Integral Models of Moduli Spaces of Shtukas with Deep Level Structures

Zur Erlangung des Grades eines Doktors der Naturwissenschaften (Dr. rer. nat.)
Genehmigte Dissertation von Patrick Bieker aus Dachau
Tag der Einreichung: 13.07.2022, Tag der Prüfung: 24.10.2022

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Date of submission: 13.07.2022
Date of thesis defense: 24.10.2022

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Abstract

We construct integral models for moduli spaces of shtukas with deep Bruhat-Tits level structures. In the Drinfeld case, we define Drinfeld level structures for Drinfeld shtukas of any rank and show that their moduli spaces are regular and admit finite flat level maps. In particular, the moduli space of Drinfeld shtukas with Drinfeld $\Gamma_0(\mathfrak{p}^n)$-level structures provides a good integral model and a relative compactification of the moduli space of shtukas with naive $\Gamma_0(\mathfrak{p}^n)$-level defined using shtukas for dilated group schemes.

For general reductive groups, we embed the moduli space of global shtukas for the deep Bruhat-Tits group scheme into the limit of the moduli spaces of shtukas for all associated parahoric group schemes. We define the integral model of the moduli space of shtukas with deep Bruhat-Tits level as the schematic image of this map and show that the integral models defined in this way admit proper, surjective and generically étale level maps as well as a natural Newton stratification. In the Drinfeld case, this general construction of integral models recovers the moduli space of Drinfeld shtukas with Drinfeld level structures.

Zusammenfassung

Wir konstruieren ganzzahlige Modelle von Modulräumen von globalen Shtukas mit tiefen Bruhat-Tits Levelstrukturen. Im Drinfeld-Fall definieren wir Drinfeld Levelstrukturen für Drinfeld Shtukas von beliebigem Rang. Wir zeigen die Regularität der zugehörigen Modulräume sowie dass die Levelabbildungen endlich flach sind. Insbesondere liefert der Modulraum von Drinfeld Shtukas mit Drinfeld $\Gamma_0(\mathfrak{p}^n)$-Levelstrukturen ein gutes ganzzahliges Modell und eine relative Kompaktifizierung des Modulraums von Shtukas mit naiven $\Gamma_0(\mathfrak{p}^n)$-Levelstrukturen definiert mithilfe von nicht konstanten Gruppenschemata.

Im Fall allgemeiner reduktiver Gruppen betten wir den Modulraum von globalen Shtukas für ein tiefes Bruhat-Tits Gruppenschema in den Limes seiner zugehörigen Modulräume von Shtukas mit parahorischem Level ein. Wir definieren unser ganzzahliges Modell für den Modulraum von Shtukas mit tiefem Level als das schematheoretische Bild dieser Abbildung und zeigen, dass die in dieser Weise definierten ganzzahligen Modelle eigentliche, surjektive und generisch étale Levelabbildungen genau wie eine natürliche Newtonstratifizierung besitzen. Im Drinfeld-Fall stimmt das allgemein definierte ganzzahlige Modell mit dem Modulraum von Drinfeldmoduln mit Drinfeldlevelstrukturen überein.

Acknowledgements

First of all I thank my advisor Timo Richarz for introducing me to this topic, his steady encouragement and his interest in my work. I thank Paul Hamacher, João Lourenço, Lenny Taelman, Eva Viehmann and Torsten Wedhorn for helpful conversations surrounding this work. I thank Gebhard Böckle and Urs Hartl for their comments on the preprint [Bie22]. I thank Tasho Kaletha for sharing a preliminary version of [KP] and Urs Hartl for sharing the revised version of [Bre18]. I thank Thibaud van den Hove and Catrin Mair for proofreading this thesis.

I express my gratitude to my family for their love and support. Thanks to the current and former members of the algebra group.

This work was supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) TRR 326 *Geometry and Arithmetic of Uniformized Structures*, project number 444845124.

Contents

1. Introduction

Moduli spaces of (global) shtukas serve as function field analogues of Shimura varieties. They were first introduced in [Dri87b] for GL_n and later generalised to arbitrary split reductive groups [Var04] and even flat affine group schemes of finite type in [AH21]. They are used to great succes in establishing a Langlands correspondence over function fields in [Dri87a] for GL_2, [Laf02] for GL_n and [Laf18] for arbitrary reductive groups. While a lot of progress has been made in understanding the geometry of moduli spaces of shtukas for general reductive groups with parahoric level, compare for example [AH14], [AH19], [Bre18], [YZ19] and [Zhu14], little is known for deeper level structures. The goal of this work is to construct good integral models of moduli spaces of shtukas with deep Bruhat-Tits level structures for general reductive groups that generalise the parahoric case, and to give an explicit moduli description of these integral models in the Drinfeld case (that means GL_r-shtukas for some fixed $r \geqslant 1$ with two legs bounded by the minuscule cocharacters $(0, \ldots, 0, -1)$ and $(1, 0, \ldots, 0)$) with $\Gamma_0(\mathfrak{p}^n)$-level structure. Let us explain the construction in more detail.

Let X be a smooth, projective and geometrically connected curve over a finite field $\mathbb{F}_q$. Let us fix an $\mathbb{F}_q$-rational point ∞ of X and let us denote $X' = X \backslash \{\infty\}$. In this introduction, we focus on the case of Drinfeld shtukas of rank 2. Roughly speaking, a Drinfeld shtuka of rank 2 is a vector bundle of rank 2 on X together with a rational isomorphism to its Frobenius twist. More precisely, a *Drinfeld shtuka* of rank 2 (with pole fixed at ∞) over an $\mathbb{F}_q$-scheme S is given by the data $\underline{\mathcal{E}} = (x, \mathcal{E}, \varphi)$, where

- $x \in X'(S)$ is the characteristic section (also called leg or paw),
- $\mathcal{E}$ is a rank r vector bundle on X_S and
- $\varphi\colon \sigma^*\mathcal{E}|_{X_S \backslash (\Gamma_x \cup \Gamma_\infty)} \xrightarrow{\cong} \mathcal{E}|_{X_S \backslash (\Gamma_x \cup \Gamma_\infty)}$ is an isomorphism of $\mathcal{O}_{X_S}$-modules away from the graphs Γ_x of x and Γ_∞ of ∞, such that φ extends to a map $\varphi\colon \sigma^*\mathcal{E}|_{X'} \to \mathcal{E}|_{X'}$ with $\mathrm{coker}(\varphi)$ supported on Γ_x and invertible on its support, and φ^{-1} extends to a map $\varphi^{-1}\colon \mathcal{E}|_{X \backslash \Gamma_x} \to \sigma^*\mathcal{E}|_{X \backslash \Gamma_x}$ with $\mathrm{coker}(\varphi^{-1})$ supported on Γ_∞ and invertible on its support.

In this sense, one can think of φ as having a zero of order 1 at x and a pole of order 1 at ∞. We denote by Sht_2 the stack of Drinfeld shtukas of rank 2, it is a Deligne-Mumford stack locally of finite type over $\mathbb{F}_q$. The projection to the characteristic section defines a map $\mathrm{Sht}_2 \to X'$, which is smooth of relative dimension 2. Drinfeld shtukas in this sense are a generalisation of Drinfeld modules (compare Proposition 2.2.3 for a precise statement). Moduli spaces of rank 2 Drinfeld modules can be thought of as a function field analogue of the moduli space of elliptic curves. In this sense, Sht_2 is a function field analogue of the modular curve.

As in the case of elliptic curves, we want to consider $\Gamma_0(\mathfrak{p}^n)$-level structures. Let us explain what this means. Let us fix a $\mathbb{F}_q$-rational point 0 of X and denote by $\mathcal{O}_0$ (respectively $\mathfrak{p} = \mathfrak{p}_0$) the completion of the local ring of X at 0 (respectively its maximal ideal). We denote for $n \in \mathbb{N}$ by $D_n = n[0] \subseteq X$ the effective Cartier divisor in X defined by (multiples of) the point 0. Then we have $D_n = \mathrm{Spec}(\mathcal{O}_0/\mathfrak{p}^n)$. For a Drinfeld shtuka $\underline{\mathcal{E}} \in \mathrm{Sht}_r(S)$ over a ($\mathbb{F}_q$-)scheme S, we denote by $\underline{\mathcal{E}}|_{D_{n,S}}$ its pullback to $D_{n,S}$. A *naive* $\Gamma_0(\mathfrak{p}^n)$-level structure on a rank 2 shtuka $\underline{\mathcal{E}}$ is given by a quotient $\mathcal{E} \twoheadrightarrow \mathcal{L}$ of $\mathcal{O}_{D_{n,S}}$-modules such that $\mathcal{L}$ is finite locally free of rank 1 on $\mathcal{O}_{D_{n,S}}$ and such that φ descends to a map $\sigma^*\mathcal{L} \to \mathcal{L}$. By the analogy with the modular curve, in the fibre of the moduli space of shtukas with $\Gamma_0(\mathfrak{p}^n)$-level structures over 0 we should expect to find $n+1$ components intersecting at supersingular points. However, it can be shown that in the non-parahoric case (in other words if $n \geqslant 2$) the moduli space of Drinfeld shtukas with naive level structures as above only has two components which moreover do not intersect in its fibre over 0. In particular, its supersingular points are missing (compare Remark 2.1.20).

The reduction modulo p of modular curves with Γ_1- and Γ_0-level structures was studied by [DR73] using a normalisation procedure. [KM85] gave an explicit moduli description of an integral model using Drinfeld level structures. This notion goes back to [Dri76], who first introduced such level structure for Drinfeld modules. The analogy to the modular curve suggests a strategy to construct good integral models in our shtuka setting, in other words, to define a good notion of level structure that also behaves as desired at a place of bad reduction: to use Drinfeld level structures.

In order to define Drinfeld level structures for shtukas, we associate to a Drinfeld shtuka its *scheme of* $\mathfrak{p}^n$*-torsion points* $\underline{\mathcal{E}}[\mathfrak{p}^n]$. This was essentially constructed in [Dri87b] and shares similar properties with the scheme of $\mathfrak{p}^n$-torsion points of a Drinfeld module (respectively the scheme of p^n-torsion points of an elliptic curve). It is a finite locally free $\mathcal{O}_0/\mathfrak{p}^n$-module scheme of rank q^2 over S. Moreover, we show that étale locally on S we find an embedding of $\underline{\mathcal{E}}[\mathfrak{p}^n]$ as a Cartier divisor into $\mathbb{A}^1_S$ by adapting a similar result for the p^n-torsion of one-dimensional p-divisible groups of [Fri19] (compare Proposition 2.2.13 and Remark 2.2.14).

This allows us to define Drinfeld $\Gamma_0(\mathfrak{p}^n)$-level structures on Drinfeld shtukas as follows.

Definition 1.0.1 (compare Definition 2.4.2 for general rank)**.** A *(Drinfeld-)* $\Gamma_0(\mathfrak{p}^n)$*-level structure* on a rank 2 shtuka $\underline{\mathcal{E}}$ is a finite locally free closed subscheme $\mathbf{H} \subseteq \underline{\mathcal{E}}[\mathfrak{p}^n]$ of rank q^n that admits a generator fppf-locally on S, that means an $\mathcal{O}_0/\mathfrak{p}^n$-linear map $\iota\colon (\mathfrak{p}^{-n}/\mathcal{O}_0) \to \underline{\mathcal{E}}[\mathfrak{p}^n](S)$ such that after the choice of an embedding $\mathcal{E}[\mathfrak{p}^n] \hookrightarrow \mathbb{A}^1_S$ we have

$$\sum_{\alpha \in \mathfrak{p}^{-n}/\mathcal{O}_0} [\iota(\alpha)] = \mathbf{H} \qquad \text{and} \qquad \sum_{\alpha \in \mathfrak{p}^{-1}/\mathcal{O}_0} [\iota(\alpha)] \subseteq \underline{\mathcal{E}}[\mathfrak{p}]$$

as Cartier divisors in $\mathbb{A}^1_S$.

Note the subtle difference compared to the definition of $\Gamma_0(p^n)$-level structures in [KM85]: In the setting of elliptic curves the second condition is automatic. However, in our setting the second condition is in particular necessary to get well-defined level maps, see Remark 2.3.4 for an explicit counterexample.

Adapting the theory of Drinfeld level structures for elliptic curves in [KM85], we obtain the following.

Theorem 1.0.2 (compare Theorem 2.4.3 for general rank)**.** *Let $n \geqslant 0$ be an integers.*

(1) The moduli stack $\mathrm{Sht}_{2,\Gamma_0(\mathfrak{p}^n)}$ of rank 2 Drinfeld shtukas with Drinfeld $\Gamma_0(\mathfrak{p}^n)$-level structures is representable by a regular Deligne-Mumford stack locally of finite type over $\mathbb{F}_q$.

(2) The level map $\mathrm{Sht}_{2,\Gamma_0(\mathfrak{p}^n)} \to \mathrm{Sht}_2$ is schematic, finite and flat. Moreover, it is finite étale away from $\mathfrak{p}$.

In particular, $\mathrm{Sht}_{2,\Gamma_0(\mathfrak{p}^n)}$ acquires the supersingular points missing in the moduli space of rank 2 Drinfeld shtukas with naive $\Gamma_0(\mathfrak{p}^n)$-level structures.

As in the case of elliptic curves in [KM85, Chapter 5], we first show the corresponding results for $\Gamma_1(\mathfrak{p}^n)$-level structures. The main step in the proof of the $\Gamma_1(\mathfrak{p}^n)$-case is the study of the deformation theory at supersingular points, where we rely on results of [Dri76]. Using the flatness of the moduli space, we construct a compatible system of level maps

$$\mathrm{Sht}_{2,\Gamma_0(\mathfrak{p}^n)} \to \mathrm{Sht}_{2,\Gamma_0(\mathfrak{p}^m)}$$

that are finite locally free and generically étale for all $m \leqslant n$.

The level maps allow us to interpret our construction in the following way in terms of the combinatorics of the Bruhat-Tits building $\mathcal{B}(\mathrm{GL}_2, K_0)$ of GL_2 over the fraction field K_0 of $\mathcal{O}_0$. Let us denote by $\Omega = [0, n]$ the standard interval of length n in the standard apartment of $\mathcal{B}(\mathrm{GL}_2, K_0)$. By Bruhat-Tits theory, for such a subset Ω of the Bruhat-Tits building we get an associated smooth affine group scheme $\mathrm{GL}_{2,\Omega}$ over $\mathcal{O}_0$ that can be glued with GL_2 to a smooth affine group scheme over X that we denote by a slight abuse of notation also by $\mathrm{GL}_{2,\Omega}$. Following [MRR20], we can view Drinfeld shtukas with naive $\Gamma_0(\mathfrak{p}^n)$-level structures as shtukas for the Bruhat-Tits group scheme $\mathrm{GL}_{r,\Omega}$ bounded by $\underline{\mu} = ((0,\dots,0,-1),(1,0,\dots,0))$. For a precise definition see Section 2.1.5 below. Let us denote by $\mathrm{Sht}_{2,\Omega}$ the moduli stack of shtukas for the Bruhat-Tits group scheme $\mathrm{GL}_{2,\Omega}$ bounded by $(0,\dots,0,-1)$ and $(1,0,\dots,0)$. To such a bounded $\mathrm{GL}_{2,\Omega}$-shtuka we can associate a Drinfeld shtuka with Drinfeld $\Gamma_0(\mathfrak{p}^n)$-level structure, this is explained in more detail below. Moreover, using the level maps, we get compatible system of maps $\mathrm{Sht}_{2,\Gamma_0(\mathfrak{p}^n)} \to \mathrm{Sht}_{2,\mathfrak{f}}$ to the moduli space of shtukas for Bruhat-Tits group schemes for all facets $\mathfrak{f} \prec \Omega$ contained in Ω.

Theorem 1.0.3 (compare Theorem 2.5.7)**.** *The map $\mathrm{Sht}_{2,\Omega} \to \varprojlim_{\mathfrak{f}\prec\Omega} \mathrm{Sht}_{2,\mathfrak{f}}$ is a quasi-compact open immersion and an isomorphism away from 0. Its schematic image in the sense of [EG21] is $\overline{\mathrm{Sht}}_{2,\Omega} = \mathrm{Sht}_{2,\Gamma_0(\mathfrak{p}^n)}$ via the maps*

$$\mathrm{Sht}_{2,\Omega} \hookrightarrow \mathrm{Sht}_{2,\Gamma_0(\mathfrak{p}^n)} \hookrightarrow \varprojlim_{\mathfrak{f}\prec\Omega} \mathrm{Sht}_{2,\mathfrak{f}}$$

constructed above. In the parahoric case $n = 1$, the map $\mathrm{Sht}_{2,\Omega} \to \mathrm{Sht}_{2,\Gamma_0(\mathfrak{p})}$ is an isomorphism.

Another way to phrase Theorem 1.0.3 is that we (relatively) compactified the level map $\mathrm{Sht}_{2,\Omega} \to \mathrm{Sht}_2$, which we saw in the example above is not proper in general, by factoring it in an open immersion with dense image followed by a finite (hence proper) and surjective map

$$\mathrm{Sht}_{2,\Omega} \hookrightarrow \mathrm{Sht}_{2,\Gamma_0(\mathfrak{p}^n)} \to \mathrm{Sht}_2 \,.$$

We can also interpret this result as follows. The theorem shows that $\overline{\mathrm{Sht}}_{2,\Omega} = \mathrm{Sht}_{2,\Gamma_0(\mathfrak{p}^n)}$ is the flat closure of the generic fibre inside $\varprojlim_{\mathfrak{f}\prec\Omega} \mathrm{Sht}_{2,\mathfrak{f}}$. In this sense, Theorem 1.0.3 suggests that a candidate for a good integral model for the moduli spaces of shtukas for a general reductive group with deep Bruhat-Tits level structure (i.e. level structures generalising $\Gamma_0(\mathfrak{p}^n)$-level structures in the GL_2-case) is the closure of the moduli stack of shtukas for the Bruhat-Tits group scheme inside the limit of all moduli stacks with corresponding parahoric level.

In the second part of this thesis, we confirm this expectation by considering the following situation. Let G be a (connected) reductive group over the function field K of X and let us fix a parahoric model $\mathcal{G} \to X$ of G. That is, $\mathcal{G}$ is a smooth affine group scheme over X with generic fibre G such that for all closed points x of X the pullback $\mathcal{G}_{\mathcal{O}_x}$ is a parahoric group scheme in the sense of [BT84]. Let Ω be a bounded subset of an apartment in the

Bruhat-Tits building of G_{K_0}, where K_0 is the completion of K at 0. As in the GL_2 case above, we get a smooth affine $\mathcal{O}_0$-group scheme $\mathcal{G}_\Omega$ that we glue with $\mathcal{G}$ outside of x_0 to obtain a (global) Bruhat-Tits group scheme $\mathcal{G}_\Omega \to X$ which is smooth and affine by construction. Without loss of generality, we may assume that Ω is convex, closed and a union of facets.

Let I be a finite set and let $\underline{\mu} = (\mu_i)_{i \in I}$ be a tuple of conjugacy classes of geometric cocharacters of G. For simplicity, we assume in this introduction that $\underline{\mu}$ is defined over the function field K of X (in general it will only be defined over a finite separable extension of K). A global $\mathcal{G}_\Omega$-shtuka over a scheme S is a $\mathcal{G}_\Omega$-bundle $\mathcal{E}$ on X_S together with an isomorphism $\varphi\colon \sigma^*\mathcal{E}|_{X_S \setminus \Gamma_{\underline{x}}} \xrightarrow{\cong} \mathcal{E}|_{X_S \setminus \Gamma_{\underline{x}}}$ away from the graph $\Gamma_{\underline{x}}$ of an I-tuple $\underline{x} \in X^I(S)$ of points of X. We denote by $\mathrm{Sht}_{\mathcal{G}_\Omega, X^I}^{\leqslant \underline{\mu}}$ the moduli space of global $\mathcal{G}_\Omega$-shtukas bounded by $\underline{\mu}$, compare Definition 3.2.7 and Construction 3.2.14 for the precise definition of boundedness conditions. Note that in the Drinfeld case $\mathrm{Sht}_{2,\Omega} = \mathrm{Sht}_{\mathrm{GL}_{2,\Omega}, X^2}^{\leqslant((0,-1),(1,0))}|_{\{\infty\} \times X'}$. In this sense, $\mathrm{Sht}_{\mathcal{G}_\Omega, X^I}^{\leqslant \underline{\mu}}$ generalises the moduli space of Drinfeld shtukas with naive $\Gamma_0(\mathfrak{p}^n)$-level structure.

While for a subset Ω' of Ω there is still a natural map $\mathrm{Sht}_{\mathcal{G}_\Omega, X^I}^{\leqslant \underline{\mu}} \to \mathrm{Sht}_{\mathcal{G}_{\Omega'}, X^I}^{\leqslant \underline{\mu}}$ by [Bre18, Theorem 3.20] (compare also Theorem 3.3.3), already in the Drinfeld case $G = \mathrm{GL}_2$, the level map $\mathrm{Sht}_{\mathrm{GL}_{2,[0,n]}, X^2}^{\leqslant((0,-1),(1,0))} \to \mathrm{Sht}_{\mathrm{GL}_2, X^2}^{\leqslant((0,-1),(1,0))}$ is neither proper nor surjective for $n \geqslant 2$ as discussed above.

We propose the following construction to relatively compactify $\mathrm{Sht}_{\mathcal{G}_\Omega, X^I}^{\leqslant \underline{\mu}}$.

Definition 1.0.4 (compare Definition 3.3.7)**.** In the situation above, that is, for a reductive group G over K, and a Bruhat-Tits group scheme $\mathcal{G}_\Omega \to X$ for a subset Ω (assumed to be convex, closed and a union of facets) of the Bruhat-Tits building for G_{K_0} at the fixed point 0 of X as above, the *integral model of the moduli space of shtukas with $\mathcal{G}_\Omega$-level structure* $\overline{\mathrm{Sht}}_{\mathcal{G}_\Omega, X^I}^{\leqslant \underline{\mu}}$ is defined to be the schematic image in the sense of [EG21] of the map

$$\mathrm{Sht}_{\mathcal{G}_\Omega, X^I}^{\leqslant \underline{\mu}} \to \varprojlim_{\mathfrak{f} \prec \Omega} \mathrm{Sht}_{\mathcal{G}_{\mathfrak{f}}, X^I}^{\leqslant \underline{\mu}},$$

where the limit is taken over all facets $\mathfrak{f}$ contained in Ω.

Clearly, in the parahoric case (that is, when Ω is a facet) we have

$$\mathrm{Sht}_{\mathcal{G}_\Omega, X^I}^{\leqslant \underline{\mu}} = \overline{\mathrm{Sht}}_{\mathcal{G}_\Omega, X^I}^{\leqslant \underline{\mu}} = \varprojlim_{\mathfrak{f} \prec \Omega} \mathrm{Sht}_{\mathcal{G}_{\mathfrak{f}}, X^I}^{\leqslant \underline{\mu}},$$

so the construction above generalises the parahoric case. Moreover, as we have seen above, this general notion of integral models in the Drinfeld case recovers the moduli space of shtukas with Drinfeld $\Gamma_0(\mathfrak{p}^n)$-level structure at 0.

The main result of this work is to show that this construction of integral models admits proper, surjective and generically finite étale level maps:

Theorem 1.0.5 (compare Proposition 3.3.6 and Theorem 3.3.8)**.** *In the situation of Definition 1.0.4, the map*

$$\mathrm{Sht}^{\leqslant\underline{\mu}}_{\mathcal{G}_\Omega,X^I} \to \varprojlim_{\mathfrak{f}\prec\Omega} \mathrm{Sht}^{\leqslant\underline{\mu}}_{\mathcal{G}_\mathfrak{f},X^I}$$

is schematic and a quasi-compact locally closed immersion. It factors into an open immersion $\mathrm{Sht}^{\leqslant\underline{\mu}}_{\mathcal{G}_\Omega,X^I} \to \overline{\mathrm{Sht}}^{\leqslant\underline{\mu}}_{\mathcal{G}_\Omega,X^I}$ followed by the closed immersion $\overline{\mathrm{Sht}}^{\leqslant\underline{\mu}}_{\mathcal{G}_\Omega,X^I} \to \varprojlim_{\mathfrak{f}\prec\Omega} \mathrm{Sht}^{\leqslant\underline{\mu}}_{\mathcal{G}_\mathfrak{f},X^I}$. The restriction of the inclusion

$$\mathrm{Sht}^{\leqslant\underline{\mu}}_{\mathcal{G}_\Omega,X^I}|_{(X\backslash\{0\})^I} \xrightarrow{\cong} \overline{\mathrm{Sht}}^{\leqslant\underline{\mu}}_{\mathcal{G}_\Omega,X^I}|_{(X\backslash\{0\})^I}$$

away from 0 is an isomorphism. Moreover, for a subset $\Omega' \prec \Omega$, there is a natural level map

$$\bar{\rho}_{\Omega',\Omega} \colon \overline{\mathrm{Sht}}^{\leqslant\underline{\mu}}_{\mathcal{G}_\Omega,X^I} \to \overline{\mathrm{Sht}}^{\leqslant\underline{\mu}}_{\mathcal{G}_{\Omega'},X^I}$$

that is schematic, proper, surjective and over $(X\backslash\{0\})^I$ is finite étale.

In the parahoric case, the level maps on moduli spaces of shtukas are also studied in [Bre18, Theorem 3.20]. However, the notion of bounds used there does not quite capture the situation we are interested in here. We discuss the notion of global bounds for global shtukas following [AH19] and give a defintion of local bounds that is compatible with the global notion. We generalise the result of [Bre18, Theorem 3.20] to include bounds in this sense (compare Theorem 3.3.3). Using the assertion in the parahoric case, we are able to deduce the result also for deep level structures.

Additionally to the existence of well-behaved level maps, we show that the Newton stratification on the special fibre of the moduli space of shtukas in the parahoric case induces a well-defined Newton stratification on the special fibre in the case of deeper level. For a reductive group H over a local field k we denote by $B(H)$ the set of σ-conjugacy classes in $H(\breve{k})$, where $\breve{k}$ is the completion of the maximal unramified extension of k. Then $B(H)$ classifies quasi-isogeny classes of local shtukas for (an integral model of) H.

We fix a tuple of pairwise distinct closed points $\underline{y} = (y_i)_{i\in I}$ in X and denote by $\overline{\mathrm{Sht}}^{\leqslant\underline{\mu}}_{\mathcal{G}_\Omega,X^I,\mathbb{F}_{\underline{y}}} = \overline{\mathrm{Sht}}^{\leqslant\underline{\mu}}_{\mathcal{G}_\Omega,X^I} \times_{X^I} \mathbb{F}_{\underline{y}}$ the special fibre over $\underline{y}$, where $\mathbb{F}_{\underline{y}}$ is the compositum of the residue fields of the points y_i of X.

Theorem 1.0.6 (compare Definition 3.4.3 and Corollary 3.4.4)**.** *Let ℓ be an algebraically closed extension of $\mathbb{F}_{\underline{y}}$. There is a well-defined map*

$$\bar{\delta}_{\mathcal{G}_\Omega} \colon \overline{\mathrm{Sht}}^{\leqslant\underline{\mu}}_{\mathcal{G}_\Omega,X^I,\mathbb{F}_{\underline{y}}}(\ell) \to \prod_{i\in I} B(G_{K_{y_i}})$$

that is compatible with the level maps in the sense that for $\Omega' \prec \Omega$ we have

$$\bar{\delta}_{\mathcal{G}_\Omega} = \bar{\delta}_{\mathcal{G}_{\Omega'}} \circ \bar{\rho}_{\Omega',\Omega}.$$

Moreover, for $\underline{b} = (b_i)_{i \in I} \in B(G_{K_{y_i}})$ the preimage of $\underline{b}$ under $\bar{\delta}_{\mathcal{G}_\Omega}$ is the set of ℓ-valued points of a locally closed substack $\mathrm{Sht}^{\leqslant \underline{\mu}, \underline{b}}_{\mathcal{G}_\Omega, X^I, \mathbb{F}_{\underline{y}}}$ *of* $\mathrm{Sht}^{\leqslant \underline{\mu}}_{\mathcal{G}_\Omega, X^I, \mathbb{F}_{\underline{y}}}$ *called the* Newton stratum *of* $\mathrm{Sht}^{\leqslant \underline{\mu}}_{\mathcal{G}_\Omega, X^I, \mathbb{F}_{\underline{y}}}$ *for $\underline{b}$.*

In the parahoric case this result is due to [Bre18, Section 5], compare also [HV11, Theorem 7.11]. In this case, the map $\bar{\delta}$ is given by associating to a point in the special fibre over $\underline{y}$ the quasi-isogeny classes of its local shtukas at the points y_i. We use the compatibility of the Newton stratification with the level maps in the parahoric case to extend this result to the case of deep level.

Moreover, we show that in the hyperspecial case the Newton stratification satisfies the strong stratification property (as for Shimura varieties). Recall that there is a natural order on $B(H)$ induced by the dominance order on cocharacters. It is well-known in the parahoric case that the closure $\overline{\mathrm{Sht}^{\leqslant \underline{\mu}, \underline{b}}_{\mathcal{G}, X^I, \mathbb{F}_{\underline{y}}}} \subseteq \bigcup_{\underline{b}' \leqslant \underline{b}} \mathrm{Sht}^{\leqslant \underline{\mu}, \underline{b}'}_{\mathcal{G}, X^I, \mathbb{F}_{\underline{y}}}$. Note that this also generalises to deeper level. We say that the Newton stratification satisfies the strong stratification property when we even have equality. However, the inclusion is strict in general.

Theorem 1.0.7 (compare Theorem 3.4.5). *Let $\mathcal{G} \to X$ be a parahoric group scheme that is hyperspecial at y_i for all $i \in I$. Then the Newton stratification at $\underline{y}$ satisfies the strong stratification property in the sense that*

$$\overline{\mathrm{Sht}^{\leqslant \underline{\mu}, \underline{b}}_{\mathcal{G}, X^I, \mathbb{F}_{\underline{y}}}} = \bigcup_{\underline{b} \leqslant \underline{b}'} \mathrm{Sht}^{\leqslant \underline{\mu}, \underline{b}'}_{\mathcal{G}, X^I, \mathbb{F}_{\underline{y}}}$$

for all $\underline{b} \in \prod_{i \in I} B(G_{y_i})$.

We deduce the closure relations from the corresponding local result in [Vie13] using the (bounded version of the) Serre-Tate theorem for shtukas. For PEL-type Shimura varieties, this result is due to [Ham15].

In order to establish the first two assertions of Theorem 1.0.5, we study the deformation theory of torsors under Bruhat-Tits group schemes. In the process, we show two results that may also be of independent interest. In the local case (and hence also for the corresponding global Bruhat-Tits group schemes), we get the not necessarily parahoric Bruhat-Tits group scheme as the limit of all its associated parahoric group schemes.

Theorem 1.0.8 (compare Theorem 3.1.3)**.** *Let G be a reductive group over a local field k and Ω a subset of the Bruhat-Tits building for G as above. Then the induced map*

$$\mathcal{G}_\Omega \xrightarrow{\cong} \varprojlim_{\mathfrak{f} \prec \Omega} \mathcal{G}_\mathfrak{f}$$

is an isomorphism of $\mathcal{O}$-group schemes, where $\mathcal{O}$ is the ring of integers of k.

We use this result on the level of group schemes to show that the moduli stack of $\mathcal{G}_\Omega$-bundles $\mathrm{Bun}_{\mathcal{G}_\Omega}$ on X embeds via an open immersion into the limit of $\mathrm{Bun}_{\mathcal{G}_\mathfrak{f}}$ over all associated parahoric group schemes.

