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The marginal stability of the metastable TAP states

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Abstract

The existing investigations on the complexity are extended. In addition to the Edward–Anderson parameter q_2 the fourth moment $q_4 = 1/N \sum_i m_i^4$ of the magnetizations m_i is included to the set of constrained variables and the constrained complexity $\Sigma(T; q_2, q_4)$ is numerically determined. The maximum of $\Sigma(T; q_2, q_4)$ (representing the total complexity) sticks at the boundary for temperatures at and below a new critical temperature. This implies marginal stability for the nearly all metastable states. The temperature dependence of the lowest value of the Gibbs potential consistent with various physical requirements is presented.

Keywords: spin-glass, TAP equation, complexity, marginal stability

(Some figures may appear in colour only in the online journal)

1. Introduction

The Thouless–Anderson–Palmer (TAP) approach [1–3] to the Sherrington and Kirkpatrick (SK) model [4] plays a central role in the attempt to understand the physics of spin glasses and related interdisciplinary problems (neural networks, computer science, theoretical biology, econophysics etc). The system exhibits meta-stability below the spin glass temperature.

The well established work of Bray and Moore (BM) [5] leads to a finite complexity below the spin glass temperature, which implies an exponential increase of the number of metastable TAP states with increasing system size. It is essential to count exclusively the physical states and neglect non-physical ones. However, the BM and subsequent work [6–8] do not completely satisfy this requirement.



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An alternative method to work out the characteristic properties of the metastable TAP states are numerical investigations [7, 9–12] based on iteration techniques or phenomenological dynamics for systems of finite size. The numerically determined fix-points are interpreted as metastable TAP states. Extrapolation to infinite size systems results in the conclusion that these states are marginally stable. Such an extrapolation procedure, however, leads generally to some uncertainty. Note in this context, that the numerical investigations are usually performed for systems with just some hundreds of spins. Increasing the system size results in a drastic reduction of the rate of finding a TAP state.

In this work the existing approaches on the complexity are extended by the additional inclusion of the fourth moment of the magnetizations to the set of constrained variables. This procedure enables a complete counting of the physical TAP states. In section 2 the results of the calculation are presented. The total complexity and various averages are discussed in some detail in section 3. Finally conclusions are drawn in section 4.

2. Calculation

More than four decades ago SK introduced the spin glass Hamiltonian

$$H = -\frac{1}{2} \sum_{i \neq j} J_{ij} S_i S_j$$

of N Ising spins ($S_i = \pm 1$). The bonds J_{ij} are independent random variables with zero means and standard deviations $N^{-1/2}$ (which fixes the spin glass temperature to $T_{\text{sg}} = 1$). According to the TAP approach [1–3] the energy

$$U = N(w - \frac{1}{2T}(1 - q_2)^2), \quad w = -\frac{1}{2N} \sum_{i \neq j} J_{ij} m_i m_j, \quad (1)$$

the entropy

$$S = \sum_i s_0(m_i) - \frac{N}{4T^2}(1 - q_2)^2, \quad s_0(m) = -\frac{1+m}{2} \ln \frac{1+m}{2} - \frac{1-m}{2} \ln \frac{1-m}{2} \quad (2)$$

and consequently the Gibbs potential $G(T, m_i) = U - TS$ are given in terms of the local magnetizations m_i and the temperature T , where q_k is defined as

$$q_k = N^{-1} \sum_i m_i^k \quad (k = 2, 4). \quad (3)$$

In general the local magnetic fields h_i are determined by the TAP equations $h_i = \partial G / \partial m_i$. This work is exclusively restricted to the case $h_i = 0$ and the TAP equations reduce to

$$G_i \equiv \frac{T}{2} \ln \frac{1+m_i}{1-m_i} - \sum_j J_{ij} m_j + \frac{1}{T}(1 - q_2)m_i = 0, \quad (4)$$

where the definition G_i is introduced for later use. As shown by the present author [2] the m_i have to satisfy the two convergence criteria

$$c_1 \equiv T^2 - 1 + 2q_2 - q_4 > 0, \quad c_2 \equiv T^2 - 2q_2 + 2q_4 > 0. \quad (5)$$

Criterion c_1 is generally accepted and is related to the de Almeida Thouless condition [13] for the SK solution. Criterion c_2 is controversial [7, 14].

