CM values of regularized theta lifts

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Zusammenfassung

In der vorliegenden Dissertation werden spezielle Werte von regularisierten Thetaliftungen an sogenannten CM-Punkten untersucht. Es wird insbesondere gezeigt, dass sich die CM-Werte von Borcherdsprodukten in Termen der Koeffizienten des holomorphen Teils gewisser harmonischer Maaß-Formen ausdrücken lassen. Es wird eine direkte Beziehung zwischen diesen Koeffizienten und arithmetischen Zykeln hergestellt, die von Kudla, Rapoport und Yang eingeführt wurden. Diese Zykel parametrisieren elliptische Kurven mit speziellen Endomorphismen. Es kann dann gezeigt werden, dass die Koeffizienten des holomorphen Teils der untersuchten Funktionen Logarithmen von algebraischen Zahlen sind und es wird eine Formel für die Primidealfaktorisierungen angegeben.

Contents

Introduction

In this thesis, special values of regularized theta lifts at complex multiplication (CM) points are studied. In particular, it is shown that CM values of Borcherds products can be expressed in terms of finitely many Fourier coefficients of certain harmonic weak Maaß forms of weight one. As it turns out, these coefficients are logarithms of algebraic integers whose prime ideal factorization is determined by cycles on an arithmetic curve that parametrize special endomorphisms of CM elliptic curves.

We begin with a motivating example and give some historical background before we describe our main results.

Singular moduli

A good starting point is the celebrated theorem of Gross and Zagier on singular moduli [GZ85], which describes the norm of the modular function $j(\tau)$ at imaginary quadratic irrationals in the complex upper half-plane $\mathbb{H}$. This function, also called Klein's j-invariant, is an invariant associated with an elliptic curve over the field of complex numbers $\mathbb{C}$. Every such elliptic curve is isomorphic to a curve of the form $E_\tau = \mathbb{C}/\Lambda_\tau$, where $\Lambda_\tau = \mathbb{Z}\tau + \mathbb{Z}$ is a lattice in $\mathbb{C}$ with $\tau \in \mathbb{H}$. There is an action of the group $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$, given by

$$\gamma\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d} \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Two elliptic curves E_τ and $E_{\tau'}$ are isomorphic over $\mathbb{C}$ if and only if $\tau' = \gamma\tau$ for some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. The j-function is an invariant of the isomorphism class of an elliptic curve and therefore invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$. Such functions are called modular forms of weight 0 or *modular functions*. The Fourier expansion of j starts with

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + \dots,$$

where $q = e(\tau) = e^{2\pi i\tau}$.

Let $d < 0$ be a negative fundamental discriminant and denote by $\mathcal{Q}_d$ the set of positive definite integral binary quadratic forms of discriminant d. For every $Q \in \mathcal{Q}_d$ given by $Q(x, y) = ax^2 + bxy + cy^2$ the unique root of $Q(\tau, 1) = 0$ in $\mathbb{H}$ is denoted $\alpha_Q \in \mathbb{H}$. These points are called CM points because the associated elliptic curve E_τ has *complex multiplication*. That is, its endomorphism ring is an order in an imaginary quadratic field. In our case the order is simply the ring of integers in $\mathbb{Q}(\sqrt{d})$. The values of $j(\tau)$ at CM points are classically called *singular moduli* [Web61, Zag02].

The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{Q}_d$ and two forms $P, Q \in \mathcal{Q}_d$ are in the same equivalence class with respect to this action if and only if the points α_P and α_Q are equivalent under the action of $\mathrm{SL}_2(\mathbb{Z})$ via linear fractional transformations on $\mathbb{H}$. The action on a binary quadratic form is given by a substitution of variables

$$Q(x,y) \mapsto Q(ax+by, cx+dy) \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

The value $j(\alpha_Q)$ is an algebraic integer of degree h_d over $\mathbb{Q}$, generating the Hilbert class field H of $k_d = \mathbb{Q}(\sqrt{d})$. Here, h_d denotes the class number of k_d and the Hilbert class field is defined to be the maximal unramified abelian extension of k_d. This statement is part of the theory of complex multiplication [Sil94, Shi94], realizing *Kronecker's Jugendtraum* (also called *Hilbert's twelfth problem*) for imaginary quadratic number fields. This result can be understood as an analog of the Kronecker-Weber theorem [Neu07, V., Korollar (1.9)], which states that every finite abelian extension of $\mathbb{Q}$ is contained in a cyclotomic extension. Such an extension is generated by "special values" of the exponential function.

Even though the arithmetic of singular moduli has already been studied in the 19th century, they came again into the focus of research much later, probably starting with the paper of Gross and Zagier, published 1985. It was not until 2002 that Zagier [Zag02] showed that the traces of singular moduli occur as Fourier coefficients of weakly holomorphic modular forms of half-integral weight. His seminal paper inspired researchers and resulted in many new results in this direction, for instance [BF06, DJ08, BO11, MP10, AE13].

To describe the result of Gross and Zagier we consider the modular function

$$\Psi(z,d) = \prod_{Q \in \mathrm{SL}_2(\mathbb{Z}) \backslash \mathcal{Q}_d} (j(z) - j(\alpha_Q))^{2/w_d},$$

for $z \in \mathbb{H}$, and where w_d is the number of roots of unity in k_d.

Gross and Zagier proved that for a negative fundamental discriminant D coprime to d, we have

$$\prod_{Q \in \mathrm{SL}_2(\mathbb{Z}) \backslash \mathcal{Q}_D} \Psi(\alpha_Q, d)^{4/w_D} = \pm \prod_{\substack{x \in \mathbb{Z},\, n, n' > 0 \\ 4nn' = dD - x^2}} n^{\epsilon(n')}, \tag{0.1}$$

where $\epsilon(n') = \pm 1$. Note that the left-hand side is equal to the absolute norm of the value $\Psi(\alpha_Q, d)^{2/w_D}$ because the set of values $\{j(\alpha_Q)\ ;\ Q \in \mathrm{SL}_2(\mathbb{Z}) \backslash \mathcal{Q}_d\}$ forms a full system of Galois conjugates over $\mathbb{Q}$.

In the special case $d = -3$, the class number h_{-3} is equal to one and the j-invariant vanishes at the unique CM point of discriminant -3 modulo the action of $\mathrm{SL}_2(\mathbb{Z})$. Thus, Equation (0.1) also gives a formula of the norm of $j(z)$ at a CM point.

The theorem of Gross and Zagier can be understood in the context of Borcherds products. These are certain meromorphic modular forms on orthogonal groups obtained via a singular theta correspondence.

Regularized theta lifts and Borcherds products

We let L be an even lattice of type $(2,n)$, that is, a free $\mathbb{Z}$-module together with a $\mathbb{Z}$-valued quadratic form Q with associated bilinear form $(\cdot,\cdot)$ which has two positive and n negative eigenvalues over $\mathbb{R}$. We write $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$ for the corresponding rational vector space. Associated with such a lattice is its discriminant group $A_L = L'/L$, where L' is the dual of L with respect to $(\cdot,\cdot)$. The quadratic form Q induces a well defined map $A_L \to \mathbb{Q}/\mathbb{Z}$, which we also denote by Q. For simplicity, let us assume that n is even. Then there is a representation ρ_L of $\mathrm{SL}_2(\mathbb{Z})$ on the group ring $S_L = \mathbb{C}[A_L]$, the so-called Weil representation [Bru02, Wei64]. The group $\mathrm{SL}_2(\mathbb{Z})$ acts on functions $f : \mathbb{H} \to S_L$ via

$$(f \mid_{k,L} \gamma)(\tau) = (c\tau + d)^{-k} \rho_L(\gamma)^{-1} f(\tau), \tag{0.2}$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. A *modular form* of weight k with representation ρ_L is a holomorphic function which is invariant under this action and additionally satisfies a certain growth condition towards the “cusp” at ∞. The growth condition can be stated as follows. A holomorphic function satisfying (0.2) admits a Fourier expansion

$$f(\tau) = \sum_{\mu \in A_L} \sum_{\substack{m \in \mathbb{Q} \\ m \equiv Q(\mu) \bmod \mathbb{Z}}} c_f(m,\mu) e(m\tau) \phi_\mu,$$

where $e(x) = e^{2\pi i x}$ and ϕ_μ denotes the standard basis vector of S_L corresponding to $\mu \in A_L$. Then f is called a weakly holomorphic modular form if $c_f(m,\mu) \neq 0$ for only finitely many $m < 0$. We denote the space of weakly holomorphic modular forms of weight k transforming with representation ρ_L by $M^!_{k,L}$. Similarly, f is called holomorphic if $c_f(m,\mu) = 0$ for all $m < 0$ and a cusp form if additionally $c_f(0,\mu) = 0$ for all $\mu \in A_L$. We denote the spaces of such forms by $M_{k,L}$ and $S_{k,L}$, respectively.

The Siegel theta function attached to L is a non-holomorphic function $\Theta_L(\tau,z)$ in two variables $\tau \in \mathbb{H}$ and $z \in \mathbb{D}$, where $\mathbb{D}$ is the hermitian symmetric domain associated with the orthogonal group $\mathrm{SO}_V(\mathbb{R})$. We can realize $\mathbb{D}$ as the Grassmannian of oriented 2-dimensional positive definite subspaces of $V(\mathbb{R}) = V \otimes_{\mathbb{Q}} \mathbb{R}$.

The function $\Theta_L(\tau,z)$ is invariant under the action of a subgroup $\Gamma_L \subset \mathrm{SO}_L$ in z and transforms as a modular form of weight $(2-n)/2$ for $\mathrm{SL}_2(\mathbb{Z})$ with representation ρ_L in the variable τ. Such theta functions can be used as an integration kernel to lift modular forms from one group to another. The underlying principle is the theory of dual reductive pairs in the sense of Howe [How79]. Borcherds [Bor98] extended this theory for the dual reductive pair $(\mathrm{SL}_2, \mathrm{O}_V)$ to weakly holomorphic modular forms.

Explicitly, for a weakly holomorphic modular form $f \in M^!_{k,L}$ with $k = (2-n)/2$, we consider the integral

$$\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \langle f(\tau), \overline{\Theta_L(\tau,z)} \rangle v^k \frac{du dv}{v^2}, \tag{0.3}$$

where $\tau = u + iv \in \mathbb{H}$ and $\langle \cdot,\cdot \rangle$ denotes the $\mathbb{C}$-bilinear pairing such that $\langle \phi_\mu, \phi_\mu \rangle = 1$ and $\langle \phi_\mu, \phi_\lambda \rangle = 0$ for $\lambda \neq \mu$. However, the integral diverges if f has a pole at the cusp.

Borcherds used the method of Harvey and Moore [HM96] to show that the integral can be regularized. We obtain a Γ_L-invariant function $\Phi_L(z,f)$ which is real analytic outside a divisor $Z(f)$ given by codimension one sub-Grassmannians of $\mathbb{D}$. The divisor $Z(f)$ depends only on the principal part of f, given by the Fourier polynomial

$$P_f(\tau) = \sum_{\mu \in A_L} \sum_{m<0} c_f(m,\mu) e(m\tau) \phi_\mu.$$

We assume that $c_f(m,\mu) \in \mathbb{Z}$ for all $m \leq 0$ and all $\mu \in A_L$. Moreover, we assume for simplicity that the constant term $c_f(0,0)$ in the Fourier expansion of f vanishes.

By working out the Fourier expansion of $\Phi_L(z,f)$, Borcherds showed that

$$\Phi_L(z,f) = -4 \log |\Psi_L(z,f)| \tag{0.4}$$

holds for a meromorphic function $\Psi_L(z,f)$ on $\mathbb{D}$ with divisor $\frac{1}{2}Z(f)$. The function $\Psi_L(z,f)$ has an infinite product expansion near the cusps, giving it the name *Borcherds product*.

Schofer [Sch09b] gave a finite formula for the weighted average value of the theta lift $\Phi_L(z,f)$ over a CM cycle. The function $\Psi_L(z,d)$ can be obtained as the Borcherds product of a weakly holomorphic modular form of weight 1/2 for the lattice $L = \mathbb{Z}^3$ with the quadratic form $Q(a,b,c) = a^2 - bc$. Together with (0.4), Schofer's result provides a new proof of (0.1). The type of L is $(2,1)$ and the associated symmetric domain can be identified with $\mathbb{H} \cup \bar{\mathbb{H}}$. To explain Schofer's result and state our first theorem, which is a generalization of his work, we will have to introduce some more notation.

It is convenient to work with an adelic setup to describe the CM points we are considering. This description goes back to Shimura [Shi94]. We let $H = \mathrm{GSpin}_V$ be the general spin group, which is a central extension of the special orthogonal group SO_V and consider these groups as algebraic groups over $\mathbb{Q}$. We denote by $\mathbb{A}_f$ the finite adeles of $\mathbb{Q}$ and let $K \subset H(\mathbb{A}_f)$ be a compact open subgroup such that K stabilizes L and acts trivially on A_L. We consider the associated *Shimura variety* with complex points

$$X_K(\mathbb{C}) = H(\mathbb{Q}) \backslash (\mathbb{D} \times H(\mathbb{A}_f)/K).$$

An example for such a variety is the modular curve $Y_0(N) = \Gamma_0(N)\backslash\mathbb{H}$. For $N=1$ it is obtained by considering the lattice L of type $(2,1)$ as above and a suitable choice of K. Here, the group $\Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z})$ is given by all matrices in $\mathrm{SL}_2(\mathbb{Z})$, such that the lower left entry is divisible by N. In particular, for $N = 1$ we have $\mathrm{SL}_2(\mathbb{Z}) = \Gamma_0(1)$. The definition of the Siegel theta function can be naturally extended to obtain a function $\Theta_L(\tau, z, h)$ on X_K for $z \in \mathbb{D}$ and $h \in H(\mathbb{A}_f)$. We can then consider the regularized theta lift $\Phi_L(z,h,f)$ as a function on X_K, as well.

CM values

Let $U \subset V(\mathbb{Q})$ be a rational 2-dimensional positive definite subspace of V. Then U defines two rational points $z_U^\pm$ in $\mathbb{D}$, given by $U(\mathbb{R})$ together with the two possible orientations.

Let $T = \mathrm{GSpin}_U$ and $K_T = K \cap T(\mathbb{A}_f)$. We obtain a CM cycle in X_K by considering the Shimura variety

$$Z(U) = T(\mathbb{Q})\backslash(\{z_U^\pm\} \times T(\mathbb{A}_f)/K_T).$$

It is possible to recover the CM points α_Q described above in this way for $Y_0(N)$.

We obtain two lattices $P = L \cap U$ and $N = L \cap U^\perp$, where $U^\perp$ is the orthogonal complement of U in V. The lattice P is 2-dimensional and positive definite and N is n-dimensional and negative definite. For simplicity, we assume in the introduction that $L = P \oplus N$. Under this assumption the theta function Θ_L splits as the tensor product $\Theta_L(\tau, z_U^\pm, h) = \Theta_P(\tau, h) \otimes \Theta_N(\tau)$ for $h \in T(\mathbb{A}_f)$.

We need to define one more object to describe our first result. A function $f : \mathbb{H} \to S_L$ is called a *harmonic weak Maaß form* of weight k and representation ρ_L if it transforms like a vector valued modular form of weight k, is harmonic with respect to the weight k Laplace operator and grows at most exponentially towards the cusp at ∞. We write $\mathcal{H}_{k,L}$ for the space of such functions.

There is an antilinear differential operator $\xi = \xi_k$ defined by

$$\xi(f)(\tau) := 2iv^k \overline{\frac{\partial}{\partial \overline{\tau}} f(\tau)}. \tag{0.5}$$

It was shown by Bruinier and Funke that $\xi : \mathcal{H}_{k,L} \longrightarrow M^!_{2-k,L^-}$ is surjective [BF04, Theorem 3.7] with kernel $M^!_{k,L}$. Here, the lattice L^- is given by the lattice L together with the quadratic form $-Q$. The associated Weil representation can be identified with the dual representation of ρ_L. We denote by $H_{k,L} \subset \mathcal{H}_{k,L}$ the subspace of those harmonic weak Maaß forms that map to a cusp form under ξ. An element $f \in \mathcal{H}_{k,L}$ admits a unique decomposition $f = f^+ + f^-$ into a *holomorphic part* f^+ and a *non-holomorphic part* f^-.

Bruinier [Bru02] showed that the additive Borcherds lift $\Phi_L(z, h, f)$ can be extended to harmonic weak Maaß forms contained in $H_{k,L}$. If $f \in H_{k,L}$ is not weakly holomorphic, then $\Phi_L(z, h, f)$ is no longer the logarithm of a meromorphic modular form but it is an automorphic Green function for the divisor $Z(f)$.

Our first main result is the following.

Theorem 1. *Let $f \in H_{1-n/2,L}$ and let $\widetilde{\Theta}_P(\tau, h) \in \mathcal{H}_{1,P^-}$ be a harmonic weak Maaß form of weight 1 with the property that $\xi(\widetilde{\Theta}_P(\tau, h)) = \Theta_P(\tau, h)$. Then for any $(z, h) \in Z(U)$ the value of $\Phi_L(z, h, f)$ is given by*

$$\begin{aligned}\Phi_L(z, h, f) = \mathrm{CT}\left(\langle f^+(\tau), \Theta_{N^-}(\tau) \otimes \widetilde{\Theta}_P^+(\tau, h)\rangle\right) \\ - \int^{\mathrm{reg}}_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \langle \overline{\xi(f)}(\tau), \Theta_{N^-}(\tau) \otimes \widetilde{\Theta}_P(\tau, h)\rangle \, v^{1+n/2} \frac{dudv}{v^2}.\end{aligned}$$

Here, $\mathrm{CT}(\cdot)$ denotes the constant term in the Fourier expansion. It is a finite sum of products of coefficients of f^+ and $\Theta_{N^-}(\tau) \otimes \widetilde{\Theta}_P^+(\tau, h)$. Note that if f is weakly holomorphic, then the regularized integral above vanishes, as $\xi(f) = 0$ in that case. In particular, we obtain a formula for CM values of Borcherds products which involves only a finite number of coefficients of $\widetilde{\Theta}_P^+(\tau,h)$ weighted by representation numbers of the lattice N and the coefficients of f^+. Moreover, note that $\widetilde{\Theta}_P(\tau,h)$ is not uniquely determined. It can be modified by adding *any* weakly holomorphic modular form and the theorem is still valid.

The average value of $\Phi_L(z,h,f)$ over the CM-cycle $Z(U)$ has been treated by Schofer [Sch09b] for weakly holomorphic modular forms and by Bruinier and Yang [BY09] for harmonic weak Maaß forms. The proof of Theorem 1 follows along the lines of the proof of Theorem 4.7 in [BY09], without averaging over the CM cycle. Using the relation $\xi(\widetilde{\Theta}(\tau,h)) = \Theta_P(\tau,h)$, it is basically an application of Stokes theorem, involving some careful growth estimates.

By the Siegel-Weil formula ([KR88], see also Section 3.1), we have that

$$\frac{|\mathrm{vol}(K_T)|}{w_{K,T}} \sum_{h \in T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K_T} \Theta_P(\tau,h) = E_P(\tau),$$

where $w_{K,T} = |K_T \cap T(\mathbb{Q})|$ and $E_P(\tau)$ is the unique normalized holomorphic Eisenstein series in $M_{1,P}$. Theorem 4.7 of [BY09] expresses the weighted sum $\Phi(Z(U),f)$ over the points in the CM cycle $Z(U)$ as

$$\Phi(Z(U),f) = \deg(Z(U)) \cdot \mathrm{CT}\left(\langle f^+(\tau), \Theta_{N^-}(\tau) \otimes \mathcal{E}_P^+(\tau)\rangle\right) - L'(\xi(f),U,0), \tag{0.6}$$

where $\mathcal{E}_P(\tau) \in \mathcal{H}_{1,P^-}$ is the derivative of an "incoherent" Eisenstein series $\hat{E}_P(\tau,s)$ at $s=0$. Moreover, $L'(\xi(f),U,0)$ is the special value at $s=0$ of the derivative of an L-function obtained by means of the convolution integral

$$L(\xi(f),U,s) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \langle \overline{\xi(f)}(\tau), \hat{E}_P(\tau,s) \otimes \Theta_{N^-}(\tau)\rangle \, v^{1+n/2} \frac{du dv}{v^2}.$$

Analogously to our functions $\widetilde{\Theta}_P(\tau,h)$ the Eisenstein series $\mathcal{E}_P(\tau)$ is a harmonic weak Maaß form contained in $\mathcal{H}_{1,P^-}$ and satisfies $\xi(\mathcal{E}_P(\tau)) = E_P(\tau)$. The relation of (0.1) to (0.6) is that it is possible to give an explicit finite formula for the coefficients $\kappa(m,\mu)$ of $\mathcal{E}_P^+(\tau)$. They are of the form $\kappa(m,\mu) = a(m,\mu)\log(p)$ for a prime p depending on m. In the case that $U \cong k$ is an imaginary quadratic field of discriminant D, and $P \cong \mathfrak{a}$ is a fractional ideal in k with quadratic form $Q(x) = \mathrm{N}(x)/\mathrm{N}(\mathfrak{a})$ given by the norm on k, Schofer [Sch09b, Theorem 4.1] gave an explicit formula for $a(m,\mu) \in \mathbb{Z}$. Using this, it is possible to recover (0.1) from Schofer's result.

Theorem 1 by itself is not very enlightening, as the coefficients of $\widetilde{\Theta}_P^+(\tau,h)$ are not explicitly known (and not unique at all). Therefore, in the second part of this thesis we will study the coefficients of appropriate choices for $\widetilde{\Theta}_P(\tau,h)$ and their arithmetic meaning.

Moduli of CM elliptic curves

There is another interpretation of the coefficients of $\mathcal{E}_P(\tau)$ in terms of the degrees of certain special cycles, observed by Kudla, Rapoport and Yang [KRY04]. We let k be an imaginary quadratic number field of discriminant $D < 0$. For simplicity, we assume in the introduction that $D = -l$ for a prime $l \equiv 3 \bmod 4$ with $l > 3$. We will remove this restriction in the body of the thesis and work with odd fundamental discriminants. We write Cl_k for the class group of k and h_k denotes the class number of k. Let $P = \mathcal{O}$ be the ring of integers in k, which is a lattice of type $(2, 0)$ together with the quadratic form $Q(x) = \mathrm{N}(x)$. The dual lattice $P' = \mathfrak{d}_k^{-1}$ is given by the inverse different in k and discriminant group P'/P is cyclic of order $|D|$.

We let C_D be the (Deligne-Mumford) moduli stack of elliptic curves with complex multiplication by the ring of integers $\mathcal{O}$ of k. The coarse moduli scheme of C_D is isomorphic to $\operatorname{Spec} \mathcal{O}_H$, where H is the Hilbert class field of k. Kudla, Rapoport and Yang define cycles $\mathcal{Z}(m)$ on this arithmetic curve given by elliptic curves with certain “special endomorphisms”. These endomorphisms only occur in positive characteristic and the cycles $\mathcal{Z}(m)$ are always supported in the fiber above a unique prime p, which is non-split in k. The authors show the identity

$$- h_k \mathcal{E}_P^+(\tau) = 2 \sum_{m \in \frac{1}{|D|}\mathbb{Z}_{>0}} \widehat{\deg}\, \mathcal{Z}(m) e(m\tau)(\phi_m + \phi_{-m}) + 2\Lambda'(\chi_D, 0)\phi_0, \tag{0.7}$$

where $\Lambda'(\chi_D, s)$ denotes the derivative of the completed Dirichlet L-function for the character $\chi_D = \left(\frac{D}{\cdot}\right)$. Here, we wrote $\phi_{\pm m}$ for the basis elements $\phi_{\pm\mu}$ with $Q(\pm\mu) + m \in \mathbb{Z}$. If m is not represented by $-Q$ modulo $\mathbb{Z}$, then the corresponding coefficient vanishes. In our case this can be phrased in simpler terms: the coefficient of index m vanishes unless $Dm \in \mathbb{Z}$ is congruent to a square modulo $|D|$. Alternatively, we could also identify the space $\mathcal{H}_{1,P}$ with the space $\mathcal{H}_1^-(|D|, \chi_D)$ of scalar valued harmonic weak Maaß forms for $\Gamma_0(|D|)$ and character χ_D such that all Fourier coefficients of index n with $\chi_D(n) = 1$ vanish. For the proof of (0.7), the authors employ the explicit formulas for $\mathcal{E}_P(\tau)$ and a theorem of Gross [Gro86], which allows them to compute the degree of the cycle $\mathcal{Z}(m)$ explicitly, as well. A comparison of these two independent results establishes the equality (0.7). This has been generalized by Bruinier and Yang [BY09, Theorem 6.5] to odd negative fundamental discriminants using the same method.

Motivated by these results, we show that a similar relation holds for the coefficients of the holomorphic part of (a suitably normalized) $\widetilde{\Theta}_P(\tau, h)$. As no explicit construction of the forms $\widetilde{\Theta}_P(\tau, h)$ is known, we use a completely different method.

To state this second result in more detail, we write the pushforward of the cycle $\mathcal{Z}(m)$ to $\operatorname{Spec} \mathcal{O}_H$ as an Arakelov divisor with vanishing archimedean contribution as

$$\mathcal{Z}(m) = \sum_{\mathfrak{P} \subset \mathcal{O}_H} \mathcal{Z}(m)_{\mathfrak{P}} \mathfrak{P},$$

where the sum runs over all prime ideals of $\mathcal{O}_H$. In our case, the arithmetic degree above can be simply defined as

$$\widehat{\deg}\,\mathcal{Z}(m) = \sum_{\mathfrak{P}\subset\mathcal{O}_H} \mathcal{Z}(m)_{\mathfrak{P}} \log \mathrm{N}_{H/\mathbb{Q}}(\mathfrak{P}).$$

After an appropriate normalization of the map $C_D \to \operatorname{Spec}\mathcal{O}_H$, we obtain the following result, which will be restated in greater generality as Theorem 5.3.12.

Theorem 2. *For every* $[\mathfrak{a}] \in \mathrm{Cl}_k$ *there is a harmonic weak Maaß form* $\widetilde{\Theta}_{\mathfrak{a}}(\tau) \in \mathcal{H}_{1,P^-}$ *with holomorphic part*

$$\widetilde{\Theta}_{\mathfrak{a}}^{+}(\tau) = \sum_{m\gg-\infty} c^{+}(\mathfrak{a},m)e(m\tau)(\phi_m + \phi_{-m})$$

satisfying the following properties.

(i) We have $\xi(\widetilde{\Theta}_{\mathfrak{a}}(\tau)) = \Theta_{\mathfrak{a}}(\tau)$ *and*

$$\frac{1}{h_k}\sum_{\mathfrak{b}\in\mathrm{Cl_k}} \widetilde{\Theta}_{\mathfrak{a}\mathfrak{b}^2}(\tau) =: \widetilde{E}_P(\tau), \tag{0.8}$$

such that $\xi(\widetilde{E}_P(\tau)) = E_P(\tau)$ *and the principal part of* $\widetilde{E}_P(\tau)$ *vanishes.*

(ii) If $\chi_D(-Dm) = 1$, *we have* $c^{+}(\mathfrak{a},m) = 0$.

(iii) For all $m \in \mathbb{Q}$ *with* $\chi_D(-Dm) \neq 1$ *we have*

$$c^{+}(\mathfrak{a},m) = -\frac{2}{r}\log|\alpha(\mathfrak{a},m)|,$$

with $r \in \mathbb{Z}$ *depending only on* D *and* $\alpha(\mathfrak{a},m) \in \mathcal{O}_H$.

(iv) Furthermore, we have for $m > 0$ *that*

$$\mathrm{ord}_{\mathfrak{P}}(\alpha(\mathfrak{a}^2,m)) = 2r\mathcal{Z}(m)_{\mathfrak{P}^{\sigma}}$$

for all prime ideals $\mathfrak{P} \subset \mathcal{O}_H$, *where* $\sigma = \sigma(\mathfrak{a}^{-1})$ *corresponds to the ideal class of* $\mathfrak{a}^{-1}$ *under the Artin map* $(\cdot, H/k)$ *of class field theory.*

(v) If $m < 0$ *with* $\chi_D(-Dm) \neq 1$, *then* $\alpha(\mathfrak{a},m) \in \mathcal{O}_H^{\times}$.

Note that the theorem implies that we can write $c^{+}(\mathfrak{a},m) = \log(\beta(\mathfrak{a},m))$, where $\beta(\mathfrak{a},m)$ is contained in a finite field extension of H.

The idea of the proof is to use a certain seesaw dual reductive pair. This concept, introduced by Kudla [Kud84], explains many identities between theta liftings that look quite surprising at first glance. We use this to relate the coefficients of $\widetilde{\Theta}_P^{+}(\tau,h)$ to Borcherds products on modular curves. We consider the lattice $L = \mathbb{Z}^3$, this time with quadratic form

$Q(a,b,c) = N(a^2 - bc)$ for a suitable choice of $N \in \mathbb{Z}_{>0}$. In particular, in the case of prime discriminant $D = -l$, we can choose $N = l$. The fundamental identity for us is of the form

$$\Phi_P(h, f) = \Phi_L(z_P^{\pm}, h, g), \tag{0.9}$$

for an appropriate $g \in M^!_{1/2,L}$. The same identity has been used by Viazovska [Via12] to obtain explicit formulas for the regularized Petersson inner products of weakly holomorphic modular forms of weight one and the theta function $\Theta_P(\tau, 1)$ for the lattice $P = \mathcal{O}$ as above.

To obtain the relation to the special cycles, we work with an integral model $\mathcal{X}_0(N)$ for the compactification $X_0(N)$ of the modular curve $Y_0(N)$. We study the value of modular functions (that is, meromorphic modular forms of weight 0), rational over $\mathbb{Q}$, with zeros and poles supported on Heegner divisors *on the integral model.* The value of such a modular function at a CM point of fundamental discriminant can be described by its pullback to the stack C_D considered above. These Heegner divisors are generalizations of the collection of points $\{\alpha_Q\}$ described above, by considering equivalence classes of integral positive definite binary quadratic forms of the form $[Na, b, c]$ modulo the group $\Gamma_0(N)$. They have a moduli interpretation which extends to the integral model $\mathcal{X}_0(N)$.

As a corollary of our results, we are also able to show that the right hand side of (0.7) is (up to a constant) the holomorphic part of a harmonic weak Maaß form $\widetilde{E}_P(\tau)$ without using explicit formulas, which might shed some new light on the identity (0.7).

Theorem 3. *Let $\widetilde{E}_P(\tau)$ be the function in* (0.8). *Then we have*

$$-h_k \widetilde{E}_P^+(\tau) = 2 \sum_{m \in \frac{1}{|D|}\mathbb{Z}_{>0}} \widehat{\deg}\, \mathcal{Z}(m) e(m\tau)(\phi_m + \phi_{-m}) + c\phi_0,$$

where $c \in \mathbb{C}$ is a constant.

The proof of the theorem is given in Section 6.1. However, we are not able to conclude directly from this statement that $\widetilde{E}_P(\tau) = \mathcal{E}_P(\tau)$, without comparing coefficients. We can only infer that $\widetilde{E}_P^+(\tau)$ differs from $\mathcal{E}_P(\tau)$ by a cusp form. If it was possible to show that $\widetilde{E}_P(\tau)$ is orthogonal to cusp forms, then this could probably be interpreted as an arithmetic version of the Siegel-Weil formula in our case.

Explicit formulas

Having Theorem 2 available, we can now use the arithmetic theory of the special cycles to obtain explicit formulas for the valuations of the numbers $\alpha(\mathfrak{a}, m)$ in Theorem 1 at primes of the Hilbert class field. For this we now finally use Gross' formula for the length of the local rings of $\mathcal{Z}(m)$ in a similar fashion as in [KRY04].

For $m \in \mathbb{Q}_{>0}$, we define a set of rational primes by

$$\mathrm{Diff}(m) = \{p < \infty \mid (-m\mathrm{N}(\mathfrak{a}), D)_p = -1\}.$$

Moreover, for $[\mathfrak{a}] \in \mathrm{Cl}_\mathrm{k}$ and $n \in \mathbb{Z}$, we let

$$\rho(n, [\mathfrak{a}]) = \#\{\mathfrak{b} \subset \mathcal{O} \ \mid \ \mathrm{N}(\mathfrak{b}) = n, \ \mathfrak{b} \in [\mathfrak{a}]\}.$$

If m is a rational number and p is a rational prime which is non-split in k, we define

$$\nu_p(m) = \begin{cases} \frac{1}{2}(\mathrm{ord}_p(m) + 1), & \text{if } p \text{ is inert in } k, \\ (\mathrm{ord}_p(m\,|D|)), & \text{if } p \text{ is ramified in } k. \end{cases}$$

Finally, we let $o(m) = 1$ if $\mathrm{ord}_l(m\,|D|) > 0$ and $o(m) = 0$, otherwise.

With the same normalization of the map $C_D \to \operatorname{Spec} \mathcal{O}_H$ as in Theorem 2 we obtain the following result.

Proposition 4. *(i) We have $\mathcal{Z}(m)_{\mathfrak{P}} = 0$ unless $|\mathrm{Diff}(m)| = 1$.*

(ii) Assume that $\mathrm{Diff}(m) = \{p\}$. Then there is a unique prime ideal $\mathfrak{P}_0 \mid p$ fixed by complex conjugation, $\bar{\mathfrak{P}}_0 = \mathfrak{P}_0$. For $\mathfrak{P} = \mathfrak{P}_0^\sigma$, where $\sigma = \sigma(\mathfrak{a})$ corresponds to the ideal class of $\mathfrak{a}$ under the Artin map $(\cdot, H/k)$, we have

$$\mathcal{Z}(m)_{\mathfrak{P}} = 2^{o(m)-1} \nu_p(m) \rho(m\,|D|\,/p, [\mathfrak{a}]^{-2}),$$

We remark that in [GZ85], the authors give an analytic and an algebraic proof of (0.1). The latter is given for prime discriminants only, but does in fact give the valuation of the function $\Psi(z, d)$ at CM points at primes of the Hilbert class field. This result has later been generalized by Dorman [Dor88].

We also remark that Duke and Li [DL12] independently obtained related results for prime discriminants using different methods. The authors show the existence of preimages of the theta functions under ξ with coefficients of the holomorphic part given by logarithms of algebraic integers together with an action of the Galois group. Based on numerical evidence, they formulated a conjecture on the prime factorization of these numbers. Theorem 2 and Proposition 4 provide a proof of this conjecture. We also remark that some of our results, in particular Theorem 1 and a preliminary version of the description of the coefficients of $\widetilde{\Theta}_P^+(\tau, h)$, were announced in a preprint [Ehl12].

Outlook

Following the philosophy of the Kudla program [Kud04], we are led to (formally) form an arithmetic generating series with coefficients in $\widehat{\mathrm{CH}}^1(C_D)$ given by

$$\hat{\Phi}_P(\tau) = \sum_{m>0} \hat{\mathcal{Z}}(m) e(m\tau)(\phi_m + \phi_{-m}). \tag{0.10}$$

Here, we completed $\mathcal{Z}(m)$ to an arithmetic divisor $\hat{\mathcal{Z}}(m) = (\mathcal{Z}(m), \Phi_{P,m})$, where $\Phi_{P,m}$ is given by the theta lift of a harmonic weak Maaß form with principal part $\frac{1}{2}q^{-m}(\phi_m + \phi_{-m})$.

It is possible to show that the degree generating series $\widehat{\deg}\,\hat{\Phi}_P(\tau)$ is modular. However, this is a "trivial" result because, in fact, the series vanishes identically! This follows for instance from Theorem 6.5 of [BY09]. But this result might nevertheless be seen as evidence for the modularity of (0.10) itself.

Proving modularity of the generating series $\hat{\Phi}_P$ might be difficult because the cycles $\hat{\mathcal{Z}}(m)$ are torsion by Theorem 2. Therefore, the usual method of passing to the Chow group with rational coefficients is not an option here. Thus, even though it might be a very interesting object to study, it is not clear at the moment how to overcome these issues. We are looking forward to investigate this problem in the future.

There are several other applications and open problems that are closely related to the presented results that we did not work out at the time of finishing this thesis. An obvious application would be to consider the values of the theta lift averaged over twisted CM cycles, by applying an automorphic character of $\mathrm{SO}_U(\mathbb{A})$ to the CM cycle $Z(U)$. The resulting cycles would be similar to the twisted Heegner divisors considered in [BO10, AE13].

Another application, which was in fact the initial motivation to study the value of Borcherds products at an individual CM point, would be to apply the methods of this thesis to "twisted Borcherds products" as in [BO10] and [BY07, Ehl10]. This requires, however, an extension of our results to non-fundamental discriminants, which will be a future project probably involving further technical difficulties.

1 Preliminaries

In this chapter the basic objects that we are concerned with in this thesis are introduced. The exposition is usually brief and does not include proofs but several references are given. However, for some facts which, to the best knowledge of the author, cannot be found in the literature, we will work out a proof.

1.1 Quadratic forms and lattices

We start by recalling some basic notions in the theory of quadratic forms since these will be ubiquitous in the present thesis. The basic references are the books by Kitaoka [Kit93], Kneser [Kne02] and Serre [Ser73].

Let R be a ring with unity 1 and let M be a finitely generated R-module.

Definition 1.1.1. A *quadratic form* is a map $Q : M \to R$ such that

(i) $Q(rx) = r^2 Q(x)$ for all $r \in R$ and all $x \in M$,

(ii) $(x, y) := Q(x + y) - Q(x) - Q(y)$ is a bilinear form.

The pair (M, Q) is called a *quadratic module* over R. If $A = k$ is a field this pair is also called a *quadratic space* over k. We will sometimes also write x^2 for $(x, x) = 2Q(x)$.

If 2 is invertible in R, the second condition implies the first one.

Example 1.1.2. Let r, s be non-negative integers. We denote by $\mathbb{R}^{r,s}$ the quadratic space over $\mathbb{R}$ given by $\mathbb{R}^{r+s}$ with the quadratic form

$$Q(x) = x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_{r+s}^2$$

for $x = (x_1, \ldots, x_n)$.

Definition 1.1.3. For a submodule $N \subset M$ of a quadratic module, we write

$$N^\perp = \{x \in M \mid (x, y) = 0 \text{ for all } y \in N\}$$

for the orthogonal complement of N in M. Similarly, for an element $x \in M$, the set $x^\perp$ is defined in the same way. We say that a quadratic module is non-degenerate if $M^\perp = \{0\}$.

Definition 1.1.4. Let (M, Q) and (M', Q') be quadratic modules. An R-linear map $\sigma : M \to M'$ is called an *isometry* if σ is injective and

$$Q'(\sigma(m)) = Q(m)$$

for all $m \in M$. Two quadratic modules M, M' are called *isometric* if there is a bijective isometry $\sigma : M \to M'$.

Definition 1.1.5. The *orthogonal group* $\mathrm{O}_V = \mathrm{O}_{(V,Q)}$ of (V, Q) is defined to be the group of all isometries of V. The *special orthogonal group* $\mathrm{SO}_V \subset \mathrm{O}_V$ is the subgroup of all isometries of determinant one.

For the rest of this section we assume that $R = k$ a field, not of characteristic 2, and we let (V, Q) be a quadratic space of dimension $n = \dim V$ over k.

Definition 1.1.6. Let $(e_i)_{i=1,\dots,n}$ be a basis of V. The *Gram matrix* of Q corresponding to this basis is the matrix $A_Q = A_V = (a_{ij})$, where $a_{ij} = (e_i, e_j)$.

For $x = \sum_{i=1}^{n} x_i e_i$, we have

$$(x, x) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j.$$

The *determinant* of V is defined as

$$\det(Q) = \det(V) = \det((V, Q)) := \det(A_Q).$$

The determinant $\det(Q)$ is well defined as an element of $k^\times/(k^\times)^2$ (if it is nonzero).

Proposition 1.1.7. *Let (V, Q) be a quadratic space over $\mathbb{R}$. There exist non-negative integers r, s, such that V is isometric to $\mathbb{R}^{r,s}$. The pair (r, s) is uniquely determined by V.*

This motivates the following definition.

Definition 1.1.8. The pair (r, s) is called the *type* of V. The integer $r - s$ is called the *signature* of V. We denote by $\mathrm{O}(r, s)$ the orthogonal group of $\mathbb{R}^{r,s}$. For a rational quadratic space V, that is, a quadratic space over $\mathbb{Q}$, the signature and type of V are defined to be the corresponding invariants of $V \otimes_{\mathbb{Q}} \mathbb{R}$.

Definition 1.1.9. A *lattice* is a finitely generated quadratic module (L, Q) over $\mathbb{Z}$. We call a lattice *integral*, if the bilinear form (x, y) takes only values in $\mathbb{Z}$ for all $x, y \in L$. We call it *even*, if the quadratic form is integral valued on L, that is, $Q(x) \in \mathbb{Z}$ for all $x \in L$.

Definition 1.1.10. For a lattice L we define the *dual lattice* L' of L as

$$L' := \{x \in L \otimes_{\mathbb{Z}} \mathbb{Q} \mid (x, y) \in \mathbb{Z} \text{ for all } y \in L\}.$$

For an integral lattice, the finite abelian group L'/L is called the *discriminant group* of L.

Lemma 1.1.11. *If L is an integral lattice and A_L is the Gram matrix corresponding to a lattice basis of L, as in Definition 1.1.6, we have*

$$|L'/L| = |\det A_L|.$$

Lemma 1.1.12. *If L is integral, the bilinear form induces a well-defined map*

$$(\cdot,\cdot) : L'/L \times L'/L \to \mathbb{Q}/\mathbb{Z}.$$

If L is even, the quadratic form Q induces a well-defined map

$$Q : L'/L \to \mathbb{Q}/\mathbb{Z}.$$

Definition 1.1.13. A tuple (A, Q), where A is a finite abelian group and $Q : A \to \mathbb{Q}/\mathbb{Z}$ is a quadratic form as in Lemma 1.1.12 is called a *finite quadratic module* or *discriminant form*.

Definition 1.1.14. Let N be the smallest positive integer, such that $NQ(\lambda) \in \mathbb{Z}$ for every $\lambda \in L'$. Then N is called the *level* of the lattice L.

Definition 1.1.15. We define the orthogonal group of a finite quadratic module $\mathrm{O}(A, Q)$ to be the group of all group homomorphisms $\sigma : A \to A$ that preserve the quadratic form. If $A = L'/L$ and the quadratic form is clear in the context we also write $\mathrm{O}(A, Q) = \mathrm{O}(L'/L)$.

1.1.1 The Clifford algebra and the spin groups

We briefly recall the definition of the general spin group and the spin group. A good reference is Kitaoka's book [Kit93]. All the facts we need are also discussed in Section 2.2 of the article of Bruinier in [BvdGHZ08], which we follow in parts. Let (V, Q) be a finitely generated quadratic module over a ring R.

The *Clifford algebra* C_V of V is an algebra containing V and R and such that taking the square v^2 of a vector $v \in V \subset C_V$ corresponds to the quadratic form $Q(v)$. It is constructed as follows. Consider the tensor algebra

$$T_V = \bigoplus_{m=0}^{\infty} V^{\otimes m}$$

and the two-sided ideal I_V generated by the set

$$\{v \otimes v - Q(v) \mid v \in V\}.$$

The Clifford algebra is defined as $C_V = T_V/I_V$.

We abbreviate $v_1 \cdots v_r = v_1 \otimes \cdots \otimes v_r$. Note that we have by definition

$$v^2 = Q(v) \text{ and } uv + vu = (u, v) \tag{1.1.1}$$

for all $u, v \in V \subset C_V$.

If V is free and $v_1, \dots, v_n$ is a basis of V, then the elements

$$v_{i_1} \cdots v_{i_r}, \quad 1 \leq i_1 < \dots < i_r \leq n \text{ and } 0 \leq r \leq n$$

form a basis of C_V. The Clifford algebra decomposes into a direct sum

$$C_V = C_V^0 \oplus C_V^1,$$

where C_V^0 is generated by elements that are a product of an even number of basis vectors of V. Similarly, C_V^1 is generated by such products of odd length. The subalgebra C_V^0 is called the *even Clifford algebra* of V. We have the *canonical automorphism*

$$J : C_V \to C_V,$$

which is induced by multiplication with -1 on V and the *canonical involution*

$${}^t : C_V \to C_V, \quad (x_1 \otimes \cdots \otimes x_m)^t = x_m \otimes \cdots \otimes x_1.$$

Moreover, the *Clifford norm* is defined as

$$\mathrm{N} : C_V \to C_V, \quad \mathrm{N}(x) = x^t x.$$

For $v \in V$, we have $\mathrm{N}(v) = Q(v)$.

The *Clifford group* is defined to be

$$\mathrm{CG}_V = \{x \in C_V \mid x \text{ invertible and } xVJ(x)^{-1} = V\}.$$

Finally, we can define the groups GSpin and Spin as

$$\begin{aligned} \mathrm{GSpin}_V &= \mathrm{CG}_V \cap C_V^0, \\ \mathrm{Spin}_V &= \{x \in \mathrm{GSpin}_V \mid \mathrm{N}(x) = 1\}. \end{aligned}$$

From now on we assume that $k = R$ is a field, not of characteristic 2. We can consider the groups GSpin_V and Spin_V as affine algebraic groups over k because the Clifford algebra satisfies a universal property. If A is a k-algebra then the group of A-valued points $\mathrm{CG}_V(A)$ of CG_V is given by $\mathrm{CG}_{V(A)}$.

One has the following exact sequence of algebraic groups:

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{GSpin}_V \longrightarrow \mathrm{SO}_V \longrightarrow 1. \tag{1.1.2}$$

Here, $\mathbb{G}_m$ denotes the multiplicative (algebraic) group.

In low dimensions, the group GSpin is rather easy to characterize. In particular, if $\dim(V) \leq 4$, we have

$$\mathrm{GSpin}_V = \{x \in C_V^0 \mid \mathrm{N}(x) \in k^\times\}.$$

1.2 The Hilbert class field

We recall only very briefly some facts from class field theory that we need later, in particular the Artin map. We refer to [Sil94, Chapter II] or [Shi94] for details. Let k be a totally imaginary field and let L be a finite abelian extension of k. Write $\mathcal{O}_L$ for the ring of integers in L. Let $\mathfrak{p}$ be a prime ideal of k that does not ramify in L. There is a unique element $\sigma_{\mathfrak{p}} \in \mathrm{Gal}(L/k)$, such that

$$\sigma_{\mathfrak{p}}(x) \equiv x^{\mathrm{N}_{L/k}(\mathfrak{p})} \bmod \mathfrak{P}$$

for all $x \in \mathcal{O}_L$ and any prime ideal $\mathfrak{P} \mid \mathfrak{p}$. If $\mathfrak{c}$ is an integral ideal of k which is divisible by all primes that ramify in L/k and $I(\mathfrak{c})$ is the group of fractional ideals that are relatively prime to $\mathfrak{c}$, then the *Artin map* is defined as

$$(\cdot, L/k) : I(\mathfrak{c}) \to \mathrm{Gal}(L/k),$$

by extending the map $\sigma_{\mathfrak{p}}$ linearly.

Now we specialize to the situation which concerns us most in this thesis. We let k be an imaginary quadratic field and H be the Hilbert class field of k. This is the maximal unramified abelian extension of k. It corresponds to the so-called ray class field for $\mathfrak{c} = 1$. By class field theory, we obtain in this case [Sil94, II, Example 3.3] an isomorphism via the Artin map

$$(\cdot, H/k) : \mathrm{Cl_k} \to \mathrm{Gal}(H/k).$$

We will use the convention that we write $\sigma(\mathfrak{a}) = \sigma([\mathfrak{a}])$ for $([\mathfrak{a}], H/k)$ for the image of the fractional ideal $\mathfrak{a}$ under this map.

1.3 Symmetric domains and Shimura varieties

A good reference for this section, giving a few more details, is the article of Bruinier in [BvdGHZ08]. Let $n \geq 0$ be an integer and let V be a quadratic space over $\mathbb{Q}$ of type $(2, n)$ with a non-degenerate quadratic form Q. We abbreviate $H := \mathrm{GSpin}_V$.

Remark 1.3.1. Our setup is basically the same as in [Kud03, Sch09b, BY09]. However, we warn the reader that we are working with a quadratic space of type $(2, n)$, whereas Kudla's setup, which also has been adopted by Bruinier, Yang and Schofer, always uses type $(n, 2)$ quadratic spaces.

Let $C \subset \mathrm{SO}_V(\mathbb{R})$ be a maximal compact subgroup of $\mathrm{SO}_V(\mathbb{R})$. Since V is a quadratic space over $\mathbb{Q}$ of type $(2, n)$, the quotient $\mathrm{SO}_V(\mathbb{R})/C$ is a symmetric space with a complex structure. There are several ways to realize $\mathrm{SO}_V(\mathbb{R})/C$. We will briefly describe two of them that will be used frequently.

1.3.1 The Grassmannian model

Consider the *Grassmannian* $\mathbb{D}$ of oriented two-dimensional positive definite subspaces of $V(\mathbb{R}) = V \otimes_{\mathbb{Q}} \mathbb{R}$. That is, we let

$$\mathbb{D} := \{z^{\pm} \mid z \subset V(\mathbb{R}), \dim z = 2,\ Q|_z > 0\}.$$

Here, for each 2-dimensional positive definite subspace $z \subset V(\mathbb{R})$, we write z^+ and z^- for z together with one of the two possible choices of orientation. The group $H(\mathbb{R})$ acts naturally on $\mathbb{D}$ and this action is transitive by Witt's theorem. The Grassmannian has two connected components and each of them is isomorphic to the symmetric space $\mathrm{SO}_V(\mathbb{R})/C$.

1.3.2 The projective model

We extend the bilinear form $\mathbb{C}$-bilinearly to the complex vector space $V(\mathbb{C})$, and define the projective space

$$P(V(\mathbb{C})) := (V(\mathbb{C}) \setminus \{0\})/\mathbb{C}^{\times}. \tag{1.3.1}$$

Then we have that

$$\mathcal{K} := \{[Z] \in P(V(\mathbb{C})) \mid (Z, Z) = 0,\ (Z, \overline{Z}) > 0\} \tag{1.3.2}$$

is a complex manifold of dimension n which has two connected components. The group $H(\mathbb{R})$ acts on $\mathcal{K}$ via $\gamma[Z] := [\gamma Z]$ for $\gamma \in H(\mathbb{R})$.

We write $Z = X + iY \in \mathcal{K}$ with $X, Y \in V(\mathbb{R})$ for the real and imaginary parts of Z.

Lemma 1.3.2 ([BvdGHZ08], Lemma 2.17)**.** *The map $\mathcal{K} \to \mathbb{D}, [Z] \mapsto \mathbb{R}X + \mathbb{R}Y$ is a real analytic isomorphism.*

Lemma 1.3.2 gives the symmetric domain $\mathbb{D}$ a complex structure.

1.3.3 Shimura varieties

We introduce some notation for the rest of this thesis. By a place of a number field k we mean an equivalence class of valuations of k. They are represented by the prime ideals (the finite places), the embeddings of k into $\mathbb{R}$ (the real archimedean places) and pairs of complex conjugate embeddings into $\mathbb{C}$ (the complex archimedean places). We indicate that a place $\mathfrak{p}$ is an archimedean place by writing $\mathfrak{p} \mid \infty$ and otherwise by $\mathfrak{p} \nmid \infty$. For a place $\mathfrak{p}$ of k we let $k_{\mathfrak{p}}$ denote the completion of k with respect to the valuation $v_{\mathfrak{p}}$ corresponding to $\mathfrak{p}$. For non-archimedean $\mathfrak{p}$, we denote by $\mathcal{O}_{\mathfrak{p}} \subset k_{\mathfrak{p}}$ the corresponding valuation ring. We consider the adeles over k, which is the restricted product

$$\mathbb{A}_k = \prod_{\mathfrak{p}}{}' k_{\mathfrak{p}}$$

with respect to $\mathcal{O}_\mathfrak{p}$. That is, the elements of $\mathbb{A}_k$ are families $(\alpha_\mathfrak{p})$ with $\alpha_\mathfrak{p} \in k_\mathfrak{p}$ and almost all $\alpha_\mathfrak{p}$ are integral. Addition and multiplication are defined component-wise. The finite adeles are denoted by

$$\mathbb{A}_{k,f} = \prod_{\mathfrak{p}\nmid\infty}{}' k_\mathfrak{p}.$$

Moreover, we denote by $\mathbb{A}_k^\times$ and $\mathbb{A}_{k,f}^\times$ the groups of (finite) ideles over k. These consist of families $(\alpha_\mathfrak{p})$ with $\alpha_\mathfrak{p} \in k^\times$ and $\alpha_\mathfrak{p} \in \mathcal{O}_\mathfrak{p}^\times$ for almost all $\mathfrak{p}$. Here, $\mathcal{O}_\mathfrak{p}^\times = k_\mathfrak{p}^\times$ for $\mathfrak{p} \mid \infty$ complex and $\mathcal{O}_\mathfrak{p}^\times = \mathbb{R}_+^\times$ for $\mathfrak{p}$ real.

If $k = \mathbb{Q}$, we denote by $\mathbb{A}$ the adeles over $\mathbb{Q}$ and by $\mathbb{A}_f$ the finite adeles.

Let $K \subset H(\mathbb{A}_f)$ be a compact open subgroup. We write X_K for the associated *Shimura variety* with complex points

$$X_K(\mathbb{C}) = H(\mathbb{Q})\backslash(\mathbb{D} \times H(\mathbb{A}_f)/K).$$

We have the following decomposition of X_K that will be useful later on.

Lemma 1.3.3 ([Mil05, Lemma 5.13])**.** *Let $\mathcal{C}$ be a set of representatives for the double coset space $H(\mathbb{Q})\backslash H(\mathbb{A}_f)/K$ and let $\mathbb{D}^+$ be a connected component of $\mathbb{D}$. Then*

$$X_K \cong \coprod_{h\in\mathcal{C}} \Gamma_h\backslash\mathbb{D}^+,$$

where Γ_h is the subgroup $hKh^{-1} \cap H(\mathbb{Q})^+$ of $H(\mathbb{Q})^+$. Here, $H(\mathbb{Q})^+$ denotes the connected component of the identity. If we endow $\mathbb{D}$ with its usual topology and $H(\mathbb{A}_f)$ with its adelic topology, this becomes a homeomorphism.

Example 1.3.4. Let N be a positive integer and consider the congruence subgroup $\Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z})$, defined by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \ \mid \ c \equiv 0 \bmod N \right\}.$$

We will describe how to obtain the modular curve $Y_0(N) := \Gamma_0(N)\backslash\mathbb{H}$. Consider the vector space $V := \{x \in M_2(\mathbb{Q}) \mid \mathrm{tr}(x) = 0\}$ and define the quadratic form by $Q(x) = -N\det(x)$. The corresponding bilinear form is $(x, y) = N\,\mathrm{tr}(xy)$. The space (V, Q) has signature $(2, 1)$.

The even part of the Clifford algebra is $C^0(V) = M_2(\mathbb{Q})$ and $H = \mathrm{GSpin}_V \cong \mathrm{GL}_2$. The action of $\gamma \in H$ on $x \in V$ is given by

$$\gamma.x = \gamma x \gamma^{-1}.$$

We have isomorphisms $\mathbb{H} \cup \overline{\mathbb{H}} \to \mathcal{K} \to \mathbb{D}$ via

$$z = x + iy \mapsto \left[\begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix}\right] \mapsto \mathbb{R}\Re\begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix} \oplus \mathbb{R}\Im\begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix}.$$

The action of $\gamma \in \mathrm{GL}_2 = \mathrm{GSpin}_V$ is explicitly given by

$$\gamma \cdot \begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix} = \frac{(cz+d)^2}{\det(\gamma)} \begin{pmatrix} \gamma z & -(\gamma z)^2 \\ 1 & -\gamma z \end{pmatrix},$$

where γz is the action via linear fractional transformations on $\mathbb{H} \cup \bar{\mathbb{H}}$. For a prime p, let

$$K_p = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}_p) \mid c \in N\mathbb{Z}_p \}$$

and

$$K = \prod_p K_p.$$

Then K is a compact open subgroup of the adelic group $H(\mathbb{A}_f)$ and by strong approximation [Bum97, Theorem 3.3.1], we have

$$H(\mathbb{A}_f) = H(\mathbb{Q})K \text{ and } H(\mathbb{A}) = H(\mathbb{Q})H(\mathbb{R})^+K.$$

Therefore, we obtain from Lemma 1.3.3 that

$$X_K \cong \Gamma \backslash \mathbb{D}^+,$$

where we choose $\mathbb{D}^+ = \mathbb{H}$ and Γ_1 is given by $\Gamma = H(\mathbb{Q}) \cap H(\mathbb{R})^+K \cong \Gamma_0(N)$. The isomorphism is explicitly given by

$$Y_0(N) \to X_K, \ \Gamma_0(N)z \mapsto H(\mathbb{Q})(z,1)K. \tag{1.3.3}$$

1.3.4 Heegner divisors

We conclude by describing a natural family of divisors on the Shimura varieties of orthogonal type that will play an important role. We also refer to [BY09], [Kud03] and [Kud97]. Let $L \subset V(\mathbb{Q})$ be an even lattice and let $K \subset H(\mathbb{A}_f)$ be an open compact subgroup such that $KL \subset L$ and K acts trivially on L'/L. We will make these assumptions throughout this section. In this situation, we consider the group

$$\Gamma_K = H(\mathbb{Q}) \cap K,$$

which is an arithmetic subgroup of $H(\mathbb{Q})$.

Let $x \in V(\mathbb{Q})$ be a vector of negative norm and denote the orthogonal complement $x^\perp \subset V(\mathbb{Q})$ by V_x. We let H_x be the stabilizer of x in H. Then $H_x \cong \mathrm{GSpin}(V_x)$ and the Grassmannian

$$\mathbb{D}_x = \{ z \in \mathbb{D} \mid z \perp x \} \subset \mathbb{D}$$

defines an analytic set of codimension one in $\mathbb{D}$.

Let $h \in H(\mathbb{A}_f)$ and consider

$$H_x(\mathbb{Q})\backslash\mathbb{D}_x \times H_x(\mathbb{A}_f)/(H_x(\mathbb{A}_f) \cap hKh^{-1}) \longrightarrow X_K \tag{1.3.4}$$

given by

$$(z, h_1) \mapsto (z, h_1 h).$$

The image of this map defines a divisor $Z(x, h)$ on X_K which is rational over $\mathbb{Q}$ [Kud97]. For $m \in \mathbb{Q}_{<0}$ consider the quadric $\Omega_m \subset V$ given by

$$\Omega_m = \{x \in V \ \mid \ Q(x) = m\}.$$

By Witt's theorem, if $\Omega_m(\mathbb{Q}) \neq \emptyset$, the orthogonal group acts transitively and thus for every $x_0 \in \Omega_m(\mathbb{Q})$ we have $\Omega_m(\mathbb{Q}) = H(\mathbb{Q})x_0$ and $\Omega_m(\mathbb{A}_f) = H(\mathbb{A}_f)x_0$. Here,

$$\Omega_m(\mathbb{A}_f) = \left(\prod_{p \nmid \infty} \Omega_m(\mathbb{Q}_p)\right) \cap V(\mathbb{A}_f)$$

and $\Omega_m(\mathbb{Q}_p) = \{x \in V(\mathbb{Q}_p) \ \mid \ Q(x) = m\}$. Moreover, for any compact open subgroup $K \subset H(\mathbb{A}_f)$, we have $\Omega_m(\mathbb{A}_f) = K\Omega_m(\mathbb{Q})$ (see Lemma 5.1 of [Kud97]).

