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On Elastic Stability and Positive Definiteness for One-Dimensional Quasicrystals

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Abstract: In this work, a proper formulation of the elastic stability and positive definiteness of the elastic energy density of one-dimensional quasicrystals is provided. Based on an appropriate Voigt notation, the Sylvester criterion is applied and the necessary and sufficient conditions for the positive definiteness of the elastic energy density imposed on the elastic constants are derived for each Laue class from 4 to 10 of one-dimensional quasicrystals. Comparisons of the obtained sets of the necessary and sufficient conditions of one-dimensional quasicrystals with the corresponding ones of crystals are made, revealing that the phason fields, which characterize the quasicrystalline structure in any case, considerably influence the number as well as the degree of the obtained inequalities.

Keywords: elastic stability; positive definiteness; one-dimensional quasicrystals; Sylvester criterion; Voigt notation

1. Introduction

Quasicrystals, an interesting class of new or novel materials, were discovered by Shechtman in 1982 [1] who received the Nobel Prize in Chemistry for the discovery of quasicrystals in 2011. The study of quasicrystalline materials has been rapidly developed due to their particular and unique properties, among others low friction coefficient, low adhesion and high wear resistance, which have increased the interest in their application in technology. A review of many promising technological applications of quasicrystals, including hydrogen storage, catalysis, prosthetic biomaterials, optical absorbers and thermoelectric devices has been given by Maciá [2]. Another recent review focusing on quasicrystalline materials produced from various non-atom building blocks, their properties and potential applications ranging from photonic crystal metamaterials to architecture/art designs and time crystals is given by Nagaoka et al. [3].

Quasicrystals belong to aperiodic crystals and possess long-range orientational order but no translational symmetry in the quasiperiodic directions [1]. In the generalized elasticity theory of quasicrystals, there are three types of quasicrystals: one-dimensional, two-dimensional and three-dimensional quasicrystals [4]. In one-dimensional quasicrystals, there is a quasiperiodic arrangement of atoms in one direction, usually in the z-direction, and a regular periodic arrangement of atoms in the plane perpendicular to this direction, that is in the xy-plane. The basis of the continuum theory of solid quasicrystals is set up by two elementary excitations; the phonons and the phasons [5,6]. In the atomistic picture, phonons are related to the translation of atoms and phasons lead to local rearrangements of atoms in a cell. The elastic behavior of one-dimensional quasicrystals was first studied by Wang et al. [7], deducing 31 point groups, which are crystallographic ones, and they

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are divided into 6 systems (monoclinic, triclinic, orthorhombic, tetragonal, trigonal and hexagonal) and 10 Laue classes.

The concept of stability goes back to Born [8] and it is an essential property of a crystal. The elastic stability is expressed by the requirement of the positive definiteness of the elastic energy density, which leads to constraints in the form of inequalities for the elastic constants, the so-called stability conditions. If the elastic stability is not realized, then the crystal is also thermodynamically not stable. One can understand the significance of the elastic stability in the following sentence written by Born [8] about the stability condition: "If this condition is not satisfied the lattice cannot be considered as a thermodynamical system; it would then break up into small parts, molecules, which form a liquid or gas, but not a crystal". Conditions for elastic stability and/or positive definiteness for crystals can be found in Nye [9], Fedorov [10], Ting [11,12] and Mouhat and Coudert [13].

The investigation of elastic stability and positive definiteness is also of high importance for the study of the behavior of quasicrystalline materials. A violation of the stability conditions results in a phase transformation [14]. The conditions of elastic stability for three-dimensional icosahedral quasicrystals were derived by Levine et al. [6], Widom [14] and Ricker et al. [15]. The conditions of positive definiteness for all two-dimensional quasicrystals with non-crystallographic symmetries were given by Wang et al. [16]. By using the molecular dynamics method, Tian et al. [17] simulated the mechanical properties of decagonal quasicrystal approximate phases Al_2Fe deriving the conditions of stability for tetragonal and orthorhombic crystal systems, which are, however, not the correct ones, as one can see, for instance, in [10,13]. The necessary and sufficient conditions for the elastic stability of one-dimensional quasicrystals obtained in Liu et al. [18], using actually the Sylvester criterion, are erroneous, since the contraction of the stiffness matrix of the elastic constants by using the Voigt notation was not properly performed. It should be noted that for the application of the Sylvester criterion, one needs to use the Voigt notation, which gives the possibility to represent (contract), for example, in the context of classical elasticity, symmetric second-rank tensors into column vectors and analogously fourth-rank elasticity tensors into 6×6 symmetric square matrices (see, e.g., [19]). In Liu et al. [18], although the fourth-rank tensor of the elastic constants of phonons is contracted to a second-rank matrix, the fourth-rank tensors of the elastic constants of phonon–phason coupling and of the elastic constants of phasons "are simply reduced to one order tensors" as written in [18], leading thereby to misleading results. A proper and consequent formulation of the Voigt notation in the context of generalized elasticity of one-dimensional quasicrystals in order to apply the Sylvester criterion for the derivation of the necessary and sufficient conditions of positive definiteness is given in this work.

This paper is devoted to the elastic stability and positive definiteness of one-dimensional quasicrystals, in the sense that the elastic energy density is positive definite. The paper is organized as follows. In Section 2, the basic equations of generalized elasticity of one-dimensional quasicrystals are given. A proper formulation for the elastic stability and positive definiteness of one-dimensional quasicrystals is provided in Section 3 using an appropriate Voigt notation. In Section 4, by applying the Sylvester criterion, the sets of the necessary and sufficient conditions of stability and positive definiteness imposed on the elastic constants are derived for each Laue class from 4 to 10 of one-dimensional quasicrystals. The article concludes with important remarks concerning the results in Section 5.

2. Basics of One-Dimensional Quasicrystals

In this section, we provide the basic equations of the linear generalized compatible elasticity theory of one-dimensional quasicrystals.

A one-dimensional quasicrystal can be generated by the projection of a four-dimensional periodic structure to the three-dimensional physical space. The four-dimensional hyperspace E^4 can be decomposed into the direct sum of two orthogonal subspaces

$$E^4 = E^3_{\parallel} \oplus E^1_{\perp} \,, \tag{1}$$

where $E^3_{\parallel}$ is the three-dimensional physical or parallel space of the phonon fields and $E^1_{\perp}$ is the one-dimensional perpendicular space of the phason fields with quasiperiodicity in the z-direction. Throughout the text, phonon fields will be denoted by $(\cdot)^{\parallel}$ and phason fields by $(\cdot)^{\perp}$. Note that all quantities (phonon and phason fields) depend on the so-called material space coordinates $\boldsymbol{x} \in \mathbb{R}^3$. Indices in the parallel space are denoted by small letters i, j, k, l with $i, j, k, l = 1, 2, 3$.

In the theory of generalized elasticity of one-dimensional quasicrystals, the *phonon displacement field* is denoted by $u^{\parallel}_k, k = 1, 2, 3$ and the *phason displacement field*, which is in the z-direction, is denoted by $u^{\perp}_3$. That is, the phason displacement field in the case of one-dimensional quasicrystals is a scalar field. The *phonon and phason distortion tensors*, $\beta^{\parallel}_{kl}$ and $\beta^{\perp}_{3l}$, are defined as the spatial gradients of $u^{\parallel}_k$ and $u^{\perp}_3$, respectively

$$\beta^{\parallel}_{kl} = \partial_l u^{\parallel}_k \,, \tag{2}$$

$$\beta^{\perp}_{3l} = \partial_l u^{\perp}_3 \,, \tag{3}$$

where $\partial_l = \partial/\partial x_l$ indicates the derivative with respect to the spatial coordinates x_l. The phonon distortion tensor $\beta^{\parallel}_{kl}$ is a tensor of rank two and the phason distortion tensor $\beta^{\perp}_{3l}$ is a tensor of rank one and transforms like a vector. The *phonon strain tensor*, which is the symmetric part of the phonon distortion tensor, is given by

$$e^{\parallel}_{kl} = \frac{1}{2}\,(\partial_l u^{\parallel}_k + \partial_k u^{\parallel}_l)\,. \tag{4}$$

It should be noticed that a corresponding quantity for the phason distortion tensor, that is a phason strain tensor, cannot be defined, since the phason distortion tensor is a two-point tensor with the index 3 “living” in the perpendicular space and the index l “living” in the parallel space (see also [20,21]). In the literature of the linear generalized elasticity theory of quasicrystals, the terminology phason strain tensor is often misleadingly used for the phason distortion tensor.