These criteria are of some importance for the present work. They result from an application of a theorem of Pastur [15], which requires the in-dependency of the variables m_i from the bonds J_{ij} . For a Gibbs potential, indeed these magnetizations m_i are the free and independent variables. Note, that for every thermodynamic stability analysis one has to study the influence of *all possible m_i values including the J_{ij} independent values*. This requirement is also essential for the integration procedure used in this work. Thus the application of the theorem of Pastur is justified (compare¹) and $c_2 > 0$ is a necessary (but probably not sufficient [17]) convergence condition for the expansion [2].

Further support for validity of both criteria result from the fact, that they are necessary to prove the positivity of the entropy $S(T, m_i)$ [16]. Simple examples leading to a negative entropy, if $c_2 < 0$, can easily be constructed (see²). Consequences of the criteria (5) to the T -dependence of q_2 and q_4 has already been presented in [18].

The present approach is related to the studies [5–7] of the complexity

$$\Sigma(T, \Omega) = N^{-1} \log \mathcal{N}(T, \Omega) \quad (6)$$

which describes the extensive number \mathcal{N} of solutions of TAP equations (4)

$$\mathcal{N}(T, \Omega) = \int_{-1}^1 \langle \prod_i dm_i \delta(G_i) \mathcal{C}(\Omega) |\det \partial G_i / \partial m_k| \rangle_J, \quad (7)$$

where G_i is defined in equation (4) and $\langle \cdot \cdot \cdot \rangle_J$ denotes the J_{ij} averaging. Constraints are considered in the term $\mathcal{C}(\Omega)$, which is chosen in this work as

$$\mathcal{C}(\Omega) = \delta(q_2 - \frac{1}{N} \sum_i m_i^2) \delta(q_4 - \frac{1}{N} \sum_i m_i^4) \delta(w - \frac{1}{2N} \sum_{ij} m_i J_{ij} m_j) \quad (8)$$

and the set of constrained variables is $\Omega = \{q_2, q_4, w\}$. The inclusion of q_4 is new, but is essential to take into account the restrictions due to criteria (5). Note that q_4 is a sum of single particle terms and the modifications due to such terms are simple. The use of w instead of the Gibbs potential is a technical tool simplifying the calculation.

The further calculation is lengthy but straightforward due to the existing previous work [5–7]. Following these investigations the Fourier representations for the arising δ -functions and the exponential representation of the determinant using anti-commuting Grassmann variables are employed to rewrite $\Sigma(T, \Omega)$ in a form which permits to perform the J_{ij} averaging via Gaussian integrations. The remaining integrations are performed by the steepest decent method which is valid due to the large N limit. Adopting the notation of [6] the calculation of the complexity finally leads to

$$\Sigma(T, \Omega) = \Sigma_0 + \log \int dm e^{\mathcal{L}(\Omega, m)}, \quad (9)$$

¹ Owen [14] investigates a different question, as he considers m_i , which depend on the J_{ij} via the TAP equations and the Pastur theorem cannot be applied. There is no disagreement to [2], which investigates the Gibbs potential with free and independent m_i . Moreover, it is generally impossible to conclude anything on the convergence of a series from a partial re-summation, as done in [14].

² Consider a special set with $m_i^2 = 1 - T$, which implies $c_1 = 0$ and $c_2 = T(3T - 2)$. For $T < 2/3$ condition c_2 is violated. The entropy is given by $S/N = s_0(\sqrt{1-T}) - 1/4$, which is negative for all temperatures smaller than 0.256.