Theorem 1.0.9 (compare Theorem 3.1.13)**.** *In the situation of Definition 1.0.4, the natural map*

$$\mathrm{Bun}_{\mathcal{G}_\Omega} \to \varprojlim_{\mathfrak{f} \prec \Omega} \mathrm{Bun}_{\mathcal{G}_\mathfrak{f}}$$

is a quasi-compact open immersion.

Note that given a compatible system of $\mathcal{G}_\mathfrak{f}$-torsors for all facets $\mathfrak{f} \prec \Omega$, it is in general not true that their limit is a torsor for $\mathcal{G}_\Omega$, as it might be impossible to construct a compatible system of sections. By controlling the deformation theory of torsors for the $\mathcal{G}_\mathfrak{f}$, we are able to show that the locus where the limit of a compatible system of $\mathcal{G}_\mathfrak{f}$-bundles on X is already a $\mathcal{G}_\Omega$-bundle on X is open.

Conclusion and Outlook

In this thesis, we construct integral models for moduli spaces of shtukas with deep Bruhat-Tits level structures that generalise the known constructions in the parahoric case and the GL_r-case with $\Gamma_0(\mathfrak{p}^n)$-level structure. Moreover, we show that our integral models admit proper, surjective and generically finite étale level maps.

In future work, we use our construction of integral models to study the local geometry of the fibres of our integral models at places with deep level structures. To this end, we aim to construct a local model also for deep level structures in order to relate the geometry of the special fibre to the combinatorics of the Bruhat-Tits building as in the parahoric case.

As an application, we can then use the insights on the geometry of the special fibre to calculate the semisimple trace of Frobenius on the sheaf of nearby cycles in order to construct elements of the Bernstein center of the (local) Hecke algebra of the reductive group G.

Organisation

This thesis is organised as follows. In Chapter 2, we consider the Drinfeld case and study the moduli space of Drinfeld shtukas with Drinfeld level structures. More precisely, in Section 1 we recall some facts on shtukas (in particular Drinfeld shtukas) and define naive $\Gamma_0(\mathfrak{p}^n)$-level structures. In Section 2, we explain the comparison with Drinfeld modules. This provides us with a way to associate group schemes to (global, local and finite) shtukas, which is what makes it possible to define Drinfeld level structures in the first place. In particular, we construct the scheme of $\mathfrak{p}^n$-division points of a Drinfeld shtuka and study its properties. In Sections 3 and 4, we define our Drinfeld (Γ_1- and Γ_0-type) level structures and prove the regularity of their moduli spaces. For this, we follow [KM85]. In Section 5, we show that the Drinfeld level structures actually provide a good (relative) compactification of the moduli space with naive level structure $\mathrm{Sht}_{r,\Omega}$. Chapter 2 is contained in [Bie22].

In Chapter 3, we consider the case of shtukas for a general reductive group with deep Bruhat-Tits level structures. In Section 1, we study (torsors under) Bruhat-Tits group schemes and show Theorems 1.0.8 and 1.0.9. In Section 2, we introduce moduli spaces of shtukas and discuss how to define boundedness conditions. In particular, we give a new definition of local bounds that is compatible in a natural way with usual notions of global bounds. In Section 3, we first prove a variant of the functoriality result for moduli spaces of shtukas of [Bre18, Theorem 3.20] showing in particular that the level maps in the parahoric case are well-behaved in our setting. We use this result to define our integral models with deep level structure and show that these models admit well-behaved level maps as well, proving Theorem 1.0.5. In Section 4, we construct a Newton stratification on the integral models with deep level.

Notation

We fix the following notation. Let $\mathbb{F}_q$ be a finite field with q elements, let p be the characteristic of $\mathbb{F}_q$. All schemes will be $\mathbb{F}_q$-schemes unless otherwise specified. Let X be a smooth projective and geometrically connected curve over $\mathbb{F}_q$ with function field K. For a closed point x of X we denote by $\mathcal{O}_{X,x}$ the local ring at x and by $\mathcal{O}_x$ its completion. Moreover, we denote by K_x the completion of K at x.

We denote by σ the (absolute) q-Frobenius endomorphism Frob_S of some $\mathbb{F}_q$-scheme S, and also the map $\sigma = id_X \times \mathrm{Frob}_S \colon X_S \to X_S$. It is always clear from context which map σ is meant.

2. Compactification of level maps of moduli spaces of Drinfeld Shtukas

2.1. Moduli spaces of shtukas and naive $\Gamma_0(\mathfrak{p}^n)$-level structures

Drinfeld Shtukas were introduced in [Dri87b] as *elliptic sheaves* and were vastly generalised to arbitrary reductive groups or even general smooth affine group schemes in [Var04] and [AH14], respectively.

We introduce naive $\Gamma_0(\mathfrak{p}^n)$-level structures on Drinfeld shtukas, present how to encode these level structures in terms of Bruhat-Tits group schemes following [MRR20] and explain, why the naive definition is not appropriate for deeper level (that means for $n > 1$).

Let us for the whole of this chapter fix two distinct $\mathbb{F}_q$-rational points ∞ and 0 of X and denote by $\mathfrak{p} = \mathfrak{p}_0$ the maximal ideal in the complete local ring $\mathcal{O}_0$ at 0. Let us also fix a uniformiser ϖ of $\mathfrak{p}$.

2.1.1. Global shtukas

We recall the definitions of global shtukas and isogenies of Drinfeld shtukas. We restrict ourselves to shtukas with two legs with one leg fixed at the point ∞.

Definition 2.1.1 ([AH14])**.** Let G be a smooth affine group scheme on X. A *global G-shtuka* over a scheme S is given by the data

$$\underline{\mathcal{E}} = (x, \mathcal{E}, \varphi\colon \sigma^*\mathcal{E} \dashrightarrow \mathcal{E}),$$

where

- $x \in X'(S)$ is a section of $X' = X \backslash \{\infty\}$,
- $\mathcal{E}$ is a G-bundle on X_S and
- $\varphi\colon \sigma^*\mathcal{E}|_{X_S \backslash (\Gamma_x \cup \Gamma_\infty)} \xrightarrow{\cong} \mathcal{E}|_{X_S \backslash (\Gamma_x \cup \Gamma_\infty)}$ is an isomorphism of G-bundles away from the graphs Γ_x of x and Γ_∞ of ∞.

The point x is called *characteristic* or *leg* of $\underline{\mathcal{E}}$. A map of G-shtukas is a tuple of maps of G-bundles compatible with the maps φ and φ'.

Note that there are several ways to bound the zeros (and poles, respectively) of G-shtukas, and in general they are not equivalent (compare Remark 2.1.7). We will mostly be interested in the case of Drinfeld shtukas, that means we consider $G = \mathrm{GL}_r$ (or corresponding Bruhat-Tits group schemes) and bounds given by the minuscule coweights $\underline{\mu} = ((0, \dots, 0, -1), (1, 0, \dots, 0))$. These admit the following explicit description.

Definition 2.1.2 ([Dri87b])**.** A *Drinfeld shtuka* of rank r over a scheme S is given by the data

$$\underline{\mathcal{E}} = (x, \mathcal{E}, \varphi),$$

where

- $x \in X'(S)$ is the characteristic section,
- $\mathcal{E}$ is a rank r vector bundle on X_S and
- $\varphi\colon \sigma^*\mathcal{E}|_{X_S \backslash (\Gamma_x \cup \Gamma_\infty)} \xrightarrow{\cong} \mathcal{E}|_{X_S \backslash (\Gamma_x \cup \Gamma_\infty)}$ is an isomorphism of $\mathcal{O}_{X_S}$-modules away from the graphs Γ_x of x and Γ_∞ of ∞, such that φ extends to a map $\varphi\colon \sigma^*\mathcal{E}|_{X'} \to \mathcal{E}|_{X'}$ with $\operatorname{coker}(\varphi)$ supported on Γ_x and invertible on its support, and φ^{-1} extends to a map $\varphi^{-1}\colon \mathcal{E}|_{X \backslash \Gamma_x} \to \sigma^*\mathcal{E}|_{X \backslash \Gamma_x}$ with $\operatorname{coker}(\varphi^{-1})$ supported on ∞ and invertible on its support.

We denote by Sht_r the stack of Drinfeld shtukas of rank r.

It is well known that Sht_r is a Deligne-Mumford stack locally of finite type over $\mathbb{F}_q$. It has a forgetful map $\mathrm{Sht}_r \to X'$ which is smooth of relative dimension $(2r - 2)$, see [Dri87b, Proposition 3.2 and 3.3].

In the context of Drinfeld shtukas, the characteristic section x is often called the *zero* of $\mathcal{E}$ while the second leg (that we fixed to be ∞) is the *pole* of $\underline{\mathcal{E}}$. By a slight abuse of notation we say that $\underline{\mathcal{E}}$ is *in characteristic* $\mathfrak{p}$ if its characteristic section factors through 0.

Remark 2.1.3. Note that once the zero and the pole of the shtuka do not intersect, we can glue $\mathcal{E}$ and $\sigma^*\mathcal{E}$ along the isomorphism φ over $X_S \backslash (\Gamma_x \cup \Gamma_{x'})$ and obtain a vector bundle $\mathcal{E}'$ together with maps

$$\varphi'\colon \mathcal{E} \hookrightarrow \mathcal{E}' \hookleftarrow \sigma^*\mathcal{E}\colon \varphi$$

of $\mathcal{O}_{X_S}$-modules that satisfy the analogous conditions on the cokernels as in our definition of Drinfeld shtukas. This notion is used in the original definition of Drinfeld shtukas in [Dri87b] and does not require the two legs of the shtuka to be disjoint. We denote by $\mathrm{Sht}_{r,X^2} \to X^2$ the stack of Drinfeld shtukas in this sense. Then $\mathrm{Sht}_r = \mathrm{Sht}_{r,X^2} \times_{X^2} (\{\infty\} \times X')$.

For $n \in \mathbb{N}$ we denote by $D_n = n[0] \subseteq X$ the effective Cartier divisor in X. Note that $D_n = \mathrm{Spec}(\mathcal{O}_0/\mathfrak{p}^n)$.

Definition 2.1.4. A map $f\colon \underline{\mathcal{E}}_1 \to \underline{\mathcal{E}}_2$ of Drinfeld shtukas is an *isogeny* if f is injective and $\mathrm{coker}(f)$ is finite locally free as $\mathcal{O}_S$-module. Moreover, we say that f is a $\mathfrak{p}^n$-isogeny, if the $\mathcal{O}_{X_S}$-module structure on $\mathrm{coker}(f)$ factors through $\mathcal{O}_{D_{n,s}}$, in other words, if $\mathrm{coker}(f)$ is $\mathfrak{p}^n$-torsion.

In order to give a criterion which $\mathcal{O}_S$-modules can arise as cokernels of $\mathfrak{p}^n$-isogenies, we use the following notion of a $\mathfrak{p}^n$*-torsion shtuka*, which are an $\mathcal{O}_0/\mathfrak{p}^n$-linear analogue of the φ-sheaves introduced by [Dri87b].

Definition 2.1.5. A $\mathfrak{p}^n$*-torsion shtuka* over S is a pair $\underline{\mathcal{F}} = (\mathcal{F}, \varphi)$ consisting of a quasi-coherent $\mathcal{O}_{D_{n,S}}$-module $\mathcal{F}$ which is finite locally free as $\mathcal{O}_S$-module and an $\mathcal{O}_{D_{n,S}}$-module homomorphism $\varphi\colon \sigma^*\mathcal{F} \to \mathcal{F}$. A map of $\mathfrak{p}^n$-torsion shtukas is a map of the underlying $\mathcal{O}_{D_{n,S}}$-modules compatible with φ. We say that a $\mathfrak{p}^n$-torsion shtuka is *étale* if φ is an isomorphism.

In [HS19] Drinfeld's φ-sheaves are also called finite shtukas. For our purposes however, the $\mathcal{O}_{D_n}$-module structure is central.

To a rank r Drinfeld shtuka $\underline{\mathcal{E}} = (x, \mathcal{E}, \varphi)$ over S we associate its $\mathfrak{p}^n$-torsion shtuka defined as the pullback of $\mathcal{E}$ to the divisor $D_{n,S}$, which is more explicitly given by $\underline{\mathcal{E}}|_{D_{n,S}} = (\mathcal{E}|_{D_{n,S}}, \varphi|_{D_{n,S}})$. Note that its underlying $\mathcal{O}_S$-module has rank nr. A second important class of examples of $\mathfrak{p}^n$-torsion shtukas are cokernels of $\mathfrak{p}^n$-isogenies of Drinfeld shtukas. Note that $\underline{\mathcal{E}}|_{D_{n,S}}$ is the cokernel of the $\mathfrak{p}^{rn}$-isogeny $\underline{\mathcal{E}}(\mathfrak{p}^n) \hookrightarrow \underline{\mathcal{E}}$, where we denote by $\underline{\mathcal{E}}(\mathfrak{p}^n) = \underline{\mathcal{E}} \otimes \mathcal{O}(D_{n,S})$ the twist of $\underline{\mathcal{E}}$ by the divisor D_n.

2.1.2. Local shtukas

We can associate to Drinfeld shtukas its local counterparts called *local shtukas* in the same way p-divisible groups are local analogues of abelian varieties. Local shtukas are introduced as Dieudonné $\mathbb{F}_q[\![\varpi]\!]$-modules in [Har05] as analogues of Dieudonné modules of p-divisible groups and are studied and generalised for example by [HV11] and [AH14].

Let us denote by $\mathbb{F}_q[\![\zeta]\!]$ the ring of formal power series in the formal variable ζ and by $\mathcal{N}ilp_{\mathbb{F}_q[\![\zeta]\!]}$ the category of schemes S over $\mathbb{F}_q[\![\zeta]\!]$ such that ζ is locally nilpotent in S. For a ring R, we denote by $R[\![\varpi]\!]$ the ring of formal power series in the formal variable ϖ and by $R(\!(\varpi)\!)$ the ring of formal Laurent series in ϖ on S. Note that for $\mathrm{Spec}(R) \in \mathcal{N}ilp_{\mathbb{F}[\![\zeta]\!]}$, we have $R(\!(\varpi)\!) = R[\![\varpi]\!]\left[\frac{1}{\varpi - \zeta}\right]$. We denote by σ the endomorphism of $R[\![\varpi]\!]$ (respectively $R(\!(\varpi)\!)$) that acts as the identity on ϖ and as $b \mapsto b^q$ on R.

Definition 2.1.6. Let $S = \mathrm{Spec}(R) \in \mathcal{N}ilp_{\mathbb{F}_q[\![\zeta]\!]}$. A *local shtuka* $\underline{\mathcal{G}} = (\mathcal{G}, \varphi)$ of rank r over S is a locally free sheaf of $R(\!(\varpi)\!)$-modules $\mathcal{G}$ of rank r together with an isomorphism $\varphi\colon \sigma^*\mathcal{G}[\frac{1}{\varpi-\zeta}] \to \mathcal{G}[\frac{1}{\varpi-\zeta}]$ of $R(\!(\varpi)\!)$-modules. The local shtuka $\underline{\mathcal{G}}$ is called *effective* if φ comes from a map $\tilde{\varphi}\colon \sigma^*\mathcal{G} \hookrightarrow \mathcal{G}$ of $R[\![\varpi]\!]$-modules and *étale* if additionally $\tilde{\varphi}$ is an isomorphism. A *quasi-isogeny* $f\colon \underline{\mathcal{G}} \to \underline{\mathcal{G}'}$ between local shtukas is an isomorphism $\mathcal{G}\left[\frac{1}{\varpi}\right] \xrightarrow{\sim} \mathcal{G}'\left[\frac{1}{\varpi}\right]$ of the underlying $R(\!(\varpi)\!)$-modules, which is compatible with φ and φ'.

We say a local shtuka $\underline{\mathcal{G}} = (\mathcal{G}, \varphi)$ is *bounded by* $(1, 0, \ldots, 0)$ if it is effective, $\mathrm{coker}(\varphi)$ is locally free of rank 1 as an R-module and $(\varpi - \zeta)$ annihilates $\mathrm{coker}(\varphi)$. Similarly, we say $\underline{\mathcal{G}}$ is bounded by $(0, \ldots, 0, -1)$ if φ^{-1} is bounded by $(1, 0, \ldots, 0)$ in the above sense. More precisely, $\underline{\mathcal{G}}$ is bounded by $(0, \ldots, 0, -1)$ if φ induces a map $\mathcal{G} \hookrightarrow \sigma^*\mathcal{G}$ with a cokernel which is locally free of rank 1 as R-module and which is annihilated by $(\varpi - \zeta)$.

Remark 2.1.7. There are several ways to define bounds for local shtukas in general, cf. [HV11, Definition 3.5 and Lemma 4.3.] and [AH14, Defintion 4.8.]. For the Drinfeld case the bound in the sense of [AH14] is also given more explicitly in [Bre18, Section 7.2.]. Note that the straightforward generalisation of our definition above does not produce the correct notion for coweights $(d, 0, \ldots, 0)$ with $d > 1$ by [HS19, Example 8.3]. In particular [HV11, Example 4.5] and [Zhu17, Example 2.1.8.] seem to be problematic.

The Newton stratification for local shtukas is defined in [HV11] as an analogue of the Newton stratification for F-isocrystals in [RR96].

Definition 2.1.8. The *Newton point* of of a local shtuka $\underline{\mathcal{G}}$ of rank r over an algebraically closed field ℓ is $(u_1, \ldots, u_r) \in \mathbb{Q}^r$ with $u_1 \geqslant \ldots \geqslant u_r$ and the u_i are the slopes associated to the corresponding isoshtuka $\underline{\mathcal{G}}\left[\varpi^{-1}\right]$ by the Dieudonné-Manin classification in the function field case [Lau96, Theorem 2.4.5].

We denote by $B(\mathrm{GL}_r)$ the *Kottwitz set* of isomorphism classes of isoshtukas over an algebraically closed field ℓ, in other words, the set of σ-conjugacy classes of invertible $(r \times r)$-matrices over $\ell(\!(\varpi)\!)$. The set $B(\mathrm{GL}_r)$ does not depend on the choice of ℓ. Recall that the Newton map $\nu_{\mathrm{GL}_r}\colon B(\mathrm{GL}_r) \to \mathbb{Q}^r$ is already injective (this fails for general reductive groups). The Bruhat order on the space of cocharacters $X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^r$ induces a partial order on $B(\mathrm{GL}_r)$. It is more explicitly given by

$$(u_1, \ldots, u_r) \leqslant (u'_1, \ldots, u'_r) \qquad \text{if} \qquad \sum_{j=1}^{i} u_j \leqslant \sum_{j=1}^{i} u'_j$$

for all $1 \leqslant i \leqslant r$ with equality in the case $i = r$. Moreover, for a dominant cocharacter μ, in other words, $\mu = (\mu_1, \ldots, \mu_r) \in \mathbb{Z}^r$ with $\mu_1 \geqslant \ldots \geqslant \mu_r$, we denote by $B(\mathrm{GL}_r, \mu) = \{[b] \in B(\mathrm{GL}_r)\colon \nu_{\mathrm{GL}_r}([b]) \leqslant \mu\}$.

For a local shtuka $\underline{\mathcal{G}}$ over a scheme $S = \mathrm{Spec}(R)$ and a geometric point s of S we denote by $[\underline{\mathcal{G}}_s]$ the associated point in $B(\mathrm{GL}_r)$ after pullback to s. Note that if $\underline{\mathcal{G}}$ is bounded by $\mu = (1, 0, \ldots, 0)$, then $[\underline{\mathcal{G}}_s]$ is contained in $B(\mathrm{GL}_r, \mu)$ for all $s \in S$. The Newton point induces a stratification in the following way.

Proposition 2.1.9 ([HV11, Theorem 7.3], compare also [RR96, Theorem 3.6]). *Let $S = \mathrm{Spec}(R)$ be an affine $\mathbb{F}_q$-scheme and $\underline{\mathcal{G}}$ be a local shtuka over S and $b \in B(\mathrm{GL}_r)$. Then the set $\{s \in S\colon [\underline{\mathcal{G}}_s] \leqslant b\}$ is a Zariski-closed subset of S. Furthermore, $\{s \in S\colon [\underline{\mathcal{G}}_s] = b\}$ is an open subset of the former.*

We denote by $S_{\leqslant b}$ the closed subscheme of S given by the reduced subscheme on $\{s \in S\colon [\underline{\mathcal{G}}_s] \leqslant b\}$ and similarly S_b the corresponding open subscheme of $S_{\leqslant b}$. Then S_b is a locally closed subscheme of S.

2.1.3. Global-to-local functor and a Serre-Tate theorem

We explain how to associate local shtukas to global shtukas. We follow the general construction of [AH14]. This is a generalisation of the construction of [BH11, Section 8] for abelian sheaves and Anderson motives.

We follow the notation of [AH14, Section 5.2.]. Let y be a closed point of X, which we assume for simplicity to be defined over $\mathbb{F}_q$. This is the only case we use later. For the general construction we refer to [AH14]. Let $\mathcal{O}_y$ be the completed local ring at y. The choice of a uniformiser ϖ_y at y defines an isomorphism $\mathcal{O}_y \cong \mathbb{F}_q [\![\varpi_y]\!]$. Let $x \in X(\mathrm{Spec}(R))$ be a section that factors through $\mathrm{Spf}(\mathcal{O}_y)$. Then ϖ_y is nilpotent in R. Let $\mathbb{D}_y = \mathrm{Spec}(\mathcal{O}_y)$ and $\hat{\mathbb{D}}_y = \mathrm{Spf}(\mathcal{O}_y)$. We denote by $\hat{\mathbb{D}}_{y,R}$ the ϖ_y-adic completion of $\mathbb{D}_y \times_{\mathbb{F}_q} \mathrm{Spec}(R)$.

By [AH14, Lemma 5.3.], the section x induces a canonical isomorphism of the formal completion of X_R along the graph Γ_x of x with $\hat{\mathbb{D}}_{y,R}$. By construction, the formal completion along Γ_x has structure sheaf $R [\![\varpi_y - \zeta]\!]$, where ζ is the image of ϖ_y in R. As ζ is nilpotent in R, $R [\![\varpi_y - \zeta]\!]$ and $R [\![\varpi_y]\!]$ are isomorphic.

We fix a pair $\underline{y} = (y_1, y_2)$ of ($\mathbb{F}_q$-rational) closed points of X with $y_1 \neq y_2$. Let $\mathcal{O}_{\underline{y}}$ be the completion of the local ring of X^2 at $\underline{y}$. We denote by $\mathrm{Sht}_r^{\underline{y}} = \mathrm{Sht}_{r,X^2} \times_{X^2} \mathrm{Spf}(\mathcal{O}_{\underline{y}})$ the substack of Sht_{r,X^2} such that the legs factor through $\mathrm{Spf}(\mathcal{O}_{y_1})$ and $\mathrm{Spf}(\mathcal{O}_{y_2})$, respectively. In particular, for points of $\mathrm{Sht}_r^{\underline{y}}$ the graphs of its legs are disjoint. Let $\underline{\mathcal{E}} = (x', x, \mathcal{E}, \varphi) \in \mathrm{Sht}_r^{\underline{y}}(R)$. The local shtuka associated to $\underline{\mathcal{E}}$ at y_i is then its pullback to $\hat{\mathbb{D}}_{y_i,R}$ for $i = 1, 2$.

Definition 2.1.10. The *global-to-local functor* associates to a global shtuka $\underline{\mathcal{E}} \in \mathrm{Sht}_r^{\underline{y}}(R)$ a pair of local shtukas (at y_1 and y_2, respectively) given by

$$\widehat{\underline{\mathcal{E}}_{y_i}} := (\mathcal{E}|_{\hat{\mathbb{D}}_{y_i,R}}, \tilde{\varphi}) \qquad \text{and} \qquad \widehat{\underline{\mathcal{E}}_{\underline{y}}} = (\widehat{\underline{\mathcal{E}}_{y_1}}, \widehat{\underline{\mathcal{E}}_{y_2}}).$$

Then, $\widehat{\underline{\mathcal{E}}_{y_i}}$ is called the *local shtuka* of $\underline{\mathcal{E}}$ at y_i.

By definition of Sht_{r,X^2}, the local shtuka at y_2 is bounded by $(1,0,\ldots,0)$ as the condition that $(\varpi_{y_2} - \zeta_{y_2})$ annihilates the cokernel in the local case directly corresponds to the fact that the cokernel is supported on the graph in the global case. Similarly, the local shtuka at y_1 is bounded by $(0,\ldots,0,-1)$.

The global-to-local functor also gives rise to a Serre-Tate theorem relating the deformation theory of global shtukas with the deformation theory of their associated local shtukas. Let $S = \mathrm{Spec}(R) \in \mathcal{N}ilp_{\mathcal{O}_{\underline{y}}}$ and let $i\colon \overline{S} = \mathrm{Spec}(R/I) \hookrightarrow S$ be a closed subscheme defined by a nilpotent ideals I. Let $\underline{\bar{\mathcal{E}}} \in \mathrm{Sht}_r^{\underline{y}}(\overline{S})$. The category $\mathrm{Def}_{\underline{\bar{\mathcal{E}}}}(S)$ is the category of deformations of $\underline{\mathcal{E}}$ to S, i.e. the category of pairs $(\underline{\mathcal{E}}, f\colon i^*\underline{\mathcal{E}} \xrightarrow{\sim} \underline{\bar{\mathcal{E}}})$ where $\mathcal{E} \in \mathrm{Sht}_r^{\underline{y}}(S)$ and f is an isomorphism of shtukas over $\overline{S}$. Similarly, for a local shtuka $\underline{\bar{\mathcal{G}}}$ bounded by $(1,0,\ldots,0)$ we define $\mathrm{Def}_{\underline{\bar{\mathcal{G}}}}^{\leqslant(1,0,\ldots,0)}(S)$ as the category of deformations of $\underline{\bar{\mathcal{G}}}$ to S, i.e. the category of pairs $(\underline{\mathcal{G}}, g\colon i^*\underline{\mathcal{G}} \xrightarrow{\sim} \underline{\bar{\mathcal{G}}})$ where $\mathcal{G}$ is a local shtuka on S bounded by $(1,0,\ldots 0)$ and g is an isomorphism of local shtukas over $\overline{S}$. Similarly, we define deformations of local shtukas bounded by $(0,\ldots,0,-1)$.

Proposition 2.1.11 (Serre-Tate Theorem for shutkas, [AH14, Theorem 5.10.])**.** *Let $\underline{\bar{\mathcal{E}}} \in \mathrm{Sht}_r^{\underline{y}}(\overline{S})$. Then the functor*

$$\widehat{(-)_{\underline{y}}}\colon \mathrm{Def}_{\underline{\bar{\mathcal{E}}}}(S) \to \mathrm{Def}^{\leqslant(0,\ldots,0,-1)}_{\widehat{\underline{\bar{\mathcal{E}}}_{y_1}}}(S) \times \mathrm{Def}^{\leqslant(1,0,\ldots,0)}_{\widehat{\underline{\bar{\mathcal{E}}}_{y_2}}}(S), \qquad (\underline{\mathcal{E}}, f) \mapsto \prod_{i=1,2} (\widehat{\underline{\mathcal{E}}_{y_i}}, \widehat{f_{y_i}})$$

induced by the global-to-local functor is an equivalence of categories.

Proof. As before, this follows directly from the unbounded case in [AH14, Theorem 5.10.] as a global GL_r-shtuka is bounded by $(0,\ldots,0,-1),(1,0,\ldots,0)$ if and only if the associated local shtukas are. ☐

The Newton stratification induces also a stratification on the special fibre of the stack of (global) Drinfeld shtukas in the sense of [Bre18, Section 4]. We continue to restrict ourselves to the case of Drinfeld shtukas with one leg fixed at ∞ as this is the only case of interest to us in the following. The following has obvious analogues also for Sht_{r,X^2}. For a closed point y of X with residue field F_y different from ∞ we set $\mathrm{Sht}_{r,\mathbb{F}_y} = \mathrm{Sht}_r \times_{X',y} \mathbb{F}_y$.

Definition 2.1.12 (compare [Neu16, Proposition 4.1.4.], [Bre18, Definition 4.12.] for the general definition)**.** Le $b_\infty \in B(\mathrm{GL}_{r,K_\infty}, (0,\ldots,0,-1))$. The locally closed and reduced substack of Sht_r where the associated local shtuka at ∞ has Newton point b_∞ is called the *Newton stratum* associated to b_∞ and is denoted by $\mathrm{Sht}_{r,b_\infty}$. In a similar fashion, for a closed point y of X and a pair

$$\underline{b_y} = (b_\infty, b_y) \in B(\mathrm{GL}_{r,K_\infty}, (0,\ldots,0,-1)) \times B(\mathrm{GL}_{r,K_y}, (1,0,\ldots,0))$$

the locally closed and reduced substack of $\mathrm{Sht}_{r,\mathbb{F}_y}$ where the associated local shtuka at ∞ has Newton point b_∞ and the associated local shtuka at y has Newton point b_y is denoted by $\mathrm{Sht}_{r,\underline{b_y}}$.

2.1.4. Isogenies of Drinfeld shtukas

We study isogenies of Drinfeld shtukas in more detail. We consider the following moduli problem of Drinfeld shtukas with chains of isogenies.

Definition 2.1.13. Let $m, r_1, \ldots, r_m \geqslant 1$ be positive integers such that $\sum_{j=1}^m r_j \leqslant r$. A *chain of* $\mathfrak{p}^n$*-isogenies of type* $(r_1,\ldots,r_m)$ on a Drinfeld shtuka $\underline{\mathcal{E}}$ over a scheme S is a flag of quotients of $\mathfrak{p}^n$-torsion shtukas

$$\underline{\mathcal{E}}|_{D_{n,S}} = \underline{\mathcal{F}}_{m+1} \twoheadrightarrow \underline{\mathcal{F}}_m \twoheadrightarrow \ldots \twoheadrightarrow \underline{\mathcal{F}}_1 \twoheadrightarrow 0$$

over S such that $\mathcal{F}_i$ has rank $n \cdot (r_1 + \ldots + r_i)$ as $\mathcal{O}_S$-module. We denote the stack of Drinfeld shtukas with chains of $\mathfrak{p}^n$-isogenies of type $(r_1,\ldots,r_m)$ by $\mathrm{Sht}_{r,(r_1,\ldots,r_m)-\mathfrak{p}^n\text{-chain}}$.

We show below that a chain of $\mathfrak{p}^n$-isogenies of type $(r_1,\ldots,r_m)$ in the sense of the definition is the same as giving a chain of actual $\mathfrak{p}^n$-isogenies of Drinfeld shtukas

$$\underline{\mathcal{E}}(\mathfrak{p}^n) = \underline{\mathcal{E}}_{m+1} \xrightarrow{f_{m+1}} \underline{\mathcal{E}}_m \xrightarrow{f_m} \ldots \xrightarrow{f_2} \underline{\mathcal{E}}_1 \xrightarrow{f_1} \underline{\mathcal{E}}_0 = \underline{\mathcal{E}}$$

such that the composition $f_{m+1} \circ \ldots \circ f_1$ is the inclusion $\underline{\mathcal{E}}(\mathfrak{p}^n) \to \underline{\mathcal{E}}$.

Proposition 2.1.14. *Let* $\underline{\mathcal{E}} \in \mathrm{Sht}_r(S)$ *be a Drinfeld shtuka. Every quotient* $\underline{\mathcal{E}}|_{D_{n,S}} \twoheadrightarrow \underline{\mathcal{F}}$*, where* $\underline{\mathcal{F}}$ *is a* $\mathfrak{p}^n$*-torsion shtuka, is the cokernel of a* $\mathfrak{p}^n$*-isogeny.*

Moreover, for two $\mathfrak{p}^n$*-isogenies* $f_1 \colon \underline{\mathcal{E}_1} \hookrightarrow \underline{\mathcal{E}}$ *and* $f_2 \colon \underline{\mathcal{E}_2} \hookrightarrow \underline{\mathcal{E}}$ *such that the cokernels factor as successive quotients* $\underline{\mathcal{E}}|_{D_{n,S}} \twoheadrightarrow \mathrm{coker}(f_1) \twoheadrightarrow \mathrm{coker}(f_2)$*, there exists a unique* $\mathfrak{p}^n$*-isogeny* $f \colon \underline{\mathcal{E}_1} \hookrightarrow \underline{\mathcal{E}_2}$ *such that* $f_1 = f_2 \circ f$ *and* $\mathrm{coker}(f) \cong \ker(\mathrm{coker}(f_1) \twoheadrightarrow \mathrm{coker}(f_2))$.

