\phi^T\right)\Psi_R\right) - \frac{\mathrm{i}}{\sqrt{2}}\left(\bar{\xi}\left(\not{D}F^T\right)\Psi_R\right) \\
&- \frac{1}{\sqrt{2}}\left(\bar{\xi}\gamma^\mu\left(D_\mu\phi\right)^T\not{D}\Psi_R\right) + \frac{\mathrm{i}}{\sqrt{2}}\left(\bar{\xi}F^T\not{D}\Psi_R\right) \\
&+ \frac{1}{\sqrt{2}}\left(\bar{\xi}\gamma^\mu\gamma^\nu\left(D_\nu D_\mu\phi\right)^\dagger\Psi_L\right) - \frac{\mathrm{i}}{\sqrt{2}}\left(\bar{\xi}(\not{D}F^\dagger)\Psi_L\right) \\
&- \frac{g}{2}\left(\bar{\xi}\gamma_\mu\gamma^5\vec{\lambda}\right)\cdot\left(\overline{\Psi_L}\gamma^\mu\vec{T}\Psi_L\right) + \frac{g}{2}\left(\bar{\xi}\gamma_\mu\gamma^5\vec{\lambda}\right)\cdot\left(\overline{\Psi_R}\gamma^\mu\vec{T}\Psi_R\right) \\
& \qquad \text{(D.12)} \\
\tilde{\delta}_\xi\left[-\sqrt{2}g\overline{\vec{\lambda}}\cdot\phi^\dagger\vec{T}\Psi_L\right] =& \frac{\mathrm{i}g}{\sqrt{2}}\left(\bar{\xi}\gamma^5\gamma^\nu\gamma^\mu\vec{F}_{\mu\nu}\cdot\phi^\dagger\vec{T}\Psi_L\right) - \sqrt{2}g\left(\bar{\xi}\vec{D}\cdot\phi^\dagger\vec{T}\Psi_L\right) \\
&- 2g\left(\bar{\xi}\Psi_R\right)\left(\overline{\vec{\lambda}}\cdot\vec{T}\Psi_L\right) - 2\mathrm{i}g\left(\bar{\xi}\phi^\dagger\vec{T}(\not{D}\phi)\cdot\vec{\lambda}_L\right) \\
&- 2g\left(\bar{\xi}\phi^\dagger\vec{T}F\cdot\vec{\lambda}\right) \qquad \text{(D.13)} \\
\tilde{\delta}_\xi\left[-\sqrt{2}g\overline{\Psi_L}\vec{T}\phi\cdot\vec{\lambda}\right] =& -2\mathrm{i}g\left(\bar{\xi}\gamma^\mu(D_\mu\phi)^\dagger\vec{T}\phi\cdot\vec{\lambda}_R\right) - 2g\left(\bar{\xi}F^\dagger\vec{T}\phi\cdot\vec{\lambda}_R\right) \\
&- 2g\left(\bar{\xi}\Psi_L\right)\left(\overline{\Psi_L}\vec{T}\cdot\vec{\lambda}\right) + \frac{\mathrm{i}g}{\sqrt{2}}\left(\bar{\xi}\gamma^5\gamma^\nu\gamma^\mu\vec{F}_{\mu\nu}\cdot\phi^T\vec{T}\Psi_R\right) \\
&- \sqrt{2}g\left(\bar{\xi}\vec{D}\cdot\phi\vec{T}\Psi_R\right) \qquad \text{(D.14)} \\
\tilde{\delta}_\xi\left[g\phi^\dagger\vec{T}\phi\cdot\vec{D}\right] =& \sqrt{2}g\left(\bar{\xi}\vec{D}\cdot\phi\vec{T}\Psi_R\right) + \sqrt{2}g\left(\bar{\xi}\vec{D}\cdot\phi^\dagger\vec{T}\Psi_L\right) \\
&- \mathrm{i}g\left(\phi^\dagger\vec{T}\phi\right)\cdot\left(\bar{\xi}\not{D}\vec{\lambda}\right) \qquad \text{(D.15)} \\
\tilde{\delta}_\xi\mathcal{W}(\phi,\Psi,F) \equiv& \,\partial_\mu\bar{\xi}\tilde{K}^\mu(\phi,\Psi,F) \qquad \text{(D.16)}
\end{aligned}$$

Now we start to examine all the produced terms. At first, all terms containing four spinors cancel each other – the two rightmost terms of (D.12) and the third term of (D.13) and (D.14), respectively. Using the Fierz identities (note again the global sign due to the presence of anticommuting spinors) yields:

$$\begin{aligned}
&- 2g\left(\overline{\xi_R}\Psi_L\right)\left(\overline{\Psi_L}\vec{T}\cdot\vec{\lambda}_R\right) \\
&\quad= \underline{\frac{g}{2}\left(\overline{\xi_R}\vec{\lambda}_R\right)\cdot\left(\overline{\Psi_L}\vec{T}\Psi_L\right)} + \frac{g}{2}\left(\overline{\xi_R}\gamma^\mu\vec{\lambda}_R\right)\cdot\left(\overline{\Psi_L}\gamma_\mu\vec{T}\Psi_L\right) \\
&\qquad + \underline{\frac{g}{4}\left(\overline{\xi_R}\sigma^{\mu\nu}\vec{\lambda}_R\right)\cdot\left(\overline{\Psi_L}\sigma_{\mu\nu}\vec{T}\Psi_L\right)} + \frac{g}{2}\left(\overline{\xi_R}\gamma^\mu\gamma^5\vec{\lambda}_R\right)\cdot\left(\overline{\Psi_L}\gamma^5\gamma_\mu\vec{T}\Psi_L\right) \\
&\qquad + \underline{\frac{g}{2}\left(\overline{\xi_R}\gamma^5\vec{\lambda}_R\right)\cdot\left(\overline{\Psi_L}\gamma^5\vec{T}\Psi_L\right)} \\
&\quad= -g\left(\overline{\xi_R}\gamma^\mu\gamma^5\vec{\lambda}_R\right)\cdot\left(\overline{\Psi_L}\gamma_\mu\gamma^5\vec{T}\Psi_L\right)
\end{aligned}$$

The underlined terms vanish here and in the following calculation, since scalar, pseudoscalar and tensor bilinears cannot couple spinors of like chirality to each other.

$$\begin{aligned}
&-2g\left(\overline{\xi_L}\Psi_R\right)\left(\overline{\vec{\lambda}_R}\vec{T}\cdot\Psi_L\right) = -2g\left(\overline{\xi_L}\Psi_R\right)\left(\overline{\Psi_R}\vec{T}\vec{\lambda}_L\right)\\
&\qquad = -g\left(\overline{\xi_L}\gamma^\mu\gamma^5\vec{\lambda}_L\right)\cdot\left(\overline{\Psi_R}\gamma_\mu\gamma^5\vec{T}\Psi_R\right)
\end{aligned}$$

For the other terms we get:

$$\begin{aligned}
&\frac{g}{2}\left(\bar{\xi}\gamma_\mu\gamma^5\vec{\lambda}\right)\cdot\left(\overline{\Psi_R}\gamma^\mu\gamma^5\Psi_R\right)\\
&\quad = \frac{g}{2}\left(\overline{\xi_L}\gamma_\mu\gamma^5\vec{\lambda}_L\right)\cdot\left(\overline{\Psi_R}\gamma^\mu\gamma^5\Psi_R\right) + \frac{g}{2}\left(\overline{\xi_R}\gamma_\mu\gamma^5\vec{\lambda}_R\right)\cdot\left(\overline{\Psi_R}\gamma^\mu\gamma^5\Psi_R\right)\\
&\quad = \frac{g}{2}\left(\overline{\xi_L}\gamma_\mu\gamma^5\vec{\lambda}_L\right)\cdot\left(\overline{\Psi_R}\gamma^\mu\gamma^5\Psi_R\right) + \frac{g}{2}\left(\overline{\xi_R}\gamma_\mu\gamma^5\vec{\lambda}_R\right)\cdot\left(\overline{\Psi_L}\gamma^\mu\gamma^5\Psi_L\right)
\end{aligned}$$

$$\begin{aligned}
&\frac{g}{2}\left(\bar{\xi}\gamma_\mu\gamma^5\vec{\lambda}\right)\cdot\left(\overline{\Psi_L}\gamma^\mu\gamma^5\Psi_L\right)\\
&\quad = \frac{g}{2}\left(\overline{\xi_L}\gamma_\mu\gamma^5\vec{\lambda}_L\right)\cdot\left(\overline{\Psi_L}\gamma^\mu\gamma^5\Psi_L\right) + \frac{g}{2}\left(\overline{\xi_R}\gamma_\mu\gamma^5\vec{\lambda}_R\right)\cdot\left(\overline{\Psi_L}\gamma^\mu\gamma^5\Psi_L\right)\\
&\quad = \frac{g}{2}\left(\overline{\xi_L}\gamma_\mu\gamma^5\vec{\lambda}_L\right)\cdot\left(\overline{\Psi_R}\gamma^\mu\gamma^5\Psi_R\right) + \frac{g}{2}\left(\overline{\xi_R}\gamma_\mu\gamma^5\vec{\lambda}_R\right)\cdot\left(\overline{\Psi_R}\gamma^\mu\gamma^5\Psi_R\right)
\end{aligned}$$

Each of the second manipulations for the two latest identities follow from the Majorana properties of the spinor field Ψ. As promised, all four terms from the last four identities cancel.

It is obvious that all terms containing the auxiliary fields D^a – the second from (D.13), the last from (D.14) and the first two from (D.15) – give zero.

Let us consider the terms containing $F^\dagger$ now. There are five of them: the first and third term in (D.11), the second and eighth from (D.12) and the second term of (D.14). As is immediately seen, the latter cancels the third term from (D.11). What remains is:

$$\begin{aligned}
&-\mathrm{i}\sqrt{2}\left(\bar{\xi}F^\dagger \not{D}\Psi_L\right) + \frac{\mathrm{i}}{\sqrt{2}}\left(\bar{\xi}F^\dagger \not{D}\Psi_L\right) - \frac{\mathrm{i}}{\sqrt{2}}\left(\bar{\xi}(F^\dagger \overleftarrow{\not{D}}')\Psi_L\right)\\
&= -\frac{\mathrm{i}}{\sqrt{2}}\left(\bar{\xi}F^\dagger \not{D}\Psi_L\right) - \frac{\mathrm{i}}{\sqrt{2}}\left(\bar{\xi}(F^\dagger \overleftarrow{\not{D}}')\Psi_L\right)\\
&= -\frac{\mathrm{i}}{\sqrt{2}}\partial_\mu\left(\bar{\xi}\gamma^\mu F^\dagger\Psi_L\right)
\end{aligned} \tag{D.17}$$

The symbol D'_μ introduced here is the covariant derivative originally acting on the righthanded spinor field Ψ_R, so it has the opposite sign compared to the covariant derivative for the lefthanded fields. Only the contributions with the partial derivatives survive.

The calculation for the parts with F^T (or F) proceeds analogously. Here the last term of (D.13) and the last one from (D.11) cancel, leaving the second term of (D.11) as well as the fourth and sixth from (D.12):

$$\begin{aligned}
&-\mathrm{i}\sqrt{2}\left(\bar{\xi}F^T \not{D}\Psi_R\right) - \frac{\mathrm{i}}{\sqrt{2}}\left(\bar{\xi}(F^T \overleftarrow{\not{D}}')\Psi_R\right) + \frac{\mathrm{i}}{\sqrt{2}}\left(\bar{\xi}F^T \not{D}\Psi_R\right)\\
&= -\frac{\mathrm{i}}{\sqrt{2}}\partial_\mu\left(\bar{\xi}\gamma^\mu F^T\Psi_R\right)
\end{aligned} \tag{D.18}$$

Herein D'_μ is the covariant derivative originally acting on the lefthanded spinor Ψ_L. Again, only the terms with the partial derivatives remain.

Next we turn our attention to the contributions containing both ϕ and $\phi^\dagger$, i.e. the second and fourth of (D.10), the fourth term of (D.13), the first of (D.14) and the rightmost of (D.15). We split the covariant derivatives in partial derivatives and the gauge field parts:

$$
\begin{aligned}
& \mathrm{i}g\left(\bar\xi(\not{D}\phi^\dagger)\gamma^5\vec{T}\phi\cdot\vec\lambda\right) - \mathrm{i}g\left(\bar\xi\gamma_\mu\gamma^5\phi^\dagger\vec{T}(D^\mu\phi)\cdot\vec\lambda\right) - 2\mathrm{i}g\left(\bar\xi\gamma^\mu\phi^\dagger\vec{T}(D_\mu\phi)\cdot\vec\lambda_L\right) \\
& - 2\mathrm{i}g\left(\bar\xi\gamma^\mu(D_\mu\phi)^\dagger\vec{T}\phi\cdot\vec\lambda_R\right) - \mathrm{i}g\left(\phi^\dagger\vec{T}\phi\right)\cdot\left(\bar\xi\not{D}\vec\lambda\right) \\
= \; & \mathrm{i}g\underline{\left(\bar\xi(\not\partial\phi^\dagger)\gamma^5\vec{T}\phi\cdot\vec\lambda\right)} - \mathrm{i}g\underline{\left(\bar\xi\phi^\dagger\vec{T}(\not\partial\phi)\gamma^5\cdot\vec\lambda\right)} - \mathrm{i}g\left(\bar\xi\phi^\dagger\vec{T}(\not\partial\phi)\cdot(1-\underline{\gamma^5})\vec\lambda\right) \\
& - \mathrm{i}g\left(\bar\xi(\not\partial\phi^\dagger)\vec{T}\phi\cdot(1+\underline{\gamma^5})\vec\lambda\right) - \mathrm{i}g\left(\bar\xi\phi^\dagger\vec{T}\phi\cdot\not\partial\vec\lambda\right) \\
& - g^2\underline{\left(\bar\xi\gamma^\mu\gamma^5\phi^\dagger\vec{T}(\vec{T}\cdot\vec{A}_\mu)\phi\cdot\vec\lambda\right)} - g^2\underline{\left(\bar\xi\gamma^\mu\gamma^5\phi^\dagger\vec{T}(\vec{T}\cdot\vec{A}_\mu)\phi\cdot\vec\lambda\right)} \\
& - g^2\left(\bar\xi\gamma^\mu\phi^\dagger\vec{T}(\vec{T}\cdot\vec{A}_\mu)\phi\cdot(1-\underline{\gamma^5})\vec\lambda\right) + g^2\left(\bar\xi\gamma^\mu\phi^\dagger(\vec{T}\cdot\vec{A}_\mu)\vec{T}\phi\cdot(1+\underline{\gamma^5})\vec\lambda\right) \\
& - \mathrm{i}g^2\left(\phi^\dagger\vec{T}\phi\right)\cdot\left(\bar\xi f^a_{\ bc}A^b_\mu\gamma^\mu\lambda^c\right) \\
= \; & -\mathrm{i}g\partial_\mu\left(\bar\xi\gamma^\mu\phi^\dagger\vec{T}\phi\cdot\vec\lambda\right) \\
& + g^2\underline{\left(\bar\xi\gamma^\mu\phi^\dagger\left[T^a,T^b\right]\phi A^a_\mu\lambda^b\right)} - \mathrm{i}g^2\underline{\left(\bar\xi\gamma^\mu\phi^\dagger T^a\phi f^a_{\ bc}A^b_\mu\lambda^c\right)}
\end{aligned}
$$

Underlined contributions cancel each other. The terms in the last line vanish due to the Lie algebra of the gauge group. We get the gradient

$$-\mathrm{i}g\partial_\mu\left(\bar\xi\gamma^\mu\phi^\dagger\vec{T}\phi\cdot\vec\lambda\right) \tag{D.19}$$

