

Classification of Immersions which are Bounded by Curves in Surfaces



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Zusammenfassung

In viele verschiedenen Gebieten der Mathematik stellt sich die Frage, ob eine gegebene Randwertfunktion auf das Innere fortgesetzt werden kann. In der Funktionentheorie besagt der Riemannsche Abbildungssatz, dass eine Einbettung $f: \mathbb{S}^1 \rightarrow \mathbb{R}^2$ zu einer diffeomorphen Funktion $F: \overline{D} \rightarrow \mathbb{R}^2$ auf die abgeschlossene Kreisscheibe fortgesetzt werden kann.

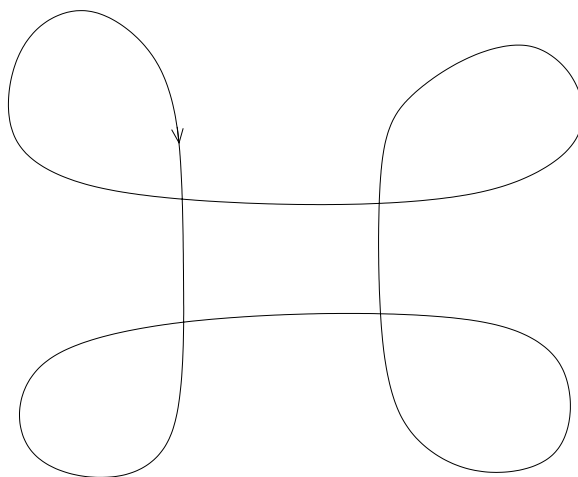
In der Theorie der Minimalflächen stellt das Plateau Problem ein Fortsetzungsproblem dar. Hier ist die Fortsetzung einer Randkurve f gesucht, die nicht nur differenzierbar sondern zusätzlich auch noch eine Minimalfläche ist.

In der Kategorie der topologischen Mannigfaltigkeiten hat *Arthur Schoenflies* gezeigt, dass eine Einbettung $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ die Sphäre in genau zwei Zusammenhangskomponenten zerteilt. Jede dieser Komponenten ist homöomorph zu \overline{D} und somit lässt sich f zu einem Homöomorphismus $F: \overline{D} \rightarrow \mathbb{S}^2$ fortsetzen (Jordan-Schoenflies Theorem).

Die vorliegende Arbeit untersucht ein Fortsetzungsproblem in der Kategorie der Immersionen. Angenommen $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow N$ ist eine Immersion von der disjunkten Vereinigung von Kreisen in eine geschlossene Fläche N . Wann existiert eine Fläche M mit Rand $\partial M = \coprod_{j=1}^m \mathbb{S}^1$ so, dass f zu einer Immersion $F: M \rightarrow N$ fortgesetzt werden kann? Weiterhin stellt sich die Frage wieviele verschiedene Fortsetzungen existieren.

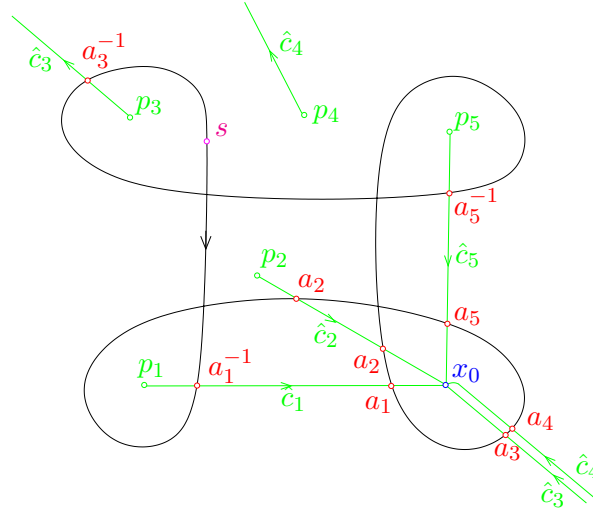
Ist $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ eine Einbettung, so liefert der Riemannsche Abbildungssatz eine Fortsetzung zu einer Einbettung $F: \overline{D} \rightarrow \mathbb{S}^2$. Sei nun $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ eine Immersion, aber keine Einbettung. Wenn f in geschlossene Einbettungen zerlegt werden kann, so kann jede dieser Einbettungen auf \overline{D} fortgesetzt werden. Diese Fortsetzungen können zu einer Fortsetzung von f zusammengeklebt werden.

Es stellt sich die Frage, wie diese Schnitte lokalisiert werden können? Betrachten wir das folgende Beispiel:



Hat diese Immersion eine Fortsetzung auf die abgeschlossene Einheitskreisscheibe? Hat sie eine Fortsetzung zu einer anderen Fläche M mit $\partial M = \mathbb{S}^1$? Wieviele verschiedene Fortsetzungen existieren?

All diese Fragen können mit Hilfe eines f zugeordneten Wortes $w(f)$ beantwortet werden. Um dieses Wort zu bekommen wählen wir einen Basispunkt x_0 und Punkte p_1, \dots, p_k in jeder Zusammenhangskomponenten von $\mathbb{S}^2 \setminus f(\mathbb{S}^1)$. Danach ziehen wir einen Strahl \hat{c}_j von p_j nach x_0 . Diese Strahlen schneiden $f(\mathbb{S}^1)$ und wir markieren die Schnittpunkte von \hat{c}_j mit $a_j^{\pm 1}$. Der Schnittpunkt ist positiv, wenn der Strahl $f(\mathbb{S}^1)$ von links nach rechts schneidet und er ist negativ wenn er von rechts nach links schneidet.



Die Strahlen \hat{c}_j und die orientierten Schnittpunkte $a_j^{\pm 1}$.

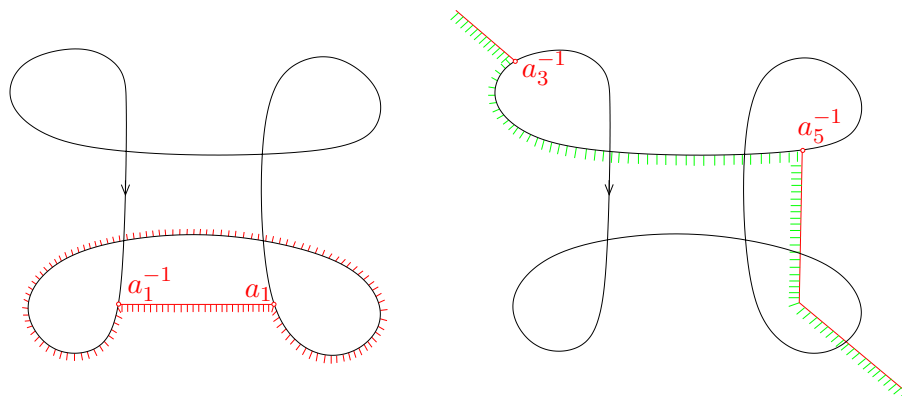
Durch die Wahl eines Startpunktes s können die Buchstaben $a_j^{\pm 1}$ als Wort aufgeschrieben werden, d.h.

$$w(f) = a_1^{-1} a_2 a_5 a_4 a_3 a_1 a_2 a_5^{-1} a_3^{-1}.$$

Dieses Wort enthält alle Informationen um die oben gestellten Fragen zu beantworten. Die Existenz von Fortsetzungen $F: \overline{D} \rightarrow \mathbb{S}^2$ steht im Zusammenhang mit der Kombinatorik der Buchstaben von $w(f)$. Spezielle Teilwörter kennzeichnen die Stellen, an denen f so zerschnitten werden kann, dass eine Einbettung abgespalten wird. Diese speziellen Teilwörter heißen Paarungen und negative Gruppen. Eine Paarung ist ein Teilwort der Form $a_j^{\pm 1} \dots a_j^{\mp 1}$ und eine negative Gruppe ein Teilwort der Form $a_j^{-1} a_i^{-1}$. Das Beispiel enthält eine Paarung und eine negative Gruppe:

$$w(f) = a_1^{-1} a_2 a_5 a_4 a_3 a_1 a_2 a_5^{-1} a_3^{-1}.$$

Die nachfolgende Abbildung zeigt, dass diese Teilwörter Einbettungen markieren, welche zu eingebetteten Kreisscheiben fortgesetzt werden können:



(a) Die durch eine Paarung gekennzeichnete Einbettung. Die Kreisscheibe ist durch die Schraffierung markiert. (b) Die durch eine negative Gruppe gekennzeichnete Einbettung. Die Kreisscheibe ist durch die Schraffierung markiert.

Bleibt nach dem Kürzen aller Paarungen und negativen Gruppen ein positives Wort, d.h. ein Wort mit positiven Buchstaben, übrig, dann hat das Wort eine Gruppierung. Das Übrigbleiben eines positiven Wortes bedeutet, dass der verbleibende Teil von f ebenfalls eine Einbettung ist und somit fortgesetzt werden kann. Nachdem alle Einbettungen fortgesetzt wurden werden die Kreisscheiben zu einer Fortsetzung von f zusammengeklebt.

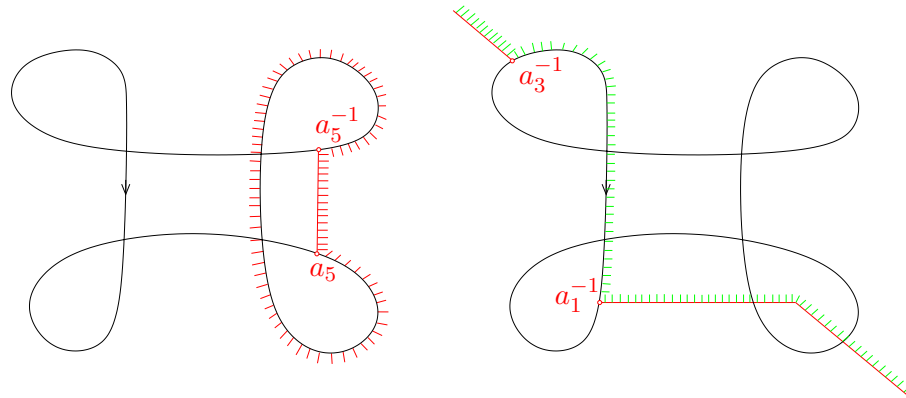
Der Ansatz einer Immersion ein Wort zuzuordnen und ein Fortsetzungsproblem anhand dieses Wortes zu untersuchen lässt sich auf *C. J. Titus* [Tit60] zurückführen. Im Gegensatz zu unserem Ansatz nutzte er ausschliesslich die Selbstschnitte der Immersion um die sogenannte Titus Schnittfolge zu definieren. Damit war er in der Lage die Frage nach der Existenz einer Fortsetzung zu einer gegebenen Immersion $f: \mathbb{S}^1 \rightarrow \mathbb{R}^2$ zu beantworten, aber er konnte nicht sagen, wieviele verschiedene Fortsetzungen es gibt.

Eine Antwort auf diese Frage im Falle immersierte Kreisscheiben in der Ebene gab *Samuel J. Blank* in seiner Dissertation von 1967 [Bla67]. Er verbesserte Titus' Ansatz dadurch, dass er die Strahlen \hat{c}_j ergänzte und deren Schnittpunkte mit $f(\mathbb{S}^1)$ untersuchte. Er behauptete das die Anzahl der Gruppierungen von $w(f)$ mit der Anzahl der verschiedenen Äquivalenzklassen von Fortsetzungen übereinstimmt (vgl. Abbildung 1 für eine weitere Fortsetzung des Beispiels.). Allerdings hat seine Dissertation einen unvollständigen Charakter und blieb unveröffentlicht.

Es war *Valentin Poénaru* der die fehlenden Beweise ergänzte und Blanks Ideen veröffentlichte [Poé69]. Die vorliegende Arbeit stellt neue kombinatorische Strukturen vor, die das Verständnis der Resultate wesentlich verbessern. Dies führt zu einer Vereinfachung und verkürzt die Beweise stark.

Die Frage ob eine Immersion $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow \mathbb{S}^2$ eine Fortsetzung zu einer Immersion $F: M \rightarrow \mathbb{S}^2$ hat blieb aber weiterhin offen. *M. L. Marx* entwickelte notwendige Bedingungen sowohl dafür, dass eine Immersion $f: \mathbb{S}^1 \coprod \mathbb{S}^1 \rightarrow \mathbb{R}^2$ mit zwei Randkomponenten auf den Kreisring fortgesetzt werden kann [Mar65], als auch dafür, dass eine Immersion $f: \mathbb{S}^1 \rightarrow \mathbb{R}^2$ auf eine Fläche M mit Rand $\partial M = \mathbb{S}^1$ und Geschlecht $g_M \in \{0, 1\}$ [Mar68] fortgesetzt werden kann.

K. D. Bailey verbesserte diese Resultate indem er Fortsetzungen $F: M \rightarrow \mathbb{R}^2$ von $f: \mathbb{S}^1 \rightarrow \mathbb{R}^2$



(c) Die durch eine Paarung gekennzeichnete Einbettung. Die Kreisscheibe ist durch die Schraffierung markiert. (d) Die durch eine negative Gruppe gekennzeichnete Einbettung. Die Kreisscheibe ist durch die Schraffierung markiert.

Abbildung 1: Eine weiter Gruppierung von $w(f) = a_1^{-1}a_2a_5a_4a_3a_1a_2a_5^{-1}a_3^{-1}$ und die zugehörige Fortsetzung.

untersuchte, bei denen M eine Fläche mit beliebigem Geschlecht ist. Er zeigte, dass eine neue Operation auf dem Wort $w(f)$, die er Versammlung nannte, zu einer vollständigen Charakterisierung immersierter Flächen M mit Rand $\partial M = \mathbb{S}^1$ in der Ebene führte [Bai75].

1986 komplettierten *C. Curley* und *D. Wolitzer* die Klassifikation durch das Ergänzen des verbleibenden Falles von Fortsetzungen auf Flächen M mit Rand $\partial M = \coprod_{j=1}^m \mathbb{S}^1$ ([CW86]).

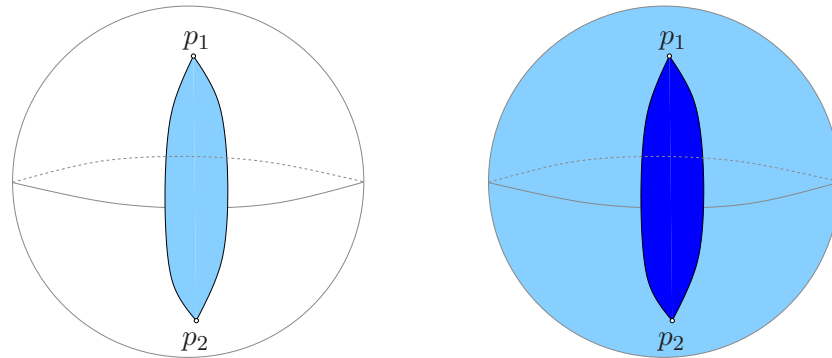
Zu dieser Zeit war das Fortsetzungsproblem für immersive Flächen in der Ebene vollständig klassifiziert. Aber das Klassifikationsproblem für immersive Flächen in beliebigen geschlossenen Flächen blieb offen.

Das einzige Resultat zu dieser Thematik kommt von *George K. Francis*, der eine Klassifikation immersierter Kreisscheiben in der Sphäre zeigt [Fra73]. Er kündigt auch ein Resultat für Immersionen $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow \mathbb{S}^2$ an, welches aber scheinbar nie erschien.

Neben der Vereinfachung der bekannten Resultate komplettiert die vorliegende Arbeit die Klassifikation in Dimension 2. Die Methoden von *C. Curley* und *D. Wolitzer* werden auf Immersionen $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow \mathbb{S}^2$ in die Sphäre übertragen. Dies führt zu einer Klassifikation von immersierten Flächen in der Sphäre. Abschliessend werden die Resultate auf Immersionen in beliebige Zielflächen verallgemeinert.

Abgesehen von der Vervollständigung der Klassifikation lieferten *Karsten Grosse-Brauckmann*, *Robert B. Kusner* und *John M. Sullivan* eine neue Motivation sich mit diesem Fortsetzungsproblem zu beschäftigen [GKS07]. Sie untersuchten spezielle Flächen konstanter mittlerer Krümmung, sogenannte k -Unduloide, durch die Zuordnung einer sphärischen Metrik, d.h. einer Metrik lokal isometrisch zu \mathbb{S}^2 . Die Vervollständigung einer solchen sphärischen Metrik ist eine stückweise Immersion $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ mit stückweise geodätischem Rand, d.h. einem sphärischen Polygon. Somit führen Fortsetzungen von diesen stückweisen Immersionen zu k -Unduloiden.

Ist der Rand glatt, so ist die Anzahl der Urbilder einer Fortsetzung $F: \overline{D} \rightarrow \mathbb{S}^2$ durch



(a) Eine Fortsetzung eines sphärischen Polygons mit Winkeln kleiner als 2π in den Ecken.

(b) Eine Fortsetzung eines sphärischen Polygons mit Winkeln zwischen 2π und 4π in den Ecken.

Abbildung 2: Fortsetzungen mit unterschiedlichen Winkeln in den Ecken.

die Randimmersion $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ bestimmt. Hat das sphärische Polygon nichtdifferenzierbare Punkte, d.h. Ecken, so ist dies nicht mehr der Fall. Hier ist der Winkel in einer Ecke nur noch modulo 2π definiert, wie in Abbildung 4 dargestellt. Somit hat ein sphärisches Polygon Verzweigungspunkte in den Ecken, d.h. eine Fortsetzung führt zu einer ganzen Familie von Fortsetzungen. Die Resultate der vorliegenden Arbeit sollten die Grundlage für eine Klassifizierung von stückweise Immersionen $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ mit Verzweigungspunkten bilden.

Eine weitere Anwendung könnte in der Klassifikation von immersierten 3-Bällen in \mathbb{R}^3 liegen. In diesem Fall wird das Fortsetzungsproblem wie folgt definiert: *Sei $f: \mathbb{S}^2 \rightarrow \mathbb{R}^3$ eine Immersion. Wann hat f eine Fortsetzung $F: B^3 \rightarrow \mathbb{R}^3$ auf den abgeschlossenen 3-Ball?*

Robert B. Kusner schlug vor, eine Homotopie $H: \mathbb{R}^2 \times [0, 1] \rightarrow \mathbb{R}^3$ von immersierten Ebenen zu untersuchen, so dass

$$\bigcup_{t \in [0, 1]} H_t(\mathbb{R}^2) \cap f(\mathbb{S}^2) = f(\mathbb{S}^2).$$

Dann besteht für jedes $t \in [0, 1]$ der Schnitt der immersierten Ebene mit $f(\mathbb{S}^2)$ aus einer Familie von Immersionen $f_t^{(1)}, \dots, f_t^{(k)}: \mathbb{S}^1 \rightarrow \mathbb{R}^2$. Eine notwendige Bedingung dafür, dass $f: \mathbb{S}^2 \rightarrow \mathbb{R}^3$ auf einen immersierten 3-Ball fortgesetzt werden kann ist, dass für jeden Schnitt $H_t(\mathbb{R}^2) \cap f(\mathbb{S}^2)$ die entsprechende Familie zu einer (möglicherweise unzusammenhängenden) immersierten Fläche in der Ebene fortgesetzt werden kann. Um daraus eine Fortsetzung für f zu erhalten muss die Familie in t differenzierbar sein. Probleme treten hierbei auf, wenn eine Immersion $f_t^{(i)}$ sich aufteilt oder mehrere Immersionen zusammenfallen. Für diese $t \in [0, 1]$ muss sichergestellt werden, dass der Übergang ebenfalls differenzierbar ist.

Darüber hinaus könnte dies zu einer Klassifikation immersierter 3-Mannigfaltigkeiten in beliebigen geschlossenen 3-Mannigfaltigkeiten und dadurch zu einer Klassifikation immersierter n -Mannigfaltigkeiten in geschlossenen n -Mannigfaltigkeiten führen.

Ein weiteres Problem, vorgeschlagen von Robert B. Kusner, ist die Frage, ob die Länge des Wortes $w(f)$ einer Immersion eine Abschätzung der Anzahl verschiedener Fortsetzungen erlaubt: Wenn n die Länge des Wortes $w(f)$ bezeichnet so stellt sich die Frage, ob eine Funktion

$\xi(n)$ existiert, die eine obere Schranke für die Anzahl der Fortsetzungen ist.

Da die Länge des Wortes leicht erhöht werden kann ohne die Anzahl der Fortsetzungen zu erhöhen sollte die Wachstumsrate mindestens exponentiell sein.

Wir beschliessen diese Einleitung mit einem Überblick über diese Arbeit. In der ersten Sektion wird das Fortsetzungsproblem formuliert und einige grundlegenden topologischen Fakten hergeleitet.

Sektion 2 verallgemeinert Blanks Ansatz auf Immersionen in die Sphäre. Paarungen und negative Gruppen werden eingeführt und eine Gruppierung wird definiert. Sektion 3 überträgt Blanks Resultate auf Immersionen $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$. Im Existenz Satz 3.1.5 und im Eindeutigkeitsatz 3.3.3 wird gezeigt, dass die Anzahl der Gruppierungen mit der Anzahl der verschiedenen Äquivalenzklassen von Fortsetzungen auf die Kreisscheibe übereinstimmt.

In Sektion 4 werden die Methoden von *C. Curley* und *D. Wolitzer* [CW86] benutzt um das Fortsetzungsproblem für Immersionen $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow \mathbb{S}^2$ auf den Fall von immersierten Kreisscheiben in der Sphäre zurückzuführen. Zunächst werden die Randkomponenten zu einer gemeinsamen Randkomponente vereinigt, so dass f eine Immersion $f^*: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ induziert. Eine neue Operation auf dem Wort $w(f)$ wird eingeführt. Diese erlaubt es eine Fläche M^* mit Rand $\partial M^* = \mathbb{S}^1$ zu konstruieren so dass f^* zu einer Immersion $F^*: M^* \rightarrow \mathbb{S}^2$ fortgesetzt werden kann. Abschliessend wird diese Fortsetzung benutzt um eine Fläche M mit Rand $\partial M = \coprod_{j=1}^m \mathbb{S}^1$ zu konstruieren, so dass f zu einer Immersion $F: M \rightarrow \mathbb{S}^2$ fortgesetzt werden kann.

In der abschliessenden Sektion werden die Resultate auf allgemeine geschlossene Zielflächen N übertragen. Chirurgie Theorie wird benutzt um N in eine Sphäre zu transformieren und somit das Fortsetzungsproblem auf den Fall von Immersionen in der Sphäre zu reduzieren. Sei $f: \mathbb{S}^1 \rightarrow N$ eine Immersion auf eine geschlossene Fläche N mit positivem Geschlecht. Eine Folge ν von Chirurgien transformieren N in eine Sphäre. Da diese Chirurgien im allgemeinen $f(\coprod_{j=1}^m \mathbb{S}^1)$ zerschneiden führt dies zu einer Familie $f_1^\nu, \dots, f_n^\nu: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ von Immersionen. Wenn all diese Immersionen eine Fortsetzung zu Immersionen $F_j^\nu: M_j^\nu \rightarrow \mathbb{S}^2$ haben, so werden diese Fortsetzungen benutzt um eine Fortsetzung von f zu erhalten.

Introduction

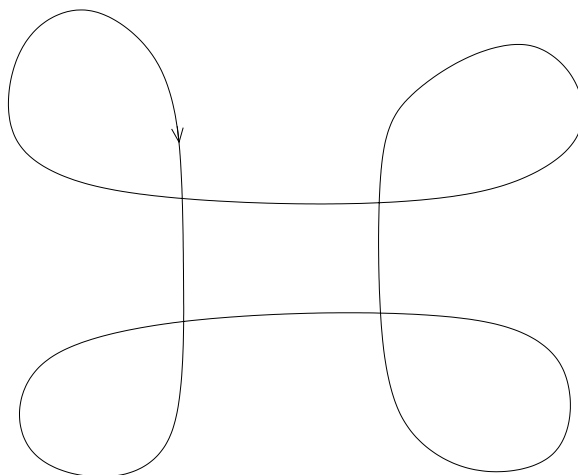
In many different areas of mathematics the problem arises whether a given function of boundary values extends to the interior. In complex analysis the Riemann Mapping Theorem states that an embedding $f: \mathbb{S}^1 \rightarrow \mathbb{R}^2$ extends to a diffeomorphic function $F: \overline{D} \rightarrow \mathbb{R}^2$, where \overline{D} denotes the closed unit disc.

An extension problem in the theory of minimal surfaces is the Plateau problem. Here an extension of a boundary curve f is sought which is not only differentiable but also is a minimal surface.

In the category of topological manifolds *Arthur Schoenflies* shows that an embedding $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ decomposes the sphere in exactly two components. Each of them is homeomorphic to \overline{D} and therefore f extends to a homeomorphism $F: \overline{D} \rightarrow \mathbb{S}^2$ (Jordan-Schoenflies Theorem).

The present work studies an extension problem in the category of immersions. Suppose $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow N$ is an immersion from the disjoint union of circles to a closed surface. When does a surface M with boundary $\partial M = \coprod_{j=1}^m \mathbb{S}^1$ exist such that f extends to an immersion $F: M \rightarrow N$? Furthermore the question arises how many different extensions of a given immersion f exist.

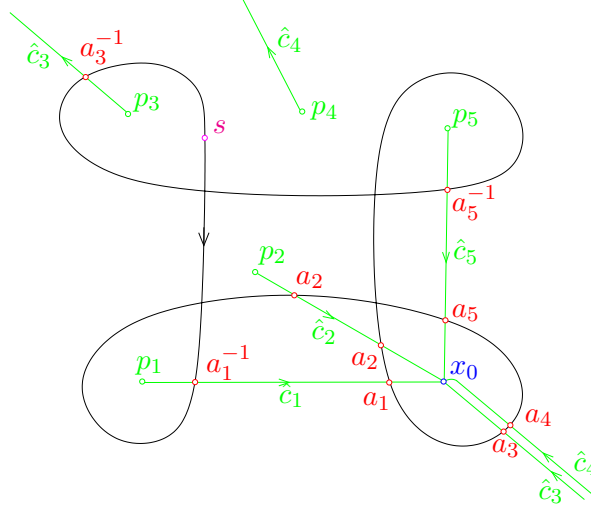
If $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ is an embedding then the Riemann Mapping Theorem provides an extension to an embedding $F: \overline{D} \rightarrow \mathbb{S}^2$. Now assume that $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ is an immersion but not an embedding. If f can be decomposed into closed embeddings then each embedding extends to \overline{D} . These extensions can be glued together to obtain an immersion $F: \overline{D} \rightarrow \mathbb{S}^2$, which extends f . The question is, how to localize these cuts? Look at the following example:



Does this immersion have an extension to the closed unit disc? Does it have an extension to another surface M with $\partial M = \mathbb{S}^1$? How many different extensions exist?

All these questions can be answered in terms of a word $w(f)$, assigned to f . To obtain this word we select a base point x_0 and points p_1, \dots, p_k in each component of $\mathbb{S}^2 \setminus f(\mathbb{S}^1)$. Afterwards we draw a ray \hat{c}_j from p_j to x_0 . These rays intersect $f(\mathbb{S}^1)$ and we denote an

intersection point of \hat{c}_j by $a_j^{\pm 1}$. The intersection point is positive if the ray crosses $f(\mathbb{S}^1)$ from left to right and negative if it crosses from right to left.



The rays \hat{c}_j and the oriented intersection points $a_j^{\pm 1}$.

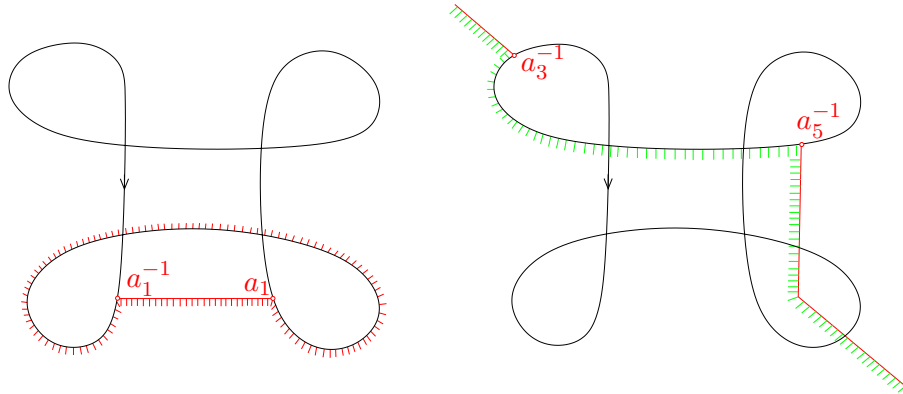
By choosing an initial point s we can write down the letters $a_j^{\pm 1}$ as a word, i.e.,

$$w(f) = a_1^{-1} a_2 a_5 a_4 a_3 a_1 a_2 a_5^{-1} a_3^{-1}.$$

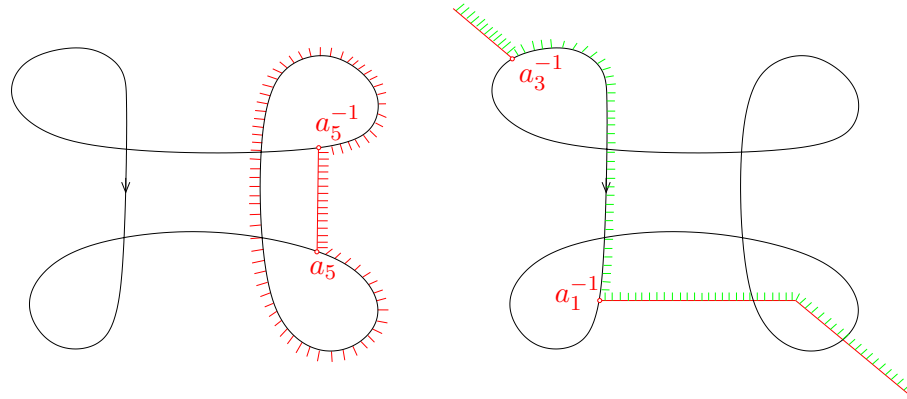
This word contains all informations to answer the questions stated above. The existence of an extension $F: \overline{D} \rightarrow \mathbb{S}^2$ is related to the combinatorics of the letters of $w(f)$. Special subwords indicate where f can be cut such that an embedding splits off. These special subwords are called pairings and negative groups. Pairings are subwords of the form $a_j^{\pm 1} \dots a_i^{\mp 1}$ and negative groups are subwords of the form $a_j^{-1} a_i^{-1}$. The example contains one pairing and one negative group:

$$w(f) = a_1^{-1} a_2 a_5 a_4 a_3 a_1 a_2 a_5^{-1} a_3^{-1}.$$

The following figure shows that these subwords mark embeddings which extend to an embedded disc:



(a) The embedding indicated by a pairing. The disc is marked by the pattern. (b) The embedding indicated by a negative group. The disc is marked by the pattern.



(c) The embedding indicated by a pairing. (d) The embedding indicated by a negative group. The immersed disc is marked by the pattern.

Figure 3: Another grouping of $w(f) = a_1^{-1}a_2a_5a_4a_3a_1a_2a_5^{-1}a_3^{-1}$ and the corresponding extension.

If a positive word, i.e., a word of positive letters, remains after cancellation of all pairings and negative groups, then the word has a grouping. In the example the word a_2 remains after the cancellation. It is positive and hence the word has a grouping. That a positive word remains indicates that the remaining part of f is an embedding as well and hence provides an extension. After extending all embeddings these immersed discs are glued together to an extension of f .

The approach to assign a word to an immersion and analyze extension problems in terms of this word was introduced by *C. J. Titus* in 1960 [Tit60]. Contrary to our approach he only used the selfintersections of the immersions to define what he called the Titus intersection sequence. That way he was able to answer the question whether a given immersion $f: \mathbb{S}^1 \rightarrow \mathbb{R}^2$ has an extension to an immersed disc but fails to answer how many different extensions exist.

An answer to this question for immersed discs in the plane gave *Samuel J. Blank* in his dissertation in 1967 [Bla67]. He improved Titus approach by adding the system of rays \hat{c}_j and studying the problem in terms of the word $w(f)$. He stated that the number of groupings is equal to the number of equivalence classes of extensions (see Figure 3 for another extension of the example). But his dissertation has an unfinished character and remained unpublished.

It was *Valentin Poénaru* who filled in missing proofs and published Blank’s ideas in 1969 [Poé69]. The present work introduced new combinatorial structures which improve the understanding of the results substantially. This leads to a simplification and shorten the proofs severely.