where

$$\Sigma_0 = -\lambda q_2 - \mu q_4 - \Delta(1 - q_2) - \frac{\Delta^2}{2\beta^2} - \frac{1}{2} \log(2\pi\beta^2 q_2) + \frac{\beta^2 v^2 q_2^2}{4} \quad (10)$$

and

$$\mathcal{L}(\Omega, m) = \lambda m^2 + \mu m^4 - \frac{[\tanh^{-1}(m) - \Delta m]^2}{2\beta^2 q_2} - \log(1 - m^2) \quad (11)$$

with $\beta = 1/T$. The new variables λ, μ and v enter via the Fourier representations of the δ -functions and are therefore conjugated to q_2, q_4 and w , respectively. Similarly Δ results from $\delta(G_i)$. The stationary of $\Sigma(T, \Omega)$ with respect to λ, Δ, μ, v finally leads to

$$q_2 = \langle\langle m^2 \rangle\rangle \quad (12)$$

$$\beta w = -\frac{\beta^2}{2} v q_2^2 - q_2 \Delta - \beta^2 q_2 (1 - q_2) \quad (13)$$

$$\Delta = -\frac{\beta^2}{2} (1 - q_2) + \frac{1}{2q_2} \langle\langle m \tanh^{-1}(m) \rangle\rangle - \frac{\beta^2}{2} v q_2 \quad (14)$$

$$q_4 = \langle\langle m^4 \rangle\rangle \quad (15)$$

with

$$\langle\langle F(m) \rangle\rangle = \frac{1}{\int dm e^{\mathcal{L}(\Omega, m)}} \int dm F(m) e^{\mathcal{L}(\Omega, m)}. \quad (16)$$

The set of equations (9)–(14) correspond to the equations (56)–(61) of [6] with the replacements $f \rightarrow w, \phi_0 \rightarrow 0, u \rightarrow v, q \rightarrow q_2, B \rightarrow 0$ and the additional terms proportional to μ resulting from the inclusion of q_4 . The apparent differences of equation (10) and equation (56) of [6] result from a simplification using equation (13). Setting $B = 0$ corresponds to an exclusion of a non-physical solution. Equation (15) is obvious and results from the stationary with respect to μ . For more details of the calculation and for the performed approximations it is referred to the previous work [5–7].

Note, however, that as long as q_2 and q_4 satisfy the criteria (5) the value of the determinant $\det \partial G_k / \partial m_l$ in equation (7) is always positive. All previous work disregards the modulus with not completely satisfying arguments.

3. Applications

3.1. Total complexity

As first application the BM work [5] for the total complexity $\Sigma_{\text{tot}}(T)$ is reanalysed, which describes the total number of TAP states.

Setting $v = 0$ in equations (9)–(15) the resulting equations determine the complexity $\Sigma(T; q_2, q_4, v = 0)$ for fixed values of q_2 and q_4 . These equations are numerical investigated for all possible values of q_2, q_4 and for all temperatures $T < 1$.

As example $\Sigma(T = 0.4; q_2, q_4, v = 0)$ is plotted in the q_2 – q_4 plane in figure 1. The region of allowed q_2 – q_4 values is restricted by $q_2^2 \leq q_4 \leq q_2$ and by the criteria (5). The cyan and the red boundaries represent the lines $c_1 = 0$ and $c_2 = 0$, respectively. The physical relevant region $c_2 > 0$ is above the red borderline. The region below the red line with $c_2 < 0$ has no physical significance.

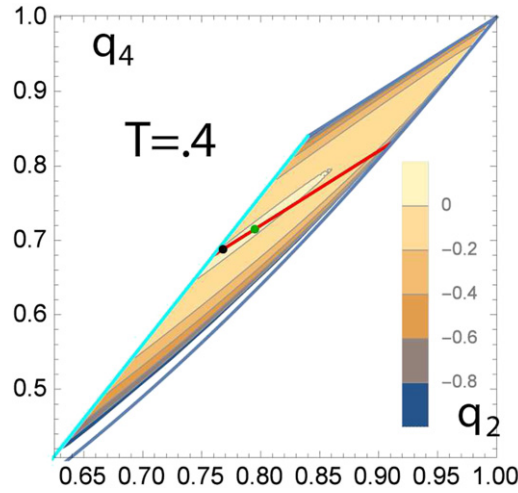


Figure 1. Contour-plot of the complexity $\Sigma(T; q_2, q_4, v = 0)$ at $T = 0.4$: the cyan and the red boundaries represent the lines $c_1 = 0$ and $c_2 = 0$. The region above the red line is the relevant area with $c_2 > 0$. The maximum value represents the total complexity $\Sigma_{\text{tot}}(T = 0.4)$ and its position is indicated by the green dot. The black dot indicates the position of g_0 .