Therefore, the *elastic energy density W for one-dimensional quasicrystals* is given by the following quadratic form in terms of the phonon strain tensor $e^{\parallel}_{ij}$ and the phason distortion $\beta^{\perp}_{3j}$

$$W = W(e^{\parallel}_{ij}, \beta^{\perp}_{3j}) = \frac{1}{2}\, C_{ijkl} e^{\parallel}_{ij} e^{\parallel}_{kl} + D_{ij3l} e^{\parallel}_{ij} \beta^{\perp}_{3l} + \frac{1}{2}\, E_{3j3l} \beta^{\perp}_{3j} \beta^{\perp}_{3l} \,, \tag{5}$$

where C_{ijkl} is the *tensor of the elastic moduli of phonons*, E_{3j3l} is the *tensor of the elastic moduli of phasons* and D_{ij3l} is the *tensor of the elastic moduli of the phonon–phason coupling* for one-dimensional quasicrystals. The constitutive tensors possess the following *major symmetries* (see [22])

$$C_{ijkl} = C_{klij} \,, \qquad E_{3j3l} = E_{3l3j} \tag{6}$$

and *minor symmetries*

$$C_{ijkl} = C_{jikl} = C_{ijlk} \,, \qquad D_{ij3l} = D_{ji3l} \,. \tag{7}$$

By using the minor symmetry of D_{ij3l}, Equation (7), the elastic energy density (5) can be written as follows

$$W = W(e^{\|}_{ij}, \beta^{\perp}_{3j}) = \frac{1}{2}\, C_{ijkl} e^{\|}_{ij} e^{\|}_{kl} + \frac{1}{2}\, D_{ij3l} e^{\|}_{ij} \beta^{\perp}_{3l} + \frac{1}{2}\, D_{kl3j} e^{\|}_{kl} \beta^{\perp}_{3j} + \frac{1}{2}\, E_{3j3l} \beta^{\perp}_{3j} \beta^{\perp}_{3l}\,, \tag{8}$$

or in matrix form

$$W = \frac{1}{2}\, \big(e^{\|}_{ij}, \beta^{\perp}_{3j}\big) \begin{pmatrix} C_{ijkl} & D_{ij3l} \\ D_{kl3j} & E_{3j3l} \end{pmatrix} \begin{pmatrix} e^{\|}_{kl} \\ \beta^{\perp}_{3l} \end{pmatrix}. \tag{9}$$

The associated linear constitutive relations are given by

$$\sigma^{\|}_{ij} = \frac{\partial W}{\partial e^{\|}_{ij}} = C_{ijkl} e^{\|}_{kl} + D_{ij3l} \beta^{\perp}_{3l}\,, \tag{10}$$

$$\sigma^{\perp}_{3j} = \frac{\partial W}{\partial \beta^{\perp}_{3j}} = D_{kl3j} e^{\|}_{kl} + E_{3j3l} \beta^{\perp}_{3l}\,, \tag{11}$$

where $\sigma^{\|}_{ij}$ denotes the *phonon stress tensor* and $\sigma^{\perp}_{3j}$ are the *phason stresses*. The phonon stress tensor is a symmetric tensor of rank two: $\sigma^{\|}_{ij} = \sigma^{\|}_{ji}$. Using Equations (10) and (11), the elastic energy density (8) can be written as

$$W = \frac{1}{2}\, \sigma^{\|}_{ij} e^{\|}_{ij} + \frac{1}{2}\, \sigma^{\perp}_{3j} \beta^{\perp}_{3j}\,, \tag{12}$$

or equivalently in terms of the phonon and phason distortion tensors

$$W = \frac{1}{2}\, \sigma^{\|}_{ij} \beta^{\|}_{ij} + \frac{1}{2}\, \sigma^{\perp}_{3j} \beta^{\perp}_{3j}\,. \tag{13}$$

The equilibrium conditions in the absence of body force densities are of the form (see, e.g., [22,23])

$$\partial_j \sigma^{\|}_{ij} = 0\,, \tag{14}$$

$$\partial_j \sigma^{\perp}_{3j} = 0\,. \tag{15}$$

Substituting Equations (3), (4), (10) and (11) into Equations (14) and (15), we obtain the *coupled homogeneous anisotropic Navier equations for the phonon and phason displacement fields*

$$C_{ijkl} \partial_j \partial_l u^{\|}_k + D_{ij3l} \partial_j \partial_l u^{\perp}_3 = 0\,, \tag{16}$$

$$D_{kl3j} \partial_j \partial_l u^{\|}_k + E_{3j3l} \partial_j \partial_l u^{\perp}_3 = 0\,. \tag{17}$$

3. Proper Formulation of Elastic Stability and Positive Definiteness for One-Dimensional Quasicrystals

In this section, we provide an appropriate formulation of the elastic stability of one-dimensional quasicrystals in the sense that the elastic energy density is positive definite.

There are different methods or criteria in order to check the positive definiteness of a quadratic form, with the most used being the eigenvalue method and the Sylvester criterion. For the eigenvalue method, one needs to use the normalized Voigt notation [24] or Mandel notation [25] or Kelvin notation [19], whereas for the Sylvester criterion one can use the Voigt notation or the normalized Voigt notation and usually the former is preferred, since it is simpler to be used. The advantage of the eigenvalue method is that it provides the number of the independent eigenvalues and consequently the independent necessary and sufficient conditions for the positive definiteness of a quadratic form. However, it often presents the difficulty that the obtained eigenvalues are not in explicit form. This is the reason that the eigenvalue method will not be used here, since the obtained results are extremely lengthy even in the simplest case of one-dimensional hexagonal quasicrystals of Laue class 10, making them rather unusable. In contrast to the eigenvalue method, it can be seen that in the case of one-dimensional quasicrystals, the Sylvester criterion works very well, since it delivers explicit results, which can be easily handled.

Following Eringen [26], we give the definition of elastic stability generalized towards one-dimensional quasicrystals (in the absence of external loads, see [13]).

Definition 1. *A one-dimensional quasicrystal is said to be stable if and only if the elastic energy density* (8) *is non-negative for all phonon strains and phason distortions. Hence, the stability condition is expressed by the positive semi-definiteness of the elastic energy density*

$$W(e^{\|}_{ij},\beta^{\perp}_{3j}) \geq 0\,, \quad \textit{for all} \quad e^{\|}_{ij} \quad \textit{and} \quad \beta^{\perp}_{3j}\,. \tag{18}$$

This definition has to be understood in the sense that it includes also the case $W = 0$, which corresponds to the nature or undeformed state and this is the state of stable equilibrium and $W > 0$ when there is any deformation (see Fedorov [10]). Then, the condition for elastic stability (18) can be alternatively written in the common known expression for positive definite elastic energy density for nonzero phonon strains and nonzero phason distortions as follows (see, e.g., Nye [9], Ting [12])

$$W(e^{\|}_{ij},\beta^{\perp}_{3j}) > 0\,, \quad \text{for every} \quad e^{\|}_{ij} \neq 0 \quad \text{and} \quad \beta^{\perp}_{3j} \neq 0\,. \tag{19}$$

The six independent components of $e^{\|}_{ij}$ and the three independent components of $\beta^{\perp}_{3j}$ can be represented by the components of a single column vector $\mathcal{B}_\nu$, $\nu = 1,\ldots,9$ in a 9-dimensional space as follows

$$\begin{aligned} \mathcal{B}_\nu &= \big(\mathcal{B}_1,\ldots,\mathcal{B}_9\big)^{\mathrm{T}} \\ &= \big(e^{\|}_{11}, e^{\|}_{22}, e^{\|}_{33}, 2e^{\|}_{23}, 2e^{\|}_{13}, 2e^{\|}_{12}, \beta^{\perp}_{31}, \beta^{\perp}_{32}, \beta^{\perp}_{33}\big)^{\mathrm{T}}\,. \end{aligned} \tag{20}$$

Here, for the phonon strain components the classical Voigt notation is used in the order of $11, 22, 33, 23, 13, 12$ (see, e.g., [9]) and for the phason distortion components $\beta^{\perp}_{3j}$ we choose the natural order $31, 32, 33$. This order is important for the contraction that follows and for the order of the components of the constitutive tensors in the corresponding matrices.

From the elastic energy density (9), the matrix of the elastic moduli, which is a symmetric 9×9 matrix

$$\mathcal{C}_{\mu\nu} = \begin{pmatrix} C_{ijkl} & D_{ij3l} \\ D_{kl3j} & E_{3j3l} \end{pmatrix} = \begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} & D_{1131} & D_{1132} & D_{1133} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} & D_{2231} & D_{2232} & D_{2233} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3313} & C_{3312} & D_{3331} & D_{3332} & D_{3333} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2313} & C_{2312} & D_{2331} & D_{2332} & D_{2333} \\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1313} & C_{1312} & D_{1331} & D_{1332} & D_{1333} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212} & D_{1231} & D_{1232} & D_{1233} \\ D_{1131} & D_{2231} & D_{3331} & D_{2331} & D_{1331} & D_{1231} & E_{3131} & E_{3132} & E_{3133} \\ D_{1132} & D_{2232} & D_{3332} & D_{2332} & D_{1332} & D_{1232} & E_{3132} & E_{3232} & E_{3233} \\ D_{1133} & D_{2233} & D_{3333} & D_{2333} & D_{1333} & D_{1233} & E_{3133} & E_{3233} & E_{3333} \end{pmatrix} \tag{21}$$

can be written in its contracted form $\mathcal{C}_{\mu\nu}, \mu, \nu = 1, \ldots, 9$ in Voigt notation as follows

$$\mathcal{C}_{\mu\nu} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} & D_{11} & D_{12} & D_{13} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} & D_{21} & D_{22} & D_{23} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} & D_{31} & D_{32} & D_{33} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} & D_{41} & D_{42} & D_{43} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} & D_{51} & D_{52} & D_{53} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} & D_{61} & D_{62} & D_{63} \\ D_{11} & D_{21} & D_{31} & D_{41} & D_{51} & D_{61} & E_{11} & E_{12} & E_{13} \\ D_{12} & D_{22} & D_{32} & D_{42} & D_{52} & D_{62} & E_{12} & E_{22} & E_{23} \\ D_{13} & D_{23} & D_{33} & D_{43} & D_{53} & D_{63} & E_{13} & E_{23} & E_{33} \end{pmatrix}. \tag{22}$$

In the representation of the elastic moduli (21), the classical 6 × 6 matrix form of C_{ijkl} was used along with the 6 × 3 matrix form of D_{ij3l}, the 3 × 6 matrix form of D_{kl3j} and the 3 × 3 matrix form of E_{3j3l} given in Agiasofitou and Lazar [27]. The matrix $\mathcal{C}_{\mu\nu}$ in Voigt notation, Equation (22), is a real symmetric matrix, $\mathcal{C}_{\mu\nu} = \mathcal{C}_{\nu\mu}$, and will be used in the study that follows.

Employing the contracted notations, the elastic energy density, Equation (9), can be written in the following real symmetric quadratic form

$$W = \frac{1}{2}\, \mathcal{B}_{\mu}\, \mathcal{C}_{\mu\nu}\, \mathcal{B}_{\nu}\,, \qquad \mu, \nu = 1, \ldots, 9\,. \tag{23}$$

A quadratic form $\mathcal{B}_{\mu}\, \mathcal{C}_{\mu\nu}\, \mathcal{B}_{\nu}$ is called a *positive definite form* if it is in general positive and can be zero only if all the $\mathcal{B}_{\mu}$ are zero [28].