The question whether an immersion $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow \mathbb{S}^2$ has an extension to an immersion $F: M \rightarrow \mathbb{S}^2$ still remained open. *M. L. Marx* developed necessary conditions for immersions $f: \mathbb{S}^1 \coprod \mathbb{S}^1 \rightarrow \mathbb{R}^2$ with two boundary components to extend to the annulus [Mar65] and for immersions $f: \mathbb{S}^1 \rightarrow \mathbb{R}^2$ to extend to a surface M with boundary $\partial M = \mathbb{S}^1$ and genus $g_M \in \{0, 1\}$ [Mar68].

K. D. Bailey improved these results by analyzing extensions $F: M \rightarrow \mathbb{R}^2$ of $f: \mathbb{S}^1 \rightarrow \mathbb{R}^2$,

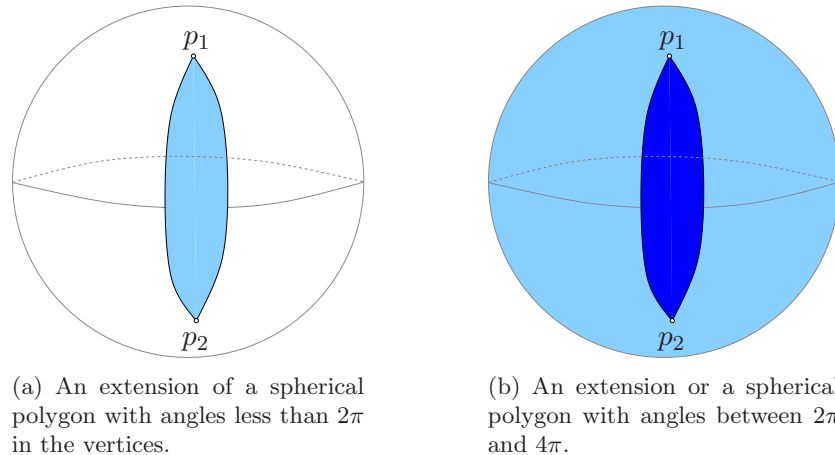


Figure 4: Extensions with different angles at the vertices.

where M is a surface of arbitrary genus. He showed that a new operation on the word $w(f)$, which he called an assemblage, leads to a complete characterization of immersed surfaces M with boundary $\partial M = \mathbb{S}^1$ in the plane [Bai75].

In 1986 *C. Curley* and *D. Wolitzer* completed the classification of immersed surfaces into the plane by including the remaining case of extensions to surfaces M with boundary $\partial M = \coprod_{j=1}^m \mathbb{S}^1$ ([CW86]).

By this time the extension problem for immersed surfaces in the plane was completely classified. But the classification problem for immersed surfaces in arbitrary closed surfaces remained open.

The only result on that topic comes from *George K. Francis* who stated a classification of immersed discs in the sphere [Fra73]. He also announced a result for immersions $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow \mathbb{S}^2$ which does not seem to be appeared.

Beside the simplification of the known results the present work completes the classification in dimension 2. The methods of *C. Curley* and *D. Wolitzer* are extended to immersions $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow \mathbb{S}^2$ to the sphere. This leads to the classification of immersed surfaces into the sphere. Finally the results are generalized to arbitrary closed target surfaces.

Apart from the completion of the classification *Karsten Grosse-Brauckmann*, *Robert B. Kusner* and *John M. Sullivan* proved a new motivation to study the extension problem [GKS07]. They analyzed special constant mean curvature surfaces, called k -unduloids, by associating spherical metrics to them, i.e., metrics locally isometric to \mathbb{S}^2 . The completion of such a spherical metric is a piecewise immersion $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ with piecewise geodesic boundary, i.e., a spherical polygon. Therefore extensions of these piecewise immersions lead to k -unduloids.

If the boundary is smooth the number of preimages of an extension $F: \overline{D} \rightarrow \mathbb{S}^2$ is determined by the boundary immersion $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$. If the spherical polygon has nondifferentiable points, i.e., vertices, this is not true. Here the angle in a vertex is only determined modulo 2π , as shown in Figure 4. Thus a spherical polygon has branchpoints at the vertices, i.e., an extension leads to a whole family of extensions. The results of the present work should build

a basis for the classification of piecewise immersions $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ with branchpoints.

Another application could be in the classification of immersed 3-balls in \mathbb{R}^3 . In this case, the extension problem reads as follows: *Suppose $f: \mathbb{S}^2 \rightarrow \mathbb{R}^3$ is an immersion. When does f have an extension $F: B^3 \rightarrow \mathbb{R}^3$ to the closed 3-ball?*

Robert B. Kusner suggested to study a homotopy $H: \mathbb{R}^2 \times [0, 1] \rightarrow \mathbb{R}^3$ of immersed planes, such that

$$\bigcup_{t \in [0, 1]} H_t(\mathbb{R}^2) \cap f(\mathbb{S}^2) = f(\mathbb{S}^2).$$

Then for each $t \in [0, 1]$ the section of the immersed plane with $f(\mathbb{S}^2)$ is a family of immersions $f_t^{(1)}, \dots, f_t^{(k)}: \mathbb{S}^1 \rightarrow \mathbb{R}^2$. A necessary condition for $f: \mathbb{S}^2 \rightarrow \mathbb{R}^3$ to extend to an immersed 3-ball is, that for each section $H_t(\mathbb{R}^2) \cap f(\mathbb{S}^2)$ the corresponding family has an extension to a (possibly disconnected) immersed surface in the plane. To obtain an extension of f the family has to be differentiable in t . Problems arise when an immersion $f_t^{(i)}$ splits into two immersions or if two immersions collapse to a single immersion. For these $t \in [0, 1]$ it has to be assured that the junction is differentiable as well.

Furthermore this could be the basis for a classification of immersed 3-manifolds in arbitrary closed 3-manifolds and hence for a classification of n -manifolds in closed n -manifolds.

Another problem suggested by Robert B. Kusner is, whether the length of the word $w(f)$ of a immersion allows an estimate of the number of different extensions: If n denotes the length of the word $w(f)$ then the question is, whether a function $\xi(n)$ exists which is an upper bound for the number of extensions.

Since the length of the word can be easily increased without increasing the number of extensions the growth rate of ξ should be at least exponential.

We conclude this introduction with an overview of the present work. In the first section the extension problem is stated and some basic topological facts are developed.

Section 2 generalizes Blank's approach to immersions to the sphere. Pairings and negative groups are introduced and a grouping is defined. Section 3 extends Blank's results to immersions $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$. In the Existence Theorem 3.1.5 and the Uniqueness Theorem 3.3.3 is shown that the number of groupings equals the number of equivalence classes of extensions of f to the disc.

In Section 4 the methods of *C. Curley* and *D. Wolitzer* [CW86] are used to reduce the extension problem for immersions $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow \mathbb{S}^2$ to the case of immersed discs in the sphere. Firstly the boundary components are joined to one boundary component such that f induces an immersion $f^*: \mathbb{S}^1 \rightarrow \mathbb{S}^2$. A new operation on the word $w(f)$ is introduced which allows to construct a surface M^* with boundary $\partial M^* = \mathbb{S}^1$ such that f^* extends to an immersion $F^*: M^* \rightarrow \mathbb{S}^2$. Finally this extension is used to construct a surface M with boundary $\partial M = \coprod_{j=1}^m \mathbb{S}^1$ such that f extends to an immersion $F: M \rightarrow \mathbb{S}^2$.

In the final Section the results are extended to arbitrary closed target surfaces N . Surgery theory is used to transform N into a sphere and hence reduce the problem to the case of immersions to the sphere. Suppose $f: \mathbb{S}^1 \rightarrow N$ is an immersion to a closed surface N of positive genus. A sequence ν of surgeries on N result in a sphere. Since these surgeries cuts

$f(\prod_{j=1}^m \mathbb{S}^1)$ in general this leads to a family of immersions $f_1^\nu, \dots, f_n^\nu: \mathbb{S}^1 \rightarrow \mathbb{S}^2$. If all these immersions extends to immersions $F_j^\nu: M_j^\nu \rightarrow \mathbb{S}^2$, these extensions are used to obtain an extension of f .

1 Basic Concepts

1.1 Normal Immersions

In this work we classify the extensions of normal immersions between surfaces. At first, we need to define a normal immersion.

Definition 1.1.1. A compact connected oriented 2-dimensional differentiable manifold is called a *surface*. A mapping $f: M \rightarrow N$ between surfaces M, N is an *immersion at* $x \in M$ if the differential $D_x: T_x M \rightarrow T_x N$ is injective for all $x \in M$.

If the mapping is an immersion at all points $x \in M$ then the mapping itself is an *immersion*.

Definition 1.1.2. An immersion $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow N$ is called *normal* if it has finitely many selfintersections, such that

- each selfintersection p is a double point, i.e., $f^{-1}(p) = \{t_1, t_2\}$ and
- the tangent vectors $f'(t_1)$ and $f'(t_2)$ of the double points are linearly independent.

Normal immersions are immersions which satisfy special conditions on the intersection points and the differential.

While we study only normal immersions it is remarkable, that every nonnormal immersion f has a small perturbation which is normal. Hence the upcoming results are true for nonnormal immersions as well.

Remark. For differentiable maps between differentiable manifolds the concept of transversality is defined. A normal immersion is a special case of a transversal map.

The main goal of this work is to classify extensions of normal immersion $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow N$.

Definition 1.1.3. Suppose M is a surface with boundary $\partial M = \coprod_{j=1}^m \mathbb{S}^1$ and N is a closed surface. A normal immersion $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow N$ *extends to* M if there exists an immersion $F: M \rightarrow N$ such that $F|_{\partial M} = f$.

The orientation is chosen such that the surface lies on the left of the boundary.

1.2 Basic Topological Facts

Suppose $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow N$ is a normal immersion and denote the components of $N \setminus f(\coprod_{j=1}^m \mathbb{S}^1)$ by X_1, \dots, X_k . If f has an extension $F: M \rightarrow N$ then the number of preimages of $F^{-1}(X_j)$ is constant. We assign natural numbers to the components indicating the difference in the number of preimages.

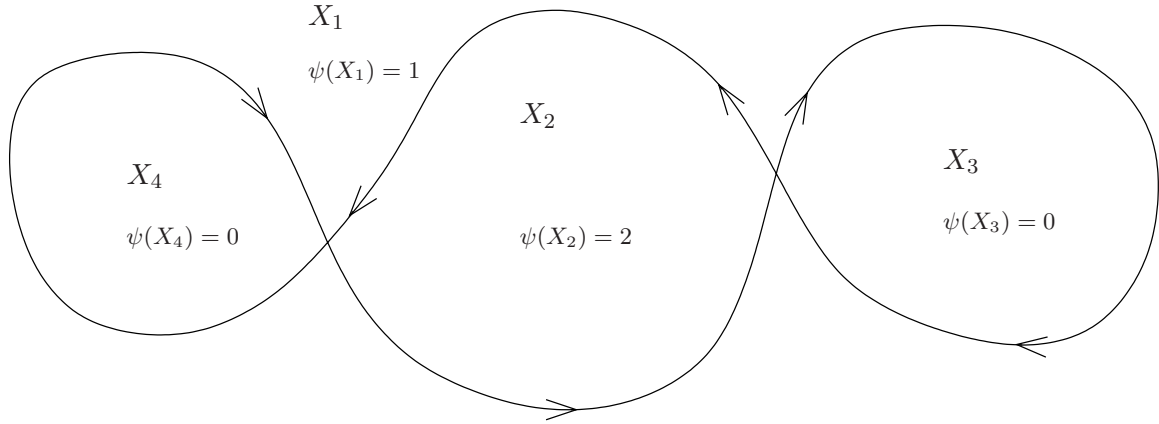
Definition 1.2.1. Suppose N is a closed surface and $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow N$ a normal immersion. Denote the components of $N \setminus f(\coprod_{j=1}^m \mathbb{S}^1)$ by X_1, \dots, X_k . A normal immersion f is called *numerable* if a continuous function $\psi: \{X_1, \dots, X_k\} \rightarrow \mathbb{Z}$ exists, such that

$$\psi(X_j) = \psi(X_i) + 1,$$

if X_j and X_i shares a common boundary and X_j lies on the left of this boundary.

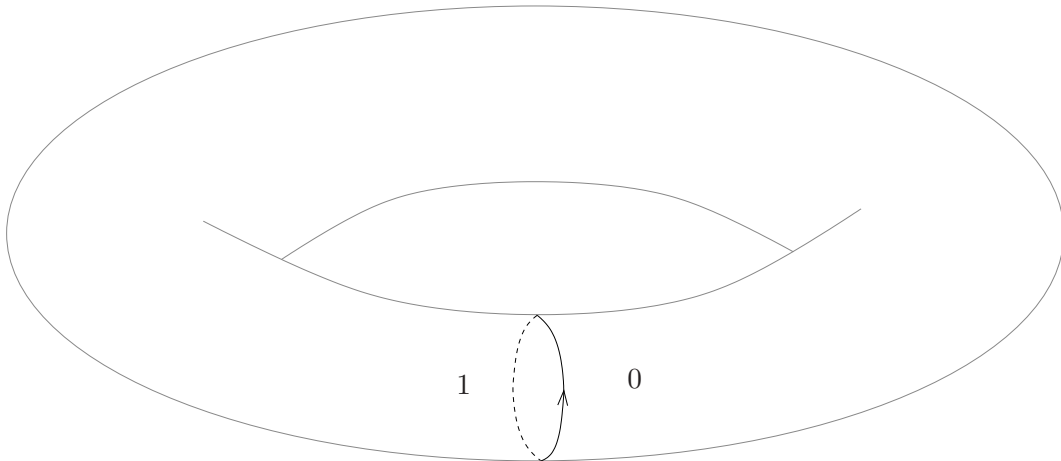
The function ψ is called a *numbering* of f . A function $\psi_n: \{X_1, \dots, X_k\} \rightarrow \mathbb{N}_0$ with $0 \in \text{im}(\psi_n)$ is called a *normal numbering* of f .

Example 1.2.2. We take the immersion $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ which is given by the following curve in stereographic projection:



The natural numbers assigned to the components define a numbering $\psi: \{X_1, X_2, X_3, X_4\} \rightarrow \mathbb{Z}$. Since ψ maps to \mathbb{N}_0 and $\psi(X_3) = 0$ the numbering is a normal one.

The next picture shows an example of an embedding where no numbering can assigned:



In [MC93] Margaret McIntyre and Grant Cairns describe an algorithm that assign a numbering to a given normal immersion $f: \mathbb{S}^1 \rightarrow \mathbb{R}^2$. We describe this algorithm for a more general normal immersion $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow N$.

Let M be a surface with boundary $\partial M = \coprod_{j=1}^m \mathbb{S}^1$. Let $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow N$ be a normal immersion to a closed surface N then the set $N \setminus f(\coprod_{j=1}^m \mathbb{S}^1)$ consists of finitely many components X_1, \dots, X_k .

If $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow N$ is numerable then the following algorithm assigns to each component a unique integer and hence defines a numbering of f .

Algorithm 1.2.3 (Numbering Algorithm). Suppose $f = (f_1, \dots, f_m): \coprod_{j=1}^m \mathbb{S}^1 \rightarrow N$ is a normal immersion, where $f_j: \mathbb{S}^1 \rightarrow N$ denotes the restriction of f to the j -th boundary component.

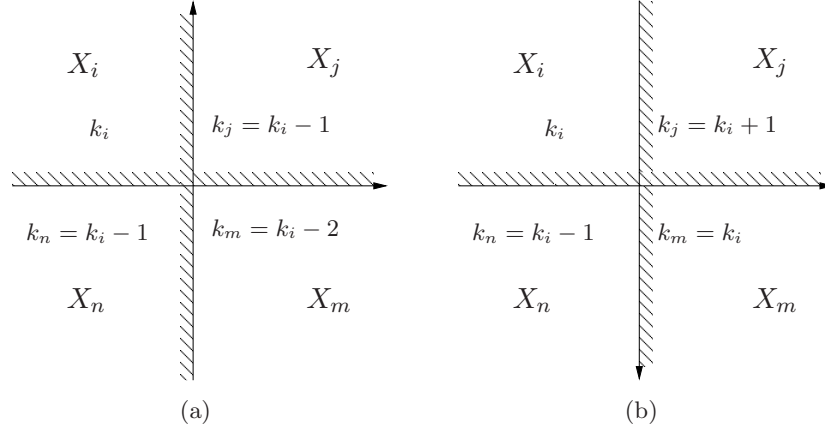


Figure 5: The path shown travels from left to right and crosses a double point. If the two components on the left are numbered as indicated, the numbering extends to the components on the right as indicated.

Step 1: Denote the components of $N \setminus f(\coprod_{j=1}^m \mathbb{S}^1)$ by X_1, \dots, X_k .

Step 2: Choose an initial point s_1^1 on $f_1(\mathbb{S}^1)$ that is not a double point.

Step 3: Assign to the component X_l on the right of s_1^1 the value $\psi(X_l) = 0$ and to the component X_i on the left the value $\psi(X_i) = 1$.

Step 4: Let s_2^1 be the first double point behind s_1^1 . Suppose the previous steps assigned the value $k_n = k_i - 1$ to the component X_n on the right. Behind the double point there is a component X_l with value k_l on the left and a component X_m with value k_m on the right. If the intersection is from right to left, then decrease the values k_l and k_m by 1 (Figure 5 (a)). If it is from left to right, then increase the values k_l and k_m by 1 (Figure 5 (b)).

Step 5: Repeat Step 4 until you reach the initial point s_1^1 .

Step 6: Pick an initial point s_1^j on the boundary component f_j , such that Steps 2-4 have assigned a number to the component on the left of s_1^j . Continue with steps 4 and 5 for each boundary component f_j , $j = 1, \dots, m$.

Since the numbering at a double point is necessary as shown in Figure 5, the algorithm assigns a numbering to each numerable immersion $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow N$. From this numbering we can derive a normal numbering by the equation

$$\psi_n(X_l) = \psi(X_l) - \min_{i=1, \dots, k} \psi(X_i). \quad (1)$$

We will now analyze when a normal immersion is numerable. For that suppose N is a closed surface of genus g and let ν_1, \dots, ν_{2g} be a set of differentiable curves whose homology classes $[\nu_1], \dots, [\nu_{2g}]$ form a generating set for the first homology group $H_1(N)$. Now let γ be an oriented, normal curve with homology class

$$[\gamma] = n_1[\nu_1] + \dots + n_{2g}[\nu_{2g}].$$

Moreover assume that the curves $\gamma, \nu_1, \dots, \nu_{2g}$ are pairwise transversal.

Lemma 1.2.4 (McIntyre, Cairns 1993). *Let X_1, \dots, X_k be the components of $N \setminus \{f \cup \nu_1 \cup \dots \cup \nu_{2g}\}$. One can associate integers to each of the components X_1, \dots, X_k such that at each segment of γ the number to the left of γ is 1 greater than the number to the right of γ , and for each $i = 1, \dots, 2g$ the number to the left of each segment of ν_i is n_i less than the number to the right of ν_i .*

The numbering is unique if we choose one of the components to be 0.

Proof. Lemma 2 of [MC93]. □

This yields the following lemma:

Lemma 1.2.5. *Let M be a compact connected oriented manifold with boundary $\partial M = \coprod_{j=1}^m \mathbb{S}^1$. Let N be a closed surface and $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow N$ a normal immersion. Then f is numerable if and only if f is homologically trivial.*

Proof. If f is homologically trivial the homology class of f is $[f] = 0$, i.e., $n_i = 0$ for all $i = 1, \dots, 2g$. Thus Lemma 1.2.4 provides a numbering of f .

If f is not homologically trivial then at least one of the numbers n_i in

$$[f] = n_1[\nu_1] + \dots + n_{2g}[\nu_{2g}]$$

is nonzero. W.l.o.g. assume that $n_1 \neq 0$. Since the curves ν_1, \dots, ν_{2g} and $f(\coprod_{j=1}^m \mathbb{S}^1)$ are transversal there is at least one component X_l of $N \setminus f(\coprod_{j=1}^m \mathbb{S}^1)$ such that $X_l \setminus \nu_1$ consists of two components X_{l1} and X_{l2} . Assume that X_{l1} is to the left of ν_1 then Lemma 1.2.4 assigns to X_{l1} the number m and to X_{l2} the number $m + n_1$. Hence two different numbers are assigned to the component X_l of $N \setminus f(\coprod_{j=1}^m \mathbb{S}^1)$, i.e., no numbering can exist. □

Corollary 1.2.6. *If N is simply connected then each normal immersion is numerable.*

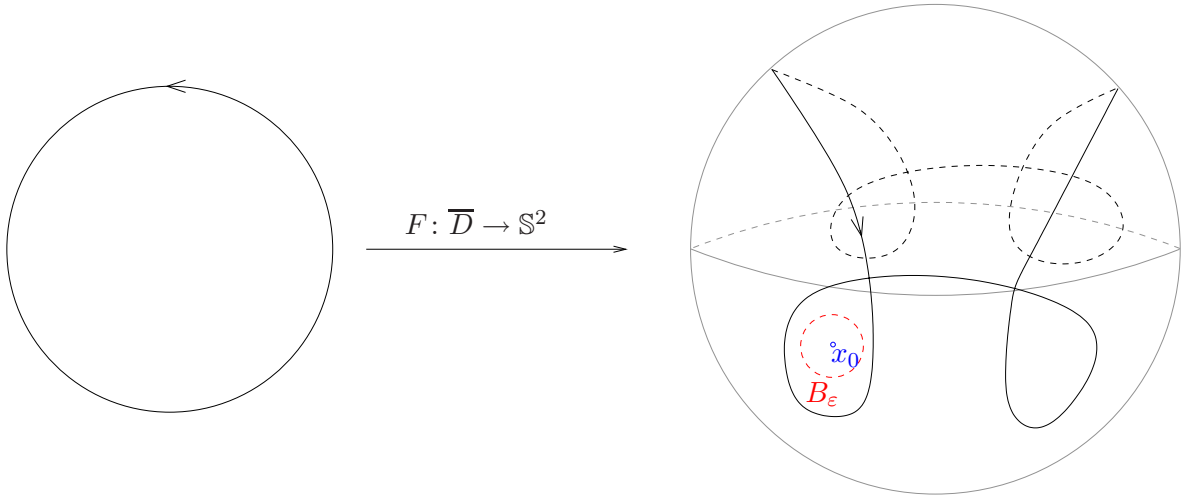
If $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow N$ extends to an immersion $F: M \rightarrow N$, the value of a normal numbering marks the difference between the number of preimages $\omega_f(X_j)$ of a component X_j under f . Indeed each time we pass the curve $f(\coprod_{j=1}^m \mathbb{S}^1)$ from left to right in Step 4 of Algorithm 1.2.3, we add a layer and each time we pass it from right to left we lose a layer of the surface (recall that the surface lies to the left of the curve $f(\coprod_{j=1}^m \mathbb{S}^1)$). In Figure 5 the surface is marked by the pattern.

The number of preimages of an extension $F: M \rightarrow N$ defines a numbering of the boundary curve $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow N$. While the number of preimages of different extensions is not constant in general, it is for $N = \mathbb{S}^2$, as we will see in the next subsection.

1.3 Preimages of an Immersed Disc in \mathbb{S}^2

Suppose $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ is a normal immersion. In this subsection we will show that the number of preimages of an extension $F: \overline{D} \rightarrow \mathbb{S}^2$ is determined by $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$.

By using Algorithm 1.2.3 and (1) we get a normal numbering $\psi_n: \{X_1, \dots, X_k\} \rightarrow \mathbb{N}_0$. We will see that this normal numbering is directly connected with the number of preimages of an extension of the normal immersion $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$.

Figure 6: Cutting out a neighborhood of x_0 .

Proposition 1.3.1. *Let $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ be a normal immersion which extends to \overline{D} and denote the components of $\mathbb{S}^2 \setminus f(\mathbb{S}^1)$ by X_1, \dots, X_k . Let $F: \overline{D} \rightarrow \mathbb{S}^2$ and $G: \overline{D} \rightarrow \mathbb{S}^2$ be extensions of f . Denote by ω_j^F the number of preimages of X_j under F and by ω_j^G the number of preimages of X_j under G . Then*

$$\omega_j^F = \omega_j^G \quad \text{for } j = 1, \dots, k.$$

Proof. For a normal immersion $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$, Algorithm 1.2.3 and (1) give a normal numbering $\psi_n: \{X_1, \dots, X_k\} \rightarrow \mathbb{N}_0$. We may assume that $\psi_n(X_1) = 0$. Then the number of preimages of X_j under F can be written as

$$\omega_j^F = \omega_1^F + \psi_n(X_j).$$

Since the numbering is independent of the extension, for the number of preimages of X_j under G an equivalent equation holds. Therefore it is sufficient to show that $\omega_1^F = \omega_1^G$.

The goal is to view the extension as an immersion to \mathbb{R}^2 and use the global Gauss-Bonnet-Theorem (see [dC76], p. 274) to determine $\omega_1^{F,G}$.

Let us pick a point $x_0 \in X_1 \subset \mathbb{S}^2$. Since the components X_j are open there is a small ball $B_\varepsilon(x_0)$ with radius $\varepsilon > 0$ around x_0 such that $B_\varepsilon(x_0) \cap f(\mathbb{S}^1) = \emptyset$ (Figure 6).

Since x_0 has ω_1^F preimages in \overline{D} , the preimage of $F(\overline{D}) \setminus B_\varepsilon(x_0)$ is $\overline{D} \setminus \{A_1, \dots, A_{\omega_1^F}\}$, where A_i denotes the components of $F^{-1}(B_\varepsilon(x_0))$. If F is not surjective $\omega_1^F = 0$ and hence there is no A_i .

In general $F^{-1}(B_\varepsilon(x_0))$ consists of ω_1^F components. Because x_0 is not in $F(\overline{D}) \setminus B_\varepsilon(x_0)$ the image $F(\overline{D} \setminus \{A_1, \dots, A_{\omega_1^F}\}) \subset \mathbb{S}^2 \setminus B_\varepsilon(x_0)$ can be projected to \mathbb{R}^2 . That way F induces an immersion $\tilde{F}: \tilde{D} \rightarrow \mathbb{R}^2$ for instance, by composing with the stereographic projection from x_0 . The boundary of $\tilde{F}(\tilde{D})$ consists of the image $C_0 := \tilde{F}(\partial\overline{D})$ of the boundary of \overline{D} and

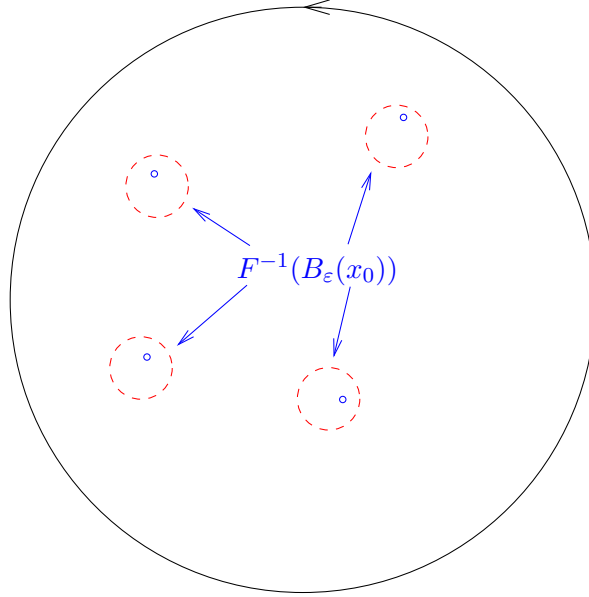


Figure 7: The preimage of $F(\overline{D}) \setminus B_\varepsilon(x_0)$

the images $C_j := \tilde{F}(\partial A_j)$ of the boundaries of the holes A_j , for $j = 1, \dots, \omega_1^F$. The global Gauss-Bonnet-Theorem leads to

$$\sum_{j=0}^{\omega_1^F} \int_{C_j} \kappa_g ds + \int_{\tilde{F}(\tilde{D})} K dA = 2\pi\chi(\tilde{F}(\tilde{D})). \quad (2)$$

We choose a positive orientation of the boundary curves $C_1, \dots, C_{\omega_1^F}$ and so the total curvature of one of these bounding curves C_j is

$$\int_{C_j} \kappa_g ds = 2\pi \text{wind}(C_j) = 2\pi,$$

where $\text{wind}(C_j)$ denotes the winding number of C_j . Therefore (2) leads to

$$2\pi \text{wind}(C_0) + 2\pi\omega_1^F + \int_{\tilde{F}(\tilde{D})} K dA = 2\pi\chi(\tilde{F}(\tilde{D})).$$

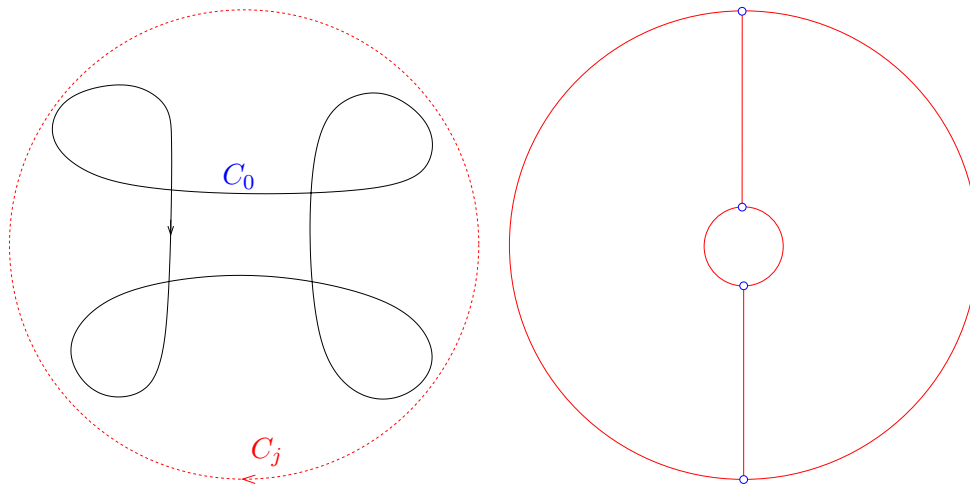
Since \mathbb{R}^2 has constant curvature $K = 0$ we get

$$2\pi \text{wind}(C_0) + 2\pi\omega_1^F = 2\pi\chi(\tilde{F}(\tilde{D})). \quad (3)$$

The Euler characteristic of a disc is 1, but how does it change if we cut out holes? Every hole leads to 2 more vertices, 4 more edges and 1 more face (Figure 8 (b)). So a disc with n holes has the Euler characteristic $\chi = 1 + 2n - 4n + n = 1 - n$.

Since \tilde{D} is a disc with ω_1^F holes (3) becomes

$$2\pi \text{wind}(C_0) + 2\pi\omega_1^F = 2\pi(1 - \omega_1^F) \quad (4)$$



(a) The image of \tilde{F} where C_0 is the image of the boundary of \overline{D} and C_j the boundary of the neighborhood of x_0 , counted with multiplicity. (b) Euler characteristic of a disc with a hole

Figure 8: Cutting out the base point

and therefore

$$\omega_1^F = \frac{1 - \text{wind}(C_0)}{2}.$$

Hence the number of preimages of F is determined by the winding number of $C_0 = F(\partial\overline{D}) = f(\mathbb{S}^1)$ and since C_0 is independent of the extension, any other extension G also satisfies

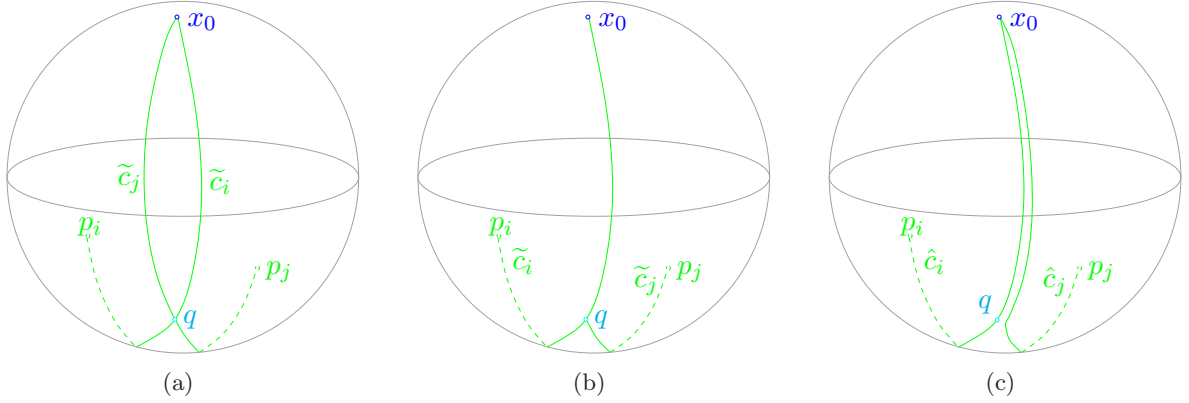
$$\omega_1^G = \frac{1 - \text{wind}(C_0)}{2} = \omega_1^F.$$

□

Definition 1.3.2. Let $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ be a normal immersion and ψ_n a normal numbering of f . For each component X_j of $\mathbb{S}^2 \setminus f(\mathbb{S}^1)$ the number

$$\omega_j := \psi_n(X_j)$$

is called the *degree of X_j* .