We let $S(V(\mathbb{A}))$ be the space of Schwartz functions on $V(\mathbb{A})$. That is, the space $S(V(\mathbb{R}))$ is the usual space of Schwartz (rapidly decreasing) functions on $V(\mathbb{R})$ and $S(V(\mathbb{Q}_p))$ is the space of locally constant functions $V(\mathbb{Q}_p) \to \mathbb{C}$ with compact support. We consider the finite dimensional subspace

$$S(V(\mathbb{A}_f)) = \bigotimes_p S(V(\mathbb{Q}_p))$$

and $S(V(\mathbb{A})) = S(V(\mathbb{A}_f)) \otimes S(V(\mathbb{R}))$.

Let L be an even lattice and $\mu \in L'/L$. We write $\hat{L} = L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$ and let $\phi_\mu \in S(V(\mathbb{A}_f))$ be the characteristic function of $\mu + \hat{L}$ for $\mu \in L'/L$. Here, $\hat{\mathbb{Z}} = \prod_{p<\infty} \mathbb{Z}_p$. Let

$$S_L = \bigoplus_{\mu \in L'/L} \mathbb{C}\phi_\mu \subset S(V(\mathbb{A}_f)).$$

Definition 1.3.5. For a Schwartz function $\varphi \in S_L$ write

$$\mathrm{supp}(\varphi) \cap \Omega_m(\mathbb{A}_f) = \bigsqcup_j K\xi_j^{-1}x_0,$$

where $\xi_j \in H(\mathbb{A}_f)$. We define the special divisor

$$Z(m, \varphi) := \sum_j \varphi(\xi_j^{-1}x_0) Z(x_0, \xi_j).$$

For $\mu \in L'/L$, briefly write $Z(m,\mu) := Z(m,\phi_\mu)$.

Lemma 1.3.6 (Proposition 5.4 in [Kud97]). *Assume that $H(\mathbb{A}_f) = H(\mathbb{Q})K$. Then we have*

$$Z(m,\varphi) = \sum_{x \in \Gamma_K \backslash \Omega_m(\mathbb{Q})} \varphi(x)\,\mathrm{pr}(\mathbb{D}_x, 1).$$

Here, $\mathrm{pr} : \mathbb{D} \times H(\mathbb{A}_f) \longrightarrow X_K$ *denotes the natural projection.*

In this context, we also let

$$L_m = \Omega_m \cap L' \text{ and } L_{m,\mu} = L_m \cap (\mu + L).$$

Definition 1.3.7. If L is an even lattice and $\lambda \in L'$, we define the *content* of λ to be the largest positive integer, such that

$$\frac{1}{n}\lambda \in L'.$$

We write $\mathrm{cont}(\lambda) = \mathrm{cont}_L(\lambda)$ for the content of λ. A vector $\lambda \in L'$ is called *primitive* (with respect to L'), if $\mathrm{cont}(\lambda) = 1$.

Remark 1.3.8. We note that primitivity is a relative condition with respect to the lattice.

With the same assumption as in Lemma 1.3.6, namely $H(\mathbb{A}_f) = H(\mathbb{Q})K$, we can also consider

$$L^0_{m,\mu} = \{\lambda \in L_{m,\mu} \mid \lambda \text{ is primitive}\}.$$

Then the group Γ_K acts on $L^0_{m,\mu}$. Thus, we may consider

$$Z^0(m,\mu) = \sum_{\Gamma_K \backslash L^0_{m,\mu}} \mathrm{pr}(\mathbb{D}_x, 1).$$

Lemma 1.3.9. *With the same assumptions as in Lemma 1.3.6, we have*

$$Z(m,\mu) = \sum_{n^2 \mid m} \sum_{\substack{\nu \in L'/L \\ n\nu = \mu}} Z^0\left(\frac{m}{n^2}, \nu\right).$$

Proof. By Lemma 1.3.6, we know that

$$Z(m,\mu) = \sum_{\lambda \in \Gamma_K \backslash L_{m,\mu}} \mathrm{pr}(\mathbb{D}_\lambda, 1).$$

Moreover, we have $\mathbb{D}_\lambda = \mathbb{D}_{n\lambda}$ for any $n \in \mathbb{Q}$. Therefore, we have to show that

$$L_m = \bigsqcup_{n^2 \mid m} \bigsqcup_{\substack{\nu \in L'/L \\ n\nu = \mu}} nL^0_{m/n^2,\nu}.$$

Let $\lambda \in L^0_{m/n^2,\nu}$ with $n\nu = \mu$. Then $Q(n\lambda) = n^2 Q(\lambda)$ and $n\lambda \equiv \mu \bmod L$. Thus, $n\lambda \in L_{m,\mu}$. In the other direction, for $\lambda \in L_{m,\mu}$ we let $n = \mathrm{cont}(\lambda)$. Then

$$\lambda' = \frac{1}{n}\lambda \in L^0_{m/n^2,\nu},$$

for some $\nu \in L'/L$ with $n\nu = \mu$.

Moreover, the union on the right hand side is clearly a disjoint union. □

1.4 The Weil representation

Let $n \geq 0$ be an integer and let V be a quadratic space over $\mathbb{Q}$ of type $(2, n)$ with a non-degenerate quadratic form Q. We follow [Kud03] and also refer to [Gel76] for details.

In this section, we let $G = \mathrm{SL}_2$, viewed as an algebraic group over $\mathbb{Q}$ and denote by $\widetilde{G}_{\mathbb{A}}$ the 2-fold metaplectic cover of $G(\mathbb{A})$. We write $\widetilde{G}_{\mathbb{R}}$ for the inverse image of $G(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$ under the covering map $\widetilde{G}_{\mathbb{A}} \to G(\mathbb{A})$. It is often useful to identify $\widetilde{G}_{\mathbb{R}}$ with

$$\left\{(g, \phi(\tau)) \mid g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathbb{R}),\ \phi\colon \mathbb{H} \to \mathbb{C} \text{ holomorphic},\ \phi^2(\tau) = c\tau + d\right\},$$

endowed with the multiplication $(g_1, \phi_1(\tau))(g_2, \phi_2(\tau)) = (g_1 g_2, \phi_1(g_2\tau)\phi_2(\tau))$.

Moreover, we let $C = \mathrm{SL}_2(\hat{\mathbb{Z}}) \subset G(\mathbb{A}_f)$ and denote by $\widetilde{C}$ the inverse image of C in $\widetilde{G}_{\mathbb{A}}$. Similarly, we let $C_\infty = \mathrm{SO}_2(\mathbb{R}) \subset G(\mathbb{R})$ and denote by $\widetilde{C}_\infty$ its inverse image in $\widetilde{G}_{\mathbb{R}}$. We let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and denote by $\widetilde{\Gamma} = \mathrm{Mp}_2(\mathbb{Z})$ the inverse image of $\Gamma \subset G(\mathbb{R})$ inside $\widetilde{G}_{\mathbb{R}}$.

Finally, we write $\widetilde{G}_{\mathbb{Q}} = s(G(\mathbb{Q}))$ for the image under the canonical section $s : G(\mathbb{Q}) \to \widetilde{G}_{\mathbb{A}}$. With this notation, we have

$$\widetilde{G}_{\mathbb{A}} = \widetilde{G}_{\mathbb{Q}} \widetilde{G}_{\mathbb{R}} \widetilde{C}$$

and $s(\Gamma) = \widetilde{G}_{\mathbb{Q}} \cap \widetilde{G}_{\mathbb{R}} \widetilde{C}$.

For $\gamma' \in \widetilde{\Gamma}$, there are unique $\gamma \in \Gamma, \gamma'' \in \widetilde{C}$ such that $s(\gamma) = \gamma'\gamma''$. The map $\widetilde{\Gamma} \to \widetilde{C}$ defined by $\gamma' \mapsto \gamma''$ is a homomorphism. Recall that we denote by $S(V(\mathbb{A}))$ the space of Schwartz-Bruhat functions on $V(\mathbb{A})$. The homomorphism $\widetilde{\Gamma} \to \widetilde{C}$ gives a representation ρ of $\widetilde{\Gamma}$ on $S(V(\mathbb{A}_f))$ by defining $\rho(\gamma') = \omega_f(\gamma'')$, where ω_f is the so-called (finite) *Weil representation*, determined by the standard additive character ψ of $\mathbb{A}/\mathbb{Q}$, such that $\psi_\infty(x) = e(x) = e^{2\pi i x}$. This is referred to as the Weil representation of $\widetilde{\Gamma}$. We will not give the most general definition of this representation. Instead we will now describe the finite dimensional subrepresentations we are interested in. For the interested reader, Gelbart [Gel76] is a good reference.

Let $L \subset V$ be an even lattice and let $\mu \in L'/L$. Recall that we write $\hat{L} = L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$ and $\phi_\mu \in S(V(\mathbb{A}_f))$ for the characteristic function of $\mu + \hat{L}$ for $\mu \in L'/L$. Recall that we consider the space

$$S_L = \bigoplus_{\mu \in L'/L} \mathbb{C}\phi_\mu \subset S(V(\mathbb{A}_f)). \tag{1.4.1}$$

We write $\langle \phi, \chi \rangle$ for the standard bilinear pairing between $S(V(\mathbb{A}))$ and its dual $S(V(\mathbb{A}))^*$. In particular

$$\langle a\phi_\mu, b\phi_\nu \rangle = ab\, \delta_{\mu,\nu}$$

for $a, b \in \mathbb{C}$ and $\mu, \nu \in L'/L$, where we identify S_L with its dual.

Remark 1.4.1. We note that the space S_L can be identified with the group ring $\mathbb{C}[L'/L]$ of the finite abelian group L'/L via $\phi_\mu \mapsto \mathfrak{e}_\mu$, if $\{\mathfrak{e}_\mu \mid \mu \in L'/L\}$ is the standard basis for $\mathbb{C}[L'/L]$. In the latter space the corresponding scalar product is conjugate-linear in the second argument.

The representation ρ of $\widetilde{\Gamma}$ acts on S_L because S_L is stable under $\omega_f \mid_{\widetilde{K}}$. Thus, we obtain a finite dimensional representation $\rho_L : \widetilde{\Gamma} \to \operatorname{Aut} S_L$. This representation can be described explicitly as follows. The group $\widetilde{\Gamma}$ is generated by

$$S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right), \quad T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right). \tag{1.4.2}$$

For these generators, we have

$$\begin{aligned} \rho_L(T)\phi_\mu &= e(Q(\mu))\phi_\mu, \\ \rho_L(S)\phi_\mu &= \frac{e(-\operatorname{sgn}(V)/8)}{\sqrt{|L'/L|}} \sum_{\nu \in L'/L} e(-(\mu,\nu))\phi_\nu. \end{aligned} \tag{1.4.3}$$

Here, $\operatorname{sgn}(V)$ denotes the signature of V, which is equal to $2-n$ in our case.

1.5 Automorphic forms

1.5.1 Scalar valued modular forms

We briefly recall the definition of modular forms for $\Gamma_0(N)$. We refer to one of the standard references for details, for instance [Kob93, Miy06, Ono04, Ste07], to mention just a few.

Let N be a positive integer and consider the congruence subgroup $\Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z})$ defined by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \bmod N \right\}.$$

Similarly, we let

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \;\middle|\; a \equiv d \equiv 1 \bmod N \right\}.$$

For an integer $k \in \mathbb{Z}$ and a matrix $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$, we introduce the *Petersson slash operator* on functions $f : \mathbb{H} \to \mathbb{C}$, defined by

$$(f \mid_k \gamma)(\tau) = (c\tau + d)^{-k} \det(\gamma)^{k/2} f(\gamma\tau), \text{ where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q}).$$

Definition 1.5.1. Let $k \in \mathbb{Z}$ and let χ be a Dirichlet character modulo N such that $\chi(-1) = (-1)^k$. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a *modular form* of weight k and character χ for $\Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z})$ if

(i) $(f \mid_k \gamma)(\tau) = \chi(d) f(\tau)$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

(ii) and f is holomorphic at the cusps.

Such a form is called a *cusp form* if it vanishes at all cusps.

We denote by $M_k(N, \chi)$ the space of all modular forms of weight k for $\Gamma_0(N)$ and character χ. Moreover, we denote by $M_k(N)$ the space of all modular forms for $\Gamma_1(N)$, defined in the analogous way. The spaces of cusp forms are denoted $S_k(N)$ and $S_k(N, \chi)$.

We have

$$M_k(N) = \sum_{\chi} M_k(N, \chi) \text{ and } S_k(N) = \sum_{\chi} S_k(N, \chi).$$

The conditions at the cusps can be phrased in terms of Fourier expansions. An element $f \in M_k(N, \chi)$ has a Fourier expansion (at the cusp ∞) of the form

$$f(\tau) = \sum_{n=0}^{\infty} a_f(n) q^n,$$

where $q = e^{2\pi i\tau} = e(\tau)$. We say that f vanishes at the cusp ∞ if $a_f(1) = 0$. The conditions at the other cusps are similar.

The spaces of modular forms come equipped with an inner product, the *Petersson inner product*. For $f, g \in S_k(N)$ it is defined by

$$(f, g)_{\Gamma_1(N)} = \int_{\Gamma_1(N)\backslash\mathbb{H}} f(\tau)\overline{g(\tau)} v^k d\mu(\tau)$$

and analogously for $S_k(N, \chi)$, denoted by $(f, g)_{\Gamma_0(N)}$. The integral converges in fact absolutely if at least one of the forms is a cusp form. Note that we do not normalize the inner product with respect to the volume of the fundamental domain. We indicate this, however, by the subscript.

We denote by $S_k^{\mathrm{old}}(N)$ the subspace of $S_k(N)$ spanned by all functions $f(t\tau)$ with $f \in S_k(t^{-1}N)$ for some divisor t of N greater than 1. Moreover, let $S_k^{\mathrm{new}}(N)$ be the orthogonal complement of $S_k^{\mathrm{old}}(N)$ with respect to the Petersson inner product. The elements of $S_k^{\mathrm{new}}(N)$ are called *newforms*. We call an element $f \in S_k^{\mathrm{new}}(N)$ *primitive* if it is normalized, that is $a_f(1) = 1$, and a common eigenfunction of all Hecke operators (cf. [Shi76], [AL70]).

Lemma 1.5.2 (Lemma 4.6.9 of [Miy06]). *If χ is a primitive Dirichlet character of conductor N, then $S_k^{\mathrm{new}}(N,\chi) = S_k(N,\chi)$.*

Lemma 1.5.3 (Theorem 4.6.8 of [Miy06]). *Let $f \in M_k(N,\chi)$ and let l be a positive integer. Let c be the conductor of χ. Assume that $a_f(n) = 0$ for all n coprime to l. Then*

(i) If $(l, N/c) = 1$, then $f(\tau) = 0$.

(ii) If $(l, N/c) \neq 1$, then there is an $f_p \in M_k(N/p,\chi)$ for all prime factors p of $(l, N/c)$, such that

$$f(\tau) = \sum_{p|(l,N/c)} f_p(p\tau).$$

If f is a cusp form, then all $f_p(\tau)$ can be taken as cusp forms.

Remark 1.5.4. In particular, if all Fourier coefficients of $f \in S_k^{new}(N,\chi)$ with index coprime to the level N vanish, then $f = 0$.

Finally, we recall the definition of the *Fricke involution*. Consider the matrix

$$W_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R}).$$

It defines an involution on $M_k(N)$.

Lemma 1.5.5 (Lemma 1, [Asa76]). *Let $f \in S_k(N,\chi)$. Then $f \mid_k W_N \in S_k(N,\bar{\chi})$. Moreover, if $f \in S_k^{\mathrm{new}}(N,\chi)$, then $f \mid_k W_N \in S_k^{\mathrm{new}}(N,\bar{\chi})$.*

1.5.2 Harmonic weak Maaß forms

We will now define automorphic forms that are no longer required to be holomorphic functions but eigenfunctions of the Laplace operator in weight k. In contrast to the wave forms studied by Maaß, a weak Maaß form is allowed to have linear exponential growth at the cusps. The main reference for this section is the fundamental article by Bruinier and Funke [BF04].

Let (V, Q) be a rational quadratic space of type (b^+, b^-) and let $L \subset V$ be an even lattice. Moreover, let $k \in \frac{1}{2}\mathbb{Z}$. For $(\gamma, \phi) \in \tilde{\Gamma}$, we define the *Petersson slash operator* on functions $f : \mathbb{H} \to S_L$ by

$$(f \mid_{k,L} (\gamma, \phi))(\tau) = \phi(\tau)^{-2k} \rho_L((\gamma, \phi))^{-1} f(\gamma\tau).$$

Definition 1.5.6. A twice continuously differentiable function $f : \mathbb{H} \to S_L$ is called a *harmonic weak Maaß form* (of weight k with respect to $\tilde{\Gamma}$ and ρ_L) if it satisfies:

(i) $f \mid_{k,L} \gamma = f$ for all $\gamma \in \tilde{\Gamma}$,

(ii) there is a $C > 0$ such that $f(\tau) = O(e^{Cv})$ as $v \to \infty$ (uniformly in u, where $\tau = u+iv$),

(iii) $\Delta_k f = 0$, where

$$\Delta_k := -v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + ikv \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right)$$

is the hyperbolic Laplace operator in weight k.

We denote the space of harmonic weak Maaß forms of weight k with respect to ρ_L by $\mathcal{H}_{k,L}$ and write $M^!_{k,L}$ for the subspace of weakly holomorphic modular forms. Moreover, we write $S_{k,L}$ and $M_{k,L}$ for the subspaces of cusp forms and holomorphic modular forms. These spaces are defined by modifying the analytic conditions appropriately. That is, elements of $S_{k,L} \subset M_{k,L} \subset M^!_{k,L}$ are all assumed to be holomorphic on the upper half-plane. Moreover, cusp forms are defined to vanish at the cusp, holomorphic modular forms are defined to be holomorphic at the cusp and weakly holomorphic modular forms are allowed to have at most a pole at the cusp.

Note that for an even, unimodular lattice we recover the definition for scalar valued modular forms and harmonic weak Maaß forms (for $\mathrm{SL}_2(\mathbb{Z})$ in this case).

We write the Fourier expansion of $f \in \mathcal{H}_{k,L}$ as

$$f(\tau) = \sum_{\mu \in L'/L} \sum_{n \in \mathbb{Q}} c_f(n, \mu, v) q^n. \tag{1.5.1}$$

Since f is harmonic with respect to the weight k Laplace operator, the coefficients $c(n, \mu, v)$ satisfy $\Delta_k c(n, \mu, v) = 0$. Computing a basis for the space of solutions to this differential equation, gives rise to a unique decomposition $f = f^+ + f^-$.

For $k \neq 1$, it is given by

$$f^+(\tau) = \sum_{\mu \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n \gg -\infty}} c_f^+(n, \mu) q^n \phi_\mu, \tag{1.5.2}$$

$$f^-(\tau) = \sum_{\mu \in L'/L} \left(c_f^-(0, \mu) v^{1-k} + \sum_{\substack{n \in \mathbb{Q} \\ n \neq 0}} c_f^-(n, \mu) W(2\pi n v) q^n \right) \phi_\mu, \tag{1.5.3}$$

where $W(a) = W_k(a) := \int_{-2a}^{\infty} e^{-t} t^{-k} \, dt = \Gamma(1-k, -2a)$.

The function f^+ is called the *holomorphic part* and f^- the *non-holomorphic part* of f. If the form f is clear from the context, we also omit the subscript and simply write $c^+(n, \mu)$ and $c^-(n, \mu)$.

For $k = 1$, the expansion of the non-holomorphic part is slightly different. Namely, the two linear independent solutions to the differential equation $\Delta_1 c(0, v) = 0$ are given by 1 and $\log(v)$. Therefore, we obtain in this case

$$f^{-}(\tau) = \sum_{\mu \in L'/L} \left(c_f^{-}(0,\mu)\log(v) + \sum_{\substack{n \in \mathbb{Q} \\ n \neq 0}} c_f^{-}(n,\mu) W(2\pi n v) q^n \right) \phi_\mu. \tag{1.5.4}$$

We collect a few more facts that can all be found in [BF04]. We denote by L^{-} the lattice given by the $\mathbb{Z}$-module L together with the quadratic form $-Q$. There is an antilinear differential operator $\xi = \xi_k : \mathcal{H}_{k,L} \to M^!_{2-k,L^-}$, defined by

$$f(\tau) \mapsto \xi(f)(\tau) := v^{k-2}\overline{L_k f(\tau)} = R_{-k} v^k \overline{f(\tau)}. \tag{1.5.5}$$

Here L_k and R_k are the Maaß lowering and raising operators,

$$R_k = 2i\frac{\partial}{\partial\tau} + kv^{-1} \quad \text{and} \quad L_k = -2iv^2\frac{\partial}{\partial\overline{\tau}}.$$

The kernel of ξ is equal to $M^!_{k,L}$ and by [BF04, Corollary 3.8], the sequences

$$\begin{tikzcd} 0 \arrow[r] & M^!_{k,L} \arrow[r] & \mathcal{H}_{k,L} \arrow[r, "\xi_k"] & M^!_{2-k,L^-} \arrow[r] & 0 \end{tikzcd} \tag{1.5.6}$$

$$\begin{tikzcd} 0 \arrow[r] & M^!_{k,L} \arrow[r] & H_{k,L} \arrow[r, "\xi_k"] & S_{2-k,L^-} \arrow[r] & 0 \end{tikzcd} \tag{1.5.7}$$

are exact.

Lemma 1.5.7. *The Fourier expansion of $\xi_k(f) \in M^!_{2-k,L^-}$ for any $f \in \mathcal{H}_{k,L}$ is given by*

$$-\sum_{\mu \in L'/L} \left(\overline{c_f^{-}(0,\mu)}(k-1-\delta_{k,1}) + \sum_{n \in \mathbb{Q}} \overline{c_f^{-}(-n,\mu)}(4\pi n)^{1-k} e(n\tau) \right) \phi_\mu.$$

Definition 1.5.8. The subspace $H_{k,L} \subset \mathcal{H}_{k,L}$ is defined to be

$$H_{k,L} := \{ f \in \mathcal{H}_{k,L} \;\mid\; \xi(f) \in S_{k,L^-} \}.$$

Alternatively, we could define this to be the space of $f \in \mathcal{H}_{k,L}$, such that there is a Fourier polynomial

$$P_f(\tau) = \sum_{\mu \in L'/L} \sum_{m<0} c_f(m,\mu) e(m\tau),$$

with

$$f - P_f(\tau) = O(1)$$

as $\Im(\tau) \to \infty$. The Fourier polynomial $P_f(\tau)$ is also called the *principal part* of f. Note that this space was denoted by $H^{+}_{k,L}$ in [BF04].

Lemma 1.5.9. *If $f \in H_{k,L}$, then the non-holomorphic part f^- of f decays exponentially as $\Im(\tau) \to \infty$.*

Proof. This follows from the standard growth estimates of the Whittaker functions. □

We have the following growth estimates for the Fourier coefficients of harmonic weak Maaß forms.

Lemma 1.5.10. *Let $f \in \mathcal{H}_{k,L}$. Then there is a constant $C > 0$ such that*

$$\begin{aligned} c_f^+(n,\mu) &= O(e^{C\sqrt{|n|}}), \quad n \to \infty, \\ c_f^-(n,\mu) &= O(e^{C\sqrt{|n|}}), \quad n \to -\infty. \end{aligned}$$

Moreover, for $f \in H_{k,L}$, we have the stronger estimate

$$c_f^-(n,\mu) = O(|n|^{k/2}), \quad n \to -\infty.$$

We denote by

$$d\mu(\tau) = \frac{du \wedge dv}{v^2}$$

the $\mathrm{SL}_2(\mathbb{R})$-invariant volume form on $\mathbb{H}$.

For $f \in S_{k,L}$ and $g \in M_{k,L}$, we define the *Petersson inner product* of f and g as

$$(f,g) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \langle f(\tau), \overline{g(\tau)}\rangle v^k d\mu(\tau).$$

We define the usual Dolbeaut operators ∂ and $\overline{\partial}$, such that we have

$$\partial(f d\tau + g d\bar{\tau}) = \left(\frac{\partial}{\partial \tau} g\right) d\tau \wedge d\bar{\tau}$$

and

$$\bar{\partial}(f d\tau + g d\bar{\tau}) = \left(\frac{\partial}{\partial \bar{\tau}} f\right) d\bar{\tau} \wedge d\tau.$$

We have $d = \partial + \overline{\partial}$ for the exterior derivative $d : \mathcal{E}^1 \to \mathcal{E}^2$, where $\mathcal{E}^k$ is the space of C^∞ differential k-forms.

Lemma 1.5.11. *In terms of differential forms, we have*

$$\bar{\partial}(f d\tau) = -v^{2-k}\overline{\xi_k(f)} d\mu(\tau) = -L_k f d\mu(\tau).$$

Using the Petersson inner product and the operator ξ, we obtain a bilinear pairing between $g \in M_{2-k,L^-}$ and $f \in H_{k,L}$ via

$$\{g,f\} := (g, \xi_k(f))_{2-k} = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \langle g, \overline{\xi_k(f)}\rangle v^{2-k} d\mu(\tau) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \langle g, L_k f\rangle d\mu(\tau). \quad (1.5.8)$$

Using Lemmas 1.5.11 and 1.5.9, the following result is essentially an application of Stoke's theorem.

Lemma 1.5.12. *Let* $f \in H_{k,L}$ *and* $g \in M_{2-k,L^-}$. *Then*

$$\{g, f\} = \sum_{\mu \in L'/L} \sum_{n \leq 0} c^+(n,\mu) b(-n,\mu),$$

which implies that the pairing only depends on the principal part of f *(and on* g*). The exact sequence* (1.5.6) *implies that the pairing between* S_{2-k,L^-} *and* $H_{k,L}/M^!_{k,L}$ *is non-degenerate.*

1.5.3 Operations on vector-valued modular forms

In this section we simply write $A_{k,L}$ for the space of S_L-valued functions that are invariant under the weight k slash operator.

Let $M \subset L$ be a sublattice of finite index, then if $f \in A_{k,L}$, it can be naturally viewed as an element of $A_{k,M}$. Indeed, we have the inclusions $M \subset L \subset L' \subset M'$ and therefore

$$L/M \subset L'/M \subset M'/M.$$

We have the natural map $L'/M \to L'/L$, $\mu \mapsto \bar{\mu}$.

Lemma 1.5.13. *There are two natural maps*

$$\mathrm{res}_{L/M} : A_{k,L} \to A_{k,M}, \quad f \mapsto f_M$$

and

$$\mathrm{tr}_{L/M} : A_{k,M} \to A_{k,L}, \quad g \mapsto g^L$$

such that for any $f \in A_{k,L}$ *and* $g \in A_{k,M}$

$$\langle f, \bar{g}^L \rangle = \langle f_M, \bar{g} \rangle.$$

They are given as follows. For $\mu \in M'/M$ *and* $f \in A_{k,L}$,

$$(f_M)_\mu = \begin{cases} f_{\bar{\mu}}, & \text{if } \mu \in L'/M, \\ 0, & \text{if } \mu \notin L'/M. \end{cases}$$

For any $\bar{\mu} \in L'/L$, *and* $g \in A_{k,M}$, *let* μ *be a fixed preimage of* $\bar{\mu}$ *in* L'/M. *Then*

$$(g^L)_{\bar{\mu}} = \sum_{\alpha \in L/M} g_{\alpha+\mu}.$$

Proof. See [Sch04, Proposition 6.9] for the map $\mathrm{res}_{L/M}$. The assertion for $\mathrm{tr}_{L/M}$ can be proved analogously. □

1.5.4 Jacobi forms and vector valued modular forms

In this section we recall the notion of a (weakly holomorphic) Jacobi form. We will only use Jacobi forms of scalar index which have been intensively studied by Eichler and Zagier [EZ85]. We refer to their book for more details.

Jacobi forms are sometimes convenient for us to use because they have a natural ring structure and we have an explicit description of the generators of the $M^!_*$-module of weakly holomorphic Jacobi forms.

Definition 1.5.14. Let $k, m \in \mathbb{Z}$ and $\varphi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Then φ is called a *holomorphic Jacobi form* of weight k and index m if

(i) $\varphi(\gamma\tau, \frac{z}{c\tau+d}) = (c\tau+d)^k e(mcz^2/(c\tau+d))\varphi(\tau, z)$, for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$,

(ii) $\varphi(\tau, z + r\tau + s) = e(-m(r^2\tau + 2rz))\varphi(\tau, z)$ for all $r, s \in \mathbb{Z}$, and

(iii) $\varphi(\tau, z)$ is holomorphic at the cusp ∞.

Note that such a φ has a Fourier expansion of the form

$$\varphi(\tau, z) = \sum_{n,r\in\mathbb{Z}} c(n, r) q^n \zeta^r,$$

where $q = e^{2\pi i\tau}$ and $\zeta = e^{2\pi iz}$. The last condition in the definition means that $c(n, r) = 0$ if the discriminant $4nm - r^2$ is negative.

A *weakly holomorphic Jacobi form* satisfies all the preceding conditions except that we only require it to be meromorphic at ∞. This means in terms of the Fourier expansion that there are only finitely many non-vanishing Fourier coefficients with negative discriminant. We denote by $J_{k,m}$ for the space of holomorphic Jacobi forms of weight k and index m and by $J^!_{k,m}$ for the space of weakly holomorphic Jacobi forms of weight k and index m.

Using the definitions, it is easy to check that

(i) If $f \in J^!_{k_1,m_1}$ and $g \in J^!_{k_2,m_2}$, then $fg \in J^!_{k_1+k_2,m_1+m_2}$.

(ii) If $f \in M^!_{k_1}(\mathrm{SL}_2(\mathbb{Z}))$ and $\varphi \in J^!_{k_2,m}$, then $f\varphi \in J^!_{k_1+k_2,m}$.

(iii) If $\varphi \in J^!_{k,m}$, then $\varphi(\tau, 0) \in M^!_k$.

For $r \in \mathbb{Z}/2m\mathbb{Z}$ we write

$$\theta_r(\tau, z) = \sum_{\substack{n\in\mathbb{Z} \\ n\equiv r \bmod 2m}} q^{\frac{n^2}{4m}} \zeta^n,$$

for the corresponding theta function.

Proposition 1.5.15. *Let $\varphi \in J^!_{k,m}$ be a weakly holomorphic Jacobi form. Then $\varphi(\tau, z)$ has a theta expansion of the form*

$$\varphi(\tau, z) = \sum_{r \bmod 2m} \varphi_r(\tau)\theta_r(\tau, z).$$

Moreover, let $L = \mathbb{Z}$ *be the lattice with quadratic form* $Q(x) = -mx^2$*. Then*

$$\Phi(\tau) = \sum_{r \bmod 2m} \varphi_r(\tau)\phi_r$$

is a vector valued modular form contained in $M^!_{k-1/2,L}$*. Here,* ϕ_r *is the characteristic function of the coset* $r + 2m\mathbb{Z}$*. This correspondence establishes an isomorphism*

$$M^!_{k-1/2,L} \cong J^!_{k,m}.$$

Proof. For holomorphic Jacobi forms, this is Theorem 5.1 of [EZ85]. It is straightforward to extend this to weakly holomorphic forms. See also [Zag02, Section 8]. □

The following lemma will be very convenient for us in Chapter 5.

Lemma 1.5.16. *With the same notation as in Proposition 1.5.15, let* $\varphi \in J^!_{k,m}$ *be a weakly holomorphic Jacobi form. We let* $\Theta_{L^-}(\tau)$ *be the theta function*

$$\Theta_{L^-}(\tau) = \sum_{n\in\mathbb{Z}} e\left(\frac{n^2}{4m}\tau\right)\phi_n = \sum_{r\in\mathbb{Z}/2m\mathbb{Z}} \Theta_{L^-,r}(\tau)\phi_r \in M_{\frac{1}{2},L^-}$$

associated with the lattice L^-*. Then we have*

$$\langle \Phi(\tau), \Theta_{L^-}(\tau)\rangle = \varphi(\tau, 0).$$

Proof. This follows directly from Proposition 1.5.15 using

$$\langle \Phi(\tau), \Theta_{L^-}(\tau)\rangle = \sum_{r\in\mathbb{Z}/2m\mathbb{Z}} \varphi_r(\tau)\Theta_{L^-,r}(\tau) = \sum_{r\in\mathbb{Z}/2m\mathbb{Z}} \varphi_r(\tau)\theta_r(\tau, 0).$$

□

From the properties stated above, it follows that the bi-graded ring

$$J^!_{ev,*} = \bigoplus_{k,m\in\mathbb{Z}} J^!_{2k,m}$$

of weakly holomorphic Jacobi forms of even weight is a module over the graded ring

$$M^!_* = M^!_*(\mathrm{SL}_2(\mathbb{Z})) = \bigoplus_{k\in\mathbb{Z}} M^!_k(\mathrm{SL}_2(\mathbb{Z})).$$

(Note that $M^!_k(\mathrm{SL}_2(\mathbb{Z})) = \{0\}$ for k odd.)

Proposition 1.5.17. *The $M^!_*$-module $J^!_{ev,*}$ of weakly holomorphic Jacobi forms of even weight is free of rank two and generated by*

$$F(\tau,z)=\phi_{-2,1}(\tau,z)=(\zeta-2+\zeta^{-1})+(-2\zeta^2+8\zeta-12+8\zeta^{-1}-2\zeta^{-2})q+\ldots\in J^!_{-2,1},$$
$$G(\tau,z)=\phi_{0,1}(\tau,z)=(\zeta+10+\zeta^{-1})+(10\zeta^2-64\zeta+108-64\zeta^{-1}+10\zeta^{-2})q+\ldots\in J^!_{0,1}.$$

Proof. The forms F and G are in fact so-called weak Jacobi forms. This is the subspace of $J^!_k$, where all Fourier coefficients with negative n vanish. It is shown in [EZ85] that these two forms are the two generators of the free module over M_* of weak Jacobi forms of even weight. The corresponding statement for weakly holomorphic Jacobi forms is directly obtained by extending M_* to $M^!_*$. Also note that there are no weakly holomorphic modular forms of odd weight on the full modular group. We also refer to [Zag02, Section 8]. □

1.5.5 Integrality and bounded denominators

As in the theory of scalar valued modular forms, it is a fundamental fact that we have an integral basis for the space of modular forms.

Theorem 1.5.18. *([McG03, Theorem 5.6]) Each of the spaces $M_{k,L}$ and $S_{k,L}$ has a basis of modular forms all of whose Fourier expansion have only integer coefficients.*

We can use the existence of an integral basis to bound the denominators of weakly holomorphic modular forms with rational Fourier coefficients, which will become important later on.

Lemma 1.5.19. *Let $f\in M^!_{k,L}$ with rational Fourier coefficients. Then the Fourier coefficients of f have bounded denominators.*

Proof. Without loss of generality, we can assume that the principal part of f has integral Fourier coefficients. Multiplying f by $\Delta(\tau)^m$ for some large enough $m\in\mathbb{Z}_{>0}$, we obtain that $f'=\Delta(\tau)^m f\in M_{12m+k,L}$. Here $\Delta(\tau)\in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$ is the unique normalized cusp form of weight 12 for the full modular group. The space $M_{12m+k,L}$ has a basis of forms with integral Fourier coefficients by Theorem 1.5.18, say $h_1,\ldots,h_s$, and f' has rational Fourier coefficients. Therefore, there are $a_1,\ldots,a_s\in\mathbb{Q}$, such that

$$f'=\sum_{i=1}^{s}a_ih_i.$$

Clearly, f' has bounded denominators. We write

$$f(\tau)=\sum_{\substack{n\in\mathbb{Q}\\ n\gg-\infty}}a(n)e(n\tau)$$

with coefficients $a(n) \in S_L \cong \mathbb{C}[L'/L]$. (Note that the coefficients $a(n)$ are vectors here.) Similarly, write

$$f'(\tau) = \sum_{n \in \mathbb{Q}_{\geq 0}} c(n)e(n\tau),$$

where

$$c(n) = \sum_{\substack{r \in \mathbb{Z} \\ 0 \leq r \leq n}} a(n-r)\tau_m(r)$$

and

$$\Delta^m(\tau) = \sum_{n=0}^{\infty} \tau_m(n)e(n\tau).$$

Now let $N_0 \in \mathbb{Z}$ such that $N_0 c(n) \in \mathbb{Z}[L'/L]$ is a vector with integral coefficients for all $n \in \mathbb{Q}$. We claim that $N_0 a(n) \in \mathbb{Z}[L'/L]$ for all $n \in \mathbb{Q}$, as well. Suppose the coefficients of f do not have bounded denominators. Then there is a *minimal* $n_0 \in \mathbb{Q}$, such that $a(n_0) \notin \mathbb{Z}[L'/L]$. Then, since we assumed that f has an integral principal part, we know that $n_0 \geq 0$. We have seen that

$$N_0 c(n_0+1) = N_0 \sum_{\substack{r \in \mathbb{Z} \\ 0 \leq r \leq n_0+1}} a(n_0+1-r)\tau_m(r) \in \mathbb{Z}[L'/L].$$

However, this is equal to

$$N_0 c(n_0+1) = N_0 a(n_0) + \sum_{\substack{r \in \mathbb{Z} \\ 1 < r \leq n_0+1}} N_0 a(n_0+1-r)\tau_m(r),$$

since $\tau_m(0) = 0$. We have that $N_0 a(n_0+1-r) \in \mathbb{Z}[L'/L]$ for $r > 1$ since n_0 is minimal. The left hand side, $N_0 c(n_0+1)$, is also contained in $\mathbb{Z}[L'/L]$ because f' has bounded denominators. Thus, we conclude that $N_0 a(n_0) \in \mathbb{Z}[L'/L]$, as well. This is a contradiction and consequently, the coefficients of f have bounded denominators. □

1.5.6 Lifting scalar valued modular forms

Let L be an even lattice of type $(2,n)$, level N and determinant $D = |L'/L|$. The group $\Gamma_0(N)$ acts on ϕ_0 via the Weil representation ρ_L by a character. It is given by

$$\chi_L\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} \left(\frac{(-1)^{\frac{n+2}{2}} D}{d}\right) & \text{if } d > 0, \\ (-1)^{\frac{n+2}{2}} \left(\frac{(-1)^{\frac{n+2}{2}} D}{-d}\right) & \text{if } d < 0. \end{cases}$$

Suppose that N is square-free. Then $2+n$ is even and the character is quadratic. Moreover, this implies that for any $f \in M_{k,L}$, the component function f_0 is a modular form in

$M_k(N,\chi_L)$. Conversely, we can lift any $f \in M_k(N,\chi_L)$ to a vector-valued modular form by defining

$$\mathcal{S}_L(f) = \sum_{\gamma\in\Gamma_0(N)\backslash\operatorname{SL}_2(\mathbb{Z})} (f\mid_k \gamma)\rho_L(\gamma^{-1})\phi_0 \in M_{k,L}. \tag{1.5.9}$$

This construction preserves any analytic properties we might impose. In particular, the lift also works for weak Maaß forms and weakly holomorphic modular forms. We also refer to [BB03], [Bun01], and [Sch11].

There is also a map that is adjoint to the lift with respect to the Petersson inner product. It is simply given by the map $F \mapsto F_0$ for $F \in M_{k,L}$.

Proposition 1.5.20. *Let $f \in S_k(N,\chi_L)$ and let $F \in M_{k,L}$. Then, we have for the Petersson inner product*

$$(\mathcal{S}_L(f), F) = (f, F_0)_{\Gamma_0(N)}.$$

Proof. Using the definitions, we obtain

$$\begin{aligned}
(\mathcal{S}_L(f), F) &= \int_{\mathcal{F}} \langle \sum_{\gamma\in\Gamma_0(N)\backslash\operatorname{SL}_2(\mathbb{Z})} (f\mid_k \gamma)\rho_L(\gamma^{-1})\phi_0, \overline{F(\tau)}\rangle v^k d\mu(\tau)\\
&= \int_{\mathcal{F}} \sum_{\gamma\in\Gamma_0(N)\backslash\operatorname{SL}_2(\mathbb{Z})} (f\mid_k \gamma)\langle\phi_0, \overline{\rho_L(\gamma)F(\tau)}\rangle v^k d\mu(\tau)\\
&= \int_{\mathcal{F}} \sum_{\gamma\in\Gamma_0(N)\backslash\operatorname{SL}_2(\mathbb{Z})} \Im(\gamma\tau)^k f(\gamma\tau)\langle\phi_0, \overline{F(\gamma\tau)}\rangle d\mu(\tau)\\
&= \sum_{\gamma\in\Gamma_0(N)\backslash\operatorname{SL}_2(\mathbb{Z})} \int_{\gamma\mathcal{F}} \Im(\tau)^k f(\tau)\overline{F_0(\tau)} d\mu(\tau).
\end{aligned}$$

The last line is equal to $(f, F_0)_{\Gamma_0(N)}$. □

1.5.7 Modular forms for orthogonal groups

In this section we define the notion of a modular form for $H(\mathbb{Q}) = \operatorname{GSpin}_V(\mathbb{Q})$. We use the same notation and setup as in Section 1.3 and write $\tilde{\mathcal{K}} = \{Z \in V(\mathbb{C}) \mid [Z] \in \mathcal{K}\}$ for the cone above $\mathcal{K}$.

Definition 1.5.21. Let $k \in \mathbb{Z}$ and let χ be a character of $H(\mathbb{Q})$.

A function $f : \tilde{\mathcal{K}} \times H(\mathbb{A}_f) \to \mathbb{C}$ is called a holomorphic modular form of weight $k \in \mathbb{Z}$ and level $K \subset H(\mathbb{A}_f)$ for $H(\mathbb{Q})$ if

(i) the function $f(z,h)$ is holomorphic with respect to z (for fixed $h \in H(\mathbb{A}_f)$),

(ii) we have $f(z,hk) = f(z,h)$, for all $k \in K$,

(iii) $f(cz,h) = c^{-k} f(z,h)$ for all $c \in \mathbb{C}\setminus\{0\}$,

(iv) $f(\gamma z, \gamma h) = \chi(\gamma) f(z, h)$ for all $\gamma \in H(\mathbb{Q})$,

(v) and f is holomorphic at the boundary.

The variety X_K is compact if and only if V is anisotropic over $\mathbb{Q}$. In this case condition (v) is empty. If the Witt rank of V (the rank of a maximal isotropic subspace) is strictly less than n, then (v) is automatically satisfied (this is the so-called Koecher-principle). This is always the case when $n > 2$.

1.6 Algebraic and arithmetic geometry

The basic reference for this section is of course Hartshorne [Har77]. We can also recommend the book of Liu [Liu02]. We provide this section for the convenience of the reader, but do not give any details. Let S be a scheme. By a scheme over S or an S-scheme, we mean a scheme X together with a morphism $\pi : X \to S$. We denote by (Sch/S) the category of schemes over S. The morphisms in this category are morphisms of S-schemes, that is, morphisms that are compatible with the given morphisms to S.

Recall that an R-module M is called flat over R if the functor $N \mapsto M \otimes_R N$ is an exact functor for N in the category of modules over R. A ring homomorphism $R \to R'$ is called flat if R' is flat as an R-module.

Definition 1.6.1. A morphism $f : X \to Y$ of schemes is called *flat* if the induced map

$$f_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$$

of stalks is flat as a homomorphism of rings.

Definition 1.6.2. Let $f : X \to Y$ be a morphism of schemes. Then f is called *étale* if it is smooth of relative dimension 0.

Proposition 1.6.3. *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent*

(i) f is étale,

(ii) f is flat and unramified,

(iii) f is flat and $\Omega_{X/Y} = 0$.

Finally, we recall geometric points, étale neighborhoods, and étale local rings. The étale topology is a Grothendieck topology on a category of schemes. A category together with a choice Grothendieck topology is called a site. For us it is sufficient to know that an open covering of an object X in the category (Sch/S) is a collection of étale morphisms $\{X_i \to X\}$ such that $\cup_i X_i \to X$ is surjective.

Definition 1.6.4. A *geometric point* of a scheme X is a morphism $\bar{x} : \mathrm{Spec}(k) \to X$, where k is an algebraically closed field.

If the unique closed point of Spec(k) maps to the closed point $x \in X$, we say that $\bar{x}$ lies over x. Moreover, we denote the field k also by $\kappa(\bar{x})$.

Geometric points are the "correct" notion of a point for the étale topology.

An étale neighborhood of a geometric point $\bar{x} : \mathrm{Spec}(k) \to X$ is given by an étale morphism $\phi : U \to X$ and a geometric point $\bar{u} : \mathrm{Spec}(k) \to U$, such that the following diagram commutes

$$\begin{tikzcd}
 & U \arrow[d, "\phi"] \\
\mathrm{Spec}(k) \arrow[ur, "\bar{u}"] \arrow[r, "\bar{x}"] & X.
\end{tikzcd}$$

A morphism of étale neighborhoods $(U, \bar{u}) \to (V, \bar{v})$ is an X-morphism $f : U \to V$ with $\bar{v} = f \circ \bar{u}$. It is possible to show that the category of étale neighborhoods of a geometric point is cofiltered and that it makes sense to define the *local ring at x for the étale topology* $\mathcal{O}_{X,\bar{x}}$ to be a colimit over étale neighborhoods $(U, \bar{u})$ of a fixed geometric point $\bar{x}$ over the sets $\mathcal{O}(U)$, where $\mathcal{O}$ is the structure sheaf on the étale site of X. We will not go into detail here, but quote the following lemma [Sta13, Tag 04HX] that gives an alternative way to understand this local ring.

Lemma 1.6.5. *Let X be a scheme and $\bar{x}$ be a geometric point of X lying over $x \in X$. Let $\kappa = \kappa(x)$, the field of definition of x and $\kappa \subset \kappa^{\mathrm{sep}} \subset \kappa(\bar{s})$ be the separable algebraic closure of κ in $\kappa(\bar{s})$. Then there is a canonical identification*

$$(\mathcal{O}_{X,x})^{\mathrm{sh}} \cong \mathcal{O}_{X,\bar{x}},$$

where $(\mathcal{O}_{X,x})^{\mathrm{sh}}$ denotes the strict henselization of the local ring $\mathcal{O}_{X,x}$ with respect to κ^{sep}.

1.6.1 Elliptic curves

In this section we recall the definition of an elliptic curve over an arbitrary base scheme. Moreover, we briefly discuss the endomorphism rings of elliptic curves. We refer to Chapter 2 of the book by Katz and Mazur [KM85] and the two books by Silverman [Sil09, Sil94] for details.

Definition 1.6.6. Let S be a scheme. A proper smooth curve $E \to S$ with geometrically connected fibers all of genus one together with a section $0 : S \to E$, is called an *elliptic curve.*

It is an important fact that an elliptic curve admits a unique structure as a group scheme [KM85, Theorem 2.1.2]. The reader might be more familiar with elliptic curves over fields. In fact, this knowledge will be sufficient (most of the time) for our purposes. However, the moduli problems that will occur later on, will make use of the more general definition we gave.

We will later discuss certain "special endomorphisms" of elliptic curves. The following result classifies the endomorphism rings of elliptic curves over fields. These descriptions are due to Deuring [Deu41].

Proposition 1.6.7 (Corollary 9.4, [Sil09]). *The endomorphism ring of an elliptic curve over a field is either $\mathbb{Z}$, an order in an imaginary quadratic field or an order in a quaternion algebra.*

We will use the notation

$$\mathbb{B} = \left(\frac{a, b}{F}\right),$$

to denote the quaternion algebra $F \oplus F\alpha \oplus F\beta \oplus F\alpha\beta$ with $\alpha^2 = a$ and $\beta^2 = b$ and $\alpha\beta = -\beta\alpha$. We say that the quaternion algebra $\mathbb{B}$ is *split* over F if $\mathbb{B}$ is isomorphic to $M_2(F)$. Here, $M_2(F)$ is the algebra of 2×2-matrices.

For a quaternion algebra $\mathbb{B}$ over $\mathbb{Q}$, we say that $\mathbb{B}$ is split at a prime p (or ∞), if $\mathbb{B} \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is split over $\mathbb{Q}_p$. Otherwise, we say that $\mathbb{B}$ is *ramified* at p (or ∞). We use the same convention over other fields.

Let E/k be an elliptic curve over a field k. If the endomorphism ring $\mathrm{End}_k(E)$ is given by an order $\mathcal{O} \subset k$ for an imaginary quadratic field k, then we say that E has *complex multiplication* (CM).

A *CM elliptic curve* over a scheme S is a pair (E, ι) consisting of an elliptic curve E/S and an action $\iota : \mathcal{O} \hookrightarrow \mathrm{End}_S(E)$ of an order $\mathcal{O}$ in an imaginary quadratic field on $\mathrm{End}_S(E)$.

If $\mathrm{End}_S(E) = \mathcal{O} \subset \mathbb{B}$ is an order in a quaternion algebra, then we say that E is *supersingular*. Supersingular elliptic curves only occur in positive characteristic (cf. VI., Theorem 6.1 of [Sil09]).

1.6.2 Deformation theory

We just briefly mention some notions in deformation theory that we be used in Chapter 4. We follow the short summary in the preliminary chapter of [DR73].

Let k be a field and let Λ be a complete Noetherian local ring with residue field k. Particular examples are $\Lambda = k$ and for k perfect and of characteristic $p > 0$, the ring of Witt vectors $W(k)$. The latter is the unique complete discrete valuation ring, which is absolutely unramified (that is it has maximal ideal (p)) with residue field k (cf. II, §5, Theorem 3 of [Ser79]).

We denote by $\mathcal{ART}$ the category of local Artin Λ-algebras with residue field k and by $\widehat{\mathcal{ART}}$ the category of complete Noetherian local Λ-algebras with residue field k. A covariant functor $F : \mathcal{ART} \to$ (Sets) is called *pro-representable* if there is an object $R \in \mathrm{Ob}\,\widehat{\mathcal{ART}}$ and an isomorphism of functors $\mathcal{ART} \to (Sets)$

$$\hat{\xi} : \mathrm{Hom}(R, A) \cong F(A).$$

Such an algebra R is uniquely determined.

1.6.3 Algebraic stacks

In this section we summarize some facts about *algebraic stacks*. We will work with the definition introduced by Deligne and Mumford [DM69] and follow a mixture of the exposition of Vistoli [Vis89, Appendix] and Fantechi [Fan01]. The most complete reference is probably the book by Laumon and Moret-Bailly [LMB00] and the original paper by Deligne and Mumford [DM69].

Roughly speaking, a stack over a scheme S is a functor that assigns a groupoid $C(T)$ to each scheme T over S which satisfies glueing-conditions with respect to étale coverings. An algebraic stack additionally admits an étale and surjective covering by a scheme (and also satisfies other technical conditions). In this thesis we will be concerned with several moduli problems that are not representable by a scheme. Algebraic stacks provide a generalization of schemes that adequately represent these moduli problems.

The exact definition of a stack is *not* necessary to understand the contents of the present thesis and the reader may prefer to skip this section. We mainly use the fact that the moduli problems we are dealing with are represented by algebraic stacks as a tool. It allows us to use the moduli description to obtain geometric results. We provide the definitions in this section for the sake of completeness. For details, the interested reader may consult the literature.

Here and throughout, let S be a regular Noetherian scheme.

Definition 1.6.8. A *groupoid over S* is a category C together with a functor

$$p_C : C \to (\mathrm{Sch}/S)$$

such that:

(i) For every arrow $f : X \to Y$ in (Sch/S) and every object η of C with $p_C(\eta) = Y$, there exists an arrow $\varphi : \xi \to \eta$ such that $p_C(\varphi) = f$. We denote this object ξ by $f^*\eta$.

(ii) If $\varphi : \xi \to \zeta$ and $\psi : \eta \to \zeta$ are arrows in C, and $h : p_C(\xi) \to p_C(\eta)$ is such that $p_C(\psi) \circ h = p_C(\varphi)$, then there is a unique arrow $\chi : \xi \to \eta$ such that $\psi \circ \chi = \varphi$ and $p_C(\chi) = h$.

For an S-scheme X we denote by $C(X)$ the category with objects given by all objects ξ of C such that $p_C(\xi) = X$ and with arrows given by arrows in C that map to the identity under p_C. The category $C(X)$ is a *groupoid*, that is, all arrows in $C(X)$ are isomorphisms.

For this reason, a groupoid over S is also called a *category fibered in groupoids* over (Sch/S).

Definition 1.6.9. A groupoid C over S is called a *stack* if

(i) for any X in (Sch/S) and any two objects ξ_1 and ξ_2 in $C(X)$, the functor

$$\mathrm{Isom}_X(\xi_1, \xi_2) : (\mathrm{Sch}/X) \to (\mathrm{Sets}),$$

which associates to a morphism $f : Y \to X$ the set of isomorphisms in $C(Y)$ between $f^*\xi_1$ and $f^*\xi_2$, is a sheaf in the étale topology.

(ii) Let $\{X_i \to X\}$ be a covering of $X \in (\mathrm{Sch}/S)$ in the étale topology, Let $\xi_1 \in C(X_i)$, and let

$$\varphi_{ij} : \xi_j \mid_{X_i \times_X X_j} \to \xi_i \mid_{X_i \times_X X_j}$$

be isomorphisms in $C(X_i \times_X X_j)$ satisfying the cocycle condition. Then there is a $\xi \in C(X)$ and there are isomorphisms $\psi_i : \xi \mid_{X_i} \to \xi_i$, such that

$$\varphi_{ij} = (\psi_i \mid_{X_i \times_X X_j}) \circ (\psi_j \mid_{X_i \times_X X_j})^{-1}.$$

Example 1.6.10. The category (Sch/S) is a stack over S. It is called the stack associated with the scheme S.

Morphisms of stacks and fiber products of stacks are defined in the usual way without problems. We refer to [LMB00] or [DM69] for details.

Definition 1.6.11. A stack C over S is called representable if it is isomorphic to the stack associated with a scheme X.

Definition 1.6.12. A morphism of stacks $C \to D$ is called *representable* if, for every morphism $R \to D$ where R is (the stack associated to) a scheme, the fiber product $R \times_D C$ is representable.

Definition 1.6.13. A stack C over S is called *algebraic* (in the sense of Deligne and Mumford) if the diagonal Δ_C is representable, quasicompact and separated and there exists an étale and surjective representable morphism $U \to C$, where U is (the stack associated to) a scheme. Such a morphism $U \to C$ is called an *atlas.*

Warning. Algebraic stacks are often also called Deligne-Mumford (DM) stacks, in contrast to Artin stacks for which one essentially replaces the word “étale” by “smooth”. We warn the reader however that [LMB00] uses the term “champ algebrique” for what we call an Artin stack.

A *substack* Z of a stack C is a morphism of stacks $Z \to C$ that is represented by an embedding of schemes. It is possible to define many of the notions from the theory of schemes in this way for algebraic stacks. For instance, a representable morphism has a property P if every morphism of schemes representing it has property P.

Definition 1.6.14. Let C be an algebraic stack over S. A *coarse moduli space* for C is an S-scheme X together with a proper morphism $p : C \to X$ such that the following holds.

(i) Every S-morphism $C \to Y$ to a scheme Y over S factors uniquely through p.

(ii) If $\bar{s} : \mathrm{Spec}(k) \to S$ is a geometric point of S, then p induces a bijection on isomorphism classes of objects in $C(\bar{s})$ with $X(\bar{s})$.

Note that definitions vary a bit but this is essentially the definition used by Deligne and Rapoport [DR73], except that they consider the more general Artin stacks in which case a coarse moduli space need only to be an algebraic space, not a scheme.

It is possible to show that algebraic stacks which are locally of finite presentation having a finite inertia stack always admit a coarse moduli scheme [Con05, DR73].

An important property of algebraic stacks is that we can define sheaves on the étale site [DM69, Definition 4.10] of a stack. A prominent example is the coherent sheaf $\mathcal{O}_C$, the *structure sheaf* of C, defined by $\mathcal{O}_C = \mathcal{O}_U$ for any atlas $U \to X$.

If C is a stack that admits a coarse moduli scheme X, we have the following important property of the local ring at a geometric point [DR73, p. 30]. We have an isomorphism

$$\operatorname{Spec}(\mathcal{O}^{\mathrm{sh}}_{C,\bar{x}})/\operatorname{Aut}(\bar{x}) \xrightarrow{\sim} \operatorname{Spec}(\mathcal{O}_{X,p(\bar{x})})$$

for any geometric point $\bar{x}$ of C. Here $\operatorname{Aut}(\bar{x}) = \operatorname{Isom}_X(\bar{x}, \bar{x})$.

1.6.4 Cycles and Chow groups

Let C be an algebraic stack that is separated and of finite type over a regular and noetherian scheme S. For this section we refer in particular to [How12a].

Let us first recall the definition of the function field of a stack. We can define a *rational function* on a stack C to be a morphism $U \to \mathbb{A}^1_S$, where U is a nonempty open substack of C. Rational functions on C then form a field, called the *function field* of C and denoted by $k(C)$.

Definition 1.6.15. A *prime cycle* on C is a nonempty integral closed substack of C. For $k \in \mathbb{Z}_{>0}$ we let $Z^k(C)$ be the free $\mathbb{Z}$-module generated by the codimension k prime cycles on C. The elements of $Z^k(C)$ are called codimension k cycles. Moreover, we define the group of *rational equivalences* in codimension k as

$$R^k(C) = \bigoplus_Z k(Z)^\times,$$

where the sum is over all prime cycles Z on C of codimension $k-1$ and $k(Z)$ is the field of rational functions on Z. We also write

$$R^*(C) = \bigoplus_{k\in\mathbb{Z}} R^k(C)$$

and similarly

$$Z^*(C) = \bigoplus_{k\in\mathbb{Z}} Z^k(C).$$

If X is a scheme and W is a closed and integral subscheme of codimension $k-1$, there is a homomorphism [Ful98, Chapter 1]

$$\operatorname{div} : k(W)^\times \to Z^k(X),$$

defined by

$$f \mapsto [\mathrm{div}(f)] = \sum_{V \subset W} \mathrm{ord}_V(f) V,$$

where the sum runs over all prime cycles on W of codimension one and the multiplicity $\mathrm{ord}_V(f)$ is the valuation of f in the local ring $\mathcal{O}_{W,V}$.

This map extends to stacks ([Vis89], [Gil84]) because there are sheaves $\mathcal{Z}^*$ and $\mathcal{R}^*$ on the étale site of C such that the map div extends to a morphism of sheaves $\mathcal{R}^* \to \mathcal{Z}^*$. Moreover, we can identify Z^* and R^* as the global sections of these sheaves and obtain the morphism $k(W) \to Z^k(C)$ as above for a stack C and a prime cycle W on C.

In this thesis we will only need the simplest case, namely codimension one cycles. The rational equivalences are given by rational functions *on* C in this case.

Similar to schemes, we can define the *proper pushforward* $f_* : Z^*(X) \to Z^*(Y)$ and a *flat pullback* $f^* Z^*(Y) \to Z^*(X)$ for a proper, respectively finite type flat, morphism $f : X \to Y$. This is shown in [Vis89, Proposition 3.7] for algebraic stacks over a field but the proposition extends to algebraic stacks over a regular Noetherian scheme S (cf. [How12a]).

1.7 Siegel theta functions

As before, let $n \geq 0$ be an integer and let V be a quadratic space over $\mathbb{Q}$ of type $(2, n)$ with a non-degenerate quadratic form Q and we write again $H = \mathrm{GSpin}_V$ and $G = \mathrm{SL}_2$.

