Valuation and risk management in life insurance

Vom Fachbereich Mathematik der
Technischen Universität Darmstadt
zur Erlangung des Grades eines
Doktors der Naturwissenschaften
(Dr. rer. nat.) genehmigte

Dissertation

von
Dipl.-Math. Tom Fischer
aus Heilbronn-Neckargartach

Referent:	Prof. Dr. J. Lehn
Korreferent:	Prof. Dr. U. Rieder
Tag der Einreichung:	30. Oktober 2003
Tag der mündlichen Prüfung:	5. Februar 2004

Darmstadt 2004
D 17

Certain forms of uncertainty make life interesting but less safe. I was always intrigued by the various forms of risk that are associated with human life and activity and how they can be mitigated for the individual by contractual risk exchange between two or more parties. Certain forms of certainty make life interesting and more safe. I was always attracted to mathematics because it allows of statements that are non-trivial and still indisputably true. These two areas of interest synthesize perfectly into actuarial/financial mathematics, which gives precise contents to notions of risk and develops methods for measuring and controlling it. [...]

Ragnar Norberg

Preliminary remark

Life insurance mathematics is perhaps the most interesting and challenging field at the interface of modern actuarial and financial mathematics. It is the intention of this Ph.D. thesis to examine and understand some particular aspects of modern life insurance which have not yet been sufficiently considered. Perhaps this work can make the gap of open questions, but also the gap between financial and actuarial mathematics, a little bit smaller.

Acknowledgements

The thesis was inspired by my work on a project on unit-linked life insurance products at Darmstadt University of Technology and the research center `caesar` in Bonn. I gratefully acknowledge supervision by Prof. Dr. J. Lehn, who is head of the project in Darmstadt, and Dr. A. May, having this function in Bonn. Mr. Lehn's and Ms. May's ongoing support enabled me to summarize my research in this Ph.D. thesis. I want to express my gratitude towards Mr. Lehn as referee for this dissertation, as well as towards Prof. Dr. U. Rieder (University of Ulm) who was willing to be co-referee. Furthermore, I would like to thank Prof. Dr. K. Ritter and Prof. Dr. E. Scheffold (both from Darmstadt University of Technology) for being additional examiners of my public defense. Many thanks to all present and former members of the Stochastics and Operations Research Group (AG 9) in Darmstadt who influenced this work by their comments and several fruitful discussions. Special thanks to Dr. A. Roehrl (Approximity GmbH, Kötzting) who supported my computer equipment. Last but not least, I gratefully acknowledge funding by the German Federal Ministry of Education and Research (BMBF, 03LEM6DA).

Tom Fischer

Contents

Chapter 1

Introduction

1.1 General motivation

The scientific roots of life insurance mathematics can be traced back to the late seventeenth century. In 1693, Edmond Halley (1656-1742) published an article with following title in the Philosophical Transactions of the Royal Society of London: *"An Estimate of the Degrees of the Mortality of Mankind, drawn from curious Tables of the Births and Funerals at the City of Breslaw; with an Attempt to ascertain the Price of Annuities upon Lives"* (Halley, 1693). This important work contains the first ever known mortality table and was of fundamental influence on mortality statistics, but also on social statistics in general. The valuation method for annuities proposed by Halley is still used today and can be found in recent textbooks - over 300 years after the first publication.

Halley had no real stochastic theory of human mortality or life insurance. Nonetheless, he intuitively proposed methods that still seemed suitable when stochastic models came in use in insurance. Finally, it was Pierre Simon de Laplace (1749-1827) who applied the theory of probability to the matter of actuarial questions (cf. Laplace (1820, 1951)). Hence, without any doubt, stochastic actuarial mathematics - and in particular life insurance mathematics - is much older than the stochastic theory of financial markets.

The rise of stochastic financial mathematics got its initial ignition in the early 1970s with the seminal works of Black, Merton and Scholes on the pricing and hedging of financial option contracts. Since then, mathematical finance developed fast and actuarial mathematics - still working with deterministic financial models - seemed to fall behind this quickly emerging branch of science.

However, soon it became clear that the results of financial mathematics would have to be applied to insurance, and especially to life insurance mathematics. New forms of contracts like unit-linked life insurance contingencies with guaranteed minimum payoffs rather had the character of financial options combined with some mortality risk than being classical contracts as known until then. The pricing of the new products clearly demanded knowledge of option-pricing techniques and actuaries began to study financial mathematics (cf. *Actuaries of the Third Kind?* - Editorial of the ASTIN Bulletin, Bühlmann (1987)). Although research on valuation of unit-linked products already started in the late 1960s, first important results were developed in the second half of the 1970s (e.g. Brennan and Schwartz, 1976) and were further developed in the 1990s (e.g. Aase and Persson, 1994). Today, this development continues and it seems to be clear that even the simplest forms of life insurance contracts (e.g. traditional contracts or pure endowments) demand for a stochastic treatment from the financial side since also interest rates are by now (fortunately) the subject of stochastic models. To some extent, life insurance mathematics has become part of financial mathematics.

Nonetheless, the merger of these two sciences is not yet completed. This becomes clear when one compares the practical work of actuaries and financial engineers in banks. It is evident that modern life insurance is not yet popular enough and many insurance companies still use classical methods when calculating their products. Furthermore, communication between actuarial and asset management departments in life insurance companies is not yet good enough.

Also on the scientific side still a lot of work remains to be done - not only when it comes to pricing or hedging of new products, but also with respect to the foundations of modern life insurance. For instance, neither the question for a reasonable set of model assumptions (axioms) nor the question for an exhaustive reasoning for the widely used valuation method have been satisfyingly considered until now. These two questions, as well as several other open problems of similar importance, are the subject of this Ph.D. thesis. In fact, the thesis considers a widespread field of topics: valuation, hedging, risk decomposition, pooling and risk capital allocation.

Beside the introduction and the conclusion, the dissertation consists of three chapters. Each of them will be motivated and summarized in the following sections. A more detailed motivation is given in each of the chapters.

Before we start with the chapter content, an important notion must be introduced. In the following, the adjective *biometric* (or biometrical) will be used extensively. In fact, *biometrics* is the science of the application of mathematical methods for data capture, planning and analysis of biological, medical or agricultural experiments. But also identifying data of human individuals like e.g. the size, colour of eyes or fingerprints is often called biometric. In this thesis, we will call all data concerning the biological and some of the social states of human individuals biometric. This can include characteristics like health, age, sex, family status, but also the ability to work. In the context of life insurance, the most important biometric information at a certain point of time will always be the age and sex of an individual, and whether the individual is alive or not.

1.2 Chapter 2 - An axiomatic approach to valuation in life insurance

Naturally, valuation is one of the most important problems of life insurance mathematics. The search for the minimum fair price, i.e. the question how a contract must at least be priced such that the insurance company is treated fairly, can not be overvalued.

In fact, the actuarial community agrees on the answer to this question. Similar to the classical case, where the minimum fair price of any payoff is the discounted expectation of this payoff (Expectation Principle), the commonly used modern valuation principle also is an expectation of the discounted payoff - but, in analogy to mathematical finance, the expectation is taken due to an equivalent martingale measure. In the case of life insurance, this measure is a product measure and further described below. However, reasoning for this product measure approach is usually not (or not satisfyingly) done by the Law of Large Numbers which is the main reason for the Expectation Principle in classical insurance mathematics. It would therefore be satisfying to find an analogous reasoning for the modern valuation method.

The consideration of a fundamental topic like valuation at the same time implies intensive examination of the underlying principles and model assumptions of life insurance mathematics. Hence, this is a crucial topic of this chapter, too.

Chapter 2 can be summarized as follows.

The classical Principle of Equivalence ensures that a life insurance company can accomplish that the mean balance per contract converges to zero almost surely for an increasing number of independent clients. In an axiomatic approach, this idea is adapted to the general case of stochastic financial markets. The implied minimum fair price of general life insurance products is then uniquely determined by the product of the given equivalent martingale measure of the financial market with the probability measure of the biometric state space. This minimum fair price (valuation principle) is in accordance with existing results. A detailed historical example about contract pricing and valuation is given.

Chapter 2 considers a discrete time framework. It is based on Fischer (2003b) which contains most of the results that are presented in the chapter.

1.3 Chapter 3 - On the decomposition of risk in life insurance

Consider a life insurance company which made a certain gain or loss during a given time interval. Clearly, the company's professional activities, respectively its balance, are influenced by two types of risk. On the one side there is financial risk evolving from the stochastics of financial markets, on the other side there is biometrical risk, e.g. mortality risk. Without any doubt, information on how much of the win or loss is caused by financial, respectively biometric events is crucial for the understanding and the management of the company.

But not only on the company-wide level, also on the single contract level this information is relevant. Usually, a client participates in financial wins belonging to his/her contract, whereas financial losses remain in the company. The decomposition of gains or risks into a biometrical and a financial part (and also the pricing of these parts) is therefore a very important question in life insurance. Furthermore, the so-called *pooling* of the biometric parts of the gains, i.e. the absorption of biometric fluctuations by the mere size of the insurance portfolio and the Law of Large Numbers without any further risk management, should be seen as *the core competence of life insurance companies*.

Chapter 3 is dedicated to this topic and related questions. A brief description follows.

Assuming a product space model for biometric and financial events, there exists a rather natural principle for the decomposition of gains of life insurance contracts into a financial and a biometric part using orthogonal projections. In a discrete time framework, the chapter shows the connection between this decomposition, locally variance-optimal hedging and the so-called pooling of biometric risk contributions. For example, the mean aggregated discounted biometric risk contribution per client converges to zero almost surely for an increasing number of clients. A general solution of Bühlmann's AFIR-problem is proposed. The stochastic discounting and risk decomposition approach of Bühlmann is briefly reviewed. Some problems arising from these techniques are discussed.

1.4 Chapter 4 - Risk capital allocation by coherent risk measures based on one-sided moments

Consider a life insurance company that drives e.g. a locally variance-optimal hedging strategy as it will be described in Chapter 3. Assume that the considered time interval is one year. As we will see, the company can under certain circumstances hedge its financial risk away, but the biometric part of the risk is not hedgeable. Therefore, the company should hold back a certain amount of risk capital to cope with probable losses arising from the biometrical changes.

A method to determine the amount of risk capital due to a certain risky payoff is usually called a *risk measure*. Risk measures became very popular in financial institutions during the 1990s when the so-called Value-at-Risk (VaR) methodology became more and more accepted by practitioners and researchers. A scientific work which still has great influence on risk measurement is the paper of Artzner et al. (1999). They developed an axiomatic approach to risk measurement, the so-called *coherent risk measures*. Unfortunately, Value-at-Risk turned out not to be coherent.

Back to the example. As the insurance company needs a certain amount of money as risk capital, this amount must somehow be distributed to the different clients implying it. Actually, each client should pay for his/her particular risk contribution. In other words, in the case of a life insurance company, we are in the need of a reasonable principle to allocate the risk capital to the individual contracts depending on the riskiness of their biometric risk contributions.

In a more general context, simply considering portfolios (i.e. sums) of random payoffs, the above problem finally results in the question for a reasonable or *fair* per-unit allocation of the risk capital (per unit of the respective payoff which is e.g. the payoff of one certain financial security and can be represented by several units in the considered portfolio). In the case of differentiable positively homogeneous risk measures (the risk depends on the portfolios which are given as finite real vectors), the gradient has figured out to be the unique reasonable allocation principle (cf. Tasche (2000), Denault (2001)).

Unfortunately, many of the established risk measures (e.g. Value-at-Risk, but also some coherent risk measures) easily encounter situations where they are not differentiable, e.g. when working with discrete probability spaces as in life insurance or credit default models.

Chapter 4 is dedicated to this problem and a possible solution. Its content can be summarized as follows.

The chapter proposes differentiability properties for positively homogeneous risk measures which ensure that the gradient can be applied for reasonable risk capital allocation on non-trivial portfolios. It is shown that these properties are fulfilled for a wide class of coherent risk measures based on the mean and the one-sided moments of a risky payoff. In contrast to quantile-based risk measures like Value-at-Risk, this class allows allocation in portfolios of very general distributions, e.g. discrete ones. Two examples show how risk capital given by the VaR can be allocated by adapting risk measures of this class to the VaR.

Chapter 4 is based on Fischer (2003a).

Chapter 2

An axiomatic approach to valuation in life insurance

2.1 Introduction

In traditional life insurance mathematics, financial markets are assumed to be deterministic. Under this assumption, the philosophy of the classical Principle of Equivalence is that a life insurance company should be able to accomplish that the mean balance per contract converges to zero almost surely for an increasing number of clients. Roughly speaking, premiums are chosen such that incomes and losses are "balanced in the mean". This idea leads to a valuation method usually called "Expectation Principle" and relies on two important ingredients: the stochastic independence of individual lifes and the Strong Law of Large Numbers. In modern life insurance mathematics, where financial markets are sensibly assumed to be stochastic and where more general products (e.g. unit-linked ones) are taken into consideration, the widely accepted valuation principle is also an expectation principle. However, the respective probability measure is different as the minimum fair price (or present value) of an insurance claim is determined by the no-arbitrage pricing method as known from financial mathematics. The respective equivalent martingale measure (EMM) is the product of the given EMM of the financial market with the probability measure of the biometric state space.

Although research on the valuation of unit-linked products already started in the late 1960s, one of the first results (for a particular type of contract) that was in its core identical to the mentioned product measure approach was Brennan and Schwartz (1976). The most recent papers mainly dedicated to valuation following this approach are Aase and Persson (1994) for the Black-

Scholes model and Persson (1998) for a simple stochastic interest rate model. A brief history of valuation in (life) insurance can be found in Møller (2002). The works Møller (2002, 2003a, 2003b) also consider valuation, but focus on hedging, resp. advanced premium principles.

Again, one should look at the assumptions underlying the considered valuation principle. Aase and Persson (1994), but also other authors, a priori suppose independence of financial and biometric events. In Aase and Persson (1994), an arbitrage-free and complete financial market ensures the uniqueness of the financial EMM. The use of the product measure as mentioned above is usually explained by the risk-neutrality of the insurer with respect to biometric risks (cf. Aase and Persson (1994), Persson (1998)). In Møller (2001), another good reason is given: the product measure coincides trivially with the so-called minimal martingale measure (cf. Schweizer, 1995b).

Apart from these reasons for the product measure approach, the aim of this chapter is the deduction of a valuation principle by an adaption of the classical demand for convergence of mean balances due to the Law of Large Numbers. This idea seems to be new. In a discrete finite time framework, it is carried out by an axiomatic approach which mainly reflects the commonly accepted assumptions of modern life insurance mathematics (as already mentioned: independence of individuals, independence of biometric and financial events, no-arbitrage pricing etc.). The resulting valuation principle is in accordance with the above mentioned results since the implied minimum fair price for general life insurance products is uniquely determined by the equivalent martingale measure given by the product of the EMM of the financial market with the probability measure of the biometric state space. In fact, due to no-arbitrage pricing, the complete price process is determined.

Under the mentioned axioms, it is shown how a life insurance company can accomplish that the mean balance per contract at any future time t converges to zero almost surely for an increasing number of customers. The respective (purely financial and self-financing) hedging strategy can be financed (the initial costs, of course) by the minimum fair premiums.

The considered hedging method is different from the risk-minimizing and mean-variance hedging strategies in Møller (1998, 2001, 2002). In fact, the method is a discrete generalization of the matching approach in Aase and Persson (1994). Although this hedging method is less sophisticated than e.g. risk minimizing strategies (which are unfortunately not self-financing), it surely is of practical use since it is easier to realize as not every single life has to be observed over the whole time axis. Examples for pricing and hedging of different types of contracts are given. A more detailed example shows for a traditional life insurance and an endowment contract the historical development of the ratio of the minimum fair annual premium per benefit. Assuming that premiums are calculated by a conservatively chosen constant technical rate of interest, the example also derives the development of the present values of these contracts.

Although the model considered in this chapter is restricted to a finite number of time steps, the approach is quite general in the sense that it does not propose particular models for the dynamics of financial securities or biometric events. The concept of a life insurance contract is introduced in a very general way and the presented methods are not restricted to a particular type of contract. Furthermore, all methods and results of the chapter can be applied to non-life insurance as long as the assumptions are also appropriate for the considered cases.

The section content is as follows. In Section 2.2, the principles which are considered to be reasonable for a modern theory of life insurance are briefly discussed in an enumerated list. Section 2.3 introduces the market model and some first axioms concerning the common probability space of financial and biometric risks. Section 2.4 contains a definition of general life insurance contracts and the statement of a generalized Principle of Equivalence. The chapter makes a difference between the classical Expectation Principle, which is a valuation method, and the Principle of Equivalence, which is an economic "fairness" argument. In Section 2.5, the case of classical life insurance mathematics is briefly reviewed. Section 2.6 contains the axiomatic approach to valuation in the general case and the deduction of the minimum fair price. Section 2.7 is about hedging, i.e. about the convergence of mean balances. In this section, examples are given, too. In Section 2.8, it is shown how parts of the results can be adapted to the case of incomplete markets. Even for markets with arbitrage opportunities some results still hold. The last section is dedicated to the numerical pricing example mentioned above and confirms the importance of modern valuation principles.

2.2 Principles of life insurance mathematics

In the author's opinion, the following eight assumptions are crucial for a modern theory of life insurance mathematics. The principles are given in an informal manner. The mathematically precise formulation follows later.

1. Independence of biometric and financial events. One of the basic assumptions is that the biometric (technical) events, for instance death or injury of persons, are independent of the events of the financial markets (cf. Aase and Persson, 1994). In contrast to reinsurance companies, where the movements on the financial markets can be highly correlated to the technical events (e.g. earthquakes), it is common sense that such effects can be neglected in the case of life insurance.

2. Complete, arbitrage-free financial markets. Except for Section 2.8, where incomplete markets are examined, complete and arbitrage-free *financial* markets are considered throughout the chapter. Even though this might be an unrealistic assumption from the viewpoint of finance, it is realistic from the perspective of life insurance. The reason is that a life insurance company usually does not invent purely financial products as this is the working field of banks. Therefore, it can be assumed that all considered *financial* products are either traded on the market, can be bought from banks or can be replicated by self-financing strategies. Nonetheless, it is self-evident that a claim which also depends on a technical event (e.g. the death of a person) *can not* be hedged by financial securities, i.e. the joint market of financial and technical risks is not complete. In the literature, completeness of financial markets is often assumed by the use of the Black-Scholes model (cf. Aase and Persson (1994), Møller (1998)). However, parts of the results of the chapter are also valid in the case of incomplete financial markets - which allows more models. In this case, financial portfolios will be restricted to replicable ones and also the considered life insurance contracts are restricted in a similar way.

3. Biometric states of individuals are independent. This is the standard assumption of classical life insurance. However, neglecting the possibility of epidemic diseases or wars, the principle still seems to be appropriate in a modern environment. Even the well-known argument that e.g. married couples bear some dependencies, for instance when both have contracts with the same company or a joint contract on two lives, is not relevant since the couples themselves will usually be independent.

4. Large classes of similar individuals. Concerning the Law of Large Numbers as applied in classical life insurance mathematics, an implicit assumption is a large number of persons under contract in a particular company. Even stronger, it can be assumed that classes of "similar" persons, e.g. of the same age, sex and health status, are large. At least, an insurance company should be able to cope with such a large class of similar persons even if all members of the class have the same kind of contract (cf. Principle 7 below).

5. Similar individuals can not be distinguished. For fairness reasons, any two individuals with similar biometric development to be expected should pay the same price for the same kind of contract. Furthermore, any activity (e.g. hedging) of an insurance company due to two individuals having the same kind of contract is assumed to be identical as long as their probable future biometric development is independently identical from the stochastic point of view.

6. No-arbitrage pricing. As we know from the theory of financial markets, an important property of a reasonable pricing system is the absence of arbitrage, i.e. the absence of riskless wins. In particular, it should not be possible to beat the market by selling and buying (life) insurance products in an existing or hypothetical reinsurance market (see e.g. Delbaen and Haezendonck, 1989). Hence, any product and cash flow will be priced under the no-arbitrage principle.

7. Minimum fair prices allow hedging such that mean balances converge to zero almost surely. The principle of independence of the biometric state spaces is closely related to the Expectation Principle of classical life insurance mathematics. In the classical case, where financial markets are assumed to be deterministic, this principle states that the present value (single net premium) of a cash flow (contract) is the expectation of the sum of its discounted payoffs. The connection between the two principles is the Law of Large Numbers. Present values or prices are determined such that for an increasing number of contracts due to independent individuals the insurer can accomplish that the mean final balance per contract, but also the mean balance at any time t, converges to zero almost surely. In analogy to the classical case, we generally demand that the minimum fair price of any contract (from the viewpoint of the insurer) should at least cover the price of a purely financial hedging strategy that lets the mean balance per contract converge to zero a.s. for an increasing number of clients.

8. Principle of Equivalence. Under a reasonable valuation principle (cf. Principle 7), the Principle of Equivalence demands that the future payments to the insurer (premiums) should be determined such that their present value equals the present value of the future payments to the insured (benefits). The idea is that the liabilities (benefits) can somehow be hedged working with the premiums. In the coming sections, this concept will be considered in detail.

REMARK 2.1. Concerning premium calculation, the classical Expectation Principle (cf. Principle 7) is usually seen as a minimum premium principle since any insurance company must be able to cope with higher expenses than the expected (cf. Embrechts, 2000). So-called *safety loads* on the minimum fair premiums can be obtained by more elaborate *premium principles*. We refer to the literature for more information on the topic (e.g. Delbaen and Haezendonck (1989); Gerber (1997); Goovaerts, De Vylder and Haezendonck (1984); Møller (2002-2003b); Schweizer (2001)). Another possibility to get safety loads is to use the Expectation Principle with a prudent *first order base* (technical base) for biometric and financial developments, e.g. conservatively chosen mortality probabilities and interest rates, which represents a worst-case scenario for the future development of the *second order base* (experience base) which is the real (i.e. observed) development (e.g. Norberg, 2001).

2.3 The model

Let $(F, \mathcal{F}_T, \mathbb{F})$ be a probability space equipped with the filtration $(\mathcal{F}_t)_{t\in\mathbb{T}}$, where $\mathbb{T} = \{0, 1, 2, \ldots, T\}$ denotes the discrete finite time axis. Assume that $\mathcal{F}_0$ is trivial, i.e. $\mathcal{F}_0 = \{\emptyset, F\}$. Let the price dynamics of d securities of a frictionless financial market be given by an adapted $\mathbb{R}^d$-valued process $S = (S_t)_{t\in\mathbb{T}}$. The d assets with price processes $(S_t^0)_{t\in\mathbb{T}}, \ldots, (S_t^{d-1})_{t\in\mathbb{T}}$ are traded at times $t \in \mathbb{T} \setminus \{0\}$. The first asset with price process $(S_t^0)_{t\in\mathbb{T}}$ is called the *money account* and has the properties $S_0^0 = 1$ and $S_t^0 > 0$ for $t \in \mathbb{T}$. The tuple $M^F = (F, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{F}, \mathbb{T}, S)$ is called a *securities market model*. A *portfolio* due to M^F is given by a d-dimensional vector $\theta = (\theta^0, \ldots, \theta^{d-1})$ of real-valued random variables θ^i $(i = 0, \ldots, d-1)$ on $(F, \mathcal{F}_T, \mathbb{F})$. A *t-portfolio* is a portfolio θ_t which is $\mathcal{F}_t$-measurable. As usual, $\mathcal{F}_t$ is interpreted as the information available at time t. Since an economic agent takes decisions due to the available information, a *trading strategy* is a vector $\theta_{\mathbb{T}} = (\theta_t)_{t\in\mathbb{T}}$ of t-portfolios θ_t. The discounted total gain (or loss) of such a strategy is given by $\sum_{t=0}^{T-1}\langle\theta_t, \overline{S}_{t+1} - \overline{S}_t\rangle$, where $\overline{S} := (S_t/S_t^0)_{t\in\mathbb{T}}$ denotes the price process discounted by the money account and $\langle .\,,.\rangle$ denotes the inner product on $\mathbb{R}^d$. One can now define

$$G = \left\{ \sum_{t=0}^{T-1} \langle \theta_t, \overline{S}_{t+1} - \overline{S}_t \rangle : \text{ each } \theta_t \text{ is a } t\text{-portfolio} \right\}. \tag{2.1}$$

G is a subspace of the space of all real-valued random variables $L^0(F, \mathcal{F}_T, \mathbb{F})$ where two elements are identified if they are equal $\mathbb{F}$-a.s. The process S satisfies the so-called *no-arbitrage condition* (NA) if $G \cap L^0_+ = \{0\}$, where L^0_+ are the non-negative elements of $L^0(F, \mathcal{F}_T, \mathbb{F})$ (cf. Delbaen, 1999). The Fundamental Theorem of Asset Pricing (Dalang, Morton and Willinger, 1990) states that the price process S satisfies (NA) if and only if there is a probability measure $\mathbb{Q}$ equivalent to $\mathbb{F}$ such that under $\mathbb{Q}$ the process $\overline{S}$ is a martingale. $\mathbb{Q}$ is called *equivalent martingale measure* (EMM), then. Moreover, $\mathbb{Q}$ can be found with bounded Radon-Nikodym derivative $d\mathbb{Q}/d\mathbb{F}$.

DEFINITION 2.2. *A* **valuation principle** π^F *on a set* Θ *of portfolios due to* M^F *is a linear mapping which maps each* $\theta \in \Theta$ *to an adapted* $\mathbb{R}$*-valued stochastic process (price process)* $\pi^F(\theta) = (\pi^F_t(\theta))_{t\in\mathbb{T}}$ *such that*

$$\pi^F_t(\theta) = \langle \theta, S_t \rangle = \sum_{i=0}^{d-1} \theta^i S^i_t \tag{2.2}$$

for any $t \in \mathbb{T}$ *for which* θ *is* $\mathcal{F}_t$*-measurable.*

For the moment, the set Θ is not specified any further.

Consider an arbitrage-free market with price process S as given above and a portfolio θ with price process $\pi^F(\theta)$. From the Fundamental Theorem it is known that the enlarged market with price dynamics $S' = ((S^0_t, \ldots, S^{d-1}_t, \pi^F_t(\theta)))_{t\in\mathbb{T}}$ is arbitrage-free if and only if there exists an EMM $\mathbb{Q}$ for $\overline{S'}$, i.e. $\mathbb{Q} \sim \mathbb{F}$ and $\overline{S'}$ a $\mathbb{Q}$-martingale. Hence, one has

$$\pi^F_t(\theta) = S^0_t \cdot \mathbf{E}_{\mathbb{Q}}[\langle \theta, S_T \rangle / S^0_T | \mathcal{F}_t]. \tag{2.3}$$

As is well-known, the no-arbitrage condition does not imply a unique price process for θ when the portfolio can not be replicated by a self-financing strategy $\theta_{\mathbb{T}}$, i.e. a strategy such that $\langle \theta_{t-1}, S_t \rangle = \langle \theta_t, S_t \rangle$ for each $t > 0$ and $\theta_T = \theta$. However, in a *complete* market M^F, i.e. a market which features a self-financing replicating strategy for any portfolio θ (cf. Lemma 2.4), the no-arbitrage condition implies unique prices (where prices are identified when equal a.s.) and therefore a unique EMM $\mathbb{Q}$. Actually, an arbitrage-free securities market model as introduced above is complete if and only if the set of equivalent martingale measures is a singleton (cf. Harrison and Kreps (1979); Taqqu and Willinger (1987); Dalang, Morton and Willinger (1990)).

DEFINITION 2.3. *A* **t-claim** *with payoff C_t at time t is a t-portfolio of the form $\frac{C_t}{S_t^0} e_0$ where C_t is a $\mathcal{F}_t$-measurable random variable and e_0 denotes the first canonical base vector in $\mathbb{R}^d$. A* **cash flow** *over the time period $\mathbb{T}$ is a vector $(\frac{C_t}{S_t^0} e_0)_{t\in\mathbb{T}}$ of t-claims.*

Interpretation. A t-claim is interpreted as a contract about the payment of the amount C_t in shares of the money account at time t. That means one can assume that the owner is actually given C_t in cash at t. The interpretation of a cash flow is obvious.

We will now introduce axioms which concern the properties of market models (not of valuation principles) that include biometric events (cf. Principles 1 to 4 of Section 2.2).

Assume to be given a filtered probability space $(B, (\mathcal{B}_t)_{t\in\mathbb{T}}, \mathbb{B})$ which describes the development of the biological states of all considered human beings. *No particular model for the development of the biometric information is chosen.*

AXIOM 1. *A common filtered probability space*

$$(M, (\mathcal{M}_t)_{t\in\mathbb{T}}, \mathbb{P}) = (F, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{F}) \otimes (B, (\mathcal{B}_t)_{t\in\mathbb{T}}, \mathbb{B}) \tag{2.4}$$

of financial and biometric events is given, i.e. $M = F \times B$, $\mathcal{M}_t = \mathcal{F}_t \otimes \mathcal{B}_t$ and $\mathbb{P} = \mathbb{F} \otimes \mathbb{B}$. Furthermore, $\mathcal{F}_0 = \{\emptyset, F\}$ and $\mathcal{B}_0 = \{\emptyset, B\}$.

As $\mathcal{M}_0 = \{\emptyset, F \times B\}$, the model implies that at time 0 the world is known for sure. The symbols $M, \mathcal{M}_t$ and $\mathbb{P}$ are introduced to shorten notation. M and $\mathcal{M}_t$ are chosen since these objects describe events of the underlying market *m*odel, whereas $\mathbb{P}$ denotes the *p*hysical probability measure. Later, $\mathbb{M}$ is used to denote a *m*artingale measure.

AXIOM 2. *A complete securities market model*

$$M^F = (F, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{F}, \mathbb{T}, {}_F S) \tag{2.5}$$

with $|\mathcal{F}_T| < \infty$ *and a unique equivalent martingale measure* $\mathbb{Q}$ *are given. The common market of financial and biometric risks is denoted by*

$$M^{F\times B} = (M, (\mathcal{M}_t)_{t\in\mathbb{T}}, \mathbb{P}, \mathbb{T}, S), \tag{2.6}$$

where $S(f,b) = {}_F S(f)$ *for all* $(f,b) \in M$.

In the following, $M^{F\times B}$ is understood as a securities market model. The notions portfolio, no-arbitrage etc. are used as above. $|\mathcal{F}_T| < \infty$ is assumed as there are no discrete time financial market models which are complete and have a really infinite state space (cf. Dalang, Morton and Willinger, 1990). However, the proofs of the chapter do not explicitly rely on this finiteness (cf. Remark 2.16). Usually, a non-deterministic financial market will be considered, i.e. $2 < |\mathcal{F}_T| < \infty$.

The following lemma is useful.

LEMMA 2.4.

- (*i*) *Any* $\mathcal{F}_t$*-measurable portfolio can be replicated by a s.f. financial strategy until* t.
- (*ii*) *Any* $\mathcal{F}_t$*-measurable payoff can be replicated by a s.f. financial strategy until* t.

