

CYCLIC MUTUALLY UNBIASED BASES

AND QUANTUM PUBLIC-KEY ENCRYPTION



Vom Fachbereich Physik
der Technischen Universität Darmstadt
zur Erlangung der Würde
eines Doktors der Naturwissenschaften
(Dr. rer. nat.)
genehmigte

D I S S E R T A T I O N


von

Dipl.-Phys. Ulrich Seyfarth

aus Hadamar

Darmstädter Dissertation
Darmstadt 2013
D17

Referent:	Prof. Dr. rer. nat. Gernot Alber
Korreferent:	Prof. Dr. rer. nat. Robert Roth
Tag der Einreichung:	Mittwoch, 28. November 2012
Tag der mündlichen Prüfung:	Montag, 11. Februar 2013



Cyclic mutually unbiased bases and quantum public-key encryption

Abstract

Based on quantum physical phenomena, quantum information theory has a potential which goes beyond the classical conditions. Equipped with the resource of complementary information as an intrinsic property it offers many new perspectives. The field of quantum key distribution, which enables the ability to implement unconditional security, profits directly from this resource. To measure the state of quantum systems itself for different purposes in quantum information theory, which may be related to the construction of a quantum computer, as well as to realize quantum key distribution schemes, a certain set of bases is necessary. A type of set which is minimal is given by a complete set of mutually unbiased bases. The construction of these sets is discussed in the first part of this work. We present complete sets of mutually unbiased bases which are equipped with the additional property to be constructed cyclically, which means, each basis in the set is the power of a specific generating basis of the set. Whereas complete sets of mutually unbiased bases are related to many mathematical problems, it is shown that a new construction of cyclic sets is related to Fibonacci polynomials. Within this context, the existence of a symmetric companion matrix over the finite field \mathbb{F}_2 is conjectured. For all Hilbert spaces which have a finite dimension that is a power of two ($d = 2^m$), the cyclic sets can be generated explicitly with the discussed methods. Results for $m = \{1, \dots, 600\}$ are given. A generalization of this construction is able to generate sets with different entanglement structures. It is shown that for dimensions $d = 2^{2^k}$ with k being a positive integer, a recursive construction of complete sets exists at least for $k \in \{0, \dots, 11\}$, where for higher dimensions a direct connection to an open conjecture in finite field theory by Wiedemann is identified. All discussed sets can be implemented directly into a quantum circuit by an invented algorithm. The (unitary) equivalence of the considered sets is discussed in detail.

In the second part of this work the security of a quantum public-key encryption protocol is discussed, which was recently published by Nikolopoulos [Nik08a], where the information of all published keys is taken into account. Lower bounds on two different security parameters are given and an attack on single qubits is introduced

which is asymptotically equivalent to the optimal attack. Finally, a generalization of this protocol is given that permits a noisy-preprocessing step and leads to a higher security against the presented attack for two leaked copies of the public key and to first results for a non-optimal implementation of the original protocol.



Zyklische komplementäre Basen und Quantenkryptographie mit öffentlichen Schlüsseln

Kurzfassung

Quantenmechanische Phänomene verleihen der Quanteninformationstheorie ein Potenzial, welches über die klassische Informationstheorie hinausgeht. Die hierin verankerte Fähigkeit, komplementäre Information zu erzeugen, bietet viele neue Möglichkeiten. Die Theorie zur Quantenschlüsselverteilung nutzt diese Information unmittelbar aus, um beweisbar sichere kryptografische Verfahren umzusetzen. Zur Realisierung einer solchen Quantenschlüsselverteilung, aber auch zur Bestimmung eines Quantenzustandes, beispielsweise um einen Quantencomputer zu realisieren, werden gewisse Mengen von Messbasen benötigt. Eine kleinstmögliche Menge dieser Messbasen ist eine sogenannte vollständige Menge von komplementären Basen. Im ersten Teil dieser Arbeit wird die Konstruktion solcher Mengen betrachtet. Diese haben die zusätzliche Eigenschaft, zyklisch zu sein, d. h. jedes Element der Menge lässt sich als Vielfaches eines bestimmten Generatorelementes der Menge erzeugen. Es wurde bereits gezeigt, dass die Theorie der komplementären Basen mit einigen anderen mathematischen Gebieten verwandt ist. Hier wird ein Zusammenhang der Konstruktion von zyklischen komplementären Basen mit Fibonaccipolynomen beleuchtet. Weiterhin wird die Existenz einer symmetrischen Begleitmatrix über dem endlichen Körper \mathbb{F}_2 vermutet. Die behandelten zyklischen Mengen von komplementären Basen können für alle endlichen Hilbertraumdimensionen explizit erzeugt werden, deren Dimension ein Vielfaches von zwei ist ($d = 2^m$); Ergebnisse für $m = \{1, \dots, 600\}$ werden aufgeführt. Eine Verallgemeinerung dieser Konstruktion ist in der Lage, Mengen zu erzeugen, welche eine alternative Struktur der Verschränkung aufweisen. Für den Fall dass die Dimension des Hilbertraumes $d = 2^{2^k}$ beträgt, wobei k eine positive ganze Zahl ist, existiert eine rekursive Erzeugungsmethode, solange $k \in \{0, \dots, 11\}$ gilt. Für alle höheren Werte von k wird diese Konstruktion mit einer offenen Vermutung von Wiedemann aus dem Bereich der Theorie endlicher Körper in Verbindung gebracht. Alle behandelten Mengen lassen sich mithilfe eines vorgestellten Algorithmus unmittelbar als Quantenschaltkreis realisieren. Die (unitäre) Äquivalenz verschiedener behandelte Mengen wird ebenfalls im Detail betrachtet.

Der zweite Teil dieser Arbeit behandelt die Sicherheit eines kürzlich von Nikolopoulos [Nik08a] vorgestellten asymmetrischen Verschlüsselungsprotokolls, welches öffentliche Quantenschlüssel verwendet, wobei der Informationsgewinn aus allen veröffentlichten Schlüsseln für die Betrachtung eines potenziellen Lauschers berücksichtigt wird. Es werden untere Schranken für zwei verschiedene Sicherheitsparameter angegeben sowie ein Angriff besprochen, welcher einfach zu realisieren ist, da er nur einzelne Qubits misst. Es wird weiterhin gezeigt, dass dieser asymptotisch äquivalent zu einem optimalen Angriff ist, welcher physikalisch schwieriger umzusetzen ist. Abschließend wird eine Verallgemeinerung des Protokolls vorgestellt, welche durch das absichtliche Einbauen von Störungen zu einer höheren Sicherheit führt. Exemplarisch wird dies für den Fall gezeigt, dass ein bzw. zwei Exemplare des öffentlichen Schlüssels vom Angreifer abgefangen werden. Diese Verallgemeinerung kann auch zur Betrachtung einer nicht idealisierten Realisierung des Ausgangsprotokolls genutzt werden.



Acknowledgments

This thesis was prepared in the context of my research activities in the group of Prof. Dr. Gernot Alber. I would like to thank him for his encouragement and support, for his advice and trust in me, and for inviting great guests.

I also like to thank Prof. Dr. Robert Roth for being the second referee of this thesis and investing his time in studying my work.

I am grateful to CASED for supporting me by a scholarship and for the opportunity to work in an interdisciplinary environment.

The problem of *constructing complete sets of cyclic mutually unbiased bases*, which is a main goal in the first part of this work, was brought to me by Christopher Charney, who visited our group in 2008 for several months and who deserves many thanks from me, also for introducing me into mathematical theories like finite field theory or representation theory.

Many thanks to Oliver Kern and Kedar Ranade for having the first ideas on the construction of the sets, for the fruitful collaboration and many nice and funny discussions.

Influenced by discussions with Luis Sánchez-Soto, who I like to thank very much, the focus of the first part moved a little towards the construction of complete sets of cyclic mutually unbiased bases with different entanglement properties.

The second part of this work is based on a manuscript of Georgios Nikolopoulos, who was also guest in our group and who had the patience to discuss with me my ideas on the security of his protocol. Let me also thank him for this very nice collaboration.

Finally, many thanks for the valuable contributions to both parts of the work which were done in collaboration with the Bachelor students Niklas Dittmann and Walid Mian.

I always enjoyed the environment in the complete group with many discussions and debates on physical topics, as well as diverse topics which are not related to physics at all. So let me thank all members and former members I met in this group.

I cannot thank Kedar Ranade enough for improving my thesis substantially by proofreading the draft very carefully, for the very fruitful collaboration, many discussions and his frequent visits.

Most importantly, I would like to thank my family, especially for lighten the load for me as much as possible, for tolerating my addiction to think about research problems almost everywhere, and also for assisting me with nearly everything in the last weeks before the submission. Thank you, Andrea! Thank you for your belief in me and your love. Many thanks to Johanna for being such a nice daughter, who sends laughs the whole day and who had to miss her daddy too much in the last few weeks. It is wonderful to have both of you by my side.

Finally, many thanks to my parents, who rendered all this possible and gave me always the liberty to follow my own path. . .



Contents

Notation	xiii
1 Introduction and outline	1
1.1 Cyclic mutually unbiased bases	3
1.2 Quantum public-key encryption	5
I Cyclic mutually unbiased bases	7
2 Introduction	9
2.1 Relations	10
2.2 Quantum state estimation	11
2.3 Quantum key distribution	12
3 Fundamentals	15
3.1 Constructions	15
3.1.1 Exponential sum analysis	16
3.1.2 Pauli operators partition	18
3.2 Fibonacci polynomials	22
3.2.1 Fibonacci polynomials over \mathbb{F}_2	24
4 Construction of cyclic sets	27
4.1 Fibonacci-based sets	28
4.1.1 Numerical construction of reduced stabilizer matrix	32
4.1.2 Reduced stabilizer matrix as symmetric companion matrix	35
4.2 Fermat sets	36
4.3 Standard form	39
4.4 Entanglement properties	40
4.5 Homogeneous sets	43
4.5.1 Group-based sets	44

4.5.2	Semigroup-based sets	46
4.6	Inhomogeneous sets	47
4.7	Unitary operator	47
4.7.1	Fibonacci-based sets	48
4.7.2	Fermat sets	51
4.7.3	More general sets	54
4.8	Gate decomposition	55
4.8.1	Fibonacci-based sets	55
4.8.2	Homogeneous sets	57
4.8.3	Inhomogeneous sets	62
4.8.4	Practical implementation	63
5	Equivalence of mutually unbiased bases	65
5.1	Heisenberg group partition sets	66
5.1.1	Fibonacci-based sets	67
5.1.2	(In-)Homogeneous sets	68
5.1.3	Different constructions	69
5.2	Class-permutation operators	69
6	Conclusions and further work	71
II Quantum public-key encryption		73
7	Introduction	75
7.1	Single-qubit-rotation protocol	76
8	Security analysis	79
8.1	Preliminary considerations	79
8.2	Security of the private key	80
8.3	Security of a message	82
8.3.1	Information about the public key	83
8.3.2	Information about the message	84
8.4	Noisy preprocessing	89
8.4.1	Single-key test attack	90
8.4.2	Double-key test attack	91
8.4.3	Protocol and security analysis	92
9	Conclusions and outlook	97
Appendix		97
A	Algebra and quantum information	99
A.1	Fundamentals	99
A.2	Finite fields	100
A.3	Hadamard matrix	100

A.4	Pauli operators	101
A.5	Bloch sphere representation	102
A.6	Clifford group	103
B	Wiedemann's conjecture proof approach	105
B.1	Unitary operator approach	105
B.1.1	Two-rowed diagonal minors	106
C	Results	109
C.1	Solutions for cyclic MUBs	109
C.1.1	Homogeneous sets with group structure	109
C.1.2	Homogeneous sets with semigroup structure	112
C.1.3	Inhomogeneous sets	115
C.1.4	Testing Wiedemann's conjecture	119
C.1.5	Triangle solutions	121
C.1.6	Companion matrix solutions	126
C.2	Fractal patterns	128
	Bibliography	133
	Index	141
	Curriculum Vitae	143



Notation

In this document, several mathematical symbols are used. To avoid misinterpretations, we list the most important symbols in the following, starting with the definition of different sets, operators, and matrices. The section ends by a description of the notation of vectors.

Sets

Symbol	Meaning
K	A general set
\mathbb{R}	Set of real numbers
\mathbb{C}	Set of complex numbers
\mathbb{N}	Set of natural numbers starting with zero
\mathbb{N}^*	Set of natural numbers starting with one
\mathbb{Z}	Set of integers
\mathbb{Z}_m	Set of integers modulo m with $m \in \mathbb{N}^*$
\mathbb{F}_p	Finite field with p elements, $p \in \mathbb{N}^*$ and p prime
\mathbb{F}_{p^m}	Finite field with p^m elements, $p, m \in \mathbb{N}^*$ and p prime
$M_m(K)$	Set of $m \times m$ matrices with entries from K
$\text{GL}_m(K)$	Group of invertible $m \times m$ matrices with entries from K

Operations

Symbol	Meaning
\oplus_m	Addition of two values modulo $m \in \mathbb{N}^*$
$\text{gcd}(a, b)$	Greatest common divisor of $a, b \in \mathbb{N}^*$
$\lfloor x \rfloor$	Largest integer not greater than $x \in \mathbb{R}$ (Gaussian floor)
$\lceil x \rceil$	Smallest integer not less than $x \in \mathbb{R}$ (Gaussian ceiling)
$\langle a \rangle$	Group generated by the element a
$a \otimes b$	Tensor product of two linear operations a, b which act on a finite dimensional Hilbert space

Matrices

Symbol	Meaning
$\mathbb{1}_m$	m -dimensional identity matrix
0_m	$m \times m$ zero matrix

Vectors

Usually, vectors are written in the form $\vec{a} = (a_1, \dots, a_m)^t$, with numbers a_1, \dots, a_m from a certain set, $m \in \mathbb{N}$ and $(\cdot)^t$ denoting the transposition of a vector or matrix. In few cases, where it is mentioned in addition, the same vector is given by \mathbf{a} for a better readability.

Introduction and outline

As long as humans populate our world, their addiction to analyze the incidents and the behavior of this world is gigantic. On the one hand, this attitude may result from the benefits this knowledge provides. On the other hand, it seems to be based on the pure exploration urge. Already a long time ago, humans started to systematize their knowledge; it leads to specifically adapted methods for the different fields and allows a better overview which is helpful in order to teach younger generations. A first approach which coined nowadays methods in the occidental culture was given by the *Platonic Academy*. The idea of science was introduced and different sciences were defined. Aristoteles was one of the first members of this Academy who was motivated in discussing natural phenomena; the physical laws were summarized systematically the first time by Isaac Newton in his famous *Principia Mathematica* in the 17th century. Later on, based on the number of results and obviously the different forms of research, natural sciences were divided amongst others into biology, chemistry, and physics. Around the 19th century, different research topics on physics were of interest, which are nowadays called the areas of *classical physics*, namely mechanics, electrodynamics, thermodynamics, and optics. With the beginning of the 20th century, *modern physics* came up with the theory of *general relativity* and *quantum mechanics*. The latter became necessary in order to avoid the ultraviolet catastrophe which came up with the Rayleigh–Jeans law which was derived to describe black-body radiation. Max Planck finally solved this problem by introducing the so-called Planck constant h . The aim of quantum mechanics is, roughly speaking, the description of physical phenomena at microscopic scales, but many descriptions succeeded this idea which are contrary to the usual expectation. There are Heisenberg’s *uncertainty principle* (*complementary variables cannot be measured perfectly*), *entanglement* (a system can have more information than the sum of the information of the subsystems), *no-cloning-theorem* (*unknown quantum systems cannot be copied perfectly*), and many more. Fundamental legitimation problems still arise from the Copenhagen interpretation which describes the measurement process as a wave-function collapse which is not convenient to the normal time evolution in quantum physics. Bell introduced his famous no-go theorem which describes a test to distinguish between system states which occur in classical

physics and systems states which occur only in quantum physics. His goal was to solve the famous Einstein-Podolsky-Rosen paradox which challenged the Copenhagen interpretation in order to ask whether quantum mechanics is complete. Along with these fundamental questions, the effects which originate from the theory of quantum mechanics (and which are also observed in experiments) have the potential for several applications.

As long as humans populate our world, their creativity in order to avoid undesired duties is unlimited. An important benefit is to save time for more important concerns, another is possibly the unattractiveness of tasks which can be schematized easily. A first important innovation were water clocks, already known in Babylon which were able to measure time automatically. Based on the steam engine, the industrial revolution started in the 18th century and helped the workers to implement larger projects. Around the same time, Charles Babbage invented the first *mechanical computer* and the concept of a programmable computer in order to automatize tedious and error-prone calculations. Ada Lovelace worked theoretically on this computer in the 19th century and is seen as the first programmer. The first digital computer was build in the 20th century and was able to solve simple mathematical problems; with the miniaturization computers became more and more powerful and helped to solve complicated mathematical problems. Numerical methods and the idea of simulating systems with computers took place into the scientific research. Nevertheless, many problems are too hard to be implementable efficiently into classical computers.

As long as humans populate our world, they have secrets which they like to share only with selected persons. Different *ciphers* were known already in the ancient Greece in order to hide information or to make information unreadable to third parties. In the 20th century, more complicated ciphers were implemented by machines, like the *Enigma* during the second world war, which was able to encrypt and to decrypt messages in a complicated but logical way. As all algorithms were broken, they became more and more complex. In 1882, a simple cipher, the *one-time pad* was described by Frank Miller¹. It encodes each letter of the message with an individual letter of the key, thus the key has to be as long as the message. It can be proven by methods of information theory, that this cipher is unbreakable, if the key is perfectly random, only used once and not leaked by a third party. Those problems seem to be unsolvable in the classical ways.

A combination of these three mentioned research areas, namely *quantum mechanics*, *computer sciences*, and *cryptography*, is given by the field of *quantum information theory*. Equipped with the effects of quantum mechanics, it seems that a new generation of computers, so-called *quantum computers*, may have the capability to expand the efficiency of classical computers dramatically. The first example is the Deutsch–Jozsa algorithm, which scales exponentially faster on a quantum computer, but is mostly of scientific interest [DJ92]. The first quantum algorithm which demonstrated the practical potential of the quantum computer was invented by Peter Shor in 1994 [Sho97] and is able to perform prime factorization in polynomial time—which does not seem to be possible for classical computers. As many common asymmetric cryptographic ciphers such as RSA are based on the difficulty of this problem, Shor’s algorithm would affect

¹Steven Bellovin figured out in 2011 that Frank Miller invented the one-time pad 35 years before Gilbert Vernam and Joseph Mauborgne, who were in general seen as the inventors [Bel11].

instantly the security of nowadays secret information—which is important for the security of credit cards, for instance.² Already in 1984, ideas of Charles Bennett and Gilles Brassard paved the way to recover a security scheme which may replace the classical cryptography by *quantum cryptography*. After Shor’s algorithm was known, security proofs for the *quantum key distribution* protocol were given. Finally, as quantum computers seem to be efficient, they are seen as an attractive candidate to simulate quantum systems for their analysis.

To realize a quantum computer, different methods and tools need to be explored. For several purposes, the construction of *complete sets of cyclic mutually unbiased bases* is relevant as will be seen in Section 1.1. These bases find their applications also in the field of quantum cryptography. Regarding the development of the field of quantum cryptography, the security of many protocols is important and should be analyzed. A discussion on the security of a recently invented quantum public-key encryption scheme is started in Section 1.2.

1.1 Cyclic mutually unbiased bases

The first part of this work deals with the problem of constructing *complete sets of cyclic mutually unbiased bases* (MUBs).

A detailed introduction into the history, the properties and the applications of MUBs is given in **Chapter 2**. Complete sets of MUBs play an important role for quantum state tomography of finite dimensional quantum systems, as they define a minimal set of measurement bases. This qualifies them for example to be a candidate which measures states of quantum registers, a part of a quantum computer. Furthermore, MUBs have a large potential in quantum cryptographic protocols. In order to increase the efficiency of these protocols it turned out that, by using higher-dimensional information carriers, complete sets of MUBs need to be constructed for higher dimensions, most suitable with a cyclicity property [Cha02, Cha05, RA06]. This cyclicity is a special property those sets may obey, which means that a whole set is generated by the powers of a single element. As it is not even known yet if complete sets exist in all complex Hilbert spaces with a finite dimension, MUBs are still a field of current research.

Whereas first ideas on complete sets of MUBs were given by Ivanović [Iva81] and Wootters and Fields [WF89], applications of cyclic sets and alternative constructions were discussed many years later by Chau [Cha05] and Gow [Gow07]. These ideas are retraced in **Chapter 3**, together with mathematical methods which are used later on.

²In 1996, another important quantum algorithm was found by Lov Grover [Gro96], which improves the search in an unsorted list quadratically whereas the classical algorithm seems to be optimal within a classical setup.

In **Chapter 4**, a construction of complete sets of cyclic MUBs is introduced. Following the methods of a systematic scheme of Bandyopadhyay *et al.* [BBRV02], we published a first work in 2010 in which we construct cyclic sets of MUBs in all even prime power dimensions [KRS10]:

Complete sets of cyclic mutually unbiased bases in even prime-power dimensions,

by Oliver Kern, Kedar S. Ranade, and Ulrich Seyfarth,
in Journal of Physics A **43**, 275305 (2010).

The introduced methods allow to reduce the problem of explicitly constructing a complete set of MUBs for a Hilbert space of dimension $d = 2^m$ in a first step from $(2^m)^2$ free variables of \mathbb{Z}_4 to m^2 variables of \mathbb{Z}_2 . With the help of a second step, an assumed reduction of the search space, the number of free variables goes below $(m/2)^2$.

A second work formalizes and extends these results as it shows the relation of the construction with so-called Fibonacci polynomials [SR12]:

Cyclic mutually unbiased bases, Fibonacci polynomials and Wiedemann's conjecture,

by Ulrich Seyfarth and Kedar S. Ranade,
in Journal of Mathematical Physics **53**, 062201 (2012).

Furthermore, we prove the existence of complete sets of cyclic MUBs for the discussed construction. Results of both manuscripts are given in Section 4.1; numerical methods which deal with the search of the solutions in the remaining space which is spanned by the free variables are summarized in Section 4.1.1. An analytical approach which may lead to a *symmetric companion matrix*³ is given in Section 4.1.2. In Section 4.2 results of the second work are presented, which show that for $m = 2^k$ with $k \in \mathbb{N}$, a complete set of cyclic MUBs for dimension $d = 2^{2^k}$ can be constructed recursively. Namely, it can be constructed from the complete set of cyclic MUBs for dimension $d = 2^{2^{k-1}}$ at least for $k \in \{1, \dots, 11\}$, which is shown in Appendix C.1.4, limited by the largest *Fermat number* for which the prime factorization is known. For all k , it is proven in that section that the problem is related to an open conjecture in finite field theory by Wiedemann [Wie88] which is still of current interest [MS96, Vol10]; an approach for a proof is given in Appendix B. In Section 4.3 a unique form of representing complete sets of cyclic MUBs in order to be able to compare different sets is given.

Discussions with L. L. Sánchez-Soto drew the author's attention to the problem of constructing complete sets of cyclic MUBs with different entanglement properties. An introduction is given in Section 4.4, a first subclass, presented in Section 4.5.1, is a generalization of the construction which was explored in the two mentioned articles. First approaches on two other classes are given in Sections 4.5.2 and 4.6. Results can be found in Appendices C.1.1, C.1.2, and C.1.3.

An important representation of sets of MUBs are sets of unitary operators. Therefore, a transformation of the different sets into a unitary operator representation is

³The conjectured construction of a symmetric companion matrix assumes, that for each polynomial with coefficients in \mathbb{F}_2 , a symmetric matrix can be constructed which has that polynomial as its characteristic polynomial.

given in Section 4.7, which is published in the first work. For those sets which can be constructed recursively, we published another manuscript [SR11]:

Construction of mutually unbiased bases with cyclic symmetry for qubit systems,

by Ulrich Seyfarth and Kedar S. Ranade,
in Physical Review A **84**, 042327 (2011).

The first part of these results is shown in Section 4.7.2, namely that the corresponding unitary operator can also be constructed recursively. Finally, the generators of these complete sets of MUBs can be implemented by a quantum circuit into an experimental setup. In the context of the Bachelor thesis of N. Dittmann, the construction of the circuit was generalized for all discussed sets of cyclic MUBs and even more general operators, which is presented in Section 4.8. As the implementation of a large cyclic set may accumulate errors, a more practical implementation for such sets is drawn in Section 4.8.4, which makes nevertheless use of the cyclic structure.

Chapter 5 deals with the equivalence of MUBs. A slight generalization of the results of [SR12] is presented and it is proved that the introduced method constructs complete sets of cyclic MUBs which are (unitary) equivalent to others like the non-cyclic sets constructed by Wootters and Fields [WF89].

Finally, the results are concluded in **Chapter 6** and an outlook on possible future research topics is given.

To keep the sections short, many results are shown in the appendices as well as basic mathematical properties. In **Appendix A**, tools from algebra and quantum information theory which are important for this work are summarized.

An approach to prove Wiedemann's conjecture is presented in **Appendix B**.

Most of the computational results are given in a compressed form in **Appendix C**. Section C.2 shows the appearance of similar fractal patterns in Fibonacci polynomials and characteristic polynomials of certain matrices which are both important for the construction of cyclic MUBs. Section C.1.5 lists generators for complete sets of cyclic MUBs for dimensions $d = 2^m$ with $m \in \{2, \dots, 600\}$ for the method introduced in [KRS10], taking advantage of the improvements of [SR12]. The results which may indicate the existence (and maybe a construction) of a symmetric companion matrix are given shortly in Section C.1.6. Then, generators of sets of cyclic MUBs with different entanglement properties are listed for four-qubit systems in Sections C.1.1 and C.1.2 and in Section C.1.3, respectively. Finally, the *Matlab* code which is used to test Wiedemann's conjecture for $k \in \{0, \dots, 11\}$ is shown in Section C.1.4.

1.2 Quantum public-key encryption

The second part of this work treats the security of an asymmetric protocol of quantum cryptography which was recently published by Nikolopoulos [Nik08a]. As quantum cryptography wants to offer *unbounded security*, detailed security analyses are essential. Since no security proof exists yet for this commonly known *quantum public-key encryption* (QPKE) protocol, a detailed analysis was published [SNA12]:

Symmetries and security of a quantum-public-key encryption based on single-qubit rotations,

by Ulrich Seyfarth, Georgios M. Nikolopoulos, and Gernot Alber,
in Physical Review A **85**, 022342 (2011).

In **Chapter 7**, an introduction into the protocol is given.

Within **Chapter 8** all new investigations on the security of the protocol are depicted. Namely, Section 8.2 deals with the security of the private key which is essential for the security of the protocol. If though, the security of encrypted messages has to be guaranteed in addition, as they can be attacked directly and in the worst case by using all copies of the public key. This message security is analyzed in Section 8.3, where the security against a used security parameter is shown for an attack which can be implemented easily. Nevertheless, it is also shown that this attack has a similar behavior as a class of very general attacks.

In the context of the Bachelor thesis of W. Mian the effect of a noisy-preprocessing step was analyzed. With two simple attacks in Sections 8.4.1 and 8.4.2 it is shown in Section 8.4.3 that the security of the message increases with the help of this method. Nikolopoulos' protocol, which is assumed to work in an *ideal* description of the world, has to be transformed into a protocol which is used in a *real* description of the world, as errors come into account. But therefore, this method can also be used in order to model errors in the construction process of the public key. Obviously, the implementation of error correction protocols needs to be discussed within further work.

The second part is concluded and an outlook is given in **Chapter 9**.

Part I

Cyclic mutually unbiased bases

Introduction

A fundamental characteristic of quantum mechanical systems is the *uncertainty principle* which was formulated by Heisenberg in 1927 [Hei27]. It describes the observation that pairs of physical variables exist which cannot be measured simultaneously with maximal precision. If this mutual influence is maximal, the variables are called *complementary*.¹ As finite-dimensional quantum mechanical systems are described by density matrices that are defined in the Hilbert space $\mathcal{H} = \mathbb{C}^d$, a set of operators exists that defines a *unitary operator basis* (and is capable to describe properties of complementary variables). Back in 1960, Schwinger derived initially a *complementary pair of operators* that is able to describe two complementary variables in an arbitrary finite dimensional Hilbert space [Sch60]. It turned out that the absolute overlap of two vectors from different bases is constant for a given dimension. Ivanović followed this path, motivated by the complete state estimation of an unknown quantum state, and figured out that the minimal number of pairwise complementary operators needed to determine the quantum state completely, equals the dimension of the Hilbert space plus one [Iva81]. His considerations result in a construction method for a *complete set* in prime Hilbert space dimensions, whereas Wootters coined the notion of *mutually unbiased bases* (MUBs) for pairwise complementary operators [Woo86]. In collaboration with Fields, he presented a construction method for complete sets of MUBs in prime power Hilbert space dimensions [WF89]. Pointing back again to Heisenberg's uncertainty principle, it was conjectured by Kraus and shown by Maassen und Uffink that the optimal solution for a certain kind of uncertainty relations is a set of MUBs [Kra87, MU88].

It still remains an open question that is under current research, how many bases a maximal set of MUBs contains for *composite* Hilbert space dimensions.² Even for the smallest such dimension, which is $d = 6$, this problem is unsolved, where the

¹A common example is the measurement outcome of the electron spin; if the spin is known to be in an eigenstate of the Pauli- σ_x operator, the outcome of a measurement in the basis of the Pauli- σ_z operator is completely undetermined.

²The term "composite dimension" refers to dimensions that cannot be represented as the power of a prime number.

largest known sets have only three elements.³ Also many different constructions to that given by Wootters and Fields were figured out (e.g. [BBRV02, KR04]).⁴ In 2005, Chau proved a theorem which predicts the existence of a cyclic group generator that can be used to generate a complete set of MUBs [Cha05];⁵ in other words, such a complete set of *cyclic* MUBs is given by the powers of a single unitary operator. It was shown by Gow [Gow07], that cyclic sets exist only for *even-prime power* dimensions. An example of such a cyclic set was already used by Gottesman in 1998 in order to transform the three Pauli operators cyclically [Got98]. The advantages of a cyclic generation of a set of MUBs were used in proofs of the quantum cryptographic six-state protocol [Lo01, GL03] and even an abstract definition of such sets was taken into account to prove generalizations of that protocol [Cha05].

The aim of the first part of this work is to continue this path by finding explicit constructions of cyclic MUBs in a straightforward way with suggestions for a direct implementation of these sets into experimental setups. After addressing the fundamental properties of MUBs, their usage in the fields of *quantum state estimation*, *quantum key distribution*, and further relations in the subsequent sections, different well-known constructions will be discussed in Section 3.1 of Chapter 3. Properties of the so-called *Fibonacci polynomials* are presented and extended in Section 3.2, which are fundamental for the generation of complete sets of cyclic MUBs in Chapter 4. Three different constructions are given in Sections 4.2, 4.5, and 4.6, that aim on sets with specific entanglement properties which are derived in Section 4.4. It turns out that another form, the *Fermat-based sets* which are discussed in Section 4.1, are related to an open conjecture in finite field theory which was given in 1988 by Wiedemann [Wie88]. Supposed this conjecture is true, an unlimited class of complete sets of cyclic MUBs can be created by the given recursive construction. Until then, the results given in Appendix C.1.4 can be used for the construction of complete sets of cyclic MUBs for dimensions $d = 2^{2^k}$ with $k \in \{0, \dots, 11\}$.⁶ By their nice form, these sets can be implemented quite easily into *quantum circuits* as shown in Section 4.8.1. For more general cyclic MUBs, another method is presented in Section 4.8.2. An improved practical implementation is suggested by a promising method in Section 4.8.4. The classification of the generated sets of MUBs is derived in Chapter 5, where the *equivalence* of the sets with known constructions is discussed. The results are summarized and an outlook is provided in Chapter 6.

2.1 Relations

From a mathematical point of view, MUBs are related to many mathematical objects of other research fields. In the context of the existence of a complete set of MUBs with $d + 1$ bases, a well-known connection to *orthogonal Latin squares* was established [Zau99, GHW04, PDB09], but also *finite projective planes* play an impor-

³There is strong evidence that these sets are maximal [Zau99, Gra04, BBE⁺07, RLE11].

⁴A recent review article on MUBs was published by Durt *et al.* [DEBZ10].

⁵The proof by Chau is based on finite field theory, whereas Gow gave a proof based on representation theory [Gow07].

⁶The limitation to $k = 11$ is limited due to the largest known prime factorization of Fermat numbers.

tant role [SPR04, Ben04, BE05]. In the context of equivalence of MUBs, relations to *symplectic spreads* [CCKS97] and *affine planes* [Kan12] are of importance.

2.2 Quantum state estimation

As already mentioned, MUBs were originally introduced in the context of quantum state estimation [Iva81, WF89] and seen as a good candidate for optimal schemes for *quantum state tomography*. For a systematic approach, we may observe the general state of a quantum system, which is defined in a d -dimensional Hilbert space $\mathcal{H} = \mathbb{C}^d$ by a density operator

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \quad (2.1)$$

which is diagonal in its orthonormal eigenbasis $\{|\psi_i\rangle\}_i$ with a normalized probability distribution, where $\sum_i p_i = 1$ holds. More generally, we can describe the state vectors $|\psi_i\rangle$ as the eigenvectors of ρ with their corresponding eigenvalues p_i . An experiment which measures the state ρ of the system in its orthonormal eigenbasis, will measure the state $|\psi_i\rangle$ with probability p_i . Since these probabilities have to be real, quantum mechanics postulates ρ to be Hermitian, thus $\rho = \rho^\dagger$.

If we are aware of the eigenbasis of ρ and have infinitely many copies of the system state ρ , we can measure infinitely many times within the eigenbasis to reconstruct the probability distribution, thus we have to solve a well-known classical problem, namely the approximation of a probability distribution by sampling. The number of free parameters of ρ is $d - 1$, since ρ can be represented by a diagonal $d \times d$ matrix with a normalized set of real eigenvalues.

Conversely, if we are not aware of the eigenbasis of the state ρ , it can still be represented by a normalized Hermitian matrix that might not be diagonal and has at most d^2 nonzero entries. By Hermiticity, those entries can be described by d^2 real parameters. Normalization of the matrix fixes another parameter and leaves $d^2 - 1$ free parameters⁷ that describe an arbitrary quantum state ρ of the d -dimensional Hilbert space \mathcal{H} . Measuring on infinitely many copies of the system in some basis $\{|\phi_i\rangle\}_i$, we learn at most $d - 1$ free parameters due to the normalization of the measurement outcome. Since the total number of free parameters of the quantum state ρ is $d^2 - 1$ and by a single measurement operator we can figure out $d - 1$ of those parameters, it turns out that we need at least $d + 1$ different bases to completely estimate the quantum state.

The set of unitary operators of the d -dimensional Hilbert space defines exactly the set of possible orthonormal bases, where the row vectors of the unitary operators are identified as the basis vectors of the corresponding orthonormal basis. Thus, in order to completely estimate the system state ρ , one has to find a set of $d + 1$ bases from the set of unitary operators.⁸ Given such a set, the application of a unitary

⁷In contrast to the $d - 1$ free parameters in the case where the system is measured in its eigenbasis, additional free parameters appear which encode the basis information.

⁸In the following, we will use this correspondency without mentioning again and declare unitary operators as bases.

transformation to all bases will not affect the amount of information that is extracted by the measurements. Hence, we are free to choose one of the bases to be the standard basis. Measuring a state that is diagonal in that standard basis with one of the d remaining bases should lead to equally probable outcomes, even if the system is in a definite state (in the standard basis). To achieve this goal, the overlap of every vector of this basis with every vector from the standard basis has to be constant. If we want to find such a set of $d + 1$ bases to reconstruct the state ρ completely, the operators have to fulfill those properties pairwise, which leads to the concept of MUBs.

Definition 2.2.1 (Mutually unbiased bases).

A set of orthonormal bases $\mathfrak{S} = \{\mathcal{B}_0, \dots, \mathcal{B}_{r-1}\}$, $r \in \mathbb{N}^*$, of the d -dimensional Hilbert space $\mathcal{H} = \mathbb{C}^d$ is called a *set of mutually unbiased bases (MUBs)*, if for every pair $(\mathcal{B}_k, \mathcal{B}_l)$ with $k \neq l$, the absolute value of the overlap of their basis vectors is constant. With $\mathcal{B}_k = \{|\psi_1^k\rangle, \dots, |\psi_d^k\rangle\}$ and $\mathcal{B}_l = \{|\psi_1^l\rangle, \dots, |\psi_d^l\rangle\}$, there holds

$$|\langle \psi_i^k | \psi_j^l \rangle| = 1/\sqrt{d}, \quad (2.2)$$

for $k, l \in \{0, \dots, r-1\}$, $k \neq l$, and $i, j \in \{1, \dots, d\}$.

Such a set of MUBs is called *complete* if no set in the same Hilbert space exists which has a higher number of elements, it is therefore a *maximal* set. For a Hilbert space of dimension d this size is in general still unknown. Nevertheless, it was shown by Wootters and Fields with geometric arguments that each set has at most $d + 1$ elements [WF89];⁹ another proof based on the connection of sets of MUBs with *pairwise orthogonal matrices* was given by Bandyopadhyay *et al.* [BBRV02].

2.3 Quantum key distribution

Another area of application for sets of MUBs emerged in 1984, when Bennett and Brassard introduced their ideas for *quantum key distribution* (QKD) [BB84]. The aim of this approach is to solve the classically unsolved problem of a secure distribution of a common secret key to two distinct parties, called *Alice* and *Bob*. The classical cipher which is known as *one-time pad* guarantees a secret transmission of a message between two parties, provided that both parties share a perfectly random bit-key, which has at least the size of the message and is used only once. Classically, this key cannot be distributed secretly from Alice to Bob, but a guaranteed secure key transmission may be implemented with the help of QKD.

In a general formulation, Bennett and Brassard use in their approach a *qubit*¹⁰ in order to send *quantum information* from Alice to Bob. At random, Alice prepares the qubit either in an eigenstate of the Pauli- σ_x or Pauli- σ_z operator.¹¹ By construction, the eigenbases of these operators are mutually unbiased in the sense of Definition

⁹Wootters and Fields give also a construction of sets with $d + 1$ elements if the dimension of the Hilbert space is a power of a prime. For none of the remaining dimensions, even for $d = 6$, a construction of $d + 1$ elements is known.

¹⁰A *qubit* denotes a two-level quantum system, which may be in any complex superposition of two possible quantum states $|0\rangle$ and $|1\rangle$, namely $\alpha|0\rangle + \beta|1\rangle$, with $\alpha, \beta \in \mathbb{C}^2$, $|\alpha|^2 + |\beta|^2 = 1$.

¹¹More information about the Pauli operators is given in Appendix A.4.

2.2.1. In the chosen basis, quantum state $|0\rangle$ or $|1\rangle$ is again taken randomly with equal probability. After the qubit is sent to Bob, he measures randomly in one of the two distinct bases. If he chooses the same basis as Alice, he will obtain the encoded bit perfectly, if not, the property of the MUBs leads to a random output.¹² Therefore, Alice and Bob communicate after the measurement of the qubits over a *classical authenticated channel*¹³ and discard those pairs of bits, where they have chosen different bases. If the transmission would be perfect, both parties would share a common key at this point, but as the transmitted single qubits encounter quantum noise on the channel, a perfect correlation between the qubits of Alice and Bob cannot be guaranteed. By declaring some of the transmitted qubits as test qubits, they can calculate the induced *bit error rate*; in principle, this error rate cannot be distinguished from an error a possible eavesdropper *Eve* induced, so it is seen as information which is leaked to her. Possible attacks can be considered within a quite general way, but for simplicity reasons one may focus on a rather straightforward, but powerful attack, which is called *intercept-and-resend attack*. In this case, Eve measures each qubit in one of the two preparation bases randomly. Consequently, in average, in every second case the basis is correct and in cases where Eve took a different basis than Alice and Bob, her probability to measure the correct result is only 1/2. Therefore, she induces an error of 25% in the case she attacks each qubit. Alice and Bob can correct this error by using classical *error correction* protocols and try to rule out the information the eavesdropper got, by so-called *privacy amplification* protocols. The *tolerable bit error rate* (BER) by using the mentioned *post-processing* protocols is limited by the corresponding proof of the protocol, which guarantees *unconditional security*. For the protocol by Bennett and Brassard a first rigorous proof was given by Mayers which allows a tolerable BER of 7.5% [May96]. Many versions of this BB84 called QKD protocol are known (see e. g. [Eke91, Ben92]) and further rigorous proofs also in order to raise this limit were given (e. g. [LC99, SP00, GL03]). Finally, Chau proved that this limit can be raised asymptotically to 20% [Cha02]. A formulation of this result which is usable in a larger context was given by Ranade and Alber [RA06].

A promising approach which uses the properties of MUBs and generalizes the BB84 protocol was considered by Bruß in 1998 and is called *six-state* protocol [Bru98]; its security was again rigorously proven [Ina00, Lo01]. The six-state protocol makes use of the third variable σ_y , that is complementary to σ_x and σ_z . So the protocol uses then a complete set of three MUBs in a Hilbert space of dimension two. By the same arguments as above, it is clear that only 1/3 of the transmitted pairs can be used. It was again proven by Chau as well as by Ranade and Alber, that the tolerable error rate is roughly 27.6%, which is clearly above the limit for the BB84 protocol [Cha02, RA06]. Considerations of possible generalizations of this protocol show that this rate can be raised asymptotically up to 50% by using complete sets

¹²Imagine, the eigenstates of the Pauli- σ_z operator are denoted by $|0\rangle$ and $|1\rangle$. If the quantum state $2^{-1/2}(|0\rangle + |1\rangle)$ (which is then an eigenstate of the Pauli- σ_x operator) is measured by the operator $|1\rangle\langle 1|$, the respective expectation value is 1/2, thus no information about the prepared state is measured.

¹³To guarantee, that the classical messages Alice and Bob receive are not sent by Eve, they have to be authenticated with the help of a previously shared key; used schemes take then advantage of *universal hash functions* as introduced by Wegman and Carter [WC81].

of MUBs and *qudits*¹⁴ as information carriers, which are defined in an d -dimensional Hilbert space [RA07]. Examinations of these protocols indicate, that a *cyclic* property of the set of MUBs is advantageous [Cha05]. Those cyclic sets have the property, that all elements within the set are given by the powers of one element of the set of bases.

¹⁴A *qudit* denotes the generalization of a qubit, namely a d -level quantum system with all possible complex superpositions of d orthogonal states for which their absolute squares sum up to one.

Fundamentals

The first natural step when aiming on the construction of complete sets of cyclic MUBs is the reconstruction of existing complete sets of MUBs with the purpose of finding steps in the construction that are suitable in order to generate cyclic sets. It may be useful to combine different aspects and ideas from different approaches in order to achieve this goal. Ultimately, these considerations may lead to new ideas that require fundamental observations of more distant aspects which become relevant. The aim of this chapter is to retrace exactly this path. Within Section 3.1, two different approaches for the construction of complete sets of MUBs will be discussed in order to get a notion of MUBs and to have a playground which enhances the potential for the construction of cyclic sets. It will turn out later in this work (cf. Section 4.1), that the properties of so-called *Fibonacci polynomials* are useful for that construction. The basis properties and their relation to the usual *Fibonacci series* will be discussed in Section 3.2, as well as advanced results that appeared in literature and own results.

3.1 Constructions

Many different constructions of complete sets of MUBs are known in literature. As later discovered by Klappenecker and Rötteler, Alltop gave a construction for all prime dimensions with $p \geq 5$ unknowingly in 1980 by solving a different problem [All80]. One year later, a construction which works for all prime dimensions was given by Ivanović [Iva81] and generalized to prime power dimensions by Wootters and Fields [WF89] many years later. This construction is based on basic observations of the properties of complete sets of MUBs and known results from number theory and field theory, respectively. Klappenecker and Rötteler gave a precise formulation of all these constructions more than a decade later, using finite fields and Galois rings more explicitly [KR04]. In the meantime, a different construction was discussed by Bandyopadhyay *et al.*, which is based on the partition of the set of Pauli operators [BBRV02].

As these two different approaches seem to be the most important constructions of complete sets of MUBs that appeared in literature, they will be discussed in the

following two sections. In Section 3.1.1, the ideas of Ivanović and Wootters and Fields are summarized, whereas Section 3.1.2 concerns with the construction suggested by Bandyopadhyay *et al.*

3.1.1 Exponential sum analysis

The first general construction of complete sets of MUBs was given by Ivanović for all finite dimensional Hilbert spaces with a prime dimension $d = p$ [Iva81]. To follow his approach, let us assume that a complete set of MUBs exists. According to Definition 2.2.1, it is clear by the usage of the scalar product, that the application of any unitary transformation to all elements of the set causes again a complete set of MUBs.¹ Therefore, w.l.o.g., if a complete set of MUBs exists, there is always another complete set, which includes the standard basis. To fulfill then Definition 2.2.1, all remaining bases should have only numbers as entries with an absolute value of $p^{-1/2}$.