For $g \in \widetilde{G}_{\mathbb{A}}$, $h \in H(\mathbb{A})$ and $\varphi \in S(V(\mathbb{A}))$, we can define a theta function

$$\theta(g, h, \varphi) = \sum_{\lambda \in V(\mathbb{Q})} (\omega(g, h)\varphi)(\lambda).$$

Here, g acts through the Weil representation $\omega(g)$ and h simply acts linearly on $S(V(\mathbb{A}))$ as $\omega(h)\varphi(x) = \varphi(h^{-1}x)$. These two actions commute and we denote $\omega(g)\omega(h)$ for simplicity by $\omega(g, h)$. We also refer to [Kud03] and the references given there. The theta function $\theta(g, h, \varphi)$ is trivially left invariant under the action of $H(\mathbb{Q})$ and left invariant under the action of $\widetilde{G}_{\mathbb{Q}}$ by Poisson summation.

The vector-valued Siegel theta functions briefly mentioned in the introduction are obtained as follows. We let $\varphi = \varphi_f \otimes \varphi_\infty$ with $\varphi_f \in S(V(\mathbb{A}_f))$ and $\varphi_\infty \in S(V(\mathbb{R}))$. At the real place, we put

$$\varphi_\infty(x, z) = e^{-2\pi(Q(x_z) - Q(x_{z^\perp}))}$$

for an element $z \in \mathbb{D}$.

Moreover, to obtain a function on $\mathbb{H}$, for $\tau \in \mathbb{H}$, we let $g_\tau \in G(\mathbb{R})$, such that $g_\tau i = \tau$. We take

$$g_\tau = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{\frac{1}{2}} & 0 \\ 0 & v^{-\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} v^{\frac{1}{2}} & uv^{-\frac{1}{2}} \\ 0 & v^{-\frac{1}{2}} \end{pmatrix}$$

and let $\tilde{g}_\tau = (g_\tau, 1) \in \widetilde{G}_{\mathbb{R}}$ (or rather $\tilde{g}_\tau = (g_\tau, v^{-\frac{1}{4}})$).

Remark 1.7.1. Note that $\hat{L} \cap V(\mathbb{Q}) = L$. The group $H(\mathbb{A}_f)$ acts on lattices $L \subset V$ by $L \mapsto hL := \cap_p (V(\mathbb{Q}) \cap h_p L_p) = V(\mathbb{Q}) \cap h\hat{L}$, where $h = (h_p) \in H(\mathbb{A}_f)$ and $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Moreover, we have $(hL)' = hL'$ for the dual lattices. Therefore, the action of h induces an isomorphism of discriminant groups:

$$\begin{tikzcd}
L'/L \arrow[r] \arrow[d, Rightarrow, "\wr"] & (hL)'/hL \arrow[d, Rightarrow, "\wr"] \\
\hat{L}'/\hat{L} \arrow[r] & (h\hat{L})'/h\hat{L}.
\end{tikzcd}$$

We choose a base point $z_0 \in \mathbb{D}$ and for each $z \in \mathbb{D}$ we let $h_z \in H(\mathbb{R})$, such that $h_z z_0 = z$.

Definition 1.7.2 (Siegel theta function)**.** For $\tau \in \mathbb{H}$, $z \in \mathbb{D}$ and $h_f \in H(\mathbb{A}_f)$ we let

$$\begin{aligned}
\theta_\mu(\tau, z, h_f) &:= v^{\frac{n-2}{4}} \theta(\tilde{g}_\tau, (h_z, h_f), \phi_\mu \otimes \varphi_\infty(z_0, \cdot)) \\
&= v^{n/2} \sum_{\lambda \in h_f(\mu + L)} e\left(Q(\lambda_z)\tau + Q(\lambda_{z^\perp})\overline{\tau}\right).
\end{aligned}$$

Using this, we obtain an S_L-valued theta function

$$\Theta_L(\tau, z, h_f) := \sum_{\mu \in L'/L} \theta_\mu(\tau, z, h_f)\phi_\mu.$$

The following theorem can be found (in a slightly more classical language) in [Bor98]. A reference that uses our (adelic) setup is [Kud03].

Theorem 1.7.3. *If $K \subset H(\mathbb{A}_f)$ is an open compact subgroup preserving L and acting trivially on L'/L, then the Siegel theta function $\Theta_L(\tau, z, h_f)$ defines a function on the Shimura variety X_K. Moreover, as a function on $\mathbb{H}$, it is a non-holomorphic vector-valued modular form of weight $(2-n)/2$, that is, for $\gamma = \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \phi(\tau)\right) \in \widetilde{\Gamma}$, we have*

$$\Theta_L(\gamma\tau, z, h_f) = \phi(\tau)^{2-n} \rho_L(\gamma) \Theta_L(\tau, z, h_f).$$

1.8 Regularized theta lifts

We recall the regularization of Harvey and Moore [HM96] used by Borcherds [Bor98]. Denote by $\mathcal{F} := \{\tau \in \mathbb{H}; |\tau| \geq 1, -1/2 \leq \Re(\tau) \leq 1/2\}$ the standard fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ and let $\mathcal{F}_T := \{\tau \in \mathcal{F};\ \Im(\tau) \leq T\}$. For $f \in H_{1-n/2,L}$, we define

$$\Phi(z, h, f) = \Phi_L(z, h, f) = \int_{\Gamma\backslash\mathbb{H}}^{\mathrm{reg}} \langle f(\tau), \overline{\Theta_L(\tau, z, h)} \rangle v^k d\mu(\tau),$$

where the regularized integral is defined as

$$\int_{\Gamma\backslash\mathbb{H}}^{\mathrm{reg}} \langle f(\tau), \overline{\Theta_L(\tau, z, h)}\rangle v^k d\mu(\tau) := \mathop{\mathrm{CT}}_{s=0} \left[\lim_{T\to\infty} \int_{\mathcal{F}_T} \langle f(\tau), \overline{\Theta_L(\tau, z, h)}\rangle v^{k-s} d\mu(\tau)\right].$$

Here, $\mathrm{CT}_{s=0}$ denotes the constant term in the Laurent expansion at $s = 0$ of the meromorphic continuation of the function in brackets defined by the limit.

Associated with f is the divisor

$$Z(f) = \sum_{\mu\in L'/L} \sum_{m<0} c^+(m, \mu) Z(m, \mu). \tag{1.8.1}$$

Theorem 1.8.1 ([Bor98], Theorem 13.3, cf. Theorem 1.3 in [Kud03]). *Let* $f \in M^!_{(2-n)/2,L}$ *with* $c(m, \mu) \in \mathbb{Z}$ *for all* $m < 0$ *and* $c(m, \mu) \in \mathbb{Q}$ *for all* $m \in \mathbb{Q}$. *There is a function* $\Psi(z, h, f)$ *on* $\mathbb{D} \times H(\mathbb{A}_f)$, *such that*

(i) $\Psi(z, h, f)$ *is a meromorphic modular form for* $H(\mathbb{Q})$ *of weight* $c_f(0, 0)/2$ *and level* K, *with some unitary multiplier system of finite order.*

(ii) The divisor of $\Psi(z, h, f)^2$ *on* X_K *is given by* $Z(f)$.

(iii) We have the identity $\Phi(z, h, f) = -2 \log\|\Psi(z, h, f)\|^2 - c_f(0, 0)(\log(2\pi) + \Gamma'(1))$, *where* $\|\Psi(z, h, f)\|^2 = |\Psi(z, h, f)|^2 \, |y|^{c_f(0,0)}$.

The modular form $\Psi(z, h, f)$ admits an expansion as an infinite product, giving it the name *Borcherds product*. We will only use the product expansion in a special case in Chapter 5. Therefore, we omit the general case and refer to Theorem 13.3 of [Bor98].

Bruinier extended the space of input functions of the theta lift to harmonic weak Maaß forms and this extension will play an important role for us. Recall that a function $G(z)$ on $\mathbb{D}$ has a logarithmic singularity along a divisor D, if every point in $\mathbb{D}$ has a small neighborhood U, such that for any meromorphic function locally defining D, we have that $G - \log|g|$ extends to a smooth function on U.

Theorem 1.8.2 ([Bru02], Theorem 2.12, Theorem 4.7, Theorem 5.5).
Let $f \in H_{1-n/2,L}$. *Then the following holds.*

(i) The function $\Phi(z, h, f)$ *is smooth on* $X_K\backslash Z(f)$ *and has a logarithmic singularity along* $-2Z(f)$.

(ii) The differential $dd^c\Phi(z, h, f)$ *extends to a smooth* $(1, 1)$ *form on* X_K. *As a current on* X_K, *we have*

$$dd^c [\Phi(z, h, f)] + \delta_{Z(f)} = [dd^c\Phi(z, h, f)].$$

(iii) We have

$$\Delta_z\Phi(z, h, f) = \frac{n}{4} c^+_f(0, 0).$$

Here, Δ_z *denotes the* $H(\mathbb{R})$*-invariant Laplacian on* $\mathbb{D}$, *normalized as in [Bru02].*

2 Theta series and Shimura varieties for spaces of type $(2, 0)$

In this chapter we collect some facts about integral positive definite binary quadratic forms and their theta functions. We will describe a precise relation between the spaces of scalar and vector valued theta series in this case and give explicit orthogonal bases for both of them. Throughout this chapter, we let $k = k_D = \mathbb{Q}(\sqrt{D})$ be the imaginary quadratic field of discriminant D and we write $\mathrm{Cl_k}$ for the ideal class group of k.

2.1 The idele class group of k

Recall our conventions from Section 1.3.3.

We let $\mathcal{O}_D \subset k$ be the order of discriminant D in k. Since we assume that D is fundamental, $\mathcal{O}_D$ is simply the ring of integers in k. We write $\mathbb{A}_k$ for the ring of adeles over k.

Definition 2.1.1. The *idele class group* of k is defined as the quotient

$$\mathcal{I}_k = k^\times \backslash \mathbb{A}_k^\times.$$

Here, $k^\times$ is embedded diagonally into $\mathbb{A}_k^\times$ and the elements of the subgroup $k^\times$ are called *principal ideles.* We also write

$$I_k = k^\times \backslash \mathbb{A}_{k,f}^\times$$

for the finite idele class group.

Theorem 2.1.2 (VI. Satz 1.3, [Neu07]). *We have a surjective homomorphism*

$$I_k \to \mathrm{Cl}_k, \quad (\alpha_\mathfrak{p})_\mathfrak{p} \mapsto \prod_{\mathfrak{p} \nmid \infty} \mathfrak{p}^{v_\mathfrak{p}(\alpha)},$$

inducing an isomorphism

$$I_k / \hat{\mathcal{O}}_D^\times = k^\times \backslash \mathbb{A}_k^\times / \hat{\mathcal{O}}_D^\times \cong \mathrm{Cl}_k,$$

where $\hat{\mathcal{O}}_D^\times$ is the subgroup

$$\hat{\mathcal{O}}_D^\times = \prod_{\mathfrak{p} \nmid \infty} \mathcal{O}_\mathfrak{p}^\times.$$

2.2 Genus theory

In this section we recall several facts about the genus theory for imaginary quadratic extensions of $\mathbb{Q}$. We continue to use the same notation as in the last section. Let t be the number of prime factors of D.

Definition 2.2.1. The group $\mathrm{Cl_k} / \mathrm{Cl_k^2}$ is called the group of *genera*. The coset $[\mathfrak{a}] \mathrm{Cl_k^2}$ is called the *genus* of the ideal class $[\mathfrak{a}]$.

We denote by $\mathrm{Cl_k^*}$ the group of characters $\mathrm{Cl_k} \to \mathbb{C}^\times$ of $\mathrm{Cl_k}$. For each prime $p \mid D$, there is a quadratic character of $\mathrm{Cl_k}$, defined by

$$\chi_p([\mathfrak{a}]) = (D, \mathrm{N}(\mathfrak{a}))_p,$$

where $(\cdot,\cdot)_p$ is the Hilbert symbol. The character is well defined on ideal classes since $(D, \mathrm{N}(a))_p = 1$ for all $a \in k^\times$ (in fact, by definition, for all $a \in k_p^\times$). We write $\mathrm{Cl_k}[2]$ for the kernel of the map $x \mapsto x^2$ in the ideal class group, that is, the group of elements of order dividing 2.

Proposition 2.2.2.

(i) We have a short exact sequence

$$1 \to \mathrm{Cl_k}[2] \to \mathrm{Cl_k} \xrightarrow{x \mapsto x^2} \mathrm{Cl_k^2} \to 1.$$

(ii) The order of $\mathrm{Cl_k} / \mathrm{Cl_k^2}$ *and* $\mathrm{Cl_k}[2]$ *is equal to* 2^{t-1}.

(iii) The dual group $(\mathrm{Cl_k} / \mathrm{Cl_k^2})^* \subset \mathrm{Cl_k^*}$ *is generated by the characters* χ_p *for* $p \mid D$ *with a single relation given by*

$$\prod_{p \mid D} \chi_p = 1.$$

(iv) The 2-torsion $\mathrm{Cl_k}[2]$ *is generated by the ideal classes* $[\mathfrak{p}]$ *for the primes* $\mathfrak{p}$ *over primes* $p \mid D$ *ramified in* k *with a single relation*

$$\prod_{p \mid D} [\mathfrak{p}] = [\mathcal{O}_D] = 1 \in \mathrm{Cl_k}.$$

Proof. This can be found in paragraph 12 of [Zag81]. □

2.3 Scalar valued theta functions

A fractional ideal $\mathfrak{a}$ of k defines an integral binary quadratic form in the following way. If $\mathfrak{a}$ is generated as a $\mathbb{Z}$-module by two elements

$$\mathfrak{a} = (\alpha, \beta) = \mathbb{Z}\alpha + \mathbb{Z}\beta,$$

then

$$Q_{\mathfrak{a}}(x,y) = \frac{\mathrm{N}(\alpha)}{\mathrm{N}(\mathfrak{a})}x^2 + \frac{\mathrm{tr}(\alpha\bar{\beta})}{\mathrm{N}(\mathfrak{a})}xy + \frac{\mathrm{N}(\beta)}{\mathrm{N}(\mathfrak{a})}y^2 = \frac{\mathrm{N}(x\alpha + y\beta)}{\mathrm{N}(\mathfrak{a})}$$

is an integral binary quadratic form of discriminant D. This induces a bijective correspondence between equivalence classes of positive definite integral binary quadratic forms of discriminant D and the class group Cl_k of k (if we also restrict to oriented bases). A good reference for this correspondence is [Zag81].

For an integral ideal $\mathfrak{a} \subset \mathcal{O}_D$, we can consider the associated theta function

$$\theta_{\mathfrak{a}}(\tau) = \sum_{a \in \mathfrak{a}} e\left(\frac{\mathrm{N}(a)}{\mathrm{N}(\mathfrak{a})}\tau\right) = 1 + \sum_{n \geq 1} \rho(n, \mathfrak{a}) e(n\tau). \tag{2.3.1}$$

Since $Q_{\mathfrak{a}}$ is positive definite, the series converges normally and defines a holomorphic modular form of weight one. It is contained in $M_1(|D|, \chi_D)$, where χ_D is the primitive Dirichlet character of conductor $|D|$ (the Kronecker symbol).

It is easy to see that the representation number $\rho(n, \mathfrak{a})$, and therefore also the theta function $\theta_{\mathfrak{a}}$, only depends on the class $[\mathfrak{a}] \in \mathrm{Cl}_k$ of $\mathfrak{a}$. Indeed, if $n = \mathrm{N}(a)/\mathrm{N}(\mathfrak{a})$, then $n = \mathrm{N}(ba)/\mathrm{N}((b)\mathfrak{a})$ for all $b \in k$.

We denote by $\Theta(k) \subset M_1(|D|, \chi_D)$ the space generated by all theta functions $\theta_{\mathfrak{a}}$ for $[\mathfrak{a}] \in \mathrm{Cl}_k$. Note that since D is a fundamental discriminant, the theta functions $\theta_{\mathfrak{a}}$ are all newforms (χ_D is primitive). The space $\Theta(k)$ has a basis consisting of primitive forms (see Section 1.5.1 for the definition and for instance [Kan11] for a proof).

Let $\psi \in \mathrm{Cl}_k^*$ be a class group character and define

$$\theta_{\psi}(\tau) = \frac{1}{w_k} \sum_{[\mathfrak{a}] \in \mathrm{Cl}_k} \psi([\mathfrak{a}]) \theta_{\mathfrak{a}}(\tau). \tag{2.3.2}$$

Here, w_k is the number of roots of unity contained in k.

We let $\bar{\mathrm{Cl}}_k^*$ be the set of equivalence classes of Cl_k^* under the relation $\psi \mapsto \bar{\psi}$.

Theorem 2.3.1 ([Kan11]).

(i) If $\psi^2 = 1$, then θ_{ψ} is an Eisenstein series.

(ii) If $\psi^2 \neq 1$, then θ_{ψ} is a primitive cuspidal newform.

(iii) The set

$$\mathcal{B}(k) = \{\theta_{\psi} \mid \psi \in \bar{\mathrm{Cl}}_k^*\}$$

is an orthogonal basis for $\Theta(k)$ with respect to the Petersson inner product.

It is in fact easy to see that we have

$$\theta_{\mathfrak{a}}(\tau) = \frac{w_k}{h_k} \sum_{\chi \in \mathrm{Cl}_k^*} \bar{\chi}([\mathfrak{a}]) \theta_{\chi}(\tau). \tag{2.3.3}$$

Definition 2.3.2. Let $\mathfrak{A} \in \mathrm{Cl_k} / \mathrm{Cl_k^2}$ be a genus and let $\mathfrak{a} \in \mathfrak{A}$. The Eisenstein series

$$E_{\mathfrak{A}}(\tau) = \frac{1}{h_k} \sum_{[\mathfrak{b}] \in \mathrm{Cl_k}} \theta_{\mathfrak{a}\mathfrak{b}^2}(\tau)$$

is called the (normalized) *genus Eisenstein series* of $\mathfrak{A}$.

Remark 2.3.3. The fact that $E_{\mathfrak{A}}(\tau)$ is an Eisenstein series is “elementary” in our case. It is also a special case of the Siegel-Weil formula, as we will see later on.

2.4 The action of $\mathrm{GSpin}_U(\mathbb{A}_f)$

In this section, we let $U = \mathfrak{a} \otimes_{\mathbb{Z}} \mathbb{Q}$ for a fractional ideal $\mathfrak{a}$ of k and we view U simply as a 2-dimensional rational quadratic space with quadratic form $Q(x) = \mathrm{N}(x)/\mathrm{N}(\mathfrak{a})$. We write $T = \mathrm{GSpin}_U$ instead of H as we will later consider U embedded into a larger quadratic space V. Recall the short exact sequence (1.1.2) of algebraic groups over the rationals

$$1 \longrightarrow \mathbb{G}_m \longrightarrow T \longrightarrow \mathrm{SO}_U \longrightarrow 1.$$

Over $\mathbb{Q}$ we have that $C_U^0 \cong \mathbb{Q}(\sqrt{-|\det(U)|})$, the even part of the Clifford algebra, is isomorphic to k. The Clifford norm corresponds to the norm $\mathrm{N}(x) = x\bar{x}$ in k. Here, $\bar{x}$ denotes complex conjugation. Moreover, the group $SO_U(\mathbb{Q})$ is isomorphic to

$$k^1 = \{x \in k \mid \mathrm{N}(x) = 1\}$$

and $T(\mathbb{Q}) \cong k^\times$ is the multiplicative group of k.

Under this identification the map $T(\mathbb{Q}) \mapsto \mathrm{SO}_U(\mathbb{Q})$ is given by $x \mapsto x/\bar{x}$. This is essentially Hilbert's theorem 90 but can also be seen directly by a short calculation using the definition of the Clifford group. To see this, we consider the orthogonal basis $\{v_1 = \mathrm{N}(\mathfrak{a}), v_2 = -\sqrt{D}\}$ of k as a vector space over $\mathbb{Q}$, where D is the discriminant of k. We have $Q(v_1) = \mathrm{N}(\mathfrak{a})$ and $Q(v_2) = \mathrm{N}(\sqrt{D})/\mathrm{N}(\mathfrak{a}) = -D/\mathrm{N}(\mathfrak{a})$.

The even Clifford algebra C_U^0 is generated (as a $\mathbb{Q}$-algebra) by 1 and $\delta = v_1 v_2$. Note that $\delta^2 = D$.

The group GSpin_U is given by all non-zero elements in C_U^0 in our case. By definition, an element $a + b\delta \in \mathrm{GSpin}_U$ acts on $x \in k = U$ via

$$(a + b\delta) \cdot x \cdot (a + b\delta)^{-1}$$

where the multiplication is in the Clifford algebra C_U. It is enough to compute this on the basis vectors v_1, v_2 of k. Using (1.1.1) it is easy to see that $\delta v_j^{-1} = -v_j^{-1}\delta$ and we obtain

$$(a + b\delta) \cdot v_j \cdot (a + b\delta)^{-1} = (a + b\delta) \cdot ((a + b\delta)v_j^{-1})^{-1} = (a + b\delta) \cdot (a - b\delta)^{-1} \cdot v_j.$$

The element $x = (a + b\delta) \cdot (a - b\delta)^{-1}$ is contained in $k^\times$. The isomorphism $k^\times \cong T(\mathbb{Q})$ is explicitly given via $a + b\sqrt{D} \mapsto a + b\delta$. Under this identification, the action of $x \in k^\times \cong \mathrm{GSpin}_U$ on k is given by multiplication with $x/\bar{x}$.

Using this, we see that $T(\mathbb{A}_f) \cong \mathbb{A}_{k,f}^\times$ is isomorphic to the multiplicative group of ideles over k. To avoid confusion, in this section we write $h.x$ for the action of $h \in T(\mathbb{A}_f)$ on x and simply hx for multiplication of adeles. Recall that the group $T(\mathbb{A}_f) = \mathrm{GSpin}_U(\mathbb{A}_f)$ acts on lattices in U. If $h = (h_p)_p \in \mathrm{GSpin}_U(\mathbb{A}_f)$ and $L = \hat{L} \cap V(\mathbb{Q})$ is a lattice in V, then $h.L = (h.\hat{L}) \cap V(\mathbb{Q}) = \prod_p (h_p.L_p)_p \cap U(\mathbb{Q})$. (See also Section 1.3.4 for the notation.)

In the following, we will examine the action of $T(\mathbb{A}_f)$ on lattices in U more closely. It is important to note that this action is different form the "natural" action on ideals (or lattices) in k. Recall that this natural action is simply given by the linear action of $\mathbb{Q}_p^\times$ on $k \otimes \mathbb{Q}_p$.

The $\mathbb{Q}_p$ vector space $k \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is an algebra with the multiplication $(a \otimes b)(c \otimes d) = ac \otimes bd$, isomorphic to $C_U^0(\mathbb{Q}_p)$. It is also isomorphic [Neu07, II, Theorem 8.3] to the product

$$\prod_{\mathfrak{p} | p} k_{\mathfrak{p}}, \tag{2.4.1}$$

where the product is over all prime ideals $\mathfrak{p}$ of k that lie above p.

For our purposes, it is enough to consider a lattice given by a fractional ideal $\mathfrak{a} \subset k$. Then the action of $x \in T(\mathbb{A}_f)$ is given as follows.

We write $\pi_{\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}}$ for a uniformizer in $\mathcal{O}_{\mathfrak{p}}$. This means that the only prime ideal in $\mathcal{O}_{\mathfrak{p}}$ is generated by $\pi_{\mathfrak{p}}$ and every element in $k_{\mathfrak{p}}$ can be written as $\pi_{\mathfrak{p}}^m u$, where $m \in \mathbb{Z}$ and $u \in \mathcal{O}_{\mathfrak{p}}^\times$. We can write

$$\mathfrak{a} = \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{a})} = (\pi_{\mathfrak{p}}^{v_{\mathfrak{p}}(\mathfrak{a})})_{\mathfrak{p}} \cap k,$$

where we view k as diagonally embedded into $\mathbb{A}_{k,f}$.

Lemma 2.4.1. *Let* $h = (h_{\mathfrak{p}})_{\mathfrak{p}} \in T(\mathbb{A}_f)$. *Then we have*

$$h.\mathfrak{a} = \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{a}) + \mu_{\mathfrak{p}}(h)},$$

where

$$\mu_{\mathfrak{p}}(h) = \begin{cases} 0, & \textit{if } \mathfrak{p} = \bar{\mathfrak{p}} \\ v_{\mathfrak{p}}(h_{\mathfrak{p}}) - v_{\bar{\mathfrak{p}}}(h_{\bar{\mathfrak{p}}}), & \textit{otherwise.} \end{cases}$$

Proof. For primes $\mathfrak{p}$ with $\mathfrak{p} = \bar{\mathfrak{p}}$, that is for inert and ramified primes, the action of $T(\mathbb{Q}_p)$ does not change the valuation $v_{\mathfrak{p}}$. In those cases $h_p \in T(\mathbb{Q}_p) \cong k_{\mathfrak{p}}^\times$ acts by multiplication with $h_p/\bar{h}_p$, where $\bar{h}_p$ denotes the image of h_p under the non-trivial Galois automorphism of the extension $k_{\mathfrak{p}}/\mathbb{Q}_p$.

If the rational prime p however splits in k as $p\mathcal{O}_D = \mathfrak{p}\bar{\mathfrak{p}}$, the action is necessarily slightly different. We have

$$k \otimes \mathbb{Q}_p \cong k_{\mathfrak{p}} \times k_{\bar{\mathfrak{p}}} \cong \mathbb{Q}_p^2,$$

as in (2.4.1). The isomorphism is given explicitly as follows. Let $\mathfrak{d} \in \mathbb{Q}_p$ with $\mathfrak{d}^2 = D$. Such a square-root exists because p is split in k and therefore D is a square modulo p. Then the isomorphism $k \otimes \mathbb{Q}_p \cong \mathbb{Q}_p^2$ is realized by

$$(a + b\sqrt{D}) \otimes c \mapsto ((a + b\mathfrak{d})c, (a - b\mathfrak{d})c) \in \mathbb{Q}_p^2.$$

Therefore,

$$T(\mathbb{Q}_p) \cong (k \otimes \mathbb{Q}_p)^\times \cong k_{\mathfrak{p}}^\times \times k_{\bar{\mathfrak{p}}}^\times \cong \mathbb{Q}_p^\times \times \mathbb{Q}_p^\times.$$

Using the same arguments as over $\mathbb{Q}$, we see that an element $x \otimes c \in T(\mathbb{Q}_p)$ acts by multiplication with $x/\bar{x} \otimes 1$. Therefore, if $x = a + b\sqrt{D}$, then this corresponds to multiplication with

$$\left(\frac{a + b\mathfrak{d}}{a - b\mathfrak{d}}, \frac{a - b\mathfrak{d}}{a + b\mathfrak{d}} \right),$$

giving the formula in the lemma. □

In particular, we see that the action of $T(\mathbb{A}_f)$ on lattices in U is really fundamentally different from multiplication in the class group! We denote the class of h under the surjective map $\mathbb{A}_{k,f}^\times \to \mathrm{Cl_k}$ in Theorem 2.1.2 by $[h]$. From the formulas above, we see that the action of $T(\mathbb{A}_f)$ on the class $[\mathfrak{a}]$ of $\mathfrak{a}$ corresponds to multiplication by the class $[h]/\overline{[h]}$. Here, $\overline{[h]}$ denotes the complex conjugate class of $[h]$. Note that $[h]/\overline{[h]} = [h]^2$ since in an imaginary quadratic field the ideal $\mathfrak{p}\bar{\mathfrak{p}}$ is a principal ideal for all prime ideals $\mathfrak{p} \subset \mathcal{O}_D$. (It is either generated by $p = \mathrm{N}(\mathfrak{p})$ or by p^2.)

Therefore, the class $[h.\mathfrak{a}] \in \mathrm{Cl_k}$ is given by $[h.\mathfrak{a}] = [h]^2[\mathfrak{a}]$, in accordance with the fact that $\mathrm{GSpin}_U(\mathbb{A}_f)$ acts on lattices in the same genus.

2.5 Vector valued theta functions

In this section, we let (P, Q) be a two-dimensional positive definite even quadratic lattice. We let $U = P \otimes_{\mathbb{Z}} \mathbb{Q}$ be the associated rational quadratic space. We will assume that (P, Q) is given by a fractional ideal in an imaginary number field k as in the last sections and only use the letter P to differentiate between the scalar valued and vector valued case. We have that the dual lattice of P is given by $P' \cong \mathfrak{d}_k^{-1}\mathfrak{a}$, where $\mathfrak{d}_k$ denotes the different ideal of k. Recall the definition of the theta function attached to P from Definition 1.7.2 and note that $\mathbb{D} = \{z_U^\pm\}$ is simply a two-point set. Therefore, we also write

$$\Theta_P(\tau, h) = \Theta_P(\tau, z_U^\pm, h) = \sum_{\beta \in P'/P} \sum_{\lambda \in h(\beta + P)} e(Q(\lambda)\tau)\phi_\beta$$

for $\tau \in \mathbb{H}$ and $h \in T(\mathbb{A}_f) \cong \mathbb{A}_{k,f}^\times$. It is easily seen that in our case $K := \hat{\mathcal{O}}_D^\times$ preserves P and acts trivially on P'/P. Therefore, the theta function $\Theta_P(\tau, h)$ is well defined on the Shimura variety

$$X_K = k^\times \backslash \{z_U^\pm\} \times \mathbb{A}_{k,f}^\times / K \cong \{z_P^\pm\} \times \mathrm{Cl_k}.$$

As a function in τ the theta function $\Theta_P(\tau,h)$ is a holomorphic, vector valued modular form contained in $M_{1,P}$ for all h. We will now describe the space generated by these theta functions in more detail.

Definition 2.5.1. We define the space of theta functions

$$\Theta(P) = \langle\, \Theta_P(\tau,h) \ \mid\ h \in \mathrm{Cl}_\mathrm{k} \rangle_{\mathbb{C}}$$

contained in $M_{1,P}$.

In the definition, we identified Cl_k with the quotient I_k/K via Theorem 2.1.2.

Remark 2.5.2. We should warn the reader that if $P = \mathfrak{a} \subset k$ is a fractional ideal, the theta function $\Theta_P(\tau,h)$ is in general not equal to the vector-valued theta function corresponding to $(h)^2\mathfrak{a}$, if (h) denotes the ideal corresponding to h (defined as in Theorem 2.1.2). This is due to the fact that $T(\mathbb{A}_f)$ also acts on the components via automorphisms, see also Remark 1.7.1 and Definition 1.7.2.

Definition 2.5.3. Let $\psi \in \mathrm{Cl}_\mathrm{k}^*$. We let

$$\Theta_P(\tau,\psi) = \sum_{h\in I_k/K} \psi(h)\Theta_P(\tau,h).$$

An easy calculation shows that the 0-th component of $\Theta_P(\tau,\psi)$ is given by

$$\Theta_P(\tau,\psi)_0 = \sum_{\mathfrak{b}\in\mathrm{Cl}_\mathrm{k}} \psi(\mathfrak{b})\theta_{\mathfrak{a}\mathfrak{b}^2}(\tau) = w_k \sum_{\chi^2=\psi} \bar{\chi}(\mathfrak{a})\theta_\chi(\tau).$$

The first equality is clear. For the second one, we find using Equation (2.3.3) that

$$\sum_{\mathfrak{b}\in\mathrm{Cl}_\mathrm{k}} \psi(\mathfrak{b})\theta_{\mathfrak{a}\mathfrak{b}^2} = \frac{w_k}{h_k}\sum_{\mathfrak{b}\in\mathrm{Cl}_\mathrm{k}} \psi(\mathfrak{b}) \sum_{\chi\in\mathrm{Cl}_\mathrm{k}^*} \bar{\chi}(\mathfrak{a}\mathfrak{b}^2)\theta_\chi = \frac{w_k}{h_k}\sum_{\chi\in\mathrm{Cl}_\mathrm{k}^*} \bar{\chi}(\mathfrak{a}) \left(\sum_{\mathfrak{b}\in\mathrm{Cl}_\mathrm{k}} \psi(\mathfrak{b})\bar{\chi}^2(\mathfrak{b}) \right) \theta_\chi.$$

The inner sum in the last line is equal to 0 unless $\chi^2 = \psi$ in which case it is equal to h_k.

In the next section, we will show that, in analogy with the scalar valued case, the functions $\Theta_P(\tau,\psi)$ and $\Theta_P(\tau,\chi)$ are orthogonal if $\chi \neq \psi$ and $\chi \neq \bar{\psi}$.

2.6 Petersson inner products of cusp forms of weight one

In this section we obtain an explicit expression for the Petersson inner products of the cusp forms contained in $\Theta(P)$. We will utilize a seesaw identity that relates these inner products to special values of the Borcherds lift for $\mathrm{O}(2,2)$. For more details on seesaw identities, we also refer to Section 5.2 and Kudla's seminal paper [Kud84].

Suppose that we are given a lattice P of signature $(2,0)$ that corresponds to the integral binary quadratic form $[A,B,C]$ of negative fundamental discriminant $D \equiv 1 \bmod 4$.

Equivalently, P corresponds to a fractional ideal $\mathfrak{a} \subset \mathcal{O}_D$ generated by A and $(B+\sqrt{D})/2$. Here, $\mathcal{O}_D \subset k$ is the ring of integers in $k = \mathbb{Q}(\sqrt{D})$.

The lattice $P \oplus P^-$ has type $(2,2)$ and level $|D|$. Recall that we write P^- for the lattice given by P together with the negative of the quadratic form. The discriminant group has order D^2. We take a $\mathbb{Z}$-basis $\{p_1, p_2\}$ of P with $Q(p_1) = A$, $Q(p_2) = C$ and bilinear form $(p_1, p_2) = B$. We use the same basis for P^-. The starting point is the following embedding.

Consider the even unimodular lattice $L = M_2(\mathbb{Z})$ with the quadratic form given by $Q(X) = -\det(X)$. The bilinear form is

$$(X,Y) = -\operatorname{tr}(XY^*), \text{ where } \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

and the type of L is $(2,2)$.

Recall our basic setup from Section 1.3. The symmetric domain $\mathbb{D}$ attached to $H = \mathrm{GSpin}_V$ can be identified with $\mathbb{H}^2 \cup \bar{\mathbb{H}}^2$ in this case via

$$(z_1, z_2) \mapsto \mathbb{R}\Re\begin{pmatrix} z_1 & -z_1z_2 \\ 1 & -z_2 \end{pmatrix} \oplus \mathbb{R}\Im\begin{pmatrix} z_1 & -z_1z_2 \\ 1 & -z_2 \end{pmatrix}. \tag{2.6.1}$$

Lemma 2.6.1. *Under this identification, the group $H = \mathrm{GSpin}_V$ for $V = L \otimes \mathbb{Q}$ can be identified with the subgroup G of $\mathrm{GL}_2 \times \mathrm{GL}_2$ defined by*

$$G = \{(g_1, g_2) \in \mathrm{GL}_2 \times \mathrm{GL}_2 \mid \det g_1 = \det g_2\},$$

which acts on $\mathbb{D}$ via fractional linear transformations in both components. The corresponding action of $(g_1, g_2) \in H$ on $x \in M_2(\mathbb{Q})$ is given by

$$(g_1, g_2).x = g_1 x g_2^{-1}.$$

Proof. Consider the orthogonal basis

$$v_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2, \; v_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; v_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; v_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

According to Example 2.10 in the second contribution to [BvdGHZ08], we have that the center $Z(C_V^0)$ of the even Clifford algebra is given by $\mathbb{Q} \oplus \mathbb{Q}$ and

$$C_V^0 = Z + Zv_1v_2 + Zv_2v_3 + Zv_1v_3.$$

We obtain an isomorphism

$$C_V^0 \cong M_2(\mathbb{Q}) \oplus M_2(\mathbb{Q})$$

via

$$1 \mapsto (I_2, I_2), \quad v_iv_j \mapsto (v_iv_j^*, v_iv_j^*).$$

Under this isomorphism, the canonical involution of C_V corresponds to

$$(A, B) \mapsto (A^*, B^*)$$

and the Clifford norm is given by

$$\mathrm{N}(A, B) = (\det(A) I_2, \det(B) I_2).$$

Therefore, $\mathrm{N}(A, B) \in \mathbb{Q}^\times$ is equivalent to $A, B \in \mathrm{GL}_2(\mathbb{Q})$ with $\det(A) = \det(B)$.

Finally, we check that under the identification (2.6.1), the action of H corresponds to fractional linear transformations. We have

$$(A, I_2).Z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 & -z_1 z_2 \\ 1 & -z_2 \end{pmatrix} = (cz_1 + d) \begin{pmatrix} \frac{az_1+b}{cz_1+d} & -\frac{az_1+b}{cz_1+d} z_2 \\ 1 & -z_2 \end{pmatrix}.$$

Therefore, under the isomorphism in Lemma 1.3.2, the class of $(A, I_2).Z$ maps to the subspace defining the point $(\frac{az_1+b}{cz_1+d}, z_2)$. The action of an element of the form (I_2, A) is analogous. □

We let $K = H(\hat{\mathbb{Z}})$, that is

$$K = H(\hat{\mathbb{Z}}) = \{(g_1, g_2) \in \mathrm{GL}_2(\hat{\mathbb{Z}}) \times \mathrm{GL}_2(\hat{\mathbb{Z}}) \mid \det g_1 = \det g_2 \in \hat{\mathbb{Z}}\}.$$

It is clear that K preserves L. By strong approximation and Lemma 1.3.3, the associated Shimura variety X_K is a product of two modular curves

$$X_K = H(\mathbb{Q}) \backslash \mathbb{D} \times H(\mathbb{A}_f)/K \cong \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H} \times \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}.$$

It turns out that the additive Borcherds lift of the constant function is equal to

$$\Phi_L(z_1, z_2, 1) = -4 \log \left| (y_1 y_2)^{1/4} \eta(z_1) \eta(z_2) \right| + c \tag{2.6.2}$$

as a function on $\mathbb{H}^2$ for a constant $c \in \mathbb{C}$. We refer to Section 5.1 of the thesis of Hofmann [Hof11] for details.

Recall that given an ideal $\mathfrak{b}$ of k which corresponds to the binary quadratic form $[a, b, c]$, there is a CM point given by the unique root of the polynomial $a\tau^2 + b\tau + c$ that lies in $\mathbb{H}$. We write $\tau(\mathfrak{b}) \in \mathbb{H}$ for this point throughout this chapter. The point $(\tau(\mathfrak{a}), \tau(\mathcal{O}_D)) \in \mathbb{H}^2$ is explicitly given by

$$(\tau(\mathfrak{a}), \tau(\mathcal{O}_D)) = \left(\frac{B + \sqrt{D}}{2A}, \frac{B + \sqrt{D}}{2} \right) \in \mathbb{H}^2.$$

It corresponds to two rational points $z_P^\pm \in \mathbb{D}$. For simplicity, we drop the sign ± indicating the orientation from our notation in this section.

A basis of $U = z_P \cap V(\mathbb{Q})$ is given by

$$\mathbb{Q}f_1 \oplus \mathbb{Q}f_2, \text{ with } f_1 = \begin{pmatrix} -1 & B \\ 0 & A \end{pmatrix} \text{ and } f_2 = \begin{pmatrix} 0 & \frac{B^2-D}{4A} \\ 1 & 0 \end{pmatrix}.$$

Indeed, we have

$$z_P = \mathbb{R}\begin{pmatrix} \frac{B}{2A} & -\frac{B^2+D}{4A} \\ 1 & -\frac{B}{2} \end{pmatrix} \oplus \mathbb{R}\sqrt{|D|}\begin{pmatrix} \frac{1}{2A} & -\frac{B}{2A} \\ 0 & -\frac{1}{2} \end{pmatrix}.$$

We obtain f_1 and f_2 as

$$f_1 = -2A\begin{pmatrix} \frac{1}{2A} & -\frac{B}{2A} \\ 0 & -\frac{1}{2} \end{pmatrix}.$$

and

$$f_2 = \begin{pmatrix} \frac{B}{2A} & -\frac{B^2+D}{4A} \\ 1 & -\frac{B}{2} \end{pmatrix} - B\begin{pmatrix} \frac{1}{2A} & -\frac{B}{2A} \\ 0 & -\frac{1}{2} \end{pmatrix}.$$

In fact, with this choice of basis, we get an isometry of even lattices.

Lemma 2.6.2. *We have an isometry of lattices* $(P,Q) \cong \mathbb{Z}f_1 \oplus \mathbb{Z}f_2 \subset L$ *given by*

$$p_1 \mapsto f_1, \quad p_2 \mapsto f_2,$$

or, equivalently of $(\mathfrak{a}, \mathrm{N}(x)/\mathrm{N}(\mathfrak{a})) \cong (P,Q)$ *given by*

$$A \mapsto f_1, \quad \frac{B+\sqrt{D}}{2} \mapsto f_2.$$

Moreover, we have for $U = \mathbb{Q}f_1 \oplus \mathbb{Q}f_2$ *that* $L \cap U = \mathbb{Z}f_1 \oplus \mathbb{Z}f_2$ *and*

$$L \cap U^{\perp} = \mathbb{Z}\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \oplus \mathbb{Z}\begin{pmatrix} 0 & \frac{B^2-D}{4A} \\ -1 & B \end{pmatrix}$$

is isometric to P^-.

Proof. It is trivial to check that $Q(f_1) = A$, $Q(f_2) = (B^2-D)/4A$ and $(f_1,f_2) = B$. Similarly, the matrices

$$\tilde{f}_1 = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}, \quad \tilde{f}_2 = \begin{pmatrix} 0 & \frac{B^2-D}{4A} \\ -1 & B \end{pmatrix}$$

satisfy $Q(\tilde{f}_1) = -A$, $Q(\tilde{f}_2) = -(B^2-D)/4A$ and $(\tilde{f}_1, \tilde{f}_2) = -B$. Moreover, f_1 and f_2 are both orthogonal to $\tilde{f}_1$ and $\tilde{f}_2$.

As for the equalities $L \cap U = \mathbb{Z}f_1 \oplus \mathbb{Z}f_2$ and $L \cap U^{\perp} = \mathbb{Z}\tilde{f}_1 \oplus \mathbb{Z}\tilde{f}_2$, the inclusions "$\subset$" are clear and the other direction is easy to see because any non-integral linear combination of these vectors has a non-integral entry. $\square$

The lemma provides an embedding of $P \oplus P^-$ into L as an orthogonal sum. We write

$T = \mathrm{GSpin}_U$ and identify it with $k^\times$ as an algebraic group over $\mathbb{Q}$, as before. We now come to the corresponding embedding on the level of orthogonal groups.

Lemma 2.6.3. *The group $T = \mathrm{GSpin}_U$ embeds into G via*

$$1 \mapsto (I_2, I_2)\,,$$

where $I_2 \in \mathrm{GL}_2$ is the identity matrix and

$$\sqrt{D} \mapsto (X, Y)\,, \text{ where } X = \begin{pmatrix} B & \frac{D-B^2}{2A} \\ 2A & -B \end{pmatrix} \text{ and } Y = \begin{pmatrix} B & \frac{D-B^2}{2} \\ 2 & -B \end{pmatrix}.$$

Similarly, the image of $T^- = \mathrm{GSpin}_{U^\perp}$ is given by

$$\sqrt{D} \mapsto (X, Y^*)\,.$$

Proof. This can easily be seen by determining the stabilizer of the point z_P as given above on $\mathbb{H}^2$. We also refer to Section 4.4 in [Shi94]. Proposition 4.6, ibid., tells us that if $\mathbb{C}/\Lambda$ is an elliptic curve with complex multiplication, $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$, then there is an embedding q of k into $M_2(\mathbb{Q})$, such that

$$q_\tau(k^\times) = \{A \in \mathrm{GL}_2^+(\mathbb{Q}) \mid A\tau = \tau\}.$$

There are exactly two embeddings with this property for a given point τ. One of them has the property

$$q_\tau(\mu)\begin{pmatrix} \tau \\ 1 \end{pmatrix} = \mu \begin{pmatrix} \tau \\ 1 \end{pmatrix}.$$

The other one, denoted q_τ^*, satisfies the same property with τ replaced by $\bar{\tau}$, that is,

$$q_\tau^*(\mu)\begin{pmatrix} \tau \\ 1 \end{pmatrix} = \bar{\mu} \begin{pmatrix} \tau \\ 1 \end{pmatrix}.$$

It is easy to check that $q_{\tau(\mathfrak{a})}(\sqrt{D}) = X$ and $q_{\tau(\mathcal{O}_D)}(\sqrt{D}) = Y$ as well as $q^*_{\tau(\mathcal{O}_D)}(\sqrt{D}) = Y^*$. Using these formulas, we see that the correct embedding in our case is given by $(\lambda, \mu) \mapsto (q_{\tau(\mathfrak{a})}(\lambda) q_{\tau(\mathfrak{a})}(\mu), q_{\tau(\mathcal{O}_D)}(\lambda) q^*_{\tau(\mathcal{O}_D)}(\mu))$ for $\lambda, \mu \in k$. ☐

Similar to the proof of Theorem 6.31 in [Shi94], we have a commutative diagram

$$\begin{tikzcd} \mathbb{Q}^2 \arrow[r, "\iota_\tau"] \arrow[d, "q_z(\mu)"'] & k^\times \arrow[d, "\mu"] \\ \mathbb{Q}^2 \arrow[r, "\iota_\tau"] & k^\times, \end{tikzcd}$$

where $\iota_\tau(x_1, x_2) = (x_1, x_2)\begin{pmatrix} \tau \\ 1 \end{pmatrix}$ and the vertical arrow on the right is given by multiplication with μ. The map q_τ extends to $\mathbb{A}_{k,f}^\times$ and $q_\tau(\mathbb{A}_f) \subset \mathrm{GL}_2(\mathbb{A}_f)$ acts on lattices in $\mathbb{Q}^2$. Similarly, we have the linear action given by an idele on the right and these actions commute with the map ι_τ in the same way. We let $\mathfrak{a}_\tau = \mathbb{Z}\tau + \mathbb{Z}$ and $q_\tau(h) = \gamma k$ for $h \in \mathbb{A}_{k,f}^\times$ and with $\gamma \in H(\mathbb{Q})$ and $k \in K$. There is an element $\mu \in k^\times$, such that

$$\gamma^{-1}\begin{pmatrix}\tau\\1\end{pmatrix} = \mu\begin{pmatrix}\tau\\1\end{pmatrix}.$$

Therefore, we have

$$h^{-1}\mathfrak{a}_\tau = \iota_\tau(\mathbb{Z}^2 q(h)^{-1}) = \iota_\tau(\mathbb{Z}^2\gamma^{-1}) = \mu\mathfrak{a}_w,$$

where $w = \gamma^{-1}\tau$.

This shows that for $g \in T(\mathbb{A}_f)$ and $h \in T^-(\mathbb{A}_f)$, we have

$$H(\mathbb{Q})((\tau(\mathfrak{a}), \tau(\mathcal{O}_D)), (g,h))K = H(\mathbb{Q})((\tau((gh)^{-1}\mathfrak{a}), \tau((g^{-1}h))), (1,1))K.$$

Here, we used the notation (h) for the ideal corresponding to h and $(h)\mathfrak{a}$ means multiplication of fractional ideals (and *not* the action of GSpin_U on lattices in U). Note that we have $K_T := K \cap T(\mathbb{A}_f) \cong \hat{\mathcal{O}}_D^\times$.

Proposition 2.6.4. *Let* $g, h \in T(\mathbb{A}_f) \cong \mathbb{A}_{k,f}^\times$ *and write* $\tau(\mathfrak{c}) = u(\mathfrak{c}) + iv(\mathfrak{c})$ *for* $\mathfrak{c} \in \mathrm{Cl_k}$. *We write* $\tau_1 = \tau((hg)^{-1}\mathfrak{a}) = u_1 + iv_1$ *and* $\tau_2 = \tau((gh^{-1})) = u_2 + iv_2$ *and obtain*

$$\Phi_P(\Theta_P(\tau, g), h) = -4\log\left|(v_1v_2)^{1/4}\eta(\tau_1)\eta(\tau_2)\right| + c,$$

where $c = -\log(2\pi) - \Gamma'(1)$.

Proof. The proposition essentially follows from the identity

$$-4\log\left|(v_1v_2)^{1/4}\eta(\tau_1)\eta(\tau_2)\right| + c = \Phi_L((z_P, (h,g)), 1),$$

which follows from (2.6.2) and our considerations above. We use the maps $\mathrm{res}_{L/(P\oplus P^-)}$ and $\mathrm{tr}_{L/(P\oplus P^-)}$ defined in Lemma 1.5.13. Note that the Siegel theta function satisfies

$$\Theta_{P\oplus P^-}(\tau, (h,g)) = \Theta_P(\tau, h) \otimes \Theta_{P^-}(\tau, g)$$

and $\Theta_{P\oplus P^-}^L = \Theta_L$. Moreover, according to Lemma 1.5.13, we have that

$$\begin{aligned}\langle f(\tau), \overline{\Theta_L(\tau, z_P, (h,g))}\rangle &= \langle f_{P\oplus P^-}(\tau), \overline{\Theta_P(\tau,h)\otimes\Theta_{P^-}(\tau,g)}\rangle\\ &= \langle f_{P\oplus P^-}(\tau), \Theta_{P^-}(\tau,h)\otimes\Theta_P(\tau,g)\rangle.\end{aligned}$$

With the embeddings defined above, we consider $P \oplus P^-$ as a sublattice of L. Then we have $P \oplus P^- \subset L = L' \subset P' \oplus (P^-)'$ and

$$L/(P\oplus P^-) \subset P'/P \oplus (P^-)'/P^- \cong P'/P \oplus P'/P.$$

By definition (Lemma 1.5.13), we have for the constant function $1 = 1\phi_{0+L}$ that

$$1_{P \oplus P^-} = \sum_{\mu \in L/(P \oplus P^-)} \phi_\mu,$$

where $\mu \in L/(P \oplus P^-) \subset P'/P \oplus P'/P$. We write $\mu = x + y$ with $x, y \in P'$. Then $\mu = x + y \in L$ is implies that $Q(\mu) = Q(x) - Q(y) \in \mathbb{Z}$. Therefore, since $|P'/P| = |D|$ is square-free, we have $x \equiv y \bmod P$ or $x \equiv -y \bmod P$.

Using our embeddings defined above, it is not hard to see that we necessarily have $x \equiv y \bmod P$. This means that

$$1_{P \oplus P^-} = \sum_{\beta \in P'/P} \phi_{\beta+P} \otimes \phi_{\beta+P^-}.$$

Thus, we have for the theta lift of the constant function 1 that

$$\begin{aligned} \Phi_L((z_P, (h, g)), 1) &= \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}}^{\mathrm{reg}} \langle 1_{P \oplus P^-}, \Theta_{P^-}(\tau, h) \otimes \Theta_P(\tau, g) \rangle d\mu(\tau) \\ &= \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}}^{\mathrm{reg}} \langle \Theta_P(\tau, g), \overline{\Theta_P(\tau, h)} \rangle v d\mu(\tau) \\ &= \Phi_P(\Theta_P(\tau, g), h). \end{aligned}$$ ☐

The identity in the proof is a very simple kind of seesaw identity and gives us a way to evaluate the Petersson inner product of the weight one cusp forms coming from theta functions. We will see another instance of a seesaw identity in Chapter 5.

If $\mathfrak{b} \in \mathrm{Cl}_\mathrm{k}$, then we write $\tau(\mathfrak{b}) = u(\mathfrak{b}) + iv(\mathfrak{b})$ for $u(\mathfrak{b}), v(\mathfrak{b}) \in \mathbb{R}$. We can now find a rather explicit expression for the Petersson inner products of vector valued theta functions in $\Theta(P)$ using Proposition 2.6.4.

Proposition 2.6.5. *For $\chi, \psi \in \mathrm{Cl}_\mathrm{k}^*$, not both trivial, we have for the Petersson inner product that*

$$(\Theta_P(\tau, \psi), \Theta_P(\tau, \chi)) = 0$$

unless $\psi = \bar{\chi}$ or $\psi = \chi$. If $\psi^2 \neq 1$ and $\psi = \bar{\chi}$, we obtain

$$(\Theta_P(\tau, \psi), \Theta_P(\tau, \bar{\psi})) = -\psi(\mathfrak{a}) h_k \sum_{\mathfrak{b} \in \mathrm{Cl}_\mathrm{k}} \psi(\mathfrak{b}) \log \left| v(\mathfrak{b}) \eta^4(\tau(\mathfrak{b})) \right|$$

and if $\psi^2 \neq 1$ but $\psi = \chi$, we have

$$(\Theta_P(\tau, \psi), \Theta_P(\tau, \psi)) = -h_k \sum_{\mathfrak{b} \in \mathrm{Cl}_\mathrm{k}} \psi(\mathfrak{b}) \log \left| v(\mathfrak{b}) \eta^4(\tau(\mathfrak{b})) \right|.$$

If $\psi = \chi$ and $\psi^2 = \chi^2 = 1$, the result is the sum of these two expressions.

Proof. Let us abbreviate

$$f(\mathfrak{b}) = v(\mathfrak{b})^{1/4}\eta(\tau(\mathfrak{b}))$$

for any fractional ideal (class) $\mathfrak{b} \subset k$. We have by definition and Proposition 2.6.4 that

$$\begin{aligned}(\Theta_P(\tau,\psi),\Theta_P(\tau,\chi)) &= \sum_{h,g\in T(\mathbb{A}_f)/K_T} \psi(g)\bar{\chi}(h)\Phi_P(\Theta_P(\tau,g),h) \\ &= -4\sum_{h,g\in T(\mathbb{A}_f)/K_T} \psi(g)\bar{\chi}(h)\log\left|f((hg)^{-1}\mathfrak{a})f(\tau((h^{-1}g)))\right|\end{aligned}$$

because for non-trivial characters the constant c does not contribute to the sum by orthogonality of characters. We split the sum above into

$$\begin{aligned}&\sum_{g,h}\psi(g)\bar{\chi}(h)\log\left|f((hg)^{-1}\mathfrak{a})\right| + \sum_{g,h}\psi(g)\bar{\chi}(h)\log\left|f((h^{-1}g))\right| \\ &\quad= \sum_g \psi(g)\chi(g)\sum_h \chi(h)\log|f(h\mathfrak{a})| + \sum_g \psi(g)\bar{\chi}(g)\sum_h \chi(h)\log|f(h)| \\ &\quad= \begin{cases} h_k\sum_h \chi(h)\log|f((h)\mathfrak{a})|, & \text{if } \psi=\bar{\chi}, \\ h_k\sum_h \chi(h)\log|f(h)|, & \text{if } \psi=\chi, \\ 0, & \text{otherwise,}\end{cases}\end{aligned}$$

as long as we do not have $\chi=\psi=\bar{\psi}$, in which case we get the sum of the two terms. For the first sum, we obtain

$$\begin{aligned}\sum_h \chi(h)\log|f((h)\mathfrak{a})| &= \sum_h \chi(h)\log\left|v((h)\mathfrak{a})^{1/4}\eta(\tau((h)\mathfrak{a}))\right| \\ &= \bar{\chi}(\mathfrak{a})\sum_h \chi(h)\log\left|v((h))^{1/4}\eta(\tau((h)))\right|\end{aligned}$$ ☐

We can now state the following theorem, which is an analog of Theorem 2.3.1 for vector valued theta functions. We choose a set of representatives of Cl_k^* modulo the relation $\chi \mapsto \bar{\chi}$ and denote this set by $\bar{\mathrm{Cl}}_k^*$.

Theorem 2.6.6.

(i) If $\psi = 1$, then $\Theta_P(\tau,\psi) = E_P(\tau)$ is an Eisenstein series, spanning space $\mathrm{Eis}_{1,P}$.

(ii) If $\psi \neq 1$, then $\Theta_P(\tau,\psi)$ is a cusp form.

(iii) The set

$$\mathcal{B}(P) = \{\Theta_P(\tau,\psi) \mid \psi \in \bar{\mathrm{Cl}}_k^*\}$$

is an orthogonal basis for $\Theta(P)$.

(iv) The dimension of the space $\Theta(P)$ *is equal to*

$$\dim \Theta(P) = \frac{h_k + 2^{t-1}}{2},$$

where h_k *is the class number of* k *and* t *is the number of distinct prime divisors of* D*, the discriminant of* k*.*

Proof. That $E_P(\tau)$ as defined above is really an Eisenstein series follows from the Siegel-Weil formula (see Theorem 3.1.2 and Lemma 3.3.3). The Eisenstein series correspond to isotropic vectors in the discriminant group P'/P and we assumed that $|P'/P| = |D|$ is square-free, which implies (i).

That $\Theta_P(\tau, \psi)$ is a cusp form for non-trivial ψ is clear.

To see that $\mathcal{B}(P)$ is a basis of $\Theta(P)$, first note that Proposition 2.6.5 implies that the set $\mathcal{B}(P)$ is linear independent. Moreover, if $\psi^2 \neq 1$ the Proposition also implies

$$(\bar{\psi}(\mathfrak{a})\Theta_P(\tau, \psi) - \Theta_P(\tau, \bar{\psi}), f(\tau)) = 0$$

for all $f \in \Theta(P)$. Therefore, $\bar{\psi}(\mathfrak{a})\Theta_P(\tau, \psi) - \Theta_P(\tau, \bar{\psi}) \in \Theta(P) \cap \Theta(P)^{\perp}$, where $\Theta(P)^{\perp}$ is the orthogonal complement of $\Theta(P)$ with respect to the Petersson inner product. Consequently, $\Theta_P(\tau, \bar{\psi}) = \bar{\psi}(\mathfrak{a})\Theta_P(\tau, \psi)$.

Finally, let A be the set of elements $x \in \mathrm{Cl}_{\mathrm{k}}$, such that $\bar{x} = x$ and let $B = \mathrm{Cl}_{\mathrm{k}} \setminus A$. Then $|\mathcal{B}(P)| = |A| + |B|/2$. Moreover, $|A| = 2^{t-1}$ by Proposition 2.2.2 and $|B| = h_k - 2^{t-1}$ which implies the assertion. $\square$

Remark 2.6.7. Note that the set $\mathcal{B}(P)$ does depend on the choice of representatives, but only up to scalar factors.

2.7 Liftings of newforms in the case of square-free level

In this section we will show some general properties of liftings of scalar valued modular forms to vector valued modular forms in the case of square-free level. We will apply these results to relate scalar valued theta series to vector valued ones. This lifting has been used by Bundschuh in his thesis [Bun01], by Bruinier and Bundschuh [BB03] and has been quite intensively studied by Scheithauer [Sch11].

Let L be an even lattice with quadratic form Q and square-free level N and let χ_L be the associated Dirichlet character. Recall the operator $\mathcal{S}_L$ defined in section 1.5.6.

Following Bundschuh [Bun01], we define a subspace of the newforms in $S_k(N, \chi_L)$. Let $A = L'/L$ and for a prime p denote by A_p the p-component of A. Moreover, write $\chi_L = \prod_{p|N} \chi_{L,p}$ as a product of characters modulo p for $p \mid N$. For each prime p_i dividing $N = p_1 \ldots p_r$, we define a an element $\varepsilon_i \in \{0, 1, -1\}$.

Definition 2.7.1. If $\dim_{\mathbb{F}_{p_i}} A_{p_i} \geq 2$ or $p_i = 2$, we define $\varepsilon_i = 0$. If $\dim_{\mathbb{F}_{p_i}} A_{p_i} = 1$, $p_i \neq 2$ and $NQ \mid_{A_{p_i}}$ represents the squares modulo p_i, we define $\varepsilon_i = 1$. Otherwise, we define $\varepsilon_i = -1$. Using these signs, we let

$$S_k^{\varepsilon_1,\dots,\varepsilon_r}(N,\chi_L) = \{f \in S_k^{\mathrm{new}}(N,\chi_L) \mid \chi_{L,p_i}(n) = -\varepsilon_i \text{ for some } i \text{ implies } c_f(n) = 0\}.$$

Remark 2.7.2. Note that we have

$$S_k^{\mathrm{new}}(N,\chi_L) = \bigoplus_{(\varepsilon_1,\dots,\varepsilon_r)\in\{\pm1\}^r} S_k^{\varepsilon_1,\dots,\varepsilon_r}.$$

We refer to the thesis of Bundschuh [Bun01, Satz 4.3.4] for details.