Proof. Let $\underline{\mathcal{F}}$ be a $\mathfrak{p}^n$-torsion shtuka as in the statement of the proposition. Let us denote by $\mathcal{E}' = \ker(\mathcal{E} \twoheadrightarrow \mathcal{F})$. As a first step, we want to show that $\mathcal{E}'$ is finite locally free of rank r on X_S. In order to do so, we may by reduction to the universal case assume that $S = \operatorname{Spec} R$ is affine and noetherian. As $\mathcal{E}' \hookrightarrow \mathcal{E}$ is an isomorphism away from $\mathfrak{p}$, it then suffices by fpqc-descent to show that the completion at 0 is finite locally free of rank r. As we assumed R to be noetherian, completion at 0 is exact. The completion $\widehat{\mathcal{E}'_0} = \mathcal{E}' \otimes \mathcal{O}_0$ of $\mathcal{E}'$ at 0 is hence given by the kernel of $\widehat{\mathcal{E}_0} \twoheadrightarrow \underline{\mathcal{F}}$. The assertion now follows from [Gen96, Lemma 2.2.8].

By the right exactness of the tensor product, the cokernel of the induced map $\sigma^*\mathcal{E}' \to \sigma^*\mathcal{E}$ is given by $\sigma^*\mathcal{F}$. By [Har19, Lemma 2.2], the map $\sigma^*\mathcal{E}' \to \sigma^*\mathcal{E}$ is thus injective, and $\sigma^*\mathcal{E}' = \ker(\sigma^*\mathcal{E} \twoheadrightarrow \sigma^*\mathcal{F})$. In particular, we obtain an induced map $\tilde{\varphi}\colon \sigma^*\mathcal{E}' \dashrightarrow \mathcal{E}'$ defined away from Γ_x and Γ_∞. As the map $\mathcal{E}' \to \mathcal{E}$ is an isomorphism away from 0, locally around ∞ we obtain a map $\mathcal{E}' \to \sigma^*\mathcal{E}'$ with cokernel supported at ∞ and of rank 1 as $\mathcal{O}_S$-module. It follows also that $\varphi'|_{X'_S}\colon \sigma^*\mathcal{E}'|_{X'_S} \to \mathcal{E}'|_{X'_S}$ is a well-defined and injective map (as φ is). Note that $(\mathcal{E}'|_{X'_S}, \varphi'|_{X'_S})$ is the associated A-motive in the sense of [Har19] where $A = \Gamma(X\backslash\{\infty\}, \mathcal{O}_X)$. By [Har19, Proposition 2.3], the A-motive $(\mathcal{E}'|_{X'_S}, \varphi'|_{X'_S})$ is effective, this means that $\operatorname{coker}(\varphi'|_{X'_S})$ is annihilated by $\mathcal{J}^n$ for some positive integer n, where $\mathcal{J}$ is the quasi-coherent sheaf of ideals defining $\Gamma_x \subseteq X_S$. Using [Har19, Proposition 5.8] we obtain that $\operatorname{coker}(\varphi'|_{X'_S})$ has rank 1 as $\mathcal{O}_S$-module. Thus, $\operatorname{coker}(\varphi'|_{X'_S})$ is already annihilated by $\mathcal{J}$, which means that $(\mathcal{E}', \varphi')$ defines a point of $\mathrm{Sht}_r(S)$.

For the second part let $f_1\colon \underline{\mathcal{E}}_1 \hookrightarrow \underline{\mathcal{E}}$ and $f_2\colon \underline{\mathcal{E}}_2 \hookrightarrow \underline{\mathcal{E}}$ be two $\mathfrak{p}^n$-isogenies as in the assertion. It follows essentially by assumption that there is a unique injective homomorphism of shtukas $f\colon \underline{\mathcal{E}}_2 \to \underline{\mathcal{E}}_1$ such that $f_2 = f_1 \circ f$. It remains to check that f is a $\mathfrak{p}^n$-isogeny. We have the short exact sequence of R-modules

$$0 \to \mathcal{E}_1/f(\mathcal{E}_2) = \operatorname{coker}(f) \hookrightarrow \mathcal{E}/f_2(\mathcal{E}_2) = \operatorname{coker}(f_2) \twoheadrightarrow \mathcal{E}/f_1(\mathcal{E}_1) = \operatorname{coker}(f_1) \to 0,$$

where the first map is f_1 and the second map is well-defined by assumption. As both f_1 and f_2 are isogenies, their cokernels are finite locally free R-modules. It follows that $\operatorname{coker}(f)$ is finite locally free as well, and thus f is an isogeny. That it is a $\mathfrak{p}^n$-isogeny is also clear. □

Remark 2.1.15. (1) Note that in the proof we really used that 0 is $\mathbb{F}_q$-rational. It would be desirable to have an analogous statement in general.

(2) Using the comparison [Har19, Theorem 5.8] with isogenies of Drinfeld modules, we get as immediate corollaries that any finite locally free closed submodule scheme with strict $\mathbb{F}_q$-action of the $\mathfrak{p}^n$-torsion of a Drinfeld module is the kernel of an isogeny and a factorisation property as in the second part of the proposition. Both of these facts seem to be only proven in the literature when the base is a field in [Leh09, 2, Lemma 3.1 and Lemma 3.2].

(3) This also shows that giving a point of $\mathrm{Sht}_{r,(r_1,\ldots,r_m)-\mathfrak{p}^n\text{-chain}}$ is the same as giving a chain of actual $\mathfrak{p}^n$-isogenies of Drinfeld shtukas

$$\underline{\mathcal{E}}(\mathfrak{p}^n) = \underline{\mathcal{E}}_{m+1} \xrightarrow{f_{m+1}} \underline{\mathcal{E}}_m \xrightarrow{f_m} \ldots \xrightarrow{f_2} \underline{\mathcal{E}}_1 \xrightarrow{f_1} \underline{\mathcal{E}}_0 = \underline{\mathcal{E}}$$

such that $\mathrm{coker}(f_i)$ has rank $n \cdot r_i$ and such that the composition $f_{m+1} \circ \ldots \circ f_1$ is the inclusion $\underline{\mathcal{E}}(\mathfrak{p}^n) \to \underline{\mathcal{E}}$.

2.1.5. Naive $\Gamma_0(\mathfrak{p}^n)$-level structures and shtukas for Bruhat-Tits group schemes

We introduce naive $\Gamma_0(\mathfrak{p}^n)$-level structures on Drinfeld shtukas and explain how to interpret them as shtukas for certain Bruhat-Tits group schemes. These naive level structures seem inadequate in the non-parahoric case (that means when $n > 1$), as their moduli spaces are missing points in the fibre over 0. In other words, the level map to Sht_r is not proper, compare Remark 2.1.20 below. The interpretation of naive level structures in terms of Bruhat-Tits group schemes allows us to give a candidate for a compactification of the level map: We can take the closure of the stack of shtukas with naive level in the product of the stacks of Drinfeld shtukas with corresponding parahoric level.

Definition 2.1.16. A *naive* $\Gamma_0(\mathfrak{p}^n)$*-level structure* on a Drinfeld shtuka $\underline{\mathcal{E}} = (\mathcal{E}, \varphi) \in \mathrm{Sht}_r(S)$ of rank r is a flag of quotients of $\mathfrak{p}^n$-torsion finite shtukas

$$\underline{\mathcal{E}}|_{D_{n,S}} = \underline{\mathcal{L}}_r \twoheadrightarrow \underline{\mathcal{L}}_{r-1} \twoheadrightarrow \ldots \twoheadrightarrow \underline{\mathcal{L}}_1 \twoheadrightarrow \underline{\mathcal{L}}_0 = 0$$

such that $\mathcal{L}_i$ is finite locally free of rank i as $\mathcal{O}_{D_{n,S}}$-module (and hence of rank in as $\mathcal{O}_S$-module).

Remark 2.1.17. By Proposition 2.1.14, a naive $\Gamma_0(\mathfrak{p}^n)$-level structure is equivalently given as a chain of isogenies

$$\underline{\mathcal{E}}(\mathfrak{p}^n) = \underline{\mathcal{E}}_r \xrightarrow{f_r} \underline{\mathcal{E}}_{r-1} \xrightarrow{f_{r-1}} \underline{\mathcal{E}}_{r-2} \to \ldots \xrightarrow{f_1} \underline{\mathcal{E}}_0 = \underline{\mathcal{E}}$$

such that $\mathrm{coker}(f_i)$ is finite locally free of rank 1 as $\mathcal{O}_{D_{n,S}}$-module for all $1 \leqslant i \leqslant r$.

We interpret naive $\Gamma_0(\mathfrak{p}^n)$-level structures on Drinfeld shtukas as shtukas for certain Bruhat-Tits group schemes in the following sense.

Definition 2.1.18. A *Bruhat-Tits group scheme* on X is a smooth affine group scheme $G \to X$ such that

(1) all fibres of G are connected,

(2) the generic fibre of G is a reductive group over K and

(3) for every closed point x of X the base change $G_{\mathcal{O}_x} = G \times_X \operatorname{Spec}(\mathcal{O}_x)$ is a Bruhat-Tits group scheme in the sense that there is a non-empty bounded subset Ω in some appartment in the Bruhat-Tits building of G_{K_x} such that $G(\mathcal{O}_x) \subseteq G(K_x)$ is the connected fixator of Ω in the sense of [BT84, (4.6.26)].

A Bruhat-Tits group scheme is *parahoric*, if the subgroups $G(\mathcal{O}_x) \subseteq G(K_x)$ in (3) are parahoric for all places of X.

Remark 2.1.19. Of particular relevance to our situation is the case where Ω is the stabiliser of a regular $(r-1)$-simplex Ω in the standard appartment of the (reduced) Bruhat-Tits building of GL_{r,K_0} with side-length n. We denote by $\mathrm{GL}_{r,\Omega} \to X$ the corresponding Bruhat-Tits group scheme that is isomorphic to GL_r away from 0 and such that $\mathrm{GL}_r(\mathcal{O}_0) \subseteq \mathrm{GL}_r(K_0)$ is the connected stabiliser of Ω.

We can more explicitly describe this subgroup by $\mathrm{GL}_{r,\Omega}(\mathcal{O}_0) = \{M \in \mathrm{GL}_r(\mathcal{O}_0) \colon M \bmod \mathfrak{p}^n \in B(\mathcal{O}_0/\mathfrak{p}^n)\}$. By [MRR20, Lemma 3.1 and Theorem 3.2], the group scheme $\mathrm{GL}_{r,\Omega}$ can thus also be interpreted as the Néron blowup of GL_r in its subgroup B of upper triangular matrices along the divisor D_n in the sense of [MRR20, Section 3.1].

By [MRR20, Theorem 4.8], giving a $\mathrm{GL}_{r,\Omega}$-torsor on X is equivalent to giving a GL_r-torsor $\mathcal{E}$ on X together with a reduction of $\mathcal{E}$ to an B-torsor over the divisor D_n of X. More explicitly, a point of $\mathrm{Bun}_{\mathrm{GL}_{r,\Omega}}(S)$ is given by a rank r vector bundle $\mathcal{E}$ on X_S together a flag of quotients of $\mathcal{E}|_{D_{n,S}}$ as in the definition of naive $\Gamma_0(\mathfrak{p}^n)$-level structures. In this sense, a naive $\Gamma_0(\mathfrak{p}^n)$-level structure on a Drinfeld shtuka $\underline{\mathcal{E}}$ defines a (B, D_n)-level structure on $\underline{\mathcal{E}}$ in the sense of [MRR20, Section 4.2.2].

A $\mathrm{GL}_{r,\Omega}$-shtuka is called *bounded by* $\underline{\mu} = ((0, \dots, 0, -1), (1, 0, \dots, 0))$ if its underlying GL_r-shtuka $(x, \mathcal{E}, \varphi)$ is bounded by $(0, \dots, 0, -1), (1, 0, \dots, 0)$, and if the flag of quotients given by the (B, D_n)-structure on the underlying vector bundle $\mathcal{E}$ is φ-stable. In other words, the $\mathrm{GL}_{r,\Omega}$-shtukas bounded by $\underline{\mu}$ in this sense are exactly the Drinfeld shtukas with naive $\Gamma_0(\mathfrak{p}^n)$-level structures in the sense above. We denote this stack of bounded $\mathrm{GL}_{r,\Omega}$-shtukas (or equivalently the stack of Drinfeld shtukas with naive $\Gamma_0(\mathfrak{p}^n)$-level structures) by $\mathrm{Sht}_{r,\Omega}$.

For a facet $\mathfrak{f}$ in the Bruhat-Tits building of GL_{r,K_0} we write $\mathfrak{f} \prec \Omega$ if $\mathfrak{f}$ is contained in the closure of Ω. In a similar fashion to the construction above, for such a facet $\mathfrak{f} \prec \Omega$ we write $\mathrm{GL}_{r,\mathfrak{f}}$ for the corresponding Bruhat-Tits group scheme and $\mathrm{Sht}_{r,\mathfrak{f}}$ for the stack of $\mathrm{GL}_{r,\mathfrak{f}}$-shtukas bounded by $\underline{\mu}$ in the sense above. By Bruhat-Tits theory, for any facet $\mathfrak{f}$ contained in Ω there is a map of group schemes $\mathrm{GL}_{r,\Omega} \to \mathrm{GL}_{r,\mathfrak{f}}$ that is the identity away from 0. By [Bre18, Corollary 3.16], we get maps $\mathrm{Sht}_{r,\Omega} \to \mathrm{Sht}_{r,\mathfrak{f}}$.

In particular, in the case $n = 1$ the set Ω is just given by the base alcove (corresponding to the standard Iwahori subgroup of matrices that upper triangular mod $\mathfrak{p}$). Hence, for an alcove $\mathfrak{f}$ its corresponding moduli space of shtukas $\mathrm{Sht}_{r,\mathfrak{f}}$ parametrises chains of isogenies of Drinfeld shtukas as in Definition 2.1.16. For a facet $\mathfrak{f}'$ of the alcove $\mathfrak{f}$ the map $\mathrm{Sht}_{r,\mathfrak{f}} \to \mathrm{Sht}_{r,\mathfrak{f}'}$ is then given by projection to some subchain of isogenies, depending on the position of $\mathfrak{f}'$. In particular, when $\mathfrak{f}'$ is a vertex, $\mathrm{Sht}_{r,\mathfrak{f}'}$ parametrises single Drinfeld shtukas and when $\mathfrak{f}'$ is an edge, $\mathrm{Sht}_{r,\mathfrak{f}'}$ parametrises pairs of Drinfeld shtukas with a certain isogeny between them.

In order to describe the maps $\mathrm{Sht}_{r,\Omega} \to \mathrm{Sht}_{r,\mathfrak{f}}$ for facets $\mathfrak{f} \prec \Omega$ more explicitly, we label the vertices in Ω by tuples $\underline{m} = (m_1, \dots, m_{r-1})$ such that $n \geqslant m_1 \geqslant \dots \geqslant m_{r-1} \geqslant 0$, edges are between vertices $\underline{m}$ and $\underline{m}'$ if and only if $0 \leqslant m_i - m_i' \leqslant 1$ for all i or $0 \leqslant m_i - m_i' \leqslant 1$ for all i. The base alcove corresponds to the simplex defined by the vertices $(0, \dots, 0), (1, 0, \dots, 0), (1, 1, 0, \dots, 0), \dots, (1, 1, \dots, 1)$. The vertex $(0, \dots, 0)$ corresponds to the constant group scheme GL_r.

Note that every facet $\mathfrak{f} \prec \Omega$ has a unique base point $\underline{m}$ (such that $m_i \leqslant x_i$ for all i and points $\underline{x} \in \mathfrak{f}$), and an orientation that we encode by an element $\tau \in \mathrm{Sym}_{r-1}$ of the symmetric group on $(r-1)$ elements. The orientation τ is chosen such that the vertices of $\mathfrak{f}$ are given by $\underline{m} + \tau(\mathbf{1}_{r-1}^{(i)})$ for $0 \leqslant i \leqslant r-1$, where $\mathbf{1}_{r-1}^{(i)} = (1, \dots, 1, 0, \dots, 0) \in \mathbb{Z}^{r-1}$ has exactly i many entries equal to 1. For a given pair $(\underline{m}, \tau)$ there clearly exists a unique alcove $\mathfrak{f}_{\underline{m},\tau}$ in the standard apartment of the Bruhat-Tits building with base point $\underline{m}$ and orientation τ.

Starting from a $\mathrm{GL}_{r,\Omega}$-shtuka $(\underline{\mathcal{E}}, (\underline{\mathcal{L}_i})) \in \mathrm{Sht}_{r,\Omega}(S)$, we construct a Drinfeld shtuka $\underline{\mathcal{E}}_{\underline{m}}$ for a vertex $\underline{m} \prec \Omega$ as follows. Assume that $S = \mathrm{Spec}(R)$ is affine and that all $\mathcal{L}_i$ are finite free as $R[\varpi]/(\varpi^n)$-modules. In this case, we can choose a basis $(e_1, \dots, e_{r-1})$ of $\mathcal{L}_{r-1} = (R[\varpi]/(\varpi^n))^{r-1}$ such that $(e_1, \dots, e_i)$ is a basis for $\mathcal{L}_i$ for all $1 \leqslant i \leqslant r-1$. We consider the quotient

$$\mathcal{L}_{r-1} \twoheadrightarrow \mathcal{L}_{\underline{m}} := R[\varpi]/(\varpi^{m_1})e_1 \oplus \dots \oplus R[\varpi]/(\varpi^{m_{r-1}})e_{r-1}.$$

As all the $\mathcal{L}_i$ are φ-stable quotients of $\mathcal{L}_{r-1}$, the matrix representation of φ with respect to $(e_1, \dots, e_{r-1})$ is upper-triangular. This shows that also $\mathcal{L}_{\underline{m}}$ is φ-stable as $m_1 \geqslant \dots \geqslant m_{r-1}$ by assumption. By a similar argument, $\mathcal{L}_{\underline{m}}$ does not depend on the choice of basis (any base change matrix is again upper-triangular). Thus, we can glue to obtain a φ-stable quotient $\mathcal{L}_{\underline{m}}$ also in the general case. We then associate to the vertex $\underline{m}$ the Drinfeld shtuka corresponding to the kernel $\underline{\mathcal{E}}_{\underline{m}} = \ker(\underline{\mathcal{E}} \twoheadrightarrow \underline{\mathcal{L}_{\underline{m}}})$ by Proposition 2.1.14. Moreover, by the second part of Proposition 2.1.14, there are also canonical isogenies associated to the edges in the Bruhat-Tits building.

Using this construction, for an alcove $\mathfrak{f}_{\underline{m},\tau} \prec \Omega$ the level map $\mathrm{Sht}_{r,\Omega} \to \mathrm{Sht}_{r,\mathfrak{f}_{\underline{m}}}$ associates to $(\underline{\mathcal{E}}, (\underline{\mathcal{L}}_i))$ the chain of isogenies

$$\underline{\mathcal{E}}_{\underline{m}}(\mathfrak{p}^n) \hookrightarrow \underline{\mathcal{E}}_{\underline{m}+\tau(\mathbf{r}-\mathbf{1}^{(1)}_{r-1})} \hookrightarrow \ldots \hookrightarrow \underline{\mathcal{E}}_{\underline{m}+\tau(\mathbf{1}^{(1)}_{r-1})} \hookrightarrow \underline{\mathcal{E}}_{\underline{m}}.$$

This means that the induced map $\mathrm{Sht}_{r,\Omega} \to \varprojlim_{\mathfrak{f}<\Omega} \mathrm{Sht}_{r,\mathfrak{f}}$ associates to a Drinfeld shtuka with naive $\Gamma_0(\mathfrak{p}^n)$-level structure a diagram $(\underline{\mathcal{E}}_{\underline{m}})_{\underline{m}}$ with the canonical isogenies as constructed above.

Remark 2.1.20. For parahoric level (in our case that means $n \leqslant 1$) [Bre18] shows that the level maps are proper and surjective. An explicit calculation for deeper level (that is for $n > 1$) shows that this is false already in the GL_2-case over $X = \mathbb{P}^1$ in general. Namely, we study the fibre over 0 using the local model of [AH19]. An explicit calculation in the local model shows that for $n = 1$ we get the familiar local picture of two copies of $\mathbb{P}^1$ intersecting transversally at supersingular points.

However, for any $n > 1$ the special fibres of the corresponding local models only contain two copies of $\mathbb{A}^1$ that do not intersect. This means in particular that $\mathrm{Sht}_{r,\Omega}$ is missing the supersingular points in the special fibre. Moreover, from the comparison with the modular curve, we might expect to find $n+1$-components two of which are reduced by [KM85, Theorem 13.4.7]. The two components we see using the naive level structure correspond to the two reduced components, but we do not get the non-reduced ones.

It turns out that requiring the quotients in the definition of naive level structures to be locally free as $\mathcal{O}_{D_{n,S}}$-modules is too restrictive and we rather should allow in the special fibre also degenerations to certain $\mathfrak{p}^n$-torsion finite shtukas which are not locally free as $\mathcal{O}_{D_{n,S}}$-modules.

The goal of this paper is to explain one way to remedy this. We show that we can explicitly describe the schematic image of the map $\mathrm{Sht}_{r,\Omega} \to \varprojlim_{\mathfrak{f}<\Omega} \mathrm{Sht}_{r,\mathfrak{f}}$ in terms of *Drinfeld level structures* and that this provides a natural compactification of the level map.

2.2. Group schemes attached to Drinfeld shtukas

In order to define Drinfeld level structures for Drinfeld shtukas, we explain how to construct a (finite locally free $\mathcal{O}_0/\mathfrak{p}^n$-module) scheme of $\mathfrak{p}^n$-torsion points $\underline{\mathcal{E}}$ of a Drinfeld shtuka. This scheme of $\mathfrak{p}^n$-torsion points serves as an analogue of the scheme of p^n-torsion points of an elliptic curve and behaves similarly in many ways. In order to study properties of $\underline{\mathcal{E}}[\mathfrak{p}^n]$ we use an explicit comparison of Drinfeld shtukas and Drinfeld modules.

2.2.1. Comparison with Drinfeld modules

We recall some facts about Drinfeld modules and show how to construct Drinfeld shtukas from them. Let $A = \Gamma(X\backslash\{\infty\}, \mathcal{O}_X)$. The point 0 of X then corresponds to a maximal ideal of A, which by a slight abuse of notation we denote by $\mathfrak{p}$ as well.

Roughly speaking, a Drinfeld A-module is an A-module structure on a (geometric) line bundle. Drinfeld modules were first introduced in [Dri76] in order to construct a Langlands correspondence in the cohomology of their moduli spaces. In this sense, Drinfeld modules (of rank 2) are function field analogues of elliptic curves in the number field case. For a more detailed treatment also compare [Leh09], [BS97] or [Lau96].

Let $\mathcal{L}$ be an invertible sheaf on S. The corresponding geometric line bundle is denoted by $\mathbb{G}_{a,\mathcal{L}} = \underline{\mathrm{Spec}}_S(\mathrm{Sym}(\mathcal{L}^{-1}))$. If $S = \mathrm{Spec}(R)$ is an affine scheme such that $\mathcal{L}$ is trivial, the corresponding line bundle is given by $\mathbb{G}_{a,R} = \mathrm{Spec}\, R[t]$. Locally, the ring of endomorphisms of a line bundle is then given by the skew-polynomial ring $R\{\tau\}$ with the commutation relation $\tau c = c^q \tau$ for $c \in R$.

Definition 2.2.1. A *Drinfeld A-module* $\mathbf{E} = (\mathbb{G}_{a,\mathcal{L}}, e)$ of rank r over a scheme S consists of an additive group scheme $\mathbb{G}_{a,\mathcal{L}}$ and a ring homomorphism $e\colon A \to \mathrm{End}(\mathbb{G}_{a,\mathcal{L}}), a \mapsto e_a$ such that e_a is finite for all $a \neq 0 \in A$ of degree $|a|^r$, where $|\cdot|$ is the normalised absolute value on K corresponding to ∞. The composition $\partial \circ e$ with the differential induces a map $S \to \mathrm{Spec}(A)$ called the *characteristic* of $\mathbf{E}$.

We denote by $\mathrm{Dr\text{-}Mod}_r$ the moduli stack of Drinfeld modules of rank r. It is a Deligne-Mumford stack of finite type over $\mathbb{F}_q$, which is smooth of relative dimension $r-1$ over $X' = \mathrm{Spec}(A)$.

When $S = \mathrm{Spec}\,\ell$ is the spectrum of a field (or more generally when the line bundle $\mathcal{L}$ is trivial), a Drinfeld module as a ring homomorphism $e\colon A \to \ell\{\tau\}$. As for Drinfeld shtukas, in a slight abuse of notation, we say $\mathbf{E}$ has characteristic $\mathfrak{p}$ if the the characteristic of $\mathbf{E}$ factors through 0, or in other words, if the kernel of the induced map $A \to \mathcal{O}_S(S)$ is $\mathfrak{p}$. We say that a Drinfeld module over a field ℓ in characteristic $\mathfrak{p}$ has height h, if the smallest non-vanishing coefficient in $e_\varpi \in \ell\{\tau\}$ has degree h, where $\varpi \in \mathfrak{p}$ is a uniformiser.

There are several ways to associate vector bundles to Drinfeld modules, for example the so-called *elliptic sheaves* due to [Dri77], for a more detailed treatment also compare [BS97], [Har05] or [Wie04], or t-motives [And86] and their generalisations, see for example [Har19]. However, a precise comparison to Drinfeld shtukas, which is certainly well-known to the experts, does not seem to be part of the literature yet. We explain how to construct Drinfeld shtukas from Drinfeld modules.

Recall that an *elliptic sheaf* $\underline{\mathcal{E}}$ over S of rank r is given by the data $(x, (\mathcal{E}_i)_{i\in\mathbb{Z}}, (s_i)_{i\in\mathbb{Z}}, (t_i)_{i\in\mathbb{Z}})$ where $x\colon S \to X' = X\backslash\{\infty\}$ is a map of schemes, $\mathcal{E}_i$ is a rank r vector bundle on $X \times S$

for every $i \in \mathbb{Z}$ and $s_i\colon \mathcal{E}_i \to \mathcal{E}_{i+1}$ and $t_i\colon \sigma^*\mathcal{E}_i \to \mathcal{E}_{i+1}$ are injective maps that satisfy some further properties. In particular, $\operatorname{coker}(s_i)$ and $\operatorname{coker}(t_i)$ are supported on ∞ and Γ_x, respectively and invertible as $\mathcal{O}_S$-modules on their support. We denote by $\mathcal{E}\ell\ell_r$ the moduli stack of elliptic sheaves. We have a well-defined map $\mathcal{E}\ell\ell_r \to \mathrm{Sht}_r$ given by the projection

$$(x,(\mathcal{E}_i),(s_i),(t_i)) \mapsto (x,\mathcal{E}_0, s_0^{-1}|_{X\setminus(\Gamma_x\cup\Gamma_\infty)} \circ t_0|_{X\setminus(\Gamma_x\cup\Gamma_\infty)}),$$

or by Remark 2.1.3 equivalently by projection to $(x,\mathcal{E}_0,\mathcal{E}_1,s_0,t_0)$. We use this second perspective for the remainder of this section as it more convenient in this context. We define a functor $\mathbb{Z}\times \text{Dr-Mod}_r \to \mathrm{Sht}_r$ by composing the equivalence $\mathbb{Z}\times\text{Dr-Mod}_r \to \mathcal{E}\ell\ell_r$ of [Dri77] with this projection.

Lemma 2.2.2. *The projection $\mathcal{E}\ell\ell_r \to \mathrm{Sht}_r$ is fully faithful.*

Proof. Let $\underline{\mathcal{E}}_\bullet = (x,(\mathcal{E}_i),(s_i),(t_i))$ and $\underline{\tilde{\mathcal{E}}}_\bullet = (x,(\tilde{\mathcal{E}}_i),(\tilde{s}_i),(\tilde{t}_i))$ be two elliptic sheaves over S. Assume that we have a map of the corresponding shtukas, in other words a pair of maps $f_0\colon \mathcal{E}_0 \to \tilde{\mathcal{E}}_0$ and $f_1\colon \mathcal{E}_1 \to \tilde{\mathcal{E}}_1$ that commute with s_0 and t_0 in the obvious way. By [Wie04, Corollary 5.4] we may then glue f_1 and $\sigma^* f_1$ to get a map $f_2\colon \mathcal{E}_2 \to \tilde{\mathcal{E}}_2$ again commuting with s_1 and t_1. Such a map is necessarily unique. We continue inductively to define maps in higher degrees. The maps in degrees smaller than 0 can be constructed as twists. $\square$

Let us denote by $b_\infty = (-1/r,\dots,-1/r) \in B(\mathrm{GL}_{r,K_\infty},(0,\dots,0,-1))$ the basic Newton polygon. Recall that we defined $\mathrm{Sht}_{r,b_\infty}$ to be the (reduced) locus in Sht_r where the local shtuka at ∞ has Newton polygon b_∞. Note that $\mathrm{Sht}_{r,b_\infty}$ is a closed substack of Sht_r as b_∞ is basic.

Proposition 2.2.3. *The functor $\mathbb{Z}\times\text{Dr-Mod}_r \to \mathrm{Sht}_r$ is schematic and a closed immersion which factors through an isomorphism*

$$\mathbb{Z}\times\text{Dr-Mod}_r \xrightarrow{\cong} \mathrm{Sht}_{r,b_\infty}.$$

Proof. As a first step we show that the locus where a Drinfeld shtuka can be extended to an elliptic sheaf is closed. Let $\underline{\mathcal{E}} = (x,\mathcal{E}_{-1},\mathcal{E}_0,s_{-1},t_{-1}) \in \mathrm{Sht}_r(S)$. As the zero and pole of $\underline{\mathcal{E}}$ do not intersect, we can repeatedly glue $\mathcal{E}_i$ and $\sigma^*\mathcal{E}_i$ to obtain a commutative diagram

$$\begin{tikzcd}
\mathcal{E}_{-1} \arrow[r, "s_{-1}"] & \mathcal{E}_0 \arrow[r, "s_0"] & \mathcal{E}_2 \arrow[r, "s_2"] & \dots \\
\sigma^*\mathcal{E}_{-1} \arrow[r] \arrow[ur, "t_{-1}"'] & \sigma^*\mathcal{E}_0 \arrow[r] \arrow[ur, "t_0"'] & \sigma^*\mathcal{E}_2 \arrow[r] \arrow[ur, "t_2"'] & \dots
\end{tikzcd}$$

If the diagram comes from an elliptic sheaf, we have by definition that $\mathcal{E}_0 \hookrightarrow \mathcal{E}_r$ identifies $\mathcal{E}_0$ with $\mathcal{E}_r(-\infty)$. In a similar fashion we get $\mathcal{E}_1 = \mathcal{E}_{r+1}(-\infty)$.