Figure 9: From \tilde{c}_j to \hat{c}_j .

2 Groupings

In this section we describe how to use Blank's algorithm [Bla67] in the case of an immersion $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ to define a word $w(f)$. The word will depend on some choices. A choice optimal for our purposes will lead to what we call a reduced word.

2.1 The Word Algorithm

Algorithm 2.1.1 (The Word Algorithm). *Suppose $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ is a normal immersion and denote the components of $\mathbb{S}^2 \setminus f(\mathbb{S}^1)$ by X_1, \dots, X_k and the degrees of X_j by ω_j .*

Step 1: *Choose a component X_j , which has a minimal degree ω_j .*

Step 2: *Pick a base point x_0 in X_j .*

Step 3: *Select in each component X_i with $i \neq j$ a point p_i , such that no two points p_i and p_k lie on the same great circle through x_0 .*

Step 4: *Connect each p_i with x_0 by a great circle \tilde{c}_i . Choose the great circle arc \tilde{c}_j , which produces the fewest intersections with $f(\mathbb{S}^1)$.*

Step 5: *If $\tilde{c}_j \cap \tilde{c}_i \neq \{x_0\}$ for $j \neq i$ then there exists a point $q \neq x_0$ which belongs to \tilde{c}_j and to \tilde{c}_i (Figure 9 (a)). Change the arc \tilde{c}_j at the point q , such that \tilde{c}_j goes from p_j along the great circle arc \tilde{c}_j to q and then along the great circle arc \tilde{c}_i to x_0 (Figure 9 (b)). Now homotopy the part between q and x_0 such that there is no intersection with \tilde{c}_i (Figure 9 (c)).*

Repeat this as long as there are intersection points between the \tilde{c}_j . At the end we get rays \hat{c}_j from p_j to x_0 with $\hat{c}_j \cap \tilde{c}_i = \{x_0\}$ for $j \neq i$. We choose an orientation of each ray \hat{c}_j such that it starts at p_j and ends in x_0 .

Step 6: *Choose an initial point s on $f(\mathbb{S}^1)$. Start at s and follow $f(\mathbb{S}^1)$ in the direction of the orientation until you reach the first intersection point of $f(\mathbb{S}^1)$ and a ray \hat{c}_j . Label the intersection point with a_j if \hat{c}_j crosses from left to right and with a_j^{-1} if \hat{c}_j crosses from right to left. Repeat this until you reach the initial point s .*

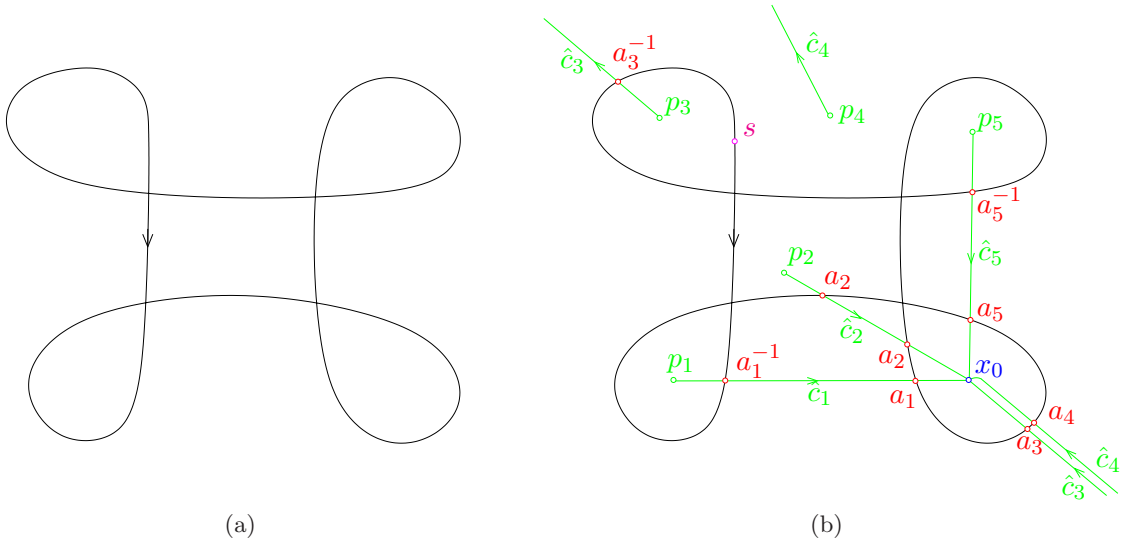


Figure 10: A Result of Algorithm 2.1.1 in projection on \mathbb{R}^2 .

Step 7: Write down all intersection points in the order they appear when you walk along $f(\mathbb{S}^1)$, starting at s .

Example 2.1.2. Figure 10 (b) shows the result of Algorithm 2.1.1 when applied to the curve in Figure 10 (a). Starting at point s we get the word

$$a_1^{-1}a_2a_5a_4a_3a_1a_2a_5^{-1}a_3^{-1}.$$

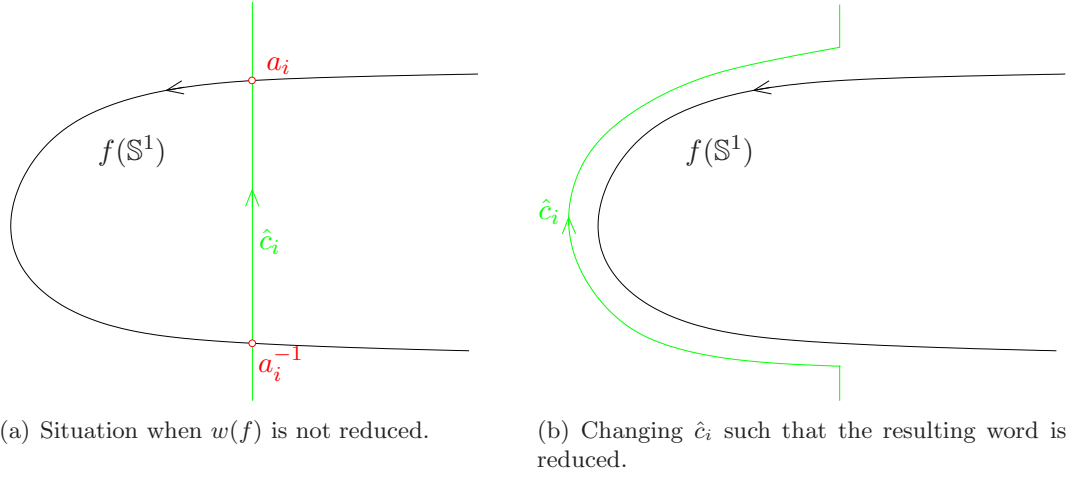
Definition 2.1.3. Let $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ be a normal immersion. A word, which results by application of Algorithm 2.1.1 to f , is called *word of f* and will be denoted by $w(f)$. It is unique up to cyclic permutations.

Remark. The uniqueness up to cyclic permutations is caused by the choice of the initial point s . The choice of a different starting point leads to a cyclic permutation of the word, since the order of the intersection points does not change. Only the choice of the first letter is different.

2.2 Reducing the Word

By applying Algorithm 2.1.1 to a normal immersion $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ we get a word $w(f)$. Since the algorithm allows many different choices (i.e., numbering the components, setting the initial point, choose another component for the base point, etc.) we can get different words for the same immersion f . We want to choose an appropriate representative of all possible words $w(f)$ optimal for our purposes.

Definition 2.2.1. A *subword* ω of $w(f)$ is a subsequence of consecutive letters of $w(f)$. A word $w(f)$ is called *reduced* if it satisfies the following conditions:

Figure 11: Reducing the word $w(f)$.

- The word $w(f)$ contains no subword of the form $a_i a_i^{-1}$ or $a_i^{-1} a_i$.
- The word $w(f)$ is not of the form $a_i \dots a_i^{-1}$ or $a_i^{-1} \dots a_i$.

The reduced word is unique up to cyclic permutations as well. If we move the starting point s along $f(\mathbb{S}^1)$ the reduced word changes by cyclic permutations.

We will now show that for each normal immersion $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ a word $w(f)$ exists which is reduced:

Lemma 2.2.2. *For every normal immersion $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ the points p_i , s and x_0 as well as the rays \hat{c}_i can be chosen such that the resulting word $w(f)$ is reduced.*

Proof. Assume the word $w(f)$ of f is not reduced.

1. Assume $w(f)$ contains a subword of the form $a_i a_i^{-1}$. This means that $f(\mathbb{S}^1)$ hits the ray \hat{c}_i two times consecutively. Since there is no other letter between a_i and a_i^{-1} the situation looks like in Figure 11 (a). In this case we can homotopy the ray \hat{c}_i such that the intersection points a_i and a_i^{-1} do not occur any more (Figure 11 (b)). If $w(f)$ contains a subword of the form $a_i^{-1} a_i$ then the situation is the same. Only the orientation of the ray \hat{c}_i in relation to $f(\mathbb{S}^1)$ has changed. But there is still no intersection of another ray and $f(\mathbb{S}^1)$ between a_i^{-1} and a_i and so we can homotopy \hat{c}_i again such that the letters $a_i^{-1} a_i$ do not occur any more.
2. Assume the word $w(f)$ is of the form $a_i^{-1} \dots a_i = a_i^{-1} \omega a_i$, where ω is the subword which contains all letters except the outer ones. This means, that the initial point s lies between a_i and a_i^{-1} . Choose a new initial point \tilde{s} which is directly after the intersection point a_i^{-1} . Now Algorithm 2.1.1 results in a word $\omega a_i a_i^{-1}$. As in 1. this word can be reduced to ω .

This shows that every time one of the conditions of Definition 2.2.1 is not satisfied we can change the rays or the initial point to get a setting for which the algorithm results in a reduced word. \square

In the following the word $w(f)$ of a normal immersion f is assumed to be reduced.

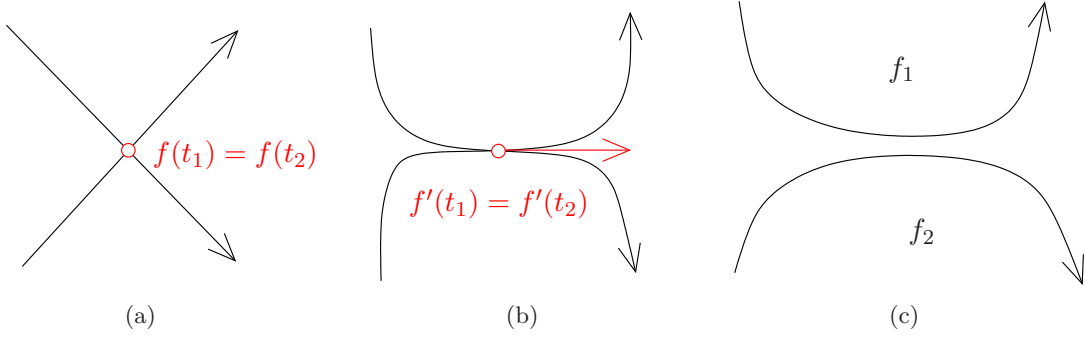


Figure 12: Decomposing a normal immersion into embeddings

2.3 Decomposing Immersions

According to [Bla67] we will give an algorithm to decompose a normal immersion $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ into embeddings.

Algorithm 2.3.1 (Decomposition Algorithm). *Let $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ be a normal immersion. If f has no selfintersections then f is an embedding itself. Assume f has a double point $f(t_1) = f(t_2)$ with $t_1, t_2 \in \mathbb{S}^1$.*

Step 1: *Choose a starting point $t_0 \in \mathbb{S}^1$ and assume that $t_1 < t_2$ which means that starting from t_0 we reach at first t_1 and then t_2 .*

Step 2: *Homotopy f in a small neighborhood of the double point such that $f'(t_1) = f'(t_2)$ (Figure 12 (b)).*

Step 3: *Decompose f into two immersions $f_1: [t_0, t_1] \cup [t_2, t_0] \rightarrow \mathbb{S}^2$ and $f_2: [t_1, t_2] \rightarrow \mathbb{S}^2$ with $f_i(t) = f(t)$. Since $f(t_1) = f(t_2)$ and $f'(t_1) = f'(t_2)$ we get two closed immersions f_1 and f_2 .*

Step 4: *Homotopy f_1 and f_2 such that $\text{im}(f_1) \cap \text{im}(f_2) = \emptyset$ (Figure 12 (c)).*

Step 5: *Repeat Steps 1 – 4 until we get a family f_1, \dots, f_n of embeddings.*

Now take a normal immersion $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ and choose a base point x_0 in a component of minimal degree. By applying the Decomposition Algorithm 2.3.1 we get a family f_1, \dots, f_n of embeddings respectively oriented circles.

Definition 2.3.2. Let $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ be a normal immersion and f_1, \dots, f_n the family of embeddings which results from the Decomposition Algorithm 2.3.1. Choose a point $x_0 \in \mathbb{S}^2 \setminus f(\mathbb{S}^1)$ in a component of minimal degree. With an orientation preserving diffeomorphism $\varphi: \mathbb{S}^2 \setminus \{x_0\} \rightarrow \mathbb{R}^2$ we denote by $\tilde{f}_i = \varphi \circ f_i$ the induced embedding to \mathbb{R}^2 .

Define the *tangent winding number* of f_i as

$$\tau(f_i) := \text{wind}(\tilde{f}_i)$$

and the *tangent winding number* of f as

$$\tau(f) = \sum_{i=1}^n \tau(f_i).$$

The following lemma ensures that the definition of the tangent winding number is well defined.

Lemma 2.3.3. *Let $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ be a normal immersion and X_1, X_2 two components of $\mathbb{S}^2 \setminus f(\mathbb{S}^1)$ of minimal degree. Denote by $\tau_1(f)$ the tangent winding number of f , if $x_0 \in X_1$, and by $\tau_2(f)$ the tangent winding number of f , if x_0 is in X_2 . Then $\tau_1(f) = \tau_2(f)$.*

Proof. Decompose f into embeddings according to Algorithm 2.3.1. W.l.o.g. f_1 is the boundary of X_1 and f_2 is the boundary of X_2 . Since both components are of minimal degree, f_1 and f_2 have the same orientation. If x_0 is in X_1 , then $\tau_1(f_1) = +1$ and $\tau_1(f_2) = -1$. If x_0 is in X_2 , then $\tau_2(f_1) = -1$ and $\tau_2(f_2) = +1$.

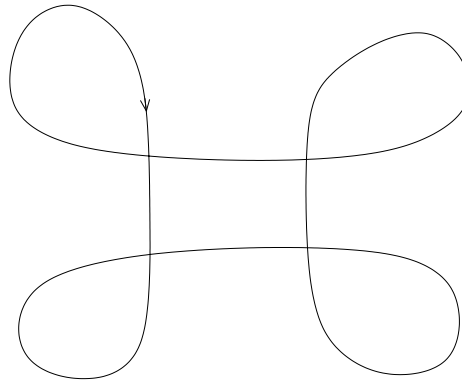
Since the other embeddings are not affected, this leads to:

$$\tau_1(f) = 1 - 1 + \sum_{j=3}^n \tau_1(f_j) = -1 + 1 + \sum_{j=3}^n \tau_2(f_j) = \tau_2(f).$$

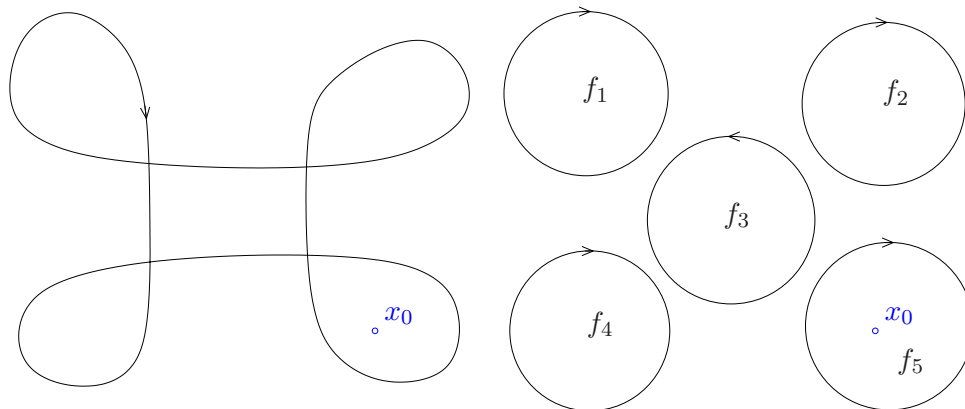
□

Example 2.3.4. 1. An oriented embedding $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ has the tangent winding number $\tau(f) = 1$. This follows directly from the fact that an oriented embedding in \mathbb{R}^2 has winding number 1 ([tD00] Theorem X.3.8, p. 375).

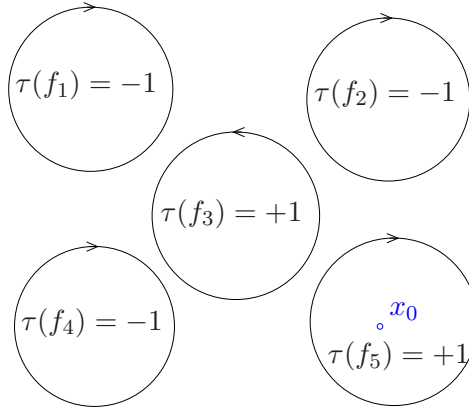
2. Take the following immersion:



Choosing a base point x_0 and applying Algorithm 2.3.1 leads to the following situation:



Assigning the tangent winding number to each embedding yield



and so the tangent winding number in this case is

$$\tau(f) = \tau(f_1) + \tau(f_2) + \tau(f_3) + \tau(f_4) + \tau(f_5) = -1.$$

Lemma 2.3.5. *Let $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ be a normal immersion and x_0 the base point in a component of minimal degree. With an orientation preserving diffeomorphism $\varphi: \mathbb{S}^2 \setminus \{x_0\} \rightarrow \mathbb{R}^2$ we denote by $\tilde{f} = \varphi \circ f$ the induced immersion to \mathbb{R}^2 . Then*

$$\tau(f) = \text{wind}(\tilde{f}).$$

Proof. Applying the Decomposition Algorithm 2.3.1 to the induced immersion \tilde{f} leads to embeddings $\tilde{f}_1, \dots, \tilde{f}_n$. Since the algorithm reparametrizes in double points where the tangent vectors are equal, the total number of rotations of the tangent vector is not changed by the algorithm. Therefore

$$\text{wind}(\tilde{f}) = \sum_{j=1}^n \text{wind}(\tilde{f}_j).$$

Now apply the Decomposition Algorithm 2.3.1 to the immersion f to get embeddings f_1, \dots, f_n . Note, that the number of embeddings is the same as before since the number of double points is the same. Hence

$$\tau(f) = \sum_{j=1}^n \tau(f_j).$$

By Definition 2.3.2 $\tau(f_j) = \text{wind}(\tilde{f}_j)$ and thus

$$\tau(f) = \sum_{j=1}^n \tau(f_j) = \sum_{j=1}^n \text{wind}(\tilde{f}_j) = \text{wind}(\tilde{f}).$$

□

Lemma 2.3.6. *For every normal immersion $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ the Decomposition Algorithm 2.3.1 results in at least one embedding f_j with tangent winding number $\tau(f_j) = 1$.*

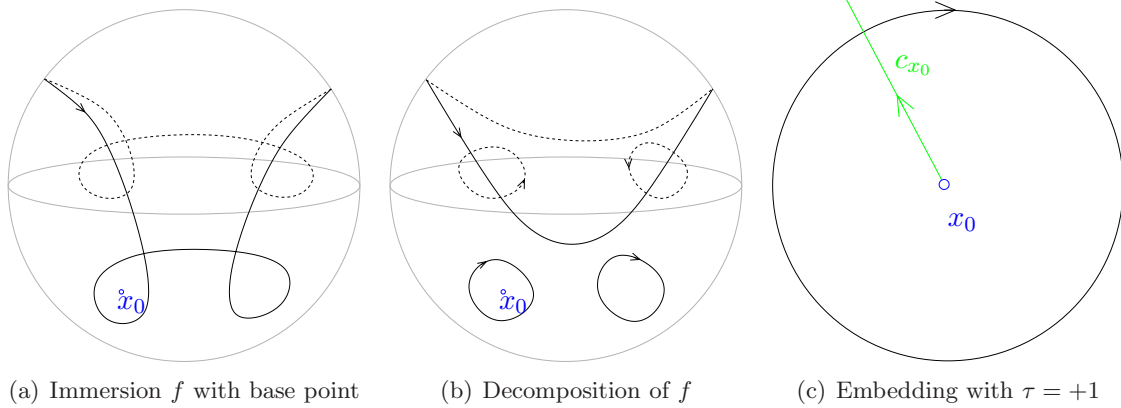


Figure 13: An embedding with tangent winding number $\tau = +1$

Proof. We choose a base point x_0 and apply the Decomposition Algorithm 2.3.1 to f to get embeddings f_1, \dots, f_n . Take a great circle arc c_{x_0} which starts in x_0 and follow it until it intersects one of the embeddings f_1, \dots, f_n . W.l.o.g. this intersection occurs with embedding f_1 . Since f_1 is a closed curve any great circle arc which starts in x_0 intersects f_1 first. Since x_0 lies in a component of minimal degree the embedding intersects the great circle arc from left to right. If it intersected in the other direction, then the component on the other side has a smaller degree in contradiction to the choice of x_0 . Hence x_0 lies on the right of f_1 and so $\tau(f_1) = +1$. \square

In the case of normal immersions to \mathbb{R}^2 a necessary condition for a normal immersion $f: \mathbb{S}^1 \rightarrow \mathbb{R}^2$ to extend is that f has winding number $+1$. In our case the necessary condition is in terms of the tangent winding number:

Proposition 2.3.7 (Necessary Extension Condition for one Boundary Component). *Every normal immersion $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ which can be extended to an immersion $F: \overline{D} \rightarrow \mathbb{S}^2$ has the tangent winding number*

$$\tau(f) = 1 - 2\omega_1 \quad \text{for } \omega_1 \in \mathbb{N}_0,$$

where ω_1 denotes the degree of the base point component X_1 .

Proof. In the proof of Proposition 1.3.1 (4) we have seen that the following is true for the degree ω_1 of the base point component X_1 :

$$2\pi \text{wind}(\tilde{f}) + 2\pi\omega_1 = 2\pi(1 - \omega_1),$$

where \tilde{f} denotes the induced immersion to \mathbb{R}^2 . Solving this for the winding number leads to

$$\text{wind}(\tilde{f}) = 1 - 2\omega_1.$$

Referring to Lemma 2.3.5 it follows that

$$\tau(f) = 1 - 2\omega_1.$$

Since ω_1 denotes the degree of the base point component X_1 , it is a nonnegative integer and hence in \mathbb{N}_0 . \square

Remark. In the case of normal immersions to \mathbb{R}^2 Blank [Bla67] uses the noncompact component of $\mathbb{R}^2 \setminus f(\mathbb{S}^1)$ as base point component. Since each extension of f is compact, the degree of the base point component is 0 in this case. If we look at a normal immersion $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$, which has a base point component of degree $\omega_1 = 0$ then $\tau(f) = 1$. Because the base point component has degree 0, the immersion f can be projected to an immersion $\tilde{f}: \mathbb{S}^1 \rightarrow \mathbb{R}^2$. According to Lemma 2.3.5 the winding number of \tilde{f} is $\text{wind}(\tilde{f}) = 1$. Thus this immersion satisfies the necessary condition in the case of \mathbb{R}^2 as well as in the case of \mathbb{S}^2 .

2.4 Intervals, Groupings and Trees

In Proposition 2.3.7 we have seen a necessary condition for a normal immersion $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ to extend to \overline{D} . In this subsection we present the tool to decide whether a normal immersion $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ with $\tau(f) = 1 - 2\omega_1$ can be extended or not.

Crucial for this decision are the properties of the reduced word, which we get from Algorithm 2.1.1. First of all we will name some special subwords which will play a central role later on.

Definition 2.4.1. For a reduced word $w(f)$ the subword

- $a_i a_j \dots a_k$ is called a *positive word* if consecutive letters are different,
- $a_i^{\pm 1} p a_i^{\mp 1}$ is called a *pairing* if p is a positive word,
- $a_j^{-1} a_i^{-1}$ is called a *negative group* if $i \neq j$.

The empty word is defined as positive.

We will show that an immersion whose reduced word is composed of these special subwords extend. If a reduced word $w(f)$ contains a subword ω , which is a pairing or a negative group, then $w(f)$ can be written as $w(f) = x_1 \omega x_2$. Then we can cancel the subword ω out of $w(f)$ to get $\tilde{w}(f) = x_1 x_2$.

Definition 2.4.2. A reduced word $w(f)$ is called *groupable* if a cancellation of pairings and negative groups exists, such that a positive word remains.

Example 2.4.3. 1. Take the word $a_2 a_3 a_1 a_4^{-1} a_2 a_1 a_4 a_3$. Canceling leads to

$$a_2 a_3 a_1 | a_4^{-1} a_2 a_1 a_4 | a_3 \rightsquigarrow a_2 a_3 a_1 a_3.$$

Since a positive word remains the word is groupable.

2. Take the word $a_2 a_3^{-1} a_1^{-1} a_4^{-1} a_2 a_1 a_4 a_3$. We can cancel in the following way:

$$a_2 | a_3^{-1} a_1^{-1} | a_4^{-1} a_2 a_1 a_4 a_3 \rightsquigarrow a_2 | a_4^{-1} a_2 a_1 a_4 | a_3 \rightsquigarrow a_2 a_3.$$

Since the positive word $a_2 a_3$ remains the word is groupable. But in this case we have another possibility to cancel pairings and negative groups:

$$a_2 a_3^{-1} | a_1^{-1} a_4^{-1} | a_2 a_1 a_4 a_3 \rightsquigarrow a_2 | a_3^{-1} a_2 a_1 a_4 a_3 | \rightsquigarrow a_2.$$

Again a positive word remains, i.e., the word is groupable in two different ways.

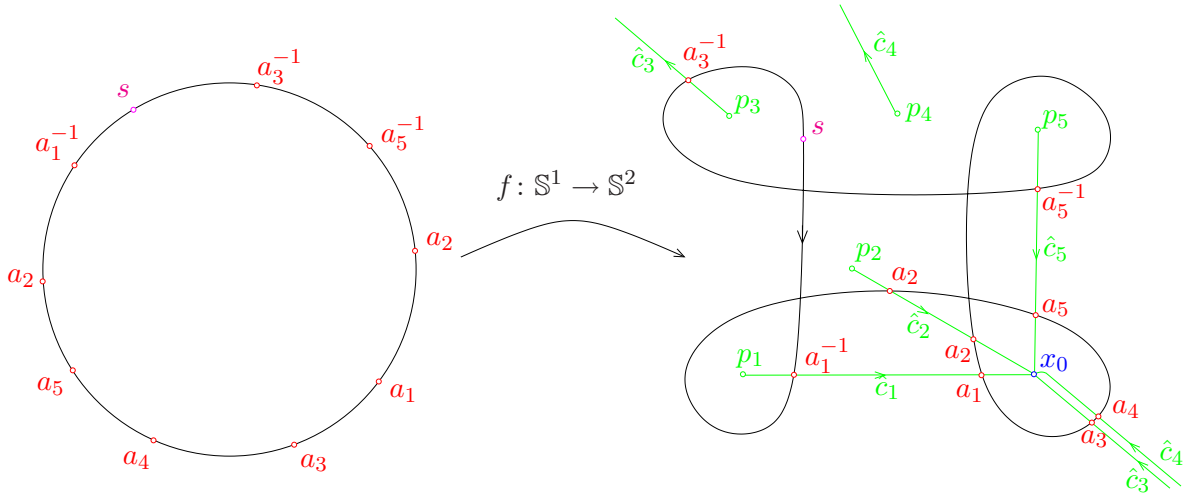


Figure 14: Intersection points in the image and preimage

The goal is to show that a normal immersion f whose reduced word $w(f)$ is groupable extends to an immersion on \overline{D} and vice versa.

Suppose $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ is a normal immersion. Then Algorithm 2.1.1 results in the word $w(f)$. We mark the preimages of the intersection points $a_i^{\pm 1}$ on \mathbb{S}^1 (Figure 14). For convenience we use the notation $a_i^{\pm 1}$ for the preimages as well.

Given the case of a pairing $a_i^{\pm 1} \dots a_i^{\mp 1}$ we can join the corresponding points in the preimage by a differentiable curve in D which is transverse to \mathbb{S}^1 . This curve is called an interval and is denoted by $[a_i^{\pm 1}, a_i^{\mp 1}]$ (Figure 15).

In the case of a negative group $a_j^{-1} a_k^{-1}$, we join the points in the preimage by an interval which contains a preimage of the base point x_0 . Of course, this can be the case only if a possible extension $F: \overline{D} \rightarrow \mathbb{S}^2$ is surjective. This means the base point component has a positive degree. In this case mark an arbitrary point in the interior of \overline{D} as x_0 . Now join a_j^{-1} and x_0 by an interval as well as x_0 and a_k^{-1} . The intervals should be chosen as transversal to \mathbb{S}^1 and differentiable in x_0 . Denote the union of these two intervals by $[a_j^{-1}, a_k^{-1}]$ (Figure 15).

If there is more than one such interval the question arises whether these intervals can be chosen, so that they are disjoint.

Definition 2.4.4. Let $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ be a normal immersion and $w(f)$ the word which arises by application of Algorithm 2.1.1. We mark the preimages of the letters of $w(f)$ in \mathbb{S}^1 . The word $w(f)$ induces a partial order on \mathbb{S}^1 via

$$a < b \quad :\Leftrightarrow \quad a \text{ arises before } b \text{ in } w(f).$$

If a and b are two letters of $w(f)$ with $a < b$ then the part of \mathbb{S}^1 which starts in a and ends in b is denoted by $\mathbb{S}_{[a,b]}^1$.

Lemma 2.4.5. Suppose $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ is a normal immersion with groupable word $w(f)$. Then for each pairing and each negative group an interval can be chosen such that all intervals are disjoint.

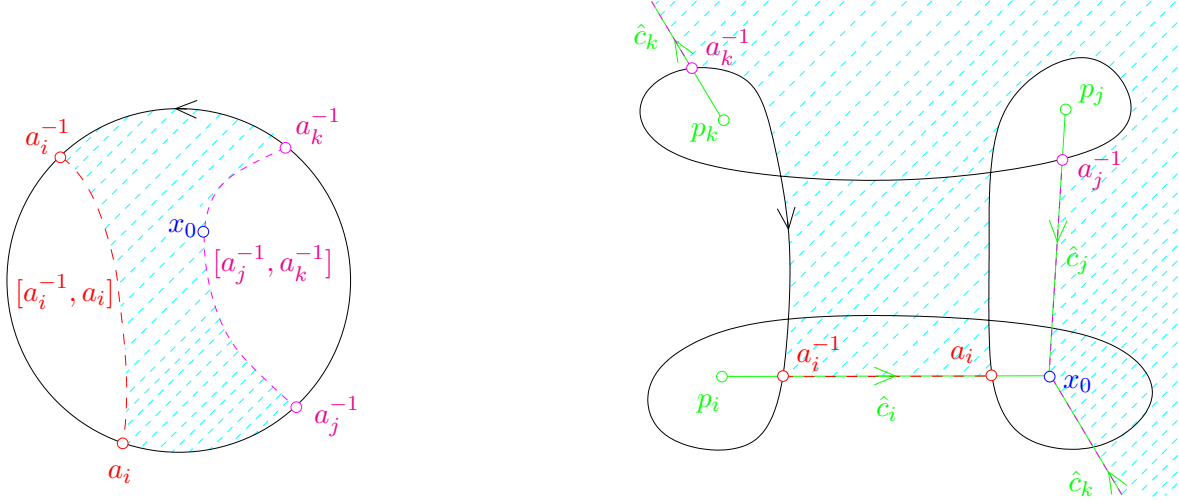


Figure 15: Extending the normal immersion to intervals. The pattern marks the disc enclosed by the intervals.

Proof. We start with the negative groups. Take a small neighborhood of the boundary, like $\mathbb{S}_{[a_j^{-1}, a_i^{-1}]}$ together with the secant from a_j^{-1} to a_i^{-1} and the interior. Since negative groups join two consecutive letters, no other interval can start or end in $\mathbb{S}_{[a_j^{-1}, a_i^{-1}]}$ and hence the neighborhoods of different negative groups are disjoint. Inside of the neighborhoods we choose intervals and since the neighborhoods are disjoint the intervals are disjoint as well.