The absolute maximum of $\Sigma(T; q_2, q_4, v = 0)$ in the q_2 - q_4 plane represents $\Sigma_{\text{tot}}(T)$ and can generally be located in the interior or on the boundary of the relevant region. For temperatures $T \geq T_1$ the maximum is within the region and for $T \leq T_1$ the maximum is located on the boundary $c_2 = 0$. The numerical value of the critical temperature T_1 is given by

$$T_1 = 0.367. \tag{17}$$

The coordinates of the maxima are determined by $\mu = 0$ and by $\partial\Sigma/\partial q_2 = 0$ or by $c_2 = 0$. At T_1 the internal maximum coincides with the boundary maximum.

In addition to figure 1, which gives an overview, some details are presented in figure 2 for $T = 0.6$, for $T = 0.37$ and for $T = 0.075$. The internal maxima and the boundary maxima are marked by a green and red dots, respectively. (On the scale of figure 1 these two minima are not separated.)

$\Sigma_{\text{tot}}(T)$ has two branches Σ^i and Σ^b resulting from the two different maxima. Σ^i is identical to BM and represents the stable branch for $T > T_1$. The quantity characteristic for the transition is c_2^i , the c_2 for the internal maximum, which tends to zero for $T \rightarrow T_1$ from above. Below T_1 criterion c_2^i is negative, the border maximum is the physical one and the branch Σ^b with $c_2^b = 0$ is relevant. Figure 3 shows the T dependence of these two branches. Both have continuations from T_1 to the irrelevant temperatures regions. The difference of their extremal values is small (in the order of 10^{-4}) and are plotted in figure 4.

For temperatures at and below T_1 the presented results are new and have the consequence that *nearly all TAP solutions are marginally stable for these temperatures*. This finding differs from the classical BM work which does not claim any marginal stability for any temperature. For temperatures above T_1 the present results agree with BM. Early claims of marginality [9–11] are based on $c_1 \rightarrow 0$ and are therefore again different from the present findings, which are a consequence of $c_2 \rightarrow 0$.