We claim that these two conditions are already sufficient for the diagram to come from an elliptic sheaf. By construction, the cokernel of s_i is supported on Γ_∞ and the cokernel of t_i is supported on Γ_x, and both are invertible on their respective supports. We first check that $s_0 = s_r \otimes \mathrm{id}_{\mathcal{O}_X(-\infty)}$. As all s_i are isomorphisms away from ∞ and the question is fpqc-local on S, it suffices to consider the completion at ∞ and we may assume that $S = \mathrm{Spec}(R)$ is affine and all $\mathcal{E}_i$ are free $R\llbracket\varpi_\infty\rrbracket$-modules of rank r. Thus, the s_i are identified with endomorphisms of $R\llbracket\varpi_\infty\rrbracket^r$ such that both $s_{r-1} \circ \ldots \circ s_0$ and $s_r \circ \ldots \circ s_1$ are multiplication by ϖ_∞ by assumption, where ϖ_∞ is a uniformiser at ∞. But as multiplication by ϖ_∞ is injective and lies in the centre of the endomorphism ring, this implies that $s_0 = s_r$ as desired. Moreover, the s_i induce isomorphism $\mathrm{coker}(t_{i-1}) \xrightarrow{\cong} \mathrm{coker}(t_i)$ for all $i \geqslant 1$, hence $t_r = t_{-1} \otimes \mathrm{id}_{\mathcal{O}_X(\infty)}$. Hence, we get inductively that $\mathcal{E}_{i+r} = \mathcal{E}_i \otimes \mathcal{O}_X(\infty)$ for all $i \geqslant 1$. The data for indices $i \leqslant 0$ is then obtained by twisting. This shows the claim.

It remains to check that the conditions of the claim are closed conditions. In order to see that the locus where $\mathcal{E}_0 = \mathcal{E}_r(-\infty)$ is closed, we argue as follows. As $\mathcal{E}_r/\mathcal{E}_0$ is supported on Γ_∞, the uniformiser ϖ_∞ at ∞ acts on $\mathcal{E}_r/\mathcal{E}_0$ and we have $\mathcal{E}_0 = \mathcal{E}_r(-\infty)$ if and only if $\varpi_\infty = 0$ in $\mathrm{End}_{\mathcal{O}_S}(\mathcal{E}_r/\mathcal{E}_0)$. Hence, the locus where $\mathcal{E}_0 = \mathcal{E}_r(-\infty)$ is represented by the vanishing locus $V(\mathcal{I})$ of the quasi-coherent ideal $\mathcal{I} = \mathrm{image}(\mathcal{E}nd_{\mathcal{O}_S}(\mathcal{E}_r/\mathcal{E}_0)^\vee \xrightarrow{\varpi_\infty^\vee} \mathcal{O}_S)$. In a similar fashion, the locus where $\mathcal{E}_1 = \mathcal{E}_{r+1}(-\infty)$ is representable by a closed subscheme of S given by the vanishing locus of a quasi-coherent sheaf of ideals $\mathcal{I}'$ in $\mathcal{O}_S$. Thus, the locus where $\underline{\mathcal{E}}$ defines a (necessarily unique) elliptic sheaf is representable by the closed subscheme $S' = V(\mathcal{I} + \mathcal{I}')$ of S. In particular, $\mathbb{Z} \times \text{Dr-Mod}_r \to \mathrm{Sht}_r$ is schematic and a closed immersion.

Moreover, it is clear that both stacks have the same geometric points, as one can easily see by the classification of bounded local shtukas over algebraically closed fields that a Drinfeld shtuka over an algebraically closed field ℓ comes from a Drinfeld module if and only if the local shtuka at ∞ is

$$\left(\ell\llbracket\varpi_\infty\rrbracket^r, \sigma \cdot \begin{pmatrix} 0 & & & \varpi_\infty^{-1} \\ 1 & & & \\ & \ddots & & \\ & & 1 & 0 \end{pmatrix}\right)$$

The second part of the assertion follows as Dr-Mod_r and the Newton stratum $\mathrm{Sht}_{r,b_\infty}$ both are reduced. □

Remark 2.2.4. (1) For the general case (when ∞ is not defined over $\mathbb{F}_q$) giving a correct definition of the truncation is more subtle, as the pole might not be supported at ∞ but at some Frobenius twist of ∞. In order to remedy this, one should include a Frobenius twist in the action of $\mathbb{Z}$ by shifts and still obtain a well-defined closed immersion $\mathbb{Z} \times \text{Dr-Mod}_r \to \text{Sht}_r$.

(2) It follows that we get an essentially surjective functor $\text{Sht}_{r,b_\infty} \to \text{Dr-Mod}_r$ that agrees with the construction from [Bre18, Proposition 7.8] (up to forgetting the level structure).

(3) By [Har19] and [HS19], the comparison is compatible with local and finite objects. More precisely, the local equivalence of [HS19, Theorem 8.3] and [Har19, Theorem 7.6] identifies the $\mathfrak{p}$-divisible module associated to a Drinfeld module $\mathbf{E}$ over $S \in \mathcal{N}ilp_{\mathbb{F}[\![\varpi]\!]}$ the local shtuka at 0 of any Drinfeld shtuka associated to $\mathbf{E}$ by the comparison. We call this local shtuka the *local shtuka at* 0 (or the *local shtuka at* $\mathfrak{p}$) of the Drinfeld module $\mathbf{E}$. By [Dri76, Proposition 1.7], the Newton polygon of the local shtuka at $\mathfrak{p}$ associated to a Drinfeld module of height h over an algebraically closed field in characteristic $\mathfrak{p}$ is given by $(1/h, \dots, 1/h, 0, \dots, 0)$.

2.2.2. Strong stratification property of the Newton stratification

The existence of supersingular Drinfeld modules then implies the non-emptiness of the basic Newton stratum in Sht_r.

Proposition 2.2.5. *Let* $\underline{b_0} = ((-1/r, \dots, -1/r), (1/r, \dots, 1/r)) \in B(\mathrm{GL}_{r,K_\infty}) \times B(\mathrm{GL}_{r,K_0})$. *Then the basic Newton stratum* $\text{Sht}_{r,\underline{b_0}} \subseteq \text{Sht}_{r,\mathbb{F}_0}$ *is non-empty.*

Proof. By Remark 2.2.4 and Proposition 2.2.3 a basic Drinfeld module in characteristic $\mathfrak{p}$, that is, a Drinfeld module of both rank and height r, defines a point in $\text{Sht}_{r,\underline{b_0}}$. But basic Drinfeld modules in characteristic $\mathfrak{p}$ exist by [KY20, Proposition 7.4.1]. □

As a next step, we study closure relations among the Newton strata. The result may be well-known to experts. The author was unable to track down a precise reference. The corresponding statement for Shimura varieties in the Siegel case is due to [Oor01] and has been generalised to the PEL case by [Ham15].

For a pair of closed points $\underline{y} = (y_1, y_2)$ of X we define a partial order on $B(\mathrm{GL}_{r,K_{y_1}}) \times B(\mathrm{GL}_{r,K_{y_2}})$ (and also on $B(\mathrm{GL}_{r,K_{y_1}}, \mu_1) \times B(\mathrm{GL}_{r,K_{y_2}}, \mu_2)$ for a pair of cocharacters μ_1, μ_2 of GL_r) by $b_{\underline{y}} = (b_{y_1}, b_{y_2}) \leqslant b'_{\underline{y}} = (b'_{y_1}, b'_{y_2})$ if $b_{y_1} \leqslant b'_{y_1}$ and $b_{y_2} \leqslant b'_{y_2}$. Let us fix the cocharacters $\mu_1 = (0, \dots, 0, -1)$ and $\mu_2 = (1, 0, \dots, 0)$ of GL_r.

Theorem 2.2.6. *The Newton stratification on* $\mathrm{Sht}_{r,\mathbb{F}_0}$ *satisfies the strong stratification property. In other words, for all* $\underline{b} \in B(\mathrm{GL}_{r,\infty}, \mu_1) \times B(\mathrm{GL}_{r,0}, \mu_2)$ *we have*

$$\overline{\mathrm{Sht}_{r,\underline{b}}} = \bigcup_{\underline{b}' \leqslant \underline{b}} \mathrm{Sht}_{r,\underline{b}'} = \mathrm{Sht}_{r,\leqslant \underline{b}}\,.$$

Moreover, all the Newton strata $\mathrm{Sht}_{r,\underline{b}}$ *for* $\underline{b} \in B(\mathrm{GL}_{r,\infty}, \mu_1) \times B(\mathrm{GL}_{r,0}, \mu_2)$ *are non-empty.*

Viehmann in [Vie20, Remark 5.6.] remarks that the assertion should follow by a similar argument as in [Ham15] for Shimura varieties of PEL type.

Proof. Let $\underline{b}^0$ correspond to $((1/r, \ldots, 1/r), (-1/r, \ldots, -1/r))$, which is the unique basic point in $B(\mathrm{GL}_{r,K_\infty}, \mu_1) \times B(\mathrm{GL}_{r,K_0}, \mu_2)$. By Proposition 2.2.5 the Newton stratum $\mathrm{Sht}_{r,\underline{b}^0}$ is non-empty. The non-emptiness of the other strata will follow from the closure relations.

Now, let $\underline{b} \in B(\mathrm{GL}_{r,K_\infty}) \times B(\mathrm{GL}_{r,K_0})$ and assume that $\mathrm{Sht}_{r,\underline{b}}$ is non-empty. We fix a point $s \in \mathrm{Sht}_{r,\underline{b}}$ and let R be its universal deformation ring. Then s lies in the closure of some $\mathrm{Sht}_{r,\underline{b}'}$ for $\underline{b}' \leqslant \underline{b}$ if and only if the same is true in the Newton stratification on $\mathrm{Spec}\, R$. By the Serre-Tate Theorem (Proposition 2.1.11) the universal deformation ring factors as $\mathrm{Spec}\, R = \mathrm{Spec}\, R_\infty \times \mathrm{Spec}\, R_0$, where R_* is the universal deformation ring of the corresponding local shtuka at $* = \infty, 0$. Under this isomorphism we have $\mathrm{Spec}(R)_{\underline{b}} = \mathrm{Spec}(R_1)_{b_\infty} \times \mathrm{Spec}(R_2)_{b_0}$, where we denote by $\mathrm{Spec}(R_*)_{b_*}$ the corresponding Newton strata in $\mathrm{Spec}\, R_*$ for $* = \infty, 0$. On $\mathrm{Spec}\, R_*$ the closure properties hold by [Vie13, Theorem 2, Lemma 21 (2)], and thus they hold on $\mathrm{Spec}\, R$. This proves the assertion. □

Remark 2.2.7. In a similar fashion, there is a Newton stratification on the moduli space of Drinfeld modules in characteristic $\mathfrak{p}$ defined via the local shtukas as defined in Remark 2.2.4 (3). The Newton stratifications are clearly compatible with the projection $\mathrm{Sht}_{r,b_\infty,\mathbb{F}_0} \to \text{Dr-Mod}_{r,\mathbb{F}_0}$ in the fibre over 0 from Remark 2.2.4 (2). Thus, the Newton stratification on Drinfeld modules retains the strong stratification property as in the theorem above.

Corollary 2.2.8. *Let* $\underline{\mathcal{E}}$ *be a Drinfeld shtuka over a complete local noetherian ring* R *with algebraically closed residue field* ℓ *such that the characteristic of* $\underline{\mathcal{E}}_\ell$ *factors through* 0*. Then there exists a Drinfeld module* $\mathbf{E}$ *over* R *such that the local shtukas at* 0 *of* $\underline{\mathcal{E}}$ *and* $\mathbf{E}$ *are isomorphic.*

Proof. The case where $R = \ell$ is a field directly follows from the non-emptiness of Newton strata of Theorem 2.2.6 and Remark 2.2.7. The case that R is local artinian then follows from the Serre-Tate Theorem 2.1.11, and the general case that R is a complete noetherian local ring with algebraically closed residue field follows from the fact that Dr-Mod_{r} is of finite type over $\mathbb{F}_q$ and [HV11, Proposition 3.16]. □

2.2.3. The $\mathfrak{p}^n$-torsion scheme of a Drinfeld shtuka.

We briefly explain how to construct a scheme of $\mathfrak{p}^n$-torsion points of a shtuka, which will play the role of the p^n-torsion points of an elliptic curve. The construction goes back to [Dri87b]. Let $\underline{\mathcal{F}}$ be a φ-sheaf (for example a $\mathfrak{p}^n$-torsion shtuka) over S. We set

$$\mathrm{Dr}_q(\underline{\mathcal{F}}) = \underline{\mathrm{Spec}}\left(\mathrm{Sym}^\bullet \mathcal{F}\right)/\mathcal{I},$$

where $\mathcal{I}$ is the ideal locally generated by the sections $v^{\otimes q} - \varphi(\sigma^* v)$. It induces a contravariant functor from the category of φ-sheaves to the category of finite locally free group schemes with $\mathbb{F}_q$-action over S. Assume $S = \mathrm{Spec}\, R$ is affine, $\mathcal{F} = R^r$ is trivial and φ is given by the Matrix (a_{ij}). Then

$$\mathrm{Dr}_q(\underline{\mathcal{F}}) = \mathrm{Spec}\left(R[Y_1, \ldots, Y_r]\Big/\left(Y_1^q - \sum_{i=1}^r a_{i1} Y_i, \ldots, Y_r^q - \sum_{i=1}^r a_{ir} Y_i\right)\right).$$

Proposition 2.2.9 ([Dri87b, Proposition 2.1], [Abr06, Theorem 2] and [HS19, Theorem 5.2])**.** *Let $\underline{\mathcal{F}} = (\mathcal{F}, \varphi)$ be a finite shtuka of rank r on S. Then the group scheme $\mathrm{Dr}_q(\underline{\mathcal{F}})$ is finite locally free of rank q^r over S, étale over S if and only if φ is an isomorphism, and radicial over S if and only if φ is locally nilpotent on S. Moreover, the functor Dr_q is $\mathbb{F}_q$-linear and exact. Its essential image is characterised by the property that the $\mathbb{F}_q$-action is strict in the sense of [Fal02].*

Note that the notion of a strict $\mathbb{F}_q$-action is a condition on the $\mathbb{F}_q$ action on the co-Lie complex of a certain deformation of the group scheme. We do not need the exact definition here and refer to [Fal02] or [HS19] for more details. In our setting the strictness of the $\mathbb{F}_q$-action will usually be automatic.

Definition 2.2.10. Let $\underline{\mathcal{E}}$ be a rank r Drinfeld shtuka over S. We denote by

$$\underline{\mathcal{E}}[\mathfrak{p}^n] = \mathrm{Dr}_q(\underline{\mathcal{E}}|_{D_{n,S}})$$

the *scheme of* $\mathfrak{p}^n$*-division points* of $\underline{\mathcal{E}}$.

The previous proposition implies that $\underline{\mathcal{E}}[\mathfrak{p}^n]$ is a finite locally free S-group scheme of rank q^{nr} with strict $\mathbb{F}_q$-action. The $\mathcal{O}_0/\mathfrak{p}^n$-module structure on $\underline{\mathcal{E}}|_{D_{n,S}}$ gives rise to a canonical $\mathcal{O}_0/\mathfrak{p}^n$-module structure on $\underline{\mathcal{E}}[\mathfrak{p}^n]$. The finite shtuka equivalence in particular induces an equivalence of quotients of $\underline{\mathcal{E}}|_{D_{n,S}}$ as $\mathfrak{p}^n$-torsion shtukas and finite locally free closed $\mathcal{O}/\mathfrak{p}^n$-module subschemes with strict $\mathbb{F}_q$-action of $\underline{\mathcal{E}}[\mathfrak{p}^n]$.

By comparison with Drinfeld modules, we get the following explicit description of the $\mathfrak{p}^n$-torsion in characteristic $\mathfrak{p}$.

Corollary 2.2.11. *Let $\underline{\mathcal{E}}$ be a Drinfeld shtuka over a complete local noetherian ring R with algebraically closed residue field ℓ such that the characteristic of $\underline{\mathcal{E}}_\ell$ factors through 0. Then there exists a Drinfeld module $\mathbf{E}$ over R such that*

$$\underline{\mathcal{E}}[\mathfrak{p}^n] \cong \mathbf{E}[\mathfrak{p}^n]$$

as $\mathcal{O}_0/\mathfrak{p}^n$-module schemes over R for all $n \in \mathbb{N}$.

Proof. This follows directly from the corresponding assertion for the local shtukas in Corollary 2.2.8. □

Proposition 2.2.12. *The scheme of $\mathfrak{p}^n$-division points of a shtuka $\underline{\mathcal{E}}$ of rank r over an algebraically closed ℓ is given by the $\mathcal{O}_0/\mathfrak{p}^n$-module scheme*

$$\underline{\mathcal{E}}[\mathfrak{p}^n] = \alpha_{q^h} \times (\mathfrak{p}^{-n}/\mathcal{O}_0)^{r-h},$$

where the operation of ϖ on α_{q^h} is given by $t \mapsto t^{q^h}$, and where h is the height of $\underline{\mathcal{E}}$ (we use the convention $h = 0$ when the characteristic of $\underline{\mathcal{E}}$ is away from 0).

Proof. We first consider the case that $\mathfrak{p}$ is away from the characteristic of $\underline{\mathcal{E}}$. Then, $\underline{\mathcal{E}}[\mathfrak{p}^n]$ is a finite étale scheme by the finite shtuka equivalence. It follows that étale locally on S the $\mathcal{O}_0/\mathfrak{p}^n$-module scheme $\underline{\mathcal{E}}[\mathfrak{p}^n]$ is constant. Over geometric points, we have that $\underline{\mathcal{E}}[\mathfrak{p}^n] \cong (\mathfrak{p}^{-n}/\mathcal{O}_0)^r$ as the corresponding étale local shtuka is trivial by [AH14, Corollary 2.9]. In characteristic $\mathfrak{p}$, by the previous Corollary 2.2.11 it suffices to check the assertion for Drinfeld modules, which then follows from [Leh09, 3, Proposition 1.5], [Leh09, 3, Proposition 1.5] and [Leh09, 2, Corollary 2.4.]. □

Even more generally, we can embed the scheme of $\mathfrak{p}$-torsion points as a closed subscheme of a smooth curve. However, this smooth curve will not be a Drinfeld module in general.

Proposition 2.2.13. *Let $\underline{\mathcal{E}}$ be a Drinfeld shtuka over S. Then étale-locally on S, the scheme of $\mathfrak{p}^n$-division points of $\underline{\mathcal{E}}$ can be embedded as a closed subscheme of a smooth curve over S. More precisely, we can étale-locally on S embed $\underline{\mathcal{E}}[\mathfrak{p}^n]$ as a closed subscheme of $\mathbb{A}^1_S$.*

Remark 2.2.14. For one-dimensional p-divisible groups a similar statement is discussed in [Fri19, Lemma 5.2.1], building on arguments from [HT01]. However, [Fri19] claims that an embedding even exists Zariski-locally on S, this seems to be problematic to us for the following reason. Let us assume that S is the spectrum of a finite field. We assume that the étale part of $\underline{\mathcal{E}}[\mathfrak{p}^n]$ is non-trivial and constant over S. Then the number of rational points of $\underline{\mathcal{E}}[\mathfrak{p}^n]$ tends to infinity as $n \to \infty$. However, the number of S-rational points on $\mathbb{A}^1_S$ is bounded. In particular, it cannot be possible to embed $\underline{\mathcal{E}}[\mathfrak{p}^n]$ into $\mathbb{A}^1_S$ for all $n \in \mathbb{N}$.

For the proof of the proposition, we essentially adapt the proof of [Fri19, Lemma 5.2.1], but we allow finite extensions on the residue fields in order to circumvent the issue discussed above. So we only get an étale local statement.

Proof. We adapt the proof of [Fri19, Lemma 5.2.1]. We first consider the case that $S = \operatorname{Spec} \ell$ is the spectrum of an algebraically closed field. In this case, the assertion follows from the explicit description of $\underline{\mathcal{E}}[\mathfrak{p}^n]$ in Proposition 2.2.12.

For the general case, we may by reduction to the universal case assume that S is locally of finite type over $\mathbb{F}_q$. As the statement is local on S, we may further assume that $S = \operatorname{Spec}(R)$ is affine and of finite type over $\mathbb{F}_q$. Then $\underline{\mathcal{E}}[\mathfrak{p}^n] = \operatorname{Spec}(B)$ is affine as well. We fix a closed point $s \in S$. By the argument above, there exists a finite extension $\mathbb{F}$ of the residue field $k(s)$ (which is finite by assumption) at s such that there exists a closed immersion $\underline{\mathcal{E}}[\mathfrak{p}^n]_{\mathbb{F}} \hookrightarrow \mathbb{A}^1_{\mathbb{F}}$ over $\mathbb{F}$, in other words a surjection $\mathbb{F}[t] \twoheadrightarrow B \otimes \mathbb{F}$. By [Stacks, Tag 00UD] there exists an étale neighbourhood $\operatorname{Spec} R' \to \operatorname{Spec} R$ of s and a point s' over s such that the extension of residue fields $k(s) \to k(s')$ is given by $k(s) \to \mathbb{F}$. We can thus lift the surjection to a map $R'[t] \to B \otimes R'$ by choosing a lift of the image of t. By Nakayma's lemma this is a surjection over some R'_f, where $f \in R'$ is not contained in the maximal ideal at s'. In other words, $\underline{\mathcal{E}}[\mathfrak{p}^n]_{\mathbb{F}} \hookrightarrow \mathbb{A}^1_{\mathbb{F}}$ extends to $\underline{\mathcal{E}}[\mathfrak{p}^n]_{R'_f} \hookrightarrow \mathbb{A}^1_{R'_f}$ over the étale neighbourhood $\operatorname{Spec}(R'_f)$ of s. □

We conclude this section by collecting some consequences on isogenies of Drinfeld shtukas. Using the finite shtuka equivalence we see that a chain of $\mathfrak{p}^n$-isogenies of type $(r_1, \ldots, r_m)$ on a Drinfeld shtuka $\underline{\mathcal{E}}$ is equivalent to the data of a flag

$$0 \subseteq \mathbf{H}_1 \subseteq \mathbf{H}_2 \subseteq \ldots \subseteq \mathbf{H}_m \subseteq \underline{\mathcal{E}}_0[\mathfrak{p}^n]$$

of finite locally free submodule schemes $\mathbf{H}_i \subseteq \underline{\mathcal{E}}[\mathfrak{p}^n]$ of rank $q^{n\cdot(r_1+\ldots+r_i)}$ over S with strict $\mathbb{F}_q$-action. In particular, $\mathbf{H}_i/\mathbf{H}_{i-1}$ has rank q^{nr_i} and has an induced strict $\mathbb{F}_q$-action.

Proposition 2.2.15. *The stack* $\operatorname{Sht}_{r,(r_1,\ldots,r_m)-\mathfrak{p}^n\text{-chain}}$ *is a Deligne-Mumford stack locally of finite type over* $\mathbb{F}_q$. *The forgetful map to* Sht_r *given by projection to* $\underline{\mathcal{E}}$ *is schematic and finite. Moreover, the forgetful map* $\operatorname{Sht}_{r,(r_1,\ldots,r_m)-\mathfrak{p}^n\text{-chain}} \to \operatorname{Sht}_r$ *is finite étale away from* 0.

Proof. Let $\underline{\mathcal{E}} \in \operatorname{Sht}_r(S)$. The functor on S-schemes

$$T \mapsto \left\{ \begin{array}{l} \text{flags of quotients } \underline{\mathcal{E}}|_{D_{n,T}} \twoheadrightarrow \underline{\mathcal{F}}_1 \twoheadrightarrow \ldots \twoheadrightarrow \underline{\mathcal{F}}_m \twoheadrightarrow 0 \text{ of } \mathfrak{p}^n\text{-torsion} \\ \text{finite shtukas such that } \mathcal{F}_i \text{ has rank } n(r_1+\ldots+r_i) \text{ as } \mathcal{O}_T\text{-module} \end{array} \right\}$$

is representable by the closed subscheme of a certain flag variety of quotients of $\underline{\mathcal{E}}|_{D_{n,S}}$ (as $\mathcal{O}_S$-module) where both the map $\sigma^*\underline{\mathcal{E}}|_{D_{n,S}} \to \underline{\mathcal{E}}|_{D_{n,S}}$ and the $\mathcal{O}_0/\mathfrak{p}^n$-module structure descend to all the $\mathcal{F}_i$. As the flag variety is projective, we see that $\mathrm{Sht}_{r,(r_1,\dots,r_m)-\mathfrak{p}^n\text{-chain}} \to \mathrm{Sht}_r$ is schematic and projective.

In order to show finiteness of the map we proceed as in the proof of [KM85, Proposition 6.5.1]. By [EGA4, Corollaire 18.12.4] it suffices to show that the map has finite fibres. Let ℓ be an algebraically closed field and let $\underline{\mathcal{E}}$ be a rank r shtuka over ℓ. It suffices to show that $\underline{\mathcal{E}}[\mathfrak{p}^n]$ only has finitely many submodule schemes. We know by Proposition 2.2.12 that for some $h \geqslant 0$ we have

$$\underline{\mathcal{E}}[\mathfrak{p}^n] \cong \alpha_{q^h} \times \left(\mathfrak{p}^{-n}/\mathcal{O}_0\right)^{r-h}.$$

As ℓ is in particular perfect, any $\mathcal{O}_0/\mathfrak{p}^n$-submodule scheme $\mathbf{H} \subseteq \underline{\mathcal{E}}[\mathfrak{p}^n]$ factors as $\mathbf{H} \cong \mathbf{H}^{\mathrm{conn}} \times \mathbf{H}^{\text{ét}}$ but for both factors (which are necessarily submodule schemes of α_{q^h} and $(\mathfrak{p}^{-n}/\mathcal{O}_0)^{r-h}$, respectively) there are only finitely many possibilities.

The étaleness away from 0 follows for example from [Var04, Lemma 3.3 a)]. $\square$

2.3. Drinfeld $\Gamma_1(\mathfrak{p}^n)$-level structures on shtukas

In this section, we introduce Γ_1-type (Drinfeld-) level structures on Drinfeld shtukas adapting similar constructions for Drinfeld modules and elliptic curves. We show that the moduli space of Drinfeld shtukas with these level structures is regular following the arguments of [KM85]. For Drinfeld modules, full Drinfeld level structures were studied extensively starting with [Dri76], compare for example also [Leh09]. For other kinds of level structures some results are known, [Sha07] considers $\Gamma_1(\mathfrak{p})$-level structures in the rank 2 case and [KY20] study level structures for arbitrary torsion modules and higher rank Drinfeld modules.

We propose a slightly different generalisation of a M-level structure on Drinfeld shtukas for a $\mathfrak{p}$-torsion $\mathcal{O}_0$-module M. In this notation $(\mathfrak{p}^{-n}/\mathcal{O}_0)^r$-structures are full level structures and in the rank 2 case $(\mathfrak{p}^{-n}/\mathcal{O}_0)$-strucutres are $\Gamma_1(\mathfrak{p}^n)$-level structures. For us, it does not seem to be a priori clear that our definition and the analogue of [KY20] agree, even for full level, as is claimed in [KY20, (4.1.2.)]. For full level structures on Drinfeld modules, this follows from a deep result on the deformation theory of [Leh09, Proposition 3.3]. We show that the two definitions agree in general in a similar fashion. One could also directly adapt the definition of [KM85], as does for example [Tae06]. However, it seems to us that this definition does not give the correct moduli space, see Remark 2.3.4.

Moreover, we define analogues of balanced level structures of [KM85] and use this notion of balanced level structure to give a definition of $\Gamma_1(\mathfrak{p}^n)$-level structure for Drinfeld shtukas of arbitrary rank and arbitrary $n \in \mathbb{N}$ in Definition 2.3.15.

2.3.1. M-Structures on Drinfeld shtukas

In order to define Drinfeld level structures we use the notion of *full sets of sections*, compare [KM85, Section 1.8]. When working with closed subschemes of a smooth curve, this can be expressed in terms of Cartier divisors by [KM85, Theorem 1.10.1]. Katz and Mazur hoped that the notion of "full sets of sections" might be useful to define level structures for higher dimensional abelian varietes. However, this notion gives rise to a moduli problem which is not even flat over $\mathbb{Z}$ in general (compare [CN90]). Nevertheless, these issues do not appear in our setting, as Proposition 2.2.13 allows us to locally work with Cartier divisors in $\mathbb{A}^1$. Note that in a similar fashion Drinfeld level structures are well-behaved when working with one-dimensional p-divisible groups, as do [HT01] and [Sch13].

Let M be a finitely generated $\mathfrak{p}^n$-torsion $\mathcal{O}_0$-module. In order to define M-structures on Drinfeld shtukas, we would like for an $\mathcal{O}_0$-module homomorphism $\iota\colon M \to \underline{\mathcal{E}}[\mathfrak{p}^n](S)$ to induce a (unique) scheme *generated* by ι, similar to the Cartier divisor generated by an $\Gamma_1(p^n)$-Drinfeld level structure on elliptic curves. In other words, we are looking for a unique finite locally free scheme over S such that the image of ι forms a full set of sections for S in the sense of [KM85, Section 1.8]. This notion is defined as follows. Let Z be a finite locally free S-scheme of rank N. A set of sections $P_1, \dots, P_N \in Z(S)$ is called *full set of sections* of Z if for every affine S-scheme $\mathrm{Spec}(R) \to S$ and every $f \in \Gamma(Z_R, \mathcal{O}_{Z_R})$ we have $\mathrm{Norm}(f) = \prod_{i=1}^N f(P_i)$. By [KM85, Theorem 1.10.1], when $Z \hookrightarrow C$ is embedded as a relative effective Cartier divisor in a smooth curve C over S, the set $P_1, \dots, P_N \in Z(S)$ is a full set of sections of Z if and only if $Z = \sum_{i=1}^N [P_i]$ as Cartier divisors in C.

Recall that in general, that given a set of sections $P_1, \dots, P_{N'} \in Z(S)$ we can neither expect that a finite locally free subscheme Z' of Z of rank N' such that $P_1, \dots, P_{N'}$ forms a full set of sections of Z' exists nor that it is unique when it exists (compare [KM85, Remark 1.10.4]). However, Proposition 2.2.13 allows us to construct such a unique scheme in the cases we are interested in.

Lemma 2.3.1. *Let $\underline{\mathcal{E}}$ be a Drinfeld shtuka over S and let M be a $\mathfrak{p}^n$-torsion module. Let $\iota\colon M \to \underline{\mathcal{E}}[\mathfrak{p}^n](S)$ be an $\mathcal{O}_0$-linear map.*

(1) *Assume there exists a closed immersion $\underline{\mathcal{E}}[\mathfrak{p}^n] \hookrightarrow C$ into a smooth curve C over S. Then there exists a unique finite locally free closed subscheme $\mathbf{H}$ of C such that the image of ι (in $C(S)$) forms a full set of sections for $\mathbf{H}$.*

(2) *There exists at most one finite locally free closed subscheme $\mathbf{H} \subseteq \underline{\mathcal{E}}[\mathfrak{p}^n]$ such that the image of ι forms a full set of sections for $\mathbf{H}$.*

(3) *There exists a (by the previous point necessarily unique) finite locally free closed subscheme $\mathbf{H}$ of $\underline{\mathcal{E}}[\mathfrak{p}^n]$ such that ι gives a full set of sections for $\mathbf{H}$ if and only if the*

following equivalent conditions are satisfied:

a) *For all étale maps $U \to S$ and all closed immersions $\underline{\mathcal{E}}[\mathfrak{p}^n]_U \hookrightarrow C$ into a smooth curve over U, the Cartier divisor defined by the image of ι in $C(U)$ is a subscheme of $\underline{\mathcal{E}}[\mathfrak{p}^n]_U$.*

b) *There exists an étale cover $\{U_i\}_{i \in I}$ of S and for each $i \in I$ a smooth curve C_i over U_i together with a closed immersion $\underline{\mathcal{E}}[\mathfrak{p}^n]_{U_i} \hookrightarrow C_i$ such that the Cartier divisor defined by the image of ι in $C_i(U_i)$ is a subscheme of $\underline{\mathcal{E}}[\mathfrak{p}^n]_{U_i}$ for all $i \in I$.*

The existence of such an $\mathbf{H}$ *is a closed condition on S, defined locally on S by finitely many equations.*

Proof. (1) This is [KM85, Theorem 1.10.1]. The scheme $\mathbf{H}$ is the Cartier divisor $\sum_{\alpha \in M} [\iota(\alpha)]$.