Proof. (i) As M^F is complete, any $\mathcal{F}_T$-measurable payoff X at T can be replicated until T. This is the usual definition of the completeness of a securities market model. Hence, there exists for any $\mathcal{F}_t$-measurable portfolio θ_t a replicating self-financing (s.f.) strategy $(\varphi_t)_{t\in\mathbb{T}}$ in M^F, i.e. $\varphi_T = \theta_t$, since $X = \langle\theta_t, S_T\rangle$ could be chosen. For no-arbitrage reasons, one must have $\pi_s^F(\theta_t) = \langle\varphi_s, S_s\rangle$ for $s \in \mathbb{T}$ and therefore $\langle\theta_t, S_s\rangle = \langle\varphi_s, S_s\rangle$ for any $s \geq t$. So, there also exists a s.f. strategy such that $\varphi_t = \theta_t$, i.e. the portfolio θ_t is replicated until t.
(ii) Due to (i), the portfolio $\theta_t = X/S_t^0 \cdot e_0$ can for any $\mathcal{F}_t$-measurable payoff X be replicated until t. Observe that $\langle\theta_t, S_t\rangle = X$. □

REMARK 2.5. S is the canonical embedding of ${}_F S$ into $(M, (\mathcal{M}_t)_{t\in\mathbb{T}}, \mathbb{P})$. We will usually use the same symbol for a random variable X in $(F, \mathcal{F}_t, \mathbb{F})$ and a random variable Y in $(M, \mathcal{M}_t, \mathbb{P})$ ($t \in \mathbb{T}$) when Y is the embedding of X into $(M, \mathcal{M}_t, \mathbb{P})$, i.e. $Y(f,b) = X(f)$ for all $(f,b) \in M$. Now, any portfolio ${}_F\theta$ of the complete financial market M^F can be replicated by some self-financing trading strategy ${}_F\theta_{\mathbb{T}} = ({}_F\theta_t)_{t\in\mathbb{T}}$. Under (NA), the unique price process $\pi^F({}_F\theta)$ of the portfolio is given by

$$\pi_t^F({}_F\theta) = {}_FS_t^0 \cdot \mathbf{E}_{\mathbb{Q}}[\langle {}_F\theta, {}_FS_T\rangle / {}_FS_T^0 | \mathcal{F}_t]. \tag{2.7}$$

Since S is the embedding of ${}_FS$ into $(M, (\mathcal{M}_t)_{t\in\mathbb{T}}, \mathbb{P})$, the embedded portfolio ${}_F\theta$ in $M^{F\times B}$ is replicated by the embedded trading strategy ${}_F\theta_{\mathbb{T}} = ({}_F\theta_t)_{t\in\mathbb{T}}$ in $M^{F\times B}$. Hence, to avoid arbitrage opportunities, any reasonable valuation principle π must feature a price process $\pi({}_F\theta)$ in $M^{F\times B}$ that fulfills $\pi_t({}_F\theta) = \pi_t^F({}_F\theta)$ $\mathbb{P}$-a.s. for any $t \in \mathbb{T}$. Since $\mathbf{E}_{\mathbb{Q}}[X|\mathcal{F}_t] = \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[X|\mathcal{F}_t \otimes \mathcal{B}_0]$ $\mathbb{P}$-a.s. for *any* random variable X in $(F, \mathcal{F}_T, \mathbb{F})$, one must have $\mathbb{P}$-a.s.

$$\begin{aligned} \pi_t({}_F\theta) &= S_t^0 \cdot \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\langle {}_F\theta, S_T\rangle / S_T^0 | \mathcal{F}_t \otimes \mathcal{B}_0] \\ &= S_t^0 \cdot \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\langle {}_F\theta, S_T\rangle / S_T^0 | \mathcal{F}_t \otimes \mathcal{B}_t]. \end{aligned} \tag{2.8}$$

Observe that $(S_t/S_t^0)_{t\in\mathbb{T}}$ is a $\mathbb{Q}\otimes\mathbb{B}$-martingale.

AXIOM 3. *There are infinitely many human individuals and we have*

$$(B, (\mathcal{B}_t)_{t\in\mathbb{T}}, \mathbb{B}) = \bigotimes_{i=1}^{\infty} (B^i, (\mathcal{B}_t^i)_{t\in\mathbb{T}}, \mathbb{B}^i), \tag{2.9}$$

where $B_H = \{(B^i, (\mathcal{B}_t^i)_{t\in\mathbb{T}}, \mathbb{B}^i) : i \in \mathbb{N}^+\}$ *is the set of filtered probability spaces which describe the development of the i-th individual* $(\mathbb{N}^+ := \mathbb{N} \setminus \{0\})$. *Each* $\mathcal{B}_0^i$ *is trivial.*

It follows that $\mathcal{B}_0$ is also trivial, i.e. $\mathcal{B}_0 = \{\emptyset, B\}$.

AXIOM 4. *For any space* $(B^i, (\mathcal{B}_t^i)_{t\in\mathbb{T}}, \mathbb{B}^i)$ *in* B_H *there are infinitely many isomorphic (identical, except for the index) ones in* B_H.

In the sense of Remark 2.1, the four axioms above define a model for the second order base.

2.4 Life insurance contracts

By definition, the biometric development has no influence on the price process S of the financial market - and vice versa. A portfolio θ that contains technical risk - that is a portfolio which is not of the form $\theta = {}_F\theta$ $\mathbb{P}$-a.s. with ${}_F\theta$ an M^F-portfolio - can not be replicated by purely financial products. Hence, relative pricing of life insurance products due to M^F is not possible. In general, life insurance policies are not traded and the possibility of the valuation of such contracts by the market is not given. The market $M^{F\times B}$ of financial and biometric risks is incomplete. Nonetheless, the products have to be priced as e.g. the insured usually have the right to dissolve any contract at any time of its duration. We are therefore in the need of a reasonable valuation principle π for the considered portfolios Θ of the market $M^{F\times B}$ and in particular for general life insurance products.

DEFINITION 2.6. *A general* **life insurance contract** *is a vector* $(\gamma_t, \delta_t)_{t\in\mathbb{T}}$ *of pairs* (γ_t, δ_t) *of t-portfolios in* Θ *(to shorten notation we drop the inner brackets of* $((\gamma_t, \delta_t))_{t\in\mathbb{T}}$*). For any* $t \in \mathbb{T}$*, the portfolio* γ_t *is interpreted as a payment of the insurer to the insurant* (**benefit**) *and* δ_t *as a payment of the insurant to the insurer* (**premium**)*, respectively taking place at* t*. The notation* $({}^i\gamma_t, {}^i\delta_t)_{t\in\mathbb{T}}$ *means that the contract depends on the i-th individual's life, i.e. for all* $(f,x),(f,y) \in M$

$$({}^i\gamma_t(f,x), {}^i\delta_t(f,x))_{t\in\mathbb{T}} = ({}^i\gamma_t(f,y), {}^i\delta_t(f,y))_{t\in\mathbb{T}} \tag{2.10}$$

whenever $p^i(x) = p^i(y)$*,* p^i *being the canonical projection of* B *onto* B^i*.*

For any contract $(\gamma_t, \delta_t)_{t\in\mathbb{T}}$ between a life insurance company and an individual, this stream of payments is from the viewpoint of the insurer equivalent to holding the portfolios $(\delta_t - \gamma_t)_{t\in\mathbb{T}}$.

Although there has not been considered any particular valuation principle until now, it is assumed that a suitable principle π is a minimum fair price in the heuristic sense given in Section 2.2, Principle 7. The properties of a minimum fair price will be further explained in Section 2.6.

AXIOM 5. *Suppose a suitable valuation principle* π *on* Θ*. For any life insurance contract* $(\gamma_t, \delta_t)_{t\in\mathbb{T}}$ *the* **Principle of Equivalence** *demands that*

$$\pi_0\left(\sum_{t=0}^{T}\gamma_t\right) = \pi_0\left(\sum_{t=0}^{T}\delta_t\right). \tag{2.11}$$

As already mentioned in Section 2.2 (Principle 8), the idea of equation (2.11) is that the liabilities $(\gamma_t)_{t\in\mathbb{T}}$ can somehow be hedged working with the premiums $(\delta_t)_{t\in\mathbb{T}}$ since their present values are identical. For the classical case, this idea is explained in the next section.

2.5 Valuation I - The classical case

In classical life insurance mathematics, the financial market is assumed to be deterministic. We realize the assumption by $|\mathcal{F}_T| = 2$, i.e. $\mathcal{F}_T = \{\emptyset, F\}$, and identify $(M, (\mathcal{M}_t)_{t\in\mathbb{T}}, \mathbb{P})$ with $(B, (\mathcal{B}_t)_{t\in\mathbb{T}}, \mathbb{B})$. As the market is assumed to be free of arbitrage, all assets must show the same dynamics. Hence, we can assume $S = (S_t^0)_{t\in\mathbb{T}}$, i.e. $d = 1$ and the only asset is the money account as a deterministic function of time. In the classical framework, it is common sense that the fair present value at time s of a $\mathbb{B}$-integrable payoff C_t at t is the conditional expectation of the discounted payoff due to $\mathcal{B}_s$, i.e. for a t-claim C_t/S_t^0 (cf. Definition 2.3) we have

$$\pi_s(C_t/S_t^0) := S_s^0 \cdot \mathbf{E}_{\mathbb{B}}[C_t/S_t^0|\mathcal{B}_s], \quad s \in \mathbb{T}. \tag{2.12}$$

Under the *Expectation Principle* (2.12), the well-known classical Principle of Equivalence is given by (2.11). As the discounted price processes are $\mathbb{B}$-martingales, the classical financial market together with a finite number of classical price processes of life insurance policies is free of arbitrage opportunities.

Let us have a closer look at the logic of valuation principle (2.12). Assume that Θ is given by the $\mathbb{B}$-integrable portfolios. Suppose Axiom 1 to 3 and consider the claims $\{(-{}^i\gamma_t)_{t\in\mathbb{T}} : i \in \mathbb{N}^+\}$ of an insurance contract from the companies point of view, where ${}^i\gamma_t$ depends on the i-th individual's life, only (cf. Definition 2.6). Furthermore, suppose that for all $t \in \mathbb{T}$ there is a $c_t \in \mathbb{R}^+$ such that

$$||{}^i\gamma_t||_2 \leq c_t \tag{2.13}$$

for all $i \in \mathbb{N}^+$, where $||.||_2$ denotes the L^2-norm on the Hilbert space $L^2(M, \mathcal{M}_T, \mathbb{P})$ of all square-integrable real functions on $(M, \mathcal{M}_T, \mathbb{P})$. Now, buy for all $i \in \mathbb{N}^+$ and all $t \in \mathbb{T}$ the portfolios $\mathbf{E}_{\mathbb{B}}[{}^i\gamma_t]$, where $\mathbf{E}_{\mathbb{B}}[{}^i\gamma_t]$ is interpreted as a financial product (a t-portfolio) which matures at time t, i.e. the payoff $\mathbf{E}_{\mathbb{B}}[{}^i\gamma_t] \cdot S_t^0$ in cash at t is bought at 0. Consider the balance of wins and losses at time t. The mean total payoff at t for the first m contracts is given by

$$\frac{1}{m}\sum_{i=1}^{m}(\mathbf{E}_{\mathbb{B}}[{}^i\gamma_t] - {}^i\gamma_t) \cdot S_t^0. \tag{2.14}$$

Clearly, (2.14) converges $\mathbb{B}$-a.s. to 0 as we can apply the Strong Law of Large Numbers by Kolmogorov's Criterion (cf. (2.13)). Furthermore, it follows directly from (2.12) that we have $\pi_0(\mathbf{E}_{\mathbb{B}}[{}^i\gamma_t]) = \pi_0({}^i\gamma_t)$ for all $i \in \mathbb{N}^+$. Hence, in the classical case, the fair present value of any claim equals (except for the different sign, perhaps) the price of a hedge at time 0 such that for an increasing number of independent claims the mean balance of claims and hedges converges to zero almost surely.

Now, consider the set of life insurance contracts $\{({}^i\gamma_t, {}^i\delta_t)_{t\in\mathbb{T}} : i \in \mathbb{N}^+\}$ with the deltas being defined in analogy to the gammas above. Since for the company a contract can be considered as a vector $({}^i\delta_t - {}^i\gamma_t)_{t\in\mathbb{T}}$ of portfolios, the analogous hedge is given by $(\mathbf{E}_{\mathbb{B}}[{}^i\gamma_t] - \mathbf{E}_{\mathbb{B}}[{}^i\delta_t])_{t\in\mathbb{T}}$. Under Axiom 5 the contract has value zero. From the Expectation Principle (2.12) we therefore obtain for all $i \in \mathbb{N}^+$

$$\sum_{t=0}^{T} \pi_0(\mathbf{E}_{\mathbb{B}}[{}^i\delta_t] - \mathbf{E}_{\mathbb{B}}[{}^i\gamma_t]) = \sum_{t=0}^{T} \pi_0({}^i\delta_t - {}^i\gamma_t) = 0. \tag{2.15}$$

Hence, under (2.12) and Axiom 1, 2, 3 and 5, a life insurance company can (without any costs at time 0) pursue a hedge such that the mean balance per contract at any time t converges to zero almost surely for an increasing number of individual contracts:

$$\frac{1}{m}\sum_{i=1}^{m}({}^i\delta_t - {}^i\gamma_t - \mathbf{E}_{\mathbb{B}}[{}^i\delta_t] + \mathbf{E}_{\mathbb{B}}[{}^i\gamma_t]) \cdot S_t^0 \overset{m\to\infty}{\longrightarrow} 0 \quad \mathbb{B}\text{-a.s.} \tag{2.16}$$

As a direct consequence, the mean of the *final* balance converges, too:

$$\frac{1}{m}\sum_{i=1}^{m}\sum_{t=0}^{T}({}^i\delta_t - {}^i\gamma_t - \mathbf{E}_{\mathbb{B}}[{}^i\delta_t] + \mathbf{E}_{\mathbb{B}}[{}^i\gamma_t]) \cdot S_T^0 \overset{m\to\infty}{\longrightarrow} 0 \quad \mathbb{B}\text{-a.s.} \tag{2.17}$$

REMARK 2.7. Roughly speaking, the Expectation Principle (2.12) implies that the price of any claim at least covers the costs of a purely financial hedge such that for an increasing number of independent claims the mean balance of claims and hedges converges to zero almost surely. Under the Equivalence Principle (2.11), the hedge of any insurance contract costs nothing at time 0, which is important as the contract itself is for free, too (cf. Eq. (2.15)).

2.6 Valuation II - The general case

Before it comes to the topic of valuation in the general case, two technical lemmas have to be proven and some further notation has to be introduced.

Let the set $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ be equipped with the usual Borel-σ-algebra and recall that a function g into $\overline{\mathbb{R}}$ is called *numeric*.

LEMMA 2.8. *Consider $n > 1$ measurable numeric functions g_1 to g_n on the product $(F, \mathcal{F}, \mathbb{F}) \otimes (B, \mathcal{B}, \mathbb{B})$ of two probability spaces. Then $g_1 = \ldots = g_n$ $\mathbb{F} \otimes \mathbb{B}$-a.s. if and only if $\mathbb{F}$-a.s. $g_1(f, .) = \ldots = g_n(f, .)$ $\mathbb{B}$-a.s.*

Proof. For any $Q \in \mathcal{F} \otimes \mathcal{B}$ it is well-known that $\mathbb{F} \otimes \mathbb{B}(Q) = \int \mathbb{B}(Q_f) d\mathbb{F}$, where $Q_f = \{b \in B : (f, b) \in Q\}$ and the function $\mathbb{B}(Q_f)$ on F is $\mathcal{F}$-measurable. As for $i \neq j$ the difference $g_{i,j} := g_i - g_j$ is measurable, the set $Q := \bigcap_{i \neq j} g_{i,j}^{-1}(0)$ is $\mathcal{F} \otimes \mathcal{B}$-measurable. Now, $g_1 = \ldots = g_n$ a.s. is equivalent to $\mathbb{F} \otimes \mathbb{B}(Q) = 1$ and this again is equivalent to $\mathbb{B}(Q_f) = 1$ $\mathbb{F}$-a.s. However, $\mathbb{B}(Q_f) = 1$ is equivalent to $g_1(f, .) = \ldots = g_n(f, .)$ $\mathbb{B}$-a.s. □

LEMMA 2.9. *Let $(g_n)_{n \in \mathbb{N}}$ and g be a sequence, respectively a function, in $L^0(F \times B, \mathcal{F} \otimes \mathcal{B}, \mathbb{F} \otimes \mathbb{B})$, i.e. the real valued measurable functions on $F \times B$, where $(F \times B, \mathcal{F} \otimes \mathcal{B}, \mathbb{F} \otimes \mathbb{B})$ is the product of two arbitrary probability spaces. Then $g_n \to g$ $\mathbb{F} \otimes \mathbb{B}$-a.s. if and only if $\mathbb{F}$-a.s. $g_n(f, .) \to g(f, .)$ $\mathbb{B}$-a.s.*

Proof. The elements of $L^0(F \times B, \mathcal{F} \otimes \mathcal{B}, \mathbb{F} \otimes \mathbb{B})$ are measurable numeric functions. Now, recall that for any sequence of real numbers $(h_n)_{n \in \mathbb{N}}$ and any $h \in \mathbb{R}$ the property $h_n \to h$ is equivalent to $\limsup h_n = \liminf h_n = h$. As the limes superior and the limes inferior of a measurable numeric function always exist and are measurable, one obtains from Lemma 2.8 that

$$\limsup_{n \to \infty} g_n = \liminf_{n \to \infty} g_n = g \quad \mathbb{F} \otimes \mathbb{B}\text{-a.s.} \tag{2.18}$$

if and only if $\mathbb{F}$-a.s.

$$\limsup_{n \to \infty} g_n(f, .) = \liminf_{n \to \infty} g_n(f, .) = g(f, .) \quad \mathbb{B}\text{-a.s.} \tag{2.19}$$

□

As we have seen in Section 2.4, there is the need for a suitable set Θ of portfolios on which a particular valuation principle will work. Furthermore, a mathematically precise description of what was called "similar" in Principle 5 (Section 2.2) has to be introduced.

DEFINITION 2.10.

(*i*) *Define*

$$\Theta = (L^1(M, \mathcal{M}_T, \mathbb{P}))^d \tag{2.20}$$

and

$$\Theta^F = (L^0(F, \mathcal{F}_T, \mathbb{F}))^d, \tag{2.21}$$

where Θ^F can be interpreted as a subset of Θ by the usual embedding since all $L^p(F, \mathcal{F}_T, \mathbb{F})$ are identical for $p \in [0, \infty]$.

(*ii*) *A set $\Theta' \subset \Theta$ of portfolios in $M^{F\times B}$ is called* **independently identically distributed due to** $(B, \mathcal{B}_T, \mathbb{B})$, *abbreviated* **B-i.i.d.**, *when for almost all $f \in F$ the random variables $\{\theta(f, .) : \theta \in \Theta'\}$ are i.i.d. on $(B, \mathcal{B}_T, \mathbb{B})$. Under Axiom 4, such sets exist and can be countably infinite.*

(*iii*) *Under the Axioms 1 to 3, a set $\Theta' \subset \Theta$ satisfies condition* **(K)** *if for almost all $f \in F$ the elements of $\{\theta(f, .) : \theta \in \Theta'\}$ are stochastically independent on $(B, \mathcal{B}_T, \mathbb{B})$ and $||\theta^j(f, .)||_2 < c(f) \in \mathbb{R}^+$ for all $\theta \in \Theta'$ and all $j \in \{0, \ldots, d-1\}$.*

Sets fulfilling condition (B-i.i.d.) or (K) are indexed with the respective symbol. A discussion of the Kolmogorov-Criterion-like condition (K) can be found below (Remark 2.23). The condition figures out to be quite weak with respect to all relevant practical purposes.

The remaining axioms which concern valuation can be stated, now. The next axiom is motivated by the demand that whenever the market with the original d securities with prices S is enlarged by a finite number of price processes $\pi(\theta)$ due to general portfolios $\theta \in \Theta$, the no-arbitrage condition (NA) should hold for the new market. This axiom corresponds to Principle 6 of Section 2.2.

AXIOM 6. *Any valuation principle π taken into consideration must for any $t \in \mathbb{T}$ and $\theta \in \Theta$ be of the form*

$$\pi_t(\theta) = S_t^0 \cdot \mathbf{E}_{\mathbb{M}}[\langle \theta, S_T \rangle / S_T^0 | \mathcal{F}_t \otimes \mathcal{B}_t] \tag{2.22}$$

for a probability measure $\mathbb{M} \sim \mathbb{P}$. Furthermore, one must have

$$\pi_t({}_F\theta) = \pi_t^F({}_F\theta) \tag{2.23}$$

$\mathbb{P}$*-a.s. for any M^F-portfolio ${}_F\theta$ and all $t \in \mathbb{T}$, where π_t^F is as in (2.7).*

Observe that due to Axiom 6 the process $(S_t/S_t^0)_{t\in\mathbb{T}}$ must be an $\mathbb{M}$-martingale. To see that use (2.22) and (2.23) with ${}_F\theta = e_{i-1}$ (i-th canonical base vector in $\mathbb{R}^d$) and apply (2.2).

The following axiom is due to the fifth and the seventh principle.

AXIOM 7. *Under the Axioms 1 - 4 and 6, a* **minimum fair price** *is a valuation principle π on Θ that must for any $\theta \in \Theta$ fulfill*

$$\pi_0(\theta) = \pi_0^F(H(\theta)) \tag{2.24}$$

where

$$H : \Theta \longrightarrow \Theta^F \tag{2.25}$$

is such that

(*i*) *$H(\theta)$ is a t-portfolio whenever θ is.*

(*ii*) *$H(^1\theta) = H(^2\theta)$ for B-i.i.d. portfolios $^1\theta$ and $^2\theta$.*

(*iii*) *for t-portfolios $\{^i\theta : i \in \mathbb{N}^+\}_{B-i.i.d.}$ or $\{^i\theta : i \in \mathbb{N}^+\}_K$ one has*

$$\frac{1}{m}\sum_{i=1}^{m}\langle {}^i\theta - H({}^i\theta), S_t\rangle \overset{m\to\infty}{\longrightarrow} 0 \quad \mathbb{P}\text{-}a.s. \tag{2.26}$$

Relation (2.25) means that the *hedge* $H(\theta)$ is a portfolio of the *financial* market. Recall, that the financial market M^F is complete and any t-portfolio features a self-financing replicating strategy until time t (cf. Lemma 2.4). However, (2.25) also implies that the hedging strategy does not react on biometric events happening after time 0. Due to (ii), as in the classical case, the *hedging method H* can not distinguish between similar (B-i.i.d.) individuals (cf. Principle 5). Property (iii) is also adopted from the classical case, where pointwise convergence is ensured by the Expectation Principle for appropriate insurance products combined with respective hedges (cf. Principle 7 and Section 2.5). Property (iii) is also related to Principle 4 in Section 2.2 as insurance companies should be able to cope with large classes of similar (B-i.i.d.) contracts.

Now, the main result of this chapter can be stated.

PROPOSITION 2.11. *Under the Axioms 1 - 4, 6 and 7, the minimum fair price π on Θ is uniquely determined by $\mathbb{M} = \mathbb{Q} \otimes \mathbb{B}$, i.e. for $\theta \in \Theta$ and $t \in \mathbb{T}$*

$$\pi_t(\theta) = S_t^0 \cdot \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\langle\theta, S_T\rangle / S_T^0 | \mathcal{F}_t \otimes \mathcal{B}_t]. \tag{2.27}$$

As already mentioned, this product measure approach to valuation is quite well established in the existing literature. However, the deduction by an axiomatic approach as well as the generality of the above result seem to be new.

Clearly, (2.12) is the special case of (2.27) in the presence of a deterministic financial market ($|\mathcal{F}_T| = 2$). As π is unique, it is at the same time the minimal valuation principle with the demanded properties. That means there is no other valuation principle under the setting of Axiom 1 - 4 that fulfills 6 and 7 and implies under the Principle of Equivalence (Axiom 5) lower premiums than (2.27). Actually, property (iii) of Axiom 7 ensures that insurance companies do not charge more than the cost of a more or less acceptable purely financial hedge for each product which is sold. So to speak, the minimum fair price is fair from the viewpoint of the insured, as well as from the viewpoint of the companies.

The following lemmas are needed in order to prove the proposition.

LEMMA 2.12. *On $(F \times B, \mathcal{F}_T \otimes \mathcal{B}_T)$ it holds that*

$$\mathbb{Q} \otimes \mathbb{B} \sim \mathbb{F} \otimes \mathbb{B}, \tag{2.28}$$

and for the Radon-Nikodym derivatives one has $\mathbb{F} \otimes \mathbb{B}$-a.s.

$$\frac{d(\mathbb{Q} \otimes \mathbb{B})}{d(\mathbb{F} \otimes \mathbb{B})} = \frac{d\mathbb{Q}}{d\mathbb{F}}. \tag{2.29}$$

Proof. For any $\mathcal{F}_T \otimes \mathcal{B}_T$-measurable set Z one has $\mathbb{Q} \otimes \mathbb{B}(Z) = 0$ if and only if $\mathbf{1}_Z = 0$ $\mathbb{Q} \otimes \mathbb{B}$-a.s. for the indicator function $\mathbf{1}_Z$ of Z. However, $\mathbf{1}_Z = 0$ $\mathbb{Q} \otimes \mathbb{B}$-a.s. if and only if $\mathbb{Q}$-a.s. $\mathbf{1}_Z(f, .) = 0$ $\mathbb{B}$-a.s. due to Lemma 2.8. But $\mathbb{Q} \sim \mathbb{F}$, i.e. $\mathbb{Q}$-a.s. and $\mathbb{F}$-a.s. are equivalent and $\mathbb{Q} \otimes \mathbb{B}(Z) = 0$ equivalent to $\mathbb{F} \otimes \mathbb{B}(Z) = 0$ follows. Hence, (2.28). For any $\mathcal{F}_T \otimes \mathcal{B}_T$-measurable set Z,

$$\mathbb{Q} \otimes \mathbb{B}(Z) = \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\mathbf{1}_Z] = \mathbf{E}_{\mathbb{Q}}[\mathbf{E}_{\mathbb{B}}[\mathbf{1}_Z]] \tag{2.30}$$

due to Fubini's Theorem. From the Fundamental Theorem $d\mathbb{Q}/d\mathbb{F}$ exists and is bounded, i.e.

$$\mathbb{Q} \otimes \mathbb{B}(Z) = \mathbf{E}_{\mathbb{F}}\left[\frac{d\mathbb{Q}}{d\mathbb{F}}\mathbf{E}_{\mathbb{B}}[\mathbf{1}_Z]\right] = \mathbf{E}_{\mathbb{F}\otimes\mathbb{B}}\left[\mathbf{1}_Z\frac{d\mathbb{Q}}{d\mathbb{F}}\right]. \tag{2.31}$$

□

LEMMA 2.13. *Under Axiom 1 and 2, one has for any $\theta \in \Theta$*

$$H^*(\theta) := \mathbf{E}_{\mathbb{B}}[\theta] \in \Theta^F. \tag{2.32}$$

There is a self-financing strategy replicating $H^*(\theta)$ and under Axiom 6

$$\pi_t(H^*(\theta)) = S_t^0 \cdot \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\langle\theta, S_T\rangle / S_T^0 | \mathcal{F}_t \otimes \mathcal{B}_0] \tag{2.33}$$

for $t \in \mathbb{T}$. Moreover, H^* fulfills properties (i), (ii) and (iii) of Axiom 7.

Proof. By Fubini's Theorem, $\mathbf{E}_{\mathbb{B}}[\theta(f,.)]$ exists $\mathbb{F}$-a.s. and $\mathbf{E}_{\mathbb{B}}[\theta]$ is $\mathbb{F}$-measurable and -integrable. Hence, by the completeness of M^F and uniqueness of $\mathbb{Q}$, the portfolio (2.32) can be replicated by the financial securities in M^F and has due to Axiom 6 and Remark 2.5 the price process

$$\pi_t(\mathbf{E}_{\mathbb{B}}[\theta]) = S_t^0 \cdot \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\langle \mathbf{E}_{\mathbb{B}}[\theta], S_T\rangle / S_T^0 | \mathcal{F}_t \otimes \mathcal{B}_0]. \tag{2.34}$$

$\langle \theta, S_T\rangle / S_T^0$ is $\mathbb{F}\otimes\mathbb{B}$-integrable, since $S_T^0 > 0$, $|\mathcal{F}_T| < \infty$ and each θ^i ($i = 0,\ldots,d-1$) is $\mathbb{F}\otimes\mathbb{B}$-integrable. By Lemma 2.12, (2.33) exists as (2.29) is bounded. Since $\mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\mathbf{E}_{\mathbb{B}}[X]|\mathcal{F}_t\otimes\mathcal{B}_0] = \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[X|\mathcal{F}_t\otimes\mathcal{B}_0]$ $\mathbb{P}$-a.s. for any $\mathbb{Q}\otimes\mathbb{B}$-integrable X (recall that $\mathcal{B}_0 = \{0, B\}$), (2.34) is identical to (2.33) $\mathbb{P}$-a.s. As we have $\mathbf{E}_{\mathbb{B}}[X] = \mathbf{E}_{\mathbb{F}\otimes\mathbb{B}}[X|\mathcal{F}_t\otimes\mathcal{B}_0]$ $\mathbb{P}$-a.s. for $\mathcal{F}_t\otimes\mathcal{B}_t$-measurable X, $H^*(\theta)$ is a t-portfolio. Property (*ii*) of Axiom 7 is obviously fulfilled. For any t-portfolios $\{{}^i\theta : i \in \mathbb{N}^+\}_K$ or $\{{}^i\theta : i \in \mathbb{N}^+\}_{B-i.i.d.}$, the Strong Law of Large Numbers (in the first case by Kolmogorov's Criterion) implies for almost all $f \in F$ that

$$\frac{1}{m}\sum_{i=1}^{m} \langle {}^i\theta(f,.) - H^*({}^i\theta)(f), S_t(f)\rangle \overset{m\to\infty}{\longrightarrow} 0 \quad \mathbb{B}\text{-a.s.} \tag{2.35}$$

Lemma 2.9 completes the proof. □

LEMMA 2.14. *Under Axiom 1 and 2, for any $\theta \in \Theta$, any $t \in \mathbb{T}$ and for $\mathbb{M} \in \{\mathbb{F}\otimes\mathbb{B}, \mathbb{Q}\otimes\mathbb{B}\}$*

$$\mathbf{E}_{\mathbb{M}}[\langle \theta - H^*(\theta), S_t\rangle] = 0. \tag{2.36}$$

Proof. By Fubini's Theorem. □

LEMMA 2.15. *Under the Axioms 1 - 4 and 6, any $H : \Theta \to \Theta^F$ fulfilling (i), (ii) and (iii) of Axiom 7 fulfills for any θ in some $\Theta_{B-i.i.d.}$*

$$\pi_t(H(\theta)) = S_t^0 \cdot \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\langle \theta, S_T\rangle / S_T^0 | \mathcal{F}_t \otimes \mathcal{B}_0], \quad t \in \mathbb{T}. \tag{2.37}$$

Roughly speaking, Lemma 2.15 states that there is no reasonable purely financial hedging method (i.e. a strategy not using biometric information) for the relevant portfolios with better convergence properties than (2.32). Even a hedging method with stronger than pointwise convergence, e.g. an additional L^p-convergence ($p \geq 1$), must follow (2.37) and has the same price process as (2.32) when fulfilling (i), (ii) and (iii) of Axiom 7.

Proof of Lemma 2.15. Consider to be given such an H as in Lemma 2.15 and a set $\{{}^i\theta, i \in \mathbb{N}^+\}_{B-i.i.d.}$ of portfolios that contains a given portfolio $\theta \in \Theta$. As any $\theta \in \Theta$ is a T-portfolio, Lemma 2.9 implies that $\mathbb{F}$-a.s.

$$\frac{1}{m}\sum_{i=1}^{m}\langle {}^i\theta(f,.) - H(\theta)(f), S_T(f)\rangle \stackrel{m\to\infty}{\longrightarrow} 0 \quad \mathbb{B}\text{-a.s.} \tag{2.38}$$

and by the Strong Law of Large Numbers one must have $\mathbb{F}$-a.s.

$$\langle H(\theta)(f), S_T(f)\rangle = \langle \mathbf{E}_{\mathbb{B}}[\theta(f,.)], S_T(f)\rangle. \tag{2.39}$$

Axiom 6 (2.23) and condition (NA) in M^F imply $\pi_t(H(\theta)) = \pi_t(\mathbf{E}_{\mathbb{B}}[\theta])$ $\mathbb{P}$-a.s. for $t \in \mathbb{T}$. Lemma 2.13 completes the proof. □

Proof of Proposition 2.11. From Lemma 2.12 one has that $\mathbb{Q}\otimes\mathbb{B} \sim \mathbb{F}\otimes\mathbb{B}$. Analogously to Lemma 2.13 one obtains that (2.27) exists. Hence, (2.27) fulfills Axiom 6 (cf. Remark 2.5 (2.8)). Furthermore, (2.27) is a minimum fair price in the sense of Axiom 7 since with $H = H^*$ one has (2.24) by

$$\mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\langle\theta, S_T\rangle/S_T^0] = \mathbf{E}_{\mathbb{Q}}[\langle H^*(\theta), S_T\rangle/S_T^0] \tag{2.40}$$

due to Fubini's Theorem and Lemma 2.13 shows that (i), (ii) and (iii) are fulfilled. Observe that (2.27) is a valuation principle since $(S_t/S_t^0)_{t\in\mathbb{T}}$ is a $\mathbb{Q}\otimes\mathbb{B}$-martingale and therefore $\pi_t(\theta_t) = \langle\theta_t, S_t\rangle$ for any t-portfolio $\theta_t \in \Theta$ (cf. Remark 2.5 and Definition 2.2). Now, uniqueness will be shown. Suppose that π is a minimum fair price in the sense of Axiom 7 and consider some $\{{}^i\theta,\ i \in \mathbb{N}^+\}_{B-i.i.d.}$. From Lemma 2.15 it is known that then $\pi_0({}^i\theta) = \pi_0(H^*({}^i\theta)) = \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\langle {}^i\theta, S_T\rangle/S_T^0]$ for all $i \in \mathbb{N}^+$. However, one can choose the set $\{{}^i\theta,\ i \in \mathbb{N}^+\}_{B-i.i.d.}$ such that ${}^1\theta = (\mathbf{1}_Z, 0, \ldots, 0)$, where $\mathbf{1}_Z$ is the indicator function of a cylinder set $Z = F' \times B_1 \times B_2 \times \ldots$ with $F' \in \mathcal{F}_T$ and $B_j \in \mathcal{B}_T^j$ for $j \in \mathbb{N}^+$ where $B_j \neq B^j$ for only finitely many j (Axiom 4 is crucial for the possibility of this choice!). Clearly, these cylinders form a $\cap$-stable generator for $\mathcal{M}_T$, the σ-algebra of the product space, and M itself is an element of this generator. One obtains $\pi_0({}^1\theta) = \mathbb{Q}\otimes\mathbb{B}(Z) = \mathbb{M}(Z)$ from (2.33) and (2.22). $\mathbb{M} = \mathbb{Q}\otimes\mathbb{B}$ follows from the coincidence of the measures on the generator. □

Axiom 7 (together with 6) could be interpreted as a *strong no-arbitrage principle* that fulfills (NA) and also precludes arbitrage-like strategies that have their origin in the Law of Large Numbers.