For odd dimensions, Ivanović uses a property of number theory, namely

$$\left| \sum_{j=0}^{p-1} e^{(2\pi i/p)(sj^2+tj)} \right| = \sqrt{p}, \quad (3.1)$$

which holds for all $t \in \mathbb{N}$, $s \in \mathbb{N}^*$ and p being an odd prime number. This expression is the absolute value of a *generalized quadratic Gauss sum* [BEW98, p. 13]. If the component l of the vector k within the basis r is denoted as $(v_k^{(r)})_l$, the standard basis is given by

$$\left(v_k^{(0)} \right)_l = \delta_{kl}, \quad (3.2)$$

with $k, l \in \{0, \dots, p-1\}$. All remaining bases, i. e. with $r \in \{1, \dots, d\}$ within a complete set of MUBs read in this construction as

$$\left(v_k^{(r)} \right)_l = \frac{1}{\sqrt{p}} e^{(2\pi i/p)(rl^2+kl)}. \quad (3.3)$$

It can easily be checked that the bases given by Equation (3.3) define unitary operators and that all of them are mutually unbiased with respect to the standard basis. To test the mutual unbiasedness of all remaining pairs of bases in the fashion of Definition 2.2.1, the expression given by Equation (3.1) appears and guarantees the expected result. Wootters and Fields used the fact, that a generalization of Equation (3.1) exists in finite field theory and turns out to be a good candidate in order to generalize this construction of complete sets of MUBs to odd prime-power dimensions [WF89]. Namely, it is known that

$$\left| \sum_{j \in \mathbb{F}_p^m} e^{(2\pi i/p) \text{tr}(sj^2+tj)} \right| = \sqrt{p^m}, \quad (3.4)$$

¹As will be seen later on, this transformation is one of the transformations which leads to an *equivalent set* of MUBs (cf. Chapter 5).

holds for $s \neq 0$ and s, t being elements of the finite field with 2^m elements [LN08].² The trace denotes the trace defined in field theory, which maps an element of the field \mathbb{F}_{p^m} to the field \mathbb{F}_p according to Definition A.2.1 with resulting properties. The non-standard bases read finally similar to Equation (3.3) as

$$\left(v_k^{(r)}\right)_l = \frac{1}{\sqrt{p^m}} e^{(2\pi i/p) \operatorname{tr}(rl^2 + kl)}, \quad (3.5)$$

with $r, k, l \in \mathbb{F}_{2^m}$. Unitarity can again be checked easily. The standard basis is given analogously to Equation (3.2), but with k, l being elements of the finite field \mathbb{F}_{p^m} .

As the left-hand side of Equation (3.4) turns to zero in the case that the characteristic of the field equals $p = 2$, Wootters and Fields have reformulated the bases defined by Equation (3.5) in a different representation in order to generate complete sets of MUBs for even prime-power dimensions as will be discussed in a brief summary in the following. In principle, every finite field \mathbb{F}_{p^m} can be seen as a vector space over the ground field \mathbb{F}_p , thus every element $\beta \in \mathbb{F}_{p^m}$ can be written in a basis as $\beta = \sum_{i=1}^m \beta_i f_i$ with f_i being the basis vectors and β_i the coefficients. The product of two basis vectors can always be expressed within the bases by a set of coefficients like $f_i f_j = \sum_{n=1}^m \alpha_{ij}^{(n)} f_n$. With the help of this argumentation, the expression l^2 which appears in the argument of the trace in Equation (3.5) can be rewritten as

$$l^2 = \left(\sum_{i=1}^m l_i f_i\right)^2 = \sum_{n=1}^m \mathbf{l}^t \alpha^{(n)} \mathbf{l} f_n, \quad (3.6)$$

with $\mathbf{l} = (l_1, \dots, l_m)^t$ on the right-hand side being a column vector and using that $\alpha^{(n)}$ is a symmetric $m \times m$ matrix. Following this notion and that the trace in Equation (3.5) can be rewritten as

$$\operatorname{tr}(rl^2 + kl) = \sum_{n=1}^m \mathbf{l}^t \alpha^{(n)} \mathbf{l} \operatorname{tr}(r f_n) + \operatorname{tr}(kl), \quad (3.7)$$

the non-standard bases of Equation (3.5) are given by

$$\left(v_{\mathbf{d}}^{(\mathbf{c})}\right)_l = \frac{1}{\sqrt{p^m}} e^{(2\pi i/p)(\mathbf{l}^t(\mathbf{c}\cdot\boldsymbol{\alpha})\mathbf{l} + \mathbf{d}^t\mathbf{l})}, \quad (3.8)$$

where \mathbf{c} is a vector of the coefficients which appear by a similar transformation of that given in Equation (3.6) from r and \mathbf{d} a vector of the coefficients which appear analogously from k . Finally, $\boldsymbol{\alpha}$ denotes a column vector of the matrices $\alpha^{(1)}, \dots, \alpha^{(n)}$. The mutual unbiasedness of the bases was proven by Wootters and Fields also in the non-field representation of Equation (3.8), but again only for odd prime-power dimensions. For even prime-power dimensions, the construction of the set of bases has to be adapted slightly as

$$\left(v_{\mathbf{d}}^{(\mathbf{c})}\right)_l = \frac{1}{\sqrt{2^m}} e^{(2\pi i/2)(\mathbf{l}^t(\mathbf{c}\cdot\boldsymbol{\alpha})\mathbf{l}/2 + \mathbf{d}^t\mathbf{l})} \quad (3.9)$$

$$= \frac{1}{\sqrt{2^m}} i^{\mathbf{l}^t(\mathbf{c}\cdot\boldsymbol{\alpha})\mathbf{l}} (-1)^{\mathbf{d}^t\mathbf{l}}, \quad (3.10)$$

²It was mentioned by Klappenecker and Rötteler that Equation (3.4) is related to a stronger version of Weil's theorem [LN08, Theorem 5.37].

and was again proven to be a complete set of MUBs (cf. also [BEW98, p. 47]). Identifying the rightmost term as an m -folded tensor product of the Hadamard matrix as given in Equations (A.8) and (A.9), leads to an abbreviated form of Equation (3.10), namely

$$(\mathbf{v}^{(c)})_l = i^{l^t(\mathbf{c}\cdot\boldsymbol{\alpha})} H^{\otimes m}, \quad (3.11)$$

where $(\mathbf{v}^{(c)})_l$ denotes the row vector with entries $(\mathbf{v}_d^{(c)})_l$.

A more formal description of the discussed complete sets of MUBs was given by Klappenecker and Rötteler [KR04], which is based on *Weil sums* in the case of odd prime-power dimensions and in even prime-power dimensions on *Galois rings*, which form by their roots and the zero element the so-called *Teichmüller set*.

3.1.2 Pauli operators partition

In 2002 a paper by Bandyopadhyay *et al.* appeared that follows another approach to construct MUBs in a way that highlights the structure of these bases [BBRV02] (cf. also [LBZ02]). In this construction, in a Hilbert space of dimension $d \in \mathbb{N}^*$, each basis is seen as the set of common eigenvectors belonging to a maximal set of commuting Pauli operators within the set of all d^2 generalized Pauli operators. If we exclude the unity operator, which commutes with all operators, we can find at most $d - 1$ pairwise commuting operators within a single set. A partition of $d + 1$ such sets results in a complete set of MUBs. These partitions exist, as the previous construction (cf. Section 3.1.1), for prime power dimensions, although, the initial approach is limited artificially, which leads to very specific sets. In this section we will introduce the ideas of the approach Chapter 4 is based on. Possible alternatives in the construction process are pointed out.

We define the generators of the *generalized Pauli operators* Z and X , acting on a state $|i\rangle$ of the d -dimensional Hilbert space $\mathcal{H} = \mathbb{C}^d$ as

$$Z|i\rangle = \omega^i|i\rangle \quad \text{and} \quad X|i\rangle = |i \oplus_d 1\rangle, \quad (3.12)$$

with $\omega = \exp(2\pi i/d)$ being the first d -th root of unity. The group which is generated by these operators, i. e. $H := \langle Z, X \rangle$, is known as (*Weyl-Heisenberg group* or sometimes *generalized Pauli group*). Since phases are not relevant for unitary operators in quantum theory, we will only refer to elements with a real and positive phase in the following, thus we factorize the group by its center $\{\pm 1, \pm i\} \mathbb{1}$ and get the *set of Pauli operators* \tilde{H} . Any element of this set is given by

$$ZX(k, l) := \begin{cases} (-i)^{kl} Z^k X^l & \text{for } d = 2, \\ Z^k X^l & \text{else,} \end{cases} \quad (3.13)$$

with $k, l \in \mathbb{N}$, whereas $Z^d = X^d = \mathbb{1}_d$ by construction. For prime-power dimensions $d = p^m$ with p being a prime number and $m \in \mathbb{N}^*$, each Pauli operator of this set can be seen as the representation of a $2m$ -dimensional vector which is an element of the finite field \mathbb{F}_p^{2m} . Then, the generalized Pauli operators can be written in a tensor-product

structure as

$$ZX(\vec{a}) = \begin{cases} (-i)^{a_1^z a_1^x} Z^{a_1^z} X^{a_1^x} \otimes \dots \otimes (-i)^{a_m^z a_m^x} Z^{a_m^z} X^{a_m^x} & \text{for } p = 2, \\ Z^{a_1^z} X^{a_1^x} \otimes \dots \otimes Z^{a_m^z} X^{a_m^x} & \text{else,} \end{cases} \quad (3.14)$$

with $\vec{a} = (a_1^z, \dots, a_m^z, a_1^x, \dots, a_m^x)^t \in \mathbb{F}_p^{2m}$. The commutation relation of two elements of this set $\vec{a}, \vec{b} \in \mathbb{F}_p^m$ is given by

$$ZX(\vec{a}) \cdot ZX(\vec{b}) = \omega^{(\vec{a}, \vec{b})_{\text{sp}}} ZX(\vec{b}) \cdot ZX(\vec{a}), \quad (3.15)$$

with the *symplectic product* $(\vec{a}, \vec{b})_{\text{sp}}$ as defined in Definition A.6.2. Thus, two Pauli operators $ZX(\vec{a})$ and $ZX(\vec{b})$ commute, if and only if the symplectic product $(\vec{a}, \vec{b})_{\text{sp}}$ equals zero, where the symplectic product is additive (cf. Corollary A.6.2).

As it is shown by Lemma A.4.1, the set of Pauli operators is an orthogonal basis of linear operators. Thus, a possible choice is to describe every projective measurement operator by this set. Since we aim on constructing MUBs which can be represented as unitary operators it is natural to restrict the consideration to projective measurements in the following. A discussion on more general measurements is given at the end of this section. As we can parametrize any unitary measurement basis uniquely with the elements of the set of Pauli operators, we will choose an exceptional set of MUBs, if each basis is given directly by the common eigenspace of a subset of the generalized Pauli operators, instead of taking a set of commuting unitary operators in general. Therefore, we partition the set of Pauli operators into disjoint classes \mathcal{C}_j , such that

$$\tilde{H} \setminus \{\mathbb{1}_d\} = \bigcup_{j=0}^d \mathcal{C}_j. \quad (3.16)$$

Each class \mathcal{C}_j is a set of $d + 1$ commuting Pauli operators and is created by a $2m \times m$ generator matrix G_j with entries in \mathbb{F}_p , as

$$\mathcal{C}'_j = \mathcal{C}_j \cup \{\mathbb{1}_d\} = \{ZX(\vec{a}) : \vec{a} = G_j \cdot \vec{c} : \vec{c} \in \mathbb{F}_p^m\}. \quad (3.17)$$

By using the generator matrix in the mentioned way, the resulting set of vectors \vec{a} forms an m -dimensional subspace of \mathbb{F}_p^{2m} and the class is therefore called *linear*.

It was shown by Bandyopadhyay *et al.*, using Equation (3.12), that the *Hilbert-Schmidt inner product* of two elements $\vec{a}, \vec{b} \in \tilde{H}$ of the set of Pauli operators, namely

$$\langle ZX(\vec{a}) | ZX(\vec{b}) \rangle_{\text{HS}} := \text{tr}(ZX(\vec{a})^\dagger ZX(\vec{b})), \quad (3.18)$$

vanishes for all $\vec{a} \neq \vec{b}$ [BBRV02, Theorem 4.2]. Pairs of matrices with a vanishing Hilbert-Schmidt inner product are called *orthogonal*. Since all operators within a single class \mathcal{C}_j commute, they have a common eigenbasis. But the Hilbert-Schmidt inner product is invariant under basis transformation and reduces for diagonal matrices to the inner product of their diagonal vectors. As at most d vectors can be found that are mutually orthogonal in a d -dimensional space (and form therefore an orthogonal basis), at most d (mutually orthogonal) elements of the set of Pauli operators can be found, that commute pairwise [BBRV02, Lemma 3.1]. These elements will always form a linear class:

Lemma 3.1.1 (Linearity of maximal class).

A class of d commuting elements of the set of Pauli operators can always be created by a generator matrix and is therefore always linear.

Proof. It was shown above that at most d elements of the set of Pauli operators commute pairwise in a d -dimensional Hilbert space. By Corollary A.6.2, also those elements $ZX(\vec{a}_k)$ and $ZX(\vec{a}_l)$ commute, which can be constructed by the linear combinations of their generating vectors \vec{a}_k and \vec{a}_l . But there is no class with more than d elements, thus there exists always a basis of m elements which we call generator, as in Equation (3.17). \square

A unitary operator basis, i. e. a basis for unitary operators, with d^2 elements that can be partitioned into $d + 1$ classes with mutual orthogonal and pairwise commuting elements, is called a *maximal commuting basis* and can be used to construct a maximal set of $d + 1$ MUBs [BBRV02, Theorem 3.2]. Conversely, also a complete set of MUBs implies the existence of a maximal commuting basis [BBRV02, Theorem 3.4]. It remains an open question to figure out all (or at least all non-equivalent) maximal commuting bases (see Chapter 5).

Therefore, in order to construct the classes \mathcal{C}_j , we need—according to Equation (3.17)—to find $d + 1$ generators G_j that partition the set of Pauli operators into disjoint classes of $d - 1$ pairwise commuting elements each³. It will turn out later on in this work that we are free to fix one of the classes to construct a certain set of MUBs (cf. Chapter 5). But still, specific separability properties of the MUBs are modified by this choice (cf. Section 4.4). A possible choice is to set the generator of the class \mathcal{C}_0 as

$$G_0 = \begin{pmatrix} \mathbb{1}_m \\ 0_m \end{pmatrix}, \quad (3.19)$$

which generates all Pauli- Z operators that obviously commute; the symbols $\mathbb{1}_m$ and 0_m refer to the identity matrix and a quadratic zero matrix, respectively, where $\mathbb{1}_m, 0_m \in M_m(\mathbb{F}_2)$. In order to obtain classes \mathcal{C}_j with $j \in \{1, \dots, d\}$ that are disjoint with the class \mathcal{C}_0 , the column vectors of their generators have to be linearly independent, which is exactly true if the block matrices (G_0, G_j) are invertible for $j \in \{1, \dots, d\}$.⁴ By Lemma A.1.2 this is true if the determinant of $G_0^z G_j^x - G_j^z G_0^x$ is not zero for $j \neq 0$, where $G_j = (G_j^z, G_j^x)^t$. Since $G_0^x = 0_m$, this equation can only hold if G_j^x is invertible. But if G_j^x is invertible, we can write all generators with $j \neq 0$ as

$$G_j = \begin{pmatrix} G_j^z \\ \mathbb{1}_m \end{pmatrix}, \quad (3.20)$$

which will be proven in Corollary 4.3.1 and called *standard form* in Section 4.3. Within this form, the elements in a single class commute, if the symplectic product of all pairs

³As the unity element appears obviously in all classes it is excluded in order to construct disjoint classes.

⁴A class with a maximal number of elements can only be created if the column vectors of a single generator are linearly independent.

of vectors of the generating set (thus, all column vectors of the generator) have a vanishing symplectic product. With $G_j^z = (\vec{a}_1^z, \dots, \vec{a}_m^z)$ follows

$$\vec{a}_k^z \vec{a}_l^x - \vec{a}_k^x \vec{a}_l^z = 0 \quad \text{for } k, l \in \{1, \dots, m\}, \quad (3.21)$$

and with $G_j^x = \mathbb{1}_m$ finally

$$a_{k,l}^z - a_{l,k}^z = 0 \quad \text{for } k, l \in \{1, \dots, m\}. \quad (3.22)$$

Thus, the matrices G_j^z with $j \in \{1, \dots, d\}$ have to be symmetric [BBRV02, Lemma 4.3]. The last point we have to achieve is that arbitrary pairs of generators G_j with $j \neq 0$ do not span the same vectors spaces, thus for $k, l \in \{1, \dots, m\}$ the determinant of $G_k^z G_l^x - G_l^z G_k^x$ does not vanish. But with $G_k^x = G_l^x = \mathbb{1}_m$ we find

$$\det(G_k^z - G_l^z) \neq 0 \quad \text{for } k, l \in \{1, \dots, m\}. \quad (3.23)$$

In summary, this leads to the following three conditions in order to construct a maximal commuting basis if we set $G_0 = (\mathbb{1}_m, 0_m)^t$:

- (1) $G_j = (G_j^z, \mathbb{1}_m)$ for $j \in \{1, \dots, d\}$.
- (2) G_j^z is symmetric.
- (3) $\det(G_k^z - G_l^z) \neq 0$ for $k, l \in \{1, \dots, d\}, k \neq l$.

If the unitary operators within a class \mathcal{C}'_j are given by

$$\mathcal{C}'_j = \{U_{j,0}, \dots, U_{j,d-1}\}, \quad (3.24)$$

with $U_{j,0}$ referring to the unity matrix $\mathbb{1}_d$, there is an orthonormal basis in which all of these operators are diagonal. This leads to a set of eigenvalues $\lambda_{j,k,l}$ where $k \in \{0, \dots, d-1\}$ indicates the operator $U_{j,k}$ and $l \in \{1, \dots, d\}$ belongs to the eigenvector index. Bandyopadhyay *et al.* have shown, that the following construction generates a complete set of MUBs from the maximal commuting basis with M_j being the unitary operator that is a common eigenbasis of the elements of \mathcal{C}'_j and serves as an element of the set of MUBs:

$$M_j = \begin{pmatrix} \lambda_{j,0,1} & \lambda_{j,0,2} & \cdots & \lambda_{j,0,d} \\ \lambda_{j,1,1} & \lambda_{j,1,2} & \cdots & \lambda_{j,1,d} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{j,d-1,1} & \lambda_{j,d-1,2} & \cdots & \lambda_{j,d-1,d} \end{pmatrix}. \quad (3.25)$$

Finally, we like to mention, that the set of generalized Pauli operators as it was chosen to construct the mutually unbiased bases which may serve as a set for completely estimating the quantum state of a system, is only a subset of the most general set of measurement operators which contains all *positive operator valued measurements (POVMs)*.⁵ Investigations on so-called *symmetric informationally complete POVMs*

⁵A POVM is a set of positive operators, $\{E_n\}$, which are defined by measurement operators M_n as $E_n = M_n^\dagger M_n$ with $\sum_n E_n = \mathbb{1}$ for $n \in \mathbb{N}^*$. The probability that outcome n occurs is given by $p(n) = \langle \psi | E_n | \psi \rangle$ if the state of the system is given by $|\psi\rangle$.

(SIC-POVMs) may expand the state estimation techniques raised by MUBs. After this concept was introduced in the seventies [LS73, Pru77], Zauner conjectured that a complete state estimation with SIC-POVMs is possible for all Hilbert space dimensions and showed this explicitly for dimensions $d \leq 5$ [Zau99]. Whereas Renes *et al.* [RBKSC04] found numerical evidences for larger dimensions, an explicit construction of a SIC-POVM for the smallest composite dimension $d = 6$ was given by Grassl [Gra04]. A formal discussion of the former results with the notion of an *extended Clifford group* was done by Appleby [App05, App09].

3.2 Fibonacci polynomials

Fibonacci polynomials play an important role in the process of constructing the so-called *cyclic MUBs*, which will be considered in Chapter 4. We will discuss those properties in detail which are necessary in order to understand the features of most of the sets of cyclic MUBs which are constructed in this work (cf. Sections 4.1 and 4.5.1). This section starts by defining the Fibonacci polynomials and enters directly into the area of important and generally available properties of these polynomials. We will then examine properties which are limited to the case where these polynomials are defined over the finite field \mathbb{F}_2 . Most of these lemma with similar proofs were done for [KRS10, SR12]. Some were already given in [WP69, Bic70, GKT97, GKW02], as well as further properties. The existence of complete sets of cyclic MUBs, using the constructions of Sections 4.1 and 4.5.1, can be proven with the presented results.

The well-known Fibonacci sequence can be generalized in a way to generate the so-called Fibonacci polynomials $F_n(x)$, which are defined recursively.

Definition 3.2.1 (Fibonacci polynomials).

The polynomial $F_n(x)$ is called the *Fibonacci polynomial* of index n , and recursively defined as

$$F_{n+1}(x) := x \cdot F_n(x) + F_{n-1}(x) \quad (3.26)$$

with $F_0 := 0$ and $F_1 := 1$.

For $x = 1$ we end up with the usual Fibonacci sequence given by the ordered set $\{F_n\}_0^\infty = \{0, 1, 1, 2, 3, 5, \dots\}$. For further investigations it is a great advantage to have a generalized recursion relation.

Lemma 3.2.1 (General recursion relation).

The Fibonacci polynomial $F_{k+l}(x)$ with $k, l \in \mathbb{N}$ can be derived with the help of the Fibonacci polynomials $F_k(x)$ and $F_l(x)$ as

$$F_{k+l}(x) = F_k(x)F_{l+1}(x) + F_{k-1}(x)F_l(x). \quad (3.27)$$

Proof. We show this formula by induction. For $l = 0$ we find $F_k(x) = F_k(x)$. Assuming that (3.27) holds, we get $F_{k+(l+1)}(x) = F_k(x)F_{(l+1)+1}(x) + F_{k-1}(x)F_{l+1}(x) = xF_{k+l}(x) + F_{k+(l-1)}(x)$ by using Equation (3.26). \square

We can use this relation in order to prove an important lemma on the divisibility properties of the Fibonacci polynomials. Beforehand, we need the auxiliary lemma which follows.

Lemma 3.2.2 (Coprime Fibonacci polynomials).

The polynomials $F_n(x)$ and $F_{n+1}(x)$ are coprime for $n \in \mathbb{N}^$.*

Proof. It is given by construction that $\gcd(F_1(x), F_2(x)) = \gcd(1, x) = 1$. If we assume that $\gcd(F_n(x), F_{n+1}(x)) = 1$ we can step the induction forward by $\gcd(F_{n+1}(x), F_{n+2}(x)) = \gcd(F_{n+1}(x), xF_{n+1}(x) + F_n(x)) = 1$, using the assumption. \square

This basic divisibility property leads to a more fundamental property of the Fibonacci polynomials.

Lemma 3.2.3 (Divisibility of Fibonacci polynomials).

The polynomial $F_n(x)$ is divisible by $F_m(x)$ if and only if m divides n , with $m, n \in \mathbb{N}^$.*

Proof. To show the implication, let us assume that $n = mm'$ with $m' \in \mathbb{N}^*$. We note that $F_m(x)$ divides $F_n(x)$ trivially for $m' = 1$. Using relation (3.27) with $k := m$ and $l := m(m' - 1)$, we proceed by induction with the assumption that $F_{m(m'-1)}(x)$ is divisible by $F_m(x)$ and see that this implies that $F_{mm'}(x) = F_{m+m(m'-1)}(x) = F_m F_{m(m'-1)+1}(x) + F_{m-1}(x) F_{m(m'-1)}(x)$ is also divisible by $F_m(x)$.

To show the converse we set $n = mm' + r$ with some remainder $r \in \mathbb{N}$ such that $r < m$. The generalized recursion relation (3.27) gives $F_n(x) = F_{r+mm'}(x) = F_r F_{mm'+1}(x) + F_{r-1}(x) F_{mm'}(x)$. Using the normal recursion relation (3.26) we get $F_n(x) = xF_r(x) \cdot F_{mm'}(x) + F_r(x) F_{mm'-1}(x) + F_{r-1}(x) F_{mm'}(x)$. By the implication, the first and the last term are divisible by $F_m(x)$. Since we assume that $F_n(x)$ is divisible by $F_m(x)$, the term $F_r(x) F_{mm'-1}(x)$ should also be divisible by $F_m(x)$ or vanish. From Lemma 3.2.2 we know that $F_{mm'}(x)$ and $F_{mm'-1}(x)$ are coprime, thus $F_{mm'-1}(x)$ is coprime to $F_m(x)$. Since we further assume that $r < m$ which implies that the degree of $F_m(x)$ is larger than the degree of $F_r(x)$, the polynomial $F_r(x)$ cannot be divisible by $F_m(x)$, thus should vanish by identifying $r = 0$. \square

Keeping these properties in mind, we can expand our investigation on the Fibonacci polynomials by discussing their coefficients. Therefore, it is useful to read off the coefficients $a_k^{(n)}$ from Equation (3.26), meaning the coefficient belonging to x^k in the polynomial $F_n(x)$, and get

$$a_k^{(n+1)} = a_{k-1}^{(n)} + a_k^{(n-1)}, \quad (3.28)$$

with $a_k^{(0)} = 0$ for $k \in \mathbb{N}$, $a_0^{(1)} = 1$ and $a_k^{(1)} = 0$ for $k \in \mathbb{N}^*$. Using this relation, we are able to show the following lemma:

Lemma 3.2.4 (Coefficients of Fibonacci polynomials).

For $F_n(x) = \sum_{k=0}^n a_k^{(n)} x^k$ and $n \in \mathbb{N}$, there holds

$$a_k^{(n)} = \begin{cases} \binom{(n+k+1)/2}{(n-k+1)/2}, & \text{if } n-k \equiv 1 \pmod{2}, \\ 0, & \text{otherwise.} \end{cases} \quad (3.29)$$

Proof. For $F_0(x) = 0$ and $F_1(x) = 1$ the statement holds. Using Equation (3.28), we have to show that

$$\binom{\frac{n+k}{2} + 1}{\frac{n-k}{2} + 1} = \binom{\frac{n+k}{2}}{\frac{n-k}{2}} + \binom{\frac{n+k}{2}}{\frac{n-k}{2} + 1} \quad (3.30)$$

holds for the first case. But this is known from Pascal's triangle. Using again Equation (3.28), we see that the second case always stays zero. \square

We can restate the result of this lemma as

$$F_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k+1} x^{n-2k-1}. \quad (3.31)$$

Since the Fibonacci polynomials define a *linear recurring sequence*, namely an *impulse response sequence* [LN08, Chapter 8], we can represent the sequence given by Equation (3.26) by the associated *companion matrix* A of its characteristic polynomial $\chi_F(Z) = Z^2 - x \cdot Z - 1$, i. e.

$$A = \begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix}. \quad (3.32)$$

With the *initial state vector*⁶ $(F_0, F_1)^t = (0, 1)^t$, we obtain the Fibonacci polynomials by

$$\begin{pmatrix} F_n(x) \\ F_{n+1}(x) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix}^n \cdot \begin{pmatrix} F_0 \\ F_1 \end{pmatrix}. \quad (3.33)$$

If we identify the entries within the companion matrix A with their associated Fibonacci polynomials, we find

$$A^n = \begin{pmatrix} F_0 & F_1 \\ F_1 & F_2(x) \end{pmatrix}^n = \begin{pmatrix} F_{n-1}(x) & F_n(x) \\ F_n(x) & F_{n+1}(x) \end{pmatrix}. \quad (3.34)$$

In the case that n is decomposable into integers as $n = k+l$, the relation $A^{k+l} = A^k \cdot A^l$, namely

$$\begin{pmatrix} F_{k+l-1}(x) & F_{k+l}(x) \\ F_{k+l}(x) & F_{k+l+1}(x) \end{pmatrix} = \begin{pmatrix} F_{k-1}(x) & F_k(x) \\ F_k(x) & F_{k+1}(x) \end{pmatrix} \cdot \begin{pmatrix} F_{l-1}(x) & F_l(x) \\ F_l(x) & F_{l+1}(x) \end{pmatrix} \quad (3.35)$$

leads directly to the generalized recursion relation (3.27).

3.2.1 Fibonacci polynomials over \mathbb{F}_2

If we limit the coefficients of the Fibonacci polynomials to be an element of the finite field which has characteristic two and is prime, namely \mathbb{F}_2 , we find further properties which are important for the construction of cyclic MUBs. A major feature of fields

⁶In the theory of linear recurring sequences, an initial state vector is defined as the vector which describes the initial state of the *feedback shift register* (cf. [LN08, Chapter 8]).

with characteristic two is the fact that the minus sign can be replaced by the plus sign. We may then create a second form of Equation (3.26), i. e.

$$F_{n-1}(x) = x \cdot F_n(x) + F_{n+1}(x). \quad (3.36)$$

This leads to Fibonacci polynomials with a negative index, being equal to their positive counterparts, such that $F_{-n}(x) = F_n(x)$. A similar result with alternating signs can be deduced for different prime fields. Some of the Fibonacci polynomials have special properties if they are defined over the field \mathbb{F}_2 .

Lemma 3.2.5 (Fibonacci polynomials $F_{2n}(x)$).

For Fibonacci polynomials that have an even index and are defined over the finite field \mathbb{F}_2 , holds $F_{2n}(x) = x F_n(x)^2$ for $n \in \mathbb{N}$. More general, we can state $F_{2^m \cdot r}(x) = F_r^{2^m}(x) x^{2^m - 1}$ with $m, r \in \mathbb{N}$.

Proof. Using the generalized recursion relation (3.27), we find $F_{n+n}(x) = F_n(x)F_{n+1}(x) + F_{n-1}(x)F_n(x)$. With Equation (3.26) this yields the requested result. The second part follows by induction over m . \square

Despite the general recursion relation (3.27), we can deduce a symmetric generalization of the normal recursion relation (3.26), that is only valid for Fibonacci polynomials over \mathbb{F}_2 .

Lemma 3.2.6 (Symmetric recursion relation).

For Fibonacci polynomials that are defined over \mathbb{F}_2 , there holds $F_{k+l}(x) + F_{k-l}(x) = x F_k(x) F_l(x)$ with $k, l \in \mathbb{N}$.

Proof. To show the lemma by induction, we start with the normal recursion relation (3.26). We can proceed the induction over l by using Equations (3.26) and (3.36) as $F_{k+(l+1)}(x) + F_{k-(l+1)}(x) = F_{k+l}(x) + F_{k-l}(x) + F_{k+(l-1)}(x) + F_{k-l-1}(x)$. With the assumption this leads to the expected result. \square

Using this lemma we can formulate a subtractive type of a recursion relation.

Lemma 3.2.7 (Subtractive recursion relation).

For Fibonacci polynomials that are defined over the finite field \mathbb{F}_2 , there holds $F_{k+1}(x)F_l(x) + F_k(x)F_{l+1}(x) = F_{|k-l|}(x)$ with $k, l \in \mathbb{N}$.

Proof. If we multiply the expression $F_{k+1}(x)F_l(x) + F_k(x)F_{l+1}(x)$ by x and apply the implication of Lemma 3.2.6, we have $F_{k-1+1}(x) + F_{k-l-1}(x)$. Using the converse of this lemma and ruling out the initially multiplied x we get the expected result. \square

With Lemma 3.2.6 in mind, we can prove the main theorem of this section.

Theorem 3.2.8 (Completeness of Fibonacci polynomials).

All polynomials of degree $m \in \mathbb{N}^$ or degree $m' \in \mathbb{N}^*$ such that m' divides m , which are irreducible over the field \mathbb{F}_2 , divide either $F_{2^m-1}(x)$ or $F_{2^m+1}(x)$, or equal x .*

Proof. If we calculate the product of x with $F_{2^m-1}(x)$ and $F_{2^m+1}(x)$ using Lemma 3.2.6, we get $F_{2^m+1}(x) + F_2(x)$, which equals $x^{2^{m+1}-1} + x = x(x^{2^m-1} + 1)^2$ by Lemma 3.2.5. But the term $x^{2^m} + x$ is, as a well-known result from finite field theory (see e.g. Lemma 2.13 of [LN08]), equal to the product of all irreducible polynomials over the field \mathbb{F}_2 , whose degree divides m . \square

This theorem enables us to well-define a property of irreducible polynomials which are defined over \mathbb{F}_2 that will be important in this work later on.

Definition 3.2.2 (Fibonacci index).

Every irreducible polynomial $p \in \mathbb{F}[x]$ of degree m has a property, namely the *Fibonacci index* (sometimes called *depth*), which is given by the minimal positive number $d \in \mathbb{N}$, such that p divides F_d .

Corollary 3.2.9 (Maximal Fibonacci index).

The Fibonacci index of an irreducible polynomial $p \in \mathbb{F}[x]$ of degree m is upper bounded by $2^m + 1$.

Proof. The statement follows directly by applying Theorem 3.2.8. \square

To limit the possible Fibonacci index of a specific irreducible polynomial, we state the following lemma which was proven by Sutner (cf. Theorem 3.1 of [Sut00]).

Lemma 3.2.10 (Irreducible factors of $F_{2^m \pm 1}$).

All irreducible polynomials which divide $F_{2^m+1}(x)$, have a linear term, whereas the linear term of those irreducible polynomials which divide $F_{2^m-1}(x)$ vanishes.

Finally, we like to mention that the structure of the Fibonacci polynomials is related to the Sierpinski triangle (see Appendix C.2) and to structures that appear with characteristic polynomials of *reduced stabilizer matrices* with *submatrix* A equal to zero, as defined in Equation (4.13).

Construction of cyclic sets

In Section 3.1 we have discussed different constructions of MUBs that refer to nice mathematical structures which lead to direct constructions of the different unitary matrices within a set of MUBs. Only the construction introduced by Bandyopadhyay *et al.* [BBRV02] breaks this rule by introducing a preceding step, namely the arrangement of the set of Pauli operators into classes of commuting elements. From a physical point of view it is useful to equip complete sets of MUBs with further features. One of these features is important both for theoretical investigations and also for experimental implementations of MUBs. It is a cyclic structure we might only find for complete sets of MUBs for Hilbert space dimensions which are a power of two [Gow07], thus $d = 2^m$ with $m \in \mathbb{N}^*$. Namely, if one of the bases within the set—which can be represented as a unitary operator—is given by U , we get a complete set of MUBs $\mathfrak{S} = \langle U \rangle = \{U, U^2, \dots, U^{d+1}\}$ with $U^{d+1} = \mathbb{1}_d$. Therefore, we will refer to these sets as *sets of cyclic mutually unbiased bases (cyclic MUBs)* from now on. It is the main aim of the first part of this work to investigate the properties of cyclic MUBs, to provide constructions and to expand these ideas. Initially the existence of the chosen construction will be proven. From a theoretical point of view, cyclic MUBs are easier to handle since they allow elegant proofs, e.g. for QKD protocols [Got98, Lo01, Cha05]. For the experimental aspect cyclic MUBs feature their advantage of being implementable by a single quantum circuit. As we restrict our quantum systems to be describable in a Hilbert space of dimension $d = 2^m$ with $m \in \mathbb{N}^*$, we can consider our systems to be systems of interacting qubits. Formally, this provides us a simple tensor product structure. Conceptually, the restriction refers to most of the valuable applications, which are built on qubits. However, extensions to general prime-power dimensions appear to be feasible by dropping some dispensable properties, but demand further investigations. In fact, it seems that the set of MUBs can only be reduced to a *set of generators* [Gow07].

This chapter is organized as follows: In Section 4.1 a method is introduced which is proven to be able to generate complete sets of cyclic MUBs for all dimensions 2^m . Numerical methods are discussed in subsections. Section 4.2 discusses a special kind of solutions for $m = 2^k$ with $k \in \mathbb{N}$ which can be constructed recursively and is related to

a mathematical conjecture of finite field theory by Wiedemann. In Section 4.3 the *standard form* is introduced which will be used later on in order to compare sets of cyclic MUBs, followed by a discussion about their entanglement properties (Section 4.4). The resulting *homogeneous* and *inhomogeneous* sets are shown in Sections 4.5 and 4.6, respectively. As all sets are given in a special representation, the construction of the corresponding unitary operator is given in Section 4.7. Finally, the implementation of all discussed sets into a quantum circuit is possible with the methods specified in Section 4.8.

4.1 Fibonacci-based sets

The first class of cyclic MUBs we create in this work is based on properties of Fibonacci polynomials which were discussed in detail in Section 3.2. Contrary to other classes, which will be discussed later on in Sections 4.5 and 4.6, the connection to Fibonacci polynomials allows a straightforward construction scheme that is based on finite field theory. The method we will use, finds solutions for the construction of Bandyopadhyay *et al.* [BBRV02] that was introduced in Section 3.1.2, with an additional cyclicity property. Therefore, we will establish a *symplectic matrix* $C \in M_{2m}(\mathbb{F}_2)$ which will be called *stabilizer matrix* in the following, that cyclically permutes the $d + 1$ classes \mathcal{C}_j of Equation (3.17). In other words, that stabilizer matrix is a matrix representation of a generator of a finite group of order $d + 1$ and generates a complete set of MUBs. This section starts by discussing the required properties of the stabilizer matrix and different approaches to achieve them, as published in [KRS10, SR12]. We then show the existence of the stabilizer matrix for each dimension $d = 2^m$ with $m \in \mathbb{N}^*$, followed by a conjecture that predicts a *symmetric companion matrix* of polynomials defined over \mathbb{F}_2 , which raises the ability of calculating the stabilizer matrix from a certain polynomial directly. A list of solutions for the symmetric stabilizer matrix is given in Appendix C.1.6 for dimensions with $m = 2, \dots, 36$. Solutions for stabilizer matrices in a certain form are listed for dimensions with $m = 2, \dots, 600$ in Appendix C.1.5. Section 4.7.1 deals with a unitary representation of the stabilizer matrix C .

As the method we choose to generate complete sets of cyclic MUBs is based on the construction of Bandyopadhyay *et al.*, we need to partition the set of Pauli operators into disjoint classes of commuting elements, as seen by Equation (3.16). For each of these classes \mathcal{C}_j with $j \in \{0, \dots, d\}$, by repeating Equation (3.17) with $p = 2$, we define a $2m \times m$ generator matrix G_j that generates the $d - 1$ elements of the class \mathcal{C}_j as well as the identity as

$$\mathcal{C}'_j = \mathcal{C}_j \cup \{\mathbb{1}_d\} = \{ZX(\vec{a}) : \vec{a} = G_j \cdot \vec{c} : \vec{c} \in \mathbb{F}_2^m\}. \quad (4.1)$$

To implement a cyclic structure into this partition, we demand that the stabilizer matrix $C \in M_{2m}(\mathbb{F}_2)$ permutes the generators as $C^k \cdot G_l = G_{k \oplus l}$, which implies that $C^{d+1} = \mathbb{1}_{2m}$ and leads to

$$\mathcal{C}'_j = \mathcal{C}_j \cup \{\mathbb{1}_d\} = \{ZX(\vec{a}) : \vec{a} = C^j \cdot G_0 \cdot \vec{c} : \vec{c} \in \mathbb{F}_2^m\}. \quad (4.2)$$

As we are free to fix one of the generators, we set $G_0 = (\mathbb{1}_m, 0_m)^t$. To guarantee disjoint classes, the generators need to span non-overlapping vector spaces, i. e. the

matrix (G_k, G_l) is invertible for all values $k, l \in \{0, \dots, d\}$ with $k \neq l$. Since the set of automorphisms of the Heisenberg group is the Clifford group and the commutation relations of $ZX(\vec{a})$ indicate a symplectic structure (see Appendix A.6), we need C to be symplectic. More precisely, the matrix $C \in M_{2m}(\mathbb{F}_2)$ is a $2m \times 2m$ representation of a *Clifford unitary* matrix. So the problem is boiled down from finding a $2^m \times 2^m$ unitary matrix to the task of finding this $2m \times 2m$ stabilizer matrix C . In summary, we have the following restrictions on the stabilizer matrix C :

- (I) C is symplectic.
- (II) $C^{d+1} = \mathbb{1}_{2m}$.
- (III) (G_k, G_l) is invertible for $k, l \in \{0, \dots, d\}$, $k \neq l$, with $G_k = C^k \cdot (\mathbb{1}_m, 0_m)^t$,

where (G_k, G_l) defines a *block matrix* in the form

$$(G_k, G_l) = \begin{pmatrix} G_k^z & G_l^z \\ G_k^x & G_l^x \end{pmatrix}, \quad \text{with} \quad G_k = \begin{pmatrix} G_k^z \\ G_k^x \end{pmatrix}. \quad (4.3)$$

These restrictions yield still a lot of freedom to create complete sets of cyclic MUBs. We will refer to alternative classes of possible solutions in Sections 4.5 and 4.6. Within this section, we limit the stabilizer matrix C to

$$C = \begin{pmatrix} B & \mathbb{1}_m \\ \mathbb{1}_m & 0_m \end{pmatrix}, \quad (4.4)$$

with $B \in M_m(\mathbb{F}_2)$. Symplecticity is given (see Definition A.6.1) when

$$C^t \cdot \begin{pmatrix} 0_m & -\mathbb{1}_m \\ \mathbb{1}_m & 0_m \end{pmatrix} \cdot C \equiv \begin{pmatrix} 0_m & -\mathbb{1}_m \\ \mathbb{1}_m & 0_m \end{pmatrix} \pmod{2}, \quad (4.5)$$

thus, the matrix B , which we will refer to as *reduced stabilizer matrix*, needs to be symmetric to fulfill Condition (I). As we consider the powers of C , namely C^n with $n \in \mathbb{N}$, we get

$$C^n = \begin{pmatrix} F_{n+1}(B) & F_n(B) \\ F_n(B) & F_{n-1}(B) \end{pmatrix}, \quad (4.6)$$

with $F_n(x)$ being the Fibonacci polynomials as defined in Section 3.2. In order to achieve Condition (II), we can read off from Equation (4.6) that for $n = d + 1$, $F_n(B)$ has to be equal to 0_m and $F_{n-1}(B)$, as well as $F_{n+1}(B)$ have to equal $\mathbb{1}_m$. But the last property follows from the first two properties by the recursion relation of the Fibonacci polynomials over \mathbb{F}_2 (see Equation (3.36)). We conclude this symmetry with the following lemma:

Lemma 4.1.1 (Symmetry of Fibonacci polynomials with reduced stabilizer matrix). *For a reduced stabilizer matrix $B \in M_m(\mathbb{F}_2)$ that creates a complete set of cyclic MUBs, there holds $F_n(B) = F_{2^m+1-n}(B)$ for $n \in \{0, \dots, 2^m + 1\}$; in particular, $F_{2^m-1}(B) = F_{2^m-1+1}(B)$.*

Proof. With $F_0(B) = 0_m$ and $F_1(B) = \mathbb{1}_m$, the recursion relation of Equation (3.26) defines all $F_n(B)$ with $n \in \mathbb{N}$. As seen above (cf. Equation (4.6)), to create a complete set of cyclic MUBs, it holds $F_{2^{m+1}}(B) = 0_m$ as well as $F_{2^m}(B) = \mathbb{1}_m$. The inverse recursion relation of Equation (3.36) defines again all $F_n(B)$ with $n \in \mathbb{N}$. By simply counting we find that $F_{2^{m-1}}(B) = F_{2^{m-1}+1}(B)$. \square

The generator matrix G_j of the class \mathcal{C}_j with $j \in \{0, \dots, N\}$ is given by

$$G_j = C^j \cdot G_0 = \begin{pmatrix} F_{j+1}(B) \\ F_j(B) \end{pmatrix}, \quad (4.7)$$

so to achieve Condition (III), we have to check that the matrices

$$(C^k \cdot G_0, C^l \cdot G_0) = \begin{pmatrix} F_{k+1}(B) & F_{l+1}(B) \\ F_k(B) & F_l(B) \end{pmatrix}, \quad (4.8)$$

with $k, l \in \{0, \dots, d\}$ and $k \neq l$ are invertible. Thus, as Lemma A.1.2 indicates, the expression $F_{k+1}(B) \cdot F_l(B) + F_{l+1}(B) \cdot F_k(B)$ has to be invertible. But this is equal to $F_{|k-l|}(B)$ as Lemma 3.2.7 shows.

Using the results of this discussion and the approach of Equation (4.4), we can reformulate Conditions (I)-(III):

- (i) B is symmetric.
- (ii) $F_k(B)$ equals $\mathbb{1}_m$ for $k = d$ and 0_m for $k = d + 1$.
- (iii) $F_k(B)$ is invertible for $k \in \{1, \dots, d\}$.

By Lemma 4.1.1 we need to check Condition (iii) only for $k \in \{1, \dots, d/2\}$, but for the subsequent discussion the chosen form is preferable. At this point, it is useful to call the properties of Fibonacci polynomials (cf. Section 3.2) back into mind in order to reduce this list of conditions. In the following, we will discuss how the structure of Fibonacci polynomials implies that Condition (iii) follows from Condition (ii). Therefore, we start with a lemma which shows that the characteristic polynomial of the reduced stabilizer matrix is irreducible, if a complete set of MUBs is created by that matrix.