Now we have selected an interval for each negative group. Each interval $[a_j^{-1}, a_i^{-1}]$ together with the boundary part $\mathbb{S}_{[a_j^{-1}, a_i^{-1}]}$ bounds a topological disc. After removing all these discs a disc remains whose corresponding word contains no negative groups.

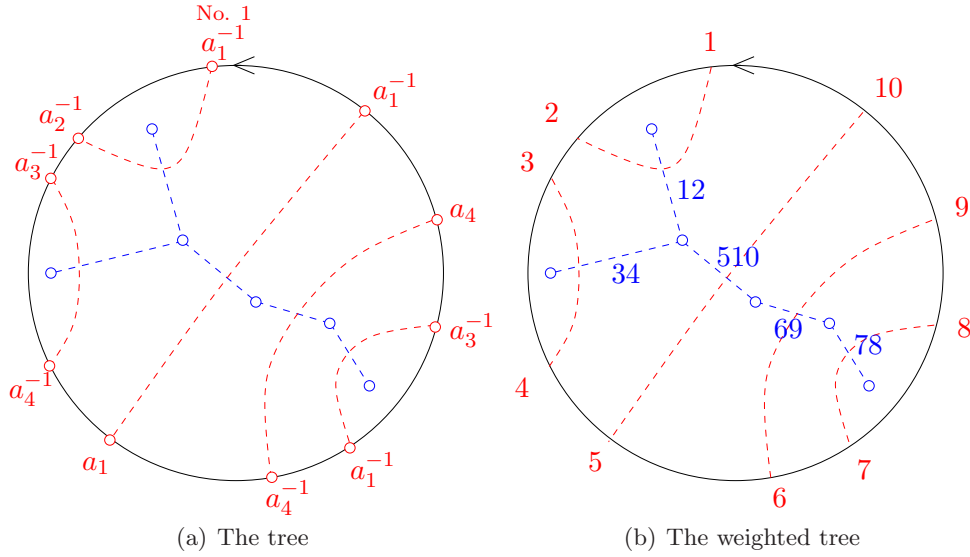
If this word contains a pairing $a_j \dots a_j^{-1}$ we connect the letters a_j and a_j^{-1} with an interval $[a_j, a_j^{-1}]$ inside of the remaining disc.

We have to rule out that two intervals of different pairings intersect. Thus we take two pairings $a_j \dots a_j^{-1}$ and $a_i \dots a_i^{-1}$. According to the definition of a pairing the boundary parts $\mathbb{S}_{[a_j, a_j^{-1}]}$ and $\mathbb{S}_{[a_i, a_i^{-1}]}$ are disjoint. Again we can choose disjoint neighborhoods of $\mathbb{S}_{[a_j, a_j^{-1}]}$ and $\mathbb{S}_{[a_i, a_i^{-1}]}$. Inside these neighborhoods we choose the intervals and hence they are disjoint. \square

Choosing the intervals to be disjoint leads to a subdivision of \overline{D} into smaller discs. If we put a vertex in each small disc and connect two vertices of adjacent discs, we get a tree which is dual to \mathbb{S}^1 together with the intervals (Figure 16).

Recall that a word is groupable if we can cancel out pairings and negative groups such that a positive word remains. We have seen in Example 2.4.3, 2. that different ways of canceling can arise. Each of these cancellations induce a decomposition of \overline{D} into smaller discs and hence a tree. Moreover we will see that different ways of canceling lead to different trees.

Take the reduced word $w(f)$ and number all letters. Remember that the reduced word is

Figure 16: Decomposing \overline{D} in smaller discs

unique up to cyclic permutations so the numbering is unique if we pick a first letter.

Now mark the preimages of the letters and their number in \mathbb{S}^1 . Since $w(f)$ is groupable there has to be a pairing or a negative group which can be canceled. Cancel it and insert the corresponding interval into \overline{D} . Continue until a positive word remains. Now \overline{D} is decomposed into smaller discs.

Put in each of these discs a vertex and connect two vertices if their small discs share a common boundary. Since the intervals can be chosen so that they are disjoint (Lemma 2.4.5) each edge intersects exactly one interval $[a, b]$. If the letter a has the number i and the letter b has the number j then label the edge with ij (Figure 16 (b)).

If $w(f)$ contains no pairing or negative group but is groupable then $w(f)$ is a positive word. In this case the decomposition consists of only one disc and hence the induced tree has only one vertex and no edge.

Lemma 2.4.6. *Suppose $f : \mathbb{S}^1 \rightarrow \mathbb{S}^2$ is a normal immersion and the corresponding reduced word is $w(f)$. If $w(f)$ is groupable each cancellation, which results in a positive word, induces a weighted tree.*

Proof. It remains to show that this graph is a tree. According to [Deo74], Theorem 3-2, we have to show that there is only one path between every two vertices to prove that the graph is a tree. Assume that there are two vertices v_1 and v_2 which are connected by two different paths p_1 and p_2 . Since both paths connect the vertices v_1 and v_2 there has to be a subpath c which is a circuit. This circuit encloses a region, so the dual of the tree has a vertex inside of the circuit. Since the tree is induced by the dual structure of the intervals this means that there has to be a point $a_j^{\pm 1}$ inside of the circuit. But there is no letter in the interior and hence such a circuit cannot occur. \square

Definition 2.4.7. Suppose $f : \mathbb{S}^1 \rightarrow \mathbb{S}^2$ is a normal immersion and $w(f)$ the corresponding reduced word. If $w(f)$ is groupable then the induced weighted tree \mathcal{G} is called a *grouping* of

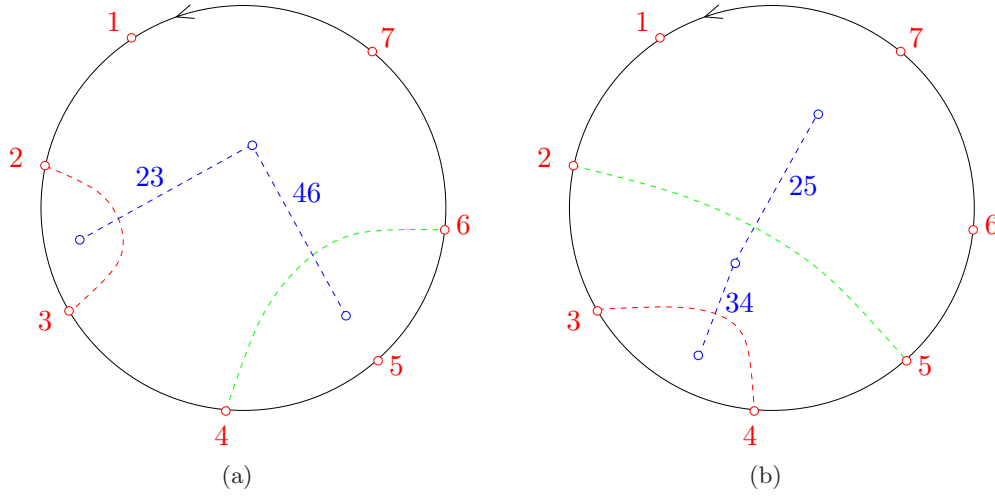


Figure 17: Two different groupings

f .

Two groupings of $w(f)$ are *equivalent* if the weighted trees are isomorphic.

Remark. Since every grouping of a word $w(f)$ results in a decomposition of \overline{D} into smaller discs we could refer to this decomposition as a grouping as well.

Example 2.4.8. Take the word

$$w(f) = a_2 a_1^{-1} a_4^{-1} a_5^{-1} a_1 a_5 a_4,$$

$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix}$

where a_2 is numbered by 1. This word can be grouped in two different ways. The first grouping is

$$a_2 | a_1^{-1} a_4^{-1} | a_5^{-1} a_1 a_5 a_4 \rightsquigarrow a_2 | a_5^{-1} a_1 a_5 | a_4 \rightsquigarrow a_2 a_4.$$

We insert an interval into \overline{D} for the negative group $a_1^{-1} a_4^{-1}$ and an interval for the pairing $a_5^{-1} a_1 a_5$. Thus we get a tree with 3 vertices and 2 edges. The edge which crosses the interval $[a_1^{-1}, a_4^{-1}]$ is labeled 23 and the edge which crosses the interval $[a_5^{-1}, a_5]$ is labeled 46 (Figure 17 (a)).

The second grouping is

$$a_2 a_1^{-1} | a_4^{-1} a_5^{-1} | a_1 a_5 a_4 \rightsquigarrow a_2 | a_1^{-1} a_1 | a_5 a_4 \rightsquigarrow a_2 a_5 a_4.$$

This grouping induces a tree with 3 vertices and 2 edges as well. But in this case the edge which crosses the interval induced by the negative group is labeled 34 and the edge which crosses the interval induced by the pairing is labeled 25 (Figure 17 (b)).

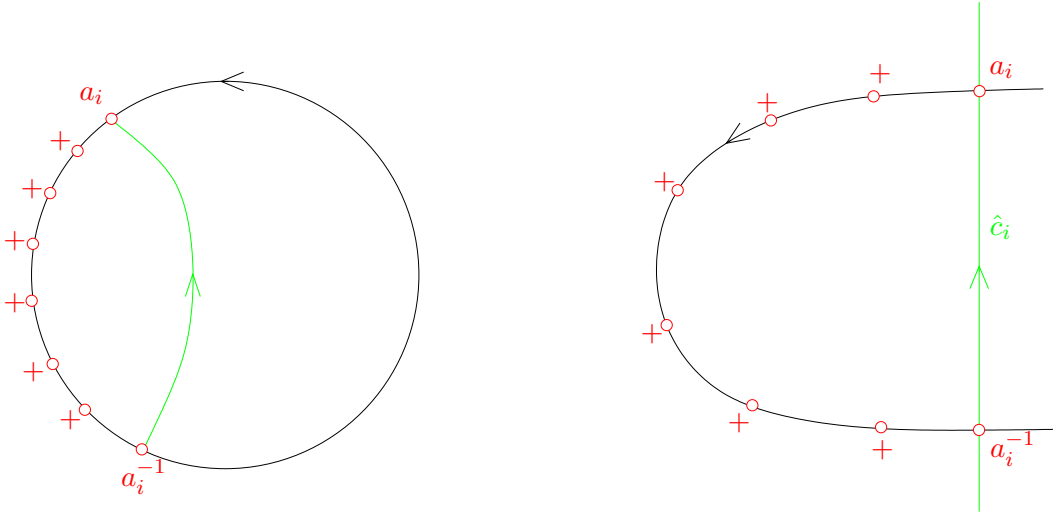


Figure 18: Extending a pairing

3 Immersed Discs in the Sphere

After some preparations in the previous section we can now look at the main goal: Extending immersions.

3.1 Existence of Extensions

For convenience we use the following speech:

Definition 3.1.1. We say that a word $w(f)$ *surrounds an immersed disc* if there is a normal immersion $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ which has $w(f)$ as a result of Algorithm 2.1.1 and which can be extended to an immersion $F: \overline{D} \rightarrow \mathbb{S}^2$. If the same holds true for an embedding, $w(f)$ *surrounds an embedded disc*.

A subword $a \dots b$ of a word $w(f)$ *surrounds an immersed disc* if there is an interval $[a, b]$, such that $\mathbb{S}_{[a,b]}^1 \cup [a, b]$ surrounds an immersed disc. Again, if the same holds true for an embedding then the subword *surrounds an embedded disc*.

We will show that a normal immersion f extends to \overline{D} if and only if the reduced word $w(f)$ is groupable. Since groupable means to cancel out pairings and negative groups in $w(f)$ we will show at first that these subwords themselves surround immersed discs.

Lemma 3.1.2. *A positive word surrounds an embedded disc.*

Proof. Decompose the normal immersion $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ with the Decomposition Algorithm 2.3.1 into embeddings f_1, \dots, f_n . According to Lemma 2.3.6 w.l.o.g. $\tau(f_1) = +1$. Assume there is an embedding with $\tau(f_j) = -1$. Since x_0 lies in a component of minimal degree the ray \hat{c}_j has to intersect f_j from right to left. This produces a negative intersection point a_j^{-1} in contradiction to the assumption that $w(f)$ is a positive word. Hence all embeddings have tangent winding number $\tau(f_j) = +1$ which leads to

$$\tau(f) = \sum_{j=1}^n \tau(f_j) = n.$$

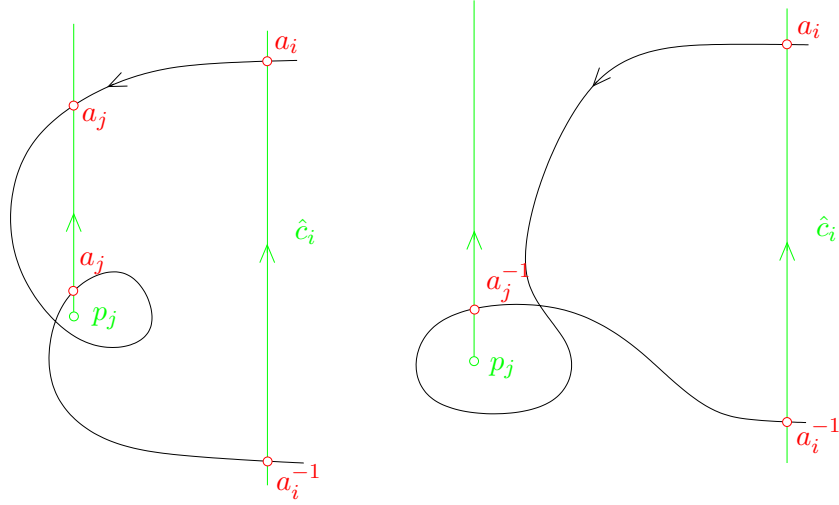


Figure 19: The immersion inside of a pairing cannot have selfintersections.

On the other hand $\tau(f) \leq 1$, according to Proposition 2.3.7. Thus $n = 1$ and hence f is an embedding itself. Therefore it extends to an embedding $F: \overline{D} \rightarrow \mathbb{S}^2$ according to the Riemann Mapping Theorem (see [Ahl53], p. 172ff). \square

Lemma 3.1.3. *A pairing surrounds an embedded disc.*

Proof. If the word $w(f)$ contains a pairing the immersion on \mathbb{S}^2 locally looks like the right side of Figure 18 whereas the left side shows the situation in the preimage.

Join the points a_i and a_i^{-1} in the preimage by an interval. The ray \hat{c}_i in \mathbb{S}^2 induces an orientation on the interval such that it starts in a_i^{-1} and ends in a_i . So we denote the interval by $[a_i^{-1}, a_i]$. Since the rays \hat{c}_j are chosen so that they do not intersect each other, the ray \hat{c}_i has no further intersection points with $f(\mathbb{S}^1)$ or another ray between the points a_i and a_i^{-1} . Therefore we can extend the immersion $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ to an immersion $\tilde{f}: \mathbb{S}^1 \cup [a_i^{-1}, a_i] \rightarrow \mathbb{S}^2$. Now take the restriction $\hat{f}: \mathbb{S}_{[a_i, a_i^{-1}]}^1 \cup [a_i^{-1}, a_i] \rightarrow \mathbb{S}^2$. We will show by contradiction that this restriction \hat{f} is an embedding. Assume that it is not an embedding. Since the ray \hat{c}_i is chosen with no selfintersections, $f(\mathbb{S}_{[a_i, a_i^{-1}]}^1)$ need to have a selfintersection. Because $\mathbb{S}_{[a_i, a_i^{-1}]}^1$ is connected $f(\mathbb{S}_{[a_i, a_i^{-1}]}^1)$ is connected as well. Hence there must be at least one loop in $f(\mathbb{S}_{[a_i, a_i^{-1}]}^1)$. This loop encloses a component X_j , that is, a ray \hat{c}_j starts at $p_j \in X_j$. This ray yields intersection points.

We have to distinguish between two cases: The first case is that the loop is on the left of $f(\mathbb{S}_{[a_i, a_i^{-1}]}^1)$ and the second that it is on the right. In the first case the ray \hat{c}_j intersects $f(\mathbb{S}_{[a_i, a_i^{-1}]}^1)$ in two consecutive points $a_j a_j = a_j^2$ and in the second case it intersects $f(\mathbb{S}_{[a_i, a_i^{-1}]}^1)$ in a negative point a_j^{-1} (Figure 19). Both times this contradicts the assumption that there is a positive word inside of the pairing. Hence \hat{f} is an embedding and extends to an embedded disc. \square

Lemma 3.1.4. *A negative group surrounds an embedded disc.*

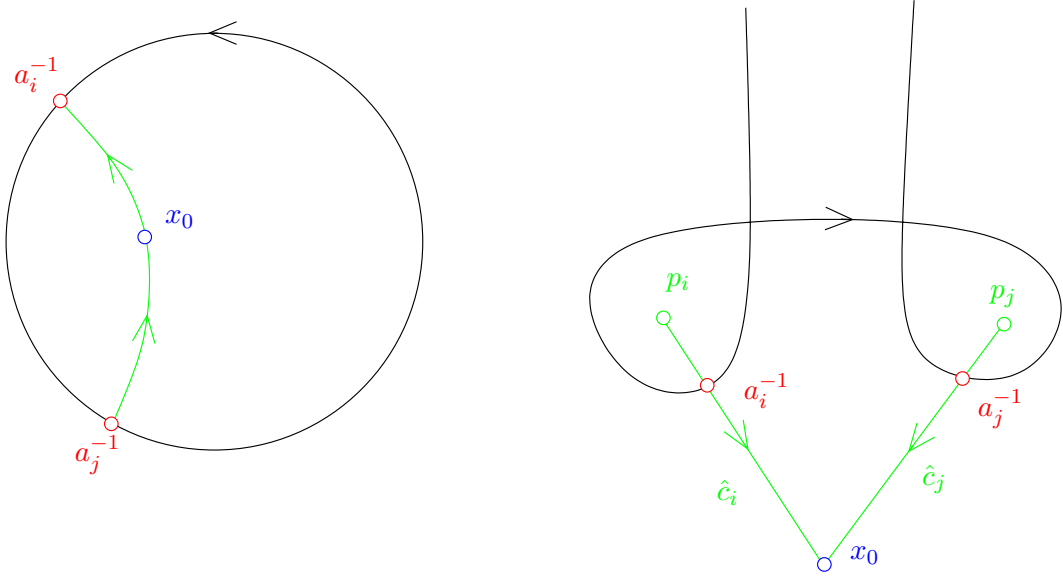


Figure 20: Extending a negative group

Proof. In the case of a negative group we locally get the situation depicted in Figure 20. Extend the immersion $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ to the interval $[a_j^{-1}, a_i^{-1}]$ and look at the restriction $\hat{f}: \mathbb{S}^1_{[a_i^{-1}, a_j^{-1}] \cup [a_j^{-1}, a_i^{-1}]} \rightarrow \mathbb{S}^2$. We will show that \hat{f} is an embedding. As in the proof of Lemma 3.1.3 we assume that it is not, hence there has to be a double point in $f(\mathbb{S}^1_{[a_i^{-1}, a_j^{-1}] \cup [a_j^{-1}, a_i^{-1}]})$. Since $[a_j^{-1}, a_i^{-1}]$ is mapped to \hat{c}_j, \hat{c}_i and x_0 the double point has to be in $f(\mathbb{S}^1_{[a_i^{-1}, a_j^{-1}]})$. Similar to the proof of Lemma 3.1.3 there has to be either consecutive intersection points $a_k a_k = a_k^2$ or a negative intersection point a_k^{-1} (Figure 21). But since no further intersection points occur in a negative group this is a contradiction and hence \hat{f} is an embedding. \square

Now we have seen that the special subwords, which were introduced in Definition 2.4.1, surround embedded discs.

Theorem 3.1.5 (Extension Theorem for Immersed Discs). *Let $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ be a normal immersion with word $w(f)$. If $\tau(f) = 1 - 2\omega_1$ and f has a grouping \mathcal{G} then f extends to an immersion $F: \overline{D} \rightarrow \mathbb{S}^2$.*

Proof. Since $w(f)$ is groupable, a positive word remains after canceling k_p pairings and k_n negative groups, i.e., the grouping \mathcal{G} has $k := k_p + k_n$ edges (Lemma 2.4.6), each of them crossing an interval I_1, \dots, I_k .

Denote the boundary points of the interval I_j by a_j and b_j , that is, $I_j = [a_j, b_j]$. These intervals decompose \overline{D} in $k + 1$ smaller discs D_1, \dots, D_{k+1} .

The boundary of the interval I_j maps to the intersection points a_j and b_j of $f(\mathbb{S}^1)$ and a ray \hat{c}_j . W.l.o.g. assume that the ray \hat{c}_j is oriented from a_j to b_j . Denote that part of \hat{c}_j , which starts at a_j and ends at b_j with $\hat{c}_{[a_j, b_j]} = \hat{c}_{I_j}$. Then there are diffeomorphisms $\varphi_j: I_j \rightarrow \hat{c}_{I_j}$, such that $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ extends to $f^*: \mathbb{S}^1 \cup I_1 \cup \dots \cup I_k \rightarrow \mathbb{S}^2$.

Have a closer look at a small disc D_j . The boundary of D_j contains n Intervals I_1, \dots, I_n .

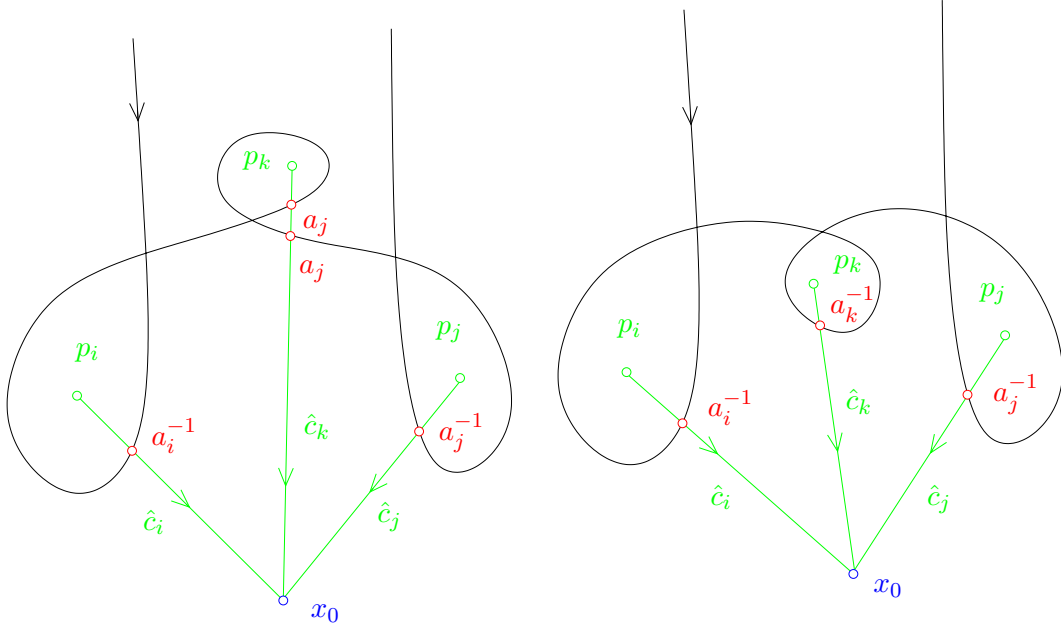


Figure 21: The immersion inside of a negative group cannot have selfintersections.

Since the intervals are disjoint and the boundary points of each interval are in \mathbb{S}^1 , the boundary of D_j contains n connected subsets $S_1, \dots, S_n \subset \mathbb{S}^1$ (Figure 22).

Each disc D_j contains a word in its boundary. Since the intervals contain no letters this word is composed of the subwords of $w(f)$ which are contained in the subsets S_k . Because $w(f)$ is groupable the cancellation process leads to a positive word. Each negative letter of $w(f)$ belongs to a pairing or negative group and so is canceled. Thus the boundary of D_j contains a positive word.

Consider the restriction $f^*|_{\partial D_j} : \partial D_j \rightarrow \mathbb{S}^2$. Since $\partial D_j \simeq \mathbb{S}^1$ we can apply Algorithm 2.1.1 to $f^*|_{\partial D_j}$. The rays \hat{c}_j^* for $f^*|_{\partial D_j}$ are a selection of the rays \hat{c}_j of f . Hence the resulting word $w(f^*|_{\partial D_j})$ is positive and according to Lemma 3.1.2 the restriction $f^*|_{\partial D_j}$ is an embedding which extends to an embedding with the help of the Riemann mapping Theorem (see [Ahl53], p. 172ff).

The Schwarz reflection principle assures that the restrictions f^* can be glued together by an analytic transformation. That way the condition of the Sewing Theorem (see [Cou50], Theorem 2.5) are fulfilled and hence the the restrictions can be glued together to an immersion $F : \overline{D} \rightarrow \mathbb{S}^2$, which is the desired extension of $f : \mathbb{S}^1 \rightarrow \mathbb{S}^2$. \square

Proposition 3.1.6. *Suppose $f : \mathbb{S}^1 \rightarrow \mathbb{S}^2$ is a normal immersion and $w(f)$ the corresponding reduced word. Denote the degree of the base point component by ω_1 . If $w(f)$ is groupable then each grouping of $w(f)$ contains exactly ω_1 negative groups.*

Proof. Since $w(f)$ is groupable, the normal immersion $f : \mathbb{S}^1 \rightarrow \mathbb{S}^2$ extends to an immersion $F : \overline{D} \rightarrow \mathbb{S}^2$. We choose the immersion which is constructed in Theorem 3.1.5.

Each negative group in the cancellation process of $w(f)$ induces an interval. Denote the intervals which are induced by negative groups by I_1, \dots, I_n . According to the construction

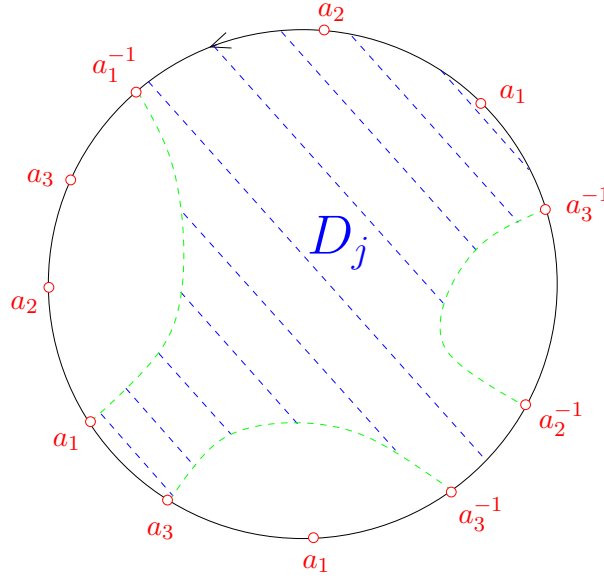


Figure 22: Restriction of f^* to a smaller disc D_j .

of F for the preimage of x_0 we have

$$F^{-1}(\{x_0\}) \subset \bigcup_{j=1}^n I_j.$$

Since each interval contains exactly one preimage of x_0 (Lemma 3.1.4) and the degree of the base point component is ω_1 , it follows that $n = \omega_1$. \square

Since ω_1 only depends on $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ (Proposition 1.3.1) we get:

Corollary 3.1.7. *Every grouping of a reduced word $w(f)$ has the same number of negative groups.*

Remark. Samuel J. Blank [Bla67] analyzed normal immersions $f: \mathbb{S}^1 \rightarrow \mathbb{R}^2$. In this case there is an outer component with degree 0 for every normal immersion, since an extension has to be compact and \mathbb{R}^2 is not compact. According to Proposition 3.1.6 no negative groups occur in the grouping. Hence only pairings are canceled out, which is in fact the same as what Samuel J. Blank showed in his proof of his theorem in the case of normal immersions $f: \mathbb{S}^1 \rightarrow \mathbb{R}^2$. So the proof of Theorem 3.1.5 includes as a special case that each normal immersion $f: \mathbb{S}^1 \rightarrow \mathbb{R}^2$ can be written as a normal immersion $\tilde{f}: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ via stereographic projection.

3.2 Ungroupable Words

We have shown in the previous subsection that a normal immersion $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ extends to an immersion $F: \overline{D} \rightarrow \mathbb{S}^2$ when the corresponding reduced word $w(f)$ is groupable. Now we will have a look at which properties of the reduced word $w(f)$ ban such an extension.

Lemma 3.2.1. *Let $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ be a normal immersion and $w(f)$ the corresponding reduced word. If $\tilde{w}(f)$ is the word which remains after canceling all pairings and negative groups, then $\tilde{w}(f)$ does not contain a subword $\omega = a_j^n$ with $n \geq 3$.*

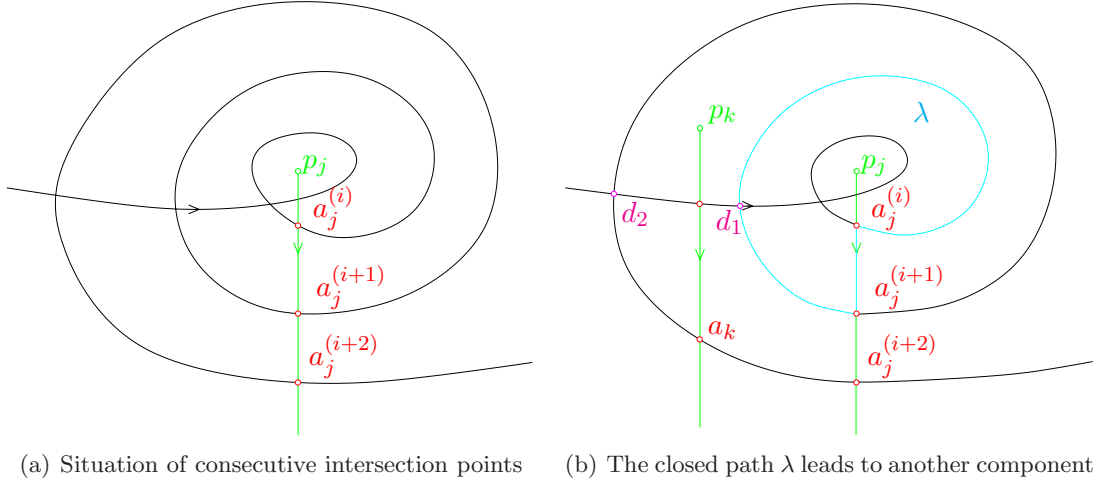


Figure 23: Consecutive Intersection points

Proof. To get consecutive intersection points a_j^n the immersion $f(\mathbb{S}^1)$ has to orbit the point p_j , because after the first intersection point a_j the image $f(\mathbb{S}^1)$ has to intersect the ray \hat{c}_j again in the same direction without intersecting another ray \hat{c}_i (Figure 23 (a)).

We denote the intersection points by $a_j^{(1)}, \dots, a_j^{(n)}$, according to their appearance in ω . Two consecutive intersection points $a_j^{(i)}$ and $a_j^{(i+1)}$ define a closed curve λ , which starts at $a_j^{(i)}$, travels along $f(\mathbb{S}^1)$ to $a_j^{(i+1)}$ and along \hat{c}_j back to $a_j^{(i)}$ (Figure 23 (b)). The first intersection point $a_j^{(1)}$ arises when the immersion $f(\mathbb{S}^1)$ intersects the ray \hat{c}_j for the first time. Therefore this first intersection point lies on the same side of λ as p_j . Since $f(\mathbb{S}^1)$ is connected it has to intersect λ at another point. This could be on the part of λ which belongs to $f(\mathbb{S}^1)$ or on the part which belongs to \hat{c}_j .

If this intersection point arises on the part which belongs to \hat{c}_j then it produces an intersection point $a_j^{\pm 1}$. If it has the same sign as the one in ω , then we have found another intersection point before $a_j^{(1)}$, which contradicts the choice of $a_j^{(1)}$ as the first point. Hence it has to be the other sign. But in this case we have a sequence $a_j a_j^{-1}$, which is a contradiction to the fact that the word is reduced. Therefore the intersection point has to arise on the part of λ which belongs to $f(\mathbb{S}^1)$ and so it is a double point of $f(\mathbb{S}^1)$.