REMARK 2.16. The proofs of the Proposition 2.11 and the Lemmas 2.4, 2.13 - 2.15 also hold for $|\mathcal{F}_T| = \infty$ when $\Theta = (L^2(M, \mathcal{M}_T, \mathbb{P}))^d$, each $S_T^i \in L^2(F, \mathcal{F}_T, \mathbb{F})$ and $S_t^0 \geq c(t) > 0$ for all $t \in \mathbb{T}$.

EXAMPLE 2.17 (Arbitrage-like trading opportunities). Consider a set $\{^i\theta, i \in \mathbb{N}^+\}_{B-i.i.d.}$ of portfolios. The minimum fair price for each portfolio is given by (2.27) ($t = 0$). If an insurance company sells the products $\{^1\theta, \ldots, {}^m\theta\}$ at that prices, it can buy hedging portfolios such that the mean balance converges to zero almost surely with m (cf. Axiom 7, (*iii*)). However, if the company charges $\pi_0(^i\theta)+\epsilon$, where $\epsilon > 0$ is an additional fee and π is as in (2.27), there still is the hedge as explained above, but the gain ϵ per contract was made at $t = 0$. Hence, the safety load ϵ makes in the limit a deterministic money making machine out of the insurance company.

Example 2.17 directly points at the main difference between pricing in life insurance mathematics and financial mathematics. In financial markets such arbitrage-like strategies are not possible as there usually are not enough independent stocks. Furthermore, the stochastic behaviour of securities is by far not as good known as the stochastics of biometric events. Indeed, practitioners say that the probabilities from the biometric probability space are almost known for sure. Hence, biometric expectations can be computed with high accuracy whereas expectations in financial markets have the character of speculation. From this point of view, any possible EMM $\mathbb{M}'$ of the market $M^{F\times B}$ obtained from free trading of portfolios in $M^{F\times B}$ should be expected to be close to $\mathbb{Q}\otimes\mathbb{B}$. Any systematic deviation could give rise to arbitrage-like trading opportunities, as we have seen above.

REMARK 2.18 (Quadratic hedging). Consider an L^2-framework, i.e. the payoff $\langle\theta_t, S_t\rangle$ of any considered t-portfolio θ_t lies in $L^2(M, \mathcal{M}_t, \mathbb{P})$. As $\mathbb{P} = \mathbb{F}\otimes\mathbb{B}$, it can easily be shown that $\mathbf{E}_{\mathbb{B}}[.]$ is the orthogonal projection of $L^2(M, \mathcal{M}_t, \mathbb{P})$ onto its purely financial (and closed) subspace $L^2(F, \mathcal{F}_t, \mathbb{F})$. Standard Hilbert space theory implies that the payoff $\langle\mathbf{E}_{\mathbb{B}}[\theta_t], S_t\rangle = \mathbf{E}_{\mathbb{B}}[\langle\theta_t, S_t\rangle]$ of the hedge $H^*(\theta_t)$ is the best L^2-approximation of the payoff $\langle\theta_t, S_t\rangle$ of the t-portfolio θ_t by a purely financial portfolio in M^F. Furthermore, it can easily be shown that $\mathbb{M} = \mathbb{Q}\otimes\mathbb{B}$ minimizes $||d\mathbb{M}/d\mathbb{P} - 1||_2$ under the constraint $\mathbf{E}_{\mathbb{B}}[d\mathbb{M}/d\mathbb{P}] = d\mathbb{Q}/d\mathbb{F}$ which is implied by Axiom 6. Under some additional technical assumptions, this property is a characterization of the so-called *minimal martingale measure* in the time continuous case (cf. Schweizer (1995b), Møller (2001)). Hence, $\mathbb{Q}\otimes\mathbb{B}$ can be interpreted as the EMM which lies "next" to $\mathbb{P} = \mathbb{F}\otimes\mathbb{B}$ due to the L^2-metric. Beside the convergence properties discussed in this chapter, these are the most important (and "natural") reasons for the use of (2.27). The hedging method H^* considered in this chapter is not the so-called *mean-variance hedge* as it is known from the literature (cf. Bouleau and Lamberton (1989), Duffie and Richardson (1991)). The difference is that the mean-variance approach generally allows for *all* self-financing trading strategies in $M^{F\times B}$, i.e. also biometric events can influence the strategy in this case. However, the ideas are of course quite similar. An overview concerning hedging approaches in insurance can be found in Møller (2002).

2.7 Hedging

In this section, it is shown in which sense an insurance company can hedge its risk by products of the financial market - proposed the market is liquid enough. The technical assumptions are quite weak.

Suppose Axiom 1 to 4 and a set of life insurance contracts $\{({}^i\gamma_t, {}^i\delta_t)_{t\in\mathbb{T}} : i \in \mathbb{N}^+\}$ with $\{{}^i\gamma_t : i \in \mathbb{N}^+\}_K$ and $\{{}^i\delta_t : i \in \mathbb{N}^+\}_K$ for all $t \in \mathbb{T}$. Following hedging method H^* of Lemma 2.13, the portfolios (or strategies replicating) $\mathbf{E}_{\mathbb{B}}[{}^i\gamma_t]$ and $-\mathbf{E}_{\mathbb{B}}[{}^i\delta_t]$ are bought at time 0 for all $i \in \mathbb{N}^+$ and all $t \in \mathbb{T}$. Consider the balance of wins and losses at any time $t \in \mathbb{T}$. For the *mean total payoff per contract at time t* we have

$$\frac{1}{m}\sum_{i=1}^{m}\langle {}^i\delta_t - {}^i\gamma_t - \mathbf{E}_{\mathbb{B}}[{}^i\delta_t - {}^i\gamma_t], S_t\rangle \overset{m\to\infty}{\longrightarrow} 0 \quad \mathbb{P}\text{-a.s.} \tag{2.41}$$

due to Lemma 2.13. In analogy to Section 2.5, also the mean *final* balance converges to zero a.s., i.e.

$$\frac{1}{m}\sum_{i=1}^{m}\sum_{t=0}^{T}\langle {}^i\delta_t - {}^i\gamma_t - \mathbf{E}_{\mathbb{B}}[{}^i\delta_t - {}^i\gamma_t], S_T\rangle \overset{m\to\infty}{\longrightarrow} 0 \quad \mathbb{P}\text{-a.s.} \tag{2.42}$$

This kind of risk management is static in the sense that no trading strategy reacts on biometric events happening after time 0. This corresponds to the considerations in the classical case (Section 2.5). It was already mentioned in Remark 2.18 that the considered hedging method is not the so-called mean-variance hedging. Another more comprehensive but not self-financing hedging approach are the so-called *risk-minimizing strategies* (e.g. Møller (1998, 2001)).

REMARK 2.19. Due to Lemma 2.14, any of the balances in (2.41) and (2.42) has expectation 0 under the physical probability measure $\mathbb{P} = \mathbb{F} \otimes \mathbb{B}$.

Until now, premium calculation has not played any role in this section. However, if the Principle of Equivalence (2.11) is applied under the minimum fair price (2.27), one obtains for all $i \in \mathbb{N}^+$

$$\sum_{t=0}^{T} \pi_0(\mathbf{E}_{\mathbb{B}}[-{}^i\delta_t + {}^i\gamma_t]) = \sum_{t=0}^{T} \pi_0({}^i\delta_t - {}^i\gamma_t) = 0. \tag{2.43}$$

REMARK 2.20. Under (2.11) and (2.27), a life insurance company can without any costs at time 0 (!) pursue a self-financing trading strategy such that the mean balance per contract at any time t converges to zero almost surely for an increasing number of individual contracts. The realization of such a hedge for real world insurance companies would demand the precise knowledge of the second order base given by the Axioms 1 to 4 (see also Remark 2.1).

Remark 2.20 is perhaps the result with the strongest practical impact. In contrast to other, more comprehensive hedging methods, the presented method has the advantage that there is no need for the hedger to take into account the biometric development of each individual. The information available at the time of contract subscription ($t = 0$) is sufficient and all strategies are self-financing.

EXAMPLE 2.21 (Traditional contracts with stochastic interest rates). Consider a life insurance contract which is for the i-th individual given by two cash flows $({}^i\gamma_t)_{t\in\mathbb{T}} = (\frac{{}^iC_t}{S_t^0}e_0)_{t\in\mathbb{T}}$ and $({}^i\delta_t)_{t\in\mathbb{T}} = (\frac{{}^iD_t}{S_t^0}e_0)_{t\in\mathbb{T}}$ with $\mathbb{T} = \{0, 1, \ldots, T\}$ in years. Assume that ${}^i\gamma_t = {}^i\delta_t = 0$ for t greater than some $T_i \in \mathbb{T}$, i.e. the contract has an expiration date T_i, and that each ${}^iC_t$ is for $t \leq T_i$ given by ${}^iC_t(f, b) = {}^ic\,{}^i\beta_t^\gamma(b^i)$ for all $(f, b) = (f, b^1, b^2, \ldots) \in M$ where ${}^ic$ is a positive constant. Let $({}^i\delta_t)_{t\in\mathbb{T}}$ be defined analogously with the variables ${}^iD_t, {}^id$ and ${}^i\beta_t^\delta$. Suppose that ${}^i\beta_t^{\gamma(\delta)}$ is $\mathcal{B}_t^i$-measurable with ${}^i\beta_t^{\gamma(\delta)}(b^i) \in \{0, 1\}$ for all $b^i \in B^i$ ($t \leq T_i$). For the following have in mind that the portfolio e_0/S_t^0 can be interpreted as the guaranteed payoff of one currency unit at time t. This kind of contract is called a *zero-coupon bond with maturity* t and its price at time $s < t$ is denoted by $p(s, t - s) = \pi_s(e_0/S_t^0)$ where $t - s$ is the time to maturity and $p(s, 0) := 1$ for all $s \in \mathbb{T}$.

1. Traditional life insurance. Suppose that for $t \leq T_i$ one has ${}^i\beta_t^\gamma = 1$ if and only if the i-th individual has died in $(t-1, t]$ and for $t < T_i$ that ${}^i\beta_t^\delta = 1$ if and only if the i-th individual is still alive at t, but ${}^i\beta_{T_i}^\delta \equiv 0$. Assume that i is alive at $t = 0$. Clearly, this contract is a life insurance with fixed annual premiums ${}^id$ and the benefit ${}^ic$ in the case of death. $\mathbf{E}_{\mathbb{B}}[{}^i\beta_t^\gamma]$ and $\mathbf{E}_{\mathbb{B}}[{}^i\beta_t^\delta]$ are mortality, respectively survival probabilities. This data can be obtained from so-called mortality tables. Usually, the notation is ${}_{t-1|1}q_x = \mathbf{E}_{\mathbb{B}}[{}^i\beta_t^\gamma]$ $(t > 0)$ and ${}_tp_x = \mathbf{E}_{\mathbb{B}}[{}^i\beta_t^\delta]$ $(0 < t < T_i)$ for an individual of age x (cf. Gerber (1997); for convenience reasons, the notation ${}_{-1|1}q_x = 0$ and ${}_0p_x = 1$ is used in the following). The hedge H^* for ${}^i\delta_t - {}^i\gamma_t$ is for $t < T_i$ given by the number of $({}^ic\,{}_{t-1|1}q_x - {}^id\,{}_tp_x)$ zero-coupon bonds with maturity t, and for $t = T_i$ by ${}^ic\,{}_{T_i-1|1}q_x$ zero-coupon bonds with maturity T_i.

2. Endowment. Assume for $t < T_i$ that ${}^i\beta_t^\gamma = 1$ if and only if the i-th individual has died in $(t-1, t]$, but ${}^i\beta_{T_i}^\gamma = 1$ if and only if i has died in $(T_i-1, T_i]$ or is still alive at T_i. Furthermore, ${}^i\beta_t^\delta = 1$ if and only if the i-th individual is still alive at $t < T_i$, but ${}^i\beta_{T_i}^\delta \equiv 0$. Assume that i is alive at $t = 0$. This contract is a so-called endowment that features fixed annual premiums ${}^id$ and the benefit ${}^ic$ in the case of death, but also the payoff ${}^ic$ when i is alive at T_i. The hedge H^* due to ${}^i\delta_t - {}^i\gamma_t$ is for $t < T_i$ given by the number of $({}^ic\,{}_{t-1|1}q_x - {}^id\,{}_tp_x)$ zero-coupon bonds with maturity t, and for $t = T_i$ by ${}^ic\,({}_{T_i-1|1}q_x + {}_{T_i}p_x)$ zero-coupon bonds with maturity T_i.

Actually, in the case of traditional contracts, all hedging can be done by zero-coupon bonds (which is also called *matching*).

EXAMPLE 2.22 (Unit-linked products). The case of a unit-linked product is interesting if and only if the product is not the sum of a traditional life insurance contract and a simple funds policy (which is often the case in practice). So, let us assume that the contract is given by a cash flow of constant premiums $({}^i\delta_t)_{t\in\mathbb{T}}$ as in Example 2.21 and a flow of benefits $({}^i\gamma_t)_{t\in\mathbb{T}}$ such that ${}^i\gamma_t(f, b) = {}^i\theta_t \cdot {}^ic\,{}^i\beta_t^\gamma(b^i)$ for all $(f, b) \in M$ where ${}^i\theta_t \in \Theta^F$ is an arbitrary purely financial t-portfolio and all other notations are the same as in the introduction of Example 2.21. For instance, one could consider a number of shares of an index, or a number of assets together with the respective European Puts which ensure a certain level of benefit (i.e. a "unit-linked product with guarantee"). The strategy due to ${}^i\delta_t - {}^i\gamma_t$ is given by ${}^ic \cdot \mathbf{E}_{\mathbb{B}}[{}^i\beta_t^\gamma]$ times the replicating strategy of ${}^i\theta_t$ minus $({}^id \cdot \mathbf{E}_{\mathbb{B}}[{}^i\beta_t^\delta])$ zero-coupon bonds maturing at time t. In particular, for ${}^i\theta_t$ being a constant portfolio, the strategy is obviously very simple as the portfolio must not be replicated, but can be bought directly.

REMARK 2.23. The technical assumption (K) which is sufficient for the convergence of (2.41) (cf. Definition 2.10 (iii)) and which is demanded at the very beginning of this section will be discussed now. In the case of traditional life insurances as in Example 2.21, the realistic condition ${}^ic, {}^id \leq const \in \mathbb{R}^+$ for all $i \in \mathbb{N}^+$ implies (K) for the sets $\{{}^i\gamma_t : i \in \mathbb{N}^+\}$ and $\{{}^i\delta_t : i \in \mathbb{N}^+\}$ for all $t \in \mathbb{T}$. In the case of unit-linked products, suppose that there are only finitely many possible portfolios ${}^i\theta_t$ for each $t \in \mathbb{T}$ (which is also quite realistic as often shares of one single funds are considered). Under this assumption, again ${}^ic, {}^id \leq const \in \mathbb{R}^+$ for all $i \in \mathbb{N}^+$ implies (K) for the sets $\{{}^i\gamma_t : i \in \mathbb{N}^+\}$ and $\{{}^i\delta_t : i \in \mathbb{N}^+\}$ for all $t \in \mathbb{T}$. Hence, (K) is no drawback for practical purposes.

2.8 Incomplete financial markets

Until now, the theory presented in this chapter assumed complete and arbitrage-free markets (cf. Axiom 2), which reduces the number of explicit market models that can be considered. However, some of the concepts work (under some restrictions) with incomplete market models.

In particular, it is now assumed that in Axiom 2 completeness of the market model M^F and uniqueness of the EMM $\mathbb{Q}$ is *not* demanded, but $\mathbb{Q} \sim \mathbb{F}$ and $d\mathbb{Q}/d\mathbb{F}$ bounded. Let us enumerate the altered axiom by 2' and define

$$\Theta^F = \{\theta : \theta \text{ replicable by a self-financing strategy in } M^F\} \tag{2.44}$$

$$\Theta = \{\theta : \theta \in (L^1(M, \mathcal{M}_T, \mathbb{P}))^d \text{ and } \mathbf{E}_{\mathbb{B}}[\theta] \in \Theta^F\}. \tag{2.45}$$

It is well-known from the theory of financial markets that *any* EMM $\mathbb{Q}$ fulfills pricing formula (2.3) for any replicable portfolio $\theta \in \Theta^F$. Now, with Θ^F and Θ as defined above and Axiom 2 replaced by 2', it can easily be checked that the Lemmas 2.12 - 2.15 still hold. Concerning Proposition 2.11, π as defined in (2.27) is for *any* financial EMM $\mathbb{Q}$ a minimum fair price. Hence, uniqueness seems to be lost. However, for any minimum fair price one still has that π_0 is unique on (2.45). The reason is that for any $\theta \in \Theta$ and any two EMM $\mathbb{Q}$ and $\underline{\mathbb{Q}}$ of M^F

$$\mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\langle\theta, S_T\rangle / S_T^0] = \mathbf{E}_{\underline{\mathbb{Q}}\otimes\mathbb{B}}[\langle\theta, S_T\rangle / S_T^0] \tag{2.46}$$

due to Fubini's Theorem and the (NA)-condition. Hence, pricing at time $t = 0$ (i.e. present values) and hedging (cf. Section 2.7) still work as in the case of complete financial markets.

In the presence of arbitrage opportunities, the existence of an equivalent martingale measure gets lost. Nonetheless, assume a financial market model M^F which is neither necessarily arbitrage-free, nor complete and suppose that there is a valuation principle π^F used in M^F on a set Θ^F of purely financial portfolios which are taken into consideration (this does not mean absence of arbitrage). Under the considered Θ^F, define Θ by (2.45) and for any $\theta \in \Theta$

$$\pi_0(\theta) = \pi_0^F(\mathbf{E}_{\mathbb{B}}[\theta]), \tag{2.47}$$

which is the price of the hedge H^* at time 0 (compare with (2.24) and (2.33) for t = 0). In an L^2-framework as in Remark 2.18, i.e. if we have for any t that $\langle \Theta, S_t \rangle \subset L^2(M, \mathcal{M}_T, \mathbb{P})$, $\mathbf{E}_{\mathbb{B}}[\theta]$ is the best approximation in Θ^F to any $\theta \in \Theta$ in the L^2-sense (cf. Remark 2.18). Even if we do not assume the L^2-framework, the properties (i), (ii) and (iii) of Axiom 7 are still fulfilled for the above defined Θ and for H^* as in (2.32). Hence, π_0 satisfies the demand for converging balances as stated in Principle 7 of Section 2.2 and the expressions (2.41) and (2.42) are still valid. For these reasons, (2.47) is a rather sensible valuation principle.

2.9 Historical pricing example

Let us consider the traditional contracts as described in Example 2.21. Due to the Equivalence Principle (2.11), we demand

$$\pi_0\left(\sum_{t=0}^{T_i} {}^i c \, {}^i\beta_t^\gamma e_0 / S_t^0\right) = \pi_0\left(\sum_{t=0}^{T_i} {}^i d \, {}^i\beta_t^\delta e_0 / S_t^0\right). \tag{2.48}$$

Now, suppose that the minimum fair price π from (2.27), respectively the valuation principle (2.47), is applied for premium calculation. Clearly,

$$\frac{{}^i d}{{}^i c} = \sum_{t=0}^{T_i} p(0,t) \cdot \mathbf{E}_{\mathbb{B}}[{}^i\beta_t^\gamma] \Big/ \sum_{t=0}^{T_i} p(0,t) \cdot \mathbf{E}_{\mathbb{B}}[{}^i\beta_t^\delta] \tag{2.49}$$

where $p(0,t)$ is the price of a zero-coupon bond as defined in Section 2.7. An important consequence of (2.49) is that the quotient ${}^i d / {}^i c$ (minimum fair premium/benefit) depends on the zero-coupon bond prices (or yield curve) at time 0. As the term structure of interest rates varies from day to day, this particularly means that ${}^i d / {}^i c$ varies from day to day and therefore depends on the day of underwriting (actually, it depends on the exact time).

Insurance companies do not determine the prices for products daily. Hence, they give rise to financial risks as the contracts may be over-valued.

Now, assume that any time value is given in fractions of years. The so-called *spot (interest) rate* $R(t,\tau)$ for the time interval $[t, t+\tau]$ is defined by

$$R(t,\tau) = -\frac{\log p(t,\tau)}{\tau}. \tag{2.50}$$

The *short rate* $r(t)$ at t is defined by $r(t) = \lim_{\tau \to 0} R(t,\tau)$, where the limit is assumed to exist. The *yield curve* at time t is the mapping with $\tau \mapsto R(t,\tau)$ for $\tau > 0$ and $0 \mapsto r(t)$. Figure 2.5 on page 44 shows the historical yield structure (i.e. the set of yield curves) of the German debt securities market from September 1972 to April 2003 (the 368 values are taken from the end of each month). The maturities' range is 0 to 28 years. The values for $\tau > 0$ were computed via a parametric presentation of yield curves (the so-called Svensson-method; cf. Schich (1997)) for which the parameters can be taken from the Internet page of the German Federal Reserve (*Deutsche Bundesbank*; `http://www.bundesbank.de`). The implied Bundesbank values R' are estimates of *discrete* interest rates on notional zero-coupon bonds based on German Federal bonds and treasuries (cf. Schich, 1997) and have to be converted to continuously compounded interest rates (as implicitly used in (2.50)) by $R = \ln(1+R')$. As an approximation for the short rate, the day-to-day money rates from the Frankfurt market (*Monatsdurchschnitt des Geldmarktsatzes für Tagesgeld am Frankfurter Bankplatz*; also available at the Bundesbank homepage) are taken and converted into continuous rates. Actually, the short rate is not used in the following but completes Figure 2.5.

Equation (2.50) shows that interest rates (yields) and zero-coupon bond prices contain the same information, namely the present value of a non-defaultable future payoff. As there is a yield curve given for any time t of the considered historical time axis, it is possible to compute the historical value of ${}^id/{}^ic$ for t (which is the date when the respective contract was signed) via (2.50) and (2.49). Doing so, one obtains

$$\frac{{}^id}{{}^ic}(t) = \sum_{\tau=0}^{T_i} p(t,\tau)\,{}_{\tau-1|1}q_x(t) \Big/ \sum_{\tau=0}^{T_i-1} p(t,\tau)\,{}_{\tau}p_x(t) \tag{2.51}$$

for the traditional life insurance and

$$\frac{{}^id}{{}^ic}(t) = \left(p(t,T_i)\,{}_{T_i}p_x(t) + \sum_{\tau=0}^{T_i} p(t,\tau)\,{}_{\tau-1|1}q_x(t) \right) \Big/ \sum_{\tau=0}^{T_i-1} p(t,\tau)\,{}_{\tau}p_x(t) \tag{2.52}$$

for the endowment (cf. Example 2.21). The values ${}_{\tau-1|1}q_x$ ($\tau > 0$) and ${}_{\tau}p_x$ ($0 < \tau < T_i$) are taken from (or computed by) the DAV (*Deutsche Aktuarvereinigung*) mortality table "1994 T" (Loebus, 1994), the value ${}_{T_i}p_x$ is computed by the table "1994 R" (Schmithals and Schütz, 1995). The reason for the different tables is that in actuarial practice mortality tables contain safety loads which depend on whether the death of a person is in (financial) favour of the insurance company, or not. In this sense, the used mortality tables are first order tables (cf. Remark 2.1). Clearly, the use of internal second order tables of real life insurance companies would be more appropriate. However, for competitive reasons they are usually not published. All probabilities mentioned above are considered to be constant in time. Especially, to make things easier, there is no "aging shift" applied to table "1994 R".

Now, consider a man of age $x = 30$ years and the time axis $\mathbb{T} = \{0, 1, \ldots, 10\}$ (in years). In Figure 2.1, the rescaled quotients (2.51) and (2.52) are plotted for the above setup. For comparison reasons: the absolute values at the starting point (September 1972) are ${}^id/{}^ic = 0.063792$ for the endowment, respectively ${}^id/{}^ic = 0.001587$ for the life insurance. The plot nicely shows the dynamics of the quotients and hence of the minimum fair premiums ${}^id$ if the benefit ${}^ic$ is assumed to be constant. The premiums of the endowment seem to be much more subject to the fluctuations of the interest rates than the premiums of the traditional life insurance. For instance, the minimum fair annual premium ${}^id$ for the 10-years endowment with a benefit of ${}^ic = 100,000$ Euros was 5,285.55 Euros at the 31st July 1974 and 8,072.26 at the 31st January 1999. For the traditional life insurance (with the same benefit), one obtains ${}^id = 152.46$ Euros at the 31st July 1974 and 168.11 at the 31st January 1999 (cf. Table 2.1).

If one assumes a discrete technical (= first order) rate of interest R'_{tech}, e.g. 0.035, which is the mean of the interest rates legally guaranteed by German life insurers, one can compute technical quotients ${}^id_{\text{tech}}/{}^ic$ by computing the technical values of zero-coupon bonds, i.e. $p_{\text{tech}}(t,\tau) = (1 + R'_{\text{tech}})^{-\tau}$, and plugging them into (2.51), resp. (2.52). If a life insurance company charges the technical premiums ${}^id_{\text{tech}}$ instead of the minimum fair premiums ${}^id$ and if one considers the valuation principle (2.27), respectively (2.47), to be a reasonable choice, the *present value* of the considered insurance contract at time t is

$$ {}^iPV = ({}^id_{\text{tech}} - {}^id) \cdot \sum_{\tau=0}^{T_i-1} p(t,\tau)\, {}_\tau p_x(t) \tag{2.53} $$

due to the Principle of Equivalence, respectively (2.48). In particular, this means that the insurance company can book the gain or loss (2.53) in the mean (or limit; cf. Example 2.17 and Remark 2.19) at time 0 as long as proper risk management (as described in Section 2.7) takes place afterwards. Thus, the present value (2.53) is a measure for the profit, or simply *the expected* *discounted profit* of the considered contract if one neglects all additional costs and the fact that first order mortality tables are used.

Figure 2.2 shows the historical development of ${}^{i}PV/{}^{i}c$ (present value/benefit) for the 10-years endowment as described above (solid line). For instance, the present value ${}^{i}PV$ of a 10-years endowment with a benefit of ${}^{i}c = 100,000$ Euros was 20,398.70 Euros at July 31, 1974. At the 31st January 1999, it was worth 2,578.55 Euros, only. The situation gets even worse in the case of a technical (or promised) rate of interest $R'_{\text{tech}} = 0.050$ (dashed line) - which is quite little in contrast to formerly promised returns of e.g. German life insurers. At the 31st January 1999, such a contract was worth -3,141.95 Euros, i.e. the contract actually produced a loss in the mean. Some present values of the 10-years traditional life insurance can be found in Table 2.1 on page 41.

All computations from above have also been carried out for a 25-years endowment, respectively life insurance (cf. Table 2.1). The corresponding figures are 2.3 and 2.4. Concerning Figure 2.3, the absolute values at the starting point (September 1972) are ${}^{i}d/{}^{i}c = 0.013893$ for the endowment, respectively ${}^{i}d/{}^{i}c = 0.002553$ for the life insurance. The minimum fair premium ${}^{i}d$ for the 25-years endowment with benefit ${}^{i}c = 100,000$ Euros was 808.39 Euros at the 31st July 1974 and 2,177.32 Euros at the 31st January 1999. For the traditional life insurance (with the same benefit), one obtains ${}^{i}d = 216.37$ Euros at the 31st July 1974 and 303.90 at the 31st January 1999. Hence, the premium-to-benefit ratio for both types of contracts seems to be more dependent on the yield structure than in the 10-years case. However, compared to the 10-years contracts, the longer running time seems to stabilize the present values of the contracts (cf. Table 2.1 and Figure 2.4). Nonetheless, they are still strongly depending on the yield structure.

The examples have shown the importance of realistic valuation principles in life insurance. Any premium calculation method and all related parameters (like e.g. technical rates of interest, which have to be determined in some way) should be carefully examined in order to be properly prepared for the fluctuations of financial markets. There is no doubt that many of the financial problems of life insurance companies that have arisen in the past few years could have been avoided by a proper use of modern valuation principles and - perhaps even more important - modern financial hedging strategies.