With the help of the algebra of covariant derivatives

$$D_\mu D_\nu - D_\nu D_\mu = -\mathrm{i}g\vec{T}\cdot\vec{F}_{\mu\nu} \tag{D.20}$$

we can rewrite the two terms with the field strength tensor of the gauge field, the first term of (D.13) and the fourth in (D.14):

$$
\begin{aligned}
& \frac{\mathrm{i}g}{\sqrt2}\left(\bar\xi\gamma^5\gamma^\nu\gamma^\mu\vec{F}_{\mu\nu}\cdot\phi^\dagger\vec{T}\Psi_L\right) + \frac{\mathrm{i}g}{\sqrt2}\left(\bar\xi\gamma^5\gamma^\nu\gamma^\mu\vec{F}_{\mu\nu}\cdot\phi^T\vec{T}\Psi_R\right) \\
= \; & -\frac{\mathrm{i}g}{\sqrt2}\left(\bar\xi\gamma^\nu\gamma^\mu\vec{F}_{\mu\nu}\cdot\phi^\dagger\vec{T}\Psi_L\right) + \frac{\mathrm{i}g}{\sqrt2}\left(\bar\xi\gamma^\nu\gamma^\mu\vec{F}_{\mu\nu}\cdot\phi^T\vec{T}\Psi_R\right) \\
= \; & +\frac{1}{\sqrt2}\left(\bar\xi\gamma^\nu\gamma^\mu\left[(D^\dagger_\nu D^\dagger_\mu - D^\dagger_\mu D^\dagger_\nu)\phi^\dagger\right]\Psi_L\right) - \frac{1}{\sqrt2}\left(\bar\xi\gamma^\nu\gamma^\mu\left[(D_\mu D_\nu - D_\nu D_\mu)\phi^T\right]\Psi_R\right)
\end{aligned}
$$

The order of spacetime indices in both terms can be understood from the fact that in the first one the commutator of covariant derivatives acts upon the Hermitean adjoint scalar field and therefore to the left, so we have to replace the operators by their Hermitean adjoints and to revert their order. For the right term there is no such effect since we simply have the transposed representation of the gauge group there. Consider all missing terms containing the scalar field or its adjoint together with the spinor field Ψ (i.e. the first and third from (D.10) and the first, third, fifth and seventh in (D.12)), and use the Dirac algebra

$$\eta^{\mu\nu} = \frac12\left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right) \tag{D.21}$$

for the following manipulations (underlined terms cancel after summing up all terms)

$$\begin{aligned}
\sqrt{2}\left(\bar{\xi}(D_\mu\phi)^\dagger D^\mu\Psi_L\right) &= \frac{1}{\sqrt{2}}\left(\bar{\xi}(\underline{\gamma^\mu\gamma^\nu}+\gamma^\nu\gamma^\mu)(D_\mu\phi)^\dagger D_\nu\Psi_L\right)\\
\sqrt{2}\left(\bar{\xi}(D_\mu\phi)^T D^\mu\Psi_R\right) &= \frac{1}{\sqrt{2}}\left(\bar{\xi}(\underline{\gamma^\mu\gamma^\nu}+\gamma^\nu\gamma^\mu)(D_\mu\phi)^T D_\nu\Psi_R\right)\\
-\frac{1}{\sqrt{2}}\left(\bar{\xi}\gamma^\mu(D_\mu\phi)^\dagger \not{D}\Psi_L\right) &= -\frac{1}{\sqrt{2}}\underline{\left(\gamma^\mu\gamma^\nu(D_\mu\phi)^\dagger D_\nu\Psi_L\right)}\\
\frac{1}{\sqrt{2}}\left(\bar{\xi}(\phi^T\overleftarrow{\not{D}}\overleftarrow{\not{D}})\Psi_R\right) &= \frac{1}{\sqrt{2}}\left(\bar{\xi}\gamma^\mu\gamma^\nu(D_\nu D_\mu\phi^T)\Psi_R\right)\\
-\frac{1}{\sqrt{2}}\left(\bar{\xi}\gamma^\mu(D_\mu\phi)^T \not{D}\Psi_R\right) &= -\frac{1}{\sqrt{2}}\underline{\left(\bar{\xi}\gamma^\mu\gamma^\nu(D_\mu\phi)^T D_\nu\Psi_R\right)}\\
\frac{1}{\sqrt{2}}\left(\bar{\xi}(\phi\overleftarrow{D}_\mu)^\dagger\gamma^\mu\overleftarrow{\not{D}}'\Psi_L\right) &= \frac{1}{\sqrt{2}}\left(\bar{\xi}\gamma^\mu\gamma^\nu(D'_\nu(D_\mu\phi)^\dagger)\Psi_L\right)
\end{aligned}$$

After adding the contributions from the field strength tensors we arrive at:

$$\begin{aligned}
&\frac{1}{\sqrt{2}}\bar{\xi}\gamma^\nu\gamma^\mu\Big\{(D_\mu\phi)^\dagger D_\nu\Psi_L+(D_\mu\phi)^T D_\nu\Psi_R+\underline{(D_\mu D_\nu\phi^T)\Psi_R}+(D'_\mu(D_\nu\phi)^\dagger)\Psi_L\\
&\qquad +(D^\dagger_\nu D^\dagger_\mu\phi^\dagger)\Psi_L-(D^\dagger_\mu D^\dagger_\nu\phi^\dagger)\Psi_L-\underline{(D_\mu D_\nu\phi^T)\Psi_R}+(D_\nu D_\mu\phi^T)\Psi_R\Big\}\\
=\;&\frac{1}{\sqrt{2}}\bar{\xi}\gamma^\nu\gamma^\mu\Big\{(D_\mu\phi)^\dagger D_\nu\Psi_L+(D_\mu\phi)^T D_\nu\Psi_R+\underline{(D^\dagger_\mu D^\dagger_\nu\phi^\dagger)\Psi_L}+(D^\dagger_\nu D^\dagger_\mu\phi^\dagger)\Psi_L\\
&\qquad -\underline{(D^\dagger_\mu D^\dagger_\nu\phi^\dagger)\Psi_L}+(D_\nu D_\mu\phi^T)\Psi_R\Big\}\\
=\;&\frac{1}{\sqrt{2}}\partial_\mu\Big\{\left(\bar{\xi}\gamma^\mu\gamma^\nu(D_\nu\phi)^\dagger\Psi_L\right)+\left(\bar{\xi}\gamma^\mu\gamma^\nu(D_\nu\phi)^T\Psi_R\right)\Big\}
\end{aligned}$$

For the second identity it has been used, that the primed covariant derivative originally acting on the righthanded fermion field and hence endowed with a positive sign in front of the gauge field, is identical to the Hermitean adjoint of the "ordinary" covariant derivative (in the fundamental representation). In the last equation all gauge field contributions from the second covariant derivatives in each term cancel, hence the final result is a total derivative. We also relabelled the indices ($\mu\leftrightarrow\nu$).

Altogether, for the de Wit-Freedman transformation of the matter Lagrangean density, we get the total derivative

$$\boxed{\begin{aligned}
\tilde{\delta}_\xi\mathcal{L}_{\rm mat} =& -\frac{\mathrm{i}}{\sqrt{2}}\partial_\mu\left(\bar{\xi}F^\dagger\gamma^\mu\Psi_L\right)-\frac{\mathrm{i}}{\sqrt{2}}\partial_\mu\left(\bar{\xi}F^T\gamma^\mu\Psi_R\right)+\frac{1}{\sqrt{2}}\partial_\mu\left(\bar{\xi}\gamma^\mu\gamma^\nu(D_\nu\phi)^\dagger\Psi_L\right)\\
&+\frac{1}{\sqrt{2}}\partial_\mu\left(\bar{\xi}\gamma^\mu\gamma^\nu(D_\nu\phi)^T\Psi_R\right)-\mathrm{i}g\partial_\mu\left(\bar{\xi}\gamma^\mu\phi^\dagger\vec{T}\phi\cdot\vec{\lambda}\right)+\partial_\mu\bar{\xi}\tilde{K}^\mu(\phi,\Psi,F)
\end{aligned}}\tag{D.22}$$

We now turn to the gauge part of the Lagrangean density, (9.6). The transformations of the various parts are

$$\begin{aligned}
\tilde{\delta}_\xi\left[\frac{\mathrm{i}}{2}\left(\overline{\lambda^a}\gamma^\mu(D_\mu\lambda)^a\right)\right] =& \frac{1}{4}\left(\bar{\xi}\gamma^5\gamma^\nu\gamma^\mu\gamma^\alpha F^a_{\mu\nu}(D_\alpha\lambda)^a\right)-\frac{1}{4}\left(\bar{\xi}\gamma^5\gamma^\nu\gamma^\mu\gamma^\alpha(D_\alpha F_{\mu\nu})^a\lambda^a\right)\\
&+\frac{\mathrm{i}}{2}\left(\bar{\xi}\gamma^\mu D^a(D_\mu\lambda)^a\right)-\frac{\mathrm{i}}{2}\left(\bar{\xi}\gamma^\mu(D_\mu D)^a\lambda^a\right)
\end{aligned}\tag{D.23}$$

$$\tilde{\delta}_\xi \left[-\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a \right] = F^{\mu\nu}_a \partial_\mu \left(\bar{\xi} \gamma_\nu \gamma^5 \lambda^a \right) + g F^{\mu\nu}_a f^a_{\ bc} \left(\bar{\xi} \gamma_\mu \gamma^5 \lambda^b \right) A^c_\nu \tag{D.24}$$

$$\tilde{\delta}_\xi \left[\frac{1}{2} D^a D^a \right] = -\mathrm{i} D^a \left(\bar{\xi} \not{D} \lambda \right)^a \tag{D.25}$$

The term $-(\mathrm{i}g/2) f^a_{\ bc} \left(\overline{\lambda^a} \gamma^\mu \lambda^c \right) \left(\bar{\xi} \gamma_\mu \gamma^5 \lambda^b \right)$ produced by the transformation of the gauge field in the covariant derivative vanishes due to the Fierz identities: The scalar, pseudoscalar and vector parts vanish because the bilinears contracted with the totally antisymmetric structure constants $f^a_{\ bc}$ are symmetric in the gauge group indices (ab) since they are built from Majorana spinors. Note that since we are using $\gamma^5 \lambda^b$ as spinor in the Fierz identity the vector part becomes the axial vector and vice versa. Only the pseudovector is antisymmetric and survives. After relabelling the indices we get

$$f^a_{\ bc} \left(\overline{\lambda^a} \gamma^\mu \lambda^c \right) \left(\bar{\xi} \gamma_\mu \gamma^5 \lambda^b \right) = +\frac{1}{2} f^a_{\ bc} \left(\overline{\lambda^a} \gamma^\mu \lambda^c \right) \left(\bar{\xi} \gamma_\mu \gamma^5 \lambda^b \right) = 0, \tag{D.26}$$

which is seen to vanish after a second Fierz transformation.

The terms containing the auxiliary field D^a (the rightmost from (D.23) and (D.25)) together yield:

$$\begin{aligned} & -\frac{\mathrm{i}}{2} \left(\bar{\xi} D^a (\not{D} \lambda)^a \right) - \frac{\mathrm{i}}{2} \left(\bar{\xi} (\not{D} D)^a \lambda^a \right) \\ = & -\frac{\mathrm{i}}{2} \left(\bar{\xi} D^a (\not{\partial} \lambda^a) \right) - \frac{\mathrm{i}}{2} \left(\bar{\xi} (\not{\partial} D^a) \lambda^a \right) - \frac{\mathrm{i}g}{2} \underline{\left(\bar{\xi} \gamma^\mu D^a f^a_{\ bc} A^b_\mu \lambda^c \right)} - \frac{\mathrm{i}g}{2} \underline{\left(\bar{\xi} \gamma^\mu f^a_{\ bc} A^b_\mu D^c \lambda^a \right)} \\ = & -\frac{\mathrm{i}}{2} \partial_\mu \left(\bar{\xi} \gamma^\mu D^a \lambda^a \right) \end{aligned} \tag{D.27}$$

Underlined terms cancel each other.

To calculate the remaining terms with the field strength tensors we need the following identity for gamma matrices,

$$[\gamma^\mu, \gamma^\nu] \gamma^\rho = -2\eta^{\mu\rho} \gamma^\nu + 2\eta^{\nu\rho} \gamma^\mu - 2\mathrm{i} \epsilon^{\mu\nu\rho\sigma} \gamma_\sigma \gamma^5, \tag{D.28}$$

which can be easily derived by expanding a general 4×4 matrix as a linear combination of the 16 gamma matrices $\mathbb{I}$, γ^5, $\sigma^{\mu\nu}$, γ^μ, $\gamma^\mu \gamma^5$. As the only available Lorentz invariant tensor coefficients are the metric $\eta^{\mu\nu}$ and the Levi-Cività tensor $\epsilon^{\mu\nu\rho\sigma}$, the three leftmost combinations are not possible. Considering the properties under parity transformation shows that only the product of metric and tensor as well as the product of the pseudovector with the Epsilon-tensor are allowed. The explicit prefactors can be calculated by inserting (121) and (123) for $(\mu\nu\rho)$.

With the identity (D.28) we are able to rewrite one of the appearing terms:

$$\begin{aligned} \left(\bar{\xi} \gamma^5 \gamma^\beta \gamma^\alpha \gamma^\mu (D_\mu F_{\alpha\beta})^a \lambda^a \right) &= \frac{1}{2} \left(\bar{\xi} \gamma^5 \left[\gamma^\beta, \gamma^\alpha \right] \gamma^\mu (D_\mu F_{\alpha\beta})^a \lambda^a \right) \\ &= -\left(\bar{\xi} \gamma^5 \gamma^\alpha (D^\beta F_{\alpha\beta})^a \lambda^a \right) + \left(\bar{\xi} \gamma^5 \gamma^\beta (D^\alpha F_{\alpha\beta})^a \lambda^a \right) \\ &\quad - \mathrm{i} \left(\bar{\xi} \gamma^5 \gamma_\sigma \gamma^5 \epsilon^{\sigma\beta\alpha\mu} (D_\mu F_{\alpha\beta})^a \lambda^a \right) \\ &= -2 \left(\bar{\xi} \gamma^5 \gamma^\alpha (D^\beta F_{\alpha\beta})^a \lambda^a \right) \end{aligned} \tag{D.29}$$

The first identity as well as the equality of the first two terms in the middle line hold due to the antisymmetry of the field strength tensor in the spacetime indices. The Bianchi identity of non-Abelian gauge theories causes the third term in the middle line to vanish.