Lemma 4.1.2 (Characteristic polynomial of reduced stabilizer matrix).

The characteristic polynomial χ_B of a reduced stabilizer matrix $B \in M_m(\mathbb{F}_2)$ with $m \in \mathbb{N}^$ is irreducible and coincides with the minimal polynomial of B , if and only if the Fibonacci index of χ_B equals either $2^m - 1$ or $2^m + 1$.*

Proof. Any irreducible polynomial p which is a factor of the characteristic polynomial of B and has in the set of all factors the smallest Fibonacci index (as defined in Section 3.2.2), annihilates B . Therefore, the Fibonacci polynomial where p appears first as a factor, annihilates B . But all irreducible polynomials which have Fibonacci index $2^m \pm 1$, have minimal degree m (see Theorem 3.2.8), which is also the dimension of B . \square

In the case that the characteristic polynomial of B , namely χ_B , has Fibonacci index $d + 1$ with $d = 2^m$, the Fibonacci polynomial $F_{d+1}(B)$ equals 0_m . With the help of the following corollary this implies that $F_d(B) = \mathbb{1}_m$.

Corollary 4.1.3 (Multiplicative order of reduced stabilizer matrix).

If the characteristic polynomial of B has Fibonacci index $d + 1$ with $d \in \mathbb{N}^$, the order¹ of χ_B and therefore the multiplicative order of B divides $d - 1$.*

Proof. As shown by Lemma 4.1.2, the characteristic polynomial of B is irreducible and has degree m . Since it is defined over the field \mathbb{F}_2 , its splitting field is isomorphic to \mathbb{F}_{2^m} , thus all roots of χ_B are elements of the multiplicative group of \mathbb{F}_{2^m} with $2^m - 1$ elements. Since B is a root of χ_B by the Hamilton-Cayley theorem, it has a multiplicative order that divides $d - 1$. \square

If we take Lemma 3.2.5 and set $r = 1$, it follows with $F_1(x) = 1$ that $F_{2^m}(x) = x^{2^m-1}$. Hence, with Corollary 4.1.3 we get $F_d(B) = \mathbb{1}_m$ for $d = 2^m$. This reduces Condition (ii) to the requirement that the characteristic polynomial of B has Fibonacci index $d + 1$. To combine this new statement with Condition (iii), the following field-theoretic proposition is essential.

Proposition 4.1.4 (Representation of fields).

Let L/K be a field extension² of order $n \in \mathbb{N}^$ and let $A \in M_n(K)$ be a matrix that has an irreducible characteristic polynomial χ_A , which is naturally an element of the polynomial ring $K[x]$. Then, the set $\{f(A) \mid f \in K[x]\}$ is isomorphic to the extension field L .*

Proof. Since one of the roots of the irreducible polynomial χ_A is A itself (by Hamilton-Cayley), we can simply adjoin this root A to the ground field K in order to obtain the extension L which is isomorphic to the polynomial ring $K[x]$ (cf. [LN08, pp. 66]). \square

Given that Condition (ii) is fulfilled, namely the Fibonacci index of the characteristic polynomial χ_B of the reduced stabilizer matrix $B \in M_m(\mathbb{F}_2)$ equals $d + 1$, no Fibonacci polynomial with a positive index smaller than $d + 1$ annihilates B . But since all of those polynomials are elements of the polynomial ring $\mathbb{F}_2[B]$ which is isomorphic to a finite field (as Proposition 4.1.4 states), they are invertible. But this is Condition (iii). Finally, we can rewrite Condition (ii) (which implies Condition (iii)) as:

(ii') The characteristic polynomial of B has Fibonacci index $d + 1$.

Hence, if we are interested in the construction of a complete set of mutually unbiased bases with a cyclic generator for a Hilbert space of dimension d —how it is discussed here—we may propose the following algorithm:

1. Find an irreducible polynomial p of degree m that divides F_{d+1} .
2. Check if p divides any Fibonacci polynomial with an index that divides $d + 1$, if so, go back to 1.

¹ The order of a non-zero polynomial $f(x) \in \mathbb{F}_{p^m}[x]$ is defined to be the smallest natural number e for which the polynomial divides $x^e - 1$ [LN08, Definition 3.2]. If the polynomial has no trivial root, the order of the polynomial equals the order of any of its roots in the multiplicative group $\mathbb{F}_{p^m}^*$ [LN08, Theorem 3.3].

²An introduction and further information about field extensions can be found in Lidl and Niederreiter [LN08, Section 1.4].

3. Find a symmetric matrix $B \in M_m(\mathbb{F}_2)$ that has p as its characteristic polynomial.

To ensure the reliability of this algorithm we need to prove the existence of an appropriate Fibonacci polynomial as well as the corresponding reduced stabilizer matrix B .

Theorem 4.1.5 (Existence of reduced stabilizer matrices).

For any dimension $d = 2^m$ with $m \in \mathbb{N}^$ there exists a reduced stabilizer matrix $B \in M_m(\mathbb{F}_2)$ that is symmetric and has an irreducible characteristic polynomial with Fibonacci index $d + 1$, hence fulfills Conditions (i) and (ii').*

Proof. We will follow the checklist given above. The number of polynomials with Fibonacci index $2^m + 1$ is given by $\frac{\phi(d+1)}{2^m}$, where ϕ denotes Euler's totient function (cf. Theorem 8 of [GKW02]); by Theorem 3.2.8 their degree is m . Since this expression is non-zero, there exists at least one polynomial for any dimension d that has Fibonacci index $d + 1$.

For the last step, we take the well-known fact that all polynomials $p \in K[x]$ have a companion matrix A , i. e. a matrix which has a characteristic polynomial that equals p .³ Finally, for every monic polynomial that is defined over a finite field $K = \mathbb{F}_q$ there exists a symmetric matrix that is similar to the companion matrix and, therefore, has the same characteristic polynomial (cf. Lemma 2 of [BT98]). \square

Having this theorem in mind, we are free in the way of how to construct the stabilizer matrix. We propose three different ways:

- Testing all symmetric matrices $B \in M_m(\mathbb{F}_2)$.
- Testing a reduced set of matrices by defining a more specific form of B .
- Creating B directly, as a *symmetric companion matrix*.

The first two suggestions need numerical methods, which will be discussed in the following paragraph, whereas the last approach aims on an analytic solution that leads to a conjecture given in the subsequent paragraph.

4.1.1 Numerical construction of reduced stabilizer matrix

Within this paragraph we concentrate on the first two suggestions given above to find an appropriate reduced stabilizer matrix B . We will estimate the runtime of the proposed algorithm (Steps 1.–3.) and compare this result with an alternative approach.

This idea is not based on the construction of any polynomial, it uses the resulting Condition (ii') indirectly and starts with testing a chosen set of possible matrices that fulfill Condition (i) by construction. The task of Condition (ii') is to ensure, that the characteristic polynomial of B appears the first time as a factor in the Fibonacci polynomial F_{d+1} with $d = 2^m$, such that $F_{d+1}(B) = 0_m$. If so, then Corollary 4.1.3 tells us that $F_d(B) = \mathbb{1}_m$, thus C^{d+1} equals $\mathbb{1}_{2^m}$. Furthermore, it has to be guaranteed

³Despite the fact that different constructions for a companion matrix exist, it is possible to define such a matrix with a simple structure uniquely.

that all Fibonacci polynomials $F_j(B)$ with $j \in \{1, \dots, d\}$ do not equal 0_m . But by Lemma 3.2.3, only such Fibonacci polynomials $F_j(B)$ divide $F_{d+1}(B)$, that have an index j that is a divisor of $d + 1$. Thus, if none of those Fibonacci polynomial $F_j(B)$ is zero, where j is a non-trivial divisor of $d + 1$, the Fibonacci index of B is $d + 1$ and Condition (ii') is fulfilled. This brings us to the following alternative algorithm:

- 1'. Take a stabilizer matrix C from a previously defined test set.
- 2'. Check if the power to $d + 1$ of this stabilizer matrix equals $\mathbb{1}_{2m}$.
- 3'. Check that for any power of C to a non-trivial divisor of $d + 1$ the off-diagonal blocks do not equal 0_m ,

whereas we construct C in the form of Equation (4.4).⁴

To be able to compare the runtime of both methods, we examine this question by calculating the computational costs. For simplicity reasons, we will take the number of *array accesses*⁵ as an appropriate measure.

For the first method (Steps 1.–3.) we have to calculate the characteristic polynomial of an $m \times m$ matrix. Therefore, we may derive the sums of all principal minors. One of these is the determinant of B , which is calculated by the Leibniz formula

$$\det(B) = \sum_{\sigma \in S_m} \text{sgn}(\sigma) \prod_{i=1}^m b_{i, \sigma(i)}, \quad (4.9)$$

where $b_{i,j}$ denotes the matrix element of B and S_m is the symmetric group of all possible permutations of the index set $\{1, \dots, m\}$. The calculation of the determinant in this way yields to $|S_m| \cdot m = m \cdot m!$ array accesses. For the next minor sum, we have to sum up all determinants where any pair (i, i) with $i \in \{1, \dots, m\}$ of row and column is deleted. This yields m principal minors with $|S_{m-1}| \cdot (m - 1) = (m - 1) \cdot (m - 1)!$ array accesses each, thus $(m - 1) \cdot m!$ array accesses. If we follow this approach and take the sum of all array accesses, we end up with

$$\left(\frac{(m - 1) \cdot m}{2} \right) \cdot m! \quad (4.10)$$

array accesses for the calculation of the characteristic polynomial of an $m \times m$ matrix.

For the alternative method (Steps 1'–3'), we need to calculate the power to $2^m + 1$ of a stabilizer matrix C as given in Equation (4.4). To do this efficiently, we can multiply the matrix C with itself and write the answer to a new variable, square this, and so on. We would end with 2^m , which we multiply by C again and write this final result. This procedure needs $3(m + 1)$ full read outs or writings of an $2m \times 2m$ matrix, thus $12m^3 + 12m^2$ array accesses. At the end, we have to check if this matrix

⁴After finishing this thesis, the author realized that only those powers of C have to be tested, which are given by $d + 1$ divided by one of the prime factors. This holds due to Lemma 3.2.3 and gives an exponential advantage over the discussed method. In the calculation of the computational costs the divisor function has to be replaced by another function with exponentially reduced values.

⁵The most time consuming operations are reading and writing elements of matrices, as the complete matrices cannot be stored directly into the processor registers. In comparison to these operations, arithmetical operations are done instantaneously.

equals $\mathbb{1}_{2^m}$ which yields in summary $12m^3 + 16m^2$ array accesses. Since we have to check all possible factors to be not equal to a block-diagonal matrix with $m \times m$ submatrices, we have to take a similar procedure $d(2^m + 1)$ times, where $d(n)$ with $n \in \mathbb{N}^*$ denotes the divisor function that counts the number of divisors of an integer n . Since any divisor of $2^m + 1$ can be represented as an m -bit number (with leading zeros), we can use the same number of array accesses as an upper bound for this calculation. We have numerical evidence, that for odd integers in the form $n = 2^m + 1$ with $m \geq 2$, the divisor function is upper bounded by \sqrt{n} (including 1 and n), as it is true for at least $m = 200$. This limits the total number of array accesses to

$$(\sqrt{2^m + 1} - 1) \cdot (12m^3 + 16m^2). \quad (4.11)$$

It is clear that the second method scales much better since the factorial in Equation (4.10) grows faster than any polynomial and obviously faster than 2^m . An explicit calculation shows that for $m \geq 7$ the second method is preferable. We are aware of the fact that we left out some terms like the generation of all divisors of a number, since they are not that much relevant in the scaling and cannot easily be taken into account. We did also not mention the fact, that we have to find a polynomial with an appropriate Fibonacci index in the first case, which is another reason to use the second method.

As a next step we should decide which set of matrices we take as our test set. If we test all possible $m \times m$ matrices which have only ones and zeros as entries, this yields m^2 free parameters, which we denote by g . Consequently, we have 2^{m^2} matrices to test. But we are able to fulfill Condition (i) by construction, by shrinking the number of free parameters, such that the resulting matrices are symmetric which leaves us $2^{(m^2+m)/2}$ matrices to test. In this case we have $g = (m^2 + m)/2$. The number of all $m \times m$ matrices B that are symmetric and invertible is given by⁶

$$a(m) = 2^{m(m+1)/2} \frac{\prod_{j=1..2 \cdot \lceil \frac{m}{2} \rceil} (1 - (1/2)^j)}{\prod_{j=1..\lceil \frac{m}{2} \rceil} (1 - (1/4)^j)}, \quad (4.12)$$

but it would be quite challenging to implement a construction that creates only these matrices with higher efficiency than creating all symmetric matrices.

A method that really shrinks the number of free parameters is published in [KRS10]. Therefore, the upper left half of the matrix is fixed to one, the lower right to zero, but with the lower-right corner being a $r \times r$ matrix A with free parameters ($r \in \{0, \dots, \lfloor m/2 \rfloor\}$). For $m = 5$ the possible reduced stabilizer matrices would look like

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & a_{11} & a_{12} \\ 1 & 0 & 0 & a_{12} & a_{22} \end{pmatrix}, \quad (4.13)$$

leaving 3 free parameters. For dimensions $m = 2$ and 3 this matrix $A = (a_{i,j})$ is a 1×1 matrix. For $m = 1$ the matrix $B = (1)$ is already a solution (A does not exist

⁶Refer to *Sloane's A086812* and Brent *et al.* [RPB88].

by construction). For most of the dimensions with $m \leq 30$, the matrix A needs only to be a 2×2 matrix. Starting with dimensions $m = 12, 20, 21, 25, 28$ it turns out, that this matrix needs to be at least a 3×3 matrix, thus having six free parameters. In higher dimensions that size has to be increased further. Comparing with the general $(m^2 + m)/2$ parameters to test, this approach is quite a good improvement, but we have no rigorous proof that the set of matrices this scheme produces, contains always a solution. Nevertheless, solutions for $m = 2, \dots, 600$ are given in Appendix C.1.5.

An approach that shrinks even more the number of possible matrices in the test set is given in the following paragraph, where we set the number of free parameters to $2m - 1$.

4.1.2 Reduced stabilizer matrix as symmetric companion matrix

To calculate a certain reduced stabilizer matrix B , we like to investigate the existence of a symmetric companion matrix, which means, given any polynomial $p \in \mathbb{F}_2[x]$ with degree m , we can directly construct a matrix that has p as its characteristic polynomial⁷. Thus, we do not have to test different matrices on their suitability. This construction aims on the first algorithm that was given above (Steps 1.–3.).

The idea is the following, based on observations on the different solutions for reduced stabilizer matrices: The reduced stabilizer matrix B is described by a set of $2m - 1$ free parameters $\{s_1, \dots, s_{2m-1}\}$ as

$$B = \begin{pmatrix} s_1 & s_2 & \cdots & s_{m-1} & s_m \\ s_2 & s_3 & & s_m & s_{m+1} \\ \vdots & & \ddots & & \vdots \\ s_{m-1} & s_m & & s_{2m-3} & s_{2m-2} \\ s_m & s_{m+1} & \cdots & s_{2m-2} & s_{2m-1} \end{pmatrix}. \quad (4.14)$$

In other words, the matrix elements $b_{i,j}$ with $i, j \in \{1, \dots, m\}$ of B are given by

$$b_{i,j} = s_{i+j-1}. \quad (4.15)$$

As we were not able to construct this companion matrix uniquely yet, we can only conjecture its existence and guess a possible solution.

Conjecture 4.1.6 (Existence of symmetric companion matrix).

For any polynomial $p \in \mathbb{F}_2^m$ with $m \in \mathbb{N}^$ there exists a symmetric companion matrix $B \in M_m(\mathbb{F}_2)$ in the form of Equation (4.14) which has characteristic polynomial $\chi_B = p$.*

A table with string representation of possible solutions for s is given for $m = \{1, \dots, 36\}$ in Appendix C.1.6. Contrary to the analytical expectation, the method of the last paragraph that finds a matrix B in the form of Equation (4.13) succeeds faster than finding the symmetric companion matrix by testing all possibilities. The reason for this contradiction is that, effectively, we find solutions for matrices A that have

⁷Ideally, this matrix would be unique.

a dimension r which is much smaller than the limit of $\lceil m/2 \rceil$. Nevertheless, from a mathematical point of view, it is quite interesting to have a candidate for a symmetric companion matrix that serves for all polynomials over \mathbb{F}_2 .

4.2 Fermat sets

A special subclass of the Fibonacci-based sets which were discussed in Section 4.1 is given by sets we denote as *Fermat sets*. These sets are only defined for dimensions $d = 2^{2^k}$ with $k \in \mathbb{N}$. Since the number of elements of a complete set of MUBs in such a dimension is given by $2^{2^k} + 1$, they are called Fermat sets, where the Fermat numbers are defined as

$$\mathcal{F}_k = 2^{2^k} + 1. \quad (4.16)$$

The outstanding feature of these sets is their recursive construction that supersedes the search of an appropriate reduced stabilizer matrix as it is common for general Fibonacci-based sets. The only caveat of these Fermat sets is that in general their completeness (the fact that a complete set of $d + 1$ MUBs is created) can only be conjectured. Within the subsequent discussion we will derive a connection of this completeness to an open conjecture on *iterated quadratic extensions of \mathbb{F}_2* that was found by D. Wiedemann in 1988 [Wie88], is of current interest [MS96, Vol10], and was tested for $k = 0, \dots, 8$ by Wiedemann himself. By the methods derived in Section 4.1 we are able to raise this limit to $k = 11$, the largest Fermat number where the complete factorization into prime numbers is currently known. It is clear, that the problem of testing if possible factors divide a Fermat number is easier than testing if the power of the stabilizer matrix C to the same factors has vanishing off-diagonal blocks of size $m \times m$. The program code we use is given in Appendix C.1.4.

The solutions we propose to obtain complete sets of cyclic MUBs for dimensions $d = 2^{2^k}$ with $k \in \mathbb{N}$ are based on Equation (4.4). We thus need to find an appropriate reduced stabilizer matrix B . In the following we add an index to B that refers to the integer m which defines the dimension as $d = 2^m$. In the case of Fermat sets the number m is defined by 2^k . If we recall some already known solutions from [KRS10], we find

$$B_{2^0} = (1), \quad B_{2^1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad B_{2^2} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \quad (4.17)$$

If we iterate this construction, we end up with the recursion

$$B_{2^{k+1}} = \begin{pmatrix} B_{2^k} & \mathbb{1}_{2^k} \\ \mathbb{1}_{2^k} & 0_{2^k} \end{pmatrix} \in M_{2^{k+1}}(\mathbb{F}_2). \quad (4.18)$$

Analogously, we define C_{2^k} as

$$C_{2^k} = \begin{pmatrix} B_{2^k} & \mathbb{1}_{2^k} \\ \mathbb{1}_{2^k} & 0_{2^k} \end{pmatrix} \in M_{2^{k+1}}(\mathbb{F}_2), \quad (4.19)$$

thus $C_{2^k} \equiv B_{2^{k+1}}$. At this point, we can already use the program code from Appendix C.1.4 to see that the recursion produces complete sets of cyclic MUBs for $k \in \{0, \dots, 11\}$. But this does not help us understanding the problem and developing ideas that may serve to prove this recursion for all k . It is clear by construction that the reduced stabilizer matrices are symmetric, thus Condition (i) of Section 4.1 is fulfilled. To be able to check if the construction is also conform with Condition (ii') we shall start with calculating the characteristic polynomial χ_B of the reduced stabilizer matrix B . It will turn out that the notion of *reciprocal* and *self-reciprocal* polynomials is of interest within this context.

Definition 4.2.1 (Reciprocal polynomial).

The *reciprocal* of a polynomial $f \in K[x]$ is defined by $f^*(x) := x^n f(x^{-1})$, where $n \in \mathbb{N}^*$ is the degree of f . If a polynomial coincides with its reciprocal, it is called *self-reciprocal*.

We can create a self-reciprocal polynomial by applying the *reciprocal operator* Q on an arbitrary polynomial f with degree n , namely

$$f^Q(x) := x^n f(x + x^{-1}). \quad (4.20)$$

The recursion we propose in Equation (4.18) is a matrix representation of the application of the reciprocal operator, starting with an appropriate polynomial.

Lemma 4.2.1 (Characteristic polynomials).

Let K be a finite field of characteristic 2 and χ_S be the characteristic polynomial of a matrix $S \in M_n(K)$ with $n \in \mathbb{N}^*$. Then $\chi_{S'} = (\chi_S)^Q$ is the characteristic polynomial of

$$S' := \begin{pmatrix} S & \mathbb{1}_n \\ \mathbb{1}_n & 0_n \end{pmatrix} \in M_{2n}(K). \quad (4.21)$$

Proof. The characteristic polynomial $\chi_{S'}$ is given by $\chi_{S'}(x) = \det(x\mathbb{1}_{2n} - S') = \det \begin{pmatrix} x\mathbb{1}_n - S & \mathbb{1}_n \\ \mathbb{1}_n & x\mathbb{1}_n \end{pmatrix}$. If we use Lemma A.1.1, we get $\chi_{S'}(x) = \det(x\mathbb{1}_n) \det(x\mathbb{1}_n - S - x^{-1}\mathbb{1}_n)$. But the first factor equals x^n and the second is given by $\chi_S(x - x^{-1})$. Thus $\chi_{S'}(x) = x^n \cdot \chi_S(x - x^{-1}) = (\chi_S)^Q(x)$. \square

We can use Lemma 4.2.1 to show that the characteristic polynomials of the B_{2^k} are factors of the Fibonacci polynomials $F_{2^{m+1}}$ with $m = 2^k$. With the help of Theorem 3.2.8 and Lemma 3.2.10, it turns out that these polynomials have to be irreducible and keep their linear term in order to be possible candidates for producing complete sets of cyclic MUBs.

Lemma 4.2.2 (Minimal polynomial of B_{2^k}).

The characteristic polynomial of B_{2^k} as defined in Equation (4.18) is given by $f^{Q^k}(x)$ with $f(x) = 1 + x$. It is irreducible and therefore minimal.

Proof. The first part follows by induction from Lemma 4.2.1. It was shown by Varshamov and Garakov [VG69] and in a more general form by Meyn [Mey90] that for a polynomial $g \in \mathbb{F}_2[x]$ and g being irreducible, g^Q is also irreducible, if and only if the linear coefficient of g does not vanish. This is true for f by construction. But for

an arbitrary polynomial $h(x) = \sum_{j=0}^n a_j x^j$ it holds $h^Q(x) = \sum_{j=0}^n \sum_{i=0}^j \binom{j}{i} a_j x^{n+j-2i}$. The only contribution to the linear coefficient from $h^Q(x)$ arises from $j = i = n - 1$ which is given by a_{n-1} . Since the reciprocal operator creates a self-reciprocal function this stays true for all k . \square

Thus, regarding Condition (ii') of Section 4.1, we have shown by Lemma 4.2.2 in combination with Lemma 3.2.10 that the Fibonacci index of the reduced stabilizer matrices B_{2^k} in the form of Equation (4.18) equals $d + 1$ or is a divisor of $d + 1$ with $d = 2^{2^k}$. We have tried to prove that the Fibonacci index is $d + 1$, but without success. A promising approach can be found in Appendix B.

However, we are able to relate the question of the Fibonacci index to an open conjecture from finite field theory, namely *Wiedemann's conjecture*, which was given already in 1988 [Wie88]. Wiedemann considered *iterated quadratic extensions* of the finite field \mathbb{F}_2 using generators x_j which are recursively defined as

$$x_{j+1} + x_{j+1}^{-1} = x_j, \quad (4.22)$$

with $j \in \mathbb{N}$ and $x_0 + x_0^{-1} = 1$, where $x_0 \in \mathbb{F}_2$. Rewriting Equation (4.22) into

$$x_{j+1}^2 + x_{j+1}x_j + 1 = 0 \quad (4.23)$$

shows that the roots of this equation are not in the field where x_j belongs to, but in its quadratic extension, due to the quadratic nature of this equation [Wie88, Theorem 1]. Accordingly, he defined the (extension) fields E_j as

$$E_{j+1} := E_j(x_{j+1}), \quad (4.24)$$

with $E_0 := \mathbb{F}_2(x_0)$. The extension field E_j is then isomorphic to the finite field $\mathbb{F}_{2^{2^j+1}}$. As x_j^{-1} is the *conjugate* of x_j with respect to the ground field, $x_j^{-1} = x_j^{2^{2^j}}$ holds [LN08, Lemma 2.14]. Thus, the multiplicative order of x_j divides the j -th Fermat number $\mathcal{F}_j = 2^{2^j} + 1$. Calculating the product of the roots as $\tilde{x}_n := x_0 x_1 \dots x_n \in E_n$, the order of \tilde{x}_n is given by the product of the orders of the x_j , since the Fermat numbers are mutually coprime. Wiedemann finally conjectured, that the orders of x_j are given by $\mathcal{F}_j = 2^{2^j} + 1$, thus $|\tilde{x}_n| = \mathbb{F}_{2^{2^n+1}} - 2$ would hold, meaning \tilde{x}_n would be a *primitive element* of E_n , generating its multiplicative subgroup E_n^* . He successfully tested his conjecture computationally for $j = 0, \dots, 8$. The matrices C_{2^k} defined in Equation (4.19) can be seen as a realization of the x_k .

Theorem 4.2.3 (Wiedemann analogy).

Wiedemann's conjecture is true, if and only if the characteristic polynomial of all reduced stabilizer matrices B_{2^k} which are defined in Equation (4.18), has Fibonacci index $2^{2^k} + 1$.

Proof. If the stabilizer matrix C_{2^k} is constructed due to Equation (4.19), $C_{2^0} + C_{2^0}^{-1} = \mathbb{1}_2$ holds obviously. Assuming that the order of C_{2^k} is given by $2^{2^k} + 1$, the characteristic polynomial of $B_{2^k} = C_{2^{k-1}}$ has Fibonacci index $2^{2^k} + 1$ (which is in accordance to Condition (ii') of Section 4.1). The characteristic polynomial of C_{2^k} over the field where $B_{2^k} = C_{2^{k-1}}$ belongs to, is then by construction given as $x^2 + C_{2^{k-1}}x + 1$.

By comparing this result with Equation (4.23), the stabilizer matrices C_{2^k} can be identified with Wiedemann's x_k ; in equivalence with Wiedemann's construction the C_{2^k} are the roots of their characteristic polynomials. \square

With Theorem 4.2.3 it is shown that the construction of complete sets of cyclic MUBs with a reduced stabilizer matrix in the form of Equation (4.19) is an instance of Wiedemann's conjecture. A potential approach to proof this conjecture is given in Appendix B. We followed Wiedemann's idea and tested his conjecture for $j = 0, \dots, 11$, limited by the largest Fermat number with a known prime factorization (cf. Appendix C.1.4).

4.3 Standard form

All cyclic sets of MUBs which are based on the ideas of Bandyopadhyay *et al.* [BBRV02] need to partition the set of Pauli operators into disjoint classes of commuting elements, as discussed in Section 4.1 that deals with Fibonacci-based sets. The Properties (I)–(III) should always be fulfilled, but the form of the stabilizer matrix C can be different to that given in Equation (4.4). Additionally, we have the freedom to choose a different generator G_0 than $G_0 = (\mathbb{1}_m, 0_m)^t$ in order to prevent the appearance of a set with all Pauli- Z operators (which will be identified as the standard basis in Section 4.7). To be able to observe a complete set of cyclic MUBs within a common picture—different and more general than by relations to Fibonacci polynomials—we introduce the so-called *standard form*, derived from the Fibonacci-based sets. Unfortunately, this form does not preserve the cyclic structure of the generators of the classes, but it can be seen as an instrument to identify certain properties of the analyzed set of MUBs. It turns out that this form is also used by Bandyopadhyay *et al.*, but not in this very general form (cf. Equations (3.19) and (3.20) of Section 3.1.2).

We start with the generators that follow Equation (4.6) and the fact that G_0 is fixed for Fibonacci-based sets. We find

$$G_0 := \begin{pmatrix} \mathbb{1}_m \\ 0_m \end{pmatrix} \quad \text{and} \quad G_j = \begin{pmatrix} F_{j+1}(B) \\ F_j(B) \end{pmatrix}, \quad (4.25)$$

for $j \in \{1, \dots, d\}$. As the class \mathcal{C}'_j consists of all Pauli operators with argument $G_j \cdot \vec{c}$ and $\vec{c} \in \mathbb{F}_2^m$ (cf. Equation (3.17)), we have a certain freedom in the choice of G_j .

Corollary 4.3.1 (Generator matrix).

The generator G_j with $j \in \mathbb{N}$ that creates a class in the form $\mathcal{C}'_j = \{ZX(\vec{a}) : \vec{a} = G_j \cdot \vec{c} : \vec{c} \in \mathbb{F}_2^m\}$ can be multiplied from the right by any invertible matrix $P \in \text{GL}_m(\mathbb{F}_2)$ to produce the same set of Pauli operators, i. e.

$$\mathcal{C}'_j \equiv \{ZX(\vec{a}) : \vec{a} = G_j \cdot P \cdot \vec{c} : \vec{c} \in \mathbb{F}_2^m\}. \quad (4.26)$$

Proof. Since the class is created by all vectors \vec{c} with $\vec{c} \in \mathbb{F}_2^m$, the invertible matrix P only permutes this set by executing the operation $P \cdot \vec{c}$. \square

Using Corollary 4.3.1, we can write the generators from above (Equation (4.25)) in a different form by multiplying G_j with $(F_j(B))^{-1}$. Recalling Proposition 4.1.4, $F_j(B)$

equals either 0_m or its inverse exists. Therefore, we will not change G_0 for this form, which yields

$$\bar{G}_0 := \begin{pmatrix} \mathbb{1}_m \\ 0_m \end{pmatrix} \quad \text{and} \quad \bar{G}_j = \begin{pmatrix} F_{j+1}(B)(F_j(B))^{-1} \\ \mathbb{1}_m \end{pmatrix}, \quad (4.27)$$

the generators of a Fibonacci-based set written in *standard form*. We introduce sets which are not based on Fibonacci polynomials within Sections 4.5 and 4.6. It will turn out, that the following definition is capable to describe them uniquely.

Definition 4.3.1 (Standard form).

Generators G_j with $j \in \{0, \dots, d\}$, $d \in \mathbb{N}^*$, which create classes that partition the set of Pauli operators in order to produce a complete set of mutually unbiased bases are in standard form, if there holds

$$\bar{G}_0 := \begin{pmatrix} \mathbb{1}_m \\ G_0^x \end{pmatrix} \quad \text{and} \quad \bar{G}_j = \begin{pmatrix} G_j^z \\ \mathbb{1}_m \end{pmatrix}, \quad (4.28)$$

with matrices $G_j^z, G_0^x \in M_m(\mathbb{F}_2)$.

The values of G_j^z and G_0^x are clear in the case of Fibonacci-based sets. We refer to sets where $G_0^x = 0_m$ as *homogeneous sets* and sets where $G_0^x \neq 0_m$ as *inhomogeneous sets*. For both sets, we can in principle reuse Conditions (I)–(III) of Section 4.1. To fulfill Condition (III) for the homogeneous sets, all sums $G_k^z + G_l^z$ with $k, l \in \{1, \dots, d\}$, $k \neq l$ have to be invertible due to Lemma A.1.2. For the inhomogeneous sets, the expressions $\mathbb{1}_m + G_0^x G_j^z$ with $j \in \{1, \dots, d\}$ have to be invertible in addition. In order to generate commuting operators by the G_j , the matrices G_j^z and G_0^x have to be symmetric [BBRV02, Lemma 4.3]. It is reasonable to discuss the appearance of the different sets in relation to the entanglement properties of the classes of Pauli operators as will be seen in Section 4.4.

4.4 Entanglement properties

Within the constraints of the approach given by Bandyopadhyay *et al.* (cf. Section 3.1.2), different sets of cyclic MUBs exist. Some of these sets were studied in 2002 by Lawrence *et al.* [LBZ02], a complete list of four different sets in $d = 2^3$ dimensions was given by Romero *et al.* [RBKSS05]. Following the consideration of the latter work we will discuss the properties of the different sets and formulate conditions on the stabilizer matrix C in order to construct a complete set of cyclic MUBs with specific properties. Therefore, we first introduce the idea which is behind the classification of different sets.

Recalling Equation (3.17), a single basis within a set of MUBs (that is constructed according to the approach of Bandyopadhyay *et al.*) is given by the eigenspace of the Pauli operators which constitute a class \mathcal{C}_j with $j \in \{0, \dots, d\}$. The elements of such a class are generated by G_j and are by construction given by a tensor product of Pauli operators of the Hilbert space $\mathcal{H} = \mathbb{C}^2$ (cf. Equations (3.17) and (3.14)). Operators which are in the same class commute [BBRV02]. With a single basis we are able to

measure those properties of the complete quantum system, which are describable by the Pauli operators of the corresponding class \mathcal{C}_j .

We can categorize a basis in the following way: If the Pauli operators of each two-dimensional subsystem, as given by the tensor product decomposition, commute separately, this basis measures properties of a fully separable system, the *separability count* is $s(\mathcal{C}_j) = m$. If the Pauli operators of two subsystems within the class do not commute separately, the basis measures properties of a system that is decomposable into $m - 2$ single-qubit subsystems and a two-qubit subsystem. For a system of m subsystems, the separability count is then $s(\mathcal{C}_j) = m - 1$. The continuation of this classification leads to a couple of different decomposition structures, which we formalize in the following way: If a complete set of MUBs is used to determine the quantum state of an m -qubit system, we can describe the decomposition structure of a certain basis by arranging the set of m qubits into different subsets, where the properties are measured on the completely entangled states of the subsets. For all different possible set structures we define the separability count, by ordering all possible structures and take the position in this list as the separability count. This ordering works as follows:

1. Order the different structures by their number of subsets, starting with the largest.
2. Order the different structures which have the same number of subsets by the size of their largest set, starting with the smallest; if they are equal, continue with the second largest set, and so on.

Finally, we describe the decomposition structure of the whole basis set by counting the different decomposition structures of the classes with a single column vector \vec{n} . The first entry of this vector, namely n_1 , describes the number of bases which are fully separable, thus, which measure properties of m completely non-entangled subsystems. Then follows n_2 which counts the number of bases that describe systems that are separable into $m - 1$ parts. To get $m - 2$ parts we have two possibilities. Following the algorithm for the ordering, n_3 gives the number of bases which describe systems that can be decomposed into $m - 2$ subsystems, with $m - 4$ single-qubit systems and two two-qubit systems; n_4 counts bases that describe $m - 3$ single-qubit systems and one triple-qubit system. This goes logically forth until we end up with a fully-entangled system. This ordering is slightly different from that in [RBKSS05], in order to classify systems with few highly-entangled subsystems as less separable than systems with many sparsely-entangled subsystems.

Since there are only three different Pauli operators for a single-qubit system, there exist at most three different bases in a complete set of MUBs, that measure properties of a fully separable system. So the maximum value of n_1 is three. The Fibonacci-based sets of Section 4.1 reach this limit: the generator of the class \mathcal{C}_0 constructs all Pauli- Z operators that can be totally decomposed into m subsystems. The same holds true for the class \mathcal{C}_d which is generated by

$$G_d = C^d \cdot G_0 = \begin{pmatrix} 0_m & \mathbb{1}_m \\ \mathbb{1}_m & B \end{pmatrix} \cdot \begin{pmatrix} \mathbb{1}_m \\ 0_m \end{pmatrix} = \begin{pmatrix} 0_m \\ \mathbb{1}_m \end{pmatrix}, \quad (4.29)$$

and contains all Pauli- X operators. Finally, the class $\mathcal{C}_{d/2}$ is generated by $G_{d/2} = (F_{d/2}(B), F_{d/2+1}(B))^t$ which equals $(F_{d/2}(B), F_{d/2}(B))^t$ (see Lemma 4.1.1). Writing

this generator in standard form as allowed by Corollary 4.3.1, we find

$$\bar{G}_{d/2} = \begin{pmatrix} \mathbb{1}_m \\ \mathbb{1}_m \end{pmatrix}, \quad (4.30)$$

that produces all Pauli- Y operators.

As an example, we generate a complete Fibonacci-based set of cyclic MUBs for a three-qubit system with the reduced stabilizer matrix

$$B_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad (4.31)$$

that generates the classes as follows:

$$\begin{aligned} \mathcal{C}_0 &= \{\mathbb{1}\mathbb{1}Z, \mathbb{1}Z\mathbb{1}, \mathbb{1}ZZ, Z\mathbb{1}\mathbb{1}, Z\mathbb{1}Z, ZZ\mathbb{1}, ZZZ\}, & s(\mathcal{C}_0) &= 3, \\ \mathcal{C}_1 &= \{\mathbb{1}YX, XXZ, XZY, YXY, YZZ, Z\mathbb{1}X, ZY\mathbb{1}\}, & s(\mathcal{C}_1) &= 1, \\ \mathcal{C}_2 &= \{\mathbb{1}ZX, XXY, XYZ, Y\mathbb{1}X, YZ\mathbb{1}, ZXZ, ZYY\}, & s(\mathcal{C}_2) &= 1, \\ \mathcal{C}_3 &= \{\mathbb{1}ZY, X\mathbb{1}Y, XZ\mathbb{1}, YXZ, YYX, ZXX, ZYZ\}, & s(\mathcal{C}_3) &= 1, \\ \mathcal{C}_4 &= \{\mathbb{1}\mathbb{1}Y, \mathbb{1}Y\mathbb{1}, \mathbb{1}YY, Y\mathbb{1}\mathbb{1}, Y\mathbb{1}Y, YY\mathbb{1}, YYY\}, & s(\mathcal{C}_4) &= 3, \\ \mathcal{C}_5 &= \{\mathbb{1}XY, XYX, XZZ, YYZ, YZX, Z\mathbb{1}Y, ZX\mathbb{1}\}, & s(\mathcal{C}_5) &= 1, \\ \mathcal{C}_6 &= \{\mathbb{1}XZ, XYY, XZX, Y\mathbb{1}Z, YX\mathbb{1}, ZYX, ZZY\}, & s(\mathcal{C}_6) &= 1, \\ \mathcal{C}_7 &= \{\mathbb{1}YZ, X\mathbb{1}Z, XY\mathbb{1}, YXX, YZY, ZXY, ZZX\}, & s(\mathcal{C}_7) &= 1, \\ \mathcal{C}_8 &= \{\mathbb{1}\mathbb{1}X, \mathbb{1}X\mathbb{1}, \mathbb{1}XX, X\mathbb{1}\mathbb{1}, X\mathbb{1}X, XX\mathbb{1}, XXX\}, & s(\mathcal{C}_8) &= 3, \end{aligned} \quad (4.32)$$

where $\mathbb{1}$ abbreviates $\mathbb{1}_2$. The numbers of decomposable systems for the classes are given by the separability counts. So the vector \vec{n} in the case of this set of MUBs is given by $\vec{n}_1 = (3, 0, 6)^t$.⁸ Other possible structures are $\vec{n}_2 = (2, 3, 4)^t$, $\vec{n}_3 = (1, 6, 2)^t$ and $\vec{n}_4 = (0, 9, 0)^t$ [RBKSS05].

To generate these structures, which do not have three totally-decomposable classes, we cannot use the already discussed Fibonacci-based sets of Section 4.1. Therefore, we have to reduce the number of constraints, which are required to produce the Fibonacci-based sets, in order to achieve a larger set of possible structures. For Fibonacci-based sets, we find a field structure:

Lemma 4.4.1 (Field structure of Fibonacci-based sets).

The submatrices G_j^z of the generators $\bar{G}_j = (G_j^z, \mathbb{1}_m)^t$ in standard form with $j \in \{1, \dots, d\}$ that are used to generate a complete set of cyclic Fibonacci-based MUBs, form a representation of the finite field \mathbb{F}_2^m .

Proof. Considering the standard form (cf. Section 4.3) of the Fibonacci-based sets, we find the 2^m generators $G_j = (F_{j+1}(B)F_j(B)^{-1}, \mathbb{1}_m)^t$. As B has an irreducible

⁸This set is commonly known as the *standard set* [GRK10].

characteristic polynomial of degree m that generates a polynomial ring and 2^m different elements $F_{j+1}(B)F_j(B)^{-1}$ exist, the set of those elements is a representation of a polynomial ring, in particular of the finite field \mathbb{F}_2^m . \square

To create cyclic sets of MUBs it is more comfortable to use homogeneous sets, as discussed in Section 4.3, since their first basis is used to create the standard basis (cf. Section 4.7). In this case the generator of the first group is given by $G_0 = (\mathbb{1}_m, 0_m)^t$. To diminish the first value of the vector \vec{n} to two, by keeping the homogeneity property, it is obvious that we lose the field property of Lemma 4.4.1. What we get is an additive group of the values G_j^z without the unity element $\mathbb{1}_m$ of the multiplication, thus, a class of all Pauli- Y operators as given by Equation (4.30) is not included. In this case, Condition (III) of Section 4.1 is fulfilled if the expression $G_l^z + G_k^z$ is invertible for $k, l \in \{1, \dots, d\}$ with $k \neq l$, but this element is again an element of the group. We can thus create this group of matrices by a generating set, say w.l.o.g. $H_k \in \text{GL}_m(\mathbb{F}_2)$ with $k \in \{1, \dots, m\}$ and

$$G_j^z = \sum_{k=1}^m h_k H_k, \quad \text{for } (h_1, \dots, h_m) \in \mathbb{F}_2^m, \quad (4.33)$$

where all G_j^z with $j \neq 0$ have to be invertible. Equation (4.33) is equivalent to the construction of Bandyopadhyay *et al.* [BBRV02, Section 4.3]. Details will be discussed in Section 4.5.

To obtain $n_1 = 1$, we need to exclude also the element 0_m which forms the class of Pauli- X operators from the set of matrices G_j^z with $j \in \{1, \dots, d\}$. The highest set structure which fits is then a semigroup, which does not allow a generation as in Equation (4.33).

Finally, we have to give up the homogeneity property to set $n_1 = 0$. To be able to prevent all fully-decomposable classes, we have to change the generator G_0 . In Section 4.3 we discussed why $\tilde{G}_0 = (\mathbb{1}_m, G_0^x)^t$ is a good choice. It is clear, that Condition (III) of Section 4.1 has to be adapted minimally. Details of the inhomogeneous sets will be discussed in Section 4.6.

4.5 Homogeneous sets

The second class of cyclic MUBs we create in this work includes the class of Fibonacci-based MUBs formally, but does not share completely their nice way of construction. This generalization is based on the Conditions (I)–(III) of Section 4.1. As seen in Section 4.4, the entanglement properties of the bases within a certain set of MUBs are different for this class of homogeneous sets. Experimentally, complete sets with a reduced number of bases that measure properties of a fully entangled system, are to be preferred.

As already stated in Section 4.4, the homogeneous sets can be build either from generators with a field structure, as done for the Fibonacci-based MUBs, with an additive group structure or an additive semigroup structure. The number of bases with a fully-decomposable structure is three, two and one, respectively, in these cases. As the construction of Fibonacci-based MUBs is already solved, we give a generalized

solution in Section 4.5.1, which generates also group-based sets. In Section 4.5.2, a first notion on the realization of semigroup-based sets is presented.

4.5.1 Group-based sets

Considering the properties of the stabilizer matrix to generate a complete set of cyclic MUBs, we use Conditions (1)–(3) of Section 3.1.2. We propose a generalized form of the stabilizer matrix as in Section 4.1 in order to implement an additive group structure of the matrices G_j^z in the standard form, that has solutions which avoid the field structure (cf. Equation (4.28)). The stabilizer matrix we propose is given by

$$C = \begin{pmatrix} B & R \\ R^{-1} & 0_m \end{pmatrix}, \quad (4.34)$$

where the reduced stabilizer matrix $B \in M_m(\mathbb{F}_2)$ appears, but also an invertible matrix $R \in \text{GL}_m(\mathbb{F}_2)$. The powers of the stabilizer matrix, as given by Equation (4.34), are

$$C^m = \begin{pmatrix} F_{n+1}(B) & F_n(B)R \\ R^{-1}F_n(B) & R^{-1}F_{n-1}(B)R \end{pmatrix}, \quad (4.35)$$

with $F_n(x)$ being the Fibonacci polynomial with index $n \in \mathbb{N}$, as defined in Equation (3.26). In standard form the generators G_j of the classes \mathcal{C}_j (cf. Equation (3.17)) look like

$$\bar{G}_0 = \begin{pmatrix} \mathbb{1}_m \\ 0_m \end{pmatrix} \quad \text{and} \quad \bar{G}_j = \begin{pmatrix} F_{j+1}(B)F_j(B)^{-1}R \\ \mathbb{1}_m \end{pmatrix}, \quad (4.36)$$

thus, the field elements G_j^z of Equation (4.27) are multiplied by the invertible matrix R . The new elements \bar{G}_j^z of Equation (4.36) form an additive group which may be written by a generating set as stated by Equation (4.33). We can define conditions which are similar to Conditions (i) and (ii') of Section 4.1. Since those conditions emerge from Conditions (1)–(3) of Section 3.1.2, we can start our discussion there. As seen by Equation (4.36), the generators have the required form, thus, Condition (1) is fulfilled by construction. For Condition (2), all elements \bar{G}_j^z with $j \in \{1, \dots, d\}$ and $d = 2^m$ have to be symmetric. Those elements are given by the polynomials $p_n(B)$, defined as $p_n(B) = F_{n+1}(B)F_n(B)^{-1}$ with $n \in \{1, \dots, d\}$, multiplied by the matrix R . By the following lemma the number of matrices to be tested on their symmetry can be reduced dramatically to two.