According to the *Sylvester criterion*: "A set of necessary and sufficient conditions for a real symmetric quadratic form to be positive definite is that all the leading principal minors of the real symmetric matrix $\mathcal{C}_{\mu\nu}$ are positive" (see, e.g., [11,29]).

For reasons of clarity, we describe here the construction of a leading principal minor as it is given, for instance, in Ting [11]. If an equal number of rows and columns is removed from a symmetric matrix $\boldsymbol{C}$, a square submatrix is obtained. The determinant of such a submatrix is called a *minor*. If the rows and columns removed have the same indices, the diagonal elements of the submatrix are the diagonal elements of the original matrix. Such an obtained submatrix is called a *principal submatrix* and its determinant is a *principal minor*. If the $r \times r$ principal submatrix is from the first r rows and first r columns of $\boldsymbol{C}$, then we have a *leading principal submatrix*. The determinant of a leading principal submatrix is a *leading principal minor*. The matrix $\mathcal{C}_{\mu\nu}$, Equation (22), is a 9 × 9 matrix and therefore has nine leading principal minors.

It has to be noted that in Liu et al. [18], the fourth-rank tensors of the elastic constants of the phonon–phason coupling and of the phasons, R_{ijkl} and K_{ijkl}, respectively, "are simply reduced to one order tensors" R_i, $i = 1, \ldots, 18$ and K_i, $i = 1, \ldots, 6$, respectively, instead of being contracted to second-rank matrices according to the Voigt notation and as was done for the fourth-rank tensor of the elastic constants of phonons C_{ijkl}.

4. Sets of Conditions of Positive Definiteness

It is significant to highlight that as far as the elastic behavior of one-dimensional quasicrystals is concerned, according to Wang et al. [7] the elastic behavior is the same for point groups belonging to the same Laue class. Therefore, the number of independent elastic constants depends only on the considered Laue class. We would like to note at this point that this is not the case, for instance, when the piezoelectric behavior is examined, since the piezoelectric properties are not the same for point groups belonging to the same Laue class (see, e.g., [27]). For the needs of elasticity of one-dimensional quasicrystals, it is enough to examine the Laue classes. For that reason, in this section, we apply the Sylvester criterion for all Laue classes of one-dimensional quasicrystals from 4 to 10, since to every Laue class corresponds a different matrix $\mathcal{C}_{\mu\nu}$ of the elastic constants. Each one of these matrices $\mathcal{C}_{\mu\nu}$ is real and symmetric.

Let us start with the case of one-dimensional quasicrystals with the highest symmetry, that is the hexagonal symmetry, and therefore the lowest number of independent elastic constants. In particular, one-dimensional hexagonal quasicrystals of Laue class 10 are the most studied one-dimensional quasicrystals to date, for which one can find values for the elastic constants based on simulations (see, e.g., [30–32]).

4.1. One-Dimensional Hexagonal Quasicrystals of Laue Class 10

A one-dimensional hexagonal quasicrystal of Laue class 10 possesses 10 independent elastic moduli [7]. In addition, a one-dimensional hexagonal quasicrystal of Laue class 10 is a transversely isotropic medium; namely, it is isotropic in the basal xy-plane (see, e.g., [21]).

If we use the classical 6 × 6 matrix form of C_{ijkl} for hexagonal crystals given in Nye [9], the 6 × 3 matrix form of D_{ij3l} and the 3 × 3 matrix form of E_{3j3l} given in Agiasofitou and Lazar [27] for one-dimensional hexagonal quasicrystals of Laue class 10, then the matrix $\mathcal{C}_{\mu\nu}$, Equation (22), reduces to the following 9 × 9 matrix in Voigt notation

$$\mathcal{C}_{\mu\nu} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 & 0 & 0 & D_{13} \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 & 0 & 0 & D_{13} \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 & 0 & 0 & D_{33} \\ 0 & 0 & 0 & C_{44} & 0 & 0 & 0 & D_{42} & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 & D_{42} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{42} & 0 & E_{11} & 0 & 0 \\ 0 & 0 & 0 & D_{42} & 0 & 0 & 0 & E_{11} & 0 \\ D_{13} & D_{13} & D_{33} & 0 & 0 & 0 & 0 & 0 & E_{33} \end{pmatrix} \tag{24}$$

with $C_{66} = (C_{11} - C_{12})/2$. The application of the Sylvester criterion to the 9 × 9 matrix (24), which is real and symmetric, leads to the following nine leading principal minors:

$$\mathcal{D}_1 = C_{11}\,, \tag{25}$$

$$\mathcal{D}_2 = (C_{11} - C_{12})(C_{11} + C_{12})\,, \tag{26}$$

$$\mathcal{D}_3 = (C_{11} - C_{12})(C_{33}C_{11} + C_{33}C_{12} - 2C_{13}^2)\,, \tag{27}$$

$$\begin{aligned} \mathcal{D}_4 &= C_{44}(C_{11} - C_{12})(C_{33}C_{11} + C_{33}C_{12} - 2C_{13}^2) \\ &= C_{44}\,\mathcal{D}_3\,, \end{aligned} \tag{28}$$

$$\begin{aligned} \mathcal{D}_5 &= C_{44}^2(C_{11} - C_{12})(C_{33}C_{11} + C_{33}C_{12} - 2C_{13}^2) \\ &= C_{44}\,\mathcal{D}_4 = C_{44}^2\,\mathcal{D}_3\,, \end{aligned} \tag{29}$$

$$\mathcal{D}_6 = \frac{1}{2}\,C_{44}^2(C_{11} - C_{12})^2(C_{33}C_{11} + C_{33}C_{12} - 2C_{13}^2)\,, \tag{30}$$

$$\mathcal{D}_7 = \frac{1}{2}\,C_{44}(C_{11} - C_{12})^2(C_{44}E_{11} - D_{42}^2)(C_{33}C_{11} + C_{33}C_{12} - 2C_{13}^2)\,, \tag{31}$$

$$\mathcal{D}_8 = \frac{1}{2}\,(C_{11} - C_{12})^2(C_{44}E_{11} - D_{42}^2)^2(C_{33}C_{11} + C_{33}C_{12} - 2C_{13}^2)\,, \tag{32}$$

$$\begin{aligned} \mathcal{D}_9 = \frac{1}{2}\,&(C_{11} - C_{12})^2(C_{44}E_{11} - D_{42}^2)^2 \\ &\Big\{\big[(C_{11} + C_{12})C_{33} - 2C_{13}^2\big]E_{33} - (C_{11} + C_{12})D_{33}^2 - 2(C_{33}D_{13} - 2C_{13}D_{33})D_{13}\Big\}\,. \end{aligned} \tag{33}$$

All leading principal minors $\mathcal{D}_1, \ldots, \mathcal{D}_9$, Equations (25)–(33), should be positive, initially leading to seven conditions:

$$C_{11} > 0\,, \quad C_{11} > C_{12}\,, \quad C_{11} > -C_{12}\,, \quad (C_{11} + C_{12})C_{33} - 2C_{13}^2 > 0\,, \quad C_{44} > 0\,, \tag{34}$$

$$C_{44}E_{11} - D_{42}^2 > 0\,, \tag{35}$$

$$\big[(C_{11} + C_{12})C_{33} - 2C_{13}^2\big]E_{33} - (C_{11} + C_{12})D_{33}^2 - 2(C_{33}D_{13} - 2C_{13}D_{33})D_{13} > 0\,. \tag{36}$$

The second and third condition of inequalities (34) can be compactly written as $C_{11} > |C_{12}|$, which already includes the first condition $C_{11} > 0$ of inequalities (34). In this sense, the first condition is redundant. The inequality $C_{11} > |C_{12}|$ is accounted as two conditions.

Therefore, the necessary and sufficient conditions which have to be fulfilled by the elastic constants so that the corresponding elastic energy density is positive definite are given by the following six inequalities:

$$C_{11} > |C_{12}|\,, \qquad (C_{11} + C_{12})C_{33} - 2C_{13}^2 > 0\,, \qquad C_{44} > 0\,, \tag{37}$$

$$C_{44}E_{11} - D_{42}^2 > 0\,, \tag{38}$$

$$\big[(C_{11} + C_{12})C_{33} - 2C_{13}^2\big]E_{33} - (C_{11} + C_{12})D_{33}^2 - 2(C_{33}D_{13} - 2C_{13}D_{33})D_{13} > 0\,. \tag{39}$$

Inequalities (37)–(39) are *the 6 necessary and sufficient conditions for elastic stability and positive definiteness for one-dimensional hexagonal quasicrystals of Laue class 10.*

It should be noticed that the first four conditions (37), which have to be satisfied only by the elastic moduli of phonons, are the same as those for hexagonal crystals [9,10,13] and thereby the limit to classical elasticity in the absence of phason fields is recovered.

In the literature of quasicrystals, the notation introduced by Ding et al. [22] (see also [7]) is often used and for that reason we rewrite the above-obtained conditions in this notation by employing the following relations connecting our notation with the notation introduced by Ding et al. [22]

$$D_{13} = R_1\,, \quad D_{33} = R_2\,, \quad D_{41} = R_4\,, \quad D_{42} = R_3\,, \tag{40}$$

$$E_{33} = K_1\,, \quad E_{11} = K_2\,. \tag{41}$$

Then, the necessary and sufficient conditions for elastic stability and positive definiteness for one-dimensional hexagonal quasicrystals of Laue class 10, Inequalities (37)–(39), are alternatively written as follows:

$$C_{11} > |C_{12}|\,, \qquad (C_{11} + C_{12})C_{33} - 2C_{13}^2 > 0\,, \qquad C_{44} > 0\,, \tag{42}$$

$$C_{44}K_2 - R_3^2 > 0\,, \tag{43}$$

$$\big[(C_{11} + C_{12})C_{33} - 2C_{13}^2\big]K_1 - (C_{11} + C_{12})R_2^2 - 2(C_{33}R_1 - 2C_{13}R_2)R_1 > 0\,. \tag{44}$$

Conditions (42)–(44) have been already given in [33] in order to check that the values of the elastic constants given in Li et al. [32] fulfill these conditions and to use them for the numerical implementation of the 10 independent components of the three-dimensional 4 × 4 Green tensor for one-dimensional hexagonal quasicrystals of Laue class 10, which has been given in a closed-form expression in [33].