Theorem 2.7.3. *Let L be an even lattice of square-free level N and $f \in S_k^{\varepsilon_1,\dots,\varepsilon_r}(N,\chi_L)$. Assume that $\dim_{\mathbb{F}_{p_i}} A_{p_i} = 1$ or $\dim_{\mathbb{F}_{p_i}} A_{p_i} \geq 2$ even for all odd p_i. If we write $\mathcal{S}_L(f) = \sum_{\mu\in L'/L} F_\mu \phi_\mu$, then we have*

$$F_0 = \nu \frac{N}{|L'/L|} f.$$

Here, $\nu = \#\{\mu \in L'/L \mid NQ(\mu) \equiv m \bmod N\}$ for any $m \in \mathbb{Z}$ with $(m,N) = 1$ is independent of m.

Proof. We follow the proof of Satz 4.3.9 in [Bun01]. Let

$$f(\tau) = \sum_{n=1}^{\infty} a(n) e(n\tau)$$

be the Fourier expansion of f (at the cusp ∞) and let

$$f \mid_k W_N = \sum_{n=1}^{\infty} a_N(n) e(n\tau).$$

Let $\mu \in L'/L$ with $(NQ(\mu), N) = 1$. In this case it is not hard to see that

$$F_\mu(\tau) = \frac{N^{1+k/2} e(\operatorname{sgn}(L)/8)}{\sqrt{|L'/L|}} \sum_{n\equiv NQ(\mu) \bmod N} a_N(n) e\left(\frac{n}{N}\tau\right). \tag{2.7.1}$$

This follows from Theorem 4.2.8 in [Bun01] and can also be deduced from explicit formulas for the Weil representation [Sch09a, Str13]. We obtain

$$\begin{aligned} F_0 \mid_k W_N &= N^{-k/2}(F_0 \mid_k S)(N\tau) = N^{-k/2} \frac{e(-\operatorname{sgn}(L)/8)}{\sqrt{|L'/L|}} \sum_{\mu\in L'/L} F_\mu(N\tau) \\ &= \frac{N}{|L'/L|} \sum_{0\neq\mu\in L'/L} \sum_{n\equiv NQ(\mu) \bmod N} a_N(n) e(n\tau) + \sum_{n=2}^{\infty} b(n) e(Nn\tau), \end{aligned} \tag{2.7.2}$$

with certain coefficients $b(n)$. By the assumptions of the theorem on the dimension of A_p over $\mathbb{F}_p$ for $p \mid N$, we have that the representation number

$$\nu(m) = |\{\mu \in L'/L \mid NQ(\mu) \equiv m \bmod N\}|$$

is in fact equal for all $m \neq 0$ with $\nu(m) \neq 0$ (cf. [Kne02, Section 13]). Therefore, if we put $\nu = \nu(m)$ for any $m \in \mathbb{Z}$ with $(m, N) = 1$ and $\nu(m) \neq 0$, the last expression simplifies to

$$F_0 \mid_k W_N = \frac{N}{|L'/L|}\nu \sum_{(n,N)=1} a_N(n)e(n\tau) + \sum_{\substack{n>1\\(n,N)>1}} b(n)e(n\tau), \tag{2.7.3}$$

Here, we used the assumption that $f \in S_k^{\varepsilon_1,\dots,\varepsilon_r}(N, \chi_L)$. Therefore, we can express the difference to $f \mid_k W_N$ as

$$F_0 \mid_k W_N - \frac{N}{|L'/L|}\nu f \mid_k W_N = \sum_{\substack{n\geq 1\\(n,N)>1}} c(n)e(n\tau)$$

for some complex numbers $c(n)$. However, we also know that F_0 and $F_0 \mid_k W_N$ are newforms by Lemma 1.5.5 and thus this difference vanishes by Remark 1.5.4. Consequently, we also have

$$F_0 = \frac{N}{|L'/L|}\nu \cdot f.$$

□

The group $\mathrm{O}(L'/L)$ acts on vector-valued modular forms by permuting the basis vectors ϕ_μ. That is, $\sigma \in \mathrm{O}(L'/L)$ acts via $\phi_\mu \mapsto \phi_{\sigma(\mu)}$. Using this action, we define the symmetrization of a modular form $f \in M_{k,L}$ as

$$f^{\mathrm{sym}}(\tau) = \sum_{\sigma \in \mathrm{O}(L'/L)} f^\sigma(\tau) = \sum_{\mu \in L'/L} \sum_{\sigma \in \mathrm{O}(L'/L)} f_\mu(\tau)\phi_{\sigma(\mu)}.$$

This function is clearly invariant under the action of $\mathrm{O}(L'/L)$. We write $M_{k,L}^{\mathrm{sym}}$ for the subspace of $M_{k,L}$ that is invariant under $\mathrm{O}(L'/L)$. The map

$$M_{k,L} \longrightarrow M_{k,L}^{\mathrm{sym}}, \quad f \mapsto f^{\mathrm{sym}}$$

is obviously surjective.

The following proposition can be found in Propositions 5.1 and 5.3 of [Sch11].

Proposition 2.7.4. *Let L be an even lattice of square-free level N. Then the orthogonal group $\mathrm{O}(L'/L)$ acts transitively on all elements of the same norm and order in L'/L. Moreover, if $F \in M_{k,L}^{\mathrm{sym}}$ and $F_0 = 0$, then $F = 0$.*

2.8 Liftings of scalar valued theta functions

As before, let $D < 0$ be an odd fundamental discriminant and let $k = \mathbb{Q}(\sqrt{D})$ be the imaginary quadratic field of discriminant D. We write $\mathcal{O}_D$ for the ring of integers in k and Cl_k for the ideal class group of k. We assume that D is odd.

Definition 2.8.1. We define the subspace of symmetric theta functions as

$$\Theta^{\mathrm{sym}}(P) = \langle\, \Theta_P^{\mathrm{sym}}(\tau, h) \mid h \in I_k/\hat{\mathcal{O}}_D^\times \cong \mathrm{Cl}_\mathrm{k} \rangle_\mathbb{C} \subset \Theta(P).$$

In the definition, we identified Cl_k again with $I_k/\hat{\mathcal{O}}_D^\times$ via Theorem 2.1.2.

Proposition 2.8.2. *Let $\mathfrak{a} \subset \mathcal{O}_D$ be an ideal and let $(P, Q) = \left(\mathfrak{a}, \frac{\mathrm{N}(x)}{\mathrm{N}(\mathfrak{a})}\right)$ be the corresponding even quadratic lattice. For $h \in I_k/\hat{\mathcal{O}}_D^\times$ corresponding to the ideal class of $\mathfrak{b} \subset \mathcal{O}_D$, we have*

$$\mathcal{S}_P(\theta_{\mathfrak{a}\mathfrak{b}^2})(\tau) = \Theta_P^{\mathrm{sym}}(\tau, h).$$

Proof. Note that in our case the level N is equal to $|D|$, the order of the discriminant group. Moreover, for $p \mid D$, the $\mathbb{F}_p$-rank of A_p is equal to one. That means the "signs" $\epsilon_1, \dots, \epsilon_t$, where t is the number of prime divisors of D in Definition 2.7.1 are all nonzero.

We first show that the 0-th components of $\mathcal{S}_P(\theta_{\mathfrak{a}\mathfrak{b}^2})(\tau)$ and $\Theta_P^{\mathrm{sym}}(\tau, h)$ agree and then the claim follows from Proposition 2.7.4.

We write

$$\theta_{\mathfrak{a}\mathfrak{b}^2} = E_\mathfrak{A}(\tau) + g_{\mathfrak{a}\mathfrak{b}^2}(\tau)$$

for a cusp form $g_{\mathfrak{a}\mathfrak{b}^2}(\tau) \in S_1(|D|, \chi_D)$. Then, it is not hard to see that in fact

$$g_{\mathfrak{a}\mathfrak{b}^2}(\tau) \in S(|D|, \chi_D)^{\varepsilon_1, \dots, \varepsilon_t}$$

for $\varepsilon_1, \dots, \varepsilon_r$ as in Definition 2.7.1 for the lattice P. Indeed, we write $\chi_P = \chi_D = \prod_{i=1}^t \chi_{p_i^*}$, where

$$\chi_{p^*}(n) = \left(\frac{p^*}{n}\right) \text{ with } p^* = \left(\frac{-1}{p}\right) p$$

for a prime divisor p of D. Then $\chi_{p_i^*}(n) = -\epsilon_i$ implies that the coefficient of index n of $\theta_{\mathfrak{a}\mathfrak{b}^2}$ and of $E_\mathfrak{A}$ vanish because the characters $\chi_{p_i^*}(n)$ are the basis of the genus characters (see Section 2.2).

Moreover, the normalized Eisenstein series $E_\mathfrak{A} \in M_1(|D|, \chi_D)$, where $\mathfrak{A}$ is the genus of $\mathfrak{a}$, lifts to

$$\mathcal{S}_L(E_\mathfrak{A}) = \nu E_P.$$

Proposition 1.5.20 shows that the lift of an Eisenstein series is again an Eisenstein series. Since the Eisenstein subspace of $M_{1,P}$ is one-dimensional, the lift of it has to be a multiple of E_P. The correct multiple can be read off from (2.7.1) and (2.7.2) in the proof of Theorem 2.7.3.

Note that under the assumptions of the Proposition, we have $\nu = |\mathrm{O}(L'/L)| = 2^{t-1}$ by Proposition 2.7.4. Thus, by Theorem 2.7.3, the 0-th component of

$$\Theta_P^{\mathrm{sym}}(\tau, h) - \mathcal{S}_P(\theta_{\mathfrak{a}\mathfrak{b}^2})(\tau)$$

vanishes. Then the lemma follows from Proposition 2.7.4 since $\Theta_P^{\mathrm{sym}}(\tau)$ and $\mathcal{S}_P(\theta_{\mathfrak{a}\mathfrak{b}^2})(\tau)$ are invariant under $\mathrm{O}(P'/P)$. ☐

Remark 2.8.3. It follows from Proposition 2.8.2 that the space $\Theta^{\mathrm{sym}}(P)$ is the space spanned by the lifts $\mathcal{S}_P(\theta_{\mathfrak{a}\mathfrak{b}^2})$ of the scalar valued theta functions $\theta_{\mathfrak{a}\mathfrak{b}^2}$ in the genus of $\mathfrak{a}$. This establishes an isomorphism between Θ_P^{sym} and the space of scalar valued theta functions in the genus of $\mathfrak{a}$.

Proposition 2.8.4. *Let $\mathcal{C} = \mathrm{Cl}_{\mathrm{k}}^*$ be the group of class group characters. Then the set*

$$\mathcal{B}^{\mathrm{sym}}(P) = \{\Theta_P^{\mathrm{sym}}(\tau, \psi) \mid \psi \in \mathcal{C}^2\}$$

spans $\Theta^{\mathrm{sym}}(P) \subset M_{1,P}$. The elements of $\mathcal{B}^{\mathrm{sym}}(P)$ are permuted by the action of $\mathrm{Aut}(\mathbb{C})$. *Moreover, $(\Theta_P^{\mathrm{sym}}(\tau, \psi), \Theta_P^{\mathrm{sym}}(\tau, \chi)) = 0$ unless $\psi = \chi$ or $\psi = \overline{\chi}$.*

Proof. Using Proposition 2.8.2, we see that $\Theta_P^{\mathrm{sym}}(\tau, \psi)$ is in fact equal to the lift of

$$w_k \sum_{\chi^2 = \psi} \bar{\chi}(\mathfrak{a})\theta_\chi.$$

Moreover, we have the relation $\bar{\psi}(\mathfrak{a})\Theta_P(\tau, \psi) = \overline{\Theta_P(\tau, \psi)} = \Theta_P(\tau, \bar{\psi})$.

The orthogonality and the action of $\mathrm{Aut}(\mathbb{C})$ follows from these relations together with Proposition 1.5.20. Alternatively, it also follows from Proposition 2.6.6, which also implies that this set spans $\Theta^{\mathrm{sym}}(P)$. ☐

Corollary 2.8.5. *Let $\bar{\mathcal{C}}^2$ be a set of representatives of $\mathcal{C}^2$ modulo the relation $\chi \mapsto \bar{\chi}$. Then the set*

$$\{\Theta_P^{\mathrm{sym}}(\tau, \psi) \mid \psi \in \bar{\mathcal{C}}^2\}$$

is an orthogonal basis of $\Theta^{\mathrm{sym}}(P)$.

Corollary 2.8.6. *Let $\psi \in \mathrm{Cl}_{\mathrm{k}}^*, \psi \neq 1$. We have*

$$w_k^2 \sum_{\chi^2 = \psi} (1 + \bar{\chi}^2(\mathfrak{a}))(\theta_\chi(\tau), \theta_\chi(\tau)) = (\Theta_P^{\mathrm{sym}}(\tau, \psi), \Theta_P^{\mathrm{sym}}(\tau, \psi)).$$

Proof. We expand the right hand side and obtain

$$\begin{aligned}(\Theta_P^{\mathrm{sym}}(\tau, \psi), \Theta_P^{\mathrm{sym}}(\tau, \psi)) &= w_k^2 (\sum_{\chi^2 = \psi} \bar{\chi}(\mathfrak{a})\theta_\chi(\tau), \sum_{\lambda^2 = \psi} \bar{\lambda}(\mathfrak{a})\theta_\lambda(\tau)) \\ &= w_k^2 \sum_{\chi^2 = \psi} (1 + \chi^2(\mathfrak{a}))(\theta_\chi(\tau), \theta_\chi(\tau)).\end{aligned}$$

☐

Corollary 2.8.7. *Suppose that* $D=-p$ *is a prime discriminant. Let* $\chi \in \mathrm{Cl}_{\mathrm{k}}^{*}$ *with* $\chi \neq 1$ *and write again* $\tau(\mathfrak{a}) = u(\mathfrak{a}) + iv(\mathfrak{a})$*. Then we have*

$$(\theta_\chi(\tau), \theta_\chi(\tau)) = -\frac{4h_k}{w_k^2} \sum_{\mathfrak{a}\in \mathrm{Cl}_{\mathrm{k}}} \chi^2(\mathfrak{a}) \log \left| v(\mathfrak{a})^{1/2} \eta^2(\tau(\mathfrak{a}))\right| .$$

Proof. We use Corollary 2.8.6 with $P \cong \mathcal{O}_D$ for $D=-p$ together with Theorem 2.6.6. Moreover, we have to use the fact that for prime discriminants, the class number is odd and therefore, the sum in Corollary 2.8.6 reduces to a single term. ☐

3 CM cycles and CM values of regularized theta lifts

In this chapter we fix a rational quadratic space (V, Q) of type $(2, n)$. Recall that we write $H = \mathrm{GSpin}_V$. We fix a compact open subgroup K of $H(\mathbb{A}_f)$ and consider the Shimura variety X_K as in Section 1.3.3. Its complex points are given by

$$X_K(\mathbb{C}) = H(\mathbb{Q})\backslash(\mathbb{D} \times H(\mathbb{A}_f)/K).$$

The type of CM cycles we consider are given as follows. Let $U \subset V$ be a 2-dimensional, positive definite rational subspace. This determines a two-point subset $\{z_U^\pm\} \subset \mathbb{D}$ given by $U(\mathbb{R})$ with the two possible choices of orientation. Denote by $V_- = U^\perp \subset V$ the n-dimensional negative definite orthogonal complement of U over $\mathbb{Q}$. Then we have a *rational* splitting

$$V = U \oplus V_-. \tag{3.0.1}$$

We obtain a cycle $Z(U)_K \subset X_K$, which is called the *CM cycle* in X_K corresponding to U. It is obtained by embedding a Shimura variety associated with U into X_K, which is given as follows. Put $T = \mathrm{GSpin}_U$, which we view as a subgroup of H acting trivially on V_-. The group $K_T = K \cap T(\mathbb{A}_f)$ is a compact open subgroup of $T(\mathbb{A}_f)$. We obtain a generically injective map

$$Z(U)_K = T(\mathbb{Q})\backslash(\{z_U^\pm\} \times T(\mathbb{A}_f)/K_T) \hookrightarrow X_K. \tag{3.0.2}$$

Here, each point is counted with multiplicity $\frac{2}{w_{K,T}}$, where we let $w_{K,T} = |(T(\mathbb{Q}) \cap K_T)|$. If the choice of K is clear from the context, we will abbreviate $Z(U) = Z(U)_K$. The following lemma follows directly from Theorem 2.1.2 and our explications in Section 2.4.

Lemma 3.0.8. *Suppose that U is isomorphic as a rational quadratic space to an imaginary quadratic field k and let $\mathcal{O}_k \subset k$ be its ring of integers. If $K_T = \hat{\mathcal{O}}_k^\times$, then $Z(U)$ is isomorphic to two copies of the ideal class group $\mathrm{Cl_k}$ of k, that is, $Z(U) \cong \mathrm{Cl_k} \times \{z_U^\pm\}$.*

3.1 The value of the theta lift at a CM cycle

Let $L \subset V$ be an even lattice. In the following sections, we study the value of the theta lift $\Phi(z, h, f) = \Phi_L(z, h, f)$ at a CM point $(z, h) \in Z(U)$, as well as the average value $\Phi(Z(U), f)$.

The latter is defined to be the weighted sum

$$\Phi(Z(U), f) = \frac{2}{w_{K,T}} \sum_{(z,h)\in \operatorname{supp} Z(U)_K} \Phi(z, h, f). \tag{3.1.1}$$

This value has been studied by Schofer [Sch09b] for the Borcherds lift of weakly holomorphic modular forms and by Bruinier and Yang [BY09] for weak Maaß forms. We review their main results.

The splitting (3.0.1) yields two lattices, P and N, defined by

$$N = L \cap V_-, \quad P = L \cap U.$$

The direct sum $P \oplus N$ is a sublattice of L of finite index.

For $z = z_U^\pm$ and $h \in T(\mathbb{A}_f)$, the Siegel theta function $\Theta_{P\oplus N}(\tau, z, h)$ splits as a product

$$\Theta_{P\oplus N}(\tau, z_U^\pm, h) = \Theta_P(\tau, z_U^\pm, h) \otimes \Theta_N(\tau). \tag{3.1.2}$$

Here $\Theta_N(\tau) = \Theta_N(\tau, 1)$ is the S_N-valued theta function of weight $n/2$ associated to the negative definite lattice N. Note that $v^{-n/2}\overline{\Theta_N(\tau)}$ is the holomorphic theta function corresponding to the positive definite lattice N^-.

Attached to P there is a so-called incoherent Eisenstein series $\hat{E}_P(\tau, s)$ of weight 1 transforming with representation $\overline{\rho_P} = \rho_{P^-}$ [KRY99, KRY04]. Here, the term "incoherent" refers to the fact that it is built from local data at each place which does not correspond to a quadratic space over $\mathbb{Q}$.

Its central value at $s = 0$ vanishes but it is the value of the derivative $\frac{\partial}{\partial s}\hat{E}_P(\tau, s)$ at $s = 0$ that carries the arithmetic data which contributes to the CM values. The function

$$\mathcal{E}_P(\tau) = \frac{\partial}{\partial s}\hat{E}_P(\tau, s)\mid_{s=0} \tag{3.1.3}$$

is a harmonic weak Maaß form of weight 1 with respect to $\overline{\rho_P}$.

If $S(q) = \sum_{n\in\mathbb{Z}} a_n q^n$ is a Laurent series in q, we write $\mathrm{CT}(S) = a_0$ for the constant term in the q-expansion.

Theorem 3.1.1. *The value of the theta lift $\Phi(z, h, f)$ at the CM cycle $Z(U)_K$ is given by*

$$\Phi(Z(U), f) = \deg(Z(U)) \left(\mathrm{CT}\left(\langle f^+_{P\oplus N}(\tau),\, \Theta_{N^-}(\tau) \otimes\, \mathcal{E}^+_P(\tau)\rangle\right) - L'(\xi(f), U, 0)\right).$$

Here, $L'(\xi(f), U, s)$ is the derivative with respect to s of the L-function defined by the convolution integral

$$L(\xi(f), U, s) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \langle \overline{\xi(f)(\tau)}, \hat{E}_P(\tau, s) \otimes \Theta_{N^-}(\tau)\rangle\, v^{1+n/2}\frac{du dv}{v^2}.$$

Proof. This is Theorem 4.7 in [BY09] with a corrected sign. □

The proof involves the Siegel-Weil formula and the standard Eisenstein series associated with P, which is defined as

$$E_P(\tau, s) = \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (\Im(\tau)^s \varphi_0) \mid_{1,P} \gamma. \tag{3.1.4}$$

The series converges for $\Re(s) > 1$ and has a meromorphic continuation to the whole complex s-plane. Note that we normalized $E_P(\tau, s)$, such that the constant term is equal to one.

As in [BY09], we fix the Tamagawa Haar measure on $\mathrm{SO}_U(\mathbb{R}) \cong \mathrm{SO}(2, \mathbb{R})$ such that $\mathrm{vol}(\mathrm{SO}_U(\mathbb{R})) = 1$. This implies that we have $\mathrm{vol}(\mathrm{SO}_U(\mathbb{Q}) \backslash \mathrm{SO}_U(\mathbb{A}_f)) = 2$. Moreover, we use the usual Haar measure on $\mathbb{A}_f^\times$. It satisfies $\mathrm{vol}(\mathbb{Z}_p^\times) = \mathrm{vol}(\hat{\mathbb{Z}}^\times) = 1$ and $\mathrm{vol}(\mathbb{Q}^\times \backslash \mathbb{A}_f^\times) = 1/2$.

Theorem 3.1.2 (Siegel-Weil formula)**.** *The Eisenstein series $E_P(\tau, s)$ is holomorphic at $s = 0$ and we have*

$$\int_{\mathrm{SO}_U(\mathbb{Q}) \backslash \mathrm{SO}_U(\mathbb{A}_f)} \Theta_P(\tau, z_U, h)\, dh = 2E_P(\tau, 0).$$

Proof. This has been proved by Kudla and Rallis [KR88]. See also Bruinier and Yang [BY09, Proposition 2.2]. ☐

Remark 3.1.3. Note that the Siegel-Weil formula provides a proof that $E_P(\tau, 0) \in M_{1,P}$. This can be seen quite explicitly using Proposition 3.3.3 in the next section.

For later reference, we write the Fourier expansion of $E_P(\tau, 0)$ as

$$E_P(\tau) = E_P(\tau, 0) = \phi_0 + \sum_{\beta \in P'/P} \sum_{\substack{n \in \mathbb{Q}_{>0} \\ n \in Q(\beta) + \mathbb{Z}}} \tilde{\rho}(n, \beta) e(n\tau) \phi_\beta. \tag{3.1.5}$$

A crucial fact is that $\mathcal{E}_P(\tau)$ maps to $E_P(\tau)$ under the ξ_1-operator. This has been stated by Bruinier and Yang [BY09, Remark 2.4] and follows directly from equation (2.19) in [BY09]. Note that with their normalization of the Eisenstein series there is a factor 1/2 missing in the remark.

The identity $\xi(\mathcal{E}_P(\tau)) = E_P(\tau, 0)$ can also be stated in terms of differential forms using Lemma 1.5.11.

Lemma 3.1.4. *We have*

$$\bar{\partial} \left(\mathcal{E}_P(\tau)\, d\tau \right) = -v \overline{E_P(\tau, 0)}\, d\mu(\tau).$$

The Fourier expansion of $\mathcal{E}_P(\tau)$ can be determined using a very general result by Kudla and Yang [KY10] on the coefficients of Eisenstein series on SL_2. The following result is Proposition 7.2 in [KY10]. In our particular case, the result goes back to Schofer [Sch09b].

We write the Fourier expansion of $\mathcal{E}_P(\tau)$ as

$$\begin{aligned}\mathcal{E}_P(\tau) = &\sum_{\beta\in P'/P}\sum_{\substack{n\in\mathbb{Q}_{>0}\\ n+Q(\beta)\in\mathbb{Z}}}\kappa(n,\beta)e(n\tau)\phi_\beta \\ &+\log(v)\phi_0 - \sum_{\beta\in P'/P}\sum_{\substack{n\in\mathbb{Q}_{<0}\\ n+Q(\beta)\in\mathbb{Z}}}\tilde{\rho}(-n,\beta)W(2\pi nv)e(n\tau)\phi_\beta.\end{aligned} \tag{3.1.6}$$

Let $D<0$ be a fundamental discriminant and h_k be the class number of $k=\mathbb{Q}(\sqrt{D})$. We denote by $\rho(n)$ the number of integral ideals of k of norm n. Write the global Hilbert symbol $\chi = (D,\cdot)_{\mathbb{A}} = \prod_p \chi_p$ as a product of local quadratic characters. Assume that $(P,Q) = (\mathfrak{a}, \frac{\mathrm{N}}{\mathrm{N}(\mathfrak{a})})$ for a fractional ideal $\mathfrak{a}\subset k$. Moreover, write

$$\Lambda(\chi_D,s) = |D|^{\frac{s}{2}}\,\pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)L(\chi_D,s)$$

for the completed L-function associated with χ_D, such that $\Lambda(1-s,\chi)=\Lambda(s)$. For $n>0$, define $\mathrm{Diff}(n)$ to be the set of primes $p<\infty$ such that $\chi_p(-n\mathrm{N}(\mathfrak{a}))=-1$. For $n<0$, let $\mathrm{Diff}(n)$ be the set of such finite primes together with ∞.

Denote by $o(n)$ the number of primes $p\mid D$ such that $\mathrm{ord}_p(nD)>0$.

Theorem 3.1.5. *Let $\beta\in P'/P$ and $n>0$ such that $n+Q(\beta)\in\mathbb{Z}$. We have $\kappa(n,\beta)=0$ unless $|\mathrm{Diff}(n)|=1$. Assume that $\mathrm{Diff}(n)=\{p\}$.*

(i) If p is inert in k, then

$$\Lambda(\chi_D,0)\kappa(n,\beta) = -2^{o(n)}(\mathrm{ord}_p(n)+1)\rho(n|D|/p)\log p.$$

(ii) If p is ramified in k, then

$$\Lambda(\chi_D,0)\kappa(n,\beta) = -2^{o(n)}(\mathrm{ord}_p(nD))\rho(n|D|)\log p.$$

Finally, we have for the constant term

$$\kappa(0,0) = -2\frac{\Lambda'(\chi_D,0)}{\Lambda(\chi_D,0)}.$$

Remark 3.1.6. Note that by the class number formula $\Lambda(\chi_D,0)=\Lambda(\chi_D,1)=\frac{2}{w_k}h_k$, where w_k is the number of roots of unity contained in k. See Section 6 of [Dav00].

Note that for $n>0$ with $n\in Q(\beta)+\mathbb{Z}$, we have $\tilde{\rho}(n,\beta)=\frac{w_k}{h_k}2^{o(n)-1}\rho(n\,|D|)$.

Remark 3.1.7. If you compare the theorem above with Theorem 4.1 in [Sch09b], note that Schofer's normalization of the completed L-function $\Lambda(\chi_D,s)$ does not include the factor $|D|^{s/2}$.

3.2 Generalities on principal parts of preimages under ξ

Before we come to the value of the theta lift at an individual CM point, we have to make some considerations regarding preimages under the ξ-operator. The reader might want to skip this section on a first reading.

For a number field F, denote the F-vector space of cusp forms in S_{2-k,L^-} with Fourier coefficients in F by $S_{2-k,L^-}(F)$. Moreover, we denote by $M^!_{k,L}(F)$ and $H_{k,L}(F)$ the subspaces of weakly holomorphic forms and weak Maaß forms with principal parts contained in $F[q^{-1}]$.

Proposition 3.2.1. *The space $H_{k,L}/M^!_{k,L}$ has a system of representatives in $H_{k,L}$ with rational principal parts.*

Proof. The pairing $\{\cdot,\cdot\}$ induces an isomorphism between $H_{k,L}(\mathbb{Q})/M^!_{k,L}(\mathbb{Q})$ and the dual space $S^*_{2-k,L^-}(\mathbb{Q})$ of $S_{2-k,L^-}(\mathbb{Q})$ mapping $\bar{f} \in H_{k,L}(\mathbb{Q})/M^!_{k,L}(\mathbb{Q})$ to the linear functional $\{\cdot, f\}$ for any representative $f \in H_{k,L}(\mathbb{Q})$ of the class $\bar{f}$.

By Theorem 1.5.18, there is a basis $f_1, \ldots, f_r$ of S_{2-k,L^-} with rational Fourier coefficients. Let $\bar{F}_1, \ldots, \bar{F}_r \in H_{k,L}(\mathbb{Q})/M^!_{k,L}(\mathbb{Q})$ be the dual basis. Thus, we have

$$H_{k,L}/M^!_{k,L} \cong S^*_{2-k,L^-} \cong S^*_{2-k,L^-}(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}.$$

We have seen that the latter space is isomorphic to $(H_{k,L}(\mathbb{Q})/M^!_{k,L}(\mathbb{Q})) \otimes_{\mathbb{Q}} \mathbb{C}$. □

Lemma 3.2.2. *Let $F \subset \mathbb{C}$ be a subfield of $\mathbb{C}$. Let $g \in S_{2-k,L^-}(F)$ and $B = \{g_1, \ldots, g_r\}$ be a basis of S_{2-k,L^-} with all Fourier coefficients contained in F. Then there is a $\widetilde{g} \in H_{k,L}$ with $\xi\widetilde{g} = g$, such that the Fourier coefficients of the principal part of $\widetilde{g}$ are contained in the ring $F[S]$ for the set $S = \{(g, g_j) \mid j \in \{1, \ldots, r\}\}$.*

Proof. Let $\mathcal{B} = \{G_1, \ldots, G_r\}$ be the dual basis of B with respect to the pairing $\{\cdot,\cdot\}$. In the proof of Proposition 3.2.1, we have shown that $\mathcal{B}$ determines a system of representatives for $H_{k,L}/M^!_{k,L}$ with principal parts in F. For all $i \in \{1, \ldots, r\}$ let $\widetilde{g_i} \in \xi_k^{-1}(S_{2-k,L^-})$, such that $\xi_k\widetilde{g_i} = g_i$ and write $\widetilde{g_i} = \sum_{j=1}^r a_{ij}G_j$. Since $\mathcal{B}$ is dual to B, we have

$$\{g_i, \widetilde{g_j}\} = a_{ji}.$$

On the other hand, we have $\{g_i, \widetilde{g_j}\} = (g_i, g_j)$. Moreover, if $g \in S_{2-k,L^-}(F)$, then we can write $g = \sum_{j=1}^r c_j g_j$ with coefficients $c_j \in F$. The form $\widetilde{g} := \sum_{j=1}^r c_j\widetilde{g_j}$ satisfies $\xi_k\widetilde{g} = g$ and the coefficient of index (γ, n) is equal to $\sum_{j=1}^r c_j \sum_{m=1}^r a_{jm}G_m(\gamma, n)$, where $G_m(\gamma, n)$ denotes the corresponding Fourier coefficient of G_m. Since G_m has only rational coefficients in its principal part, the c_j are contained in F and $a_{jm} = (g_j, g_m)$, the claim follows. □

3.3 The value of $\Phi(z, f)$ at an individual CM point

We are now interested in computing the value of the theta lift $\Phi(z, f)$ at a CM point. Let $K_P \subset K_T \subset T(\mathbb{A}_f)$ be a compact open subgroup such that K_P preserves P and acts trivially on P'/P. Consider the Shimura variety

$$Z(U)_{P,K} = T(\mathbb{Q})\backslash(\{z_U^\pm\} \times T(\mathbb{A}_f)/K_P).$$

This is isomorphic to two identical copies of the “class group”

$$C_{P,K} = T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K_P \tag{3.3.1}$$

and defines a cover of the CM cycle $Z(U)_K$ with $[C_K : C_{P,K}]$ branches.

Since K_P acts trivially on P'/P, the value $\Theta_P(\tau, z_U^\pm, h)$ is well defined for an element $h \in C_{P,K}$. As a function of τ, we have $\Theta_P(\tau, z_U^\pm, h) \in M_{1,P}$.

As before, we would like to apply Stoke's theorem to compute $\Phi(z, h, f)$ for $(z, h) \in Z(U)_{P,K}$. The existence of a preimage of each theta function is clear by the exact sequence (1.5.6). However, such a preimage is only unique up to weakly holomorphic modular forms. Therefore, we will use the following “natural” normalizations.

Write the Fourier expansion of $\Theta_P(\tau, z_U^\pm, h)$ as

$$\Theta_P(\tau, h) := \Theta_P(\tau, z_U^\pm, h) = \sum_{\beta\in P'/P} \sum_{n\geq 0} a_P(h, n, \beta)e(n\tau)\phi_\beta. \tag{3.3.2}$$

Since P is positive definite $a_P(h, 0, \beta) = 0$ for $\beta \neq 0$ and $a_P(h, 0, 0) = 1$. The constant coefficient of the holomorphic Eisenstein series $E_P(\tau, 0)$ is also equal to 1. Thus, we have the decomposition

$$\Theta_P(\tau, h) = E_P(\tau, 0) + g_P(\tau, h), \tag{3.3.3}$$

where for each $h \in C_{P,K}$ the form $g_P(\tau, h) \in S_{1,P}$ is a cusp form of weight 1.

We let $s_1, \dots, s_r \in S_{1,P}$ be a basis with integral Fourier coefficients. Moreover, by Lemma 3.2.2, there are weak Maaß forms $S_1, \dots, S_r \in H_{1,P}$, such that $\xi_1 S_i = s_i$ and each coefficient in the principal part of S_i is of the form $\sum_{j=1}^r a_{ij} \cdot (s_i, s_j)$ with $a_{ij} \in \mathbb{Q}$.

We define coefficients $a_i(h)$ by $\sum_{i=1}^r a_i(h)s_i = g_P(\tau, h)$. Using the same set of coefficients, we put $\widetilde{g}_P(\tau, h) := \sum_{i=1}^r a_i(h)S_i$.

Remark 3.3.1. Choosing the preimages this way is depends on the choice of a basis of $S_{1,P}$ and of the representatives of the dual basis. In this sense, the choice is unique up to an element in $M^!_{1,P^-}(\mathbb{Q})$. After our normalization, the forms $\widetilde{g}_P(\tau, h)$ are unique up to an element in $M^!_{1,P^-}(R)$, where $R = \mathbb{Q}[S]$ for the set $S := \{(s_i, s_j) \mid i, j \in \{1, \dots, r\}\}$.

The following proposition summarizes our construction.

Proposition 3.3.2. *Let $\widetilde{E}_P \in H_{1,P}$ with vanishing principal part such that $\xi(\widetilde{E}_P) = E_P$ holds. For $h \in C_{P,K}$ define*

$$\widetilde{\Theta}_P(\tau, h) := \widetilde{E}_P(\tau) + \widetilde{g}_P(\tau, h).$$

Then $\xi_1\widetilde{\Theta}_P(\tau,h) = \Theta_P(\tau, z_U^{\pm}, h)$ *and we have*

$$\sum_{h\in C_{P,K}} \widetilde{\Theta}_P(\tau,h) = \frac{w_P}{\operatorname{vol}(K_P)}\widetilde{E}_P(\tau)$$

and

$$\frac{w_P}{\operatorname{vol}(K_P)} = |C_{P,K}|\,.$$

Here, $w_P = \#(T(\mathbb{Q})\cap K_P)$*.*

For the proof, we quote the following Lemma[1] of Schofer [Sch09b, Lemma 2.13].

Lemma 3.3.3. *Let* $B(h)$ *be a function on* $T(\mathbb{A}_f)$ *depending only on the image of* h *in* $\mathrm{SO}_U(\mathbb{A}_f)$*. Assume that* B *is invariant under* K_P *and* $T(\mathbb{Q})$*. Then*

$$2\frac{\operatorname{vol}(K_P)}{w_P}\sum_{h\in C_{P,K}} B(h) = \int_{\mathrm{SO}_U(\mathbb{Q})\backslash \mathrm{SO}_U(\mathbb{A}_f)} B(h)dh.$$

Proof of Proposition 3.3.2. Setting $B(h) = \Theta_N(\tau,(z_U^+,h)) = \Theta_N(\tau,(z_U^-,h))$ in Lemma 3.3.3 we get

$$\sum_{h\in C_{P,K}} \Theta_P(\tau,(z_U^{\pm},h)) = \frac{w_P}{2\operatorname{vol}(K_P)}\int_{\mathrm{SO}_U(\mathbb{Q})\backslash \mathrm{SO}_U(\mathbb{A}_f)} \Theta_P(\tau,(z_U^+,h))dh.$$

The latter integral is equal to $2E_P(\tau,0)$ by the Siegel-Weil formula (Theorem 3.1.2). Therefore, since $\Theta_P(\tau,z_U^{\pm},h) = E_P(\tau,0)+g_P(\tau,h)$, we have indeed

$$\sum_{h\in C_{P,K}} g_P(\tau,h) = 0.$$

Consequently,

$$\sum_{h\in C_{P,K}} a_i(h) = 0$$

for all $h\in C_{P,K}$, since the s_i are linearly independent. It follows that

$$\xi_1\widetilde{g}_P(\tau,h) = g_P(\tau,h) \text{ from the definition of } \widetilde{g}_P(\tau,h) \text{ and}$$
$$\sum_{h\in C_{P,K}} \widetilde{g}_P(\tau,h) = 0.$$

Thus, we obtain

$$\sum_{h\in C_{P,K}} \widetilde{\Theta}_P(\tau,h) = |C_{P,K}|\cdot \widetilde{E}_P(\tau).$$

---

[1] Note that the factor $2/w_P$ is missing in [Sch09b].

The second identity follows from Lemma 3.3.3 with $B(h) = 1$. □

Lemma 1.5.11 becomes the following statement in our situation.

Lemma 3.3.4. *We have the equality of differential forms*

$$\bar{\partial}(\widetilde{\Theta}_P(\tau, h)d\tau) = -v\overline{\Theta_P(\tau, z_U^{\pm}, h)}d\mu(\tau).$$

Note that we can always assume that L splits as $L = P \oplus N$ without loss of generality because we can replace f by $f_{P\oplus N}$, yielding

$$\langle f, \Theta_L\rangle = \langle f_{P\oplus N}, \Theta_P \otimes \Theta_N\rangle,$$

since $\Theta_{P\oplus N} = \Theta_P \otimes \Theta_N$, and $\Theta_L = (\Theta_{P\oplus N})^L$ by [BY09].

We express the theta integral in a way that is convenient for the following calculations.

Lemma 3.3.5. *We have*

$$\Phi(z_U^{\pm}, h, f) = \lim_{T\to\infty}\left(\int_{\mathcal{F}_T} \langle f_{P\oplus N}(\tau), \Theta_{N^-}(\tau) \otimes \overline{\Theta_P(\tau, z_U^{\pm}, h)}\rangle v d\mu(\tau) - A_0 \log(T)\right),$$

where

$$A_0 = \mathrm{CT}\left(\langle f^+_{P\oplus N}(\tau),\, \Theta_{N^-}(\tau) \otimes \phi_{0+P}\rangle\right).$$

Proof. This is Lemma 4.5 of [BY09]. The proof is quite short and so we give a few more details for the sake of completeness. Note that a similar statement in the case of signature $(2, 0)$ can be found in Lemma 2.19 of [Sch09b]. The idea of the proof is based on the proof Proposition 2.5 of [Kud03].

We assume, as described above, that $L = P \oplus N$ and consider the regularized integral

$$\Phi(z_U^{\pm}, h, f) = \int_{\mathcal{F}}^{\mathrm{reg}} \langle f(\tau), \Theta_{N^-}(\tau) \otimes \overline{\Theta_P(\tau, z_U^{\pm}, h)}\rangle v d\mu(\tau).$$

We split the integral into two parts, corresponding to the decomposition $f = f^+ + f^-$ and obtain

$$\Phi(z_U^{\pm}, h, f) = \int_{\mathcal{F}}^{\mathrm{reg}} \langle f^+(\tau), \Theta_{N^-}(\tau) \otimes \overline{\Theta_P(\tau, z_U^{\pm}, h)}\rangle v d\mu(\tau) \tag{3.3.4}$$

$$+ \int_{\mathcal{F}} \langle f^-(\tau), \Theta_{N^-}(\tau) \otimes \overline{\Theta_P(\tau, z_U^{\pm}, h)}\rangle v d\mu(\tau). \tag{3.3.5}$$

Here, the second integral converges absolutely due to the exponential decay of f^- as

$\Im(\tau) \to \infty$, see Lemma 1.5.9. We rewrite the first integral as

$$\begin{aligned}
&\int_{\mathcal{F}}^{\mathrm{reg}} \langle f^+(\tau), \Theta_{N^-}(\tau) \otimes \overline{\Theta_P(\tau, z_U^{\pm}, h)}\rangle v d\mu(\tau) \\
&= \operatorname*{CT}_{s=0} \lim_{T\to\infty} \int_{\mathcal{F}_T} \langle f^+(\tau), \Theta_{N^-}(\tau) \otimes \overline{\Theta_P(\tau, z_U^{\pm}, h)}\rangle v^{1-s} d\mu(\tau) \\
&= \int_{\mathcal{F}_1} \langle f^+(\tau), \Theta_{N^-}(\tau) \otimes \overline{\Theta_P(\tau, z_U^{\pm}, h)}\rangle v d\mu(\tau) \qquad (3.3.6) \\
&\quad + \operatorname*{CT}_{s=0} \lim_{T\to\infty} \int_{\mathcal{F}_T - \mathcal{F}_1} \langle f^+(\tau), \Theta_{N^-}(\tau) \otimes \overline{\Theta_P(\tau, z_U^{\pm}, h)}\rangle v^{1-s} d\mu(\tau). \qquad (3.3.7)
\end{aligned}$$

The integral in (3.3.6) is finite and we write the second contribution (3.3.7) as

$$\operatorname*{CT}_{s=0} \lim_{T\to\infty} \int_1^T C(v) v^{-s-1} dv, \tag{3.3.8}$$

where $C(v)$ is given by

$$\int_{u=-1/2}^{1/2} \langle f^+(u+iv), \Theta_{N^-}(u+iv) \otimes \overline{\Theta_P(u+iv, z_U^{\pm}, h)}\rangle du.$$

That is,

$$C(v) = \sum_{\substack{\beta\in P'/P \\ \nu\in N'/N}} \sum_{m\in\mathbb{Q}} c_f^+(m, \beta+\nu) \sum_{\substack{\lambda\in P+\beta \\ \kappa\in N+\nu \\ Q(\lambda)+Q(\kappa)=m}} e^{-2\pi v(Q(\lambda)-Q(\kappa))}$$

Now consider the integral

$$\int_1^\infty (C(v) - A_0) v^{-1} dv. \tag{3.3.9}$$

Note that we have

$$C(v) - A_0 = \sum_{\substack{\beta\in P'/P \\ \nu\in N'/N}} \sum_{m\in\mathbb{Q}} c_f^+(m, \beta+\nu) \sum_{\substack{\lambda\in P+\beta,\, \lambda\neq 0 \\ \kappa\in N+\nu \\ Q(\lambda)+Q(\kappa)=m}} e^{-2\pi v(Q(\lambda)-Q(\kappa))}.$$

We see that there are no terms left with exponent equal to 0. Indeed, if $Q(\lambda) = Q(\kappa)$, then $0 \leq Q(\lambda) = Q(\kappa) \leq 0$ and thus $\lambda = \kappa = 0$ because P and N are definite. It follows from the growth estimates of the Fourier coefficients of f^+ that the integral (3.3.9) converges absolutely.

Therefore, (3.3.8) is equal to

$$\operatorname*{CT}_{s=0} \lim_{T\to\infty} \left(\int_1^T (C(v) - A_0) v^{-1} dv + \int_1^T A_0 v^{-s-1} dv \right). \tag{3.3.10}$$

The second term in (3.3.10) does not contribute to the result because it is equal to

$$\int_1^T A_0 v^{-s-1} dv = -A_0 \frac{1}{s}\left(T^{-s}-1\right).$$

For $\Re(s) > 0$, taking the limit $T \to \infty$, we obtain

$$\frac{A_0}{s}.$$

Thus, the constant term in the Laurent expansion is equal to zero.

Finally, the first term of (3.3.10) is equal to

$$\lim_{T\to\infty} \int_1^T (C(v)-A_0)v^{-1}dv = \lim_{T\to\infty}\left(\int_1^T C(v)v^{-1}dv - A_0\log(T)\right).$$

This finishes the proof. □

Using the same techniques as Bruinier and Yang [BY09], we obtain the following theorem which is central to all of our applications.

Theorem 3.3.6. *Let $f \in H_{1-n/2,L}$. Then the value of $\Phi(z,h,f)$ for any $(z,h) \in Z(U)_{P,K}$ is given by*

$$\begin{aligned}\Phi(z,h,f) &= \mathrm{CT}\left(\langle f^+_{P\oplus N}(\tau), \Theta_{N^-}(\tau)\otimes \widetilde{\Theta}^+_P(\tau,h)\rangle\right)\\ &\quad - \int^{\mathrm{reg}}_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \langle \overline{\xi(f_{P\oplus N})}(\tau), \Theta_{N^-}(\tau)\otimes\widetilde{\Theta}_P(\tau,h)\rangle v^{1+n/2} d\mu(\tau).\end{aligned}$$

Here, the integral is regularized by taking the limit

$$\lim_{T\to\infty}\int_{\mathcal{F}_T} \langle \overline{\xi(f_{P\oplus N})}(\tau), \Theta_{N^-}(\tau)\otimes\widetilde{\Theta}_P(\tau,h)\rangle v^{1+n/2} d\mu(\tau).$$

Remark 3.3.7. Note that for $f \in M^!_{1-n/2,L}$ the second summand does not occur since $\xi(f)=0$ in that case. Moreover, we should remark that the regularized integral can also be written as

$$\begin{aligned}&\int^{\mathrm{reg}}_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \langle L_{1-n/2}(f_{P\oplus N}), \Theta_{N^-}(\tau)\otimes\widetilde{\Theta}_P(\tau,h)\rangle d\mu(\tau)\\ &= \int^{\mathrm{reg}}_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \langle \overline{\xi(f_{P\oplus N})}, \Theta_{N^-}(\tau)\otimes\widetilde{\Theta}_P(\tau,h)\rangle v^{1+n/2} d\mu(\tau),\end{aligned}$$

Proof of Theorem 3.3.6. Assume again that $L = P\oplus N$. According to Lemma 3.3.5, we write

$$\Phi(z,h,f) = \lim_{T\to\infty}\left(I_T(z,h,f) - A_0\log(T)\right), \tag{3.3.11}$$

where

$$\begin{aligned} I_T(z,h,f) &= \int_{\mathcal{F}_T} \langle f(\tau), \Theta_{N^-}(\tau) \otimes \overline{\Theta_P(\tau, z_U^{\pm}, h)} \rangle v d\mu(\tau) \\ &= -\int_{\mathcal{F}_T} \langle f(\tau), \Theta_{N^-}(\tau) \otimes \overline{\partial}\, \widetilde{\Theta}_P(\tau, h) \rangle d\tau \\ &= -\int_{\mathcal{F}_T} d\left(\langle f(\tau), \Theta_{N^-}(\tau) \otimes \widetilde{\Theta}_P(\tau, h) \rangle d\tau \right) \\ &\quad + \int_{\mathcal{F}_T} \langle \overline{\partial}\, f(\tau), \Theta_{N^-}(\tau) \otimes \widetilde{\Theta}_P(\tau, h) \rangle d\tau. \end{aligned}$$

Here, we have used Lemma 3.3.4.

For the first integral, we apply Stoke's theorem and obtain

$$\begin{aligned} \int_{\mathcal{F}_T} d\left(\langle f(\tau), \Theta_{N^-}(\tau) \otimes \widetilde{\Theta}_P(\tau, h) \rangle \right) d\tau &= \int_{\partial\, \mathcal{F}_T} \langle f(\tau), \Theta_{N^-}(\tau) \otimes \widetilde{\Theta}_P(\tau, h) d\tau \rangle \\ &= -\int_{iT}^{iT+1} \langle f(\tau), \Theta_{N^-}(\tau) \otimes \widetilde{\Theta}_P(\tau, h) d\tau \rangle, \end{aligned}$$

since the integrand is an $\mathrm{SL}_2(\mathbb{Z})$-invariant differential form and thus the integral over the equivalent pieces of $\partial\, \mathcal{F}_T$ cancel. We split the first integral into three pieces, insert this splitting into (3.3.11) and regroup to obtain

$$\begin{aligned} \Phi(z,h,f) = &\lim_{T\to\infty} \int_{iT}^{iT+1} \langle f^+(\tau), \Theta_{N^-}(\tau) \otimes \widetilde{\Theta}_P^+(\tau, h) \rangle d\tau && (3.3.12) \\ &+ \lim_{T\to\infty} \left(\int_{iT}^{iT+1} \langle f^+(\tau), \Theta_{N^-}(\tau) \otimes \widetilde{\Theta}_P^-(\tau, h) \rangle d\tau - A_0 \log(T) \right) && (3.3.13) \\ &+ \lim_{T\to\infty} \int_{iT}^{iT+1} \langle f^-(\tau), \Theta_{N^-}(\tau) \otimes \widetilde{\Theta}_P(\tau, h) \rangle d\tau && (3.3.14) \\ &+ \lim_{T\to\infty} \int_{\mathcal{F}_T} \langle \overline{\partial}\, f(\tau), \Theta_{N^-}(\tau) \otimes \widetilde{\Theta}_P(\tau, h) \rangle d\tau. && (3.3.15) \end{aligned}$$

Each of the limits above exist.

The limit in (3.3.14) is equal to zero due to the exponential decay of $f^-(\tau)$. In detail, we have

$$\begin{aligned} &\int_{iT}^{iT+1} \langle f^-(\tau), \Theta_{N^-}(\tau) \otimes \widetilde{\Theta}_P(\tau, h) \rangle d\tau \\ &= \int_0^1 \langle f^-(u+iT), \Theta_{N^-}(u+iT) \otimes \widetilde{\Theta}_P(u+iT, h) \rangle du. \end{aligned}$$

First note that for two vector-valued forms g, h transforming with representations ρ and $\bar{\rho}$

the q-expansion of $\langle g, h\rangle$ has integral exponents (the exponents satisfy $n - Q(\mu) \in \mathbb{Z}$ and $m + Q(\mu) \in \mathbb{Z}$ which yields $n + m \in \mathbb{Z}$). We write the Fourier expansion of the integrand as

$$\langle f^-(\tau), \Theta_{N^-}(\tau) \otimes \widetilde{\Theta}_P(\tau, h)\rangle = \sum_{n\in\mathbb{Z}} a(n, v)e(n\tau).$$

and insert this to obtain

$$\begin{aligned}&\int_0^1 \langle f^-(u + iT), \Theta_{N^-}(u + iT) \otimes \widetilde{\Theta}_P(u + iT, h)\rangle du\\ &= \sum_{n\in\mathbb{Z}} a(n, iT)e(inT) \int_0^1 e^{2\pi i n u} du.\end{aligned}$$

The integral above is equal to 0 for all $n \in \mathbb{Z}\backslash\{0\}$ and is equal to 1 for $n = 0$. Consequently,

$$\begin{aligned}&\lim_{T\to\infty} \int_{iT}^{iT+1} \langle f^-(\tau), \Theta_{N^-}(\tau) \otimes \widetilde{\Theta}_P(\tau, h)\rangle d\tau\\ &= \lim_{T\to\infty} a(0, iT) = \lim_{T\to\infty} \sum_{n\in\mathbb{Z}_{>0}} c_f^-(-n, 0)W(-2\pi nT)c_g(n, 0),\end{aligned}$$

where $g(\tau) = \Theta_{N^-}(\tau) \otimes \widetilde{\Theta}_P(\tau, h)$. Here, we have used that $f \in H_{1-n/2,L}$ and therefore $c_f^-(n, 0) = 0$ for $n \geq 0$. Note that since Θ_{N^-} is a holomorphic modular form, the function g satisfies the bounds for the growth of the Fourier coefficients in Lemma 1.5.10. Moreover, using the asymptotic expansion of the incomplete Gamma function (cf. [DLMF], 8.11.2), we see that $W = W_k$ satisfies

$$W_k(s) = O(|s|^{-k} e^{2s}), s \to -\infty.$$

Using these estimates for f and g in the respective weights, we obtain that there is an $N \in \mathbb{Z}_{>0}$ and a constant $C > 0$, such that for all $n \geq N$, we have

$$c_f^-(-n, 0)W(-2\pi nT)c_g(n, 0) = O(e^{-nCT}).$$

Thus, for every $T > 0$, the constant term in the Fourier expansion of the function

$$\langle f^-(u + iT), \Theta_{N^-}(u + iT) \otimes \widetilde{\Theta}_P(u + iT, h)\rangle$$

can be bounded by

$$|a(0, iT)| \leq c\frac{r(T)}{1 - r(T)} \text{ with } r(T) = e^{-CT},$$

where $c, C > 0$ are constants. Therefore, in the limit $T \to \infty$, we have

$$\lim_{T\to\infty} |a(0, iT)| = 0,$$

which finally shows that (3.3.14) vanishes.

Similarly, we see that the limit in (3.3.13) is equal to zero, as well. To see this note that $\widetilde{\Theta}_P^-(\tau,h)$ has a Fourier expansion of the form $\log(v)\phi_{0+P} + \widetilde{\Theta}_P^{--}(\tau,h)$, where $\widetilde{\Theta}_P^{--}(\tau,h)$ decays exponentially as $v \to \infty$ and we can argue as above.

Using the same argument once more, we see that (3.3.12) is the constant term

$$\mathrm{CT}\left(\langle f^+_{P\oplus N}(\tau), \Theta_{N^-}(\tau) \otimes \widetilde{\Theta}^+_P(\tau,h)\rangle\right).$$

Note that this is now really a finite sum.

Finally, by Lemma 1.5.11, we see that (3.3.15) is equal to

$$\begin{aligned}&\lim_{T\to\infty}\int_{\mathcal{F}_T}\langle \overline{\partial} f(\tau), \Theta_{N^-}(\tau)\otimes\widetilde{\Theta}_P(\tau,h)\rangle d\tau\\ &= -\lim_{T\to\infty}\int_{\mathcal{F}_T}\langle L_{1-n/2}f(\tau), \Theta_{N^-}(\tau)\otimes\widetilde{\Theta}_P(\tau,h)\rangle d\mu(\tau).\end{aligned}$$

This is exactly the definition of the regularized integral in the statement of the theorem. We still have to justify that this limit exists. However, this now follows from the vanishing of (3.3.13) and (3.3.14) and the fact that $\Phi(z,h,f)$ is defined at (z,h). That is, we have shown that

$$\begin{aligned}&\lim_{T\to\infty}\int_{\mathcal{F}_T}\langle \overline{\xi(f)}, \Theta_{N^-}(\tau)\otimes\widetilde{\Theta}_P(\tau,h)\rangle v^{1+n/2} d\mu(\tau)\\ &= -\Phi(z,h,f) + \mathrm{CT}\left(\langle f^+_{P\oplus N}(\tau), \Theta_{N^-}(\tau)\otimes\widetilde{\Theta}^+_P(\tau,h)\rangle\right)\end{aligned}$$

and therefore, the limit exists. □

Remark 3.3.8. Note that the formula holds for *any* preimage of $\Theta_P(\tau,h)$ under ξ. Just looking at the formula on the right hand side however does not immediately reveal this independence from the choice of a particular preimage. For instance, consider the simplest case possible where the signature of L is equal to $(2,0)$ and $f \in M_{1,L}$ is holomorphic. The domain $\mathbb{D}$ has just two points in this case and the regularized lift is independent of z. We obtain by the theorem that

$$\Phi(h,f) = \int^{\mathrm{reg}}_{\Gamma\backslash\mathbb{H}} \langle f, \Theta_L(\tau,h)\rangle d\mu(\tau) = \mathrm{CT}(\langle f, \widetilde{\Theta}^+_L(\tau,h)\rangle).$$

Now let us take the sum over the CM cycle, as in Theorem 3.1.1 and define for a moment

$$\widetilde{E}_L(\tau) = \frac{1}{|C_{P,K}|}\sum_{h\in C_{P,K}}\widetilde{\Theta}_L(\tau,h).$$

Then we obtain

$$\Phi(Z(U),f) = \deg(Z(U))\,\mathrm{CT}(\langle f(\tau), \widetilde{E}_L(\tau)\rangle).$$

Clearly, for every $g \in M_{1,P^-}$, we have that

$$\mathrm{CT}(\langle f, \widetilde{E}_L(\tau) + g\rangle) = \mathrm{CT}(\langle f, \widetilde{E}_L(\tau)\rangle) + \mathrm{CT}(\langle f, g\rangle)$$

and it might not seem obvious that this is independent of g. In particular, taking $f = E_L(\tau)$ the second term above is just the constant term of g. So this implies at least that there is no Eisenstein series in M_{1,P^-}. However, this is absolutely clear: since $f \in M_{1,P}$ and $g \in M_{1,P^-}$, we have in fact that $\langle f, g\rangle \in M_2(\mathrm{SL}_2(\mathbb{Z})) = \{0\}$.

In general, the independence of the right hand side in Theorem 3.3.6 can in some sense be seen as an extension of the pairing (1.5.8) to our case. In fact, we can state the following corollary, which probably also generalizes to other weights.

Corollary 3.3.9. *Let $f \in H_{1,L}$ and $g \in M^!_{1,L^-}$. We can define a regularized bilinear pairing*

$$H_{1,L} \times M^!_{1,L^-} \to \mathbb{C}$$

by

$$\{f, g\} := \int^{\mathrm{reg}}_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \langle \overline{\xi(f(\tau))}, g(\tau)\rangle d\mu(\tau).$$

We have

$$\{f, g\} = \mathrm{CT}\left(\langle f^+(\tau), g(\tau)\rangle\right).$$

Comparing our theorem with the result by Schofer (Theorem 3.1.1 for weakly holomorphic f), we obtain the following identity.

Corollary 3.3.10. *Let $f \in M^!_{1-n/2,L}$ be a weakly holomorphic modular form. Then,*

$$\Phi(Z(U), f) = \deg(Z(U))\left(\mathrm{CT}\left(\langle f_{P\oplus N}(\tau), \Theta_{N^-}(\tau) \otimes \widetilde{E}^+_P(\tau)\rangle\right)\right).$$

Of course, this identity also holds with $\mathcal{E}^+_P(\tau)$ in place of $\widetilde{E}^+_P(\tau)$.