Lemma 4.5.1 (Symmetry of Fibonacci polynomials multiplied with invertible matrix).

For invertible matrices $A, B \in \text{GL}_m(K)$, all polynomials $p(A) \in K[x]$ multiplied with B from the right are symmetric, if and only if B and AB are symmetric.

Proof. Let B and AB be symmetric matrices. Then for any matrix $A^k B$ with $k \in \mathbb{N}$ there holds

$$(A^k B)^t = B^t A^t (A^{k-1})^t = AB^t A^t (A^{k-2})^t = \dots = A^k B. \quad (4.37)$$

Any polynomial in A multiplied with B is then a sum of symmetric matrices. But sums of symmetric matrices are again symmetric. The converse is obvious, if we consider the cases $k = 0$ and $k = 1$, thus $\mathbb{1}_m B$ and AB . \square

Using Lemma 4.5.1, Condition (2) of Section 3.1.2 is fulfilled, if and only if R and BR are symmetric. Condition (3) is also correct, since the difference of two group elements leads again to a group element. If all three conditions are fulfilled, the stabilizer matrix C has to be symplectic, but as a precaution we can check this easily by using Corollary A.6.1 of Appendix A.6. The product $R(R^{-1})^t$ equals $\mathbb{1}_m$ as R is a symmetric matrix. B^tR^{-1} is symmetric, since its transpose $(R^{-1})^tB$ is the product of a polynomial in R (cf. Proposition 4.1.4) with the symmetric matrix B , therefore, we can apply Lemma 4.5.1. To get a complete set of MUBs, the similarity of Equation (4.36) with Equation (4.27) indicates, that the characteristic polynomial of B has to have a Fibonacci index of $d + 1$. Finally, we find the following conditions:

- (i) R and BR are symmetric.
- (ii) The characteristic polynomial of B has Fibonacci index $d + 1$.

The polynomials $p_n(B)$ have field structure, but multiplied by the matrix R this structure disappears. Only an additive group structure is left, as wanted. For $R = \mathbb{1}_m$ or more generally for $R = p_n(B) \neq 0_m$ with $n \in \{1, \dots, d\}$, we find the Fibonacci-based sets. Thus, for appropriate values of $R \neq p_n(B)$ we find complete sets of cyclic MUBs that have exactly two bases with a totally-decomposable structure, namely \mathcal{C}_0 and \mathcal{C}_d .

According to Equation (4.34), we find 126 different pairs (B, R) for a system of three qubits that have a decomposition structure of the bases as given in [RBKSS05], namely $\vec{n} = (2, 3, 4)^t$. One of the solutions is given by

$$B_{(2,3,4)} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad R_{(2,3,4)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (4.38)$$

It is expected that most of the 126 solutions can be transformed into each other with transformations that originate from simple symmetry considerations. One of those transformations corresponds only to sets which are not related to the Fibonacci-based sets and was indirectly discussed above. If we replace the matrix R in the construction of Equation (4.34) by the product of a polynomial p_l in B , with the old matrix R , we get

$$C = \begin{pmatrix} B & p_l(B)R \\ R^{-1}p_l(B)^{-1} & 0_m \end{pmatrix}, \quad (4.39)$$

with generators in standard form as

$$\bar{G}_0 = \begin{pmatrix} \mathbb{1}_m \\ 0_m \end{pmatrix} \quad \text{and} \quad \bar{G}_j = \begin{pmatrix} F_{j+1}(B)F_j(B)^{-1}p_l(B)R \\ \mathbb{1}_m \end{pmatrix}. \quad (4.40)$$

Since all polynomials in B with maximal degree $m - 1$ appear in the groups \bar{G}_j , the polynomial $p_l(B)$ with $l \in \{1, \dots, d\}$ permutes the original generators \bar{G}_j . Since there are only $2^m - 1$ non-zero polynomials with degree smaller than m , we found a symmetry to reduce the solutions by a factor of $2^m - 1$. In the case discussed above, we can divide 126 by $2^3 - 1$ and get 18. We would get cyclic MUBs with

the same entanglement properties by relabeling the different qubits, which lead to $m!$ permutation operators, in the case of three qubits to six possible permutations. But it is not clear if this symmetry is uncorrelated with the symmetry that appears from multiplying the matrix R with a polynomial in B . A complete explanation of the appearance of 126 solutions may be an interesting issue but is above the scope of this work.

We list seven systems with a different decomposition structure for four-qubit systems in Appendix C.1.1. Some of the sets do not appear in the list given in [RBKSS05]. This is in accordance to a statement in [Law11, Section IV D], which claims an incompleteness of the list of 16 MUBs in the former article; the complete classification lists 34 sets [GRK10].

4.5.2 Semigroup-based sets

In order to obtain complete sets of cyclic MUBs which contain only one class of operators that is fully separable into qubit systems, we need to find a form of the stabilizer matrix C which is even more general than the form of Section 4.5.1 and is given by Equation (4.34). A hypothetical idea would be the following: To construct the Fibonacci sets of Section 4.1, the stabilizer matrix $C \in M_{2m}(\mathbb{F}_2)$ was completely determined by the reduced stabilizer matrix $B \in M_m(\mathbb{F}_2)$, for the group-based sets of Section 4.5.1, a second matrix $R \in \text{GL}_m(\mathbb{F}_2)$ became relevant. To construct semigroup-based sets we can imagine that a third matrix will be relevant. Nevertheless, we have not been able to find an appropriate generalization of Equation (4.34) so far to reach this goal. Conversely, from a top-down point of view, we are able to construct at least some solutions within an obtainable computation time: If we assume that the upper left submatrix of C remains invertible, we find by the properties which are given by Corollary A.6.1, that for any symplectic matrix $C = \begin{pmatrix} s & t \\ u & v \end{pmatrix}$, with $s, t, u, v \in M_m(K)$ and s invertible, there holds

$$v = (s^t)^{-1}(1 + u^t t). \quad (4.41)$$

Thus, we should be able to construct the semigroup-based sets with the three matrices $s, t, u \in M_m(\mathbb{F}_2)$. A similar observation is possible by assuming that u is invertible, but does not lead to a logical generalization of the group-based sets, as we are not able to require a certain value for the characteristic polynomial of s . A cyclic version of the corresponding three-qubit system in [RBKSS05] is generated by

$$C_{(1,6,2)} = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (4.42)$$

However, as a suitable form is not found yet, computational results based on these approaches and Conditions (1)–(3) of Section 3.1.2 are given in Appendix C.1.2.

4.6 Inhomogeneous sets

The third and last class of cyclic MUBs that are discussed in this work does not contain any class of operators that is fully separable into qubit systems in its decomposition structure which was introduced in Section 4.4. Accordingly, these sets need to have a more general standard form as given in Definition 4.3.1, thus, the generator \bar{G}_0 does not equal $(\mathbb{1}_m, 0_m)^t$ as for the Fibonacci-based sets and the homogeneous sets in general. The approach we choose in order to create an inhomogeneous set is the following: At first, a symplectic matrix $C_0 \in M_{2m}(\mathbb{F}_2)$ is taken which produces a more general generator \bar{G}_0 from the matrix $(\mathbb{1}_m, 0_m)^t$. Secondly, the ordinary symplectic stabilizer matrix C is taken.

If we take the generator $\bar{G}_0^{(h)}(\mathbb{1}_m, 0_m)^t$ of the homogeneous sets, we can produce any generator \bar{G}_0 as

$$\bar{G}_0 = C_0 \bar{G}_0^{(h)} = \begin{pmatrix} \mathbb{1}_m & t \\ G_0^x & v \end{pmatrix} \begin{pmatrix} \mathbb{1}_m \\ 0_m \end{pmatrix} = \begin{pmatrix} \mathbb{1}_m \\ G_0^x \end{pmatrix}, \quad (4.43)$$

with $t, v \in M_m(\mathbb{F}_2)$, where t and v can be chosen according to fulfill the properties derived by Corollary A.6.1 to let C_0 be symplectic. For the first of these three properties, G_0^x has to be symmetric, which has also be true in order to create a class of commuting elements in \mathcal{C}_0 , corresponding to the derived Property (2) of Section 3.1.2.

The following discussion gives a first idea of the construction of inhomogeneous sets. The results can be seen as a proof-of-principle that legitimates the approach. In order to generate a class which is not fully decomposable, we entangle two qubit systems with indices $i, j \in \{1, m\}$, $i \neq j$ by setting $(G_0^x)_{i,j} = (G_0^x)_{j,i} = 1$, which has the expected effect, regarding the class generation which is introduced by Equation (3.17). Following then the ideas of Section 4.5.2, we found some of the solutions for four-qubit systems and list them in Appendix C.1.3. A cyclic generator for the three-qubit system that equals the system which is generated in [RBKSS05] is given by

$$C_{(0,9,0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (4.44)$$

where the corresponding generator G_0 equals

$$(G_0^x)_{(0,9,0)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (4.45)$$

with $G_0 = (\mathbb{1}_m, G_0^x)^t$.

4.7 Unitary operator

So far, we have discussed different sets of cyclic MUBs in the view of a partition of the set of Pauli operators, as introduced in Section 3.1.2. The corresponding stabilizer

matrices were introduced in Sections 4.1, 4.5, and 4.6 and serve as generators of the set of classes which partition the set of Pauli operators. But, for measuring quantum states, we need to transform those matrices into unitary operators, in the fashion of Section 3.1.2. This section starts by deriving that transformation and leads to a comfortable form of the unitary operators which result from Fibonacci-based sets (Section 4.7.1). Section 4.7.2 recalls the iterative construction of the Fermat sets of Section 4.2 in order to derive again an iterative construction of these MUBs, but this time on the level of unitary operators as published in [SR11]. The section closes by discussing shortly the generation of the unitary operator in cases of a more general stabilizer matrix (Section 4.7.3).

For the derivation of the unitary operator we use the theory of the *stabilizer formalism*⁹, as could have already been figured out by the definition of the stabilizer matrix $C \in M_{2m}(\mathbb{F}_p)$. Within this theory, substitutes of the usual Pauli operators are defined, namely the *logical Pauli operators*. The usual Pauli operators $ZX(\vec{a})$ were defined in Equation (3.14). With $\vec{z}_k \in \mathbb{F}_p^{2m}$ being a vector that has a one at position k and zeros else and $\vec{x}_k \in \mathbb{F}_p^{2m}$ a vector with a one at position $m+k$ and zeros else, we can define the *physical operators*, namely a unitary representation of \mathbb{F}_p^{2m} , as $Z_k = ZX(\vec{z}_k)$ and $X_k = ZX(\vec{x}_k)$; using a stabilizer matrix A results in the *logical operators* $\bar{Z}_k = ZX(A\vec{z}_k)$ and $\bar{X}_k = ZX(A\vec{x}_k)$ with $k \in \{1, \dots, m\}$, $m \in \mathbb{N}^*$ and p prime. The physical Pauli- Z operators produce a local phase factor, whereas physical Pauli- X operators induce bit (or dit¹⁰) flips. Therefore, the logical ground state $|0\rangle_L$ is naturally defined as the joint eigenstate of all logical Pauli- Z operators with eigenvalue $+1$. The other logical states can be derived by applying bit (or dit) flips as

$$|j\rangle_L = \bar{X}_1^{j_1} \cdots \bar{X}_m^{j_m} |0\rangle_L, \quad (4.46)$$

with $\{j_1, \dots, j_m\}$ being the bit (or dit) decomposition of $j \in \mathbb{F}_p^m$. At first, we focus on the construction of the unitary operator that generates a Fibonacci-based set of MUBs.

4.7.1 Fibonacci-based sets

In order to facilitate the following consideration, we use the inverse of the stabilizer matrix for the Fibonacci-based sets (cf. Equation (4.4)), which, in the case of qubits, is obviously given by

$$C^{-1} = \begin{pmatrix} 0_m & \mathbb{1}_m \\ \mathbb{1}_m & B \end{pmatrix}. \quad (4.47)$$

This leads to the logical operators

$$\bar{Z}_k = ZX(C^{-1}\vec{z}_k) \quad \text{and} \quad \bar{X}_k = ZX(C^{-1}\vec{x}_k), \quad (4.48)$$

⁹The theory of the stabilizer formalism was developed by Gottesman [Got96]. See also his Ph. D. thesis [Got97].

¹⁰A dit denotes a variable defined over \mathbb{Z}_d , generalizing a bit.

with $\bar{Z}_k = X_k$ for $k \in \{1, \dots, m\}$. Thus, the logical ground state is (up to a global phase) given by

$$|0\rangle_L = 2^{-m/2} \sum_{j \in \mathbb{F}_2^m} |j\rangle. \quad (4.49)$$

For the construction of a complete set of MUBs for a dimension of the Hilbert space which is a power of two, the set of Pauli operators is partitioned into $d + 1 = 2^m + 1$ disjoint classes of commuting operators (cf. Equation (3.16)). The common eigenbases of those classes are mutually unbiased, as shown by Bandyopadhyay *et al.* [BBRV02]. In order to get a cyclic set, the transformation that brings a class \mathcal{C}_l with $l \in \{0, \dots, d\}$ of operators to the class $\mathcal{C}_{(l+1) \bmod (d+1)}$, has to be of multiplicative order $d + 1$. By construction, the class \mathcal{C}_0 of the Fibonacci-based sets appears as

$$\mathcal{C}_0 = \left\{ ZX \begin{pmatrix} \vec{a}_z \\ \vec{0} \end{pmatrix} \middle| \vec{a}_z \in \mathbb{F}_2^m \setminus \{0\} \right\}. \quad (4.50)$$

For a complete set of cyclic MUBs there exists a unitary operator $U \in M_d(\mathbb{C})$, such that $U\mathcal{C}_l U^\dagger = \mathcal{C}_{(l+1) \bmod (d+1)}$ and $U^{d+1} = \mathbb{1}_d$. With $\mathbb{1}_d$ being the common eigenbasis of the operators of the class \mathcal{C}_0 , the columns of U^l form the vectors of the eigenbases of \mathcal{C}_l . Since U maps Pauli operators onto Pauli operators, it is called a *Clifford unitary operator*. Properties of these operators and their relation to symplectic matrices is given in Appendix A.6. As mentioned above, we will calculate the inverse of the unitary operator, thus U^\dagger . By definition, the operator U^\dagger transforms a physical state into the corresponding logical state, namely $U^\dagger |j\rangle = |j\rangle_L$ which yields with Equation (4.46)

$$U^\dagger |j\rangle = e^{-i\Psi} \prod_{k=1}^m ZX(C^{-1} \vec{x}_k)^{j_k} |0\rangle_L, \quad (4.51)$$

and with Equation (4.49) finally

$$U^\dagger = 2^{-m/2} e^{-i\Psi} \sum_{i, j \in \mathbb{F}_2^m} \prod_{k=1}^m ZX(C^{-1} \vec{x}_k)^{j_k} |i\rangle \langle j|. \quad (4.52)$$

The global phase factor $e^{-i\Psi}$ with $\Psi \in [0, 2\pi)$ is fixed by assuming that $(U^\dagger)^{d+1} = \mathbb{1}_d$, but the phases of the elements of the set of Pauli operators within a class \mathcal{C}_j are arbitrary. Thus, this global phase factor does not follow from the stabilizer formalism. To calculate the expression of Equation (4.52), we plug in C^{-1} in the form of Equation (4.47) with $B = (b_{ij})$. The general k -th factor looks like

$$ZX(C^{-1} \vec{x}_k) = \left(\bigotimes_{t=1}^{k-1} X^{b_{tk}} \right) \otimes (-i)^{b_{kk}} ZX^{b_{kk}} \otimes \left(\bigotimes_{t=k+1}^m X^{b_{tk}} \right). \quad (4.53)$$

Applying the product, we find for a single tensor factor of the qubit at position k the term

$$(-i)^{b_{kk}j_k} X^{b_{1k}j_1 + \dots + b_{k-1,k}j_{k-1}} \cdot Z^{j_k} \cdot X^{X_{kk}j_k + \dots + b_{mk}j_m}, \quad (4.54)$$

that can, by shifting the operator Z to the left, be rewritten as

$$i^{b_{kk}j_k} (-1)^{b_{1k}j_1j_k + \dots + b_{kk}j_kj_k} Z^{j_k} \cdot X^{X_{1k}j_k + \dots + b_{mk}j_m}. \quad (4.55)$$

To be able to write the tensor product in a short form, we define the abbreviations

$$p_j^* := i^{\sum_{k=1}^m b_{kk}j_k} (-1)^{\sum_{k=1}^m b_{1k}j_1j_k + \dots + b_{kk}j_kj_k}, \quad (4.56)$$

$$\tilde{X}_j := \bigotimes_{k=1}^m X^{b_{1k}j_1 + \dots + b_{mk}j_m}, \quad (4.57)$$

which lead to

$$U^\dagger = 2^{-m/2} e^{-i\Psi} \sum_{i,j \in \mathbb{F}_2^m} p_j^* \cdot \left(\bigotimes_{k=1}^m Z^{j_k} \right) \cdot \tilde{X}_j |i\rangle \langle j|. \quad (4.58)$$

Only the factor \tilde{X}_j acts on $\sum_i |i\rangle$, but it has no argument in i and keeps the sum invariant. Therefore—summarized over i —it equals the unity operator and vanishes. The tensor product $\bigotimes_{k=1}^m Z^{j_k}$ can be identified with $(-1)^{i \cdot j}$ that equals Sylvester's Hadamard matrix (cf. Equation A.7). Thus we end up with the short form of U^\dagger as

$$U^\dagger = e^{-i\Psi} H^{\otimes m} \cdot P^*, \quad (4.59)$$

the m -folded tensor product of the *normalized Hadamard matrix* (cf. Equation (A.6)) which is defined as

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad (4.60)$$

and the diagonal *phase system* matrix which is given by $P^* = \text{diag}((p_j^*)_{j \in \mathbb{F}_2^m})$. With B being symmetric and the quadratic form

$$\langle j|B|j\rangle = \sum_{k,l=1}^m b_{kl}j_kj_l, \quad (4.61)$$

where $\langle j|B|j\rangle \in \mathbb{Z}_4$ holds, as it is only taken as the power of a root of order four, we finally find

$$p_j^* = i^{\langle j|B|j\rangle} (-1)^{\sum_{k=1}^m b_{kk}j_k}. \quad (4.62)$$

For the sake of completeness, the original unitary operator U is accordingly given by

$$U = e^{i\Psi} P \cdot H^{\otimes m}, \quad (4.63)$$

with $P = \text{diag}((p_j)_{j \in \mathbb{F}_2^m})$ and $p_j = (-i)^{\langle j|B|j\rangle} (-1)^{\sum_{k=1}^m b_{kk}j_k}$.¹¹ The consideration of the unitary operators U for the derived stabilizer matrices that follow from the matrices A in Appendix C.1 leads to an unproven conjecture.

¹¹ The additional minus sign comparing to the results of [KRS10, SR12] is due to the fact that we to construct U here and not U^\dagger , which does not matter in principle.

Conjecture 4.7.1 (Spectrum of generators of MUBs).

The spectrum of a unitary operator U that generates a complete set of MUBs is non-degenerate; it consists of all roots of unity of order $d + 1$ with a single exception.

It is clear, that the eigenvalues of U have to be roots of unity of order $d + 1$ in order to fulfill $U^{d+1} = \mathbb{1}_d$. But U is a $d \times d$ matrix, thus if it is non-degenerate, it has d different eigenvalues. Multiplying a valid U by any root of unity of order $d + 1$ will result again in a valid U . We are free to exclude the trivial root 1 from the spectrum which results in a spectrum that is symmetric to the real axis. In this case, the trace of U is given by $\text{tr} U = -1$. This way, we can calculate the global phase $e^{i\Psi}$ of U by inserting Equation (4.63) as

$$e^{i\Psi} = -\text{tr}(P^* H^{\otimes m}). \quad (4.64)$$

The consideration of derived unitary operators that generate a complete set of cyclic MUBs, where the eigenvalue -1 is excluded from the spectrum, results again in a conjecture.

Conjecture 4.7.2 (Global phase of generators of MUBs).

The global phase of the generator U in the form of Equation (4.63) of a complete set of cyclic MUBs, where the eigenvalue $+1$ is chosen to be excluded from the spectrum of U , provided Conjecture 4.7.1 is true, reads

$$e^{i\Psi} = \begin{cases} \frac{-1-i}{\sqrt{2}}, & \text{for } m \text{ odd,} \\ -i, & \text{for } m \text{ even.} \end{cases} \quad (4.65)$$

It is surprising that the global phase $e^{i\Psi}$ seems to depend not on the stabilizer matrix C at all, but only on the number of qubits, thus, on the dimension $d = 2^m$. Since by construction of the phase factors p_j (cf. Equation (4.62)), the product $P \cdot H^{\otimes m}$ leads to a matrix with entries that are roots of unity of order four (and a global real normalization factor), for even m the unitary matrix U has only roots of unity of order four as entries, whereas for odd m it has only entries that are principal roots of order eight (so not of order four). The complex conjugation of the phase factor in relation to the results of [KRS10] comes from the fact that here are operators ZX used, in [KRS10] operators XZ , but C looks the same. As an advantage, the approach used here results in a symmetric arrangement of the classes \mathcal{C}_j , for example \mathcal{C}_0 consists only of Pauli- Z operators and \mathcal{C}_d of Pauli- X operators.

4.7.2 Fermat sets

In Section 4.2 it was shown that for dimensions $d = 2^{2^k}$ with $k \in \mathbb{N}$, a complete set of cyclic MUBs can be build recursively, thus, using the complete set of cyclic MUBs from the next smaller dimension. This procedure in terms of the stabilizer matrix C was shown in Equation (4.19). Considering the unitary matrix U which results from C for Fibonacci-based sets as given by Equation (4.63), we find an analogous recursive construction, but this time in terms of U . And, comparably to the definition of C_{2^k} of Equation (4.19), we will refer to the unitary operator U of dimension 2^m with $m = 2^k$ in the following as U_{2^k} .

To simplify matters, we define the matrix V by

$$V = P \cdot \bar{H}^{\otimes m}, \quad (4.66)$$

which has, compared to the unitary operator U of Equation (4.63), a factor of one for the global phase and with the *Hadamard matrix* as given in Equation A.5, namely

$$\bar{H} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad (4.67)$$

no normalization factor. Thus, V is not a unitary matrix, but U can be derived from V simply as $U = V/(-\text{tr } V)$. To calculate a matrix V_m with a B_m as discussed in Section 4.2, we have

$$V_m = \text{diag}(((-i)^{\langle i|B_m|i \rangle} (-1)^{\sum_{k=1}^m b_{kk} i_k})_{i \in \mathbb{F}_2^m}) \cdot \bar{H}^{\otimes m}, \quad (4.68)$$

using Equation (4.62). By Equation (4.18), the only value on the diagonal of B_{2^k} that does not vanish, is b_{11} . We can therefore abbreviate Equation (4.68) as

$$V_m = \text{diag}(((-i)^{\langle i|B_m|i \rangle} (-1)^{i_1})_{i \in \mathbb{F}_2^m}) \cdot \bar{H}^{\otimes m}. \quad (4.69)$$

The same construction holds for V_{2m} ; we may consider the new vector $i' = (i'_1, \dots, i'_{2m})^t \in \mathbb{F}_2^{2m}$ as $i' = (i, j)^t = (i_1, \dots, i_m, j_1, \dots, j_m)^t \in \mathbb{F}_2^{2m}$, which leads to

$$V_{2m} = \text{diag}(((-i)^{\langle i,j|B_{2m}|i,j \rangle} (-1)^{i_1})_{i,j \in \mathbb{F}_2^m}) \cdot \bar{H}^{\otimes 2m}. \quad (4.70)$$

The matrix element $\langle i, j|B_{2m}|i, j \rangle$ with $B_{2m} = \begin{pmatrix} B_m & \mathbb{1}_m \\ \mathbb{1}_m & 0_m \end{pmatrix}$ equals $\langle i|B_m|i \rangle + 2(i \cdot j)$,¹² thus, we find for Equation (4.70):

$$V_{2m} = \text{diag}(((-i)^{\langle i|B_m|i \rangle} (-1)^{i_1} (-1)^{i \cdot j})_{i,j \in \mathbb{F}_2^m}) \cdot \bar{H}^{\otimes 2m}. \quad (4.71)$$

The factor $(-1)^{i \cdot j}$ defines the $2^m \times 2^m$ dimensional Hadamard matrix $\bar{H}^{\otimes m}$, the same, that is used to describe V_m . By definition, i represents the lower bit half and j the higher bit half of i' as $i' = (i, j)^t$. We identify i as the row index and j as the column index of $H^{\otimes m}$, the factors, namely $(-i)^{\langle i|B_m|i \rangle} (-1)^{i_1}$ are invariant of j , thus repeat for each j . The phase vector of V_{2m} arises, by calculating V_m and concatenating its column vectors such that

$$\begin{pmatrix} p_1 \\ p_2 \\ \dots \\ p_{2^{2m}} \end{pmatrix} = \begin{pmatrix} (\vec{v}_1)_m \\ (\vec{v}_2)_m \\ \dots \\ (\vec{v}_{2^m})_m \end{pmatrix}, \quad (4.72)$$

where $(\vec{v}_k)_m$ is the k -th column vector of V_m . We will refer to the mapping of V_m to the phase vector P_{2m} of V_{2m} as the *chop map* \mathcal{M} , i. e. $\mathcal{M}(V_m) = P_{2m}$. This defines the recursion relation

$$V_{2m} = \text{diag}(\mathcal{M}(V_m)) \cdot \bar{H}^{\otimes 2m}. \quad (4.73)$$

¹²It has to be taken care, that $2(i \cdot j) \in \mathbb{Z}_4$ holds and does therefore not vanish, as it is part of the exponent of $-i$ in Equation (4.70).

For $k = 0$, thus $m = 2^k = 1$, we have with $B_1 = (1)$:

$$V_1 = \text{diag}((-i)^{0 \cdot 1 \cdot 0}(-1)^0, (-i)^{1 \cdot 1 \cdot 1}(-1)^1) \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (4.74)$$

$$= \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}. \quad (4.75)$$

This implies with $\mathcal{M}(V_1) = (1, i, 1, -i)^t$:

$$V_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ i & -i & i & -i \\ 1 & 1 & -1 & -1 \\ -i & i & i & -i \end{pmatrix}, \quad (4.76)$$

and can be continued at least to V_{2048} as checked in Appendix C.1.4, and if Wiedemann's conjecture is true, for all V_{2^k} with $k \in \mathbb{N}$.

To calculate the unitary operator U_m as $U_m = V_m / (-\text{tr } V_m)$, we need to analyze the matrix V_m in detail. If $i \in \mathbb{F}_2$ represents the rows and $j \in \mathbb{F}_2$ the columns of V_1 , we find with Equation (4.75),

$$(V_1)_{i,j} = \begin{cases} 1 \cdot (-1)^{i \cdot j}, & \text{for } i \equiv 0 \pmod{2}, \\ i \cdot (-1)^{i \cdot j}, & \text{for } i \equiv 1 \pmod{2}. \end{cases} \quad (4.77)$$

Taking Equation (4.73) into account, V_2 reads as

$$(V_2)_{i,j} = \begin{cases} 1 \cdot (-1)^{i_1 \cdot i_2 + i \cdot j}, & \text{for } i \equiv 0 \pmod{2}, \\ i \cdot (-1)^{i_1 \cdot i_2 + i \cdot j}, & \text{for } i \equiv 1 \pmod{2}, \end{cases} \quad (4.78)$$

with $i = \{i_1, \dots, i_m\}$ in general. As seen above, the lengths of the vectors i and j are one for V_1 . For V_2 the vectors are two bit long, for V_4 four bit and so forth. Using Equation (4.71) in order to recognize Hadamard matrices in the construction of V_m recursively, we find

$$(V_m)_{i,j} = \begin{cases} 1 \cdot (-1)^x, & \text{for } i \equiv 0 \pmod{2}, \\ i \cdot (-1)^x, & \text{for } i \equiv 1 \pmod{2}, \end{cases} \quad (4.79)$$

with

$$x = i_1 \cdot i_2 + (i_1, i_2) \cdot (i_3, i_4)^t + \dots + (i_1, \dots, i_{m/2}) \cdot (i_{m/2+1}, \dots, i_m)^t + i \cdot j. \quad (4.80)$$

The trace of V_m is given by the sum of the diagonal values, where $j = i$ holds. We will calculate the real and the imaginary part separately.

For the real part of the trace of V_m , the first bit value of i has to be zero, thus we find

$$\mathcal{R} \{\text{tr } V_m\} = \sum_{i \in \mathbb{F}_2^m, i_1=0} (V_m)_{i,i} = \sum_{i \in \mathbb{F}_2^m, i_1=0} (-1)^x, \quad (4.81)$$

with $x = 0 \cdot i_2 + (0, i_2) \cdot (i_3, i_4)^t + \dots + (0, i_2, \dots, i_{m/2}) \cdot (i_{m/2+1}, \dots, i_m)^t + i \cdot i$. We arrange the sum in pairs, where for one element of the pair $i_{m/2+1} = 0$ holds and for the other element $i_{m/2+1} = 1$. Since $i_1 = 0$ holds for all terms in the sum, x differs only in the last term for the two elements of a pair, where $i \cdot i$ gives the Hamming weight of i , that differs by one for the different cases. So for one element of each pair, x is an even number, for the other element it is odd and as exponents of -1 can be taken modulo two, the expression $(-1)^0 + (-1)^1 = 0$ is the sum of both elements and equals zero. Since all elements can be paired this way, the real part of the trace of V_m vanishes.

For the imaginary part of the trace of V_m the value of i_1 equals one, we find

$$\mathcal{I}\{\text{tr } V_m\} = \sum_{i \in \mathbb{F}_2^m, i_1=1} (V_m)_{i,i} = \sum_{i \in \mathbb{F}_2^m, i_1=1} (-1)^x, \quad (4.82)$$

with $x = 1 \cdot i_2 + (1, i_2) \cdot (i_3, i_4)^t + \dots + (1, i_2, \dots, i_{m/2}) \cdot (i_{m/2+1}, \dots, i_m)^t + i \cdot i$. We will consider two cases: For those elements i , where at least one of the bit values of $\{i_2, \dots, i_{m/2}\}$ equals zero, we can arrange the elements analogously in pairs as done for the real part. The second case treats all remaining elements, namely those where all bit values of $\{i_2, \dots, i_{m/2}\}$ equal one, which produces terms with

$$x = 1 \cdot 1 + (1, 1) \cdot (1, 1)^t + \dots + (1, \dots, 1) \cdot (i_{m/2+1}, \dots, i_m)^t + i \cdot i. \quad (4.83)$$

For $m = 1$, we have only one term and this adds -1 to the imaginary part. For $m = 2$, the imaginary part equals $\sum_{i_2=0}^1 (-1)^{1 \cdot i_2 + (1, i_2) \cdot (1, i_2)^t}$ which leads to -2 . For all $m > 2$ we can use the following observation: For each of the $2^{m/2}$ elements with $i = (1, \dots, 1, i_{m/2+1}, \dots, i_m)$ the two rightmost terms of Equation (4.83) give the same result modulo two, since the number of ones in the left part is even. As m is a power of two, all remaining terms without the first one result in powers of two and the first one equals one. Therefore, the sum of all those elements provides a contribution of $-2^{m/2}$ to the imaginary part of the trace of V_m . Thus, we have

$$\text{tr } V_m = -i2^{m/2}. \quad (4.84)$$

Taking all results together, we finally find for the unitary operator U_m which results for the Fermat-based sets,

$$U_{2m} = -i2^{-m} \text{diag}(\mathcal{M}(V_m)) \cdot \bar{H}^{\otimes 2m}. \quad (4.85)$$

The difference of the operators to those given in [SR11] is due to the replacement of XZ operators by ZX operators which was motivated in Section 4.7.1. Using the results of this section may lead to a proof of Wiedemann's conjecture by the analogy that was proven in Theorem 4.2.3. An idea is given in Appendix B.

4.7.3 More general sets

The construction of a unitary operator that generates a complete set of cyclic MUBs which is based on a more general stabilizer matrix, like those discussed in Sections 4.5 and 4.6 is possible in a similar way as was done for the Fibonacci-based sets in

Section 4.7.1 by using the same steps. But, presumably, the resulting form will not be decomposable as simple as in Equation (4.63). The search for a nice construction form is beyond the scope of this work; nevertheless, a physical implementation of general stabilizer matrices will be discussed in Section 4.8 that is more applicable than the representation as a unitary matrix with a simple construction.

4.8 Gate decomposition

We have seen so far, that the construction of complete sets of cyclic MUBs for a Hilbert space of dimension $d = 2^m$ with $m \in \mathbb{N}^*$ is based on the construction of a stabilizer matrix (cf. Sections 4.1, 4.2, 4.5, and 4.6), which can be represented as a unitary matrix (cf. Section 4.7). What still lacks is the implementation of the individual bases into an experimental setup (e. g. a quantum computer). A common general description of an implementation is the idea of a *quantum circuit*. This is a scheme, similar to a classical circuit, that gives a description of the behavior of a unitary operation that can be seen as a recipe on how to implement the operation in *quantum gates*. The challenge in deriving this circuit from the operation is to get a minimal number of those gates where each gate acts on a minimal number of qubits.¹³ It was shown that single-qubit gates and two-qubit gates produce a *universal set*, which can be used to implement any unitary operation [DiV95, DBE95, Llo95, BMP⁺00]. It is important to mention that the application of gates in quantum circuits have to be read from left to right, rather than the application of quantum operations on quantum states. Within this section, we provide a simple generation of a quantum circuit for the Fibonacci-based sets of Section 4.1 and show the minimal form of this circuit in the case of the Fermat sets of Section 4.2. Finally, an approach for constructing a quantum circuit from a more general stabilizer matrix is given. A short discussion on performance and error optimizations closes this section.

4.8.1 Fibonacci-based sets

In the case of Fibonacci-based sets, it turns out that the construction of an appropriate quantum circuit that implements the generator of a complete set of cyclic MUBs can be read out from the reduced stabilizer matrix $B \in M_m(\mathbb{F}_2)$ as it was introduced in Equation (4.4). But, honestly speaking, this fact results from the direct construction of the unitary operator, as given by Equation (4.63). Since the global phase cannot be measured and is therefore irrelevant in the physical implementation, we have to generate a quantum circuit for the operator

$$U' = P \cdot H^{\otimes m}. \quad (4.86)$$

Obviously, the application of the Hadamard matrix $H^{\otimes m}$ can be done by applying H to each of the m qubits individually. The phase system is given by $P = \text{diag}((p_j)_{j \in \mathbb{F}_2^m})$, where the binary representation of j refers to a specific quantum state. The bit j_k

¹³An advantage of cyclic MUBs is that a single circuit can be used to implement a complete set of MUBs, where for a non-cyclic set, $d + 1$ different circuits have to be implemented, as was done e. g. by Klimov *et al.* [KMFS08].

addresses then with $k \in \{1, \dots, m\}$ the qubit at position k . This phase system depicts directly the implementation into a quantum circuit. More precisely, we can rewrite the phase factor as it was given in Equation (4.56), where we find

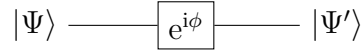
$$\begin{aligned} p_j &= (-i)^{\sum_{k,l=1}^m b_{kl} j_k j_l} (-1)^{\sum_{k=1}^m b_{kk} j_k} \\ &= \prod_{k>l=1}^m ((-1)^{j_k j_l})^{b_{kl}} \cdot \prod_{k=1}^m (i^{j_k})^{b_{kk}}. \end{aligned} \quad (4.87)$$

The second term of Equation (4.87) tells us, that a phase of i is applied to the qubit with index k , if it is in state $|1\rangle_k \langle 1|$ and $b_{kk} = 1$; in the case $|0\rangle_k \langle 0|$ nothing happens. The first term is only relevant in the case that the qubits with index k and l are in the state $|1\rangle_k \langle 1| \otimes |1\rangle_l \langle 1|$ (which means that $j_k = j_l = 1$) and if $b_{kl} = 1$. The described actions can be performed by the following elementary gates:

- The single-qubit **Phase** gate acting on qubit k which is defined as

$$\text{Phase}_k(e^{i\phi}) = |0\rangle_k \langle 0| + e^{i\phi} |1\rangle_k \langle 1|, \quad (4.88)$$

and visualized as

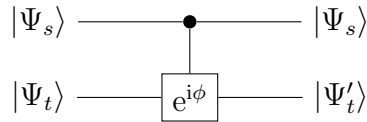


with phase $\phi \in [0, 2\pi)$.

- The two-qubit controlled-**Phase** gate

$$\begin{aligned} \text{CPhase}_{s \rightarrow t}(e^{i\phi}) &= |0\rangle_s \langle 0| \otimes |0\rangle_t \langle 0| + |0\rangle_s \langle 0| \otimes |1\rangle_t \langle 1| \\ &+ |1\rangle_s \langle 1| \otimes |0\rangle_t \langle 0| + e^{i\phi} |1\rangle_s \langle 1| \otimes |1\rangle_t \langle 1|, \end{aligned} \quad (4.89)$$

with control qubit s , target qubit t and phase $\phi \in [0, 2\pi)$. For a two-qubit state $|\Psi_s, \Psi_t\rangle$, the **CPhase** gate induces in general a transformation $|\Psi_s, \Psi_t\rangle \rightarrow |\Psi_s, \Psi'_t\rangle$ and is visualized by the following circuit:



We can use these two gates to implement the phase factor of Equation (4.87), namely by identifying the gates from the reduced stabilizer matrix B as:

- For $b_{kk} = 1$, apply $\text{Phase}_k(i)$.
- For $b_{kl} = 1$ and $k > l$, apply $\text{CPhase}_{k \rightarrow l}(-1)$.

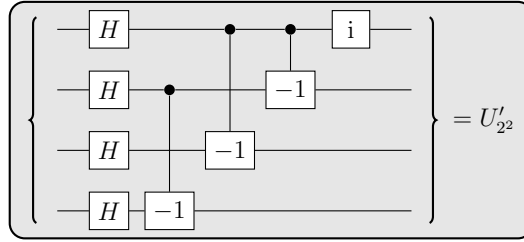


Figure 4.1: Quantum circuit for the generator U'_{22} of a complete set of cyclic MUBs for a four qubit system.

As an example, a possible reduced stabilizer matrix that can be used to generate a complete system of cyclic MUBs for a quantum system of four qubits, is given by the recursive construction of Equation (4.18) with

$$B_{2^2} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \quad (4.90)$$

Following the results given above, this solution results in one **Phase** gate, three **CPhase** gates and four **Hadamard** gates, as depicted in Figure 4.1. Accordingly, in the case of the recursive construction of Fermat sets given by Equation (4.18), the implementation of the generator of a complete set of cyclic MUBs for m qubits needs one **Phase** gate, $m - 1$ **CPhase** gates and m **Hadamard** gates, which is optimal in the number of two-qubit gates.¹⁴ In the case of the Fibonacci-based sets in the Form of Equation (4.13), an implementation would need not more than $\lceil m/2 \rceil + 5$ **Phase** gates, $(\lceil m/2 \rceil^2 - \lceil m/2 \rceil)/2 + 10$ **CPhase** gates and m **Hadamard** gates for $m \in \{1, \dots, 600\}$ as can be seen in Appendix C.1.5. This does not seem to be optimal at all. Also the proposed method to find a symmetric companion matrix (cf. Conjecture 4.1.6) does not seem to be capable to solve this problem optimally. Therefore, further investigations on how to reduce the number of non-vanishing entries of the reduced stabilizer matrices are essential in order to reduce the number of gates needed.

4.8.2 Homogeneous sets

To implement homogeneous sets into quantum circuits we cannot use similar methods as for the Fibonacci-based sets as long as no construction of the unitary matrix is given in an analogous form to that of Equations (4.63) and (4.87). Therefore, we will introduce a method that is capable to generate the quantum circuit of nearly any matrix $C \in M_{2m}(\mathbb{F}_2)$ that is symplectic. At first, symplectic representations of different quantum gates will be discussed; thereafter, it will be shown how a large set of

¹⁴As the bases of a complete set of cyclic MUBs for an m qubit system measure the complete state of this system, a possible quantum circuit should link all qubits together, which can be done by controlled gates. If not all qubits were linked together, then only information on subsystems could be measured. Consequently, if for controlled gates only two qubit gates are used, the minimum number of these gates is $m - 1$.

symplectic matrices with entries in \mathbb{F}_2 can be decomposed into a product of symplectic matrices in order to become directly implementable by corresponding quantum gates. Quantum circuits derived by this method for the homogeneous sets which are discussed, are shown in Appendices C.1.1 and C.1.2.

Let us define an arbitrary symplectic matrix $C \in M_{2m}(\mathbb{F}_2)$ in block-matrix form as

$$C = \begin{pmatrix} s & t \\ u & v \end{pmatrix}, \quad (4.91)$$

with submatrices $s, t, u, v \in M_m(\mathbb{F}_2)$. As stated in Appendix A.6, elements of the Clifford group are automorphisms of the set of Pauli operators. By their group property, these elements are invertible, so also their representation as symplectic matrices. Therefore, any symplectic matrix has full rank, thus, the row vectors within the submatrix $(u, v)^t$ are linearly independent. For the group-based sets defined by Equation (4.34) this fact can be seen also in the stabilizer matrix directly, since $v := 0_m$ and $u := R^{-1}$ is invertible. The same holds for the submatrix (s, u) , but is not that obvious. According to this observation, it is in general possible to obtain by elementary row and column operations a transformed matrix where the submatrix u is invertible. Within this section, we limit the consideration only to those cases where u is invertible. To apply these necessary operations and to implement the resulting matrix, different quantum gates have to be considered.

By the construction of the classes \mathcal{C}_j with $j \in \{0, \dots, d\}$ as given in Equation (4.2), each class consists of Pauli operators. The matrix C then realizes by its symplecticity property that the set of a certain class of Pauli operators is transformed to another set of operators, i.e. the next class. Therefore, a circuit of quantum gates which represents the matrix C has to implement transformations of a set of Pauli operators to another set. As we can see the whole circuit as the product of basic transformations (i.e. the quantum gates), the implication of each quantum gate can be analyzed by the consideration of its truth table of Pauli operators. To explain the idea, we analyze the Hadamard gate as an example:

The Hadamard operator is defined by Equation (A.6) as

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}; \quad \text{with } H^\dagger = H, \quad (4.92)$$

it acts on the Pauli operators X, Y and Z , which are listed in Appendix A.4, as $HXH^\dagger = Z$, $HYH^\dagger = -Y$ and $HZH^\dagger = X$. To construct a complete set of cyclic MUBs, the eigenbases of the classes \mathcal{C}_j are relevant, but the phases of the elements are not relevant. A quantum gate, represented as a symplectic matrix $C_{\mathbb{H}}$, that implements the Hadamard operator, should then act on the Pauli operators for a single qubit, represented as elements of the finite field \mathbb{F}_2^2 , as

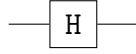
$$C_{\mathbb{H}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C_{\mathbb{H}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad C_{\mathbb{H}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (4.93)$$

Obviously, any symplectic matrix will set the zero vector to the zero vector, which is in accordance to the unitary representation given above. The representation of the

Hadamard operator as a symplectic matrix is finally given by

$$C_{\mathbf{H}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (4.94)$$

with the corresponding visual representation as the Hadamard gate.



This gate acts only on a single qubit, so the unitary operator on a multi-qubit system with m qubits that applies a Hadamard gate on the i -th qubit, is given by

$$U_{\mathbf{H}}^{(i)} = \left(\bigotimes_{l=1}^{i-1} \mathbb{1}_2 \right) \otimes H \otimes \left(\bigotimes_{l=i+1}^m \mathbb{1}_2 \right), \quad (4.95)$$

its representation as a symplectic matrix $C_{\mathbf{H}}^{(i)} = \left(c_{\mathbf{H}}^{(i)} \right)_{kl}$ has entries

$$\left(c_{\mathbf{H}}^{(i)} \right)_{kl} = \begin{cases} \delta_{kl} & \text{for } k, l \notin \{i, m+i\}, \\ 1 - \delta_{kl} & \text{for } k, l \in \{i, m+i\}. \end{cases} \quad (4.96)$$

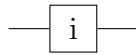
Another single-qubit gate is the **Phase** gate, that was already mentioned in Section 4.8.1, but we will again concentrate on the **Phase-i** gate, with a phase of $\pm i$.¹⁵ The unitary matrix of the **Phase-i** gate is given by

$$U_{\mathbf{P}_{\pm i}} = \begin{pmatrix} 1 & 0 \\ 0 & \pm i \end{pmatrix}, \quad (4.97)$$

and can be represented by the symplectic matrix

$$C_{\mathbf{P}_{\pm i}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad (4.98)$$

with the following visual representation.