We collect the remaining terms:

$$\begin{aligned}
&\frac{1}{4}\left(\bar{\xi}\gamma^5\gamma^\beta\gamma^\alpha\gamma^\mu F^a_{\alpha\beta}(D_\mu\lambda)^a\right) - \frac{1}{4}\left(\bar{\xi}\gamma^5\gamma^\beta\gamma^\alpha\gamma^\mu(D_\mu F_{\alpha\beta})^a\lambda^a\right) + F^{\mu\nu}_a\partial_\mu\left(\bar{\xi}\gamma_\nu\gamma^5\lambda^a\right)\\
&+ gF^{\mu\nu}_a f^a_{\ bc}\left(\bar{\xi}\gamma_\mu\gamma^5\lambda^b\right)A^c_\nu\\
=\ &\frac{1}{4}\left(\bar{\xi}\gamma^5\gamma^\beta\gamma^\alpha\gamma^\mu F^a_{\alpha\beta}(\partial_\mu\lambda^a)\right) + \underline{\frac{g}{4}\left(\bar{\xi}\gamma^5\gamma^\beta\gamma^\alpha\gamma^\mu F^a_{\alpha\beta}f^a_{\ bc}A^b_\mu\lambda^c\right)}\\
&+ \frac{1}{4}\left(\bar{\xi}\gamma^5\gamma^\beta\gamma^\alpha\gamma^\mu(\partial_\mu F^a_{\alpha\beta})\lambda^a\right) + \underline{\frac{g}{4}\left(\bar{\xi}\gamma^5\gamma^\beta\gamma^\alpha\gamma^\mu f^a_{\ bc}A^b_\mu F^c_{\alpha\beta}\lambda^a\right)}\\
&- \frac{1}{2}\left(\bar{\xi}\gamma^5\gamma^\beta\gamma^\alpha\gamma^\mu(D_\mu F_{\alpha\beta})^a\lambda^a\right) + F^{\mu\nu}_a\partial_\mu\left(\bar{\xi}\gamma_\nu\gamma^5\lambda^a\right) + gF^{\mu\nu}_a f^a_{\ bc}\left(\bar{\xi}\gamma_\mu\gamma^5\lambda^b\right)A^c_\nu\\
\stackrel{(D.28)}{=}\ &\frac{1}{4}\partial_\mu\left(\bar{\xi}\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^5 F^a_{\alpha\beta}\lambda^a\right) + F^{\mu\nu}_a\partial_\mu\left(\bar{\xi}\gamma_\nu\gamma^5\lambda^a\right) + \underline{gF^{\mu\nu}_a f^a_{\ bc}\left(\bar{\xi}\gamma_\mu\gamma^5\lambda^b\right)A^c_\nu}\\
&+ \left(\bar{\xi}(\partial_\beta F^{\beta\alpha}_a)\gamma_\alpha\gamma^5\lambda^a\right) + \underline{gF^{\alpha\beta}_c f^a_{\ bc}\left(\bar{\xi}\gamma^5\gamma_\alpha\lambda^a\right)A^b_\beta}\\
=\ &\frac{1}{4}\partial_\mu\left(\bar{\xi}\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^5 F^a_{\alpha\beta}\lambda^a\right) + \partial_\mu\left(\bar{\xi}F^{\mu\nu}_a\gamma_\nu\gamma^5\lambda^a\right)
\end{aligned} \tag{D.30}$$

The divergence to which the gauge part of the Lagrangean density of an SYM theory is transformed under a de Wit-Freedman transformation finally is:

$$\boxed{\tilde{\delta}_\xi\mathcal{L}_{\text{gauge}} = \partial_\mu\left(\bar{\xi}F^{\mu\nu}_a\gamma_\nu\gamma^5\lambda^a\right) + \frac{1}{4}\partial_\mu\left(\bar{\xi}\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^5 F^a_{\alpha\beta}\lambda^a\right) - \frac{\mathrm{i}}{2}\partial_\mu\left(\bar{\xi}\gamma^\mu D^a\lambda^a\right)} \tag{D.31}$$

The last point is a possible Fayet-Iliopoulos contribution. Its transformation can be written down immediately:

$$\boxed{\tilde{\delta}_\xi\mathcal{L}_{\text{FI}} = -\mathrm{i}\partial_\mu\left(\bar{\xi}\gamma^\mu\zeta^a\lambda^a\right)} \tag{D.32}$$

In the long run we have to construct the current from the contribution of the SUSY transformation of the Lagrangean density and the "Noether" part. The terms produced by the SUSY transformation of the superpotential are identical to those found for theories without gauge symmetry, (D.7). We can think of the problem as having gauged the global symmetry in the models discussed earlier. As the superpotential contains no derivatives of the fields, this contribution to the Lagrangean density is then of course also locally invariant. The difference between the ordinary SUSY transformations and the de Wit–Freedman transformations can be written as a local gauge transformation with special scalar and spinor fields as gauge parameters (cf. [26]). Since the superpotential is invariant under SUSY transformations as well as under gauge transformations and also under the above mentioned special gauge transformations, it is invariant under de Wit–Freedman transformations. As the superpotential contains no derivatives it does not contribute to the current,

$$\tilde{K}^\mu_{\text{local}} = \tilde{K}^\mu_{\text{global}}. \tag{D.33}$$

The Noether part of the supersymmetric current is:

$$\sum_{\text{all fields}}\frac{\partial_R\mathcal{L}}{\partial(\partial_\mu\Lambda)}\tilde{\delta}_\xi\Lambda = -\bar{\xi}N^\mu \tag{D.34}$$

Finally, we have the contributions:

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\tilde{\delta}_\xi\phi = \sqrt{2}(D^\mu\phi)^\dagger\left(\bar{\xi}\Psi_L\right) \tag{D.35}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)}\tilde{\delta}_\xi \phi^\dagger = \sqrt{2}(D^\mu\phi)\left(\bar{\xi}\Psi_R\right) \tag{D.36}$$

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi)}\tilde{\delta}_\xi \Psi = & \frac{1}{\sqrt{2}}\Big(\bar{\xi}\gamma^\nu\gamma^\mu\left((D_\nu\phi)^T\mathcal{P}_R + (D_\nu\phi)^\dagger\mathcal{P}_L\right)\Psi\Big) \\ & - \frac{\mathrm{i}}{\sqrt{2}}\left(\bar{\xi}\gamma^\mu\left(F^T\mathcal{P}_R + F^\dagger\mathcal{P}_L\right)\Psi\right)\end{aligned} \tag{D.37}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \lambda^a)}\tilde{\delta}_\xi \lambda^a = -\frac{1}{4}\left(\bar{\xi}\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^5 F^a_{\alpha\beta}\lambda^a\right) - \frac{\mathrm{i}}{2}\left(\bar{\xi}\gamma^\mu D^a\lambda^a\right) \tag{D.38}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^a_\nu)}\tilde{\delta}_\xi A^a_\nu = F^{\mu\nu}_a\left(\bar{\xi}\gamma_\nu\gamma^5\lambda^a\right) \tag{D.39}$$

Adding these terms to the contributions from the variation of the Lagrangean density (D.22), (D.31), (D.32), we arrive at the supersymmetric current for supersymmetric Yang-Mills theories, which has the form

$$\boxed{\begin{aligned}\mathcal{J}^\mu = & -\sqrt{2}\gamma^\nu\gamma^\mu(D_\nu\phi)^T\Psi_R - \sqrt{2}\gamma^\nu\gamma^\mu(D_\nu\phi)^\dagger\Psi_L - \mathrm{i}\gamma^\mu\zeta^a\lambda^a \\ & + \frac{1}{2}\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^5 F^a_{\alpha\beta}\lambda^a - \mathrm{i}g\gamma^\mu\left(\phi^\dagger\vec{T}\phi\right)\cdot\vec{\lambda} \\ & - \mathrm{i}\sqrt{2}\gamma^\mu\left(\frac{\partial f(\phi)}{\partial\phi}\right)^T\Psi_L - \mathrm{i}\sqrt{2}\gamma^\mu\left(\frac{\partial f(\phi)}{\partial\phi}\right)^\dagger\Psi_R\end{aligned}}\;. \tag{D.40}$$

D.3 Proof of SYM current conservation

In this section we want to check the conservation of the supersymmetric current for super-Yang–Mills theories given in (D.40). For this purpose, we list the equations of motion of all participating fields:

$$(D_\mu D^\mu)\phi = -\sqrt{2}g\overline{\vec{\lambda}}\cdot\vec{T}\Psi_L + g\vec{T}\phi\cdot\vec{D} - \frac{1}{2}\left(\frac{\partial^3 f(\phi)}{\partial\phi^3}\right)^\dagger\left(\overline{\Psi_L}\Psi_R\right) + F^\dagger\left(\frac{\partial^2 f(\phi)}{\partial\phi^2}\right)^\dagger \tag{D.41}$$

$$(D_\mu D^\mu\phi)^\dagger = -\sqrt{2}g\overline{\Psi_L}\vec{T}\cdot\vec{\lambda} + g\phi^\dagger\vec{T}\cdot\vec{D} - \frac{1}{2}\left(\frac{\partial^3 f(\phi)}{\partial\phi^3}\right)\left(\overline{\Psi_R}\Psi_L\right) + F\left(\frac{\partial^2 f(\phi)}{\partial\phi^2}\right) \tag{D.42}$$

$$\mathrm{i}\not{D}\Psi_L = \sqrt{2}g\vec{T}\phi\cdot\vec{\lambda}_R + \left(\frac{\partial^2 f(\phi)}{\partial\phi^2}\right)^\dagger\Psi_R \tag{D.43}$$

$$\mathrm{i}\not{D}\Psi_R = \sqrt{2}g\phi^\dagger\vec{T}\cdot\vec{\lambda}_L + \left(\frac{\partial^2 f(\phi)}{\partial\phi^2}\right)\Psi_L \tag{D.44}$$

$$\mathrm{i}(\not{D}\lambda)^a = \sqrt{2}g\phi^\dagger T^a\Psi_L + \sqrt{2}gT^a\phi\Psi_R \tag{D.45}$$

$$D^a = -\zeta^a - g\left(\phi^\dagger T^a\phi\right) \tag{D.46}$$

$$F = -\left(\frac{\partial f(\phi)}{\partial\phi}\right)^\dagger \tag{D.47}$$

$$F^\dagger = -\left(\frac{\partial f(\phi)}{\partial\phi}\right) \tag{D.48}$$

$$\partial_\mu F^{\mu\nu}_a = gf_{abc}F^{\nu\rho}_c A^b_\rho - \left(\mathrm{i}g\phi^\dagger T^a\right)(D^\nu\phi) + \mathrm{i}g(D^\nu\phi)^\dagger(T^a\phi)$$

$$-\frac{g}{2}\left(\overline{\Psi_L}\gamma^\nu T^a\Psi_L\right)+\frac{g}{2}\left(\overline{\Psi_R}\gamma^\nu T^a\Psi_R\right)-\frac{\mathrm{i}g}{2}\left(\overline{\lambda^b}f_{abc}\gamma^\nu\lambda^c\right) \tag{D.49}$$

The divergence of the current is:

$$\begin{aligned}\partial_\mu\mathcal{J}^\mu = &-\sqrt{2}\gamma^\nu\gamma^\mu\left[\partial_\mu(D_\nu\phi)^T\right]\Psi_R-\sqrt{2}\gamma^\nu\gamma^\mu\left[\partial_\mu(D_\nu\phi)^\dagger\right]\Psi_L-\sqrt{2}\gamma^\nu(D_\nu\phi)^T\not{\partial}\Psi_R\\ &-\sqrt{2}\gamma^\nu(D_\nu\phi)^\dagger\not{\partial}\Psi_L-\mathrm{i}\zeta^a\not{\partial}\lambda^a+\frac{1}{2}\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^5(\partial_\mu F^a_{\alpha\beta})\lambda^a-\frac{1}{2}\gamma^\alpha\gamma^\beta\gamma^5F^a_{\alpha\beta}\not{\partial}\lambda^a\\ &-\mathrm{i}g(\not{\partial}\phi^\dagger)\vec{T}\phi\cdot\vec{\lambda}-\mathrm{i}g\phi^\dagger\vec{T}(\not{\partial}\phi)\cdot\vec{\lambda}-\mathrm{i}g\left(\phi^\dagger\vec{T}\phi\right)\cdot\not{\partial}\vec{\lambda}\\ &-\sqrt{2}\,\mathrm{i}\left(\frac{\partial^2f(\phi)}{\partial\phi^2}\right)^T(\not{\partial}\phi)\Psi_L-\sqrt{2}\,\mathrm{i}\left(\frac{\partial^2f(\phi)}{\partial\phi^2}\right)^\dagger(\not{\partial}\phi^\dagger)\Psi_R\\ &-\sqrt{2}\,\mathrm{i}\left(\frac{\partial f(\phi)}{\partial\phi}\right)^T\not{\partial}\Psi_L-\sqrt{2}\,\mathrm{i}\left(\frac{\partial f(\phi)}{\partial\phi}\right)^\dagger\not{\partial}\Psi_R\end{aligned} \tag{D.50}$$

For the first two terms we calculate:

$$\begin{aligned}\partial_\mu(D_\nu\phi)^T&=\partial_\mu\partial_\nu\phi^T-\mathrm{i}g\vec{T}^T\vec{A}_\nu\partial_\mu\phi^T-\mathrm{i}g\vec{T}^T(\partial_\mu\vec{A}_\nu)\phi^T\\ \partial_\mu(D_\nu\phi)^\dagger&=\partial_\mu\partial_\nu\phi^\dagger+\mathrm{i}g\vec{T}\vec{A}_\nu\partial_\mu\phi^\dagger+\mathrm{i}g\vec{T}(\partial_\mu\vec{A}_\nu)\phi^\dagger\end{aligned}$$

This is split up into a symmetric and an antisymmetric part; note that the first term, which only contains partial derivatives, has no antisymmetric contribution. The symmetrization and antisymmetrization respectively can be transferred to the gamma matrices; using the Dirac algebra, the first two terms of the current's divergence read:

$$\begin{aligned}&-\sqrt{2}\left(\partial^\mu D_\mu\phi\right)^T\Psi_R-\sqrt{2}\left(\partial^\mu D_\mu\phi\right)^\dagger\Psi_L+\frac{\mathrm{i}g}{\sqrt{2}}\left[\gamma^\nu,\gamma^\mu\right]\vec{T}^T\vec{A}_\nu\partial_\mu\phi^T\\ &-\frac{\mathrm{i}g}{\sqrt{2}}\left[\gamma^\nu,\gamma^\mu\right]\vec{T}\vec{A}_\nu\partial_\mu\phi^\dagger+\frac{\mathrm{i}g}{2\sqrt{2}}\left[\gamma^\nu,\gamma^\mu\right]\vec{T}^T(\partial_\mu\vec{A}_\nu-\partial_\nu\vec{A}_\mu)\phi^T\\ &-\frac{\mathrm{i}g}{2\sqrt{2}}\left[\gamma^\nu,\gamma^\mu\right]\vec{T}(\partial_\mu\vec{A}_\nu-\partial_\nu\vec{A}_\mu)\phi^\dagger\end{aligned} \tag{D.51}$$

In the first two terms of (D.51) we complete the squares of the covariant derivatives, insert the equations of motion (D.41) and (D.42) for the scalar field and get

$$\begin{aligned}\mathbf{I}=&+2g\left(\overline{\vec{\lambda}}\cdot\vec{T}^T\Psi_L\right)\Psi_R-\sqrt{2}g\phi^T\vec{T}^T\cdot\vec{D}\Psi_R-\sqrt{2}F^\dagger\left(\frac{\partial^2f(\phi)}{\partial\phi^2}\right)^\dagger\Psi_R\\ &+2g\left(\overline{\Psi_L}\vec{T}\cdot\vec{\lambda}\right)\Psi_L-\sqrt{2}g\phi^\dagger\vec{T}\cdot\vec{D}\Psi_L-\sqrt{2}F^T\left(\frac{\partial^2f(\phi)}{\partial\phi^2}\right)\Psi_L\\ &-\sqrt{2}\,\mathrm{i}g\vec{T}^T\vec{A}^\mu(D_\mu\phi)^T\Psi_R+\sqrt{2}\,\mathrm{i}g\vec{T}\vec{A}^\mu(D_\mu\phi)^\dagger\Psi_L\\ &+\frac{\mathrm{i}g}{\sqrt{2}}\left[\gamma^\nu,\gamma^\mu\right]\vec{T}^T\vec{A}_\nu\partial_\mu\phi^T\Psi_R-\frac{\mathrm{i}g}{\sqrt{2}}\left[\gamma^\nu,\gamma^\mu\right]\vec{T}\vec{A}_\nu\partial_\mu\phi^\dagger\Psi_L\\ &+\frac{\mathrm{i}g}{2\sqrt{2}}\left[\gamma^\nu,\gamma^\mu\right]\vec{T}^T(\partial_\mu\vec{A}_\nu-\partial_\nu\vec{A}_\mu)\phi^T\Psi_R-\frac{\mathrm{i}g}{2\sqrt{2}}\left[\gamma^\nu,\gamma^\mu\right]\vec{T}(\partial_\mu\vec{A}_\nu-\partial_\nu\vec{A}_\mu)\phi^\dagger\Psi_L\end{aligned} \tag{D.52}$$

Here we already used the vanishing of $(\overline{\Psi_L}\Psi_R)\Psi_R$ and $(\overline{\Psi_R}\Psi_L)\Psi_L$ as third powers of a Grassmann odd spinor. In the case of a general (nonrenormalizable) superpotential with more than three matter superfields, this contribution vanishes due to the Schouten identity.