(2) This is clear from the previous point, as étale-locally on S, $\underline{\mathcal{E}}[\mathfrak{p}^n]$ admits an embedding into a smooth curve over S by Proposition 2.2.13.

(3) It is clear that the existence of an $\mathbf{H}$ implies condition (a), and that condition (a) implies (b) using Proposition 2.2.13. Let us now assume that condition (b) is satisfied. We denote by $\mathbf{H}_i$ the Cartier divisor in C_i defined by ι. We can glue the $\mathbf{H}_i$ to form a finite locally free scheme $\mathbf{H}$ over S by the uniqueness in the previous point. It is clear that ι forms a full set of sections for $\mathbf{H}$, this can be checked étale-locally on S.

In order to check that the locus of existence of $\mathbf{H}$ is closed in S, we may choose an étale cover $\{U_i\}$ of S together with embeddings of $\underline{\mathcal{E}}[\mathfrak{p}^n]$ into a smooth curve over U_i. The assertion on U_i follows from [KM85, Key Lemma 1.3.4].

□

We can now give our definition of M-structures for shtukas.

Definition 2.3.2. Let $\underline{\mathcal{E}} \in \mathrm{Sht}_r(S)$ be a rank r shtuka over S. Let M be a finitely generated $\mathcal{O}_0/\mathfrak{p}^n$-module. A *$M$-structure* on $\underline{\mathcal{E}}$ is an $\mathcal{O}_0$-module homomorphism $\iota\colon M \to \underline{\mathcal{E}}[\mathfrak{p}^n](S)$ such that there exists a finite locally free subscheme $\mathbf{H}$ of $\underline{\mathcal{E}}[\mathfrak{p}]$ of rank $|M[\mathfrak{p}]|$ such that the image of the restriction $\iota|_{M[\mathfrak{p}]}$ of ι to the $\mathfrak{p}$-torsion forms a full set of sections for $\mathbf{H}$ in the sense of [KM85, Section 1.8].

Remark 2.3.3. Note that in the theory of Drinfeld modules the modules M and $\underline{\mathcal{E}}[\mathfrak{p}]$ would usually be considered as $\mathfrak{p}$-torsion A-modules, where $A = \Gamma(X \backslash \{\infty\}, \mathcal{O}_X)$. As $A/\mathfrak{p}^n \cong \mathcal{O}_0/\mathfrak{p}^n$ this does not give a different notion of level structures. In our context, working with $\mathcal{O}_0$- instead of A-modules seems more natural and should stress that the level structure only depends on local data of $\underline{\mathcal{E}}$ at 0 and in particular not on the choice of ∞.

Remark 2.3.4. Our definition is an analogue of Drinfeld's original definition of full level structures in [Dri76]. The definition in [KM85] for elliptic curves is slightly different. The direct analogue of their definition asserts that there is a finite locally free subgroup scheme $\mathbf{H} \subseteq \underline{\mathcal{E}}[\mathfrak{p}^n]$ of rank $|M|$ such that ι is a full set of sections for $\mathbf{H}$ (instead of the corresponding assertion only for the $\mathfrak{p}$-torsion). We show that this is implied by our definition in Proposition 2.3.10. For full level structures on Drinfeld modules, [Leh09, 3, Proposition 3.1] and [Wie10] show that the two notions are equivalent. However, this is not true in general as we can see in the following example. Consider $X = \mathbb{P}^1_{\mathbb{F}_q}$, and $S = D_2 = 2[0] = \mathrm{Spec}(\mathbb{F}_q[\zeta]/(\zeta^2))$ viewed as an X-scheme via the canonical inclusion. Then the map

$$\varphi\colon \sigma^*\mathcal{O}_{X_S}^2 \xrightarrow{\left(\begin{smallmatrix} 0 & \varpi-\zeta \\ 1 & -\zeta \end{smallmatrix}\right)} \mathcal{O}_{X_S}^2$$

defines a rank 2 Drinfeld shtuka over S. Its schemes of $\mathfrak{p}$- and $\mathfrak{p}^2$-torsion points are given by

$$\underline{\mathcal{E}}[\mathfrak{p}] = \mathrm{Spec}\left(R[t]/(t^{q^2} + \zeta t^q + \zeta t)\right) \qquad \text{and} \qquad \underline{\mathcal{E}}[\mathfrak{p}^2] = \mathrm{Spec}\left(R[t]/(t^{q^4} + \zeta t^{q^3} + \zeta t^{q^2})\right),$$

respectively. Then the constant zero map $\iota\colon \mathfrak{p}^{-2}/\mathcal{O}_0 \to R, \varpi^{-2} \mapsto 0$ defines the subscheme of $\mathrm{Spec}(R[t]/(t^{q^2})) \subseteq \underline{\mathcal{E}}[\mathfrak{p}^2]$. However, the restriction of ι to $\mathfrak{p}^{-1}/\mathcal{O}_0$ induces the subscheme $\mathrm{Spec}(R[t]/(t^{q^2}))$ of $\underline{\mathcal{E}}[\mathfrak{p}^2]$, which is not a subscheme of $\underline{\mathcal{E}}[\mathfrak{p}]$. Thus, ι is not an $\mathfrak{p}^{-2}/\mathcal{O}_0$-structure in the sense of our definition. Hence, the definition of [KM85] does not yield well-defined level maps in our setting and thus does not seem to be adequate here.

We also do not require the subscheme defined by ι to be a subgroup scheme as we show this is already automatic below in Proposition 2.3.10. Moreover, we show that it is even automatically an $\mathcal{O}_0$-module subscheme.

Proposition 2.3.5. *Let $\underline{\mathcal{E}} \in \mathrm{Sht}_r(S)$ be a Drinfeld shtuka over S and let M be a finitely generated $\mathcal{O}_0/\mathfrak{p}^n$-module. Let $\iota\colon M \to \underline{\mathcal{E}}[\mathfrak{p}^n](S)$ be a M-structure on $\underline{\mathcal{E}}$ and let $M' \subseteq M$ be a submodule. The restriction of ι to M' defines a M'-structure on $\underline{\mathcal{E}}$.*

Proof. Étale locally, the Cartier divisor defined by the restriction of ι to $M'[\mathfrak{p}]$ is a closed subscheme of the Cartier divisor defined by $\iota|_{M[\mathfrak{p}]}$, which in turn is a closed subscheme of $\underline{\mathcal{E}}[\mathfrak{p}]$ by assumption. The assertion follows from Lemma 2.3.1. □

In the étale case, we have the following descriptions of M-structures.

Proposition 2.3.6. *Let $\underline{\mathcal{E}} \in \mathrm{Sht}_r(S)$ be a Drinfeld shtuka over S and let M be a finitely generated $\mathcal{O}_0/\mathfrak{p}^n$-module. Let $\iota\colon M \to \underline{\mathcal{E}}[\mathfrak{p}^n](S)$ be an $\mathcal{O}_0$-module homomorphism. The following are equivalent:*

(1) For every geometric point $\operatorname{Spec}\ell \to S$, the induced homomorphism

$$\iota_\ell\colon M \to \underline{\mathcal{E}}[\mathfrak{p}^n](\ell)$$

is injective.

(2) The map ι defines a locally free closed subscheme of $\underline{\mathcal{E}}[\mathfrak{p}^n]$ which is finite étale over S.

(3) The map ι defines a closed immersion of $\mathcal{O}_0/\mathfrak{p}^n$-module S-schemes

$$M_S \hookrightarrow \underline{\mathcal{E}}[\mathfrak{p}^n]$$

and ι is a full set of sections for the image of this map (as subscheme of $\underline{\mathcal{E}}[\mathfrak{p}^n]$).

If the equivalent conditions (1)-(3) are satisfied, ι is a M-structure on $\underline{\mathcal{E}}$. Moreover, when S is connected, these conditions are equivalent to saying that $M \to \mathbf{H}(S)$ is an isomorphism of $\mathcal{O}_0$-modules for some constant closed finite locally free $\mathcal{O}_0$-module subscheme $\mathbf{H}$ of $\underline{\mathcal{E}}$.

Moreover, these conditions are automatically satisfied when the characteristic of $\underline{\mathcal{E}}$ is away from 0 and ι is a M-structure on $\underline{\mathcal{E}}$.

Proof. We adapt the proof of an analogous assertion for elliptic curves in [KM85, Lemma 1.4.4].
(2) $\Leftrightarrow$ (3): This follows directly from the set-theoretic analogue in [KM85, Proposition 1.8.3].
(1) $\Leftrightarrow$ (3): The map ι defines a map of $\mathcal{O}_0$-module schemes $M_S \to \underline{\mathcal{E}}[\mathfrak{p}^n]$. We may work étale-locally on S and assume that we can embed $\underline{\mathcal{E}}[\mathfrak{p}^n]$ into a smooth curve C over S. Let us denote by D the Cartier divisor in C defined by ι. We can check that the natural map $M_S \to D$ is an isomorphism on geometric points as in the proof of [KM85, Lemma 1.4.4.], and this is clearly satisfied if and only if ι is injective on geometric points.

Let us now assume that (1)-(3) are satisfied. By (3) the restriction of ι to $M[\mathfrak{p}]$ defines a subscheme of $\underline{\mathcal{E}}[\mathfrak{p}]$. Thus, ι is a M-structure on $\underline{\mathcal{E}}$.

Now assume that $\mathfrak{p}$ is away from the characteristic. Let ι be a M-structure on $\underline{\mathcal{E}}$. We check that condition (1) is satisfied. Let $\operatorname{Spec}\ell \to S$ be a geometric point of S. By Proposition 2.2.12, we have an $\mathcal{O}_0$-linear isomorphism $\underline{\mathcal{E}}[\mathfrak{p}^n](\ell) \cong (\mathfrak{p}^{-n}/\mathcal{O}_0)^r$. Now ι_ℓ is injective if and only if the restriction $\iota_\ell|_{M[\mathfrak{p}]}$ is injective, as multiplying a non-trivial element m in the kernel of ι_ℓ by the maximal power of ϖ that does not kill m produces a non-trivial element in the kernel of $\iota_\ell|_{M[\mathfrak{p}]}$. The injectivity of $\iota_\ell|_{M[\mathfrak{p}]}$ follows by assumption and the implication (2) $\Rightarrow$ (1) in the $n = 1$ case. $\square$

Remark 2.3.7. If the characteristic is away from 0, Drinfeld level structures agree with corresponding classical level structures. Let $\underline{\mathcal{E}} \in \mathrm{Sht}_r(S)$ be such that the characteristic is away from 0. By the previous Proposition 2.3.6, a full level structure is an isomorphism of étale $\mathcal{O}_0/\mathfrak{p}^n$-module schemes over S

$$\left(\mathfrak{p}^{-n}/\mathcal{O}_0\right)_S^r \xrightarrow{\cong} \underline{\mathcal{E}}[\mathfrak{p}^n]$$

by. By [Dri87b, Proposition 2.2], this is the same as giving a trivialisation of $\underline{\mathcal{E}}|_{D_{n,S}}$.

2.3.2. Regularity of the moduli stack of shtukas with M-structures

We show the main result on M-structures: the corresponding moduli problem gives rise to a Deligne-Mumford stack, which we show to be regular following the corresponding result on elliptic curves in [KM85].

Proposition 2.3.8. *Let $\underline{\mathcal{E}}$ be a Drinfeld shtuka over S and let M be a finitely generated $\mathcal{O}_0/\mathfrak{p}^n$-module. The functor on S-schemes*

$$T \mapsto \{M\text{-structures on } \underline{\mathcal{E}}_T\}$$

is representable by a finite S-scheme locally of finite presentation. Moreover, it is finite étale over S if $\mathfrak{p}$ is away from the characteristic of $\underline{\mathcal{E}}$ and can in this case étale locally on S be represented by the constant S-scheme

$$S \times \{\text{injective } \mathcal{O}_0\text{-module homomorphisms } M \hookrightarrow (\mathfrak{p}^{-n}/\mathcal{O}_0)^r\}.$$

Proof. We proceed as in the proof of the corresponding assertions for elliptic curves in [KM85, Proposition 1.6.2, Proposition 1.6.4 and Corollary 3.7.2]. By the classification of finitely generated modules over principal ideal domains, there exists an isomorphism of $\mathcal{O}_0/\mathfrak{p}^n$-modules $M \cong (\mathcal{O}_0/\mathfrak{p}^{n_1}) \oplus \ldots \oplus (\mathcal{O}_0/\mathfrak{p}^{n_m})$ for some $m \geqslant 0$ and integers $1 \leqslant n_i \leqslant n$ for $1 \leqslant i \leqslant m$. Using this isomorphism, we find for a scheme T over S that

$$\mathrm{Hom}_{\mathcal{O}_0}(M, \underline{\mathcal{E}}[\mathfrak{p}^n](T)) = \prod_i \mathrm{Hom}_{\mathcal{O}_0/\mathfrak{p}^{n_i}}(\mathcal{O}_0/\mathfrak{p}^{n_i}, \underline{\mathcal{E}}[\mathfrak{p}^{n_i}](T)) = \prod_i \underline{\mathcal{E}}[\mathfrak{p}^{n_i}](T).$$

The functor of M-structures on $\underline{\mathcal{E}}$ is clearly represented by the closed subscheme of $\prod_i \underline{\mathcal{E}}[\mathfrak{p}^{n_i}](T)$ over which the universal homomorphism defines a M-structure on $\underline{\mathcal{E}}$. This is a closed subscheme defined locally by finitely many equations by Lemma 2.3.1.

Now assume that the characteristic of $\underline{\mathcal{E}}$ is away from 0. By the above, it suffices to show that the scheme is formally étale. Therefore, let $T_0 \subseteq T$ be a closed subscheme defined by a locally nilpotent sheaf of ideals. Let ι_0 be a M-structure on $\underline{\mathcal{E}}_{T_0}$. Then ι_0 factors through $\underline{\mathcal{E}}[\mathfrak{p}^n](T_0)$. As $\mathfrak{p}$ is away from the characteristic of $\underline{\mathcal{E}}$, the scheme $\underline{\mathcal{E}}[\mathfrak{p}^n]$ is étale over S. We construct a map $\iota\colon M \to \underline{\mathcal{E}}[\mathfrak{p}^n](T)$ where we associate to $m \in M$ the unique lift of $\iota_0(m)$ to $\underline{\mathcal{E}}[\mathfrak{p}^n](T)$. As lifts are unique, the $\mathcal{O}_0$-linearity of ι_0 implies that ι is an $\mathcal{O}_0$-linear map as well. We check that ι defines a M-structure on $\underline{\mathcal{E}}$. Using Proposition 2.3.6 this can be done on geometric points, but the geometric points of T_0 and T agree.

For the second claim, we may assume that $\underline{\mathcal{E}}[\mathfrak{p}^n] \cong (\mathfrak{p}^{-n}/\mathcal{O}_0)^r_S$ by Proposition 2.2.12. So, the claim follows from Proposition 2.3.6 (3). □

We denote by $\mathrm{Sht}_{r,M}$ the stack parametrising shtukas of rank r with a M-structure as defined above.

Theorem 2.3.9. *Let M be a submodule of $(\mathfrak{p}^{-n}/\mathcal{O}_0)^r$. The stack $\mathrm{Sht}_{r,M}$ is a regular Deligne-Mumford stack locally of finite type over $\mathbb{F}_q$. Moreover, the forgetful map $\mathrm{Sht}_{r,M} \to \mathrm{Sht}_r$ is schematic and finite flat. It is finite étale away from 0.*

Proof. By Proposition 2.3.8, the forgetful map to Sht_r is schematic and finite. In particular, $\mathrm{Sht}_{r,M}$ is a Deligne-Mumford stack locally of finite type over $\mathbb{F}_q$ (since Sht_r is a DM-stack locally of finite type over $\mathbb{F}_q$). Also by Proposition 2.3.8, the forgetful map is finite étale away from 0.

We proceed as in the proof of [KM85, Theorems 5.1.1 and 5.2.1]. Again since Sht_r is a smooth DM-stack of dimension $(2r-1)$ over $\mathbb{F}_q$, we find an étale presentation $S \to \mathrm{Sht}_r$ by a $(2r-1)$-dimensional smooth scheme S over $\mathbb{F}_q$. We denote by $T = S \times_{\mathrm{Sht}_r} \mathrm{Sht}_{r,M}$.

We denote by $U \subseteq S$ the set of points in $s \in S$ such that the local rings at all points in T over s are regular. Then U is open in S, as its complement is the image under a finite (hence closed) map of the non-regular locus in T, which is closed in T as T is locally of finite type over a perfect field.

In order to show that $U = S$, it suffices to show that all closed points of S are contained in U. As the map $T \to S$ is étale away from 0, clearly all points away from 0 are contained in U. It remains to check that all closed points in the fibre over 0 are in U. By passing to the completion of the strict henselisation we are reduced to showing that for all ℓ-valued points $\bar{s}$ of S in the fibre over 0, where ℓ is an algebraic closure of $\mathbb{F}_q$, the complete local rings at all ℓ-valued points of T over $\bar{s}$ are regular. Note that the completion of the strict henselisation at a closed point $s \in S$ over 0 is then given by $\widehat{\mathcal{O}^{\mathrm{sh}}_{S,s}} = \widehat{\mathcal{O}}_{S_{\ell[\![\varpi]\!]},\bar{s}}$, where $S_{\ell[\![\varpi]\!]}$ denotes the base change $S \times_X \mathrm{Spec}(\ell[\![\varpi]\!])$.

Let us fix some ℓ-valued point $\bar{s}$ of S over 0 and let $\underline{\mathcal{E}}_0 \in \mathrm{Sht}_r(\ell)$ be the corresponding shtuka. By [Stacks, Tag 07N9], the disjoint union of the spectra of the completions of all local rings at ℓ-valued points of $T_{\ell[\![\varpi]\!]}$ over $\bar{s}$ is given by the scheme

$$T' = T_{\ell[\![\varpi]\!]} \times_{S_{\ell[\![\varpi]\!]}} \mathrm{Spec}(\widehat{\mathcal{O}}_{S_{\ell[\![\varpi]\!]},\bar{s}}).$$

As S is étale over Sht_r, we get by [Dri76, Proposition 3.3] that

$$\widehat{\mathcal{O}}_{S_{\ell[\![\varpi]\!]},\bar{s}} \cong \ell[\![\varpi, T_1, \ldots, T_{2r-2}]\!],$$

which identifies the pullback of the universal shtuka to $\mathrm{Spec}(\widehat{\mathcal{O}}_{S_{\ell[\![\varpi]\!]}})$ with the universal deformation $\underline{\mathcal{E}}$ of $\underline{\mathcal{E}}_0$. In particular, $T' = (\mathrm{Sht}_{r,M})_{\underline{\mathcal{E}}} := \mathrm{Sht}_{r,M} \times_{\mathrm{Sht}_r} \mathrm{Spec}(\widehat{\mathcal{O}}_{S_{\ell[\![\varpi]\!]},\bar{s}})$, where we interpret $\underline{\mathcal{E}}$ as the corresponding $\mathrm{Spec}(\widehat{\mathcal{O}}_{S_{\ell[\![\varpi]\!]},\bar{s}})$-valued point of Sht'_r.

Thus, by construction, T' only depends on the scheme of $\mathfrak{p}^n$-torsion points of $\underline{\mathcal{E}}$ (and thus on its local shtuka at 0 by [Har19, Theorem 7.6]), which in turn by the Serre-Tate Theorem (Proposition 2.1.11) only depends on the local shtuka at 0 of $\underline{\mathcal{E}}_0$, which are classified up to isomorphism by their Newton polygons. In particular, $\bar{s}$ is contained in U if and only if U contains all points in the fibre over 0 in the same Newton stratum. By Theorem 2.2.6 it thus suffices to show that U contains a basic point.

Let thus $\bar{s}$ be a basic point in characteristic $\mathfrak{p}$ (recall that such a point exists by Proposition 2.2.5) corresponding to a shtuka $\underline{\mathcal{E}}_0$. By Proposition 2.2.12, we get that $\underline{\mathcal{E}}_0[\mathfrak{p}^n](\ell) = \{0\}$ for all $n \in \mathbb{N}$, so in particular the only possible M-structure is the zero map (which is readily checked to be a M-structure as M is a submodule of $(\mathfrak{p}^{-n}/\mathcal{O}_0)^r$), so there is exactly one point lying over $\bar{s}$. This means that T' is the spectrum of the complete local ring pro-representing the deformation functor of the unique point lying over $\bar{s}$. Note that since $\underline{\mathcal{E}}$ is basic, the associated divisible module at 0 is formal (in the sense of [Dri76, §1] or [HS19, Definition 1.1]). As M-structures only depend on the local shtukas, the Serre-Tate Theorem is also compatible with M-structures and we are thus reduced to showing that the deformation functor of formal modules with M-structures is representable by a r-dimensional regular local ring.

This can be shown as in [Dri76, Proposition 4.3]. We sketch the argument. We write $M = (\mathfrak{p}^{-n_1}/\mathcal{O}_0) \oplus \ldots \oplus (\mathfrak{p}^{-n_{r'}}/\mathcal{O}_0)$. Note that by assumption $r' \leqslant r$. The lemma in the proof of [Dri76, Proposition 4.3] shows that the deformation functor on formal modules with $(\mathfrak{p}^{-1}/\mathcal{O}_0)^{r'}$-structure is pro-represented by a complete regular local ring R_1 finite flat over $R_0 = \ell[\![\varpi, T_1, \ldots, T_{r-1}]\!]$ whose maximal ideal is generated by $\iota(e_1), \ldots \iota(e_{r'}), T_{r'}, \ldots, T_{r-1}$, where $e_1, \ldots, e_{r'}$ is a basis of $(\mathfrak{p}^{-1}/\mathcal{O}_0)^{r'}$ and ι is the universal $(\mathfrak{p}^{-1}/\mathcal{O}_0)^{r'}$-structure. This settles the case that M is $\mathfrak{p}$-torsion. As a next step we show the claim for $M[\mathfrak{p}^m]$ by induction on m. Let us assume that a complete regular local ring R_m finite flat over R_{m-1} pro-represents the deformation functor of formal modules with $M[\mathfrak{p}^m]$-level and that $\iota_m(\varpi^{-\min\{n_1,m\}}), \ldots, \iota_m(\varpi^{-\min\{n_{r'},m\}}), T_{r'}, \ldots, T_r$ forms a system of local parameters for R_m, where ι_m is the universal $M[\mathfrak{p}^m]$-level structure. Let us denote by $i_1, \ldots, i_j$ the indices such that $n_i > m$. Then

$$R_{m+1} = R_m\left[\!\left[\tilde{T}_{i_1}, \ldots, \tilde{T}_{i_j}\right]\!\right]/(e_\varpi(\tilde{T}_{i_1}) - \iota_m(\varpi^{-\min\{n_{i_1},m\}}), \ldots, e_\varpi(\tilde{T}_{i_j}) - \iota_m(\varpi^{-\min\{n_{i_j},m\}})).$$

This also shows that R_{m+1} is regular and finite flat over R_m. Moreover, it has a system of local parameters as desired. $\square$

The regularity allows us to collect some consequences. We show that a M-structure defines an $\mathcal{O}_0$-module subscheme $\mathbf{H} \subseteq \underline{\mathcal{E}}[\mathfrak{p}^n]$.

Proposition 2.3.10. *Let $\underline{\mathcal{E}} \in \mathrm{Sht}_r(S)$ and let $\iota\colon M \to \underline{\mathcal{E}}[\mathfrak{p}^n](S)$ be a M-structure on $\underline{\mathcal{E}}$ for some submodule $M \subseteq (\mathfrak{p}^{-n}/\mathcal{O}_0)^r$.*

(1) There exists a (necessarily unique) finite locally free closed subscheme $\mathbf{H} \subseteq \underline{\mathcal{E}}[\mathfrak{p}^n]$ of rank $|M|$ such that the image of ι forms a full set of sections for $\mathbf{H}$.

Moreover, for each submodule M' of M the restriction of ι to M' defines a M'-structure on $\underline{\mathcal{E}}$ and in particular there exists a finite locally free closed subscheme $\mathbf{H}_{M'}$ of $\underline{\mathcal{E}}$ of rank M' such that $\iota|_{M'}$ forms a full set of sections for $\mathbf{H}_{M'}$.

(2) $\mathbf{H}$ is an $\mathcal{O}_0$-module subscheme of $\underline{\mathcal{E}}[\mathfrak{p}^n]$ such that the $\mathbb{F}_q$-module structure on $\mathbf{H}$ is strict.

We call $\mathbf{H}$ the *subscheme generated by* ι and the map ι a *M-generator of* $\mathbf{H}$. To be more precise, when we say that a map ι is a M-generator of a finite locally free closed subscheme $\mathbf{H} \subseteq \underline{\mathcal{E}}[\mathfrak{p}^n]$ of rank $|M|$, we really mean both that ι gives a full set of sections of $\mathbf{H}$ *and* that ι is a M-structure on $\underline{\mathcal{E}}$ (recall that the first condition does not imply the second one, compare Remark 2.3.4).

For full level structures on Drinfeld modules the assertion is essentially shown in [Leh09, 3, Proposition 3.3.]. The proposition also implies that for general M-structures on Drinfeld modules our definition agrees with the one given in [KY20].

Proof. That the restriction to M' defines a M'-structure is Proposition 2.3.5 and thus, the second statement in (1) follows from the first. In order to show both the first part of (1) and (2), we may assume by reduction to the universal case that S is locally noetherian and flat over X'. Both assertions are true away from 0 by Proposition 2.3.6. It thus remains to show that the conditions are closed in both cases. For the first part of (1) this follows from Lemma 2.3.1.

For (2) we may additionally assume that $S = \mathrm{Spec}(R)$ is affine and that we can embed $\underline{\mathcal{E}}[\mathfrak{p}^n]$ in $\mathbb{A}^1_R$ as the assertion is étale local on S (for the strictness of the $\mathbb{F}_q$-action this is [Har19, Lemma 4.4]). The locus where the group structure on $\underline{\mathcal{E}}$ restricts to a group structure on $\mathbf{H}$ is closed by the argument of [KM85, Corollary 1.3.7].

By the discussion above, we can write $\underline{\mathcal{E}}[\mathfrak{p}^n] = \mathrm{Spec}(R[t]/(f))$ for some monic polynomial $f \in R[t]$ and $\mathbf{H} = \mathrm{Spec}(R[t]/(g))$ for some monic polynomial $g \in R[t]$ such that $f \in (g)$. Then the $\mathcal{O}_0/\mathfrak{p}^n$-module structure restricts to $\mathbf{H}$ if and only if for each $a \in \mathcal{O}_0/\mathfrak{p}^n$ the map e_a induces a map

$$\begin{tikzcd} R[t]/(f) \arrow[r, "e_a"] \arrow[d, two heads] & R[t]/(f) \arrow[d, two heads] \\ R[t]/(g) \arrow[r, dashed] & R[t]/(g), \end{tikzcd}$$

in other words, that $e_a(g)(t) = g(e_a^\sharp(t)) \equiv 0 \mod g$, where $e_a^\sharp(t) \in R[t]$ is a polynomial defining the map e_a. Thus, the locus where $\mathbf{H}$ is an $\mathcal{O}_0/\mathfrak{p}^n$-module subscheme is the closed subscheme of $\operatorname{Spec}(R)$ where all the coefficients of the remainders of $g(e_a^\sharp(t))$ modulo g vanish. Note that this is clearly independent of the choice of $e_a^\sharp$.

It remains to show that the locus where the $\mathbb{F}_q$-action is strict is closed. As $\underline{\mathcal{E}}[\mathfrak{p}^n]$ carries a strict $\mathbb{F}_q$-action by construction, we have a lift of the $\mathbb{F}_q$-action to $\underline{\mathcal{E}}[\mathfrak{p}^n]^\flat = \operatorname{Spec}(R[t]/(tf))$ by [HS19, Lemma 4.4]. By the same argument as for the $\mathcal{O}_0/\mathfrak{p}^n$-module structure, the $\mathbb{F}_q$-action restricts to a map on the deformation $\mathbf{H}^\flat = \operatorname{Spec}(R[t]/(tg)) \subseteq \underline{\mathcal{E}}[\mathfrak{p}^n]^\flat$. That it induces the correct operation on the co-Lie complex of $(\mathbf{H}, \mathbf{H}^\flat)$ is again a closed condition. □

We now define M-cyclic isogenies.

Definition 2.3.11. Let $\underline{\mathcal{E}} \in \operatorname{Sht}_r(S)$ and let M be a finitely generated $\mathfrak{p}^n$-torsion module.

(1) A *M-generator* of a finite locally free subscheme $\mathbf{H} \subseteq \underline{\mathcal{E}}[\mathfrak{p}^n]$ is a M-structure ι on $\underline{\mathcal{E}}_{S'}$ such that the subscheme of $\underline{\mathcal{E}}[\mathfrak{p}^n]_{S'}$ defined by ι is $\mathbf{H}_{S'}$.

(2) A finite locally free subscheme $\mathbf{H} \subseteq \underline{\mathcal{E}}[\mathfrak{p}^n]$ is called *M-cyclic* if there is an fppf cover $S' \to S$ such that $\mathbf{H}_{S'}$ admits a M-generator.

(3) A $\mathfrak{p}^n$-isogeny of Drinfeld shtukas $f\colon \underline{\mathcal{E}} \to \underline{\mathcal{E}}'$ is called *M-cyclic* if $\operatorname{Dr}_q(\operatorname{coker}(f))$ is M-cyclic.

Note that a M-cyclic subscheme necessarily has rank $|M|$. We also use the term $\mathfrak{p}^n$*-cyclic* as abbreviation for $(\mathfrak{p}^{-n}/\mathcal{O}_0)$-cyclic submodule schemes or isogenies, respectively.

Lemma 2.3.12. *Let M be a submodule of $(\mathfrak{p}^{-n}/\mathcal{O}_0)^r$. Every M-cyclic subscheme $\mathbf{H} \subseteq \underline{\mathcal{E}}[\mathfrak{p}^n]$ is an $\mathcal{O}_0$-module subscheme with strict $\mathbb{F}_q$-action.*

Proof. All of the assertions can be checked fppf-locally on the base, where they follow from Lemma 2.3.10. □

We collect two representability results.

Proposition 2.3.13. *Let $\underline{\mathcal{E}} \in \mathrm{Sht}_r(S)$ and assume that its characteristic is away from 0. Let M be a finitely generated $\mathcal{O}_0/\mathfrak{p}^n$-module. Then the functor on S-schemes*

$$T \mapsto \{M\text{-cyclic subgroups of } \underline{\mathcal{E}}[\mathfrak{p}^n]_T\}.$$

is representable by a finite étale S-scheme. Moreover, étale locally on S the functor is representable by the constant S-scheme

$$S \times \{\text{submodules of } (\mathfrak{p}^{-n}/\mathcal{O}_0)^r \text{ isomorphic to } M\}.$$

Proof. We proceed as in the proof of [KM85, Theorem 3.7.1]. By descent for finite locally free schemes and closed immersions, and the fact that cyclicity is local for the fppf-topology by definition, the functor is a fppf (and hence an étale) sheaf. By étale descent, it thus suffices to show the second statement. We may assume that $\underline{\mathcal{E}}[\mathfrak{p}^n] \cong (\mathfrak{p}^{-n}/\mathcal{O}_0)^r_S$ by Proposition 2.2.12. By the argument in the proof of [KM85, Theorem 3.7.1], over a connected base T any finite locally free closed subgroup scheme of a constant scheme is itself constant. The claim follows from the explicit description in Proposition 2.3.6. □

Proposition 2.3.14. *Let $\underline{\mathcal{E}} \in \mathrm{Sht}_r(S)$, let M be a finitely generated $\mathcal{O}_0/\mathfrak{p}^n$-module and let $\mathbf{H} \subseteq \underline{\mathcal{E}}[\mathfrak{p}^n]$ be a finite locally free closed $\mathcal{O}_0$-module subscheme of rank $|M|$. We consider its functor of generators, i.e. the functor on S-schemes*

$$T \mapsto \{M\text{-generators of } \mathbf{H}_T \text{ in the sense of Definition 2.3.11}\}.$$

It is representable by a finite scheme of finite presentation over S denoted by $\mathbf{H}^\times$. Moreover, it is finite étale when $\mathbf{H}$ is étale (in particular, when the characteristic of $\underline{\mathcal{E}}$ is away from 0).