The multi-qubit operators can be built in a similar fashion as for the **Hadamard** gate which are given by Equations (4.95) and (4.96) and yield in the representation as a symplectic matrix, acting on the i -th qubit,

$$\left(c_{\mathbf{P}_{\pm i}}^{(i)} \right)_{kl} = \delta_{kl} + \delta_{ki} \delta_{l(m+i)}. \quad (4.99)$$

As a next step, we consider controlled two-qubit gates like the **CPhase** gate that was already introduced in Section 4.8.1. For these gates, an operation is applied to the *target* qubit if and only if the state of the *control* qubit equals $|1\rangle$. Here, we concentrate

¹⁵It turns out, that a phase of $+i$ leads to the same result as a phase of $-i$, if we do only concentrate on equivalence classes of Pauli operators, thus ignoring additional phases.

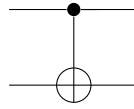
on the **CNot** gate and the **controlled-Z** gate. The unitary operator of the **CNot** gate, which is equivalent to a **controlled-X** gate, is given by

$$U_{\text{CNot}}^{(1,2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (4.100)$$

The symplectic representation derived from the truth table of Pauli operators of this operation is given by

$$C_{\text{CNot}}^{(1,2)} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad (4.101)$$

and has also a visual representation.



The multi-qubit operator for a **CNot** gate with control qubit c and target qubit t , represented as a symplectic matrix, is given by

$$\left(C_{\text{CNot}}^{(c,t)} \right)_{kl} = \delta_{kl} + \delta_{kc} \delta_{lt} + \delta_{k(t+m)} \delta_{l(c+m)}. \quad (4.102)$$

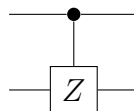
The last gate we want to consider is the **controlled-Z** gate, which is a special version of the **controlled-Phase** gate, with a fixed phase factor of -1 . We will denote this gate by **CZ** in the following. Its unitary representation is given by

$$U_{\text{CZ}}^{(1,2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (4.103)$$

and its symplectic representation by

$$C_{\text{CZ}}^{(1,2)} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (4.104)$$

The visualization of the **CZ** gate is the following circuit.



Within a multi-qubit environment, the symplectic representation of the CZ gate with control qubit named c and target qubit t is accordingly given by

$$\left(c_{\text{CZ}}^{(c,t)} \right)_{kl} = \delta_{kl} + \delta_{kc} \delta_{l(t+m)} + \delta_{kt} \delta_{l(c+m)}. \quad (4.105)$$

Armed with all those gates we can implement any stabilizer matrix $C \in M_{2m}(\mathbb{F}_2)$ in the form of Equation (4.91) with u invertible by a quantum circuit. The process we suggest is realized in two parts. In the first part, the stabilizer matrix is transformed into a special form by symplectic operations such that u is mapped to $\mathbb{1}_m$, in the second part, this form can be written as a product of symplectic matrices that can be implemented directly.

Lemma 4.8.1 (Gaussian elimination).

Any symplectic matrix $C \in M_{2m}(\mathbb{F}_2)$ in the form of Equation (4.91) with u invertible, can be transformed by applying CNot operations in the form of Equation (4.102) from the left in order to obtain a symplectic matrix in the form

$$C' = \begin{pmatrix} s' & t' \\ \mathbb{1}_m & v' \end{pmatrix}. \quad (4.106)$$

Proof. If u is an invertible matrix, it can be diagonalized by the Gaussian elimination, that uses elementary row or column operations. By multiplying a CNot operation in the manner of Equation (4.102) from the left to the symplectic matrix C , within the block of the submatrix u , the row which has the index of the control qubit is added and stored to the row which has the index of the target qubit. Two rows i, j of u can be swapped by applying $C_{\text{CNot}}^{(i,j)} C_{\text{CNot}}^{(j,i)} C_{\text{CNot}}^{(i,j)}$, as the elements of u have characteristic two. Therefore, all elementary row operations can be realized with CNot operations.¹⁶ \square

The transformation may also be realized by multiplying CNot operations from the right which induces elementary column operations, respectively. Lemma 4.8.1 leads implicitly to the following corollary:

Corollary 4.8.2 (Gate commutation).

Two quantum gates which have representations as symplectic matrices in the form

$$C_1 = \begin{pmatrix} \mathbb{1}_m & a_1 \\ 0_m & \mathbb{1}_m \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} \mathbb{1}_m & a_2 \\ 0_m & \mathbb{1}_m \end{pmatrix}, \quad (4.107)$$

with $a_1, a_2 \in M_m(\mathbb{F}_2)$, commute.

Proof. The application of C_1 to C_2 (or vice versa) gives

$$C_1 C_2 = \begin{pmatrix} \mathbb{1}_m & a_1 + a_2 \\ 0_m & \mathbb{1}_m \end{pmatrix}. \quad (4.108)$$

Therefore, the Phase- i gates and the CZ gates commute. \square

¹⁶The third operation, namely the multiplication of any row with a non-zero scalar, is given by the identity operation in \mathbb{F}_2 , as no non-zero element exists besides 1.

The resulting matrix of Lemma 4.8.1 can be decomposed into a product of matrices that can be used to read off the required CZ gates, Phase gates and Hadamard gates directly.

Lemma 4.8.3 (Stabilizer matrix decomposition).

Any symplectic matrix $C \in M_{2m}(\mathbb{F}_2)$ in the form of Equation (4.106), can be factorized in matrices that describe the product of CZ gates, Phase gates, and an m -folded tensor product of the Hadamard matrix as

$$C' = \begin{pmatrix} s' & t' \\ \mathbb{1}_m & v' \end{pmatrix} = \begin{pmatrix} \mathbb{1}_m & s' \\ 0_m & \mathbb{1}_m \end{pmatrix} \begin{pmatrix} 0_m & \mathbb{1}_m \\ \mathbb{1}_m & 0_m \end{pmatrix} \begin{pmatrix} \mathbb{1}_m & v' \\ 0_m & \mathbb{1}_m \end{pmatrix}, \quad (4.109)$$

which implies $t' = \mathbb{1}_m + s'v'$.

Proof. Corollary A.6.1 of Appendix A.6 states the conditions on C' to be a symplectic matrix. From the first condition follows, that s' has to be symmetric. From the last condition follows then with $u' = \mathbb{1}_m$ that $t' = \mathbb{1}_m + s'v'$. Inserting t' into the second condition shows finally, that also v' has to be symmetric.

The given equation appears naturally, where the middle term corresponds to Hadamard operations given by Equation (4.96). The remaining two terms can be created using Corollary 4.8.2 the following way: Each non-zero element on the diagonal of s' and v' , respectively, refers to a Phase-i gate, as can be seen from Equation (4.99). As s' and v' are symmetric matrices, their off-diagonal, non-zero elements refer to CZ gates, according to Equation (4.105). \square

To implement s' and v' , at most $2m$ Phase-i gates and at most $m^2 - m$ CZ gates are needed. The form derived by Equation (4.8.3) requires exactly m Hadamard gates. An upper bound to realize the Gaussian elimination as discussed in Lemma 4.8.1 is the implementation of m^2 different CNot gates, as any m -bit vector can be build by maximally m XOR operations within the set of m linearly independent vectors. For $v' = 0_m$ follows $t' = \mathbb{1}_m$, thus a Fibonacci-based set. The quantum circuit which appears then by applying Lemma 4.8.3 is the same as the one given by Figure 4.1.

Results of the semigroup sets (cf. Appendix C.1.2) indicate, that the sets of MUBs can be limited to sets which have an invertible $m \times m$ submatrix in the upper right corner (i. e. the matrix t of Equation (4.91) is invertible). If this is not the case, one may find a set of quantum gates that realizes an appropriate transformation. Implementations of complete sets of MUBs for three qubit systems are shown by Figure 4.2 and Figure 4.3; quantum circuits for four qubit systems are listed in Appendices C.1.1 and C.1.2.

4.8.3 Inhomogeneous sets

Quantum circuits for inhomogeneous sets can in principle be built in the same way as discussed for the homogeneous sets in the former section. But these sets do not include the standard basis as shown in Section 4.6, i. e. the generator G_0 of the first class \mathcal{C}_0 is not given by $G'_0 = (\mathbb{1}_m, 0_m)^t$; in standard form it reads $\bar{G}_0 = (\mathbb{1}_m, G_0^x)^t$ (cf. Equation (4.28)). Any symplectic matrix C_0 that transforms the generator G'_0 to the generator \bar{G}_0 can be taken to implement the quantum circuit which produces the class

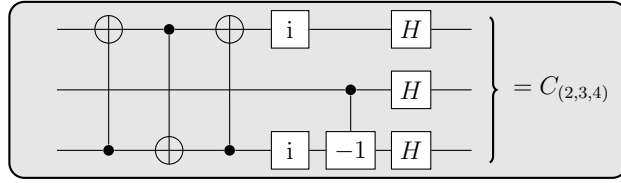


Figure 4.2: Quantum circuit for the generator $C_{(2,3,4)}$ of Equation (4.38) of a complete, homogeneous set of group-based cyclic MUBs for a three qubit system.

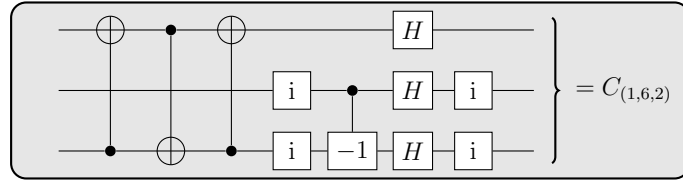


Figure 4.3: Quantum circuit for the generator $C_{(1,6,2)}$ of Equation (4.42) of a complete, homogeneous set of semigroup-based cyclic MUBs for a three qubit system.

\mathcal{C}_0 . In order to generate a class with commuting elements, G_0^x has to be a symmetric matrix (cf. [BBRV02, Lemma 4.3]). The symplectic transformation $C_0 G_0' = \tilde{G}_0$ equals

$$C_0 = \begin{pmatrix} \mathbb{1}_m & 0_m \\ G_0^x & \mathbb{1}_m \end{pmatrix}, \quad (4.110)$$

and can be rewritten using Hadamard gates similarly to Lemma 4.8.3 as

$$C_0 = \begin{pmatrix} 0_m & \mathbb{1}_m \\ \mathbb{1}_m & 0_m \end{pmatrix} \begin{pmatrix} \mathbb{1}_m & G_0^x \\ 0_m & \mathbb{1}_m \end{pmatrix} \begin{pmatrix} 0_m & \mathbb{1}_m \\ \mathbb{1}_m & 0_m \end{pmatrix}. \quad (4.111)$$

The quantum circuit is then given by a Hadamard gate on each individual qubit (to implement the rightmost matrix), followed by Phase- i gates on each diagonal non-zero value of G_0^x and CZ gates on each off-diagonal, upper triangular value of G_0^x , as done for Lemma 4.8.3 (to implement the matrix in the middle) and again Hadamard gates on each individual qubit (to implement the leftmost matrix). Since the application of two consecutively applied Hadamard operations is the unity operation, we can omit Hadamard gates on those qubits, that are involved neither in any Phase- i nor in any CZ gate.

Resulting quantum circuits for complete sets of MUBs on systems with four qubits are given in Appendix C.1.3, the quantum circuit for the three-qubit system which is generated by $C_{(0,9,0)}$ of Equation (4.44) is shown in Figure 4.4.

4.8.4 Practical implementation

For a practical implementation of a complete set of cyclic MUBs, a quantum circuit like the example of Figure 4.1 would be able to transform a quantum state by an operation that shifts one of the $d+1$ mutually unbiased bases to the next. Thus, after this process, variables that are complementary to the former ones, could be measured in the computational basis. To be able to measure in a specific basis of these $d+1$

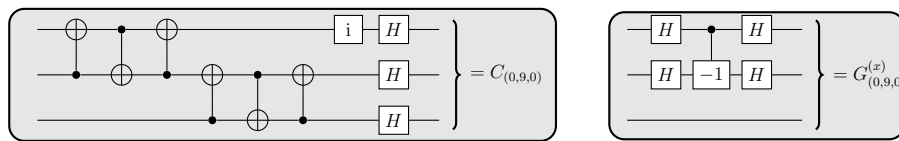


Figure 4.4: Quantum circuit for the generator $C_{(0,9,0)}$ of Equation (4.44) and the lower submatrix of the corresponding generator of the class \mathcal{C}_0 , namely $G_{(0,9,0)}^{(x)}$ of Equation (4.45), of a complete, inhomogeneous set of cyclic MUBs for a three qubit system.

different bases, the quantum circuit has to be used several times, in order to achieve the desired set of variables. In average, this would cause $d/2$ applications of the quantum circuit. Consequently, for a large number of bases, namely a high dimension d of the Hilbert space, we propose to decompose the index of the bases binary and create for each power of two an individual circuit. This raises the size of the experimental setup, but reduces the number of gates which are used in average to create a specific basis.

To give an example, for a dimension $d = 2^{10}$, a complete set of MUBs has 1025 different bases. Following the suggestion, there would be ten different quantum circuits, thus any basis can be reached by using five of them in average. This is a huge improvement over using 512 times the same circuit in average if only the generator of the cyclic set of MUBs is implemented.

Depending on the dimension, it could also be favored to decompose the index to any other number system; fortunately, all those approaches can be realized by using the powers of the stabilizer matrix.

Equivalence of mutually unbiased bases

A set of mutually unbiased bases defines a set of bases, where the absolute value of the overlap of all pairs of vectors that are taken from two different bases within the set is constant and equals $d^{-1/2}$, where d is the dimension of the corresponding Hilbert space $\mathcal{H} = \mathbb{C}^d$, as was seen in Definition 2.2.1. From this definition it does not make any sense to take the ordering of the bases into account, as well as the ordering of the basis vectors of any basis. Also the multiplication of any basis vector with a phase factor preserves the mutual unbiasedness which is finally even invariant under any unitary transformation which is applied to the whole set of bases. By choosing a unitary transformation which is the inverse of any unitary operator \mathcal{B}_j of the set, this basis becomes the standard basis, thus, each set of MUBs we find, can be transformed into a set that includes the standard basis. We sum up these naturally appearing arguments to define the *equivalence* of different sets of MUBs.

Definition 5.0.1 (Equivalence of mutually unbiased bases).

Two sets of mutually unbiased bases, namely $\mathfrak{S} = \{\mathcal{B}_0, \dots, \mathcal{B}_{r-1}\}$ and $\mathfrak{S}' = \{\mathcal{B}'_0, \dots, \mathcal{B}'_{r-1}\}$ with $r \in \mathbb{N}^*$, are said to be equivalent, if there holds

$$\mathcal{B}'_j = U\mathcal{B}_{\pi(j)}W_j, \quad (5.1)$$

with a unitary matrix $U \in M_d(\mathbb{C})$, a permutation π on $\{0, \dots, r-1\}$ and monomial matrices¹ W_j that are given by the product of a permutation matrix and a diagonal phase matrix, for $j \in \{0, \dots, r-1\}$.

It has to be kept in mind that this definition of equivalence does not distinguish between the different sets of cyclic MUBs, that were discussed in Sections 4.1, 4.5 and 4.6. Their equivalence in a Hilbert space of dimension $d = 2^3$ was shown already in [RBKSS05]. To distinguish those sets, the consideration of the entanglement properties is essential, as discussed in Section 4.4.

¹A monomial matrix is defined to be a square matrix which has in every row and every column exactly one entry that differs from zero.

Nevertheless, on this mathematical level of equivalence, different sets were discussed in the literature [CKKS97, GR09, Kan12]. It is still an open problem how many inequivalent sets exist and how they are related. Within the following section, we will discuss the equivalence of sets of MUBs which are based on the construction of Bandyopadhyay *et al.* [BBRV02] and compare them with other constructions. The construction of the Fibonacci-based sets of MUBs allows us to construct a set of operators that realize permutations of the bases which is shown in the subsequent section.

5.1 Heisenberg group partition sets

To analyze the equivalence of sets of MUBs which are based on the construction of Bandyopadhyay *et al.*, we can use their construction principles in order to simplify the comparison of two different sets. Following the explanation given in Section 3.1.2, each element U_j of the set of MUBs \mathfrak{S} which is generated by the powers of a unitary matrix U as $\mathfrak{S} = \langle U \rangle$, is given by the set of common eigenvectors of a corresponding class of Pauli operators \mathcal{C}_j (cf. Equation (3.25)) with $j \in \{0, \dots, r-1\}$ and $r \in \mathbb{N}^*$. For a complete set of MUBs we have $r = d + 1$. By the construction of Equation (3.17), a class \mathcal{C}_j is a set of elements $ZX(\vec{a})$ of the set of Pauli operators. Analogously to the set of unitary matrices, we can define the set $\mathfrak{C} = \{\mathcal{C}_0, \dots, \mathcal{C}_{r-1}\}$ of classes \mathcal{C}_j , that defines a complete set of MUBs, but without determining the order of the basis vectors or their phase within a single class. To define equivalence on this level, there is no need for matrices W_j as introduced by Definition 5.0.1. We can formulate the following lemma:

Lemma 5.1.1 (Equivalence of mutually unbiased bases).

Two sets of MUBs which are characterized by $\mathfrak{C} = \{\mathcal{C}_0, \dots, \mathcal{C}_{r-1}\}$ and $\mathfrak{C}' = \{\mathcal{C}'_0, \dots, \mathcal{C}'_{r-1}\}$ are equivalent, if there holds

$$\mathcal{C}'_j = U\mathcal{C}_{\pi(j)}U^\dagger, \quad (5.2)$$

with a unitary matrix $U \in \mathbb{C}^d$, a permutation π on $\{0, \dots, r-1\}$, and $r \in \mathbb{N}^$.*

Proof. Starting with the Definition 5.0.1 on the equivalence of MUBs, the matrices W_j do not play any role if we observe sets of Pauli operators \mathcal{C}_j and $\mathcal{C}'_{\pi(j)}$ that do neither have any ordering nor keep any phase. The equivalence has to be formulated the given way, since in the definition, we transform an eigenvector v as Uv ; so, considering v is an eigenvector of a matrix A , then Uv is an eigenvector of UAU^\dagger . \square

As the classes of both sets \mathfrak{C} and \mathfrak{C}' contain only Pauli operators, the unitary transformation U has to be an element of the Clifford group (cf. Appendix A.6), that maps by definition Pauli operators onto Pauli operators. A certain class \mathcal{C}_j can be written in terms of a $2m \times (d-1)$ -matrix, where each column represents a Pauli operator, namely one of the vectors \vec{a} given in Equation (3.17). It can be seen within that formula, that the $d-1$ elements can be generated by the (non-unique) so-called

$2m \times m$ generator matrix G_j , thus $\mathcal{C}_j = \langle G_j \rangle$.² Consequently, equivalence of two sets \mathcal{C} and \mathcal{C}' is given, if and only if

$$\langle G'_j \rangle = f \langle G_{\pi(j)} \rangle Q_j, \quad (5.3)$$

with a symplectic matrix f (that maps Pauli operators onto Pauli operators), a permutation Q_j and a permutation π on $\{0, \dots, d\}$. In the case, we write the generators in standard form as discussed in Section 4.3, the generators G_j and G'_j become unique. Then the permutation matrix Q_j can be set to unity, thus neglected. For the transformation matrix f we are free to assume the block-matrix form

$$f = \begin{pmatrix} s & t \\ u & v \end{pmatrix}. \quad (5.4)$$

As it is shown by Corollary A.6.1, this matrix is symplectic if and only if $s^t u$ and $t^t v$ are symmetric and $v^t s - t^t u = \mathbb{1}_m$. With the methods introduced so far, we are able to prove the equivalence of all Fibonacci-based complete sets of MUBs for a specific dimension $d = 2^m$ with $m \in \mathbb{N}^*$.

5.1.1 Fibonacci-based sets

To construct a Fibonacci-based set, we have to follow simply the Steps 1.–3. that were given in Section 4.1. It turns out, that neither the choice of the reduced stabilizer matrix B is unique, nor the polynomial with Fibonacci index $d+1$. Within this section we will show, that all Fibonacci-based sets which are based on the same polynomial, are equivalent, independent of the choice of B . In a second step it is shown, that also a different choice of the polynomial will create an equivalent set of cyclic MUBs.

Applying the generator of a set to all elements rotates this set, so we can choose $\pi(0) = 0$ for Equation (5.3), which implies $\langle G_0 \rangle = \langle G'_0 \rangle$ and in standard form $\bar{G}_0 = \bar{G}'_0$. According to the construction of the Fibonacci-based sets, this generator is given by

$$G_0 = \begin{pmatrix} \mathbb{1}_m \\ 0_m \end{pmatrix}, \quad (5.5)$$

which can also be seen in the derivation of the standard form in Equation (4.27). Thus, the choice for the permutation π maps the class of Pauli- Z operators of the set \mathcal{C} to the class of Pauli- Z operators of the set \mathcal{C}' . Using Equation (5.3), we have to demand that $u = 0_m$; to keep f symplectic, we need to set $v = (s^t)^{-1}$. If both sets represent complete sets of MUBs, we can represent their generators in standard form, where the elements G_j^z as given in Equation (4.28) form a matrix representation of the finite field \mathbb{F}_2^m ; the same holds true for the elements $G_j'^z$. We can formulate a lemma on equivalent MUBs:

Lemma 5.1.2 (Equivalence of certain Fibonacci-based sets).

The choice of different reduced stabilizer matrices $B \in M_m(\mathbb{F}_2)$ with the same characteristic polynomial that has Fibonacci index $2^m + 1$ in the construction of Fibonacci-based MUBs leads to equivalent sets.

²Normally, the symbol $\langle A \rangle$ refers to the set, which is generated by the powers of A . Adopting the abstract meaning of this symbol, the expression $\langle G_j \rangle$ refers to a generation which is specified by Equation (3.17).

Proof. Two reduced stabilizer matrices $B, B' \in M_m(\mathbb{F}_2)$, which have to be symmetric according to Condition (i), formulated in Section 4.1, are related as $B' = sBs^t$ with an orthogonal matrix $s \in M_m(\mathbb{F}_2)$, i. e. $s^{-1} = s^t$. If we set the transformation matrix as

$$f = \begin{pmatrix} s & 0_m \\ 0_m & s \end{pmatrix}, \quad (5.6)$$

this matrix is symplectic. Applied to the generators of the classes \mathcal{C}_j of \mathfrak{C} in standard form, we get $f\tilde{G}_j = (sp_j(B), s)^t$, where $p_j(B)$ is the polynomial $F_{j+1}(B)(F_j(B))^{-1}$. As seen by Corollary 4.3.1, the transformed generators can be multiplied from the right by any invertible, block-diagonal matrix, in particular also by f^{-1} . This leads to $f\tilde{G}_jf^{-1} = (sp_j(B)s^t, \mathbb{1}_m)^t \equiv \tilde{G}'_j$, the generators of the classes of the set \mathfrak{C}' , in standard form. The case of $j = 0$ can be shown the same way. \square

We can use this lemma to give a general theorem on the equivalence of Fibonacci-based sets.

Theorem 5.1.3 (Equivalence of Fibonacci-based sets).

All Fibonacci-based complete sets of MUBs are equivalent.

Proof. It was shown by Lemma 5.1.2, that all reduced stabilizer matrices with the same characteristic polynomial with Fibonacci index $d+1$ result in equivalent sets of MUBs. In the case that the two stabilizer matrices B and B' have different characteristic polynomials p and p' , respectively, with Fibonacci index $d+1$, we can use the following consideration: In general, the two different irreducible polynomials have degree m and are defined over the ground field \mathbb{F}_2 . We pick a pair of their roots, namely β and β' , respectively. The adjunction of β to \mathbb{F}_2 results in the field $\mathbb{F}_2(\beta) \cong \mathbb{F}_2[x]/p\mathbb{F}_2[x] \cong \mathbb{F}_{2^m}$, a similar consideration holds for β' . But as both extensions are *Galois extensions*, the elements in $\mathbb{F}_2(\beta)$ and in $\mathbb{F}_2(\beta')$ are equivalent up to a permutation [Bos06, Chapter 4.1]. The elements of those fields appear in the Pauli- Z part of the generators in standard form (cf. Equation (4.27)), thus both roots result in the same set of generators. If we chose a symmetric matrix $B \in M_m(\mathbb{F}_2)$ to represent the root β , we have the freedom shown in Lemma 5.1.2, where the minimal polynomial of B equals the minimal polynomial of β . \square

In other words, Theorem 5.1.3 shows, that the minimal polynomials of different elements of the finite field \mathbb{F}_{2^m} are possible minimal polynomials of a polynomial ring $\mathbb{F}_2[x]$ which is congruent to the field itself; the chosen element only needs to have a minimal polynomial with degree m (which is irreducible by definition).

5.1.2 (In-)Homogeneous sets

To show the equivalence of all sets which are based on the construction of Bandyopadhyay *et al.* in a similar fashion as for the Fibonacci-based sets, further investigations are essential and exceed the scope of this work. As already stated above, it was shown in [RBKSS05] that all discussed sets in Hilbert space dimension $d = 2^3$ are equivalent

and the equivalence of different sets in higher dimensions was indicated. Considerations in the fashion of Section 5.1.1 would also lead to explicit transformations between the different sets. In a first step, one may prove the equivalence of homogeneous sets with an additive group structure in the Z component of the class generators to Fibonacci-based sets and in a second step their equivalence with sets that possess only an additive semi-group structure (cf. Sections 4.5.1 and 4.5.2). As inhomogeneous sets do not contain the standard basis, they are by Definition 5.0.1 always equivalent to sets with a standard basis, where a unitary operation for an arbitrary basis of this set exists, that transforms this set into a homogeneous set.

5.1.3 Different constructions

We can relate the Fibonacci-based sets to different constructions by following the discussion given in [BBRV02]: Theorem 4.4 introduces a construction that is equivalent to the standard form of homogeneous sets (cf. Equation (4.28)). In Section 4.3 the construction is limited to sets with an additive group structure (cf. Equation (4.33)). Finally, at the end of this section, the group structure is fixed in a way to obtain a field structure. Bandyopadhyay *et al.* took this construction from the Wootters and Fields construction [WF89], indicating their equivalence.³ Godsil and Roy [GR09] relate these sets also with the sets constructed by Klappenecker and Rötteler [KR04] and again to those of Bandyopadhyay *et al.*

5.2 Class-permutation operators

Starting with the definition of equivalence given by Definition 5.0.1, the question arises whether the permutation π on the indexing of the different bases can be realized by a symplectic transformation. In the case of Fibonacci-based sets we discovered a set of operators that seems to be capable to implement the set of permutations by a set of symplectic transformations. We call this set the set of *class-permutation operators*. The construction is quite intuitive, to follow the ideas, we start with the definition of the first class-permutation operator A_1 , that can be deduced from the generator of a Fibonacci-based set in the form of Equation (4.4) as

$$A_1 = \begin{pmatrix} \mathbb{1}_m & B \\ 0_m & \mathbb{1}_m \end{pmatrix}, \quad (5.7)$$

with the same reduced stabilizer matrix $B \in M_m(\mathbb{F}_2)$. Since the generator G_j with $j \in \{0, \dots, d\}$ of any class is given by Equation (4.7), the application of A_1 to this class generator leads to

$$A_1 G_j = \begin{pmatrix} \mathbb{1}_m & B \\ 0_m & \mathbb{1}_m \end{pmatrix} \begin{pmatrix} F_{j+1}(B) \\ F_j(B) \end{pmatrix} = \begin{pmatrix} F_{j+1}(B) + B F_j(B) \\ F_j(B) \end{pmatrix} = \begin{pmatrix} F_{j-1}(B) \\ F_j(B) \end{pmatrix}. \quad (5.8)$$

³In fact, the similarity of the phase factor in Equation (3.11) and Equation (4.62) is a strong evidence of this equivalence.

Using Lemma 4.1.1, we finally find

$$A_1 G_j = \begin{pmatrix} F_{d+1-j+1}(B) \\ F_{d+1-j}(B) \end{pmatrix} \equiv G_{d+1-j}, \quad (5.9)$$

with $d = 2^m$. The first class-permutation operator A_1 inverts therefore the ordering of the classes \mathcal{C}_j , where $G_0 \rightarrow G_{d+1} \equiv G_0$ keeps its position. The symplecticity of the operator A_1 is obvious and can easily be checked with the help of Corollary A.6.1. In this fashion, we may construct further class-permutation operators as

$$A'_l = \begin{pmatrix} F_l(B) & F_{l+1}(B) \\ 0_m & F_l(B) \end{pmatrix}, \quad (5.10)$$

with $l \in \{1, \dots, d\}$, where in this case the properties of Corollary A.6.1 are not fulfilled. To obtain a set of symplectic operators we multiply this set from the right by a block-diagonal matrix A_l^\times and get

$$A_l := A'_l \cdot A_l^\times = \begin{pmatrix} F_l(B) & F_{l+1}(B) \\ 0_m & F_l(B) \end{pmatrix} \begin{pmatrix} (F_l(B))^{-1} & 0_m \\ 0_m & (F_l(B))^{-1} \end{pmatrix} \quad (5.11)$$

$$= \begin{pmatrix} \mathbb{1}_m & F_{l+1}(B)(F_l(B))^{-1} \\ 0_m & \mathbb{1}_m \end{pmatrix}. \quad (5.12)$$

By Proposition 4.1.4 the mentioned polynomials in B exist; we also like to point out the striking similarity of these operators with the generator matrices in standard form given by Equation (4.27). If we apply these class-permutation operators to an arbitrary generator in standard form, we get for $A_l \cdot \bar{G}_0 = \bar{G}_0$ and for $j \neq 0$,

$$A_l \cdot \bar{G}_j = \begin{pmatrix} F_{j+1}(B)(F_j(B))^{-1} + F_{l+1}(B)(F_l(B))^{-1} \\ \mathbb{1}_m \end{pmatrix}. \quad (5.13)$$

Thus, the operator A_l adds a fixed polynomial of B to the Pauli- Z operator part of the generator G_j in standard form. As the set of all generator matrices contains all polynomials in B , the operator A_l permutes these generators according to fundamental finite field theory. Since we have d different such class-permutation operators A_l , we realize all basic permutations which are possible this way. Finally, the set of elements of a field is invariant under the addition of any of its elements, so we can change the position of an arbitrary generator to any new position with this set of transformations. But it is not clear if all permutations π on $\{1, \dots, d\}$ can be realized. If so, the permutations can be expanded on the set $\{0, \dots, d\}$ by adding the MUB generator C to the set of permutation operators, since it permutes the classes cyclically, namely

$$C \cdot G_j = G_{(j+1) \bmod (d+1)}. \quad (5.14)$$

For general homogeneous sets of MUBs, an appropriate set of class-permutation operators can be constructed by naturally expanding Equation (5.13) for homogeneous sets, which results in the set of operators

$$A_l^{\text{hom}} := \begin{pmatrix} \mathbb{1}_m & F_{l+1}(B)(F_l(B))^{-1}R \\ 0_m & \mathbb{1}_m \end{pmatrix}, \quad (5.15)$$

and leads to the same effective permutations as for the Fibonacci-based sets.

Conclusions and further work

Mutually unbiased bases (MUBs) find their applications in the fields of *quantum state estimation* and *quantum key distribution*. By their relation to mathematical objects like *orthogonal Latin squares*, *symplectic spreads* and many more, considerations result in immediate effects in these neighboring fields. Many (even fundamental) questions on MUBs are still unsolved. The most famous are the question of the existence of a set with more than three MUBs in dimension six (and how large sets may be in all other non-prime power dimensions), as well as the number of inequivalent sets in general.

What we treat in this work is the problem of explicitly constructing *cyclic mutually unbiased bases* in a straightforward way. Those bases have advantages in theoretical tasks as well as in experimental implementations. They provide also a reduced formulation of a concrete set of MUBs. It turns out that the introduced *Fibonacci-based sets* have nice properties and are related to well-known algebraic questions and may solve some of them if a construction for the *symmetric companion matrix* is found. Generalizations indicate the potential of this approach and produce sets with different *entanglement properties* which can be observed in the *standard form*. A relation to an open conjecture given by Wiedemann 1988 in finite field theory is discovered and serves a realization of this conjecture which may lead to a proof, as well as a recursive construction of cyclic MUBs in an infinite subset of the dimensions. Most fortunately, such a construction is not only found in terms of finite field theory, which can be used immediately for a discussed implementation in *quantum circuits*, but also in terms of unitary matrices. The algorithmical formulation of an implementation strategy that may be suitable for all MUBs based on the investigations of this work generalizes these observations and leads to feasible implementations. Approaches to prove the conjecture of Wiedemann, a positive test of this conjecture for $k \in \{1, \dots, 11\}$ in dimensions $d = 2^{2^k}$ (which is limited by the largest known prime-number factorization of Fermat numbers) and resulting cyclic MUBs for the Fibonacci-based sets for dimensions $d = 2^m$ with $m = \{1, \dots, 600\}$ complete the considerations. The examination of the construction schemes relates the discussed MUBs to known constructions on the level of *equivalence* of the measurement behavior.

Obviously, several questions remain open, but lead with possible generalizations

of the derived results to promising future investigations. At first, it seems to be possible to construct a symmetric companion matrix in order to generate the cyclic MUBs directly from the algebraic results. For more general sets, the derivation of a relation between the entanglement properties and the form of the stabilizer matrix would be valuable. The equivalence transformations between all sets would show their equivalence and be useful in order to observe the transformations of their properties. Indications on generalizations of the Fermat-based sets and the use of the doubling scheme for MUBs over the field of real numbers, should be followed. Considerations for odd-prime power dimensions will probably lead to somehow different constructions as they refer to finite fields with an odd characteristic (that behave entirely different comparing with an even characteristic), but would again expand the dimensions where sets of MUBs with a reduced set of generators can be created. Finally, it would be nice to know if the derived methods can be used in a similar way to construct inequivalent sets of MUBs by replacing the set of Pauli operators with different sets.

Part II

Quantum public-key encryption

Introduction

The foundation for the field of quantum cryptography was laid in the early seventies of the last century, when Wiesner had his prospective ideas on *quantum money* that cannot be counterfeited, based on a physical uncertainty principle (and the fact, that an unknown quantum state cannot be cloned perfectly [WZ82]). Shortly after this work appeared in 1983¹, the field emerged with the *quantum key distribution* (QKD) protocol of Bennett and Brassard [BB84] which is used for symmetric encryption schemes. Different modifications of this protocol were discussed [Eke91, Ben92] and different security proofs were given later [May96, LC99, SP00, GL03].² With the new millennium, the potential benefit of the properties of quantum physics was also discussed for further cryptographic primitives. Examples are *quantum digital signatures* [GC01], *quantum fingerprinting* [BCWdW01] and *quantum direct communication* [BF02]. Another primitive is referred to as *quantum public-key encryption* (QPKE) (see [OTU00]), which will be of interest here.³

In contrast to the QKD, the aim of this scheme is to reduce the number of keys needed in a network with a large number of parties which want to exchange secret information pairwise. To implement a QKD protocol in a network where all parties should be able to communicate securely, the total number of keys scales quadratically with the number of parties, where in the QPKE scheme this number scales linearly.⁴

The discussion in this Part of the work is based on a scheme given by Nikolopoulos [Nik08a] which achieves its security from so-called *quantum one-way functions* (qOWF). At first, we will redraw within this chapter the protocol itself. In Chapter 8, a detailed analysis of the security of this protocol against a powerful *individual attack* is given as well as an idea to enhance the security against attacks by a *noisy*

¹The manuscript by Wiesner on quantum money was rejected by different journals and finally presented at a conference [Wie83].

²A more detailed discussion on the development of symmetric QKD protocols is given in Part I in Section 2.3.

³ Many propositions for such a scheme were given [KKNY05, HKK08, Kak06, Nik08a].

⁴A different approach, where QKD protocols are used, implements a *key-distribution center* (KDC), which is a trusted third party, but obviously also an attractive target.

preprocessing method. The investigations will be summarized finally in Chapter 9.

7.1 Single-qubit-rotation protocol

A proposition for a QPKE protocol was given by Nikolopoulos in 2008 [Nik08a] which can be described in a nice way and therefore used to calculate the robustness against different attacks and easily expanded with preprocessing steps, error performance and so on. The security of the protocol is based on a *quantum one-way function*. Alice chooses a classical private key randomly and uses the mentioned function in order to create as much public keys—which are given as qubit strings—as wanted (and proved to be secure against certain classes of attacks). A party Bob, which wants to send a message to Alice, can apply for such a key and perform a determined transformation on each qubit depending on the corresponding bit value of the message and send it back. As Alice knows by the public key the initial states of the qubits, she is able to extract the information Bob encrypted. We will sketch in this section the basic properties of this protocol; a more substantial security analysis will be given in Sections 8.2 and 8.3.

The protocol starts by creating a classical private key which is the integer string $\mathbf{k} = (k_1, \dots, k_N)$ with $N \in \mathbb{N}^*$ and uses a security parameter $n \in \mathbb{N}^*$, $n \gg 1$. The key generation part of the protocol is the following:

1. Take a (public) random positive integer $n \gg 1$.
2. Generate a (private) random integer string $\mathbf{k} = (k_1, \dots, k_N)$ with $N \in \mathbb{N}^*$, where each integer is individually chosen randomly and independently from \mathbb{Z}_{2^n} .
3. Prepare $T' \in \mathbb{N}^*$ copies of the (public) N -qubit *public-key* state from the *private key* \mathbf{k} as

$$|\Psi_{\mathbf{k}}(\theta_n)\rangle = \bigotimes_{j=1}^N |\psi_{k_j}(\theta_n)\rangle, \quad (7.1)$$

with

$$|\psi_{k_j}(\theta_n)\rangle = \cos\left(\frac{k_j\theta_n}{2}\right) |0_z\rangle + \sin\left(\frac{k_j\theta_n}{2}\right) |1_z\rangle, \quad (7.2)$$

where $|0_z\rangle$ and $|1_z\rangle$ refer to the eigenstates of the Pauli operator σ_z .

An appropriate lower limit on the value of n depends on the number of provided public keys; it will be derived in Section 8.2. A copy of the public key $|\Psi_{\mathbf{k}}(\theta_n)\rangle$ is simply given by the tensor product of N individual public key qubits $|\psi_{k_j}(\theta_n)\rangle$. Any such qubit can be represented as a *Bloch vector* (cf. Appendix A.17)

$$\mathbf{R}_j(\theta_n) = \cos(k_j\theta_n)\sigma_z + \sin(k_j\theta_n)\sigma_x, \quad (7.3)$$

with the Pauli operators σ_z and σ_x . Therefore, each chosen bit of the private key k_j results in a qubit that is in the state $|\psi_{k_j}(\theta_n)\rangle$, where the protocol defines the angles to be equidistant and determined by

$$\theta_n := \pi/2^{n-1}. \quad (7.4)$$

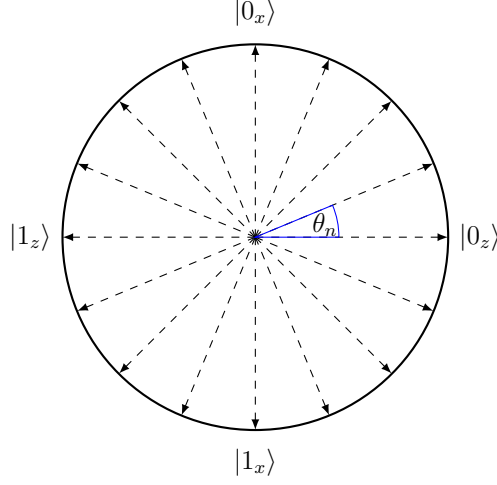


Figure 7.1: For $n = 4$, a single private key bit has one of 16 different values, which is transformed into one of these 16 equidistant states of the corresponding public-key qubit within the z - x -plane of the Bloch sphere.

In other words, the public key is a representation of the private key, where each integer k_j is represented by the angle $\pi/2^{n-1}$ and implemented as a rotation of a qubit within the z - x -plane of the Bloch sphere as can be seen in Figure 7.1. This mapping is also called a *quantum one-way function*, where the probability is large to generate the quantum state from the private key and the probability is low to reconstruct the private key perfectly from a single copy of the quantum state of the public key, if the initial transformation is unknown.

Alice sends the copies of the public key directly to parties like Bob that want to send encrypted messages back to Alice. Alice does not delegate the distribution to a so-called *key distribution center* (KDC).⁵ Bob may use one of those keys in order to encrypt a message \mathbf{m} which has a length of at most N bits. For simplicity and w. l. o. g. we consider a message which is exactly N bit long, thus $\mathbf{m} = (m_1, \dots, m_N)$.

The encryption of the protocol is defined as follows: For each bit m_j of the message with $j \in \{1, \dots, N\}$, apply the operator \mathcal{E}_{m_j} to the qubit $|\psi_{k_j}(\theta_n)\rangle$ of the public key, where this operator is given by

$$\mathcal{E}_{m_j} = \mathcal{R}_y(m_j\pi), \quad (7.5)$$

with $\mathcal{R}_y(\alpha) = e^{-i\alpha/2\sigma_y}$ being a rotation around the σ_y -axis of the Bloch sphere. A rotation is therefore applied to the original public key state, if the corresponding bit value of the message is one and the original state is left untouched if the value equals zero, resulting in the (quantum) cipher state

$$|\mathbf{X}_{\mathbf{k}, \mathbf{m}}(\theta_n)\rangle = \bigotimes_{j=1}^N \mathcal{E}_{m_j} |\psi_{k_j}(\theta_n)\rangle = \bigotimes_{j=1}^N |\chi_{k_j, m_j}(\theta_n)\rangle. \quad (7.6)$$

In the last step, this cipher state is sent by Bob back to Alice who is able to undo the initial rotations and measure in the Pauli- σ_z basis. This part of the protocol reads as:

⁵Such a KDC which distributes public keys has to be *unconditionally trusted*.

1. Undo the initial rotations on the cipher state of Equation (7.6):

$$|\mathbf{M}_m\rangle = \bigotimes_{j=1}^N \mathcal{R}_y^{-1}(k_j \theta_n) |X_{k_j}(\theta_n)\rangle = \bigotimes_{j=1}^N |M_{m_j}\rangle. \quad (7.7)$$

2. Measure each qubit of the resulting state in the eigenbasis of the Pauli- σ_z operator to extract the sent message.

Finally, Alice has received the classical message from Bob.

But conversely, if the cipher state is leaked to the eavesdropper Eve, she may measure each qubit in a random basis and obtain already by this simple attack the correct bit value of each message bit with a probability of $3/4$. To avoid similar attacks, Nikolopoulos introduced a second security parameter

$$s \in \mathbb{N}^*. \quad (7.8)$$

In this generalization, each message bit m_j is encoded first into an s -bit codeword \mathbf{w}_j and subsequently encrypted into s qubits of the public key. Therefore, the public key needs to have at least s times the length of the message, again we concentrate for simplicity on the case that this ratio is given exactly. The codeword is chosen as follows: If the message bit m_j is zero, take a codeword \mathbf{w}_j of length s randomly from the set of codewords with even parity. Appropriately, if the message bit m_j equals zero, the codeword is randomly chosen from the set of codewords with odd parity. In principle, the message is only extended by this procedure, so we can use the already defined steps of the protocol by applying the replacement

$$m_j \rightarrow \mathbf{w}_j. \quad (7.9)$$

Nikolopoulos derived an upper bound [Nik08a], that limits the number of public keys Alice can distribute by keeping the security of the private key against a powerful individual attack which is of interest in Section 8.3. Discussions on a *chosen-plaintext* attack, a *forward-search* attack (cf. also [NI09]) and a *chosen-ciphertext* were given by Nikolopoulos [Nik08a]. A very general attack on the private key as well as a powerful and more direct attack on the encoded message by using the information of all distributed keys will be discussed in Chapter 8, and leads to a limit for the number of public keys which may be distributed in order to achieve a certain security.

Security analysis

For the QPKE protocol presented in Section 7.1, different attacks were already discussed by Nikolopolous and Ioannou [Nik08a, NI09], but do not always benefit from the complete set of available public keys. The aim of this chapter is to take advantage of the complete set and to derive the strength of practical attacks. In Section 8.1, preliminary considerations will be discussed. For an asymmetric protocol, it is clear that the security of the private key is the most important goal in order to guarantee the secrecy of the communicated messages, as if it is broken, all messages which use that key will be leaked. In Section 8.2, this security will be discussed. Direct attacks on the message will be investigated subsequently in Section 8.3, where it will be shown that a certain individual attack¹ possesses an almost equal strength as collective attacks.² Finally, in Section 8.4, a potential extension of the protocol will be considered which uses a noisy-preprocessing step in order to reduce the information an eavesdropper may leak. Obviously, this step influences considerably the setup of the scheme, but it is in principle a generalization which takes partly the effect of noise into account.