For all terms in (D.50) containing derivatives of the matter fermions (i.e. the third and fourth and the two rightmost), we insert the equations of motion for the fermions (D.43) and (D.44), in which we must bring the gauge field term from the covariant derivative to the right hand side. Furthermore we replace the derivatives of the superpotential with respect to the scalar fields by the equations of motion for the auxiliary fields (D.47) and (D.48). This results in:

$$\begin{aligned}
&+\sqrt{2}\,\mathrm{i}g\gamma^\nu(D_\nu\phi)^T\vec{T}^T\vec{\not{A}}\Psi_R+2\mathrm{i}g\gamma^\nu(D_\nu\phi)^T\phi^\dagger\vec{T}\cdot\vec{\lambda}_L+\sqrt{2}\,\mathrm{i}\gamma^\nu(D_\nu\phi)^T\left(\frac{\partial^2 f(\phi)}{\partial\phi^2}\right)\Psi_L\\
&-\sqrt{2}\,\mathrm{i}g\gamma^\nu(D_\nu\phi)^\dagger\vec{T}\vec{\not{A}}\Psi_L+2\mathrm{i}g\gamma^\nu(D_\nu\phi)^\dagger\vec{T}\phi^T\cdot\vec{\lambda}_R+\sqrt{2}\,\mathrm{i}\gamma^\nu(D_\nu\phi)^\dagger\left(\frac{\partial^2 f(\phi)}{\partial\phi^2}\right)^\dagger\Psi_R\\
&-\sqrt{2}gF^\dagger\vec{T}\vec{\not{A}}\Psi_L+\sqrt{2}gF^\dagger\vec{T}\phi^T\cdot\vec{\lambda}_R+\sqrt{2}F^\dagger\left(\frac{\partial^2 f(\phi)}{\partial\phi^2}\right)^\dagger\Psi_R\\
&+\sqrt{2}gF^T\vec{T}^T\vec{\not{A}}\Psi_R+\sqrt{2}gF^T\phi^\dagger\vec{T}\cdot\vec{\lambda}_L+\sqrt{2}F^T\left(\frac{\partial^2 f(\phi)}{\partial\phi^2}\right)\Psi_L
\end{aligned} \tag{D.53}$$

The two terms $-\sqrt{2}\,\mathrm{i}\left(\partial^2 f(\phi)/\partial\phi^2\right)^T(\not{\partial}\phi)\Psi_L$ and $-\sqrt{2}\,\mathrm{i}\left(\partial^2 f(\phi)/\partial\phi^2\right)^\dagger(\not{\partial}\phi^\dagger)\Psi_R$, i.e. the terms in the fourth line of (D.50), cancel the partial derivatives of the scalar fields from the covariant ones in (D.53) so that only the gauge field terms remain. We yield as a second contribution to the current's divergence:

$$\begin{aligned}
\mathbf{II}=&+\sqrt{2}\,\mathrm{i}g\gamma^\nu(D_\nu\phi)^T\vec{T}^T\vec{\not{A}}\Psi_R+2\mathrm{i}g\gamma^\nu(D_\nu\phi)^T\phi^\dagger\vec{T}\cdot\vec{\lambda}_L+\sqrt{2}g\vec{\not{A}}\vec{T}^T\phi^T\left(\frac{\partial^2 f(\phi)}{\partial\phi^2}\right)\Psi_L\\
&-\sqrt{2}\,\mathrm{i}g\gamma^\nu(D_\nu\phi)^\dagger\vec{T}\vec{\not{A}}\Psi_L+2\mathrm{i}g\gamma^\nu(D_\nu\phi)^\dagger\vec{T}\phi^T\cdot\vec{\lambda}_R-\sqrt{2}g\vec{\not{A}}\vec{T}\phi^\dagger\left(\frac{\partial^2 f(\phi)}{\partial\phi^2}\right)^\dagger\Psi_R\\
&-\sqrt{2}gF^\dagger\vec{T}\vec{\not{A}}\Psi_L+\sqrt{2}gF^\dagger\vec{T}\phi^T\cdot\vec{\lambda}_R+\sqrt{2}F^\dagger\left(\frac{\partial^2 f(\phi)}{\partial\phi^2}\right)^\dagger\Psi_R\\
&+\sqrt{2}gF^T\vec{T}^T\vec{\not{A}}\Psi_R+\sqrt{2}gF^T\phi^\dagger\vec{T}\cdot\vec{\lambda}_L+\sqrt{2}F^T\left(\frac{\partial^2 f(\phi)}{\partial\phi^2}\right)\Psi_L
\end{aligned} \tag{D.54}$$

For the fifth, seventh and tenth term in (D.50) we simply have to insert the gaugino equation of motion (D.45). Again we must transfer the gauge field term to the right hand side, though it cancels in combination with the Fayet–Iliopoulos constant due to the condition (9.7). This immediately gives a third contribution to the divergence:

$$\begin{aligned}
\mathbf{III}=&-\sqrt{2}g\zeta^a\phi^\dagger T^a\Psi_L-\sqrt{2}g\zeta^a T^a\phi^T\Psi_R+\mathrm{i}g^2 f^a_{\ bc}\left(\phi^\dagger T^a\phi\right)\not{A}^b\lambda^c\\
&-\sqrt{2}g^2\left(\phi^\dagger T^a\phi\right)\phi^\dagger T^a\Psi_L-\sqrt{2}g^2\left(\phi^\dagger T^a\phi\right)\phi^T T^a\Psi_R+\frac{g}{2}\gamma^\alpha\gamma^\beta\gamma^5F^a_{\alpha\beta}f^a_{\ bc}\not{A}^b\lambda^c\\
&+\frac{\mathrm{i}g}{\sqrt{2}}\gamma^\alpha\gamma^\beta\gamma^5F^a_{\alpha\beta}\phi^\dagger T^a\Psi_L+\frac{\mathrm{i}g}{\sqrt{2}}\gamma^\alpha\gamma^\beta\gamma^5F^a_{\alpha\beta}\phi^T T^a\Psi_R
\end{aligned} \tag{D.55}$$

At first, we leave the eighth and ninth term from the divergence of the current unchanged:

$$\mathbf{IV}=-\,\mathrm{i}g(\not{\partial}\phi^\dagger)\vec{T}\phi\cdot\vec{\lambda}-\mathrm{i}g\phi^\dagger\vec{T}(\not{\partial}\phi)\cdot\vec{\lambda} \tag{D.56}$$

For further manipulations on the last remaining term with the derivative of the field strength tensor for the gauge field we use again the gamma matrix identity (D.28). This happens in the second step of the following calculation:

$$\begin{aligned}\frac{1}{2}\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^5\left(\partial_\mu F^a_{\alpha\beta}\right)\lambda^a &= \frac{1}{4}\left[\gamma^\alpha,\gamma^\beta\right]\gamma^\mu\gamma^5\left(\partial_\mu F^a_{\alpha\beta}\right)\lambda^a \\ &= \frac{1}{2}\gamma^\alpha\gamma^5\left(\partial^\beta F^a_{\alpha\beta}\right)\lambda^a - \frac{1}{2}\gamma^\beta\gamma^5\left(\partial^\alpha F^a_{\alpha\beta}\right)\lambda^a \\ &\qquad - \frac{\mathrm{i}}{2}\epsilon^{\alpha\beta\mu\sigma}\gamma_\sigma\left(\partial_\mu F^a_{\alpha\beta}\right)\lambda^a \\ &= \gamma^\alpha\gamma^5\left(\partial^\beta F^a_{\alpha\beta}\right)\lambda^a - \frac{\mathrm{i}}{2}\epsilon^{\alpha\beta\mu\sigma}\gamma_\sigma\left(\partial_\mu F^a_{\alpha\beta}\right)\lambda^a\end{aligned}$$

The insertion of the gauge field's equation of motion in the form (D.49) for the first term yields a fifth contribution to the divergence of the current:

$$\begin{aligned}\mathbf{V} = &- g\gamma^\alpha\gamma^5 f^a_{\ bc}F^c_{\alpha\rho}\lambda^a A^\rho_b + \mathrm{i}g\gamma^\alpha\gamma^5\phi^\dagger T^a(D_\alpha\phi)\lambda^a - \mathrm{i}g\gamma^\alpha\gamma^5(D_\alpha\phi)^\dagger T^a\phi\lambda^a \\ &+ \frac{g}{2}\left(\overline{\Psi_L}\gamma_\alpha T^a\Psi_L\right)\gamma^\alpha\gamma^5\lambda^a - \frac{g}{2}\left(\overline{\Psi_R}\gamma_\alpha T^a\Psi_R\right)\gamma^\alpha\gamma^5\lambda^a + \frac{\mathrm{i}g}{2}\left(\overline{\lambda^b}f_{abc}\gamma_\alpha\lambda^c\right)\gamma^\alpha\gamma^5\lambda^a \\ &- \frac{\mathrm{i}}{2}\epsilon^{\sigma\alpha\beta\mu}\gamma_\sigma\left(\partial_\mu F^a_{\alpha\beta}\right)\lambda^a\end{aligned} \tag{D.57}$$

As a next step we use the equation of motion for the auxiliary field D^a, (D.46), to see that the second and fifth term of (D.52) cancel the first, the second, the fourth and the fifth term from (D.55).

With the help of relation (D.26) and the discussion above this equation, we can set the term with three gauginos, the penultimate in (D.57), equal to zero.

Now we consider the contributions with two matter fermions and one gaugino. These are the first and fourth term in (D.52) as well as the fourth and fifth from (D.57). By the Fierz identity these four terms add up to zero in the same manner as discussed below (D.16).

We may rewrite the last term of (D.57) as follows

$$-\frac{\mathrm{i}}{2}\epsilon^{\alpha\beta\mu\sigma}\gamma_\sigma\left(\partial_\mu F^a_{\alpha\beta}\right)\lambda^a = +\frac{\mathrm{i}g}{2}\epsilon^{\alpha\beta\mu\sigma}\gamma_\sigma f^a_{\ bc}A^b_\mu F^c_{\alpha\beta}\lambda^a, \tag{D.58}$$

as the antisymmetrized covariant derivative of the field strength tensor vanishes due to the Bianchi identity. We manipulate the sixth term from (D.55) with help of the identity (D.28):

$$-\frac{g}{4}\left[\gamma^\alpha,\gamma^\beta\right]\gamma^\mu\gamma^5 F^a_{\alpha\beta}f^a_{\ bc}A^b_\mu\lambda^c = g\gamma^\beta\gamma^5 F^a_{\alpha\beta}f^a_{\ bc}A^\alpha_b\lambda^c + \frac{\mathrm{i}g}{2}\gamma_\sigma\epsilon^{\alpha\beta\mu\sigma}F^a_{\alpha\beta}f^a_{\ bc}A^b_\mu\lambda^c$$

It is obvious that this cancels the term in the last equation as well as the first term of (D.57).

It is also clear that the sum of the third and sixth term out of (D.52) and the ninth and last one from (D.54) is zero.

To rewrite the third and sixth term in (D.54) we remember that the superpotential in an SYM theory must be a gauge invariant function of the fields. Hence it is constrained by the condition

$$\left(\frac{\partial f(\phi)}{\partial\phi}\right)T^a\phi = 0 \tag{D.59}$$

Differentiating with respect to ϕ and using the equation of motion for F yields the relation (similarly for $\phi^\dagger$):

$$\left(\frac{\partial^2 f(\phi)}{\partial\phi^2}\right) T^a\phi = -\left(\frac{\partial f(\phi)}{\partial\phi}\right) T^a = F^\dagger T^a \tag{D.60}$$

$$\left(\frac{\partial^2 f(\phi)}{\partial\phi^2}\right)^\dagger T^a\phi^\dagger = -\left(\frac{\partial f(\phi)}{\partial\phi}\right)^\dagger T^a = F T^a \tag{D.61}$$

Now we see that these two terms cancel against the seventh and tenth of (D.54).

By (D.59) the eighth and penultimate term out of (D.54) vanish, too.

We want to inspect the terms containing derivatives of the scalar fields now. These are the first, second, fourth and fifth term from (D.54), the two remaining ones out of (D.57) – the second and the third – as well as the seventh up to the tenth term of (D.52). Consider first all of these terms containing gauginos. We calculate:

$$\begin{aligned}
&2\mathrm{i}g\gamma^\nu(D_\nu\phi)^T\phi^\dagger\vec{T}\cdot\vec{\lambda}_L - \mathrm{i}g\phi^\dagger\vec{T}(\not{\partial}\phi^T)\cdot\vec{\lambda} + \mathrm{i}g\gamma^\nu\gamma^5\phi^\dagger\vec{T}(D_\nu\phi)^T\vec{\lambda}\\
=\;&\underline{2\mathrm{i}g(\not{\partial}\phi^T)\phi^\dagger\vec{T}\cdot\vec{\lambda}_L} + 2g^2\gamma^\nu\phi^\dagger T^b T^a A^b_\nu\phi^T\lambda^a_L + \underline{\mathrm{i}g(\not{\partial}\phi^T)\phi^\dagger\vec{T}\cdot\gamma^5\vec{\lambda}}\\
&+ g^2\gamma^\nu\phi^\dagger T^b T^a A^b_\nu\phi^T\gamma^5\lambda^a - \underline{\mathrm{i}g\phi^\dagger\vec{T}(\not{\partial}\phi^T)\cdot\vec{\lambda}}\\
=\;&g^2\gamma^\nu\phi^\dagger T^a T^b A^b_\nu\phi^T\lambda^a
\end{aligned}$$

The underlined terms with the spacetime derivatives all vanish, solely the gauge field parts are left. An analogous calculation works for the complex conjugated scalar fields:

$$\begin{aligned}
&2\mathrm{i}g\gamma^\nu(D_\nu\phi)^\dagger\vec{T}\phi^T\cdot\vec{\lambda}_R - \mathrm{i}g(\not{\partial}\phi^\dagger)\vec{T}\phi^T\cdot\vec{\lambda} - \mathrm{i}g\gamma^\nu\gamma^5(D_\nu\phi)^\dagger\vec{T}\phi^T\vec{\lambda}\\
=\;&-g^2\gamma^\nu\phi^\dagger T^b T^a\phi^T A^b_\nu\lambda^a
\end{aligned}$$

Summing up the last two resulting expressions yields

$$g^2\gamma^\nu\phi^\dagger\left[T^a,T^b\right]\phi^T A^b_\nu\lambda^a = \mathrm{i}g^2\gamma^\nu\phi^\dagger f_{abc}T^c\phi^T A^b_\nu\lambda^a \tag{D.62}$$

and hence cancel the third term of (D.55).