4.2. One-Dimensional Hexagonal Quasicrystals of Laue Class 9

One-dimensional hexagonal quasicrystals of Laue class 9 possess 11 independent elastic moduli. If we use the classical 6×6 matrix form of C_{ijkl} for classical hexagonal crystals given in Nye [9], the 6×3 matrix form of D_{ij3l} and the 3×3 matrix form of E_{3j3l} given in Agiasofitou and Lazar [27] for one-dimensional hexagonal quasicrystals of

Laue class 9, then the matrix $\mathcal{C}_{\mu\nu}$, Equation (22), reduces to the following 9×9 matrix in Voigt notation

$$\mathcal{C}_{\mu\nu} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 & 0 & 0 & D_{13} \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 & 0 & 0 & D_{13} \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 & 0 & 0 & D_{33} \\ 0 & 0 & 0 & C_{44} & 0 & 0 & D_{41} & D_{42} & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 & D_{42} & -D_{41} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{41} & D_{42} & 0 & E_{11} & 0 & 0 \\ 0 & 0 & 0 & D_{42} & -D_{41} & 0 & 0 & E_{11} & 0 \\ D_{13} & D_{13} & D_{33} & 0 & 0 & 0 & 0 & 0 & E_{33} \end{pmatrix} \tag{45}$$

with $C_{66} = (C_{11} - C_{12})/2$. The nine leading principal minors corresponding to the 9×9 matrix (45) are:

$$\mathcal{D}_1 = C_{11}\,, \tag{46}$$

$$\mathcal{D}_2 = (C_{11} - C_{12})(C_{11} + C_{12})\,, \tag{47}$$

$$\mathcal{D}_3 = (C_{11} - C_{12})(C_{33}C_{11} + C_{33}C_{12} - 2C_{13}^2)\,, \tag{48}$$

$$\begin{aligned} \mathcal{D}_4 &= C_{44}\,(C_{11} - C_{12})(C_{33}C_{11} + C_{33}C_{12} - 2C_{13}^2) \\ &= C_{44}\,\mathcal{D}_3\,, \end{aligned} \tag{49}$$

$$\begin{aligned} \mathcal{D}_5 &= C_{44}^2\,(C_{11} - C_{12})(C_{33}C_{11} + C_{33}C_{12} - 2C_{13}^2) \\ &= C_{44}\,\mathcal{D}_4 = C_{44}^2\,\mathcal{D}_3\,, \end{aligned} \tag{50}$$

$$\mathcal{D}_6 = \frac{1}{2}\,C_{44}^2\,(C_{11} - C_{12})^2(C_{33}C_{11} + C_{33}C_{12} - 2C_{13}^2)\,, \tag{51}$$

$$\mathcal{D}_7 = \frac{1}{2}\,C_{44}\,(C_{11} - C_{12})^2(C_{44}E_{11} - D_{42}^2 - D_{41}^2)(C_{33}C_{11} + C_{33}C_{12} - 2C_{13}^2)\,, \tag{52}$$

$$\mathcal{D}_8 = \frac{1}{2}\,(C_{11} - C_{12})^2(C_{44}E_{11} - D_{42}^2 - D_{41}^2)^2(C_{33}C_{11} + C_{33}C_{12} - 2C_{13}^2)\,, \tag{53}$$

$$\begin{aligned} \mathcal{D}_9 = \frac{1}{2}\,&(C_{11} - C_{12})^2(C_{44}E_{11} - D_{42}^2 - D_{41}^2)^2 \\ &\Big\{\big[(C_{11} + C_{12})C_{33} - 2C_{13}^2\big]E_{33} - (C_{11} + C_{12})D_{33}^2 - 2(C_{33}D_{13} - 2C_{13}D_{33})D_{13}\Big\}\,. \end{aligned} \tag{54}$$

The application of the Sylvester criterion leads to the following six conditions:

$$C_{11} > |C_{12}|\,, \qquad (C_{11} + C_{12})C_{33} - 2C_{13}^2 > 0\,, \qquad C_{44} > 0\,, \tag{55}$$

$$C_{44}E_{11} - \big(D_{41}^2 + D_{42}^2\big) > 0\,, \tag{56}$$

$$\big[(C_{11} + C_{12})C_{33} - 2C_{13}^2\big]E_{33} - (C_{11} + C_{12})D_{33}^2 - 2(C_{33}D_{13} - 2C_{13}D_{33})D_{13} > 0\,. \tag{57}$$

Inequalities (55)–(57) are *the 6 necessary and sufficient conditions for elastic stability and positive definiteness for one-dimensional hexagonal quasicrystals of Laue class 9.*

In the absence of phason fields, one recovers the classical conditions for hexagonal crystal systems, that is, inequalities (55) (see, e.g., [9,10,13]).

Using the relations (40) and (41), inequalities (55)–(57) are alternatively written:

$$C_{11} > |C_{12}|\,, \qquad (C_{11} + C_{12})C_{33} - 2C_{13}^2 > 0\,, \qquad C_{44} > 0\,, \tag{58}$$

$$C_{44}K_2 - \big(R_3^2 + R_4^2\big) > 0\,, \tag{59}$$

$$\big[(C_{11} + C_{12})C_{33} - 2C_{13}^2\big]K_1 - (C_{11} + C_{12})R_2^2 - 2(C_{33}R_1 - 2C_{13}R_2)R_1 > 0\,. \tag{60}$$

An interesting observation is that the highest degree of the polynomials in the obtained inequalities for one-dimensional hexagonal quasicrystals for both Laue classes 10 and 9 is three. Therefore, in the case of one-dimensional hexagonal quasicrystals one has to deal with a cubic system of algebraic inequalities (cubic stability criteria), whereas in the case of hexagonal crystals one has to deal with a quadratic system of inequalities (quadratic stability criterion) (see also [13]).

4.3. One-Dimensional Trigonal Quasicrystals of Laue Class 8

The next system of one-dimensional quasicrystals that is examined is the trigonal system. Let us start with the Laue class 8 possessing 12 independent elastic moduli. If we use the classical 6 × 6 matrix form of C_{ijkl} for trigonal crystals of Laue class $\bar{3}m$ given in Nye [9], the 6 × 3 matrix form of D_{ij3l} and the 3 × 3 matrix form of E_{3j3l} given in Agiasofitou and Lazar [27] for one-dimensional trigonal quasicrystals of Laue class 8, then the matrix $\mathcal{C}_{\mu\nu}$, Equation (22), can be represented by the following 9 × 9 matrix in Voigt notation

$$\mathcal{C}_{\mu\nu} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 & 0 & D_{12} & D_{13} \\ C_{12} & C_{11} & C_{13} & -C_{14} & 0 & 0 & 0 & -D_{12} & D_{13} \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 & 0 & 0 & D_{33} \\ C_{14} & -C_{14} & 0 & C_{44} & 0 & 0 & 0 & D_{42} & 0 \\ 0 & 0 & 0 & 0 & C_{44} & C_{14} & D_{42} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{14} & C_{66} & D_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{42} & D_{12} & E_{11} & 0 & 0 \\ D_{12} & -D_{12} & 0 & D_{42} & 0 & 0 & 0 & E_{11} & 0 \\ D_{13} & D_{13} & D_{33} & 0 & 0 & 0 & 0 & 0 & E_{33} \end{pmatrix} \tag{61}$$

with $C_{66} = (C_{11} - C_{12})/2$. The nine leading principal minors corresponding to the 9 × 9 matrix (61) are:

$$\mathcal{D}_1 = C_{11}\,, \tag{62}$$

$$\mathcal{D}_2 = (C_{11} - C_{12})(C_{11} + C_{12})\,, \tag{63}$$

$$\mathcal{D}_3 = (C_{11} - C_{12})(C_{33}C_{11} + C_{33}C_{12} - 2C_{13}^2)\,, \tag{64}$$

$$\mathcal{D}_4 = (C_{11}C_{44} - C_{12}C_{44} - 2C_{14}^2)(C_{33}C_{11} + C_{33}C_{12} - 2C_{13}^2)\,, \tag{65}$$

$$\begin{aligned} \mathcal{D}_5 &= C_{44}\,(C_{11}C_{44} - C_{12}C_{44} - 2C_{14}^2)(C_{33}C_{11} + C_{33}C_{12} - 2C_{13}^2) \\ &= C_{44}\,\mathcal{D}_4\,, \end{aligned} \tag{66}$$

$$\mathcal{D}_6 = \frac{1}{2}\,(C_{11}C_{44} - C_{12}C_{44} - 2C_{14}^2)^2(C_{33}C_{11} + C_{33}C_{12} - 2C_{13}^2)\,, \tag{67}$$

$$\begin{aligned} \mathcal{D}_7 &= \frac{1}{2}\,(C_{11}C_{44} - C_{12}C_{44} - 2C_{14}^2)(C_{33}C_{11} + C_{33}C_{12} - 2C_{13}^2) \\ &\quad \Big\{\big[(C_{11} - C_{12})C_{44} - 2C_{14}^2\big]E_{11} - (C_{11} - C_{12})D_{42}^2 - 2(C_{44}D_{12} - 2C_{14}D_{42})D_{12}\Big\} \\ &= \frac{1}{2}\Big\{\big[(C_{11} - C_{12})C_{44} - 2C_{14}^2\big]E_{11} - (C_{11} - C_{12})D_{42}^2 - 2(C_{44}D_{12} - 2C_{14}D_{42})D_{12}\Big\}\,\mathcal{D}_4\,, \end{aligned} \tag{68}$$

$$\begin{aligned} \mathcal{D}_8 &= \frac{1}{2}\,(C_{33}C_{11} + C_{33}C_{12} - 2C_{13}^2) \\ &\quad \Big\{\big[(C_{11} - C_{12})C_{44} - 2C_{14}^2\big]E_{11} - (C_{11} - C_{12})D_{42}^2 - 2(C_{44}D_{12} - 2C_{14}D_{42})D_{12}\Big\}^2\,, \end{aligned} \tag{69}$$

$$\begin{aligned} \mathcal{D}_9 &= \frac{1}{2}\Big\{\big[(C_{11} - C_{12})C_{44} - 2C_{14}^2\big]E_{11} - (C_{11} - C_{12})D_{42}^2 - 2(C_{44}D_{12} - 2C_{14}D_{42})D_{12}\Big\}^2 \\ &\quad \Big\{\big[(C_{11} + C_{12})C_{33} - 2C_{13}^2\big]E_{33} - (C_{11} + C_{12})D_{33}^2 - 2(C_{33}D_{13} - 2C_{13}D_{33})D_{13}\Big\}\,. \end{aligned} \tag{70}$$