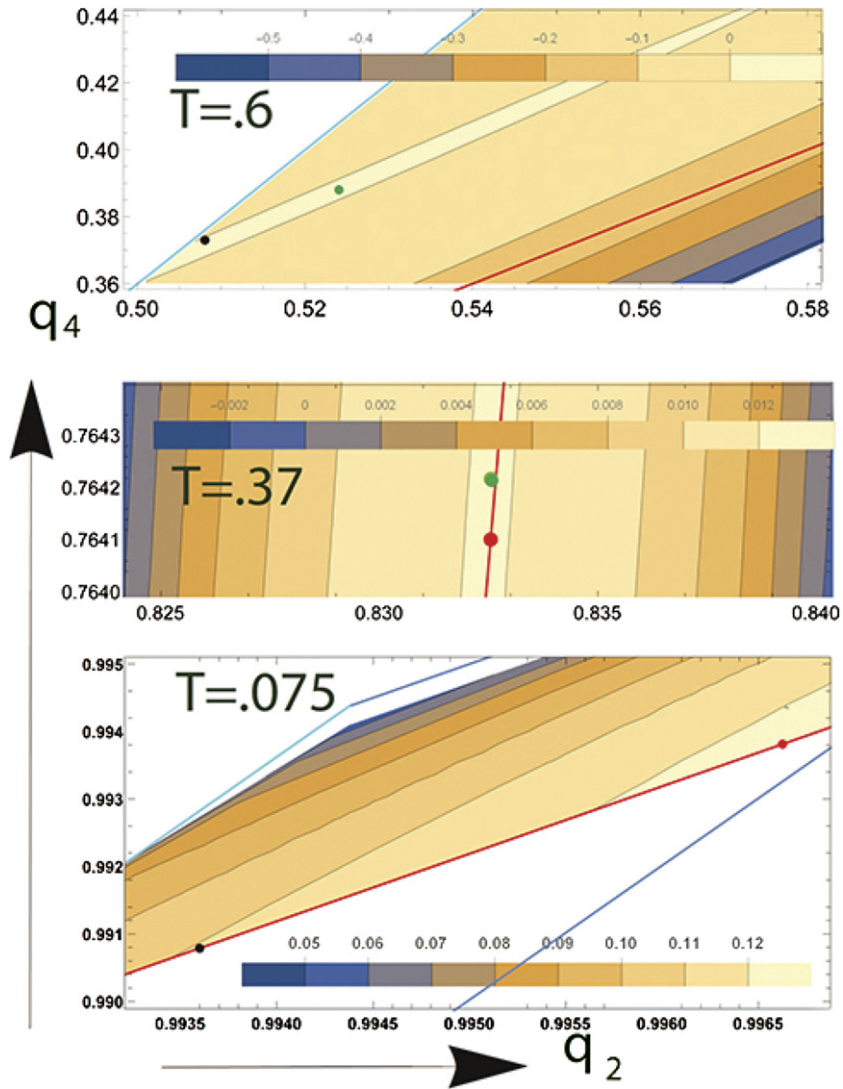


Figure 2. Contour-plot of the complexity $\Sigma(T; q_2, q_4, v = 0)$ far above, near and far below the critical temperature $T_1 = 0.367$: red dots mark boundary maxima and green dots denote maxima in the interior compatible with both criteria (5). Black dots indicate the position of g_0 .

The extension of BM worked out in [7] and the present work have a common origin, namely the projector term $P_{ij} = -\frac{2}{N}\beta m_i m_j$ of the Hessian $\partial G_i / \partial m_j = \chi_{ij}^{-1} + P_{ij}$. Such a term determines the lowest eigenvalue of $\partial G_i / \partial m_j$, which is non-negative under the condition $c_0 \equiv T^2 - 2T/N \sum_{ij} m_i \chi_{ij} m_j \geq 0$ (compare [7]). This condition has to be satisfied for every individual TAP solution and should in principle be used to eliminate non-physical solutions from the complexity counting. The approach [7] does not include this requirements as only an average value $\bar{c}_0 = 0$ of a large number of TAP states is calculated, which contains contributions resulting from $c_0 \neq 0$. The elimination of non-physical individual states with $c_0 < 0$ has

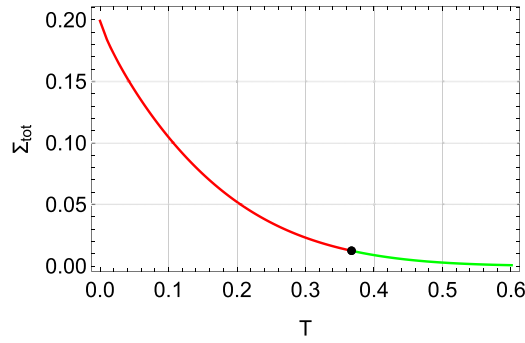


Figure 3. Total complexity $\Sigma_{\text{tot}}(T)$ versus temperature T : the red branch result from maxima on the boundary ($T < T_1$) and the green branch result from internal maxima ($T > T_1$). The black dot marks the critical point T_1 . (For $T > 0.6$ the approximative expansion of [5] is very accurate.)

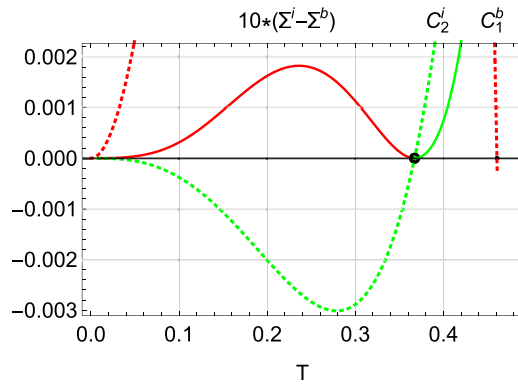


Figure 4. Separation of the extrema: plotted is the difference of the internal maximum Σ^i to the boundary maximum Σ^b (red and green full lines). The corresponding criteria c_2^i and c_1^b are represented by dotted red and green lines. The black dot marks the critical temperature T_1 .

a feedback to the average procedure and consequently to the \bar{c}_0 value. States with $c_0 > 0$ do not imply marginality. These objections together with missing influence of condition c_2 raise some doubts on the results of [7]. Recall that the numerical results of the extension [7] agree with BM in contrast to the present work which differs from BM for $T < T_1$.