The first and fourth term of (D.54) can be rewritten with the help of the Dirac algebra:

$$\begin{aligned}
\sqrt{2}\,\mathrm{i}g\gamma^\nu\gamma^\mu(D_\nu\phi)^T\vec{T}^T\vec{A}_\mu\Psi_R &= \sqrt{2}\,\mathrm{i}g\Big(\frac{1}{2}\{\gamma^\nu,\gamma^\mu\}+\frac{1}{2}[\gamma^\nu,\gamma^\mu]\Big)(D_\nu\phi)^T\vec{T}^T\vec{A}_\mu\Psi_R\\
&= +\sqrt{2}\,\mathrm{i}g\vec{T}^T\vec{A}^\mu(D_\mu\phi)^T\Psi_R\\
&\quad+\frac{\mathrm{i}g}{\sqrt{2}}[\gamma^\nu,\gamma^\mu](\partial_\nu\phi)^T\vec{T}^T\vec{A}_\mu\Psi_R\\
&\quad+\frac{g^2}{\sqrt{2}}[\gamma^\nu,\gamma^\mu]A^a_\nu T^b T^a A^b_\mu\phi^T\Psi_R
\end{aligned}$$

Of the newly established terms the first cancels the seventh term in (D.52) while the second one eliminates the ninth from (D.52). For the complex conjugated fields we get analogously

$$\begin{aligned}
-\sqrt{2}\,\mathrm{i}g\gamma^\nu\gamma^\mu(D_\nu\phi)^\dagger\vec{T}\vec{A}_\mu\Psi_L &= -\sqrt{2}\,\mathrm{i}g\vec{T}\vec{A}^\mu(D_\mu\phi)^\dagger\Psi_L\\
&\quad-\frac{\mathrm{i}g}{\sqrt{2}}[\gamma^\nu,\gamma^\mu](\partial_\nu\phi^\dagger)\vec{T}\vec{A}_\mu\Psi_L\\
&\quad+\frac{g^2}{\sqrt{2}}\phi^\dagger[\gamma^\nu,\gamma^\mu]A^a_\nu T^a T^b A^b_\mu\Psi_L
\end{aligned}$$

A similar cancellation takes place here: the first term cancels the eighth of (D.52), the second one the tenth out of (D.52). The third terms from the last and the penultimate identity are further manipulated with taking attention to their antisymmetrization:

$$\begin{aligned}\frac{g^2}{\sqrt{2}}\left[\gamma^\nu,\gamma^\mu\right]A^a_\nu A^b_\mu T^b T^a \phi^T \Psi_R &= \frac{g^2}{2\sqrt{2}}\left[\gamma^\nu,\gamma^\mu\right]\left(A^a_\nu A^b_\mu - A^a_\mu A^b_\nu\right)T^b T^a \phi^T \Psi_R \\ &= \frac{g^2}{2\sqrt{2}}\left[\gamma^\nu,\gamma^\mu\right]A^a_\nu A^b_\mu \left[T^b,T^a\right]\phi^T\Psi_R \\ &= \frac{\mathrm{i}g^2}{2\sqrt{2}}\left[\gamma^\nu,\gamma^\mu\right]A^a_\nu A^b_\mu f_{bac} T^c \phi^T \Psi_R\end{aligned} \tag{D.63}$$

Analogously:

$$\frac{g^2}{\sqrt{2}}\phi^\dagger\left[\gamma^\nu,\gamma^\mu\right]A^a_\nu T^a T^b A^b_\mu \Psi_L = \frac{\mathrm{i}g^2}{2\sqrt{2}}\left[\gamma^\nu,\gamma^\mu\right]\phi^\dagger A^a_\nu A^b_\mu f_{abc} T^c \Psi_L \tag{D.64}$$

Together with the last two terms from (D.52) this yields

$$+\frac{\mathrm{i}g}{2\sqrt{2}}\left[\gamma^\nu,\gamma^\mu\right]T^a F^a_{\mu\nu}\phi^T\Psi_R - \frac{\mathrm{i}g}{2\sqrt{2}}\left[\gamma^\nu,\gamma^\mu\right]T^a F^a_{\mu\nu}\phi^\dagger\Psi_L \quad , \tag{D.65}$$

which cancels the last remaining terms, the two rightmost ones in (D.55), so after all we get the desired result

$$\boxed{\partial_\mu \mathcal{J}^\mu = 0.} \tag{D.66}$$

Appendix E

Summary of models

E.1 The Wess-Zumino model

Action:

$$S_{WZ} = \int d^4x \left\{ \frac{1}{2} \left[\hat{\Phi}^\dagger \hat{\Phi} \right]_D + \left[\mu \hat{\Phi} + \frac{m}{2} \hat{\Phi}^2 + \frac{\lambda}{3!} \hat{\Phi}^3 + \text{h.c.} \right]_F \right\} \tag{E.1}$$

As is shown in [1], the contribution to the superpotential linear in the superfields can always be eliminated by a redefinition of the superfields. So in the sequel we set $\mu \equiv 0$.

The quadratic and cubic part of the superpotential yield:

$$\frac{m}{2} \left[\hat{\Phi}^2 + \text{h.c.} \right]_F = m\phi F + m\phi^* F^* - \frac{m}{2} \left(\psi\psi + \bar{\psi}\bar{\psi} \right), \tag{E.2}$$

$$\frac{\lambda}{3!} \left[\hat{\Phi}^3 + \text{h.c.} \right]_F = \frac{\lambda}{2} \phi^2 F + \frac{\lambda}{2} (\phi^*)^2 F^* - \frac{\lambda}{2} \phi\psi\psi - \frac{\lambda}{2} \phi^* \bar{\psi}\bar{\psi} \tag{E.3}$$

The kinetic part is:

$$\frac{1}{2} \left[\hat{\Phi}^\dagger \hat{\Phi} \right]_D = (\partial_\mu \phi)^* (\partial^\mu \phi) + \frac{\mathrm{i}}{2} \bar{\psi} \bar{\sigma}^\mu \partial_\mu \psi + \frac{\mathrm{i}}{2} \psi \sigma^\mu \partial_\mu \bar{\psi} + |F|^2 \tag{E.4}$$

The equation of motion for the auxiliary field is:

$$F^* = -m\phi - \frac{\lambda}{2} \phi^2 \quad . \tag{E.5}$$

After introducing the bispinor notation

$$\Psi = \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} \tag{E.6}$$

for the fermionic degrees of freedom (the Yukawa coupling terms will soon be brought into bispinor form) we get the Lagrangean density

$$\mathcal{L}_{WZ} = (\partial_\mu \phi)^* (\partial^\mu \phi) + \frac{1}{2} \overline{\Psi} \left(\mathrm{i}\partial\!\!\!/ - m \right) \Psi - \frac{\lambda}{2} \phi\psi\psi - \frac{\lambda}{2} \phi^* \bar{\psi}\bar{\psi} - |F|^2 \tag{E.7}$$

The scalar part of the superpotential reads

$$|F|^2 = m^2 |\phi|^2 + \frac{\lambda^2}{4} \left(|\phi|^2 \right)^2 + \frac{1}{2} m\lambda |\phi|^2 (\phi + \phi^*) \tag{E.8}$$

After splitting the complex scalar field into real and imaginary part (cf. chapter 2),

$$\phi = \frac{1}{\sqrt{2}} (A + \mathrm{i}B),$$

it is easier to write the Yukawa interactions in the bispinor language.

$$\begin{aligned} \phi\psi\psi + \phi^* \bar{\psi}\bar{\psi} &= \frac{1}{2\sqrt{2}} \overline{\Psi} \left(1 - \gamma^5 \right) \Psi \ (A + \mathrm{i}B) + \frac{1}{2\sqrt{2}} \overline{\Psi} \left(1 + \gamma^5 \right) \Psi \ (A - \mathrm{i}B) \\ &= \frac{1}{\sqrt{2}} \overline{\Psi} \Psi A - \frac{\mathrm{i}}{\sqrt{2}} \overline{\Psi} \gamma^5 \Psi B \quad . \end{aligned} \tag{E.9}$$

Finally, the whole Lagrangean density for the Wess-Zumino model reads:

$$\begin{aligned} \mathcal{L}_{WZ} = & \frac{1}{2} \left(\partial_\mu A \partial^\mu A - m^2 A^2 \right) + \frac{1}{2} \left(\partial_\mu B \partial^\mu B - m^2 B^2 \right) + \frac{1}{2} \overline{\Psi} \left(\mathrm{i}\partial\!\!\!/ - m \right) \Psi \\ & - \frac{\lambda}{2\sqrt{2}} \overline{\Psi} \Psi A + \frac{\mathrm{i}\lambda}{2\sqrt{2}} \overline{\Psi} \gamma^5 \Psi B - \frac{\lambda^2}{16} A^4 - \frac{\lambda^2}{16} B^4 - \frac{\lambda^2}{8} A^2 B^2 \\ & - \frac{1}{2\sqrt{2}} m\lambda A^3 - \frac{1}{2\sqrt{2}} m\lambda A B^2 \end{aligned} \tag{E.10}$$

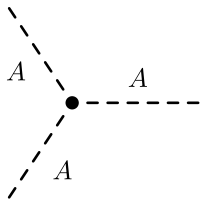
We also state the SUSY transformations for the component fields:

$$[\mathrm{i}\bar{\xi}Q, A] = \left(\bar{\xi}\Psi\right) \tag{E.11a}$$

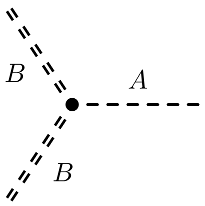
$$[\mathrm{i}\bar{\xi}Q, B] = \left(\mathrm{i}\bar{\xi}\gamma^5\Psi\right) \tag{E.11b}$$

$$[\mathrm{i}\bar{\xi}Q, \Psi] = -\left(\mathrm{i}\partial\!\!\!/ + m\right)(A + \mathrm{i}\gamma^5 B)\xi - \frac{\lambda}{2\sqrt{2}}\left(A^2 - B^2\right)\xi - \frac{\mathrm{i}\lambda}{\sqrt{2}}AB\gamma^5\xi \tag{E.11c}$$

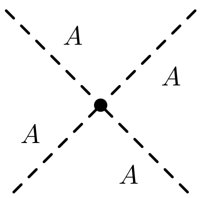
The vertices of the WZ model:



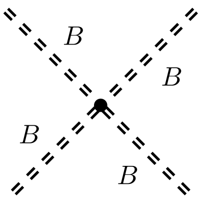
$$-\mathrm{i}\frac{3}{\sqrt{2}}m\lambda$$



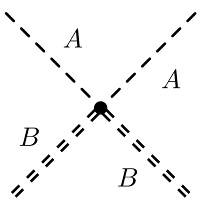
$$-\mathrm{i}\frac{1}{\sqrt{2}}m\lambda$$



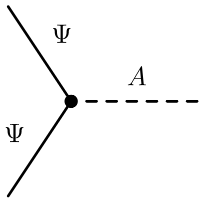
$$-\frac{3}{2}\mathrm{i}\lambda^2$$



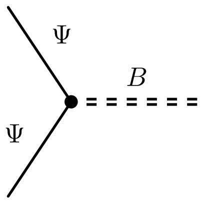
$$-\frac{3}{2}\mathrm{i}\lambda^2$$



$$-\frac{1}{2}\mathrm{i}\lambda^2$$



$$-\frac{\mathrm{i}\lambda}{\sqrt{2}}$$



$$-\frac{\lambda}{\sqrt{2}} \cdot \gamma^5$$

E.2 A toy model

Action:

$$\mathcal{S}_{\rm toy} = \int d^4x \left\{ \frac{1}{2}\left[\hat{\Phi}_1^\dagger \hat{\Phi}_1\right]_D + \frac{1}{2}\left[\hat{\Phi}_2^\dagger \hat{\Phi}_2\right]_D + \left[m\hat{\Phi}_1\hat{\Phi}_2 + g\hat{\Phi}_1\hat{\Phi}_1\hat{\Phi}_2 + \text{h.c.}\right]_F \right\} \tag{E.12}$$

Equations of motion for the auxiliary fields:

$$F_1^* = -m\phi_2 - 2g\phi_1\phi_2, \qquad F_2^* = -m\phi_1 - g\phi_1^2 \tag{E.13}$$

Lagrangean density in components,

$$\begin{aligned}\mathcal{L}_{\rm toy} = \;& (\partial_\mu \phi_1^*)(\partial^\mu \phi_1) + (\partial_\mu \phi_2^*)(\partial^\mu \phi_2) + \mathrm{i}\partial_\mu \bar{\psi}_1 \bar{\sigma}^\mu \psi_1 \\ & + \mathrm{i}\psi_2 \sigma^\mu \partial_\mu \bar{\psi}_2 + \mathrm{i}\partial_\mu \bar{\psi}_1 \bar{\sigma}^\mu \psi_1 + \mathrm{i}\psi_2 \sigma^\mu \partial_\mu \bar{\psi}_2 \\ & - |F_1|^2 - |F_2|^2 - \Big(m\psi_1\psi_2 + g\psi_1\psi_1\phi_2 + 2g\psi_1\psi_2\phi_1 + \text{h.c.}\Big).\end{aligned} \tag{E.14}$$

Superpotential:

$$\begin{aligned}|F|^2 = & \; m^2|\phi_1|^2 + m^2|\phi_2|^2 + mg|\phi_1|^2(\phi_1 + \phi_1^*) \\ & + 2mg|\phi_2|^2(\phi_1+\phi_1^*) + 4g^2|\phi_1|^2|\phi_2|^2 + g^2\left(|\phi_1|^2\right)^2\end{aligned} \tag{E.15}$$

The structure of the scalar interaction terms makes the following redefinitions reasonable:

$$\begin{aligned}\phi_1 &\longrightarrow \frac{1}{\sqrt{2}}(A + \mathrm{i}B) \\ \phi_1^* &\longrightarrow \frac{1}{\sqrt{2}}(A - \mathrm{i}B) \\ \phi_2 &\longrightarrow \phi \\ \phi_2^* &\longrightarrow \phi^*\end{aligned} \tag{E.16}$$

To diagonalize the mass terms of the fermions, we introduce a Dirac bispinor

$$\Psi \equiv \begin{pmatrix} \psi_1 \\ \bar{\psi}_2 \end{pmatrix} \quad . \tag{E.17}$$

So our Lagrangean density now looks like

$$\mathcal{L}_{\rm toy} = \mathcal{L}_{\rm kin} + \mathcal{L}_{\rm pot} + \mathcal{L}_{\rm Yukawa} \qquad , \tag{E.18}$$

with

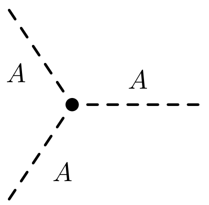
$$\begin{aligned}\mathcal{L}_{\rm kin} = & \;\frac{1}{2}\Big[(\partial_\mu A)(\partial^\mu A) - m^2A^2\Big] + \frac{1}{2}\Big[(\partial_\mu B)(\partial^\mu B) - m^2 B^2\Big] \\ & + (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2|\phi|^2 + \overline{\Psi}\,(\mathrm{i}\partial\!\!\!/ - m)\,\Psi\end{aligned} \tag{E.19}$$

$$\begin{aligned}\mathcal{L}_{\rm pot} = & -\frac{1}{\sqrt{2}} mg\left(A^2+B^2\right)A - 2\sqrt{2}mg|\phi|^2 A - 2g^2|\phi|^2\left(A^2+B^2\right) \\ & - \frac{g^2}{4}\left(A^4 + 2A^2B^2 + B^4\right)\end{aligned} \tag{E.20}$$

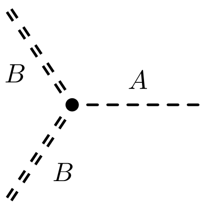
$$\mathcal{L}_{\text{Yukawa}} = -\sqrt{2}g\overline{\Psi}\Psi A + \sqrt{2}\mathrm{i}g\overline{\Psi}\gamma^5\Psi B - g\overline{\Psi^c}\mathcal{P}_L\Psi\phi - g\overline{\Psi}\mathcal{P}_R\Psi^c\phi^* \tag{E.21}$$

The Feynman rules seem to be obvious, but there are some delicacies, so we write the vertices down in a graphical notation as in the Wess-Zumino model. The point that is (in the bispinor formalism (E.14)) easily overlooked is the crucial symmetry factor two stemming from the Weyl spinors ψ and $\bar{\psi}$ appearing quadratically in the couplings to the scalar ϕ, which can be seen by applying functional derivatives for deriving the Feynman rules.