2.10 Figures and tables

Date	1974/07/31	1999/01/31
Traditional life insurance: 10 years		
Techn. premium ${}^{i}d_{\text{tech}}$ ($R'_{\text{tech}} = 0.035$)	168.94	
Techn. premium ${}^{i}d_{\text{tech}}$ ($R'_{\text{tech}} = 0.050$)	165.45	
Minimum fair annual premium ${}^{i}d$	152.46	168.11
Present value ${}^{i}PV$ ($R'_{\text{tech}} = 0.035$)	108.90	7.17
Present value ${}^{i}PV$ ($R'_{\text{tech}} = 0.050$)	85.84	-22.80
Traditional life insurance: 25 years		
Techn. premium ${}^{i}d_{\text{tech}}$ ($R'_{\text{tech}} = 0.035$)	328.02	
Techn. premium ${}^{i}d_{\text{tech}}$ ($R'_{\text{tech}} = 0.050$)	303.27	
Minimum fair annual premium ${}^{i}d$	216.37	303.90
Present value ${}^{i}PV$ ($R'_{\text{tech}} = 0.035$)	1,009.56	376.84
Present value ${}^{i}PV$ ($R'_{\text{tech}} = 0.050$)	785.80	-9.83
Endowment: 10 years		
Techn. premium ${}^{i}d_{\text{tech}}$ ($R'_{\text{tech}} = 0.035$)	8,372.65	
Techn. premium ${}^{i}d_{\text{tech}}$ ($R'_{\text{tech}} = 0.050$)	7,706.24	
Minimum fair annual premium ${}^{i}d$	5,285.55	8,072.26
Present value ${}^{i}PV$ ($R'_{\text{tech}} = 0.035$)	20,398.70	2,578.55
Present value ${}^{i}PV$ ($R'_{\text{tech}} = 0.050$)	15,995.27	-3,141.95
Endowment: 25 years		
Techn. premium ${}^{i}d_{\text{tech}}$ ($R'_{\text{tech}} = 0.035$)	2,760.85	
Techn. premium ${}^{i}d_{\text{tech}}$ ($R'_{\text{tech}} = 0.050$)	2,255.93	
Minimum fair annual premium ${}^{i}d$	808.39	2,177.32
Present value ${}^{i}PV$ ($R'_{\text{tech}} = 0.035$)	17,655.42	9,118.39
Present value ${}^{i}PV$ ($R'_{\text{tech}} = 0.050$)	13,089.53	1,228.34

Table 2.1: Selected (extreme) values due to different contracts for a 30 year old man (fixed benefit: ${}^{i}c = 100,000$ Euros)

Figure 2.1: Rescaled plot of the quotient ${}^{i}d/{}^{i}c$ (minimum fair annual premium/benefit) for the 10-years endowment (solid), resp. life insurance (dashed), for a 30 year old man

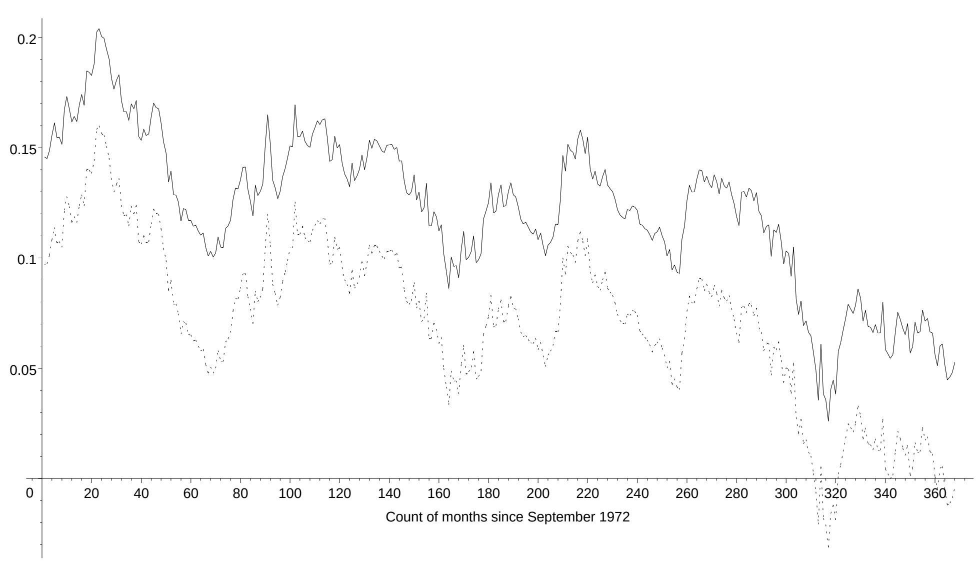

Figure 2.2: ${}^{i}PV/{}^{i}c$ (present value/benefit) for the 10-years endowment under a technical interest rate of 0.035 (solid) and 0.050 (dashed)

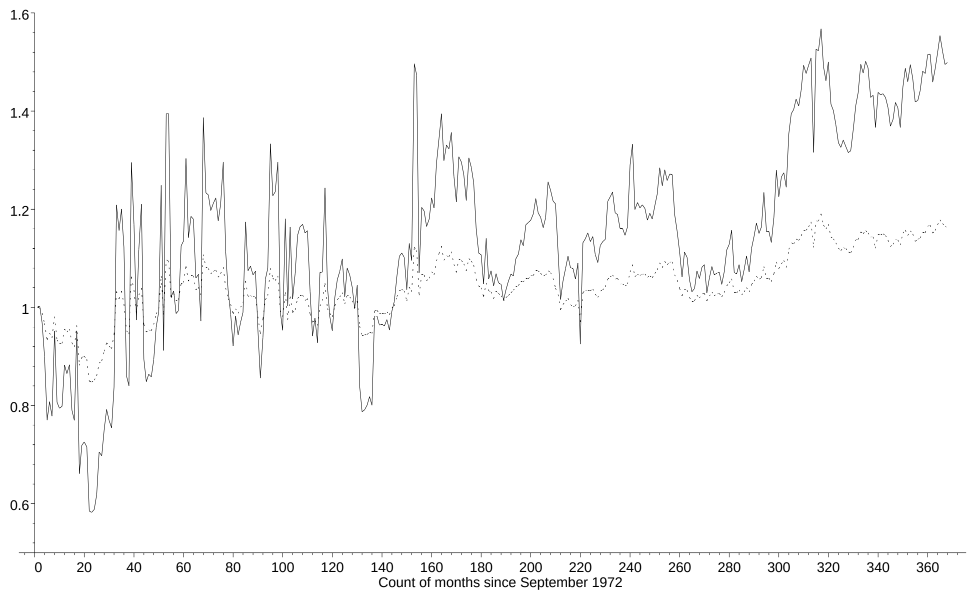

Figure 2.3: Rescaled plot of the quotient ${}^{i}d/{}^{i}c$ (minimum fair annual premium/benefit) for the 25-years endowment (solid), resp. life insurance (dashed), for a 30 year old man

Figure 2.4: ${}^{i}PV/{}^{i}c$ (present value/benefit) for the 25-years endowment under a technical interest rate of 0.035 (solid) and 0.050 (dashed)

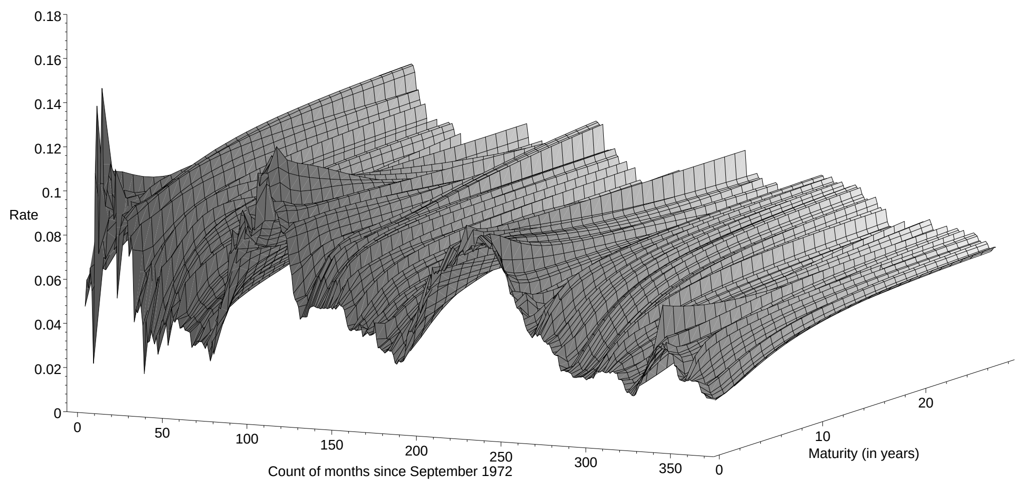

Figure 2.5: Historical yields of the German debt securities market

Chapter 3

On the decomposition of risk in life insurance

3.1 Introduction

Modern life insurance has to cope with two different kinds of risk. On the one side, there is *biometric risk* which is the classical subject of life insurance mathematics. On the other side, there is *financial risk* which comes to life insurance by financial markets, for example by stochastic interest rates or products like unit-linked life insurance policies. The modern actuary - called the *Actuary of the Third Kind* in Bühlmann (1987) - has to deal with both types of risk.

Life insurance mathematics has developed fast during the last twenty years and for many particular problems, for instance pricing, hedging and bonus theory, solutions have been developed. Nonetheless, the problem of the decomposition of gains (or risks) into biometric and financial parts has not yet been sufficiently considered, especially not with respect to the needs of modern life insurance, i.e. in the presence of stochastic financial markets. As already mentioned in Section 1.3, information on how much of the win or loss of an insurance company during a certain time interval is caused by financial, respectively biometric events is crucial for the understanding and the management of the company. Also on the single contract level risk decomposition is important as a client usually participates in financial wins belonging to his/her contract (= bonus payments), whereas financial losses remain in the company. For these reasons, risk decomposition and the understanding how biometric risk contributions can be pooled and coped with by the respective companies, which should actually be their core competence, is the subject of this chapter.

It must be mentioned that the above explained bonus problem is usually considered in a different context which comes from the practical needs of real life insurance companies (compare Norberg (1999, 2001) and Remark 3.6). Due to the more theoretical context of this chapter, we will *not* treat bonus theory in the usual sense, here. Differences will become clear at a later stage. However, a review of existing bonus theory with consideration of the results of this chapter may be a topic of future research.

In particular, there is the following connection between the risk decomposition proposed in this chapter, the pooling of biometric risks and locally variance-optimal hedging:

Under the assumption of a complete arbitrage-free financial market and a product space model for the biometric and financial events, the alternation $PV_t - PV_s$ of the present value (computed by the minimum fair price, cf. Chapter 2) of a life insurance contract from time s to time t ($s < t$) (called *gain* or *risk*; a precise definition follows later) is uniquely decomposed into a biometric and a financial part such that the financial part can from time s on be replicated by a self-financing purely financial trading strategy and the biometric residual is L^2- (and therefore variance-) minimal and has expectation 0 conditioned on s. The decomposition is done by means of orthogonal projections. Under certain reasonable assumptions, the biometric part of the gains does not depend on the investment strategy of the company. Furthermore, it is shown how a certain purely financial self-financing strategy of price 0 at s, which hedges away the financial part (except for a non-stochastic residual, seen from s), leads to the locally variance-optimal present value at time t seen from s. PV_t is then exactly how PV_s would have developed when invested into a riskless bond (maturing at t), plus the remaining biometric risk contribution. Reiteration of the locally variance-optimal hedge for a contract which was fairly priced at the time of underwriting, i.e. which had the present value zero then, implies that (under some restrictions) the mean discounted total gain from the first m contracts converges to zero almost surely for $m \to \infty$ when clients are independent. Actually, this is a corollary of a proposition that proves that the mean aggregated discounted biometric risk contribution per client converges to zero a.s. for an increasing number of independent clients. This property can for good reasons be called "pooling".

The section content is as follows. After the introduction, the second section introduces a model that is similar to the one used in Chapter 2. The difference is the finiteness of the biometric state space. A lemma on the replication of portfolios in the proposed product space framework is given. Section 3.3 motivates the central problems that are considered in this chapter, i.e. the decomposition of gains, pooling and the so-called AFIR-problem (cf. Bühlmann, 1995) which concerns the pricing and hedging of the positive financial parts of the gains. A list of four reasonable properties for the desired risk decomposition is compiled. Section 3.4 explains the role of the investment portfolio or strategy of an insurance company. It is shown that the financial risk of a life insurance company actually depends on its trading strategy. This seems to be obvious - nonetheless, the fact is for instance completely neglected by the so-called stochastic discounting method (Bühlmann, 1992). Section 3.5 is dedicated to a principle for the unique decomposition of gains into a biometric (technical) and a financial part. This principle fulfills the four properties mentioned above. Orthogonality plays a fundamental role, here. In Section 3.6 and 3.7, several implications of the presented method are deduced and discussed. In particular, a locally variance-optimal hedging method which is related to the proposed decomposition is considered. Some of the results have already been mentioned above. We also propose a general solution of the AFIR-problem. Section 3.8 shows that in a certain setup the mean accumulated discounted biometric risk contribution per contract converges to zero a.s. for an increasing number of individuals under contract. This is an important result concerning (actually, to some extent, *defining*) the "pooling" of biometric risks. Section 3.9 is on the open problem of multiperiod risk decomposition. Section 3.10 is a short review of the stochastic discounting and risk decomposition approach of Bühlmann. Some problems arising from these techniques are discussed. In the Appendix, several lemmas concerning conditional expectations can be found.

3.2 The model

We use the definitions and Axiom 1 and 2 of Section 2.3. Furthermore, the valuation principle π of Proposition 2.11 is used, i.e. the value of any $\mathcal{F}_t \otimes \mathcal{B}_t$-measurable ($t \in \{0, 1, \ldots, T\} = \mathbb{T}$) and $\mathbb{F} \otimes \mathbb{B}$-integrable portfolio θ_t at time

$s \leq t$ is supposed to be

$$\begin{aligned} \pi_s(\theta_t) &= S_s^0 \cdot \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\langle\theta_t, S_T\rangle / S_T^0 | \mathcal{F}_s \otimes \mathcal{B}_s] \\ &= S_s^0 \cdot \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\langle\theta_t, S_t\rangle / S_t^0 | \mathcal{F}_s \otimes \mathcal{B}_s]. \end{aligned} \tag{3.1}$$

The second line follows from the fact that $(S_t/S_t^0)_{t\in\mathbb{T}}$ is a $\mathbb{Q}$- and therefore a $\mathbb{Q}\otimes\mathbb{B}$-martingale. For a deduction of (3.1) and an explanation of the concept of a valuation principle see Chapter 2. Please note that we propose a complete, arbitrage-free financial market model M^F with a unique EMM $\mathbb{Q}$.

We do not apply Axiom 3 and 4 of Chapter 2 as finite biometric state spaces are sufficient for the most considerations in this chapter. Actually, we will usually consider only *one* life in finite time, except for Section 3.8. For the development of the biometric information we propose a filtration $(\mathcal{B}_t)_{t\in\mathbb{T}}$ with $|\mathcal{B}_T| < \infty$. As the financial market is complete, we also assume $|\mathcal{F}_T| < \infty$ (cf. Chapter 2) and therefore $|\mathcal{F}_T \otimes \mathcal{B}_T| < \infty$. In particular, for $t \in \mathbb{T}$ one has that $L^p(F \times B, \mathcal{F}_t \otimes \mathcal{B}_t, \mathbb{F}\otimes\mathbb{B})$ denotes the same set for all $p \in [0,\infty]$, namely the set of all real-valued measurable functions on $(F \times B, \mathcal{F}_t \otimes \mathcal{B}_t, \mathbb{F}\otimes\mathbb{B})$. The set Θ of portfolios in $M^{F\times B}$ which are taken into consideration is therefore given by

$$\Theta = (L^0(F \times B, \mathcal{F}_T \otimes \mathcal{B}_T, \mathbb{F}\otimes\mathbb{B}))^d \tag{3.2}$$

and the M^F-portfolios analogously by $\Theta^F = (L^0(F, \mathcal{F}_T, \mathbb{F}))^d$.

We will encounter situations where it is more comfortable to use a valuation principle directly defined for payoffs instead for portfolios.

DEFINITION 3.1. *For any $s \leq t$, $s,t \in \mathbb{T}$ and any $X \in L^0(M, \mathcal{M}_t, \mathbb{P})$*

$$\Pi_s^t(X) := \pi_s(X/S_t^0 \cdot e_0) = S_s^0 \cdot \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[X/S_t^0 | \mathcal{F}_s \otimes \mathcal{B}_s]. \tag{3.3}$$

Actually, (3.3) is well-defined, as the conditional expectation exists.

Please note that Lemma 2.4 showed that any $\mathcal{F}_t$-measurable portfolio and any $\mathcal{F}_t$-measurable payoff can be replicated until t by a s.f. financial strategy. The following lemma will be useful.

LEMMA 3.2. *For all $s \leq t$ and any $X \in L^0(M, \mathcal{F}_t \otimes \mathcal{B}_s, \mathbb{P})$ there exists a $\mathcal{F}_t \otimes \mathcal{B}_s$-measurable portfolio θ such that $X = \langle\theta, S_t\rangle$ and $\Pi_s^t(X) = \pi_s(\theta)$.*

Proof. Due to (ii) of Lemma 2.4, there exists a $\mathcal{F}_t$-measurable portfolio ξ with $\langle\xi, S_t\rangle = 1$. Now, chose $\theta = X\xi$. Clearly, $\langle\theta, S_t\rangle = X$ and the proof follows from (3.3). □

The next lemma will play an important role, later.

LEMMA 3.3. *Under the model assumptions and valuation principles as above, any t-portfolio $\theta_t \in (L^0(M, \mathcal{F}_t \otimes \mathcal{B}_s, \mathbb{P}))^d$, respectively any $\mathcal{F}_t \otimes \mathcal{B}_s$-measurable payoff X, which has the value $\pi_s(\theta_t)$, resp. $\Pi_s^t(X)$, at s $(0 \leq s < t \leq T)$ can be replicated by a purely financial s.f. strategy which starts at time s and costs $\pi_s(\theta_t)$, resp. $\Pi_s^t(X)$, at s.*

Here, a **purely financial self-financing strategy** which starts and has the price $P \in L^0(M, \mathcal{F}_s \otimes \mathcal{B}_s, \mathbb{P})$ at time s is understood as a vector of portfolios $(\varphi_r)_{s \leq r \leq t}$ such that φ_r is $\mathcal{F}_r \otimes \mathcal{B}_s$-measurable, $\langle \varphi_{r-1}, S_r \rangle = \langle \varphi_r, S_r \rangle$ for $s < r \leq t$ and $\pi_s(\varphi_s) = P$.

Proof. At first, we prove the portfolio case. Due to Lemma 2.4 there exists for any $\mathcal{F}_t$-measurable ${}_F\theta_t$ a replicating s.f. strategy $(\varphi_r)_{0 \leq r \leq t}$ in M^F such that $\varphi_t = {}_F\theta_t$ and $\pi_s({}_F\theta_t) = \pi_s(\varphi_s) = \langle \varphi_s, S_s \rangle$ for $s < t$. Naturally, a strategy starting at s that replicates ${}_F\theta_t$ can start with the random portfolio φ_s. For all $b \in \mathbb{B}$, the M^F-portfolio $\theta_t(., b)$ is $\mathcal{F}_t$-measurable. This implies the existence of M^F-strategies $({}^b\varphi_t)_{0 \leq r \leq t}$ as above for all $b \in \mathbb{B}$ (i.e. ${}^b\varphi_t = \theta_t(., b)$). However, $\mathcal{B}_s$ is finite and therefore there exists a set $\mathcal{B}_s^{\min}$ of minimal sets in $\mathcal{B}_s$ which is a partition of B. By contradiction it can easily be shown that for any $\epsilon \in \mathcal{B}_s^{\min}$ and $b_1, b_2 \in \epsilon$ one has $\theta_t(., b_1) = \theta_t(., b_2)$. Define φ_r on $M = F \times B$ by

$$\varphi_r : (f, b) \mapsto {}^b\varphi_r(f). \tag{3.4}$$

Since $({}^b\varphi_r)_{0 \leq r \leq t}$ replicates $\theta_t(., b)$, we can assume $\varphi_r(., b_1) = \varphi_r(., b_2)$ for $b_1, b_2 \in \epsilon \in \mathcal{B}_s^{\min}$ $(s \leq r \leq t)$. Hence, the inverse image of any measurable set due to φ_r is a finite union of sets of the form $A \times \epsilon$ where $A \in \mathcal{F}_r$ and $\epsilon \in \mathcal{B}_s^{\min}$. So, $\varphi_r \in (L^0(M, \mathcal{F}_r \otimes \mathcal{B}_s, \mathbb{P}))^d$ for $s \leq r \leq t$. Furthermore, $\langle \varphi_{r-1}, S_r \rangle = \langle \varphi_r, S_r \rangle$ for $s < r \leq t$ is clear as $\langle \varphi_{r-1}(., b), S_r \rangle = \langle \varphi_r(., b), S_r \rangle$ for all b by definition. Using Lemma 2.8 of Chapter 2, the proof is completed by the fact that for all $b \in B$ one has $\varphi_t(., b) = \theta_t(., b)$ and $\mathbb{F}$-a.s.

$$\begin{aligned} \langle \varphi_s(., b), S_s \rangle &= S_s^0 \cdot \mathbf{E}_{\mathbb{Q}}[\langle \theta_t(., b), S_t \rangle / S_t^0 | \mathcal{F}_s] \\ &\overset{\text{Lemma 3.27}}{=} S_s^0 \cdot \mathbf{E}_{\mathbb{Q} \otimes \mathbb{B}}[\langle \theta_t, S_t \rangle / S_t^0 | \mathcal{F}_s \otimes \mathcal{B}_s](., b) \\ &= \pi_s(\theta_t)(., b) \end{aligned} \tag{3.5}$$

Note that for the use of Lemma 3.27 (Section 3.11) we needed that $|\mathcal{B}_s| < \infty$ (the lemma is used with $\mathcal{F} = \mathcal{B}_s$, $\mathcal{B} = \mathcal{F}_t$ and $\mathcal{B}' = \mathcal{F}_s$). The case for payoffs follows from Lemma 3.2. $\square$

REMARK 3.4. Lemma 3.3 is the only result of this chapter where the finiteness of the biometric state space is explicitly used in the proof. Note, that finiteness of $\mathcal{F}_T$ was not explicitly used, but indirectly for the existence of conditional expectations. It is not clear, whether (or how) the lemma can be proven for infinite biometric state spaces (for portfolios in $(L^1(M, \mathcal{F}_t \otimes \mathcal{B}_s, \mathbb{P}))^d$). Fortunately, finite state spaces are sufficient for all practical purposes.

3.3 Gains in life insurance - the AFIR-problem

Consider a general life insurance contract $(\gamma_t, \delta_t)_{t \in \mathbb{T}}$ as defined in Chapter 2 and any valuation principle π. From the viewpoint of the insurer, the contract is equivalent to the portfolios $(\delta_t - \gamma_t)_{t \in \mathbb{T}}$. A first guess for the minimum fair price or *present value* of the contract at time t is therefore

$$\sum_{r \in \mathbb{T}} \pi_t(\delta_r - \gamma_r). \tag{3.6}$$

Due to (3.6), the company's *gain* G_t obtained in the time interval $[s, t]$ due to $(\gamma_t, \delta_t)_{t \in \mathbb{T}}$ and π is the difference

$$G_{s,t} = \sum_{r \in \mathbb{T}} \pi_t(\delta_r - \gamma_r) - \sum_{r \in \mathbb{T}} \pi_s(\delta_r - \gamma_r) \tag{3.7}$$

of the values of the contract at time t and s.

REMARK 3.5. The notions *gain* and *risk* are almost identically used in this chapter. Clearly, a random gain can also be negative (i.e. can be a *loss*) and therefore be considered as a risk. The subject which is meant by the two expressions is a difference of (random) present values belonging to two different points of time (cf. (3.7)).

Now, $G_{s,t}$ is presumed to have two components:

1. a financial component $G^F_{s,t}$ and

2. a biometric (technical) component $G^B_{s,t}$,

such that

$$G_{s,t} = G^F_{s,t} + G^B_{s,t}. \tag{3.8}$$

Bühlmann (1995) states that from the philosophy of life insurance it would be clear that the company has to pool technical gains or losses (due to the Law of

Large Numbers), whereas financial wins should be given to the insurant (e.g. as *bonus*). However, financial losses must be realized by the insurer. Actually, this is almost like real life insurance companies commonly work. Hence, it is important to have a reasonable decomposition of the e.g. yearly gains.

The so-called *AFIR-problem*, formulated in Bühlman (1995), is the question how the claim of the insurant on the financial wins $(G^F_{s,t})^+$ should be priced and how it can be hedged.

REMARK 3.6 (Bonus). In fact, Bühlman (1995) does not consider the gain (3.7) but a gain discounted to the beginning of the time interval s. The differences will become clear in Section 3.10 (Eq. (3.75)). However, our approach to risk decomposition is inspired by Bühlmann's. Both approaches differ from the considerations usually taking place in bonus theory. There, the *technical surplus* is defined as the difference between the second order retrospective reserve and the first order reserve (cf. Remark 2.1 and Norberg (1999, 2001)). As the first order base is chosen conservatively, this surplus is systematically positive and must be distributed to the insured for legal reasons. However, for the purposes of this chapter, we stay in the second order base and do not treat the bonus problem in the above sense (see also Section 3.1).

As already mentioned above, *pooling* should be seen as the core competence of life insurance companies. The idea is, that the *pool* should consist of biometric gains and losses such that a growing number of independent individuals which are taken into consideration implies that the mean (accumulated) biometric risk contribution per client converges to zero almost surely by the Strong Law of Large Numbers (this will be specified later). For this reason, one should also demand that biometric parts of gains have expectation zero. In this sense, *an insurance company copes with the pool by its mere existence and growing size.* No further hedging is expected to take place.

Chapter 2 showed that at least in the presence of stochastic financial markets such convergence properties (as mentioned above) are not necessarily trivial and must therefore be carefully examined. The precise understanding of the pooling idea is developed in Section 3.8.

As we work with complete financial markets, there exists no real financial risk in our model since any purely financial payoff or portfolio can be replicated for a certain price (which may therefore be seen as the only risk). For this reason we demand that the financial part $G^F_{s,t}$ in (3.8) can be replicated ongoing from time s. To simplify things, we further assume that the increase of biometric information in $(s, t]$ is not used for trading and hedging purposes. For $s = t - 1$ this is inevitable. We therefore call such a decomposition a *one-period decomposition* even if $s < t - 1$. In the case of $s < t - 1$ think of a real company. For example, premiums and claims are paid (or registered) monthly, the asset portfolio however is traded daily or almost secondly. Hence, new biometric information is not taken into account during the month, but at its end. This justifies the suggested approach. Due to Lemma 3.3, we therefore demand $G^F_{s,t} \in L^0(M, \mathcal{F}_t \otimes \mathcal{B}_s, \mathbb{P})$. Furthermore, $G^F_{s,t}$ should not be arbitrarily chosen, but *close* to $G_{s,t}$ - such that the non-hedgeable part $G^B_{s,t}$ is *small* (e.g. due to the L^2-norm).

In summary, we can compile the following short list of properties the desired decomposition should have.

1. $G^F_{s,t} \in L^0(M, \mathcal{F}_t \otimes \mathcal{B}_s, \mathbb{P})$, i.e. $G^F_{s,t}$ is replicable by a purely financial s.f. strategy starting at s (cf. Lemma 3.3).
2. $G^F_{s,t}$ close to $G_{s,t}$ (e.g. in L^2).
3. $\mathbf{E}[G^B_{s,t}] = 0$.
4. Biometric parts can be pooled (as heuristically explained above).

3.4 The role played by the insurer's portfolio

Before it comes to the matter of risk decomposition in the next section, we have to carry out some further analysis with respect to Equation (3.7).

Indeed, philosophical problems can arise from this definition since the analysis of the gains process not only requires pricing of future cash flows, but also *pricing of past cash flows*. In a deterministic financial framework, this is no problem as any investment develops like $(S^0_t)_{t\in\mathbb{T}}$ which is known in advance for sure (cf. Section 2.5). That means a payment C_r in cash at r will (for sure!) be worth $C_r \cdot S^0_t / S^0_r$ at $t > r$. However, if one has a stochastic financial market with more than one asset, one could invest C_r in several completely different assets or strategies. So, looking back, one needs to know which strategy was chosen. Therefore, any valuation approach which does not take trading strategies into account (like the stochastic discounting approach, cf. Section 3.10) should be carefully examined for its adequacy.

Note that in Chapter 2 the focus is on the suitable valuation of portfolios (and not complete contracts) in the context of life insurance. Therefore, the development of the portfolios at later stages (when trading takes place) was not considered there.

To meet the demands pointed out above, some new notation has to be introduced. Any r-portfolio δ_r which is paid as a premium to the insurer at time $r \in \mathbb{T}$ is seen together with the self-financing $M^{F\times B}$-strategy $(\delta_{r,t})_{t\geq r}$ starting at r which describes how the insurance company works with the premiums after receiving them (here, trading also takes biometric information into account). Observe that one has

$$\delta_{r,t} \in (L^0(M, \mathcal{F}_t \otimes \mathcal{B}_t, \mathbb{P}))^d \quad \text{for } t \geq r. \tag{3.9}$$

Defining

$$\delta_{r,t} = \delta_r \quad \text{for } t \leq r \tag{3.10}$$

the vector $(\delta_{r,t})_{t\in\mathbb{T}}$ contains all information concerning the premium δ_r received in r by the insurance company. Hence, $\pi_t(\delta_r) = \pi_t(\delta_{r,t})$ for all $r, t \in \mathbb{T}$ with $t \leq r$, but for $t > r$ we may have $\delta_r \neq \delta_{r,t}$ as vectors of random variables and $\pi_t(\delta_r) \neq \pi_t(\delta_{r,t})$ which means that the insurance company worked with the premium after receiving it.

Exactly the same considerations are suitable for the claims γ_r, $r \in \mathbb{T}$ where a vector $(\gamma_{r,t})_{t\in\mathbb{T}}$ provides the respective information. However, as the claims are usually payments of the insurer to the insurant, one might ask for the sense of a trading strategy for losses which have taken place. Actually, if these losses would not have taken place, the company would have invested the money into some strategy. For instance, if the company has an investment portfolio and any incomes or losses are just understood as an up- or downsizing of this portfolio (where the relative weights of the different assets are kept constant), then it is clear that losses exactly develop like this portfolio (apart from the negative sign).

EXAMPLE 3.7. It is assumed that the investment portfolio of the considered insurance company basically follows a self-financing trading strategy $(\zeta_t)_{t\in\mathbb{T}}$ in $M^{F\times B}$ (which means that the company can react on biometric events). Any incomes or losses of the company are assumed to be realized by up- or downsizing the respective portfolio (at that time) by a certain factor. To make things easier, simply assume an additional asset S^d with $S^d_t = \langle \zeta_t, S_t \rangle$ in $M^{F\times B}$. This does *not* affect completeness or absence of arbitrage in the "old" M^F which still only has d assets. Now, any portfolio $\theta_t \in \Theta$ which is a gain or a loss (e.g. a premium or claim) of the company, has the following price at $s \in \mathbb{T}$ from the viewpoint of the company (here, π is as in (3.1)):

$$\pi_s(\pi_t(\theta_t)/S_t^d \cdot e_d), \tag{3.11}$$

e_d being the $(d+1)$-th canonical base vector of $\mathbb{R}^{d+1}$. The reason is that at time t, when the portfolio is handed over, the company invested its present value $\pi_t(\theta_t)$ in $\pi_t(\theta_t)/S_t^d$ shares of S^d (which represents its trading strategy/overall portfolio). This clarifies (3.11) for $s \geq t$. However, as

$$\pi_s(\theta_t) = \pi_s(\pi_t(\theta_t)/S_t^d \cdot e_d), \tag{3.12}$$

(3.11) is also correct for $s < t$.

Using the introduced notation, the **present value** of a life insurance contract at time t can now more precisely (cf. (3.6)) be written as

$$\begin{aligned} PV_t &= PV_t((\gamma_{r,t}, \delta_{r,t})_{r\in\mathbb{T}}) \\ &= \sum_{r\in\mathbb{T}} \pi_t(\delta_{r,t} - \gamma_{r,t}) \\ &= \underbrace{\sum_{r<t} \pi_t(\delta_{r,t} - \gamma_{r,t})}_{\text{value of past stream}} + \underbrace{\sum_{r\geq t} \pi_t(\delta_r - \gamma_r)}_{\text{value of future stream}}. \end{aligned} \tag{3.13}$$

Hence, the evolution of the present value (3.13) (more precise, the present value of the past stream) of any life insurance contract depends on the asset management of the particular company. The definition of the **gains** obtained in $[s,t]$ must be altered to

$$\begin{aligned} G_{s,t} &= PV_t - PV_s \\ &= \sum_{r\in\mathbb{T}} \pi_t(\delta_{r,t} - \gamma_{r,t}) - \sum_{r\in\mathbb{T}} \pi_s(\delta_{r,s} - \gamma_{r,s}). \end{aligned} \tag{3.14}$$

The expression

$$R_t' := -\pi_t(\delta_t) - \sum_{r>t} \pi_t(\delta_r - \gamma_r) \tag{3.15}$$

is usually called the *reserve* at time t and traditionally only considered under the condition that the respective individual is still living. The difference $\pi_t(\gamma_t)$ to the negative value of the future stream in (3.13) is caused by the classical convention that benefits at time t and premiums at time $t-1$ are considered to be due to the same time interval $(t-1,t]$ (cf. Gerber, 1997).

Under the valuation principle (3.1), the following decomposition of the premium $\pi_t(\delta_t)$ can easily be deduced.

$$\begin{aligned}\pi_t(\delta_t) &= \pi_t(R'_{t+1}/S^0_{t+1}\cdot e_0) - R'_t \; + \; \pi_t(\gamma_{t+1}) \\ &= \underbrace{\Pi_t^{t+1}(R'_{t+1}) - R'_t}_{\text{savings premium}} \; + \; \underbrace{\pi_t(\gamma_{t+1})}_{\text{risk premium}},\end{aligned} \tag{3.16}$$

i.e. the premium in t can be seen as the sum of a part which is together with the reserve R'_t at t the t-value of the future reserve R'_{t+1} and one part which is exactly the t-value of the claim (or risk) γ_{t+1} at $t+1$. Actually, this is the generalization of a well-known classical relationship (cf. Gerber, 1997).

In the general context presented in this chapter, the negative value of the future stream in (3.13) may be a more appropriate choice for the reserve, i.e.

$$R_t := -\sum_{r\geq t}\pi_t(\delta_r - \gamma_r). \tag{3.17}$$

In contrast to the previous section, one could also be interested in the consideration of a *technical gain* (in this context *not* the biometric gain!), which is (in some analogy to Gerber (1997)) defined as the difference of the trading gains from the reserve R_{t-1} and the cash $\pi_{t-1}(\delta_{t-1}-\gamma_{t-1})$, minus the new reserve R_t. The philosophy behind that approach is, that the insurance company somehow compensates at any time t the difference between the value of the past stream and the future stream, such that the new present value of the contract is zero. Of course, such a policy requires some additional reserves that can compensate the respective gains and losses. Furthermore, the analysis of such technical gains requires the precise knowledge of how $R_{t-1}+\pi_{t-1}(\delta_{t-1}-\gamma_{t-1})$ is invested in the market. In particular, one could realize the compensation at $t-1$ by assuming a strategy $(\xi_t)_{t\in\mathbb{T}}$ such that $\xi_s = 0$ for $s < t-1$,

$$\pi_{t-1}(\xi_{t-1}) = -PV_{t-1} = -\sum_{r\in\mathbb{T}}\pi_{t-1}(\delta_{r,t-1}-\gamma_{r,t-1}) \tag{3.18}$$

and $(\xi_t)_{t\in\mathbb{T}}$ s.f. after time $t-1$. Observe that

$$R_{t-1} \;=\; \underbrace{\sum_{r<t-1}\pi_{t-1}(\delta_{r,t-1}-\gamma_{r,t-1})}_{\text{value of past stream}} \; + \; \underbrace{\pi_{t-1}(\xi_{t-1})}_{\text{compensation}}. \tag{3.19}$$

The technical gain during the time interval $[t-1,t]$ would then be defined as

$$G^{\text{tech}}_{t-1,t} = \sum_{r\le t-1} \pi_t(\delta_{r,t} - \gamma_{r,t}) + \pi_t(\xi_t) - R_t. \tag{3.20}$$

When calculating reserves with first order bases, technical gain and surplus are similar constructions (cf. Remark 3.6).