The application of the Sylvester criterion leads to the following six inequalities:

$$C_{11} > |C_{12}|\,, \quad (C_{11}+C_{12})C_{33} - 2C_{13}^2 > 0\,, \quad (C_{11}-C_{12})C_{44} - 2C_{14}^2 > 0\,, \tag{71}$$

$$\big[(C_{11}-C_{12})C_{44} - 2C_{14}^2\big]E_{11} - (C_{11}-C_{12})D_{42}^2 - 2(C_{44}D_{12} - 2C_{14}D_{42})D_{12} > 0\,, \tag{72}$$

$$\big[(C_{11}+C_{12})C_{33} - 2C_{13}^2\big]E_{33} - (C_{11}+C_{12})D_{33}^2 - 2(C_{33}D_{13} - 2C_{13}D_{33})D_{13} > 0\ . \tag{73}$$

Inequalities (71)–(73) are *the 6 necessary and sufficient conditions for elastic stability and positive definiteness for one-dimensional trigonal quasicrystals of Laue class 8.*

It has to be noted that the condition $\mathcal{D}_5 > 0$ gives $C_{44} > 0$, which can, however, be extracted from the condition $(C_{11}-C_{12})C_{44} - 2C_{14}^2 > 0$, which arrives from $\mathcal{D}_4 > 0$. Hence, the condition $C_{44} > 0$ is in this sense redundant.

It should be noticed that the first four inequalities (71), which have to be satisfied only by the elastic moduli of phonons, are the same as those for trigonal crystals of Laue class $\bar{3}m$ [10] and thereby the limit to classical elasticity in the absence of phason fields is recovered.

Moreover, one can observe that the above set of inequalities (71)–(73) constitutes a cubic system of algebraic inequalities and therefore trigonal quasicrystals of Laue class 8 have a cubic stability criterion.

4.4. One-Dimensional Trigonal Quasicrystals of Laue Class 7

The other class of one-dimensional trigonal quasicrystals is the Laue class 7 with 15 independent elastic moduli. By using the classical 6 × 6 matrix form of C_{ijkl} for trigonal crystals of Laue class $\bar{3}$ given in Nye [9], the 6 × 3 matrix form of D_{ij3l} and the 3 × 3 matrix form of E_{3j3l} given in Agiasofitou and Lazar [27] for one-dimensional trigonal quasicrystals of Laue class 7, the matrix $\mathcal{C}_{\mu\nu}$, Equation (22), is represented by the following 9 × 9 matrix in Voigt notation

$$\mathcal{C}_{\mu\nu} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & 0 & D_{11} & D_{12} & D_{13} \\ C_{12} & C_{11} & C_{13} & -C_{14} & -C_{15} & 0 & -D_{11} & -D_{12} & D_{13} \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 & 0 & 0 & D_{33} \\ C_{14} & -C_{14} & 0 & C_{44} & 0 & -C_{15} & D_{41} & D_{42} & 0 \\ C_{15} & -C_{15} & 0 & 0 & C_{44} & C_{14} & D_{42} & -D_{41} & 0 \\ 0 & 0 & 0 & -C_{15} & C_{14} & C_{66} & D_{12} & -D_{11} & 0 \\ D_{11} & -D_{11} & 0 & D_{41} & D_{42} & D_{12} & E_{11} & 0 & 0 \\ D_{12} & -D_{12} & 0 & D_{42} & -D_{41} & -D_{11} & 0 & E_{11} & 0 \\ D_{13} & D_{13} & D_{33} & 0 & 0 & 0 & 0 & 0 & E_{33} \end{pmatrix} \tag{74}$$

with $C_{66} = (C_{11}-C_{12})/2$. The nine leading principal minors corresponding to the 9 × 9 matrix (74) are:

$$\mathcal{D}_1 = C_{11}\,, \tag{75}$$

$$\mathcal{D}_2 = (C_{11}-C_{12})(C_{11}+C_{12})\,, \tag{76}$$

$$\mathcal{D}_3 = (C_{11}-C_{12})(C_{33}C_{11} + C_{33}C_{12} - 2C_{13}^2)\,, \tag{77}$$

$$\mathcal{D}_4 = (C_{11}C_{44} - C_{12}C_{44} - 2C_{14}^2)(C_{33}C_{11} + C_{33}C_{12} - 2C_{13}^2)\,, \tag{78}$$

$$\mathcal{D}_5 = C_{44}\,(C_{11}C_{44} - C_{12}C_{44} - 2C_{14}^2 - 2C_{15}^2)(C_{33}C_{11} + C_{33}C_{12} - 2C_{13}^2)\,, \tag{79}$$

$$\mathcal{D}_6 = \frac{1}{2}\,(C_{11}C_{44} - C_{12}C_{44} - 2C_{14}^2 - 2C_{15}^2)^2(C_{33}C_{11} + C_{33}C_{12} - 2C_{13}^2)\,, \tag{80}$$

$$\begin{aligned} \mathcal{D}_7 = {} & \frac{1}{2}\,(C_{11}C_{44} - C_{12}C_{44} - 2C_{14}^2 - 2C_{15}^2)(C_{33}C_{11} + C_{33}C_{12} - 2C_{13}^2) \\ & \Big\{\big[(C_{11}-C_{12})C_{44} - 2(C_{14}^2 + C_{15}^2)\big]E_{11} - (C_{11}-C_{12})(D_{41}^2 + D_{42}^2) \\ & - 2(C_{44}D_{12} - 2C_{14}D_{42})D_{12} - 2(C_{44}D_{11} - 2C_{14}D_{41})D_{11} + 4C_{15}(D_{11}D_{42} - D_{12}D_{41})\Big\}\,, \end{aligned} \tag{81}$$

$$\begin{aligned}\mathcal{D}_8 = {}& \frac{1}{2}\,(C_{33}C_{11} + C_{33}C_{12} - 2C_{13}^2) \\ & \Big\{\big[(C_{11} - C_{12})C_{44} - 2(C_{14}^2 + C_{15}^2)\big]E_{11} - (C_{11} - C_{12})\big(D_{41}^2 + D_{42}^2\big) \\ & - 2(C_{44}D_{12} - 2C_{14}D_{42})D_{12} - 2(C_{44}D_{11} - 2C_{14}D_{41})D_{11} + 4C_{15}(D_{11}D_{42} - D_{12}D_{41})\Big\}^2\,, \end{aligned} \tag{82}$$

$$\begin{aligned}\mathcal{D}_9 = {}& \frac{1}{2}\,\Big\{\big[(C_{11} - C_{12})C_{44} - 2(C_{14}^2 + C_{15}^2)\big]E_{11} - (C_{11} - C_{12})\big(D_{41}^2 + D_{42}^2\big) \\ & - 2(C_{44}D_{12} - 2C_{14}D_{42})D_{12} - 2(C_{44}D_{11} - 2C_{14}D_{41})D_{11} + 4C_{15}(D_{11}D_{42} - D_{12}D_{41})\Big\}^2 \\ & \Big\{\big[(C_{11} + C_{12})C_{33} - 2C_{13}^2\big]E_{33} - (C_{11} + C_{12})D_{33}^2 - 2(C_{33}D_{13} - 2C_{13}D_{33})D_{13}\Big\}\,. \end{aligned} \tag{83}$$

The application of the Sylvester criterion leads to the following six inequalities:

$$C_{11} > |C_{12}|\,, \quad (C_{11} + C_{12})C_{33} - 2C_{13}^2 > 0\,, \quad (C_{11} - C_{12})C_{44} - 2(C_{14}^2 + C_{15}^2) > 0\,, \tag{84}$$

$$\begin{aligned}&\big[(C_{11} - C_{12})C_{44} - 2(C_{14}^2 + C_{15}^2)\big]E_{11} - (C_{11} - C_{12})\big(D_{41}^2 + D_{42}^2\big) - 2(C_{44}D_{12} - 2C_{14}D_{42})D_{12} \\ &- 2(C_{44}D_{11} - 2C_{14}D_{41})D_{11} + 4C_{15}(D_{11}D_{42} - D_{12}D_{41}) > 0\,, \end{aligned} \tag{85}$$

$$\big[(C_{11} + C_{12})C_{33} - 2C_{13}^2\big]E_{33} - (C_{11} + C_{12})D_{33}^2 - 2(C_{33}D_{13} - 2C_{13}D_{33})D_{13} > 0\,. \tag{86}$$

Inequalities (84)–(86) are *the 6 necessary and sufficient conditions for elastic stability and positive definiteness for one-dimensional trigonal quasicrystals of Laue class 7.*

It has to be noted that the conditions $(C_{11} - C_{12})C_{44} - 2C_{14}^2 > 0$ (for $\mathcal{D}_4 > 0$) and $C_{44} > 0$ (for $\mathcal{D}_5 > 0$) can be extracted from the condition $(C_{11} - C_{12})C_{44} - 2(C_{14}^2 + C_{15}^2) > 0$ and in this sense they are redundant.