8.1 Preliminary considerations

All values which refer to security issues of the protocol will be considered in the following in the case of a *single run* of the protocol, meaning the information of a single bit which is transmitted. For reasons of simplicity, we will omit the index j which refers to the specific bit, in the following. In some cases, averages of quantities will be taken into account, but in principle a message has a finite length and ensuring the security of a single bit is a stronger constraint.

To visualize this difference, let us compare the conditional probability of success of

¹Individual attacks denote attacks where the transmitted quantum systems are measured individually. Collective attacks denote attacks where information about the quantum systems is collected, e.g. by entangling them to ancilla systems and measuring them after the protocol is finished and information of post-processing protocols can be taken into account.

²The ideas discussed in those sections were published in 2012 [SNA12].

an eavesdropping strategy in both cases. Within a single run of the protocol, values for the key \mathbf{k} and the codeword \mathbf{w} as defined in Equations (7.1) and (7.9) are fixed, which lead to a mean probability of success³ by averaging over all possible keys, namely

$$\bar{P}(\text{suc}|\mathbf{w}) = \sum_{\mathbf{k}} P(\mathbf{k})P(\text{suc}|\mathbf{k}, \mathbf{w}) = \frac{1}{2^{nN}} \sum_{\mathbf{k}} P(\text{suc}|\mathbf{k}, \mathbf{w}), \quad (8.1)$$

where the keys are uniformly distributed over the set $\{0, 1\}^{nN}$. Additionally, an optimal eavesdropping strategy has to be symmetric with respect to the codewords, as the value for a one-bit message is given by $m \in \{0, 1\}$ with equal probability and the conditional probability of obtaining the codeword \mathbf{w} is $P(\mathbf{w}|m) = 2^{-(s-1)}$ (cf. Equation (7.8)). Thus the probability of any codeword to occur is given by

$$P(\mathbf{w}) = \sum_m P(\mathbf{w}|m)/2 = 2^{-s}. \quad (8.2)$$

8.2 Security of the private key

As given by the third step of the key generation part of the protocol which is introduced in Section 7.1, T' copies of the public key exist. According to Eve's chosen strategy, she may use for practical reasons $\tau < T'$ copies of the public key.⁴ Obviously, she can obtain those keys directly from Alice, as they are publicly available. Without measuring the state of these copies, she holds *a priori* a mixed quantum state, given by

$$\rho_{\text{prior}}^{(\tau)} = \frac{1}{2^n} \sum_{k'=0}^{2^n-1} (|\psi_{k'}(\theta_n)\rangle\langle\psi_{k'}(\theta_n)|)^{\otimes\tau} \quad (8.3)$$

$$= \frac{1}{2^n} \sum_{k'=0}^{2^n-1} (|\phi_{k'}^{(\tau)}(\theta_n)\rangle\langle\phi_{k'}^{(\tau)}(\theta_n)|), \quad (8.4)$$

where we have abbreviated the τ -qubit state as $|\phi_{k'}^{(\tau)}(\theta_n)\rangle := |\psi_{k'}(\theta_n)\rangle^{\otimes\tau}$. To bring Equation (8.4) into a nice form, we can take advantage of the following consideration: A possible basis for a single qubit is given by the set $\{|0_z\rangle, |1_z\rangle\}$ of states, which are eigenstates of the Pauli- σ_z operator. If we treat a tensor product of τ single qubits, the basis vectors are consequently given by the set

$$\{|i\rangle : i \in \mathbb{Z}_{2^\tau}\}, \quad (8.5)$$

where $|i\rangle$ represents an eigenstate of the τ -folded tensor product of Pauli- σ_z operators $\sigma_z^{\otimes\tau}$. Eve knows from the construction of the protocol, that these τ copies of the

³Probabilities which are denoted in the following in general by $P(\text{suc}|X)$, refer to the probability for an attacker to eavesdrop successfully, given the event X . The abbreviation ‘‘suc’’, meaning ‘‘success’’, is chosen for a better readability.

⁴If Eve wants to attack a message, already one key is consumed for the encryption; if her aim is to attack the key, she may in principle use all T' copies of the public key, but this knowledge does not help her to gain any profitably information.

qubit (which individually represent the public key) are equal. The state $|\phi_{k'}^{(\tau)}(\theta_n)\rangle$ is therefore invariant under a permutation of the copies. Thus, a decomposition of that state in the basis defined by Equation (8.5) will have equal parameters in those terms, where the number of ones in the string i coincides. The state $|\phi_{k'}^{(\tau)}(\theta_n)\rangle$ can then be represented by the basis

$$|l\rangle = \left(\sum_{i=1}^{2^\tau} \delta_{l,H(i)} |i\rangle \right) / \sqrt{\binom{\tau}{l}}, \quad (8.6)$$

where the *Hamming weight* $H(i)$ is the number of ones in the string i . These $\tau + 1$ different states $|l\rangle$ (with $l \in \{0, \dots, \tau\}$) refer to subspaces of the 2^τ dimensional Hilbert space.

We can use this basis in order to reformulate the state of Equation (8.4) with

$$|\phi_{k'}^{(\tau)}(\theta_n)\rangle = \sum_{l=0}^{\tau} \sqrt{\binom{\tau}{l}} f_{\tau,l}(k'\theta_n) |l\rangle, \quad (8.7)$$

where

$$f_{\tau,l}(k'\theta_n) = \left(\cos \left(\frac{k'\theta_n}{2} \right) \right)^{\tau-l} \left(\sin \left(\frac{k'\theta_n}{2} \right) \right)^l. \quad (8.8)$$

Introducing the coefficients $C_{l,l'}$ with

$$C_{l,l'} = \frac{1}{2^n} \sqrt{\binom{\tau}{l} \binom{\tau}{l'}} \sum_{k'=0}^{2^n-1} f_{\tau,l}(k'\theta_n) f_{\tau,l'}^*(k'\theta_n), \quad (8.9)$$

leads finally to

$$\rho_{\text{prior}}^{(\tau)} = \sum_{l,l'=0}^{\tau} C_{l,l'} |l\rangle\langle l'|. \quad (8.10)$$

We can use this form of the *a priori* state Eve may hold, in order to obtain an upper bound on the amount of information about the private key, Eve may extract from the state $\rho_{\text{prior}}^{(\tau)}$. Therefore, we need the *Holevo quantity* χ which limits the average amount of information I_{av} that can be extracted from an unknown state ρ . It is given by

$$\chi = S(\rho) - \sum_k p_k S(\rho_k) \geq I_{\text{av}}, \quad (8.11)$$

where $S(\rho)$ denotes the usual *von-Neumann entropy* of a state ρ , where the system is in a mixed state with states ρ_k and corresponding probabilities p_k .⁵ As seen in

⁵Discussions, derivations, and proofs on quantities of quantum information theory can be found in the book of Nielsen and Chuang [NC00].

Equation (8.10), the τ -qubit state $\rho_{\text{prior}}^{(\tau)}$ can be expressed in terms of a $\tau+1$ dimensional Hilbert space, therefore, its von-Neumann entropy is again upper bounded as⁶

$$I_{\text{av}} \leq \chi \leq S(\rho_{\text{prior}}^{(\tau)}) \leq \log_2(\tau + 1). \quad (8.12)$$

By the construction of the private key and the resulting public key as given in Section 7.1, the state of the public key is in one of 2^n different states, which means that this key is secure, as long as the information Eve may gain from her measurements is much smaller than the number of bits which describe the information, namely, as long as

$$n \gg \log_2(\tau + 1) \quad (8.13)$$

holds. As in principle T' copies of the public key exist and at least one is consumed to encrypt a message, this limit reads in terms of $T' = \tau + 1$ as

$$n \gg \log_2(T'). \quad (8.14)$$

With the help of this limitation, the security of the private key can be guaranteed. Nevertheless, direct attacks on the message are possible and will be discussed in Section 8.3.

8.3 Security of a message

The security of the sent message in the protocol of Nikolopoulos relies essentially on the security of the private key that can be guaranteed according to the limitations given in Equation (8.14). Thus, if the private key can be reconstructed perfectly, a potential eavesdropper *Eve* may use this information in order to obtain messages which are sent with this key. But as the mentioned limitation should be achieved by construction, the optimal strategy for Eve may be different. Instead of recovering the private key perfectly, she may recover the key approximately. As the private key is in one-to-one correspondence to the public key, we can easily discuss this issue in terms of the public key. Let us assume first that Eve manages to receive $T' - 1$ copies of the public key as well as an encrypted message which uses the same key. Since the integers of the corresponding private key are chosen individually and uniformly distributed, we are free to consider a single-bit message.

The strategy of Eve is then to estimate the state of the public-key qubits, by taking into account that each qubit is prepared in a state which lies on the z - x plane of the Bloch sphere (cf. Appendix A.5). The security parameter s , which was introduced in Equation (7.8), requires an s -bit codeword, which encodes a bit of the message and is subsequently encrypted into s qubits of the public key. At first, we will consider Eve's attack on a single qubit, which encodes only a single bit of the codeword \mathbf{w} and finally

⁶Simulations suggest that the eigenvalues of Equation (8.10) are given by $\lambda_i = 2^{-\tau} \binom{\tau}{i}$. Then $\rho_{\text{prior}}^{(\tau)}$ would be the entropy of a binomial distribution with mean $\tau/2$ and variance $\tau/4$ and upper bounded by the Gaussian distribution; this would lead to a lower upper bound of I_{av} , namely, $I_{\text{av}} \leq S(\rho_{\text{prior}}^{(\tau)}) \leq \frac{1}{2} \log_2 \tau + \frac{1}{2} \log_2(\pi e/2)$.

generalize this approach on the complete ciphertext of the codeword. As already seen in Part I of this work, an optimal choice to estimate an unknown state with a minimal number of single-qubit measurements is realized by using mutually unbiased bases. Taking the limitation to the z - x plane of the Bloch sphere into account, Eve's best measurement strategy is to use the eigenbases of the Pauli- σ_z and Pauli- σ_x operators equally often. To simplify matters, we assume that the number of distributed keys is given by

$$T' = 2T + 1. \quad (8.15)$$

This results in an estimation of the public key and defines an estimation of the basis that was used to encrypt the message. If Eve uses this basis estimation to measure the ciphertext, the fidelity of the basis which results from the private key and the estimation of the bases gives the success probability of Eve's attack.

Eve can apply the two different experiments (measuring in the eigenbasis of the Pauli- σ_z or the Pauli- σ_x operator, respectively) and will receive either the measurement outcome zero or one. This outcome depends in average on the expectation value which is given by testing $|0_z\rangle$ on the single-qubit state $|\psi_k(\theta_n)\rangle$, that is introduced in Equation (7.2), namely

$$\langle\psi_k(\theta_n)|0_z\rangle\langle 0_z|\psi_k(\theta_n)\rangle, \quad (8.16)$$

which yields the probability

$$p_0^{(z)}(k) = \cos^2\left(\frac{k\theta_n}{2}\right), \quad (8.17)$$

to measure a zero in the eigenbasis of the σ_z operator. To measure a zero in the eigenbasis of the σ_x operator, we find accordingly

$$p_0^{(x)}(k) = \cos^2\left(\frac{\pi}{4} - \frac{k\theta_n}{2}\right). \quad (8.18)$$

In both cases, the probabilities to measure a one are obviously given by $p_1^{(z)}(k) = 1 - p_0^{(z)}(k)$ and $p_1^{(x)}(k) = 1 - p_0^{(x)}(k)$, respectively. If we denote the number of outcomes of a zero in the eigenbasis of the σ_z operator, by $T_0^{(z)}$ and the number of outcomes of a zero in the eigenbasis of the σ_x operator, by $T_0^{(x)}$, Eve's attack leads to one out of $(T + 1)^2$ ordered pairs $(T_0^{(z)}, T_0^{(x)})$.

8.3.1 Information about the public key

In a first step, this special attack yields information about the public key (and therefore about the private key). The *a posteriori* probability for Eve to guess a certain bit value k' with the help of the ordered pair $(T_0^{(z)}, T_0^{(x)})$ is given by *Bayes law* as

$$p(k'|T_0^{(z)}, T_0^{(x)}) = \frac{p(T_0^{(z)}, T_0^{(x)}|k')}{2^n p(T_0^{(z)}, T_0^{(x)})}. \quad (8.19)$$

To calculate the *a priori* probability of the right hand side of this expression, we can use the quantities defined in Equation (8.17) and (8.18), keeping in mind all possible permutations of the measurement outcomes, which leads to

$$p(T_0^{(z)}, T_0^{(x)}|k') = \prod_{b \in \{z,x\}} \binom{T}{T_0^{(b)}} \left(p_0^{(b)}(k')\right)^{T_0^{(b)}} \left(p_1^{(b)}(k')\right)^{T-T_0^{(b)}}. \quad (8.20)$$

The probability for Eve to obtain a certain ordered pair $(T_0^{(z)}, T_0^{(x)})$ independently of the private-key state is given by the sum of the *a priori* probability over all possible values of k' , namely

$$p(T_0^{(z)}, T_0^{(x)}) = \frac{1}{2^n} \sum_{k'=0}^{2^n-1} p(T_0^{(z)}, T_0^{(x)}|k'). \quad (8.21)$$

The information which Eve can accumulate in average by evaluating the discussed measurement outcome is given by the difference of the Shannon entropies before and after the measurements, thus,

$$I_{\text{av}} = H_{\text{prior}} - \langle H_{\text{post}} \rangle \quad (8.22)$$

$$= n + \sum_{T_0^{(z)}, T_0^{(x)}} p(T_0^{(z)}, T_0^{(x)}) \sum_{k'=0}^{2^n-1} p(k'|T_0^{(z)}, T_0^{(x)}) \log_2 p(k'|T_0^{(z)}, T_0^{(x)}). \quad (8.23)$$

As the *a priori* state is uniformly distributed and can be described by n bits, the corresponding *a priori* entropy is given by n .

8.3.2 Information about the message

Alternatively, the measurement outcome $(T_0^{(z)}, T_0^{(x)})$ can be used by Alice to estimate the public-key state. With the help of the *a posteriori* probability, that a certain value k may have been the private key, as it was introduced in Equation (8.19), we can formulate the *a posteriori* state which Eve has after the measurement, taking all possible values for k into account. This τ -qubit state is given by

$$\rho_{\text{post}}^{(\tau)}(T_0^{(z)}, T_0^{(x)}) = \sum_{k'=0}^{2^n-1} p(k'|T_0^{(z)}, T_0^{(x)}) |\phi_{k'}^{(\tau)}(\theta_n)\rangle \langle \phi_{k'}^{(\tau)}(\theta_n)|. \quad (8.24)$$

Although Eve intercepted τ raw copies of the public key, she is only interested in a single-qubit state, which can be obtained by tracing out $\tau - 1$ states and results in

$$\rho_{\text{post}}^{(1)}(T_0^{(z)}, T_0^{(x)}) = \sum_{k'=0}^{2^n-1} p(k'|T_0^{(z)}, T_0^{(x)}) |\psi_{k'}(\theta_n)\rangle \langle \psi_{k'}(\theta_n)|. \quad (8.25)$$

Analogously to Equation (7.3) we can represent the single-qubit *a posteriori* state by the Bloch vector (cf. Equation A.17)

$$\tilde{\mathbf{R}} = \sum_{k'=0}^{2^n-1} p(k'|T_0^{(z)}, T_0^{(x)}) (\cos(k'\theta_n)\sigma_z + \sin(k'\theta_n)\sigma_x), \quad (8.26)$$

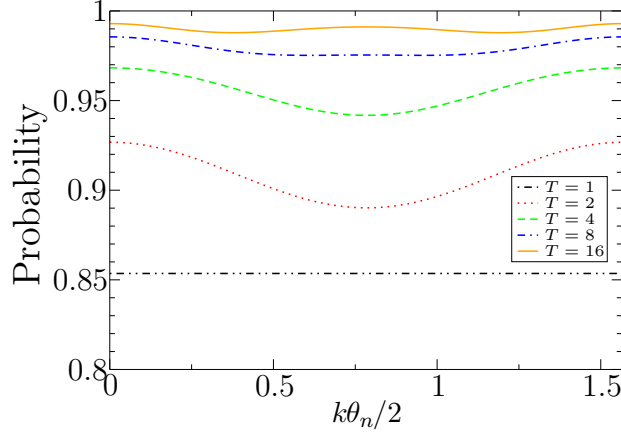


Figure 8.1: Conditional probability $P(\text{suc}|w_j, k)$ that Eve successfully attacks the public key for $n = 10$ and different values of T .

where this vector represents in general a mixed state and has therefore a length which is given by $|\tilde{\mathbf{R}}| \leq 1$.

In principle, we may compare this estimated Bloch vector now with the original Bloch vector and calculate the probability that both coincide. However, we have to call back into mind, that a single bit is encoded by an s -bit codeword \mathbf{w} and subsequently encrypted into s qubits defining the security parameter that was formulated in Equation (7.8). Following this construction, the first step is to calculate the probability that Eve correctly estimates the bit w_j of the codeword $\mathbf{w} = \{1, \dots, s\}$, which is given by

$$P(\text{suc}|w_j, k, T_0^{(z)}, T_0^{(x)}) = \cos^2(\Omega_j/2), \quad (8.27)$$

where Ω_j denotes the angle between the two Bloch vectors $\tilde{\mathbf{R}}_j$ and \mathbf{R}_j and the index j refers to the qubit that encrypts the bit w_j . Solving this approach leads to the expression

$$P(\text{suc}|w_j, k, T_0^{(z)}, T_0^{(x)}) = \frac{1}{2} + \frac{\tilde{\mathbf{R}}_j \cdot \mathbf{R}_j}{2|\tilde{\mathbf{R}}_j|}, \quad (8.28)$$

with the scalar product

$$\tilde{\mathbf{R}}_j \cdot \mathbf{R}_j = \sum_{k'=0}^{2^n-1} p(k'|T_0^{(z)}, T_0^{(x)}) \cos((k' - k)\theta_n). \quad (8.29)$$

The ordered pair $(T_0^{(z)}, T_0^{(x)})$ is an output of the measurements by Eve which is not observed by Alice. A more meaningful quantity is therefore given by

$$P(\text{suc}|w_j, k) = \sum_{T_0^{(z)}, T_0^{(x)}} P(\text{suc}, w_j, k, T_0^{(z)}, T_0^{(x)}) p(T_0^{(z)}, T_0^{(x)}|k) \quad (8.30)$$

and visualized in Figure 8.1 for different values of T . Plotted against the key value \mathbf{k} , the probability shows an oscillatory behavior around its mean value, with decreasing

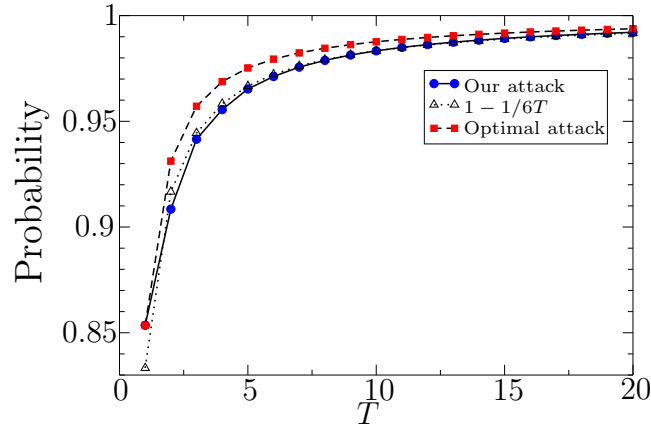


Figure 8.2: Conditional probability $\bar{P}(\text{suc}|w_j)$ that Eve successfully attacks the message for $n = 10$ for the discussed attack, the asymptotic behavior of this attack and the optimal collective attack [DBE98], plotted against T .

amplitude for a larger number of available copies. The mean value can be calculated as

$$\bar{P}(\text{suc}|w_j) = \frac{1}{2^n} \sum_{k=0}^{2^n-1} P(\text{suc}|w_j, k) \quad (8.31)$$

and scales for $T > 1$ like

$$\bar{P}(\text{suc}|w_j) \lesssim 1 - \frac{1}{6T}, \quad (8.32)$$

which is given together with the optimal probability that is discussed below, in Figure 8.2. The comparison with the *optimal* probability of successfully estimating the prepared state which takes advantage of a collective measurement method (and is therefore harder to get implemented) shows the good performance of this approach, where this optimal probability is given by [DBE98]

$$\bar{P}_{\text{opt}}(\text{suc}|w_j) = \frac{1}{2} + \frac{1}{2^{2T+1}} \sum_{i=0}^{2T-1} \sqrt{\binom{2T}{i} \binom{2T}{i+1}}, \quad (8.33)$$

and scales as

$$\bar{P}_{\text{opt}}(\text{suc}|w_j) \sim 1 - \frac{1}{8T}. \quad (8.34)$$

Even this scaling can be reached by measurements on individual qubits, as it was shown by Bagan *et al.* [BBMT02] with an attack which is similar to the discussed one. Contrary to the attack given in [NI09], the probability of success does not depend on the value w_j and can be implemented using only measurement devices instead of the more complicated quantum operations which are used in the mentioned approach.

Hence, we are well prepared to discuss the final step of this attack, namely considering the probability of correctly guessing the message bit m . As seen in Section 7.1,

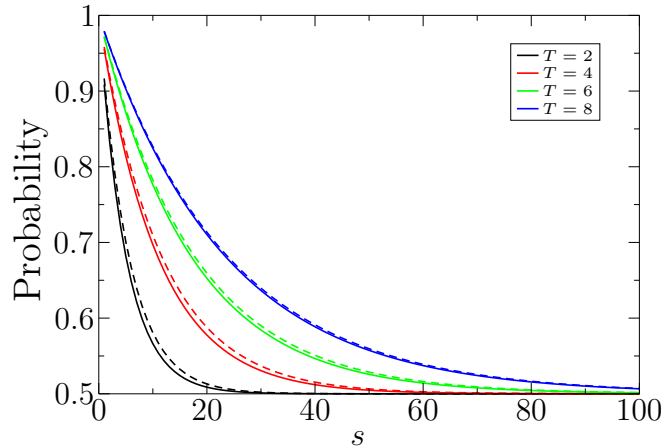


Figure 8.3: Conditional probability $\bar{P}_s(\text{suc}|m, \mathbf{w})$ that Eve successfully attacks the message for $n = 10$ and different values of T , plotted against the security parameter s . The solid lines refer to the exact results of Equation (8.35) and the dashed lines to the upper bound given by Equation (8.43).

the bit m is encoded into the parity of an s -bit codeword \mathbf{w} , where all appropriate codewords are equally probable. As seen in Equation (8.31), Eve has a certain probability for each bit of the codeword to succeed. But even though, namely in cases where Eve fails an even number of times to predict the correct value of a bit w_j of the codeword, the parity she gets results in the same and therefore correct message bit m . Additionally, as we have already seen in Figure 8.1, the amplitude of the oscillation of the success probability becomes negligible for large values of T , at least it is an order of magnitude smaller than the mean. Therefore, we will not gain a much better notion by observing the mean value of the success probability, as it should be comparable to the results within a single-run experiment. Following the structure of the codeword, Eve has a mean probability of success, given a certain codeword and a certain message bit,

$$\bar{P}_s(\text{suc}|m, \mathbf{w}) = \sum_{\substack{\alpha=0, \\ \text{even}}}^s \binom{s}{\alpha} (1 - \bar{P}(\text{suc}|w_j))^\alpha (\bar{P}(\text{suc}|w_j))^{s-\alpha}, \quad (8.35)$$

where α denotes the number of incorrectly estimated bits of the codeword. Since this value does not depend on the message bit and all codewords have the same probability, we finally find

$$\bar{P}_s(\text{suc}) = \bar{P}_s(\text{suc}|m, \mathbf{w}). \quad (8.36)$$

Within Figure 8.3 we show the behavior of this probability for different values of T against the security parameter s . As expected, this parameter produces a monotonic decrease in the success probability of Eve and can be tightly upper bounded with the help of the following lemma that can be used in order to generalize Equation (8.32).

Lemma 8.3.1 (Success probabilities with increasing security parameter).

If the probability (or an upper bound) of successfully eavesdropping a single-bit message encoded by a codeword of length $s = 1$ is given by

$$Q^{(1)} = \frac{1}{2} + \frac{\lambda}{2}, \quad (8.37)$$

with $\lambda \in \mathbb{C}$, the corresponding quantity for a single-bit message, encoded by an s -bit codeword, such that the parity of the codeword equals the message bit and all possible codewords are equally probable, is given by

$$Q^{(s)} = \frac{1}{2} + \frac{\lambda^s}{2}. \quad (8.38)$$

Proof. As an even number of wrong results in the attack leads to a correct estimate of the parity of the codeword which represents the sent message, a generalization of Equation (8.37) to a codeword of length s is given by

$$Q^{(s)} = \sum_{\substack{\alpha=0, \\ \text{even}}}^s \binom{s}{\alpha} (1 - Q^{(1)})^\alpha (Q^{(1)})^{s-\alpha}. \quad (8.39)$$

Alternatively, this generalization can also be written iteratively as

$$Q^{(s)} = Q^{(1)}Q^{(s-1)} + (1 - Q^{(1)})(1 - Q^{(s-1)}). \quad (8.40)$$

If for $Q^{(1)}$ Equation (8.37) holds and we assume that also Equation (8.38) is given, we find for $Q^{(s+1)}$, using the iterative construction,

$$Q^{(s+1)} = \left(\frac{1}{2} + \frac{1}{2}\right) \left(\frac{1}{2} + \frac{\lambda^s}{2}\right) + \left(\frac{1}{2} - \frac{1}{2}\right) \left(\frac{1}{2} - \frac{\lambda^s}{2}\right) \quad (8.41)$$

$$= \left(\frac{1}{2} + \frac{\lambda^{s+1}}{2}\right), \quad (8.42)$$

which completes the induction and provides the expected result. \square

If we set $Q^{(1)}$ to the value of Equation (8.32), we can use Lemma 8.3.1 with $\lambda = 1 - 1/(3T)$ and get finally a generalized upper bound for the correct guessing of the message⁷ by

$$Q^{(s)} := \tilde{P}_s(\text{suc}) = \frac{1}{2} + \frac{1}{2} \left(1 - \frac{1}{3T}\right)^s. \quad (8.43)$$

Finally, to achieve a security against the introduced attack, we have to show that for any small positive value $\varepsilon \in (0, 1/2]$, we find a finite value of the security parameter s , such that the probability of successfully estimating goes below the sum of the probability of randomly guessing with that value. This notion of security is also known

⁷The message is encoded to the parity of an s -bit codeword as introduced in Section 7.1.

as *asymptotic security*. Thus, to find a security parameter which fulfills $\bar{P}_s(\text{suc}) < 1/2 + \varepsilon$, we find, using Equation (8.43),

$$s > \frac{1 + \log_2 \varepsilon}{\log_2 \left(1 - \frac{1}{3T}\right)}, \quad (8.44)$$

which is also fulfilled if there holds

$$s > 3T(1 + \log_2 \varepsilon). \quad (8.45)$$

This easily implementable attack is stronger than the *forward-search* attack which was already discussed in the original work of the protocol [Nik08a, NI09] and uses complicated quantum operations as *Fourier transformations* and permutations on large sets of qubits. Similarly to Equation (8.43), an upper bound for its probability of successfully eavesdropping (which does not vary from run to run) is given by [NI09]

$$\tilde{P}_s^{\text{FS}}(\text{suc}) = \frac{1}{2} + \frac{1}{2} \left(1 - \frac{1}{2T}\right)^s. \quad (8.46)$$

The success probability of that attack leads to a lower bound for the security parameter in the notion of Equation (8.45) as

$$s^{\text{FS}} > 2T(1 + \log_2 \varepsilon), \quad (8.47)$$

and differs only by a factor of 2/3 with the new investigated attack.⁸

8.4 Noisy preprocessing

In order to increase the robustness of the protocol discussed in Section 7.1 against various attacks like those investigated in Sections 8.2 and 8.3 and potentially others, we show in the following exemplarily the behavior of a modified protocol against two simple attacks. Conversely to the original key-generation Step 3 of the protocol, we assume that the states are not taken from the z - x -plane of the Bloch sphere. One may think of an *affine plane* which would imply states on a certain degree of latitude, areas on the surface which are limited by two degrees of latitude, or even the complete surface of the sphere. Obviously, as we like to leave the measurement process untouched, *post-processing* (namely *error correction*) steps will be obligatory. Even within the original protocol they have to be added in order to get a practical implementation, as errors in quantum states will always occur. Possibly, the introduced steps may help in order to model the influence of errors in the original protocol. Nevertheless, as this consideration is based on two simple attacks, the results can only be seen as limited, but indicate the positive behavior and motivate a subsequent analysis.

⁸The differences of Equations (8.46) and (8.47) with their counterparts in [NI09] are due to the fact that within the attack discussed here, T is defined to refer to the half of the number of keys which are used in order to attack the encrypted message. The missing factor of two in [SNA12, Equation (30)], comparing to Equation (8.47), is based on a misprint.

8.4.1 Single-key test attack

The first attack we use in order to compare a notion of the security of the original protocol with the security of the modified protocol is in principle a single-copy version of the attack, which was discussed in Section 8.3 and published in the same work [SNA12]. If Eve intercepts a ciphertext and a corresponding copy of the public key, she can attack the message bitwise by a simple experiment. For each bit of the codeword \mathbf{w} as introduced in Section 7.1, she measures both corresponding qubits in the same, but in principle arbitrary basis within the z - x -plane, which leads to a success probability similar to that of Equation (8.17). As rotations of the measurement axis around the σ_y axis lead only to a constant term in the argument of the \cos^2 and the possible states are arranged symmetrically with even probabilities, the success probability is independent of the chosen axis for a random chosen public key bit, so we will take the σ_z axis and have

$$p_0(k) = \cos^2 \left(\frac{k\theta_n}{2} \right), \quad (8.48)$$

to measure a value of zero on a qubit. As a codeword bit of one generates a flip on the Pauli- σ_y axis, we have to calculate the probability of measuring this flip—without knowing the correct basis—which is given by

$$\begin{aligned} P^{(1)}(\text{suc}|k, w_j) &= \begin{cases} p_1^{(\text{key})}(k)p_0^{(\text{ct})}(k + 2^{n-1}) + p_0^{(\text{key})}(k)p_1^{(\text{ct})}(k + 2^{n-1}) & \text{for } w_j = 1, \\ p_0^{(\text{key})}(k)p_0^{(\text{ct})}(k) + p_1^{(\text{key})}(k)p_1^{(\text{ct})}(k) & \text{for } w_j = 0, \end{cases} \end{aligned} \quad (8.49)$$

where $p^{(\text{key})}$ belongs to the measurement on the key and $p^{(\text{ct})}$ to the measurement on the ciphertext. Using Equation (8.48), we find

$$P^{(1)}(\text{suc}|k) = \sin^4 \left(\frac{k\theta_n}{2} \right) + \cos^4 \left(\frac{k\theta_n}{2} \right). \quad (8.50)$$

With the help of Lemma 8.3.1 we can generalize this expression to the parity of the codeword \mathbf{w} and finally get

$$P_s^{(1)}(\text{suc}|k) = \frac{1}{2} + \frac{(1 + \cos(2k\theta_n))^s}{2^{s+1}} \quad (8.51)$$

$$= \frac{1 + \cos^{2s}(k\theta_n)}{2}. \quad (8.52)$$

For a mean probability by averaging over all possible private key states, we find with Equation (8.50) a probability of

$$\bar{P}^{(1)}(\text{suc}) = \frac{1}{2^n} \sum_{k=1}^{2^n} P(\text{suc}|k) = \frac{3}{4}, \quad (8.53)$$

for Eve to measure the correct value of a single bit of the codeword \mathbf{w} and with Equation (8.51) a probability of

$$\bar{P}_s^{(1)}(\text{suc}) = \frac{1}{2} + \frac{(1/2)^s}{2}, \quad (8.54)$$

to measure correctly the parity of the codeword.

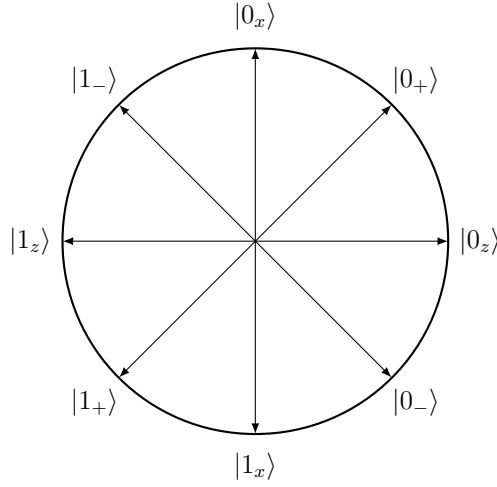


Figure 8.4: The four bases which are defined in order to implement the double-key test attack.

8.4.2 Double-key test attack

The second attack we discuss takes measurement results on two public keys and the corresponding ciphertext into account, so we assume Eve holds those three qubit-strings. For each qubit triple that belongs to the bit of a codeword, she measures the first key in the σ_z basis as for the single-key test attack, but measures the second key in the orthogonal σ_x basis. This defines one out of two bases which is taken as the measurement basis for the ciphertext qubit.

In detail: If the measurements of σ_z and σ_x give the same output, that basis is chosen, which bisects the inner angle; in the converse situation, a basis is chosen which bisects the outer angle. This attack is another formulation of the attack discussed in Section 8.3 for $T = 1$. For the probabilities of measuring the value zero or one from a copy of the key, we can take the expressions of Equations (8.17) and (8.18). The probability of getting the correct value of the codeword bit by measuring the message within the resulting basis is given by the expectation value of this basis with the corresponding state of the ciphertext. Following these ideas we find

$$P^{(2)}(\text{suc}|k, \theta_n) = p_0^{(z)}(k)p_0^{(x)}(k)|\langle 0_+|\psi_k(\theta_n)\rangle|^2 + p_1^{(z)}(k)p_1^{(x)}(k)|\langle 1_+|\psi_k(\theta_n)\rangle|^2 \\ + p_0^{(z)}(k)p_1^{(x)}(k)|\langle 0_-|\psi_k(\theta_n)\rangle|^2 + p_1^{(z)}(k)p_0^{(x)}(k)|\langle 1_-|\psi_k(\theta_n)\rangle|^2, \quad (8.55)$$

where $|0_+\rangle$ and $|1_+\rangle$ refer to outcomes 0 and 1 for the basis which bisects the inner angle between $|0_x\rangle$ and $|0_z\rangle$ and $|0_-\rangle$ and $|1_-\rangle$ to outcomes 0 and 1 for the basis which bisects the inner angle between $|0_z\rangle$ and $|1_x\rangle$. Figure 8.4 illustrates this structure. With $|\langle 0_+|\psi_k(\theta_n)\rangle|^2 = \cos^2(\pi/8 - k\theta_n/2)$ and $|\langle 0_-|\psi_k(\theta_n)\rangle|^2 = \cos^2(-\pi/8 - k\theta_n/2)$, we find

$$P^{(2)}(\text{suc}) = p(\text{suc}|k, \theta_n) = \frac{1}{4}(2 + \sqrt{2}) \approx 0.85, \quad (8.56)$$

which does neither depend on n , nor on k , as the plot of Figure 8.1 already indicated. For correctly estimating the message bit, Eve needs to guess correctly the parity of

the codeword \mathbf{w} , therefore we can take again advantage of Lemma 8.3.1 and find for this attack

$$P_s^{(2)}(\text{suc}) = \frac{1}{2} + \frac{(\sqrt{2}/2)^s}{2}. \quad (8.57)$$

8.4.3 Protocol and security analysis

We extend the protocol which was discussed in Section 7.1 by a (private) displacement parameter $a \in [-1, 1]$ which moves the plane that is shown in Figure 7.1 into the positive or negative σ_y direction. The parameter is defined to be one if the plane touches the surface only at $|1_y\rangle$ and equals minus one if it touches the opposite pole. It is linear on the σ_z axis in the representation of the Bloch sphere. Obviously, there is another representation of that parameter as an angle $\alpha \in [-\pi/2, \pi/2]$ with $a = \sin(\alpha)$, which is used in the following. The implementation into the protocol is achieved by generalizing the state $|\psi_k(\theta_n)\rangle$ which is defined in the key generation part in Step 3 as

$$|\psi_k(\theta_n, \alpha)\rangle = \mathcal{R}_y(k\theta_n)\mathcal{R}_x(\alpha)|0_z\rangle \quad (8.58)$$

$$\begin{aligned} &= \left(\cos\left(\frac{k\theta_n}{2}\right) \cos\frac{\alpha}{2} + i \sin\left(\frac{k\theta_n}{2}\right) \sin\frac{\alpha}{2} \right) |0_z\rangle \\ &\quad + \left(\sin\left(\frac{k\theta_n}{2}\right) \cos\frac{\alpha}{2} - i \cos\left(\frac{k\theta_n}{2}\right) \sin\frac{\alpha}{2} \right) |1_z\rangle. \end{aligned} \quad (8.59)$$

The probability of Eve to correctly estimate the transmitted bit using the single-key test attack (which was discussed in Section 8.4.1) is given by

$$\bar{P}_E^{(1)}(\text{suc}|\alpha) = \sum_{k=0}^{2^n-1} (|\langle \xi | \psi_k(\theta_n, \alpha) \rangle|^2)^2 + (1 - |\langle \xi | \psi_k(\theta_n, \alpha) \rangle|^2)^2 = \frac{1}{8}(5 + \cos(2\alpha)) \quad (8.60)$$

for an arbitrary measurement basis in the z - x -plane, averaged over all 2^n possible states. For $\alpha = 0$ this result coincides as expected with the value derived in Equation (8.53). As Alice does not know the transmitted message bit and the possible states are not within a single plane, we assume for simplicity that she measures on an axis which is in the plane, where the states are defined for $\alpha = 0$. This leads to a probability for Alice to get the correct bit value, given by

$$\bar{P}_A(\text{suc}|\alpha) = |\langle 0_z | \mathcal{R}_x(\alpha) | 0_z \rangle|^2 = \cos^2\left(\frac{\alpha}{2}\right). \quad (8.61)$$

If we further assume, that an *error correction* protocol exists which corrects as many bits as are distributed correctly in average, the probability for Eve to find the correct bit value in the relevant cases is given by the relative success probability

$$\bar{P}^{(1)}(\text{suc}|\alpha) = \frac{\bar{P}_E^{(1)}(\text{suc}|\alpha)}{\bar{P}_A(\text{suc}|\alpha)} = \frac{5 + \cos(2\alpha)}{8 \cos^2(\alpha/2)}. \quad (8.62)$$

Thus, to minimize the information Eve may gain by the single-key test attack for this modified protocol, we find

$$\alpha_{\min}^{(1)} = \pm 2 \arccos \sqrt[4]{\frac{3}{4}}, \quad (8.63)$$

which leads to

$$\bar{P}^{(1)}(\text{suc}|\alpha_{\min}^{(1)}) = \sqrt{3} - 1 \approx 0.732, \quad (8.64)$$

and is below the value $\bar{P}^{(1)}(\text{suc}) = 3/4$ for $\alpha = 0$.

Alternatively, we can use the double-key test attack (which was discussed in Section 8.4.2) to check if the discussed tendency becomes stronger or weaker in the case that Eve is able to catch more copies of the key. The probability for Eve to estimate the correct bit value can be calculated using Equation (8.55), but replacing $|\psi_k(\theta_n)\rangle$ by $|\psi_k(\theta_n, \alpha)\rangle$ and deriving $p_v^{(b)}$ as

$$p_v^{(b)}(k) = |\langle v_b | \psi_k(\theta_n, \alpha) \rangle|^2, \quad (8.65)$$

with $v \in \{0, 1\}$ and $b \in \{z, x\}$, which leads to

$$\bar{P}_E^{(2)}(\text{suc}|\alpha) = \frac{1}{2} + \frac{\sqrt{2}}{4} \cos^2 \alpha. \quad (8.66)$$

As the probability for Alice to get the correct bit value does not depend on Eve's attack and therefore not the number of keys she uses for the attack, the probability for Eve to get the correct bit value in the relevant cases is given by

$$\bar{P}^{(2)}(\text{suc}|\alpha) = \frac{\bar{P}_E^{(2)}(\text{suc}|\alpha)}{\bar{P}_A(\text{suc}|\alpha)} = \frac{1 + (\cos^2 \alpha)/\sqrt{2}}{2 \cos^2 \frac{\alpha}{2}}. \quad (8.67)$$

Analogously to Equation (8.63) we can find the minimum of this probability in order to minimize the information gain for Alice by

$$\alpha_{\min}^{(2)} = \pm 2 \arccos \left((\sqrt{2} - 1)^{3/4} + (\sqrt{2} - 1)^{3/4}/\sqrt{2} \right), \quad (8.68)$$

which leads to

$$\bar{P}^{(2)}(\text{suc}|\alpha_{\min}^{(2)}) = \sqrt{2} \left(\sqrt{1 + \sqrt{2}} - 1 \right) \approx 0.783, \quad (8.69)$$

and is considerably below Eve's success probability for $\alpha = 0$ which is given by

$$\bar{P}^{(2)}(\text{suc}|\alpha = 0) \approx 0.854. \quad (8.70)$$

For a security parameter $s > 1$, we can again use Lemma 8.3.1 in order to derive the success probability for Eve and find

$$\bar{P}_s^{(2)}(\text{suc}|\alpha_{\min}^{(2)}) = \frac{1}{2} + \frac{1}{2} \left(\frac{5 - 4 \cos^2 \frac{\alpha}{2} + \cos^2 \alpha}{4 \cos^2 \frac{\alpha}{2}} \right)^s, \quad (8.71)$$

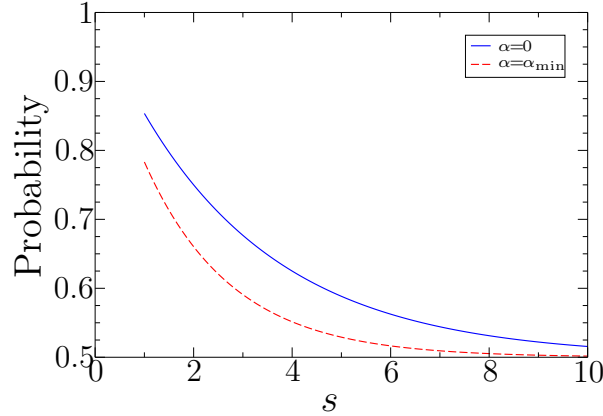


Figure 8.5: Conditional probability $\bar{P}_s^{(2)}(\text{suc}|\alpha)$ against the security parameter s that Eve successfully attacks the public key for $\alpha = 0$ and $\alpha = \alpha_{\min}$, using the double-key test attack.⁹

which decreases faster than for $\alpha = 0$ as can be seen in Figure 8.5. Finally, we may imagine cases where α is not fixed, for example to test the robustness of the protocol against certain kinds of errors or simply to adapt the protocol. We shortly calculate two situations: At first, α is a random parameter in the interval $[-\pi/2, \pi/2]$. In the second case, we average over those values of α where Eve has a success probability that is not larger than for $\alpha = 0$. In the first case, we find for the single-key test attack, using Equation (8.62),

$$\bar{P}^{(1)}(\text{suc}) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \frac{5 + \cos(2\alpha)}{8 \cos^2(\alpha/2)} d\alpha \approx 0.773, \quad (8.72)$$

which is above the value of $3/4$ for $\alpha = 0$, but for the double-key test attack, using Equation (8.67), we have

$$\bar{P}^{(2)}(\text{suc}) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \frac{1 + (\cos^2 \alpha)/\sqrt{2}}{2 \cos^2 \frac{\alpha}{2}} d\alpha \approx 0.830, \quad (8.73)$$

which is slightly below the value for $\alpha = 0$ which was given in Equation (8.70). If this tendency is monotonic, the protocol can in principle be enhanced by robustness and security, if α is taken to be arbitrary.¹⁰

For the second case we find

$$\bar{P}^{(1)}(\text{suc}|\bar{\alpha}_{\min}^{(1)}) = \frac{3}{2\pi} \int_{-\pi/3}^{\pi/3} \frac{5 + \cos(2\alpha)}{8 \cos^2(\alpha/2)} d\alpha \approx 0.740, \quad (8.74)$$

⁹Please keep in mind that s is a discrete parameter, the presentation is taken to get a better notion of the behavior of the probabilities.

¹⁰This statement holds as long as the number of public key copies is larger than two.

which is below $3/4$ and

$$\bar{P}^{(2)}(\text{suc}|\bar{\alpha}_{\min}^{(2)}) = \frac{1}{2\alpha_{\lim}^{(2)}} \int_{-\alpha_{\lim}^{(2)}}^{\alpha_{\lim}^{(2)}} \frac{1 + (\cos^2 \alpha)/\sqrt{2}}{2 \cos^2 \frac{\alpha}{2}} d\alpha \approx 0.816, \quad (8.75)$$

which is below the value for $\alpha = 0$ for the double-key test attack, with $\alpha_{\lim}^{(2)} = 2 \arccos\left(\frac{\sqrt{1+\sqrt{2}}}{2}\right)$.