3.5 Orthogonal risk decomposition

In the framework of Section 3.2, the payoffs $\langle\theta_t, S_t\rangle$ of all t-portfolios θ_t are the Hilbert space $L^0(M, \mathcal{M}_t, \mathbb{P})$ with the scalar product $(X,Y) = \mathbf{E}_{\mathbb{P}}[XY]$ (cf. Lemma 3.2). Clearly, the analogous set $L^0(F, \mathcal{F}_t, \mathbb{F})$ of purely financial payoffs is a closed subspace of $L^0(M, \mathcal{M}_t, \mathbb{P})$. It can be shown (and was in a similar context mentioned in Chapter 2) that the operator $\mathbf{E}_{\mathbb{B}}[.]$ is the orthogonal projection of $L^0(M, \mathcal{M}_t, \mathbb{P})$ onto $L^0(F, \mathcal{F}_t, \mathbb{F})$. Thus, since $\mathbf{E}_{\mathbb{B}}[\langle\theta_t, S_t\rangle] = \langle\mathbf{E}_{\mathbb{B}}[\theta_t], S_t\rangle$ for all $t \in \mathbb{T}$, $\mathbf{E}_{\mathbb{B}}[\theta]$ is the best *purely financial* approximation to any $\theta \in \Theta$ in the L^2-sense (concerning the respective payoffs).

In contrast to Chapter 2, the present chapter intends to consider trading strategies which also take biometric events later then time 0 into account. For this reason, the following problem is of interest.

Consider a t-portfolio θ in the market $M^{F\times B}$. Assume that all information until some time $s < t$ is given. What is the best approximation (in the L^2-sense) of θ that can be reached by a purely financial trading strategy starting from s and being given *all* information up to s? As surely expected and shown by the following two lemmas, it is $\mathbf{E}_{\mathbb{P}}[\theta|\mathcal{F}_t \otimes \mathcal{B}_s]$.

Have in mind that $\mathbb{P}$-a.s.

$$\mathbf{E}_{\mathbb{P}}[\langle\theta, S_t\rangle|\mathcal{F}_t \otimes \mathcal{B}_s] = \langle\mathbf{E}_{\mathbb{P}}[\theta|\mathcal{F}_t \otimes \mathcal{B}_s], S_t\rangle. \tag{3.21}$$

LEMMA 3.8. *Under the notation of Section 3.2, consider the Hilbert space $L^0(M, \mathcal{M}_t, \mathbb{P})$ and for $s < t$ its closed subspace $L^0(M, \mathcal{F}_t \otimes \mathcal{B}_s, \mathbb{P})$. For any $X \in L^0(M, \mathcal{M}_t, \mathbb{P})$ one has the orthogonal decomposition*

$$P_{s,t}(X) = \mathbf{E}_{\mathbb{P}}[X|\mathcal{F}_t \otimes \mathcal{B}_s] \tag{3.22}$$

and

$$Q_{s,t}(X) = X - P_{s,t}(X) \tag{3.23}$$

due to the subspaces $L^0(M, \mathcal{F}_t \otimes \mathcal{B}_s, \mathbb{P})$ and $L^0(M, \mathcal{F}_t \otimes \mathcal{B}_s, \mathbb{P})^\perp$. The orthogonal projection (3.22) of X is the (uniquely determined) closest point in $L^0(M, \mathcal{F}_t \otimes \mathcal{B}_s, \mathbb{P})$ to X due to the L^2-norm.

Proof. Lemma 3.28. □

From now on presume π to be as in Equation (3.1).

LEMMA 3.9. *Let $X \in L^0(M, \mathcal{M}_t, \mathbb{P})$ and $P_{s,t}(X)$ as in* (3.22)*. Then*

$$\Pi_s^t(X) = \Pi_s^t(P_{s,t}(X)), \tag{3.24}$$

and the payoff $P_{s,t}(X)$ at t can ongoing from time s be replicated by a purely financial s.f. strategy of price (3.24) *at s.*

Proof. Lemma 3.3 proves the existence of the replication. By Lemma 3.29,

$$\begin{aligned} \Pi_s^t(P_{s,t}(X)) &= S_s^0 \cdot \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\mathbf{E}_{\mathbb{F}\otimes\mathbb{B}}[X|\mathcal{F}_t \otimes \mathcal{B}_s]/S_t^0|\mathcal{F}_s \otimes \mathcal{B}_s] \qquad (3.25)\\ &= S_s^0 \cdot \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[X|\mathcal{F}_t \otimes \mathcal{B}_s]/S_t^0|\mathcal{F}_s \otimes \mathcal{B}_s] \;=\; \Pi_s^t(X). \end{aligned}$$

□

REMARK 3.10. Lemma 3.9 is a further justification for the valuation principle Π (on the payoffs, but also for π on the portfolios; cf. Lemma 3.2 and 3.3) as an approximation price.

For X in any $L^2(P, \mathcal{P}, \mathbb{P})$ and $Y \in L^2(P, \mathcal{P}', \mathbb{P})$ with σ-algebras $\mathcal{P}' \subset \mathcal{P}$ one has $\sqrt{\mathrm{Var}(X-Y)} = ||X - Y - \mathbf{E}[X-Y]||_2 \leq ||X-Y||_2$. So, if $X - Y$ is L^2-minimal (for fixed X and variable Y as above) we must have $||X - Y - \mathbf{E}[X-Y]||_2 = ||X-Y||_2$ since $Y + \mathit{const}$ is also an element of $L^2(P, \mathcal{P}', \mathbb{P})$. Variance-optimality of $X - Y$ follows immediately. Hence, if $s = 0$ and $t > 0$ then $P_{t,0}(X)$ is not only the unique L^2-optimal, but also a variance-optimal hedge of the payoff X when the increase of biometric information during $(0, t]$ is not used for hedging purposes.

Please note that the results in the existing literature on variance-optimal hedging can not be directly applied to our problems when explicit hedging strategies are desired. For instance, in a discrete time framework Schweizer (1995a) assumes a constant money account and only one stochastic asset. Furthermore, in our setup only arising financial information is used for hedging.

One-period decomposition. We use

$$G_{s,t}^F = P_{s,t}(G_{s,t}) = \mathbf{E}_{\mathbb{F}\otimes\mathbb{B}}[G_{s,t}|\mathcal{F}_t \otimes \mathcal{B}_s] \tag{3.26}$$

and

$$G_{s,t}^B = Q_{s,t}(G_{s,t}) = G_{s,t} - P_{s,t}(G_{s,t}) \tag{3.27}$$

as financial, respectively biometric (technical) part of any $G_{s,t} \in L^0(F \times B, \mathcal{F}_t \otimes \mathcal{B}_t, \mathbb{F} \otimes \mathbb{B})$ (cf. (3.14)) whenever the increase of biometric information between s and t is not used for hedging purposes.

REMARK 3.11. Due to Lemma 3.8, (3.22) and (3.23), resp. (3.26) and (3.27), is the unique decomposition which splits a payoff X into a replicable (by a purely financial strategy starting at s, cf. Lemma 3.3) and a non-replicable part such that the replicable one is L^2-closest to X and the residual (non-replicable part) hence L^2-minimal. Observe that

$$\mathbf{E}_{\mathbb{P}}[G^B_{s,t}|\mathcal{F}_s \otimes \mathcal{B}_s] = \mathbf{E}_{\mathbb{P}}[G^B_{s,t}] = 0. \tag{3.28}$$

Therefore, the first three properties which are listed at the end of Section 3.3 are fulfilled and the tightening of the second property as above induces that the first and the second one directly imply (3.26) and (3.27). One also has $\Pi^t_s(G^B_{s,t}) = 0$ due to (3.24).

The results so far obtained rely on the fact that we work with L^2-spaces. However, one could also use (3.26) and (3.27) as financial, respectively biometric part of $G_{s,t}$ when $|\mathcal{F}_T \otimes \mathcal{B}_T| = \infty$ and $G_{s,t} \in L^1(M, \mathcal{F}_t \otimes \mathcal{B}_t, \mathbb{P})$.

Concerning pooling, note that the projection $\mathbf{E}_{\mathbb{F}\otimes\mathbb{B}}[.|\mathcal{F}_t \otimes \mathcal{B}_s]$ $(t > s)$, is so to speak a generalization of the projection $\mathbf{E}_{\mathbb{B}}[.]$ which was considered in Chapter 2 for other reasons. However, in Chapter 2 the convergence of mean balances belonging to "pools" consisting of portfolios of the form ${}^i\theta - \mathbf{E}_{\mathbb{B}}[{}^i\theta]$ was shown. Since the use of arising biometric information for trading was not allowed there and ${}^i\theta - \mathbf{E}_{\mathbb{B}}[{}^i\theta]$ therefore is a biometric part of a portfolio in our sense, we actually have a first glimpse of what "pooling" can mean. The differences to the results in Chapter 2 will become clear in Section 3.8.

With the decomposition proposed in this section, the following general solution of Bühlmann's AFIR-problem can be stated.

Solution of the AFIR-problem. The minimum fair price of the t-claim with payoff $(G^F_{t-1,t})^+$ at time $s \leq t-1$ is

$$\Pi^t_s((G^F_{t-1,t})^+) = S^0_s \cdot \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[(G^F_{t-1,t})^+/S^0_t|\mathcal{F}_s \otimes \mathcal{B}_s], \tag{3.29}$$

where $(G^F_{t-1,t})^+$ is given by (3.14) and (3.26).

The respective replicating strategy for $(G^F_{t-1,t})^+$ depends on the contract and might be difficult to determine.

3.6 Time-local properties

Until now, L^2-, respectively variance-optimality of hedges was considered globally, i.e. from the viewpoint of time 0. We will now derive that certain optimality properties also hold from the viewpoint of later time stages.

First, we reconsider the L^2-minimality. Whenever Y^* minimizes $||X-Y||_2$ for fixed $X \in L^2(P, \mathcal{P}, \mathbb{P})$ and $Y \in L^2(P, \mathcal{P}', \mathbb{P})$ with $\mathcal{P}'$ a sub-σ-algebra of $\mathcal{P}$, one has $Y^* = \mathbf{E}[X|\mathcal{P}']$ by Lemma 3.28. However, Lemma 3.30 gives that

$$\mathbf{E}[(X-Y)^2|\mathcal{P}'] \leq \mathbf{E}[(X-Z)^2|\mathcal{P}'] \tag{3.30}$$

for any $Z \in L^2(P, \mathcal{P}', \mathbb{P})$ if and only if $Y = \mathbf{E}[X|\mathcal{P}']$ $\mathbb{P}$-a.s. Therefore, the orthogonal risk decomposition considered in Section 3.5 is so to speak L^2-optimal from the viewpoint of s.

PROPOSITION 3.12 (Locally variance-optimal hedge). *Suppose* $X \in L^0(M, \mathcal{M}_t, \mathbb{P})$ *and let* $\mathcal{Y}$ *be the set of all payoffs* $Y \in L^0(M, \mathcal{F}_t \otimes \mathcal{B}_s, \mathbb{P})$ *at* t *which are produced by purely financial s.f. strategies with price 0 at* s*. Then the minimization problem*

$$\min_{Y \in \mathcal{Y}} Var[X - Y|\mathcal{F}_s \otimes \mathcal{B}_s] \tag{3.31}$$

has the unique solution Y^* *which is determined by the payoff of the s.f. strategy that replicates* $P_{s,t}(X)$ *and sells for* $\Pi^t_s(P_{s,t}(X))$ *zero-coupon bonds with time to maturity* $t-s$ *at* s*.*

$\mathbb{P}$-a.s. identical solutions are identified, here. The conditional variance is defined in the Appendix, Definition 3.31.

Proof. Lemma 3.9 proved that $P_{s,t}(X)$ can be replicated and Lemma 3.32 implies that any $Y^* = P_{s,t}(X) + C \in \mathcal{Y}$, $C \in L^0(M, \mathcal{F}_s \otimes \mathcal{B}_s, \mathbb{P})$, would be a solution of (3.31) as long as the price at s is also allowed to be different from 0. However, the only investment at s with such a payoff C at t can be in zero-coupon bonds (or any asset behaving like a zero-coupon bond between s and t) with maturity date t as they have constant payoffs at t seen from s. Uniqueness of Y^* follows from the demand for price 0 at s, i.e. one *must* invest $-\Pi^t_s(P_{s,t}(X))$ in zero-coupon bonds. □

3.7 Implications

In this section we will derive several implications of the proposed decomposition (3.26) and (3.27).

Let us again consider the gains (3.14) arising from a life insurance contract.

PROPOSITION 3.13. *For $t > 0$, the biometric part $G^B_{t-1,t}$ of the gain $G_{t-1,t}$ per period is not depending on the particular trading strategy, since*

$$G^B_{t-1,t} = Q_{t-1,t}\left(\sum_{r\geq t}\pi_t(\delta_r - \gamma_r)\right). \tag{3.32}$$

Proof. From (3.13), one has

$$PV_t = \sum_{r<t}\pi_t(\delta_{r,t} - \gamma_{r,t}) + \sum_{r\geq t}\pi_t(\delta_r - \gamma_r) \tag{3.33}$$

$$PV_{t-1} = \sum_{r<t-1}\pi_{t-1}(\delta_{r,t-1} - \gamma_{r,t-1}) + \sum_{r\geq t-1}\pi_{t-1}(\delta_r - \gamma_r). \tag{3.34}$$

Obviously, PV_{t-1} is $\mathcal{F}_t \otimes \mathcal{B}_{t-1}$-measurable, and for any $r < t$ also $\pi_t(\delta_{r,t} - \gamma_{r,t})$ is since

$$\pi_t(\delta_{r,t} - \gamma_{r,t}) = \langle \delta_{r,t} - \gamma_{r,t}, S_t\rangle = \langle \delta_{r,t-1} - \gamma_{r,t-1}, S_t\rangle. \tag{3.35}$$

By (3.22), (3.23) and (3.27), (3.32) follows. □

The proposition has pointed out that only the financial part $G^F_{t-1,t}$ of $G_{t-1,t}$ depends on financial trading. However, (3.32) does not mean that $G^B_{t-1,t}$ does not depend on the market. In fact, it can be strongly depending, but the company is apart from its influence on $G^B_{t-1,t}$ by the contract design not responsible for $G^B_{t-1,t}$, i.e. after time 0, the part $G^B_{t-1,t}$ of the gains $G_{t-1,t}$ can not be influenced by the company, anymore.

Section 3.5 showed that the financial part $G^F_{s,t}$ of any gain $G_{s,t}$ of a life insurance contract can be replicated by a purely financial s.f. strategy starting at s (cf. Lemma 3.9). But, *how much costs the hedge of the claim with payoff $G^F_{s,t} = P_{s,t}(G_{s,t})$?* The answer given in the following proposition is a central result of this chapter.

PROPOSITION 3.14. *The price of the t-claim $G^F_{s,t} = P_{s,t}(G_{s,t})$ at time $s < t$ is*

$$\Pi^t_s(G^F_{s,t}) = (1 - p(s, t-s))PV_s, \tag{3.36}$$

where $p(s, t-s)$ denotes the price of a zero-coupon bond with time to maturity $t - s$ at time s, i.e. $p(s, t-s) := S^0_s \cdot \mathbf{E}_{\mathbb{Q}}[1/S^0_t|\mathcal{F}_s]$.

Proof. Due to Lemma 3.29 and the fact that S_t is $\mathcal{F}_t \otimes \mathcal{B}_s$-measurable, one has for any $\theta \in \Theta$

$$P_{s,t}(\pi_t(\theta)) = S_t^0 \cdot \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\langle\theta, S_T\rangle / S_T^0 | \mathcal{F}_t \otimes \mathcal{B}_s] \tag{3.37}$$

and one gets by (3.3)

$$\Pi_s^t(P_{s,t}(\pi_t(\theta))) = S_s^0 \cdot \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\langle\theta, S_T\rangle / S_T^0 | \mathcal{F}_s \otimes \mathcal{B}_s] = \pi_s(\theta). \tag{3.38}$$

On the other side, for any $\theta \in \Theta$ one has $P_{s,t}(\pi_s(\theta)) = \pi_s(\theta)$ and

$$\begin{aligned} \Pi_s^t(P_{s,t}(\pi_s(\theta))) &= \pi_s(\theta) \cdot S_s^0 \cdot \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[1/S_t^0 | \mathcal{F}_s \otimes \mathcal{B}_s] \\ &= \pi_s(\theta) \cdot p(s, t-s). \end{aligned} \tag{3.39}$$

Observe that

$$\pi_s(\delta_{r,t} - \gamma_{r,t}) = \pi_s(\delta_{r,s} - \gamma_{r,s}) \tag{3.40}$$

for no-arbitrage reasons. The definition of $G_{s,t}$ in (3.14), the linearity of the valuation operators π, Π as well as the linearity of (3.38) and (3.39) in θ imply (3.36). □

We will now interpret (3.36) from the economic point of view by using the following corollaries of Proposition 3.14.

COROLLARY 3.15. *Starting at s, the payoff $G_{s,t}^F$ at t can be replicated by a purely financial s.f. strategy with price* (3.36) *at time s.*

Proof. Lemma 3.3. □

COROLLARY 3.16 (Locally variance-optimal present value). *Given a life insurance contract with present value PV_s at time s,*

$$PV_t = p(s, t-s)^{-1} PV_s + G_{s,t}^B \tag{3.41}$$

is the locally variance-optimal present value (seen from s) for time t which can be achieved by a purely financial s.f. strategy starting and being for free at s.

Proof. Lemma 3.32 (3.101) implies that PV_t is locally variance-optimal if and only if $PV_t - PV_s$ is. We therefore apply Proposition 3.12 to this difference. The optimal PV_t (3.41) is therefore achieved by replication of the payoff $-G_{s,t}^F$ (cf. Corollary 3.15) and investing the negative price, i.e. (3.36), in zero-coupon bonds with maturity t. □

Hence, an insurance company can reduce the risk of its business in the sense that in any time period $[s, t]$ it can accomplish the maximum sure wins possible in the market starting from an initial capital PV_s, but must bear a remaining biometric fluctuation risk (with conditional expectation $\mathbf{E}_{\mathbb{P}}[G^B_{s,t}|\mathcal{F}_s \otimes \mathcal{B}_s] = 0$) which can not be influenced by trading if $s = t - 1$ (cf. Proposition 3.13).

Seen from time s, the present value under the locally variance-optimal hedge develops like a riskless investment in the mean.

The two corollaries are strong arguments for the proposed decomposition (3.26) and (3.27). If the company wants to, it can theoretically hedge away the financial part $G^F_{s,t}$ of the gain $G_{s,t}$ - except for an outstanding (and usually positive) rest $(p(s, t-s)^{-1} - 1)PV_s$ which is not random from the viewpoint of time s and which actually is the return of the safely invested negative cost of the hedge (the negative cost of the hedge is (3.36)). More precise, (3.36) *is the cost of the capital PV_s at time s for the time period* $[s, t]$ when PV_s is financed by zero-coupon bonds.

To make things more clear: If one borrows the amount PV_s at s (e.g. to work with it at the stock exchange), the fixed(!) amount which must be paid back at time t can easily be computed as

$$\frac{PV_s}{S^0_s \cdot \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[1/S^0_t|\mathcal{F}_s \otimes \mathcal{B}_s]} = \frac{PV_s}{p(s, t-s)}. \tag{3.42}$$

Hence,

$$\left(p(s, t-s)^{-1} - 1\right) PV_s \tag{3.43}$$

must be gained during $[s, t]$ to avoid losses. The cost of doing this (= the cost of the capital PV_s at time s) is $(1 - p(s, t-s))PV_s$ as this amount has to be invested into zero-coupon bonds with time to maturity $t - s$ at time s to have the sure return (3.43) at t.

From the economic point of view, it is absolutely reasonable that the replication of $G^F_{s,t}$ costs something. Otherwise, it would be possible to obtain the same returns from an initial capital zero as from any other positive initial capital just by following self-financing trading strategies.

COROLLARY 3.17. *Starting with a present value PV_s at time s, the present value of a contract develops like*

$$PV_t = PV_s \cdot \prod_{r=s}^{t-1} p(r, 1)^{-1} + \sum_{r=s+1}^{t} G^B_{r-1,r} \cdot \prod_{u=r}^{t-1} p(u, 1)^{-1}, \tag{3.44}$$

when the locally variance-optimal hedge of Corollary 3.16 is applied in each period (the product over an empty index set is 1).

Proof. Reiterate Corollary 3.16. □

Clearly, $\prod_{r=s}^{t-1} p(r,1)^{-1}$ is the value of a strategy at time t, where beginning at s one currency unit is repeatedly invested in immediately maturing zero-coupon bonds, i.e. in bonds with time to maturity 1. For very small time intervals (e.g. $1 \mathrel{\widehat{=}} 1$ month or even less) one can consider this strategy as a so-called *locally riskless (short rate) money account*. In the literature often exactly this money account is used as the discounting factor.

REMARK 3.18. The hedging possibilities described in the Corollaries 3.15-3.17 do not necessarily demand complete financial markets. Actually, the existence of such strategies depends on the particular structure of the portfolios in the underlying insurance contract. Hedging of particular contracts in incomplete markets could be possible. Again, it should be clear that the realization of such hedging strategies for real world insurance companies would demand the precise knowledge of the second order base defined by the Axioms 1 and 2.

3.8 Pooling - a convergence property

In this section, a convergence property of the mean accumulated discounted biometric risk contribution per contract will be deduced. The considered type of convergence is different and somehow more general than the one in Chapter 2. There, the impact of the Law of Large Numbers was examined for an exploding number of clients and a finite time horizon, only. This time, it can also be assumed that the number of the company's clients at any time t is bounded, but an infinite time axis is given. Under both assumptions, an insurance company can pool biometric risk contributions and benefit from the growing number of independent individuals which have a diversifying influence on the portfolio.

It is necessary to extend the model assumptions.

Consider a sequence of securities market models as proposed in Section 2.3, excluding Axiom 4. That means, for $t \in \mathbb{N}^+$ the common model of financial and biometric risks up to time t is given by

$$ {}^tM^{F\times B} = (M, (\mathcal{M}_s)_{s\in\{0,\dots,t\}}, \mathbb{P}, \{0,\dots,t\}, {}^tS), \tag{3.45} $$

where

$$ {}^{t}M^{F} = (F, (\mathcal{F}_s)_{s\in\{0,\dots,t\}}, \mathbb{F}, \{0,\dots,t\}, {}_{F}{}^{t}S) \tag{3.46} $$

is a complete financial market together with a unique equivalent martingale measure $\mathbb{Q}$. We assume that the market models (3.45) are embedded into each other in the sense that ${}^{t+1}M^{F\times B}$ extends ${}^{t}M^{F\times B}$ by one step of time, and $F, \mathbb{F}, B, \mathbb{B}$ and $\mathbb{Q}$ are identical for all t. In particular, ${}^{s}S_r = {}^{t}S_r$ for $r \le s \le t$, i.e. we can assume to be given a price process $(S_t)_{t\in\mathbb{N}}$ for the d securities on the whole time axis $\mathbb{N}$. $(F \times B, \mathcal{F}_\infty \otimes \mathcal{B}_\infty, \mathbb{F} \otimes \mathbb{B})$ denotes the underlying probabilistic universe. We can have $|\mathcal{F}_\infty \otimes \mathcal{B}_\infty| = \infty$, here. For the biometric probability spaces we propose that $|\mathcal{B}^i_t| < \infty$ for all $i \in \mathbb{N}^+$, $t \in \mathbb{N}$, which surely is no drawback for all practical purposes.

The existence of such sequences of models seems to be natural - e.g. for the financial parts ${}^{t}M^{F}$ one could think of a binomial model (Cox-Ross-Rubinstein) which is extended further and further by additional nodes.

REMARK 3.19. Please note that for any $i, t \in \mathbb{N}^+$ the filtered probability space $(F\times B^i, (\mathcal{F}_s\otimes\mathcal{B}^i_s)_{s\in\{0,\dots,t\}}, \mathbb{F}\otimes\mathbb{B}^i)$ fulfills the model assumptions of Section 3.2 and can in the obvious way be embedded into the larger model described above. Hence, all results (on hedging, risk decomposition etc.) of the previous sections can be applied to this subspace and to particular contracts or portfolios working on it.

The insurance contracts are modeled, now. As an infinite time axis is considered, several things will be altered.

We assume that all considered individuals ($i \in \mathbb{N}^+$) will for sure be born and will have a contract with the respective company. We do not intend to develop birth or canvassing models, here. The next assumption is a maximum lifetime Δ for the human beings (e.g. $\Delta \mathrel{\widehat{=}} 150$ years). For all individuals i a maximum date of death ($T_i \in \mathbb{N}^+$) is supposed. Only the living can be contracted.

Now, consider a life insurance contract $({}^{i}\gamma_t, {}^{i}\delta_t)_{0\le t\le T_i}$, $T_i \in \mathbb{N}^+$, in some ${}^{T}M^{F\times B}$ with $T_i \le T$, i.e. ${}^{i}\gamma_t = {}^{i}\delta_t = 0$ for $t > T_i$ when the contract is considered on the time scale $\mathbb{N}$. Let us define

$$ A^i_t = \{i \text{ signs at } t\} \ \in \ \mathcal{F}_t \otimes \mathcal{B}^i_t, \tag{3.47} $$

i.e. A^i_t is the event that a contract between i and the company is established at t. In the obvious way, $A^i_t \in \mathcal{F}_t \otimes \mathcal{B}_t$. So, $({}^{i}\gamma_t, {}^{i}\delta_t)_{t\in\mathbb{N}}$ should be seen as the *meta-contract* (in fact, this is a sum) that contains all the *sub-contracts* that i will probably sign in the future. Actually, the meta-contract exists by its definition throughout the whole time axis - even before the birth and after the death of the respective individual. The sub-contract signed at t is assumed to start immediately, even if the first claims or premiums equal zero.

Under the assumptions made so far, the date of birth, date of death, kind of insurance sub-contract or duration of this sub-contract are stochastic. Also the number of individuals under contract at a certain time is stochastic. What is assumed for sure is that the individual i (a) will have a contract with our company one day, (b) will die before T_i, and (c) has a maximum life span Δ. A more general model which also includes canvassing is beyond the scope of this dissertation.

Clearly, $\{A_t^i : 0 \le t < T_i\}$ is a partition of $F \times B$. One has

$$\sum_{t=0}^{T_i-1} \mathbf{1}_{A_t^i} = 1 \quad \text{and} \quad \sum_{t=0}^{T_i-1} \mathbb{P}(A_t^i) = 1. \tag{3.48}$$

Furthermore, ${}^i\gamma_s = {}^i\delta_s = 0$ on A_t^i for $s < t$ and $s > t + \Delta$. Hence, $\mathbf{1}_{A_t^i}{}^i\gamma_s$ is $\mathcal{M}_s$-measurable for all $t, s \in \mathbb{N}$ (analogously, $\mathbf{1}_{A_t^i}{}^i\delta_s$). From the definition of A_t^i it is clear that $\mathbf{1}_{A_t^i}(f, .)$ ($f \in F$) depends on the i-th biometric probability space, only.

Assume that each portfolio ${}^i\gamma_t$ or ${}^i\delta_t$ can only in the null-th component be different from zero, i.e. any portfolio of the contract is given in terms of the reference asset with price process $(S_t^0)_{t\in\mathbb{N}}$ (compare Example 3.7). This assumption does not affect the trading strategies of the company. There is no necessity to consider particular strategies (cf. Section 3.3) in this section as we are interested in the biometric parts of the gains due to one time period, only (cf. Proposition 3.13).

Now, assume to be given an infinite set of life insurance meta-contracts $\{({}^i\gamma_t, {}^i\delta_t)_{0\le t\le T_i} : i, T_i \in \mathbb{N}^+\}$ as above. As in Chapter 2, ${}^i\delta_t$ and ${}^i\gamma_t$ only depend on the i-th individual and M^F, i.e. the biometric events concerning i depend on $(B^i, (\mathcal{B}_t^i)_{t\in\mathbb{N}}, \mathbb{B}^i)$, only. Furthermore, we assume for all elements

$$\theta \in \{{}^i\gamma_t : i \in \mathbb{N}^+, t \in \mathbb{N}\} \cup \{{}^i\delta_t : i \in \mathbb{N}^+, t \in \mathbb{N}\} \tag{3.49}$$

that

$$|\theta^0| \le c \in \mathbb{R}^+ \quad \mathbb{P}\text{-a.s.} \tag{3.50}$$

Of course, this is a much stronger condition than (K) in Chapter 2. Nonetheless, analogously to the discussion in Chapter 2, this condition is no drawback for all relevant practical purposes (cf. Example 3.25 below).

PROPOSITION 3.20. *Under the above assumptions,*

$$\frac{1}{m}\sum_{i=1}^{m}\sum_{t=0}^{T_i-1}\mathbf{1}_{A_t^i}\sum_{r=t+1}^{T_i}{}^iG^B_{r-1,r}/S^0_r \overset{m\to\infty}{\longrightarrow} 0 \quad \mathbb{P}\text{-}a.s. \tag{3.51}$$

Interpretation. The mean aggregated discounted biometric risk contribution per client converges to zero a.s. for an increasing number of independent clients. The proposition explains to some extent what should be understood as the core competence of life insurance companies. Due to the Strong Law of Large Numbers they can aggregate the biometric parts of the risks over time and individuals and accomplish balanced wins and losses in the mean. Naturally, only risk contributions arising *after* the signing of a particular sub-contract are considered, therefore the contributions are split using the $\mathbf{1}_{A_t^i}$. The division by the reference asset in (3.51) is necessary as e.g. inflation influences have to be avoided at this point. Otherwise, the use of the Law of Large Numbers would not be possible.

COROLLARY 3.21. *Assume that $(S^0_t)_{t\in\mathbb{N}}$ is the price process of the locally riskless money account and that the insurance company sells fairly priced contracts, only, i.e. $\mathbf{1}_{A_t^i}{}^iPV_t = 0$ for $0 \le t < T_i$ when ${}^iPV_t$ denotes the present value* (cf. (3.13)) *of the i-th meta-contract at t. Under the hedge of Corollary 3.17, started at the beginning of each sub-contract,*

$$\frac{1}{m}\sum_{i=1}^{m}{}^iPV_{T_i}/S^0_{T_i} \overset{m\to\infty}{\longrightarrow} 0 \quad \mathbb{P}\text{-}a.s. \tag{3.52}$$

Interpretation. (3.52) is the mean discounted total gain (= discounted present value at T_i) of the first m contracts that converges to zero almost surely.

Proof. That the respective hedge can be applied follows from Remark 3.19. On $\mathbf{1}_{A_t^i}$ we have that ${}^iPV_t = 0$ and hence (cf. (3.44))

$$\mathbf{1}_{A_t^i}{}^iPV_{T_i} = \mathbf{1}_{A_t^i}\sum_{r=t+1}^{T_i} G^B_{r-1,r}\cdot\prod_{u=r}^{T_i-1} p(u,1)^{-1}. \tag{3.53}$$

Furthermore, $S_t^0 = \prod_{u=0}^{t-1} p(u,1)^{-1}$ and hence

$$\left(\prod_{u=r}^{T_i-1} p(u,1)^{-1} \right) \Bigg/ S_{T_i}^0 = 1/S_r^0. \tag{3.54}$$

From (3.48), (3.53) and (3.54) we get

$${}^iPV_{T_i}/S_{T_i}^0 = \sum_{t=0}^{T_i-1} \mathbf{1}_{A_t^i} \sum_{r=t+1}^{T_i} {}^iG_{r-1,r}^B/S_r^0, \tag{3.55}$$

and hence (3.52) by (3.51). □

Note, that the result in Proposition 3.20 does not depend on the distribution of the contracts on the time axis. For instance, the result is valid for a growing number of clients over an infinite time interval, e.g. when $|\{i : T_i \le t\}| < \infty$ for all $t \in \mathbb{N}$, as well as for an infinite number of contracts in a bounded time interval, e.g. when $\sup_{i\in\mathbb{N}} T_i < \infty$, or when every contract is signed at $t = 0$ as in the following corollary.