It should be noticed that the first four inequalities (84), which have to be satisfied only by the elastic moduli of phonons, are the same as those for trigonal crystals of Laue class $\overline{3}$ [13] and thereby the limit to classical elasticity in the absence of phason fields is recovered.

In addition, the system of the obtained algebraic inequalities of the stability criterion for one-dimensional trigonal quasicrystals of Laue class 7 remains cubic.

4.5. One-Dimensional Tetragonal Quasicrystals of Laue Class 6

The next system of one-dimensional quasicrystals studied is the tetragonal system. Let us start with Laue class 6, which possesses 11 independent elastic moduli. If we use the classical 6 × 6 matrix form of C_{ijkl} for tetragonal crystals of Laue class $4/mmm$ given in Nye [9], the 6 × 3 matrix form of D_{ij3l} and the 3 × 3 matrix form of E_{3j3l} given in Agiasofitou and Lazar [27] for one-dimensional tetragonal quasicrystals of Laue class 6, then the matrix $\mathcal{C}_{\mu\nu}$, Equation (22), reduces to the following 9 × 9 matrix in Voigt notation

$$\mathcal{C}_{\mu\nu} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 & 0 & 0 & D_{13} \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 & 0 & 0 & D_{13} \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 & 0 & 0 & D_{33} \\ 0 & 0 & 0 & C_{44} & 0 & 0 & 0 & D_{42} & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 & D_{42} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{42} & 0 & E_{11} & 0 & 0 \\ 0 & 0 & 0 & D_{42} & 0 & 0 & 0 & E_{11} & 0 \\ D_{13} & D_{13} & D_{33} & 0 & 0 & 0 & 0 & 0 & E_{33} \end{pmatrix}. \tag{87}$$

The nine leading principal minors corresponding to the 9 × 9 matrix (87) are:

$$\begin{aligned}
\mathcal{D}_1 &= C_{11}\,, && (88)\\
\mathcal{D}_2 &= (C_{11}-C_{12})(C_{11}+C_{12})\,, && (89)\\
\mathcal{D}_3 &= (C_{11}-C_{12})(C_{33}C_{11}+C_{33}C_{12}-2C_{13}^2)\,, && (90)\\
\mathcal{D}_4 &= C_{44}\,(C_{11}-C_{12})(C_{33}C_{11}+C_{33}C_{12}-2C_{13}^2)\\
&= C_{44}\,\mathcal{D}_3\,, && (91)\\
\mathcal{D}_5 &= C_{44}^2\,(C_{11}-C_{12})(C_{33}C_{11}+C_{33}C_{12}-2C_{13}^2)\\
&= C_{44}\,\mathcal{D}_4 = C_{44}^2\,\mathcal{D}_3\,, && (92)\\
\mathcal{D}_6 &= C_{44}^2C_{66}\,(C_{11}-C_{12})(C_{33}C_{11}+C_{33}C_{12}-2C_{13}^2)\\
&= C_{66}\,\mathcal{D}_5\,, && (93)\\
\mathcal{D}_7 &= C_{44}C_{66}\,(C_{44}E_{11}-D_{42}^2)(C_{11}-C_{12})(C_{33}C_{11}+C_{33}C_{12}-2C_{13}^2)\,, && (94)\\
\mathcal{D}_8 &= C_{66}\,(C_{44}E_{11}-D_{42}^2)^2(C_{11}-C_{12})(C_{33}C_{11}+C_{33}C_{12}-2C_{13}^2)\,, && (95)\\
\mathcal{D}_9 &= C_{66}\,(C_{44}E_{11}-D_{42}^2)^2(C_{11}-C_{12})\\
&\quad\Big\{\big[(C_{11}+C_{12})C_{33}-2C_{13}^2\big]E_{33}-(C_{11}+C_{12})D_{33}^2-2(C_{33}D_{13}-2C_{13}D_{33})D_{13}\Big\}\,. && (96)
\end{aligned}$$

The application of the Sylvester criterion leads to the following seven inequalities:

$$\begin{aligned}
&C_{11} > |C_{12}|\,, \qquad (C_{11}+C_{12})C_{33}-2C_{13}^2 > 0\,, \qquad C_{44} > 0\,, \qquad C_{66} > 0\,, && (97)\\
&C_{44}E_{11}-D_{42}^2 > 0\,, && (98)\\
&\big[(C_{11}+C_{12})C_{33}-2C_{13}^2\big]E_{33}-(C_{11}+C_{12})D_{33}^2-2(C_{33}D_{13}-2C_{13}D_{33})D_{13} > 0\,. && (99)
\end{aligned}$$

Inequalities (97)–(99) are *the 7 necessary and sufficient conditions for elastic stability and positive definiteness for one-dimensional tetragonal quasicrystals of Laue class 6.*

It should be noticed that the first five inequalities (97), which have to be satisfied only by the elastic moduli of phonons, are the same as those for tetragonal crystals of Laue class $4/mmm$ [10,13] and thereby the limit to classical elasticity in the absence of phason fields is recovered.

The highest degree of the polynomials in inequalities (97)–(99) is three. Hence, the set of the inequalities (97)–(99) constitutes a cubic system of algebraic inequalities.

Using the relations (40) and (41), inequalities (97)–(99) are alternatively written as follows:

$$\begin{aligned}
&C_{11} > |C_{12}|\,, \qquad (C_{11}+C_{12})C_{33}-2C_{13}^2 > 0\,, \qquad C_{44} > 0\,, \qquad C_{66} > 0\,, && (100)\\
&C_{44}K_2-R_3^2 > 0\,, && (101)\\
&\big[(C_{11}+C_{12})C_{33}-2C_{13}^2\big]K_1-(C_{11}+C_{12})R_2^2-2(C_{33}R_1-2C_{13}R_2)R_1 > 0\,. && (102)
\end{aligned}$$

4.6. One-Dimensional Tetragonal Quasicrystals of Laue Class 5

The next examined Laue class of one-dimensional tetragonal quasicrystals is Laue class 5, which possesses 13 independent elastic moduli. If we use the classical 6 × 6 matrix form of C_{ijkl} for tetragonal crystals of Laue class $4/m$ given in Nye [9], the 6 × 3 matrix form of D_{ij3l} and the 3 × 3 matrix form of E_{3j3l} given in Agiasofitou and Lazar [27] for one-dimensional tetragonal quasicrystals of Laue class 5, then the matrix $\mathcal{C}_{\mu\nu}$, Equation (22), reduces to the following 9 × 9 matrix in Voigt notation

$$\mathcal{C}_{\mu\nu}=\begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} & 0 & 0 & D_{13} \\ C_{12} & C_{11} & C_{13} & 0 & 0 & -C_{16} & 0 & 0 & D_{13} \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 & 0 & 0 & D_{33} \\ 0 & 0 & 0 & C_{44} & 0 & 0 & D_{41} & D_{42} & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 & D_{42} & -D_{41} & 0 \\ C_{16} & -C_{16} & 0 & 0 & 0 & C_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{41} & D_{42} & 0 & E_{11} & 0 & 0 \\ 0 & 0 & 0 & D_{42} & -D_{41} & 0 & 0 & E_{11} & 0 \\ D_{13} & D_{13} & D_{33} & 0 & 0 & 0 & 0 & 0 & E_{33} \end{pmatrix}. \tag{103}$$

The nine leading principal minors corresponding to the 9×9 matrix (103) are:

$$\mathcal{D}_1 = C_{11}\,, \tag{104}$$

$$\mathcal{D}_2 = (C_{11}-C_{12})(C_{11}+C_{12})\,, \tag{105}$$

$$\mathcal{D}_3 = (C_{11}-C_{12})(C_{33}C_{11}+C_{33}C_{12}-2C_{13}^2)\,, \tag{106}$$

$$\begin{aligned}\mathcal{D}_4 &= C_{44}\,(C_{11}-C_{12})(C_{33}C_{11}+C_{33}C_{12}-2C_{13}^2) \\ &= C_{44}\,\mathcal{D}_3\,,\end{aligned} \tag{107}$$

$$\begin{aligned}\mathcal{D}_5 &= C_{44}^2\,(C_{11}-C_{12})(C_{33}C_{11}+C_{33}C_{12}-2C_{13}^2) \\ &= C_{44}\,\mathcal{D}_4 = C_{44}^2\,\mathcal{D}_3\,,\end{aligned} \tag{108}$$

$$\mathcal{D}_6 = C_{44}^2\,(C_{11}C_{66}-C_{12}C_{66}-2C_{16}^2)(C_{33}C_{11}+C_{33}C_{12}-2C_{13}^2)\,, \tag{109}$$

$$\mathcal{D}_7 = C_{44}\,(C_{44}E_{11}-D_{42}^2-D_{41}^2)(C_{11}C_{66}-C_{12}C_{66}-2C_{16}^2)(C_{33}C_{11}+C_{33}C_{12}-2C_{13}^2)\,, \tag{110}$$

$$\mathcal{D}_8 = (C_{44}E_{11}-D_{42}^2-D_{41}^2)^2(C_{11}C_{66}-C_{12}C_{66}-2C_{16}^2)(C_{33}C_{11}+C_{33}C_{12}-2C_{13}^2)\,, \tag{111}$$

$$\begin{aligned}\mathcal{D}_9 = {}& (C_{44}E_{11}-D_{42}^2-D_{41}^2)^2(C_{11}C_{66}-C_{12}C_{66}-2C_{16}^2) \\ & \Big\{\big[(C_{11}+C_{12})C_{33}-2C_{13}^2\big]E_{33}-(C_{11}+C_{12})D_{33}^2-2(C_{33}D_{13}-2C_{13}D_{33})D_{13}\Big\}\,.\end{aligned} \tag{112}$$

The application of the Sylvester criterion leads to the following seven inequalities:

$$C_{11} > |C_{12}|\,, \quad (C_{11}+C_{12})C_{33}-2C_{13}^2 > 0\,, \quad C_{44} > 0\,, \quad (C_{11}-C_{12})C_{66}-2C_{16}^2 > 0\,, \tag{113}$$

$$C_{44}E_{11}-\big(D_{41}^2+D_{42}^2\big) > 0\,, \tag{114}$$

$$\big[(C_{11}+C_{12})C_{33}-2C_{13}^2\big]E_{33}-(C_{11}+C_{12})D_{33}^2-2(C_{33}D_{13}-2C_{13}D_{33})D_{13} > 0\,. \tag{115}$$

Inequalities (113)–(115) are *the 7 necessary and sufficient conditions for elastic stability and positive definiteness for one-dimensional tetragonal quasicrystals of Laue class 5.*

It should be noticed that the first five inequalities (113), which have to be satisfied only by the elastic moduli of phonons, are the same as those for tetragonal crystals of Laue class $4/m$ [13] and thereby the limit to classical elasticity in the absence of phason fields is recovered.