Concluding, we have seen in these two simple test attack scenarios, that a noisy-preprocessing in the introduced way is able to enhance the security of the original protocol. Even if this additional parameter α is taken in a larger interval, the security against these test attacks increases.¹¹ Finally, the new parameter can be used as a first approach in order to model a protocol with errors in the preparation process of Alice, as Alice does not need the parameter α for the considered measurement of the message.

¹¹For the case of two copies of the public key, the test attack coincides by construction with the quite general attack discussed in Section 8.3.

Conclusions and outlook

Quantum cryptographic primitives expand the field of classical cryptography and provide perspectives which have the potential to influence existing schemes fundamentally. Whereas in classical cryptography the security is based on *computational complexity*, quantum cryptographic statements are based on physically unsolvable problems. This leads to complete *provably secure* cryptographic schemes. The emerging field of quantum public-key encryption offers new ways in order to benefit from the fundamental characteristics of quantum physics for cryptographic schemes.

Within the second part of this work, the recently invented *single-qubit-rotation protocol* is introduced and the *security of the private key* as well as the *security of a message* which is transmitted with the help of the protocol are discussed. A powerful and practical attack is used to consider the security of messages. Even though it attacks individual qubits, its asymptotic behavior is comparable with collective attacks. Finally, a possible extension of the protocol that implements a kind of *noisy preprocessing* is discussed and its advantage against simple test attacks is considered. The results indicate a positive effect of this method also for more general attacks and its ability to model noise.

As a next step, a practical version of the protocol could be considered which contains *error correction* methods, an exact formulation of noise effects as well as preprocessing steps. Further analytical expressions of important quantities would be useful to generalize Eve's success probabilities.

A

Algebra and quantum information

A.1 Fundamentals

Within this section we arrange some known results from algebra that are relevant for this work and were done for [KRS10] and [SR12].

Lemma A.1.1 (Determinant decomposition).

Let R be a ring. Then there holds

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(D) \det(A - BD^{-1}C) \quad (\text{A.1})$$

if D is invertible, for $A, B, C, D \in M_n(R)$.

Proof. If D is invertible, we are free to decompose the matrix as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \mathbb{1}_n & B \\ 0_n & D \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0_n \\ D^{-1}C & \mathbb{1}_n \end{pmatrix}. \quad (\text{A.2})$$

Since the determinant of block-triangular matrices is the product of the determinants of the blocks on the diagonal, the statement follows. \square

Lemma A.1.2 (Block matrix invertibility).

Let R be a commutative ring and $A, B, C, D \in R$ be commuting elements. Then the block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is invertible, if and only if $AD - BC$ is invertible.

A proof is given by Sylvester [Sil00].

A.2 Finite fields

A field with a finite number of elements is called a *finite field* or *Galois field*. It turns out that a finite field exists only if the number of elements equals the power of a prime, which is called the *order* of the field. As for a given order, the corresponding finite field is unique up to isomorphisms, it can be denoted as

$$\mathbb{F}_q = \mathbb{F}_{p^m}, \quad (\text{A.3})$$

where \mathbb{F}_{p^m} is a field with p^m elements, thus with order p^m . The prime number p is called the *characteristic* of the field and m is a natural number. For $m = 1$, the field is called *prime field*. An important linear transformation from \mathbb{F}_{q^n} to its corresponding prime field, namely \mathbb{F}_q , with $q = p^m$ and $n, m \in \mathbb{N}^*$, is the field trace $\text{tr}_{L/K}(\alpha)$ which maps an element of a field L to a field K , where K has not to be a prime field, but the order of K has to be a divisor of L .

Definition A.2.1 (Field trace).

The trace of an element $\alpha \in L := \mathbb{F}_{q^n}$ to a certain ground field $K := \mathbb{F}_q$ with $q = p^m$, p prime and $m, n \in \mathbb{N}^*$, is given by

$$\text{tr}_{L/K}(\alpha) = \alpha + \alpha^q + \alpha^{q^2} + \dots + \alpha^{q^{n-1}}. \quad (\text{A.4})$$

A.3 Hadamard matrix

Hadamard matrices appear often in considerations of the quantum information theory. They also have applications in coding theory.

Definition A.3.1 (Hadamard matrix).

Matrices with entries ± 1 and whose rows are pairwise orthogonal, are called *Hadamard matrices*.

It then follows obviously by this definition, that the product of a Hadamard matrix with its transpose yields n times the unity matrix, where n is the dimension. In quantum information theory, the most often used Hadamard matrix is defined for two-level systems and given by

$$\bar{H} := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (\text{A.5})$$

In many cases, a normalized version, which is then also the representation of a unitary operator, reads as

$$H := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad (\text{A.6})$$

thus, it is an orthogonal matrix. Whereas different construction schemes for Hadamard matrices are known and their existence is only conjectured for dimensions which are multiples of four¹, we will only use the *Sylvester construction* which is given as follows.

¹The existence of Hadamard matrices for the dimension one and two seems to be an exception.

Starting with $\bar{H}_1 = (1)$ and $\bar{H}_2 = \bar{H}$, any matrix \bar{H}_{2^m} with $m > 1$ and $m \in \mathbb{N}^*$ is given by

$$\bar{H}_{2^m} = \begin{pmatrix} \bar{H}_{2^{m-1}} & \bar{H}_{2^{m-1}} \\ \bar{H}_{2^{m-1}} & -\bar{H}_{2^{m-1}} \end{pmatrix}, \quad (\text{A.7})$$

and equals the tensor product $\bar{H}_{2^m} = \bar{H}_2 \otimes \bar{H}_{2^{m-1}}$. The matrix \bar{H}_{2^m} can then be derived directly with the help of the m -folded tensor product of \bar{H} as

$$\bar{H}_{2^m} = \bar{H}^{\otimes m}. \quad (\text{A.8})$$

Another representation of this matrix considers its entries and it holds

$$(\bar{H}_{2^m})_{i,j} = (-1)^{i \cdot j}, \quad (\text{A.9})$$

with $i, j \in \mathbb{F}_{2^m}$, which can be read off directly from the construction of Sylvester given in Equation (A.7). The same arguments hold analogously for the normalized version, where an additional factor of $2^{-m/2}$ appears for the matrix H_{2^m} .

A.4 Pauli operators

For a complex Hilbert space of dimension two, the three Pauli operators σ_x, σ_y , and σ_z (or X, Y , and Z) are defined as

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{A.10})$$

For $x \rightarrow 1, y \rightarrow 2$, and $z \rightarrow 3$ it holds

$$\sigma_i \sigma_j = \delta_{ij} \mathbb{1}_2 + i \sum_{k=1}^3 \epsilon_{ijk} \sigma_k, \quad (\text{A.11})$$

for $i, j \in \{1, 2, 3\}$ and with ϵ_{ijk} denoting the fully antisymmetric operator, also known as *Levi-Civita* symbol. It can be checked easily, that the eigenbases of the Pauli operators are mutually unbiased.

For a d -dimensional complex Hilbert space with $d \in \mathbb{N}^*$, the generators of the *generalized Pauli operators* are defined by

$$Z|i\rangle = \omega^i|i\rangle \quad \text{and} \quad X|i\rangle = |i \oplus_d 1\rangle, \quad (\text{A.12})$$

with $\omega = \exp(2\pi i/d)$ being the first d -th root of unity. The set of Pauli operators which is generated by Z and X can be used as a basis in order to represent a linear operator of the complex Hilbert space $\mathcal{H} = \mathbb{C}^d$.

Lemma A.4.1 (Orthogonality of generalized Pauli operators).

The set of generalized Pauli operators is an orthogonal operator basis in the sense of Hilbert-Schmidt.

Proof. Using the Hilbert-Schmidt inner product,

$$\langle Z^\alpha X^\beta | Z^\gamma X^\delta \rangle_{\text{HS}} = \text{tr}(Z^{\alpha-\gamma} X^{\beta-\delta}) = \begin{cases} 1 & \text{for } \alpha - \gamma = \beta - \delta = 0, \\ 0 & \text{else,} \end{cases} \quad (\text{A.13})$$

holds for $\alpha, \beta, \gamma, \delta \in \{1, \dots, d\}$. Thus, the d^2 operators $Z^\alpha X^\beta$ form an orthogonal operator basis. \square

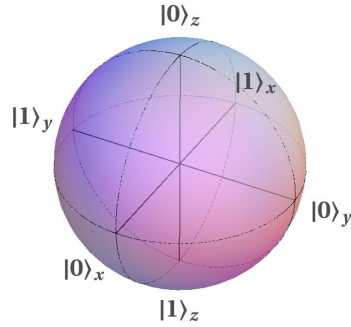


Figure A.1: Illustration of a Bloch sphere, where a vector, the Bloch vector, starting at the origin of the sphere and pointing inside or on the surface, is capable to represent any state of a two-dimensional quantum system (qubit). The axes refer to the three Pauli operators, where their points on the surface are given by the corresponding eigenstates.

A.5 Bloch sphere representation

Each state of a two-dimensional quantum system can be represented as a *Bloch vector* by three real parameters in the *Bloch sphere*, where the axes are defined by the three Pauli matrices of Equation (A.10). A pure quantum state which is written in the eigenbasis of the Pauli- σ_z operator, e. g.

$$|\Psi\rangle = \cos\left(\frac{\alpha}{2}\right) |0_z\rangle + e^{i\varphi} \sin\left(\frac{\alpha}{2}\right) |1_z\rangle \quad (\text{A.14})$$

leads to a density operator

$$\rho = |\Psi\rangle\langle\Psi| = \frac{1}{2} \begin{pmatrix} 1 + \cos\alpha & \cos\varphi \sin\alpha - i \sin\varphi \sin\alpha \\ \cos\varphi \sin\alpha + i \sin\varphi \sin\alpha & 1 - \cos\alpha \end{pmatrix} \quad (\text{A.15})$$

$$= \frac{1}{2} (\mathbb{1}_2 + (\cos\varphi \sin\alpha)\sigma_x + (\sin\varphi \sin\alpha)\sigma_y + (\cos\alpha)\sigma_z), \quad (\text{A.16})$$

where basic trigonometric identities were used. The Bloch vector is given by

$$\mathbf{R} = (\cos\varphi \sin\alpha)\sigma_x + (\sin\varphi \sin\alpha)\sigma_y + (\cos\alpha)\sigma_z, \quad (\text{A.17})$$

where the length of this vector equals one. Thus, all pure states are placed on the surface of the sphere. As mixed states are given by $\sum_i p_i \rho_i$ with $\sum_i p_i = 1$, classical probabilities $p_i \in \mathbb{R}$ and pure states ρ_i with $i \in \mathbb{N}$, they lead to a sum of vectors in the Bloch sphere. Obviously, the resulting vector will have a length smaller than one, therefore being placed inside the sphere. The zero-vector represents then the *completely mixed state*. Figure A.1 illustrates the Bloch sphere, where the extremal points on the axes are usually labeled by the eigenstates of the corresponding basis vectors. A similar representation in higher dimensions is in principle not possible.

A.6 Clifford group

Unitary matrices which map the set of Pauli operators onto the set of Pauli operators are called *Clifford unitary operators* and can be represented as a symplectic matrix $A \in M_{2m}(\mathbb{F}_p)$ with $m \in \mathbb{N}^*$ describing the number of qudits, where the dimension of the Hilbert space is given by $d = p^m$ with p prime; the positive integer p defines the Hilbert space dimension of a single qudit. Within this section we give some properties of the symplectic vector space.

Definition A.6.1 (Symplectic matrix).

A matrix $A \in M_{2m}(\mathbb{F}_p)$ is called *symplectic* if there holds

$$A^t S A = S \quad \text{for} \quad S = \begin{pmatrix} 0_m & \mathbb{1}_m \\ -\mathbb{1}_m & 0_m \end{pmatrix}. \quad (\text{A.18})$$

Corollary A.6.1 (Symplectic matrix properties).

The matrix $A \in M_{2m}(\mathbb{F}_p)$ is symplectic, if and only if there holds

$$s^t u = u^t s, \quad t^t v = v^t t, \quad \text{and} \quad s^t v - u^t t = \mathbb{1}_m \quad \text{for} \quad A = \begin{pmatrix} s & t \\ u & v \end{pmatrix}, \quad (\text{A.19})$$

for $s, t, u, v \in M_m(\mathbb{F}_p)$.

Proof. The three properties follow by simply applying Definition A.6.1 to the matrix A . \square

Definition A.6.2 (Symplectic product).

For two vectors $\vec{a}, \vec{b} \in \mathbb{F}_p^{2m}$, the *symplectic product* is defined as

$$(\vec{a}, \vec{b})_{\text{sp}} := \sum_{k=1}^m a_k^z b_k^x - a_k^x b_k^z \quad \text{mod } p, \quad (\text{A.20})$$

where a^z refers to the first half of the vector entries and a^x to the second half.

Corollary A.6.2 (Bilinearity of symplectic product).

For the symplectic product holds

$$(\vec{a} + \vec{b}, \vec{c} + \vec{d})_{\text{sp}} = (\vec{a}, \vec{c})_{\text{sp}} + (\vec{a}, \vec{d})_{\text{sp}} + (\vec{b}, \vec{c})_{\text{sp}} + (\vec{b}, \vec{d})_{\text{sp}} \quad (\text{A.21})$$

with $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d} \in K$.

Proof. The proof follows simply by using the definition of the symplectic product that is given by Equation (A.20). \square

Definition A.6.3 (Symplectic standard basis).

The vectors $\vec{z}_k, \vec{x}_k \in \mathbb{F}_p^{2m}$ with $k \in \{1, \dots, m\}$ form a *symplectic basis* of \mathbb{F}_p^{2m} , if there holds $(\vec{z}_k, \vec{z}_l)_{\text{sp}} = (\vec{x}_k, \vec{x}_l)_{\text{sp}} = 0$ and $(\vec{z}_k, \vec{x}_l)_{\text{sp}} = \delta_{kl}$ for $k, l \in \{1, \dots, m\}$. For the standard basis the vectors \vec{z}_k are defined to have a one at position k and zeros else, the vectors \vec{x}_k have accordingly a one at position $m + k$ and zeros else.

Lemma A.6.3 (Symplectic basis).

The images $\vec{z}'_k = C\vec{z}_k$ and $\vec{x}'_k = C\vec{x}_k$ of the symplectic standard basis with $k \in \{1, \dots, m\}$ and the matrix $C \in M_{2m}(\mathbb{F}_p)$ form again a symplectic basis, if and only if C is a symplectic matrix in the sense of Definition A.6.1.

Proof. For a symplectic standard basis there holds $(\vec{z}_k, \vec{z}_l)_{\text{sp}} = 0$ for all $k, l \in \{1, \dots, m\}$. The transformation of these properties with a matrix

$$C = \begin{pmatrix} s & t \\ u & v \end{pmatrix} \quad (\text{A.22})$$

may cause a new symplectic basis, if there holds $(C\vec{z}_k, C\vec{z}_l)_{\text{sp}} = 0$. With \vec{v}_k being the k -th row vector of $\nu \in \{s, t, u, v\}$ and \vec{z}^z being the z -part of the vector \vec{z} and \vec{x}^x being the x -part of the vector \vec{x} , we find

$$(C\vec{z}_k, C\vec{z}_l)_{\text{sp}} = \sum_{i=1}^m (\vec{s}_i \cdot \vec{z}_k^z)(\vec{u}_i \cdot \vec{z}_l^z) - (\vec{u}_i \cdot \vec{z}_k^z)(\vec{s}_i \cdot \vec{z}_l^z) \quad (\text{A.23})$$

$$= (\vec{z}_k^z)^t \cdot s^t \cdot u \cdot \vec{z}_l^z - (\vec{z}_k^z)^t \cdot u^t \cdot s \cdot \vec{z}_l^z. \quad (\text{A.24})$$

Since this equation has to be zero for all $k, l \in \{1, \dots, m\}$, the expression $s^t u = u^t s$ has to be fulfilled. Similarly, from $(\vec{x}_k, \vec{x}_l)_{\text{sp}} = 0$ follows that $t^t v = v^t t$. From $(\vec{z}_k, \vec{x}_l)_{\text{sp}} = \delta_{kl}$ we get

$$(C\vec{z}_k, C\vec{x}_l)_{\text{sp}} = \sum_{i=1}^m (\vec{s}_i \cdot \vec{z}_k^z)(\vec{v}_i \cdot \vec{x}_l^x) - (\vec{v}_i \cdot \vec{z}_k^z)(\vec{s}_i \cdot \vec{x}_l^x) \quad (\text{A.25})$$

$$= (\vec{z}_k^z)^t \cdot s^t \cdot v \cdot \vec{x}_l^x - (\vec{z}_k^z)^t \cdot u^t \cdot t \cdot \vec{x}_l^x. \quad (\text{A.26})$$

In order to set this equation equal to δ_{ij} , the expression $s^t v - u^t t = \mathbb{1}_m$ needs to be true. But by Corollary A.6.1 these three conditions hold if and only if C is a symplectic matrix. \square

B

Wiedemann's conjecture proof approach

Within this chapter, an approach is given that seems to be a promising candidate for proving Wiedemann's conjecture of [Wie88], where the proven analogy of Theorem 4.2.3 is the origin of. The idea is based on the construction of the unitary operator U as it was given in Section 4.7.

B.1 Unitary operator approach

By Theorem 4.2.3 it was proven, that Wiedemann's conjecture is true, if and only if the recursive construction of Equation (4.18) leads to a complete set of cyclic MUBs, which is given when the multiplicative order of the stabilizer matrix C_{2^k} of Equation (4.19) equals $2^{2^k} + 1$ for all $k \in \mathbb{N}$. As the recursive construction of the unitary generator $U_{2^k} = V_{2^k} / (-\text{tr } V_{2^k})$ that is given by Equation (4.73), is a representation of the stabilizer matrix (cf. Equation (4.68)), Wiedemann's conjecture is analogously proven if it is shown that the multiplicative order of U_{2^k} equals $2^{2^k} + 1$. We start with the following corollary that follows from Conjecture 4.7.1:

Corollary B.1.1 (Characteristic polynomial of Fermat sets).

If the characteristic polynomial of the unitary operator $U_{2^k} = V_{2^k} / (-\text{tr } V_{2^k})$ with V_m as given in Equation (4.79), equals $\sum_{l=0}^d x^l$ with $d = 2^{2^k}$ for all $k \in \mathbb{N}$, U_{2^k} generates a complete set of cyclic MUBs and Wiedemann's conjecture is true. This would also prove Conjecture 4.7.1 in the case of Fermat sets.

Proof. If the multiplicative order of U_{2^k} is $d + 1$, its eigenvalues have to be roots of unity of order $d + 1$, at least one of the eigenvalues has to be a principal root. The converse of Conjecture 4.7.1 is always true, namely if U_{2^k} has d different eigenvalues, it has order $d + 1$. Thus, if we can show that U_{2^k} has d different eigenvalues of order $d + 1$, it generates a complete set of cyclic MUBs. If we regard the characteristic polynomial $\chi_{U_{2^k}}$ of the unitary generator matrix with d different eigenvalues of order $d + 1$ and

exclude, as allowed, the eigenvalue 1 from the set, we get

$$\chi_{U_{2^k}}(x) = \prod_{l=1}^d (x - e^{2\pi i l / (d+1)}). \quad (\text{B.1})$$

If we multiply Equation (B.1) by $x - 1$, the result is equal to $x^{d+1} - 1$, which, divided again by $x - 1$, equals $\sum_{l=0}^d x^l$ as expected. \square

So we can prove Wiedemann's conjecture by calculating the characteristic polynomial of U_{2^k} , which is given by

$$\chi_{U_{2^k}}(x) := \det(x\mathbb{1}_d - U_{2^k}) \quad (\text{B.2})$$

$$= x^d - a_1 x^{d-1} + \dots + (-1)^{d-1} x^1 a_{d-1} + (-1)^d x^0 a_d; \quad (\text{B.3})$$

see e. g. Jacobson [Jac96, p. 196]. The value a_1 is defined as the sum of all diagonal elements, thus the trace of U_{2^k} . The value a_2 is the sum of the two-rowed diagonal minors and so on; a_d is then the determinant. With $U_{2^k} = V_{2^k} / (-\text{tr } V_{2^k})$ and $\text{tr } V_{2^k} = -i2^{m/2}$ as shown in Equation (4.84), we find

$$\chi_{U_{2^k}}(x) = \det(x\mathbb{1}_d - \alpha V_{2^k}) \quad (\text{B.4})$$

$$= x^d - a'_1 \alpha x^{d-1} + \dots + (-1)^{d-1} \alpha^{d-1} x^1 a'_{d-1} + (-1)^d \alpha^d x^0 a'_d, \quad (\text{B.5})$$

with $\alpha := -i2^{-m/2}$. In order to fulfill the requirements of Corollary B.1.1, all the coefficients c_l of the characteristic polynomial $\chi_{U_{2^k}}$ of U_{2^k} have to be one with $\chi_{U_{2^k}}(x) = \sum_{l=0}^d c_l x^{d-l}$. Therefore, we should find for the minor sums a'_l of V_{2^k} , that $a'_l = (-i2^{m/2})^l$ holds for $l \in \{1, \dots, d\}$. For the trace a'_1 this was already shown in Equation (4.84). We will refer to a'_l , which is the sum of the l -rowed diagonal minors, as tr_l in the following. Since the proof is not finished, we can only discuss the calculation of the two-rowed diagonal minors, but we expect that the higher orders may be derived in a similar manner.

B.1.1 Two-rowed diagonal minors

The sum of the two-rowed diagonal minors of the matrix V_{2^k} is given by

$$\text{tr}_2(V_{2^k}) := \sum_{j=2}^d \sum_{i=1}^{j-1} ((V_{2^k})_{i,i} \cdot (V_{2^k})_{j,j} - (V_{2^k})_{i,j} \cdot (V_{2^k})_{j,i}). \quad (\text{B.6})$$

To calculate this sum, we will separate the sum into the diagonal and the anti-diagonal term. We like to recall that the elements of V_{2^k} are by construction limited as $(V_{2^k})_{i,j} \in \{\pm 1, \pm i\}$.

In those cases, where we pick a real and an imaginary element from the diagonal, their product would be imaginary. But we have seen in the calculation of the trace in Equation (4.84), that the sum of all real elements on the diagonal is zero. So the sum of these cases adds a zero to $\text{tr}_2(V_{2^k})$.

Whenever we take two real elements from the diagonal, they equal either -1 or $+1$, but we know their sum is zero and $d/2$ elements are real. All their products give

$$\frac{d^2 - 4d}{32} \cdot 1 \cdot 1 + \frac{d^2 - 4d}{32} \cdot (-1) \cdot (-1) + \frac{d^2}{16} \cdot (-1) \cdot 1 = -\frac{d}{4}. \quad (\text{B.7})$$

The last case holds when two imaginary elements from the diagonal axis are multiplied. Since the sum of the 2^{m-1} imaginary elements gives $-i2^{m/2}$ (cf. Equation (4.84)), there occur $2^{m-2} - 2^{m/2-1}$ elements on the diagonal which are given by $+i$ and $2^{m-2} + 2^{m/2-1}$ elements on the diagonal which are given by $-i$. In analogy, we can calculate the sum of the pairwise products of all imaginary elements on the diagonal as

$$\frac{(c^-)^2 - c^-}{2} \cdot i^2 + \frac{(c^+)^2 - c^+}{2} \cdot (-i)^2 + (c^- \cdot c^+) \cdot i \cdot (-i) = -\frac{d}{4}, \quad (\text{B.8})$$

with abbreviations $c^- := (2^{m-2} - 2^{m/2-1})$ and $c^+ := (2^{m-2} + 2^{m/2-1})$.

In order to calculate the contributions of the anti-diagonal term of Equation (B.6), some preliminary considerations are needed. Both factors of the term, namely $(V_{2^k})_{i,j}$ and $(V_{2^k})_{j,i}$ have a factor of $(-1)^{i \cdot j}$ or $(-1)^{j \cdot i}$, respectively, according to Equation (4.79). Since their product is zero, also the product of $(-1)^{i \cdot 0}$ and $(-1)^{j \cdot 0}$ is zero. Thus, we have

$$-(V_{2^k})_{i,j} \cdot (V_{2^k})_{j,i} = -(V_{2^k})_{i,0} \cdot (V_{2^k})_{j,0}. \quad (\text{B.9})$$

By the recursion relation given in Equation (4.73) it is clear, that the set of elements of the first column of V_{2^k} equals the set of elements of $V_{2^{k-1}}$. By construction, for any V_{2^k} , every second column has imaginary elements and in each column, except for the first, the number of positive and negative elements is equal.¹

The product of two elements from the first column, with one real and one imaginary element then vanishes, as the sum of imaginary elements in the first column is zero.

If we take two imaginary elements from the first column of V_{2^k} , we have an equivalent calculation as for the real elements on the diagonal to sum up the products of all pairs. With the same number of positive and negative imaginary elements the sum gives, involving the minus sign of Equation (B.9) and analogously to Equation (B.7),

$$-\left(\frac{d^2 - 4d}{32} \cdot i \cdot i + \frac{d^2 - 4d}{32} \cdot (-i) \cdot (-i) + \frac{d^2}{16} \cdot (-i) \cdot i \right) = -\frac{d}{4}. \quad (\text{B.10})$$

For the negative sum of the products of two real elements of the first column of V_{2^k} we remember, that the set of elements of this column equals the set of all elements of $V_{2^{k-1}}$. This matrix is given by the multiplication of a phase vector with Sylvester's Hadamard matrix. The rows with even index (starting with 0), all have only real entries. Therefore, the number of $+1$ and -1 is equal in all columns, but not in the first, where only $+1$ appears. Thus, there are $2^{m-2} + 2^{m/2-1}$ entries with value $+1$ and $2^{m-2} - 2^{m/2-1}$ entries with value -1 . The sum of their products can be calculated analogously to Equation (B.8) as

$$-\left(\frac{(c^-)^2 - c^-}{2} \cdot 1^2 + \frac{(c^+)^2 - c^+}{2} (-1)^2 + (c^- \cdot c^+) \cdot 1 \cdot (-1) \right) = -\frac{d}{4}. \quad (\text{B.11})$$

¹The Hadamard matrix \bar{H} has an equal number of $+1$ and -1 in each column, except for the first, where only $+1$ appears (cf. Appendix A.3).

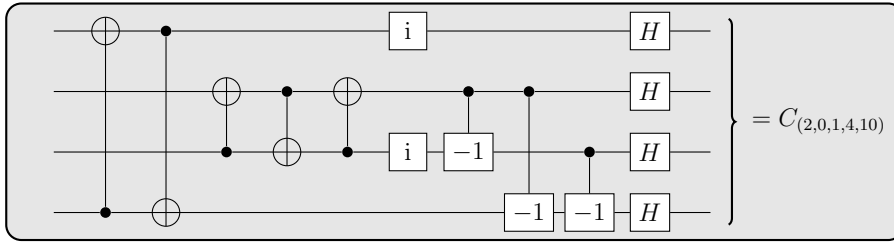
Summing up four times $-d/4$ in the above discussion leads to

$$\mathrm{tr}_2(V_{2^k}) = -d = -2^m, \quad (\text{B.12})$$

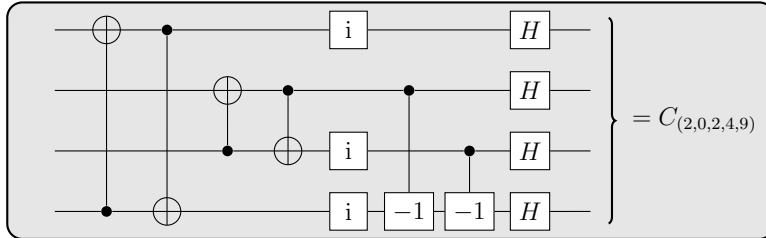
which equals $\mathrm{tr}(V_{2^k})^2$ and is the expected result (cf. Equation (4.84)).

Continuing this process, maybe by using a complete induction which shows that $\mathrm{tr}_{l+1}(V_{2^k})$ is given by $\mathrm{tr}_1(V_{2^k}) \cdot \mathrm{tr}_l(V_{2^k})$, could lead to a proof of Wiedemann's conjecture.

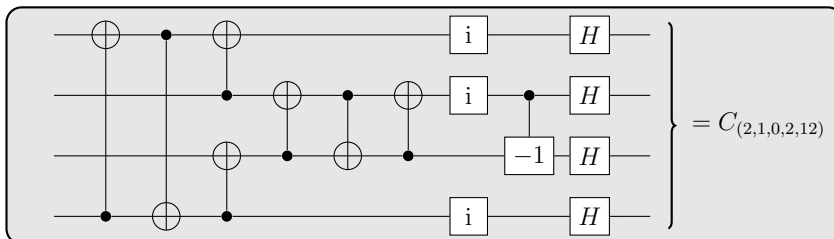
$$B_{(2,0,1,4,10)} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad R_{(2,0,1,4,10)} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (\text{C.2})$$



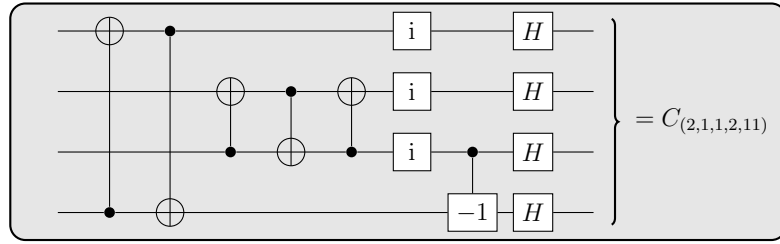
$$B_{(2,0,2,4,9)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad R_{(2,0,2,4,9)} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (\text{C.3})$$



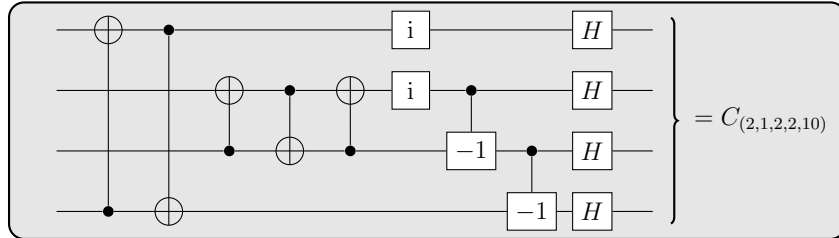
$$B_{(2,1,0,2,12)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad R_{(2,1,0,2,12)} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (\text{C.4})$$



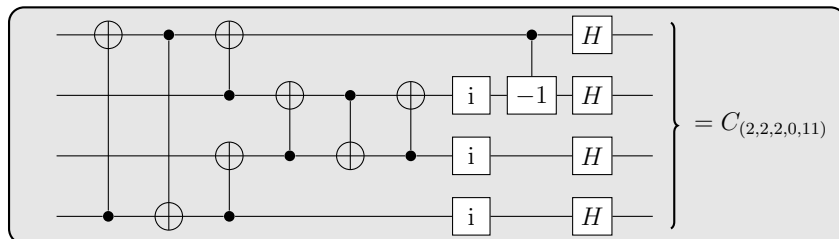
$$B_{(2,1,1,2,11)} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad R_{(2,1,1,2,11)} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (\text{C.5})$$



$$B_{(2,1,2,2,10)} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad R_{(2,1,2,2,10)} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (\text{C.6})$$



$$B_{(2,2,2,0,11)} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad R_{(2,2,2,0,11)} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (\text{C.7})$$



The listed sets are not unique. For each system we found the following number of different solutions (which may give rise to possible symmetries):

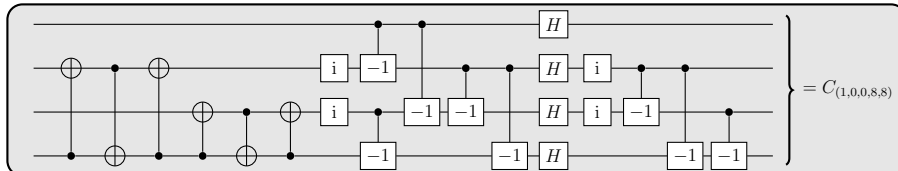
$$\begin{aligned}
\vec{n} = (3, 0, 2, 0, 12) &\Rightarrow 1 \cdot 4 \cdot 15 \cdot 4! \text{ solutions,} \\
\vec{n} = (2, 0, 0, 4, 11) &\Rightarrow 4 \cdot 4 \cdot 15 \cdot 4! \text{ solutions,} \\
\vec{n} = (2, 0, 1, 4, 10) &\Rightarrow 6 \cdot 4 \cdot 15 \cdot 4! \text{ solutions,} \\
\vec{n} = (2, 0, 2, 4, 09) &\Rightarrow 1 \cdot 4 \cdot 15 \cdot 4! \text{ solutions,} \\
\vec{n} = (2, 1, 0, 2, 12) &\Rightarrow 6 \cdot 4 \cdot 15 \cdot 4! \text{ solutions,} \\
\vec{n} = (2, 1, 1, 2, 11) &\Rightarrow 6 \cdot 4 \cdot 15 \cdot 4! \text{ solutions,} \\
\vec{n} = (2, 1, 2, 2, 10) &\Rightarrow 2 \cdot 4 \cdot 15 \cdot 4! \text{ solutions,} \\
\vec{n} = (2, 2, 2, 0, 11) &\Rightarrow 2 \cdot 4 \cdot 15 \cdot 4! \text{ solutions.}
\end{aligned}$$

As discussed in Section 4.5, the factor of $15 = 2^4 - 1$ can be explained by a freedom in the choice of R (cf. Equation (4.39)). It is not clear, if this is uncorrelated to all permutations of the indexing of the four qubits, which would explain the occurrence of the factor $4!$.

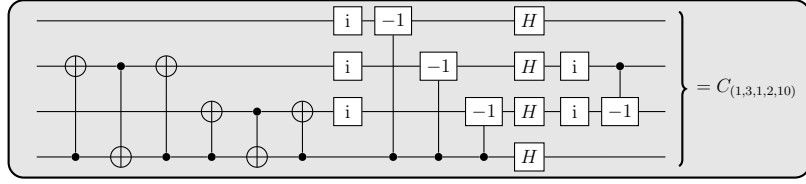
C.1.2 Homogeneous sets with semigroup structure

Homogeneous sets with an additive semigroup structure in the G_j^z components of the class generators for $j \in \{1, \dots, d\}$ were introduced in Section 4.5.2. Here we give an incomplete list of stabilizer matrices for the existing systems, having the vector \vec{n} as their index which indicates the entanglement properties of the bases of the specific set, as defined in Section 4.4 for the dimension $d = 2^4$. A representation of the solely existing stabilizer matrix that creates the semigroup structure for $d = 2^3$ was given already in Section 4.5.2. For each stabilizer matrix, the corresponding quantum circuit is derived by the methods of Section 4.8.2 and shown beneath.

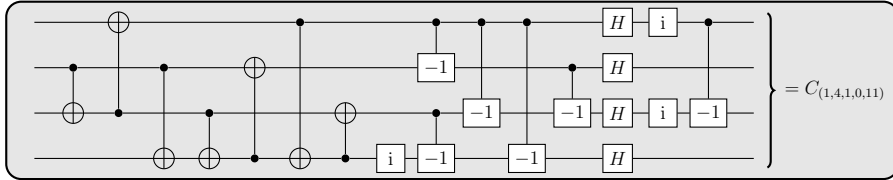
$$C_{(1,0,0,8,8)} = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}. \quad (\text{C.8})$$



$$C_{(1,3,1,2,10)} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}. \quad (\text{C.15})$$



$$C_{(1,4,1,0,11)} = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}. \quad (\text{C.16})$$



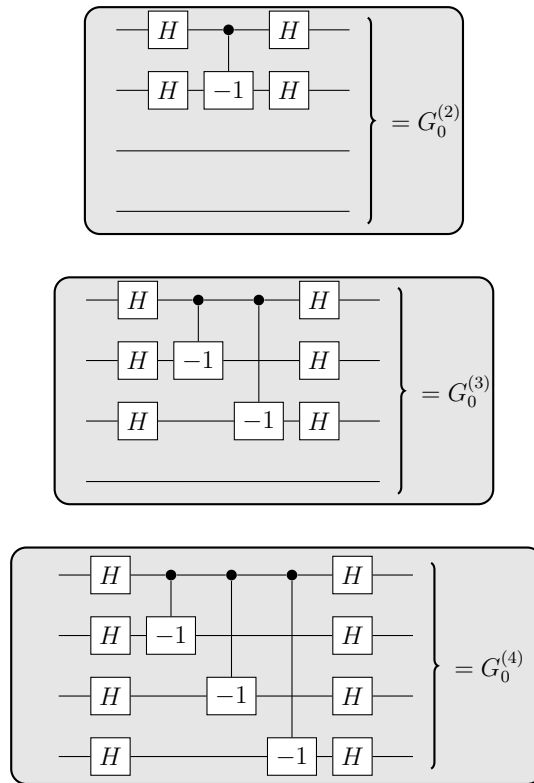
Again, the listed sets are not unique as in Section C.1.1. But as there is not a reduced form of the stabilizer matrix C similar to Equation (4.34), it is useless to discuss the number of solutions for each set.

C.1.3 Inhomogeneous sets

Inhomogeneous sets were introduced in Section 4.6. Here we list stabilizer matrices and the corresponding generators G_0 of the first class of operators \mathcal{C}_0 for selected sets of MUBs with specific entanglement properties of their bases which are indicated by the vector \vec{n} as their index. By construction, the generators G_0 have a specific form, thus, in order to keep this section short, we abbreviate these generators as

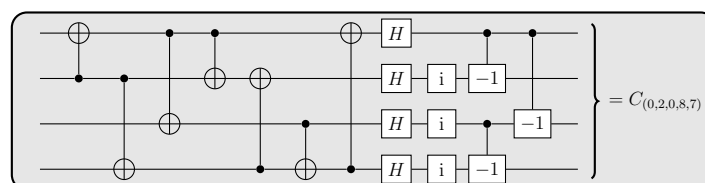
$$G_0^{(i)} = \left(0_m + \sum_{k=2}^i \mathbb{1}_m (|1\rangle\langle k| + |k\rangle\langle 1|) \right), \quad (\text{C.17})$$

with $i \in \{2, \dots, m\}$. Their implementations as a quantum circuit are shown in the following.

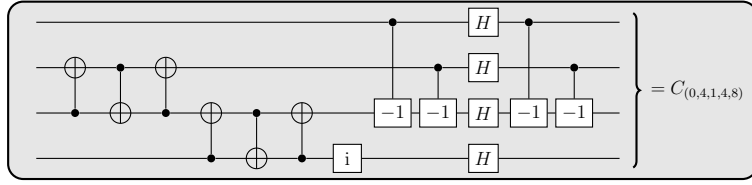


The stabilizer matrices and generators found for four-qubit systems, as well as the corresponding quantum circuits (cf. Section 4.8.3) are given as follows:

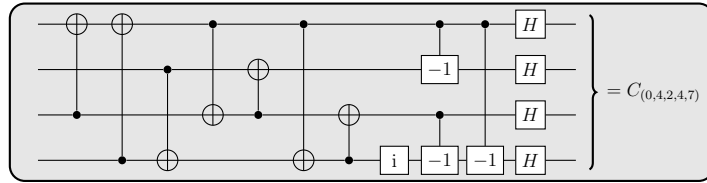
$$C_{(0,2,0,8,7)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad G_0^{(3)}. \quad (\text{C.18})$$



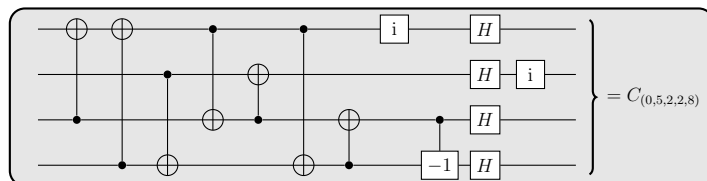
$$C_{(0,4,1,4,8)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad G_0^{(3)}. \quad (\text{C.22})$$



$$C_{(0,4,2,4,7)} = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad G_0^{(2)}. \quad (\text{C.23})$$



$$C_{(0,5,2,2,8)} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad G_0^{(2)}. \quad (\text{C.24})$$



C.1.4 Testing Wiedemann's conjecture

Wiedemann tested his conjecture with dimensions $d = 2^{2^k}$ for $k \in \{0, \dots, 8\}$ in [Wie88]. As claimed in Section 4.2, we are able to raise this test up to $k = 11$. Since the testing procedure is too complicated to be done by hand, we offer the used *Matlab* code to provide maximal insight into the calculation. It should be mentioned that the method *divisor_list* is generated by a *Mathematica* code, since *Matlab* cannot handle very long integers that easily. This code takes the factors of a Fermat number, calculates all divisors, writes them in a binary representation and reverses that string to have the least significant bit on the left hand side. Finally, these divisors are provided within a list that can be used for the *Matlab* code. The following code uses the parameter *ksize* that equals the variable k .

```

function [ ] = TestWiedemann( ksize )

% testing generator for 2^ksize qubits
size=2^ksize;

% initialize matrices
Zero=zeros( [ size , size ] );
C=zeros( [ 2*size , 2*size ] );

% create unity matrices One=1_{size} and One2=1_{2*size}
One = eye( size );
One2 = eye( 2*size );

% create matrix C
maxrow=0;
row=1;
for i=1:2*size
    C( i , row)=1;
    C( row , i)=1;
    row=row+1;
    if ( row>maxrow )
        row=1;
        if ( maxrow==0 )
            maxrow=1;
        else
            maxrow=maxrow*2;
        end
    end
end

% test if C^{2^{m+1}}=One2
CC=C;
for l=1:size
    CC=mod(CC*CC, 2);
end

```

```

% if yes...
if ( mod( CC*C,2 ) == One2)
    % load divisors of 2m+1 in binary
    % representation whereas the
    % least significant bit is on the left
    divisor_list=divisors_bin(size);

    wrong=0;

    for i=1:length(divs)

        % initialize testmatrix
        test=One2;

        % load factor
        factor=divisor_list{i};

        % calculate the power of C given by the loaded
        % factor and write the result to the testmatrix
        CC=C;
        for j=1:length(factor)
            if (factor(j)=='1')
                test=mod(test*CC,2);
            end
            CC=mod(CC*CC,2);
        end

        % check if off-diagonal blocks equal zero
        if (test(1:size,size+1:2*size) == Zero)
            if (test(size+1:2*size,1:size) == Zero)
                % if yes, Wiedemanns conjecture would be wrong
                wrong=1;
                break
            end
        end
    end

    % if Wiedemanns conjecture has never been tested wrong, it
    % fits for the tested dimension
    if (wrong == 0)
        fprintf( 'Wiedemanns_conjecture_tested_correct ');
        fprintf( ' for 2^%d_qubits!\n\n', ksize);
    else
        fprintf( 'Wiedemanns_conjecture_is_wrong!\n\n');
    end
end
end

```

C.1.5 Triangle solutions

For the method we introduced in Chapter 4.1.1 we are able to find reduced stabilizer matrices B in the form of Equation (4.13). Here we list solutions for the submatrix A for dimensions $d = 2^m$ with $m = \{2, \dots, 600\}$.¹ The runtime of that algorithm is much shorter than expected in Section 4.1.1, since the number of divisors of an integer $2^m + 1$ with $m \in \mathbb{N}^*$ seems in most times to be far away from the supposed limit of $\sqrt{2^m + 1}$, which is the leading factor in Equation (4.11) for large values of m . Table C.1 gives solutions for the matrices A , where the number in the leftmost column indicates the number of qubits m for the solution of A in the subsequent column. Moving a column to the right increments the number of qubits by one.

2	0	0	1	0	0	1
8	$\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}$	0	$\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}$	0	$\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix}$	1
14	0	$\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}$	1	1	0	1
20	$\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}$	0	$\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix}$
26	0	$\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{smallmatrix}$	0	0	$\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{smallmatrix}$
32	$\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}$	$\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{smallmatrix}$	0	$\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}$
38	$\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}$	$\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{smallmatrix}$	$\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}$	0	$\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{smallmatrix}$
44	$\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}$	$\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}$	1	$\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{smallmatrix}$	1
50	0	0	$\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix}$	$\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}$
56	$\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{smallmatrix}$	$\begin{smallmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}$
62	$\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}$	$\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{smallmatrix}$	$\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{smallmatrix}$	$\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{smallmatrix}$
68	$\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{smallmatrix}$
74	$\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix}$	1	$\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}$	$\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}$	1
80	$\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{smallmatrix}$	$\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}$	$\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{smallmatrix}$
86	0	$\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{smallmatrix}$	0	0	$\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{smallmatrix}$
92	$\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{smallmatrix}$

¹The used algorithm keeps the matrix A as small as possible and is able to generate solutions for even higher dimensions.