COROLLARY 3.22. *When every contract* ($i \in \mathbb{N}^+$) *is signed at* $t = 0$*,*

$$\frac{1}{m} \sum_{i=1}^{m} \sum_{t=1}^{T_i} {}^iG_{t-1,t}^B/S_t^0 \overset{m\to\infty}{\longrightarrow} 0 \quad \mathbb{P}\text{-}a.s. \tag{3.56}$$

Proof. $\mathbf{1}_{A_0^i} = 1$ for $i \in \mathbb{N}^+$, then. □

REMARK 3.23. The convergence properties (3.51), (3.52) and (3.56) are additional arguments in favour of the proposed decomposition of gains. In fact, Proposition 3.20 and its corollaries have shown that (3.26) and (3.27) fulfill the four desired properties which where listed at the end of Section 3.3.

Proof of Proposition 3.20. For any θ as above we have

$$\pi_t(\theta) = S_t^0 \cdot \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\theta^0 | \mathcal{F}_t \otimes \mathcal{B}_t]. \tag{3.57}$$

In the following, we use the substitution

$$f_{r-1,r,s}^i := \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[{}^i\delta_s^0 - {}^i\gamma^0{}_s | \mathcal{F}_r \otimes \mathcal{B}_r] - \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[{}^i\delta_s^0 - {}^i\gamma_s^0 | \mathcal{F}_r \otimes \mathcal{B}_{r-1}]. \tag{3.58}$$

Observe that for $t < r$

$$\begin{aligned} \mathbf{1}_{A_t^i} f_{r-1,r,s}^i &= \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\mathbf{1}_{A_t^i}({}^i\delta_s^0 - {}^i\gamma^0{}_s) | \mathcal{F}_r \otimes \mathcal{B}_r] \\ &\quad - \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[\mathbf{1}_{A_t^i}({}^i\delta_s^0 - {}^i\gamma_s^0) | \mathcal{F}_r \otimes \mathcal{B}_{r-1}]. \end{aligned} \tag{3.59}$$

By (3.32), we have for any $i \in \mathbb{N}^+$

$$\begin{aligned}
& \sum_{t=0}^{T_i-1} \mathbf{1}_{A_t^i} \sum_{r=t+1}^{T_i} {}^iG^B_{r-1,r}/S^0_r \qquad (3.60)\\
= & \sum_{t=0}^{T_i-1} \mathbf{1}_{A_t^i} \sum_{r=t+1}^{T_i} 1/S^0_r \cdot Q_{r-1,r} \left(\sum_{s=r}^{T_i} \pi_r({}^i\delta_s - {}^i\gamma_s) \right)\\
= & \sum_{t=0}^{T_i-1} \mathbf{1}_{A_t^i} \sum_{r=t+1}^{T_i} 1/S^0_r \cdot Q_{r-1,r} \left(\sum_{s=r}^{T_i} S^0_r \cdot \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[{}^i\delta^0_s - {}^i\gamma^0_s | \mathcal{F}_r \otimes \mathcal{B}_r] \right)\\
= & \sum_{t=0}^{T_i-1} \mathbf{1}_{A_t^i} \sum_{r=t+1}^{T_i} \sum_{s=r}^{T_i} f^i_{r-1,r,s}\\
= & \sum_{t=0}^{T_i-1} \mathbf{1}_{A_t^i} \sum_{r=t+1}^{t+\Delta} \sum_{s=r}^{t+\Delta} f^i_{r-1,r,s} \ ,
\end{aligned}$$

where the first equation uses (3.32), the third Lemma 3.29 and the last one (3.59) and the fact that ${}^i\gamma_s = {}^i\delta_s = 0$ on A_t^i for $s > t + \Delta$. For $f \in F$ define

$$(A_t^i)_f := \{b \in B : (f, b) \in A_t^i\}. \tag{3.61}$$

For any $f \in F$ the set $\{(A_t^i)_f : 0 \leq t < T_i\}$ is a partition of B. Clearly, $\mathbf{1}_{A_t^i}(f, .) = \mathbf{1}_{(A_t^i)_f}$. Hence, for fixed $f \in F$, the random variables $\mathbf{1}_{A_t^i}(f, .) \sum_{r=t+1}^{t+\Delta} \sum_{s=r}^{t+\Delta} f^i_{r-1,r,s}(f, .)$ for $0 \leq t < T_i$ are orthogonal due to the L^2-norm on $L^2(B, \mathcal{B}_{T_i}, \mathbb{B})$. Furthermore,

$$||\mathbf{1}_{A_t^i}(f, .)||_2^2 = \mathbf{E}_{\mathbb{B}}[(\mathbf{1}_{(A_t^i)_f})^2] = \mathbb{B}((A_t^i)_f) \tag{3.62}$$

and therefore

$$\sum_{t=0}^{T_i-1} ||\mathbf{1}_{A_t^i}(f, .)||_2^2 = 1. \tag{3.63}$$

From (3.50) one obtains

$$\left| \sum_{r=t+1}^{t+\Delta} \sum_{s=r}^{t+\Delta} f^i_{r-1,r,s}(f, .) \right| \leq 4c\Delta^2. \tag{3.64}$$

Therefore, with (3.60), (3.63) and (3.64),

$$\begin{aligned} & \left|\left|\sum_{t=0}^{T_i-1} \mathbf{1}_{A_t^i}(f,.) \sum_{r=t+1}^{T_i} {}^iG^B_{r-1,r}(f,.)/S^0_r(f)\right|\right|_2^2 & (3.65) \\ = & \sum_{t=0}^{T_i-1} \left|\left|\mathbf{1}_{A_t^i}(f,.) \sum_{r=t+1}^{t+\Delta} \sum_{s=r}^{t+\Delta} f^i_{r-1,r,s}(f,.)\right|\right|_2^2 \\ \leq & \sum_{t=0}^{T_i-1} (4c\Delta^2)^2 ||\mathbf{1}_{A_t^i}(f,.)||_2^2 \\ = & (4c\Delta^2)^2. \end{aligned}$$

Furthermore, (3.59) and (3.60) prove that $\mathbb{F}$-a.s.

$$\mathbf{E}_{\mathbb{B}}\left[\sum_{t=0}^{T_i-1} \mathbf{1}_{A_t^i} \sum_{r=t+1}^{T_i} {}^iG^B_{r-1,r}/S^0_r\right] = 0. \tag{3.66}$$

Hence, the Strong Law of Large Numbers (Kolmogorov's Criterion for fixed f) and Lemma 2.9 imply (3.51). □

REMARK 3.24. As it makes no difference whether the expectation in (3.66) is taken due to $\mathbb{B}$ or $\mathbb{B}^i$, it is easy to prove by Fubini's Theorem that the biometric risk contributions $\sum_{t=0}^{T_i-1} \mathbf{1}_{A_t^i} \sum_{r=t+1}^{T_i} {}^iG^B_{r-1,r}/S^0_r$ are pairwise uncorrelated.

EXAMPLE 3.25. Consider life insurance contracts which are for the i-th individual given by two cash flows $({}^i\gamma_t)_{t\in\mathbb{T}_i} = (\frac{{}^iC_t}{S^0_t}e_0)_{t\in\mathbb{T}_i}$ and $({}^i\delta_t)_{t\in\mathbb{T}_i} = (\frac{{}^iD_t}{S^0_t}e_0)_{t\in\mathbb{T}_i}$ with $\mathbb{T}_i = \{0, 1, \ldots, T_i\}$ in years. Assume that each ${}^iC_t$ is given by ${}^iC_t(f, b) = {}^ic_t(f)\, {}^i\beta^\gamma_t(b^i)$ for all $(f, b) = (f, b^1, b^2, \ldots) \in M$ where ${}^ic_t$ is a positive $\mathcal{F}_t$-measurable function. Let $({}^i\delta_t)_{t\in\mathbb{T}}$ be defined analogously with the variables ${}^iD_t, {}^id$ and ${}^i\beta^\delta_t$. Suppose that ${}^i\beta^{\gamma(\delta)}_t$ is $\mathcal{B}^i_t$-measurable with ${}^i\beta^{\gamma(\delta)}_t(b^i) \in \{0, 1\}$ for all $b^i \in B^i$. Clearly, (3.50) is fulfilled if

$${}^ic_t, {}^id_t \leq c \cdot S^0_t \tag{3.67}$$

for all $t \in \mathbb{T}_i$ and all $i \in \mathbb{N}^+$. If $S^0_t \geq 1$ for all $t \in \mathbb{T}_i$ (which is quite realistic), this condition is fulfilled by constants ${}^ic_t, {}^id_t \leq c$ (cf. Example 2.21). However, (3.67) allows the adjustment of premiums and claims to a possible inflation without the loss of (3.50) when one assumes that the money account $(S^0_t)_{t\in\mathbb{T}_i}$ would reflect such an inflation. Hence, (3.50) is an acceptable condition from the practical point of view.

Proposing that insurance companies reasonably price contracts and are willing to drive financial hedging strategies, we have seen that they can benefit in different ways from the biometric diversification by means of the Law of Large Numbers. One possibility is a huge number of independent individuals/contracts during a finite time interval (see also Chapter 2). Another possibility is a huge number of independent individuals/contracts over a large or infinite time interval where the number of contracts running during a finite time interval may be small. Roughly speaking, a huge insurance company which never goes bankrupt is the best proposition for an optimal benefit from the Law of Large Numbers in life insurance.

3.9 Multiperiod decomposition

The multiperiod decomposition of gains is perhaps of less importance in practice since insurance companies usually consider time intervals of one year (as balances are computed yearly) and do not use in-between arising biometric information for hedging purposes (cf. Section 3.3). However, the multiperiod decomposition, i.e. the decomposition of gains obtained over a time interval in which also biometric information was used for trading, is an interesting theoretical problem which is unfortunately beyond the scope of this thesis.

Ongoing from the L^2-considerations in the previous sections, one could try to define the financial part of the multiperiod decomposition as solution of the following minimization problem.

Let $X \in L^0(M, \mathcal{F}_t \otimes \mathcal{B}_t, \mathbb{P})$ and $\mathcal{Y}$ be the set of all payoffs $Y \in L^0(M, \mathcal{F}_t \otimes \mathcal{B}_t, \mathbb{P})$ at t which are produced by *all* self-financing strategies which start and have a certain price $P \in L^0(M, \mathcal{F}_s \otimes \mathcal{B}_s, \mathbb{P})$ at s. The solution Y^* of the minimization problem

$$\min_{Y \in \mathcal{Y}} ||X - Y||_2 \tag{3.68}$$

is then taken as financial part of X (if the solution exists and is unique).

Observe the analogy to the definition of the one-period decomposition (cf. Remark 3.11).

Again, (3.68) is different from the minimization problems which are usually studied in the literature. Furthermore, it is not clear whether a reasonable form of a possible solution Y^* (compared to (3.22)) can be deduced in our framework. We must leave this topic open and postpone it to future research. Nonetheless, a pragmatic approach to the problem could be the use of Corollary

3.17.

As $G^B_{r,r+1}$ does not depend on the trading strategy (cf. Proposition 3.13), the right summand in (3.44) could be used as one (more or less) reasonable way to compute the multiperiod biometric part of any gain $G_{s,t}$ when biometric information arising during $(s,t]$ was used. One has to point out that the financial part of this decomposition *is not* necessarily the solution of the minimization problem (3.68). The two approaches should be expected to be different as long as one does not know more about possible solutions of (3.68).

3.10 A review of Bühlmann's approach

For the sake of completeness, we discuss Bühlmann's approach to stochastic discounting and risk decomposition in this section.

Bühlmann (1992, 1995) considers a (life) insurance policy as a vector X of payoffs X_t at $t \in \mathbb{T} = \{0, 1, \dots, T\}$. In fact, $t = 0$ is excluded in Bühlmann (1992), but included in Bühlmann (1995). Positive numbers are interpreted as payments from the insurer to the insurant. We do not consider any portfolios in this section. The notion *valuation principle* is replaced by the *valuation* Q of Bühlmann, which is the price for X "made and to be paid" at $t = 0$. Q is defined as a continuous linear functional on the vectors $(X_t)_{t\in\mathbb{T}}$ of some not further specified $L^2(M, \mathcal{M}, \mathbb{P})^{|\mathbb{T}|}$, which is a Hilbert space with the scalar product

$$(X, Y) = \sum_{t\in\mathbb{T}} \mathbf{E}[X_t Y_t]. \tag{3.69}$$

Indeed, and despite of the fact that Bühlmann later uses a certain filtration for the dynamics of information, at this point Q is defined on $(L^2(M, \mathcal{M}, \mathbb{P})^{|\mathbb{T}|}$. Actually, this gives rise to some interesting questions and we will return to this topic, soon.

Under the assumptions made, one obtains by a standard representation theorem of continuous linear functionals in Hilbert spaces a representation of Q by expectations, i.e.

$$Q[X] = \mathbf{E}\left[\sum_{t=0}^{T} \varphi_t X_t\right] \tag{3.70}$$

for some $\varphi \in L^2(M, \mathcal{M}, \mathbb{P})^{|\mathbb{T}|}$. In Bühlmann (1992), the φ_t are called *stochastic discount functions*. After that, a filtration $(M_t)_{t\in\mathbb{T}}$ is defined by

$$\mathcal{M}_t = \sigma(X_0, \dots, X_t; \varphi_0, \dots, \varphi_t), \quad t \in \mathbb{T}. \tag{3.71}$$

The abstract random variables φ_t - a priori only known to be in $L^2(M, \mathcal{M}, \mathbb{P})$ - are used to define an information structure (history) which is later used to represent the development of information in the real word. From the economic point of view, this is a problematic assumption. In fact, the information structure should be fixed a priori (e.g. generated by the development of the given price processes of assets in a financial market), i.e. *before* any price operator is introduced. Furthermore, (3.71) depends on one single cash flow X, only.

Nonetheless, prices at time t are now defined by

$$Q[X|\mathcal{M}_t] = \frac{1}{\varphi_t} \mathbf{E}\left[\sum_{s=0}^{T} \varphi_s X_s \middle| \mathcal{M}_t \right]. \tag{3.72}$$

One immediately obtains the following decomposition of the value of the contract in prices of the past and the future payment stream:

$$Q[X|\mathcal{M}_t] \quad = \quad \underbrace{\sum_{s=0}^{t} \frac{\varphi_s}{\varphi_t} X_s}_{\text{past stream}} \quad + \quad \underbrace{\frac{1}{\varphi_t} \mathbf{E}\left[\sum_{s=t+1}^{T} \varphi_s X_s \middle| \mathcal{M}_t \right]}_{\text{future stream}}. \tag{3.73}$$

As a consequence, any payment at some $s < t$ develops in the same way (seen from t), independent of the investment strategy. This result - which is astonishing from the economic point of view when there are more assets than only one in the market - has its mathematical roots in the problematic assumptions concerning the information structure of the model.

First, it is important to note (and was already mentioned) that Bühlmann's equilibrium justification of (3.72) crucially depends on the fact that Q is defined on the *whole* $L^2(M, \mathcal{M}, \mathbb{P})^{|\mathbb{T}|}$. However, using an economic equilibrium argument, it is problematic to explicitly use cash flows which cannot have any *real* equivalent. For instance, payments at times s that are conditioned on events at time $t > s$ play an important role in Bühlmann (1992; p. 114, step b). Clearly, Q should be defined on some

$$L^2(M, \mathcal{M}_0, \mathbb{P}) \times \ldots \times L^2(M, \mathcal{M}_T, \mathbb{P}) \tag{3.74}$$

with $\mathcal{M}_0 \subset \ldots \subset \mathcal{M}_T \subset \mathcal{M}$ being an increasing series of a priori given σ-algebras.

The second problem is (3.71) and was already discussed above. Additionally it should be remarked that being given any information structure $(\mathcal{M}_t)_{t\in\mathbb{T}}$ in advance, i.e. before computing the φ_t (as it should be reasonably assumed), it is not at all clear whether the φ_t would be $\mathcal{M}_t$-measurable. However, this is a crucial presumption for the representation (3.72) and a reasonable interpretation of (3.73).

For these reasons it is problematic to use the stochastic discounting approach as explained above. Nonetheless, we continue the description.

Ongoing from the definitions,

$$L_t(X) = \frac{\varphi_t}{\varphi_{t-1}} Q[X|\mathcal{M}_t] - Q[X|\mathcal{M}_{t-1}] \tag{3.75}$$

is defined as *annual loss* in $(t-1, t]$, discounted to the beginning of the interval (time in years; cf. Bühlmann, 1995). Then, the following definitions take place:

$$\mathcal{G}_t = \sigma(X_0, \ldots, X_{t-1}; \varphi_0, \ldots, \varphi_t), \tag{3.76}$$

$$R[X|\mathcal{M}_t] = \frac{1}{\varphi_t} \mathbf{E}\left[\sum_{s=t+1}^{T} \varphi_s X_s \,\middle|\, \mathcal{M}_t \right], \tag{3.77}$$

which is the prospective reserve, and

$$R^+[X|\mathcal{G}_t] = \frac{1}{\varphi_t} \mathbf{E}\left[\sum_{s=t}^{T} \varphi_s X_s \,\middle|\, \mathcal{G}_t \right]. \tag{3.78}$$

Now, a certain martingale sequence for the filtration

$$\mathcal{M}_0 \subset \mathcal{G}_1 \subset \mathcal{M}_1 \subset \mathcal{G}_2 \subset \mathcal{M}_2 \subset \ldots \tag{3.79}$$

is considered. The members of this sequence due to the $\mathcal{M}_t$ are discounted sums of annual losses. From $\mathcal{M}_{t-1}$ to $\mathcal{G}_t$ the "claims experience" is identical, from $\mathcal{G}_t$ to $\mathcal{M}_t$ the "financial base" remains unchanged (cf. Bühlmann, 1995). Considering differences of this martingale, the decomposition $L_t = L_t^F + L_t^B$ is proposed by

$$L_t^B = \frac{\varphi_t}{\varphi_{t-1}} X_t + \frac{\varphi_t}{\varphi_{t-1}} R[X|\mathcal{M}_t] - \frac{\varphi_t}{\varphi_{t-1}} R^+[X|\mathcal{G}_t] \tag{3.80}$$

and

$$L_t^F = \frac{\varphi_t}{\varphi_{t-1}} R^+[X|\mathcal{G}_t] - R[X|\mathcal{M}_{t-1}]. \tag{3.81}$$

Observe, that one has

$$L_t(X) = \frac{\varphi_t}{\varphi_{t-1}} X_t + \frac{\varphi_t}{\varphi_{t-1}} R[X|\mathcal{M}_t] - R[X|\mathcal{M}_{t-1}]. \tag{3.82}$$

The problem with this decomposition is that one could choose

$$\mathcal{G}_t' = \sigma(X_0, \ldots, X_t; \varphi_0, \ldots, \varphi_{t-1}) \tag{3.83}$$

instead of $\mathcal{G}_t$ and get a quite similar, but different result. There is no explicit reason for $\mathcal{G}_t$ given in Bühlmann (1995). Finally, it is not clear whether there is an economic interpretation of (3.78).

3.11 Appendix

LEMMA 3.26. *For X in any $L^2(P, \mathcal{P}, \mathbb{P})$ and any sub-σ-algebra $\mathcal{P}' \subset \mathcal{P}$*

$$(\mathbf{E}[X|\mathcal{P}'])^2 \le \mathbf{E}[X^2|\mathcal{P}'] \quad \mathbb{P}\text{-}a.s. \tag{3.84}$$

Hence, $||\mathbf{E}[X|\mathcal{P}']||_2 \le ||X||_2 < \infty$ and therefore $\mathbf{E}[X|\mathcal{P}'] \in L^2(P, \mathcal{P}', \mathbb{P})$.

Proof. (3.84) is a well-known corollary of Jensen's inequality. □

LEMMA 3.27. *For X in any $L^1(F \times B, \mathcal{F} \otimes \mathcal{B}, \mathbb{F} \otimes \mathbb{B})$ with $|\mathcal{F}| < \infty$ and a σ-algebra $\mathcal{B}' \subset \mathcal{B}$ one has $\mathbb{F}$-a.s.*

$$\mathbf{E}_{\mathbb{F}\otimes\mathbb{B}}[X|\mathcal{F} \otimes \mathcal{B}'](f, .) = \mathbf{E}_{\mathbb{B}}[X(f, .)|\mathcal{B}'] \quad \mathbb{B}\text{-}a.s. \tag{3.85}$$

Proof. From Fubini's Theorem one has for all $F_1 \in \mathcal{F}$, $B_1 \in \mathcal{B}'$ that

$$\int_{F_1} \int_{B_1} X d\mathbb{B} d\mathbb{F} = \int_{F_1} \int_{B_1} \mathbf{E}_{\mathbb{F}\otimes\mathbb{B}}[X|\mathcal{F} \otimes \mathcal{B}'] d\mathbb{B} d\mathbb{F}. \tag{3.86}$$

Therefore it holds for all $B_1 \in \mathcal{B}'$ $\mathbb{F}$-a.s. that

$$\int_{B_1} X(f, .) d\mathbb{B} = \int_{B_1} \mathbf{E}_{\mathbb{F}\otimes\mathbb{B}}[X|\mathcal{F} \otimes \mathcal{B}'](f, .) d\mathbb{B}. \tag{3.87}$$

Hence, (3.87) for all $B_1 \in \mathcal{B}'$ on a set $F(B_1) \in \mathcal{F}$ with measure 1. As $A := \bigcap_{B_1 \in \mathcal{B}'} F(B_1)$ is a finite intersection (since $|\mathcal{F}| < \infty$), $\mathbb{F}(A) = 1$. So for all $f \in A$ one has for all $B_1 \in \mathcal{B}'$ (3.87). This implies (3.85). □

The following lemma can in several forms be found in the literature.

LEMMA 3.28. *Consider the Hilbert space $L^2(P, \mathcal{P}, \mathbb{P})$, where $(P, \mathcal{P}, \mathbb{P})$ is an arbitrary probability space, and for some σ-algebra $\mathcal{P}' \subset \mathcal{P}$ the closed subspace $L^2(P, \mathcal{P}', \mathbb{P})$. For any $X \in L^2(P, \mathcal{P}, \mathbb{P})$ one has the orthogonal decomposition*

$$P(X) = \mathbf{E}[X|\mathcal{P}'] \tag{3.88}$$

and

$$Q(X) = X - P(X) \tag{3.89}$$

due to the subspaces $L^2(P, \mathcal{P}', \mathbb{P})$ and $L^2(P, \mathcal{P}', \mathbb{P})^\perp$. In particular, $\mathbf{E}[X|\mathcal{P}']$ is the unique $Y \in L^2(P, \mathcal{P}', \mathbb{P})$ which minimizes $||X - Y||_2$.

Proof. By Lemma 3.26, $P(X) \in L^2(P, \mathcal{P}', \mathbb{P})$. It remains to prove that for any $X \in L^2(P, \mathcal{P}, \mathbb{P})$ the vector $Q(X)$ is orthogonal to any $Y \in L^2(P, \mathcal{P}', \mathbb{P})$:

$$\mathbf{E}[YQ(X)] = \mathbf{E}[\mathbf{E}[YQ(X)|\mathcal{P}']] = \mathbf{E}[Y\mathbf{E}[Q(X)|\mathcal{P}']] = 0. \tag{3.90}$$

The minimality property is a standard result (e.g. Rudin, 1987). □

LEMMA 3.29. *In the framework of Section 3.2, respectively Section 2.3,* $L^p(M, \mathcal{F}_t \otimes \mathcal{B}_s, \mathbb{P}) \subset L^p(M, \mathcal{F}_t \otimes \mathcal{B}_s, \mathbb{M})$ *for* $s, t \in \mathbb{T}$, $s \leq t$ *and* $p \in [1, \infty]$. *Furthermore, for* $X \in L^p(M, \mathcal{M}_t, \mathbb{P})$

$$\mathbf{E}_{\mathbb{F}\otimes\mathbb{B}}[X|\mathcal{F}_t \otimes \mathcal{B}_s] = \mathbf{E}_{\mathbb{Q}\otimes\mathbb{B}}[X|\mathcal{F}_t \otimes \mathcal{B}_s]. \tag{3.91}$$

Proof. By the Fundamental Theorem the Radon-Nikodym-derivative

$$d\mathbb{M}/d\mathbb{P} = d(\mathbb{Q} \otimes \mathbb{B})/d(\mathbb{F} \otimes \mathbb{B}) = d\mathbb{Q}/d\mathbb{F} \tag{3.92}$$

(cf. Lemma 2.12) is bounded. This proves the first part of the lemma. For the second part one applies Lemma 3.27 as well as Lemma 2.8 and obtains $\mathbb{F} \otimes \mathbb{B}$-a.s.

$$\mathbf{E}_{\mathbb{F}\otimes\mathbb{B}}[X|\mathcal{F}_t \otimes \mathcal{B}_s](f, b) = \mathbf{E}_{\mathbb{B}}[X(f, .)|\mathcal{B}_s](b) \tag{3.93}$$

Replacing $\mathbb{F}$ by $\mathbb{Q}$ proves (3.91). □

LEMMA 3.30. *Presume any* $X \in L^2(P, \mathcal{P}, \mathbb{P})$, $Y \in L^2(P, \mathcal{P}', \mathbb{P})$ *and σ-algebras* $\mathcal{P}'' \subset \mathcal{P}' \subset \mathcal{P}$. *It holds that*

$$\mathbf{E}[(X - Y])^2|\mathcal{P}''] \leq \mathbf{E}[(X - Z)^2|\mathcal{P}''] \quad \mathbb{P}\text{-a.s.} \tag{3.94}$$

for all $Z \in L^2(P, \mathcal{P}', \mathbb{P})$ *if and only if* $Y = \mathbf{E}[X|\mathcal{P}']$ $\mathbb{P}$*-a.s.*

Proof. One has

$$\begin{aligned} & \mathbf{E}[(X - \mathbf{E}[X|\mathcal{P}'])^2|\mathcal{P}''] \qquad (3.95)\\ = \; & \mathbf{E}[\, \mathbf{E}[\, X^2 - 2X\mathbf{E}[X|\mathcal{P}'] + \mathbf{E}[X|\mathcal{P}']^2 \;|\mathcal{P}'] \;|\mathcal{P}''] \\ = \; & \mathbf{E}[\, \mathbf{E}[\, X^2|\mathcal{P}'] \; - \mathbf{E}[X|\mathcal{P}']^2 \;|\mathcal{P}''] \\ = \; & \mathbf{E}[\, X^2 - \mathbf{E}[X|\mathcal{P}']^2 \;|\mathcal{P}'']. \end{aligned}$$

Furthermore,

$$\mathbf{E}[(X - Z)^2|\mathcal{P}''] = \mathbf{E}[\, X^2 - 2\mathbf{E}[X|\mathcal{P}']Z + Z^2 \;|\mathcal{P}'']. \tag{3.96}$$

One therefore gets for the difference of (3.96) and (3.95)

$$\mathbf{E}[\, (\mathbf{E}[X|\mathcal{P}'] - Z)^2 \;|\mathcal{P}''] \geq 0. \tag{3.97}$$

Hence, $Y = \mathbf{E}[X|\mathcal{P}']$ fulfills (3.94) for all $Z \in L^2(P, \mathcal{P}', \mathbb{P})$. However, any other candidate for Y must fulfill

$$-\mathbf{E}[\, (Y - \mathbf{E}[X|\mathcal{P}'])^2 \;|\mathcal{P}''] \geq 0, \tag{3.98}$$

which can be derived from (3.94) setting $Z = \mathbf{E}[X|\mathcal{P}']$. Hence,

$$||Y - \mathbf{E}[X|\mathcal{P}']||_2^2 \leq 0 \tag{3.99}$$

and therefore $Y = \mathbf{E}[X|\mathcal{P}']$ $\mathbb{P}$-a.s. □

DEFINITION 3.31. *For a random variable Z in any $L^2(P, \mathcal{P}, \mathbb{P})$ its* **conditional variance** *due to some sub-σ-algebra $\mathcal{P}' \subset \mathcal{P}$ is defined by*

$$Var[X|\mathcal{P}'] = \mathbf{E}[(X - \mathbf{E}[X|\mathcal{P}'])^2|\mathcal{P}']. \tag{3.100}$$

For instance, when $\mathcal{P}$ is the information at some time t and $\mathcal{P}'$ at time $s < t$, the interpretation of (3.100) as "the variance of X seen from s" is obvious.

LEMMA 3.32. *Propose some σ-algebras $\mathcal{P}'' \subset \mathcal{P}' \subset \mathcal{P}$. For any $X \in L^2(P, \mathcal{P}, \mathbb{P})$ and $Z \in L^2(P, \mathcal{P}'', \mathbb{P})$*

$$Var[X + Z|\mathcal{P}''] = Var[X|\mathcal{P}'']. \tag{3.101}$$

Presume $X \in L^2(P, \mathcal{P}, \mathbb{P})$ and $Y \in L^2(P, \mathcal{P}', \mathbb{P})$. It holds that

$$Var[X - Y|\mathcal{P}''] \leq Var[X - Z|\mathcal{P}''] \tag{3.102}$$

for all $Z \in L^2(P, \mathcal{P}', \mathbb{P})$ if and only if $Y = \mathbf{E}[X|\mathcal{P}'] + C$ $\mathbb{P}$-a.s. for some $C \in L^2(P, \mathcal{P}'', \mathbb{P})$.

Proof. (3.101) is clear. For the left side of (3.102) one has

$$\mathbf{E}[\ (X - Y - \mathbf{E}[X - Y|\mathcal{P}''])^2\ |\mathcal{P}''], \tag{3.103}$$

analogously the right side for Z. For $Y = \mathbf{E}[X|\mathcal{P}'] + C$ where $C \in L^2(P, \mathcal{P}'', \mathbb{P})$, the left side of (3.102) is identical to

$$\mathbf{E}[\ (X - \mathbf{E}[X|\mathcal{P}'])^2\ |\mathcal{P}''] \tag{3.104}$$

since $\mathbf{E}[\ X - \mathbf{E}[X|\mathcal{P}'] - C\ |\mathcal{P}''] = -C$. This implies the backward direction by Lemma 3.30 since $Z + \mathbf{E}[X - Z|\mathcal{P}''] \in L^2(P, \mathcal{P}', \mathbb{P})$ due to the Jensen-Lemma 3.26. However, any other candidate Y must fulfill

$$\begin{aligned} 0 &\leq \mathbf{E}[\ (X - \mathbf{E}[X|\mathcal{P}'])^2\ |\mathcal{P}''] - \mathbf{E}[\ (X - Y - \mathbf{E}[X - Y|\mathcal{P}''])^2\ |\mathcal{P}''] \\ &= -\mathbf{E}[\ (Y + \mathbf{E}[X - Y|\mathcal{P}''] - \mathbf{E}[X|\mathcal{P}'])^2\ |\mathcal{P}'']. \end{aligned} \tag{3.105}$$

Therefore,

$$Y = \mathbf{E}[X|\mathcal{P}'] - \mathbf{E}[X - Y|\mathcal{P}''] \quad \mathbb{P}\text{-a.s.} \tag{3.106}$$

But (3.106) if and only if $Y = \mathbf{E}[X|\mathcal{P}'] + C$ $\mathbb{P}$-a.s. for some $C \in L^2(P, \mathcal{P}'', \mathbb{P})$. □

Chapter 4

Risk capital allocation by coherent risk measures based on one-sided moments

4.1 Introduction

From the works of Denault (2001) and Tasche (2000) it is known that differentiability of risk measures is crucial for risk capital allocation in portfolios. The reason is that in the case of differentiable positively homogeneous risk measures the gradient due to asset weights has figured out to be the unique reasonable per-unit allocation principle. After a short introduction to risk measures at the end of the present section, the approaches of Denault (2001) and Tasche (2000) to this result are briefly reviewed in Section 4.2 of this chapter. However, in contrast to the mentioned result, it is known that in practice quantile-based risk measures like the widely used Value-at-Risk methodology or the so-called Expected Shortfall encounter situations, e.g. in the case of insurance claims, credit portfolios or digital options, where probability distributions are discrete and the risk measures are not differentiable anymore (cf. Tasche, 2000). Furthermore, Section 4.3 of this chapter shows that at least in the case of subadditive positively homogeneous risk measures differentiability on all portfolios actually is not desirable since the risk measures become linear and minimal in this case. As a solution, we define weaker differentiability properties (also Section 4.3). For positively homogeneous (and in particular coherent) risk measures these properties allow allocation by the gradient on all relevant portfolios. Excluded are portfolios that contain only one type of assets. However, in these cases the allocation problem is trivial. In Section 4.4, we introduce a wide class of coherent risk measures based on the mean and the one-sided moments of a risky payoff. In order to construct the class, it is shown that weighted sums of coherent risk measures are again coherent. Hence, it is possible to "mix" coherent risk measures. For example, one could consider the arithmetic mean of the maximum-loss-principle and a semi-deviation-like risk measure - both are members of the given class. An important result of Section 4.4 is that the constructed risk measures (expected and maximum loss excluded) are examples for the weakened differentiability properties of Section 4.3. In contrast to quantile-based risk measures, members of this class allow allocation in portfolios of very general distributions, e.g. discrete ones. Furthermore, for any fixed random payoff X risk measures of this class can be chosen such that the risk capital due to X equals any value between the expected and the maximum loss of X. In Section 4.5, two numerical examples show how this property can be used to choose a particular risk measure of the class which assigns the same risk capital to a given portfolio as VaR does. As a consequence, the risk capital originally given by the VaR can be allocated by the gradient due to the chosen risk measure. Section 4.6 compares the notation of this chapter with the one used in Tasche (2000), respectively Denault (2001). In addition to the mentioned results of the chapter, some of the lemmas proven in the technical appendix could be interesting in themselves.