Since we can choose consecutive intersection points in $n-1$ ways, we get $n-1$ closed curves $\lambda_1, \dots, \lambda_{n-1}$, each producing a double point of $f(\mathbb{S}^1)$. All double points have to lie on one side of the ray \hat{c}_j , otherwise there would be a new intersection point $a_j^{\pm 1}$. This means between every two consecutive intersection points there is a double point. Start at an intersection point $a_j^{(i)}$ and travel along $f(\mathbb{S}^1)$ until a double point d_1 is reached. Follow $f(\mathbb{S}^1)$ along the branch that passes through the next intersection point $a_j^{(i+1)}$ until we reach the next double point d_2 . Since there is another branch of $f(\mathbb{S}^1)$ which joins the double points without passing $a_j^{(i+1)}$, this curve surrounds a component X_k of $\mathbb{S}^2 \setminus f(\mathbb{S}^1)$ (Figure 23 (b)). Hence there must be a ray \hat{c}_k which starts in this component. Since the ray \hat{c}_k cannot intersect the ray \hat{c}_j it has to intersect $f(\mathbb{S}^1)$ somewhere between the double point d_2 and the next intersection point $a_j^{(i+2)}$. That is, a new intersection point a_k arises before $a_j^{(i+2)}$.

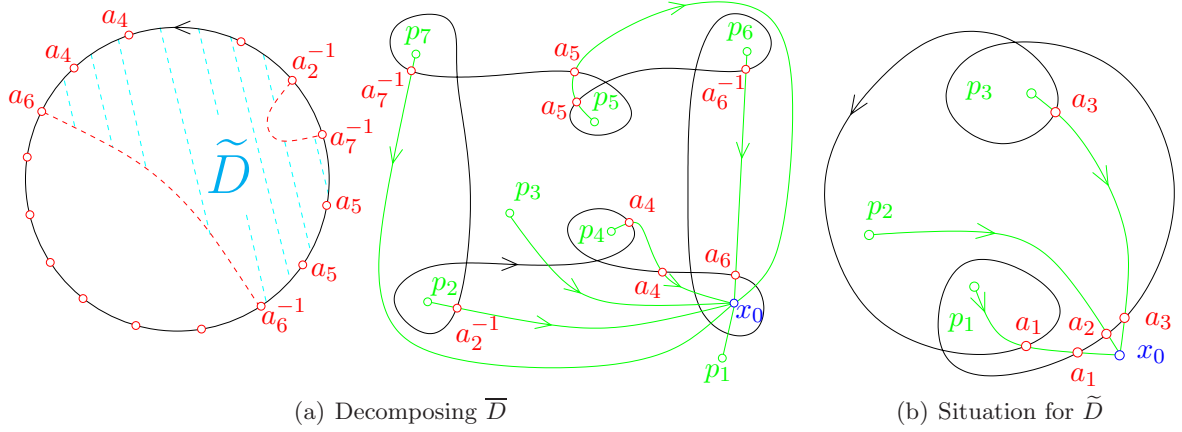


Figure 24: The remaining disc \tilde{D} , containing subwords a_j^2 .

Hence there cannot be more than two equal consecutive intersection points, i.e., $v = a_j^n$ can only occur in $w(f)$ with $n \leq 2$. \square

Lemma 3.2.2. *Let $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ be a normal immersion with word $w(f)$. Denote by $\tilde{w}(f)$ the word which remains after canceling all pairings and negative groups. If $\tilde{w}(f)$ contains a subword $v = a_j^2$ then f cannot be extended.*

Proof. Take the reduced word $w(f)$ and cancel all pairings and negative groups. Each time a pairing or a negative group is canceled an interval is inserted which bounds a small disc together with a part of \mathbb{S}^1 . According to Lemmas 3.1.3 and 3.1.4 this bounds an immersed disc. After canceling all pairings and negative groups a disc \tilde{D} remains, which has $\tilde{w}(f)$ as a word in the boundary (Figure 24).

Assume that there are n subwords a_1^2, \dots, a_n^2 in $\tilde{w}(f)$. Then the restriction \tilde{f} of f to \tilde{D} is a circle with n loops added (Figure 24 (b)). Applying the Decomposition Algorithm 2.3.1 to the immersion leads to $n+1$ embeddings. W.l.o.g. embedding \tilde{f}_0 is the circle and embeddings $\tilde{f}_1, \dots, \tilde{f}_n$ are the added loops. Then the following is true for the \tilde{f}

$$\tau(\tilde{f}) = \sum_{j=0}^n \tau(\tilde{f}_j) = \tau(\tilde{f}_0) + \sum_{j=1}^n \tau(\tilde{f}_j) = 1 + \sum_{j=1}^n \tau(\tilde{f}_j).$$

Since each loop produces two consecutive positive letters, the tangent winding number of each embedding $\tilde{f}_1, \dots, \tilde{f}_n$ is $\tau(\tilde{f}_j) = 1$. Hence

$$\tau(\tilde{f}) = 1 + \sum_{j=1}^n \tau(\tilde{f}_j) = 1 + n,$$

and according to Proposition 2.3.7 this immersion cannot be extended for $n \neq 0$. \square

Lemma 3.2.3. *Let $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ be a normal immersion with word $w(f)$. Denote by $\tilde{w}(f)$ the word which remains after canceling all pairings and negative groups. If $\tilde{w}(f)$ contains negative letters but no subword $v = a_j^2$ then f cannot be extended.*

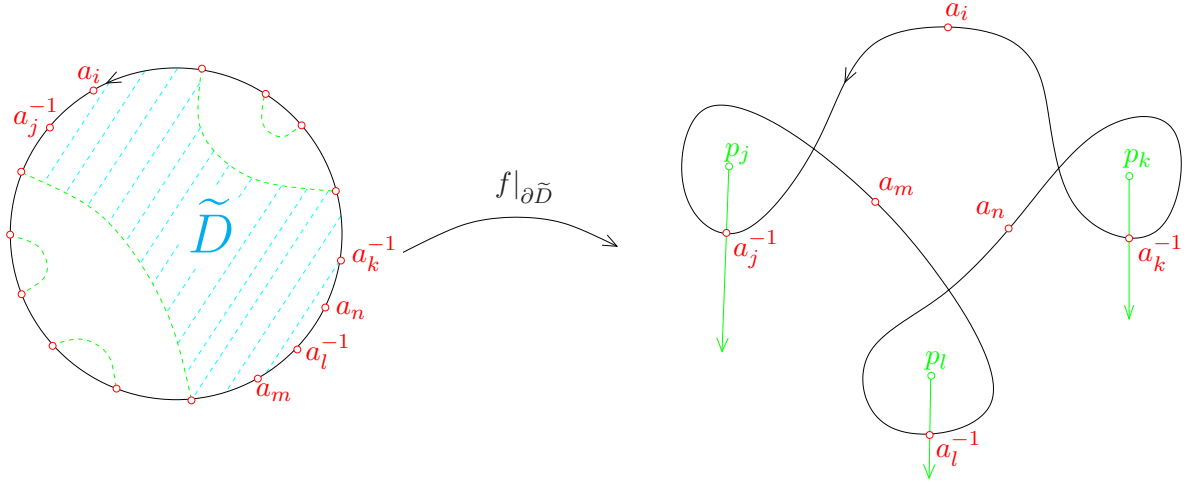


Figure 25: The remaining disc \tilde{D} with single loops added

Proof. Assume $\tilde{w}(f)$ contains n negative letters. Since $\tilde{w}(f)$ contains no negative groups, each negative letter is separated by at least one positive letter.

Canceling all pairings and negative groups leads to a decomposition of \overline{D} and a remaining disc \tilde{D} with $\tilde{w}(f)$ in the boundary. Since all negative letters are separated and no subword a_j^2 occurs the restriction $f|_{\partial\tilde{D}}$ maps to a circle with n single loops added (Figure 25). Decomposing $f|_{\partial\tilde{D}}$ into embeddings with Algorithm 2.3.1 leads to $n + 1$ embeddings f_0, \dots, f_n . W.l.o.g. f_0 is mapped to the circle and f_1, \dots, f_n to the added loops. Since f_1, \dots, f_n produce negative letters the tangent winding number is $\tau(f_j) = -1$, for $j = 1, \dots, n$. Thus for the tangent winding number of $f|_{\partial\tilde{D}}$ we have

$$\tau(f|_{\partial\tilde{D}}) = \sum_{j=0}^n \tau(f_j) = \tau(f_0) + \sum_{j=1}^n \tau(f_j) \stackrel{\tau(f_j)=-1}{=} \tau(f_0) - n.$$

Since f_0 maps to the circle $\tau(f_0) = +1$ and therefore $\tau(f|_{\partial\tilde{D}}) = 1 - n$. According to Proposition 2.3.7 f cannot be extended if n is odd.

If n is even then Proposition 2.3.7 shows that the degree ω_1 of the base point component is

$$\omega_1 = \frac{n}{2}.$$

If $f|_{\partial\tilde{D}}$ can be extended then $\tilde{w}(f)$ is groupable and hence contains ω_1 negative groups (Proposition 3.1.6). Since $\tilde{w}(f)$ contains no negative groups by assumption, $f|_{\partial\tilde{D}}$ fails to extend and hence f cannot be extended. \square

Now we can show that a normal immersion $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ with an ungroupable reduced word $w(f)$ cannot be extended to an immersion $F: \mathbb{S}^1 \rightarrow \mathbb{S}^2$.

Theorem 3.2.4. *Let $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ be a normal immersion and $w(f)$ the reduced word which is the result of Algorithm 2.1.1. If $\tau(f) \neq 1 - 2\omega_1$ or $w(f)$ is not groupable then f cannot be extended to an immersion $F: \overline{D} \rightarrow \mathbb{S}^2$.*

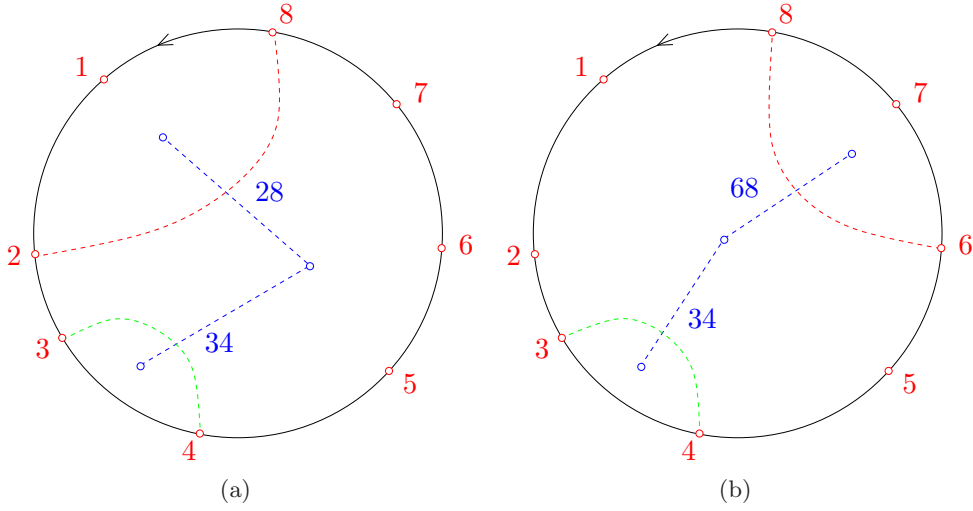


Figure 26: Different groupings with isomorphic trees, but different labels.

Proof. If the tangent winding number $\tau(f)$ of f is not equal to $1 - 2\omega_1$ then Proposition 2.3.7 shows that f cannot be extended.

So assume f has the tangent winding number $\tau(f) = 1 - 2\omega_1$ but the word $w(f)$ is not groupable. Cancel all pairings and negative groups of $w(f)$ and denote the remaining word by $\tilde{w}(f)$. Since $w(f)$ is not groupable $\tilde{w}(f)$ is not a positive word.

If $\tilde{w}(f)$ contains a subword a_j^2 then f cannot be extended according to Lemma 3.2.2. Hence we can assume that $\tilde{w}(f)$ contains no subword a_j^2 . But since $\tilde{w}(f)$ is not a positive word it has to contain negative letters. In this case f cannot be extended according to Lemma 3.2.3. \square

3.3 Uniqueness of Extensions

Recently we have seen that a normal immersion $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ can be extended to an immersion $F: \overline{D} \rightarrow \mathbb{S}^2$ if and only if the corresponding reduced word $w(f)$ is groupable. In this section we will show that the number of different groupings is equivalent to the number of different extensions.

Definition 3.3.1. Suppose $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ is a normal immersion and $F_1, F_2: \overline{D} \rightarrow \mathbb{S}^2$ are extensions of f . Two extensions are *equivalent* if there exists an orientation preserving diffeomorphism $\varphi: \overline{D} \rightarrow \overline{D}$ such that $F_1 = F_2 \circ \varphi$ and $\varphi|_{\mathbb{S}^1} = \text{id}_{\mathbb{S}^1}$.

Remark. In the proof of Theorem 3.1.5 an extension is constructed to a given normal immersion with groupable reduced word $w(f)$. In fact, this extension defines an equivalence class of extensions. If we talk about extensions then we talk about the equivalence classes. As representative we pick an extension which is constructed in the proof of Theorem 3.1.5.

Recall that a grouping \mathcal{G} is a weighted tree (Definition 2.4.7) which induces a decomposition of \overline{D} by intervals. These intervals correspond to a sequence of cancelled pairings and negative groups of $w(f)$ until a positive word remains. Thus two groupings are equivalent if they are isomorphic as weighted trees (Definition 2.4.7).

Example 3.3.2. 1. The word

$$\begin{matrix} a_2 a_3 a_4^{-1} a_5^{-1} a_1 a_3 a_5 a_3^{-1} \\ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \end{matrix}$$

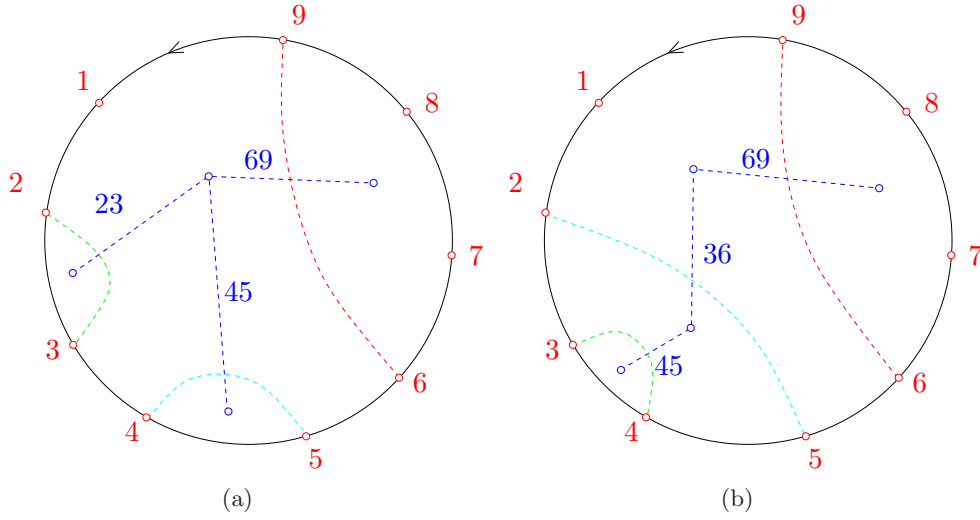


Figure 27: Different groupings with nonisomorphic trees

has two different groupings. The first one is

$$a_2 a_3 |a_4^{-1} a_5^{-1}| a_1 a_3 a_5 a_3^{-1} \rightsquigarrow a_2 |a_3 a_1 a_3 a_5 a_3^{-1}| \rightsquigarrow a_2.$$

This grouping is shown in Figure 26 (a). The second grouping is

$$a_2 a_3 |a_4^{-1} a_5^{-1}| a_3 a_1 a_5 a_3^{-1} \rightsquigarrow a_2 a_3 |a_3 a_1 a_5 a_3^{-1}| \rightsquigarrow a_2 a_3.$$

The second grouping is shown in 26 (b). In this case the trees are isomorphic but have different weights.

2. Another example is the word

$$a_4 a_3^{-1} a_5^{-1} a_2^{-1} a_1^{-1} a_3 a_4 a_1 a_3^{-1}.$$

This word has two different groupings as well. The first one is

$$a_4 |a_3^{-1} a_5^{-1}| a_2^{-1} a_1^{-1} a_3 a_4 a_1 a_3^{-1} \rightsquigarrow a_4 |a_2^{-1} a_1^{-1}| a_3 a_4 a_1 a_3^{-1} \rightsquigarrow a_4 |a_3 a_4 a_1 a_3^{-1}| \rightsquigarrow a_4.$$

This grouping is shown in Figure 27 (a). The second grouping is

$$a_4 a_3^{-1} |a_5^{-1} a_2^{-1}| a_1^{-1} a_3 a_4 a_1 a_3^{-1} \rightsquigarrow a_4 |a_3^{-1} a_1^{-1}| a_3 a_4 a_1 a_3^{-1} \rightsquigarrow a_4 |a_3 a_4 a_1 a_3^{-1}| \rightsquigarrow a_4.$$

The second grouping is shown in 27 (b). In this case not only the weights are different but the trees are not isomorphic either.

Theorem 3.3.3 (Uniqueness Theorem for Immersed Discs). *Suppose $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ is a normal immersion and $w(f)$ the corresponding reduced word. Then two different groupings define two different equivalence classes of extensions of $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$.*

Proof. Theorem 3.1.5 shows that a grouping \mathcal{G} defines an equivalence class of extensions. Assume $w(f)$ has two different groupings $\mathcal{G}_1, \mathcal{G}_2$, i.e., the weighted trees are not isomorphic.

We will show the claim by contradiction. Thus assume that $F_1, F_2: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ are two equivalent extensions of $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$. Then an orientation preserving diffeomorphism $\varphi: \overline{D} \rightarrow \overline{D}$ exists, such that $F_1 = F_2 \circ \varphi$ and $\varphi|_{\mathbb{S}^1} = \text{id}_{\mathbb{S}^1}$.

Since f extends the reduced word $w(f)$ has a grouping \mathcal{G}_1 which induces a decomposition of \overline{D} by intervals \mathcal{I}_1 . Assume that this grouping induces the extension F_1 . Since $\varphi|_{\mathbb{S}^1} = \text{id}_{\mathbb{S}^1}$ the intervals \mathcal{I}_1 are mapped to homotopic intervals \mathcal{I}_2 by φ . Thus the decomposition of \overline{D} by intervals \mathcal{I}_2 induce the extension F_2 . Since the intervals \mathcal{I}_2 are still disjoint, these intervals \mathcal{I}_2 induce a tree \mathcal{G}_2 which is isomorphic to the tree of \mathcal{G}_1 . Since φ does not change the boundary of the intervals the weighted trees are isomorphic as well.

Hence different weighted trees, and thus different groupings, define nonequivalent extensions. \square

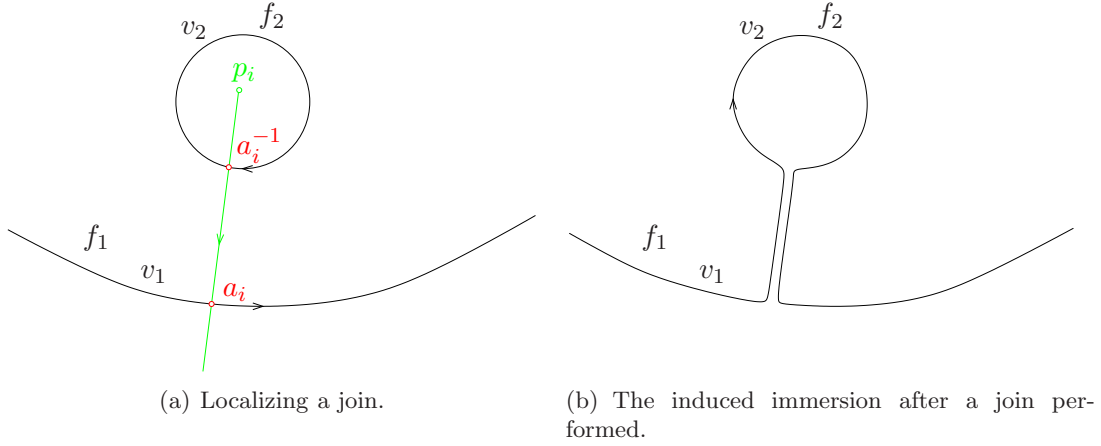


Figure 28: A join in the image

4 Immersed Surfaces in the Sphere

In Section 3 we characterized the normal immersions $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ which extend to immersions $F: \overline{D} \rightarrow \mathbb{S}^2$. In this section we analyze whether a normal immersion $f = \coprod_{j=1}^m \mathbb{S}^1 \rightarrow \mathbb{S}^2$ extends to a surface M of genus g_M with boundary $\partial M = \coprod_{j=1}^m \mathbb{S}^1$. The results in this section are based on the methods of Curley/Wolitzer [CW86], who studied the problem for normal immersions to the plane.

4.1 Immersed Surfaces with m Boundary Components in the Sphere

Suppose M is a surface with m boundary components, i.e., $\partial M = \coprod_{j=1}^m \mathbb{S}^1$. When does a normal immersion $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow \mathbb{S}^2$ extend to an immersion $F: M \rightarrow \mathbb{S}^2$?

Denote the restriction of f to the j -th boundary component by $f_j: \mathbb{S}^1 \rightarrow \mathbb{S}^2$, i.e., $f = (f_1, \dots, f_m)$. Then Algorithm 2.1.1 can be applied to each f_j to obtain a word $w(f_j)$ for each boundary component.

Definition 4.1.1. Suppose $f = (f_1, \dots, f_m): \coprod_{j=1}^m \mathbb{S}^1 \rightarrow \mathbb{S}^2$ is a normal immersion. The word $w_j := w(f_j)$ resulting from applying Algorithm 2.1.1 to the j -th boundary component f_j is called the j -th boundary word of f . The word $w(f)$ of f is the set of boundary words, i.e.,

$$w(f) = \{w(f_1), \dots, w(f_m)\}.$$

Remark. If $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ has only one boundary component then our definition coincides with Definition 2.1.3.

Suppose $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow \mathbb{S}^2$ has an extension $F: M \rightarrow \mathbb{S}^2$ to an immersion of a surface M with boundary $\partial M = \coprod_{j=1}^m \mathbb{S}^1$. Since M is connected any two boundary components can be joined by a path $\lambda: [0, 1] \rightarrow F(M)$. Cutting along λ yields a surface M' with $m - 1$ boundary components.

But how can we localize such a path? Since λ connects two boundary components f_1 and f_2 , it enters the surface at $\lambda(0)$ and leaves it at $\lambda(1)$. In terms of letters of $w(f)$ this reads as $\lambda(0) = a_i^{-1} \in w(f_2)$ and $\lambda(1) = a_i \in w(f_1)$ (Figure 28 (a)). Since these intersection points

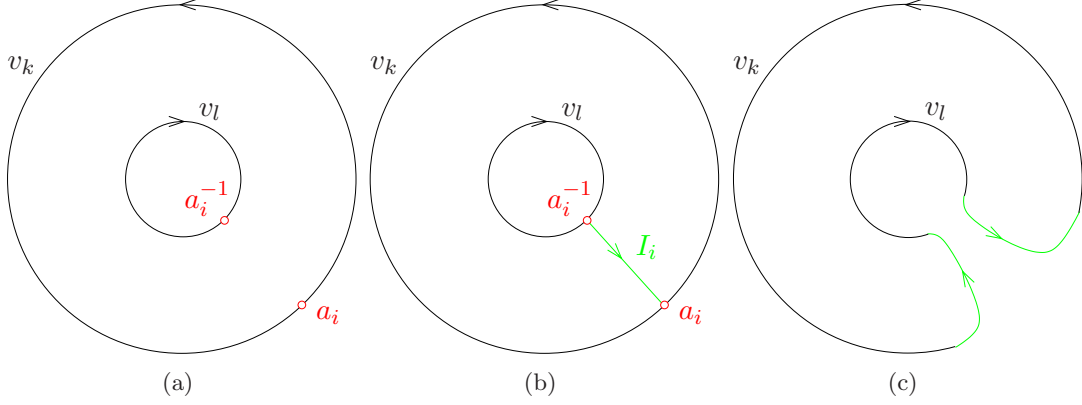


Figure 29: A join in the preimage

are connected by a ray \hat{c}_i , the path λ can be chosen as $\lambda: [0, 1] \rightarrow \hat{c}_i$ (Figure 28 (b) shows the result after cutting along λ).

Definition 4.1.2. Suppose $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow \mathbb{S}^2$ is a homologically trivial normal immersion with word $w(f)$. A path $\lambda: [0, 1] \rightarrow \mathbb{S}^2$ is called a *join of f* if

- λ is an embedding,
- $\lambda(0) = a_i^{-1} \in w_s$ and
- $\lambda(1) = a_i \in w_t$ with $t \neq s$.

A join induces an operation on the word $w(f)$: If λ joins the boundary words $w_k = v_k a_i^{\pm 1}$ and $w_l = v_l a_i^{\mp 1}$ by a_i , then the join induces a new word $\tilde{w}_{kl} = v_k v_l$. Indeed, in the preimage $a_i^{-1} \in w_k$ and $a_i \in w_l$ can be connected by an interval $I_i := [a_i^{-1}, a_i]$, such that $\lambda: I_i \rightarrow \hat{c}_i$ (Figure 29 (b)). Cutting along λ equals cutting along $[a_i^{-1}, a_i]$ in the preimage, such that the two boundary components are joined to one new boundary component f_{kl} . The boundary word of the new boundary component f_{kl} reads as follows:

$$w_{kl} = v_k a_i a_i^{-1} v_l a_i^{-1} a_i.$$

The reduced word is $\tilde{w}_{kl} = v_k v_l$ (Figure 29 (c)).

Example 4.1.3. Let $w(f) = \{w_1 := a_2 a_1 a_7 a_6 a_5 a_7^{-1} a_5 a_4 a_3, w_2 := a_2^{-1} a_3^{-1}, w_3 := a_5^{-1} a_6^{-1}\}$. Then w_1 and w_2 can be joined by a_2 . The join is

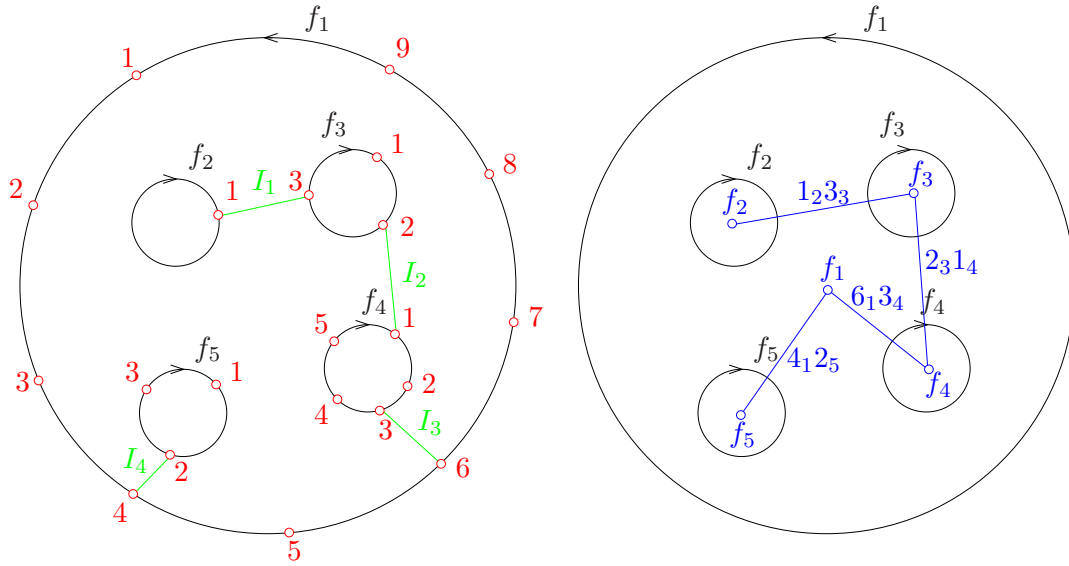
$$w_{12} = a_3^{-1} a_1 a_7 a_6 a_5 a_7^{-1} a_5 a_4 a_3.$$

This word can be joined with w_3 by the letter a_5 . Thus we get

$$w_{123} = a_3^{-1} a_1 a_7 a_6 a_6^{-1} a_7^{-1} a_5 a_4 a_3.$$

Reducing the word leads to

$$\tilde{w}_{123} = a_1 a_5 a_4.$$



(a) The boundary components with numbered letters and joins. (b) The graph with the boundary components as vertex set. The edges correspond to the joins. The labels denote which letters of the boundary words are joined.

Figure 30: The graph induced by a joining

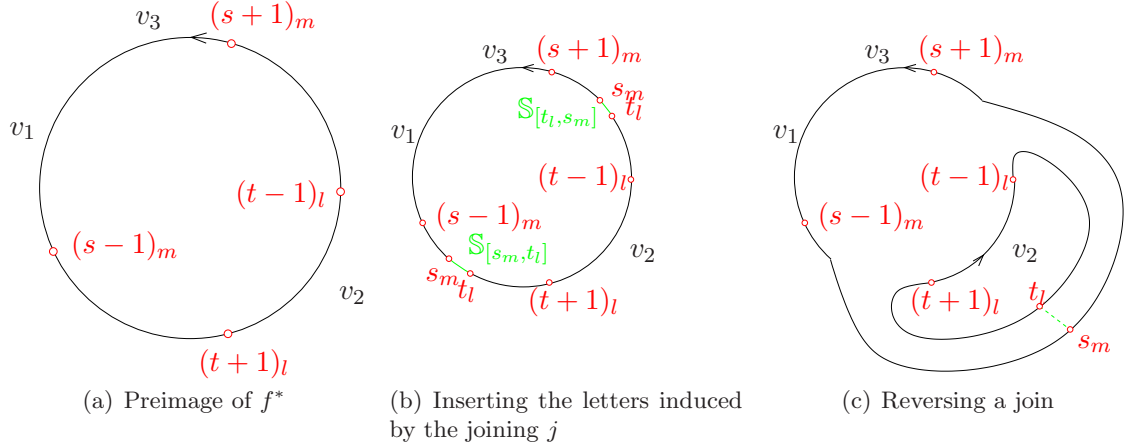
Each join decreases the number of boundary words of $w(f)$ by one. In general there are several ways to join boundary words of $w(f)$. If a boundary word contains the letter a_i twice, then each a_i can be joined with a letter a_i^{-1} of another boundary word. The two joins are different because they lead to different extensions, as we will see in Subsection 4.4.

On the other hand the order of joins does not matter. Different joins can be traced by a tree with m vertices representing the boundary components f_1, \dots, f_m . If a join exists connecting f_k and f_l then the tree has an edge which connects the vertices f_k and f_l . To distinguish between different joins the edges are labeled in the following way: Pick a first letter for each boundary word. If a join exists connecting the boundary components by the j -th letter of the boundary word w_k and the i -th letter of the boundary word w_l then label the corresponding edge of the graph by $j_k i_l$ (Figure 30).

Definition 4.1.4. Suppose $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow \mathbb{S}^2$ is a normal immersion with word $w(f)$ which can be joined to one word by pairwise disjoint joins. Then the induced weighted graph \mathcal{J} is called a *joining* of f . The result $w(f^*)$ of the word $w(f)$ and the joins is called the *joined word* of $w(f)$ and \mathcal{J} .

To obtain a normal immersion f^* with one boundary component we will make use of the graph associated with a joining. The edge of the joining \mathcal{J} indicates which boundary components are joined and the labels indicate at which letter the join connects. Therefore performing all joins of a joining \mathcal{J} lead to a normal immersion f^* with one boundary component and an extension of f^* will yield an extension of f .

Lemma 4.1.5. Let $f = (f_1, \dots, f_m): \coprod_{j=1}^m \mathbb{S}^1 \rightarrow \mathbb{S}^2$ be a normal immersion. If f has a joining \mathcal{J} then this joining induces a normal immersion $f^*: \mathbb{S}^1 \rightarrow \mathbb{S}^2$, such that $w(f^*)$ is the

Figure 31: Recovering f from f^* and the joining \mathcal{J} .

joined word of $w(f)$ and \mathcal{J} .

Proof. If the joining \mathcal{J} connects f_k and f_l by an edge a join $\lambda: [a_j^{-1}, a_j] \rightarrow \hat{c}_j$ exists, which connects the boundary components f_k and f_l . Cutting along λ joins the two boundary components to one new boundary component. Since a joining connects each boundary component with another boundary component, performing the joins lead to a normal immersion $f^*: \mathbb{S}^1 \rightarrow \mathbb{S}^2$, such that the word $w(f^*)$ of f^* is the joined word of $w(f)$ and \mathcal{J} (Figure 29). \square

The goal is to derive an extension of f from an extension of f^* and the joining \mathcal{J} . For that it is not enough to know which boundary components are joined by \mathcal{J} , we need to refer to the letters that are joined. This very information is coded in the labels of the joining \mathcal{J} .