4 Moduli of CM elliptic curves and CM values of modular functions

In this chapter we will ultimately study special values of modular functions for the congruence subgroup $\Gamma_0(N)$ with divisors supported on Heegner divisors (on $Y_0(N) = \Gamma_0(N)\backslash\mathbb{H}$). By a modular function we mean a meromorphic modular form of weight 0. In particular, (after some justification) we can apply everything in this chapter to Borcherds products. There exist however modular functions with their divisor given by a linear combination of Heegner divisors on $Y_0(N)$ that are not obtained as Borcherds products. This has been remarked by Borcherds [Bor99, Example 5.2] and follows from Theorem 7.7 of [BY09]. The authors show that

$$\langle y(f), y(f)\rangle_{NT} = \frac{2\sqrt{N}}{\pi\|g\|^2}L'(G,1).$$

Here, $y(f)$ is a Heegner divisor associated with a harmonic weak Maaß form as in (1.8.1) (with a suitable degree zero extension to the cusps), G is a normalized cuspidal newform of weight 2 and level N and $\langle\cdot,\cdot\rangle_{NT}$ denotes the Neron-Tate height pairing on the Jacobian of $X_0(N)$. The relation between f and G is that G corresponds to $\xi(f) = g$ under the Shimura correspondence. This clearly shows that if $G \neq 0$, then f is not weakly holomorphic and therefore the divisor $y(f)$ cannot be obtained from a Borcherds product. There are, however, modular forms G of weight 2 with $L'(G,1) = 0$. Assuming the Birch and Swinnerton-Dyer conjecture, examples are given by all cusp forms corresponding to elliptic curves over $\mathbb{Q}$ with Mordell-Weill group of rank greater than one. Explicit examples with $L'(G,1) = 0$ are also known. For instance, Gross and Zagier give examples of elliptic curves of conductor 714877 and 5077 [GZ86].

We remark that Bruinier [Bru12] proved a converse theorem for Borcherds products which shows that the case of modular curves is quite exceptional in this regard.

The geometric methods we use in this chapter are completely different from the analytic methods used in Chapter 3. In Chapter 5, the combination of these fundamentally different tools will culminate in an arithmetic description of the coefficients of the harmonic weak Maaß forms of weight one with theta functions as their shadow (their image under ξ).

4.1 Moduli of CM elliptic curves

In this section we gather some facts about the moduli stack of CM elliptic curves and special 0-cycles on it. These cycles will play an important role in the description of the values of modular functions on $Y_0(N)$.

Here and throughout, if X is a stack over a base scheme S and T is an S-scheme, we abbreviate

$$X_{/T} = X \times_S T.$$

If $T = \operatorname{Spec} R$ with R a ring, we simply write $X_{/R}$ for $X_{/\operatorname{Spec} R}$.

Here and for the rest of this chapter, let D be a negative fundamental discriminant and denote by $k = k_D$ the imaginary quadratic field of discriminant $D < 0$. We write $\mathcal{O}_D$ for the ring of integers of k and let $H = H_D$ be the Hilbert class field of k (see Section 1.2). We remark that our methods should also work for non-fundamental discriminants but some of the calculations would certainly get more involved.

We consider the moduli problem Which assigns to a base scheme S over $\mathcal{O}_D$ the category $C_D^+(S)$ of pairs (E, ι), where

(i) E is an elliptic curves over S with complex multiplication $\iota : \mathcal{O}_D \hookrightarrow \operatorname{End}(E)$,

(ii) such that the induced map

$$\operatorname{Lie}(\iota) : \mathcal{O}_D \to \operatorname{End}_{\mathcal{O}_S}(\operatorname{Lie} E) = \mathcal{O}_S, \tag{4.1.1}$$

coincides with the structure map $S \to \operatorname{Spec}(\mathcal{O}_D)$.

The morphisms in this category are isomorphisms respecting the actions. We will not go into detail with the second condition. We just remark that the $\mathcal{O}_D$-actions come in pairs ι^+, ι^- and only one them satisfies the normalization condition (4.1.1). We will always denote the normalized embedding by ι^+. For instance, over $\mathbb{C}$ this corresponds to the condition that $\iota^+(\lambda)$ for $\lambda \in \mathcal{O}_D$ corresponds to multiplication with λ of the invariant differential of E (and not with its complex conjugate).

Moreover, we denote by $C_D = \operatorname{cRes}_{\mathcal{O}_D/\mathbb{Z}}(C_D^+)$ the restriction of coefficients of C_D^+ to $\mathbb{Z}$ (in the sense of Grothendieck). That is, the structure map of C_D is given by $C_D^+ \to \operatorname{Spec}(\mathcal{O}_D) \to \operatorname{Spec}(\mathbb{Z})$. This describes the moduli problem without the normalization (4.1.1).

Proposition 4.1.1. *The moduli problem C_D^+ is represented by an algebraic stack, also denoted by C_D^+, which is smooth of relative dimension 0 and proper over* $\operatorname{Spec} \mathcal{O}_D$. *If R is a discrete valuation ring with algebraically closed residue field $\mathbb{F}$, the reduction map*

$$C_D^+(R) \to C_D^+(\mathbb{F})$$

is surjective. Consequently, C_D is also represented by an algebraic stack of relative dimension 0 *over* $\operatorname{Spec} \mathbb{Z}$, *which is finite and proper.*

Proof. This is a consequence of the canonical lifting theorem [How13, Lan08]. Properness follows from the fact that all points of C_D in characteristic 0 have potentially good reduction and the valuative criterion of properness. See [KRY99, Section 5] or [BHY13] for details. □

Lemma 4.1.2. *The coarse moduli scheme* $\mathbf{C}_D^+$ *of* C_D^+ *is isomorphic to* $\operatorname{Spec}\mathcal{O}_H$ *as a scheme over* $\mathcal{O}_D$*. Consequently, the coarse moduli scheme* $\mathbf{C}_D$ *of* C_D *is isomorphic to* $\operatorname{Spec}\mathcal{O}_H$ *as a scheme over* $\mathbb{Z}$*.*

Proof. This is [KRY99, Corollary 5.4]. There, the authors prove the corresponding isomorphism for C_D^+, whose coarse moduli scheme is $\operatorname{Spec}\mathcal{O}_H \to \operatorname{Spec}\mathcal{O}_D$. ☐

We denote by

$$\mathrm{pr} : C_D^+ \to \mathbf{C}_D^+$$

the canonical map to the coarse moduli scheme.

Proposition 4.1.3. *Let* $\xi \in C_D^+$ *be a geometric point and let* $\mathrm{pr}(\xi) = \bar{\xi}$ *be the corresponding point of* $\mathbf{C}_D^+$*. Then*

$$\hat{\mathcal{O}}_{C_D^+,\xi} = \hat{\mathcal{O}}_{\mathbf{C}_D^+,\bar{\xi}},$$

where $\hat{\mathcal{O}}_{C_D^+,\xi}$ *and* $\hat{\mathcal{O}}_{\mathbf{C}_D^+,\bar{\xi}}$ *denote the completions of the étale local rings at* ξ *and* $\bar{\xi}$*, respectively.*

Proof. This is Corollary 5.2 in [KRY99]. ☐

Corollary 4.1.4. *The stacks* C_D *and* C_D^+ *are integral, that is, irreducible and reduced.*

Proof. Since $\operatorname{Spec}\mathcal{O}_H$ is irreducible, the irreducibility follows from Lemma (2.3) in [Vis89]. Moreover, by Proposition 4.1.3 we have that for every geometric point $\xi \in C_D^+$ the completed étale local ring is $\hat{\mathcal{O}}_{C_D^+,\xi} = \hat{\mathcal{O}}_{\mathbf{C}_D^+,\bar{\xi}}$ and the latter is isomorphic to the completion of the strict henselization of the completion $\mathcal{O}_{H,\mathfrak{P}}$ of $\mathcal{O}_H$ at some prime ideal $\mathfrak{P} \subset \mathcal{O}_H$. Since $\mathcal{O}_{H,\mathfrak{P}}$ does not have any non-zero nilpotent elements, it is reduced. ☐

We now describe the geometric points of C_D in every characteristic. The following construction is very important for us.

Recall that over $\mathbb{C}$, we have a canonical bijection

$$C_D^+(\mathbb{C}) \cong k^\times \backslash \mathbb{A}_{k,f}^\times / \hat{\mathcal{O}}_D^\times, \tag{4.1.2}$$

given by the theory of complex multiplication [Sil94]. To an idele $h \in \mathbb{A}_{k,f}^\times$ that corresponds to the ideal class $[(h)]$, the bijection associates the (isomorphism class of the) elliptic curve with complex points

$$E(\mathbb{C}) = \mathbb{C}/(h).$$

Moreover, if $(E,\iota) \in C_D^+(\mathbb{C})$ is given by $(\mathbb{C}/\Lambda, \iota)$, then multiplication with $h \in \mathbb{A}_{k,f}^\times$ on the right hand side of (4.1.2) corresponds to

$$E \mapsto (h) \otimes_{\mathcal{O}_D} E,$$

where $(h) \otimes_{\mathcal{O}_D} E$ is the elliptic curve over $\mathbb{C}$ with complex points

$$((h) \otimes_{\mathcal{O}_D} E)(\mathbb{C}) \cong \mathbb{C}/(h)\Lambda.$$

This defines an action of the class group $\mathrm{Cl_k}$ on the set of isomorphism classes of CM elliptic curves (with CM by $\mathcal{O}_D$) over $\mathbb{C}$.

Now let $(E, \iota) \in C_D^+(S)$ for a scheme S and let $h \in \mathbb{A}_{k,f}^\times$, corresponding to the ideal (h). Then we can define a functor from the category of S-schemes to the category of $\mathcal{O}_D$-modules by

$$T \mapsto (h) \otimes_{\mathcal{O}_D} E(T).$$

This functor is in fact represented by an elliptic curve over S and the construction is called the *Serre construction*. We denote the elliptic curve representing this functor by $h.E = (h) \otimes_{\mathcal{O}_D} E$. For details, the reader may consult [How12b], [Con04, Section 7].

We follow the description given in [KRY99] to describe the geometric points of C_D^+ in positive characteristic.

Proposition 4.1.5 (Corollary 5.5 of [KRY99]). *Let $\mathfrak{p}$ be a prime ideal of k and let $\overline{\kappa(\mathfrak{p})}$ denote an algebraic closure of the residue field $\kappa(\mathfrak{p})$. We have a bijection*

$$C_D^+(\overline{\kappa(\mathfrak{p})}) \cong k^\times \backslash \mathbb{A}_{k,f}^\times / \hat{\mathcal{O}}_D^\times.$$

The action by the Frobenius automorphism over $\kappa(\mathfrak{p})$ on the left hand side corresponds to the translation by an idele of the form $(1, \ldots, 1, \pi, 1, \ldots)$, where π is a uniformizer at $\mathfrak{p}$.

The proposition establishes a simply transitive action of the class group $\mathrm{Cl_k}$ on the points $C_D^+(\bar{\mathbb{F}})$ over any algebraically closed field $\bar{\mathbb{F}}$. We also have a bijection on geometric points $C_D^+(\bar{\mathbb{F}}) \cong \operatorname{Spec} \mathcal{O}_H(\bar{\mathbb{F}})$. On the points $\operatorname{Spec} \mathcal{O}_H(\bar{\mathbb{F}})$, we have an action of the Galois group $\mathrm{Gal}(H/k)$.

Fix a morphism $\mathrm{pr} : C_D^+ \to \operatorname{Spec} \mathcal{O}_H$. Then pr induces an isomorphism

$$\mathrm{pr}_{\mathbb{F}} : C_D^+(\bar{\mathbb{F}}) \cong \operatorname{Spec} \mathcal{O}_H(\bar{\mathbb{F}})$$

on geometric points over any algebraically closed field $\bar{\mathbb{F}}$. For $\bar{\mathbb{F}} = \bar{k}$ or $\bar{\mathbb{F}} = \overline{\kappa(\mathfrak{p})}$, the group $\mathrm{Gal}(H/k)$ acts on both sides. It acts naturally on the right hand side and the action on the left hand side is given via the isomorphism

$$k^\times \backslash \mathbb{A}_{k,f}^\times / \hat{\mathcal{O}}_D^\times \cong \mathrm{Gal}(H/k), h \mapsto \sigma((h))$$

given by the Artin map of class field theory (see Section 1.2) and the action of the idele class group given above.

The next proposition shows the compatibility of these bijections and group actions.

Proposition 4.1.6. *With the notation as above, the coarse moduli space map* pr *is compatible with these actions. More precisely, we have*

$$\mathrm{pr}_{\mathbb{F}}\, h.(E, \iota) = (\mathrm{pr}_{\mathbb{F}}(E, \iota))^{\sigma^{-1}(h)}.$$

Proof. The compatibility over $\mathbb{C}$ is contained in the main theorem of complex multiplication [Sil94, II, Theorem 8.2]. (Note that our normalization of the action of the class group is different from the one used by Silverman.) Precisely, we have for an elliptic curve $E = \mathbb{C}/\Lambda$ that $h.E = \mathbb{C}/(h)\Lambda$ and $j(h.E) = j^{\sigma(h^{-1})}(E)$. Therefore, if we assume without loss of generality that $\mathrm{pr}_{\mathbb{C}}(E,\iota) = \lambda : \mathcal{O}_H \hookrightarrow \mathbb{C}$ is the embedding given by $j \mapsto j(E)$, then $\mathrm{pr}_{\mathbb{C}}(h.(E,\iota)) = j \mapsto j(h.E) = j^{\sigma^{-1}(h)}(E) = \lambda^{\sigma^{-1}(h)}$.

We will use this to prove the statement over $\kappa(\mathfrak{p})$ for a fixed prime $\mathfrak{p}$ of k. Fix an isomorphism $\mathbb{C}_{\mathfrak{p}} \cong \mathbb{C}$, where $\mathbb{C}_{\mathfrak{p}}$ is the completion of an algebraic closure of $k_{\mathfrak{p}}$. Then we obtain isomorphisms

$$C_D^+(\mathbb{C}) \cong C_D^+(\mathbb{C}_{\mathfrak{p}}) \cong C_D^+(\overline{\kappa(\mathfrak{p})})$$

by Proposition 4.1.1. All of these bijections are compatible with the Serre construction. Similarly, we have bijections

$$\operatorname{Spec} \mathcal{O}_H(\mathbb{C}) \cong \operatorname{Spec} \mathcal{O}_H(\mathbb{C}_{\mathfrak{p}}) \cong \operatorname{Spec} \mathcal{O}_H(\overline{\kappa(\mathfrak{p})}).$$

The key part is now that

$$\begin{tikzcd}
C_D^+(\mathbb{C}) \arrow[r] \arrow[d] & C_D^+(\mathbb{C}_{\mathfrak{p}}) \arrow[r] \arrow[d] & C_D^+(\overline{\kappa(\mathfrak{p})}) \arrow[d] \\
\operatorname{Spec} \mathcal{O}_H(\mathbb{C}) \arrow[r] & \operatorname{Spec} \mathcal{O}_H(\mathbb{C}_{\mathfrak{p}}) \arrow[r] & \operatorname{Spec} \mathcal{O}_H(\overline{\kappa(\mathfrak{p})})
\end{tikzcd}$$

is commutative and the bijection $C_D^+(\mathbb{C}) \cong \operatorname{Spec} \mathcal{O}_H(\mathbb{C})$ is compatible with the actions, as stated above. The bijections in the lower row are compatible with the action of the Galois group. Consequently, the bijection $C_D^+(\bar{\mathbb{F}}_p) \to \operatorname{Spec} \mathcal{O}_H(\overline{\kappa(\mathfrak{p})})$ is compatible with the two actions, as well. □

4.2 Special endomorphisms: characteristic zero

In this section we give an interpretation of the coefficients of the theta functions studied in Chapter 2 in terms of certain special endomorphisms. This gives a geometric description of the non-holomorphic part of the function $\widetilde{\Theta}_P(\tau, h)$.

The idea goes back to Gross and has been described by Kudla, Rapoport and Yang [KRY99, Section 6] in this context.

Let $E = E_{\mathfrak{a}} = \mathbb{C}/\mathfrak{a}$ be an elliptic curve with complex multiplication by $\mathcal{O}_D$. Here $\mathfrak{a} \subset k$ is a fractional ideal. Without loss of generality, we may assume that $\mathfrak{a}$ is integral. We write $\mathfrak{d}_k$ for the different ideal. Choose a basis (ω_1, ω_2) of $\mathfrak{a}$. If we consider $E = \mathbb{C}/\mathfrak{a}$ simply as a real torus, denoted by $\mathrm{E}^{\mathrm{top}}$, then its endomorphism ring $\mathcal{J} = \mathrm{End}(\mathrm{E}^{\mathrm{top}})$ is a maximal order in the split quaternion algebra $\mathrm{End}(\mathrm{E}^{\mathrm{top}}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong M_2(\mathbb{Q}) = V$. The reduced norm on V is given by the determinant $Q(X) = \det(X)$. It corresponds to the degree map on $\mathrm{End}(\mathrm{E}^{\mathrm{top}})$. The quaternion algebra is a quadratic space of type $(2,2)$ over $\mathbb{Q}$ with a non-degenerate quadratic form given by the determinant. See also Section 2.6.

In our simple case, we can also make this completely explicit (cf. [Gro84]). We choose the basis

$$\omega_1 = \frac{-B+\sqrt{D}}{2}, \quad \omega_2 = A$$

with $A \in \mathbb{Z}_{>0}$ and $B \in \mathbb{Z}$. Then $C = \frac{D-B^2}{4A} \in \mathbb{Z}$, as well and $[A, B, C]$ is an integral binary quadratic form of discriminant D. We can assume that $[A, B, C]$ is primitive, that is $\gcd(A, B, C) = 1$. Moreover, the complex point corresponding to $E_{\mathfrak{a}}$ is then given by $\alpha_{\mathfrak{a}} = \omega_1/\omega_2$. In terms of this basis, multiplication by $\sqrt{D}$ in $\mathrm{End}(\mathrm{E}^{\mathrm{top}})$ is given by the matrix

$$X = \begin{pmatrix} -B & 2A \\ \frac{D-B^2}{2A} & B, \end{pmatrix}$$

and this determines an embedding $\iota : k \hookrightarrow M_2(\mathbb{Q}) = V$ inducing and an embedding

$$\iota : \mathcal{O}_D \hookrightarrow M_2(\mathbb{Z}) = \mathcal{J}.$$

This embedding is optimal in the sense of Eichler [Eic55], that is $\mathcal{J} \cap \iota(k) = \iota(\mathcal{O}_D)$. We can write

$$V = \iota(k) \oplus \iota(k)Y$$

for an element $Y \in \mathcal{J}$ with $\mathrm{tr}(Y) = 0$ and $Q(Y) = -1$. With our choice of basis, we have

$$Y = \begin{pmatrix} -1 & 0 \\ -\frac{B}{A} & 1 \end{pmatrix}.$$

We consider the lattice $L(\mathrm{E}^{\mathrm{top}}, \iota)$ of *special endomorphisms* in V defined by

$$L(\mathrm{E}^{\mathrm{top}}, \iota) = \{x \in \mathrm{End}(\mathrm{E}^{\mathrm{top}}) \mid \iota(\alpha)x = x\iota(\bar{\alpha}) \text{ for all } \alpha \in \mathcal{O}_D\}. \tag{4.2.1}$$

Using the embedding given above, we can determine this lattice explicitly. It is easy to verify that $L(\mathrm{E}^{\mathrm{top}}, \iota) = \iota(\mathcal{O}_D)^{\perp} \cap \mathcal{J}$ is given by

$$L(\mathrm{E}^{\mathrm{top}}, \iota) = \mathbb{Z}\begin{pmatrix} A & 0 \\ -B & -A \end{pmatrix} \oplus \mathbb{Z}\begin{pmatrix} 0 & A \\ \frac{B^2-D}{4A} & 0 \end{pmatrix}.$$

Note the similarity to the embedding and the calculations in Section 2.6. The lattice $L(\mathrm{E}^{\mathrm{top}}, \iota)$ is a two-dimensional negative definite lattice with quadratic form $Q(X) = \det(X)$. We have that $L(\mathrm{E}^{\mathrm{top}}, \iota) = \iota(\mathfrak{a})Y$.

If we let $U = \iota(k)$, then the group $T = \mathrm{GSpin}_U$ is isomorphic to $k^\times$ as in Section 2.4 and acts on lattices in U. Let $\mathcal{J}'$ be another order in which $\mathcal{O}_D$ is optimally embedded, say by $\iota' : k \hookrightarrow V$.

Then there is an idele $h \in T(\mathbb{A}_f) \cong \mathbb{A}_{k,f}^\times$, such that

$$\mathcal{J}' = h(\mathcal{J} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}})h^{-1} \cap V$$

by the Chevalley-Hasse-Noether theorem [Eic55, Satz 7]. Therefore, if

$$\mathfrak{b} = (h) = (h\mathcal{O}_D \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}) \cap k$$

is the ideal corresponding to h, then the orthogonal complement of $\iota'(\mathcal{O}_D)$ in $\mathcal{J}'$ is given by

$$\mathcal{J}' \cap \iota'(\mathcal{O}_D)^{\perp} = \iota(\mathfrak{b}\bar{\mathfrak{b}}^{-1}\mathfrak{a})Y = \iota(h.\mathfrak{a})Y,$$

where $h.\mathfrak{a}$ denotes the action of $h \in T(\mathbb{A}_f)$ on the lattice $\mathfrak{a}$.

We are led to the following definition, analogous to Definition 2.5 in [KY13].

Definition 4.2.1. Let $\mathcal{Z}^{\mathrm{top}}(m, \mathfrak{a}, \beta)$ be the category of triples (E, ι, x), such that

(i) $(E, \iota) \in C_D(\mathbb{C})$,

(ii) $x \in L(\mathrm{E}^{\mathrm{top}}, \iota)\iota(\mathfrak{d}_k^{-1}\mathfrak{a})$

(iii) $x + \iota(\beta) \in \mathrm{End}(\mathrm{E}^{\mathrm{top}}, \iota)\iota(\mathfrak{a})$,

(iv) $Q(x) = m\mathrm{N}(\mathfrak{a})$.

The forgetful functor “forget ι” defines a map $\mathrm{pr} : \mathcal{Z}^{\mathrm{top}}(m, \mathfrak{a}, \beta) \to C_D(\mathbb{C})$ with finite fibers. Let us write $H = k(j)$ and fix an embedding λ of k into $\mathbb{C}$. Then every embedding $\sigma : H \hookrightarrow \mathbb{C}$ that is compatible with λ determines an elliptic curve E_σ over $\mathbb{C}$ with complex multiplication by $\mathcal{O}_D$ and j-invariant $\sigma(j)$, unique up to isomorphism. These h_k embeddings form a system of representatives of the archimedean places of H_D. This identification realizes a bijection of complex points $\operatorname{Spec}\mathcal{O}_H(\mathbb{C}) \cong C_D^+(\mathbb{C})$.

In the other direction, we denote by $\sigma(E, \iota)$ the equivalence class (modulo complex conjugation) of the complex embedding of H that corresponds to (E, ι).

Using these identifications, we define for $m \in \mathbb{Q}_{<0}$ an Arakelov divisor $\mathcal{Z}(m, \mathfrak{a}, \beta)$ on $\operatorname{Spec}\mathcal{O}_H$ by

$$\mathcal{Z}(m, \mathfrak{a}, \beta) = \sum_{(E,\iota,x)\in\mathcal{Z}^{\mathrm{top}}(m,\mathfrak{a},\beta)} \sigma(E, \iota) = \sum_{\sigma:H\hookrightarrow\mathbb{C}} n_\sigma(m, \mathfrak{a}, \beta)\sigma,$$

where the multiplicities $n_\sigma(m, \mathfrak{a}, \beta)$ on the right-hand side are defined by the left-hand side and the sum is over all archimedean places σ of H. Note that it does not matter if we consider the stack C_D or C_D^+ in this context as the Galois group $\mathrm{Gal}(k/\mathbb{Q})$ consists only of complex conjugation.

Now if $P \cong \mathfrak{a}$ with quadratic form $\mathrm{N}(x)/\mathrm{N}(\mathfrak{a})$, then we have for the theta function $\Theta_P(\tau, h)$ that

$$\Theta_P(\tau, h) = \phi_0 + \sum_{\beta\in P'/P} \sum_{m>0} n_{\sigma_{\mathfrak{a}}(h)}(-m, \mathfrak{a}, \beta)e(m\tau)\phi_\beta$$

and accordingly we have for the non-holomorphic part of $\widetilde{\Theta}_P$ that

$$\widetilde{\Theta}_P^-(\tau, h) = \log(v)\phi_0 - \sum_{\beta\in P'/P} \sum_{m<0} n_{\sigma_{\mathfrak{a}}(h)}(m, \mathfrak{a}, \beta)W(2\pi mv)e(m\tau)\phi_\beta.$$

Here, $\sigma_{\mathfrak{a}}(h)$ corresponds to the following place of H. There is a place $\sigma_{\mathfrak{a}}$ of H, such that $\sigma_{\mathfrak{a}}(j)$ is the j-invariant of $\mathbb{C}/\mathfrak{a}$. Then $\sigma_{\mathfrak{a}}(h) = \sigma_{\mathfrak{a}} \circ \sigma(h)$, where $\sigma(h)$ is the image of the ideal (h) under the Artin map.

4.3 Special endomorphisms: characteristic p

In this section we define divisors that will only play a role in characteristic p and we eventually prove in Chapter 5 that, similar to the situation described in the last section, they are intimately related to the harmonic weak Maaß forms $\widetilde{\Theta}_P(\tau, h)$, but this time to the coefficients of the holomorphic part.

For $(E, \iota) \in C_D(S)$ we write $\mathcal{O}_E = \mathrm{End}_S(E)$ and consider the lattice $L(E, \iota)$ of *special endomorphisms*

$$L(E, \iota) = \{x \in \mathcal{O}_E \mid \iota(\alpha)x = x\iota(\bar{\alpha}) \text{ for all } \alpha \in \mathcal{O}_D \text{ and } \operatorname{tr} x = 0\}$$

as in Definition 5.7 of [KRY99]. It is equipped with the positive definite quadratic form $\mathrm{N}(x) := \deg(x) = -x^2$. For $S = \operatorname{Spec} \mathbb{C}$ or $S = \operatorname{Spec} \bar{\mathbb{F}}_p$ for a prime p that is split in k, we have that $L(E, \iota)$ is zero.

For non-split primes, $L(E, \iota)$ is a positive definite lattice of rank 2 in $\mathcal{O}_E$ and (E, ι) is supersingular. In this case $\mathcal{O}_E$ is a maximal order in the quaternion algebra $\mathbb{B}_p$ over $\mathbb{Q}$, which is ramified exactly at p and ∞. We also refer to Section 1.6.1.

Fix a fractional ideal $\mathfrak{a} \subset k$ and let $\mu \in \mathfrak{d}_k^{-1}\mathfrak{a}/\mathfrak{a}$ and $m \in \mathbb{Q}_{>0}$. The following moduli problem has been studied in [KRY99] and [BY09] and generalized in [KY13].

Consider the moduli problem, which assigns to a scheme S the category $\mathcal{Z}(S)$ of triples (E, ι, x), where:

(i) $(E, \iota) \in C_D(S)$,

(ii) $x \in L(E, \iota)\mathfrak{d}_k^{-1}\mathfrak{a}$, such that

$$\mathrm{N}(x) = m\mathrm{N}(\mathfrak{a}), \quad x + \mu \in \mathcal{O}_E\mathfrak{a}.$$

Here, we also wrote $\mathrm{N}(x)$ for the reduced norm in $\mathbb{B}_p$. If $\mathcal{Z}(S)$ is non-empty, then we have $m + Q(\mu) = m + \mathrm{N}(\mu)/N(\mathfrak{a}) \in \mathbb{Z}$.

Lemma 4.3.1 (Lemma 6.2 in [BY09]). *The moduli problem $\mathcal{Z}$ is represented by an algebraic stack $\mathcal{Z}(m, \mathfrak{a}, \mu)$ of dimension 0 and the forgetful map $\psi : \mathcal{Z}(m, \mathfrak{a}, \mu) \to C_D$ defined by $(E, \iota, x) \mapsto (E, \iota)$ is finite and étale.*

For $m \in \mathbb{Q}_{>0}$, we define a set of rational primes by

$$\mathrm{Diff}(m) = \{p < \infty \mid (-m\mathrm{N}(\mathfrak{a}), D)_p = -1\}. \tag{4.3.1}$$

Remark 4.3.2. By the product formula for the Hilbert symbol [Ser73, III.2, Theorem 3], we have

$$\prod_{p\leq\infty}(-m\mathrm{N}(\mathfrak{a}), D)_p = 1.$$

But since $(-m\mathrm{N}(\mathfrak{a}), D)_\infty = -1$, the cardinality of $\mathrm{Diff}(m)$ is odd. Moreover, if $p \in \mathrm{Diff}(m)$, then p is non-split.

Lemma 4.3.3. *(i) If* $|\mathrm{Diff}(m)| > 1$*, then* $\mathcal{Z}(m) = \emptyset$.

(ii) If $\mathrm{Diff}(m) = \{p\}$*, then* p *is non-split in* k *and* $\mathcal{Z}(m)(\bar{\mathbb{F}}_q) = \emptyset$ *for* $q \neq p$.

Proof. If there is an element $(E, \iota, x) \in \mathcal{Z}(m)$, then this shows that we have an isomorphism of quaternion algebras

$$\mathbb{B}_p \cong \left(\frac{D, -m\mathrm{N}(\mathfrak{a})}{\mathbb{Q}}\right).$$

However, since $\mathbb{B}_p$ is ramified exactly at p and ∞, this is equivalent to

$$(D, -m\mathrm{N}(\mathfrak{a}))_v = \begin{cases} -1, & v = p, \infty, \\ 1, & \text{otherwise.} \end{cases}$$

This condition is equivalent to $\mathrm{Diff}(m) = \{p\}$. □

In the notation of [Vis89, Section 3], the stack $\mathcal{Z}(m, \mathfrak{a}, \mu)$ defines a 0-cycle $\psi_*[\mathcal{Z}(m, \mathfrak{a}, \mu)]$ on C_D since the forgetful map is proper, as seen above.

Moreover, note that the map $\mathrm{pr} : C_D \to \operatorname{Spec} \mathcal{O}_H$ is also proper by Lemma 4.1.2. Therefore, we can consider the proper pushforward $\mathrm{pr}_*[\mathcal{Z}(m, \mathfrak{a}, \mu)]$ to $\operatorname{Spec} \mathcal{O}_H$. In our case

$$\mathrm{pr}_*[\mathcal{Z}(m, \mathfrak{a}, \mu)] = \frac{1}{w_k}[\mathrm{pr}(\mathcal{Z}(m, \mathfrak{a}, \mu))]$$

because the automorphism group of a general geometric point of C_D is $\mathcal{O}_D^\times$ [KRY99].

By abuse of notation, we also denote by $\mathcal{Z}(m, \mathfrak{a}, \mu)$ the corresponding divisor on the coarse moduli scheme and simply write

$$\mathcal{Z}(m, \mathfrak{a}, \mu) = \sum_{\mathfrak{P}\subset\mathcal{O}_H} \mathcal{Z}(m, \mathfrak{a}, \mu)_{\mathfrak{P}}\mathfrak{P}.$$

If there is no confusion possible, we simply write $\mathcal{Z}(m)$ or $\mathcal{Z}(m)_{\mathfrak{P}}$. Note that the multiplicities above are *the same* for the pushforward from C_D^+ and from C_D.

In what follows, we will find formulas for the multiplicities $\mathcal{Z}(m, \mathfrak{a}, \mu)_{\mathfrak{P}}$. From now on, fix a prime p that is non-split in k and assume that $m \equiv Q(\mu) \bmod \mathbb{Z}$. Let $p_0 \in \mathbb{Z}$ be a prime with $p_0 \nmid 2pD$ such that if p is inert in k, we have

$$(D, -pp_0)_v = \begin{cases} -1, & v = p, \infty, \\ 1, & \text{otherwise,} \end{cases}$$

and if p is ramified in k, we have

$$(D, -p_0)_v = \begin{cases} -1, & v = p, \infty, \\ 1, & \text{otherwise.} \end{cases}$$

With this choice, put

$$\kappa_p = \begin{cases} pp_0 & \text{if } p \text{ is inert in } k, \\ p_0 & \text{if } p \text{ is ramified in } k, \end{cases}$$

and let $\mathfrak{p}_0$ be a fixed prime ideal of $\mathcal{O}_D$ lying above p_0. Here, $(\cdot,\cdot)_v$ denotes the v-adic Hilbert symbol. The existence of such a prime p_0 follows essentially from Dirichlet's theorem, see also [Ser73, III, Theorem 4].

Lemma 4.3.4. *The genus of* $[\mathfrak{p}_0]$ *is well defined.*

Proof. This is clear since the symbols $(D, \cdot)_v$ form a basis for the genus characters, see Section 2.2. ☐

We can write $\mathbb{B}_p = k \oplus k y_0$, where $y_0^2 = \kappa_p$, similar to the situation in the last section. Here, the decomposition is orthogonal with respect to the bilinear form corresponding to the reduced norm of $\mathbb{B}_p$. We write $[\gamma, \delta]$ for the element $\gamma + \delta y_0 \in \mathbb{B}_p$.

Proposition 4.3.5. *Let p be a prime that is non-split in k and let $(E, \iota) \in C_D(\bar{\mathbb{F}}_p)$. Then $L(E, \iota)$ is a projective $\mathcal{O}_D$-module of rank 1 and there is a fractional ideal $\mathfrak{b} \subset k$, such that*

$$L(E, \iota) \cong \mathfrak{b}\bar{\mathfrak{b}}^{-1}\mathfrak{p}_0^{-1} y_0.$$

Here $y_0 \in \mathcal{O}_E$ with $\mathrm{N}(y_0) = \kappa_p$. Moreover, if $h \in \mathbb{A}_{k,f}^\times$, then

$$L(h.(E, \iota)) = (h)\overline{(h)}^{-1} L(E, \iota).$$

Proof. The first statement is Proposition 5.13 in [KRY99]. The second follows from a similar description as in Section 4.2 of the action of the ideles on maximal orders in the quaternion algebra $\mathbb{B}_p$, as described in detail in Section 5 of [KRY99]. ☐

Recall that there are two actions of $\mathbb{A}_{k,f}^\times$ on ideals of k. One is given by the multiplication by the ideal (h) corresponding to the idele h and the other one is given by the action of $\mathbb{A}_{k,f}^\times \cong T(\mathbb{A}_f)$, where $T = \mathrm{GSpin}_U$ for the quadratic space $U = k$ with quadratic form given by the norm on k. We described this action in detail in Section 2.4. To avoid confusion, we will denote the action of h as an element of $T(\mathbb{A}_f)$ by $h.\mathfrak{a} = (h)(\bar{h})^{-1}\mathfrak{a}$ for any fractional ideal $\mathfrak{a} \subset k$. Also recall that the action of $T(\mathbb{A}_f)$ induces an isomorphism on discriminant groups and the element $h.\lambda$ for $\lambda \in \mathfrak{d}_k^{-1}\mathfrak{a}/\mathfrak{a}$ defines an element in $\mathfrak{d}_k^{-1}(h.\mathfrak{a})/(h.\mathfrak{a})$.

Proposition 4.3.6. *Let $h \in \mathbb{A}_{k,f}^\times$ and write $\sigma = \sigma(h)$ for the element of* $\mathrm{Gal}(H/k)$ *under the Artin map. Then we have*

$$\mathcal{Z}(m, \mathfrak{a}, \mu)_{\mathfrak{P}^\sigma} = \mathcal{Z}(m, h^{-1}.\mathfrak{a}, h.\mu)_{\mathfrak{P}}.$$

Proof. This follows from Propositions 4.1.6 and 4.3.5. □

Lemma 4.3.7. *Let* $(E_0, \iota_0) \in C_D(\overline{\mathbb{F}}_p)$ *such that* $[L(E_0, \iota_0)]^{-1} = [\mathfrak{p}_0]$*, where* $[L(E_0, \iota_0)]$ *denotes the class of the rank one* $\mathcal{O}_D$*-module in* $\mathrm{Pic}(\mathcal{O}_D)$*. Then the maximal order* $\mathrm{End}(E_0)$ *of* $\mathbb{B}_p$ *can be described in the following way.*

If p *is inert in* k*, let* $\mathfrak{c}_0 = \mathfrak{p}_0\mathfrak{d}_k$*. If* p *is ramified and* $\mathfrak{p} \subset \mathcal{O}_D$ *is the prime above* p*, let* $\mathfrak{c}_0 = \mathfrak{p}_0\mathfrak{p}^{-1}\mathfrak{d}_k$*.*

There exists a generator λ_0 *of* $\mathfrak{d}_k^{-1}\mathfrak{c}_0/\mathfrak{c}_0$ *with*

$$\mathrm{N}(\lambda_0) \equiv -\kappa_p \bmod \mathrm{N}(\mathfrak{c}_0),$$

such that

$$\mathrm{End}(E_0, \iota_0) = \mathcal{O}_{\mathfrak{c}_0, \lambda_0, \mathbb{B}_p},$$

where

$$\mathcal{O}_{\mathfrak{c}_0, \lambda_0, \mathbb{B}_p} = \{[\gamma, \delta] \mid \gamma \in \mathfrak{d}_k^{-1},\ \delta \in \mathfrak{c}_0^{-1},\ \gamma + \lambda\delta \in \mathcal{O}_D\}$$

is a maximal order in $\mathbb{B}_p$*.*

Moreover, if $(E, \iota) = h.(E_0, \iota_0)$*, we have*

$$\mathrm{End}(E, \iota) = \mathcal{O}_{h.\mathfrak{c}_0, h.\lambda, \mathbb{B}_p}.$$

Proof. See Lemma 3.3 and Lemma 7.1 of [KY13]. The result has also been described by Dorman [Dor89, Dor88]. □

Remark 4.3.8. There are 2^t possible choices for generator of $\mathfrak{d}_k^{-1}\mathfrak{c}_0/\mathfrak{c}_0$, with the required norm, where t is the number of prime divisors of D. However, there are only 2^{t-1} inequivalent ones. Here, we consider the orders to be equivalent if they are conjugate by an element of $k^\times$. Indeed, λ and $-\lambda$ yield such equivalent conjugate orders.

However, not knowing the specific λ that corresponds to the chosen point (E_0, ι_0) results in an ambiguity in Proposition 4.3.9 below, cf [Dor88]. We will resolve this issue later on by taking the relative norm to the fixed field of all elements of order dividing 2 in the Galois group $\mathrm{Gal}(H/k)$.

In the most general case that we consider, the multiplicities involve representation numbers with additional congruences that we will define now. For a fractional ideal $\mathfrak{a}$ of k, we let $\mathfrak{c}_\mathfrak{a} = \mathfrak{a}\bar{\mathfrak{a}}^{-1}\mathfrak{c}_0$. Moreover, we let $\lambda_\mathfrak{a} = a.\lambda_0 \in \mathfrak{d}_k^{-1}\mathfrak{c}_\mathfrak{a}/\mathfrak{c}_\mathfrak{a}$, where $a \in \mathbb{A}_{k,f}^\times$ is an idele determining $\mathfrak{a}$ and λ_0 is given in Lemma 4.3.7. Note that a is only unique up to an element of $\hat{\mathcal{O}}_D^\times$ but $\lambda_\mathfrak{a}$ is a well defined element of $\mathfrak{d}_k^{-1}\mathfrak{c}_\mathfrak{a}/\mathfrak{c}_\mathfrak{a}$ since $\hat{\mathcal{O}}_D^\times$ acts trivially on $\mathfrak{d}_k^{-1}\mathfrak{c}_\mathfrak{a}/\mathfrak{c}_\mathfrak{a}$.

For $n \in \mathbb{Q}_{>0}$ and $\mu \in \mathfrak{d}_k^{-1}\mathfrak{a}/\mathfrak{a}$, we let

$$\rho_0(n, \mathfrak{a}, \mu) = \#\{x \in \mathfrak{c}_\mathfrak{a}^{-1}\mathfrak{a} = \mathfrak{c}_0^{-1}\bar{\mathfrak{a}} \mid \mathrm{N}(x) = n, \lambda_\mathfrak{a}x + \mu \in \mathfrak{a}\}. \tag{4.3.2}$$

We need to define one more quantity to describe the multiplicities $\mathcal{Z}(m, \mathfrak{a}, \mu)_\mathfrak{P}$.

Let p be a prime which is non-split in k and define

$$\nu_p(m) = \begin{cases} \frac{1}{2}(\operatorname{ord}_p(m)+1), & \text{if } p \text{ is inert in } k, \\ \operatorname{ord}_p(m\,|D|), & \text{if } p \text{ is ramified in } k. \end{cases} \tag{4.3.3}$$

Note that the prime ideals $\mathfrak{P} \mid p$ of H correspond to the irreducible components of $\operatorname{Spec} \mathcal{O}_H(\bar{\mathbb{F}}_p)$. We let $\mathfrak{P}_0$ be the prime ideal such that $\operatorname{pr}_{\mathbb{F}_p}(E_0, \iota_0)$ with (E_0, ι_0) as in Lemma 4.3.7 lies in the irreducible component corresponding to $\mathfrak{P}_0$.

Proposition 4.3.9. *Suppose that* $\operatorname{Diff}(m) = \{p\}$.

(i) We have $\mathcal{Z}(m, \mathfrak{a}, \mu)_{\mathfrak{P}} = 0$ *unless* $m + Q(\mu) \in \mathbb{Z}$.

(ii) For $m + Q(\mu) \in \mathbb{Z}$, *we have*

$$\mathcal{Z}(m, \mathfrak{a}, \mu)_{\mathfrak{P}_0} = \frac{\nu_p(m)}{w_k} \rho_0 \left(\frac{m}{\kappa_p} \mathrm{N}(\mathfrak{a}), \mathfrak{a}, \mu \right).$$

Proof. First note that our cycles correspond to those studied in [KY13] which are generalizations of those in [KRY04]. The cycle $\mathcal{Z}(m, \mathfrak{a}, \mu)$ corresponds to $\mathcal{Z}(m\,|D|\,; \mathfrak{d}_k \mathfrak{a}^{-1}, \lambda', \lambda'\mu)$ for a generator $\lambda' \in \mathfrak{a}^{-1}/\mathfrak{a}^{-1}\mathfrak{d}_k$.

The push-forward $\operatorname{pr}_*[\mathcal{Z}(m)]$ is given by a formal sum

$$\sum_{\mathfrak{P} \subset \mathcal{O}_H} n_{\mathfrak{P}} \mathfrak{P}.$$

We will now determine the multiplicities. Fix a rational prime p and a prime ideal $\mathfrak{P} \subset \mathcal{O}_H$ over p. Moreover, fix any geometric point $\xi = (E_0, \iota_0) \in C_D(\overline{\kappa(\mathfrak{P})})$. Using Propositon 4.1 in [KY13], we see that the length $\lg \hat{\mathcal{O}}_{\mathcal{Z}(m),\xi}$ of the completed local ring is given by

$$\lg \hat{\mathcal{O}}_{\mathcal{Z}(m),\xi} = \nu_p(m). \tag{4.3.4}$$

Note that in the notation of [KY13], we have $\partial = \mathfrak{d}_k = \partial_\lambda$ and $\Delta = D$. Moreover, $\operatorname{ord}_p(m) = \operatorname{ord}_p(m\,|D|)$ for $p \nmid D$. Therefore,

$$n_{\mathfrak{P}} = \frac{\nu_p(m)}{w_k} \cdot \#\{x \in L(\xi) \mid (\xi, x) \in \mathcal{Z}(m)(\bar{\mathbb{F}}_p)\}.$$

Thus, what is left is to count the number of endomorphisms x, such that $(E_0, \iota_0, x) = (\xi, x) \in \mathcal{Z}(m)(\overline{\kappa(\mathfrak{P})})$. That is, we need to count the number of $x \in L(E_0, \iota_0)\mathfrak{d}_k^{-1}\mathfrak{a}$, such that

$$\mathrm{N}(x) = m\mathrm{N}(\mathfrak{a}), \quad x + \mu \in \mathcal{O}_{E_0}\mathfrak{a}.$$

The endomorphism ring $\mathcal{O}_{E_0} = \operatorname{End}(E_0)$ is a maximal order contained in the quaternion algebra $\mathbb{B}_p = k \oplus k y_0$, where $y_0^2 = \kappa_p$. By Lemma 4.3.7, we have

$$\mathcal{O}_{E_0} = \mathcal{O}_{\mathfrak{c}_0,\lambda_0,\mathbb{B}} = \{[\gamma,\delta] \mid \gamma \in \mathfrak{d}_k^{-1},\ \delta \in \mathfrak{c}_0^{-1},\ \gamma + \lambda_0\delta \in \mathcal{O}_D\}.$$

This implies that

$$L(E_0,\iota_0) = \mathcal{O}_0 \cap k y_0 = \mathfrak{c}_0^{-1}\mathfrak{d}_k y_0.$$

An element $x \in L(E_0,\iota_0)\mathfrak{d}_k^{-1}\mathfrak{a}$ is therefore of the form $x = \alpha y_0$ for

$$\alpha \in \mathfrak{c}_0^{-1}\bar{\mathfrak{a}}.$$

By Proposition 7.1 of [KY13], we have that

$$\mathfrak{a}\mathcal{O}_{\mathfrak{c}_\mathfrak{a},\lambda_\mathfrak{a},\mathbb{B}_p} = a\mathcal{O}_{\mathfrak{c}_\mathfrak{a},\lambda_\mathfrak{a},\mathbb{B}_p} = \mathcal{O}_{\mathfrak{c}_0,\lambda_0,\mathbb{B}_p} a = \mathcal{O}_{\mathfrak{c}_0,\lambda_0,\mathbb{B}_p}\mathfrak{a},$$

where $a \in \mathbb{A}_{k,f}^\times$ is an idele determining $\mathfrak{a}$. Note that this does not depend on the choice of such an a because the order is invariant under the action of $\hat{\mathcal{O}}_D^\times$.

Consequently, the condition $\mu + \alpha y_0 \in \mathcal{O}_{E_0}\mathfrak{a}$ is equivalent to $\mu \in \mathfrak{d}_k^{-1}\mathfrak{a}$ and $\alpha \in \mathfrak{c}_0^{-1}\bar{\mathfrak{a}}$ such that $\mu + \lambda_\mathfrak{a}\alpha \in \mathfrak{a}$. The norm of α is required to be $\mathrm{N}(\alpha) = (m/\kappa_p)\mathrm{N}(\mathfrak{a})$. This yields the representation number $\rho_0((m/\kappa_p)\mathrm{N}(\mathfrak{a}),\mathfrak{a},\mu)$ and ends the proof. □

We can avoid the ambiguity in the formulas above by taking the quotient $\mathrm{Cl_k}/\mathrm{Cl_k}[2]$ by the subgroup $\mathrm{Cl_k}[2]$ of elements of order dividing 2. This corresponds to calculating the valuation at primes $\ell \subset \mathcal{O}_L$, where $L \subset H$ is the subfield fixed by all elements of order 2 in $\mathrm{Gal}(H/k)$. We obtain a 0-cycle on $\operatorname{Spec}\mathcal{O}_L$ via the projection $\operatorname{Spec}\mathcal{O}_H \to \operatorname{Spec}\mathcal{O}_L$.

For an ideal class $[\mathfrak{c}] \in \mathrm{Cl_k}$ and a positive integer n we define the representation number

$$\rho(n,[\mathfrak{c}]) = |\{\mathfrak{b} \subset \mathcal{O}_D \mid \mathrm{N}(\mathfrak{b}) = n,\ \mathfrak{b} \in [\mathfrak{c}]\}|.$$

We obtain the following, more convenient result.

Proposition 4.3.10. *Let $L \subset H$ be the fixed field of* $\mathrm{Gal}(H/k)[2]$*, where H is the Hilbert class field of k. Let $[\mathfrak{c}] \in \mathrm{Cl_k}$ be an ideal class and let σ correspond to $[\mathfrak{c}]$ under the Artin map. Moreover, let $\mathfrak{f} \subset \mathcal{O}_L$ be the prime ideal below $\mathfrak{P}_0$. We have for $m + \mathbb{Q}(\mu) \in \mathbb{Z}$ that*

$$\mathcal{Z}(m,\mathfrak{a},\mu)_{\mathfrak{f}^\sigma} = 2^{o(m)-1}\nu_p(m)\rho(m\,|D|\,/p,[\mathfrak{c}]^{-2}[\mathfrak{c}_0\mathfrak{a}]),$$

where $\nu_p(m)$ is given before Proposition 4.3.9 and $o(m)$ is the number of primes $p \mid D$ such that $\mathrm{ord}_p(m\,|D|) > 0$.

Proof. It is enough to consider the case $[\mathfrak{c}] = [\mathcal{O}_D]$, that is, to determine the multiplicity for the prime $\mathfrak{f}$. The general formula follows by the action of the Galois group given in Proposition 4.3.6. We need to calculate the sum

$$f\sum_{\mathfrak{P}|\mathfrak{f}} \mathcal{Z}(m,\mathfrak{a},\mu)_\mathfrak{P} = \sum_{\tau\in\mathrm{Gal}(H/L)} \mathcal{Z}(m,\mathfrak{a},\mu)_{\mathfrak{P}_0^\tau},$$

where $f = 2$ if p is ramified in k and D is not a prime and $f = 1$, otherwise. (This is the ramification degree of $\mathfrak{P} \mid \mathfrak{f}$.) According to Proposition 4.3.9 and Proposition 4.3.6, this is equal to

$$\sum_{\tau \in \mathrm{Gal}(H/L)} \mathcal{Z}(m, \mathfrak{a}, \mu)_{\mathfrak{P}_0^\tau} = \frac{1}{w_k} \nu_p(m) \sum_{\substack{h \in \mathcal{C} \\ h^2 = 1}} \rho_0 \left(\frac{m}{\kappa_p} \mathrm{N}(\mathfrak{a}), h^{-1}.\mathfrak{a}, h.\mu \right),$$

where $\mathcal{C} = k^\times \backslash \mathbb{A}_{k,f}^\times / \hat{\mathcal{O}}_k^\times \cong \mathrm{Cl}_k$ acts as $\mathrm{GSpin}_U(\mathbb{A}_f)$ for $U = k$ as described above (see Section 2.4 for details). The elements of order less or equal to 2 in the class group Cl_k correspond to the prime divisors of D. See also Section 2.2. If $h^2 = 1$, then $h.\mathfrak{a} = \mathfrak{a}$ because $\mathfrak{p}/\bar{\mathfrak{p}} = \mathcal{O}_D$ for prime divisors of $\mathfrak{d}_k$. As h ranges over $\mathcal{C}[2]$, $h.\mu$ runs through a set of representatives of all $\beta \in \mathfrak{d}_k^{-1}\mathfrak{a}/\mathfrak{a}$ with $\mathrm{N}(\beta) \equiv \mathrm{N}(\mu) \bmod \mathrm{N}(\mathfrak{a})$ modulo the action of ± 1. Each of these β is counted with multiplicity

$$\begin{cases} 2^{o(m)-1} & \text{if } o(m) \geq 1, \\ 1 & \text{otherwise.} \end{cases}$$

Finally, if $\alpha \in \mathfrak{c}_0^{-1}\bar{\mathfrak{a}}$ with $\mathrm{N}(\alpha) = (m/\kappa_p)\mathrm{N}(\mathfrak{a})$, then $\tilde{\mathfrak{a}} = \alpha \mathfrak{c}_0 \bar{\mathfrak{a}}^{-1} \subset \mathcal{O}_D$ is an integral ideal with

$$\mathrm{N}(\tilde{\mathfrak{a}}) = \frac{m}{\kappa_p} \cdot \frac{\kappa_p \left|D\right|}{p} = \frac{m \left|D\right|}{p}$$

which lies in the class $[\tilde{\mathfrak{a}}] = [\mathfrak{c}_0 \mathfrak{a}]$. In this correspondence $\alpha \mapsto \tilde{\mathfrak{a}}$, each ideal occurs with multiplicity

$$\frac{w_k}{2} \cdot \begin{cases} 2 & \text{if } o(m) \geq 1, \\ 1 & \text{otherwise,} \end{cases}$$

because $-\mu$ is in the set $\{h.\mu \mid h^2 = 1\}$ if and only if $o(m) = o(\mu) \geq 1$. □

Proposition 4.3.11. *Let $D = -l$ for a prime $l \equiv 3 \bmod 4$. Let p be a prime that is non-split in k and fix an embedding of $H = k(j)$ into $\mathbb{C}$. We normalize the projection map* $\mathrm{pr} : C_D^+ \to \mathrm{Spec}\,\mathcal{O}_H$*, such that over $\mathbb{C}$ every CM elliptic curve (E, ι) maps to the embedding $j \mapsto j(E)$ of $\mathcal{O}_H$ into $\mathbb{C}$. There is a unique prime ideal $\mathfrak{P} \mid p$ of H fixed by complex conjugation, $\bar{\mathfrak{P}} = \mathfrak{P}$ and we have*

$$\mathcal{Z}(m, \mathfrak{a}, \mu)_{\mathfrak{P}} = 2^{o(m)-1} \nu_p(m) \rho(m \left|D\right| / p, [\mathfrak{a}]).$$

Proof. First note that the class number of h_k is odd [Zag81]. Since p is non-split in k, the unique prime $\mathfrak{p} \subset \mathcal{O}_D$ above p splits completely in H. Therefore, the number of primes of $\mathcal{O}_H$ above p is odd and there is at least one prime fixed by complex conjugation. Let $\mathfrak{P}$ be such a prime. Let τ denote complex conjugation $x \mapsto \bar{x}$. Since $\sigma \circ \tau = \tau \circ \sigma^{-1}$ for all $\sigma \in \mathrm{Gal}(H/k)$, we have

$$\mathfrak{P}^{\sigma(\mathfrak{b}^{-1})} = \overline{\mathfrak{P}^{\sigma(\mathfrak{b})}}$$

for every $\mathfrak{b}$. Suppose that $\mathfrak{Q}$ is another prime above $\mathfrak{p}$ with $\mathfrak{Q} = \bar{\mathfrak{Q}}$ and $\mathfrak{Q} = \mathfrak{P}^{\sigma(\mathfrak{b})}$. Then it is easy to see that $\mathfrak{P}^{\sigma(\mathfrak{b}^2)} = \mathfrak{P}$. Thus, since $\mathrm{Gal}(H/k)$ acts transitively on the set of primes above $\mathfrak{p}$ and $\mathfrak{p}$ is totally split in H, we have that $\sigma(\mathfrak{b}^2)$ is the identity and thus $[\mathfrak{b}]^2 = [\mathcal{O}_D]$. Since h_k is odd, this implies that $[\mathfrak{b}] = [\mathcal{O}_D]$ and therefore $\mathfrak{P}$ is the only prime fixed by complex conjugation.

Let W be the completion of the maximal unramified extension of $\mathcal{O}_{H,\mathfrak{P}}$ (here, $\mathcal{O}_{H,\mathfrak{P}}$ is the completion of $\mathcal{O}_H$ with respect to $\mathfrak{P}$). Fix an algebraic closure $\overline{\kappa(\mathfrak{p})}$ of the residue field $\kappa(\mathfrak{p}) = \mathcal{O}_D/\mathfrak{p}$. The ring W is a complete discrete valuation ring with maximal ideal π and its residue field W/π is algebraically closed and therefore isomorphic to $\overline{\kappa(\mathfrak{P})} \cong \overline{\kappa(\mathfrak{p})}$ (see Corollary 1 of Chapter II in [Ser79]). Recall the diagram in the proof of Proposition 4.1.6. We can consider a similar diagram with W in place of $\mathbb{C}_{\mathfrak{p}}$.

$$\begin{tikzcd}
C_D^+(\mathbb{C}) \arrow[r] \arrow[d] & C_D^+(W) \arrow[r] & C_D^+(\overline{\kappa(\mathfrak{p})}) \arrow[d] \\
\operatorname{Spec}\mathcal{O}_H(\mathbb{C}) \arrow[r] & \operatorname{Spec}\mathcal{O}_H(W) \arrow[r] & \operatorname{Spec}\mathcal{O}_H(\overline{\kappa(\mathfrak{p})})
\end{tikzcd}$$

Here, the bijection $C_D^+(\mathbb{C}) \to C_D^+(W)$ is given as follows. A CM elliptic curve $(E,\iota) \in \mathbb{C}_D^+(\mathbb{C})$ maps to an elliptic curve $(\tilde{E},\iota)$ over W with j-invariant $j(\tilde{E}) = j(E)$. Such an elliptic curve with good reduction exists by the theorem of Serre and Tate [ST68, GZ85] and is unique up to W-isomorphism. Moreover, note that all the maps involved are bijections is in fact a consequence of the canonical lifting theorem [How13, Lan08]. We normalize the map pr, such that over $\mathbb{C}$ every CM elliptic curve E maps to the embedding $j \mapsto j(E)$ of $\mathcal{O}_H$ into $\mathbb{C}$. Note that this, of course, restricts to an automorphism of H and therefore determines a map $\mathcal{O}_H \to W$, corresponding to $\tilde{E}$.

Now let E be an elliptic curve over W with j-invariant $j(E) = j(E_{\mathcal{O}_D})$. Using the description above, we see that the reduction of E maps to the geometric point of $\operatorname{Spec}\mathcal{O}_H(\overline{\kappa(\mathfrak{p})})$ determined by $\mathfrak{P}$ as it is the unique valuation such that the image of $j(E_{\mathcal{O}_D})$ is contained in $\mathbb{F}_p$.

As in Lemma 3.5 of [GZ85], we have that $\mathrm{End}_{W/\pi}(E)$ is isomorphic to $\mathcal{O}_{\mathcal{O}_D,\lambda,\mathbb{B}_p}$, where λ is any of the two possible $\lambda \in \mathcal{O}_D/\mathfrak{d}_k$ with $\mathrm{N}(\lambda) \equiv -p \bmod |D|$. Therefore, $\mathfrak{P} = \mathfrak{P}_0^\sigma$, where $\sigma = \sigma(\mathfrak{b})$ with $[\mathfrak{b}]^2 = [\mathfrak{c}_0] = [\mathfrak{p}_0]$.

Thus, we obtain according to Proposition 4.3.10 that

$$\mathcal{Z}(m,\mathfrak{a},\mu)_{\mathfrak{P}} = \mathcal{Z}(m,\mathfrak{a},\mu)_{\mathfrak{P}_0^\sigma} = 2^{o(m)-1}\nu_p(m)\rho(m\,|D|\,/p,[\mathfrak{a}]). \qquad \square$$

We can now also give a formula for the *Arakelov degree* $\widehat{\deg}\,\mathcal{Z}(m,\mathfrak{a},\mu)$.

Following [KY13], we define

$$\begin{aligned}\widehat{\deg}\ \mathcal{Z}(m,\mathfrak{a},\mu) &= \sum_p \log p \sum_{x\in\mathcal{Z}(m,\mathfrak{a},\mu)(\bar{\mathbb{F}}_p)} \frac{1}{|\mathrm{Aut}_{C_D}(\varphi(x))|}\,\mathrm{lg}(x)\\ &= \frac{1}{w_k}\sum_p \log(p) \sum_{x\in\mathcal{Z}(m,\mathfrak{a},\mu)(\bar{\mathbb{F}}_p)} \mathrm{lg}(x)\end{aligned}$$

and the sum runs over all rational primes. Here, we define

$$\mathrm{lg}(x) = \text{length of } \mathcal{O}_{\mathcal{Z}(m,\mathfrak{a},\mu),x} = \text{length of } \hat{\mathcal{O}}_{\mathcal{Z}(m,\mathfrak{a},\mu),x}.$$

This definition can also be expressed as $\widehat{\deg}\,\mathcal{Z}(m,\mathfrak{a},\mu) = \widehat{\deg}\,\mathrm{pr}_*[\mathcal{Z}(m,\mathfrak{a},\mu)]$, where the latter is the usual Arakelov degree of an arithmetic divisor on the arithmetic curve given by $\operatorname{Spec}\mathcal{O}_H$. We use Proposition 4.3.10 to calculate this degree by using that the degree map is compatible with pushforward. We have proved the following result, which is one of the results of [KRY99, KY13].