3.2. Averages

Next some physical interesting averages for the Gibbs potential g and the energy u per spin are calculated. According to equations (1) and (2) these averages are given by

$$g = w - \frac{\beta}{4}(1 - q_2)^2 - T \langle \langle s_0(m) \rangle \rangle \tag{18}$$

and by

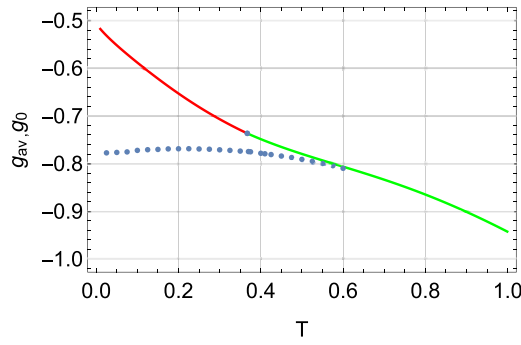


Figure 5. Gibbs potential: T -dependence of g_{av} (full lines) and g_0 (dots).

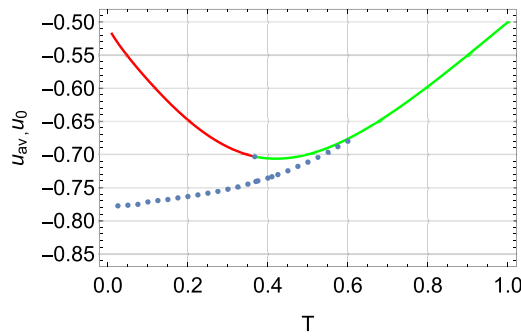


Figure 6. Energy: T -dependence of u_{av} (full lines) and u_0 (dots).

$$u = w - \frac{\beta}{2}(1 - q_2)^2. \quad (19)$$

Let us first consider the averages over all TAP states with equal weights. These ‘white’ averages g_{av} and of u_{av} are determined by the extremal values of the parameters of the total complexity $\Sigma_{\text{tot}}(T)$. The temperature dependence for $T \leq 1$ of g_{av} and of u_{av} is plotted in figures 5 and in 6, respectively. The strange increase of u_{av} with decreasing T results from the fast increasing number of TAP solutions with high energies. These white averages have therefore no physical significance or any relevance for low temperatures.

There is an interesting, alternative averaging, that leads to the lowest value g_0 of the Gibbs potential consistent with both, the existence condition of TAP solutions $\Sigma \geq 0$ and the validity of the criteria (5). To attack this problem the complete set of equations (9)–(15), (18) and (19) is needed. Keeping q_2, q_4 and v constant the parameters λ, Δ , and μ are determined numerically with equations (12), (14) and (15). Repeating this procedure for all possible values of q_2, q_4 and v the dependence of the complexity Σ , of w and of the Gibbs potential g on these quantities is obtained according to equations (10), (13) and (18). Finally the minimum g_0 of the Gibbs potential is determined in the region of the allowed values of q_2, q_4 and Σ . The findings are a vanishing complexity $\Sigma = 0$ for all temperatures, which ensures at least the presence of one TAP state in the thermodynamic limit $N \rightarrow \infty$.

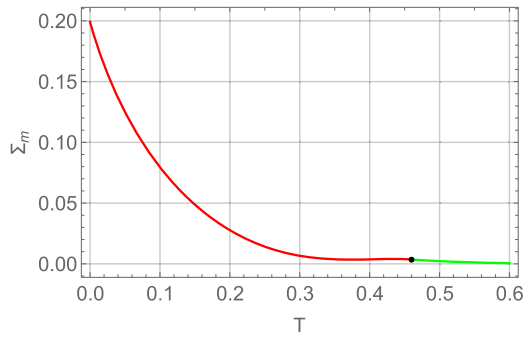


Figure 7. Complexity Σ_m versus temperature T : the red branch result from maxima on the boundary ($T < T_{av}^m$) and the green branch result from internal maxima ($T > T_{av}^m$). The black dot marks the critical temperature T_{av}^m .