Proposition 4.1.6. *Let $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow \mathbb{S}^2$ be a normal immersion and suppose $w(f)$ has a joining \mathcal{J} . Let $f^*: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ be the normal immersion induced by the joining \mathcal{J} . If f^* has an extension $F^*: M^* \rightarrow \mathbb{S}^2$, where M^* is a surface with $\partial M^* = \mathbb{S}^1$, then the joining \mathcal{J} induces a quotient map π and an extension $F: M \rightarrow \mathbb{S}^2$, where $M = \pi(M^*)$ is a surface with $\partial M = \coprod_{j=1}^m \mathbb{S}^1$.*

Proof. Since $w(f^*)$ is the joined word of $w(f)$ and \mathcal{J} , each letter is the y -th letter of a boundary word w_x of $w(f)$. Hence each letter of $w(f^*)$ has a unique label x_y . If the joining \mathcal{J} connects the boundary words w_k and w_l by an edge with label $s_k t_l$ then $w(f^*)$ can be written as (Figure 31 (a))

$$w(f^*) = v_1 \dots (s-1)_k (t+1)_l v_2 (t-1)_l (s+1)_k v_3.$$

Insert the letters $s_k t_l$ such that the following word occurs (Figure 31 (b)):

$$v_1 (s-1)_k s_k t_l (t+1)_l v_2 (t-1)_l t_l s_k (s+1)_k v_3$$

Denote the part $s_k t_l$ in the preimage of f^* by $\mathbb{S}_{[s_k, t_l]}^1$ and the part $t_l s_k$ by $\mathbb{S}_{[t_l, s_k]}^1$ (Figure 31 (b)). Identify $\mathbb{S}_{[s_k, t_l]}^1$ and $\mathbb{S}_{[t_l, s_k]}^1$ and denote the corresponding quotient map by $\tilde{\pi}$ (Figure 31 (c)). Proceeding this for every join leads to a quotient map π .

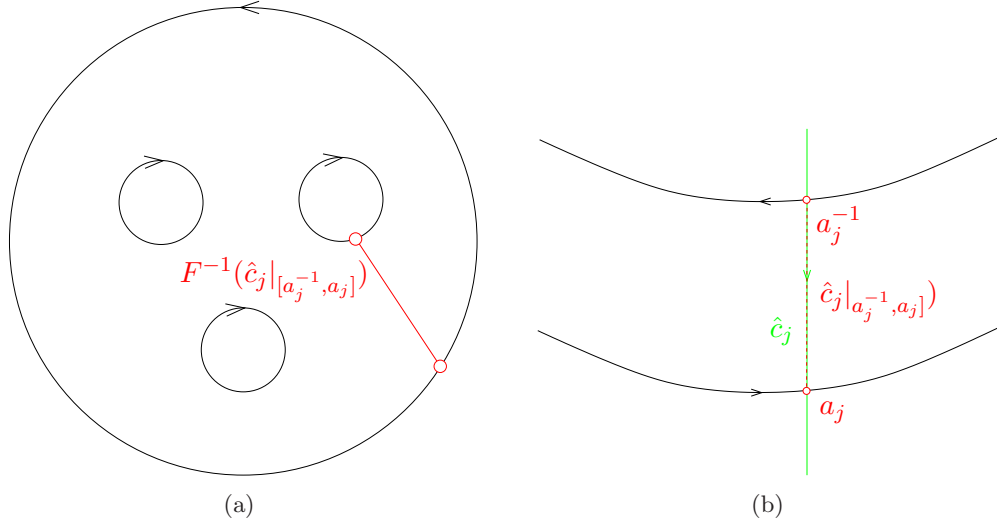


Figure 32: Localizing a join

By assumption an extension $F^*: M^* \rightarrow \mathbb{S}^2$ of $f^*: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ exists. The quotient map π extends to M^* such that $\pi(M^*) = M$ with $\partial M = \coprod_{j=1}^m \mathbb{S}^1$. Hence a unique continuous function F exists, such that the following diagram commutes:

$$\begin{array}{ccc}
 M^* & \xrightarrow{F^*} & \mathbb{S}^2 \\
 \pi \downarrow & \nearrow F & \\
 M & &
 \end{array}$$

According to the Whitehead Approximation Theorem (see [Lee02], Theorem 10.21) F is homotopic relative ∂M to a differentiable function and the Sewing Theorem (see [Cou50], Theorem 2.5) assures the existence of an immersion. \square

It remains to show that existence of a joining is a necessary condition for a normal immersion $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow \mathbb{S}^2$ to have an extension $F: M \rightarrow \mathbb{S}^2$ to a surface M with $\partial M = \coprod_{j=1}^m \mathbb{S}^1$:

Proposition 4.1.7. *Suppose M is a surface with $\partial M = \coprod_{j=1}^m \mathbb{S}^1$. If a normal immersion $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow \mathbb{S}^2$ extends to a surface M then f has a joining \mathcal{J} .*

Proof. Take an intersection point a_j^{-1} of a ray \hat{c}_j and one of the boundary components. A negative letter indicates that \hat{c}_j enters the surface and hence it has to leave the surface at a point a_j of \hat{c}_j and another boundary component. Since M is connected, the preimage $F^{-1}(\hat{c}_j|_{[a_j^{-1}, a_j]})$ connects two boundary components of M , i.e., these two boundary components can be joined by a_j .

Thus if $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow \mathbb{S}^2$ has an extension $F: M \rightarrow \mathbb{S}^2$ the m boundary components can be joined to one word, i.e., f has a joining. \square

4.2 Immersed Surfaces with Nonzero Genus in the Sphere

In the first subsection we have seen how a normal immersion with m boundary components extends if an induced normal immersion $f^*: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ has an extension. Section 3 analyzes

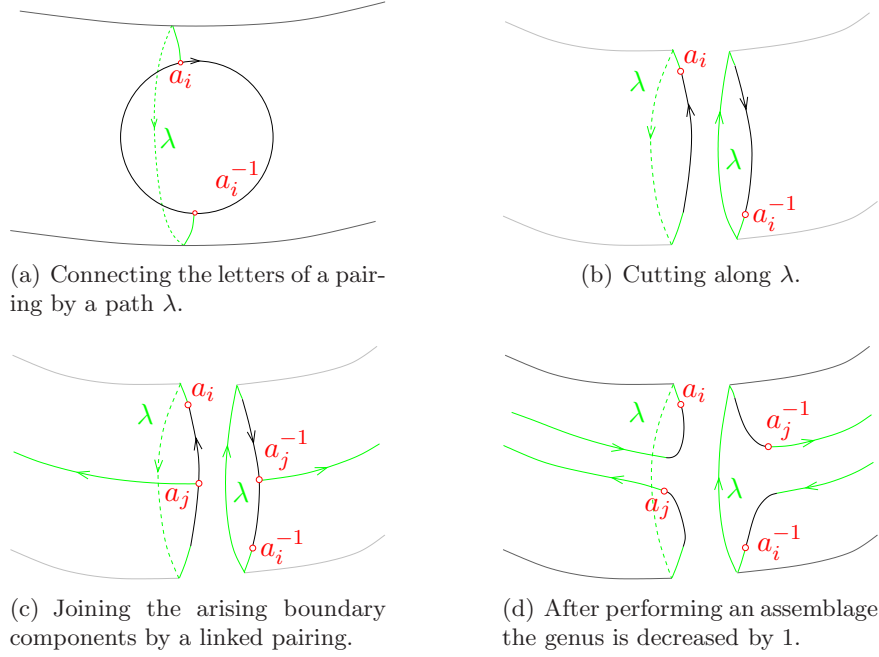


Figure 33: An assemblage shown on a handle of a tori

when f^* has an extension to an immersed disc $F: \overline{D} \rightarrow \mathbb{S}^2$, hence to a surface of genus 0. But when does f^* have an extension to a surface M of nonzero genus?

Since Subsection 4.1 reduces the case of m boundary components to the case of one boundary component, it suffices to study normal immersions $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$.

Suppose $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ is a normal immersion with word $w(f)$ which has an extension to an immersion $F: M \rightarrow \mathbb{S}^2$. Suppose $g_M > 0$. Then the boundary ∂M can be connected by a path λ , such that $M \setminus \lambda$ is still connected (Figure 33 (a)). Cutting along this path yields a surface of genus $g_M - 1$ and one additional boundary component (Figure 33 (b)).

Since f has an extension, these two boundary components can be joined (Proposition 4.1.7) to obtain a surface \widetilde{M} with genus $g_{\widetilde{M}} = g_M - 1$ (Figure 33 (c) and (d)).

To localize these paths we naturally use the word $w(f)$. If $w(f)$ contains a letter a_j and a_j^{-1} , then these two points can be connected by an interval $[a_j^{-1}, a_j] := F^{-1}(\hat{c}_j|_{[a_j^{-1}, a_j]})$. Thus $w(f)$ can be written as

$$w(f) = \bar{v}_1 a_j^{\pm 1} \bar{v}_2 a_j^{\mp 1}.$$

Cutting along $[a_j^{-1}, a_j]$ yields a surface with two boundary components. Hence $w(f)$ is decomposed into two boundary words $w_1 = \bar{v}_1$ and $w_2 = \bar{v}_2$. Since these boundary words can be joined they contain letters $a_i^{\pm 1}$ (Definition 4.1.2).

Hence $w(f)$ can be written as

$$w(f) = v_1 a_i^{\pm 1} v_2 a_j^{\pm 1} v_3 a_i^{\mp 1} v_4 a_j^{\mp 1}.$$

Definition 4.2.1. Let $w(f)$ be the reduced word of a normal immersion $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$. Two

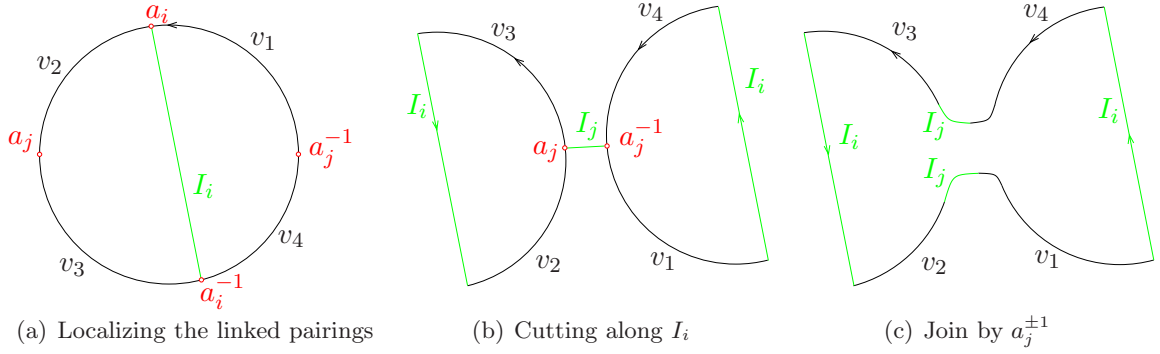


Figure 34: An assemblage in the preimage

pairings are *linked* if they overlap, that is, $w(f)$ can be written as

$$w(f) = v_1 a_i^{\pm 1} v_2 a_j^{\pm 1} v_3 a_i^{\mp 1} v_4 a_j^{\mp 1}.$$

An *assemblage* of the word $w(f)$ is a replacement of $w(f)$ by the word $v_1 v_4 v_3 v_2$.

The next lemma shows that an assemblage induces a normal immersion \tilde{f} , such that $w(\tilde{f}) = v_1 v_4 v_3 v_2$.

Lemma 4.2.2. *Let $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ be a normal immersion with word $w(f)$. If*

$$w(f) = v_1 a_i^{\pm 1} v_2 a_j^{\pm 1} v_3 a_i^{\mp 1} v_4 a_j^{\mp 1}$$

contains an assemblage then this assemblage induces a normal immersion $\tilde{f}: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ with word $w(\tilde{f}) = v_1 v_4 v_3 v_2$.

Proof. Insert an interval I_i for the pairing $a_i \dots a_i^{-1}$ (Figure 34 (a)). Map the interval I_i to the ray \hat{c}_i . Cutting along this segment $\hat{c}_i|_{[a_i^{-1}, a_i]}$ of \hat{c}_i results in a surface with genus $g_M - 1$ and two boundary components $\mathbb{S}^1 \amalg \mathbb{S}^1$. Since the pairings of the assemblage are linked the letters a_j, a_j^{-1} are in distinct boundary words. Thus the induced immersion has the word $\{v_3 v_2 a_j, a_j^{-1} v_1 v_4\}$ (Figure 34 (b)).

Joining these two boundary components by a_j leads to a surface \tilde{M} , which has only one boundary component according to Proposition 4.1.6. The induced immersion $\tilde{f}: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ has the word $w(\tilde{f}) = v_1 v_4 v_3 v_2$, which is the same as $w(f)$ after the assemblage (Figure 34 (c)). \square

Recall that we want to obtain an extension of f from an extension of the induced normal immersion \tilde{f} . Hence we need to recover f out of \tilde{f} . Since \tilde{f} is induced by an assemblage we need to know which letters of $w(f)$ build the assemblage. Pick a first letter of $w(f)$ and number all letters, starting by 1 for the first letter. Assume that $w(f)$ can be written as

$$w(f) = v_1 a_i^{\pm 1} v_2 a_j^{\pm 1} v_3 a_i^{\mp 1} v_4 a_j^{\mp 1}.$$

$\begin{matrix} & & k & & l & & m & & n \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \end{matrix}$

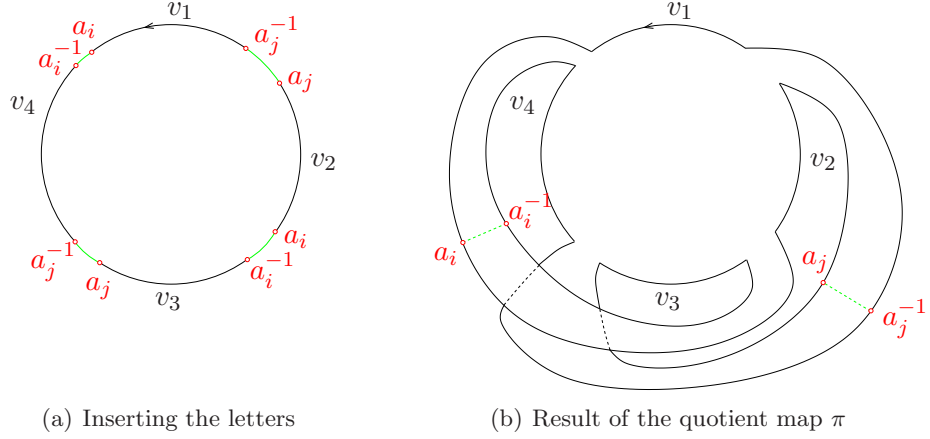


Figure 35: Recovering f from \tilde{f} and an assembling \mathcal{A} .

Then the assemblage $\dots a_i^{\pm 1} \dots a_j^{\pm 1} \dots a_i^{\mp 1} \dots a_j^{\mp 1}$ is denoted by $km.ln$. If $w(f)$ has p assemblages then each assemblage has a representation as $km.ln$ and all assemblages are denoted by the directed set

$$\mathcal{A} := (k_1 m_1 . l_1 n_1, \dots, k_p m_p . l_p n_p).$$

Definition 4.2.3. Let $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ be a normal immersion with word $w(f)$. If $w(f)$ allows p assemblages then the induced directed set

$$\mathcal{A} := (k_1 m_1 . l_1 n_1, \dots, k_p m_p . l_p n_p)$$

of p assemblages is called an *assembling of f* .

Similar to a joining in the previous subsection an assembling \mathcal{A} contains the informations needed to obtain an extension of f out of an extension of \tilde{f} , as follows:

Proposition 4.2.4. *Suppose $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ is a normal immersion with word $w(f)$ and \mathcal{A} an assembling of f . Let $\tilde{f}: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ be the normal immersion induced by the assembling \mathcal{A} . If \tilde{f} has an extension $\tilde{F}: \overline{D}^1 \rightarrow \mathbb{S}^2$ then f has an extension $F: M \rightarrow N$, where M is a surface of genus $g_M = |\mathcal{A}|$ and $\partial M = \mathbb{S}^1$.*

Proof. Since \mathcal{A} is a finite set it satisfies to prove the case $|\mathcal{A}| = 1$, i.e., $\mathcal{A} = \{km.ln\}$. Since \tilde{f} is the induced normal immersion according to Lemma 4.2.2 the word $w(\tilde{f})$ can be written as

$$w(\tilde{f}) = v_1 v_4 v_3 v_2.$$

Assume w.l.o.g. that $k < l < m < n$ and that the k -th and l -th letter of $w(f)$ are positive. Insert two copies of each letter to the preimage, such that the following word occurs (Figure 35 (a)):

$$v_1 a_i a_i^{-1} v_4 a_j^{-1} a_j v_3 a_i^{-1} a_i v_2 a_j a_j^{-1}.$$

Each of the letters $a_i, a_i^{-1}, a_j, a_j^{-1}$ is inserted twice. Identify the particular two copies and the included segment of \mathbb{S}^1 and denote the corresponding quotient map by π (Figure 35 (b)).

Since the quotient map identifies the boundary on the right it extends to \overline{D} , such that $\pi(\overline{D}) = M$ is a surface of genus $g_M = 1$ and boundary $\partial M = \mathbb{S}^1$. Hence a unique continuous function F exists, such that the following diagram commutes:

$$\begin{array}{ccc} \overline{D} & \xrightarrow{\tilde{F}} & \mathbb{S}^2 \\ \pi \downarrow & \nearrow F & \\ M & & \end{array}$$

According to the Whitehead Approximation Theorem (see [Lee02], Theorem 10.21) F is homotopic relative to ∂M to a differentiable function and the Sewing Theorem (see [Cou50], Theorem 2.5) supports the existence of an immersion. \square

The next proposition shows that the existence of a (nonempty) assembling is necessary for f to extend to a surface M of genus $g_M > 0$.

Proposition 4.2.5. *Suppose $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ is a normal immersion which has an extension $F: M \rightarrow \mathbb{S}^2$ to a surface M of genus $g_M > 0$ and boundary $\partial M = \mathbb{S}^1$. Then $w(f)$ provides an assembling \mathcal{A} with $|\mathcal{A}| = g_M$.*

Proof. Since $g_M > 0$ the preimage M contains a curve $\lambda_1: [0, 1] \rightarrow M$ with $\lambda_1(0), \lambda_1(1) \in \partial M$, such that $M \setminus \lambda_1$ is connected. W.l.o.g. λ_1 can be mapped to a ray \hat{c}_i in \mathbb{S}^2 . Since $\lambda_1(0), \lambda_1(1) \in \partial M$ the image $F(\lambda_1(\{0, 1\})) \in \{a_i, a_i^{-1}\}$. We may assume that $F(\lambda_1(0)) = a_i^{-1}$, i.e., \hat{c}_i enters the surface at $F(\lambda_1(0))$. Since λ_1 ends in $\lambda_1(1)$ in the preimage, the ray \hat{c}_i leaves the surface at $F(\lambda_1(1))$, i.e., $F(\lambda_1(1)) = a_i$.

Pick two points x_1, x_2 in the boundary ∂M , which are separated by $\lambda_1(0), \lambda_1(1)$. Since $g_M > 0$ a curve $\lambda_2: [0, 1] \rightarrow M$ exists with

$$\begin{aligned} \lambda_2(0) &= x_1, \\ \lambda_2(1) &= x_2 \text{ and} \\ \lambda_1 \cap \lambda_2 &= \emptyset. \end{aligned}$$

With the same arguments as before the image of these points are a_j and a_j^{-1} . Since two consecutive points of $(\lambda_1(0), \lambda_1(1), \lambda_2(0), \lambda_2(1))$ belong to different curves these preimages define an assemblage of $w(f)$. \square

4.3 Existence of Immersed Surfaces in the Sphere

In this subsection we will study, when a normal immersion $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow \mathbb{S}^2$ has an extension $F: M \rightarrow \mathbb{S}^2$ to a surface M of genus g_M and $\partial M = \coprod_{j=1}^m \mathbb{S}^1$.

In Proposition 2.3.7 we have seen that a necessary condition for f to extend to an immersed disc is that the tangent winding number $\tau(f) = 1 - 2\omega_1$, with ω_1 the degree of the base point component. So at first we will show how this condition generalizes to the case of surfaces of genus g_M and m boundary components.

The effect of a joining to the tangent winding number is the following:

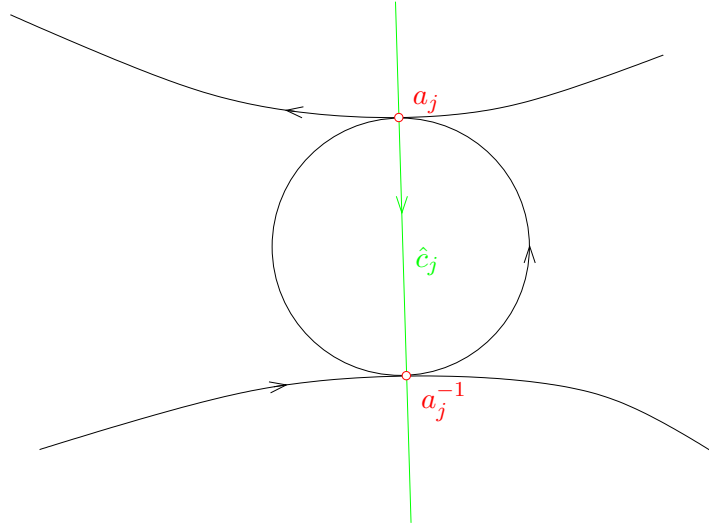


Figure 36: The tangent winding number after a join

Lemma 4.3.1. *Let $f = (f_1, \dots, f_m): \coprod_{j=1}^m \mathbb{S}^1 \rightarrow \mathbb{S}^2$ be a normal immersion with m boundary components. Assume the word $w(f)$ contains two boundary words w_1, w_2 which can be joined. Denote by f^* the induced normal immersion after the join. Then the following is true for the tangent winding number:*

$$\tau(f^*) = \tau(f) + 1.$$

Proof. If two boundary words of $w(f)$ can be joined then one boundary component contains a letter a_j and another one a letter a_j^{-1} . Joining means to insert an interval which joins both boundary components. Another way to look at this approach is to insert a circle C which touches the first boundary component at a_j and the second at a_j^{-1} (Figure 36).

Thus the tangent winding number of the joined immersion f^* is

$$\tau(f^*) = \sum_{j=1}^m \tau(f_j) + \tau(C)$$

and due to the orientation of C , $\tau(C) = +1$ and hence $\tau(f^*) = \tau(f) + 1$. \square

An assembling has the following effect on the tangent winding number:

Lemma 4.3.2. *Suppose $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ is a normal immersion with word $w(f)$. If $w(f)$ has an assembling \mathcal{A} then for the induced normal immersion $\tilde{f}: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ we have*

$$\tau(f) = \tau(\tilde{f}).$$

Proof. If $w(f)$ contains an assemblage a path λ connects two points of the preimage. If we cut along this path λ then the resulting normal immersion has tangent winding number

$\tau(f) + \tau(C_i)$, where C_i is the circle which arises from traveling along λ twice in different directions. Since the path λ is attached to the left of \mathbb{S}^1 the tangent winding number of C_i is $\tau(C_i) = -1$. Thus the resulting normal immersion has tangent winding number $\tau(f) - 1$.

To complete the assemblage a join is made. According to Lemma 4.3.1 the tangent winding number of \tilde{f} is

$$\tau(\tilde{f}) = \tau(f) + \tau(C_i) + 1 = \tau(f).$$

□

Thus the necessary condition in the general case is:

Proposition 4.3.3 (Necessary Extension Condition). *Suppose $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow \mathbb{S}^2$ is a normal immersion which has an extension $F: M \rightarrow \mathbb{S}^2$ to a surface M of genus g_M and boundary $\partial M = \coprod_{j=1}^m \mathbb{S}^1$. Then the tangent winding number of f is*

$$\tau(f) = 1 - 2\omega_1 - (m - 1),$$

where ω_1 denotes the degree of the base point component.

Proof. If f has an extension then Proposition 4.1.7 shows that a joining \mathcal{J} exists which induces a normal immersion $f^*: \mathbb{S}^1 \rightarrow \mathbb{S}^2$. According to Lemma 4.3.1 the tangent winding number of f^* is

$$\tau(f^*) = \tau(f) + m - 1. \quad (5)$$

Since f^* has an extension the assembling \mathcal{A} induces a normal immersion $\tilde{f}: \widetilde{M} \rightarrow \mathbb{S}^2$ which can be extended to an immersion $F: \widetilde{M} \rightarrow \mathbb{S}^2$ (Proposition 4.2.5). According to Proposition 4.2.4 \widetilde{M} has genus $g_{\widetilde{M}} = 0$ and one boundary component, i.e., \widetilde{M} is a disc. This case is treated in the previous sections and we know from Proposition 2.3.7 that

$$\tau(\tilde{f}) = 1 - 2\omega_1.$$

Together with Lemma 4.3.2 this leads to

$$\tau(f) \stackrel{(5)}{=} \tau(f^*) - (m - 1) \stackrel{4.3.2}{=} \tau(\tilde{f}) - (m - 1) \stackrel{2.3.7}{=} 1 - 2\omega_1 - (m - 1).$$

□

In Section 3 it is shown that a grouping \mathcal{G} is fundamental for the normal immersion to extend to \overline{D} . To decide whether a normal immersion extends to a surface of genus g with m boundary components we need a more general definition of a grouping:

Definition 4.3.4. Let $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow \mathbb{S}^2$ be a normal immersion with m boundary components and word $w(f) = \{w_1, \dots, w_m\}$ which has a joining \mathcal{J} . If a groupable word remains after n assemblages then $w(f)$ is called n -groupable.

If $w(f)$ is n -groupable then the triple $(\mathcal{J}, \mathcal{A}, \mathcal{G})$ of a joining \mathcal{J} , an assembling \mathcal{A} and a grouping \mathcal{G} is called an n -grouping of f .

As in the case of a genus 0 surface with one boundary component (Theorem 3.1.5) this leads to an existence theorem:

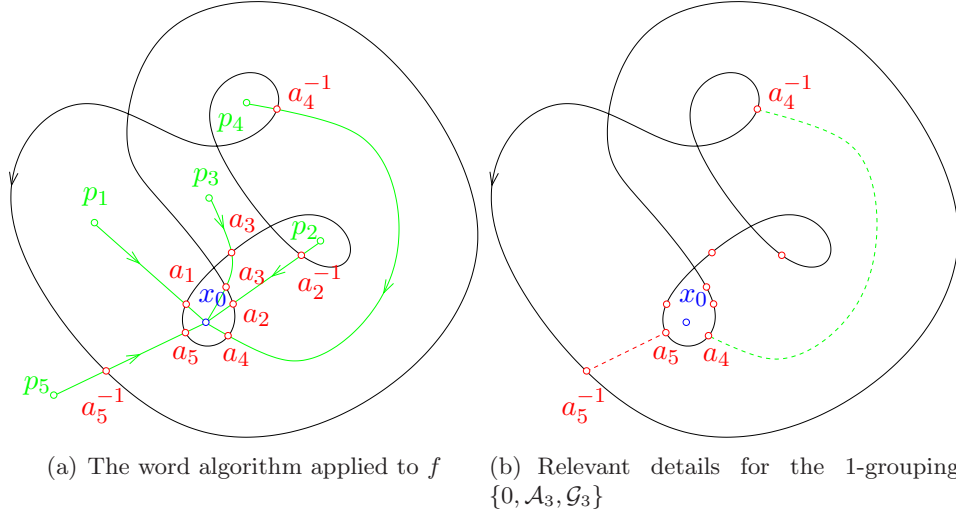


Figure 37: A normal immersion with multiple extensions

Theorem 4.3.5 (Extension Theorem for Immersed Surfaces in the Sphere). *Let $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow \mathbb{S}^2$ be a normal immersion with m boundary components and word $w(f)$. If $\tau(f) = 1 - 2\omega_1 - (m - 1)$ and $w(f)$ has an n -grouping $(\mathcal{J}, \mathcal{A}, \mathcal{G})$ then f has an extension $F: M \rightarrow \mathbb{S}^2$ to a surface M with $\dim(H_1(M)) = 2n + m - 1$ and $\partial M = \coprod_{j=1}^m \mathbb{S}^1$.*

Proof. Denote by $\text{Imm}(A, B)$ the space of normal immersions $f: A \rightarrow B$. Then an n -grouping $(\mathcal{J}, \mathcal{A}, \mathcal{G})$ induces the following sequence:

$$f \in \text{Imm}(\coprod_{j=1}^m \mathbb{S}^1, \mathbb{S}^2) \xrightarrow{\mathcal{J}} f^* \in \text{Imm}(\mathbb{S}^1, \mathbb{S}^2) \xrightarrow{\mathcal{A}} \tilde{f} \in \text{Imm}(\mathbb{S}^1, \mathbb{S}^2).$$

Since $w(f)$ is n -groupable the corresponding word $w(\tilde{f})$ of \tilde{f} is groupable. Therefore \tilde{f} has an extension $\tilde{F}: \overline{D} \rightarrow \mathbb{S}^2$, according to Theorem 3.1.5. Since \tilde{f} is a result of f^* and the assembling \mathcal{A} , Proposition 4.2.4 ensures the existence of an extension $F^*: M^* \rightarrow \mathbb{S}^2$ of \tilde{f} to a surface M^* with genus $g_M = |\mathcal{A}| = n$ and $\partial M^* = \mathbb{S}^1$.

But f^* itself is the result of f and the joining \mathcal{J} and hence Proposition 4.1.6 shows the existence of an extension $F: M \rightarrow \mathbb{S}^2$ of $f: \partial M \rightarrow \mathbb{S}^2$ to a surface M of genus g_M and boundary $\partial M = \coprod_{j=1}^m \mathbb{S}^1$. According to [Hen79], §26 the first homology group of M has dimension

$$\dim(H_1(M)) = 2g_M + m - 1 = 2n + m - 1.$$

□

Remark. Of course this Theorem is a generalization of Theorem 3.1.5. In this case f has one boundary component and M has genus $g_M = 0$. Thus the joining \mathcal{J} and the assembling \mathcal{A} are empty. Hence Theorem 4.3.5 proves the existence of an extension to a surface M of genus $g_M = 0$ and boundary $\partial M = \mathbb{S}^1$, i.e., a disc.

Example 4.3.6. Take the normal immersion $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ shown in Figure 37 (a). It has the word (Example 2.1.2)

$$w(f) = a_5^{-1} a_3 a_2 a_4 a_5 a_1 a_3 a_2^{-1} a_4^{-1}.$$

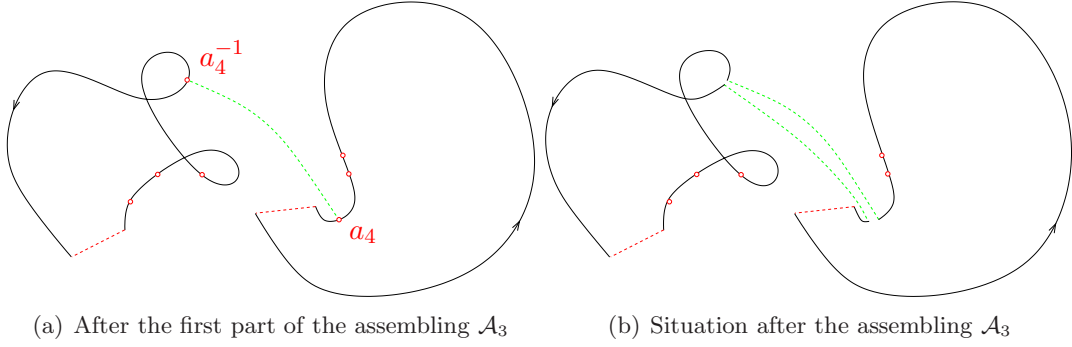


Figure 38: An extension to a torus with one boundary component

This word has the following n -groupings:

$$\begin{array}{llll}
 a_5^{-1} a_3 a_2 a_4 a_5 a_1 a_3 a_2^{-1} a_4^{-1} & \xrightarrow{\mathcal{G}_1} & a_1 a_3 & \text{0-grouping } (0, 0, \mathcal{G}_1) \\
 a_5^{-1} a_3 a_2 a_4 a_5 a_1 a_3 a_2^{-1} a_4^{-1} & \xrightarrow{\mathcal{G}_2} & a_3 & \text{0-grouping } (0, 0, \mathcal{G}_2) \\
 a_5^{-1} a_3 a_2 a_4 a_5 a_1 a_3 a_2^{-1} a_4^{-1} \xrightarrow{\mathcal{A}_3} a_4^{-1} a_1 a_3 a_4 a_3 & \xrightarrow{\mathcal{G}_3} & a_3 & \text{1-grouping } (0, \mathcal{A}_3, \mathcal{G}_3) \\
 a_5^{-1} a_3 a_2 a_4 a_5 a_1 a_3 a_2^{-1} a_4^{-1} \xrightarrow{\mathcal{A}_4} a_1 a_3 a_2^{-1} a_3 a_2 & \xrightarrow{\mathcal{G}_4} & a_1 a_3 & \text{1-grouping } (0, \mathcal{A}_4, \mathcal{G}_4) \\
 a_5^{-1} a_3 a_2 a_4 a_5 a_1 a_3 a_2^{-1} a_4^{-1} \xrightarrow{\mathcal{A}_5} a_5^{-1} a_3 a_5 a_1 a_3 & \xrightarrow{\mathcal{G}_5} & a_1 a_3 & \text{1-grouping } (0, \mathcal{A}_5, \mathcal{G}_5)
 \end{array}$$

The 0-groupings show that according to Theorem 3.3.3 this immersion can be extended to $M = \overline{D}$ in two different ways.