Corollary 4.3.12. *Assume that $m + Q(\mu) \in \mathbb{Z}$ and $\mathrm{Diff}(m) = \{p\}$. Then we have*

$$\widehat{\deg}\ \mathcal{Z}(m,\mathfrak{a},\mu) = 2^{o(m)-1}(\mathrm{ord}_p(m)+1)\rho(m\,|D|\,/p, [[\mathfrak{c}_0\mathfrak{a}]])\log(p),$$

where $\rho(n, [[\mathfrak{b}]])$ is the number of integral ideals of $\mathcal{O}_D$ of norm n in the genus of $\mathfrak{b}$.

4.4 The modular curve $Y_0(N)$

Recall the basic setup from Example 1.3.4. In particular, we let

$$V := \{x \in M_2(\mathbb{Q}) \mid \mathrm{tr}(x) = 0\}$$

with quadratic form $Q(x) = -N\det(x)$. The corresponding bilinear form is given by $(x,y) = N\,\mathrm{tr}(xy)$. Moreover, recall that we have an isomorphism

$$Y_0(N) \to X_K,\ \Gamma_0(N)z \mapsto H(\mathbb{Q})(z,1)K,$$

where K is defined as in Example 1.3.4, that is

$$K_p = \{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}_p) \bigm| c \in N\mathbb{Z}_p\}$$

and $K = \prod_p K_p$. In V we have the even lattice

$$L = \left\{\begin{pmatrix} b & -\frac{a}{N}\\ c & -b\end{pmatrix} \ \middle|\ a,b,c\in\mathbb{Z}\right\}.$$

The dual lattice of L is given by

$$L' = \left\{ \begin{pmatrix} \frac{b}{2N} & -\frac{a}{N} \\ c & -\frac{b}{2N} \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}.$$

The discriminant group L'/L is cyclic of order $2N$ and we can identify the corresponding finite quadratic module with the group $\mathbb{Z}/2N\mathbb{Z}$ together with the quadratic form $x^2/4N$, valued in $\frac{1}{4N}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$. The isomorphism of finite quadratic modules is explicitly given by

$$\mathbb{Z}/2N\mathbb{Z} \to L'/L, \qquad r \mapsto \mu_r = \begin{pmatrix} \frac{r}{2N} & 0 \\ 0 & -\frac{r}{2N} \end{pmatrix} =: \operatorname{diag}(r/2N, -r/2N).$$

The following Lemma is easy to check. We also refer to [BY09].

Lemma 4.4.1. *The group K preserves the lattice L and acts trivially on L'/L.*

4.5 An integral model for the modular curve

We recall some of the properties of the integral model for the modular curve $Y_0(N)$ and its compactification $X_0(N)$. These models have been intensively studied by Deligne, Rapoport [DR73], Katz and Mazur [KM85]. We refer to these references and [GZ86] for details.

The stack $\mathcal{Y}_0(N)$ ($\mathcal{X}_0(N)$) over $\mathbb{Z}$ represents the moduli problem that assigns to any base scheme S the cyclic isogenies of degree N of (generalized) elliptic curves $\pi : E \to E'$ over S such that $\ker \pi$ meets every irreducible component of each geometric fiber. On complex points, we have $\mathcal{Y}_0(N)(\mathbb{C}) = Y_0(N)(\mathbb{C})$ and $\mathcal{X}_0(N)(\mathbb{C}) = X_0(N)(\mathbb{C})$.

Here, the condition that $A = \ker \pi$ is cyclic of degree N means that locally on S there is a point P such that

$$A = \sum_{a=1}^{N} [aP]$$

as a Cartier divisor on E. This becomes the usual condition that A is locally isomorphic to $\mathbb{Z}/N\mathbb{Z}$, when N is invertible in S. We will always assume that N is square-free. In this case the condition means that A is locally free of rank N.

The cusps correspond to certain degenerated elliptic curves [DR73]. We will not give a precise definition of these as we will mostly work on the substack $\mathcal{Y}_0(N)$.

Theorem 4.5.1 (Theorems 1.2.1 and 3.2.7 of [Con07]). *Let N be square-free. Then the stack $\mathcal{X}_0(N)$ is a proper flat Deligne-Mumford stack over $\mathbb{Z}$. It is regular and has geometrically connected fibers of pure dimension one. Moreover, the stack $\mathcal{X}_0(N)$ is smooth over $\mathbb{Z}[1/N]$.*

Here and throughout, we assume that N is square-free.

The stack $\mathcal{Y}_0(N)$ has a coarse moduli scheme which we will describe now. The N-th *modular polynomial* (or *modular equation*) is a bivariate polynomial defined by

$$\Phi_N(j, Y) = \prod_{\gamma \in M_N} (Y - j \circ \gamma). \tag{4.5.1}$$

Here, the set M_N is given as follows. Let

$$D_N = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \gcd(a, b, c, d) = 1,\ ad - bc = N \right\}.$$

The group $\Gamma_0(N)$ acts on D_N and the set M_N above is taken to be a system of representatives for $\Gamma_0(N)\backslash D_N$. We have $\Phi_N(j, Y) \in \mathbb{Z}[j][Y]$ which is why we will write $\Phi_N(X, Y)$ for the bivariate polynomial with integer coefficients. The modular functions $j(\tau)$ and $j_N(\tau) := j(N\tau)$ satisfy $\Phi_N(j, j_N) = 0$.

Proposition 4.5.2 ([DR73], VI. Proposition 6.5)**.** *The coarse moduli scheme* $\mathbf{Y}_0(N)$ *of* $\mathcal{Y}_0(N)$ *is given by the spectrum of the normalization of* $\mathbb{Z}[X, Y]/(\Phi_N(X, Y))$.

From this description we can obtain detailed information on the special fibers of $\mathbf{Y}_0(N)$ (and $\mathbf{X}_0(N)$). The fiber above p with $p \mid N$ is reducible and singular [GZ86]. When N is square-free, it has two irreducible components. We will denote these two components by $\mathcal{F}_0^{(p)}$ and $\mathcal{F}_\infty^{(p)}$. They are characterized by the property that the horizontal divisor $\underline{\infty}$, that is obtained as the closure of the cusp ∞, intersects $\mathcal{F}_\infty$ and similarly, the divisor $\underline{0}$ intersects $\mathcal{F}_0$. This behavior is reflected by Kronecker's congruence, which is quite simple in our case:

$$\Phi_N(X, Y) \equiv \Phi_{N/p}(X^p, Y)\Phi_{N/p}(X, Y^p) \bmod p. \tag{4.5.2}$$

We refer to Gross and Zagier [GZ86] and also Deligne and Rapoport [DR73] for details.

4.5.1 Integral extensions of Heegner divisors

The following moduli problem describes a natural extension of the divisor $Z(m, \mu)$ to the stack $\mathcal{X}_0(N)$. We follow [BY09], Section 7.3.

Definition 4.5.3. Let $m \in \mathbb{Q}_{<0}$ and $r \in \mathbb{Z}$ such that $4Nm = \Delta \equiv r^2 \bmod 4N$. The integer Δ is a negative discriminant and we denote by $\mathcal{O}_\Delta$ the order of $k = \mathbb{Q}(\sqrt{\Delta})$ of discriminant Δ. The ideal $\mathfrak{n} = (N, \frac{r+\sqrt{\Delta}}{2})$ has norm N. Moreover, let

$$\mu = \mu_r = \begin{pmatrix} \frac{r}{2N} & 0 \\ 0 & -\frac{r}{2N} \end{pmatrix}.$$

We define $\mathcal{Z}(m, \mu)$ to be the Deligne-Mumford stack representing the moduli problem which assigns to a base scheme S over $\mathbb{Z}$ the set of pairs $(\pi : E \to E', \iota)$, such that

(i) $\pi : E \to E'$ is a cyclic isogeny of two elliptic curves E and E' over S of degree N,

(ii) $\iota : \mathcal{O}_\Delta \hookrightarrow \mathrm{End}(\pi) = \{\alpha \in \mathrm{End}(E) : \pi\alpha\pi^{-1} \in \mathrm{End}(E')\}$ is an $\mathcal{O}_\Delta$-action on π such that $\iota(\mathfrak{n}) \ker \pi = 0$.

There is a natural morphism

$$\mathcal{Z}(m, \mu) \longrightarrow \mathcal{X}_0(N),$$

given by the forgetful map

$$(\pi : E \to E', \iota) \mapsto \pi : E \to E'.$$

Note that $\mathcal{Z}(m, \mu)$ does not intersect the boundary $\mathcal{X}_0(N) \backslash \mathcal{Y}_0(N)$ [Con04].

Lemma 4.5.4. *The forgetful map $\mathcal{Z}(m, \mu) \longrightarrow \mathcal{X}_0(N)$ is finite and étale. Therefore, $\mathcal{Z}(m, \mu)$ defines a divisor on $\mathcal{X}_0(N)$ and this divisor is horizontal.*

Proof. That the forgetful map is finite and étale has been stated in [BY09, Section 7.3]. The map is representable by a finite and unramified morphism. This follows essentially from the rigidity of endomorphisms of elliptic curves (this also shows that the stack is a Deligne-Mumford stack). That the forgetful map defines a horizontal divisor is clear because it is flat and $\mathcal{X}_0(N)$ is flat over $\operatorname{Spec} \mathbb{Z}$. The result follows from the following Lemma. □

Lemma 4.5.5. *As divisors in the complex fiber, we have*

$$\mathcal{Z}(m, \mu)(\mathbb{C}) = Z(m, \mu).$$

Therefore, $\mathcal{Z}(m, \mu)$ is the flat closure of $Z(m, \mu)$.

Proof. Let $z \in Z(m, \mu)$. By Lemma 1.3.6, we can assume that $z = H(\mathbb{Q})(\tau, 1)K$ with $\tau \in \mathbb{D}_\lambda$ for some $\lambda \in L_{m,\mu}$. This implies that τ satisfies a quadratic equation of the form

$$a\tau^2 + b\tau + c = 0$$

with $N \mid a$ and $b^2 - 4ac = \Delta = 4Nm$. Thus,

$$\tau = \frac{-b + \sqrt{\Delta}}{2a} \in \mathbb{Q}(\sqrt{\Delta})$$

and $a\tau$ is contained in the order $\mathcal{O}_\Delta$ of discriminant Δ. Moreover, the ideal $\mathfrak{a} = (a, a\tau) \subset \mathcal{O}_\Delta$ is a proper ideal of $\mathcal{O}_\Delta$. Consequently, the elliptic curve

$$E_\tau = \mathbb{C}/\mathfrak{a}$$

has complex multiplication by (at least) $\mathcal{O}_\Delta$, that is $\mathcal{O}_\Delta \subset \mathrm{End}_{\mathbb{C}}(E_\tau)$. Furthermore, the ideal $\mathfrak{a}\mathfrak{n}^{-1} = (a/N, a\tau)$, where $\mathfrak{n} = (N, a\tau)$, is also a proper $\mathcal{O}_\Delta$-ideal and the natural map

$$E_\tau = \mathbb{C}/\mathfrak{a} \longrightarrow \mathbb{C}/\mathfrak{a}\mathfrak{n}^{-1} =: E'_\tau$$

is a cyclic N-isogeny. Thus,

$$(E_\tau \to E'_\tau, \iota) \in \mathcal{Z}(m,\mu)(\mathbb{C}),$$

where we write ι for the natural inclusion $\mathcal{O}_\Delta \hookrightarrow \mathrm{End}_{\mathbb{C}}(E_\tau)$.

The other inclusion is similar. □

We briefly mention the relation to the Heegner points as defined by Birch [Bir75], Gross [Gro84] and Gross-Zagier [GZ86]. A Heegner point on $X_0(N)(\mathbb{C})$ is described by the data $(\mathcal{O}, \mathfrak{n}, [\mathfrak{a}])$, where $\mathcal{O} \subset k$ is an order, $\mathfrak{n} \subset \mathcal{O}$ is a proper $\mathcal{O}$-ideal with quotient $\mathcal{O}/\mathfrak{n}$ cyclic of order N and $[\mathfrak{a}]$ is the class of some invertible $\mathcal{O}$-module in $\mathrm{Pic}(\mathcal{O})$. The Heegner point corresponding to this data is given by the diagram

$$\mathbb{C}/\mathfrak{a} \to \mathbb{C}/\mathfrak{a}\mathfrak{n}^{-1}.$$

If we choose an oriented basis (ω_1, ω_2) of $\mathfrak{a}$, such that $\mathfrak{a}\mathfrak{n}^{-1} = (\omega_1, \omega_2/N)$, then the point in $X_0(N)(\mathbb{C}) \cong \Gamma_0(N)\backslash\mathbb{H}$ is given by the orbit of $\tau = \omega_1/\omega_2$ [Gro84].

Finally, notice that we have a decomposition of $\mathcal{Z}(m,\mu)$ similar to Lemma 1.3.9. We let $\mathcal{Z}^0(m,\mu)$ be the Deligne-Mumford stack representing the moduli problem assigning to a base scheme S the set of pairs $(\pi : E \to E', \iota)$, such that

(i) $(\pi : E \to E', \iota) \in \mathcal{Z}(m,\mu)(S)$,

(ii) $(\pi : E \to E', \iota) \notin \mathcal{Z}(m/f^2,\nu)(S)$ for all $f > 1$ and ν with $f\nu = \mu$.

Note that over $\mathbb{C}$ the second condition is equivalent to $\mathrm{End}_{\mathbb{C}}(E) = \mathrm{End}_{\mathbb{C}}(E') = \mathcal{O}_\Delta$. Moreover, this is compatible with our definition of $Z^0(m,\mu)$ in Section 1.3.4. By definition, we have a decomposition

$$\mathcal{Z}(m,\mu) = \sum_{f^2 \mid m} \sum_{\substack{\nu \in L'/L \\ f\nu = \mu}} \mathcal{Z}^0\left(\frac{m}{f^2}, \nu\right). \tag{4.5.3}$$

4.6 CM values of modular functions

The starting point for our study of CM values of modular functions with zeros and poles supported on Heegner divisors is the following Lemma.

Lemma 4.6.1 ([BY09], Lemma 7.10)**.** *Let D be a negative fundamental discriminant and assume that $D \equiv 1 \bmod 4$. Let $m \in \mathbb{Q}_{<0}$ and $r \in \mathbb{Z}$ such that $4Nm = D \equiv r^2 \bmod 4N$. There is an isomorphism of stacks*

$$\mathbf{j}_D : C_D \to \mathcal{Z}(m,\mu_r), \quad (E,\iota) \mapsto (\pi : E \to E/E[\mathfrak{n}], \iota).$$

Here, we denote by $E[\mathfrak{n}]$ the kernel of the multiplication-by-n map. Combining the map $\mathbf{j}_D$ with the forgetful map $\mathcal{Z}(m,\mu) \to \mathcal{X}_0(N)$ yields a map $C_D \to \mathcal{X}_0(N)$, still denoted $\mathbf{j}_D$.

Note that this map also depends on the choice of r. For simplicity, we do not reflect this in the notation.

From now on, we fix $m_0, m_1 \in \mathbb{Q}_{>0}$ and write $D_i = 4Nm_i$. We assume that m_0, m_1 satisfy the properties of Definition 4.5.3. In particular, there are $r_0, r_1 \in \mathbb{Z}$, such that $r_i^2 \equiv D_i \bmod 4N$ and we let $\mu_i = \mu_{r_i}$. We write $\mathfrak{n}_i$ for the corresponding ideals of norm N in $\mathcal{O}_{D_i}$ generated by N and $\frac{r_i+\sqrt{D_i}}{2}$. Moreover, assume that D_0 is a fundamental discriminant with $D_0 \equiv 1 \bmod 4$ and $D_0 D_1$ is not a perfect square so that $\mathcal{Z}(m_0, \mu_0)$ and $\mathcal{Z}(m_1, \mu_1)$ intersect properly.

For a CM elliptic curve $(E, \iota) \in C_{D_0}(S)$, we define

$$\mathcal{O}_{E,\mathfrak{n}_0} := \mathrm{End}_S(E \to E/E[\mathfrak{n}_0]) = \{\alpha \in \mathrm{End}_S(E) \mid \pi\alpha\pi^{-1} \in \mathrm{End}_S(E/E[\mathfrak{n}_0])\}.$$

We are interested in the intersection of $\mathcal{Z}(m_0, \mu_0)$ and $\mathcal{Z}(m_1, \mu_1)$ on $\mathcal{X}_0(N)$ or, equivalently, in the pullback of $Z(m_0, \mu_0)$ under $\mathbf{j}_D$. The stack $\mathbf{j}_D^* \mathcal{Z}(m_1, \mu_1)$ represents the following moduli problem. For a base scheme S, consider the category $\mathcal{M}(m_1, \mu_1, \mathfrak{n}_0)(S)$ of triples (E, ι, ϕ), where

(i) $(E, \iota) \in C_{D_0}(S)$,

(ii) $\phi : \mathcal{O}_{D_1} \hookrightarrow \mathcal{O}_{E,\mathfrak{n}_0}$ is an action of $\mathcal{O}_{D_1}$, such that

(iii) $\phi(\mathfrak{n}_1)E[\mathfrak{n}_0] = 0$.

We consider the fiber product diagram

$$\begin{tikzcd}
\mathbf{j}_D^* \mathcal{Z}(m_1, \mu_1) = \mathcal{Z}(\mu_1, m_1) \times_{\mathcal{X}_0(N)} C_{D_0} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & C_{D_0} \arrow[d] \\
\mathcal{Z}(m_1, \mu_1) \arrow[r] & \mathcal{X}_0(N).
\end{tikzcd}$$

Lemma 4.6.2. *The map*

$$\varphi : \mathbf{j}_D^* \mathcal{Z}(m_1, \mu_1) \to \mathcal{M}(m_1, \mu_1, \mathfrak{n}_0),$$

given by

$$\xi \mapsto (E, \iota, \phi),$$

where $\pi_2(\xi) = (E, \iota)$ *and* $\pi_1(\xi) = (E \to E/E[\mathfrak{n}_0], \phi)$ *is an isomorphism of stacks.*

Proof. This can be found in [BY09, Section 7.3]. It is clear that the described map is well defined and injective over any scheme S. In the other direction, suppose that $(E, \iota, \phi) \in \mathcal{M}(m_1, \mu_1)(S)$. Then $(E, \iota) \in C_{D_0}(S)$ and $(E \to E/E[\mathfrak{n}_0], \phi) \in \mathcal{Z}(m_1, \mu_1)(S)$ by definition. Thus, we obtain maps $\mathcal{M}(m_1, \mu_1, \mathfrak{n}_0) \to \mathcal{Z}(m_1, \mu_1)$ and $\mathcal{M}(m_1, \mu_1, \mathfrak{n}_0) \to C_{D_0}$. By the universal property of the fiber product, we obtain a unique map $\tilde{\varphi}$ that makes the following diagram commutative.

$$\begin{tikzcd}
\mathcal{M}(m_1, \mu_1, \mathfrak{n}_0) \arrow[dr, "\tilde{\varphi}"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \mathcal{Z}(m_1, \mu_1) \times_{\mathcal{X}_0(N)} C_{D_0} \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] & C_{D_0} \arrow[d] \\
& \mathcal{Z}(m_1, \mu_1) \arrow[r] & \mathcal{X}_0(N)
\end{tikzcd}$$

Therefore, $\xi = \tilde{\varphi}((E, \iota, \phi)) \in \mathbf{j}_D^* \mathcal{Z}(m_1, \mu_1)$ with $\varphi(\xi) = (E, \iota, \phi)$ by the definition of φ. $\square$

Lemma 4.6.3. *We have the identity*

$$\mathbf{j}_D^* \mathcal{Z}(m_1, \mu_1) = \sum_{\substack{n \equiv r_0 r_1 \bmod 2N \\ n^2 \leq D_0 D_1}} \mathcal{Z}\left(\frac{D_0 D_1 - n^2}{4N \left| D_0 \right|}, \mathfrak{n}_0, \frac{n + r_1 \sqrt{D_0}}{2\sqrt{D_0}} \right),$$

of elements in $Z^1(C_{D_0})$. The sum runs over all integers (positive and negative) satisfying the congruence.

Proof. This is a direct consequence of Lemma 7.12 in [BY09]. There, the authors prove an identity on geometric points over algebraically closed fields in any characteristic. In our notation, the lemma stated there becomes

$$\mathbf{j}_D^* \mathcal{Z}(m_1, \mu_1)(\overline{\mathbb{F}}_p) \cong \mathcal{M}(m_1, \mu_1, \mathfrak{n}_0)(\overline{\mathbb{F}}_p) \cong \bigsqcup_{\substack{n \equiv r_0 r_1 \bmod 2N \\ n^2 \leq D_0 D_1}} \mathcal{Z}\left(\frac{D_0 D_1 - n^2}{4N \left| D_0 \right|}, \mathfrak{n}_0, \frac{n + r_1 \sqrt{D_0}}{2\sqrt{D_0}} \right)(\overline{\mathbb{F}}_p).$$

Here, the second isomorphism is given by

$$(E, \iota, \phi) \mapsto (E, \iota, x),$$

where x is given by

$$x = \phi\left(\frac{r_1 + \sqrt{D_1}}{2} \right) - \iota\left(\frac{n + r_1 \sqrt{D_0}}{2\sqrt{D_0}} \right).$$

In order to prove the equality of divisors, we need to show that the completed étale local rings are isomorphic if $(E, \iota, \phi) \mapsto (E, \iota, x)$. This follows from the fact that the completed étale local rings pro-represent the same formal deformation functor.

Let p be non-split in k_{D_0} and let $W = W(\bar{\mathbb{F}}_p)$ be the Witt ring of $\bar{\mathbb{F}}_p = \mathbb{F}$ (see Chapter II of [Ser79]). Let R be a locally complete artinian W-algebra with residue field $\mathbb{F}$. Let $(E, \iota, \phi) \in \mathbf{j}_D^* \mathcal{Z}(m_1, \mu_1)(\mathbb{F}) = \mathcal{Z}_1(\mathbb{F})$ and

$$(E, \iota, x) \in \mathcal{Z}\left(\frac{D_0 D_1 - n^2}{4N \left| D_0 \right|}, \mathfrak{n}_0, \frac{n + r_1 \sqrt{D_0}}{2\sqrt{D_0}} \right) = \mathcal{Z}_2(\mathbb{F})$$

for some $n \in \mathbb{Z}$ with $n \equiv r_0 r_1 \bmod 2N$ and $n^2 \leq D_0 D_1$ such that

$$\phi\left(\frac{r_1+\sqrt{D_1}}{2}\right) = \iota\left(\frac{n+r_1\sqrt{D_0}}{2\sqrt{D_0}}\right) + x.$$

Suppose that (E, ι, ϕ) lifts to $(\tilde{E}, \tilde{\iota}, \tilde{\phi})$ in $\mathbf{j}_D^* \mathcal{Z}(m_1, \mu_1)(R)$. Clearly, since ϕ lifts to $\tilde{\phi}$, we must have that $\pi \circ \tilde{\phi} = \phi$ and also $\pi \circ \tilde{\iota} = \iota$. Therefore, we have that

$$\tilde{x} = \tilde{\phi}\left(\frac{r_1+\sqrt{D_1}}{2}\right) - \tilde{\iota}\left(\frac{n+r_1\sqrt{D_0}}{2\sqrt{D_0}}\right)$$

is a lift of x and thus $(\tilde{E}, \tilde{\iota}, \tilde{x})$ is a lift of (E, ι, x). We have by the rigidity of endomorphisms of elliptic curves that the reduction map $\pi : \mathrm{End}(\tilde{E}) \to \mathrm{End}(E)$ is injective and therefore, $\tilde{\phi}$ and $\tilde{x}$ are both uniquely determined. The other direction is similar. We obtain that $\xi = (E, \iota, \phi)$ lifts to R if and only if $\zeta = (E, \iota, x)$ lifts to R.

The completed étale local ring $\hat{\mathcal{O}}_{\mathcal{Z}_1,\xi}$ pro-represents the formal deformation functor of ξ and $\hat{\mathcal{O}}_{\mathcal{Z}_2,\zeta}$ the one of ζ. We have just shown that these functors are isomorphic. Therefore, we have $\mathrm{Hom}_W(\hat{\mathcal{O}}_{\mathcal{Z}_1,\xi}, R) \cong \mathrm{Hom}_W(\hat{\mathcal{O}}_{\mathcal{Z}_2,\zeta}, R)$. In particular, this is true for $R = \hat{\mathcal{O}}_{\mathcal{Z}_1,\xi}/\mathfrak{m}_1^n$ and $R = \hat{\mathcal{O}}_{\mathcal{Z}_2,\zeta}/\mathfrak{m}_2^n$, where $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are the corresponding maximal ideals. Thus, we have $\hat{\mathcal{O}}_{\mathcal{Z}_1,\xi} \cong \hat{\mathcal{O}}_{\mathcal{Z}_2,\zeta}$.

Moreover, both define the same divisor because the completed local ring $\hat{\mathcal{O}}_{C,\alpha}$ for a point $\alpha = (E, \iota)$ pro-represents the formal deformation functor of α. In particular, we have a commutative diagram

$$\begin{tikzcd}
\hat{\mathcal{O}}_{\mathcal{Z}_1,\xi} \arrow[dd, "\cong"'] & \\
& \hat{\mathcal{O}}_{C_D,\alpha} \arrow[ul] \arrow[dl] \\
\hat{\mathcal{O}}_{\mathcal{Z}_2,\zeta} &
\end{tikzcd}$$

where all morphisms are morphisms of complete local rings. We also refer to Section 3.2 of [How12b] for more details on the connection between local rings and deformation functors in our situation. □

Proposition 4.6.4.

(i) We have an equality of function fields:
$k(\mathcal{X}_0(N)) = k(\mathcal{X}_0(N)_{/\mathbb{Q}}) = k(\mathbf{X}_0(N)_{/\mathbb{Q}}) = k(\mathbf{X}_0(N))$.

(ii) Moreover, the function field is given by $k(\mathcal{X}_0(N)) = \mathbb{Q}(j, j_N)$, *where* $j_N(\tau) = j(N\tau)$.

(iii) The restriction of $f \in k(\mathcal{X}_0(N))$ *to the generic fiber* $\mathcal{X}_0(N)_{/\mathbb{Q}}$ *is given by a rational meromorphic modular function for* $\Gamma_0(N)$.

Proof. (i) On an affine subscheme $\operatorname{Spec}(A)$, the fiber product $\operatorname{Spec}(A)\times_{\mathbb{Z}}\operatorname{Spec}(\mathbb{Q})$ is just the tensor product $\operatorname{Spec}(A\otimes_{\mathbb{Z}}\mathbb{Q})$. Therefore, locally on any atlas $U\to\mathcal{X}_0(N)$, the first equality holds. Thus, it also holds for $\mathcal{X}_0(N)$. The second equality follows from Proposition 2.4 of [Vis89], which states that the field extension of functions fields is purely inseparable. Since we are in characteristic 0, this implies $k(\mathcal{X}_0(N)_{/\mathbb{Q}})=k(\mathbf{X}_0(N)_{/\mathbb{Q}})$. The last equality follows in the same way as the first one.

(ii) The generic fiber $\mathcal{X}_0(N)_{/\mathbb{Q}}$ is a model for $X_0(N)$ over $\mathbb{Q}$. In fact, it is the canonical model, which has been studied by Shimura [Shi94]. By Proposition 6.72 [Shi94], we have that $\mathbb{C}k(\mathcal{X}_0(N)_{/\mathbb{Q}})=\mathbb{C}(X_0(N))=\mathbb{C}(j,j_N)$. See Proposition 2.10 of [Shi94]. For further details we refer the reader to the example on page 156, ibid.

(iii) This follows from the first two items. □

A rational function $f\in k(\mathcal{X}_0(N))$ defines a rational function on C_{D_0} via pullback $\mathbf{j}_D^*(f\mid_{\mathbf{j}_D(C_{D_0})})$, which makes sense as long as $\mathbf{j}_D(C_{D_0})$ is not contained in the divisor of f. The element $\mathbf{j}_D^*f$ then defines an element of $k(C_{D_0})\cong k(\mathbf{C}_{D_0})=H_0$. Here, H_0 is the Hilbert class field of $k_{D_0}=\mathbb{Q}(\sqrt{D_0})$. The fact that $k(C_{D_0})\cong k(\mathbf{C}_{D_0})$ holds follows again from Proposition 2.4 of [Vis89].

Now let $f\in k(\mathcal{X}_0(N))$ be a modular function such that its divisor is a linear combination of the Heegner divisors $\mathcal{Z}(m,\mu)$. That is, there are integers $c(m,r)$, such that

$$\operatorname{div}(f)=\sum_{r \bmod 2N}\ \sum_{\substack{m\in\mathbb{Q}_{<0}\\ 4Nm\equiv r^2 \bmod 4N}} c(m,r)\mathcal{Z}(m,\mu_r)+C(f),$$

where $C(f)$ is supported at the boundary. We have by definition (cf. the proof of Proposition 3.7 in [Vis89]) that

$$\operatorname{div}(\mathbf{j}_D^*f)=\mathbf{j}_D^*(\operatorname{div}(f)).$$

Finally, we normalize the map $\mathrm{pr}: C_{D_0}\to\operatorname{Spec}\mathcal{O}_{H_0}$ in the following way. (Recall that it is only unique up to an automorphism of $\mathcal{O}_{H_0}$.) We fix an embedding of H_0 into $\mathbb{C}$ and an ideal

$$\mathfrak{n}_0=\left(N,\frac{r+\sqrt{D_0}}{2}\right)\subset\mathcal{O}_{D_0}.$$

of norm N. Consider the Heegner point z_0 given by $\pi:\mathbb{C}/\mathcal{O}_{D_0}\to\mathbb{C}/\mathfrak{n}^{-1}$. Then we require that pr is chosen such that $\mathrm{pr}_*\mathbf{j}_D^*(f^{w_{k_0}})\in H_0$ is equal to $f(z_0)$, where w_{k_0} is the number of roots of unity in k_{D_0}. Note that $f(z_0)\in H_0$ defines a divisor on $\operatorname{Spec}\mathcal{O}_{H_0}$ given by

$$\sum_{\mathfrak{P}\subset\mathcal{O}_{H_0}}\operatorname{ord}_{\mathfrak{P}}(f(z_0))\mathfrak{P}. \tag{4.6.1}$$

Here, the sum is over all nonzero prime ideals of $\mathcal{O}_{H_0}$.

Proposition 4.6.5. *Let $D_0 < 0$ be a odd fundamental discriminant and normalize* pr *as described above. Let $f \in k(\mathcal{X}_0(N))$ with*

$$\operatorname{div}(f) = \sum_{r \bmod 2N} \sum_{m \in \mathbb{Q}_{<0} 4Nm \equiv r^2 \bmod 4N} c(m,r)\mathcal{Z}(m, \mu_r) + C(f),$$

and assume that $\operatorname{div}(f)$ *and* $\mathcal{Z}(m_0, \mu_0)$ *intersect properly. Then we have*

$$\operatorname{ord}_{\mathfrak{P}}(f(z_0)) = w_{k_0} \sum_{r \bmod 2N} \sum_{m \in \mathbb{Q}_{<0}} c(m,r) \sum_{\substack{n \equiv r_0 r \bmod 2N \\ n^2 \leq -4NmD_0}} \mathcal{Z}\left(|m| - \frac{n^2}{4N\,|D_0|}, \mathfrak{n}_0, \frac{n + r\sqrt{D_0}}{2\sqrt{D_0}}\right)_{\mathfrak{P}}$$

for every prime $\mathfrak{P}$ of the Hilbert class field H_0.

Proof. This follows directly from our considerations above and the fact that the pullback of $\operatorname{div}(f)$ as a Cartier divisor (Equation (4.6.1)) agrees with the pullback as a Weil divisor, corrected by the multiplicity as explained above. The pullback as a Weil divisor is described in Lemma 4.6.3. Also note that the image of C_{D_0} does not intersect the boundary. □

Remark 4.6.6. The formulas in Proposition 4.3.9 and 4.3.10 provide explicit formulas for the quantities in Proposition 4.6.5. Also, note the similarities to the Gross-Zagier formula on singular moduli. In fact, the Proposition provides a generalization of the Gross-Zagier theorem to modular functions on $X_0(N)$ with zeros and poles supported on a linear combination of the Heegner divisors $\mathcal{Z}(m, \mu)$ on the stack $\mathcal{X}_0(N)$.

We should also remark, that we have basically used arithmetic intersection theory here, even though we did not make that explicit. Since we are only dealing with principal divisors, it is not necessary to introduce the even more involved arithmetic cycles and their intersection theory. We preferred to just use this ad-hoc adaption of the pullback of divisors which is sufficient for our purposes.

Example 4.6.7. To illustrate the situation, let us consider a simple example. Let $N = 1$ and suppose we would like to compute the value of $(j(\alpha_{-3}) - j(\alpha_{-7})) = -j(\alpha_{-7})$, where we denote by α_D the unique CM point of discriminant D on $X_0(1)$ (in both cases the class number is equal to one).

We consider the functions $f_3 = (j(\tau) - j(\alpha_{-3}))^2 = j(\tau)$ and $f_7 = (j(\tau) - j(\alpha_{-7}))^2$. We can compute the same value by computing the pullback $\operatorname{div}(f_3)$ to C_{-7} or by computing the pullback of $\operatorname{div}(f_7)$ to C_{-3}. On the stack $\mathcal{X}_0(1)$ these divisors correspond to $3\mathcal{Z}(-3/4, 1/2)$ and $\mathcal{Z}(-7/4, 1/2)$ because the points in the divisor $\mathcal{Z}(D/4, 1/2)$ are counted with multiplicity $2/w_D$, where w_D is the number of roots of unity in k_D.

We obtain for the pullback of f_7 that

$$\frac{1}{6}\operatorname{ord}_p\left(f_7(\alpha_{-3})\right) = \mathcal{Z}\left(\frac{5}{3}, \mathcal{O}_{-3}, \frac{\pm 1 + \sqrt{-3}}{2\sqrt{-3}}\right)_p + \mathcal{Z}\left(1, \mathcal{O}_{-3}, \frac{\pm 3 + \sqrt{-3}}{2\sqrt{-3}}\right)_p.$$

Using Proposition 4.3.11, we see that the first summand only contributes when $p = 5$ and contributes twice $1/2$ in that case. Similarly, the second summand is equal to 1 and contributes twice for $p = 3$. And indeed, it is well known that

$$f_7(\alpha_{-3}) = j(\alpha_{-7})^2 = 3375^2 = 3^6 5^6.$$

We double-check the formula and consider f_3 to obtain

$$\frac{1}{2}\operatorname{ord}_p(f_3(\alpha_{-7})) = 3\mathcal{Z}\left(\frac{5}{7}, \mathcal{O}_{-7}, \frac{\pm 1 + \sqrt{-3}}{2\sqrt{-3}}\right)_p + 3\mathcal{Z}\left(\frac{3}{7}, \mathcal{O}_{-7}, \frac{\pm 3 + \sqrt{-3}}{2\sqrt{-3}}\right)_p.$$

Again, it is easy to see that both terms contribute twice $3 \cdot 1/2$ for $p = 3$ or $p = 5$, adding up to a total multiplicity of 3 for both primes on the right hand side.

As a generalization of the example, we recover the Theorem of Gross and Zagier on singular moduli [GZ85] and its generalization by Dorman [Dor88]. Let D_0 be a negative fundamental discriminant and suppose that D_0 is odd. Moreover, let D_1 be a negative discriminant, coprime to D_0. Consider the modular function

$$\Psi(z) = \prod_{Q \in \mathrm{SL}_2(\mathbb{Z}) \backslash \mathcal{Q}_{D_1}} (j(z) - j(\alpha_Q))^{2/w_{k_1}}.$$

Here, $\mathcal{Q}_{D_1}$ is the set of quadratic forms of discriminant D_1 as in the Introduction and w_{k_1} is the number of roots of unity in $k_1 = \mathbb{Q}(\sqrt{D_1})$

Theorem 4.6.8 (Gross-Zagier, Dorman). *Let H_0 be the Hilbert class field of $k_0 = \mathbb{Q}(\sqrt{D_0})$. Moreover, let L be the fixed field of $\mathrm{Gal}(H_0/k_0)[2]$, let p be a rational prime and $\mathfrak{f} \subset \mathcal{O}_L$ be the prime ideal below $\mathfrak{P}_0 \mid p$ (see Section 4.3). Then we have*

$$\operatorname{ord}_{\mathfrak{f}^\sigma}(\Psi(z_0)) = \frac{w_{k_0}}{4} \sum_{\substack{n \in \mathbb{Z} \\ n \equiv 1 \bmod 2}} 2^{o(n)} \bar{\nu}_p\left(\frac{D_0 D_1 - n^2}{4|D_0|}\right) \rho\left(\frac{D_0 D_1 - n^2}{4p}, [\mathfrak{c}]^{-2}[\mathfrak{c}_0]\right),$$

for $\sigma = \sigma(\mathfrak{c})$. Here, $\bar{\nu}_p(x) = \nu_p(x)$ if $\mathrm{Diff}(x) = \{p\}$ and $\bar{\nu}_p(x) = 0$, otherwise.

To conclude this chapter, we show that there is a rather simple criterion to decide if the divisor of a modular function on $\mathcal{X}_0(N)$ is horizontal if N is square-free. It is reflected in the classical Kronecker congruence of the modular equation (4.5.2).

Recall that the fiber $\mathbf{X}_p$ above p for $p \nmid N$ of $\mathbf{X}_0(N)$ is irreducible and that $\mathbf{X}_p$ has two components $\mathcal{F}_0^{(p)}$ and $\mathcal{F}_\infty^{(p)}$ for $p \mid N$. Each of these components induces a valuation on $k(\mathcal{X}_0(N))$, denoted by ν_p for $p \nmid N$ and $\nu_{p,\infty}$ and $\nu_{p,0}$ for $p \mid N$.

For a modular function $f \in \mathbb{Q}(j, j_N)$, denote by

$$f_\infty(\tau) = \sum_{n \gg -\infty} c_\infty(n) q^n$$

and

$$f_0(\tau) = f_\infty\left(\frac{-1}{\tau}\right) = \sum_{n \gg -\infty} c_0(n) q^{n/N}$$

the Fourier expansions of f at the cusps ∞ and 0, respectively. We have the following explicit variant of the *q-expansion principle*.

Proposition 4.6.9. *Let* $f \in \mathbb{Q}(j, j_N)$ *be a modular function for* $\Gamma_0(N)$*,* p *be a prime and set* $a = \inf\{\mathrm{ord}_p\, c_\infty(n)\}$ *and* $b = \inf\{\mathrm{ord}_p\, c_0(n)\}$*.*

(i) If $p \nmid N$*, then* $a = \nu_p(f)$*.*

(ii) If $p \mid N$ *then* $a = \nu_{p,\infty}(f)$ *and* $b = \nu_{p,0}(f)$*.*

Proof. Evaluation at the Tate curve (cf. [DR73, DI95])

$$\bar{\mathcal{G}}_m^q / q^{\mathbb{Z}}$$

over $\mathbb{Z}[[q]]$ gives a homomorphism

$$\tau_\infty : \mathrm{Spec}(\mathbb{Z}[[q]]) \to \mathcal{X}_0(N)$$

and evaluation at the Tate curve

$$\bar{\mathcal{G}}_m^{q^{1/N}} / q^{\mathbb{Z}}$$

over $\mathbb{Z}[[q^{1/N}]]$ accordingly

$$\tau_0 : \mathrm{Spec}(\mathbb{Z}[[q^{1/N}]]) \to \mathcal{X}_0(N).$$

Combining these maps with the geometric points given by $q \mapsto 0$ for τ_∞ and $q^{1/N} \mapsto 0$ for τ_0, we obtain the cuspidal sections $\mathrm{Spec}(\mathbb{Z}) \to \mathcal{X}_0(N)$ corresponding to the cusps ∞ and 0, respectively. The pullbacks $\tau_\infty^* f$ and $\tau_0^* f$ of $f \in k(\mathcal{X}_0(N))$ are given by the Fourier expansion of f at ∞ and 0, respectively. The valuation at p on $\mathbb{Q}((q))$ and $\mathbb{Q}((q^{1/N}))$ is given by the canonical extension of the p-adic valuation on $\mathbb{Q}$. It agrees with the valuations given by a and b in the statement of the Proposition. If $p \nmid N$ they coincide since the fiber of $\mathcal{X}_0(N)$ above p is irreducible in this case.

We refer to Theorem VI. 3.10 on page 163 and Corollary 3.12 in [DR73] for more details. Moreover, in Section 3.16, ibid., the case $\mathcal{X}_0(p)$ for a prime p is discussed in more detail. □

5 The holomorphic part of $\widetilde{\Theta}_P$

Recall our setup from Section 3.3. An important ingredient for Theorem 3.3.6 was the existence of a harmonic weak Maaß form $\widetilde{\Theta}_P^+(\tau,h)$, such that $\xi(\widetilde{\Theta}_P^+(\tau,h)) = \Theta_P(\tau,h)$ for a two-dimensional positive definite lattice P. This chapter is devoted to finding a description of the coefficients of the holomorphic part of (suitable normalizations) of these harmonic weak Maaß forms in the case that P corresponds to a fractional ideal of an imaginary quadratic field of square-free discriminant.

5.1 Embedding into a modular curve

We start by describing the CM cycles on the modular curve $X_0(N)$. Recall the setup from Section 4.4. We basically use the same setup as in Section 7.1 of [BY09], except that Bruinier and Yang work in signature $(1,2)$.

Recall that for $m \in \mathbb{Q}$ and $\mu \in L'/L$, we use the notation

$$L_{m,\mu} := \Omega_m(\mathbb{Q}) \cap \operatorname{supp}(\phi_\mu) = \{x \in \mu + L \mid Q(x) = m\}.$$

Let $m \in \mathbb{Q}_{<0}$ and $\mu \in L'/L$ such that $m - Q(\mu) \in \mathbb{Z}$ and let $r \in \mathbb{Z}$ with $\mu \equiv \mu_r \bmod L$. Then $D = 4Nm \in \mathbb{Z}$ is a negative discriminant such that $D \equiv r^2 \bmod 4N$. Using this notation, we put

$$\lambda_r = \begin{pmatrix} \frac{r}{2N} & \frac{1}{N} \\ \frac{D-r^2}{4N} & -\frac{r}{2N} \end{pmatrix} \in L_{m,\mu_r}. \tag{5.1.1}$$

The subspace $U = \lambda_r^\perp \subset V(\mathbb{Q})$ is two-dimensional and positive definite.

As in section 7.1 of [BY09], we obtain lattices $\mathcal{P}$ and $\mathcal{N}$ given by

$$\mathcal{P} := L \cap U = \mathbb{Z}\begin{pmatrix} 1 & 0 \\ -r & -1 \end{pmatrix} \oplus \mathbb{Z}\begin{pmatrix} 0 & -\frac{1}{N} \\ \frac{D-r^2}{4N} & 0, \end{pmatrix} \tag{5.1.2}$$

and a negative definite, one-dimensional lattice

$$\mathcal{N} = L \cap \mathbb{Q}\lambda_r = \mathbb{Z}\frac{2N}{t}\lambda_r \text{ with dual } \mathcal{N}' = \mathbb{Z}\frac{t}{D}\lambda_r. \tag{5.1.3}$$

Here, $t = (r, 2N)$.

From now on, assume that $D < 0$ is a *fundamental* discriminant and let $k_D = \mathbb{Q}(\sqrt{D})$. The following lemma is crucial for us.

Lemma 5.1.1. *With the same notation as above, we have an isometry of lattices*

$$(\mathcal{P}, Q) \cong \left(\mathfrak{n}, \frac{\mathrm{N}x}{\mathrm{N}\mathfrak{n}}\right)$$

with

$$\mathfrak{n} = \left(N, \frac{r+\sqrt{D}}{2}\right) \subset k_D.$$

Proof. We have seen above that the lattice $\mathcal{P}$ is generated by

$$p_1 = \begin{pmatrix} 1 & 0 \\ -r & -1 \end{pmatrix} \quad \text{with } Q(p_1) = N,$$

and

$$p_2 = \begin{pmatrix} 0 & -\frac{1}{N} \\ \frac{D-r^2}{4N} & 0 \end{pmatrix} \quad \text{with } Q(p_2) = \frac{r^2 - D}{4N}$$

and bilinear form

$$(p_1, p_2) = r.$$

Therefore, $p_1 \mapsto N$, $p_2 \mapsto \frac{r+\sqrt{D}}{2}$ is an isometry of (P, Q) and $(\mathfrak{n}, \frac{\mathrm{N}x}{\mathrm{N}\mathfrak{n}})$, with

$$\mathfrak{n} = \left(N, \frac{r+\sqrt{D}}{2}\right),$$

as claimed. □

We note that the two points $z_U^{\pm}$ corresponding to λ_r satisfy the quadratic equation

$$\frac{r^2 - D}{4}\tau + r\tau + 1 = 0$$

in terms of coordinates of $\mathbb{H} \cup \bar{\mathbb{H}}$. In terms of the moduli description (see also Sections 4.5 and 4.6), the $\Gamma_0(N)$-orbit of these correspond to

$$\mathbb{C}/\mathcal{O}_D \to \mathbb{C}/\mathfrak{n}^{-1},$$

where $\mathcal{O}_D$ is the order of discriminant D in k_D. This is easy to check: We obtain the quadratic equation from

$$N \operatorname{tr}\left(\begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix} \lambda_r\right) = 0$$

and we see that the point τ satisfying this equation is of the form ω_1/ω_2 for $\omega_1 = 1$ and $\omega_2 = -\frac{r+\sqrt{D}}{2}$. This data exactly corresponds to the point $\mathbb{C}/\mathcal{O}_D \to \mathbb{C}/\mathfrak{n}^{-1} \in \mathcal{X}_0(N)(\mathbb{C})$.

Recall that we write $T = \mathrm{GSpin}_U$ as in Chapter 3 and we consider T as a subgroup of $H = \mathrm{GSpin}_V$, acting trivially on $U^{\perp}$.

Lemma 5.1.2. *We have that* $T = \mathrm{GSpin}_U \cong k_D^\times$ *via*

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sqrt{D} \mapsto \begin{pmatrix} r & -2 \\ \frac{r^2-D}{2} & -r \end{pmatrix}.$$

Proof. This is done in exactly the same way as Lemma 2.6.3. □

Lemma 5.1.3. *We have for* K *as in Section 4.4 that*

$$K_T := K \cap T(\mathbb{A}_f) \cong \hat{\mathcal{O}}_D^\times.$$

Proof. It is clear that $\hat{\mathcal{O}}_D^\times \subset \mathbb{A}_{k,f}^\times$ preserves $\mathfrak{n}$ and that every subgroup of $\mathbb{A}_{k,f}^\times$ preserving $\mathfrak{n}$ is contained in $\hat{\mathcal{O}}_D^\times$. Therefore, the image of K_T is contained in $\hat{\mathcal{O}}_D^\times$. The other inclusion follows directly from Lemma 5.1.2 since the image of $(\mathcal{O}_D \otimes \mathbb{Z}_p)^\times$ under the map given there is obviously contained in K_p. □

Recall the setup from Section 3.3. Note that K_T acts trivially on $\mathcal{P}'/\mathcal{P}$ and thus we can take $K_{\mathcal{P}} = K_T$ and we have $C_{P,K} = T(\mathbb{A}_f)/K_T$, which is isomorphic to $I_k/\hat{\mathcal{O}}_D^\times \cong \mathrm{Cl}_k$ for $k = k_D$ in our case.

Now we specialize the setup for our application. From now on and for the rest of this section we fix a negative fundamental discriminant D and a integral ideal $\mathfrak{a} \subset \mathcal{O}_D$ of the imaginary quadratic field $k = k_D$ of discriminant D with ring of integers $\mathcal{O}_D$. We assume throughout that D is odd. Consider the positive definite lattice $(P, Q) \cong \left(\mathfrak{a}, \frac{\mathrm{N}x}{\mathrm{N}\mathfrak{a}}\right)$ and write $\mathfrak{a} = \left(A, \frac{B+\sqrt{D}}{2}\right)$ with $A, B \in \mathbb{Z}$, $A > 0$. That is, the ideal $\mathfrak{a}$ is generated by A and $\frac{B+\sqrt{D}}{2}$. This is equivalent to saying that P (or $\mathfrak{a}$) corresponds to the positive definite integral binary quadratic form $[A, B, C]$ of discriminant $D = B^2 - 4AC$, with $C \in \mathbb{Z}$ determined by A, B and D. We will use the construction above to embed the lattice P into the lattice L for $N = A\,|D|$.

Assumption 5.1.4. Without loss of generality, we will assume that $(A, D) = 1$. If A is not coprime to D, we can replace $[A, B, C]$ with an equivalent form. The associated discriminant forms are isomorphic. Therefore, we can identify the corresponding theta functions, as well. An integral binary quadratic form of discriminant D represents infinitely many primes (cf. Theorem 9.12 of [Cox89]). Thus, we may in fact choose A to be prime, represented by $[A, B, C]$, not dividing the discriminant D.

Under these assumptions, there are $E, F \in \mathbb{Z}$ with $2AE+BF = 1$. (Note that $(A, D) = 1$ implies $(2A, B) = 1$ because D is odd.) Using this, we have for $R := FD$ that

$$R^2 \equiv D \bmod 4A\,|D|. \tag{5.1.4}$$

Indeed, we have $R^2 \equiv 0 \bmod D$ and $F^2D^2 = F^2D \cdot D = F^2(B^2 - 4AC)D \equiv D \bmod 4A$.

Warning. Note that the definition of R depends of course on the ideal $\mathfrak{a}$ we started with. The notation does not reflect this dependence in order to not become too clumsy. We hope this does not cause any confusion.

With this setup, we put $M := \frac{-1}{4A} = \frac{D}{4A|D|}$ and let λ_R as in (5.1.4). Note that we obtain in this special case

$$\mathcal{N} = L \cap \mathbb{Q}\lambda_R = \mathbb{Z}2A\lambda_R \text{ with dual } \mathcal{N}^{'} = \mathbb{Z}\lambda_R. \tag{5.1.5}$$

In contrast to the general case, in our situation the lattice L splits as $L = \mathcal{P} \oplus \mathcal{N}$. This follows from the fact that the discriminant group $\mathcal{N}'/\mathcal{N}$ is isomorphic to $\mathbb{Z}/2A\mathbb{Z}$ and the following lemma.

Lemma 5.1.5. *With the same notation as above, we have an isometry of lattices*

$$(\mathcal{P}, Q) \cong \left(\mathfrak{b}, \frac{\mathrm{N}x}{\mathrm{N}\mathfrak{b}}\right)$$

with

$$\mathfrak{b} = \left(A\left|D\right|, \frac{R+\sqrt{D}}{2}\right) = \mathfrak{d}_k\mathfrak{a}.$$

In particular $[\mathfrak{b}] = [\mathfrak{a}] \in \mathrm{Cl}(D)$.

Proof. The first part is contained in Lemma 5.1.1. It is only left to show that $\mathfrak{b} = \mathfrak{d}_k\mathfrak{a}$. Clearly, $A\sqrt{D} \in \mathfrak{a}$ and therefore the first generator $A\left|D\right|$ of $\mathfrak{b}$ is contained in $\mathfrak{d}_k\mathfrak{a}$. Moreover, since $2AE + BF = 1$, we have that

$$\sqrt{D}\left(EA + F\frac{B+\sqrt{D}}{2}\right) = \sqrt{D}\frac{2EA + FB + F\sqrt{D}}{2} = \frac{\sqrt{D} + FD}{2} = \frac{R+\sqrt{D}}{2}$$

is contained in $\mathfrak{d}_k\mathfrak{a}$. Therefore, $\mathfrak{b} \subset \mathfrak{d}_k\mathfrak{a}$. For the other inclusion, we easily see that

$$A\sqrt{D} = 2A\frac{R+\sqrt{D}}{2} + A\left|D\right|F$$

is contained in $\mathfrak{b}$. Moreover, the other generator of $\mathfrak{d}_k\mathfrak{a}$

$$\sqrt{D}\frac{B+\sqrt{D}}{2}$$

is also contained in $\mathfrak{b}$ since we can write

$$\sqrt{D}\frac{B+\sqrt{D}}{2} = \frac{B\sqrt{D}+D}{2} = \frac{(2AE+BF)D + B\sqrt{D}}{2}$$

and this is equal to

$$-A\left|D\right|E + B\frac{R+\sqrt{D}}{2},$$

which is clearly contained in $\mathfrak{b}$. This shows the other inclusion and finishes the proof. □

In particular, $\mathcal{P}$ and P have isomorphic discriminants groups. This also means that their theta functions can be identified.

We will now see that the special divisor $Z(M, \mu_R)$ (Definition 1.3.5) is given by the CM cycle $Z(U)$.

Proposition 5.1.6. *With the same notation as above we have*

$$Z(U) = Z(M, \mu_R).$$

Remark 5.1.7. For D coprime to N, this was stated as Proposition 7.2 in [BY09]. We need to verify that it also holds in our situation.

Proof. We will show

$$\Omega_M(\mathbb{A}_f) \cap \operatorname{supp}(\phi_{\mu_R}) = K\lambda_R$$

by proving that $\Omega_M(\mathbb{Q}_p) \cap \operatorname{supp}(\phi_{\mu_R}) = K_p\lambda_R$ for every prime p. This shows the equality since then Definition 1.3.5 is equivalent to

$$Z(M, \mu_R) = Z(\lambda_R, 1) = Z(U).$$

(Note that the definition (3.0.2) of the cycle $Z(U)$ is exactly the same as $Z(\lambda_R, 1)$ in (1.3.4).) Let $\gamma \in \Omega_M(\mathbb{Q}_p) \cap \operatorname{supp}(\phi_{\mu_R})$ be primitive and write

$$\gamma = \begin{pmatrix} \frac{b}{2N} & -\frac{a}{N} \\ c & -\frac{b}{2N} \end{pmatrix}$$

with $a, b, c \in \mathbb{Z}_p$ such that $b^2 - 4Nac = D$.

1. Let us first suppose that p divides $N = A\,|D|$. Then $b \equiv 0 \bmod D$ since μ_R corresponds to FD in $\mathbb{Z}/2A\,|D|\,\mathbb{Z}$ as above. There are two cases to consider.
    1.1. First assume that p divides D. Then $(p, A) = 1$ by our assumption on A. However, since $b^2/D + 4Aac = 1$ but p still divides b^2/D, we must have that $(p, a) = (p, c) = 1$.
    1.2. In the other case, when p divides A, we must have that $(p, b) = 1$ since $(D, A) = 1$. If p is not coprime to a, we replace γ by

    $$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} .\gamma = \begin{pmatrix} \frac{b+2Nc}{2N} & -\frac{b+a+Nc}{N} \\ c & -\frac{b+2NC}{2N} \end{pmatrix}.$$

    In particular, $b + a + Nc$ is now coprime to p.

    Therefore, we may assume that p is coprime to a. The matrix

    $$\begin{pmatrix} 1 & 0 \\ 0 & -a \end{pmatrix}$$

is thus contained in K_p and we have

$$\begin{pmatrix}1 & 0\\ 0 & -a\end{pmatrix}.\gamma = -\frac{1}{a}\begin{pmatrix}1 & 0\\ 0 & -a\end{pmatrix}\gamma\begin{pmatrix}-a & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\frac{b}{2N} & 1/N\\ -ac & -\frac{b}{2N}\end{pmatrix}.$$

We may therefore assume that $a = -1$. We have that

$$\gamma_R = \begin{pmatrix}1 & 0\\ \frac{b-R}{2} & 1\end{pmatrix}$$

is contained in K_p since $b - R \equiv 0 \bmod 2N$ and it has determinant 1. Moreover, we calculate

$$\begin{aligned}\begin{pmatrix}1 & 0\\ \frac{b-R}{2} & 1\end{pmatrix}.\gamma &= \begin{pmatrix}1 & 0\\ \frac{b-R}{2} & 1\end{pmatrix}\begin{pmatrix}\frac{b}{2N} & \frac{1}{N}\\ c & -\frac{b}{2N}\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{R-b}{2} & 1\end{pmatrix}\\ &= \begin{pmatrix}\frac{b}{2N} & \frac{1}{N}\\ \frac{b(b-R)+4Nc}{4N} & \frac{(b-R)-b}{2N}\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{R-b}{2} & 1\end{pmatrix}\\ &= \begin{pmatrix}\frac{R}{2N} & \frac{1}{N}\\ \frac{-R^2+b^2+4Nc}{4N} & -\frac{R}{2N}\end{pmatrix}\\ &= \lambda_R.\end{aligned}$$

In the last step we have used that $b^2 + 4Nc = D$ since $a = -1$ in this case.

2. Finally, if p is coprime to N, we have that D is coprime to p and therefore one of a, b, c is coprime to p. If $(p, a) = 1$, we can directly proceed as above. If $(p, b) = 1$, we can first apply step 1.2. and are reduced to the case $(p, a) = 1$. Finally, if $(p, c) = 1$ we can switch a and c using the matrix

$$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \in K_p,$$

since p is coprime to N. □

Remark 5.1.8. We note that $Z(M, \mu_R)$ is equal to the usual Heegner divisor $P_{D,R} + P_{D,-R}$ as in [GKZ87]. This can be easily checked using Lemma 1.3.6.

It is important for us to understand the action of $T(\mathbb{A}_f) \cong \mathbb{A}_{k,f}^\times$ on modular functions precisely. It is given by the following Lemma.

Lemma 5.1.9. *Let $h \in \mathbb{A}_{k,f}^\times$ and let $f \in \mathbb{Q}(X_0(N))$ be a rational modular function. Let $t \in T(\mathbb{A}_f)$ be the image of h and write $t = \gamma k$ for $\gamma \in H(\mathbb{Q})$ and $k \in K$. Let $z \in Z(M, \mu_R)$ be a CM point. Then we have*

$$f(z)^{\sigma(h)} = f(\gamma^{-1} z).$$

Proof. This is the first item of Theorem 6.31 of [Shi94]. There, it is shown that for any modular function f (as in [Shi94]), we have $f^{\sigma(h)}(z) = f^{\tau(t^{-1})}(z)$. The map τ is defined on page 149, ibid., and it acts on $j(z)$ as $j^{\tau(x)} = j \circ \alpha$, if $x = u\alpha$ for $u \in K$ and $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. This implies if $t = \gamma k$, then $t^{-1} = k^{-1}\gamma^{-1}$ and therefore

$$j^{\sigma(h)}(z) = j^{\tau(t^{-1})}(z) = j(\gamma^{-1} z). \qquad \square$$

5.2 A seesaw identity

The motivation for this and the next section is the following. Let us write the Fourier expansion of $\widetilde{\Theta}_P^+(\tau, h)$ as

$$\widetilde{\Theta}_P^+(\tau, h) = \sum_{\beta \in P'/P} \sum_{m \gg \infty} c_P^+(h, m, \beta) e(m\tau) \phi_\beta.$$

Suppose that there is a weakly holomorphic modular form $f \in M^!_{1,\mathcal{P}}$, with principal part

$$P_f = q^{-m}(\phi_\beta + \phi_{-\beta})$$

for $m > 0$. (Note that such a form often does *not* exist but we will deal with this problem in the next section.) On the one hand we obtain by Theorem 3.3.6 that

$$\begin{aligned}\Phi_P(h, f) &= \int_{\Gamma\backslash\mathbb{H}} \langle f(\tau), \overline{\Theta_P(\tau, h)} \rangle v d\mu(\tau) \\ &= \mathrm{CT}\langle f(\tau), \widetilde{\Theta}_P^+(\tau, h)\rangle = c_P^+(h, m, \beta) + c_P^+(h, m, -\beta) + \text{“error term”},\end{aligned}$$

where the “error term” is a contribution of the pairing of the principal part of $\widetilde{\Theta}_P$ with the coefficients of positive index of f. Let us ignore this term for the moment.