98	0001 0011 0101 1111	0010 0011 1110 0101	0010 0100 1010 0000	1	0001 0010 0111 1010	0001 0001 0010 1100
104	0001 0110 0111 1010	110 111 010	0001 0111 0111 1111	0001 0111 0111 1110	001 010 100	1
110	0001 0010 0101 1010	0011 0111 1111 1111	101 000 101	0	011 111 111	001 010 101
116	100 010 001	0001 0110 0100 1001	0010 0011 1101 0111	0011 0110 1110 1001	100 010 001	001 000 100
122	0001 0010 0111 1010	0001 0101 0001 1111	011 101 111	1001 0000 0000 1001	101 000 101	0001 0111 0101 1111
128	010 110 001	0010 0011 1110 0101	110 100 000	0	0001 0010 0101 1011	101 001 110
134	0	1011 0110 1101 1011	1100 1111 0111 0111	0001 0101 0000 1100	0010 0100 1000 0001	110 101 010
140	0001 0100 0000 1001	0001 0110 0100 1001	0100 1100 0001 0011	01 11	100 000 001	0001 0110 0101 1010
146	0001 0010 0100 1000	0011 0101 1010 1101	011 101 111	0111 1111 1100 1100	100 001 010	001 001 111
152	0010 0110 1110 0000	0001 0000 0001 1010	0001 0011 0110 1100	0	100 000 001	0111 1111 1100 1100
158	0	001 010 100	0001 0000 0000 1001	0010 0001 1001 0111	0001 0110 0101 1010	0001 0001 0010 1100
164	0001 0011 0101 1111	0001 0001 0001 1110	0001 0010 0101 1011	0001 0110 0100 1001	0010 0000 1010 0000	0001 0001 0000 1100
170	0001 0100 0000 1001	0001 0010 0111 1010	0001 0011 0110 1100	0	0	0100 1010 0101 0011
176	01 11	0001 0010 0101 1011	0001 0010 0010 1100	0001 0100 0000 1001	001 001 111	0111 1101 1010 1100
182	0100 1000 0001 0011	1001 0100 0001 1011	0100 1011 0101 0110	01 11	0	0001 0000 0010 1001
188	0011 0111 1111 1111	0001 0110 0100 1001	10 01	0010 0111 1101 0110	0010 0000 1010 0000	0010 0001 1001 0111
194	0	0011 0111 1110 1100	0001 0111 0110 1101	1010 0111 1110 0100	0001 0111 0100 1101	1001 0100 0010 1001
200	0001 0001 0001 1110	0001 0010 0111 1010	10 00	0010 0010 1111 0011	0001 0010 0111 1010	0001 0010 0111 1011
206	1000 0000 0001 0011	0011 0110 1111 1011	011 101 111	0	0	0011 0010 1111 1010

212	0 0 0 1 0 1 0 0 0 0 0 0 1 0 0 1	0 0 0 1 0 0 0 1 0 0 1 0 1 1 0 1	0 1 0 0 1 1 0 1 0 0 1 0 0 1 0 1	0 1 1 1 1 0 0 1 1 0 1 1 1 1 1 1	0 0 1 0 1 0 1 0 1	1 1 0 1 1 1 0 1 0
218	0 0 0 1 0 0 0 1 0 0 0 1 1 1 1 0	0 0 0 1 0 0 1 0 0 1 1 1 1 0 1 0	0 1 0 0 1 0 1 0 0 1 0 1 0 0 1 1	0	1 0 0 0 0 1 0 1 0	0 1 0 0 1 1 1 0 0 1 1 0 0 0 0 1
224	0 0 0 1 0 1 1 1 0 1 1 1 1 1 1 0	0 1 0 0 1 1 1 1 0 1 0 1 0 1 1 0	1 1 0 1 0 0 0 0 0	0 0 0 1 0 1 0 0 0 0 0 0 1 0 0 1	1 0 1 0 0 0 1 0 1 1 1 1 0 0 1 1	0 0 1 0 1 0 1 0 0
230	0 0 0 1 0 1 0 1 0 0 0 0 1 1 0 0	0	1 0 0 1	0 0 0 0 1 0 0 0 0 0 0 0 0 1 0 0 0 1 0 1 1 0 0 1 1	1 1 1 0 1 1 1 1 1 1 1 1 0 1 1 0	0 0 1 1 0 1 0 1 1 0 1 1 1 1 1 1
236	0 0 0 1 0 1 1 1 0 1 0 1 1 1 1 0	1 0 1 0 1 0 1 0 1	0 0 1 1 0 0 0 1 1 0 0 1 1 1 1 1	1 0 0 0 0 1 0 1 0 0 0 1 0 1 1 0	0 1 1 1 0 0 1 0 1	0 1 0 0 1 1 0 0 0 0 1 0 0 0 0 0
242	0 0 0 1 0 0 1 1 0 1 0 1 1 1 1 1	0	0 0 1 0 1 0 1 0 0	0 1 0 0 1 1 1 0 0 1 1 0 0 0 0 1	0 0 0 1 0 1 1 0 0 1 1 1 1 0 1 0	0 0 1 0 0 0 0 1 1 0 1 0 0 1 0 1
248	0 0 0 1 0 1 1 1 0 1 0 1 1 1 1 0	0 0 1 1 0 1 0 1 1 0 1 1 1 1 1 0	0 0 0 1 0 0 0 0 0 0 0 0 1 0 0 1	0	0 0 1 1 0 0 1 0 1 1 1 1 1 0 1 0	0 0 0 1 0 1 0 0 0 0 0 1 1 0 1 0
254	0 0 0 1 0 0 1 0 0 1 0 0 1 0 0 0	0 0 0 1 0 1 1 0 0 1 1 1 1 0 1 0	1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0	0 1 1 1 1 1 0 1 1 0 1 1 1 1 1 1	0 0 1 1 0 0 0 1 1 0 1 0 1 1 0 0	0 1 1 0 1 1 0 0 1 0 0 1 0 0 1 0
260	1 0 0 0 0 0 1 1 0 1 1 1 0 1 1 1	0	0 0 1 0 0 0 1 0 0	1 0 0 1 0 1 1 0 0 1 0 1 1 0 1 0	1 0 0 1 0 1 1 0 0 1 0 1 1 0 1 0	1 0 1 1 0 0 1 1 1 1 1 0 1 1 0 1
266	0 1 1 0	0 0 0 1 0 0 0 1 0 0 1 0 1 1 0 1	0 0 0 1 0 1 1 1 0 1 1 1 1 1 1 1	0 0 1 1 0 0 0 1 1 0 1 0 1 1 0 1	0	0 0 1 0 0 1 0 1 1 0 1 0 0 1 0 1
272	0 0 0 1 0 1 0 0 0 0 0 0 1 0 0 1	0 0 1 0 0 1 0 0 1 0 0 0 0 0 0 0	0 0 1 1 0 0 1 1 1 1 1 1 1 1 1 1	1 1 1 0 1 1 0 1 1 0 0 0 0 1 0 1	0 1 0 0 1 1 1 1 0 1 1 1 0 1 1 0	1 1 0 1 1 1 0 1 0
278	0	1 0 0 0 0 1 0 1 0	1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0	0	0 1 0 0 1 1 1 1 0 1 1 1 0 1 1 1	1 0 1 0 0 1 1 1 1
284	0 1 1 1 0 1 1 1 1	1 1 0 0 1 0 1 1 0 1 0 1 0 1 1 0	0 1 0 0 1 0 1 0 0 1 1 1 0 0 1 0	0 0 1 0 0 0 0 0 1 0 0 0 0 0 0 1	1 0 0 0 0 0 0 0 1	1 0 0 1 0 0 0 0 0 0 1 0 1 0 0 1
290	0 0 0 1 0 0 0 1 0 0 0 0 1 1 0 0	1 0 0 1 0 0 0 0 0 0 0 1 1 0 1 0	1 0 0 1	0	0 0 0 1 0 0 1 1 0 1 0 0 1 1 0 0	0 0 0 0 1 0 0 0 1 1 0 0 0 0 0 0 1 0 1 0 1 1 0 0 0
296	0 0 1 1 0 0 0 0 1 0 1 0 1 0 0 0	0 1 0 1 0 0 0 0 1	0 0 0 0 1 0 0 0 1 1 0 0 0 0 0 0 1 0 1 0 1 1 0 0 1	0 0 0 0 1 0 0 1 0 1 0 1 0 1 1 0 0 1 1 0 1 1 1 0 0	0 0 0 1 0 1 0 0 0 0 1 0 1 0 0 0	0 0 0 1 0 0 1 0 0 1 1 0 1 0 0 0
302	0 1 0 0 1 1 1 1 0 1 1 1 0 1 1 1	0 1 1 0 1 1 0 0 1 0 1 0 0 0 0 1	0 0 1 1 0 0 0 1 1 0 1 0 1 1 0 1	0 0 0 1 0 1 1 0 0 1 0 0 1 0 0 1	0 1 0 1 1 0 1 1 0 1 0 0 1 1 0 1	0 0 1 1 0 0 0 1 1 0 1 0 1 1 0 0
308	0 0 1 0 0 0 0 0 1 0 1 0 0 0 0 0	0	0 0 0 1 0 1 0 0 0 0 0 0 1 0 0 1	0 0 1 1 0 1 0 1 1 0 1 1 1 1 1 1	1 0 0 1 0 1 1 0 0 1 0 0 1 0 0 0	0 0 1 0 0 1 0 1 1 0 1 1 0 1 1 1
314	0 0 0 0 1 0 0 0 0 1 0 0 1 1 0 0 0 1 0 0 1 1 0 0 1	0 0 1 0 0 1 0 1 1 0 1 1 0 1 1 0	1 1 0 1 1 1 0 1 0	0 1 1 1 1 1 1 1 1	0 1 0 1 1 0 0 1 0 0 1 0 1 1 0 0	0 0 1 1 0 1 0 0 1 0 1 0 1 0 0 0

320	0011 0000 1011 1010	0010 0010 1110 0001	1000 0001 0001 0111	0	00001 00011 00011 01111 11111	0010 0011 1110 0101
326	0	0011 0000 1010 1000	0010 0101 1010 0101	0	100 010 001	00001 00011 00101 01011 11110
332	111 110 100	1001 0011 0100 1100	10 01	101 010 100	1011 0010 1111 1011	0100 1100 0010 0000
338	0	0110 1111 1110 0100	00001 00000 00000 00000 10000	0010 0100 1010 0001	0011 0000 1001 1011	1
344	1001 0110 0101 1010	0111 1001 1011 1111	001 010 100	0001 0101 0000 1100	0001 0000 0001 1010	0010 0100 1000 0000
350	0	001 010 101	0001 0100 0001 1010	1	0	1100 1100 0001 0010
356	01 10	1011 0001 1001 1110	00001 00000 00011 00110 10101	0	0001 0000 0010 1001	0001 0010 0110 1000
362	00001 00100 01101 00001 10110	0010 0100 1000 0000	101 001 110	1001 0100 0010 1001	001 010 100	0110 1110 1110 0001
368	0110 1101 1011 0111	00001 00001 00101 00001 11111	0110 1100 1001 0010	00001 00001 00111 00110 11100	0010 0110 1101 0010	0011 0100 1010 1000
374	1000 0100 0010 0001	0	00001 00000 00001 00001 10110	0001 0100 0000 1001	1011 0101 1011 1110	0111 1101 1011 1111
380	1111 1000 1001 1010	10 00	001 000 100	1001 0100 0010 1000	0010 0011 1110 0101	00001 00011 00000 01010 11000
386	0010 0010 1110 0001	0001 0000 0010 1001	0100 1100 0010 0001	010 110 001	1011 0101 1000 1101	0010 0101 1010 0101
392	0001 0101 0000 1100	1010 0101 1010 0100	0011 0101 1010 1101	0011 0011 1110 1100	1010 0011 1111 0111	111 110 100
398	0	0010 0100 1000 0001	10 01	0011 0001 1011 1111	0011 0100 1001 1010	1
404	0011 0000 1010 1000	1001 0010 0101 1010	0001 0111 0111 1111	00001 00101 01111 00100 11100	0010 0101 1010 0101	0110 1110 1110 0001
410	0	0	010 100 001	0110 1101 1011 0111	0	001 001 111
416	1010 0100 1010 0000	1001 0111 0100 1101	1000 0100 0000 0001	0011 0111 1110 1100	0001 0000 0010 1001	1010 0011 1100 0101

422	0 0 1 0 0 1 0 1 1 0 0 1 0 1 1 0	1 1 1 1 1 0 1 1 1 1 0 0 1 1 0 1	0 1 1 1 0 1 1 1 1	0 0 0 1 0 0 0 0 0 1 0 0 1 1 0 1 0 1 1 0 0 1 0 0 1	0	0 0 0 1 0 0 0 1 0 1 0 1 1 0 0 1 0 0 1 0 0 1 0 0 1
428	0 1 1 1 1 1 1 0 1 1 0 0 1 0 0 0	0 1 1 0 1 1 1 1 1 1 1 0 0 1 0 0	0 0 1 0 0 0 0 1 1 0 0 0 0 1 0 0	0	0 0 1 0 0 1 0 0 1 0 1 0 0 0 0 1	0 0 1 1 0 0 0 0 1 0 1 1 1 0 1 1
434	0 0 0 0 1 0 0 0 0 1 0 0 0 1 1 0 0 1 0 0 1 1 1 0 0	1 0 1 1 0 1 0 1 1 0 0 0 1 1 0 1	1 0 0 0 0 1 0 1 0 0 1 0 0 1 0 0	1 0 0 0 0 1 0 1 0 0 1 0 0 1 0 1	0	0 0 0 0 1 0 1 0 1 1 0 0 1 0 0 0 1 0 1 0 1 1 0 0 0
440	0 1 1 0 1 1 1 0 1 1 1 0 0 0 0 1	0	0 1 1 1 1 1 0 0 1 0 1 0 1 0 0 1	0	0 0 1 1 0 0 0 1 1 0 1 0 1 1 0 0	0 0 1 0 0 0 0 0 1 0 1 1 0 0 1 0
446	0 0 0 1 0 0 0 1 0 0 0 1 1 1 1 1	1 0 0 1 0 1 0 1 0 0 0 1 1 1 1 1	0 0 0 0 1 0 0 0 0 1 0 0 0 1 0 0 0 1 0 0 1 1 0 0 0	0 0 1 0 0 0 0 1 1 0 0 1 0 1 1 1	0 0 1 0 0 1 0 0 1 0 0 0 0 0 0 1	1 0 0 1 0 0 0 0 0 0 0 0 1 0 0 0
452	0 0 1 1 0 0 1 1 1 1 1 1 1 1 1 1	0	0 0 1 0 0 0 0 0 1 0 0 1 0 0 1 1	1 0 1 0 1 0 1 0 0	0 1 1 1 1 1 0 0 1 0 1 0 1 0 0 1	1
458	0 0 0 1 0 1 0 0 0 0 0 1 1 0 1 0	0 0 0 1 0 0 1 1 0 1 0 1 1 1 1 1	0 0 1 0 0 1 0 1 1 0 0 0 0 1 0 0	1 0 0 1 0 1 1 1 0 1 1 0 1 1 0 1	0 0 0 1 0 0 0 0 0 0 0 1 1 0 1 0	0 0 1 0 0 0 0 1 1 0 0 0 0 1 0 0
464	0 1 1 1 0 1 1 1 1	0 0 0 0 1 0 0 1 1 1 0 1 0 1 1 0 1 1 0 0 1 1 1 0 1	1 1 0 1 1 1 0 1 0	0 0 1 0 0 1 0 1 1 0 0 1 0 1 1 0	0 0 0 1 0 0 1 1 0 1 0 0 1 1 0 0	0 0 1 0 0 1 0 0 1 0 1 0 0 0 0 0
470	0 0 0 0 1 0 1 0 1 0 0 0 0 1 1 0 1 1 0 1 1 0 1 1 1	0 0 0 0 1 0 0 1 0 1 0 1 0 0 0 0 0 0 1 0 1 1 0 0 0	1 1 0 0 1 1 0 0 0 0 0 0 0 0 0 1	0	0 0 0 1 0 0 0 0 0 0 0 0 1 1 0 1 0 1 1 1 0 0 0 1 0	0 0 0 0 1 0 1 0 1 1 0 0 0 1 0 0 1 1 1 0 1 1 0 0 1
476	1 0 1 0 1 0 1 0 0	0 0 1 1 0 1 0 0 1 0 1 1 1 0 1 0	1 1 1 1 1 0 1 0 0	0 0 1 1 0 1 0 1 1 0 1 1 1 1 1 1	1 0 1 0 0 0 0 0 1 0 1 1 0 0 1 1	1 0 1 0 0 0 0 0 1 0 0 1 0 0 1 0
482	0 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1	1 1 1 1 1 0 0 0 1 0 0 1 1 0 1 1	0 1 1 0 1 0 0 0 1 0 1 0 0 0 0 1	0 0 0 0 1 0 1 0 1 1 0 0 1 0 0 0 1 0 1 0 1 1 0 0 0	0 1 1 0 1 0 1 0 1 1 1 0 0 0 0 1	0 1 0 1 1 0 0 0 1
488	0 0 1 0 0 0 1 0 1 1 1 0 0 0 0 1	0 0 1 0 0 0 1 0 1 1 0 1 0 0 1 1	1 0 0 0 0 1 1 0 0 1 1 1 0 0 1 0	0	0 0 1 1 0 1 0 0 1 0 0 1 1 0 1 0	0 0 0 0 1 0 0 0 0 1 0 0 1 1 0 0 0 1 1 1 1 1 0 1 1
494	1 0 0 1 0 1 0 1 0 0 1 0 1 1 0 0	0	0 0 0 1 0 0 0 1 0 0 1 0 1 1 0 0	0 0 0 0 1 0 0 0 1 0 0 0 1 1 0 0 1 1 0 1 1 0 0 1 1	1 0 1 0 0 0 1 0 1 1 1 0 0 0 0 1	0 0 1 1 0 1 0 1 1 0 1 1 1 1 1 1
500	0 0 1 1 0 0 1 1 1 1 1 0 1 1 0 0	0 0 0 1 0 1 0 1 0 0 0 0 1 1 0 0	0 0 0 0 1 0 0 0 0 0 0 0 0 1 1 0 0 1 0 0 1 0 1 0 0	1 0 0 0 0 1 1 1 0 1 0 0 0 1 0 0	1 0 0 0 0 1 0 1 0	1 0 1 0 0 1 1 1 1
506	0 0 0 1 0 0 0 0 1 0 0 0 1 0 0 1 1 0 1 1 0 0 0 1 0	1 1 1 1 1 0 0 0 1 0 1 0 1 0 0 1	0 0 0 0 1 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 1 0 0 0 0	1 0 1 1 0 1 0 1 1 0 0 1 1 1 1 1	0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 1	0 0 1 0 0 0 1 0 1 1 1 1 0 0 1 1
512	1 0 0 0 0 1 0 1 0 0 1 0 0 1 0 1	1 0 0 1 0 0 0 0 0 0 0 0 1 0 0 0	0 1 0 0 1 1 0 1 0 0 0 1 0 1 1 0	0	0 1 0 0 1 0 1 0 0 1 1 1 0 0 1 0	0 0 1 1 0 1 0 0 1 0 1 1 1 0 1 0
518	0 0 0 0 1 0 0 0 1 0 0 0 0 1 0 0 1 1 1 0 1 0 0 0 1	0	1 0 0 1	0 0 0 0 1 0 1 0 0 1 0 0 0 0 0 0 0 0 1 0 1 1 0 0 1	0 0 0 0 1 0 1 0 1 0 0 0 0 0 1 0 1 0 0 0 1 0 1 0 0	1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

to scale worse. According to the representation of the solutions, as in Equation (4.15), we write the solutions in Table C.2 as a binary string, starting from left with s_1 and ending with the last entry that equals one.

m	s
2	11
4	1101
5	01101
6	001101
7	1101001
8	11010001
9	011001001
10	0001100001
11	01100100001
12	001101000001
13	1100101000001
14	10100110000001
15	110100010000001
16	1101000100000001
17	10011000100000001
18	000000011000000001
19	1110100001000000001
20	00011000010000000001
21	111101000010000000001
22	0001111000100000000001
23	01100100000100000000001
24	001101000001000000000001
25	1000011000001000000000001
26	01100000100010000000000001
27	110010100000010000000000001
28	1010011000000100000000000001
29	01010001000000100000000000001
30	110110111000001000000000000001
31	0000110100000001000000000000001
32	11010001000000010000000000000001
33	010110001000000010000000000000001
34	0100100011000000100000000000000001
35	10110100100000000100000000000000001
36	100111001000000001000000000000000001

Table C.2: Solutions for the symmetric companion matrix which is discussed in Section 4.1.2 and represents a reduced stabilizer matrix B in order to construct a complete set of Fibonacci-based cyclic MUBs (cf. Section 4.1) for dimensions $d = 2^m$ with $m = 2, \dots, 36$. The solutions are represented as a binary string, starting from left with s_1 and ending with the last entry that equals one, corresponding to Equation (4.15).

C.2 Fractal patterns

In Section 3.2, properties of the Fibonacci polynomials over \mathbb{F}_2 are discussed. The structure of these polynomials is shown in Table C.3 and related to the Sierpinski triangle. Also the characteristic polynomials of matrices over \mathbb{F}_2 , that have only ones in the upper left half and zeros otherwise (thus, setting A to zero in Equation (4.13)), show an equivalent pattern (see Table C.4). Removing from Table C.3 every second line and every second column leads to (an obviously smaller version of) Table C.4. Both are related to Pascal's triangle over \mathbb{F}_2 (see Table C.5). To have a good impression, we encode ones by “#” and zeros by a space.



Table C.3: Fibonacci polynomials over \mathbb{F}_2 , where “#” encodes a coefficient that equals one in the polynomial, and a space encodes a zero. The rightmost coefficient is the highest, the rows indicate the index j of the Fibonacci polynomial $F_j(x)$, starting with 1 and ending with 70.



Bibliography

- [All80] W. Alltop, *Complex sequences with low periodic correlations*, Information Theory, IEEE Transactions on **26** (1980), no. 3, 350–354.
- [App05] D. M. Appleby, *SIC-POVMs and the Extended Clifford Group*, Journal of Mathematical Physics **46** (2005), 052107, arXiv:0412001 [quant-ph].
- [App09] D. M. Appleby, *Properties of the extended Clifford group with applications to SIC-POVMs and MUBs*, preprint (2009), arXiv:09095233 [quant-ph].
- [BB84] C. H. Bennett and G. Brassard, *Quantum Cryptography: Public Key Distribution and Coin Tossing*, in Proceedings of the IEEE International Conference on Computers, Systems & Signal Processing, Bangalore, India, 1984, pp. 175–179.
- [BBE⁺07] I. Bengtsson, W. Bruzda, Å. Ericsson, J.-Å. Larsson, W. Tadej, and K. Życzkowski, *Mutually unbiased bases and Hadamard matrices of order six*, Journal of Mathematical Physics **48** (2007), 052106.
- [BBMT02] E. Bagan, M. Baig, and R. Muñoz-Tapia, *Optimal Scheme for Estimating a Pure Qubit State via Local Measurements*, Physical Review Letters **89** (2002), 277904, arXiv:0205026v2 [quant-ph].
- [BBRV02] S. Bandyopadhyay, P. O. Boykin, V. Roychowdhury, and F. Vatan, *A New Proof for the Existence of Mutually Unbiased Bases*, Algorithmica **45** (2002), no. 4, 512–528, arXiv:0103162 [quant-ph].
- [BCWdW01] H. Buhrman, R. Cleve, J. Watrous, and R. de Wolf, *Quantum Fingerprinting*, Physical Review Letters **87** (2001), 167902, arXiv:0102001v1 [quant-ph].

- [BE05] I. Bengtsson and Å. Ericsson, *Mutually Unbiased Bases and the Complementarity Polytope*, Open Systems & Information Dynamics **12** (2005), 107–120, arXiv:0410120 [quant-ph].
- [Bel11] S. M. Bellovin, *Frank Miller: Inventor of the One-Time Pad*, Cryptologia **35** (2011), no. 3, 203–222.
- [Ben92] C. H. Bennett, *Quantum cryptography using any two nonorthogonal states*, Physical Review Letters **68** (1992), 3121–3124.
- [Ben04] I. Bengtsson, *MUBs, Polytopes, and Finite Geometries*, Foundations of Probability and Physics - 3, in AIP Conference Proceedings, vol. 750, AIP, 2004, arXiv:0406174 [quant-ph], pp. 63–69.
- [BEW98] B. C. Berndt, R. J. Evans, and K. S. Williams, *Gauss and Jacobi Sums*, Canadian Mathematical Society Series of Monographs and Advanced Texts, vol. 21, A Wiley-Interscience-Publication, 1998.
- [BF02] K. Boström and T. Felbinger, *Deterministic Secure Direct Communication Using Entanglement*, Physical Review Letters **89** (2002), 187902, arXiv:0209040v2 [quant-ph].
- [Bic70] M. Bicknell, *A Primer for the Fibonacci Numbers part VII: Introduction to Fibonacci Polynomials and their Divisibility Properties*, Fibonacci Quarterly **8** (1970), 407–420.
- [BMP⁺00] P. O. Boykin, T. Mor, M. Pulver, V. Roychowdhury, and F. Vatan, *A new universal and fault-tolerant quantum basis*, Information Processing Letters **75** (2000), no. 3, 101–107.
- [Bos06] S. Bosch, *Algebra*, 6th ed., Springer-Lehrbuch, Springer, 2006.
- [Bru98] Dagmar Bruß, *Optimal Eavesdropping in Quantum Cryptography with Six States*, Physical Review Letters **81** (1998), 3018–3021, arXiv:9805019v2 [quant-ph].
- [BT98] J. V. Brawley and T. C. Teitloff, *Similarity to Symmetric Matrices over Finite Fields*, Finite Fields and Their Applications **4** (1998), 261–274.
- [CCKS97] A. R. Calderbank, P.J. Cameron, W. M. Kantor, and J. J. Seidel, *Z₄-Kerdock Codes, Orthogonal Spreads, and Extremal Euclidean Line-Sets*, Proceedings of the London Mathematical Society **75** (1997), no. 2, 436–480.
- [Cha02] H. F. Chau, *Practical scheme to share a secret key through a quantum channel with a 27.6% bit error rate*, Physical Review A **66** (2002), 060302, arXiv:0205060 [quant-ph].
- [Cha05] H. F. Chau, *Unconditionally secure key distribution in higher dimensions by depolarization*, Transactions on Information Theory **51** (2005), no. 4, 1451–1468, arXiv:0212055v2 [quant-ph].

- [DBE95] D. Deutsch, A. Barenco, and A. Ekert, *Universality in Quantum Computation*, Proceedings of the Royal Society A **449** (1995), no. 1937, 669–677, arXiv:9505018v1 [quant].
- [DBE98] R. Derka, V. Bužek, and A. K. Ekert, *Universal Algorithm for Optimal Estimation of Quantum States from Finite Ensembles via Realizable Generalized Measurement*, Physical Review Letters **80** (1998), 1571–1575, arXiv:9707028v2 [quant-ph].
- [DEBZ10] T. Durt, B.-G. Englert, I. Bengtsson, and K. Życzkowski, *On mutually unbiased bases*, International Journal of Quantum Information **8** (2010), no. 4, 535–640, arXiv:1004.3348v2 [quant-ph].
- [DiV95] D. P. DiVincenzo, *Two-bit gates are universal for quantum computation*, Physical Review A **51** (1995), 1015–1022, arXiv:9407022 [cond-mat].
- [DJ92] D. Deutsch and R. Jozsa, *Rapid Solution of Problems by Quantum Computation*, Proceedings of the Royal Society A **439** (1992), 553–558.
- [Eke91] A. K. Ekert, *Quantum cryptography based on Bell’s theorem*, Physical Review Letters **67** (1991), 661–663.
- [GC01] D. Gottesman and I. Chuang, *Quantum Digital Signatures*, preprint (2001), arXiv:0105032v2 [quant-ph].
- [GHW04] K. S. Gibbons, M. J. Hoffman, and W. K. Wootters, *Discrete phase space based on finite fields*, Physical Review A **70** (2004), 062101, arXiv:0401155 [quant-ph].
- [GKT97] J. Goldwasser, W. Klostermeyer, and G. Trapp, *Characterizing switch-setting problems*, Linear Multilinear Algebra **43** (1997), 121–135.
- [GKW02] J. Goldwasser, W. Klostermeyer, and H. Ware, *Fibonacci Polynomials and Parity Domination in Grid Graphs*, Graphs and Combinatorics **18** (2002), 271–283.
- [GL03] D. Gottesman and H.-K. Lo, *Proof of security of quantum key distribution with two-way classical communications*, IEEE Transactions on information theory **49** (2003), no. 2, 457–475, arXiv:0105121v2 [quant-ph].
- [Got96] D. Gottesman, *Class of quantum error-correcting codes saturating the quantum Hamming bound*, Physical Review A **54** (1996), 1862–1868, arXiv:9604038v2 [quant-ph].
- [Got97] D. Gottesman, *Stabilizer Codes and Quantum Error Correction*, Ph.D. thesis, Caltech, 1997, arXiv:9705052v1 [quant-ph].
- [Got98] D. Gottesman, *Theory of fault-tolerant quantum computation*, Physical Review A **57** (1998), 127–137.

- [Gow07] R. Gow, *Generation of mutually unbiased bases as powers of a unitary matrix in 2-power dimensions*, preprint (2007), arXiv:0703333v2 [math].
- [GR09] C. Godsil and A. Roy, *Equiangular lines, mutually unbiased bases, and spin models*, European Journal of Combinatorics **30** (2009), no. 1, 246–262, arXiv:0511004 [quant-ph].
- [Gra04] M. Grassl, *On SIC-POVMs and MUBs in Dimension 6*, in Proceedings of ERATO Conference on Quantum Information Science, September 2004, arXiv:0406175 [quant-ph], pp. 60–61.
- [GRK10] A. Garcia, J. L. Romero, and A. B. Klimov, *Generation of bases with definite factorization for an n -qubit system and mutually unbiased sets construction*, Journal of Physics A **43** (2010), no. 38, 385301.
- [Gro96] L. K. Grover, *A fast quantum mechanical algorithm for database search*, Proceedings of the twenty-eighth annual ACM symposium on Theory of computing, STOC '96, 1996, arXiv:9605043v3 [quant-ph], pp. 212–219.
- [Hei27] W. Heisenberg, *Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik*, Zeitschrift für Physik **43** (1927), no. 3-4, 172–198.
- [HKK08] M. Hayashi, A. Kawachi, and H. Kobayashi, *Quantum measurements for hidden subgroup problems with optimal sample complexity*, Quantum Information and Computation **1** (2008), no. 8, 345–358, arXiv:0604174v3 [quant-ph].
- [Ina00] H. Inamori, *Security of EPR-based Quantum Key Distribution using three bases*, preprint (2000), arXiv:0008076 [quant-ph].
- [Iva81] I. D. Ivanović, *Geometrical description of quantum state determination*, Journal of Physics A **14** (1981), 3241–3245.
- [Jac96] N. Jacobson, *Basic Algebra I*, 2nd ed., W. H. Freeman, San Francisco, 1996.
- [Kak06] S. Kak, *A Three-Stage Quantum Cryptography Protocol*, Foundations of Physics Letters **19** (2006), 293–296, arXiv:0503027v2 [quant-ph].
- [Kan12] W. M. Kantor, *MUBs inequivalence and affine planes*, Journal of Mathematical Physics **53** (2012), no. 3, 032204, arXiv:1104.3370v3 [quant-ph].
- [KKNY05] A. Kawachi, T. Koshihara, H. Nishimura, and T. Yamakami, *Computational Indistinguishability Between Quantum States and Its Cryptographic Application*, Advances in Cryptology – EUROCRYPT 2005 (Ronald Cramer, ed.), Lecture Notes in Computer Science, vol. 3494, Springer Berlin / Heidelberg, 2005, arXiv:0403069v4 [quant-ph], pp. 268–284.

- [KMFS08] A. B. Klimov, C. Muñoz, A. Fernández, and C. Saavedra, *Optimal quantum-state reconstruction for cold trapped ions*, Physical Review A **77** (2008), 060303, arXiv:0805.3496v1 [quant-ph].
- [KR04] A. Klappenecker and M. Rötteler, *Constructions of Mutually Unbiased Bases*, Finite Fields and Applications (Gary Mullen, Alain Poli, and Henning Stichtenoth, eds.), in Lecture Notes in Computer Science, vol. 2948, Springer Berlin / Heidelberg, 2004, arXiv:0309120v1 [quant-ph], pp. 262–266.
- [Kra87] K. Kraus, *Complementary observables and uncertainty relations*, Physical Review D **35** (1987), no. 10, 3070–3075.
- [KRS10] O. Kern, K. S. Ranade, and U. Seyfarth, *Complete sets of cyclic mutually unbiased bases in even prime-power dimensions*, Journal of Physics A **43** (2010), 275305, arXiv:0912.4661v1 [quant-ph].
- [Law11] J. Lawrence, *Entanglement patterns in mutually unbiased basis sets*, Physical Review A **84** (2011), 022338, arXiv:1103.3818v2 [quant-ph].
- [LBZ02] J. Lawrence, Č. Brukner, and A. Zeilinger, *Mutually unbiased binary observable sets on N qubits*, Physical Review A **65** (2002), 032320.
- [LC99] H.-K. Lo and H. F. Chau, *Unconditional Security of Quantum Key Distribution over Arbitrarily Long Distances*, Science **283** (1999), 2050–2056, arXiv:9803006 [quant-ph].
- [Llo95] Seth Lloyd, *Almost Any Quantum Logic Gate is Universal*, Physical Review Letters **75** (1995), 346–349.
- [LN08] R. Lidl and H. Niederreiter, *Finite Fields (Encyclopedia of Mathematics and its Applications)*, 2nd ed., Translations of Mathematical Monographs, Cambridge University Press, 2008.
- [Lo01] H.-K. Lo, *Proof of unconditional security of six-state quantum key distribution scheme*, Quantum Information and Computation **1** (2001), no. 2, 81–94.
- [LS73] P. W. H. Lemmens and J. J. Seidel, *Equiangular Lines*, Journal of Algebra **24** (1973), 494–512.
- [May96] D. Mayers, *Quantum key distribution and string oblivious transfer in noisy channels*, in Proceedings of Crypto'96: Advances in Cryptology, Springer-Verlag, January 1996, pp. 343–357.
- [Mey90] H. Meyn, *On the construction of irreducible self-reciprocal polynomials over finite fields*, Applicable Algebra in Engineering, Communication and Computing **11** (1990), 43–53.
- [MS96] G. L. Mullen and I. Shparlinski, *Open problems and conjectures in finite fields*, Finite Fields and Applications **233** (1996), 243–268.

- [MU88] H. Maassen and J. B. M. Uffink, *Generalized entropic uncertainty relations*, Physical Review Letters **60** (1988), no. 12, 1103–1106.
- [NC00] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, 1 ed., Cambridge University Press, 2000.
- [NI09] G. M. Nikolopoulos and L. M. Ioannou, *Deterministic quantum-public-key encryption: Forward search attack and randomization*, Physical Review A **79** (2009), 042327, arXiv:4744v1 [quant-ph].
- [Nik08a] G. M. Nikolopoulos, *Applications of single-qubit rotations in quantum public-key cryptography*, Physical Review A **77** (2008), 032348, arXiv:0801.2840v1 [quant-ph]. See also Erratum [Nik08b].
- [Nik08b] G. M. Nikolopoulos, *Erratum: Applications of single-qubit rotations in quantum public-key cryptography*, Physical Review A **78** (2008), 019903.
- [OTU00] T. Okamoto, K. Tanaka, and S. Uchiyama, *Quantum Public-Key Cryptosystems*, Proceedings of the 20th Annual International Cryptology Conference on Advances in Cryptology (London, UK, UK), CRYPTO '00, Springer-Verlag, 2000, pp. 147–165.
- [PDB09] T. Paterek, B. Dakić, and Č. Brukner, *Mutually unbiased bases, orthogonal Latin squares, and hidden-variable models*, Physical Review A **79** (2009), 012109, arXiv:0804.2193 [quant-ph].
- [Pru77] E. Prugovečki, *Information-theoretical aspects of quantum measurement*, International Journal of Theoretical Physics **16** (1977), 321–331.
- [RA06] K. S. Ranade and G. Alber, *Asymptotic correctability of Bell-diagonal quantum states and maximum tolerable bit-error rates*, Journal of Physics A **39** (2006), 1701–1716, arXiv:0510041v1 [quant-ph].
- [RA07] K. S. Ranade and G. Alber, *Asymptotic correctability of Bell-diagonal qudit states and lower bounds on tolerable error probabilities*, Journal of Physics A **40** (2007), 139–153, arXiv:0609196 [quant-ph].
- [RBKSC04] J. M. Renes, R. Blume-Kohout, A. J. Scott, and C. M. Caves, *Symmetric Informationally Complete Quantum Measurements*, Journal of Mathematical Physics **45** (2004), 2171–2180, arXiv:0310075 [quant-ph].
- [RBKSS05] J. L. Romero, G. Björk, A. B. Klimov, and L. L. Sánchez-Soto, *Structure of the sets of mutually unbiased bases for N qubits*, Physical Review A **72** (2005), 062310.
- [RLE11] P. Raynal, X. Lü, and B.-G. Englert, *Mutually unbiased bases in six dimensions: The four most distant bases*, Physical Review A **83** (2011), 062303, arXiv:1103.1025v1 [quant-ph].

- [RPB88] B. D. McKay R. P. Brent, *On Determinants of Random Symmetric Matrices over Z_m* , *Ars Combinatoria* **26** (1988), no. A, 57–64, arXiv:1004.5440v1 [math.CO].
- [Sch60] J. Schwinger, *Unitary operator bases*, *Proceedings of the National Academy of Sciences* **46** (1960), 570–579.
- [Sho97] P. W. Shor, *Polynomial-Time Algorithms for Prime Factorization and Discrete Logarithms on a Quantum Computer*, *SIAM Journal on Computing* **26** (1997), no. 5, 1484–1509, arXiv:9508027v2 [quant-ph].
- [Sil00] J. R. Silvester, *Determinants of Block Matrices*, *The Mathematical Gazette* **84** (2000), no. 501, 460–467.
- [SNA12] U. Seyfarth, G. M. Nikolopoulos, and G. Alber, *Symmetries and Security of a Quantum-public-key Encryption Based on Single-qubit Rotations*, *Physical Review A* **85** (2012), no. 2, 022342, arXiv:1202.3921v1 [quant-ph].
- [SP00] P. W. Shor and J. Preskill, *Simple Proof of Security of the BB84 Quantum Key Distribution Protocol*, *Physical Review Letters* **85** (2000), 441–444, arXiv:0003004 [quant-ph].
- [SPR04] M. Saniga, M. Planat, and H. Rosu, *Mutually unbiased bases and finite projective planes*, *Journal of Optics B* **6** (2004), L19–L20, arXiv:0403057 [math-ph].
- [SR11] U. Seyfarth and K. S. Ranade, *Construction of mutually unbiased bases with cyclic symmetry for qubit systems*, *Physical Review A* **84** (2011), no. 4, 042327, arXiv:1106.3515v2 [quant-ph].
- [SR12] U. Seyfarth and K. S. Ranade, *Cyclic mutually unbiased bases, Fibonacci polynomials and Wiedemann’s conjecture*, *Journal of Mathematical Physics* **53** (2012), no. 6, 062201, arXiv:1104.0202v2 [quant-ph].
- [Sut00] K. Sutner, *σ -Automata and Chebyshev-polynomials*, *Theoretical Computer Science* **230** (2000), no. 1–2, 49–73.
- [VG69] R. Varshamov and G. Garakov, *On the theory of selfdual polynomials over a Galois field (Russian)*, *Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie* **13** (1969).
- [Vol10] J. F. Voloch, *Elements of high order on finite fields from elliptic curves*, *Bulletin of the Australian Mathematical Society* **81** (2010), no. 3, 425–429.
- [WC81] M. N. Wegman and J. L. Carter, *New hash functions and their use in authentication and set equality*, *Journal of Computer and System Sciences* **22** (1981), no. 3, 265–279.

- [WF89] W. K. Wootters and B. D. Fields, *Optimal state-determination by mutually unbiased measurements*, *Annals of Physics* **191** (1989), no. 2, 363–381.
- [Wie83] S. Wiesner, *Conjugate coding*, *SIGACT News* **15** (1983), no. 1, 78–88.
- [Wie88] D. Wiedemann, *An iterated quadratic extension of GF(2)*, *Fibonacci Quarterly* **26** (1988), no. 4, 290–295.
- [Woo86] W. K. Wootters, *Quantum mechanics without probability amplitudes*, *Foundations of Physics* **16** (1986), 391–405.
- [WP69] W. Webb and E. Parberry, *Divisibility Properties of Fibonacci Polynomials*, *Fibonacci Quarterly* **7** (1969), 457–463.
- [WZ82] W. K. Wootters and W. H. Zurek, *A single quantum cannot be cloned*, *Nature* **299** (1982), no. 5886, 802–803.
- [Zau99] G. Zauner, *Quantendesigns – Grundzüge einer nichtkommutativen Designtheorie*, Ph.D. thesis, University of Vienna, 1999, available online at <http://www.mat.univie.ac.at/~neum/papers/physpapers.htm>.



Index

- Asymptotic security, 89
- Bloch vector, 76, 102
- Chop map, 52
- Class-permutation operators, 69
- Clifford unitary, 29
- Clifford unitary operator, 49, 103
- Companion matrix, 24
- Error correction, 92
- Extended Clifford group, 22
- Fermat sets, 36, 51
- Fibonacci
 - index, 26, 31
 - polynomials, 22, 29
 - sets, 29, 48
- Forward-search attack, 89
- Gauss sum, 16
- Generalized Pauli group, 18
- Generalized Pauli operators, 18, 101
- Hadamard matrix, 50, 52, 100
- Hamming weight, 81
- Heisenberg group, 18
- Hilbert-Schmidt inner product, 19
- Holevo quantity, 81
- Homogeneous sets, 40
- Impulse response sequence, 24
- Inhomogeneous sets, 40
- Initial state vector, 24
- Linear recurring sequence, 24
- Logical operator, 48
- Logical Pauli operator, 48
- Maximal commuting basis, 20
- Mutually unbiased bases, 9, 83
- One-time pad, 12
- Phase system, 50
- Physical operator, 48
- Quantum circuit, 55
- Quantum key distribution, 12
- Quantum one-way function, 76
- Quantum state tomography, 11
- Reciprocal operator, 37
- Reciprocal polynomial, 37
- Reduced stabilizer matrix, 29
- Separability count, 40
- SIC-POVM, 22
- Stabilizer formalism, 48
- Stabilizer matrix, 28
- Standard form, 20, 39, 44, 70
- Symmetric companion matrix, 28, 32, 35, 127
- Symplectic, 103
- Symplectic basis, 103
- Symplectic product, 19, 103

Universal set, 55

Weyl-Heisenberg group, 18

Wiedemann's conjecture, 38



Curriculum Vitae

Personal Data

Ulrich Thiemo Gerd Seyfarth

Email: ulrich.seyfarth@physik.tu-darmstadt.de

Born: June 29th, 1982 in Hadamar (Germany)

Ph. D. Studies

since 2008 Ph.D. studies at Technische Universität Darmstadt (supervision by Prof. Dr. G. Alber)

Higher Education

2002–2008 Diploma in Physics, Technische Universität Darmstadt, Institute of Applied Physics

2006–2007 Studies at ENPG, UJF and CNRS, Grenoble, France

Civilian Service

2001–2002 Mental-health charity for children, Klinik Rheinhöhe, Wiesbaden

Education

1998–2001 Peter-Paul-Cahensly-Schule, Secondary School, Limburg
Higher Education Entrance Qualification
Principal Subjects: Mathematics and Computer Engineering

1992–1998 Tilemannschule, Secondary School, Limburg

1988–1992 Theodor-Heuss-Schule, Primary School, Limburg

List of Publications

- [1] Ulrich Seyfarth and Kedar S. Ranade. Cyclic mutually unbiased bases, Fibonacci polynomials and Wiedemann's conjecture. *Journal of Mathematical Physics*, **53**, 062201 (2012); arXiv:1104.0202v2 [quant-ph].
- [2] Ulrich Seyfarth, Georgios M. Nikolopoulos and Gernot Alber. Symmetries and security of a quantum-public-key encryption based on single-qubit rotations. *Physical Review A*, **85**, no. 2, 022342 (2012); arXiv:1202.3921v1 [quant-ph].
- [3] Ulrich Seyfarth and Kedar S. Ranade. Construction of mutually unbiased bases with cyclic symmetry for qubit systems. *Physical Review A*, **84**, no. 2, 042327 (2011); arXiv:1106.3515v2 [quant-ph].
- [4] Oliver Kern, Kedar S. Ranade, and Ulrich Seyfarth. Complete sets of cyclic mutually unbiased bases in even prime-power dimensions. *Journal of Physics A*, **43**, no. 27, 275305 (2010); arXiv:0912.4661v1 [quant-ph].
- [5] Ulrich Seyfarth. *Quantenkryptographie mit kontinuierlichen Variablen – Diskussion verschiedener Realisierungen mit Qudits*. Diploma thesis, TU Darmstadt (2008); arXiv:0903.1955v1 [quant-ph].