Given a probability space $(\Omega, \mathcal{A}, \mathbb{Q})$, we will consider the vector space $L^p(\Omega, \mathcal{A}, \mathbb{Q})$, or just $L^p(\mathbb{Q})$, for $1 \leq p \leq \infty$. Even though $L^p(\mathbb{Q})$ consists of equivalence classes of p-integrable random variables, we will treat its elements as random variables. Due to the context, no confusion should arise. The notation will be as follows. We have $||X||_p = (\mathbf{E}_{\mathbb{Q}}[|X|^p])^{\frac{1}{p}}$ and $||X||_\infty =$ ess.sup$\{|X|\}$. Recall, that $L^p(\mathbb{Q}) \subset L^q(\mathbb{Q})$ if $1 \leq q < p \leq \infty$, since $||.||^q \leq ||.||^p$. X^- is defined as $\max\{-X, 0\}$. We denote $\sigma_p^-(X) = ||(X - \mathbf{E}_{\mathbb{Q}}[X])^-||_p$. Now, let $U \subset \mathbb{R}^n$ for $n \in \mathbb{N}^+ = \mathbb{N} \setminus \{0\}$ be open and positively homogeneous, i.e. for $u \in U$ we have $\lambda u \in U$ for all $\lambda > 0$. A function $f : U \to \mathbb{R}$ is called positively homogeneous (or homogeneous of degree one) if $f(\lambda u) = \lambda f(u)$ for all $\lambda > 0$, $u \in U$. When f is also differentiable at every $u = (u_1, \ldots, u_n) \in U$, we obtain the well-known Euler Theorem

$$f(u) = \sum_{i=1}^{n} u_i \frac{\partial f}{\partial u_i}(u). \tag{4.1}$$

We consider a one-period framework, that means we have the present time

0 and a future time horizon T. Between 0 and T no trading is possible. We assume "risk" to be given by a random payoff X, i.e. a random variable in $L^p(\mathbb{Q})$ representing a cash flow at T. We want to consider a risk measure $\rho(X)$ to be the extra minimum cash added to X that makes the position acceptable for the holder or a regulator. For this reason, we state the following definition.

DEFINITION 4.1. *A* **risk measure** *on $L^p(\mathbb{Q})$, $1 \leq p \leq \infty$, is defined by a functional $\rho : L^p(\mathbb{Q}) \to \mathbb{R}$.*

We now give a definition of coherent risk measures. For a further motivation and interpretation of this axiomatic approach to risk measurement we refer to the article of Artzner et al. (1999).

DEFINITION 4.2. *A functional $\rho : L^p(\mathbb{Q}) \to \mathbb{R}$, where $1 \leq p \leq \infty$, is called a* **coherent risk measure (CRM)** *on $L^p(\mathbb{Q})$ if the following properties hold.*

(M) *Monotonicity: If $X \geq 0$ then $\rho(X) \leq 0$.*

(S) *Subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$.*

(PH) *Positive homogeneity: For $\lambda \geq 0$ we have $\rho(\lambda X) = \lambda\rho(X)$.*

(T) *Translation: For constants a we have $\rho(a + X) = \rho(X) - a$.*

As we work without interest rates - in contrast to Artzner et al. (1999) - there is no discounting factor in Definition 4.2. A generalization of CRM to the space of all random variables on a probability space can be found in Delbaen (2000). However, having $p \geq 1$ prevents us from being forced to allow infinitely high risks. See Delbaen (2000) for details on this topic.

The scientific discussion about suitable properties of risk measures continues. Especially in the context of actuarial mathematics (a risk measure can be seen as an insurance premium principle and vice versa) alternative approaches exist (e.g. Goovaerts, Kaas and Dhaene, 2003). For the purposes of this chapter, we stay in the framework of positively homogeneous or coherent risk measures. A deeper discussion about properties of risk measures in different economic contexts is beyond the scope of this thesis.

4.2 Risk capital allocation by the gradient

Let us consider the payoff $X(u) := \sum_{i=1}^{n} u_i X_i \in L^p(\mathbb{Q})$ of a portfolio $u = (u_i)_{1 \leq i \leq n} \in \mathbb{R}^n$ consisting of assets (or subportfolios) with payoffs $X_i \in L^p(\mathbb{Q})$.

DEFINITION 4.3. *A* **portfolio base** *in* $L^p(\mathbb{Q})$ *is a vector* $B \in (L^p(\mathbb{Q}))^n$, $n \in \mathbb{N}^+$. *The components of* B *do not have to be linearly independent.*

Having $B = (X_1, \ldots, X_n)$, a risk measure ρ on the payoffs $L^p(\mathbb{Q})$ implies a risk measure ρ_B on the portfolios $\mathbb{R}^n$. In particular, we define $\rho_B : \mathbb{R}^n \to \mathbb{R}$ by

$$\rho_B : u \mapsto \rho(X(u)). \tag{4.2}$$

If ρ_B is obtained from a CRM ρ on $L^p(\mathbb{Q})$ and X_n is the only constant component in B and not equal zero, ρ_B is also called coherent (cf. Denault, 2001). If ρ fulfills axiom (S) and (PH) in Definition 4.2, ρ_B is subadditive and positively homogeneous on $\mathbb{R}^n$.

Due to diversification effects (or subadditivity of the risk measure), the total risk of a portfolio is usually assumed to be less then the sum of the risks of each subportfolio, i.e. we often have $\rho_B(u) < \sum_{i=1}^n \rho_B(u_i e_i)$, where e_i is the i-th canonical unit vector in $\mathbb{R}^n$. The so-called allocation problem is the question, how much risk capital should be allocated to each of the subportfolios $u_i e_i$ and hence how the subportfolios should benefit from the diversification. However, as identical payoffs should be treated identically, this question is equivalent to the search for a reasonable per-unit allocation principle.

DEFINITION 4.4. *Given a portfolio base* B *and a risk measure* ρ_B *on* $\mathbb{R}^n$ *a* **per-unit allocation** *in* $u \in \mathbb{R}^n$ *is a vector* $(a_i(\rho_B, u))_{1 \le i \le n}$, *such that*

$$\sum_{i=1}^n u_i a_i(\rho_B, u) = \rho_B(u). \tag{4.3}$$

In Denault (2001) the author drives the attention of the reader to a result of Aubin in the theory of coalitional games with fractional players. Aubin's theorem states that in the case of a positively homogeneous, convex and differentiable cost function the core of such a game (Aubin uses the prefix *fuzzy*) consists of one element: the gradient of the cost function due to the normed weights of the players (Aubin, 1979). From this result, it is immediate that in the case of a subadditive and positively homogeneous risk measure (e.g. a coherent one), which is differentiable at a portfolio $u \in \mathbb{R}^n$, the gradient $(\frac{\partial \rho_B}{\partial u_i}(u))_{1 \le i \le n}$ is the *unique fair* per-unit allocation. To derive this statement from Aubin's result, the notion of cost functions in game theory has to be replaced by our notion of a risk measure. The players of the game are given by the certain $u_i X_i$, coalitions of fractional players are given by portfolios v

with $0 \leq v \leq u$, where the given portfolio u can without loss of generality be assumed to be positive. Note that convexity and subadditivity are equivalent under positive homogeneity. The core of such a game contains all per-unit allocations $(a_i(\rho_B, u))_{1\leq i\leq n}$, such that for all coalitions v with $0 \leq v \leq u$ we have $\sum_{i=1}^n v_i a_i(\rho_B, u) \leq \rho_B(v)$. That means, no sub-coalition v of u features less stand-alone risk than the risk the coalition v would have been charged by the respective per-unit allocation due to u. In this sense, the elements of the core are fair allocations. For the sake of completeness, it should be mentioned that in the case of a positively homogeneous risk measure the core of the game is identical to the subdifferential of ρ_B at u. If ρ_B is also convex or subadditive, the core is nonempty, convex and compact (Aubin, 1979). However, in this general case uniqueness of the core gets lost. For differentiable CRM Denault proved that the Aumann-Shapley value, which is the above gradient, features certain coherence properties (Denault, 2001). For a deeper study of the connections between the theory of convex games and coherent risk measures we refer to Delbaen (2002).

In the case of just positively homogeneous risk measures, the theory of convex games is no longer suitable to model the allocation problem. However, it is still possible to talk about reasonable allocations. Tasche (2000) considers the so-called return on risk-adjusted capital (RORAC) of the payoff $X(u)$ of a portfolio u, which he defines by $f(u) = \mathbf{E}_{\mathbb{Q}}[X(u)]/\rho_B(u)$. Note, that what we called risk measure is denoted economic capital by Tasche, whereas he defines risk as fluctuation risk from the mean. Now, the idea is to call a per-unit allocation *suitable for performance measurement* with ρ_B, when $(a_i(\rho_B, u))_{1\leq i\leq n}$ gives the right signals for local changes in the portfolio. More precise, if $\mathbf{E}_{\mathbb{Q}}[X_i]/a_i(\rho_B, u) > f(u)$, there should be an $\varepsilon_0 > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$ we have $f(u - \varepsilon e_i) < f(u) < f(u + \varepsilon e_i)$. Analogously, for $\mathbf{E}[X_i]/(a_i(\rho_B, u)) < f(u)$ we demand $f(u - \varepsilon e_i) > f(u) > f(u + \varepsilon e_i)$. Tasche shows that in the case of differentiable positively homogeneous risk measures the unique per-unit allocation $(a_i(\rho_B, u))_{1\leq i\leq n}$ that is continuous on $\mathbb{R}^n$ and suitable for performance measurement due to the risk adjusted return function is the gradient $(\frac{\partial \rho_B}{\partial u_i}(u))_{1\leq i\leq n}$ (Tasche, 2000).

In both approaches, Denault's and Tasche's, the relationship between total risk and risk contribution per unit is established by the Euler Theorem

$$\rho_B(u) = \sum_{i=1}^n u_i \frac{\partial \rho_B}{\partial u_i}(u). \tag{4.4}$$

The per-unit risk contribution equals the marginal risk. So, concerning risk capital allocation due to a (subadditive) positively homogeneous risk measure on $L^p(\mathbb{Q})$, it would be desirable to have ρ_B to be differentiable on $\mathbb{R}^n$ for every portfolio base $B \in (L^p(\mathbb{Q}))^n$ for all $n \in \mathbb{N}^+$.

4.3 Differentiability properties

As the Value-at-Risk methodology is widely used in practice, marginal risks of VaR have been considered in several papers. In the Gaussian case we refer to the works of Garman (1996) and (1997), in the general case of continuous distributions to Gouriéroux, Laurent and Scaillet (2000). The perhaps more sophisticated (but also quantile-based) Expected Shortfall (called Tail-VaR by some authors) is considered in Scaillet (2000). Despite of the results in the case of continuous distributions, having a quantile-based risk measure ρ like VaR or Expected Shortfall, it is known that ρ_B is not differentiable on $\mathbb{R}^n$ in general. Roughly speaking, for differentiability at least one of the X_i has to possess a continuous density (Tasche, 2000). Hence, it is a problem to deal with discrete spaces $(\Omega, \mathcal{A}, \mathbb{Q})$ like e.g. in the case of credit portfolios, insurance claims or digital options. It will be shown in Section 4.4 that the step to moment based risk measures avoids this difficulty. Beside the differentiability difficulties, it is also known that VaR is not subadditive (Artzner et al., 1999). As diversification is not rewarded, this is a major drawback.

However, even if risk measures are differentiable on $\mathbb{R}^n$, this can imply some problems. To understand what kind of problems can arise, we state a proposition which connects differentiability with linearity and minimality of subadditive positively homogeneous risk measures:
We have seen that it would be desirable to have ρ_B to be differentiable on $\mathbb{R}^n$ for every portfolio base $B \in (L^p(\mathbb{Q}))^n$ for all $n \in \mathbb{N}^+$. Considering the initial ρ on $L^p(\mathbb{Q})$, this implies the existence of Gâteaux-derivatives, i.e. derivatives due to directions on $L^p(\mathbb{Q})$.

PROPOSITION 4.5. *Let $\mathcal{S}$ be a subset of the four axioms given in Definition 4.2,* (PH) *and* (S) *being contained in $\mathcal{S}$. For a risk measure ρ on $L^p(\mathbb{Q})$, $1 \le p \le \infty$, that fulfills the axioms $\mathcal{S}$, the following properties are equivalent: (i) ρ is Gâteaux-differentiable on $L^p(\mathbb{Q})$, (ii) ρ is linear, (iii) ρ is minimal due to $\mathcal{S}$, i.e. there is no risk measure $\rho' \neq \rho$ fulfilling $\mathcal{S}$ such that $\rho'(X) \le \rho(X)$ for all $X \in L^p(\mathbb{Q})$. Differentiability of ρ on $L^p(\mathbb{Q})$ implies (i), (ii) and (iii).*

COROLLARY 4.6. *A continuous coherent risk measure ρ on $L^p(\mathbb{Q})$ is Gâteaux-differentiable on $L^p(\mathbb{Q})$, $1 < p < \infty$, if and only if there exists a probability measure $\mathbb{Q}_\rho \sim \mathbb{Q}$ on Ω, such that $\rho(X) = -\mathbf{E}_{\mathbb{Q}_\rho}[X]$.*

In particular, Proposition 4.5 is true for coherent risk measures. The proof of 4.5 is omitted since equivalence of (i) and (ii) can be shown by a simple application of the axioms (PH) and (S). Since subadditive positively homogeneous risk measures are sub-linear functionals, the well-known proof for equivalence of (ii) and (iii) in the general sub-linear case can easily be adapted to our cases. The corollary follows from the duality of the $L^p(\mathbb{Q})$ spaces.

As the two statements are also true for subspaces of $L^p(\mathbb{Q})$, we face the following problem: If ρ_B is a differentiable risk measure on $\mathbb{R}^n$ which fulfills $\mathcal{S}$ (e.g. coherence), it is easy to show that ρ_B is linear. Therefore, ρ_B features no diversification effects. We also obtain that ρ is linear on the linear span $\langle B\rangle$ of the components of B, which implies that ρ is minimal on $\langle B\rangle$ due to $\mathcal{S}$ (coherence). Hence, differentiability on the whole $\mathbb{R}^n$ might be not useful.

Now, consider a portfolio base $B = (X_1, \ldots, X_n)$ and a portfolio $u = u_i e_i = (0, \ldots, 0, u_i, 0, \ldots, 0)$, $u_i \in \mathbb{R}$, $1 \le i \le n$. In this case the allocation problem is trivial, since by (4.3) the risk capital allocated to X_i - which is the only asset - is simply $\rho_B(u)/u_i$. The following definition is motivated by this consideration.

DEFINITION 4.7. *Consider a portfolio base $B = (X_1, \ldots, X_n) \in (L^p(\mathbb{Q}))^n$, $n \in \mathbb{N}^+$, $1 \le p \le \infty$, and a portfolio $u \in \mathbb{R}^n$. Define $U_e = \bigcup_{i=1}^n \langle e_i\rangle$, where $\langle e_i\rangle \subset \mathbb{R}^n$ is the linear span of e_i. We propose to call a (subadditive) positively homogeneous risk measure ρ on $L^p(\mathbb{Q})$* **suitable for risk capital allocation** *by the gradient due to the portfolio base B if the function $\rho_B : \mathbb{R}^n \to \mathbb{R}$ with $\rho_B : u \mapsto \rho(X(u))$ is differentiable on the open set $\mathbb{R}^n \setminus U_e$.*

4.4 A class based on one-sided moments

We define a class of coherent risk measures which depend on the mean and the one-sided higher moments of a risky position.

LEMMA 4.8. *For $1 \le p \le \infty$ and $0 \le a \le 1$, the risk measure $\rho_{p,a}$ with*

$$\rho_{p,a}(X) = -\mathbf{E}_{\mathbb{Q}}[X] + a \cdot \sigma_p^-(X) = -\mathbf{E}_{\mathbb{Q}}[X] + a \cdot ||(X - \mathbf{E}_{\mathbb{Q}}[X])^-||_p \qquad (4.5)$$

is coherent on $L^p(\mathbb{Q})$.

Delbaen (2002) shows that these risk measures can be obtained by the set of probability measures (also called *generalized scenarios*, compare Artzner et al. (1999)) $P = \{1 + a(g - \mathbf{E}[g]) \mid g \geq 0; ||g||_q \leq 1\}$, where $q = p/(p-1)$ and probability measures are identified with their densities. In Delbaen (2000) we find another type of risk measures that are connected to higher moments.

Proof of Lemma 4.8. The L^p-norm on the right side of (4.5) is finite, since $X \in L^p(\mathbb{Q})$. Axiom (T) and (PH) are obvious. From Minkowski's inequality and the inequality $(a+b)^- \leq a^- + b^-$ for $a, b \in \mathbb{R}$, we obtain axiom (S). Axiom (M): Let $X \geq 0$. We have $X - \mathbf{E}_{\mathbb{Q}}[X] \geq -\mathbf{E}_{\mathbb{Q}}[X]$, therefore $(X - \mathbf{E}_{\mathbb{Q}}[X])^- \leq \mathbf{E}_{\mathbb{Q}}[X]$ and hence $||(X - \mathbf{E}_{\mathbb{Q}}[X])^-||_\infty = \text{ess.sup}\{(X - \mathbf{E}_{\mathbb{Q}}[X])^-\} \leq \mathbf{E}_{\mathbb{Q}}[X]$. Since $||(X - \mathbf{E}_{\mathbb{Q}}[X])^-||_p \leq ||(X - \mathbf{E}_{\mathbb{Q}}[X])^-||_\infty$ for $p \in [1, \infty]$, we get $||(X - \mathbf{E}_{\mathbb{Q}}[X])^-||_p \leq \mathbf{E}_{\mathbb{Q}}[X]$. Recalling $0 \leq a \leq 1$, this completes the proof. □

The L^p-norms imply that $\rho_{q,a} \leq \rho_{p,a}$ if $q < p$. The following result is on weighted sums of coherent risk measures and generalizes the trivial fact that convex sums of CRM are again CRM .

LEMMA 4.9. *Let $I \subset \mathbb{R}$ be an index set and $(\rho_i)_{i \in I}$ be a family of coherent risk measures respectively defined on $L^{p(i)}(\mathbb{Q})$, where $p : I \to [1, \infty]$. Let $(\rho_i)_{i \in I}$ be point-wise uniformly bounded on $L^{\sup p(I)}(\mathbb{Q})$ in the sense that there is a function $b : L^{\sup p(I)}(\mathbb{Q}) \to \mathbb{R}_0^+$ such that for each $X \in L^{\sup p(I)}(\mathbb{Q})$ we have $|\rho_i(X)| \leq b(X)$ for all $i \in I$. Let R be a random variable with range I that is defined on a probability space Ω' with measure $\mathbb{P}$. Now, if for all $X \in L^{\sup p(I)}(\mathbb{Q})$ the mapping $\rho_{R(.)}(X) : \Omega' \to \mathbb{R}$ is measurable,*

$$\rho(X) = \mathbf{E}_{\mathbb{P}}[\rho_R(X)] \tag{4.6}$$

defines a coherent risk measure on $L^{\sup p(I)}(\mathbb{Q})$.

Proof. ρ is well-defined, since for each $X \in L^{\sup p(I)}(\mathbb{Q})$ we know from $|\rho_i(X)| \leq b(X)$ and the measurability assumption, that $\rho_R(X)$ is a bounded random variable and therefore $\mathbb{P}$-integrable. Now, the coherence axioms are obvious by the properties of $\mathbf{E}_{\mathbb{P}}$. □

Using Lemma 4.9, the result of Lemma 4.8 can be generalized.

PROPOSITION 4.10. *Let P be a random variable on a probability space $(\Omega', \mathbb{P})$ with range $P(\Omega') \subset [1, p]$ and assume that $1 \leq p \leq \infty$ and $0 \leq a \leq 1$. The risk measure*

$$\rho(X) = -\mathbf{E}_{\mathbb{Q}}[X] + a \cdot \mathbf{E}_{\mathbb{P}}[\sigma_P^-(X)] \tag{4.7}$$

is coherent on $L^p(\mathbb{Q})$*. We have* $-\mathbf{E}_{\mathbb{Q}}[X] \leq \rho(X) \leq \text{ess.sup}\{-X\}$.

Proof. Due to Lemma 4.8 we consider a family $(\rho_{i,a})_{i\in[1,p]}$ of coherent risk measures given by (4.5), respectively defined on $L^i(\mathbb{Q})$. Now, let $b(X) = |\mathbf{E}_{\mathbb{Q}}[X]| + ||(X - \mathbf{E}_{\mathbb{Q}}[X])^-||_p$. Clearly, $|\rho_i(X)| \leq b(X)$ for all $1 \leq i \leq p$. For all $X \in L^p(\mathbb{Q})$ the mapping $\rho_{P(.),a}(X) : \Omega' \to \mathbb{R}$ is measurable, since $P(.)$ is measurable and for all $Y \in L^p(\mathbb{Q})$ the mapping $q \mapsto ||Y||_q$ is measurable on $P(\Omega')$ as it is continuous due to the relative topology on $P(\Omega')$ in $\mathbb{R} \cup \{\infty\}$ with the canonical topology (cf. Lemma 4.19). We obtain coherence of (4.7) by Lemma 4.9. The last statement follows from $||.||_p \leq ||.||_\infty$ and $\sigma_\infty^- = \text{ess.sup}\{(X - \mathbf{E}_{\mathbb{Q}}[X])^-\} = \text{ess.sup}\{-X + \mathbf{E}_{\mathbb{Q}}[X]\}$. □

REMARK 4.11. An immediate consequence of Lemma 4.19 is that for any X the risk measure ρ can be chosen such that $\rho(X)$ equals any value $v \in [-\mathbf{E}_{\mathbb{Q}}[X], \text{ess.sup}\{-X\}]$, i.e. any value between the expected loss and the maximum loss. In particular, for $X \not\equiv$ *const* a.s. and $v \in [-\mathbf{E}_{\mathbb{Q}}[X] + \sigma_1^-(X), \text{ess.sup}\{-X\}]$ there is a unique $p^* = p^*(v) \in [1, \infty]$ such that $\rho_{p^*,1}(X) = -\mathbf{E}_{\mathbb{Q}}[X] + \sigma_{p^*}^-(X) = v$.

EXAMPLE 4.12. $\rho(X) = -\mathbf{E}_{\mathbb{Q}}[X] + a_1\sigma_1^- + a_2\sigma_2^- + \ldots + a_\infty\sigma_\infty^-$, where $a_p \geq 0$ for $p \in \{1, 2, 3, \ldots, \infty\}$ and $a_\infty + \sum_{p=1}^{\infty} a_p \leq 1$ is a coherent risk measure on $L^q(\mathbb{Q})$, where $q := \sup\{p | a_p > 0\}$ (we use the convention $0 \cdot (\pm\infty) = (\pm\infty) \cdot 0 = 0$). In particular, $a_2 = a_\infty = \frac{1}{2}$ could be interpreted as a coherent "mixture" of the semi-deviation and the maximum-loss-principle.

DEFINITION 4.13. *For* $B \in (L^p(\mathbb{Q}))^n$*,* $n \in \mathbb{N}^+$*,* $1 < p < \infty$*, the set* $U_C(B)$ *denotes the set of all* $u \in \mathbb{R}^n$ *for which* $\sum_{i=1}^n u_i X_i \equiv$ *const.*

LEMMA 4.14. *The set* $\mathbb{R}^n \setminus U_C(B)$ *is open in* $\mathbb{R}^n$*.*

Proof. The linear mapping $X(.) : \mathbb{R}^n \to L^p(\mathbb{Q})$, where $u \mapsto X(u)$, is bounded, since $||X(u)||_p \leq \sum_{i=1}^n |u_i| \cdot ||X_i||_p \leq ||u|| \cdot \sum_{i=1}^n ||X_i||_p$. Hence, $X(.)$ is continuous on $\mathbb{R}^n$. The set C of all constant elements of $L^p(\mathbb{Q})$ is closed, since $L^p(\mathbb{Q})$ is a Banach-space due to the theorem of Riesz-Fischer and every Cauchy-sequence of constant elements in $L^p(\mathbb{Q})$ converges to a constant limit in $L^p(\mathbb{Q})$ (due to L^p-norm). Since $X(.)$ is continuous, $[X(.)]^{-1}(C) = U_C(B)$ is closed and $\mathbb{R}^n \setminus U_C(B)$ open. □

We can now state a result on differentiability of the class of coherent risk measures that was introduced in Proposition 4.10.

PROPOSITION 4.15. *Assume $B \in (L^p(\mathbb{Q}))^n$, $n \in \mathbb{N}^+$, $1 < p < \infty$ and $0 \leq a \leq 1$. Let $1 < P \leq p$ be a random variable on a probability space with measure $\mathbb{P}$. The risk measures ρ_B implied by* (4.7) *are differentiable on $\mathbb{R}^n \setminus U_C(B)$. The partial derivatives are*

$$\begin{aligned} \frac{\partial \rho_B}{\partial u_i}(u) &= -\mathbf{E}_{\mathbb{Q}}[X_i] + a \cdot \mathbf{E}_{\mathbb{P}}[\sigma_P^-(X(u))^{1-P} \cdot \\ &\qquad \mathbf{E}_{\mathbb{Q}}[(-X_i + \mathbf{E}_{\mathbb{Q}}[X_i]) \cdot ((X(u) - \mathbf{E}_{\mathbb{Q}}[X(u)])^-)^{P-1}]]. \end{aligned} \tag{4.8}$$

The proof of Proposition 4.15 is rather technical and therefore given in the Appendix. We want to show that the risk measures (4.7) actually can not be differentiable at some $u \in U_C(B)$. Suppose $u \in U_C(B)$, $a > 0$ and the risk measure defined by (4.5), which is the special case $P \equiv p$. We have $\rho_{p,a}(u) = -\mathbf{E}_{\mathbb{Q}}[X(u)]$, since $X(u) \equiv \mathbf{E}_{\mathbb{Q}}[X(u)]$. Easily we obtain the two different one-sided partial derivatives $-\mathbf{E}_{\mathbb{Q}}[X_i] + a \cdot ||(\pm X_i \mp \mathbf{E}_{\mathbb{Q}}[X_i])^-||_p$ in u, but $||(X_i - \mathbf{E}_{\mathbb{Q}}[X_i])^-||_p \neq ||(-X_i + \mathbf{E}_{\mathbb{Q}}[X_i])^-||_p$ in general. So, we have no differentiability in general.

COROLLARY 4.16. *Under the assumptions of 4.15, the risk measures ρ implied by* (4.7) *are suitable for risk capital allocation by the gradient due to the portfolio base B if the components $X_1, \ldots, X_n$ of B are linearly independent and $X_n \not\equiv 0$ is constant. The per-unit allocations are given by* (4.8).

Proof. $U_C(B) = \langle (0, \ldots, 0, 1) \rangle \subset U_e$. □

Corollary 4.16 is the main result on risk capital allocation by the considered class of coherent risk measures. No assumptions concerning the underlying probability space $(\Omega, \mathcal{A}, \mathbb{Q})$ have been made, discrete spaces can be taken into consideration. The assumption of linear independence is quite weak as it should be no problem to find a vector base in a real market. Even the particular choice of the portfolio base B is not important as the gradient is an aggregation invariant allocation principle (Denault, 2001). The reason is that if we have two different portfolio bases B and B' as given in Corollary 4.16 with $\langle B \rangle = \langle B' \rangle$, there exists a linear isomorphism A on $\mathbb{R}^n$ such that we have $X(u) \equiv X'(u')$ and $\rho_B(u) = \rho_{B'}(u')$ for every $u = Au' \in \mathbb{R}^n$. We therefore obtain from standard analysis for any two equivalent portfolios v and v' with $v = Av'$

$$\sum_{i=1}^n v'_i \frac{\partial \rho_{B'}}{\partial u'_i}(u') = \sum_{i=1}^n v_i \frac{\partial \rho_B}{\partial u_i}(u). \tag{4.9}$$

So, the risk capital allocated to equivalent subportfolios, i.e. subportfolios with the same payoff in $L^p(\mathbb{Q})$, is identical.

4.5 Application

In this section, two examples illustrate how risk capital given by the Value-at-Risk can be allocated using the risk measures from Section 4.4. In particular, we use a risk measure of type $\rho_{p,1}(X) = -\mathbf{E}_{\mathbb{Q}}[X] + \sigma_p^-(X)$ as given in (4.5). We define the Value-at-Risk by

$$\mathrm{VaR}_\alpha(X) = -\inf\{x : \mathbb{Q}(X \leq x) > \alpha\}. \tag{4.10}$$

As long as $\mathrm{VaR}_\alpha(X) \geq -\mathbf{E}_{\mathbb{Q}}[X] + \sigma_1^-(X)$, we know from Remark 4.11 that there is a unique $p^* \in [1, \infty]$ such that $\rho_{p^*,1}(X) = \mathrm{VaR}_\alpha(X)$. Since the risk measure $\rho_{p^*,1}$ $(1 < p^* < \infty)$ is suitable for risk capital allocation (cf. Corollary 4.16), the amount $\mathrm{VaR}_\alpha(X)$ can be allocated by allocation due to $\rho_{p^*,1}$, i.e. for a portfolio base B as given in 4.16 and ρ_B^* corresponding to $\rho_{p^*,1}$ (cf. (4.2)), we have

$$\mathrm{VaR}_\alpha(X(u)) = \rho_B^*(u) = \sum_{i=1}^{n} u_i \frac{\partial \rho_B^*}{\partial u_i}(u). \tag{4.11}$$

EXAMPLE 4.17 (Discrete distributions). Suppose two stochastically independent payoff variables X_1, X_2 with discrete distributions as given in Table 4.1. The portfolio base is given by $B = (X_1, X_2, 1)$. X_1 and X_2

x	$\mathbb{Q}(X_1 = x)$	$\mathbb{Q}(X_2 = x)$
0.0	0.78	0.96
-0.5	0.20	0.02
-1.0	0.02	0.02

Table 4.1: Distribution of X_1, X_2

could be interpreted as one unit of a credit engagement. Obviously, X_1 bears higher risks as losses are more probable. We consider the portfolio $u = (u_1, u_2, u_3) = (1000, 1000, 0)$. Easily we compute $\mathrm{VaR}_{0.05}(X(u)) = 500$. To allocate the given risk capital, we adjust $\rho_B(u)$ by choosing p^*, such that $\rho_{p^*,1}(X(u)) = \mathrm{VaR}_{0.05}(X(u)) = 500$. We obtain $p^* \approx 2.9157$. From the discrete version of (4.8) ($|\Omega| = 9$, $P \equiv p^*$, $a = 1$) we obtain $\frac{\partial \rho_B^*}{\partial u_1}(u) \approx 0.31504$ and $\frac{\partial \rho_B^*}{\partial u_2}(u) \approx 0.18496$. The risk capital allocated to $u_1 X_1$ is 315.04, for $u_2 X_2$ it is 184.96. To check what happens for a more conservative VaR, we compute $\mathrm{VaR}_{0.01}(X(u))$, which is 1000. We obtain $p^* \approx 9.4355$ and the risk capital allocated to $u_1 X_1$ is 477.98, for $u_2 X_2$ it is 522.02. It is interesting that in the second case more risk capital is allocated to X_2, which seems to bear less risk. However, the relative difference is quite small compared to the first case. This seems to be reasonable as we have $\mathrm{VaR}_{0.01}(u_1 X_1) = \mathrm{VaR}_{0.01}(u_2 X_2) = 1000$.