On the other hand, there is a different expression for the theta lift $\Phi_P(h, f)$ in terms of a CM value of the theta lift for the lattice L. The basic principle goes back to Kudla [Kud84] who realized that many previously mysterious identities between theta lifts can be understood in the context of “seesaw dual reductive pairs”. In our situation, the seesaw diagram is

$$\begin{tikzcd} \mathrm{SL}_2 \times \mathrm{SL}_2 \arrow[d, dash] \arrow[dr, dash] & \mathrm{O}(2,1) \arrow[d, dash] \arrow[dl, dash] \\ \mathrm{SL}_2 & \mathrm{O}(2) \times \mathrm{O}(1). \end{tikzcd}$$

Here, the vertical lines are given by diagonal restriction/embedding maps and the diagonal lines are the corresponding theta lifts. We will apply this as follows. We consider the theta lift from SL_2 to $\mathrm{O}(2,1)$ and then restrict to an embedded $\mathrm{O}(2) \times \mathrm{O}(1)$, where we just view the resulting function as a function on $\mathrm{O}(2)$. The diagram shows that this is the same as the lift from $\mathrm{SL}_2 \times \mathrm{SL}_2$ to $\mathrm{O}(2)$ (after embedding diagonally).

Explicitly, this works as follows in our situation. We will use the embedding defined above to obtain an expression for the regularized theta lift in the case of signature $(2, 0)$ as a CM value of a modular function on $X_0(N)$ for $N = A\,|D|$. For D a prime discriminant, a similar setup is used in [Via12].

Consider the theta lift Φ_L corresponding to L defined as

$$\Phi_L(z,h,g) = \int_{\Gamma\backslash\mathbb{H}}^{\mathrm{reg}} \langle g, \overline{\Theta_L(\tau,z)}\rangle v^{\frac{1}{2}} d\mu(\tau),$$

for $(z,h) \in X_K$. Since the lattice L splits as $L = \mathcal{P} \oplus \mathcal{N}$, the Weil representation ρ_L is isomorphic to the tensor product $\rho_{\mathcal{P}} \otimes \rho_{\mathcal{N}}$. In particular, we have that $M^!_{1,\mathcal{P}} \otimes M^!_{-\frac{1}{2},\mathcal{N}}$ is a subspace of $M^!_{\frac{1}{2},L}$. At a point $(z_U^{\pm}, h) \in Z(U)$ corresponding to the splitting $\mathcal{P} \oplus \mathcal{N}$ we obtain for an element $f \otimes \varphi$ in $M^!_{1,\mathcal{P}} \otimes M^!_{-\frac{1}{2},\mathcal{N}}$ that

$$\begin{aligned}\Phi_L((z_P,h), f\otimes\varphi) &= \int_{\Gamma\backslash\mathbb{H}}^{\mathrm{reg}} \langle (f\otimes\varphi)(\tau), (\overline{\Theta_{\mathcal{P}} \otimes \Theta_{\mathcal{N}}})(\tau,h)\rangle v^{\frac{1}{2}} d\mu(\tau)\\ &= \int_{\Gamma\backslash\mathbb{H}}^{\mathrm{reg}} \langle f(\tau), \overline{\Theta_{\mathcal{P}}(\tau,h)}\rangle \langle \varphi(\tau), \Theta_{\mathcal{N}^-}(\tau)\rangle v d\mu(\tau).\end{aligned}$$

Now we choose a specific function φ. Since $\mathcal{N}'/\mathcal{N} = \mathbb{Z}/2A\mathbb{Z}$ with quadratic form $-x^2/4A$, the space $M^!_{k,\mathcal{N}}$ is isomorphic to the space of weakly holomorphic Jacobi forms $J^!_{k+\frac{1}{2},A}$ of weight $k+\frac{1}{2}$ and index A. See also Section 1.5.4. In particular, $M^!_{-\frac{1}{2},\mathcal{N}}$ is isomorphic to $J^!_{0,A}$. It follows from Proposition 1.5.17, that $J^!_{0,A}$ is generated as a $\mathbb{C}$-vector space by elements of the form

$$\sum_{j=0}^{A} \psi_j F^j G^{A-j}, \tag{5.2.1}$$

where $\psi_j \in M^!_{2j}(\mathrm{SL}_2(\mathbb{Z}))$ and F, G are the generators of the ring of weak Jacobi forms of even weight as in Proposition 1.5.17. We will frequently identify these forms as vector valued modular forms in $M^!_{-5/2,\mathcal{N}}$ and $M^!_{-1/2,\mathcal{N}}$ via the theta development of Jacobi forms (see Section 1.5.4). Under this identification, the following relation is easy to obtain.

Lemma 5.2.1. *We have* $\left\langle F^j(\tau)G^{A-j}(\tau), \Theta_{\mathcal{N}^-}(\tau)\right\rangle = 12^A \delta_{0,j}$.

Proof. It follows from Lemma 1.5.16 that

$$\left\langle F^j(\tau)G^{A-j}(\tau), \Theta_{\mathcal{N}^{-1}}(\tau)\right\rangle = (F^jG^{A-j})(\tau,0).$$

This is of course equal to

$$(F^jG^{A-j})(\tau,0) = F^j(\tau,0)G^{A-j}(\tau,0) = F(\tau,0)^jG(\tau,0)^{A-j}.$$

The Fourier expansion of F and G start with

$$F(\tau,0) = 0 + O(q) \quad \text{and} \quad G(\tau,0) = 12 + O(q).$$

The functions $F(\tau,0)$ and $G(\tau,0)$ are holomorphic modular forms for $\mathrm{SL}_2(\mathbb{Z})$ of weights -2 and 0, respectively by the transformation properties of Jacobi forms and the growth condition for weak Jacobi forms (see Section 1.5.4 and Theorem 1.3 of [EZ85]). Thus, we have $F(\tau,0)=0$ and $G(\tau,0)=12$ and the lemma follows. □

Using the lemma, we obtain that

$$\langle \psi_j F^j G^{A-j}(\tau), \Theta_{\mathcal{N}^-}(\tau)\rangle = 12^A \psi_0(\tau). \tag{5.2.2}$$

Consequently,

$$\begin{aligned}\Phi_L((z_U,h), f\otimes \sum_{j=0}^{A} \psi_j F^j G^{A-j}) &= 12^A \int_{\Gamma\backslash\mathbb{H}}^{\mathrm{reg}} \psi_0(\tau)\langle f(\tau), \overline{\Theta_P(\tau,h)}\rangle v d\mu(\tau) \\ &= 12^A \Phi_P(h, f\cdot\psi_0).\end{aligned} \tag{5.2.3}$$

5.2.1 A special basis for $S_{1,\mathcal{P}^-}$

In this subsection we consider a special basis of the space of cusp forms $S_{1,\mathcal{P}^-}$. We will use this basis to define a set of weakly holomorphic modular forms in $M^!_{1,\mathcal{P}}$ that will be convenient to use in combination with the seesaw identity described in the last section. We need to care about the space $S_{1,\mathcal{P}^-}$ because, according to the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & M^!_{1,\mathcal{P}} \arrow[r] & H_{1,\mathcal{P}} \arrow[r, "\xi_1"] & S_{1,\mathcal{P}^-} \arrow[r] & 0, \end{tikzcd}$$

this is the space of obstructions for the existence of a weakly holomorphic modular form with prescribed principal part.

The following lemma gives a basis for $S_{k,L}$ that has a “simple” structure, similar to a q-expansion basis starting with increasing powers of q for scalar valued forms.

Lemma 5.2.2. *Let L be an even lattice. Then there is a basis $\{g_1,\dots,g_d\}$ of $S_{k,L}(\mathbb{Q})$, where $\dim S_{k,L}=d$ with the following property. There are rational numbers $n_1\le\dots\le n_d$ and elements $\mu_1,\dots,\mu_d\in P'/P$, such that we have for their Fourier coefficients*

$$c_{g_j}(n_l,\mu_l)=\delta_{j,l}.$$

Proof. Let $\{\tilde g_1,\dots,\tilde g_d\}$ be an integral basis of $S_{k,L}(\mathbb{Q})$. We construct the new basis by induction. More precisely, for every $r\le d$, we construct a set

$$M_r=\{(n_j,\mu_j) \mid j\in\{1,\dots,r\}\}$$

and a basis

$$B_r=\{g_1^{(r)},\dots,g_d^{(r)}\}$$

of $S_{k,L}$, such that B_r satisfies the conditions of the Lemma with respect to M_r. That is,

(i) $n_1 \leq \ldots \leq n_r$, and

(ii) $c_j^{(r)}(n_l, \mu_l) = \delta_{j,l}$ for all $(n_j, \mu_j) \in M_r$.

Here and throughout the proof, we write

$$g_j^{(r)}(\tau) = \sum_{\mu} \sum_{m} c_j^{(r)}(m, \mu) e(m\tau) \phi_\mu.$$

We start by letting n_1 be the minimal index such that one of the forms in the basis has a non-vanishing Fourier coefficient of index (m, μ) for some $\mu \neq 0$, that is

$$n_1 = \min\{m \in \mathbb{Q}_{>0} \mid \exists j, \mu : \; c_{\tilde{g}_j}(m, \mu) \neq 0\}.$$

By reordering, we can assume that $c_{\tilde{g}_1}(n_1, \mu_1) \neq 0$ for some μ_1. We fix the pair (n_1, μ_1), put $M_1 = \{(n_1, \mu_1)\}$ and set

$$g_1^{(1)} = \frac{1}{c_{\tilde{g}_1}(n_1, \mu_1)} \tilde{g}_1.$$

Then we let

$$g_j^{(1)} = \tilde{g}_j - c_{\tilde{g}_j}(n_1, \mu_1) g_1^{(1)}$$

for $j \in \{2, \ldots, d\}$. Clearly, $c_1^{(1)}(n_1, \mu_1) = 1$ and $B_1 = \{g_1^{(1)}, g_2^{(1)}, \ldots, g_d^{(1)}\}$ is a basis of $S_{k,L}$ satisfying the condition with respect to M_1.

Now suppose we already constructed M_r and B_r as above. Now we construct B_{r+1} and M_{r+1}. Define

$$n_{r+1} := \min\{m \in \mathbb{Q}_{>0} \mid \exists j \in \{r+1, \ldots, d\} \; \exists \mu : \; c_j^{(r)}(m, \mu) \neq 0\}.$$

By reordering, we can assume that we have $c_{r+1}^{(r)}(n_{r+1}, \mu_{r+1}) \neq 0$ for some $\mu_{r+1} \neq 0$. We fix the pair (n_{r+1}, μ_{r+1}), set

$$M_{r+1} = M_r \cup \{(n_{r+1}, \mu_{r+1})\}$$

and normalize

$$g_{r+1}^{(r+1)} = \frac{1}{c_{r+1}^{(r)}(n_{r+1}, \mu_{r+1})} g_{r+1}^{(r)}.$$

Moreover, we replace $g_j^{(r)}$ by

$$g_j^{(r+1)} = g_j^{(r)} - c_j^{(r)}(n_{r+1}, \mu_{r+1}) g_{r+1}^{(r+1)}.$$

The lemma follows then for $r = d$. □

Definition 5.2.3. From now on, we fix $n_1 \leq \ldots \leq n_d \in \mathbb{Q}$, $\beta_1, \ldots, \beta_d \in P'/P$ and $\mathcal{B} = \{g_1, \ldots, g_d\}$ as in Lemma 5.2.2, such that $\mathcal{B}$ is a basis for the space $S_{1,\mathcal{P}^-}(\mathbb{Q})$. We write

$$g_j(\tau) = \sum_{\beta \in P'/P} \sum_{m \in \mathbb{Q}_{>0}} a_j(m, \beta) e(m\tau)$$

for the Fourier expansion of g_j.

Proposition 5.2.4. *For $m \in \mathbb{Q}_{>0}$ and $\beta \in P'/P$ with $m + Q(\beta) \in \mathbb{Z}$ and $(m, \beta) \neq (n_j, \beta_j)$ for all j, there is a weakly holomorphic modular form $f_{m,\beta} \in M^!_{1,\mathcal{P}}$ with principal part*

$$q^{-m}(\phi_\beta + \phi_{-\beta}) - \sum_{j=1}^{d} a_j(m, \beta) q^{-n_j}(\phi_{\beta_j} + \phi_{-\beta_j}), \tag{5.2.4}$$

constant term $c_{f_{m,\beta}}(0,0) = 0$, and only rational Fourier coefficients.

Proof. The existence of a weakly holomorphic modular form $\tilde{f}_{m,\beta} \in M^!_{1,\mathcal{P}}$ with this principal part follows from the construction of our basis and the exact sequence (1.5.6).

The rationality of the Fourier coefficients can be easily achieved using Theorem 1.5.18. Just multiply any such form $\tilde{f}_{m,\beta}$ with $\Delta(\tau)^{m_0}$, where $m_0 = \max\{m, n_1, \dots, n_d\}$, to obtain an element of $M_{1+12m_0,\mathcal{P}}$. Since this space has a basis of forms with integral Fourier coefficients, there is an element $g \in M_{1+12m_0,\mathcal{P}}(\mathbb{Q})$ with only rational Fourier coefficients, such that $\tilde{f}_{m,\beta}\Delta^{m_0} - g = O(q^{m_0})$. Consequently, we have $\tilde{g} = g/\Delta^{m_0} \in M^!_{1,\mathcal{P}}$ and $\tilde{g}$ has the correct principal part and only rational Fourier coefficients.

Finally, we can define

$$f_{m,\beta} = \tilde{g} - c_{\tilde{g}}(0,0)E_P,$$

to complete the proof. □

Remark 5.2.5. Note that $f_{m,\beta}$ is not unique. First of all, the basis of cusp forms is not unique. But even given a fixed basis of $S_{1,\mathcal{P}^-}$ as above, $f_{m,\beta}$ is only unique up to addition of cusp forms. In fact, we will use this freedom in the proof of Theorem 5.3.12.

Definition 5.2.6. By Lemma 1.5.19 $f_{m,\beta}$ has Fourier coefficients with bounded denominators. For each $\beta \in P'/P$ and $m \in \mathbb{Q}_{>0}$ with $m + Q(\beta) \in \mathbb{Z}$ we let $c_{m,\beta} \in \mathbb{Z}$, such that $c_{m,\beta} f_{m,\beta}$ has only integral Fourier coefficients.

We have that

$$12^A c_{m,\beta} \int_{\Gamma\backslash\mathbb{H}}^{\mathrm{reg}} \langle f_{m,\beta}(\tau), \overline{\Theta_P(\tau, h)} \rangle v \, d\mu(\tau) = \Phi_L((z_U, h), c_{m,\beta} f_{m,\beta} \otimes G^A). \tag{5.2.5}$$

By the theorem of Borcherds (Theorem 1.8.1), the value on the right hand side is essentially the logarithm of a special value of a rational function on $Y_0(N)$ (or rather on its compactification $X_0(N)$) which is defined over $\mathbb{Q}$, as long as the coefficients of the input function (here $c_{m,\beta} f_{m,\beta}$) are all integers. Moreover,

$$\int_{\Gamma\backslash\mathbb{H}}^{\mathrm{reg}} \langle f_{m,\beta}(\tau), \overline{\Theta_P(\tau, h)} \rangle v \, d\mu(\tau) = -\frac{2}{12^A \, c_{m,\beta}} \log \left| \Psi_L((z_U, h), c_{m,\beta} f_{m,\beta} \otimes G^A) \right|^2. \tag{5.2.6}$$

By CM theory (Lemma 5.3.3), an integral power of the value $\Psi_L((z_U, h), c_{m,\beta} f_{m,\beta} \otimes G^A)$ on the right is contained in the Hilbert class field H of k. On the left hand side, we essentially obtain the coefficient $c_P^+(h, m, \beta)$ we are interested in and some “error terms”.

5.3 The arithmetic pullback and the coefficients of the holomorphic part

In this section we will determine the prime ideal factorization of the algebraic number $\Psi_L((z_U, h), c_{m,\beta} f_{m,\beta} \otimes G^A)$ using the results of Chapter 4.

Our basic setup is the following. Given $P \cong \mathfrak{a}$, we let $\mathfrak{b} = \mathfrak{d}_k \mathfrak{a}$ as in Lemma 5.1.5 and put $N = A\,|D|$, where (P, Q) corresponds to the integral binary quadratic form $[A, B, C]$. As before, we assume that $(A, D) = 1$. Moreover, we let M, R be as in Section 5.1.

Definition 5.3.1. For a harmonic weak Maaß form $f \in H_{\frac{1}{2},L}(\mathbb{Z})$, we write

$$\mathcal{Z}(f) = \sum_{\mu \in L'/L} \sum_{m<0} c_f^+(m, \mu) \mathcal{Z}(m, \mu)$$

for the divisor associated with f on $\mathcal{Y}_0(N)$.

For $f \in H_{\frac{1}{2},L}(\mathbb{Z})$ the pair $\widehat{\mathcal{Z}}^c(f) = (\mathcal{Z}^c(f), \Phi_L(\cdot, g))$ defines an arithmetic divisor on $\mathcal{X}_0(N)$. Here, $\mathcal{Z}^c(f) = \mathcal{Z}(f) + C(f)$ is a suitable extension of $\mathcal{Z}(f)$ to $\mathcal{X}_0(N)$ where $C(f)$ is supported at the cusps.

Lemma 5.3.2. *Let $f \in M^!_{1/2,L}(\mathbb{Z})$ with constant coefficient $c_f(0, 0) = 0$ and $c_f(m, \mu) \in \mathbb{Q}$ for all $m \in \mathbb{Q}$ and $\mu \in L'/L$. Then there exists an integer M_f, such that the Borcherds product $\Psi_L(z, h, M_f \cdot f)$ defines a meromorphic modular function contained in $\mathbb{Q}(j, j_N)$.*

Proof. Since the Fourier coefficients of f have bounded denominators by Lemma 1.5.19, replacing f by an integral multiple $f' = M \cdot f$ we obtain only integral coefficients. We view $\Psi_L(z, h, f')$ as a meromorphic function on $X_0(N)$ and simply write $\Psi(z, f')$ for this function. Recall that $\Psi(z, f')$ has an infinite product expansion of the form

$$\Psi(z, f') = e((\rho_{f'}, z)) \prod_{n=1}^{\infty} (1 - e(nz))^{c_{f'}(n^2/4N, n)},$$

which converges for $\Im(z)$ large enough and where $\rho_{f'}$ is the corresponding Weyl vector at the cusp ∞. We refer to Borcherds [Bor98, Theorem 13.3] and Bruinier and Ono [BO10, Theorem 6.1] for details.

Moreover, the multiplier system of $\Psi(z, f')$ has finite order, which can be shown using the embedding trick ([Bor98, Lemma 8.1], [Bor00]). Together with the integrality of the $c_{f'}(m, \mu)$ this implies that the Fourier expansions of $\Psi(z, f')$ have rational coefficients at all cusps (for square-free N). Thus, $\Psi(z, M'f)$ for some $M' \in \mathbb{Z}$ is contained in the field $\mathbb{Q}(X_0(N)) = \mathbb{Q}(j, j_N)$ by the q-expansion principle. $\square$

The lemma implies that the arithmetic divisor $(\operatorname{div}(\Psi_L(\cdot, M_f f)), -\log|\Psi(\cdot, M_f f)|^2)$ associated with the Borcherds lift of f is principal.

Lemma 5.3.3. *The CM value $\Psi_L(z_U^\pm, h, M_f \cdot f)$ is contained in the Hilbert class field H of k for every $(z_U^\pm, h) \in Z(U)$ and we have*

$$\Psi_L(z_U^\pm, h, M_f \cdot f) = \Psi_L^{\sigma(h)}(z_U^\pm, 1, M_f \cdot f).$$

Proof. If we write $h = \gamma k \in H(\mathbb{Q})K$, then we have according to Lemma 5.1.9 that

$$\Psi_L(z_U^\pm, h, f) = \Psi_L(\gamma^{-1} z_U^\pm, 1, f) = \Psi_L^{\sigma(h)}(z_U^\pm, 1, f). \qquad \square$$

Recall that we studied the morphism

$$\mathbf{j}_D : C_D \to \mathcal{X}_0(N)$$

in Section 4.6. As a starting point, we will show that $\operatorname{div}(\Psi_L(z, h, f))$ is a horizontal divisor.

Theorem 5.3.4. *Let $f \in M^!_{1/2,L}(\mathbb{Z})$ be a weakly holomorphic modular form with only integral Fourier coefficients and assume that N is square-free. Suppose that the multiplier system of $\Psi_L(z, h, f)$ is trivial. Then the divisor of the rational function defined by $\Psi_L(z, h, f)$ on $\mathcal{Y}_0(N)$ is equal to $\mathcal{Z}(f)$.*

Remark 5.3.5. Note that in our setup $N = A\,|D|$ is always square-free. In the statement above we just wanted to emphasize that this holds in more general situations for all square-free N.

Proof. The triviality of the multiplier system then implies that $\Psi(z) := \Psi_L(z, h, f) \in \mathbb{Q}(j, j_N)$, as we have seen. The product expansion implies that the Fourier expansion of $\Psi(z, f)$ at the cusp ∞ has coprime integral coefficients under our assumption on g.

It is enough to show that $\operatorname{div}(\Psi(z))$ does not contain any vertical components. This can be checked on the coarse moduli scheme $\mathbf{X}_0(N)$ of $\mathcal{X}_0(N)$. In characteristic p, where p is coprime to N, we have that $\mathbf{X}_0(N)_p$ is irreducible. Therefore, the fibers for $p \nmid N$ cannot occur in the divisor.

For $p \mid N$, the scheme $\mathbf{X}_0(N)_p$ has two connected components intersecting in each supersingular point. The Fricke involution W_N is contained in $\operatorname{GSpin}_V(\mathbb{Q})$ and its image belongs to $\operatorname{SO}^+(L)$ (cf. [BO10]). We denote by σ_N the image of W_N in $\operatorname{O}(L'/L)$. It acts on L'/L as $\mu \mapsto -\mu$ and therefore,

$$-4\log|(\Psi \mid W_N)(z)| = \Phi\left(\frac{-1}{Nz}, f\right) = \Phi(z, f_N) = -4\log|\Psi(z, f_N)|,$$

where $f_N = f^{\sigma_N}$.

We have that $c_f(m, \mu) = c_f(m, -\mu)$ by the action of the center of $\operatorname{Mp}_2(\mathbb{Z})$. Therefore, $f_N = f$ and $\Psi \mid W_N(z) = \pm\Psi(z)$ because $W_N^2 = 1$. This implies that the Fourier expansion of $\Psi(z)$ at the cusp 0 has also coprime integral Fourier coefficients. Thus, none of the two connected components of $\mathbf{X}_0(N)_p$ contribute to the divisor of $\Psi(z)$ by Proposition 4.6.9. □

Remark 5.3.6. We note that the statement of the theorem is not obvious in general (that is, for instance for Borcherds products on other Shimura varieties of orthogonal or unitary type). It also depends on the choice of an integral model.

The pullback $\mathbf{j}_D^*\mathcal{Z}(g) = \mathbf{j}_D^* \operatorname{div}(\Psi(z,f))$ can be described in terms of the 0-cycles on C_D that we studied in Section 4.3. In Section 4.6, we have seen how this works in principle. We will use the methods developed there in our special situation to calculate the pullback and thus obtain a formula for the $\mathfrak{P}$-valuations of the value of $\Psi(z,f)$ at a CM point.

Recall the definition of the cycles $\mathcal{Z}(m,\mathfrak{b},\beta)$ from Section 4.6. Note that

$$\mathfrak{d}_k^{-1}\mathfrak{b}/\mathfrak{b} = \mathfrak{a}/\mathfrak{d}_k\mathfrak{a} \cong \mathfrak{d}_k^{-1}\mathfrak{a}/\mathfrak{a}$$

and fix the isomorphism to be given by $x \mapsto x/\sqrt{D}$. Under this isomorphism, we may then identify $\mathcal{Z}(m,\mathfrak{b},\beta)$ with $\mathcal{Z}(m,\mathfrak{a},\beta/\sqrt{D})$. We will use this identification freely without mentioning it.

For $f \in M^!_{1,\mathcal{P}}$, we write

$$\mathcal{Z}(\mathfrak{a},f) = \sum_{\beta\in P'/P}\sum_{m>0} c_f(-m,\beta)\mathcal{Z}(m,\mathfrak{a},\beta).$$

The following lemma is the arithmetic counterpart of the analytic seesaw identity from Section 5.2.

Lemma 5.3.7. *Let $f \in M^!_{1,\mathcal{P}}(\mathbb{Z})$ and $g \in M_{-\frac{1}{2},\mathcal{N}}(\mathbb{Z})$ with*

$$g = \sum_{j=0}^{A} \psi_j F^j G^{A-j},$$

where $\psi_j \in M^!_{2j}(\mathrm{SL}_2(\mathbb{Z}))$ for every j. We have that

$$\mathbf{j}_D^*\mathcal{Z}(f\otimes g) = 12^A\mathcal{Z}(\mathfrak{a},\psi_0 f) \in Z^1(\mathcal{O}_H).$$

For the convenience of the reader, we restate Lemma 4.6.3 in our special situation.

Lemma 5.3.8. *Let $m_1 \in \mathbb{Q}_{<0}$ such that $D_1 = 4Nm_1 = 4\,|D|\,Am_1 < 0$ is a discriminant and let $\mu_1 \in L'/L$. Moreover, let $r_1 \in \mathbb{Z}/2N\mathbb{Z}$, such that $\mu_1 = \mu_{r_1} = \operatorname{diag}(r_1/2N, -r_1/2N)$. As elements of $Z^1(C_D)$, we have*

$$\mathbf{j}_D^*\mathcal{Z}(m_1,\mu_1) = \sum_{\substack{n\equiv Fr_1 \bmod 2A\\ n^2\le D_1/D}} \mathcal{Z}\left(|m_1| - \frac{n^2}{4A}, \mathfrak{a}, r_1\left(\frac{1+F\sqrt{D}}{2\sqrt{D}}\right)\right).$$

Proof. According to Lemma 4.6.3, we have

$$\mathbf{j}_D^*\mathcal{Z}(m_1,\mu_1) = \sum_{\substack{n\equiv Rr_1 \bmod 2N\\ n^2\leq DD_1}} \mathcal{Z}\left(|m_1| - \frac{n^2}{4A\,|D|^2}, \mathfrak{b}, \frac{n+r_1\sqrt{D}}{2\sqrt{D}}\right).$$

Recall that $R = FD$ and $2AE + BF = 1$. Thus, replacing n by Dn yields

$$\mathbf{j}_D^*\mathcal{Z}(m_1,\mu_1) = \sum_{\substack{n\equiv Fr_1 \bmod 2A\\ n^2\leq D_1/D}} \mathcal{Z}\left(|m_1| - \frac{n^2}{4A}, \mathfrak{b}, \frac{Dn+r_1\sqrt{D}}{2\sqrt{D}}\right).$$

Note that for $n \equiv Fr_1 \bmod 2A$, we have

$$\frac{n-Fr_1}{2}\sqrt{D} \in \mathfrak{d}_k\mathfrak{a} = \mathfrak{b}.$$

Therefore, we obtain the result. □

Proof of Lemma 5.3.7. By Lemma 5.2.1, we have

$$\sum_{\beta\in P'/P} \langle f\otimes g, \phi_\beta\otimes\Theta_{\mathcal{N}^-}\rangle\phi_\beta = 12^A f\psi_0.$$

Therefore,

$$12^A c_{f\psi_0}(-m,\beta) = \sum_{n\in\mathbb{Z}} c_{f\otimes g}\left(-m - \frac{n^2}{4A}, \beta+\mu_{\tilde{n}}\right).$$

Here $\tilde{n} = -Dn$ so that $\mu_{\tilde{n}}$ corresponds to $n \bmod 2A$.

By Lemma 5.3.8, we have

$$\begin{aligned}\mathbf{j}_D^*\mathcal{Z}(f\otimes g,\mathfrak{a}) = &\sum_{\beta\in P'/P}\sum_{\nu\in N'/N}\sum_{m_1<0} c_g(m_1,\beta+\nu)\\ &\times \sum_{\substack{n\equiv Fr_1 \bmod 2A\\ n^2\leq |D_1/D|}} \mathcal{Z}\left(|m_1| - \frac{n^2}{4A}, \mathfrak{a}, r_1\left(\frac{1+F\sqrt{D}}{2\sqrt{D}}\right)\right).\end{aligned}$$

Here $\beta+\nu = \mu_1 = \mu_{r_1}$, where μ_{r_1} is as above. Thus, we obtain

$$\begin{aligned}\mathbf{j}_D^*\mathcal{Z}(f\otimes g,\mathfrak{a}) &= \sum_{\beta\in P'/P}\sum_{m>0}\sum_{n\in\mathbb{Z}} c_{f\otimes g}\left(-m-\frac{n^2}{4A},\beta+\mu_{\tilde{n}}\right)\mathcal{Z}\left(m,\mathfrak{a},\beta\right)\\ &= 12^A\sum_{m>0}\sum_{\beta\in P'/P} c_{f\psi_0}(-m,\beta)\mathcal{Z}\left(m,\mathfrak{a},\beta\right).\end{aligned}$$

Here, we have used that the generator $A/\sqrt{D}$ of $\mathfrak{d}_k^{-1}\mathfrak{a}$ maps to $2A \bmod 2N$ using the isometry in Lemma 5.1.5 and our identification of $r \bmod 2N$ with μ_r. Consequently,

$$2A\frac{1+F\sqrt{D}}{2\sqrt{D}} = \frac{A}{\sqrt{D}} + AF \equiv \frac{A}{\sqrt{D}} \bmod \mathfrak{a}.$$

$\square$

The following Proposition follows from Lemma 5.3.7 together with Proposition 4.6.5. It is the key statement that links the analytic seesaw identity with the arithmetic one, yielding a purely arithmetic description of the CM values of the Borcherds products Ψ_L and thus leading to an arithmetic interpretation of the coefficients of the harmonic weak Maaß forms $\widetilde{\Theta}_P^+(\tau,h)$.

Proposition 5.3.9. *Fix the normalization of* $\mathrm{pr}: C_D \to \operatorname{Spec}\mathcal{O}_H$ *as in Proposition 4.6.5 with* $\mathfrak{n} = \mathfrak{b} = \mathfrak{d}_k\mathfrak{a}$. *Let* $f \in M^!_{1,\mathcal{P}}$ *and* $g \in M_{-\frac{1}{2},\mathcal{N}}$ *as in Lemma 5.3.7 and suppose that the multiplier system of* $\Psi_L((z,h), f\otimes g)$ *is trivial. Then we have*

$$\operatorname{ord}_{\mathfrak{P}}(\Psi_L((z_U,h), f\otimes g)) = 12^A\frac{w_k}{2}\sum_{\beta\in P'/P}\sum_{m>0} c_{ff_0}(-m,\beta)\mathcal{Z}(m,h.\mathfrak{a},h^{-1}.\beta)_{\mathfrak{P}}. \tag{5.3.1}$$

Proof. The identity for $h=1$ is clear from Proposition 4.6.5. Then, we apply Lemma 5.3.3 together with Lemma 4.3.6. $\square$

The following Proposition is crucial for the proof of the main theorem in this section. It shows that we do not have to deal with bad intersection in order to obtain arithmetic information about the coefficients of $\widetilde{\Theta}_P^+(\tau,h)$.

Proposition 5.3.10. *Let* $f \in M^!_{1,\mathcal{P}}$ *with* $c_f(0,0)=0$ *and* $H \in M^!_{1/2,L}$ *such that*

$$\langle H(\tau), \phi_0\otimes\Theta_{\mathcal{N}^-}(\tau)\rangle = f_{0+\mathcal{P}}(\tau).$$

Then $\mathcal{Z}(H)$ *and* $\mathcal{Z}(M,\mu_R)$ *intersect properly.*

Proof. If the divisors $\mathcal{Z}(m,\mu)$ and $\mathcal{Z}(n,\nu)$ do not intersect properly, then Dd is a perfect square for $D=-4Nm$ and $d=-4Nn$.

In our case this means that improper intersection might occur if there is a coefficient $c_H(m,\mu)$ with $-4Nm = Dn^2$ for some $n\in\mathbb{Z}$.

On the one hand, we have by assumption

$$\mathrm{CT}(\langle H(\tau), \phi_0\otimes\Theta_{\mathcal{N}^-}(\tau)\rangle) = c_f(0,0) = 0.$$

On the other hand,

$$\mathrm{CT}(\langle H(\tau), \phi_0\otimes\Theta_{\mathcal{N}^-}(\tau)\rangle) = \sum_{n\in\mathbb{Z}} c_H\left(\frac{-n^2}{4A}\right), \tag{5.3.2}$$

by the definition of the theta function.

Therefore, the total multiplicity of improper intersection is zero. More precisely, according to the decomposition in Lemma (4.5.3), we have

$$\mathcal{Z}\left(\frac{n^2D}{4N},\mu\right)=\sum_{r\mid n}\sum_{r\nu=\mu}\mathcal{Z}^0\left(\frac{r^2D}{4N},\nu\right).$$

This implies that

$$\mathcal{Z}\left(\frac{n^2D}{4N},\mu\right)\cap\mathcal{Z}\left(\frac{D}{4N},\mu\right)=\mathcal{Z}\left(\frac{D}{4N},\mu\right)$$

since $\mathcal{Z}\left(\frac{D}{4N},\mu\right)=\mathcal{Z}^0\left(\frac{D}{4N},\mu\right)$ because D is square-free. Therefore, (5.3.2) implies that the improper intersection is given by

$$\sum_{n\in\mathbb{Z}}c_H\left(\frac{n^2D}{4N}\right)\mathcal{Z}\left(\frac{n^2D}{4N},\mu\right)\cap\mathcal{Z}\left(\frac{D}{4N},\mu\right)=\mathcal{Z}^0\left(\frac{D}{4N},\mu\right)\sum_{n\in\mathbb{Z}}c_H\left(\frac{-n^2}{4A}\right)=0.$$

□

Remark 5.3.11. An alternative approach to avoid improper intersection would be to show that it is always possible to subtract a weakly holomorphic modular form $H\in M^!_{1/2,L}$, such that $\tilde{H}=f_{m,\beta}\otimes G^A-H$ satisfies

$$c_{\tilde{H}}\left(\frac{n^2D}{4A},\mu\right)=0$$

for all $n\in\mathbb{Z}$ and all $\mu\in L'/L$. This is in fact possible but tedious and, as the proposition shows, completely unnecessary for our applications.

Theorem 5.3.12. *We assume that the lattice P is given by a fractional ideal $\mathfrak{a}\subset k_D$ with quadratic form $Q(x)=\mathrm{N}(x)/\mathrm{N}(\mathfrak{a})$.*

For every $h\in C_{P,K}\cong\mathrm{Cl}_k$ there is a harmonic weak Maaß form $\widetilde{\Theta}_P(\tau,h)\in\mathcal{H}_{1,P^-}$ with holomorphic part

$$\widetilde{\Theta}^+_P(\tau,h)=\sum_{\beta\in P'/P}\sum_{m\gg-\infty}c^+_P(h,m,\beta)e(m\tau)\phi_\beta$$

satisfying the following properties.

(i) We have $\xi(\widetilde{\Theta}_P(\tau,h))=\Theta_P(\tau,h)$ and

$$\frac{1}{h_k}\sum_{h\in C_{P,K}}\widetilde{\Theta}_P(\tau,h)=:\widetilde{E}_P(\tau),\tag{5.3.3}$$

such that $\xi(\widetilde{E}_P(\tau))=E_P(\tau)$ and the principal part of $\widetilde{E}_P(\tau)$ vanishes.

(ii) For all $\beta \in P'/P$ and all $m \in \mathbb{Q}$ with $m \equiv -Q(\beta) \bmod \mathbb{Z}$, we have

$$c_P^+(h, m, \beta) = -2r(m, \beta)^{-1} \log |\alpha(h, m, \beta)|, \tag{5.3.4}$$

with $r(m, \beta) \in \mathbb{Z}$ and $\alpha(h, \beta, m) \in \mathcal{O}_H$.

(iii) Moreover, if $m > 0$, then

$$\mathrm{ord}_{\mathfrak{P}}(\alpha(h, m, \beta)) = r(m, \beta) w_k \mathcal{Z}(m, h.\mathfrak{a}, h^{-1}.\beta)_{\mathfrak{P}}$$

for all prime ideals $\mathfrak{P} \subset \mathcal{O}_H$.

(iv) For $m < 0$, we have $\alpha(h, m, \beta) \in \mathcal{O}_H^\times$.

Proof. The existence of $\widetilde{\Theta}_P(\tau, h)$ such that (i) is satisfied is clear from Proposition 3.3.2.

Let $F_{m,\beta} \in M^!_{-1/2,L}$ such that (5.2.2) is satisfied for $F_{m,\beta}$ with $\psi_0 = 1$ and $f = f_{m,\beta}$. We first show that we can assume that the constant term of $F_{m,\beta}$ vanishes. Let us take $F_{m,\beta} = f_{m,\beta} \otimes G^A$, the simplest choice. Then we consider two cases.

The first case is $A = 1$. There is a $g \in M_2^!(\mathrm{SL}_2(\mathbb{Z}))$ with only integral Fourier coefficients such that we have $g(\tau) = q^{-1} + O(q)$. The weakly holomorphic modular form $E_P \otimes Fg \in M^!_{1/2,L}$ has a non-vanishing constant term

$$c_{E_P}(1, 0)c_F(0, 0) + c_F(1, 0).$$

In fact, $c_{E_P}(1, 0) = 2/h_k$ and we have $c_F(0, 0) = 10$ and $c_F(1, 0) = 108$.

Therefore, replacing $F_{m,\beta}$ by

$$F_{m,\beta} - \frac{c_{F_{m,\beta}}(0, 0)h_k}{108h_k + 20} E_P \otimes Fg.$$

eliminates the constant term and does not change the pullback and seesaw identities (Lemma 5.3.7 and (5.2.3)).

In the other case, when $A > 1$, we can take $g \in M_{2A}(\mathrm{SL}_2(\mathbb{Z}))$ with $g(\tau) = 1 + O(q)$. This time, we can consider $E_P \otimes F^A g$ which has constant term $c_{F^A}(0, 0) \neq 0$. Thus, we can replace $F_{m,\beta}$ by

$$F_{m,\beta} - \frac{c_{F_{m,\beta}}(0, 0)}{c_{F^A}(0, 0)} E_P \otimes F^A g.$$

Therefore, we can now assume without loss of generality that the constant coefficient of $F_{m,\beta}$ vanishes.

Moreover, we will now show that we can choose $\widetilde{\Theta}_P(\tau, h)$ such that the finite set of coefficients $c_P^+(h, n_j, \beta_j)$ for $j = 1, \ldots, d$ satisfy

$$c_P^+(h, n_j, \beta_j) = -2r(n_j, \beta_j)^{-1} \log |\alpha(h, n_j, \beta_j)|$$

with

$$\operatorname{ord}_{\mathfrak{P}}(\alpha(h,n_j,\beta_j)) = r(n_j,\beta_j) w_k \mathcal{Z}(n_j, h.\mathfrak{a}, h^{-1}.\beta_j)_{\mathfrak{P}}.$$

Here, the indices n_j, β_j for $j = 1, \ldots, d$ belong to our special basis of $S_{1,\mathcal{P}}$ as in Lemma 5.2.2 and $r(n_j,\beta_j) = r$ can be chosen to not depend on j. We choose $r \in \mathbb{Z}_{>0}$ such that for all primes $\mathfrak{P}$ of $\mathcal{O}_H$ with $\mathfrak{P} \mid p$ for any $p \in \bigcup_{j=1}^{d} \operatorname{Diff}(n_j)$, we have that there is an element $\alpha(1,n_j,\beta_j) \in \mathcal{O}_H$ with

$$\operatorname{ord}_{\mathfrak{P}}(\alpha(1,n_j,\beta_j)) = r w_k \mathcal{Z}(n_j, \mathfrak{a}, \beta_j)_{\mathfrak{P}}$$

for all $\mathfrak{P}$. Note that r is a divisor of h_H, the class number of H. Using this, we let

$$\alpha(h,n_j,\beta_j) = \alpha(1,n_j,\beta_j)^{\sigma(h)},$$

where $\sigma(h)$ corresponds to the class of the idele h under the Artin map.

As in Proposition 3.3.2, we let $\widetilde{E}_P(\tau) \in \mathcal{H}_{1,\mathcal{P}^-}$ with $\xi(\widetilde{E}_P(\tau)) = E_P(\tau)$ and vanishing principal part. We obtain the existence of harmonic Maaß forms $\widetilde{\Theta}_P(\tau,h)$, such that (5.3.3) is satisfied. Replacing $\widetilde{\Theta}_P(\tau,h)$ by

$$\widetilde{\Theta}_P(\tau,h) - \sum_{j=1}^{d} \frac{2}{r(n_j,\beta_j)} \log|\alpha(h,n_j,\beta_j)|\, g_j(\tau)$$

and accordingly $\widetilde{E}_P(\tau)$ by

$$\widetilde{E}_P(\tau) - \sum_{h \in C_{P,K}} \sum_{j=1}^{d} \frac{2}{r(n_j,\beta_j)} \log|\alpha(h,n_j,\beta_j)|\, g_j(\tau),$$

we obtain that the coefficients $c_P^+(h,n_j,\beta_j)$ satisfy the assertion for all j and all h and (5.3.3) is satisfied.

We will now first turn to the coefficients of the principal part and prove (5.3.4) for $m < 0$ and *(iv)*. Recall that we write

$$\Theta_P(\tau,h) = E_P(\tau) + g_P(\tau,h)$$

with $g_P(\tau,h) \in S_{1,\mathcal{P}}$. Accordingly, as in Lemma 3.3.2, we write

$$\widetilde{\Theta}_P(\tau,h) = \widetilde{E}_P(\tau) + \widetilde{g}_P(\tau,h),$$

where $\widetilde{g}_P(\tau,h) \in H_{1,\mathcal{P}}$ and $\widetilde{E}_P(\tau)$ maps to E_P under ξ and has vanishing principal part. Using Lemma 3.2.2 we see that every coefficient in the principal part of $\widetilde{\Theta}_P(\tau,h)$ is a rational linear combination of terms of the form

$$(g, g_P(\tau,h)) = (g, \Theta_P(\tau,h)),$$

where $g \in S_{1,\mathcal{P}}(\mathbb{Q})$ is a cusp form with rational Fourier coefficients. Therefore, fixing $h \in C_{P,K}$, it is enough to show that

$$(g, \Theta_P(\tau, h)) = c \log |\alpha| , \tag{5.3.5}$$

with $c \in \mathbb{Q}$ and $\alpha \in \mathcal{O}_H^\times$. This follows easily from the fact that

$$(g, \Theta_P(\tau, h)) = \Phi_P(h, g).$$

The last quantity is equal to

$$12^A \Phi_P(h, g) = -2 \log \left| \Psi_L((z_U, h), g \otimes G^A) \right|^2$$

by (5.2.3). Thus, replacing g by $s \cdot g$ for an appropriate integer s if necessary, we have by (5.3.1) that

$$\alpha := \Psi_L((z_U, h), s \cdot g \otimes G^A)^2 \in \mathcal{O}_H^\times.$$

Thus, we obtain (5.3.5) with $c = -2/(12^A s)$ and this finishes the proof of (5.3.4) for $m < 0$ and also shows *(iv)*.

We continue to prove (5.3.4) for $m > 0$ and $h = 1$ and the formula in *(iii)*. The case $h \neq 1$ then follows by the reciprocity laws in Lemma 5.1.9 and Proposition 4.3.6.

Recall Equation (5.2.6):

$$\int_{\Gamma\backslash\mathbb{H}}^{\mathrm{reg}} \langle f_{m,\beta}(\tau), \overline{\Theta_P(\tau, h)} \rangle v \, d\mu(\tau) = -\frac{4}{12^A \, c_{m,\beta}} \log |\Psi_L((z_U, h), c_{m,\beta} F_{m,\beta})| . \tag{5.3.6}$$

By Theorem 3.3.6, the left hand side is equal to

$$\begin{aligned} \{\widetilde{\Theta}_P(\tau, h), f_{m,\beta}\} &= 2c_P^+(h, m, \beta) \\ &\quad + \sum_{\gamma \in P'/P} \sum_{n>0} c_P^+(h, -n, \gamma) c_{f_{m,\beta}}(n, \gamma) - 2 \sum_{j=1}^{d} c_P^+(h, \beta_j, n_j) a_j(m, \beta). \end{aligned} \tag{5.3.7}$$

We let $M_{m,\beta} \in \mathbb{Z}_{>0}$, such that for $r(m, \beta) := 12^A \cdot r \cdot M_{m,\beta}$ the following properties are satisfied.

(i) The weakly holomorphic modular form $M_{m,\beta} \cdot r \cdot F_{m,\beta}$ has an integral principal part,

(ii) we have $\Psi_L(z, h, M_{m,\beta} \cdot r \cdot F_{m,\beta}) \in \mathbb{Q}(j, j_N)$

(iii) and there is an $\epsilon \in \mathcal{O}_H^\times$, such that

$$\frac{r(m, \beta)}{4} \sum_{\gamma \in P'/P} \sum_{n>0} c_P^+(1, -n, \gamma) c_{f_{m,\beta}}(n, \gamma) = \log |\epsilon| .$$

We replace $c_{m,\beta}$ by $r \cdot M_{m,\beta}$ and the right hand side of (5.3.6) now becomes

$$-4r(m,\beta)^{-1}\log\left|\tilde{\alpha}(1,m,\beta)\right|, \tag{5.3.8}$$

with $\tilde{\alpha}(1,m,\beta) := \Psi((z_U,1), M_{m,\beta}\cdot r\cdot F_{m,\beta}) \in H$.

Using (5.3.6), (5.3.7), (5.3.8) and that $r(m,\beta) = 12^A M_{m,\beta}\cdot r(n_j,\beta_j)$, we obtain

$$\begin{aligned} c_P^+(1,m,\beta) = &-\frac{2}{r(m,\beta)}\log\left|\tilde{\alpha}(1,m,\beta)\right| - \frac{1}{2}\sum_{\gamma\in P'/P}\sum_{n>0} c_P^+(1,-n,\gamma)c_{f_{m,\beta}}(n,\gamma) \\ &- \frac{2\cdot 12^A M_{m,\beta}}{r(m,\beta)}\sum_{j=1}^{d} a_j(m,\beta)\log\left|\alpha(1,n_j,\beta_j)\right|. \end{aligned}$$

This implies that

$$c_P^+(1,m,\beta) = -\frac{2}{r(m,\beta)}\log\left|\epsilon\,\tilde{\alpha}(1,m,\beta)\prod_{j=1}^{d}\alpha(1,n_j,\beta_j)^{12^A M_{m,\beta}a_j(m,\beta)}\right|.$$

Note that the number in the absolute value is algebraic and contained in H. To see this, note that the denominator of $a_j(m,\beta)$ is bounded by $c_{m,\beta}$. This implies that

$$\alpha(1,n_j,\pm\beta_j)^{12^A M_{m,\beta}a_j(m,\beta)} \in \mathcal{O}_H$$

for all j. This shows (5.3.4) for $m>0$.

Finally, let

$$\alpha(1,m,\beta) = \epsilon\,\tilde{\alpha}(1,m,\beta)\prod_{j=1}^{d}\alpha(1,n_j,\beta_j)^{12^A M_{m,\beta}a_j(m,\beta)} \in H.$$

Then $\alpha(1,m,\beta)$ satisfies

$$\begin{aligned} \operatorname{ord}_{\mathfrak{P}}(\alpha(1,m,\beta)) = &\operatorname{ord}_{\mathfrak{P}}(\tilde{\alpha}(1,m,\beta)) \\ &+ 12^A w_k M_{m,\beta}\sum_j a_j(m,\beta)r(n_j,\beta_j)\mathcal{Z}(n_j,\mathfrak{a},\beta_j)_{\mathfrak{P}}. \end{aligned} \tag{5.3.9}$$

By Proposition 5.3.10, we know that $\mathcal{Z}(F_{m,\beta})$ and $\mathcal{Z}(M,\mu_R)$ intersect properly and we have by (5.3.1) that

$$\operatorname{ord}_{\mathfrak{P}}(\tilde{\alpha}(1,m,\beta)) = r(m,\beta)\frac{w_k}{2}\sum_{\beta\in P'/P}\sum_{m>0} c_{f_{m,\beta}}(-m,\beta)\mathcal{Z}(m,\mathfrak{a},\beta)_{\mathfrak{P}}$$

which we can further expand to

$$\begin{aligned}\operatorname{ord}_{\mathfrak{P}}(\tilde{\alpha}(1,m,\beta)) = r(m,\beta)w_k\,\mathcal{Z}(m,\mathfrak{a},\beta)_{\mathfrak{P}}& \\ - r(m,\beta)w_k\sum_{j=1}^{d}a_j(m,\beta)\mathcal{Z}(n_j,\mathfrak{a},\beta_j)_{\mathfrak{P}},&\end{aligned} \tag{5.3.10}$$

where we used that $\mathcal{Z}(n,\mathfrak{a},-\beta)_{\mathfrak{P}} = \mathcal{Z}(n,\mathfrak{a},\beta)_{\mathfrak{P}}$.

Plugging (5.3.10) into (5.3.9) and using and $r(m,\beta) = 12^A M_{m,\beta} r(n_j,\beta_j)$ we conclude

$$\operatorname{ord}_{\mathfrak{P}}(\alpha(1,m,\beta)) = r(1,m,\beta)w_k\,\mathcal{Z}(m,\mathfrak{a},\beta)_{\mathfrak{P}},$$

which proves *(iii)*. □

Remark 5.3.13. In Proposition 2.6.5, we gave a very explicit formula for the Petersson inner products occurring in the Theorem.

Remark 5.3.14. Note that if we restrict to the subspace of symmetric theta functions $\Theta_P^{\mathrm{sym}}(\tau,\psi)$ we can also show that the Petersson inner products involved in the principal part are units in the Hilbert class field using yet another method (at least in the case of prime discriminant). The point is that the Petersson norm $(\theta_\chi,\theta_\chi)$ is known to be proportional to the special value at $s=1$ of the derivative $L'(s,\chi)$. Stark's conjecture for imaginary quadratic fields is in fact a theorem (proven by Stark himself [Sta75]) and expresses this special value in terms of units in the Hilbert class field! Also note that Stark in fact uses modular functions in the proof of this case of his conjecture.

We will now show that the integers $r(m,\beta)$ in the theorem can be bounded so that we can choose an integer r only depending on P, such that the theorem is satisfied for r in place of $r(m,\beta)$. For simplicity, we restrict to the case of prime discriminants although the argument should generalize.

Proposition 5.3.15. *Suppose that $|P'/P| = l$ for a prime $l \equiv 3 \bmod 4$. Then there is an $r \in \mathbb{Z}_{>0}$ only depending on D and $\mathfrak{a}$, such that all statements of Theorem 5.3.12 hold with $r(m,\beta) = r$ for all m and β.*

Proof. Recall that our convention in the case of prime discriminants is that $N = |D| = l$. The discriminant group L'/L has order $2l$ and $O(L'/L) = \{\pm 1\}$.

In the case of prime discriminant there is only one genus and therefore, we can assume without loss of generality that $P = \mathcal{O}$ and $A = 1$. For $f \in M^!_{1/2,L}(\mathbb{Z})$, we let $\rho_{f,\infty} \in \mathbb{R}$ be the Weyl vector associated with f at the cusp ∞, defined by

$$\rho_{f,\infty} = \frac{\sqrt{N}}{8\pi}\int_{\mathcal{F}}^{\mathrm{reg}} \langle f(\tau), \overline{\Theta_K(\tau)}\rangle v^{1/2} d\mu(\tau)$$

with $K = \mathbb{Z}$ together with the quadratic form $x^2/4N$. Note that $L'/L \cong K'/K$ and that we have

$$\Theta_K(\tau) = \sum_{n\in\mathbb{Z}} e\left(\frac{n^2}{4N}\right)\phi_{\frac{n}{2N}+\mathbb{Z}}.$$

The significance of the Weyl vector is that the divisor of $\Psi_L(z,f)$ on $X_0(N)$ is given by $Z(f)+C(f)$, where $C(f)=\rho_{f,\infty}(\infty)+\rho_{f,0}(0)$. In our case it is not hard to show that we have $\rho_{f,\infty}=\rho_{f,0}$.

We assume that the constant term of f vanishes, $c_f(m,\mu)\in\mathbb{Q}$ for all $m\in\mathbb{Q}$ and $\mu L'/L$ and $c_f(m,\mu)\in\mathbb{Z}$ for $m\leq 0$. Since the multiplier system σ of $\Psi_L(z,f)$ is of finite order M, we have that $M\cdot(Z(f)+C(f))$ is the divisor of a rational function on $X_0(N)$. Note that this also implies that $\deg(Z(f)+C(f))=0$. Conversely, as in the proof of Theorem 6.2 in [BO10], we have that if $M\cdot(Z(f)+C(f))$ is the divisor of a rational function on $X_0(N)$ then σ^M is trivial. If $\rho_{f,\infty}\in\mathbb{Z}$ then $Z(f)+C(f)$ defines a rational point in the Jacobian J of $X_0(N)$ because the Heegner divisors are defined over $\mathbb{Q}$ and so are the cusps of $\Gamma_0(N)$ for N square-free. The group of rational points $J(\mathbb{Q})$ is a finitely generated abelian group by the Mordell-Weil theorem. Therefore, $M\mid M(N)$, where $M(N)$ is the order of the torsion subgroup of $J(\mathbb{Q})$.

Theorem 4.5 of [BF06] (see also Theorem 5.5 of [AE13] for the specialization to $\Gamma_0(N)$) implies that there is a function $\widetilde{\Theta}_K(\tau)\in\mathcal{H}_{3/2,L^-}$ with $\xi_{3/2}(\widetilde{\Theta}_K)=\Theta_K$ and holomorphic part

$$\frac{2\pi}{6\sqrt{N}}\phi_0+\frac{2\pi}{\sqrt{N}}\sum_{\mu\in K'/K}\sum_{\substack{m\in\mathbb{Q}_{>0}\\ m+Q(\mu)\in\mathbb{Z}}} H(m,L)q^m\phi_\mu,$$

where

$$H(m,L)=\sum_{\substack{\lambda\in L+\mu\\ Q(\lambda)=-m}}\frac{1}{\left|\overline{\Gamma}_\lambda\right|},$$

and $\overline{\Gamma}_\lambda$ is the stabilizer of λ in $\overline{\Gamma}=\Gamma_0(N)/\{\pm 1\}$. Note that $3H(m,L)\in\mathbb{Z}$ for all m. The function $\widetilde{\Theta}_K(\tau)$ is obtained as a certain theta lift of the constant function 1 using the Kudla-Millson theta kernel in signature $(1,2)$. In fact, $\widetilde{\Theta}_K(\tau)$ is proportional to the non-holomorphic Eisenstein series $E_{3/2}(\tau)$ transforming with representation ρ_{K^-}.

Consequently, using the paring (1.5.8), we obtain for the Weyl vector

$$\rho_{f,\infty}=\frac{1}{4}\sum_{\mu\in L'/L}\sum_{m\in\mathbb{Q}_{>0}} c_f(-m,\mu)H(m,L).$$

Thus, $12\rho_{f,\infty}\in\mathbb{Z}$ for all $f\in M^!_{1/2,L}$ with integral principal part.

Recall the numbers $n_1,\ldots,n_d$ that belong to our special basis of $S_{1,\mathcal{P}^-}$ constructed in Section 5.2.1. Let $n>\max\{n_1,\ldots,n_d\}$ be a fixed integer and $S=\{n+Q(\beta)\mid\beta\in P'/P\}$. For $x\in S$ and $\beta\in P'/P$, such that $x\equiv Q(\beta)\bmod\mathbb{Z}$, we let $F_{x,\beta}$ as in the beginning of the proof of Theorem 5.3.12. From our discussion above, it follows that there is an $r\in\mathbb{Z}_{>0}$, such that the following properties are satisfied.

(i) The functions $r \cdot f_{x,\beta}$ and $r \cdot F_{x,\beta}$ have integral Fourier coefficients for all $x \in S$ and

(ii) we have that $\rho_{rF,\infty} \in \mathbb{Z}$ for all $F \in M^!_{1/2,L}$ with $c_F(m,\mu) \in \mathbb{Z}$ for $m \leq 0$.

(iii) Moreover, for all $m \leq n$, we have

$$c_P^+(h,m,\beta) = -\frac{2}{r}\log|\alpha(h,m,\beta)|$$

with $\alpha(h,m,\beta) \in \mathcal{O}_H$ and $\operatorname{ord}_{\mathfrak{P}}(\alpha(h,m,\beta)) = w_k \cdot r \cdot \mathcal{Z}(m, h.\mathfrak{a}, h^{-1}.\beta)_{\mathfrak{P}}$.

We will also assume that $M(N) \mid r$, so that $\Psi_L((z,h), rF_{x,\beta}) \in \mathbb{Q}(j, j_N)$.

Now let $m \in \mathbb{Q}$ and $\beta \in P'/P$, such that $m > n$ and $m \equiv Q(\beta) \bmod \mathbb{Z}$. Then $m = x + n'$ for some $x \in S$ and $n' \in \mathbb{Z}_{>0}$. We let $f(\tau) = j^{n'}(\tau) f_{x,\beta}(\tau)$, where $j(\tau)$ is the j-invariant. Moreover, we let $F(\tau) \in M^!_{1/2,L}$ given by

$$F = f \otimes G - \frac{c_{f\otimes G}(0,0)}{20h_k + 108} E_P \otimes Fg$$

as in the beginning of the proof of Theorem 5.3.12. We have $c_F(0,0) = 0$. Note that G has integral Fourier coefficients. By (i), we have that rF has integral Fourier coefficients and by (ii) the Weyl vectors of $r \cdot F$ are integral and by our assumption $M(N) \mid r$, the function $\Psi_L((z,h),F)$ is contained in $\mathbb{Q}(j, j_N)$.

As in the proof of Theorem 5.3.12, we have on one hand

$$\Phi_P(f,h) = 2c_P^+(h,m,\beta) + \sum_{\gamma \in P'/P} \sum_{m' < m} c_P^+(h,m',\gamma) c_f(-m',\gamma)$$

and on the other hand

$$\Phi_P(f,h) = \frac{-4}{r}\log|\Psi_L((z_U,h),F)|.$$

By our assumptions we have that $\Psi_L((z_U,h),F) \in \mathcal{O}_H$. The statement of the proposition now easily follows by induction on $n' \in \mathbb{Z}_{>0}$. □

Remark 5.3.16. Let $l \geq 5$ be a prime, let n be the numerator of $(l-1)/12$ and consider the Jacobian J of $X_0(l)$. It follows from a Theorem of Mazur [Maz77] that the torsion subgroup $J(\mathbb{Q})_{tors}$ is cyclic of order n. Together with this explicit result, Proposition 5.3.15 describes an algorithm to determine the number r for a given prime discriminant $D = -l$ explicitly.

We collect some consequences of Theorem 5.3.12 and other results we have proved in the course of this thesis.

Corollary 5.3.17. *(i) If* $|\operatorname{Diff}(m)| \neq 1$, *then* $c_P^+(h,m,\beta) = -\frac{2}{r_m}\log|\epsilon|$ *for* $\epsilon \in \mathcal{O}_H^\times$.