The 1-groupings induce extensions to a torus with a disc removed. We will show by means of the 1-grouping $(0, \mathcal{A}_3, \mathcal{G}_3)$ how this extension can be derived from the immersion itself.

In this case the assembling \mathcal{A}_3 has the linked pairings $a_5^{-1} \dots a_5$ and $a_2 \dots a_2^{-1}$. Mark the relevant parts of the rays \hat{c}_j (Figure 37 (b)) and cut along the part of \hat{c}_5 . The result is shown in Figure 38 (a).

To complete the assembling join along a_4 to get a normal immersion $\tilde{f}: \mathbb{S}^1 \rightarrow \mathbb{S}^2$, shown in Figure 38 (b). An equivalent immersion is shown in Figure 39. Since after the assembling \mathcal{A}_3 a groupable word remains, the normal immersion $\tilde{f}: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ extends to an immersion $\tilde{F}: \overline{D} \rightarrow \mathbb{S}^2$. Thus f has an extension $F: M \rightarrow \mathbb{S}^2$, where M is a torus with a disc removed, i.e., $\partial M = \mathbb{S}^1$ and $\dim(H_1(M)) = 2$.

Since $\tilde{F}: \overline{M} \rightarrow \mathbb{S}^2$ is not surjective it has an extension to \mathbb{R}^2 as well. Hence f can be extended to an immersed torus with one boundary component in the plane.

4.4 Uniqueness of Immersed Surfaces in the Sphere

At this point we know when a normal immersion $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow \mathbb{S}^2$ has an extension $F: M \rightarrow \mathbb{S}^2$. Theorem 4.3.5 classifies the preimage M up to diffeomorphism. Again, the question arises how many different extensions exist. As in Definition 3.3.1 two extensions $F_1: M \rightarrow \mathbb{S}^2$ and $F_2: M \rightarrow \mathbb{S}^2$ are equivalent if an orientation preserving diffeomorphism $\varphi: M \rightarrow M$ exists, such that $F_1 = F_2 \circ \varphi$ and $\varphi|_{\partial M} = \text{id}_{\partial M}$.

We know, that an n -grouping $(\mathcal{J}, \mathcal{A}, \mathcal{G})$ is composed of a joining \mathcal{J} , an assembling \mathcal{A} and a grouping \mathcal{G} of $w(f)$. The joining \mathcal{J} is a weighted graph with the boundary components as vertex set (Definition 4.1.4), the assembling \mathcal{A} is a directed set (Definition 4.2.3) and the grouping \mathcal{G} is a weighted tree (Definition 2.4.7). As in the case of normal immersions $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ these structures distinguish between different n -groupings:

Definition 4.4.1. *Two n -groupings $(\mathcal{J}_1, \mathcal{A}_1, \mathcal{G}_1)$ and $(\mathcal{J}_2, \mathcal{A}_2, \mathcal{G}_2)$ are equivalent if $\mathcal{A}_1 = \mathcal{A}_2$ and the pairs $\mathcal{J}_1, \mathcal{J}_2$ and $\mathcal{G}_1, \mathcal{G}_2$ are isomorphic.*

Firstly we will show that it is well defined:

Lemma 4.4.2. *Suppose $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow \mathbb{S}^2$ is an extendable normal immersion. Let $F_1: M \rightarrow \mathbb{S}^2$ and $F_2: M \rightarrow \mathbb{S}^2$ be two extensions of f with equivalent n -groupings $(\mathcal{J}_1, \mathcal{A}_1, \mathcal{G}_1)$ and $(\mathcal{J}_2, \mathcal{A}_2, \mathcal{G}_2)$. Then F_1 and F_2 are equivalent extensions.*

Proof. An n -grouping $(\mathcal{J}, \mathcal{A}, \mathcal{G})$ induces the following sequence:

$$f \in \text{Imm}(\coprod_{j=1}^m \mathbb{S}^1, \mathbb{S}^2) \xrightarrow{\mathcal{J}} f^* \in \text{Imm}(\mathbb{S}^1, \mathbb{S}^2) \xrightarrow{\mathcal{A}} \tilde{f} \in \text{Imm}(\mathbb{S}^1, \mathbb{S}^2).$$

The induced mappings are quotient maps, which are defined in terms of the words $w(f)$ and $w(f^*)$. Since the joinings \mathcal{J}_1 and \mathcal{J}_2 are equivalent the same letters of $w(f)$ are joined by them. Therefore the induced quotient maps are equal and we denote it by $\pi_{\mathcal{J}}$. The same holds for the equivalent assemblies \mathcal{A}_1 and \mathcal{A}_2 . Also in that the same letters are assembled. Again the induced quotient maps are equal and we denote it by $\pi_{\mathcal{A}}$. This leads to

$$\tilde{f} = \pi_{\mathcal{A}} \circ \pi_{\mathcal{J}} \circ f.$$

That equivalent groupings induce equivalent extensions is shown in Theorem 3.3.3. Thus \tilde{f} has two equivalent extensions \tilde{F}_1 and \tilde{F}_2 , i.e.,

$$\tilde{F}_1 = \tilde{F}_2 \circ \tilde{\varphi},$$

where $\tilde{\varphi}$ is an orientation preserving diffeomorphism which is the identity on the boundary. Since the quotient maps $\pi_{\mathcal{J}}$ and $\pi_{\mathcal{A}}$ extends to the extensions this yields

$$\begin{aligned} \tilde{F}_1 &= \pi_{\mathcal{A}} \circ \pi_{\mathcal{J}} \circ F_1 \\ \tilde{F}_2 &= \pi_{\mathcal{A}} \circ \pi_{\mathcal{J}} \circ F_2 \end{aligned}$$

and hence

$$F_1 = F_2 \circ \varphi,$$

where $\varphi: M \rightarrow M$ is an orientation preserving diffeomorphism with $\varphi|_{\partial M} = \text{id}_{\partial M}$. \square

In Proposition 4.1.6 we have seen that a normal immersion $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow \mathbb{S}^2$ has an extension $F: M \rightarrow \mathbb{S}^2$ if the joining induces an extendable normal immersion f^* . We will now show that two different joinings lead to nonequivalent extensions.

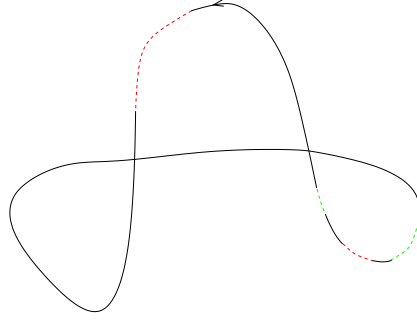


Figure 39: Resulting immersion $\tilde{f}: \tilde{M} \rightarrow \mathbb{S}^2$

Lemma 4.4.3. *Suppose $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow \mathbb{S}^2$ is a normal immersion with word $w(f)$. Then two different joinings define two different equivalence classes of extensions of f .*

Proof. Assume that \mathcal{J}_1 and \mathcal{J}_2 are two different joinings of $w(f)$. These joinings induce two normal immersions $f_1^*, f_2^*: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ with different words $w(f_1^*) \neq w(f_2^*)$. Therefore no diffeomorphism $\varphi: M^* \rightarrow M^*$ with $\varphi|_{\partial M^*} = \text{id}_{\partial M^*}$ exists, such that $f_1^* = f_2^* \circ \varphi|_{\partial M^*}$ (Theorem 3.3.3). Thus the nonequivalent extensions F_1^* and F_2^* lead to nonequivalent extensions F_1 and F_2 of f . \square

Now we will show that two different assemblings induce different extensions. In Proposition 4.2.4 is shown that a normal immersion $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ has an extension $F: M \rightarrow \mathbb{S}^2$ if an assembling \mathcal{A} induces an extendable normal immersion $\tilde{f}: \mathbb{S}^1 \rightarrow \mathbb{S}^2$.

Lemma 4.4.4. *Suppose $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ is a normal immersion with word $w(f)$. Then two different assemblings $\mathcal{A}_1, \mathcal{A}_2$ define two different equivalence classes of extensions of f .*

Proof. Let $\tilde{f}_1, \tilde{f}_2: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ be the normal immersions induced by the assemblings \mathcal{A}_1 and \mathcal{A}_2 and denote the corresponding extensions by $\tilde{F}_1, \tilde{F}_2: \overline{D} \rightarrow \mathbb{S}^2$.

According to Proposition 4.2.4 these extensions induce extensions $F_j: M_j \rightarrow \mathbb{S}^2$ to surfaces M_j of genus $g_{M_j} = |\mathcal{A}_j|$. If $|\mathcal{A}_1| \neq |\mathcal{A}_2|$ the surfaces M_1 and M_2 are not diffeomorphic (see [Ful95], Theorem 17.4) and hence the extensions F_1 and F_2 are different.

If $|\mathcal{A}_1| = |\mathcal{A}_2|$ each assembling induces a commutative diagram, such that for each quotient map π_j a unique extension F_j exists (Proposition 4.2.4):

$$\begin{array}{ccc} \overline{D} & \xrightarrow{\tilde{F}_j} & \mathbb{S}^2 \\ \pi_j \downarrow & \nearrow F_j & \\ M_j & & \end{array}$$

Since the assemblings are different, the quotient maps are different and hence the extensions F_1 and F_2 are different as well. \square

Theorem 4.4.5 (Uniqueness Theorem for Immersed Surfaces in the Sphere). *Suppose $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow \mathbb{S}^2$ is a normal immersion with word $w(f)$ and tangent winding number $\tau(f) = 1 - 2\omega_1 - (m - 1)$. Then two different n -groupings define two different equivalence classes of extensions.*

Proof. In Theorem 4.3.5 is shown that an n -grouping induces an equivalence class of extensions. Recall, that each n -grouping $(\mathcal{J}, \mathcal{A}, \mathcal{G})$ induce the following sequence on the space of normal immersions:

$$f \in \text{Imm}(\coprod_{j=1}^m \mathbb{S}^1, \mathbb{S}^2) \xrightarrow{\mathcal{J}} f^* \in \text{Imm}(\mathbb{S}^1, \mathbb{S}^2) \xrightarrow{\mathcal{A}} \tilde{f} \in \text{Imm}(\mathbb{S}^1, \mathbb{S}^2).$$

Let $(\mathcal{J}_1, \mathcal{A}_1, \mathcal{G}_1)$ and $(\mathcal{J}_2, \mathcal{A}_2, \mathcal{G}_2)$ be two different n -groupings of $w(f)$. Thus one of the parts of the first n -grouping is different to the according part of the second. If the joinings are different then Lemma 4.4.3 shows that the induced extensions are not equivalent. If the assemblings are different, then Lemma 4.4.4 proves the nonequivalence of the induced extensions.

If the joinings and assemblings are equal then the groupings has to be different. In this case the claim is shown in Theorem 3.3.3. \square

5 Immersed Surfaces in Closed Surfaces

At this point we have classified the extensions of a normal immersion $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow \mathbb{S}^2$. But what happens if we replace the target surface \mathbb{S}^2 by an arbitrary closed surface N ?

5.1 Immersed Surfaces with m Boundary Components in Closed Surfaces

We know from the beginning that a necessary condition for a normal immersion $f: \mathbb{S}^1 \rightarrow N$ to extend is that f is numerable (Definition 1.2.1).

Since in the previous subsections $N = \mathbb{S}^2$, hence simply connected, each normal immersion $f: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ is numerable. Now we consider a general target N which may not be simply connected. Then Lemma 1.2.5 shows that numerability is equivalent to homological triviality. We will reduce the case of a general target N to the case of \mathbb{S}^2 .

Recall that in the case of $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow \mathbb{S}^2$ a necessary condition for f to extend is the existence of a joining (Proposition 4.1.7). A joining is a weighted graph whose edges correspond to paths $\lambda_1, \dots, \lambda_{k-1}$ which connect the boundary components. Each path satisfies $\lambda_j(0) = a_i^{-1}$ and $\lambda_j(1) = a_i$ and hence if f has an extension to an immersion $F: M \rightarrow N$ then $\lambda_j \subset F(M)$.

The normal numbering ψ_n of f assigns to each component X_j of $N \setminus f(\coprod_{j=1}^m \mathbb{S}^1)$ a natural number, such that the number of preimages of any extension F is uniquely determined up to a constant ω_1 , as in Subsection 1.2.

Since $\lambda_j(0) = a_i^{-1}$, the intersection of such a path λ_j and the components of minimal degree satisfies

$$\lambda_j \cap \bigcup_{X_i: \psi_n(X_i)=0} X_i = \emptyset. \quad (6)$$

Therefore, in the case of a normal immersion $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow N$, if we can choose $m-1$ paths $\lambda_1, \dots, \lambda_{m-1}$ such that (6) is satisfied these paths induce a graph with the boundary components f_1, \dots, f_m of f as a vertex set, see Definition 4.1.4.

Definition 5.1.1. Suppose N is a closed surface of genus $g_N > 0$ and $f = (f_1, \dots, f_m): \coprod_{j=1}^m \mathbb{S}^1 \rightarrow N$ is a homologically trivial normal immersion. An embedded path $\lambda: [0, 1] \rightarrow N$ with $\lambda(0) \in f_i(\mathbb{S}^1)$ and $\lambda(1) \in f_j(\mathbb{S}^1)$, for $i \neq j$, satisfying (6) is called a *join* of f_i and f_j .

Assume $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow N$ is a normal immersion which has an extension $F: M \rightarrow N$ to a surface M . Since M is connected and F is continuous $F(M)$ is connected as well. Therefore we expect that each boundary component f_i can be joined with another boundary component f_j .

Since numerability is a necessary condition for f to extend, a normal numbering $\psi_n: \{X_1, \dots, X_k\} \rightarrow \mathbb{N}_0$ exists. If we define

$$S_{>0} := \bigcup_{\psi_n(X_i) > 0} \overline{X_i},$$

then a path $\lambda: [0, 1] \rightarrow S_{>0}$ satisfies (6) and vice versa. Pick a point $p \in f_i(\mathbb{S}^1)$ and a point $q \in f_j(\mathbb{S}^1)$. Each point is in the boundary of two components of $N \setminus f(\coprod_{j=1}^m \mathbb{S}^1)$ and according

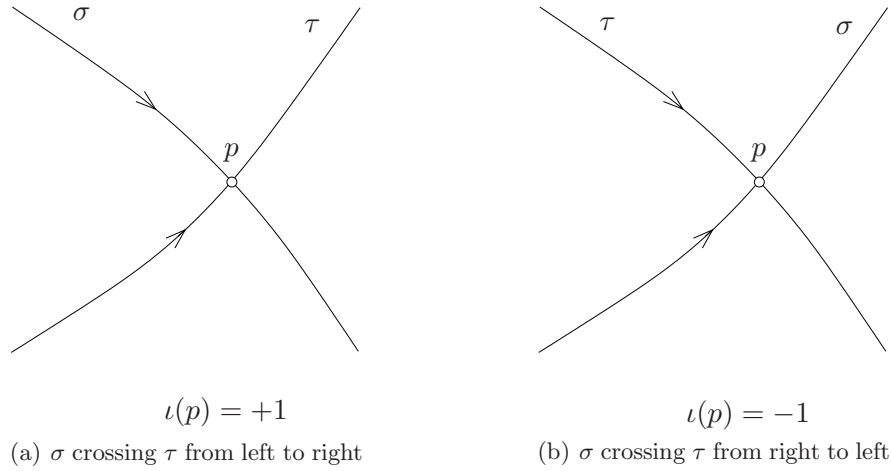


Figure 40: Intersection Number

to the normal numbering at least one of these components is a subset of $S_{>0}$. If $S_{>0}$ is connected a path $\lambda: [0, 1] \rightarrow S_{>0}$ exists connecting p and q , i.e., λ joins f_i and f_j .

For each X_j with $\psi_n(X_j) = 0$ the preimage $F^{-1}(X_j)$ is a disjoint union of open sets in M . By definition $F^{-1}(X_j) \cap F^{-1}(X_i) = \emptyset$ for $i \neq j$ and thus $F^{-1}(S_{>0}^c)$ is a disjoint union of open subsets. Hence $M \setminus F^{-1}(S_{>0}^c)$ is connected and since F is continuous $S_{>0} = F(M \setminus F^{-1}(S_{>0}^c))$ is connected as well.

Therefore the existence of a join for each boundary component is a necessary condition for f to extend. In the case $N = \mathbb{S}^2$ we use the letters of the word $w(f)$ to localize joining paths. The key was, that the letters of $w(f)$ represent oriented intersection points. Also in a closed surface of arbitrary genus we can define oriented intersection points (Figure 40):

Definition 5.1.2. Suppose N is a closed surface of genus $g_N > 0$ and $[\sigma], [\tau] \in H_1(N)$ are two homology classes which are represented by transversal differentiable loops σ, τ . For a point $p \in \sigma \cap \tau$ the *intersection number* is defined by (Figure 40)

$$\iota(p) := \begin{cases} +1 & : \sigma \text{ crosses } \tau \text{ from left to right,} \\ -1 & : \sigma \text{ crosses } \tau \text{ from right to left.} \end{cases}$$

An *intersection pairing* of $[\sigma], [\tau]$ is

$$\langle \sigma, \tau \rangle := \sum_{p \in \sigma \cap \tau} \iota(p).$$

The intersection pairing of two homology classes is independent of the choice of the representatives and therefore defines a skew symmetric bilinear form on the first homology group (see [Ful95], Lemma 18.6):

$$\begin{array}{ccc} H_1(N) \times H_1(N) & \longrightarrow & \mathbb{Z} \\ [\sigma] \times [\tau] & \longmapsto & \langle \sigma, \tau \rangle \end{array}$$

Since it is independent of the choice of a representative, we can use special representatives:

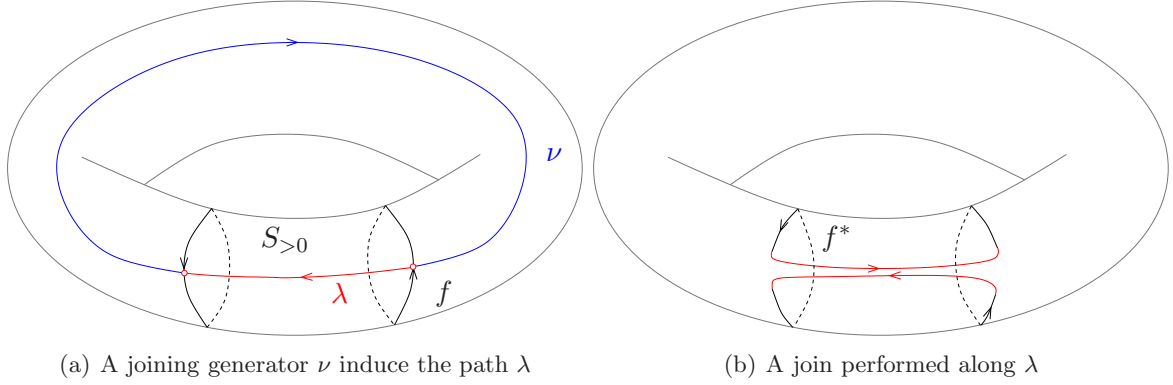


Figure 41: A joining in an arbitrary closed surface

Definition 5.1.3. Suppose N is a closed surface of genus $g_N > 0$. An embedded loop ν is called a *generator of $H_1(N)$* if ν is homologous to a basis element of $H_1(N)$.

Lemma 5.1.4. Suppose N is a closed surface of genus $g_N > 0$ and $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow N$ is a homologically trivial immersion. Then each generator ν of $H_1(N)$ satisfies

$$\langle f(\coprod_{j=1}^m \mathbb{S}^1), \nu \rangle = 0.$$

Proof. Pick a point $x \in N$ with $x \notin f(\coprod_{j=1}^m \mathbb{S}^1) \cup \nu$. Since $0 = [f(\coprod_{j=1}^m \mathbb{S}^1)] \in H_1(N)$ the constant loop $\lambda_x: \mathbb{S}^1 \rightarrow \{x\} \subset N$ is homologous to $f(\coprod_{j=1}^m \mathbb{S}^1)$. Therefore

$$\langle f(\coprod_{j=1}^m \mathbb{S}^1), \nu \rangle = \langle \lambda_x, \nu \rangle = 0,$$

because $\lambda_x \cap \nu = \emptyset$. \square

The Lemma shows that a generator ν has either no intersection points with a homologically trivial normal immersion or has an equal number of positive and negative intersection points. Hence the intersection pairing can be used to find joins:

Definition 5.1.5. Suppose N is a closed surface and $f = (f_1, \dots, f_m): \coprod_{j=1}^m \mathbb{S}^1 \rightarrow N$ is a homologically trivial normal immersion. Then a generator ν_{ij} of $H_1(N)$ is called a *joining generator of f* if $i \neq j$ exists, such that $\nu_{ij} \cap f_i(\mathbb{S}^1) \neq \emptyset \neq \nu_{ij} \cap f_j(\mathbb{S}^1)$.

Lemma 5.1.6. Suppose N is a closed surface of genus $g_N > 0$ and $f = (f_1, \dots, f_m): \coprod_{j=1}^m \mathbb{S}^1 \rightarrow N$ is a homologically trivial normal immersion. If ν_{ij} is a joining generator of f then a join λ of f_i and f_j exists.

Proof. Since ν_{ij} is a joining generator of f , intersection points $p \in \nu_{ij} \cap f_i(\mathbb{S}^1)$ and $q \in \nu_{ij} \cap f_j(\mathbb{S}^1)$ exist such that $i \neq j$ and $\iota(p) + \iota(q) = 0$. W.l.o.g. assume that $\iota(p) = -1$, i.e., ν_{ij} enters a component X_l with $\psi_n(X_l) > 0$ at p . On the other hand ν_{ij} leaves a component X_m with $\psi(X_m) > 0$ at q because $\iota(q) = +1$.

Since $S_{>0}$ is connected a path $\lambda: [0, 1] \rightarrow S_{>0}$ with $\lambda(0) = p$ and $\lambda(1) = q$ exists. Hence λ is a join of f_i and f_j (Figure 41). \square

As in Subsection 4.1 the goal is to reduce the case of multiple boundary components to that of one boundary component. Thus the existence of a join reduces the number of boundary components:

Lemma 5.1.7. *Suppose N is a closed surface, $f = (f_1, \dots, f_m): \coprod_{j=1}^m \mathbb{S}^1 \rightarrow N$ a homologically trivial normal immersion and $\lambda: [0, 1] \rightarrow N$ a join of f . Then a homologically trivial immersion $f^*: \coprod_{j=1}^{m-1} \mathbb{S}^1 \rightarrow N$ with one boundary component less exists, such that*

- $f(\coprod_{j=1}^m \mathbb{S}^1) \subset f^*(\coprod_{j=1}^{m-1} \mathbb{S}^1)$ and
- $f^*(\coprod_{j=1}^{m-1} \mathbb{S}^1) \setminus f(\coprod_{j=1}^m \mathbb{S}^1) \subset \lambda$.

Proof. Let $t_i := f_i^{-1}(\lambda(0))$, $t_j := f_j^{-1}(\lambda(1))$ and $t_0 \in \mathbb{S}^1 \setminus \{t_i, t_j\}$. Then the join f_{ij} of f_i and f_j is defined by

$$f_{ij} := f_i|_{[t_0, t_i]} + \lambda|_{[0, 1]} + f_j|_{\mathbb{S}^1} - \lambda|_{[0, 1]} + f_i|_{(t_i, t_0]}.$$

Since the other boundary components are not affected this join induces an immersion $f^* = (f_1, \dots, f_{i-1}, f_{ij}, f_{i+1}, \dots, f_{j-1}, f_{j+1}, \dots, f_m): \coprod_{j=1}^{m-1} \mathbb{S}^1 \rightarrow N$. The following is true for the homology class of f_{ij} :

$$\begin{aligned} [f_{ij}] &= [f_i|_{[t_0, t_i]}] + [\lambda|_{[0, 1]}] + [f_j|_{\mathbb{S}^1}] - [\lambda|_{[0, 1]}] + [f_i|_{(t_i, t_0]}] \\ &= [f_i|_{[t_0, t_i]}] + [f_j|_{\mathbb{S}^1 \setminus t_j}] + [f_i|_{(t_i, t_0]}] \\ &= [f_i] + [f_j] \end{aligned}$$

and hence $[f^*] = [f] = 0$. □

As in Subsection 4.1 we track the joins by a graph which has the boundary components f_1, \dots, f_m as vertex set. Each join induces a labeled edge in this graph. Recall that these labeled edges are used to derive an extension of f from the induced normal immersion f^* with one boundary component (see Proposition 4.1.6).

If λ is a join of f_i and f_j then the vertices f_i and f_j are connected by an edge. Pick an arbitrary intersection point p_1 in $f_i \cap \nu_{ij}$ and number the remaining intersection points as they appear along the oriented loop ν_{ij} . Number the intersection points of $f_j \cap \nu_{ij}$ in the same way. Assign the weight l^n to the edge connecting f_i and f_j if the induced path λ_{ij} of ν_{ij} connects the l -th intersection point of f_i and the n -th intersection point of f_j .

Definition 5.1.8. If $m - 1$ joins $\lambda_1, \dots, \lambda_{m-1}: [0, 1] \rightarrow S_{>0}$ exist, such that each boundary component f_1, \dots, f_m is joined by a join λ_j , then the induced graph \mathcal{J} is called a *joining* of f .

Since a joining has no effect on the target the proofs of Subsection 4.1 hold for normal immersions to a closed surface of arbitrary genus. Thus a joining \mathcal{J} induces a homologically trivial normal immersion $f^*: \mathbb{S}^1 \rightarrow N$ (Lemma 5.1.7 and Lemma 4.1.5). Again f has an immersed extension $F: M \rightarrow N$, where M is a surface with $\partial M = \coprod_{j=1}^m \mathbb{S}^1$, if and only if f^* has an immersed extension $F^*: M^* \rightarrow N$ to a surface M^* with $\partial M^* = \mathbb{S}^1$ (Proposition 4.1.6).

According to Lemma 4.4.3 different joinings induce nonequivalent extensions of f . To decide whether a homologically trivial normal immersion $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow N$ has an extension to a surface M with $\partial M = \coprod_{j=1}^m \mathbb{S}^1$ we have to study homologically trivial normal immersion $f^*: \mathbb{S}^1 \rightarrow N$.

5.2 Existence of Immersed Surfaces in Closed Surfaces

We will reduce the problem of extending immersions to closed surfaces to the case of immersions to the sphere. Thus assume that a surface N has genus $g_N > 0$ and hence nontrivial homology $H_1(N)$. To transform N into a sphere we use basic facts from surgery theory. To decrease the genus of N we cut along a generator ν of $H_1(N)$ and attach two discs to the obtained boundary components. This is called a 1-surgery of ν in surgery theory. Proposition 4.33 of [Ran02] shows that 1-surgery leads to a closed surface of genus $g_N - 1$. Note that for each 1-surgery on a closed surface there is an inverse dual 0-surgery which returns the original surface N (see [Ran02], Chapter 2).

Suppose $f: \mathbb{S}^1 \rightarrow N$ is a normal immersion and ν is a generator of $H_1(N)$. If $\nu \cap f(\mathbb{S}^1) = \emptyset$ then a 1-surgery of ν has no effect on the image of f . But if $\nu \cap f(\mathbb{S}^1) \neq \emptyset$ then the 1-surgery will cut $f(\mathbb{S}^1)$. Thus $f(\mathbb{S}^1) \setminus \nu$ is not connected and consists of several components $\bar{f}_1, \dots, \bar{f}_n$. We use ν to connect the endpoints of the immersions \bar{f}_j to obtain a family $f_1, \dots, f_n: \mathbb{S}^1 \rightarrow N$ of immersions (Figure 42).

The advantage of this family is, that it is separated by the generator ν . That way the 1-surgery of ν has no effect on the family and therefore induces a family $f'_1, \dots, f'_n: \mathbb{S}^1 \rightarrow N^\nu$, where the target is replaced by the result N^ν of the 1-surgery on N .

We describe how to obtain such a family of immersions. When ν intersects $f(\mathbb{S}^1)$ we know from Lemma 5.1.4 that each positive intersection point leads to a negative intersection point. If we denote the sign of the intersection points in order they appear along the oriented loop $f(\mathbb{S}^1)$, we get a sequence of numbers ± 1 adding up to 0. Thus there has to be two consecutive intersection points p and q with $\iota(p) + \iota(q) = 0$. These consecutive intersection points are used to define iteratively the family f_1, \dots, f_n , as follows:

Lemma 5.2.1. *Suppose N is a closed surface of genus $g_N > 0$ and $f: \mathbb{S}^1 \rightarrow N$ a homologically trivial normal immersion. Let ν be a generator of $H_1(N)$ intersecting $f(\mathbb{S}^1)$. Then a family $f_1, \dots, f_n: \mathbb{S}^1 \rightarrow N$ of homologically trivial immersions exists, such that*

- $f_i(\mathbb{S}^1) \subset f(\mathbb{S}^1) \cup \nu$ for all $i = 1, \dots, n$ and
- $f_i(\mathbb{S}^1) \setminus \nu$ is connected for all $i = 1, \dots, n$.

Proof. Pick two consecutive intersection points p and q in the order they appear on $f(\mathbb{S}^1)$ with $\iota(p) + \iota(q) = 0$. W.l.o.g. assume that $\iota(f(q)) = -1$ and connect p and q by an interval $I^\nu := [q, p]$, oriented from q to p . Define $S_1 := \mathbb{S}^1_{[p,q]} \cup I^\nu$ and $S' := \mathbb{S}^1_{[q,p]} \cup I^\nu$.

Denote the part of ν which starts at $f(q)$ and ends in $f(p)$ by $\nu|_{[f(q), f(p)]}$ and let $\tilde{\varphi}: [0, 1] \rightarrow \nu|_{[f(q), f(p)]}$ be a smooth embedding with $\tilde{\varphi}(0) = f(q)$ and $\tilde{\varphi}(1) = f(p)$. Then

$$f|_{\mathbb{S}^1_{[p,q]}} + \tilde{\varphi}$$

represents a cycle homologous to an integer multiple of ν (Figure 42), i.e.,

$$[f|_{\mathbb{S}^1_{[p,q]}} + \tilde{\varphi}] = \alpha[\nu] \quad \text{with } \alpha \in \mathbb{Z}.$$

The reason is that ν decomposes the 2-chain, whose boundary is f , into two 2-chains. These 2-chains have $f|_{\mathbb{S}^1_{[p,q]}} + \tilde{\varphi} - \alpha\nu$ and $f|_{\mathbb{S}^1_{[q,p]}} - \tilde{\varphi} - \alpha\nu$ as boundaries.

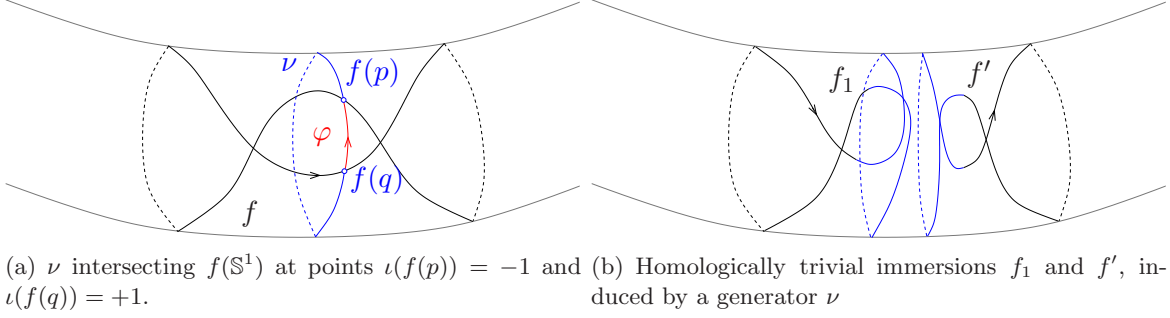


Figure 42: Family of homologically trivial immersion, induced by a generator ν .