Proof. We adapt the proof of [KM85, Proposition 1.6.5]. The functor clearly is representable by the closed subscheme of $\mathrm{Hom}(M, \mathbf{H})$ over which the universal homomorphism is a M-structure on $\underline{\mathcal{E}}$ (which is a closed condition locally defined by finitely many equations by Proposition 2.3.8) and over which the subscheme defined by the universal homomorphism is $\mathbf{H}$, which is also a closed condition given by finitely many equations by Lemma 2.3.1 and [KM85, Corollary 1.3.5].

If $\mathbf{H}$ is étale, we show as in the proof of Proposition 2.3.8 that $\mathbf{H}^\times$ is formally étale. □

2.3.3. Balanced level structures for shtukas

Definition 2.3.15. Let $m \in \mathbb{N}$ and let $r_1, \ldots, r_m$ be positive integers such that $\sum_{i=1}^m r_i \leqslant r$. A *balanced $\mathfrak{p}^n$-level structure of type* $(r_1, \ldots, r_m)$ on a Drinfeld shtuka $\underline{\mathcal{E}}$ over S is a chain of isogenies

$$\underline{\mathcal{E}}(\mathfrak{p}^n) = \underline{\mathcal{E}}_{m+1} \xrightarrow{f_{m+1}} \underline{\mathcal{E}}_m \xrightarrow{f_m} \ldots \xrightarrow{f_2} \underline{\mathcal{E}}_1 \xrightarrow{f_1} \underline{\mathcal{E}}_0 = \underline{\mathcal{E}}$$

such that the composition $f_{m+1} \circ \ldots \circ f_1$ is the inclusion $\underline{\mathcal{E}}(\mathfrak{p}^n) \to \underline{\mathcal{E}}$, together with $(\mathfrak{p}^{-n}/\mathcal{O}_0)^{r_i}$-generators of $\mathrm{Dr}_q(\mathrm{coker}(f_i)) \subseteq \underline{\mathcal{E}}_i[\mathfrak{p}^n]$ for all $1 \leqslant i \leqslant m$ in the sense of Definition 2.3.2. We denote by $\mathrm{Sht}_{r,\mathfrak{p}^n\text{-bal-}(r_1,\ldots,r_m)}$ the stack parametrising Drinfeld shtukas together with a balanced $\mathfrak{p}^n$-level structure of type $(r_1, \ldots, r_m)$.

A $\Gamma_1(\mathfrak{p}^n)$*-level structure* on a Drinfeld shtuka of rank r is a balanced $\mathfrak{p}^n$-level structure of type $\mathbf{1}_r = (1, \ldots, 1) \in \mathbb{Z}^r$. We denote by $\mathrm{Sht}_{r,\Gamma_1(\mathfrak{p}^n)} = \mathrm{Sht}_{r,\mathfrak{p}^n\text{-bal-}\mathbf{1}_r}$ the stack of Drinfeld shtukas with a $\Gamma_1(\mathfrak{p}^n)$-level structure.

As for Drinfeld shtukas with chains of isogenies, we see that a balanced $\mathfrak{p}^n$-level structure of type $(r_1, \ldots, r_m)$ on a Drinfeld shtuka $\underline{\mathcal{E}}$ is equivalent to the data of a flag

$$0 \subseteq \mathbf{H}_1 \subseteq \mathbf{H}_2 \subseteq \ldots \subseteq \mathbf{H}_m \subseteq \underline{\mathcal{E}}[\mathfrak{p}^n]$$

of finite locally free submodule schemes $\mathbf{H}_i \subseteq \underline{\mathcal{E}}[\mathfrak{p}^n]$ of rank $n \cdot (r_1 + \ldots + r_i)$ with strict $\mathbb{F}_q$-action together with $(\mathfrak{p}^{-n}/\mathcal{O}_0)^{r_i}$-generators of $\mathbf{H}_i/\mathbf{H}_{i-1}$ for all $1 \leqslant i \leqslant m$.

Lemma 2.3.16. *The stack* $\mathrm{Sht}_{r,\mathfrak{p}^n-(r_1,\ldots,r_m)-\mathrm{bal}}$ *is representable by a Deligne-Mumford stack locally of finite type over* $\mathbb{F}_q$*. The projection* $\mathrm{Sht}_{r,\mathfrak{p}^n-(r_1,\ldots,r_m)-\mathrm{bal}} \to \mathrm{Sht}_r$ *is schematic and finite. Moreover, it is finite étale away from* 0*.*

Proof. We have a well-defined map of stacks

$$\mathrm{Sht}_{r,\mathfrak{p}^n-(r_1,\ldots,r_m)-\mathrm{bal}} \to \mathrm{Sht}_{r,(r_1,\ldots,r_m)-\mathfrak{p}^n\text{-chain}}.$$

This map is schematic, finite and moreover finite étale away from 0 by Proposition 2.3.14. The assertions then follow from Proposition 2.2.15. □

Proposition 2.3.17. *The Deligne-Mumford stack* $\mathrm{Sht}_{r,\mathfrak{p}^n-(r_1,\ldots,r_m)-\mathrm{bal}}$ *is regular.*

Proof. As in the proof of Theorem 2.3.9 it suffices to check that the deformation functor of the $\mathfrak{p}$-divisible module of a basic point over 0 with balanced $\mathfrak{p}^n$-level structure of type $(r_1, \ldots, r_m)$ is pro-representable by a regular local ring. By [KM85, Proposition 5.2.2] it suffices to show that the maximal ideal is generated by r elements.

Let $(\mathbf{G}_0, (\mathbf{H}_{0,i}, \iota_{0,i})_{1 \leqslant i \leqslant m})$ be the $\mathfrak{p}$-divisible module of a basic Drinfeld shtuka of rank r over an algebraically closed field ℓ in the fibre over 0 together with a balanced $\mathfrak{p}^n$-level structure of type $(r_1, \ldots, r_m)$ on $\mathbf{G}_0$. Note that $\mathbf{G}_0$ is automatically formal and the level structure is unique, all the $\iota_{0,i}$ are the zero map. Then by the Serre-Tate theorem the deformation functor of $(\mathbf{G}_0, (\mathbf{H}_{0,i}, \iota_{0,i})_{1 \leqslant i \leqslant m})$ is representable by a complete local ring denoted by B. Let $(\mathbf{G}, ((\mathbf{H}_i), (\iota_i))_{1 \leqslant i \leqslant m})$ be the universal deformation over B. For every $1 \leqslant i \leqslant m$ we choose a basis $e_1^{(i)}, \ldots, e_{r_i}^{(i)}$ of $(\mathfrak{p}^{-n}/\mathcal{O}_0)^{r_i}$. We claim that the maximal ideal of B is generated by $\iota_i(e_j^{(i)})$ for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant r_i$ and $T_{r_1+\ldots+r_m}, \ldots, T_{r-1}$. We proceed as in the proof of [KM85, Theorem 5.3.2., (5.3.5.)]. We need to check that for every artin local $\ell[\![\varpi]\!]$-algebra R such that $T_{r_1+\ldots+r_m}, \ldots, T_{r-1}$ vanish in R every deformation $(\tilde{\mathbf{G}}, ((\tilde{\mathbf{H}}_i), (\tilde{\iota}_i))_{1\leqslant i\leqslant m})$ on R such that all $\tilde{\iota}_i$ are the constant zero maps is itself constant.

Using [KM85, Lemma 1.11.3] we see inductively that the zero map is an $(\mathfrak{p}^{-n}/\mathcal{O}_0)^{r_1+\ldots+r_i}$-structure on $\mathbf{H}_i$ for all $1 \leqslant i \leqslant m$. In particular, the zero map is an $(\mathfrak{p}^{-n}/\mathcal{O}_0)^{r_1+\ldots+r_i}$-structure on $\tilde{\mathbf{G}}$ and thus $\mathbf{G}$ is constant by the proof of Theorem 2.3.9. $\square$

We collect some consequences. We start by constructing balanced level structures from $(\mathfrak{p}^{-n}/\mathcal{O}_0)^{r'}$-structures. Let $m \in \mathbb{N}$ and let $r_1, \ldots, r_m$ be positive integers such that $r' = \sum_{i=1}^m r_i \leqslant r$. Let $(\underline{\mathcal{E}}, \iota)$ be a Drinfeld shtuka together with an $(\mathfrak{p}^{-n}/\mathcal{O}_0)^{r'}$-structure on S. For $1 \leqslant i \leqslant m$ the restriction of ι restricted to the first $r_1 + \ldots + r_i$ components is an $(\mathfrak{p}^{-n}/\mathcal{O}_0)^{r_1+\ldots+r_i}$-structure by Proposition 2.3.5 and thus defines an $\mathcal{O}_0/\mathfrak{p}^n$-submodule scheme $\mathbf{H}_i$ of $\underline{\mathcal{E}}[\mathfrak{p}^n]$ by Proposition 2.3.10. We denote by ι_i the induced map $(\mathfrak{p}^{-n}/\mathcal{O}_0)^{r_i} \to \mathbf{H}_i/\mathbf{H}_{i-1}(S)$.

Proposition 2.3.18. *Let $(\underline{\mathcal{E}}, \iota)$ be a rank r Drinfeld shtuka together with an $(\mathfrak{p}^{-n}/\mathcal{O}_0)^{r'}$-structure over S. Using the notation as above, the flag of finite locally free closed submodule schemes $0 = \mathbf{H}_0 \subseteq \mathbf{H}_1 \subseteq \ldots \subseteq \mathbf{H}_m \subseteq \underline{\mathcal{E}}[\mathfrak{p}^n]$ together with the maps $(\iota_i)_{1\leqslant i\leqslant m}$ defines a balanced $\mathfrak{p}^n$-level structure of type $(r_1, \ldots, r_m)$ on $\underline{\mathcal{E}}$.*

Proof. We follow the proof of [KM85, Theorem 5.5.2.]. By reduction to the universal case and Theorem 2.3.9 we may assume that S is flat and affine over X'. The assertion is clear when the characteristic of $\underline{\mathcal{E}}$ is away from 0. The condition that ι_i generates $\mathbf{H}_i/\mathbf{H}_{i-1}$ is closed and thus the assertion follows by flatness. $\square$

The proposition can also be applied in the following more general situation. Let $\underline{\mathcal{E}}$ be a Drinfeld shtuka together with a balanced $\mathfrak{p}^n$-level structure of type $(r_1, \ldots, r_m)$ denoted by $((\mathbf{H}_i), (\iota_i))$. Let $1 \leqslant m' \leqslant m$ and for each $1 \leqslant j \leqslant m'$ let i_j and $r'_{i_1+\ldots+i_{j-1}+1}, \ldots, r'_{i_1+\ldots+i_j}$ be positive integers such that $\sum_{i=1}^{i_j} r'_{i_1+\ldots+i_{j-1}+i} = r_j$. By applying Proposition 2.3.18 to each ι_j for $1 \leqslant j \leqslant m'$, we obtain a well-defined balanced $\mathfrak{p}^n$-level structure of type $(r'_1, \ldots, r'_{i_1+\ldots+i_{m'}})$ on $\underline{\mathcal{E}}$. This construction thus induces a map of stacks

$$\mathrm{Sht}_{r,\mathfrak{p}^n-(r_1,\ldots,r_m)-\mathrm{bal}} \to \mathrm{Sht}_{r,\mathfrak{p}^n-(r'_1,\ldots,r'_{i_1+\ldots+i_{m'}})-\mathrm{bal}}. \tag{2.1}$$

Corollary 2.3.19. *The map (2.1) is finite locally free of constant rank. In particular, fppf-locally on the base, any balanced $\mathfrak{p}^n$-level structure of type $(r'_1, \ldots, r'_{i_1+\ldots+i_{m'}})$ on $\underline{\mathcal{E}}$ can be extended to a balanced $\mathfrak{p}^n$-level structure of type $(r_1, \ldots, r_m)$ on $\underline{\mathcal{E}}$.*

Proof. We follow the proof of [KM85, Corollaries 5.5.3. & 5.5.4.] As both the stacks $\mathrm{Sht}_{r,\mathfrak{p}^n-(r_1,\ldots,r_m)-\mathrm{bal}}$ and $\mathrm{Sht}_{r,\mathfrak{p}^n-(r'_1,\ldots,r'_{i_1+\ldots+i_{m'}})-\mathrm{bal}}$ are regular r-dimensional, the map (2.1) is necessarily finite flat. The degree can be computed in the étale case, where it is clearly constant. The second assertion follows immediatly. $\square$

Corollary 2.3.20. *Let $\underline{\mathcal{E}}$ be a Drinfeld shtuka over S together with a balanced $\mathfrak{p}^n$-level structure of type $(r_1, \ldots, r_m)$ denoted by $(\mathbf{H}_i, \iota_i)_{1\leqslant i\leqslant m}$. Then $\mathbf{H}_i$ is $(\mathfrak{p}^{-n}/\mathcal{O}_0)^{r_1+\ldots+r_i}$-cyclic.*

Proof. We use induction on i. For $i = 1$ the assertion is clear by definition. Let now $i > 1$ and let us assume that $\mathbf{H}_{i-1}$ is $(\mathfrak{p}^{-n}/\mathcal{O}_0)^{r_1+\ldots+r_{i-1}}$-cyclic. As the question is fppf-local on S, we may assume that $\mathbf{H}_{i-1}$ admits a generator over S. Then $0 \subseteq \mathbf{H}_{i-1} \subseteq \mathbf{H}_i \subseteq \underline{\mathcal{E}}[\mathfrak{p}^n]$ together with the generators of $\mathbf{H}_{i-1}$ and $\mathbf{H}_i/\mathbf{H}_{i-1}$ defines a balanced $\mathfrak{p}^n$-level structure of type $(r_1 + \ldots + r_{i-1}, r_i)$ on $\underline{\mathcal{E}}$. By Corollary 2.3.19, it can be completed fppf-locally to an $(\mathfrak{p}^{-n}/\mathcal{O}_0)^{r_1+\ldots+r_i}$-structure. But this exactly means that $\mathbf{H}_i$ admits a generator fppf-locally on S. $\square$

2.4. Drinfeld $\Gamma_0(\mathfrak{p}^n)$-level structures on shtukas

We are now in a position to discuss $\Gamma_0(\mathfrak{p}^n)$-level structures on Drinfeld shtukas. We closely follow the exposition of [KM85, Chapter 6] for elliptic curves and adapt the arguments to suit our situation.

2.4.1. Main theorem on $\mathfrak{p}^n$-cyclic submodule schemes

The goal of this section is to show the following analogue of [KM85, Theorems 6.1.1 and 6.4.1].

Theorem 2.4.1 (Main Theorem on $\mathfrak{p}^n$-Cyclic Modules)**.** *Let $\underline{\mathcal{E}} \in \mathrm{Sht}_r(S)$ be a Drinfeld shtuka of rank r over a scheme S. Let $\mathbf{H} \subseteq \underline{\mathcal{E}}[\mathfrak{p}^n]$ be a finite locally free $\mathcal{O}_0/\mathfrak{p}^n$-submodule scheme of rank q^n over S.*

(1) Suppose that $\mathbf{H}$ is $\mathfrak{p}^n$-cyclic and admits a generator ι. Let $D \subseteq \mathbf{H}$ be the finite locally free subscheme of $\mathbf{H}$ of rank $q^{n-1}(q-1)$ defined by the restriction of ι to $(\mathfrak{p}^{-n}/\mathcal{O}_0)^\times = (\mathfrak{p}^{-n}/\mathcal{O}_0)\backslash(\mathfrak{p}^{-(n-1)}/\mathcal{O}_0)$. Then $D = \mathbf{H}^\times$ as subschemes of $\mathbf{H}$.

(2) $\mathbf{H}$ is $\mathfrak{p}^n$-cyclic if and only if its scheme of generators $\mathbf{H}^\times$ is finite locally free over S of rank

$$q^{n-1}(q-1).$$

(3) *Cyclicity of* $\mathbf{H}$ *is a closed condition, in the sense that there is a closed subscheme* $W \subseteq S$ *locally of finite presentation over* S *such that for any* $T \to S$ *the pullback* $\mathbf{H}_T$ *is* $\mathfrak{p}^n$*-cyclic if and only if* $T \to S$ *factors through* W.

Proof. Assertion (3) follows from (2) by the flattening stratification as in the proof of [KM85, Theorem 6.4.1]. We sketch the argument. As a first step we note that in the case where $S = \mathrm{Spec}(k)$ is the spectrum of a field and $\mathbf{H}$ is not cyclic, we have $\mathbf{H}^\times = \varnothing$. Namely, any generator of $\mathbf{H}$ can be defined over a finite extension of k, but by assumption $\mathbf{H}$ does not admit a generator after any finite extension of k. Hence, $\mathbf{H}^\times$ does not have any field valued points and is thus the empty scheme.

As the question is Zariski-local on S and both $\mathbf{H}$ and $\underline{\mathcal{E}}[\mathfrak{p}^n]$ are of finite presentation over S, we may assume that $S = \mathrm{Spec}(R)$ is affine and Noetherian. By the above argument, the maximal rank of $\mathbf{H}^\times$ over S is $q^{n-1}(q-1)$. By the flattening stratification, the locus where $\mathbf{H}^\times$ has rank $q^{n-1}(q-1)$ and hence $\mathbf{H}$ is cyclic by (2) is closed.

It is also clear that (2) follows from (1) as in [KM85, Theorem 6.1.1]. Namely, if $\mathbf{H}^\times$ is finite locally free of rank $q^{n-1}(q-1)$, the diagonal map $\mathbf{H}^\times \to \mathbf{H}^\times_{\mathbf{H}^\times}$ is a section of $\mathbf{H}^\times$ after base change along $\mathbf{H}^\times \to S$. Hence, $\mathbf{H}$ admits a generator after the fppf base change to $\mathbf{H}^\times$ and is thus $\mathfrak{p}^n$-cyclic. Conversely, assume that $\mathbf{H}$ is $\mathfrak{p}^n$-cyclic. The question is fppf-local on S, we may thus assume $\mathbf{H}$ admits a generator. The assertion in this case follows from (1).

It thus remains to show (1). We adapt the proof of [KM85, Theorem 6.1.1]. The assertion is certainly true when the characteristic of $\underline{\mathcal{E}}$ is away from $\mathfrak{p}$ by Proposition 2.3.6.

As a first step we show that $D \subseteq \mathbf{H}^\times$. It is clear by definition that $D \subseteq \mathbf{H}$. By reduction to the universal case and using Theorem 2.3.9, it suffices to consider the case when S is Noetherian and flat over X' and as the question is local on S, we may further assume that $S = \mathrm{Spec}(R)$ is affine. It follows that D is then also flat over X'. In order to show that $D \subseteq \mathbf{H}^\times$, we show that the tautological section of $\mathbf{H}$ over D induced by the inclusion $D \hookrightarrow \mathbf{H}$ is a generator of $\mathbf{H}_D$. This is certainly true away from $\mathfrak{p}$. The claim follows from the flatness of D over X' and the fact that the locus where $D \to \mathbf{H}_D$ is a generator is closed in D by Lemma 2.3.1. Hence, $D \hookrightarrow \mathbf{H}$ factors over $\mathbf{H}^\times$ and the induced map $D \subseteq \mathbf{H}^\times$ is necessarily a closed immersion.

In order to show that the closed immersion $D \hookrightarrow \mathbf{H}^\times$ is an isomorphism, we introduce the following two auxiliary moduli problems.

$$\mathcal{X}_1(S) = \left\langle (\underline{\mathcal{E}}, \mathbf{H}, \iota_1, P)\colon \begin{array}{l} \underline{\mathcal{E}} \in \mathrm{Sht}_r(S),\ \mathbf{H} \subseteq \underline{\mathcal{E}}[\mathfrak{p}^n] \text{ an } \mathfrak{p}^n\text{-cyclic submodule scheme,} \\ \iota_1 \in \mathbf{H}^\times(S), P \in D(S) \text{ such that } \{\alpha P\colon \alpha \in (\mathcal{O}_0/\mathfrak{p}^n)^\times\} \\ \text{is a full set of sections for } D \end{array} \right\rangle$$

$$\mathcal{X}_2(S) = \left\langle (\underline{\mathcal{E}}, \mathbf{H}, \iota_1, \iota_2)\colon \begin{array}{l} \underline{\mathcal{E}} \in \mathrm{Sht}_r(S),\ \mathbf{H} \subseteq \underline{\mathcal{E}}[\mathfrak{p}^n] \text{ an } \mathfrak{p}^n\text{-cyclic submodule scheme,} \\ \iota_1, \iota_2 \in \mathbf{H}^\times(S) \end{array} \right\rangle$$

It is clear that $\mathcal{X}_1$ and $\mathcal{X}_2$ are stacks, and both map to $\mathrm{Sht}_{r,(\mathfrak{p}^{-n}/\mathcal{O}_0)}$ by forgetting P and ι_2, respectively. This maps are clearly schematic and finite as they are representable by (a closed subscheme of) the finite schemes D and $\mathbf{H}^\times$, respectively. Note that both $\mathcal{X}_1$ and $\mathcal{X}_2$ have a unique point lying over a supersingular point of Sht_r over an algebraically closed field in characteristic $\mathfrak{p}$.

Since $D \subseteq \mathbf{H}^\times$ there is a natural map $\mathcal{X}_1 \to \mathcal{X}_2$ over $\mathrm{Sht}_{r,(\mathfrak{p}^{-n}/\mathcal{O}_0)}$ which is an isomorphism away from $\mathfrak{p}$ as noted above. We show that the map is an isomorphism. By an argument as in the proof of Theorem 2.3.9 (compare also [KM85, Theorem 6.2.1]), it suffices to check it is an isomorphism at the completed local rings at the unique points lying over supersingular points over algebraically closed fields in characteristic $\mathfrak{p}$.

Let $\underline{\mathcal{E}}_0 \in \mathrm{Sht}_r(\ell)$ be a supersingular rank r Drinfeld shtuka over some algebraically closed field ℓ in characteristic $\mathfrak{p}$ and let $\underline{\mathcal{E}}$ be its universal formal deformation over $\tilde{B} = \ell[\![\varpi, T_1, \ldots, T_{r-1}, T_r, \ldots, T_{2r-2}]\!]$. We denote by $B = \ell[\![\varpi, T_1, \ldots, T_{r-1}]\!]$. Note that B pro-represents the deformation functor of the local shtuka of $\underline{\mathcal{E}}_0$ at $\mathfrak{p}$. Recall that $(\mathrm{Sht}_{r,(\mathfrak{p}^{-n}/\mathcal{O}_0)})_{\underline{\mathcal{E}}} = \mathrm{Spec}(\tilde{B}_0)$ is an affine scheme, which is finite over $\tilde{B}$ and that $\tilde{B}_0$ is a complete regular noetherian ring by Theorem 2.3.9. Then $\mathcal{X}_{1,\underline{\mathcal{E}}} = \mathrm{Spec}(\tilde{B}_1)$ and $\mathcal{X}_{2,\underline{\mathcal{E}}} = \mathrm{Spec}(\tilde{B}_2)$ are affine schemes finite over $\tilde{B}_0$ (and therefore also over $\tilde{B}$). Thus, $\tilde{B}_1$ and $\tilde{B}_2$ are complete, local and noetherian rings. We have to check that the map $\tilde{B}_2 \to \tilde{B}_1$ is an isomorphism. By the Serre-Tate theorem (which is clearly compatible with all the relevant level structures as they only depend on the $\mathfrak{p}^n$-torsion), we can write $\tilde{B}_i = B_i \otimes_\ell \ell[\![T_r, \ldots T_{2r-2}]\!]$ for some complete, local and noetherian rings B_i finite over B. Moreover, B_0 is regular. It clearly suffices to check that $B_2 \to B_1$ is an isomorphism. Note that since $D \subseteq \mathbf{H}^\times$ is a closed immersion, we obtain that the map $B_2 \to B_1$ is surjective.

By Corollary 2.2.11, we may assume that we can identify $\underline{\mathcal{E}}[\mathfrak{p}^n]$ in an A-linear fashion with the $\mathfrak{p}^n$-torsion of a Drinfeld A-module $\mathbf{E}$ with trivial underlying vector bundle (as the base B is local). We have the following explicit descriptions of the rings B_0, B_1 and B_2. As B-algebras we find

$$B_0 = B[P]/\mathcal{I},$$

where $\mathcal{I}$ is the ideal expressing the fact that the map $\iota\colon \mathfrak{p}^{-n}/\mathcal{O}_0 \to B[P] = \mathbf{E}(B[P])$ defined by $\varpi^{-n} \mapsto P$ is a well-defined $(\mathfrak{p}^{-n}/\mathcal{O}_0)$-structure. More precisely, $\mathcal{I}$ is generated by $e_{\varpi^n}(P), e_a(P)$, where $(\varpi^n, a) = \mathfrak{p}^n$ in A, (this implies that $P \in \underline{\mathcal{E}}[\mathfrak{p}^n](B_0)$ and thus that the map can be extended A-linearly to a well-defined map ι) and the equations defining the condition that $\sum_{\alpha\in\mathfrak{p}^{-1}/\mathcal{O}_0}[\iota(\alpha)] = \sum_{\alpha\in\mathcal{O}_0/\mathfrak{p}}[e_{\alpha\varpi^{n-1}}(P)]$ is a subscheme of $\underline{\mathcal{E}}[\mathfrak{p}]$ (this condition is defined by finitely many equations by [KM85, Lemma 1.3.4.]). Recall that B_0 is a regular local ring with maximal ideal generated by $P, T_1, \ldots, T_{r-1}$ by the proof of Theorem 2.3.9. In a similar fashion the rings B_1 and B_2 are given as B_0-algebras as

$$B_1 = B_0[Q]/\mathcal{J},$$

where $\mathcal{J}$ is the principal ideal generated by $\prod_{\alpha\in\mathcal{O}_0/\mathfrak{p}^n}(Q - e_\alpha(P))$, and

$$B_2 = B_0[Q]/\mathcal{K},$$

where $\mathcal{K}$ is the ideal expressing the fact that Q defines an $(\mathfrak{p}^{-n}/\mathcal{O}_0)$-structure as above and defines the same submodule scheme as P, i.e. $\mathcal{K}$ is generated by $e_{\varpi^n}(Q), e_a(Q)$, where $(\varpi, a) = \mathfrak{p}$ in A, the equations defining the condition that $\sum_{\alpha\in\mathcal{O}_0/\mathfrak{p}}[e_{\alpha\varpi^{n-1}}(Q)]$ is a subscheme of $\underline{\mathcal{E}}[\mathfrak{p}]$ and the coefficients of the polynomial $\prod_{\alpha\in\mathcal{O}_0/\mathfrak{p}^n}(t - e_\alpha(Q)) - \prod_{\alpha\in\mathcal{O}_0/\mathfrak{p}^n}(t - e_\alpha(P))$.

By [KM85, Lemma 6.3.4.], the multiplication by Q on B_1 is injective. We denote by K the kernel of the map $B_2 \to B_1$. Applying the snake lemma to the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & B_2 & \longrightarrow & B_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & B_2 & \longrightarrow & B_1 & \longrightarrow & 0,
\end{array}$$

where the vertical maps are given by multiplication by Q, yields the short exact sequence (using the injectivity of multiplication by Q on B_1)

$$0 \to K/QK \to B_2/QB_2 \to B_1/QB_1 \to 0.$$

By Nakayama's Lemma K vanishes if and only if K/QK vanishes. It thus suffices to show that

$$B_2/QB_2 \to B_1/QB_1$$

is an isomorphism.

We proceed as in [KM85, Lemma 6.3.5]. From the explicit description of B_1 and B_2 above we get that

$$B_1/QB_1 = B_0/\overline{\mathcal{J}} \qquad \text{and} \qquad B_2/QB_2 = B_0/\overline{\mathcal{K}},$$

where $\overline{\mathcal{J}}$ is the ideal generated by $\prod_\alpha e_\alpha(P)$ and $\overline{\mathcal{K}}$ is the ideal generated by the coefficiens of the polynomial $t^{q^n} - \prod_\alpha (t - e_\alpha(P))$, and the reductions of $e_{\varpi^n}(Q), e_a(Q)$, where $(\varpi, a) = \mathfrak{p}$ and the equations defining the condition that $\sum_{\alpha \in \mathcal{O}_0/\mathfrak{p}} [e_{\alpha\varpi^{n-1}}(Q)]$ is a subscheme of $\underline{\mathcal{E}}[\mathfrak{p}]$ modulo Q. It suffices to show that $\prod_\alpha e_\alpha(P) \in \overline{\mathcal{K}}$. We show that it is (up to multiplication by a unit in B_0) the coefficient of the term of degree $q^n - q^{n-1}(q-1)$ of $t^{q^n} - \prod_\alpha (t - e_\alpha(P))$. This coefficient is the sum of all $q^{n-1}(q-1)$-fold products of distinct elements of the set $\{e_\alpha(P) \colon \alpha \in \mathcal{O}_0/\mathfrak{p}^n\}$.

By the definition of Drinfeld modules it follows that $e_\alpha(P)$ is of the form (unit in B)$\cdot P$ for $\alpha \in (\mathcal{O}_0/\mathfrak{p}^n)^\times$ and of the form (elt in $\max(B)) \cdot P$ for $\alpha \in \mathfrak{p}$. Thus, both $\prod_\alpha e_\alpha(P)$ and the coefficient of the term of degree $q^n - q^{n-1}(q-1)$ of are of the form (unit in $B) \cdot P^{\phi(\mathfrak{p}^n)}$. $\square$

It would be desirable to have a similar statement also for other types of level structures.

2.4.2. $\Gamma_0(\mathfrak{p}^n)$-level structures on Drinfeld shtukas

Definition 2.4.2. A $\Gamma_0(\mathfrak{p}^n)$-*level structure* on a Drinfeld shtuka $\underline{\mathcal{E}}$ over a scheme S is a chain of $\mathfrak{p}^n$-cyclic isogenies

$$\underline{\mathcal{E}}_r = \underline{\mathcal{E}}(\mathfrak{p}^n) \xrightarrow{f_r} \underline{\mathcal{E}}_{r-1} \xrightarrow{f_{r-1}} \underline{\mathcal{E}}_{r-2} \to \ldots \xrightarrow{f_1} \underline{\mathcal{E}}_0 = \underline{\mathcal{E}}$$

such that the composition $f_r \circ \ldots \circ f_1$ is the inclusion $\underline{\mathcal{E}}_0(\mathfrak{p}^n) \hookrightarrow \underline{\mathcal{E}}_0$. We denote the stack of Drinfeld shtukas with $\Gamma_0(\mathfrak{p}^n)$-level structures by $\mathrm{Sht}_{r,\Gamma_0(\mathfrak{p}^n)}$.