Using the relations (40) and (41), conditions (113)–(115) for elastic stability and positive definiteness are alternatively written as follows:

$$C_{11} > |C_{12}|\,, \quad (C_{11}+C_{12})C_{33}-2C_{13}^2 > 0\,, \quad C_{44} > 0\,, \quad (C_{11}-C_{12})C_{66}-2C_{16}^2 > 0\,, \tag{116}$$

$$C_{44}K_2-(R_3^2+R_4^2) > 0\,, \tag{117}$$

$$\big[(C_{11}+C_{12})C_{33}-2C_{13}^2\big]K_1-(C_{11}+C_{12})R_2^2-2(C_{33}R_1-2C_{13}R_2)R_1 > 0\,. \tag{118}$$

As can be seen above, the stability criterion for one-dimensional tetragonal quasicrystals of Laue class 5 is also a cubic one.

4.7. One-Dimensional Orthorhombic Quasicrystals of Laue Class 4

The one-dimensional orthorhombic quasicrystals of Laue class 4 possess 17 independent elastic moduli. By using the classical 6×6 matrix form of C_{ijkl} for orthorhombic crystals given in Nye [9], the 6×3 matrix form of D_{ij3l} and the 3×3 matrix form of E_{3j3l} given in Agiasofitou and Lazar [27] for one-dimensional orthorhombic quasicrystals of Laue class 4, the matrix $\mathcal{C}_{\mu\nu}$, Equation (22), is written in the following 9×9 matrix in Voigt notation

$$\mathcal{C}_{\mu\nu} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 & 0 & 0 & D_{13} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 & 0 & 0 & D_{23} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 & 0 & 0 & D_{33} \\ 0 & 0 & 0 & C_{44} & 0 & 0 & 0 & D_{42} & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 & D_{51} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{51} & 0 & E_{11} & 0 & 0 \\ 0 & 0 & 0 & D_{42} & 0 & 0 & 0 & E_{22} & 0 \\ D_{13} & D_{23} & D_{33} & 0 & 0 & 0 & 0 & 0 & E_{33} \end{pmatrix}. \tag{119}$$

The nine leading principal minors corresponding to the 9×9 matrix (119) are:

$$\mathcal{D}_1 = C_{11}\,, \tag{120}$$

$$\mathcal{D}_2 = C_{11}C_{22} - C_{12}^2\,, \tag{121}$$

$$\mathcal{D}_3 = (C_{11}C_{22} - C_{12}^2)C_{33} + (2C_{12}C_{13} - C_{11}C_{23})C_{23} - C_{13}^2C_{22}\,, \tag{122}$$

$$\begin{aligned} \mathcal{D}_4 &= C_{44}\left[(C_{11}C_{22} - C_{12}^2)C_{33} + (2C_{12}C_{13} - C_{11}C_{23})C_{23} - C_{13}^2C_{22}\right] \\ &= C_{44}\,\mathcal{D}_3\,, \end{aligned} \tag{123}$$

$$\begin{aligned} \mathcal{D}_5 &= C_{44}C_{55}\left[(C_{11}C_{22} - C_{12}^2)C_{33} + (2C_{12}C_{13} - C_{11}C_{23})C_{23} - C_{13}^2C_{22}\right] \\ &= C_{55}\,\mathcal{D}_4\,, \end{aligned} \tag{124}$$

$$\begin{aligned} \mathcal{D}_6 &= C_{44}C_{55}C_{66}\left[(C_{11}C_{22} - C_{12}^2)C_{33} + (2C_{12}C_{13} - C_{11}C_{23})C_{23} - C_{13}^2C_{22}\right] \\ &= C_{66}\,\mathcal{D}_5\,, \end{aligned} \tag{125}$$

$$\begin{aligned} \mathcal{D}_7 &= C_{44}C_{66}\,(C_{55}E_{11} - D_{51}^2)\left[(C_{11}C_{22} - C_{12}^2)C_{33} + (2C_{12}C_{13} - C_{11}C_{23})C_{23} - C_{13}^2C_{22}\right] \\ &= C_{66}\,(C_{55}E_{11} - D_{51}^2)\,\mathcal{D}_4\,, \end{aligned} \tag{126}$$

$$\begin{aligned} \mathcal{D}_8 &= C_{66}\,(C_{55}E_{11} - D_{51}^2)(C_{44}E_{22} - D_{42}^2)\left[(C_{11}C_{22} - C_{12}^2)C_{33} + (2C_{12}C_{13} - C_{11}C_{23})C_{23} - C_{13}^2C_{22}\right] \\ &= C_{66}\,(C_{55}E_{11} - D_{51}^2)(C_{44}E_{22} - D_{42}^2)\,\mathcal{D}_3\,, \end{aligned} \tag{127}$$

$$\begin{aligned} \mathcal{D}_9 &= C_{66}\,(C_{55}E_{11} - D_{51}^2)(C_{44}E_{22} - D_{42}^2) \\ &\quad \Big\{\Big[(C_{11}C_{22} - C_{12}^2)C_{33} + (2C_{12}C_{13} - C_{11}C_{23})C_{23} - C_{13}^2C_{22}\Big]E_{33} \\ &\quad - (C_{11}C_{22} - C_{12}^2)D_{33}^2 - (C_{11}C_{33} - C_{13}^2)D_{23}^2 - (C_{22}C_{33} - C_{23}^2)D_{13}^2 \\ &\quad + 2(C_{12}C_{33} - C_{13}C_{23})D_{13}D_{23} + 2(C_{13}C_{22} - C_{12}C_{23})D_{13}D_{33} + 2(C_{23}C_{11} - C_{12}C_{13})D_{23}D_{33}\Big\}\,. \end{aligned} \tag{128}$$

The application of the Sylvester criterion leads to the following nine inequalities:

$$C_{11} > 0\,, \qquad C_{11}C_{22} - C_{12}^2 > 0\,, \tag{129}$$

$$(C_{11}C_{22} - C_{12}^2)C_{33} + (2C_{12}C_{13} - C_{11}C_{23})C_{23} - C_{13}^2C_{22} > 0\,, \tag{130}$$

$$C_{44} > 0\,, \qquad C_{55} > 0\,, \qquad C_{66} > 0\,, \tag{131}$$

$$C_{55}E_{11} - D_{51}^2 > 0\,, \qquad C_{44}E_{22} - D_{42}^2 > 0\,, \tag{132}$$

$$\begin{aligned}&\Big[(C_{11}C_{22} - C_{12}^2)C_{33} + (2C_{12}C_{13} - C_{11}C_{23})C_{23} - C_{13}^2C_{22}\Big]E_{33}\\ &- (C_{11}C_{22} - C_{12}^2)D_{33}^2 - (C_{11}C_{33} - C_{13}^2)D_{23}^2 - (C_{22}C_{33} - C_{23}^2)D_{13}^2\\ &+ 2(C_{12}C_{33} - C_{13}C_{23})D_{13}D_{23} + 2(C_{13}C_{22} - C_{12}C_{23})D_{13}D_{33} + 2(C_{23}C_{11} - C_{12}C_{13})D_{23}D_{33} > 0\,.\end{aligned} \tag{133}$$

Inequalities (129)–(133) are *the 9 necessary and sufficient conditions for elastic stability and positive definiteness for one-dimensional orthorhombic quasicrystals of Laue class 4.*

It should be noticed that the first six inequalities (129)–(131), which have to be satisfied only by the elastic moduli of phonons, are the same as those for orthorhombic crystals (Laue class *mmm*) [13] and thereby the limit to classical elasticity in the absence of phason fields is recovered.

It can be seen that in the case of one-dimensional orthorhombic quasicrystals of Laue class 4, which possess 17 independent elastic constants, the highest degree of the polynomials in the inequalities increases to 4 and it appears in inequality (133), which comes from the ninth leading principal minor, $\mathcal{D}_9$ (see Equation (128)). Hence, one-dimensional orthorhombic quasicrystals of Laue class 4 are governed by a quartic stability criterion.

As the symmetry in the quasicrystalline structure becomes lower, the number of the independent elastic constants increases. These are the cases of one-dimensional monoclinic quasicrystals of Laue classes 2 and 3, which posses 25 and 27 independent elastic constants, respectively, and the one-dimensional triclinic quasicrystals, which posses 45 independent elastic constants [7]. The method explained above can also be applied in these two systems (monoclinic and triclinic). However, these two systems appear very rarely in the literature of quasicrystals, making them less important, at least to date.

Finally, it has to be mentioned that the obtained sets of conditions for elastic stability and positive definiteness for one-dimensional quasicrystals of Laue classes 4 to 10 differ from the corresponding sets of conditions given in Liu et al. [18], due to the use of an improper Voigt-type representation of the elastic constants with phason contribution in the contracted matrix in [18].