Let $\varphi: I^\nu \rightarrow \nu$ be a smooth function with $\varphi(q) = f(q)$ and $\varphi(p) = f(p)$, such that $[\varphi - \tilde{\varphi}] = -\alpha[\nu]$ and define the continuous functions

$$\begin{aligned} f_1 &:= f|_{\mathbb{S}_{[p,q]}^1} + \varphi, \\ f' &:= f|_{\mathbb{S}_{[q,p]}^1} - \varphi. \end{aligned}$$

Since $\mathbb{S}_{[p,q]}^1 \cup I^\nu \simeq \mathbb{S}^1 \simeq \mathbb{S}_{[q,p]}^1 \cup I^\nu$ the domain of both functions is \mathbb{S}^1 . Hence for the image of f_1 we have

$$f_1(\mathbb{S}^1) = f|_{\mathbb{S}_{[p,q]}^1} \cup \varphi(I^\nu) \subset f(\mathbb{S}^1) \cup \nu.$$

Since p and q are consecutive intersection points, $\mathbb{S}_{[p,q]}^1$ contains no further intersection points. Hence $f|_{\mathbb{S}_{[p,q]}^1} \cap \nu = \{p, q\}$ and therefore $f_1 \setminus \nu = f|_{\mathbb{S}_{(p,q)}^1}$ is connected. The homology class of f_1 satisfies

$$\begin{aligned} [f_1] &= [f|_{\mathbb{S}_{[p,q]}^1} + \varphi] = [f|_{\mathbb{S}_{[p,q]}^1} + \tilde{\varphi} - \tilde{\varphi} + \varphi] \\ &= [f|_{\mathbb{S}_{[p,q]}^1} + \tilde{\varphi} + \varphi - \tilde{\varphi}] \\ &= [f|_{\mathbb{S}_{[p,q]}^1} + \tilde{\varphi}] + [\varphi - \tilde{\varphi}] \\ &= \alpha[\nu] - \alpha[\nu] = 0. \end{aligned}$$

Since $f_1 + f' = f|_{\mathbb{S}_{[p,q]}^1} + \varphi + f|_{\mathbb{S}_{[q,p]}^1} - \varphi = f$ and $[f] = 0$ it follows that $[f'] = 0$ as well.

Hence f' satisfies the conditions of the lemma itself and can be decomposed into two immersions f_2 and f'' , satisfying the conditions. Since the number of intersection points is finite this leads to a decomposition of f into the desired family of immersions $f_1, \dots, f_n: \mathbb{S}^1 \rightarrow N$. \square

Each immersion f_j of this family lies to one side of ν . Hence a 1-surgery of ν does not disconnect f_j and therefore it leads to a family f_1', \dots, f_n' of immersions to the surface N^ν resulting from the 1-surgery.

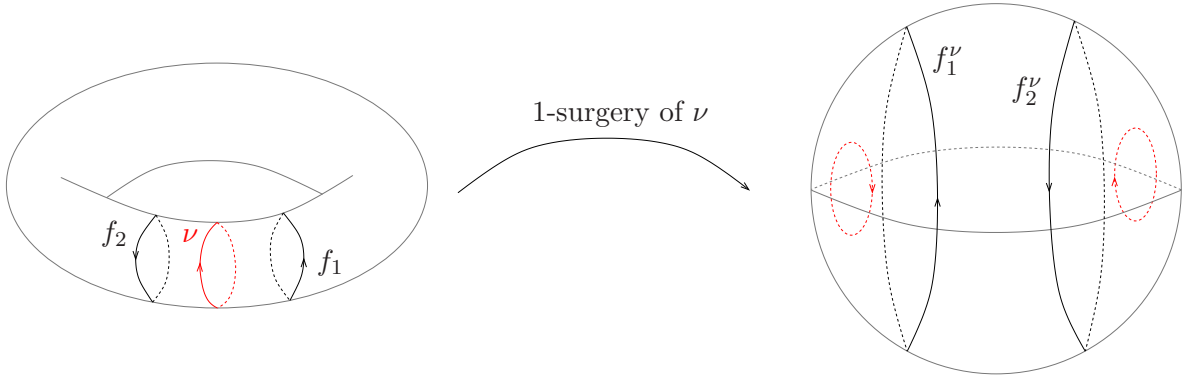


Figure 43: Family of immersions induced by a 1-surgery of ν .

Corollary 5.2.2. *Suppose N is a closed surface of genus $g_N > 0$ and $f: \mathbb{S}^1 \rightarrow N$ is a homologically trivial normal immersion. Let ν be a generator of $H_1(N)$. Then (f, ν) induces a family of normal immersion $f_1^\nu, \dots, f_n^\nu: \mathbb{S}^1 \rightarrow N^\nu$ such that*

- N^ν is a closed surface of genus $g_{N^\nu} = g_N - 1$.
- f_j^ν is homologically trivial for $j = 1, \dots, n$.

Proof. Let ν be a generator of $H_1(N)$ and denote by N^ν the closed surface which is the result of the 1-surgery of ν on N . After Proposition 4.33 in [Ran02] the genus of the resulting surface is $g_{N^\nu} = g_N - 1$.

According to Lemma 5.2.1 ν decomposes f into a family $f_1, \dots, f_n: \mathbb{S}^1 \rightarrow N$, such that each immersion f_j lies to one side of ν . Thus the 1-surgery does not disconnect the immersions f_j and hence we can replace the target N by the surface N^ν , resulting from the 1-surgery, to obtain the family $f_1^\nu, \dots, f_n^\nu: \mathbb{S}^1 \rightarrow N^\nu$. Since $0 = [f_j] \in H_1(N)$ and $H_1(N^\nu) \triangleleft H_1(N)$ the normal immersions f_j^ν are homologically trivial as well. \square

When we use surgery theory to obtain a sphere from a closed surface N of positive genus g_N we reduce the problem of extending immersions to N to the problem of extending immersions to \mathbb{S}^2 . To transform N into a sphere a sequence of g_N 1-surgeries is needed. At first we will show how to obtain an extension of f from an extension of the family f_1^ν, \dots, f_n^ν as a result of a single 1-surgery.

Proposition 5.2.3. *Suppose N is a closed surface of genus $g_N > 0$ and $f: \mathbb{S}^1 \rightarrow N$ a homologically trivial normal immersion. Let ν represent a generator of $H_1(N)$ and $f_1^\nu, \dots, f_n^\nu: \mathbb{S}^1 \rightarrow N^\nu$ be the family induced by (f, ν) . If $f_j^\nu: \mathbb{S}^1 \rightarrow N^\nu$ has an extension $F_j^\nu: M_j^\nu \rightarrow N^\nu$, with $\partial M_j^\nu = \mathbb{S}^1$, for $j = 1, \dots, n$ then the dual 0-surgery ν^* of ν induces a surface M with $\partial M = \mathbb{S}^1$, such that $f: \mathbb{S}^1 \rightarrow N$ can be extended to an immersion $F: M \rightarrow N$.*

Proof. The dual 0-surgery ν^* of ν removes two disjoint open discs D_1, D_2 from N^ν and identifies the boundaries $\partial D_1, \partial D_2$ such that $\pi(\partial D_j) = \nu$ for the corresponding quotient map $\pi: N^\nu \setminus (D_1 \cup D_2) \rightarrow N$.

According to Lemma 5.2.1 each immersion of the family $f_1, \dots, f_n: \mathbb{S}^1 \rightarrow N$ satisfies $f_j(\mathbb{S}^1) \setminus \nu \subset f(\mathbb{S}^1)$ and $f_j(\mathbb{S}^1) \setminus \nu$ is connected. Hence the following is true for the family f_1^ν, \dots, f_n^ν resulting from the 1-surgery of ν :

$$f_j^\nu(\mathbb{S}^1) \cap (D_1 \cup D_2) = \emptyset \quad \text{for } j = 1, \dots, n.$$

The quotient map $\pi: N^\nu \setminus (D_1 \cup D_2) \rightarrow N$ identifies the parts of the immersions f_1^ν, \dots, f_n^ν which are in the boundary $\partial D_1 \cup \partial D_2$. Therefore it induces a quotient map π_ν on the disjoint union $\coprod_{j=1}^n M_j^\nu \setminus F_j^{\nu-1}(D_1 \cup D_2)$ by identifying points $p_i \in \partial(M_i^\nu \setminus F_i^{\nu-1}(D_1 \cup D_2))$ and $p_j \in \partial(M_j^\nu \setminus F_j^{\nu-1}(D_1 \cup D_2))$ with

- $\pi(f_i^\nu(p_i)), \pi(f_j^\nu(p_j)) \subset \nu$ and
- $\pi(f_i^\nu(p_i)) = \pi(f_j^\nu(p_j))$.

Denote the image of π_ν by $M := \text{im}(\pi_\nu)$. Then a unique continuous function $\bar{F}: M \rightarrow N$ exists, such that the following diagram commutes:

$$\begin{array}{ccc} \coprod_{j=1}^n M_j^\nu \setminus F_j^{\nu-1}(D_1 \cup D_2) & \xrightarrow{\coprod_{j=1}^n f_j^\nu} & N^\nu \setminus (D_1 \cup D_2) \\ \pi_\nu \downarrow & & \downarrow \pi \\ M & \xrightarrow{\bar{F}} & N \end{array}$$

According to the Whitehead Approximation Theorem on Manifolds (see [Lee02], Theorem 10.21) \bar{F} is homotopic relative to ∂M to a differentiable function and the Sewing Theorem (see [Cou50], Theorem 2.5) assures the existence of an immersion $F: M \rightarrow N$.

It remains to show that $F|_{\partial M} = f$. Since $\pi_\nu^{-1}(\partial M) \subset \partial(\coprod_{j=1}^n M_j^\nu \setminus F_j^{\nu-1}(D_1 \cup D_2))$ the restriction to ∂M induces the commutative diagram

$$\begin{array}{ccc} \partial(\coprod_{j=1}^n M_j^\nu \setminus F_j^{\nu-1}(D_1 \cup D_2)) & \xrightarrow{\coprod_{j=1}^n f_j^\nu} & N^\nu \setminus (D_1 \cup D_2) \\ \pi_\nu \downarrow & & \downarrow \pi \\ \partial M & \xrightarrow{F|_{\partial M}} & N \end{array}$$

and therefore $F|_{\partial M} = f$. □

Suppose N has genus g_N . Then a sphere can be obtained from N by g_N successive 1-surgeries of generators ν_1, \dots, ν_{g_N} of $H_1(N)$. If $\nu_j \cap f(\mathbb{S}^1) = \emptyset$ then the 1-surgery has no effect on $f(\mathbb{S}^1)$ and hence $f: \mathbb{S}^1 \rightarrow N$ induces a normal immersion $f_1^\nu: \mathbb{S}^1 \rightarrow N^\nu$, with N^ν being the result of N and the 1-surgery of ν_j .

Hence if $\nu = (\nu_1, \dots, \nu_{g_N})$ is a sequence of generators with $\nu_j \cap f(\mathbb{S}^1) = \emptyset$ for each $j = 1, \dots, g_N$ then the normal immersion $f: \mathbb{S}^1 \rightarrow N$ induces a normal immersion f^ν by replacing the target surface N with \mathbb{S}^2 . In that case f has an extension $F: M \rightarrow N$ if and only if f^ν has an extension $F^\nu: M^\nu \rightarrow \mathbb{S}^2$. The extension F can be obtained from F^ν by a sequence of dual 0-surgeries $\nu^* := (\nu_{g_N}^*, \dots, \nu_1^*)$ of ν .

Each 1-surgery kills homology classes, i.e., $H_1(N^\nu) \triangleleft H_1(N)$. For this reason we have to assure, that $0 \neq [\nu_j] \in H_1(N^\nu)$ for $j = 2, \dots, g_N$:

Lemma 5.2.4. *Suppose N is a closed surface of genus $g_N > 0$. Then a sequence $\nu := (\nu_1, \dots, \nu_{g_N})$ of g_N generators of $H_1(N)$ exists with*

$$\langle \nu_i, \nu_j \rangle = 0 \quad \text{for } i \neq j \quad (7)$$

such that the 1-surgeries of ν_1, \dots, ν_{g_N} transform N into a sphere.

Proof. For each basis element $[\nu] \in H_1(N)$ exactly one basis element $[\bar{\nu}] \in H_1(N)$ exists such that $\langle \nu, \bar{\nu} \rangle = 1$, see [Ful95], Proposition 18.13. Hence a sequence of g_N generators ν_1, \dots, ν_{g_N} can be chosen, such that (7) is satisfied.

A 1-surgery of ν_1 gives a closed surface N^ν of genus $g_{N^\nu} = g_N - 1$. Thus the first homology group of N^ν has dimension $\dim(H_1(N^\nu)) = 2(g_N - 1)$. The basis elements which are killed by the 1-surgery of ν_1 are $[\nu_1]$ and the unique basis element $\bar{\nu}_1$ with $\langle \nu_1, \bar{\nu}_1 \rangle = 1$ (see [Ran02], Example 4.20).

Since $\langle \nu_1, \nu_2 \rangle = 0$, the basis element $[\nu_2] \in H_1(N)$ is still a basis element of $H_1(N^\nu)$. Therefore the sequence of 1-surgeries $\nu = (\nu_1, \dots, \nu_{g_N})$ leads to a closed surface of genus $g_N - g_N = 0$, i.e., a sphere. \square

Definition 5.2.5. Suppose N is a closed surface of genus $g_N > 0$. A sequence $\nu = (\nu_1, \dots, \nu_{g_N})$ of generators of $H_1(N)$ satisfying (7) is called a *surgery sequence of N* , if ν_j minimizes the total number of intersection points $\nu_j \cap f(\mathbb{S}^1)$ for all $j = 1, \dots, g_N$.

According to Lemma 5.2.4 a sphere is obtained after the 1-surgeries of ν . Corollary 5.2.2 shows, that each 1-surgery of ν_j induces a family $f_1^{\nu_j}, \dots, f_l^{\nu_j}$ of homologically trivial immersions. After a small perturbation these immersions are normal and hence Corollary 5.2.2 again shows, that the surgery sequence ν leads iteratively to a family $f_1^\nu, \dots, f_n^\nu: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ of immersion.

In terms of the family of immersions f_1^ν, \dots, f_n^ν it can be decided, whether f extends or not:

Proposition 5.2.6. *Suppose N is a closed surface of genus g_N and $f: \mathbb{S}^1 \rightarrow N$ is a homologically trivial normal immersion. Let ν be a surgery sequence and $f_1^\nu, \dots, f_n^\nu: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ the induced family of immersions.*

If each immersion $f_j^\nu: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ has an extension $F_j^\nu: M_j^\nu \rightarrow \mathbb{S}^2$, with M_j^ν a surface with boundary $\partial M_j^\nu = \mathbb{S}^1$, then a surface M exists, such that f has an extension to an immersion $F: M \rightarrow N$.

Proof. Denote by $f_1^{\nu \setminus \nu_{g_N}}, \dots, f_l^{\nu \setminus \nu_{g_N}}$ the family of immersions induced by the sequence of 1-surgeries of $\nu_1, \dots, \nu_{g_N-1}$. Then ν_{g_N} induces the family f_1^ν, \dots, f_n^ν from $f_1^{\nu \setminus \nu_{g_N}}, \dots, f_l^{\nu \setminus \nu_{g_N}}$. According to Proposition 5.2.3 the extensions F_j^ν and the dual 0-surgery $\nu_{g_N}^*$ induce extensions $F_i^{\nu \setminus \nu_{g_N}}$ of $f_i^{\nu \setminus \nu_{g_N}}$.

Thus the extensions $F_j^\nu: M_j^\nu \rightarrow N$ and the sequence $\nu_{g_N}^*, \dots, \nu_1^*$ of dual 0-surgeries induces an extension $F: M \rightarrow N$. \square

This leads to the main result of this section:

Theorem 5.2.7 (Extension Theorem for Immersed Surfaces in Closed Surfaces). *Suppose N is a closed surface and $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow N$ is a homologically trivial normal immersion. If f has a joining \mathcal{J} then f can be extended to a surface M with $\partial M = \coprod_{j=1}^m \mathbb{S}^1$, if the induced family $f_1^\nu, \dots, f_n^\nu: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ has extensions $F_j^\nu: M_j^\nu \rightarrow \mathbb{S}^2$.*

Proof. The joining \mathcal{J} and a surgery sequence ν induce the following sequence on the spaces of normal immersions:

$$\text{Imm}(\coprod_{j=1}^m \mathbb{S}^1, N) \xrightarrow{\mathcal{J}} \text{Imm}(\mathbb{S}^1, N) \xrightarrow{\nu} \coprod_{j=1}^n \text{Imm}(\mathbb{S}^1, \mathbb{S}^2)$$

If each immersion $f_j^\nu: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ has an extension $F_j^\nu: M_j^\nu \rightarrow \mathbb{S}^2$ then $f^*: \mathbb{S}^1 \rightarrow N$ has an extension $F^*: M^* \rightarrow N$ (Proposition 5.2.6). According to Proposition 4.1.6 the joining \mathcal{J} induces an extension $F: M \rightarrow N$ of f , where M is a surface with boundary $\partial M = \coprod_{j=1}^m \mathbb{S}^1$. \square

5.3 Uniqueness of Immersed Surfaces in Closed Surfaces

We have reduced the problem of extending immersions to closed surfaces N to the problem of extending immersions to the sphere. For that N is transformed into a sphere by a surgery sequence ν . But how does a change in the surgery sequence ν affect the extension of f ?

Since surgery is an operation on the target surface we expect a change in the surgery sequence not to have an effect on the extension.

Lemma 5.3.1. *Suppose N is a closed surface of genus $g_N > 0$ and $\nu, \bar{\nu}$ are generators of $H_1(N)$ with $\langle \nu, \bar{\nu} \rangle = 1$ and with an equal number of intersection points with $f(\mathbb{S}^1)$. Let $f: \mathbb{S}^1 \rightarrow N$ be a homologically trivial normal immersion and $f_1^\nu, \dots, f_n^\nu: \mathbb{S}^1 \rightarrow N'$ and $f_1^{\bar{\nu}}, \dots, f_n^{\bar{\nu}}: \mathbb{S}^1 \rightarrow N'$ the induced families of homologically trivial immersions to a closed surface N' of genus $g_{N'} = g_N - 1$.*

If the families have equivalent extensions, i.e., F_j^ν is equivalent to $F_j^{\bar{\nu}}$, then the induced extensions $F^\nu: M^\nu \rightarrow N$ and $F^{\bar{\nu}}: M^{\bar{\nu}} \rightarrow N$ of f are equivalent as well.

Proof. Since the induced families are equivalent for all $j = 1, \dots, n$, diffeomorphisms $\varphi_j: M_j^\nu \rightarrow M_j^{\bar{\nu}}$ exist, such that

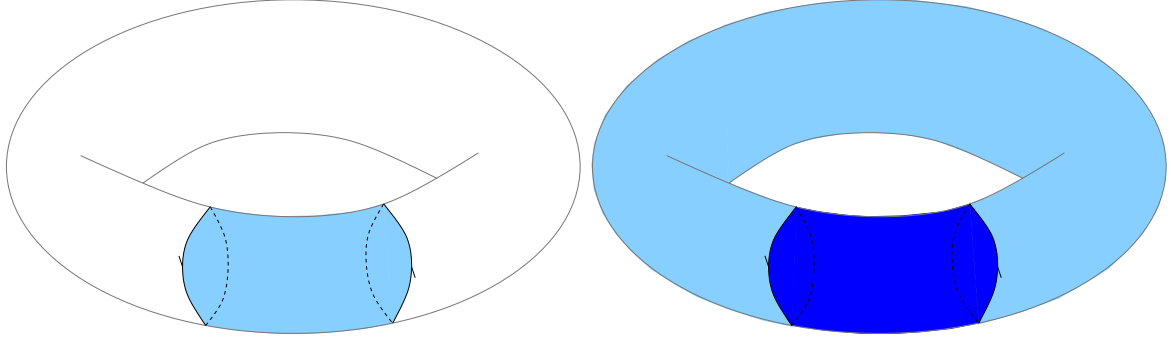
$$F_j^\nu = F_j^{\bar{\nu}} \circ \varphi_j, \quad \text{with } \varphi_j|_{\partial M_j^\nu} = \text{id}_{\partial M_j^\nu}.$$

Recall that the dual 0-surgery ν^* of ν induces a commutative diagram (Proposition 5.2.3):

$$\begin{array}{ccc} \coprod_{j=1}^n M_j^\nu \setminus F_j^{\nu^{-1}}(D_1 \cup D_2) & \xrightarrow{\coprod_{j=1}^n F_j^\nu} & N' \setminus (D_1 \cup D_2) \\ \pi_\nu \downarrow & & \downarrow \pi \\ M & \xrightarrow{F^\nu} & N \end{array}$$

Since $\bar{\nu}$ induces an equivalent diagram the quotient maps π_ν and $\pi_{\bar{\nu}}$ induce the commutative diagram

$$\begin{array}{ccc} \coprod_{j=1}^n M_j^\nu \setminus F_j^{\nu^{-1}}(D_1 \cup D_2) & \xrightarrow{\coprod_{j=1}^n \varphi_j} & \coprod_{j=1}^n M_j^{\bar{\nu}} \setminus F_j^{\bar{\nu}^{-1}}(D_1 \cup D_2) & (8) \\ \pi_\nu \downarrow & & \downarrow \pi_{\bar{\nu}} & \\ M^\nu & \xrightarrow{\varphi} & M^{\bar{\nu}} & \\ & \searrow F^\nu & \swarrow F^{\bar{\nu}} & \\ & & N & \end{array}$$



(a) An extension of f with one preimage in the small part. (b) An extension of f with two preimages in the small part.

Figure 44: Two extensions with different number of preimages

with $\varphi: M^\nu \rightarrow M^{\overline{\nu}}$ being a unique continuous function. Since $\coprod_{j=1}^n \varphi_j$ is a diffeomorphism φ is bijective according to the Whitehead Approximation Theorem on Manifolds (cf [Lee02], Theorem 10.21) it is homotopic rel. ∂M^ν to a diffeomorphism.

If we look at the restriction to the boundary, diagram (8) becomes

$$\begin{array}{ccc}
 \partial(\coprod_{j=1}^n M_j^\nu \setminus F_j^{\nu-1}(D_1 \cup D_2)) & \xrightarrow{\coprod_{j=1}^n \text{id}_{\partial M_j^\nu}} & \partial(\coprod_{j=1}^n M_j^{\overline{\nu}} \setminus F_j^{\overline{\nu}-1}(D_1 \cup D_2)) \\
 \pi_\nu \downarrow & & \downarrow \pi_{\overline{\nu}} \\
 \mathbb{S}^1 & \xrightarrow{\varphi|_{\mathbb{S}^1}} & \mathbb{S}^1 \\
 f \searrow & & \swarrow f \\
 & N &
 \end{array}$$

and hence $\varphi|_{\mathbb{S}^1} = \text{id}_{\mathbb{S}^1}$. Therefore the extensions F^ν and $F^{\overline{\nu}}$ are equivalent. \square

While in the simply connected case $N = \mathbb{S}^2$ the number of preimages of an extension $F: M \rightarrow \mathbb{S}^2$ of $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow \mathbb{S}^2$ depends only on f (Proposition 1.3.1), in the case of a closed surface N of genus $g_N > 0$ this is not true in general (Figure 44). Therefore the question arise how to distinguish extensions with a different number of preimages.

Suppose $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow N$ is a homologically trivial immersion and denote the components of $N \setminus f(\mathbb{S}^1)$ by X_1, \dots, X_k . Since $0 = [f] \in H_1(N)$ a 2-chain exists which has f as a boundary, i.e.,

$$f = \partial(n_1 X_1 + \dots + n_k X_k) = n_1 \partial X_1 + \dots + n_k \partial X_k.$$

If f has an extension $F: M \rightarrow N$, then

$$M = n_1 F^{-1}(X_1) + \dots + n_k F^{-1}(X_k).$$

Since a 2-chain $n_1 X_1 + \dots + n_k X_k$ is not unique in general, each 2-chain with boundary f induces an extension $F: M \rightarrow N$. These extensions can be distinguished by the first homology group $H_1(F(M))$.

Lemma 5.3.2. *Let N be a closed surface of genus $g_N > 0$, $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow N$ a homologically trivial normal immersion and denote the components of $N \setminus f(\coprod_{j=1}^m \mathbb{S}^1)$ by X_1, \dots, X_k . If $M := n_1 X_1 + \dots + n_k X_k$ and $\widetilde{M} := \widetilde{n}_1 X_1 + \dots + \widetilde{n}_k X_k$ are two different 2-chains with $\partial M = \partial \widetilde{M} = f$ then two nonequivalent extensions $F: M \rightarrow N$ and $\widetilde{F}: \widetilde{M} \rightarrow N$ exist.*

Proof. Since f is homologically trivial it has a normal numbering $\psi_n: \{X_1, \dots, X_k\} \rightarrow \mathbb{N}_0$ (Lemma 1.2.5), such that

$$n_j - n_i = \widetilde{n}_j - \widetilde{n}_i = \psi_n(X_j) - \psi_n(X_i).$$

Hence the coefficients differ by a constant c . W.l.o.g. assume that $n_j = \widetilde{n}_j + c$, then

$$\begin{aligned} M &= n_1 X_1 + \dots + n_k X_k \\ &= (\widetilde{n}_1 + c) X_1 + \dots + (\widetilde{n}_k + c) X_k \\ &= \widetilde{n}_1 X_1 + \dots + \widetilde{n}_k X_k + c(X_1 + \dots + X_k) \\ &= \widetilde{M} + cN. \end{aligned}$$

Since $\partial M = \partial \widetilde{M} = \coprod_{j=1}^m \mathbb{S}^1$ for the dimension of the first homology group of $F(M)$, the following equation is true:

$$\begin{aligned} 2g_{F(M)} + m - 1 &\stackrel{[\text{Hen79}]}{=} \dim(H_1(F(M))) \\ &= \dim(H_1(\widetilde{F}(\widetilde{M}))) + c \dim(H_1(N)) \stackrel{[\text{Hen79}]}{=} 2g_{\widetilde{F}(\widetilde{M})} + cg_N + m - 1. \end{aligned}$$

And since $g_N > 0$ the homology groups $H_1(F(M))$ and $H_1(\widetilde{F}(\widetilde{M}))$ are different. Therefore $F(M)$ and $\widetilde{F}(\widetilde{M})$ are not diffeomorphic (see [Bre97], Corollary IV.4.3), i.e., the extensions F and \widetilde{F} are not equivalent. \square

We have seen that joinings \mathcal{J} , the induced family f_1^ν, \dots, f_n^ν of a surgery sequence ν , and the first homology group $H_1(F(M))$ of the preimage distinguish between nonequivalent extensions.

Definition 5.3.3. Suppose N is a closed surface of genus $g_N > 0$ and $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow N$ a homologically trivial normal immersion, which has a joining \mathcal{J} and a surgery sequence ν , such that the induced family $f^\nu = (f_1^\nu, \dots, f_n^\nu): \mathbb{S}^1 \rightarrow \mathbb{S}^2$ has an extension $F^\nu = (F_1^\nu, \dots, F_n^\nu): M_j^\nu \rightarrow \mathbb{S}^2$. Denote the extension induced by \mathcal{J} and ν by $F: M \rightarrow N$. Then the triple $(\mathcal{J}, F^\nu, H_1(F(M)))$ is called an *extension triple of f* .

Two extension triples $(\mathcal{J}_1, F_1^{\nu_1}, H_1(F_1(M_1)))$ and $(\mathcal{J}_2, F_2^{\nu_2}, H_2(F_2(M_2)))$ are *equivalent* if \mathcal{J}_1 and \mathcal{J}_2 are isomorphic, $F_1^{\nu_1}$ and $F_2^{\nu_2}$ are equivalent and $H_1(F_1(M_1)) = H_1(F_2(M_2))$.

This leads to the Uniqueness Theorem in the general case:

Theorem 5.3.4 (Uniqueness Theorem for Immersed Surfaces in Closed Surfaces). *Suppose N is a closed surface of genus $g_N > 0$ and $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow N$ is a homologically trivial normal immersion. Then two different extension triples induce nonequivalent extensions.*

Proof. If $H_1(F_1(M_1)) \neq H_1(F_2(M_2))$ then $F_1(M_1)$ and $F_2(M_2)$ are not diffeomorphic, i.e., the extensions are not equivalent.

If the joinings are different, then Lemma 4.4.3 shows that F_1 and F_2 are not equivalent. If $H_1(F(M_1)) = H_1(F(M_2))$ and \mathcal{J}_1 is isomorphic to \mathcal{J}_2 , then the extensions of the induced families F_1^ν and F_2^ν are not equivalent. In this case Lemma 5.3.1 shows the claim. \square

Remark. The results of the present work can be summarized in terms of mappings between the spaces of normal immersions. If $\text{Imm}(\mathbb{S}^1, \mathbb{S}^2)$ denotes the space of normal immersions between the unit circle and the sphere and $\text{Imm}(\overline{\mathbb{D}}, \mathbb{S}^2)$ the space of normal immersions between the closed unit disc and the sphere then a grouping \mathcal{G} induces a mapping between them:

$$\begin{array}{c} \text{Imm}(\overline{\mathbb{D}}, \mathbb{S}^2) \\ \downarrow \mathcal{G} \\ \text{Imm}(\mathbb{S}^1, \mathbb{S}^2) \end{array}$$

Theorem 4.3.5 shows that a normal immersion $f: \coprod_{j=1}^m \mathbb{S}^1 \rightarrow \mathbb{S}^2$ has an extension $F: M \rightarrow \mathbb{S}^2$ to a surface M of genus $g_M = n$ and with boundary $\partial M = \coprod_{j=1}^m \mathbb{S}^1$, if a joining \mathcal{J} and an assembling \mathcal{A} with $|\mathcal{A}| = n$ exist, such that the induced normal immersion $\tilde{f}: \mathbb{S}^1 \rightarrow \mathbb{S}^2$ has an extension to an immersed disc. This induces the following commutative diagram:

$$\begin{array}{ccccc} \text{Imm}(M, \mathbb{S}^2) & \xrightarrow{\mathcal{J}} & \text{Imm}(M^*, \mathbb{S}^2) & \xrightarrow{\mathcal{A}} & \text{Imm}(\overline{\mathbb{D}}, \mathbb{S}^2) \\ \downarrow \text{Prop. 4.1.6} & & \downarrow \text{Prop. 4.2.4} & & \downarrow \text{Thm. 3.1.5} \\ \text{Imm}(\coprod_{j=1}^m \mathbb{S}^1, \mathbb{S}^2) & \xrightarrow{\mathcal{J}} & \text{Imm}(\mathbb{S}^1, \mathbb{S}^2) & \xrightarrow{\mathcal{A}} & \text{Imm}(\mathbb{S}^1, \mathbb{S}^2) \end{array}$$

The extension triple $(\mathcal{J}, F^\nu, H_1(F(M)))$ completes the characterization and finally leads to the following commutative diagram, which summarizes the results on the existence of extensions of the present work:

$$\begin{array}{ccccccc} \text{Imm}(M, N) & \xrightarrow{\mathcal{J}} & \text{Imm}(M^*, N) & \xrightarrow{\nu} & \text{Imm}(\widetilde{M}, \mathbb{S}^2) & \xrightarrow{\mathcal{A}} & \text{Imm}(\overline{\mathbb{D}}, \mathbb{S}^2) \\ \downarrow \text{Prop. 4.1.6} & & \downarrow \text{Prop. 5.2.6} & & \downarrow \text{Prop. 4.2.4} & & \downarrow \text{Thm. 3.1.5} \\ \text{Imm}(\coprod_{j=1}^m \mathbb{S}^1, N) & \xrightarrow{\mathcal{J}} & \text{Imm}(\mathbb{S}^1, N) & \xrightarrow{\nu} & \text{Imm}(\mathbb{S}^1, \mathbb{S}^2) & \xrightarrow{\mathcal{A}} & \text{Imm}(\mathbb{S}^1, \mathbb{S}^2) \end{array}$$

That the horizontal mappings are injective is shown in Theorems 4.4.5 (for the joining \mathcal{J} and the assembling \mathcal{A}), Theorem 5.3.4 (for the extensions F_j^ν) and Theorem 3.3.3 (for the grouping \mathcal{G}).

Since all spaces are spaces of normal immersion (and hence not connected) the vertical maps are covering maps. For that look at a normal immersion $f \in \text{Imm}(\mathbb{S}^1, \mathbb{S}^2)$. This immersion has the reduced word $w(f)$. This word is an invariant for the component since each regular homotopy which changes the word passes a nonnormal immersion.

That means that the number of extensions is locally constant for every neighbourhood U in $\text{Imm}(\mathbb{S}^1, \mathbb{S}^2)$ making the preimage of U a disjoint union of open sets where the number of sets equals the number of extensions.

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