Using the finite shtuka equivalence, a $\Gamma_0(\mathfrak{p}^n)$-level structure on $\underline{\mathcal{E}}$ is equivalently given by a flag

$$0 = \mathbf{H}_0 \subseteq \mathbf{H}_1 \subseteq \mathbf{H}_2 \subseteq \ldots \subseteq \mathbf{H}_r = \underline{\mathcal{E}}[\mathfrak{p}^n]$$

of finite locally free submodule schemes $\mathbf{H}_i \subseteq \underline{\mathcal{E}}[\mathfrak{p}^n]$ of rank $n \cdot i$ with strict $\mathbb{F}_q$-action such that $\mathbf{H}_i/\mathbf{H}_{i-1}$ is $\mathfrak{p}^n$-cyclic for all $1 \leqslant i \leqslant r$. In particular, a $\Gamma_0(\mathfrak{p}^n)$-level structure can fppf-locally on the base be extended to a $\Gamma_1(\mathfrak{p}^n)$-structure by definition. By Proposition 2.3.18, such a level structure can fppf-locally be extended to a $(\mathfrak{p}^{-n}/\mathcal{O}_0)^{r-1}$-structure on $\underline{\mathcal{E}}$. We call such an extension a $(\mathfrak{p}^{-n}/\mathcal{O}_0)^{r-1}$-*generator* of the $\Gamma_0(\mathfrak{p}^n)$-level structure.

We can now show one of our main theorems, that Drinfeld $\Gamma_0(\mathfrak{p}^n)$-level structures produce a regular moduli problem.

Theorem 2.4.3. *The stack $\mathrm{Sht}_{r,\Gamma_0(\mathfrak{p}^n)}$ is a regular Deligne-Mumford stacks locally of finite type over $\mathbb{F}_q$. The forgetful map $\mathrm{Sht}_{r,\Gamma_0(\mathfrak{p}^n)} \to \mathrm{Sht}_r$ is schematic and finite flat. It is finite étale away from 0. Moreover, the forgetful map $\mathrm{Sht}_{r,\Gamma_1(\mathfrak{p}^n)} \to \mathrm{Sht}_{r,\Gamma_0(\mathfrak{p}^n)}$ is schematic, faithfully flat and locally of finite presentation.*

Proof. We follow the proof of [KM85, Theorem 6.6.1]. As cyclicity is a closed condition by Theorem 2.4.1, $\mathrm{Sht}_{r,\Gamma_0(\mathfrak{p}^n)}$ is the closed substack of $\mathrm{Sht}_{r,\Gamma_1(\mathfrak{p}^n)}$ over which the universal isogeny is cyclic. Thus, the forgetful map $\mathrm{Sht}_{r,\Gamma_0(\mathfrak{p}^n)} \to \mathrm{Sht}_r$ is schematic and finite. It follows that $\mathrm{Sht}_{r,\Gamma_0(\mathfrak{p}^n)}$ is a Deligne-Mumford stack of finite type over $\mathbb{F}_q$. Moreover, the forgetful map $\mathrm{Sht}_{r,\Gamma_1(\mathfrak{p}^n)} \to \mathrm{Sht}_{r,\Gamma_0(\mathfrak{p}^n)}$ is representable by the scheme of generators $(\mathbf{H}_1/\mathbf{H}_0)^\times \times (\mathbf{H}_2/\mathbf{H}_1)^\times \ldots \times (\mathbf{H}_r/\mathbf{H}_{r-1})^\times$ and thus in particular finite flat by Theorem 2.4.1. Since $\mathrm{Sht}_{r,\Gamma_1(\mathfrak{p}^n)}$ is regular by Theorem 2.3.17, it follows that $\mathrm{Sht}_{r,\Gamma_0(\mathfrak{p}^n)}$ is also regular by [AK70, VII, Theorem 4.8]. Thus, also the map $\mathrm{Sht}_{r,\Gamma_0(\mathfrak{p}^n)} \to \mathrm{Sht}_r$ is finte flat by Miracle Flatness [Mat86, §23]. It is finite étale away from 0 by Proposition 2.3.14. $\square$

Remark 2.4.4. In a similar fashion we can also show the regularity of the moduli stack of Drinfeld shtukas together with a chain of $\mathfrak{p}^n$-cyclic isogenies of length $r' < r$. In other words, a chain of $\mathfrak{p}^n$-isogenies

$$\underline{\mathcal{E}}_{r'+1} = \underline{\mathcal{E}}_0(\mathfrak{p}^n) \overset{f_{r'+1}}{\to} \underline{\mathcal{E}}_{r'} \overset{f_{r'}}{\to} \underline{\mathcal{E}}_{r'-1} \to \ldots \overset{f_1}{\to} \underline{\mathcal{E}}_0 = \underline{\mathcal{E}}$$

such that $f_1, \ldots, f_{r'}$ are $\mathfrak{p}^n$-cyclic. The following corollaries have also obvious analogues in this setting. Note that we could generalise the argument to moduli spaces of other kinds of cyclic isogenies provided we had an analogue of Theorem 2.4.1.

Using the flatness of our moduli problems, we show that there are well-defined level maps.

Corollary 2.4.5. *Let $\underline{\mathcal{E}} \in \mathrm{Sht}_r(S)$ and let $(\mathbf{H}_i)_i$ be a $\Gamma_0(\mathfrak{p}^n)$-level structure on $\underline{\mathcal{E}}$. Let $\underline{n} = (n_1, \ldots, n_{r-1})$ with $0 \leqslant n_{r-1} \leqslant \ldots \leqslant n_1 \leqslant n$. Then there is a canonical subscheme $\mathbf{H}_{\underline{n}} \subseteq \mathbf{H}_{r-1}$, which is an $\mathcal{O}_0/\mathfrak{p}^n$-module subscheme $\mathbf{H}_{\underline{n}} \subseteq \underline{\mathcal{E}}[\mathfrak{p}^{\max\{n_i\}}]$. Fppf-locally on S, the scheme $\mathbf{H}_{\underline{n}}$ is defined for any $(\mathfrak{p}^{-n}/\mathcal{O}_0)^{r-1}$-generator $\iota\colon (\mathfrak{p}^{-n}/\mathcal{O}_0)^{r-1} \to \underline{\mathcal{E}}(S)$ of $(\mathbf{H}_i)_i$ by the restriction $\iota|_{(\mathfrak{p}^{-n_1}/\mathcal{O}_0)\oplus\ldots\oplus(\mathfrak{p}^{-n_{r-1}}/\mathcal{O}_0)}$ as in Proposition 2.3.10.*

Proof. We follow the proof of [KM85, Theorem 6.7.2]. It suffices to construct $\mathbf{H}_{\underline{n}}$ fppf-locally on S. We may thus assume that $(\mathbf{H}_i)_i$ admits a $(\mathfrak{p}^{-n}/\mathcal{O}_0)^{r-1}$-generator. Let ι and ι' be two such $(\mathfrak{p}^{-n}/\mathcal{O}_0)^{r-1}$-generators of $(\mathbf{H}_i)_i$. By Proposition 2.3.10 the restrictions to $(\mathfrak{p}^{-n_1}/\mathcal{O}_0) \oplus \ldots \oplus (\mathfrak{p}^{-n_{r-1}}/\mathcal{O}_0)$ of both ι and ι' define closed submodule schemes $\mathbf{H}_{\underline{n}}, \mathbf{H}'_{\underline{n}} \subseteq \underline{\mathcal{E}}[\mathfrak{p}^m]$. We have to check $\mathbf{H}_{\underline{n}} = \mathbf{H}'_{\underline{n}}$.

By reduction to the universal case we may assume that S is noetherian and flat over X', as the moduli space of Drinfeld shtukas with $\Gamma_0(\mathfrak{p}^n)$-level together with two generators is given by

$$\mathrm{Sht}_{r,(\mathfrak{p}^{-n}/\mathcal{O}_0)^{r-1}} \times_{\mathrm{Sht}_{r,\Gamma_0(\mathfrak{p}^n)}} \mathrm{Sht}_{r,(\mathfrak{p}^{-n}/\mathcal{O}_0)^{r-1}},$$

which is flat over X' by Theorem 2.4.3 and Corollary 2.3.19. In this case equality of closed subschemes of $\underline{\mathcal{E}}[\mathfrak{p}^m]$ is a closed condition by [KM85, Lemma 6.7.3]. The assertion is clear away from 0 and thus follows from the flatness of S in the general case. $\square$

Via the finite shtuka equivalence and Proposition 2.1.14 the submodule scheme $\mathbf{H}_{\underline{n}} \subseteq \underline{\mathcal{E}}[\mathfrak{p}^n]$ corresponds to a $\mathfrak{p}^n$-isogeny $\underline{\mathcal{E}}_{\underline{n}} \hookrightarrow \underline{\mathcal{E}}$. For $1 \leqslant m \leqslant n$ we denote by $\underline{m}^{(i)} = (m, \dots, m, 0, \dots, 0)$ with i non-zero entries. Then

$$0 = \mathbf{H}_{\underline{m}^{(0)}} \subseteq \mathbf{H}_{\underline{m}^{(1)}} \subseteq \dots \subseteq \mathbf{H}_{\underline{m}^{(r-1)}} \subseteq \underline{\mathcal{E}}[\mathfrak{p}^m]$$

is a $\Gamma_0(\mathfrak{p}^m)$ level structure on $\underline{\mathcal{E}}$. This shows that we have a well-defined level map $\mathrm{Sht}_{r,\Gamma_0(\mathfrak{p}^n)} \to \mathrm{Sht}_{r,\Gamma_0(\mathfrak{p}^m)}$ for all $0 \leqslant m \leqslant n$ which is automatically finite flat. We show that we can also construct this level map by taking closures without making explicit reference to the generators.

Corollary 2.4.6. *Let S be a scheme which is flat over X'. Let $\underline{\mathcal{E}}$ be a Drinfeld shtuka over S and Let $(\mathbf{H}_i)_i$ be a $\Gamma_0(\mathfrak{p}^n)$-level structure on $\underline{\mathcal{E}}$. For every $1 \leqslant i \leqslant r$ the canonical submodule scheme $\mathbf{H}_{\underline{m}^{(i)}} \subseteq \mathbf{H}_i$ is the schematic closure of $\mathbf{H}_i|_{S \times_{X'} (X' \setminus \{0\})}[\mathfrak{p}^m]$ in $\underline{\mathcal{E}}[\mathfrak{p}^m]$.*

Proof. From the explicit descriptions away from $\mathfrak{p}$ it is clear that $\mathbf{H}_{\underline{m}^{(i)}}$ is given by the $\mathfrak{p}^m$-torsion of $\mathbf{H}_i$ away from 0. The assertion then follows from the fact that $\mathbf{H}_{\underline{m}^{(i)}}$ is flat over S and closed in $\underline{\mathcal{E}}[\mathfrak{p}^m]$. □

Motivated by the discussion in Section 2.1.5, we also construct additional level maps. Recall that the $\Gamma_0(\mathfrak{p}^n)$-level corresponds to a standard $(r-1)$-simplex Ω of sidelength n in the standard appartment of the Bruhat-Tits building of GL_r. We want to construct level maps corresponding to inclusions of sub-$(r-1)$-simplices (of smaller sidelength). Recall that we enumerated alcoves in the standard apartment by its basepoint $\underline{m}$ and its orientation given by a permutation τ. In a similar fashion, a $(r-1)$-subsimplex of Ω is determined by its basepoint $\underline{m} = (m_1, \dots, m_{r-1})$ with $m_1 \geqslant \dots \geqslant m_{r-1}$, a sidelength n' and an orientation given by some $\tau \in \mathrm{Sym}_{r-1}$. Note that the simplex with basepoint $\underline{m}$, sidelength $\tilde{n}$ and orientation $\tau \in \mathrm{Sym}_{r-1}$ is contained in Ω if and only if $\underline{m} + \underline{\tilde{n}}_\tau^{(i)} \prec \Omega$. Let us denote by $\underline{\tilde{n}}_\tau^{(i)} \in \mathbb{Z}^{r-1}$ the vector containig $\tilde{n}$ in entries $\tau(1), \dots, \tau(i)$ and 0 otherwise.

Corollary 2.4.7. *Let $\underline{\mathcal{E}} \in \mathrm{Sht}_r(S)$ and let $(\mathbf{H}_i)_i$ be a $\Gamma_0(\mathfrak{p}^n)$-level structure on $\underline{\mathcal{E}}$. Let $\underline{m} = (m_1, \dots, m_{r-1})$ with $0 \leqslant m_{r-1} \leqslant \dots \leqslant m_1 \leqslant n$. Let $0 \leqslant \tilde{n} \leqslant n$ such that $\underline{m} + \underline{\tilde{n}}_\tau^{(i)} \prec \Omega$ for all i. Then the flag of quotients*

$$0 \subseteq \mathbf{H}_{\underline{m}+\underline{\tilde{n}}_\tau^{(1)}}/\mathbf{H}_{\underline{m}} \subseteq \mathbf{H}_{\underline{m}+\underline{\tilde{n}}_\tau^{(2)}}/\mathbf{H}_{\underline{m}} \subseteq \dots \subseteq \mathbf{H}_{\underline{m}+\underline{\tilde{n}}_\tau^{(r-1)}}/\mathbf{H}_{\underline{m}} \subseteq \underline{\mathcal{E}}_{\underline{m}}[\mathfrak{p}^{\tilde{n}}]$$

defines a $\Gamma_0(\mathfrak{p}^{\tilde{n}})$-level structure on the Drinfeld shtuka $\underline{\mathcal{E}}_{\underline{m}}$, which we denote by $(\mathbf{H}_{\underline{m}+\underline{\tilde{n}}_\tau^{(i)}}/\mathbf{H}_{\underline{m}})_i$. In case that $(\mathbf{H}_i)_i$ admits a $(\mathfrak{p}^{-n}/A)^{r-1}$-generator ι, $(\mathbf{H}_{\underline{m}+\underline{\tilde{n}}_\tau^{(i)}}/\mathbf{H}_{\underline{m}})_i$ is generated by

$$\iota_{\underline{m},\tilde{n},\tau} \colon (\mathfrak{p}^{-\tilde{n}}/A)^{r-1} \to \mathbf{H}_{\underline{m}+\underline{\tilde{n}}^{(r-1)}}/\mathbf{H}_{\underline{m}}(S).$$

defined using the isomorphism

$$(\mathfrak{p}^{-\tilde{n}}/A)^{r-1}$$
$$\cong \left((\mathfrak{p}^{-m_{\tau(1)}-\tilde{n}}/A) \oplus \ldots \oplus (\mathfrak{p}^{-m_{\tau(r-1)}-\tilde{n}}/A)\right) / \left((\mathfrak{p}^{-m_{\tau(1)}}/A) \oplus \ldots \oplus (\mathfrak{p}^{-m_{\tau(r-1)}}/A)\right).$$

Moreover, the canonical subscheme from Corollary 2.4.5 for $\underline{m'} = (m'_1, \ldots, m'_{r-1})$ with $\tilde{n} \geqslant m'_1 \geqslant \ldots \geqslant m'_{r-1} \geqslant 0$ is given by

$$(\mathbf{H}_{\underline{m}+\underline{\tilde{n}}_\tau^{(i)}}/\mathbf{H}_{\underline{m}})_{\underline{m'}} \cong \mathbf{H}_{\underline{m}+\tau(\underline{m'})}/\mathbf{H}_{\underline{m}}.$$

Proof. We follow the proof of [KM85, Theorem 6.7.4]. The question is fppf-local on S, so we can assume that $\mathbf{H}$ has a generator. By reduction to the universal case, we may further assume that S is flat over X' and noetherian by Theorem 2.3.9. Note that all assertions are clear away from 0. It thus suffices to show that the locus, where they are satisfied is closed.

The locus where each $\mathbf{H}_{\underline{m}+\underline{\tilde{n}}_\tau^{(i)}}/\mathbf{H}_{\underline{m}+\underline{\tilde{n}}_\tau^{(i-1)}}$ is $\mathfrak{p}^{\tilde{n}}$-cyclic is closed in S by Theorem 2.4.1 (3). This shows the first claim. For the second claim, the locus where $\iota_{\underline{m},\tilde{n},\tau}$ is a generator of $(\mathbf{H}_{\underline{m}+\underline{\tilde{n}}_\tau^{(i)}}/\mathbf{H}_{\underline{m}})_i$ is closed by [KM85, Proposition 1.9.1]. Moreover, the condition that $(\mathbf{H}_{\underline{m}+\underline{\tilde{n}}_\tau^{(i)}}/\mathbf{H}_{\underline{m}})_{\underline{m'}} \cong \mathbf{H}_{\underline{m}+\tau(\underline{m'})}/\mathbf{H}_{\underline{m}}$ is closed by [KM85, Lemma 6.7.3]. This shows the last claim. □

Associating the $\Gamma_0(\mathfrak{p}^{\tilde{n}})$-level structure $(\mathbf{H}_{\underline{m}+\underline{\tilde{n}}_\tau^{(i)}}/\mathbf{H}_{\underline{m}})_i$ on $\underline{\mathcal{E}_{m}}$ to $(\mathbf{H}_i)_i$ as in the previous corollary defines a map of stacks

$$F_{\underline{m},\tilde{n},\tau}\colon \mathrm{Sht}_{r,\Gamma_0(\mathfrak{p}^n)} \to \mathrm{Sht}_{r,\Gamma_0(\mathfrak{p}^{\tilde{n}})}.$$

Proposition 2.4.8. *The level map $F_{\underline{n},m,\tau}$ is schematic and finite locally free.*

Proof. Note that by Theorem 2.4.3 the map $F_{\underline{0},0}$ is schematic and finite locally free. As a first step we show that $F_{\underline{n},0}\colon \mathrm{Sht}_{r,\Gamma_0(\mathfrak{p}^n)} \to \mathrm{Sht}_r$ is schematic and finite locally free for all $\underline{m}$. In order to show that the map is representable by a finite scheme, we consider the auxiliary moduli problem $\mathrm{Sht}_{r,\underline{m}-\mathrm{isog},\Gamma_0(\mathfrak{p}^n)}$ parametrising a Drinfeld shtuka $\underline{\mathcal{E}}$, a $\mathfrak{p}^n$-isogeny $f\colon \underline{\mathcal{E}} \hookrightarrow \underline{\mathcal{E}}'$ such that $\mathrm{coker}(f)$ has rank $\sum_{i=1}^{r-1} m_i$ as $\mathcal{O}_S$-module, and a $\Gamma_0(\mathfrak{p}^n)$-level structure $(\mathbf{H}_i)_i$ on $\underline{\mathcal{E}}'$. The projection to $\underline{\mathcal{E}}$ defines then a map of stacks $\mathrm{Sht}_{r,\underline{m}-\mathrm{isog},\Gamma_0(\mathfrak{p}^n)} \to \mathrm{Sht}_r$ which is schematic and finite by Proposition 2.2.15 and Theorem 2.4.3. We also have a map $\mathrm{Sht}_{r,\Gamma_0(\mathfrak{p}^n)} \to \mathrm{Sht}_{r,\underline{m}-\mathrm{isog},\Gamma_0(\mathfrak{p}^n)}$ sending $(\underline{\mathcal{E}}, (\mathbf{H}_i)_i)$ to $(\underline{\mathcal{E}_{m}}, \underline{\mathcal{E}}, (\mathbf{H}_i)_i)$, which identifies $\mathrm{Sht}_{r,\Gamma_0(\mathfrak{p}^n)}$ with the substack of $\mathrm{Sht}_{r,\underline{m}-\mathrm{isog},\Gamma_0(\mathfrak{p}^n)}$ where $\underline{\mathcal{E}} = \underline{\mathcal{E}'_{m}}$. By [KM85, Lemma 6.7.3], this is schematic and representable by a closed immersion. The composition of the maps $\mathrm{Sht}_{r,\Gamma_0(\mathfrak{p}^n)} \to \mathrm{Sht}_{r,\underline{m}-\mathrm{isog},\Gamma_0(\mathfrak{p}^n)} \to \mathrm{Sht}_r$ is clearly given by $F_{\underline{m},0}$, which is thus schematic and finite.

Note that we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Sht}_{r,\Gamma_0(\mathfrak{p}^n)} \arrow[rr, "F_{\underline{m},\tilde{n}}"] \arrow[dr, "F_{\underline{m},0}"'] & & \mathrm{Sht}_{r,\Gamma_0(\mathfrak{p}^{\tilde{n}})} \arrow[dl, "F_{\underline{0},0}"] \\
& \mathrm{Sht}_r &
\end{tikzcd}$$

with vertical arrows that are schematic and finite. In order to see that $F_{\underline{m},\tilde{n},\tau}$ is schematic we argue as follows. We fix a map $S' \to \mathrm{Sht}_{r,\Gamma_0(\mathfrak{p}^{\tilde{n}})}$ from some S-scheme S', in other words a Drinfeld shtuka $(\underline{\mathcal{E}}, (\mathbf{H}_i)_i)$ together with a $\Gamma_0(\mathfrak{p}^{\tilde{n}})$-level structure. By composition with $F_{\underline{0},0}$, we get a map $S' \to \mathrm{Sht}_r$. By the discussion above, $S'' = S' \times_{\mathrm{Sht}_r} \mathrm{Sht}_{r,\Gamma_0(\mathfrak{p}^n)}$ is representable by a finite S-scheme. Let $(\underline{\mathcal{E}}', (\mathbf{H}_i')_i)$ denote the corresponding $\Gamma_0(\mathfrak{p}^n)$-level structure. Then the fibre product $S' \times_{\mathrm{Sht}_{r,\Gamma_0(\mathfrak{p}^{\tilde{n}})}} \mathrm{Sht}_{r,\Gamma_0(\mathfrak{p}^n)}$ is the locus where the image of $(\underline{\mathcal{E}}', (\mathbf{H}_i')_i)$ under $F_{\underline{m},\tilde{n},\tau}$ is given by $(\underline{\mathcal{E}}, (\mathbf{H}_i)_i)$. By [KM85, Lemma 6.7.3], this is representable by a closed subscheme of S''.

As both $F_{\underline{m},0}$ and $F_{\underline{0},0}$ are finite, it is immediate that $F_{\underline{m},\tilde{n},\tau}$ is finite as well. As both $\mathrm{Sht}_{r.\Gamma_0(\mathfrak{p}^n)}$ and $\mathrm{Sht}_{r,\Gamma_0(\mathfrak{p}^{\tilde{n}})}$ are regular and $(2r-1)$-dimensional, the level map is flat by miracle flatness. □

2.5. Comparison with naive level structures and Bruhat-Tits theory

We compare the Drinfeld level structures defined above with naive $\Gamma_0(\mathfrak{p}^n)$-level structures. The naive $\Gamma_0(\mathfrak{p}^n)$-level structures seem inadequate when $n > 1$ as the fibre above 0 is missing points (compare Remark 2.1.20). We construct a map from the stack of Drinfeld shtukas with naive $\Gamma_0(\mathfrak{p}^n)$-level structure to our stack of Drinfeld shtukas with Drinfeld $\Gamma_0(\mathfrak{p}^n)$-level which is an open immersion and an isomorphism away from 0. Moreover, we show that the two notions of level structures agree in the parahoric case. In this sense, the Drinfeld level structures provide a compactification of the level maps.

Recall that we defined a naive $\Gamma_0(\mathfrak{p}^n)$-level structure on a Drinfeld shtuka $\underline{\mathcal{E}} = (\mathcal{E}, \varphi)$ of rank r as a flag of quotients as $\mathfrak{p}^n$-torsion finite shtukas

$$\underline{\mathcal{E}}|_{D_{n,S}} = \underline{\mathcal{L}}_r \twoheadrightarrow \underline{\mathcal{L}}_{r-1} \twoheadrightarrow \ldots \twoheadrightarrow \underline{\mathcal{L}}_1 \subseteq \underline{\mathcal{L}}_0 = 0$$

such that $\mathcal{L}_i$ is finite locally free of rank i as $\mathcal{O}_{D_{n,S}}$-module. Equivalently, using Proposition 2.1.14, a naive $\Gamma_0(\mathfrak{p}^n)$-level structure is a chain of $\mathfrak{p}^n$-isogenies

$$\underline{\mathcal{E}}(\mathfrak{p}^n) = \underline{\mathcal{E}}_r \xrightarrow{f_r} \underline{\mathcal{E}}_{r-1} \xrightarrow{f_{r-1}} \underline{\mathcal{E}}_{r-2} \to \ldots \xrightarrow{f_1} \underline{\mathcal{E}}_0 = \underline{\mathcal{E}}$$

such that $\mathrm{coker}(f_i)$ is finite locally free of rank 1 as $\mathcal{O}_{D_{n,S}}$-module.

Lemma 2.5.1. *Let $\underline{\mathcal{E}} = (\mathcal{E}, \varphi)$ be a Drinfeld shtuka of rank r over S and let $\mathcal{E}|_{D_{n,S}} \twoheadrightarrow \mathcal{L}$ be a quotient $\mathfrak{p}^n$-torsion finite shtuka such that $\mathcal{L}$ is finite locally free of rank 1 as $\mathcal{O}_{D_{n,S}}$-module. Then $\mathrm{Dr}_q(\underline{\mathcal{L}}) \subseteq \underline{\mathcal{E}}[\mathfrak{p}^n]$ is an $\mathfrak{p}^n$-cyclic submodule scheme.*

Proof. We denote by $\underline{\mathcal{L}}^{(i)} = \underline{\mathcal{L}}|_{D_{i,S}}$ for $1 \leqslant i \leqslant n$. Then $\underline{\mathcal{L}}^{(i)}$ is a locally free $\mathcal{O}_{D_{i,S}}$-module of rank 1, and consequently a locally free $\mathcal{O}_S$-module of rank i. Thus,

$$\underline{\mathcal{L}} = \underline{\mathcal{L}}^{(n)} \twoheadrightarrow \underline{\mathcal{L}}^{(n-1)} \twoheadrightarrow \ldots \twoheadrightarrow \underline{\mathcal{L}}^{(2)} \twoheadrightarrow \underline{\mathcal{L}}^{(1)} \twoheadrightarrow 0$$

corresponds via the finite shtuka equivalence to a flag of finite locally free submodule schemes with strict $\mathbb{F}_q$-action

$$0 \subseteq \mathbf{H}^{(1)} \subseteq \ldots \subseteq \mathbf{H}^{(n-1)} \subseteq \mathbf{H}^{(n)} \subseteq \underline{\mathcal{E}}[\mathfrak{p}^n],$$

where we denote by $\mathbf{H}^{(i)} = \mathrm{Dr}_q(\underline{\mathcal{L}}^{(i)})$. It is clear that $\mathbf{H}^{(i)} \subseteq \mathbf{E}[\mathfrak{p}^i]$ by construction. As a next step, we inductively construct a generator of $\mathbf{H}^{(i)}$ fppf-locally on S.

We may assume that $S = \mathrm{Spec}(R)$ is affine and that $\mathcal{L}$ is a free $\mathcal{O}_{D_{n,S}} = R[\varpi]/(\varpi^n)$-module of rank 1. Then, $\mathcal{L}^{(i)} \cong R[\varpi]/(\varpi^i)$. We choose the standard basis $1, \varpi, \ldots, \varpi^{i-1}$ of $\mathcal{L}^{(i)}$ as R-module. As a map of finite free $R[\varpi]/(\varpi^i)$-modules, φ is given by multplication by an element $\alpha = \sum_{j=0}^{i-1} \alpha_j \varpi^j \in R[\varpi]/(\varpi^i)$, and thus, its matrix as an R-linear map with respect to the standard basis is given by

$$\begin{pmatrix} \alpha_0 & & & \\ \alpha_1 & \alpha_0 & & \\ \vdots & & \ddots & \\ \alpha_i & \alpha_{i-1} & \ldots & \alpha_0 \end{pmatrix}.$$

It follows that

$$\mathbf{H}^{(i)} = \mathrm{Dr}_q(\underline{\mathcal{L}}^{(i)}) = \mathrm{Spec}\left(R[t_0, \ldots, t_{i-1}]/(t_0^q - \sum_{j=0}^{i-1} \alpha_j t_j, t_1^q - \sum_{j=1}^{i-1} \alpha_{j-1} t_j, \ldots, t_{i-1}^q - \alpha_0 t_{i-1}) \right).$$

As the question is fppf-local on R, we may assume that R contains a root β_0 of the polynomial $t^{q-1} - \alpha_0$, a root β_1 of the polynomial $t^q - \alpha_0 t_1 - \alpha_1 \beta_0$ and inductively a root β_j of the polynomial $t^q - \alpha_0 t - \alpha_1 \beta_{j-1} - \ldots - \alpha_j \beta_0$ for all $0 \leqslant j \leqslant i-1$. Then $(\beta_{i-1}, \beta_{i-2}, \ldots, \beta_1, \beta_0)$ is a section of $\mathbf{H}^{(i)}$ over R by construction. We claim that the map

$$\begin{aligned} \iota^{(i)} \colon \mathfrak{p}^{-i}/\mathcal{O}_0 &\to \mathbf{H}^{(i)}(R) \\ \varpi^{-i} &\mapsto (\beta_{i-1}, \beta_{i-2}, \ldots, \beta_1, \beta_0) \end{aligned}$$

is a $\mathfrak{p}^{-i}/\mathcal{O}_0$-generator of $\mathbf{H}^{(i)}$. We proceed by induction on i.

Let $i = 1$. In this case $\mathbf{H}^{(1)} = \mathrm{Spec}(R[t]/(t_0^q - \alpha_0 t_0))$. In particular, $\mathbf{H}^{(1)}$ can be embedded in $\mathbb{A}^1_R$. Then $\mathfrak{p}^{-1}/\mathcal{O}_0 \to \mathbf{H}^{(1)}(R)$ given by $\varpi^{-1} \mapsto \beta$ is a generator of $\mathbf{H}^{(1)}$, as $\prod_{a \in \mathbb{F}_q}(t - a\beta) = t^q - \beta^{q-1}t = t^q - \alpha t$.

Let us now assume that the claim is true for $i \geqslant 1$. Note that the subscheme $\mathbf{H}^{(i)} \subseteq \mathbf{H}^{(i+1)}$ is given by the locus where $t_i = 0$ by construction. Note that the map $\iota^{(i+1)}|_{\mathfrak{p}^{-i}/\mathcal{O}_0} = \iota^{(i)}$ is given by $\varpi^{-i} \mapsto (\beta_{i-1}, \beta_{i-2}, \ldots, \beta_1, \beta_0, 0)$. Thus, it factors through $\mathbf{H}^{(i)}$ and is a generator of $\mathbf{H}^{(i)}$ by hypothesis.

Moreover, the quotient $\mathbf{H}^{(i+1)}/\mathbf{H}^{(i)}$ is then given by the canonical inclusion

$$R[t_i]/(t_i^q - \alpha_0 t_i) \to R[t_0, \ldots, t_{i-1}, t_i]/(t_0^q - \sum_{j=0}^{i} \alpha_j t_j, \ldots, t_i^q - \alpha_0 t_i).$$

Moreover, the image of the section $(\beta_0, \ldots, \beta_{i-1})$ of $\mathbf{H}^{(i+1)}$ in the quotient $\mathbf{H}^{(i+1)}/\mathbf{H}^{(i)}$ is β_0. In particular, the map

$$\iota^{(i+1)} \mod \mathfrak{p}^{(i)} \colon \mathfrak{p}^{-1}/\mathcal{O}_0 \cong (\mathfrak{p}^{-(i+1)}/\mathcal{O}_0)/(\mathfrak{p}^{-i}/\mathcal{O}_0) \to \left(\mathbf{H}^{(i+1)}/\mathbf{H}^{(i)}\right)(R)$$

is well-defined and sends ϖ^{-1} to β_0 and is thus a generator of $\mathbf{H}^{(i+1)}/\mathbf{H}^{(i)}$ by the discussion of the case $i = 1$ above. By [KM85, Lemma 1.11.3] it follows that $\iota^{(i+1)}$ is a full set of sections of $\mathbf{H}^{(i+1)}$.