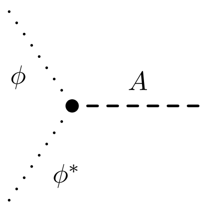
Feynman rules of the toy model:



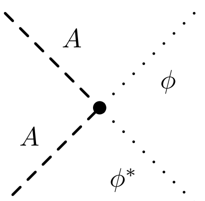
$-3\sqrt{2}\mathrm{i}mg$



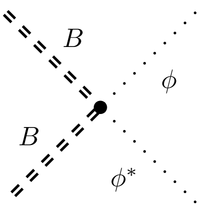
$-\sqrt{2}\mathrm{i}mg$



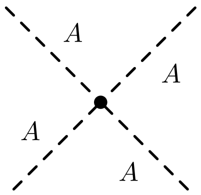
$-2\sqrt{2}\mathrm{i}mg$



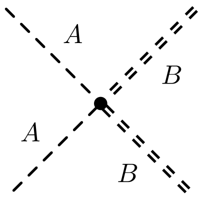
$-4\mathrm{i}g^2$



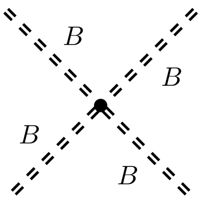
$-4\mathrm{i}g^2$



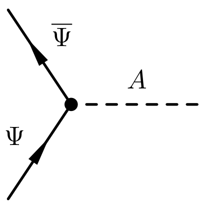
$-6\mathrm{i}g^2$



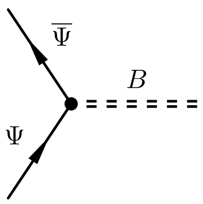
$-2\mathrm{i}g^2$



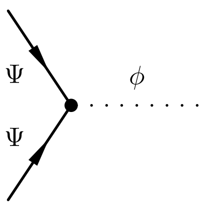
$-6\mathrm{i}g^2$



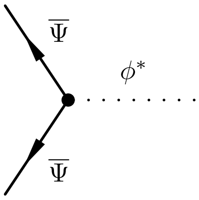
$-\sqrt{2}\mathrm{i}g$



$-\sqrt{2}g\gamma^5$



$-2\mathrm{i}g\mathcal{P}_L$



$-2\mathrm{i}g\mathcal{P}_R$

For the last two vertices compare the remark at the end of the next section.

E.3 The O'Raifeartaigh model

Action:

$$S_{OR} = \int d^4x \left\{ \frac{1}{2} \sum_{i=1}^{3} \left[\hat{\Phi}_i^\dagger \hat{\Phi}_i \right]_D + \left[\lambda \hat{\Phi}_1 + m \hat{\Phi}_2 \hat{\Phi}_3 + g \hat{\Phi}_1 \hat{\Phi}_2 \hat{\Phi}_2 + \text{h.c.} \right]_F \right\} \tag{E.22}$$

The equations of motion for the auxiliary fields:

$$F_1^* = -\lambda - g\phi_2^2, \tag{E.23}$$

$$F_2^* = -m\phi_3 - 2g\phi_1\phi_2, \tag{E.24}$$

$$F_3^* = -m\phi_2. \tag{E.25}$$

The kinetic part produces the massless kinetic terms for the fields $\phi_i, \psi_i, i = 1, 2, 3$ and their complex conjugates. Again the superpotential (after inserting the equations of motion for the auxiliary fields) consists of $-|F|^2$ and the Yukawa couplings. To diagonalize the mass terms of the fermions, we introduce the Dirac bispinor

$$\Psi = \begin{pmatrix} \psi_2 \\ \bar{\psi}_3 \end{pmatrix} \text{ with } \mathrm{i}\psi_3\sigma^\mu\partial_\mu\bar{\psi}_3 + \mathrm{i}\psi_2\sigma^\mu\partial_\mu\bar{\psi}_3 - m\left(\psi_2\psi_3 + \bar{\psi}_2\bar{\psi}_3\right) = \overline{\Psi}\left(\mathrm{i}\partial\!\!\!/ - m\right)\Psi. \tag{E.26}$$

The spinor from the first superfield remains massless and can be extended to a Majorana spinor field χ. The Lagrangean density then reads

$$\begin{aligned} \mathcal{L}_{OR} = \overline{\Psi}\left(\mathrm{i}\partial\!\!\!/ - m\right)\Psi + \frac{\mathrm{i}}{2}\overline{\chi}\partial\!\!\!/\chi + \sum_{i=1}^{3}\left(\partial_\mu\phi_i^*\right)\left(\partial^\mu\phi_i\right) - \sum_{i=1}^{3}|F_i|^2 \\ - 2g\left(\psi_1\psi_2\phi_2 + \bar{\psi}_1\bar{\psi}_2\phi_2^*\right) - g\left(\psi_2\psi_2\phi_1 + \bar{\psi}_2\bar{\psi}_2\phi_1^*\right) \end{aligned} \tag{E.27}$$

In the next step we calculate the superpotential:

$$\begin{aligned} \mathcal{W} &= |F_1|^2 + |F_2|^2 + |F_3|^2 \\ &= \left|\lambda + g\phi_2^2\right|^2 + |m\phi_3 + 2g\phi_1\phi_2|^2 + m^2|\phi_2|^2 \\ &= \lambda^2 + \lambda g\left(\phi_2^2 + (\phi_2^*)^2\right) + g^2\left(|\phi_2|^2\right)^2 + m^2|\phi_3|^2 + 2gm\left(\phi_1\phi_2\phi_3^* + \phi_1^*\phi_2^*\phi_3\right) \\ &\qquad + 4g^2|\phi_1|^2|\phi_2|^2 + m^2|\phi_2|^2 \end{aligned} \tag{E.28}$$

As for the WZ model the complex scalar field ϕ_2 is split up into real and imaginary part , $\phi_2 = \frac{1}{\sqrt{2}}(A + \mathrm{i}B)$, which changes its kinetic parts to

$$(\partial_\mu\phi_2^*)(\partial^\mu\phi_2) = \frac{1}{2}(\partial_\mu A)(\partial^\mu A) + \frac{1}{2}(\partial_\mu B)(\partial^\mu B). \tag{E.29}$$

The terms quadratic in the scalar fields in the superpotential yield:

$$\begin{aligned} &\lambda g\left(\phi_2^2 + (\phi_2^*)^2\right) + m^2|\phi_3|^2 + m^2|\phi_2|^2 \\ &\qquad = \frac{1}{2}\left(m^2 + 2\lambda g\right)A^2 + \frac{1}{2}\left(m^2 - 2\lambda g\right)B^2 + m^2|\phi_3|^2 \end{aligned} \tag{E.30}$$

Altogether, the scalar part consists of a complex scalar field with zero mass, ϕ_1, a complex scalar field with mass m, ϕ_3, as well as two real (scalar and pseudoscalar) fields,A and B, with masses $\sqrt{m^2+2\lambda g}$ and $\sqrt{m^2-2\lambda g}$. In the following we rename the field ϕ_3 by ϕ and substitute the notation Φ for the massive field ϕ_3. The final form of the kinetic part then looks like:

$$\begin{aligned}\mathcal{L}_{\text{kin}} = \overline{\Psi}\left(\mathrm{i}\partial\!\!\!/ - m\right)\Psi + \frac{\mathrm{i}}{2}\overline{\chi}\partial\!\!\!/\chi + \frac{1}{2}\left[(\partial_\mu A)(\partial^\mu A) - (m^2+2\lambda g)A^2\right] \\ + \frac{1}{2}\left[(\partial_\mu B)(\partial^\mu B) - (m^2-2\lambda g)B^2\right] + (\partial_\mu \phi^*)(\partial^\mu \phi) \\ + (\partial_\mu \Phi^*)(\partial^\mu \Phi) - m^2 |\Phi|^2 \end{aligned} \tag{E.31}$$

The remaining part of the scalar potential rewritten with the physical fields:

$$\begin{aligned}\mathcal{L}_{\text{scalar}} = -\lambda^2 - \frac{g^2}{4}\left(A^4+2A^2B^2+B^4\right) - 2g^2|\phi|^2\left(A^2+B^2\right) \\ - \sqrt{2}gmA\left(\phi\Phi^*+\phi^*\Phi\right) - \sqrt{2}\mathrm{i}gmB\left(\phi\Phi^*-\phi^*\Phi\right)\end{aligned} \tag{E.32}$$

The term $-\lambda^2$ provides a contribution to the cosmological constant.

Finally we have to transform the Yukawa couplings; again we meet the complication of vertices with fermions whose arrows both point to the vertex or both away from it. This, as mentioned for our toy model, can be handled within the general formalism of [19]. The Yukawa terms have the structure:

$$\begin{aligned}\mathcal{L}_{\text{Yukawa}} = -\frac{g}{\sqrt{2}}A\Big(\overline{\chi}(1-\gamma^5)\Psi + \overline{\Psi}(1+\gamma^5)\chi\Big) \\ - \frac{\mathrm{i}g}{\sqrt{2}}B\Big(\overline{\chi}(1-\gamma^5)\Psi - \overline{\Psi}(1+\gamma^5)\chi\Big) \\ - \frac{g}{2}\overline{\Psi^c}(1-\gamma^5)\Psi\phi - \frac{g}{2}\overline{\Psi}(1+\gamma^5)\Psi^c\phi^*\end{aligned} \tag{E.33}$$

Particle	Mass	Description
A	$\sqrt{m^2+2\lambda g}$	neutral scalar
B	$\sqrt{m^2-2\lambda g}$	neutral pseudoscalar
χ	0	neutral fermion (Goldstino)
$\Phi^{(*)}$	m	charged scalar
$\Psi, \overline{\Psi}$	m	charged fermion
$\phi^{(*)}$	0	charged scalar

Table E.1: *Physical particle content of the OR model.*

For further discussion of the OR model in the text we write down the equations of motion and the SUSY transformations of all fields:

$$
\begin{aligned}
(\mathrm{i}\partial\!\!\!/ - m)\,\Psi &= \sqrt{2}g\,(A-\mathrm{i}B)\,\mathcal{P}_R\chi + 2g\phi^*\mathcal{P}_R\Psi^c \\
\mathrm{i}\partial\!\!\!/\chi &= \sqrt{2}g\,(A+\mathrm{i}B)\,\mathcal{P}_L\Psi + \sqrt{2}g\,(A-\mathrm{i}B)\,\Psi^c \\
\left(\Box + m^2 + 2\lambda g\right) A &= -\sqrt{2}g\,(\overline{\chi}\mathcal{P}_L\Psi) - \sqrt{2}g\,\left(\overline{\Psi}\mathcal{P}_R\chi\right) \\
&\quad - g^2A^3 - g^2AB^2 - 4g^2A\,|\phi|^2 - \sqrt{2}gm\,(\phi\Phi^* + \phi^*\Phi) \\
\left(\Box + m^2 - 2\lambda g\right) B &= -\sqrt{2}\mathrm{i}g\,(\overline{\chi}\mathcal{P}_L\Psi) + \sqrt{2}\mathrm{i}g\,\left(\overline{\Psi}\mathcal{P}_R\chi\right) \\
&\quad - g^2B^3 - g^2A^2B - 4g^2B\,|\phi|^2 - \sqrt{2}\mathrm{i}gm\,(\phi\Phi^* - \phi^*\Phi) \\
\Box\,\phi &= -\sqrt{2}gm\,(A-\mathrm{i}B)\,\Phi - g\left(\overline{\Psi}\mathcal{P}_R\Psi^c\right) - 2g^2\phi\left(A^2+B^2\right) \\
\left(\Box + m^2\right)\Phi &= -\sqrt{2}gm\,(A+\mathrm{i}B)\,\phi
\end{aligned}
\tag{E.34}
$$

The missing equations can be obtained by complex or by charge conjugation. For the fermionic fields the SUSY transformation laws are

$$\delta_\xi\chi = -\sqrt{2}\mathrm{i}\partial\!\!\!/\,(\phi\mathcal{P}_R + \phi^*\mathcal{P}_L)\,\xi - \sqrt{2}\lambda\xi - \frac{g}{\sqrt{2}}\left(A^2 - B^2\right)\xi - \sqrt{2}\mathrm{i}gAB\gamma^5\xi \tag{E.35}$$

$$\delta_\xi\Psi = -(\mathrm{i}\partial\!\!\!/ + m)\,(A+\mathrm{i}B)\,\mathcal{P}_R\xi - \sqrt{2}\,(\mathrm{i}\partial\!\!\!/ + m)\,\Phi^*\mathcal{P}_L\xi - 2g\phi^*\,(A-\mathrm{i}B)\,\mathcal{P}_L\xi \tag{E.36}$$

$$\delta_\xi\Psi^c = -(\mathrm{i}\partial\!\!\!/ + m)\,(A-\mathrm{i}B)\,\mathcal{P}_L\xi - \sqrt{2}\,(\mathrm{i}\partial\!\!\!/ + m)\,\Phi\mathcal{P}_R\xi - 2g\phi\,(A+\mathrm{i}B)\,\mathcal{P}_R\xi, \tag{E.37}$$

while for the scalar fields we have

$$\delta_\xi A = \left(\bar{\xi}\mathcal{P}_L\Psi\right) + \left(\bar{\xi}\mathcal{P}_R\Psi^c\right) \tag{E.38}$$

$$\delta_\xi B = -\mathrm{i}\left(\bar{\xi}\mathcal{P}_L\Psi\right) + \mathrm{i}\left(\bar{\xi}\mathcal{P}_R\Psi^c\right) \tag{E.39}$$

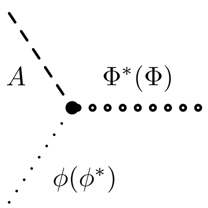
$$\delta_\xi\phi = \sqrt{2}\left(\bar{\xi}\mathcal{P}_L\chi\right) \tag{E.40}$$

$$\delta_\xi\phi^* = \sqrt{2}\left(\bar{\xi}\mathcal{P}_R\chi\right) \tag{E.41}$$

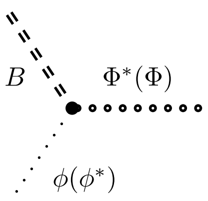
$$\delta_\xi\Phi = \sqrt{2}\left(\bar{\xi}\mathcal{P}_L\Psi^c\right) \tag{E.42}$$

$$\delta_\xi\Phi^* = \sqrt{2}\left(\bar{\xi}\mathcal{P}_R\Psi\right) \quad . \tag{E.43}$$

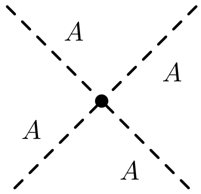
We denote the Feynman rules for the OR model graphically:



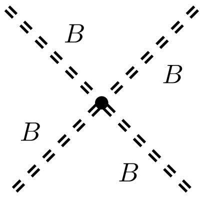
$-\sqrt{2}\mathrm{i}gm$



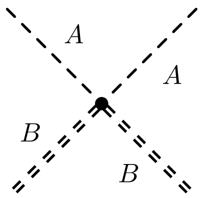
$\pm\sqrt{2}gm$



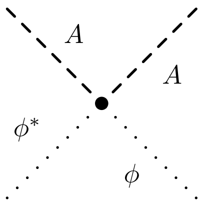
$-6\mathrm{i}g^2$



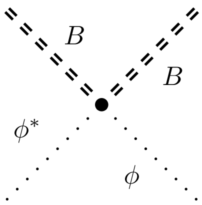
$-6\mathrm{i}g^2$



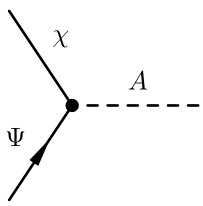
$-2\mathrm{i}g^2$



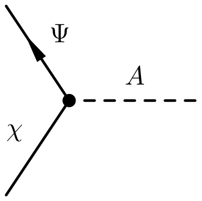
$-4\mathrm{i}g^2$



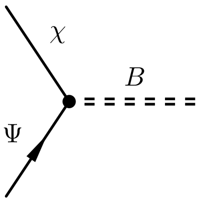
$-4\mathrm{i}g^2$



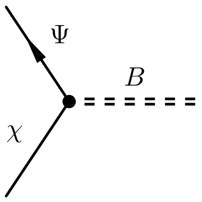
$-\sqrt{2}\mathrm{i}g\mathcal{P}_L$



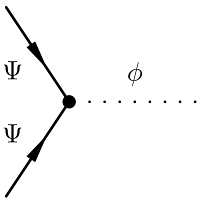
$-\sqrt{2}\mathrm{i}g\mathcal{P}_R$



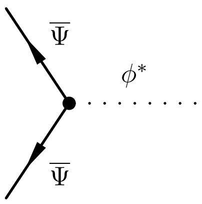
$+\sqrt{2}g\mathcal{P}_L$



$-\sqrt{2}g\mathcal{P}_R$



$-2\mathrm{i}g\mathcal{P}_L$



$-2\mathrm{i}g\mathcal{P}_R$

Remark:

Again there is the phenomenon known from the discussion of the toy model in the foregoing section, which always appears in supersymmetric models with mixings between the fermionic components of more than one superfield, called the problem of "clashing arrows". The examples in the MSSM where charged gauginos and Higgsinos are combined into the charginos are more prominent. Of course, in the last two vertices we can reverse the arrow of one of the Dirac fermions, and label the lines by the charge conjugated field Ψ^c instead of Ψ. For the structure of the bispinor products to be intact this would indeed be the better solution. We again have a symmetry factor two in the last two vertices as in the toy model, which is perhaps most clearly seen in the formulation with two-component spinors, but can also be understood when looking at the two factors of Ψ or $\overline{\Psi}$ appearing in the vertex – here this description is better suited than the one with the charge conjugated field.