The resulting g_0 and the corresponding energy u_0 are plotted in figures 5 and in 6, respectively. Both quantities exhibit the expected temperature dependence. Note that the location of g_0 is again on the boundary $c_2 = 0$ for $T < T_1$ (compare figure 2). The low temperature behavior ($T \approx 0.1$) of g_0 is similar to the results of the replica approach [19]. This is remarkable as the later approach uses the entire ∞ -replica symmetry breaking solution of the SK and sophisticated numerical methods.

3.3. Postulated marginality

Early numerical investigations [7, 9–11] claim marginal metastable states based on $c_1 \rightarrow 0$ in the thermodynamic limit. The recent work of Aspelmeier and Moore [12] have numerically studied the N -dependence of the two lowest eigenvalues of the Hessian. They found that both eigenvalues tend to zero in the thermodynamic limit. Thus there is no isolated eigenvalue caused by the projector term, which implies again $c_1 \rightarrow 0$ in the thermodynamic limit (compare subsection 3.1). Motivated by all these findings $c_1 = 0$ is postulated a priori in this subsection and the resulting consequences for the present approach are worked out.

The first quantity of interest is the complexity Σ_m , which determines the total number of TAP states with the constraint $c_1 = 0$. This complexity Σ_m is given by the absolute maximum of $\Sigma(T; q_2, q_4 = 2q_2 - 1 + T^2, v = 0)$, as function of q_2 . Again two temperature regimes exist, which are separated by a critical temperature $T_{av}^m = 0.459$. For $T > T_{av}^m$ the maximum is located within the allowed q_2 -interval $(1 - 1.5T^2) < q_2 < 1$ and for $T \leq T_{av}^m$ the maximum sticks at the endpoint $q_2 = (1 - 1.5T^2)$. The numerical results for Σ_m are consistent to $0 \leq \Sigma_m \leq \Sigma_{tot}$ for all temperatures, which ensures the existence of at least one TAP state with the postulated marginality. Note that this last conclusion is a priori not obvious. Figure 7 shows the temperature dependence of Σ_m for $T < 0.6$.

Analogue to subsection 3.2 the set of extremal parameters of the maximum determine the averages performed with all TAP states satisfying $c_1 = 0$. Together with equations (18) and (19) this leads directly to the averages of the Gibbs potential g_{av}^m and the energy u_{av}^m performed with these states. Both quantities g_{av}^m and u_{av}^m are plotted in figure 8. The results g_{av}^m and u_{av}^m are not very useful for low temperatures similar to the above findings for g_{av} and u_{av} .

The lowest value of the Gibbs potential g_0^m consistent with $c_1 = 0, c_2 \geq 0$ and $\Sigma \geq 0$ is again determined analogue to procedure of subsection 3.2. As before a vanishing complexity and two temperature regions are found with a sticking below a different critical temperature

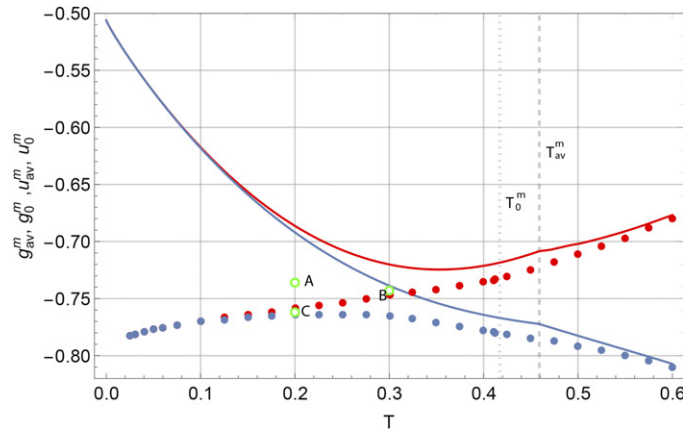


Figure 8. Gibbs potential and energy: T -dependence of g_{av}^m (full, blue line), u_{av}^m (full, red line), g_0^m (blue dots) and u_0^m (red dots). The green points A, B and C represent published values of the Gibbs potential according to [7], to [12] and to [11], respectively.