EXAMPLE 4.18 (Continuous distributions). Although continuous distributions are considered in this example, we assume that (4.5) are the risk measures of choice. A possible scenario could be the situation where these risk measures are intended to be used internally where at the same time external regulatory requirements define the minimum risk capital by the VaR-method. We assume to be given a portfolio base $B = (X_1, X_2, 1)$ with

$$\begin{aligned} X_1 &\sim n_1 \cdot v_1 \cdot (\exp(\sigma_1 Z_1) - 1) \\ X_2 &\sim \sqrt{n_2} \cdot v_2 \cdot \sigma_2 Z_2, \end{aligned} \tag{4.12}$$

where Z_1, Z_2 are assumed to be standard normally distributed with correlation $r \geq 0$. X_1 could be interpreted as the log-normal payoff of a portfolio of n_1 (identical) financial assets minus the price $n_1 v_1$ at which they were bought. The expected value of one asset is $v_1 \cdot \exp(\sigma_1^2/2)$. X_2 could be interpreted as an approximation of the sum of n_2 i.i.d. payoffs with expectation 0 and standard deviation $v_2 \cdot \sigma_2$, e.g. coming from a balanced credit portfolio or the liabilities of an insurance company. In particular, we assume $\sigma_1 = 0.2$, $\sigma_2 = 0.1$, $n_1 = 10^6$, $n_2 = 10000$, $v_1 = 200$, $v_2 = 10^6$ and $r = 0.8$ ($r > 0$ is reasonable in the case of a credit portfolio). The "external" risk measure is assumed to be given by the 5%-VaR. The portfolio base is $B = (X_1, X_2, 1)$ and the portfolio $(1, 1, 0)$, i.e. the considered overall payoff is the sum $X = X_1 + X_2$. The expectation of X_1 is $4.04 \cdot 10^6$ (i.e. a mean return of 2%) and the standard deviation $41.2 \cdot 10^6$ (rounded values). For X_2 we have expectation 0 and $10 \cdot 10^6$ for the standard deviation (also rounded). All non-trivial computation, e.g. for $\mathrm{VaR}_{0.05}(X)$ and $\sigma_p^-(X)$, is done by the classical Monte-Carlo method, i.e. Z_1 and Z_2 are simulated and the VaR-quantile and the non-trivial integrals in (4.5) and (4.8) are obtained from the simulated empirical distributions. We get $\mathrm{VaR}_{0.05}(X) \approx 70 \cdot 10^6$. The calibration of $\rho_{p,1}(X)$ is done by the bisection method ($\rho_{p,1}(X)$ is monotone in p). We start with the interval $[1, 30]$, where p^* is assumed to be contained in, and go on 16 steps which corresponds to a theoretical error for p^* of less then $(30 - 1) \cdot 2^{-16} \approx 0.44 \cdot 10^{-3}$ (neglecting the Monte-Carlo error). For each integral $200 \cdot 10^6$ pseudo-random values of Z_1, respectively Z_2, are computed. We obtain $p^* \approx 10.05$ and $\rho_{p^*,1}(X) \approx 70.01 \cdot 10^6$. Computation of the partial derivatives gives $\frac{\partial \rho_B^*}{\partial u_1}(u) \approx 53.55 \cdot 10^6$ and $\frac{\partial \rho_B^*}{\partial u_2}(u) \approx 16.38 \cdot 10^6$, i.e. a sum $69.93 \cdot 10^6 \approx 70 \cdot 10^6$. As we have assumed X_2 to be the sum of n_2 i.i.d. payoffs, we obtain the fair risk capital $\frac{\partial \rho_B^*}{\partial u_2}(u)/n_2 \approx 1638$ for each individual payoff.

4.6 Comparison of the notation of Denault, Fischer and Tasche

As each of the three articles uses a particular notation, it is useful to have a direct comparison of variables and expressions corresponding to each other (see Table 4.2). The notation in Fischer (2003a) is the same as in this chapter.

Denault (2001)	Fischer (2003a)	Tasche (2000)
$\frac{X_i}{\Lambda_i}$	X_i	C_i
Λ_i	u_i	u_i
X_i	$u_i X_i$	$u_i C_i$
λ_i	v_i	
$\frac{\lambda_i}{\Lambda_i} X_i$	$v_i X_i$	
$r(\Lambda)$	$\rho_B(u)$	$r(u) - \sum_{i=1}^n u_i m_i$
	$\mathbf{E}_{\mathbb{Q}}[X_i]$	m_i
	$\mathbf{E}_{\mathbb{Q}}[X_i] - X_i$	X_i
	$\rho_B(u) + \mathbf{E}_{\mathbb{Q}}[X(u)]$	$r(u)$
k_i	$a_i(\rho_B, u)$	$a_i(u) - m_i$
	$f(u)$	$g(u)$

Table 4.2: Different notation

A remark on Tasche's approach (in Tasche's notation, Fischer's notation in brackets): Please note, that

$$m_i r(u) > a_i(u) m' u \tag{4.13}$$

is equivalent to

$$\frac{m_i}{a_i(u) - m_i} > \frac{m'u}{r(u) - m'u} = g(u) \quad (= f(u)), \tag{4.14}$$

and

$$\sum_i u_i a_i(u) = r(u) \tag{4.15}$$

equivalent to

$$\sum_i u_i (a_i(u) - m_i) = r(u) - \sum_i u_i m_i \tag{4.16}$$

$$\left(\text{or } \sum_i u_i a_i(\rho_B, u) = \rho_B(u) \right). \tag{4.17}$$

4.7 Appendix

LEMMA 4.19. *Let $P \subset [1,\infty]$ and $X \in L^{\sup P}(\mathbb{Q})$. The mapping $||X||_{(.)}$: $P \to [0,\infty)$, $p \mapsto ||X||_p$, is continuous due to the relative topology on P in $\mathbb{R} \cup \{\infty\}$ with the canonical topology.*

Proof. The case $P \subset [1,\infty)$ *and* X essentially bounded can be deduced from results in Bourbaki (1965). However, a general proof is needed.

The case $X \equiv 0$ is trivial, therefore we assume $||X||_p > 0$. Since $||X||_{(.)}$ is a real function which is monotone on P, it suffices to show that from the convergence $p_n \to p$ of a sequence $(p_n)_{n\in\mathbb{N}}$ in P there follows $||X||_{p_n} \to ||X||_p$. We first prove the case $p = \infty$, where $\infty \in P$ is assumed. For any $\varepsilon > 0$ there exists some $A \in \mathcal{A}$ with $\mathbb{Q}(A) > 0$ such that

$$|X(\omega)| \geq \text{ess.sup}\{|X|\} - \varepsilon \tag{4.18}$$

for all $\omega \in A$. Now, as $||.||_\infty := \text{ess.sup}(.)$, we have

$$\begin{aligned} \text{ess.sup}\{|X|\} &\geq ||X||_{p_n} \\ &\geq \left(\int_A (\text{ess.sup}\{|X|\} - \varepsilon)^{p_n} d\mathbb{Q}\right)^{\frac{1}{p_n}} \\ &= (\text{ess.sup}\{|X|\} - \varepsilon)(\mathbb{Q}(A))^{\frac{1}{p_n}}. \end{aligned} \tag{4.19}$$

We obtain

$$\text{ess.sup}\{|X|\} \geq \lim_{p_n\to\infty} ||X||_{p_n} \geq \text{ess.sup}\{|X|\} - \varepsilon \tag{4.20}$$

and hence

$$\text{ess.sup}\{|X|\} = ||X||_\infty = \lim_{p_n\to\infty} ||X||_{p_n} \tag{4.21}$$

by definition of $||.||_\infty$. Now, assume $1 \leq p < \infty$ and $p \in P$. We have

$$|X(\omega)|^{p_n} \leq \max\{|X(\omega)|^{\sup P}, 1\}. \tag{4.22}$$

By dominated convergence, we obtain

$$\int |X(\omega)|^{p_n} d\mathbb{Q}(\omega) \longrightarrow \int |X(\omega)|^p d\mathbb{Q}(\omega), \tag{4.23}$$

i.e. $||X||_{p_n}^{p_n} \longrightarrow ||X||_p^p$. The triangle inequality gives us

$$\begin{aligned} &\big|\, ||X||_{p_n} - ||X||_p \big| \\ \leq &\left| \sqrt[p_n]{||X||_{p_n}^{p_n}} - \sqrt[p]{||X||_{p_n}^{p_n}} \right| + \left| \sqrt[p]{||X||_{p_n}^{p_n}} - \sqrt[p]{||X||_p^p} \right|. \end{aligned} \tag{4.24}$$

The right part of the sum converges to zero as the p-th root is a continuous function. The left part converges to zero for the following reasons. As we know, $a_n := ||X||_{p_n}^{p_n}$ converges to $a := ||X||_p^p > 0$. Now,

$$\begin{aligned} & \left| \sqrt[p_n]{||X||_{p_n}^{p_n}} - \sqrt[p]{||X||_{p_n}^{p_n}} \right| \qquad (4.25) \\ = & \ | \sqrt[p_n]{a_n} - \sqrt[p]{a_n}| \\ = & \ |\exp\{\ln\{a_n\}/p\}| \cdot |\exp\{(1/p_n - 1/p)\ln\{a_n\}\} - 1| \,. \end{aligned}$$

The first factor is bounded, since a_n converges to $a > 0$, the second one converges to zero as $(1/p_n - 1/p)\ln\{a_n\}$ converges to zero and the exponential function is continuous. □

The proof of Proposition 4.15 needs the following technical lemmas.

LEMMA 4.20. *Let U be an open subset of $\mathbb{R}^n$, $n \in \mathbb{N}^+$, and $f : U \times \Omega \to \mathbb{R}$ be a function with following properties:*

a) $\omega \mapsto f(u, \omega)$ is $\mathbb{Q}$-integrable for all $u \in U$.

b) $u \mapsto f(u, \omega)$ is in any $u \in U$ partially differentiable with respect to u_i.

c) There exists a $\mathbb{Q}$-integrable function $h_U \geq 0$ on Ω with $\left|\frac{\partial f}{\partial u_i}(u, \omega)\right| \leq h_U(\omega)$ for all $(u, \omega) \in U \times \Omega$.

The function $\varphi(u) = \int f(u, \omega) d\mathbb{Q}(\omega)$ on U is partially differentiable with respect to u_i. The mapping $\omega \mapsto \frac{\partial f}{\partial u_i}(u, \omega)$ is $\mathbb{Q}$-integrable and for $u \in U$

$$\frac{\partial \varphi}{\partial u_i}(u) = \int \frac{\partial f}{\partial u_i}(u, \omega) d\mathbb{Q}(\omega). \qquad (4.26)$$

The proof by the dominated convergence theorem is well-known.

LEMMA 4.21. *Define $U = \triangle u_1 \times \cdots \times \triangle u_n \subset \mathbb{R}^n$, where for all $i \in \{1, \ldots, n\}$ $\triangle u_i$ is a nonempty, bounded and open interval in $\mathbb{R}$. Let $X(u) = \sum_{i=1}^n u_i X_i$ be a sum of real-valued random variables $X_i \in L^p(\mathbb{Q})$ with $u = (u_1, \ldots, u_n) \in U$, $n \in \mathbb{N}^+$ and $1 < p < \infty$. Let $y(u)$ be a real-valued function that is differentiable, bounded and for which $y(u) < \text{ess.sup}\{-X(u)\}$ on U. The partial derivatives $\frac{\partial y}{\partial u_i}(u)$ are also assumed to be bounded on U. Under this assumptions, $||(X(u) + y(u))^-||_p$ is differentiable on U.*

Proof. Define $g(u,\omega) = (X(u,\omega) + y(u))^-$. For $1 \le i \le n$ we will prove existence and continuity of the partial derivatives of $||g(u)||_p$.

Existence: We have $||g(u)||_p = \left(\int g(u,\omega)^p \, d\mathbb{Q}(\omega)\right)^{1/p}$. Now, if we can apply Lemma 4.20 to g^p (where f from 4.20 corresponds to g^p) and if $g(u)$ is *not* constant 0 for every $u \in U$, we obtain for every i

$$\frac{\partial ||g(u)||_p}{\partial u_i}(u) = \int \frac{\partial g^p}{\partial u_i}(u)\, d\mathbb{Q} \cdot \frac{1}{p} \cdot \left(\int g(u)^p \, d\mathbb{Q}\right)^{\frac{1}{p}-1}. \tag{4.27}$$

Note, that for $u \in U$ we have $g(u) > 0$ on a set of measure greater 0, since $y(u) < \text{ess.sup}\{-X(u)\}$. Therefore the right integral in (4.27) is greater 0 (no division by zero!). We are going to check the points a) to c) from Lemma 4.20. Ad a). $\omega \mapsto g(u,\omega)^p$ is $\mathbb{Q}$-integrable, since $X(u) \in L^p(\mathbb{Q})$ and $y(u) \in \mathbb{R}$. Ad b). First, we consider the function $[(.)^-]^p : \mathbb{R} \to \mathbb{R}_0^+$, $x \mapsto (x^-)^p$. Clearly, this function is differentiable for $1 < p < \infty$. Now, $g(u,\omega)^p = [(\sum_{i=1}^n u_i X_i(\omega) + y(u))^-]^p$ - as a combination of a differentiable and a partially differentiable function - is partially differentiable at u_i. We obtain

$$\frac{\partial g^p}{\partial u_i}(u,\omega) \quad = \quad -\left(X_i(\omega) + \frac{\partial y}{\partial u_i}(u)\right) \cdot p \cdot g(u,\omega)^{p-1}. \tag{4.28}$$

Ad c). There exist positive constants a and b, such that for all $j \in \{1,\dots,n\}$ we have $|\frac{\partial y}{\partial u_j}(u)| \le a$ and $|y(u)| \le b$ on U. Now, define

$$u_{\max}(U) = \sup\{|u_j'| : u_j' \in \triangle u_j,\, j \in \{1,\dots,n\}\}, \tag{4.29}$$

which is finite, and

$$k_U(\omega) = n \cdot u_{\max}(U) \cdot \max_j\{|X_j(\omega)|\} + b. \tag{4.30}$$

Clearly, $k_U(\omega) \ge g(u,\omega)$. Now define

$$h_U(\omega) = (|X_i(\omega)| + a) \cdot p \cdot (k_U(\omega))^{p-1}. \tag{4.31}$$

Comparing this to (4.28), we clearly obtain

$$0 \le \left|\frac{\partial g^p}{\partial u_i}(u,\omega)\right| \le h_U(\omega) \tag{4.32}$$

for all $(u,\omega) \in U \times \Omega$. Concerning integrability of (4.31), we know that $(|X_i(\omega)|+a)\cdot p$ is p-integrable, since X_i is. We also know that $(k_U(\omega))^{p-1}$ is $\frac{p}{p-1}$-integrable. The latter statement follows from the fact that every single $|X_j(\omega)|$ is p-integrable and therefore $k_U(\omega)$ - as a multiple of the maximum plus a constant - is p-integrable. We further have $1/p+(p-1)/p = 1$. As an immediate consequence of Hölder's inequality, the product $h_U(\omega)$ of $(|X_i(\omega)| + a) \cdot p$ and $(k_U(\omega))^{p-1}$ is integrable.

Continuity: Consider a sequence $(u_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} u_n = u$ in $U = \triangle u_1 \times \cdots \times \triangle u_n$. Now, substitute u by u_n in (4.27). For fix $\omega \in \Omega$ it follows from the definition of $g(u)$ and (4.28) that the substituted expressions under the integrals in (4.27) converge (pointwise in ω) to the original expressions (in u). Now have in mind, that h_U (4.32) dominates the left integrand of (4.27) and $(k_U)^p$ (4.30) dominates the right one. As h_U and $(k_U)^p$ are integrable, it follows from the dominated convergence theorem that the substituted integrals themselves converge to the original integrals. Hence, (4.27) is continuous in u. □

LEMMA 4.22. *Assume $B \in (L^p(\mathbb{Q}))^n$, $n \in \mathbb{N}^+$, $1 < p < \infty$. Suppose $0 \leq a \leq 1$. The risk measures $\rho_B(u)$ implied by* (4.5) *are differentiable on $\mathbb{R}^n \setminus U_C(B)$. The partial derivatives are*

$$\frac{\partial \rho_B}{\partial u_i}(u) = -\mathbf{E}_{\mathbb{Q}}[X_i] + a \cdot \sigma_p^-(X(u))^{1-p} \cdot \mathbf{E}_{\mathbb{Q}}[(-X_i + \mathbf{E}_{\mathbb{Q}}[X_i]) \cdot ((X(u) - \mathbf{E}_{\mathbb{Q}}[X(u)])^-)^{p-1}]. \tag{4.33}$$

Proof. As $\mathbb{R}\setminus U_C(B)$ is open, it can be seen as union of bounded n-dimensional open intervals U. We focus on the $L^p(\mathbb{Q})$-norm expression in $\rho_B(u)$. Define $y(u) = -\mathbf{E}_{\mathbb{Q}}[X(u)]$. Now, the requirements of Lemma 4.21 are satisfied, since $-\mathbf{E}_{\mathbb{Q}}[X(u)] <$ ess.sup$\{-X(u)\}$ as long as $X(u) \not\equiv$ *const*. We obtain that the risk measure is differentiable in U and

$$\frac{\partial \rho_B}{\partial u_i}(u) = -\mathbf{E}_{\mathbb{Q}}[X_i] + \int \frac{\partial g^p}{\partial u_i}(u) d\mathbb{Q} \cdot a \cdot \frac{1}{p} \cdot ||g(u)||_p^{1-p} \quad . \tag{4.34}$$

As (4.34) does not depend on the choice of the particular $U \subset \mathbb{R}^n \setminus U_C(B)$, $\rho_B(u)$ is differentiable on $\mathbb{R}^n \setminus U_C(B)$. Since by definition $||g(u)||_p = \sigma_p^-(X(u))$, we obtain (4.33) by combining (4.28) with (4.34). □

Proof of Proposition 4.15. We use the notation from the proofs of the Lemmas 4.21 and 4.22. Assume $U = \triangle u_1 \times \cdots \times \triangle u_n$ to be a bounded nonempty n-dimensional open interval in $\mathbb{R}^n \setminus U_C(B)$, where for all $i \in \{1, \ldots, n\}$ $\triangle u_i$ is an open interval. Consider equation (4.7). We have

$$\mathbf{E}_{\mathbb{P}}[\sigma_P^-(X(u))] = \int ||g(u)||_{P(\omega')} d\mathbb{P}(\omega') \ . \tag{4.35}$$

We prove the existence and continuity of the partial derivatives of (4.35).

Existence: Again, we are going to check the points a) to c) from Lemma 4.20 (f corresponds to $||g(u)||_{P(\omega')}$). Ad a). $\omega' \mapsto ||g(u)||_{P(\omega')}$ is integrable, since $||g(u)||_{P(\omega')} \leq ||g(u)||_p < \infty$. Ad b). Since $P(\omega')$ is fix, it follows from the proof of Lemma 4.22 (Eq. (4.34)), that $u \mapsto ||g(u)||_{P(\omega')}$ is in every point $u \in U$ partially differentiable with respect to u_i. Ad c). From (4.34) we get

$$\frac{\partial f}{\partial u_i}(u,\omega') = \int \frac{\partial g^{P(\omega')}}{\partial u_i}(u)d\mathbb{Q} \cdot \frac{a}{P(\omega')} \cdot ||g(u)||_{P(\omega')}^{1-P(\omega')}. \tag{4.36}$$

From (4.28) we obtain

$$\frac{\partial g^{P(\omega')}}{\partial u_i}(u,\omega) = -(X_i(\omega) - \mathbf{E}_{\mathbb{Q}}[X_i]) \cdot P(\omega') \cdot g(u,\omega)^{P(\omega')-1}. \tag{4.37}$$

As $g(u,\omega)^{P(\omega')-1}$ is $\frac{P(\omega')}{P(\omega')-1}$-integrable, we get from Hölder's inequality

$$\begin{aligned} \left|\int \frac{\partial g^{P(\omega')}}{\partial u_i}(u,\omega)d\mathbb{Q}\right| &\leq \left|\left|\frac{\partial g^{P(\omega')}}{\partial u_i}(u,\omega)\right|\right|_1 \qquad (4.38) \\ &\leq ||(X_i - \mathbf{E}_{\mathbb{Q}}[X_i])||_{P(\omega')} \cdot P(\omega') \cdot ||g(u)||_{P(\omega')}^{P(\omega')-1}. \end{aligned}$$

Combining this with (4.36), we obtain

$$\begin{aligned} \left|\frac{\partial f}{\partial u_i}(u,\omega')\right| &\leq ||(X_i - \mathbf{E}_{\mathbb{Q}}[X_i])||_{P(\omega')} \cdot a \qquad (4.39) \\ &\leq ||(X_i - \mathbf{E}_{\mathbb{Q}}[X_i])||_p \cdot a \equiv const. \end{aligned}$$

Choosing $h_U(\omega') = ||(X_i - \mathbf{E}_{\mathbb{Q}}[X_i])||_p \cdot a$, this completes the proof of c). From the arbitrariness of $U \subset \mathbb{R}^n \setminus U_C(B)$, we obtain partial differentiability of ρ on $\mathbb{R}^n \setminus U_C(B)$. Equation (4.8) follows from the combination of Lemma (4.20) with the result (4.33) of Lemma 4.22.

Continuity: As we know from the proof of Lemma 4.22, expression (4.36) is continuous on $\mathbb{R}^n \setminus U_C(B)$. By (4.39), dominated convergence proves continuity of the partial derivatives. □

Chapter 5

Conclusion

In this thesis, several aspects of modern life insurance mathematics have been considered. In particular, the following topics were discussed: valuation, hedging, risk decomposition, pooling and risk capital allocation.

Chapter 2 has shown that a modern theory of life insurance can be based on a set of eight principles or seven mathematical axioms. The widely used modern valuation principle (or minimum fair price) is an implication of these axioms coming from the demand for converging mean balances under certain, rather rudimentary hedges which must be able to be financed by the minimum fair prices. As in the classical case, the Law of Large Numbers plays a fundamental role, here. A first glimpse of what is called "pooling" could be catched.

In Chapter 3, it became clear how strong the connection between hedging, risk decomposition and pooling is. For instance, under certain assumptions the reiteration of the so-called locally variance-optimal hedge for a fairly priced contract (under the minimum fair price of Chapter 2) implies that the mean discounted total gain of the first m contracts converges to zero almost surely for $m \to \infty$ when clients are independent. However, under the hedge, this mean gain is exactly the mean accumulated discounted biometric risk contribution of the first m contracts (cf. Proposition 3.20 and its corollaries).

Remarkable with Proposition 3.20 is that it does not matter how the contracts under consideration are distributed on the time axis and whether the time axis is finite or not. Hence, the proposition gives a very satisfying interpretation of what should be understood as pooling of biometric risk contributions in life insurance.

A self-citation should be allowed here.

Proposing that insurance companies reasonably price contracts and are willing to drive financial hedging strategies, we have seen that they can benefit in different ways from the biometric diversification by means of the Law of Large Numbers. One possibility is a huge number of independent individuals/contracts during a finite time interval (see also Chapter 2). Another possibility is a huge number of independent individuals/contracts over a large or infinite time interval where the number of contracts running during a finite time interval may be small. Roughly speaking, a huge insurance company which never goes bankrupt is the best proposition for an optimal benefit from the Law of Large Numbers in life insurance (cf. Section 3.8).

In summary, Chapter 2 and 3 developed an appealing framework (theory) in which life insurance mathematics in discrete time can be done. The framework and the generality of the deduced results seem to be new. An adaption of the results to continuous time models must be postponed to future research. Also more practical problems like an integration or review of existing bonus theory in the proposed model should be considered then.

Clearly, the fourth chapter stands out as its possible applications are not restricted to life insurance. However, the importance of questions like risk capital determination and allocation will also grow in life insurance. The definition of what should be called a *suitable* risk measure for risk capital allocation is in particular of interest when the considered probability spaces are discrete - as in the life insurance models in Chapter 2 and 3.

The proposed examples which depend on one-sided moments should not only be seen as risk measures. For instance, the so-called safety loads in insurance are often nothing else than a moment (think of the variance or standard deviation premium principle) which is added to an expectation (minimum fair price). Therefore, some established premium principles are astonishing similar to the proposed risk measures and it should be a topic of future research to examine how premium principles or safety loads for single contracts can be integrated in risk management approaches for complete companies. Perhaps, one day, a premium or safety load charged in practice will just be the risk capital allocated to the respective contract by an overall risk measure used on the company level (compare also "Premium Calculation from Top-down" in Goovaerts, Kaas, Dhaene and Tang (2003), and the references therein).

It is still a long way until integrated risk management will have been studied sufficiently by the scientific community. It will take even longer until it will be fully accepted by practitioners in life and non-life insurance companies all over the world. The author hopes that some results of this dissertation can contribute to this challenge.

> When you look out the other way toward the stars you realize it's an awful long way to the next watering hole.

Loren W. Acton

Bibliography

[1] Aase, K.K., Persson, S.-A. (1994) - *Pricing of Unit-linked Life Insurance Policies*, Scandinavian Actuarial Journal 1994 (1), 26-52

[2] Artzner, P., Delbaen, F., Eber, J.-M., Heath, D. (1999) - *Coherent measures of risk*, Mathematical Finance 9 (3), 203-228

[3] Aubin, J.-P. (1979) - *Mathematical Methods of Game and Economic Theory*, North-Holland, Amsterdam

[4] Bouleau, N., Lamberton, D. (1989) - *Residual risks and hedging strategies in Markovian markets*, Stochastic Processes and their Applications 33, 131-150

[5] Bourbaki, N. (1965) - *Éléments de Mathématique, Livre VI, Intégration, Chapitres 1, 2, 3 et 4*, Hermann, Paris

[6] Brennan, M.J., Schwartz, E.S. (1976) - *The pricing of equity-linked life insurance policies with an asset value guarantee*, Journal of Financial Economics 3, 195-213

[7] Bühlmann, H. (1987) - *Editorial*, ASTIN Bulletin 17 (2), 137-138

[8] Bühlmann, H. (1992) - *Stochastic discounting*, Insurance: Mathematics and Economics 11 (2), 113-127

[9] Bühlmann, H. (1995) - *Life Insurance with Stochastic Interest Rates* in Ottaviani, G. (Ed.) - *Financial Risk in Insurance*, Springer

[10] Dalang, R.C., Morton, A., Willinger, W. (1990) - *Equivalent martingale measures and no-arbitrage in stochastic securities market models*, Stochastics and Stochastics Reports 29 (2), 185-201

[11] Delbaen, F. (1999) - *The Dalang-Morton-Willinger Theorem*, ETH Zürich,
http://www.math.ethz.ch/~delbaen/ftp/teaching/DMW-Theorem.pdf

[12] Delbaen, F. (2000) - *Coherent risk measures on general probability spaces*, Preprint, ETH Zürich, http://www.math.ethz.ch/~delbaen/ftp/preprints/RiskMeasuresGeneralSpaces.pdf

[13] Delbaen, F. (2002) - *Coherent Risk Measures*, Monograph, Scuola Normale Superiore, Pisa

[14] Delbaen, F., Haezendonck, J. (1989) - *A martingale approach to premium calculation principles in an arbitrage free market*, Insurance: Mathematics and Economics 8, 269-277

[15] Denault, M. (2001) - *Coherent Allocation of Risk Capital*, Journal of Risk 4 (1), 7-21

[16] Duffie, D., Richardson, H.R. (1991) - *Mean-variance hedging in continuous time*, Annals of Applied Probability 1, 1-15

[17] Embrechts, P. (2000) - *Actuarial versus financial pricing of insurance*, Risk Finance 1 (4), 17-26

[18] Fischer, T. (2003a) - *Risk capital allocation by coherent risk measures based on one-sided moments*, Insurance: Mathematics and Economics 32 (1), 135-146

[19] Fischer, T. (2003b) - *An axiomatic approach to valuation in life insurance*, Preprint, TU Darmstadt

[20] Garman, M. (1996) - *Improving on VaR*, RISK 9 (5), 61-63

[21] Garman, M. (1997) - *Taking VaR to Pieces*, RISK 10 (10), 70-71

[22] Gerber, H.U. (1997) - *Life Insurance Mathematics*, 3rd ed., Springer

[23] Goovaerts, M., De Vylder, E., Haezendonck, J. (1984) - *Insurance Premiums*, North-Holland, Amsterdam

[24] Goovaerts, M., Kaas, R., Dhaene, J. (2003) - *Economic capital allocation derived from risk measures*, North American Actuarial Journal 7 (2)

[25] Goovaerts, M., Kaas, R., Dhaene, J., Tang, Q. (2003) - *Some New Classes of Consistent Risk Measures*, Working Paper, KU Leuven, http://www.econ.kuleuven.ac.be/tew/academic/actuawet/pdfs/gkdt2.pdf

[26] Gouriéroux, C., Laurent, J.-P., Scaillet, O. (2000) - *Sensitivity analysis of Values at Risk*, Journal of Empirical Finance 7, 225-245

[27] Halley, E. (1693) - *An Estimate of the Degrees of the Mortality of Mankind, drawn from curious Tables of the Births and Funerals at the City of Breslaw; with an Attempt to ascertain the Price of Annuities upon Lives*, Philosophical Transactions 196, The Royal Society of London, 596-610
A transcription can be found under
http://www.pierre-marteau.com/contributions/boehne-01/halley-mortality-1.html

[28] Harrison, J.M., Kreps, D.M. (1979) - *Martingales and arbitrage in multiperiod securities markets*, Journal of Economic Theory 20, 381-408

[29] Laplace, P.S. (1820) - *Oeuvres, Tome VII, Théorie analytique des probabilités*, 3e édition

[30] Laplace, P.S. (1951) - *A Philosophical Essay on Probabilities*, Dover Publications, New York

[31] Loebus, N. (1994) - *Bestimmung einer angemessenen Sterbetafel für Lebensversicherungen mit Todesfallcharakter*, Blätter der DGVM, Bd. XXI

[32] Møller, T. (1998) - *Risk-minimizing hedging strategies for unit-linked life insurance contracts*, ASTIN Bulletin 28, 17-47

[33] Møller, T. (2001) - *Risk-minimizing hedging strategies for insurance payment processes*, Finance and Stochastics 5, 419-446

[34] Møller, T. (2002) - *On valuation and risk management at the interface of insurance and finance*, British Actuarial Journal 8 (4), 787-828.

[35] Møller, T. (2003a) - *Indifference pricing of insurance contracts in a product space model*, Finance and Stochastics 7 (2), 197-217

[36] Møller, T. (2003b) - *Indifference pricing of insurance contracts in a product space model: applications*, Insurance: Mathematics and Economics 32 (2), 295-315

[37] Norberg, R. (1999) - *A theory of bonus in life insurance*, Finance and Stochastics 3 (4), 373-390

[38] Norberg, R. (2001) - *On bonus and bonus prognoses in life insurance*, Scandinavian Actuarial Journal 2001 (2), 126-147

[39] Persson, S.-A. (1998) - *Stochastic Interest Rate in Life Insurance: The Principle of Equivalence Revisited*, Scandinavian Actuarial Journal 1998 (2), 97-112

[40] Rudin, W. (1987) - *Real and complex analysis*, Third Edition, McGraw-Hill

[41] Scaillet, O. (2000) - *Nonparametric estimation and sensitivity analysis of expected shortfall*, forthcoming in Mathematical Finance

[42] Schich, S.T. (1997) - *Schätzung der deutschen Zinsstrukturkurve*, Diskussionspapier 4/97, Volkswirtschaftliche Forschungsgruppe der Deutschen Bundesbank

[43] Schmithals, B., Schütz, U. (1995) - *Herleitung der DAV-Sterbetafel 1994 R für Rentenversicherungen*, Blätter der DGVM, Bd. XXI

[44] Schweizer, M. (1995a) - *Variance-Optimal Hedging in Discrete Time*, Mathematics of Operations Research 20, 1-32

[45] Schweizer, M. (1995b) - *On the Minimal Martingale Measure and the Föllmer-Schweizer Decomposition*, Stochastic Analysis and Applications 13, 573-599

[46] Schweizer, M. (2001) - *From actuarial to financial valuation principles*, Insurance: Mathematics and Economics 28 (1), 31-47

[47] Taqqu, M.S., Willinger, W. (1987) - *The analysis of finite security markets using martingales*, Advances in Applied Probability 19, 1-25

[48] Tasche, D. (2000) - *Risk contributions and performance measurement*, Working Paper, TU München,
http://www-m4.mathematik.tu-muenchen.de/m4/pers/tasche/riskcon.pdf

Wissenschaftlicher Werdegang des Autors

Geboren	am 16. Juni 1975 in Heilbronn-Neckargartach
1994-99	Mathematikstudium mit Nebenfach Physik an der Universität Heidelberg
Dez. 1999	Diplom in Mathematik; Diplomarbeit über maßtheoretische dynamische Systeme
Jan. 2000 - Aug. 2001	Wissenschaftlicher Mitarbeiter am Institut für International Vergleichende Wirtschafts- und Sozialstatistik der Universität Heidelberg
seit Sept. 2001	Wissenschaftlicher Mitarbeiter am Fachbereich Mathematik der Technischen Universität Darmstadt (AG Stochastik und Operations Research)