E.4 An Abelian toy model

Action:

$$S = \int d^4x \left\{ \frac{1}{2} \left[\hat{\Phi}^\dagger \exp(-\mathcal{V}) \hat{\Phi} \right]_D + \frac{1}{2} \mathrm{Re} \left[\overline{W_R} W_L \right]_F \right\} \tag{E.44}$$

Equations of motion for the auxiliary fields:

$$F \equiv 0 \tag{E.45a}$$

$$D = -e|\phi|^2 \tag{E.45b}$$

By introducing the bispinor notation for matter fermion and gaugino [1]

$$\Psi = \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}, \qquad \lambda = \mathrm{i}\gamma^5 \begin{pmatrix} \lambda \\ \bar{\lambda} \end{pmatrix}, \tag{E.46}$$

splitting the complex scalar field in real and imaginary part A and B as well as integrating out the auxiliary fields D and F (the latter vanishing identically), we find the Lagrangean density in components as well as the Feynman rules:

$$\begin{aligned} \mathcal{L} = \frac{1}{2}(\partial_\mu A)(\partial^\mu A) + \frac{1}{2}(\partial_\mu B)(\partial^\mu B) + \frac{\mathrm{i}}{2}\overline{\Psi}\not{\partial}\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\mathrm{i}}{2}\bar{\lambda}\not{\partial}\lambda \\ + eG_\mu \left(B\partial^\mu A - A\partial^\mu B \right) + \frac{e^2}{2} G_\mu G^\mu \left(A^2 + B^2 \right) - e \left(\overline{\Psi}\lambda \right) A \\ - \mathrm{i}e \left(\overline{\Psi}\gamma^5 \lambda \right) B - \frac{e}{2} \overline{\Psi} \not{G} \gamma^5 \Psi - \frac{e^2}{8} \left(A^4 + B^4 + 2A^2 B^2 \right) \end{aligned} \tag{E.47}$$

Following the derivation from the foregoing appendix we can immediately write down the current:

$$\begin{aligned} \mathcal{J}^\mu = -(\not{\partial}A)\gamma^\mu\Psi - \mathrm{i}(\not{\partial}B)\gamma^\mu\gamma^5\Psi + \mathrm{i}eA\not{G}\gamma^\mu\gamma^5\Psi \\ - eB\not{G}\gamma^\mu\Psi + \frac{1}{2}[\gamma^\alpha, \gamma^\beta]\gamma^\mu\gamma^5(\partial_\alpha G_\beta)\lambda - \frac{\mathrm{i}e}{2} \left(A^2 + B^2 \right) \gamma^\mu \lambda \end{aligned} \tag{E.48}$$

Equations of motion for all fields:

$$\Box A = -2eG_\mu \partial^\mu B - eB\partial_\mu G^\mu + e^2 G_\mu G^\mu A - e\overline{\Psi}\lambda - \frac{e^2}{2} \left(A^3 + AB^2 \right) \tag{E.49a}$$

$$\Box B = 2eG_\mu \partial^\mu A + eA\partial_\mu G^\mu + e^2 G_\mu G^\mu B - \mathrm{i}e\overline{\Psi}\gamma^5\lambda - \frac{e^2}{2} \left(B^3 + BA^2 \right) \tag{E.49b}$$

$$\mathrm{i}\not{\partial}\Psi = eA\lambda + \mathrm{i}eB\gamma^5\lambda + e\not{G}\gamma^5\Psi \tag{E.49c}$$

$$\mathrm{i}\not{\partial}\lambda = eA\Psi + \mathrm{i}eB\gamma^5\Psi \tag{E.49d}$$

$$\partial^\nu F_{\nu\mu} = e \left(A\partial_\mu B - B\partial_\mu A \right) - e^2 G_\mu \left(A^2 + B^2 \right) + \frac{e}{2} \overline{\Psi}\gamma_\mu\gamma^5\Psi \tag{E.49e}$$

The charge generates the SUSY transformations of the fields:

$$[\mathrm{i}\bar{\xi}Q, A] = \left(\bar{\xi}\Psi \right) \tag{E.50a}$$

$$[\mathrm{i}\bar{\xi}Q, B] = \left(\mathrm{i}\bar{\xi}\gamma^5\Psi \right) \tag{E.50b}$$

[1] The strange prefactor of $\mathrm{i}\gamma^5$ is due to the conventions originally introduced by Wess and Bagger to define two-component gaugino fields, cf. the appendix of [2]

$$[\mathrm{i}\bar{\xi}Q, \Psi] = -\mathrm{i}(\not{\partial} - \mathrm{i}e\not{G}\gamma^5)(A + \mathrm{i}\gamma^5 B)\xi \tag{E.50c}$$

$$[\mathrm{i}\bar{\xi}Q, \lambda] = -\frac{\mathrm{i}}{2}F_{\alpha\beta}\gamma^\alpha\gamma^\beta\gamma^5\xi - \frac{e}{2}\left(A^2 + B^2\right)\xi \tag{E.50d}$$

$$[\mathrm{i}\bar{\xi}Q, G_\mu] = -\left(\bar{\xi}\gamma_\mu\gamma^5\lambda\right) \tag{E.50e}$$

The gauge transformations with transformation parameter $\theta = \theta(x)$ are (note that this model has a chiral charge, the gauge boson couples to the axial vector of the fermion, and scalar and pseudoscalar are interchanged by gauge transformations):

$$\delta A = -e\theta B \tag{E.51a}$$

$$\delta B = +e\theta A \tag{E.51b}$$

$$\delta\Psi = -\mathrm{i}e\gamma^5\theta\Psi \tag{E.51c}$$

$$\delta G_\mu = \partial_\mu\theta \tag{E.51d}$$

$$\delta\lambda = 0 \tag{E.51e}$$

The "covariance of the equations of motions" is important for the Green functions to have the same poles. We just show one example:

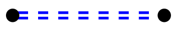
$$\begin{aligned}
[\mathrm{i}\bar{\xi}Q, \Box A] = {} & 2\mathrm{i}e\left(\bar{\xi}(\not{\partial}B)\gamma^5\lambda\right) + 2e\left(\bar{\xi}\gamma^5 G_\mu\partial^\mu\Psi\right) - \mathrm{i}e\left(\bar{\xi}B\gamma^5\not{\partial}\lambda\right) \\
& + e(\partial_\mu G^\mu)\left(\bar{\xi}\gamma^5\Psi\right) - 2\mathrm{i}e^2 A\left(\bar{\xi}\not{G}\gamma^5\lambda\right) + \mathrm{i}e^2 G_\mu G^\mu\left(\bar{\xi}\Psi\right) \\
& + e\bar{\xi}\Big(\left(\not{\partial} - \mathrm{i}e\not{G}\gamma^5\right)\left(A - \mathrm{i}\gamma^5 B\right)\Big)\lambda + \frac{e^2}{2}\left(\bar{\xi}[\gamma^\alpha, \gamma^\beta]\gamma^5(\partial_\alpha G_\beta)\Psi\right) \\
& + \frac{\mathrm{i}e^2}{2}\left(A^2 + B^2\right)\left(\bar{\xi}\Psi\right) - \frac{3\mathrm{i}e^2}{2}A^2\left(\bar{\xi}\Psi\right) - \frac{\mathrm{i}e^2}{2}B^2\left(\bar{\xi}\Psi\right) \\
& + e^2 AB\left(\bar{\xi}\gamma^5\Psi\right) \\
= {} & \mathrm{i}e\left(\bar{\xi}(\not{\partial}B)\gamma^5\lambda\right) + 2e\left(\bar{\xi}\gamma^5 G_\mu\partial^\mu\Psi\right) - \mathrm{i}e\left(\bar{\xi}B\gamma^5\not{\partial}\lambda\right) \\
& + e(\partial_\mu G^\mu)\left(\bar{\xi}\gamma^5\Psi\right) - \mathrm{i}e^2 A\left(\bar{\xi}\not{G}\gamma^5\lambda\right) + \mathrm{i}e^2 G_\mu G^\mu\left(\bar{\xi}\Psi\right) \\
& + e\left(\bar{\xi}(\not{\partial}A)\lambda\right) + e^2 B\left(\bar{\xi}\not{G}\lambda\right) + \frac{e^2}{2}\left(\bar{\xi}[\gamma^\alpha, \gamma^\beta]\gamma^5(\partial_\alpha G_\beta)\Psi\right) \\
& - \mathrm{i}e^2 A^2\left(\bar{\xi}\Psi\right) + e^2 AB\left(\bar{\xi}\gamma^5\Psi\right)
\end{aligned} \tag{E.52}$$

$$\begin{aligned}
\Box\,[\mathrm{i}\bar{\xi}Q, A] = {} & \mathrm{i}\left(\bar{\xi}\not{\partial}\not{\partial}\Psi\right) \\
= {} & eA\left(\bar{\xi}\not{\partial}\lambda\right) + e\left(\bar{\xi}(\not{\partial}A)\lambda\right) - \mathrm{i}eB\left(\bar{\xi}\gamma^5\not{\partial}\lambda\right) + \mathrm{i}e\left(\bar{\xi}(\not{\partial}B)\gamma^5\lambda\right) \\
& + e\left(\bar{\xi}(\not{\partial}\not{G})\gamma^5\Psi\right) + e\left(\bar{\xi}\gamma^\mu\not{G}\gamma^5\partial_\mu\Psi\right) \\
= {} & -\mathrm{i}e^2 A^2\left(\bar{\xi}\Psi\right) + e^2 AB\left(\bar{\xi}\gamma^5\Psi\right) + e\left(\bar{\xi}(\not{\partial}A)\lambda\right) - \mathrm{i}e\left(\bar{\xi}B\gamma^5\not{\partial}\lambda\right) \\
& + \mathrm{i}e\left(\bar{\xi}(\not{\partial}B)\gamma^5\lambda\right) + \frac{e}{2}\left(\bar{\xi}[\gamma^\alpha, \gamma^\beta]\gamma^5(\partial_\alpha G_\beta)\Psi\right) + e(\partial_\mu G^\mu)\left(\bar{\xi}\gamma^5\Psi\right) \\
& + 2e\left(\bar{\xi}\gamma^5 G_\mu\partial^\mu\Psi\right) - \mathrm{i}e^2 A\left(\bar{\xi}\not{G}\gamma^5\lambda\right) + e^2 B\left(\bar{\xi}\not{G}\lambda\right) \\
& + \mathrm{i}e^2(G_\mu G^\mu)\left(\bar{\xi}\Psi\right)
\end{aligned} \tag{E.53}$$

$$\Longrightarrow [\bar{\xi}Q, \Box A] = \Box\,[\bar{\xi}Q, A] \tag{E.54}$$

To clear our notation with respect to the propagators – especially in comparison with the ghost propagators in the third part of the text – we write down the propagators for the particles of the model:

$A(-p)$  $A(p)$ $\quad = \dfrac{\mathrm{i}}{p^2 + \mathrm{i}\epsilon}$ (E.55a)

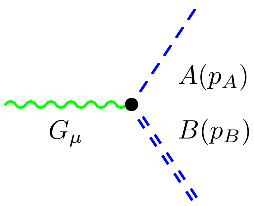
$B(-p)$  $B(p) \quad = \dfrac{\mathrm{i}}{p^2 + \mathrm{i}\epsilon}$ (E.55b)

$G_\mu(-p)$  $G_\nu(p) \quad = \dfrac{-\mathrm{i}\eta_{\mu\nu}}{p^2 + \mathrm{i}\epsilon}$ (E.55c)

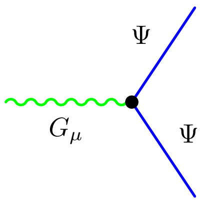
$\Psi(-p)$  $\overline{\Psi}(p) \quad = \dfrac{\mathrm{i}\not{p}}{p^2 + \mathrm{i}\epsilon}$ (E.55d)

$\lambda(-p)$  $\overline{\lambda}(p) \quad = \dfrac{\mathrm{i}\not{p}}{p^2 + \mathrm{i}\epsilon}$ (E.55e)

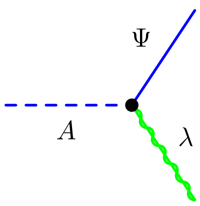
Vertices of our Abelian toy model (all momenta incoming):



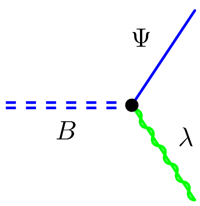
$e(p_A - p_B)_\mu$



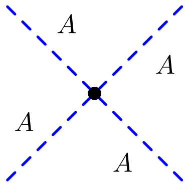
$\mathrm{i}e\gamma^5\gamma_\mu$



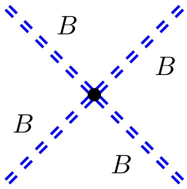
$-\,\mathrm{i}e$



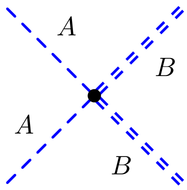
$e\gamma^5$



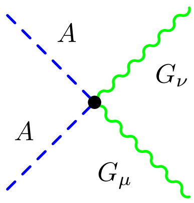
$-3\mathrm{i}e^2$



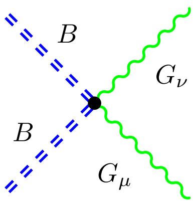
$-3\mathrm{i}e^2$



$-\mathrm{i}e^2$



$2\mathrm{i}e^2\eta_{\mu\nu}$



$2\mathrm{i}e^2\eta_{\mu\nu}$

E.5 Supersymmetric Yang–Mills theory (SYM)

The Lagrangean density without gauge fixing is:

$$
\begin{aligned}
\mathcal{L} = {}& (D_\mu\phi_+)^\dagger(D^\mu\phi_+) + (D_\mu\phi_-)^\dagger(D^\mu\phi_-) + \frac{\mathrm{i}}{2}\overline{\psi}_{+,i}(\not{D}\psi_+)_i + \frac{\mathrm{i}}{2}\overline{\psi}_{-,i}(\not{D}\psi_-)_i \\
& + |F_+|^2 + |F_-|^2 - \sqrt{2}g(\overline{\lambda^a}\phi^\dagger_{+,i}T^a_{ij}\mathcal{P}_L\psi_{+,j}) - \sqrt{2}g(\overline{\psi}_{+,i}T^a_{ij}\phi_{+,j}\mathcal{P}_R\lambda^a) \\
& - \sqrt{2}g(\overline{\lambda^a}\phi^\dagger_{-,i}(-T^{a\,*})_{ij}\mathcal{P}_L\psi_{-,j}) - \sqrt{2}g(\overline{\psi}_{-,i}(-T^{a\,*})_{ij}\phi_{-,j}\mathcal{P}_R\lambda^a) \\
& + g(\phi^\dagger_{+,i}T^a_{ij}\phi_{+,j})D^a + g(\phi^\dagger_{-,i}(-T^{a\,*})_{ij}\phi_{-,j})D^a - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a \\
& + \frac{\mathrm{i}}{2}\overline{\lambda^a}(\not{D}\lambda)^a + \frac{1}{2}(D^aD^a) + m\phi_{+,i}F_{-,i} + m\phi^\dagger_{+,i}F^\dagger_{+,i} \\
& + m\phi_{-,i}F_{-,i} + m\phi^\dagger_{-,i}F^\dagger_{-,i} - m\left(\psi_+\psi_- + \bar{\psi}_+\bar{\psi}_-\right)
\end{aligned}
\tag{E.56}
$$