5. Concluding Remarks and Discussion

In this work, a proper formulation for the elastic stability and positive definiteness of the elastic energy density of one-dimensional quasicrystals has been given. By using the Sylvester criterion, the sets of necessary and sufficient conditions for the positive definiteness imposed on the elastic constants have been derived for each Laue class of one-dimensional quasicrystals from 4 to 10. It should be pointed out that in every study demanding the use of the elastic constants, it is important to check in advance if the elastic constants fulfill the necessary and sufficient conditions for the positive definiteness given in this work, since it is difficult to find values of the elastic constants based on experiments or atomistic calculations in the literature of one-dimensional quasicrystals.

Considering the results on the whole, the following interesting remarks can be made:

- Even if the highest symmetry is considered, that is the case of hexagonal quasicrystals, the highest degree of the obtained inequalities for positive definiteness is three and it is due to phason and phonon–phason coupling elastic constants, whereas in hexagonal crystals the highest degree of the obtained inequalities is two. It is obvious and it is expected that the phason fields increase the complexity of the obtained sets of the inequalities. In particular, the highest degree of inequalities in quasicrystals is one degree greater than the highest degree of inequalities in crystals. How much the complexity increases concerning the degree and the number of inequalities between quasicrystals and crystals can be seen in Table 1.

- In Table 1, it can be seen that hexagonal, trigonal and tetragonal quasicrystal systems have cubic stability criteria, whereas hexagonal, trigonal and tetragonal crystal systems have quadratic stability criteria. Orthorhombic quasicrystals have quartic stability criteria, whereas orthorhombic crystals have cubic stability criteria.
- In Table 1, it can be seen that in the case of hexagonal, trigonal and tetragonal quasicrystals, there are two additional inequalities due to the phason contributions and as the symmetry decreases and the number of the independent elastic moduli increases, that is, in the case of orthorhombic quasicrystals, the number of the additional inequalities due to phason contributions increases to three.
- In the obtained sets of the conditions for elastic stability and positive definiteness of one-dimensional quasicrystals, it can be seen that the highest degree of the inequalities with phason contributions is one degree greater than the highest degree of the inequalities with only phonon contributions.
- It is interesting to observe that the last inequality of the sets for elastic stability and positive definiteness, which comes from the ninth leading principal minor $\mathcal{D}_9$, that corresponds to the determinant of the considered matrix is the same condition for the hexagonal, trigonal and tetragonal systems of one-dimensional quasicrystals, that is, Inequalities (39), (57), (73), (86), (99) and (115).

Although the Sylvester criterion cannot assure the number of the independent necessary and sufficient conditions, it can be assumed that the above number is the number of independent necessary and sufficient conditions for positive definiteness based on the knowledge of those conditions from crystals and observing that the two or three additional inequalities due to the phason contributions cannot be redundant due to their complexity.

Table 1. Degree and number of inequalities between one-dimensional quasicrystals and crystals.

	One-Dimensional Quasicrystals			**Crystals**		
System-Laue Class	**Degree**	**Number**	**Stability Criteria**	**Degree**	**Number**	**Stability Criteria**
Hexagonal-10, 9	3	6	cubic	2	4	quadratic
Trigonal-8, 7	3	6	cubic	2	4	quadratic
Tetragonal-6, 5	3	7	cubic	2	5	quadratic
Orthorhombic-4	4	9	quartic	3	6	cubic

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References

1. Shechtman, D.; Blech, I.; Gratias, D.; Cahn, J.W. Metallic phase with long-range orientational order and no translational symmetry. *Phys. Rev. Lett.* **1984**, *53*, 1951–1953. [CrossRef]
2. Maciá, E. The role of aperiodic order in science and technology. *Rep. Prog. Phys.* **2006**, *69*, 397–441. [CrossRef]
3. Nagaoka, Y.; Schneider, J.; Zhu, H.; Chen, O. Quasicrystalline materials from non-atom building blocks. *Matter* **2023**, *6*, 30–58. [CrossRef]
4. Fan, T.Y. *Mathematical Theory of Elasticity of Quasicrystals and Its Applications*; Science Press: Beijing, China; Springer: Berlin/Heidelberg, Germany, 2011.
5. Bak, P. Symmetry, stability and elastic properties of icosahedral incommensurate crystals. *Phys. Rev. B* **1985**, *32*, 5764–5772. [CrossRef]
6. Levine, D.; Lubensky, T.C.; Ostlund, S.; Ramaswamy, S.; Steinhardt, P.J.; Toner, J. Elasticity and dislocations in pentagonal and icosahedral quasicrystals. *Phys. Rev. Lett.* **1985**, *54*, 1520–1523. [CrossRef] [PubMed]
7. Wang, R.; Yang, W.; Hu, C.; Ding, D.-H. Point and space groups and elastic behaviours of one-dimensional quasicrystals. *J. Phys. Condens. Matter* **1997**, *9*, 2411–2422. [CrossRef]
8. Born, M. On the stability of crystal lattices. I. *Math. Proc. Camb. Philos. Soc.* **1940**, *36*, 160–172. [CrossRef]
9. Nye, J.F. *Physical Properties of Crystals*; Oxford University Press: Oxford, UK, 1957.
10. Fedorov, F.I. *Theory of Elastic Waves in Crystals*; Plenum Press: New York, NY, USA, 1968.
11. Ting, T.C.T. *Anisotropic Elasticity: Theory and Applications*; Oxford University Press: Oxford, UK, 1996.
12. Ting, T.C.T. Positive definiteness of anisotropic elastic constants. *Math. Mech. Solids* **1996**, *1*, 301–314. [CrossRef]
13. Mouhat, F.; Coudert, F.-X. Necessary and sufficient elastic stability conditions in various crystal systems. *Phys. Rev. B* **2014**, *90*, 224104. [CrossRef]
14. Widom, M. Elastic stability and diffuse scattering in icosahedral quasicrystals. *Philos. Mag. Lett.* **1991**, *64*, 297–305. [CrossRef]
15. Ricker, M.; Bachteler, J.; Trebin, H.-R. Elastic theory of icosahedral quasicrystals-application to straight dislocations. *Eur. Phys. J. B* **2001**, *23*, 351–363. [CrossRef]
16. Wang, J.; Zhang, R.; Ding, D.H.; Wang, R. Positive-definite conditions of elastic constants of two-dimensional quasicrystals with noncrystallographic symmetries. *Acta Crystallogr. A* **1999**, *55*, 558–560. [CrossRef] [PubMed]
17. Tian, Y.; Guo, J.; Chen, W.; Wang, Q.; Wang, J.; Liu, G. Molecular dynamics simulation of mechanical properties of decagonal quasicrystal approximate phase Al_2Fe. *Mater. Today Commun.* **2024**, *39*, 108693. [CrossRef]
18. Liu, Z.; Li, H.; Fan, C.; Luo, W. Necessary and sufficient elastic stability conditions in 21 quasicrystal Laue classes. *Eur. J. Mech. A Solids* **2017**, *65*, 30–39. [CrossRef]
19. Vannucci, P. *Anisotropic Elasticity*; Springer Nature: Singapore, 2018.
20. Lazar, M.; Agiasofitou, E. Fundamentals in generalized elasticity and dislocation theory of quasicrystals: Green tensor, dislocation key-formulas and dislocation loops. *Phil. Mag.* **2014**, *94*, 4080–4101. [CrossRef]
21. Lazar, M.; Agiasofitou, E. Eshelbian mechanics of novel materials: Quasicrystals. *J. Micromech. Mol. Phys.* **2016**, *1*, 1640008. [CrossRef]
22. Ding, D.-H.; Wang, W.; Hu, C.; Yang, R. Generalized elasticity theory of quasicrystals. *Phys. Rev. B* **1993**, *48*, 7003–7010. [CrossRef]
23. Hu, C.; Wang, R.; Ding, D.H. Symmetry groups, physical property tensors, elasticity and dislocations in quasicrystals. *Rep. Prog. Phys.* **2000**, *63*, 1–39. [CrossRef]
24. Mehrabadi, M.M.; Cowin, S.C. Eigentensors of linear anisotropic elastic materials. *Q. J. Mech. Appl. Math.* **1990**, *43*, 15–41. [CrossRef]
25. Brannon, R.M. *Rotation, Reflection and Frame Changes. Orthogonal Tensors in Computational Engineering Mechanics*; IOP Publishing Ltd.: Bristol, UK, 2018.
26. Eringen, A.C. *Microcontinuum Field Theories. I: Foundations and Solids*; Springer: New York, NY, USA, 1999.
27. Agiasofitou, E.; Lazar, M. On the constitutive modelling of piezoelectric quasicrystals. *Crystals* **2023**, *13*, 1652. [CrossRef]
28. Jeffreys, H.; Jeffreys, B. *Methods of Mathematical Physics*; Cambridge University Press: Cambridge, UK, 1972.
29. Csáki, F. A concise proof of Sylvester's theorem. *Period. Polytech. Electr. Eng.* **1970**, *14*, 105–112.
30. Li, X.Y. Fundamental solutions of penny-shaped and half-infinite plane cracks embedded in an infinite space of one-dimensional hexagonal quasi-crystal under thermal loading. *Proc. R. Soc. A* **2013**, *469*, 20130023. [CrossRef]
31. Wu, Y.F.; Chen, W.Q.; Li, X.Y. Indentation on one-dimensional hexagonal quasicrystals: General theory and complete exact solutions. *Philos. Mag.* **2013**, *93*, 858–882. [CrossRef]

32. Li, X.Y.; Li, P.D.; Wu, T.H.; Shi, M.X.; Zhu, Z.W. Three-dimensional fundamental solutions for one-dimensional hexagonal quasicrystal with piezoelectric effect. *Phys. Lett. A* **2014**, *378*, 826–834. [CrossRef]
33. Lazar, M.; Michelitsch, T.; Agiasofitou, E. Three-dimensional Green tensor of one-dimensional hexagonal quasicrystals. *Crystals* **2024**, *14*, 1034. [CrossRef]