(ii) If $\operatorname{Diff}(m) = \{p\}$, *then* $\operatorname{ord}_{\mathfrak{P}}(\alpha(h,m,\beta)) = 0$ *for all primes* $\mathfrak{P} \nmid p$.

(iii) The Shimura reciprocity

$$\alpha(h, m, \beta) = \alpha(1, m, \beta)^{\sigma(h)}$$

holds, where $\sigma(h)$ corresponds to h under the Artin map.

(iv) The explicit formula in Proposition 4.3.9 for the valuation at primes above p holds.

(v) In particular, let L be the subfield of H fixed by all elements of order less or equal than two and let $\mathfrak{f} \subset \mathcal{O}_L$ be the prime ideal below the fixed prime $\mathfrak{P}_0$ (see Lemma 4.3.7 on page 89). We have

$$\begin{aligned}\frac{\operatorname{ord}_{\mathfrak{f}}(\mathrm{N}_{H/L}(\alpha(h, m, \beta)))}{r_m \cdot w_k} &= \mathcal{Z}(m, h.\mathfrak{a}, h^{-1}.\beta)_{\mathfrak{f}} \\ &= 2^{o(m)-1}\nu_p(m)\rho(m\,|D|\,/p, [h]^2[\mathfrak{c}_0\mathfrak{a}]),\end{aligned}$$

where $\nu_p(m)$ is given in (4.3.3), $\mathfrak{c}_0$ is also defined in Lemma 4.3.7 and $o(m)$ is the number of primes $p \mid D$ such that $\operatorname{ord}_p(m\,|D|) > 0$.

(vi) If $D = -l$ is prime, then there is a unique prime $\mathfrak{P} \mid p$ that is fixed by complex conjugation. We have

$$\frac{\operatorname{ord}_{\mathfrak{P}}(\alpha(h, m, \beta))}{r_m \cdot w_k} = 2^{o(m)-1}\nu_p(m)\rho(m\,|D|\,/p, [h]^2[\mathfrak{a}]^2).$$

Using the explicit formulas for $\mathcal{E}_P(\tau)$ we obtain another corollary.

Corollary 5.3.18. *With the same notation as in Theorem 5.3.12, we have*

$$\widetilde{E}_P(\tau) = \mathcal{E}_P(\tau),$$

where $\mathcal{E}_P(\tau) = \frac{\partial}{\partial s}\hat{E}_P(\tau, s)\mid_{s=0}$ is defined in (3.1.3).

Proof. Recall the explicit formulas for the coefficients of $\mathcal{E}_P(\tau)$ from Theorem 3.1.5. First we note that the principal part of $\mathcal{E}_P(\tau)$ vanishes. Since the formulas also show that $\xi_1(\mathcal{E}_P(\tau)) = E_P(\tau)$, the difference $\mathcal{E}_P(\tau) - \widetilde{E}_P(\tau)$ is holomorphic. Write the Fourier expansion of $\widetilde{E}_P^+(\tau)$ as

$$\widetilde{E}_P^+(\tau) = \sum_{\beta \in P'/P} \sum_{m \geq 0} \tilde{c}_P(m, \beta) e(m\tau)\phi_\beta.$$

Proposition 4.3.10 provides explicit formulas for $\mathcal{Z}(m, \mathfrak{a}, \beta)_{\mathfrak{P}}$. If $\operatorname{Diff}(m) = \{p\}$, then

$$\begin{aligned}\frac{2h_k}{w_k}\,\tilde{c}_P^+(m, \beta) &= -4r(m, \beta)^{-1}w_k^{-1}\sum_h \log|\alpha(h, m, \beta)| \\ &= -2r(m, \beta)^{-1}w_k^{-1}\sum_{\sigma \in \operatorname{Gal}(H/k)} \log|\alpha(1, m, \beta)^\sigma|^2.\end{aligned}$$

We have

$$\sum_{\sigma\in\mathrm{Gal}(H/k)} \log\left|\alpha(1,m,\beta)^\sigma\right|^2 = \sum_{\mathfrak{P}|p} \log\left|\mathrm{N}_{H/k}(\mathfrak{P})^{n_\mathfrak{P}}\right|^2 = \sum_{\mathfrak{P}|p} \log \mathrm{N}_{H/\mathbb{Q}}(\mathfrak{P})^{n_\mathfrak{P}}$$

where

$$\begin{aligned}\sum_{\mathfrak{P}} n_\mathfrak{P} &= r(m,\beta)\frac{w_k}{2}2^{o(m)}\nu_p(m) \sum_{[\mathfrak{c}]\in\mathrm{Cl_k}\,/\,\mathrm{Cl_k^2}} \rho(m\left|D\right|/p, [\mathfrak{c}]^2[\mathfrak{c}_0\mathfrak{a}]) \\ &= r(m,\beta)\frac{w_k}{2}2^{o(m)}\nu_p(m)\rho(m\left|D\right|/p).\end{aligned}$$

Note that all the factors cancel and we obtain

$$\frac{2h_k}{w_k}\,\tilde{c}_P^+(m,\beta) = -2^{o(m)}\nu_p(m)\rho(m\left|D\right|/p).$$

This shows that $\tilde{c}_P(m,\beta)=\kappa(m,\beta)$ for all $m>0$ and all $\beta\in P'/P$ by comparing with the formulas given in Theorem 3.1.5. Consequently, $\tilde{c}_P(0,0)=\kappa(0,0)$ as well. ☐

Corollary 5.3.19. *The constant term of $\widetilde{\Theta}_P(\tau,h)$ is given by*

$$c_P^+(h,0,0) = -\sum_{\beta\in L'/L}\sum_{m>0} c_P^+(h,-m,\beta)\tilde{\rho}(m,\beta) - 2\frac{\Lambda'(\chi_D,0)}{\Lambda(\chi_D,0)},$$

where $\tilde{\rho}(m,\beta)$ is the coefficient of index (m,β) of $E_P(\tau)$.

5.4 The scalar valued case

In this section, we finish our study of the harmonic weak Maaß forms that map to theta functions of weight one by providing the corresponding statement for the corresponding scalar valued theta series.

Corollary 5.4.1. *Let $D<0$ with $D\equiv 1 \bmod 4$ be a fundamental discriminant and let $\mathfrak{A}\in\mathrm{Cl_k}\,/\,\mathrm{Cl_k^2}$ be a genus. For every $\mathfrak{a}\in\mathfrak{A}$, there is a $\widetilde{\theta}_\mathfrak{a}\in\mathcal{H}_1(\left|D\right|,\chi_D)$, where χ_D is given by the Kronecker symbol, with the following properties.*

(i) We have $\xi(\widetilde{\theta}_\mathfrak{a})=\theta_\mathfrak{a}$ and

$$\sum_{\mathfrak{b}\in\mathrm{Cl_k}} \widetilde{\theta}_{\mathfrak{b}^2\mathfrak{a}} = h_k\,\widetilde{E}_\mathfrak{A},$$

where $\xi(\widetilde{E}_\mathfrak{A})=E_\mathfrak{A}$ and the principal part of $\widetilde{E}_\mathfrak{A}$ vanishes.

(ii) The constant term of $\widetilde{\theta}_\mathfrak{a}^+$ is equal to

$$c_\mathfrak{a}^+(0) = -\sum_{n>0} c_\mathfrak{a}^+(-n)\tilde{\rho}(n) - 2\frac{\Lambda'(\chi_D,0)}{\Lambda(\chi_D,0)},$$

where $\tilde{\rho}(n) = \tilde{\rho}(n, 0)$*.*

(iii) For all $n > 0$ *we have*

$$c_{\mathfrak{a}}^{+}(n) = -\frac{2}{r_n} \log |\alpha(\mathfrak{a}, n)| ,$$

where $\alpha(\mathfrak{a}, n) \in \mathcal{O}_H$ *and* $r_n \in \mathbb{Z}_{>0}$*.*

(iv) Finally, for all $n < 0$*, we have that*

$$c_{\mathfrak{a}}^{+}(n) = \frac{-2}{r_n} \log |\alpha(\mathfrak{a}, n)| ,$$

where $\alpha(\mathfrak{a}, n) \in \mathcal{O}_H^{\times}$*.*

Proof. All statements follow from Theorem 5.3.12 by considering

$$\widetilde{\theta}_{\mathfrak{a}\mathfrak{b}^2}(\tau) = \widetilde{\Theta}_{P,0}(\tau, h) \in \mathcal{H}_1(|D| , \chi_D)$$

where $P = \mathfrak{a}$, $(h) = \mathfrak{b}$ and

$$\widetilde{\Theta}_P(\tau, h) = \sum_{\beta \in P'/P} \widetilde{\Theta}_{P,\beta}(\tau, h)\phi_\beta.$$

Since we have $\theta_{\mathfrak{a}}(\tau) = \Theta_{P,0}(\tau)$, we conclude $\xi(\widetilde{\theta}_{\mathfrak{a}}) = \theta_{\mathfrak{a}}(\tau)$. □

Corollary 5.4.2. *We use the same notation as in Theorem 5.4.1.*

(i) We have $\operatorname{ord}_{\mathfrak{P}}(\alpha(\mathfrak{a}, n)) = 0$*, unless* $|\mathrm{Diff}(n)| = 1$*.*

(ii) If $\mathrm{Diff}(n) = \{p\}$*, then*

$$\operatorname{ord}_{\mathfrak{P}_0}(\alpha(\mathfrak{a}, n)) = r \cdot w_k \cdot \nu_p(n)\rho(n |D| /p, [\mathfrak{a}\mathfrak{c}_0]),$$

where $\mathfrak{P}_0$ *is the distinguished prime in Lemma 4.3.7.*

(iii) The Shimura reciprocity

$$\operatorname{ord}_{\mathfrak{P}_0^{\sigma}}(\alpha(\mathfrak{a}, n)) = \alpha(\mathfrak{b}^{-2}\mathfrak{a}, n),$$

holds, where $\sigma = \sigma(\mathfrak{b})$*.*

(iv) For the coefficients in the principal part, we have that $\log |\alpha(\mathfrak{a}, n)| \in \mathcal{O}_H^{\times}$*.*

Remark 5.4.3. Note that we were able to give a relatively simple formula for the $\mathfrak{P}$-valuations of the coefficients in comparison to the vector valued case, where this was only possible after passing to the fixed field of elements of order two in the class group. In fact, considering scalar valued theta series corresponds to "forgetting" the additional congruence conditions that the vector valued forms know about.

Remark 5.4.4. In a related matter, we should also remark that in the preprint [Ehl12], this issue is not properly handled in the introduction. Theorem 1.1, ibid., is only valid for prime discriminants. Additional weight factors (or vector valued harmonic weak Maaß forms) are necessary to make the statement hold for non-prime fundamental discriminants.

5.5 The conjecture of Duke and Li

In [DL12], the authors study the scalar valued case. They construct the preimages under ξ of scalar valued theta functions of weight one and prime level using Poincaré series. They also obtain that the coefficients are given by rational multiplies of logarithms of algebraic numbers and that they satisfy a Shimura reciprocity type identity. Motivated by numerical experiments, the authors were led to formulate the following conjecture.

We work with the same notation as on the last section. Let $k = \mathbb{Q}(\sqrt{-l})$ for a prime $l \equiv 3 \bmod 4$, $l > 3$ and let p be a prime that is non-split in k. We consider the lattice $P = \mathcal{O}_k$ in k. Moreover, let $\mathfrak{P}_0 \mid p$ be the unique prime above p fixed by complex conjugation. Let $\mathfrak{P}^{\sigma_{\mathfrak{b}}} = \mathfrak{P}_0$ for a fractional ideal $\mathfrak{b}$. Then their conjecture (p. 6 of [DL12]) is that for $n \in \mathbb{Z}_{>0}$ with $\chi_p(n) \neq 1$, we have

$$c^+_{\mathfrak{a}^2}(n) = -\frac{2}{r} \log \left| u(\mathfrak{a}^2, n) \right|$$

with

$$\operatorname{ord}_{\mathfrak{P}}(u(\mathfrak{a}^2, n)) = 2r \sum_{m \geq 1} \rho\left(\frac{n}{p^m}, [\mathfrak{a}\mathfrak{b}]^2\right),$$

for $r \in \mathbb{Z}$ independent of n and dividing $24h_k h_H$.

Our results imply this conjecture regarding the valuations completely, but we were not able to confirm the explicit bound $24h_k h_H$ for r.

We will now show that the representation numbers involved in the conjecture agree with the formulas we gave. Suppose that $p \neq l$ and that $\operatorname{Diff}(n) = \{p\}$. Then $\chi_p(-Dn) = -1$ and we only have a non-zero contribution if $\operatorname{ord}_p(n) > 0$. We have that $\operatorname{ord}_p(n)$ is necessarily odd. More precisely, we have

$$\rho\left(\frac{n}{p^m}, [\mathfrak{c}]\right) = \begin{cases} 0, & \text{if } m \text{ is even,} \\ \rho\left(n/p, [\mathfrak{c}]\right), & \text{if } m \text{ is odd.} \end{cases}$$

Since p is inert, there are no ideals of norm p. Therefore, the map $\mathfrak{c} \mapsto p\mathfrak{c}$ establishes a bijection between ideals of norm $\mathrm{N}(\mathfrak{c})$ and ideals of norm $p^2\mathrm{N}(\mathfrak{c})$, leaving the class invariant. Thus, we obtain the contribution $\rho(n\left|D\right|/p, [\mathfrak{c}]) = \rho(n/p, [\mathfrak{c}])$ times $\frac{1}{2}(\operatorname{ord}_p(n) + 1)$, as in our formula.

The case $p = l$ is similar, but this time we obtain $\operatorname{ord}_p(n) \cdot \rho(n\left|D\right|/p, [\mathfrak{c}])$ because there is a unique ideal of norm p. Note that our theorem also covers the case $p = 3$ and we provide a generalization to non-prime discriminants.

6 Applications and examples

6.1 Two arithmetic generating series

As a first application of Theorem 5.3.12, we will show the "modularity" of a certain generating series. Recall the definition of the special 0-cycles $\mathcal{Z}(m, \mathfrak{a}, \mu)$ on C_D and consider the generating series of their arithmetic degrees

$$\hat{E}^+(\tau) = \sum_{\mu \in P'/P} \sum_{m>0} \widehat{\deg}\, \mathcal{Z}(m, \mathfrak{a}, \mu) e(m\tau) \phi_\mu,$$

for $P \cong \mathfrak{a}$, as usual. Here, for $\mathcal{Z} = \mathcal{Z}(m, \mathfrak{a}, \mu)$, the arithmetic degree is defined as

$$\widehat{\deg}\, \mathcal{Z}(m, \mathfrak{a}, \mu) = \sum_{p} \sum_{x \in \mathcal{Z}(\mathbb{F}_p)} \frac{1}{\mathrm{Aut}(x)} \lg \hat{\mathcal{O}}_{\mathcal{Z},x} \log p,$$

and $\lg \hat{\mathcal{O}}_{\mathcal{Z},x}$ denotes the length of the complete local ring at x. Here, the sum runs over all rational primes p. There is no archimedean contribution in this case.

We would like to show that this generating series is (up to the constant term) the holomorphic part of a harmonic weak Maaß form. In fact, this is a known result, mainly due to Kudla, Rapoport and Yang [KRY04]. However, the known proof relies on the explicit formulas for the coefficients of $\mathcal{E}_P$ and on those for the arithmetic degrees, as well.

We will now show that this result is a corollary of Theorem 5.3.12 which did not use any explicit formulas. We only used the explicit formulas later in Corollary 5.3.18.

We have in fact shown the following result.

Theorem 6.1.1. *There exists a harmonic weak Maaß form $\widetilde{E}_P(\tau) \in M_{1,P^-}$ with vanishing principal part, such that*

$$-\frac{h_k}{w_k} \widetilde{E}_P^+(\tau) = \hat{E}^+(\tau) + c\phi_0,$$

where $c \in \mathbb{C}$ is a constant.

Proof. Consider the sum

$$\widetilde{E}_P(\tau) = \frac{1}{h_k} \sum_{h \in C_k/K} \widetilde{\Theta}_P(\tau, h).$$

We obtain by Theorem 5.3.12 that the holomorphic part of $\widetilde{E}_P(\tau)$ has a vanishing principal part and we have

$$\begin{aligned}\frac{h_k}{w_k}\widetilde{E}_P^+(\tau) &= \frac{1}{w_k}\sum_{\beta\in P'/P}\sum_{m\geq 0}\sum_{h\in C_{P,K}} c_P^+(h,m,\beta)e(m\tau)\phi_\beta \\ &= -\frac{1}{w_k}\sum_{\beta\in P'/P}\sum_{m\geq 0} r(m,\beta)^{-1}\sum_{h\in C_{P,K}} \log|\alpha(h,m,\beta)|^2 e(m\tau)\phi_\beta,\end{aligned}$$

where $\alpha(h,m,\beta)\in\mathcal{O}_H$.

Moreover, for $m>0$ and for every prime $\mathfrak{P}$ of the Hilbert class field H of k, we have

$$\operatorname{ord}_{\mathfrak{P}}\left(\prod_h \alpha(h,m,\beta)\right) = r(m,\beta)w_k\sum_h \mathcal{Z}(m,h.\mathfrak{a},h^{-1}.\beta)_{\mathfrak{P}}.$$

By Proposition 4.3.6, we have for $\sigma=\sigma(h)$ that

$$\mathcal{Z}(m,\mathfrak{a},\beta)_{\mathfrak{P}^\sigma} = \mathcal{Z}(m,h^{-1}.\mathfrak{a},h.\beta)$$

and thus

$$\sum_{h\in C_k/K}\mathcal{Z}(m,h^{-1}.\mathfrak{a},h.\beta)_{\mathfrak{P}} = \sum_{\sigma\in\operatorname{Gal}(H/k)}\mathcal{Z}(m,\mathfrak{a},\beta)_{\mathfrak{P}^\sigma}.$$

This shows in fact that

$$\sum_h \log|\alpha(h,m,\beta)|^2 = \log \mathrm{N}_{H/\mathbb{Q}}(\alpha(1,m,\beta)).$$

But since $\mathcal{Z}(m,\mathfrak{a},\beta)$ is supported at a unique prime p, we also have that

$$\widehat{\deg}\,\mathcal{Z}(m,\mathfrak{a},\beta) = \sum_{\sigma\in\operatorname{Gal}(H/k)}\mathcal{Z}(m,\mathfrak{a},\beta)_{\mathfrak{P}^\sigma}\log\mathrm{N}_{H/\mathbb{Q}}(\mathfrak{P}).$$

Here, we used that p is non-split in k. Then we find

$$\begin{aligned}\sum_{h\in C_k/K}\log|\alpha(h,m,\beta)|^2 &= w_k r(m,\beta)\sum_{\sigma\in\operatorname{Gal}(H/k)}\mathcal{Z}(m,\mathfrak{a},\beta)_{\mathfrak{P}^\sigma}\log\mathrm{N}_{H/\mathbb{Q}}(\mathfrak{P}) \\ &= w_k r(m,\beta)\widehat{\deg}\,\mathcal{Z}(m,\mathfrak{a},\beta),\end{aligned}$$

which implies the statement of the proposition. □

Note that the non-holomorphic part has a similar interpretation. This follows from Section 4.2 and agrees with the results in [KRY99].

As mentioned in the Introduction, we should also be able to form a generating series related to the cycles $\mathcal{Z}(m,\mathfrak{a},\mu)$, that is a holomorphic modular form. For this, we have to add an archimedean part (a "Green function") and form the formal power series

$$\hat{\Phi}(\tau) = \sum_{\mu \in P'/P} \sum_{m>0} \hat{\mathcal{Z}}(m, \mathfrak{a}, \mu) e(m\tau) \phi_\mu. \tag{6.1.1}$$

This should then be a holomorphic cusp form, contained in S_{1,P^-}, which would be quite an interesting object, as the forms in this space correspond to non-dihedral Galois representations.

We try to keep the setup as simple as possible and define the completed cycles as Arakelov divisors [Neu07, Kapitel III] on $\operatorname{Spec} \mathcal{O}_H$, so that we are in a more or less "classical" situation. We let

$$\hat{\mathcal{Z}}(m, \mathfrak{a}, \mu) = \sum_{\mathfrak{P} \subset \mathcal{O}_H} \mathcal{Z}(m, \mathfrak{a}, \mu)_{\mathfrak{P}} \mathfrak{P} + \sum_{\sigma} \lambda(m, \mu, \sigma) \sigma.$$

Here, σ runs over the archimedean places of H and we define $\lambda(m, \mu, \sigma) \in \mathbb{R}$ as follows. For $m \in \mathbb{Z}_{>0}$ and $\mu \in P'/P$, we let $g_{m,\mu} \in H_{1,P}$ with principal part

$$P_g(\tau) = \frac{1}{2} q^{-m} (\phi_\mu + \phi_{-\mu})$$

and $c^+_{g_{m,\mu}}(0, 0) = 0$. Then we let

$$\lambda(m, \mu, \sigma) = \frac{1}{w_k} \Phi_P(g_{m,\beta}(\tau), h(\sigma)),$$

where h corresponds to $\sigma = \sigma(h)$ under the Artin map. Then we define the formal S_P-valued power series $\hat{\Phi}$ via (6.1.1).

Note that it is a consequence of Theorem 5.3.12, that $\widehat{\deg}\, \hat{\mathcal{Z}}(m, \mathfrak{a}, \mu) = 0$. Thus, the degree generating series is trivially modular.

To show modularity of this generating series in general would require to show that the pairing with any weakly holomorphic modular form f of weight one transforming with representation ρ_P vanishes in the Chow group. As we do not want to use rational coefficients, we have to restrict to the space of weakly holomorphic modular forms with integral principal part. A successful strategy to show modularity might be to refine the explicit construction of modular forms of weight 1/2 to be used with the seesaw identity in Chapter 5 to guarantee that the multiplier system of $\Psi_L(z, F)$ is trivial for $F \in M^!_{1/2,L}$ corresponding to $f \in M^!_{1,P}$. We did not work out the details at the time of finishing this thesis but hope to be able to further investigate this problem in the future.

6.1.1 Numerical evidence

As a numerical example, we can consider the case $D = -283$ because the prime 283 is the smallest one where there is a non-zero cusp form in S_{1,P^-} [Ser77]. We use `sage` [S+13] and `magma` [BCP97] to perform the following computations.

Let $k = \mathbb{Q}(\sqrt{-283})$ with ring of integers $\mathcal{O}$ and consider $P = \mathcal{O}$ with quadratic form $Q(x) = \mathrm{N}(x)$. The class number h_k of k is equal to 3. The space S_{1,P^-} can be identified with the space $S_1^-(283, \chi_{-283})$ of scalar valued modular forms such that all Fourier coefficients of index n with $\chi_{-283}(n) = 1$ vanish. Here, χ_{-283} is given by the Kronecker symbol. We will therefore simply consider the 0-th component of the generating series $\hat{\Phi}(\tau)$.

We denote by H the Hilbert class field of k. The class number h_H is equal to 4 and the structure of the class group is $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. We embed H into $\mathbb{C}$, such that $H = k(j)$ and j is mapped to $j(\mathcal{O}) \approx -8.96113233868328017909862973e22 \in \mathbb{R}$.

The space $S_1^-(283, \chi_{-283})$ is one-dimensional [Ser77]. We use `magma` to compute the Fourier expansion of the normalized generator g. It starts with

$$g(\tau) = q^2 - q^3 - q^5 + q^{12} - q^{14} - q^{18} + q^{19} + O(q^{20}).$$

If the generating series $\hat{\Phi}(\tau)$ is modular, we expect a direct relation to g. More precisely, we should be able to obtain g from $\hat{\Phi}(\tau)$ by passing to the class group of H under the map $\widehat{\mathrm{CH}}^1(\mathcal{O}_H)$ and then apply a linear functional. On the one hand, this does not seem to be the case because if we fix an embedding of H into $\mathbb{C}$ and then take the primes above $2, 3$ fixed by complex conjugation, we see that they define the same class in Cl_H, but the coefficients of g of index 2 and 3 are not equal. On the other hand, we have to take additive characters of the group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ in order to relate vanishing in the class group to vanishing Fourier coefficients. But this suggests that we should rather look at the reduction of g modulo 2.

The image of $\mathcal{Z}(m) = \mathcal{Z}(m, \mathcal{O}, 0)$, understood as an element of Cl_H, under any non-trivial linear map $\mathrm{Cl}_H \to \mathbb{Z}/2\mathbb{Z}$ is equal to $1 \in \mathbb{Z}/2\mathbb{Z}$ if the class does not vanish and 0, otherwise. We computed the data of the cycles $\mathcal{Z}(m)$ using Proposition 4.3.11, and generated the corresponding classes in the ideal class group of H for $m \leq 5000$. The coefficients of g modulo 2 all agree with this data. In Table 6.1 on page 144, we list the first 80 coefficients of g, as computed with `magma` and the corresponding data for the cycles $\mathcal{Z}(m)$.

The table is structured as follows. We first list m and the coefficient $c_g(m)$ of g of index m. and the quantity $\nu(m) = \nu_p(m)$ if $\mathrm{Diff}(m) = \{p\}$ and $\nu(m) = 0$, otherwise. Then again for $\mathrm{Diff}(m) = \{p\}$, ρ_0 denotes the number of ideals of norm m/p of k in the trivial class $[\mathcal{O}]$ and ρ_1 is the number of ideals of norm m/p in each of the other two ideal classes. The last column lists the class in Cl_H corresponding to the cycle $\mathcal{Z}(m)$. It is obtained as follows.

Suppose that $\mathrm{Diff}(m) = \{p\}$. Then the unique ideal $\mathfrak{p} \subset k$ above p is split completely in H. The class of $\mathcal{Z}(m)$ is obtained as the class of $\mathfrak{P}_0^{\rho_0}(\mathfrak{P}_1\mathfrak{P}_2)^{\rho_1}$, where $\mathfrak{P}_0$ is the unique ideal above $\mathfrak{p}$ that is fixed by complex conjugation. As representatives for the classes in Cl_H, we choose $[\mathcal{O}_H]$ and $[\mathfrak{P}_{2,1}], [\mathfrak{P}_{2,2}]$, $[\mathfrak{P}_{2,3}]$, where $2\mathcal{O}_H = \mathfrak{P}_{2,1}\mathfrak{P}_{2,2}\mathfrak{P}_{2,3} \subset \mathcal{O}_H$, such that, under the embedding specified above $\mathfrak{P}_{2,1}$ is fixed under complex conjugation. Note that $[\mathfrak{P}_{2,1}] = [\mathfrak{P}_{2,2}][\mathfrak{P}_{2,3}]$. This data completely determines the resulting classes. In our example, only the class of $\mathfrak{P}_{2,1}$ occurs.

Giving just one example is not enough to make a precise conjecture, but at least it makes it even more plausible that the generating series $\hat{\Phi}$ is indeed a modular form.

6.1.2 Speculation: A pullback formula

We note that our techniques should also allow us to relate this generating series to the corresponding generating series for the special cycles on a modular curve, which is modular by the Gross-Kohnen-Zagier theorem [GKZ87, Bor99, KRY06]. We only sketch the idea here and are not very precise.

We consider the lattice L for $\Gamma_0(N)$ considered in the last chapter and let

$$\hat{\Phi}_L(\tau) = c_1\phi_0 + \sum_{\mu\in L'/L}\sum_{m>0}\hat{\mathcal{Z}}(-m,\mu)e(m\tau)\phi_\mu \tag{6.1.2}$$

be the generating series of the arithmetic divisors $\hat{\mathcal{Z}}(-m,\mu)$. Here $\hat{\mathcal{Z}}(-m,\mu)$ is given as a pair $(\mathcal{Z}(-m,\mu),\Phi^L_{m,\mu}(z))$, where $\Phi^L_{m,\mu}(z)=\Phi_L(G_{m,\mu},z)$ is the lift of a harmonic weak Maaß form $G_{m,\mu}$ in $H_{1/2,L}$ with principal part $\frac{1}{2}q^{-m}(\phi_\mu+\phi_{-\mu})$. The constant term c_1 is the first Chern class of the line bundle of modular forms of weight 1.

We calculated the pullback of $\mathcal{Z}(-m,\mu)$ to the moduli stack C_D in Section 4.6 Using the methods that we used in Chapter 5, it should be possible to express the pullback of the generating series in (6.1.2) as

$$\mathbf{j}_D^*\hat{\Phi}_L(\tau)=\hat{\Phi}(\tau)\otimes\Theta_{\mathcal{N}}.$$

Here, the pullback of the Green function is given by evaluating at the CM cycle or in our setup rather

$$\mathbf{j}_D^*\Phi^L_{m,\mu}(z)=\sum_\sigma\Phi^L_{m,\mu}((z_P,\sigma(h)))\sigma.$$

We have already seen that the non-archimedean part of the pullback is indeed given by such an expression. The identity for the Green functions is not so clear. We sketch how we could proceed. We define for $f\in A_{k,\mathcal{P}\oplus\mathcal{N}}$ and $g\in A_{l,\mathcal{N}}$ the operator

$$\langle f,g\rangle_{L,P}=v^l\sum_{\beta\in P'/P}\left(\sum_{\nu\in N'/N}f_{\beta+\nu}\overline{g}_\nu\right)\phi_\beta,$$

which maps $A_{k,P\oplus N}\times A_{l,N}$ to $A_{k-l,P}$. Here, we simply denoted by $A_{k,L}$ the space of S_L-valued functions transforming with representation ρ_L and the other ones analogously. We have that

$$\mathbf{j}_D^*\Phi^L_{m,\mu}(z)=\sum_\sigma\Phi_L((z_P,h(\sigma)),G_{m,\mu})\sigma=\sum_\sigma\Phi_P((z_P,h(\sigma)),\langle G_{m,\mu},\Theta_{\mathcal{N}}\rangle_{L,P}).$$

This is basically the seesaw identity we have used in a special case in Chapter 5. The problem here is that $\langle G_{m,\mu},\Theta_N\rangle_{L,P}$ is not a harmonic weak Maaß form. If it was nevertheless possible to show an identity like

$$\Phi_P((z_P, h(\sigma)), \langle G_{m,\beta+\nu}, \Theta_N\rangle_{L,P}) \overset{??}{=} \sum_n \Phi_P(g_{m-n^2/4N,\beta}, h(\sigma)),$$

then this would imply the desired equality. At least the principal parts of the sum of the input functions on the right hand side and the input on the left hand side agree. But Theorem 3.3.6 has to be checked carefully.

We note that even with such an identity of generating series at hand, it is probably not possible to show the modularity of the generating series $\hat{\Phi}$ using that $\hat{\Phi}_L$ is known to be modular. This is because we have to pass to the Chow group with rational coefficients and since the class group is finite, this does not give us any non-trivial information about $\hat{\Phi}$.

6.2 A numerical example

6.2.1 Setup

The smallest prime $p \equiv 3 \bmod 4$ such that $k = \mathbb{Q}(\sqrt{-p})$ has class number $h_k > 1$ is $p = 23$. In this case $h_k = 3$. Therefore, the structure of the class group is obviously $\mathrm{Cl_k} \cong C_3$. The three ideal classes can, for instance, be represented by the ideals $\mathcal{O} = (1), \mathfrak{a} = \big(2, (2+\sqrt{-23})/2\big)$ and $\bar{\mathfrak{a}}$. The two corresponding scalar valued theta functions have Fourier expansions starting with

$$\begin{aligned}\theta_{\mathcal{O}}(\tau) &= 1 + 2q + 2q^4 + 4q^6 + 4q^8 + 2q^9 + 4q^{12} + O(q^{15})\\ \theta_{\mathfrak{a}}(\tau) &= 1 + 2q^2 + 2q^3 + 2q^4 + 2q^6 + 2q^8 + 2q^9 + 4q^{12} + 2q^{13} + O(q^{15}).\end{aligned}$$

The difference

$$g(\tau) = \frac{1}{2}(\theta_{\mathcal{O}}(\tau) - \theta_{\mathfrak{a}}(\tau)) = q - q^2 - q^3 + q^6 + q^8 - q^{13} + O(q^{15})$$

can easily be identified as $\eta(\tau)\eta(23\tau)$. It is a normalized newform and obviously does not have any zeroes on $\mathbb{H}$. The space $M_1(\Gamma_0(23), \chi_{-23})$ is 2-dimensional and spanned by g and the genus Eisenstein series

$$\begin{aligned}E(\tau) &= \frac{1}{3}(\theta_{\mathcal{O}}(\tau) + 2\theta_{\mathfrak{a}}(\tau))\\ &= 1 + \frac{2}{3}q + \frac{4}{3}q^2 + \frac{4}{3}q^3 + 2q^4 + \frac{8}{3}q^6 + \frac{8}{3}q^8 + 2q^9 + 4q^{12} + \frac{4}{3}q^{13} + O(q^{15}).\end{aligned}$$

6.2.2 The Petersson norm of g

The Petersson norm of g, can be identified as the theta lift of $\frac{3}{4}g$. We can rewrite this statement as

$$(g,g)_{\Gamma_0(23)} = (g, \frac{3}{4}(\theta_{\mathcal{O}} - E))_{\Gamma_0(23)} = \frac{3}{4}(\mathcal{S}_L(g), \Theta(\tau,1)).$$

Here, $\Theta(\tau,h)$ denotes the vector valued theta function corresponding to the lattice $\mathcal{O}$ with quadratic form $\mathrm{N}(x)$. In Section 5.2, we identified this theta lift as the CM value of the Borcherds lift

$$(\mathcal{S}_L(g), \Theta(\tau,h)) = -\frac{2}{12}\log\left|\Psi((z_{23},h), \mathcal{S}_L(g)\otimes G)\right|^2,$$

where the point $(z_{23},1)$ can be identified with

$$\frac{-23+\sqrt{-23}}{64} \in \mathbb{H}$$

under the isomorphism $X_K \cong Y_0(23)$.

The product $\mathcal{S}_L(g)\otimes G$ can be identified with a scalar valued modular form of weight $1/2$ for $\Gamma_0(92)$ having a Fourier expansion starting with

$$f(\tau) := \sum_{\mu}(\mathcal{S}_L(g)\otimes G)_\mu(92\tau) = 2q^{-19}+2q^{-15}+2q^{-11}+2q+20q^4-20q^8+2q^9-20q^{12}+O(q^{15}).$$

Consequently, the divisor of $\Psi((z,h),\mathcal{S}_L(g)\otimes G)^2$ on $\mathbb{H}$ is equal to

$$Z\left(\frac{19}{92},(\pm1,1)\right) - Z\left(\frac{15}{92},(\pm5,1)\right) - Z\left(\frac{11}{92},(\pm7,1)\right).$$

It follows from Proposition 4.6.5, that the value $\Psi((z,h),\mathcal{S}_L(g)\otimes G)$ is a *unit* in the Hilbert class field H of k and we would like to test this assertion and the constructions in the previous chapter. Using the method described in the proof of Proposition 5.3.15, it is easy to see that the Weyl vectors at ∞ and 0 are equal to 4. We obtain

$$\begin{aligned}\Psi((z,h),\mathcal{S}_L(g)\otimes G)^2 &= q^4\prod_{n=1}^{\infty}(1-q^n)^{c_f(n^2)}\\ &= q^4-2q^5-19q^6+38q^7+174q^8-342q^9-1029q^{10}+O(q^{11}).\end{aligned}$$

The function $f_{23}(\tau) = \frac{\theta_{\mathcal{O}}(\tau)}{g(\tau)} - 3$ is a modular function for $\Gamma_0(23)$ with trivial character and has a Fourier expansion starting with

$$f_{23}(\tau) = q^{-1}+4q+7q^2+13q^3+19q^4+33q^5+47q^6+74q^7+106q^8+154q^9+O(q^{10}).$$

It is invariant under the Fricke involution. In fact, the extension of $\Gamma_0(23)$ by the Fricke involution has genus zero and f_{23} generates the function field of $X_0(23)$. Using this information it is not hard to see that

$$\Psi((z,1),\mathcal{S}_L(g)\otimes G)^2 = P(f_{23}(z)),$$

where P is the polynomial given by

$$P(x) = \prod_{z\in\operatorname{div}(\Psi^2)} (x - f_{23}(z)) = \prod_{d>0}\prod_{z\in Z(d/92,d)} (x - f_{23}(z))^{c_f(-d)}.$$

(Again, note that since f_{23} is invariant under the Fricke involution and therefore we do not consider *both*, $Z(m,\mu)$ and $Z(m,-\mu)$, in the product above.) We use `sage` [S+13] to compute this in our case and obtain

$$P(x) = \frac{(x+3)^2}{(x^2+3x+3)^2(x+1)^2}.$$

Fix an embedding of H into $\mathbb{C}$ and let $\alpha \in H$ be the unit which is the unique real root of the polynomial $X^3 - X - 1$ with complex embedding equal to $1.324717\ldots$. Then the value of f_{23} at z_{23} is given by $-\alpha - 2$ and we obtain that

$$\Psi((z_{23},1),\mathcal{S}_L(g)\otimes G)^2 = \alpha^{-24}.$$

Collecting all of the factors we can see that

$$(g,g)_{\Gamma_0(23)} = 3\log|\alpha|.$$

This is exactly the example that Stark gives on page 91 of [Sta75].

6.2.3 Testing the pullback formula from Section 4.6

Now we can go ahead and use f_{23} to produce modular functions on $X_0(23)$ which are invariant under the Fricke involution and have a divisor given by Heegner divisors to test Proposition 4.6.5. For instance, the class number of $\mathbb{Q}(\sqrt{-43})$ is equal to one and the value of f at the corresponding CM point is 1. The divisor of $f_{23}(z) - 1$ is equal to $\frac{1}{2}(Z(-43/92, 43) + Z(-43/92, -43))$. From the formula in Proposition 4.6.5, we expect that the value of $f_{23}(\tau) - 1$ at z_{23} has a positive valuation only at primes of H lying above 5 because

$$\frac{43}{92} - \frac{1}{4} = \frac{5}{23}$$

is the only positive term that occurs and we have $\operatorname{Diff}(5/23) = 5$. The exact valuation can now be obtained from Proposition 4.3.11. We have to calculate the multiplicities of $\mathcal{Z}(5/23, \mathcal{O}, \pm\beta)$ for $\beta \in P'/P$ with $Q(\beta) + 5/23 \in \mathbb{Z}$. Since $\frac{5\cdot 23}{23\cdot 5} = 1$, $\nu_5(5/23) = 1$ and $o(5/23) = 0$, only one prime can occur, namely the unique one that is fixed by complex conjugation. It occurs with multiplicity $2\cdot 2\cdot 1/2 = 2$, according to the formula. We can confirm this using `sage` [S+13] again. The value in question is $-\alpha - 3$. It is the unique real root of $x^3 + 9x^2 + 26x + 25$. Using the following steps, we can confirm the formula in this case. First, we initialize the imaginary quadratic field and the Hilbert class field.

```
sage: k.<a> = QuadraticField(-23)
sage: k.class_number()
3
sage: H.<b> = k.hilbert_class_field()
sage: bc = b.complex_embeddings()[0]
sage: Habs.<c> = NumberField(H.absolute_polynomial(), embedding=bc)
sage: G = Habs.galois_group()
sage: default_embedding = Habs.specified_complex_embedding()
sage: cc = G.complex_conjugation()
```

Here, `Habs` is used to consider the Hilbert class field as an absolute field extension of $\mathbb{Q}$. Now we assign the correct value to α.

```
sage: p = x^3-x-1
sage: for z,m in p.roots(Habs):
....:     if cc(z)==z:
....:         alpha = z
....:         break
....:
sage: default_embedding(alpha)
1.32471795724475? + 0.?e-14*I
```

We compute the prime factorization of the ideal generated by $-\alpha - 3$.

```
sage: id=Habs.ideal(-eps-3)
sage: factors = id.factor()
sage: for P,m in factors:
....:     print "norm:", P.norm().factor(), " ; multiplicity:", m
....:
norm: 5^2 ; multiplicity: 2
```

This shows that there is a unique prime in the factorization and it has multiplicity 2. Finally, we check if this is indeed the unique prime fixed by complex conjugation.

```
sage: P=id.factor()[0][0]
sage: cc(P)==P
True
```

m	$c_g(m)$	$\nu(m)$	ρ_0	ρ_1	class
1	0	0	0	0	$[\mathcal{O}]$
2	1	1	1	0	$[\mathfrak{P}_{2,1}]$
3	-1	1	1	0	$[\mathfrak{P}_{2,1}]$
4	0	0	0	0	$[\mathcal{O}]$
5	-1	1	1	0	$[\mathfrak{P}_{2,1}]$
6	0	0	0	0	$[\mathcal{O}]$
7	0	0	0	0	$[\mathcal{O}]$
8	0	2	1	0	$[\mathfrak{P}_{2,1}]$
9	0	0	0	0	$[\mathcal{O}]$
10	0	0	0	0	$[\mathcal{O}]$
11	0	0	0	0	$[\mathcal{O}]$
12	1	1	1	0	$[\mathfrak{P}_{2,1}]$
13	0	0	0	0	$[\mathcal{O}]$
14	-1	1	0	1	$[\mathfrak{P}_{2,1}]$
15	0	0	0	0	$[\mathcal{O}]$
16	0	0	0	0	$[\mathcal{O}]$
17	0	1	1	0	$[\mathcal{O}]$
18	-1	1	1	0	$[\mathfrak{P}_{2,1}]$
19	1	1	1	0	$[\mathfrak{P}_{2,1}]$
20	1	1	1	0	$[\mathfrak{P}_{2,1}]$
21	1	1	0	1	$[\mathfrak{P}_{2,1}]$
22	1	1	0	1	$[\mathfrak{P}_{2,1}]$
23	0	0	0	0	$[\mathcal{O}]$
24	0	0	0	0	$[\mathcal{O}]$
25	0	0	0	0	$[\mathcal{O}]$
26	1	1	0	1	$[\mathfrak{P}_{2,1}]$
27	0	2	1	0	$[\mathfrak{P}_{2,1}]$
28	0	0	0	0	$[\mathcal{O}]$
29	0	0	0	0	$[\mathcal{O}]$
30	-2	1	0	0	$[\mathcal{O}]$
31	1	1	1	0	$[\mathfrak{P}_{2,1}]$
32	-1	3	1	0	$[\mathfrak{P}_{2,1}]$
33	-1	1	0	1	$[\mathfrak{P}_{2,1}]$
34	0	0	0	0	$[\mathcal{O}]$
35	1	1	0	1	$[\mathfrak{P}_{2,1}]$
36	0	0	0	0	$[\mathcal{O}]$
37	0	1	1	0	$[\mathcal{O}]$
38	0	0	0	0	$[\mathcal{O}]$
39	-1	1	0	1	$[\mathfrak{P}_{2,1}]$
40	0	0	0	0	$[\mathcal{O}]$
41	0	0	0	0	$[\mathcal{O}]$
42	0	0	0	0	$[\mathcal{O}]$
43	1	1	1	0	$[\mathfrak{P}_{2,1}]$
44	0	0	0	0	$[\mathcal{O}]$
45	1	1	1	0	$[\mathfrak{P}_{2,1}]$
46	-1	1	0	1	$[\mathfrak{P}_{2,1}]$
47	-1	1	1	0	$[\mathfrak{P}_{2,1}]$
48	1	1	1	0	$[\mathfrak{P}_{2,1}]$
49	0	0	0	0	$[\mathcal{O}]$
50	-1	1	1	0	$[\mathfrak{P}_{2,1}]$
51	0	0	0	0	$[\mathcal{O}]$
52	0	0	0	0	$[\mathcal{O}]$
53	0	1	1	0	$[\mathcal{O}]$
54	0	0	0	0	$[\mathcal{O}]$
55	-1	1	0	1	$[\mathfrak{P}_{2,1}]$
56	0	2	0	1	$[\mathfrak{P}_{2,1}]$
57	0	0	0	0	$[\mathcal{O}]$
58	-1	1	0	1	$[\mathfrak{P}_{2,1}]$
59	0	0	0	0	$[\mathcal{O}]$
60	0	0	0	0	$[\mathcal{O}]$
61	0	0	0	0	$[\mathcal{O}]$
62	0	0	0	0	$[\mathcal{O}]$
63	0	0	0	0	$[\mathcal{O}]$
64	0	0	0	0	$[\mathcal{O}]$
65	-1	1	0	1	$[\mathfrak{P}_{2,1}]$
66	0	0	0	0	$[\mathcal{O}]$
67	0	1	1	0	$[\mathcal{O}]$
68	0	1	1	0	$[\mathcal{O}]$
69	1	1	0	1	$[\mathfrak{P}_{2,1}]$
70	0	0	0	0	$[\mathcal{O}]$
71	0	0	0	0	$[\mathcal{O}]$
72	0	2	1	0	$[\mathfrak{P}_{2,1}]$
73	0	0	0	0	$[\mathcal{O}]$
74	0	0	0	0	$[\mathcal{O}]$
75	1	1	1	0	$[\mathfrak{P}_{2,1}]$
76	-1	1	1	0	$[\mathfrak{P}_{2,1}]$
77	0	0	0	0	$[\mathcal{O}]$
78	0	0	0	0	$[\mathcal{O}]$
79	0	1	1	0	$[\mathcal{O}]$
80	1	1	1	0	$[\mathfrak{P}_{2,1}]$

Table 6.1: The table displays the arithmetic data of the cycles $\mathcal{Z}(m)$ for $D = -283$ alongside with the coefficients $c_g(m)$ of g.

List of Symbols

(f, g)	The Petersson inner product of f and g (vector valued modular forms), 29
$(f, g)_\Gamma$	The Petersson inner product of f and g, with respect to a fundamental domain of Γ, 25
(a, b)	$= \gcd(a, b)$, the greatest common divisor of a and b
$\langle \cdot, \cdot \rangle$	The $\mathbb{C}$-bilinear pairing between $S(V(\mathbb{A}))$ and its dual, 24
$(\cdot, \cdot)_p$	The p-adic Hilbert symbol, 46
$\vert_k$	The Petersson slash operator, 25
$\vert_{k,L}$	The Petersson slash operator on vector valued functions, 26
$\left(\frac{a,b}{F}\right)$	A quaternion algebra over F, 38
A	In Chapter 5: a fixed positive integer, coprime to D, 109
$\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$	Fractional ideals
$\mathbb{A}$	The adeles over $\mathbb{Q}$, 18
$\mathbb{A}_f$	The finite adeles over $\mathbb{Q}$, 19
$\mathbb{A}_k$	The adeles over k, 18
$\mathbb{A}_{k,f}$	The finite adeles over k, 19
B	In Chapter 5: a fixed integer, 109
$\mathbb{B}_p$	The quaternion algebra over $\mathbb{Q}$ which is exactly ramified at p and ∞, 86
$\mathbb{C}$	The field of complex numbers
C	In Chapter 5: a fixed integer, 109
$\mathcal{C}_D$	The moduli stack over $\operatorname{Spec} \mathbb{Z}$ of elliptic curves with CM by $\mathcal{O}_D$, 80
$\mathcal{C}_D^+$	The moduli stack over $\operatorname{Spec} \mathcal{O}_D$ of elliptic curves with CM by $\mathcal{O}_D$, 80

$\mathbf{C}_D$	The coarse moduli scheme of C_D, 81
$\mathbf{C}_D^+$	The coarse moduli scheme of C_D^+, 81
χ_L	The character of the Weil representation ρ_L, 34
χ_D	The primitive character of conductor D, 47
Cl_{k}	The class group of the field k, 45
$\mathrm{Cl}_{\mathrm{k}}^*$	The group of class group characters, 46
$c_P(h,m,\beta)$	The coefficient of index (m,β) of $\widetilde{\Theta}_P^+(\tau,h)$, 113
$C_{P,K}$	A certain cover of the CM cycle, 70
CT	The constant term in the Fourier expansion of the Laurent series S, 66
C_V	The Clifford algebra of V, 15
C_V^0	The even part of the Clifford algebra of V, 16
$\mathfrak{d}_k$	The different ideal of k, 83
D	A negative fundamental discriminant, usually assumed to be odd, 45
$\mathrm{diag}(a_1,\dots,a_n)$	The $n \times n$ diagonal matrix with diagonal $(a_1,\dots,a_n)$, 95
$\mathrm{Diff}(m)$	A finite set of primes, 86
$\mathbb{D}$	The symmetric domain, usually identified with the Grassmannian, 18
$\mathbb{D}_x$	$= \{z \in \mathbb{D} \mid z \perp x\}$, 20
E	An elliptic curve, 80
$e(x)$	$= e^{2\pi i x}$
$E_P(\tau)$	The holomorphic Eisenstein series in $M_{1,P}$, equal to $E_P(\tau,0)$, 67
$\widetilde{E}_P(\tau)$	A preimage of $E_P(\tau)$ under the ξ-operator, 70
$\mathcal{E}_P(\tau)$	A harmonic weak Maaß form, the value of the derivative at $s=0$ of $\hat{E}_P(\tau,s)$, 66
$\hat{E}_P(\tau,s)$	An incoherent Eisenstein series attached to P, 66
f^+	The holomorphic part of f, 27
f^-	The non-holomorphic part of f, 27

$\mathcal{F}$	The standard fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$, 43
$\mathcal{F}_T$	A truncated fundamental domain, 43
f^L	The image of f under the trace map tr, 30
f_M	The image of f under the restriction map res, 30
$\mathbb{F}_q$	A finite field with q elements
$\overline{\mathbb{F}}_q$	An algebraic closure of a finite field with q elements
γ	Often an element of SL_2, GL_2, 19
$\Gamma_0(N)$	A congruence subgroup, 19
$\Gamma_1(N)$	A congruence subgroup, 24
$\mathbb{G}_m$	The multiplicative group (as an algebraic group), 16
GSpin_V	The general spin group, 16
(h)	The ideal corresponding to an idele $h \in \mathbb{A}_{k,f}^\times$, 50
$[h]$	The ideal class of the ideal (h) associated with $h \in \mathbb{A}_{k,f}^\times$, 50
$\mathbb{H}$	The complex upper half-plane, $\mathbb{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$
H	$= \mathrm{GSpin}_V$, 17
	Also short for H_D, the Hilbert class field k_D, 80
$H_{k,L}(F)$	The subspace of harmonic weak Maaß forms with principal part in $F[q^{-1}]$, 69
h_k	The class number of the number field k, 59
$\mathcal{H}_{k,L}$	The space of harmonic weak Maass forms of weight k and representation ρ_L, 27
$H_{k,L}$	Harmonic weak Maaß forms $f \in \mathcal{H}_{k,L}$ such that $\xi_k(f) \in S_{k,L}$, 28
$\mathcal{I}_k$	The idele class group of the field k, 45
I_k	The finite idele class group of the field k, 45
$\Im(z)$	The imaginary part of z
$\mathbf{j}_D$	An isomorphism of stacks, 98

K	An open compact subgroup of $H(\mathbb{A}_f)$, 19
k	Often a field, sometimes a half-integer (a weight)
k_D	$= \mathbb{Q}(\sqrt{D})$, 80
$\kappa(\mathfrak{p})$	The residue field of $\mathfrak{p}$, 82
L	A lattice, 14
L^-	The quadratic module given by L together with the quadratic form $-Q$, 26
L'	The dual lattice of L, 15
$L(\mathrm{E}^{\mathrm{top}}, \iota)$	The lattice of special endomorphisms of $(\mathrm{E}^{\mathrm{top}}, \iota)$, 84
$L(E, \iota)$	The lattice of special endomorphisms of (E, ι), 86
$L_{m,\mu}$	A subset of $\mu + L$, 22
λ_r	A special element in $L_{m,\mu}$, 107
$M^!_{k,L}(F)$	The subspace of weakly holomorphic modular forms with principal part in $F[q^{-1}]$, 69
$M_{k,L}$	The space of holomorphic modular forms of weight k and representation ρ_L, 27
$M^!_{k,L}$	The space of weakly holomorphic modular forms of weight k and representation ρ_L, 27
$M_k(N, \chi)$	The space of modular forms for $\Gamma_0(N)$ and character χ, 25
$M_n(R)$	The ring of $n \times n$ matrices with entries in R, 19
μ_r	A special element in L'/L corresponding to $r \in \mathbb{Z}/2N\mathbb{Z}$, 96
$\mathcal{N}$	A one-dimensional negative definite lattice given by $\mathcal{N} = L \cap U^{\perp}$, 107
N	A natural number, usually the level of a lattice L, 15
	Particularly, $N = A\,\lvert D\rvert$ in Chapter 5, 109
	Also a negative definite lattice, 66
$\mathrm{N}(x)$	A norm map, 16
$\mathrm{O}_V, \mathrm{O}_{(V,Q)}$	The orthogonal group of (V, Q), 14

$\mathcal{O}_{\mathfrak{c}_0,\lambda_0,\mathbb{B}_p}$	A certain order in the quaternion algebra $\mathbb{B}_p$, 89
$\mathcal{O}_D$	The order of discriminant D in $k_D = \mathbb{Q}(\sqrt{D})$, 45
$\hat{\mathcal{O}}_D^\times$	$\prod_{\mathfrak{p}\nmid\infty} \mathcal{O}_{\mathfrak{p}}^\times$, 45
$\mathcal{O}_H$	The ring of integers in $H = H_D$, 81
$\hat{\mathcal{O}}_{X,x}$	The completion of the étale local ring at x, 81
$\mathrm{O}(L'/L)$	The orthogonal group of the discriminant form L'/L, 15
$o(m)$	The number of primes $p \mid D$ such that $\mathrm{ord}_p(mD) > 0$, 91
$\mathfrak{P}$	A prime ideal of H_D above $\mathfrak{p}$, 81
$\mathfrak{p}$	A prime ideal, 17
p	A rational prime, 20
p_0	An auxiliary prime, 88
$\mathfrak{p}_0$	A prime ideal of k above p_0, 88
$\mathfrak{P}_0$	A certain prime of $\mathcal{O}_H$, 90
$P_f(\tau)$	The principal part of f, 28
ϕ_μ	The characteristic function of $\mu + L$, 21
$\Phi(Z(U), f)$	The average value of the theta lift on the CM cycle $Z(U)$, see equation (3.1.1), 66
P	A two-dimensional positive definite lattice, usually assumed to be isomorphic to a fractional ideal $\mathfrak{a}$ in an imaginary quadratic field, 50
	Often obtained as $P = L \cap U$, 66
$\mathcal{P}$	A two-dimensional positive definite lattice given by $\mathcal{P} = L \cap U$, 107
Q	A quadratic form, 13
q	Usually $q = e^{2\pi i\tau}$
$\mathbb{Q}_p$	The field of p-adic numbers
r	An integer such that $D \equiv r^2 \bmod 4N$, 98
$\Re(z)$	The real part of z

ρ_L	The Weil representation associated with L, 24
$\tilde{\rho}(n,\beta)$	The coefficient of index (n,β) of $E_P(\tau)$, 67
$\rho(n,[\mathfrak{c}])$	The number of integral ideals of norm n in the class $[\mathfrak{c}]$, 91
R	A fixed integer, $R = FD$ with $R^2 \equiv D \bmod 4A\,\lvert D\rvert$, 109
S	The matrix $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ or the element $\left(\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right), \sqrt{\tau}\right) \in \mathrm{Mp}_2(\mathbb{Z})$, 24
	Often also a scheme
$S_{k,L}(F)$	The subspace of cusp forms with all coefficients contained in the number field F, 69
$\sigma(\mathfrak{a})$	The image of $\mathfrak{a}$ under the Artin map $(\cdot, H/k)$, 17
$S_k(N,\chi)$	The space of cusp forms of weight k for $\Gamma_0(N)$ and character χ, 25
S_L	The span of the characteristic functions ϕ_μ in $S(V(\mathbb{A}_f))$, 21
$\mathrm{SO}_V, \mathrm{SO}_{(V,Q)}$	The special orthogonal group of (V,Q), 14
$S(V(Q_p))$	The space of Schwartz-Bruhat functions on $V(\mathbb{Q}_p)$, 21
$S(V(\mathbb{A}))$	The space of Schwartz functions on $V(\mathbb{A})$, 21
τ	$\tau = u + iv \in \mathbb{H}$
t	The number of prime divisors of D, 46
T	The matrix $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ or the element $\left(\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right), 1\right) \in \mathrm{Mp}_2(\mathbb{Z})$, 24
	Also GSpin_U, 48
$\tau(\mathfrak{a})$	The CM point in $\mathbb{H}$ corresponding to the ideal $\mathfrak{a}$, 53
$\theta_{\mathfrak{a}}$	The scalar valued theta function corresponding to the ideal (class) $\mathfrak{a}$, 47
$\Theta(k)$	The space of scalar valued theta functions for the field k, 47
$\Theta(P)$	The space of vector valued theta functions in $M_{1,P}$, 51
$\Theta_P(\tau,h)$	The vector valued theta function corresponding to the positive definite lattice P, 50
$\widetilde{\Theta}_P(\tau,h)$	A preimage of $\Theta_P(\tau,h)$ under the ξ-operator, 71
tr	The trace map, 19
U	A two-dimensional positive definite quadratic space, 48

V	A rational quadratic space over $\mathbb{Q}$, of signature $(2, n)$, 17
$W(a)$	$= W_k(a) = \int_{-2a}^{\infty} e^{-t} t^{-k}\, dt = \Gamma(1-k, -2a)$, 27
w_k	The number of roots of unity contained in k, 47
$w_{K,T}$	$= \lvert (T(\mathbb{Q}) \cap K_T) \rvert$, 65
$X_0(N)$	The compactification of the modular curve $Y_0(N)$, 95
$\mathscr{X}_0(N)$	The moduli stack of cyclic isogenies of degree N of generalized elliptic curves, 95
$\mathbf{X}_0(N)$	The coarse moduli scheme of $\mathscr{X}_0(N)$, 96
ξ	Usually, $\xi = \xi_k$, a differential operator, 28
X_K	A Shimura variety, 19
$Y_0(N)$	The modular curve $\Gamma_0(N)\backslash\mathbb{H}$, 19
$\mathscr{Y}_0(N)$	The moduli stack of cyclic isogenies of degree N of elliptic curves, 95
$\mathbf{Y}_0(N)$	The coarse moduli scheme of $\mathscr{Y}_0(N)$, 96
$\mathbb{Z}$	The ring of integers
z	A complex number, $z = x + iy$ with $x, y \in \mathbb{R}$, 19
$\mathbb{Z}_{>0}$	The set of positive integers
$\hat{\mathbb{Z}}$	$= \prod_{p<\infty} \mathbb{Z}_p$, 21
$Z(m, \varphi)$	A special divisor, 22
$Z(m, \mu)$	The special divisor corresponding to the function ϕ_μ, 22
$\mathscr{Z}(m, \mathfrak{a}, \mu)$	For $m > 0$ a special cycle on C_D, supported at a single prime, 86
	Also the corresponding divisor on $\operatorname{Spec} \mathcal{O}_H$, 87
$\mathscr{Z}(m, \mathfrak{a}, \mu)_{\mathfrak{P}}$	The multiplicity of $\mathscr{Z}(m, \mathfrak{a}, \mu)$ at $\mathfrak{P}$, 87
$\mathscr{Z}(m, \mu)$	A Deligne-Mumford stack, defining a divisor on $\mathscr{X}_0(N)$, 97
$\mathbb{Z}_p$	The p-adic integers
$\mathscr{Z}^{\mathrm{top}}(m, \mathfrak{a}, \mu)$	For $m < 0$ a special cycle on C_D, 85
	Also the corresponding Arakelov divisor on $\operatorname{Spec} \mathcal{O}_H$, 85

$z_U^{\pm}$	The two CM points corresponding to U, 70
$Z(U)_K$	A CM cycle, see equation (3.0.2), 65
$Z(U)$	A CM cycle, short for $Z(U)_K$, see equation (3.0.2), 65

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