$T_0^m = 0.417$. The temperature dependence of g_0^m together with the corresponding energy u_0^m is plotted in figure 8.

For $T = 0.2$ the numerical value is for the Gibbs potential is $g_0^m = -0.7644$. This value is in remarkable agreement with the value of $\hat{g}_0^m = -0.07619$ found by the dynamical investigation [11] with $c_1 \geq 0$ and with a vanishing complexity. The result \hat{g}_0^m is added to figure 8 as point C together with the points A and B found by further investigations [7, 12]. Even though $c_1 \geq 0$ is satisfied no results for the complexity are available for the points A and B. Nevertheless these results are compatible with the present work.

According to the above constructions $g_0 < g_0^m$ holds for $0 < T < 1$. The numerical differences, however, are small and would not clearly be resolved on a scale of figures 5 or 8. The questions why the states with lowest Gibbs potentials g_0 are not reached by dynamical relaxation or if there are alternative dynamical paths to g_0 are interesting, but are beyond the scope of this work.

4. Conclusion

The presented investigation of the complexity is based the inclusion of q_4 to the set of constrained variables. This extension together with a strict regard of the validity criteria for the TAP equations leads to new results in low temperatures regime. Numerically the differences to the existing approaches are rather small, the interpretations and conclusions, however, differ considerably. At and below a critical temperature T_1 *nearly all TAP states are marginal stable with $c_2 = 0$* , a property not found in previous theories on the total complexity. Marginal stability implies a vanishing eigenvalue of the Hessian and the divergence of a mode of the susceptibility matrix [2]. The system is critical for all temperatures below T_1 and shows critical slowing down effects.

In addition to this findings consequences for averages over all metastable TAP states and averages over states with the lowest value of the Gibbs potential have been worked out for all temperatures. Moreover the influence of a different kind of marginality $c_1 = 0$, as found by supplementary numerical investigations [7, 9–12], has been worked out.

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References

- [1] Thouless D J, Anderson P W and Palmer R G 1977 *Phil. Mag.* **35** 593
- [2] Plefka T 1982 *J. Phys. A: Math. Gen.* **15** 1971
- [3] Plefka T 2002 *Europhys. Lett.* **58** 892
- [4] Sherrington D and Kirkpatrick S 1975 *Phys. Rev. Lett.* **35** 1972
- [5] Bray A J and Moore M A 1980 *J. Phys. C* **13** L469
- [6] Cavagna A, Giardina I, Parisi G and Mézard M 2003 *J. Phys. A: Math. Gen.* **36** 1175
- [7] Aspelmeier T, Bray A J and Moore M A 2004 *Phys. Rev. Lett.* **92** 087203
- [8] Müller M, Leuzzi L and Crisanti A 2006 *Phys. Rev. B* **74** 134431
- [9] Bray A J and Moore M A 1979 *J. Phys. C* **12** L441
- [10] Plefka T 2002 *Phys. Rev. B* **65** 224206
- [11] Plefka T 2003 arXiv:cond-mat/0310782
- [12] Aspelmeier T and Moore M A 2019 *Phys. Rev. E* **100** 032127
- [13] de Almeida J R L and Thouless D J 1978 *J. Phys. A: Math. Gen.* **11** 983
- [14] Owen J C 1982 *J. Phys. C* **15** L1071
- [15] Pastur L A 1974 *Russ. Math. Surv.* **28** 1
- [16] Plefka T 1982 *Phys. Lett. A* **90** 262
- [17] Kistler N 2018 private communication
- [18] Plefka T 1982 *J. Phys. A: Math. Gen.* **15** L251
- [19] Crisanti A and Rizzo T 2002 *Phys. Rev. E* **65** 046137