The generators of the gauge group fulfill the Lie algebra

$$
[T^a, T^b] = \mathrm{i}f_{abc}T^c, \qquad [(-T^a)^*, (-T^b)^*] = \mathrm{i}f_{abc}(-T^c)^* \tag{E.57}
$$

For the auxiliary fields the equations of motion are

$$
F_{+,i} = -m\phi^\dagger_{-,i} \qquad F^\dagger_{+,i} = -m\phi_{-,i} \tag{E.58a}
$$

$$
F_{-,i} = -m\phi^\dagger_{+,i} \qquad F^\dagger_{-,i} = -m\phi_{+,i} \tag{E.58b}
$$

$$
D^a = -g(\phi^\dagger_+T^a\phi_+) + g(\phi^\dagger_-T^{a\,*}\phi_-) \tag{E.58c}
$$

We diagonalize the mass terms of the fermions by introducing the bispinors

$$
\Psi_i = \begin{pmatrix}\psi_{+,i}\\ \bar{\psi}_{i,-}\end{pmatrix}, \qquad \overline{\Psi}_i = \left(\psi_{-,i}, \bar{\psi}_{+,i}\right) \quad . \tag{E.59}
$$

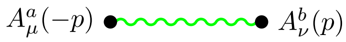
By the redefinitions of the fermion fields and after integrating out all auxiliary fields we get the Lagrangean density (with gauge-fixing and Faddeev-Popov terms)

$$
\begin{aligned}
\mathcal{L} = {}& (D_\mu\phi_+)^\dagger(D^\mu\phi_+) - m^2|\phi_+|^2 + (D_\mu\phi_-)^\dagger(D^\mu\phi_-) - m^2|\phi_-|^2 \\
& + \overline{\Psi}(\mathrm{i}\not{D} - m)\Psi + \frac{\mathrm{i}}{2}\overline{\lambda^a}(\not{D}\lambda)^a - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a - \sqrt{2}\,\mathrm{i}g\phi^\dagger_{+,i}T^a_{ij}(\overline{\lambda^a}\mathcal{P}_L\Psi_j) \\
& - \sqrt{2}\,\mathrm{i}g\phi_{-,i}T^a_{ij}(\overline{\lambda^a}\mathcal{P}_R\Psi_j) + \sqrt{2}\,\mathrm{i}g(\overline{\Psi}_i\mathcal{P}_R\lambda^a)T^a_{ij}\phi_{+,j} \\
& + \sqrt{2}\,\mathrm{i}g(\overline{\Psi}_i\mathcal{P}_L\lambda^a)T^a_{ij}\phi^\dagger_{-,j} - \frac{g^2}{2}\left(\phi^\dagger_{+,i}T^a_{ij}\phi_{+,j}\right)\left(\phi^\dagger_{+,k}T^a_{kl}\phi_{+,l}\right) \\
& - \frac{g^2}{2}\left(\phi_{-,i}T^a_{ij}\phi^\dagger_{-,j}\right)\left(\phi_{-,k}T^a_{kl}\phi^\dagger_{-,l}\right) + g^2\left(\phi^\dagger_{+,i}T^a_{ij}\phi_{+,j}\right)\left(\phi_{-,k}T^a_{kl}\phi^\dagger_{-,l}\right) \\
& - \frac{1}{2\xi}(\partial^\mu A^a_\mu)(\partial^\nu A^a_\nu) + \mathrm{i}\overline{c}^a\partial_\mu(D^\mu c)^a - \mathrm{i}\overline{c}^a(\bar{\epsilon}\not{\partial}\lambda^a) + \frac{\mathrm{i}\xi}{2}\overline{c}^a(\bar{\epsilon}\gamma^\mu\epsilon)\partial_\mu\overline{c}^a
\end{aligned}
\tag{E.60}
$$

The propagators for the particles of the model are

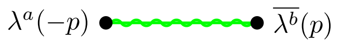
$\phi_{+,i}(-p)$  $\phi^\dagger_{+,j}(p) \quad = \dfrac{\mathrm{i}\delta_{ij}}{p^2 - m^2 + \mathrm{i}\epsilon}$ (E.61a)

$\phi^\dagger_{-,i}(-p)$  $\phi_{-,j}(p) \quad = \dfrac{\mathrm{i}\delta_{ij}}{p^2 - m^2 + \mathrm{i}\epsilon}$ (E.61b)

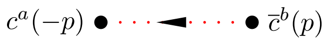


$$= \frac{-\mathrm{i}\delta_{ab}}{p^2+\mathrm{i}\epsilon}\left(\eta_{\mu\nu}-(1-\xi)\frac{p_\mu p_\nu}{p^2}\right) \xrightarrow{\xi\to 1} \frac{-\mathrm{i}\eta_{\mu\nu}\delta_{ab}}{p^2+\mathrm{i}\epsilon} \tag{E.61c}$$

$$= \frac{\mathrm{i}\delta_{ij}(\not{p}+m)}{p^2-m^2+\mathrm{i}\epsilon} \tag{E.61d}$$

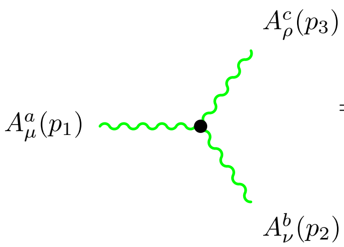


$$= \frac{\mathrm{i}\delta_{ab}\not{p}}{p^2+\mathrm{i}\epsilon} \tag{E.61e}$$

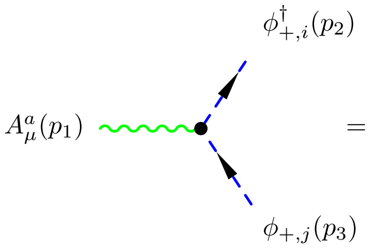


$$= \frac{-\delta_{ab}}{p^2+\mathrm{i}\epsilon} \tag{E.61f}$$

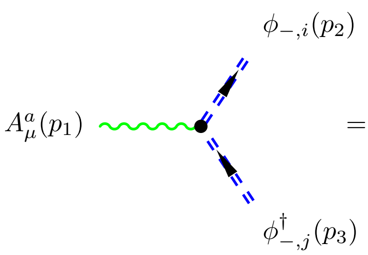
The 3-vertices are (all momenta incoming):



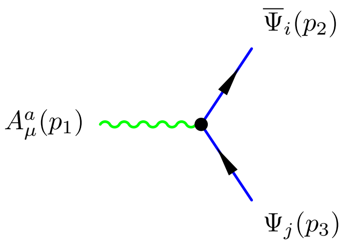
$$\begin{aligned} &= gf_{abc}[\eta_{\mu\nu}\,(p_1-p_2)_\rho+\eta_{\nu\rho}\,(p_2-p_3)_\mu \\ &\quad +\eta_{\rho\mu}\,(p_3-p_1)_\nu] \end{aligned} \tag{E.62a}$$



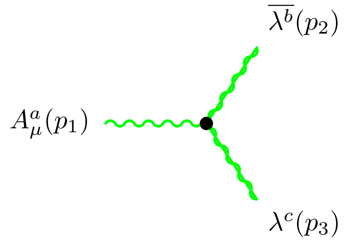
$$= \mathrm{i}g\,(p_3-p_2)_\mu\,T^a_{ij} \tag{E.62b}$$



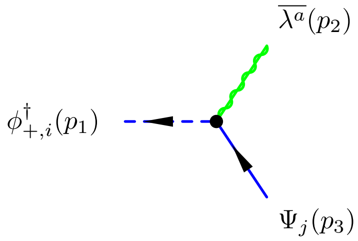
$$= \mathrm{i}g\,(p_3-p_2)_\mu\,T^a_{ij} \tag{E.62c}$$



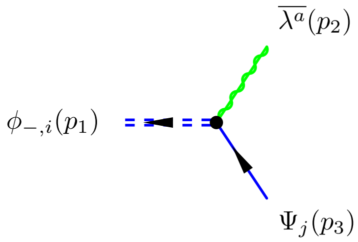
$$= \mathrm{i}g\gamma_\mu T^a_{ij} \tag{E.62d}$$



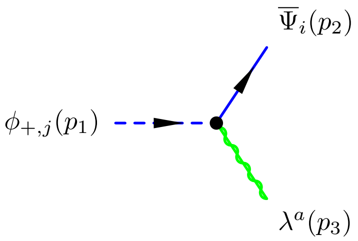
$$= g\gamma_\mu f_{abc} \tag{E.62e}$$



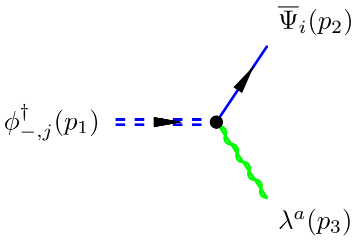
$$= \frac{g}{\sqrt{2}}\left(1-\gamma^5\right)T^a_{ij} \tag{E.62f}$$



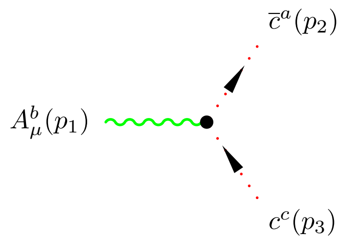
$$= \frac{g}{\sqrt{2}}\left(1+\gamma^5\right)T^a_{ij} \tag{E.62g}$$



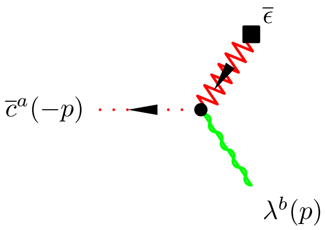
$$= -\frac{g}{\sqrt{2}}\left(1+\gamma^5\right)T^a_{ij} \tag{E.62h}$$



$$= -\frac{g}{\sqrt{2}}\left(1-\gamma^5\right)T^a_{ij} \tag{E.62i}$$

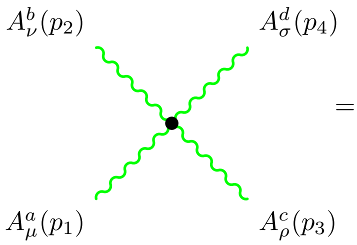


$$= -\mathrm{i}g f_{abc} p_{2,\mu} \tag{E.62j}$$

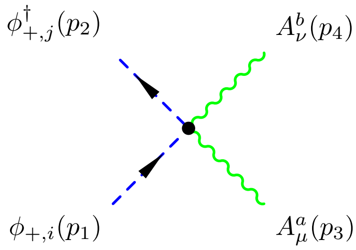


$$= -\mathrm{i}\not{p}\,\delta_{ab} \tag{E.62k}$$

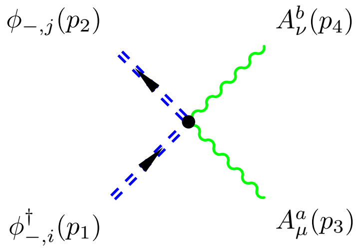
We have the following 4-vertices



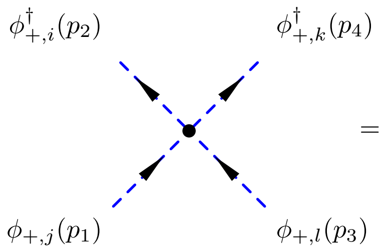
$$\begin{aligned} = -\mathrm{i}g^2[&f_{abe}f_{cde}\,(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}) \\ &+f_{ace}f_{bde}\,(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}) \\ &+f_{ade}f_{bce}\,(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma})] \end{aligned} \tag{E.63a}$$



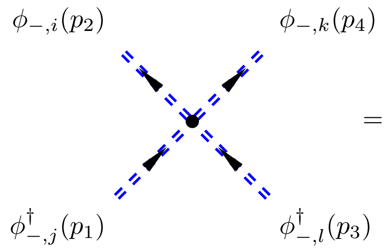
$$= \mathrm{i}g^2\eta_{\mu\nu}\left\{T^a, T^b\right\}_{ij} \tag{E.63b}$$



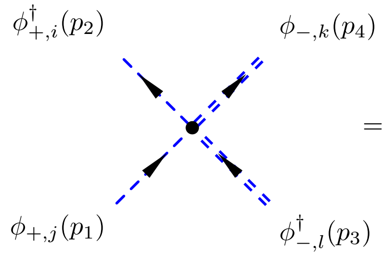
$$= \mathrm{i}g^2\eta_{\mu\nu}\left\{T^a, T^b\right\}_{ij} \tag{E.63c}$$



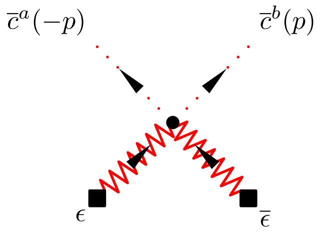
$$= -\frac{\mathrm{i}g^2}{4}\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right) \tag{E.63d}$$



$$= -\frac{\mathrm{i}g^2}{4}\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right) \tag{E.63e}$$



$$= \frac{\mathrm{i}g^2}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right) \tag{E.63f}$$



$$= \xi \not{p}\,\delta_{ab} \tag{E.63g}$$

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LEBENSLAUF

Jürgen Reuter
Bahnhofstraße 28
D-63128 Dietzenbach

Persönliche Daten

Geburt	27. April 1974 in Frankfurt am Main
Eltern	Gerhard Reuter, Bauschlossermeister Wilhelmine Reuter, Versicherungskauffrau
Familienstand	ledig
Staatsangehörigkeit	deutsch

Ausbildungsdaten

1980 - 1984	Aue-Schule in Dietzenbach
1984 - 1986	Ernst-Reuter-Schule in Dietzenbach
1986 - 1993	Heinrich-Mann-Schule in Dietzenbach Abitur 1993 mit der Gesamtnote 1.0
1993 - 1994	Beginn des Physikstudiums an der TH Darmstadt
1994 - 1995	Zivildienst in Dietzenbach
1995 - 1999	Physikstudium an der TH/TU Darmstadt

Okt. 1996	Diplom-Vorprüfung Physik mit der Gesamtnote "sehr gut"
Okt. 1998	Aufnahme in die Arbeitsgruppe Prof. P. Manakos. Arbeitsschwerpunkt: Quantenfeldtheorie, speziell Störungstheorie mittels Helizitätsamplituden, später supersymmetrische Feldtheorien
	Leitung von Übungsgruppen in theoretischer Physik sowie von Tutorien für das Vordiplom in theoretischer Physik und Mathematik (regelmäßig bis 2001/2002)
Sept. 1999	Abschluß als Diplomphysiker mit der Gesamtnote "mit Auszeichnung bestanden" Thema der Diplomarbeit: *Helizitätsamplituden leichter Fermionen bei hohen Energien*
	Teilnahme an der 31. Herbstschule für Hochenergie-Physik in Maria Laach
Nov. 1999	Beginn der Doktorarbeit über supersymmetrische Feldtheorien und das MSSM in der Arbeitsgruppe Manakos
Mai 2002	Voraussichtlicher Abschluß der Doktorarbeit an der TU Darmstadt Thema der Doktorarbeit: *Supersymmetry of Scattering Amplitudes and Green Functions in Perturbation Theory*

EIDESSTATTLICHE ERKLÄRUNG

Hiermit erkläre ich an Eides Statt, daß ich die vorliegende Dissertation selbständig verfaßt und keine anderen als die angegebenen Hilfsmittel verwendet habe. Es wurde bislang kein weiterer Promotionsversuch unternommen.

Darmstadt, im Mai 2002

(Jürgen Reuter)