Thus, $(\mathbf{H}^{(1)}, \iota)$ is a generator of $\mathbf{H}^{(i+1)}$ in the sense of Definition 2.3.2 and $\mathbf{H}^{(i+1)}$ is $\mathfrak{p}^{i+1}$-cyclic. This shows the claim. $\square$

Proposition 2.5.2. *Let $\underline{\mathcal{E}} = (\mathcal{E}, \varphi)$ be a Drinfeld shtuka over S and let*

$$\mathcal{E}|_{D_{n,S}} = \mathcal{L}_r \twoheadrightarrow \mathcal{L}_{r-1} \twoheadrightarrow \ldots \twoheadrightarrow \mathcal{L}_1 \twoheadrightarrow \mathcal{L}_0 = 0$$

be a naive $\Gamma_0(\mathfrak{p}^n)$-level structure on $\underline{\mathcal{E}}$. Then

$$0 \subseteq \mathrm{Dr}_q(\underline{\mathcal{L}}_1) \subseteq \ldots \mathrm{Dr}_q(\underline{\mathcal{L}}_{r-1}) \subseteq \underline{\mathcal{E}}[\mathfrak{p}^n]$$

is a Drinfeld $\Gamma_0(\mathfrak{p}^n)$-level structure on $\underline{\mathcal{E}}$ in the sense of Definition 2.4.2.

Proof. This follows directly from Lemma 2.5.1. $\square$

Recall that a Drinfeld shtuka with a naive $\Gamma_0(\mathfrak{p}^n)$-level structure is a bounded global $\mathrm{GL}_{r,\Omega}$-shtuka for the Bruhat-Tits group scheme $\mathrm{GL}_{r,\Omega}$ as defined in Remark 2.1.19. In particular, we thus constructed a map of Deligne-Mumford stacks

$$\mathrm{Sht}_{r,\Omega} \to \mathrm{Sht}_{r,\Gamma_0(\mathfrak{p}^n)}\,. \tag{2.2}$$

As a next step, we show that the map (2.2) is an isomorphism in the case $n = 1$.

Proposition 2.5.3. *Let $\underline{\mathcal{E}} \in \mathrm{Sht}_r(S)$. Then every $\Gamma_0(\mathfrak{p})$-level structure on $\underline{\mathcal{E}}$ comes from a naive $\Gamma_0(\mathfrak{p})$-level structure.*

Proof. It suffices to show that for an $\mathfrak{p}$-cyclic submodule scheme $\mathbf{H} \subseteq \underline{\mathcal{E}}[\mathfrak{p}]$, the corresponding finite shtuka $\underline{M}_q(\mathbf{H})$ is finite locally free of rank 1 as $R \otimes \mathcal{O}_0/\mathfrak{p} \cong R$-module. But this is clear by construction. $\square$

Lemma 2.5.4. *Let $\underline{\mathcal{E}} \in \mathrm{Sht}_r(S)$ and assume its characteristic is away from 0. Then every $\Gamma_0(\mathfrak{p}^n)$-level structure on $\underline{\mathcal{E}}$ comes from a naive $\Gamma_0(\mathfrak{p}^n)$-level structure.*

Proof. Let $(\mathbf{H}_i)_{1\leqslant i\leqslant r}$ be a $\Gamma_0(\mathfrak{p}^n)$-structure on $\underline{\mathcal{E}}$. As the characteristic of $\underline{\mathcal{E}}$ is away from 0, all the $\mathbf{H}_i$ are finite étale over S. As the claim is fppf-local on the base, we may choose a $(\mathfrak{p}^{-n}/\mathcal{O}_0)^{r-1}$-generator of $(\mathbf{H}_i)_{1\leqslant i\leqslant r}$. By Proposition 2.3.6, the $\mathbf{H}_i$ are then given by

$$0 \subseteq (\mathfrak{p}^{-n}/\mathcal{O}_0)_S \subseteq (\mathfrak{p}^{-n}/\mathcal{O}_0)_S^2 \subseteq \ldots \subseteq (\mathfrak{p}^{-n}/\mathcal{O}_0)_S^{r-1} \subseteq \underline{\mathcal{E}}[\mathfrak{p}^n].$$

By the finite shtuka equivalence, this corresponds to the flag of quotients

$$\underline{\mathcal{E}}|_{D_{n,S}} \twoheadrightarrow \mathcal{O}_{D_{n,S}}^{r-1} \twoheadrightarrow \ldots \twoheadrightarrow \mathcal{O}_{D_{n,S}} \twoheadrightarrow 0,$$

where the map $\mathcal{O}_{D_{n,S}}^{i+1} \to \mathcal{O}_{D_{n,S}}^{i}$ is given by the projection to the first i components and the Frobenius-linear map on $\mathcal{O}_{D_{n,S}}^{i}$ is the trivial one. This is clearly a naive $\Gamma_0(\mathfrak{p}^n)$-level structure. $\square$

Proposition 2.5.5. *The map (2.2) is schematic and a quasi-compact open immersion.*

Proof. By construction, $\mathrm{Sht}_{r,\Omega}$ is identified with the substack of $\mathrm{Sht}_{r,\Gamma_0(\mathfrak{p}^n)}$ where all the $\mathbf{H}_i$ correspond to finite locally free $\mathcal{O}_{D_{n,S}}$-modules of rank i via the finite shtuka equivalence, or equivalently the locus, where all $\mathbf{H}_i/\mathbf{H}_{i-1}$ correspond to finite locally free $\mathcal{O}_{D_{n,S}}$-modules of rank 1.

In order to show that this condition is representable by an open subscheme, we work locally and assume that $S = \mathrm{Spec}(R)$ is affine. Let M be a $R[\varpi]/(\varpi^n)$-module such that M is free of rank n as R-module. Then M is locally free of rank 1 as $R[\varpi]/(\varpi^n)$-module if and only if it is generated by a single element.

Let $q \subseteq R$ be a prime ideal such that $M \otimes \kappa(q)$ is a one-dimensional vector space over the residue field $\kappa(q)$ of $R[\varpi]/(\varpi^n)$ at (q,ϖ) ($\kappa(q)$ is also the residue field of R at q). By Nakayama's Lemma, there exists a $a \in (R[\varpi]/(\varpi^n))\backslash(q,\varpi)$ such that $M[a^{-1}]$ is free of rank 1. Let $a_0 = a(0)$ be the constant term of a. Then $M[a^{-1}] = M[a_0^{-1}]$. Hence, the principal open $D(a_0) \subseteq \mathrm{Spec}(R)$ is an open neighbourhood of q such that $M[a_0^{-1}]$ is locally free of rank 1 as $R[a_0^{-1}][\varpi]/\varpi^n$-module over $D(a)$. Hence, the condition is representable by an open immersion on the base scheme. $\square$

By Propositions 2.4.8 and 2.5.3, we can interpret the level maps to $\Gamma_0(\mathfrak{p})$-level structures as maps

$$F_{\underline{m},1,\tau}\colon \operatorname{Sht}_{r,\Gamma_0(\mathfrak{p}^n)} \to \operatorname{Sht}_{r,\mathfrak{f}_{\underline{m},\tau}},$$

where $\mathfrak{f}_{\underline{m},\tau}$ is the alcove in the Bruhat-Tits building corresponding to $\underline{m}$ and τ. This system of maps is compatible with level maps to parahoric levels given by smaller facets by Corollary 2.4.5 and thus define a map

$$F_\Omega\colon \operatorname{Sht}_{r,\Gamma_0(\mathfrak{p}^n)} \to \varprojlim_{\mathfrak{f}<\Omega} \operatorname{Sht}_{r,\mathfrak{f}}. \tag{2.3}$$

Proposition 2.5.6. *The map F_Ω is a closed immersion.*

Proof. As all the maps $\operatorname{Sht}_{r,\Gamma_0(\mathfrak{p}^n)} \to \varprojlim_{\mathfrak{f}<\Omega} \operatorname{Sht}_{r,\mathfrak{f}}$ are schematic and finite, so is their limit. Moreover, by the explicit moduli description it is clear that the map F_Ω is a monomorphism. □

Theorem 2.5.7. *The map $\operatorname{Sht}_{r,\Omega} \to \varprojlim_{\mathfrak{f}<\Omega} \operatorname{Sht}_{r,\mathfrak{f}}$ is schematic and representable by a quasi-compact open immersion that is an isomorphism away from 0. Its schematic image in the sense of [EG21] is*

$$\overline{\operatorname{Sht}}_{r,\Omega} = \operatorname{Sht}_{r,\Gamma_0(\mathfrak{p}^n)}$$

via the maps

$$\operatorname{Sht}_{r,\Omega} \hookrightarrow \operatorname{Sht}_{r,\Gamma_0(\mathfrak{p}^n)} \hookrightarrow \varprojlim_{\mathfrak{f}<\Omega} \operatorname{Sht}_{r,\mathfrak{f}}$$

constructed above. In the parahoric case $n = 1$, the map $\operatorname{Sht}_{r,\Omega} \to \operatorname{Sht}_{r,\Gamma_0(\mathfrak{p})}$ is an isomorphism.

Proof. The assertion for the parahoric case is Proposition 2.5.3. That the inclusion

$$\operatorname{Sht}_{r,\Omega} \to \varprojlim_{\mathfrak{f}<\Omega} \operatorname{Sht}_{r,\mathfrak{f}}$$

is schematic and representable by a quasi-compact locally closed immersion follows from Propositions 2.5.5 and 2.5.6. That the image of $\operatorname{Sht}_{r,\Omega}$ in $\operatorname{Sht}_{r,\Gamma_0(\mathfrak{p}^n)}$ is dense follows from the fact that the inclusion (2.2) is an isomorphism away from 0 by Lemma 2.5.4 together with the flatness of $\operatorname{Sht}_{r,\Gamma_0(\mathfrak{p}^n)}$ over X' from Theorem 2.4.3.

In order to see that the map $\operatorname{Sht}_{r,\Omega} \to \varprojlim_{\mathfrak{f}<\Omega} \operatorname{Sht}_{r,\mathfrak{f}}$ is already open, we follow the proof of Proposition 2.5.5. One can again check that a point $(\mathcal{E}_{\underline{m}})_{\underline{m}} \in \varprojlim_{\mathfrak{f}<\Omega} \operatorname{Sht}_{r,\mathfrak{f}}$ comes from $\operatorname{Sht}_\Omega$ if and only if the cokernels of the isogenies $\underline{\mathcal{E}}_{(n,\dots,n,0,0,\dots,0)} \hookrightarrow \underline{\mathcal{E}}_{(n,\dots,n,n,0,\dots,0)}$ are locally free of rank 1 as $\mathcal{O}_{D_{n,S}}$-modules. By the argument in the proof of Proposition 2.5.5, this condition is representable by an open subscheme. □

3. Integral models of moduli spaces of shtukas with deep Bruhat-Tits level structures

3.1. Torsors under Bruhat-Tits group schemes

We show that a Bruhat-Tits group scheme is the limit of all corresponding parahoric group schemes and use this observation to show that the induced map on the level of $\mathrm{Bun}_{\mathcal{G}}$ is an open immersion. We first discuss (pseudo-)torsors for limits of groups.

3.1.1. Pseudo-torsors for limits of groups

We use the following result on pseudo-torsors under limits of groups. For a sheaf of groups $\underline{G}$ on a site $\mathcal{C}$ we denote by $\mathrm{PTor}_{\underline{G}}$ the category of $\underline{G}$*-pseudo-torsors* for $\underline{G}$ with $\underline{G}$-equivariant maps. In other words, an object of $\mathrm{PTor}_{\underline{G}}$ is given by a sheaf E on $\mathcal{C}$ together with a (right) action $E \times \underline{G} \to E$ of $\underline{G}$ such that the induced map $E \times \underline{G} \to E \times E$ given by $(e, g) \mapsto (e, eg)$ is an isomorphism. A map $f \colon \underline{G} \to \underline{G}'$ of sheaves of groups on $\mathcal{C}$ induces a functor $f_* \colon \mathrm{PTor}_{\underline{G}} \to \mathrm{PTor}_{\underline{G}'}$ given by $E \mapsto E \times^{\underline{G}} \underline{G}'$, where the action of $\underline{G}'$ is by right multiplication in the second factor. Moreover, the canonical map $(\mathrm{id}_E, \mathbf{1}_{\underline{G}'}) \colon E \to E \times^{\underline{G}} \underline{G}'$ is $\underline{G}$-equivariant for the $\underline{G}$-action on $E \times^{\underline{G}} \underline{G}'$ via f on the second factor.

A $\underline{G}$-pseudo-torsor E is a G*-torsor* if for every object U on $\mathcal{C}$ there is a cover $\{U_i \to U \colon i \in I\}$ of U in $\mathcal{C}$ such that $\Gamma(U_i, E) \neq \varnothing$. We denote by $\mathfrak{B}(\underline{G})$ the full subcategory of $\mathrm{PTor}_{\underline{G}}$ of $\underline{G}$-torsors on $\mathcal{C}$. The map f_* for a map of sheaves of groups $f \colon \underline{G} \to \underline{G}'$ restricts to a map $f_* \colon \mathfrak{B}(\underline{G}) \to \mathfrak{B}(\underline{G}')$.

Lemma 3.1.1. *Let I be a finite partially ordered set and let $(\underline{G}_i)_{i \in I}$ be a diagram of sheaves of groups over I. Let $\underline{G} = \varprojlim_{i \in I} \underline{G}_i$. Then $\underline{G}$ is a sheaf of groups on $\mathcal{C}$ together with a compatible system of projection maps $f_i \colon \underline{G} \to \underline{G}_i$. The functor*

$$\varprojlim_{i \in I} f_{i,*} \colon \mathrm{PTor}_{\underline{G}} \to \varprojlim_{i \in I} \mathrm{PTor}_{\underline{G}_i}, \qquad E \mapsto (E \times^{\underline{G}} \underline{G}_i)_{i \in I}$$

has a right-adjoint given by

$$\lim\colon \left(\varprojlim_{i\in I} \mathrm{PTor}_{\underline{G_i}}\right) \to \mathrm{PTor}_{\underline{G}}, \qquad (E_i)_{i\in I} \mapsto \varprojlim_{i\in I} E_i.$$

Moreover, the restriction $\varprojlim_{i\in I} f_{i,*}\colon \mathfrak{B}(\underline{G}) \to \varprojlim_{i\in I} \mathfrak{B}(\underline{G_i})$ to the full subcategory of torsors is fully faithful.

Proof. As a first step, we show that $\varprojlim_{i\in I} E_i$ is indeed a pseudo-torsor for $\underline{G}$. The sheaf of groups $\underline{G}$ acts on E_i by the action induced by f_i, and all these actions are compatible by the observation above that the reduction maps are equivariant. Hence, $\varprojlim_{i\in I} E_i$ carries a canonical $\underline{G}$-action. As all the E_i are pseudo-torsors under $\underline{G_i}$, the induced map

$$\begin{gathered}\left(\varprojlim_{i\in I} E_i\right) \times \underline{G} \to \left(\varprojlim_{i\in I} E_i\right) \times \left(\varprojlim_{i\in I} E_i\right)\\ ((e_i)_{i\in I}, g) \mapsto ((e_i)_{i\in I}, (e_i f_i(g))_{i\in I})\end{gathered}$$

is an isomorphism, so $\varprojlim_{i\in I} E_i$ is a $\underline{G}$-pseudo-torsor.

As a next step, we show that the limit is right adjoint to the family of projections. Let $(F_i)_{i\in I} \in \varprojlim_{i\in I} \mathrm{PTor}_{\underline{G_i}}$. A $\underline{G}$-equivariant map $E \to F_i$ factors as $E \to E \times^{\underline{G}} \underline{G_i} \to F_i$ for a unique $\underline{G_i}$-equivariant map $E \times^{\underline{G}} \underline{G_i} \to F_i$. Hence, we get

$$\mathrm{Hom}_{\mathrm{PTor}_{\underline{G}}}(E, \varprojlim_{i\in I} F_i) = \mathrm{Hom}_{\varprojlim_{i\in I} \mathrm{PTor}_{\underline{G_i}}}((E \times^{\underline{G}} \underline{G_i})_{i\in I}, (F_i)_{i\in I}).$$

In order to see that the restriction to $\mathfrak{B}(\underline{G})$ is fully faithful, we check that the unit of the adjunction $E \mapsto \varprojlim_{i\in I} E \times^{\underline{G}} \underline{G_i}$ is an isomorphism for $E \in \mathfrak{B}(\underline{G})$. We can do so locally, so we may assume that E is trivial. As all maps $E \to E \times^{\underline{G}} \underline{G_i}$ are $\underline{G}$-equivariant, choosing a trivialisation of E induces a compatible choice of trivialisations of all $E \times^{\underline{G}} \underline{G_i}$. Hence, the map $E \to \varprojlim_{i\in I} E \times^{\underline{G}} \underline{G_i}$ is given by $\underline{G} \to \varprojlim_{i\in I} \underline{G_i}$, which is an isomorphism by construction. $\square$

Remark 3.1.2. Note that given a compatible family of $\underline{G_i}$-torsors $(E_i)_{i\in I} \in \varprojlim_{i\in I} \mathfrak{B}(G_i)$, their limit will in general not be a $\underline{G}$-torsor, as it might not be possible to produce a compatible system of sections for $(E_i)_{i\in I}$. For example, consider $G_1 = G_2 = \{e\}$ the trivial group and $G_3 = \mathbb{Z}/2$. Then $G_1 \times_{G_3} G_2 = \{e\}$ is again the trivial group. Let us moreover consider the sets $E_1 = E_2 = \{*\}$ and $E_3 = \{a_1, a_2\}$. Then E_i is a trivial G_i-torsor for all $i = 1, 2, 3$. However, under the maps $f_i\colon E_i \to E_3, * \mapsto a_i$ for $i = 1, 2$, the fibre product $E_1 \times_{E_3} E_2$ is empty, hence in particular not a torsor under the trivial group.

3.1.2. Deep Bruhat-Tits group schemes are limits of parahoric group schemes

Let us briefly recall some facts from Bruhat-Tits theory [BT72; BT84]. In this section, let k be a discretely valued henselian field with ring of integers $\mathcal{O}$. We denote by $\mathfrak{m} \subseteq \mathcal{O}$ its maximal ideal and by $\mathbb{F} = \mathcal{O}/\mathfrak{m}$ its residue field. Moreover, we denote by k^{ur} the maximal unramified extension inside some fixed algebraic closure of k, by $\mathcal{O}^{\mathrm{ur}}$ its ring of integers and by $\breve{k}$ (respectively $\breve{\mathcal{O}}$) the completion of k^{ur} (respectively $\mathcal{O}^{\mathrm{ur}}$).

Let G be a (connected) reductive group over k such that G is quasi-split over k^{ur}. Note that G is automatically quasi-split over k^{ur} when the cohomological dimension of k^{ur} is at most 1 by a theorem of Steinberg. This includes in particular the case $k = \mathbb{F}(\!(\varpi)\!)$ for a finite field $\mathbb{F}$ we are interested in later. Let us fix a maximal k-split torus $S \subseteq G$. We denote by $\mathcal{B}(G/k)$ the corresponding (reduced) Bruhat-Tits building and by $\mathcal{A} = \mathcal{A}(G, S, k) \subseteq \mathcal{B}(G, k)$ the apartment corresponding to S. Let $\Phi = \Phi(G, S)$ be the set of roots of G with respect to S and let $\Phi^+ \subseteq \Phi$ be a system of positive roots. We denote by $\Phi^- = -\Phi^+$ and by $\Phi^+_{\mathrm{nd}} \subseteq \Phi^+$ (respectively by $\Phi^-_{\mathrm{nd}} \subseteq \Phi^-$) the subset of non-divisible positive (respectively negative) roots.

We consider the space of affine functionals $\mathcal{A}^*$ on $\mathcal{A}$ and the set of affine roots $\Psi = \Psi(G, S) \subseteq \mathcal{A}^*$ of G with respect to S. For an affine functional $\psi \in \mathcal{A}^*$, let $\mathcal{H}_\Psi \subseteq \mathcal{A}$ be the vanishing hyperplane for ψ and let $\mathcal{H}_{\psi \geqslant 0} = \{x \in \mathcal{A} \colon \psi(x) \geqslant 0\}$ (respectively $\mathcal{H}_{\psi \leqslant 0} = \{x \in \mathcal{A} \colon \psi(x) \leqslant 0\}$) be the corresponding half-spaces. For an affine functional $\psi \in \mathcal{A}^*$, we denote by $\dot{\psi}$ its gradient. By construction, for $\psi \in \Psi$ we have $\dot{\psi} \in \Phi$.

For a non-empty bounded subset $\Omega \subseteq \mathcal{A}$, we consider the corresponding (local) Bruhat-Tits group scheme[1] $\mathcal{G}_\Omega$ constructed in [BT84, § 5.1.9 (resp. § 4.6.26)]. It is the unique smooth affine $\mathcal{O}$-group scheme with generic fibre G, connected special fibre and $\mathcal{G}_\Omega(\mathcal{O}^{\mathrm{ur}}) = G(k^{\mathrm{ur}})^0_\Omega$, where $G(k^{\mathrm{ur}})^0_\Omega$ is the "connected" (pointwise) stabiliser of Ω.

For a bounded subset $\Omega \subseteq \mathcal{A}$, we denote by $\mathrm{cl}(\Omega) = \bigcap_{\psi \in \Psi, \Omega \subseteq \mathcal{H}_{\psi \geqslant 0}} \mathcal{H}_{\psi \geqslant 0}$ the intersection of all half-spaces containing Ω. Then the corresponding Bruhat-Tits group scheme does not change when replacing Ω by $\mathrm{cl}(\Omega)$, compare [BT84, § 4.6.27]. Hence, we may always assume $\Omega = \mathrm{cl}(\Omega)$ in the following. By construction, $\mathrm{cl}(\Omega)$ is convex. For two bounded subsets Ω, Ω' of $\mathcal{A}(G, S, k)$ with $\Omega = \mathrm{cl}(\Omega)$, we write $\Omega' \prec \Omega$ if Ω' is contained in Ω. In this case, we obtain an induced homomorphism of $\mathcal{O}$-group schemes $\rho_{\Omega', \Omega} \colon \mathcal{G}_\Omega \to \mathcal{G}_{\Omega'}$ whose restriction to the generic fibre is given by the identity on G. Below, we often take limits over the partially ordered set $\{\mathfrak{f} \prec \Omega\}$ of facets contained in Ω ordered by inclusion. This poset is connected as $\Omega = \mathrm{cl}(\Omega)$ is connected.

For a root $a \in \Phi$ and Ω as above, we denote by $U_{a,\Omega} \subseteq G(k)$ the corresponding root

[1] In the literature it is often additionally required that Ω is contained in a facet. We explicitly allow Ω to not be contained in the closure of a facet (this will be the interesting case later) and call $\mathcal{G}_\Omega$ with Ω contained in the closure of a facet a *parahoric* (Bruhat-Tits) group scheme.

subgroup and by $\mathcal{U}_{a,\Omega}$ its integral model, which is a smooth affine $\mathcal{O}$-group scheme. As for the $\mathcal{G}_\Omega$, the group scheme $\mathcal{U}_{a,\Omega}$ only depends on $\mathrm{cl}(\Omega)$ and for $\Omega' \prec \Omega$ there is a natural map $\mathcal{U}_{a,\Omega} \to \mathcal{U}_{a,\Omega'}$. These integral models are used to construct the *big open cell*

$$\prod_{a \in \Phi^-_{\mathrm{nd}}} \mathcal{U}_{a,\Omega} \times \mathcal{Z} \times \prod_{a \in \Phi^+_{\mathrm{nd}}} \mathcal{U}_{a,\Omega} \hookrightarrow \mathcal{G}_\Omega,$$

which is an open immersion by [BT84, § 4.6.2], where $\mathcal{Z}$ is an integral model of the centraliser Z of S. Note that when G is quasi-split, $T = Z$ is a maximal torus in G.

The main result of this section is the following theorem.

Theorem 3.1.3. *Let G be a reductive group over k such that G is quasi-split over the maximal unramified extension k^{ur} of k. Let $\Omega \subseteq \mathcal{A}(G,S,k)$ be a bounded subset with $\Omega = \mathrm{cl}(\Omega)$. The map*

$$\rho = \varprojlim_{\mathfrak{f} \prec \Omega} \rho_{\mathfrak{f},\Omega} \colon \mathcal{G}_\Omega \to \varprojlim_{\mathfrak{f} \prec \Omega} \mathcal{G}_{\mathfrak{f}}$$

induced by the $\rho_{\mathfrak{f},\Omega}$ for facets $\mathfrak{f} \prec \Omega$ is an isomorphism of $\mathcal{O}$-group schemes.

We need some results on the deformation theory of torsors under (limits of) Bruhat-Tits group schemes. For us, torsors are always taken with respect to the fppf-topology. However, torsors for smooth affine group schemes are always representable by a (necessarily smooth affine) scheme and thus have sections étale locally. The deformation theory of such sections of torsors can be controlled by the (dual of) the invariant differentials $\omega_{\mathcal{G}/\mathcal{O}} = e^*\Omega_{\mathcal{G}/\mathcal{O}}$, where $e\colon \mathcal{O} \to \mathcal{G}$ is the identity section, due to the following result.

Lemma 3.1.4. *Let $\mathcal{G}$ be a smooth affine $\mathcal{O}$-group scheme and let R be an $\mathcal{O}$-algebra with an ideal I of square $I^2 = 0$. We denote by $\overline{R} = R/I$ and $r\colon \mathcal{O} \to \overline{R}$ the induced map. Let $\mathcal{E}$ be a $\mathcal{G}$-torsor over R. Let $\gamma \in \mathcal{E}(\overline{R})$ be a section of $\mathcal{E}$. Then the set of all lifts of γ to R is a torsor under $\mathfrak{g}_{(R,I)} = r^*\omega^\vee_{\mathcal{G}/\mathcal{O}} \otimes_{\overline{R}} I$.*

Proof. This is essentially a special case of [SGA1, Exposé III, Corollaire 5.2]. Recall that $\mathcal{E}$ is representable by a smooth affine $\mathcal{O}$-scheme. In particular, there exist lifts of γ to R, so $\mathcal{E}$ is trivial. So let us fix a lift γ' of γ and a trivialisation of $\mathcal{E}$ that identifies the section γ' with the unit in $\mathcal{G}_R$. By [SGA1, Exposé III, Corollaire 5.2], the set of lifts of γ is then a torsor under

$$\gamma^*\Omega^\vee_{\mathcal{E}/R} \otimes_{\overline{R}} I \cong r^*e^*\Omega^\vee_{\mathcal{G}/\mathcal{O}} \otimes_{\overline{R}} I = r^*\omega^\vee_{\mathcal{G}/\mathcal{O}} \otimes_{\overline{R}} I.$$

□

We use the following lemma to relate the deformation theory problem to the combinatorics in the Bruhat-Tits building.

Lemma 3.1.5 (compare [BT84, § 4.6.41]). *Assume that G is quasi-split. Let $\psi \in \mathcal{A}^*$ be an affine functional with gradient $a = \dot{\psi}$. Let $\Omega \subseteq \mathcal{A}$ be a bounded subset such that $\Omega \subseteq \mathcal{H}_{\psi \leqslant 0}$. Let moreover $\Omega' \prec \Omega$ such that $\Omega' \subseteq \mathcal{H}_{\psi}$. Then the natural map $\omega^{\vee}_{\mathcal{U}_{a,\Omega}/\mathcal{O}} \to \omega^{\vee}_{\mathcal{U}_{a,\Omega'}/\mathcal{O}}$ is an isomorphism.*

Proof. By assumption, we have $U_{a,\Omega} = U_{a,\Omega'}$ as subgroups of $G(k)$. Hence, the induced maps on integral models and consequently on invariant differentials are isomorphisms. □

Note that in the situation of the lemma when $\Omega \cap \mathcal{H}_{\psi<0} \neq \varnothing$ the induced map on Lie algebras for the negative root groups

$$\mathrm{Lie}(\mathcal{U}_{-a,\Omega,\mathbb{F}}) = \omega^{\vee}_{\mathcal{U}_{-a,\Omega}/\mathcal{O}} \otimes_{\mathcal{O}} \mathbb{F} \to \mathrm{Lie}(\mathcal{U}_{-a,\Omega',\mathbb{F}}) = \omega^{\vee}_{\mathcal{U}_{-a,\Omega'}/\mathcal{O}} \otimes_{\mathcal{O}} \mathbb{F}$$

in the special fibre of $\mathrm{Spec}(\mathcal{O})$ typically (in particular when a is non-divible and $2a$ is not a root) is the zero map by [BT84, § 4.6.41].

Let $(\mathcal{E}_{\mathfrak{f}})_{\mathfrak{f}\prec\Omega} \in \varprojlim_{\mathfrak{f}\prec\Omega} \mathfrak{B}(\mathcal{G}_{\mathfrak{f}})(R)$ be a compatible system of $\mathcal{G}_{\mathfrak{f}}$-torsors. We use the previous two lemmas to construct compatible lifts of sections of $\mathcal{E}_{\Omega} = \varprojlim_{\mathfrak{f}\prec\Omega} \mathcal{E}_{\mathfrak{f}}$. This serves two purposes: On the one hand, we use this result for the trivial torsors $\mathcal{E}_{\mathfrak{f}} = \mathcal{G}_{\mathfrak{f}}$ to show that we can lift sections from the special fibre of $\varprojlim_{\mathfrak{f}\prec\Omega} \mathcal{G}_{\mathfrak{f}}$ in the proof of Theorem 3.1.3 and on the other hand, we use it in the proof of Proposition 3.1.10, which gives a criterion when $\mathcal{E}_{\Omega}$ is actually a $\mathcal{G}_{\Omega}$-torsor. For a subset $\Omega' \prec \Omega$ we denote by $\mathcal{E}_{\Omega'} = \varprojlim_{\mathfrak{f}\prec\Omega'} \mathcal{E}_{\mathfrak{f}}$.

Lemma 3.1.6. *Assume that G is quasi-split. Let R be an $\mathcal{O}$-algebra with an ideal I of square $I^2 = 0$. We denote by $\overline{R} = R/I$.*

(1) Let $\Omega_1, \Omega_2 \prec \Omega$ be two bounded subsets such that $\Omega_1 = \mathrm{cl}(\Omega_1)$, $\Omega_2 = \mathrm{cl}(\Omega_2)$ and that $\Omega_1 \cap \Omega_2$ is contained in an affine root hyperplane $\mathcal{H}_{\psi}$ for some $\psi \subseteq \Psi$. Assume moreover that $\Omega_1 \cup \Omega_2$ is convex and that $\Omega_1 \subseteq \mathcal{H}_{\psi\geqslant 0}$ and $\Omega_2 \subseteq \mathcal{H}_{\psi\leqslant 0}$ lie in different half-spaces.

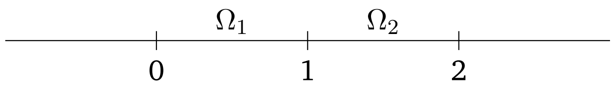
Assume that the assertion of Theorem 3.1.3 holds for $\mathcal{G}_{\Omega_1}$ and $\mathcal{G}_{\Omega_2}$. Assume that there is a section $\gamma \in \mathcal{E}_{\Omega_1\cup\Omega_2}(\overline{R})$ and deformations $\gamma_{\Omega_1} \in \mathcal{E}_{\Omega_1}(R)$ and $\gamma_{\Omega_2} \in \mathcal{E}_{\Omega_2}(R)$ of the images of γ in $\mathcal{E}_{\Omega_1}$ and $\mathcal{E}_{\Omega_2}$, respectively. Then there exists a deformation $\gamma_{\Omega_1\cup\Omega_2} \in \mathcal{E}_{\Omega_1\cup\Omega_2}(R)$ of γ.

(2) Let now $\Omega' = \mathrm{cl}(\Omega') \prec \Omega$ and let $a \in \Phi^+_{\mathrm{nd}}$ and let $\psi_1 < \psi_2 < \ldots < \psi_m$ be the affine roots with gradient $\dot{\psi}_i = a$ such that $\Omega \cap \mathcal{H}_{\psi_i} \neq \varnothing$. We denote by $\Omega_i = \overline{(\Omega \cap \mathcal{H}_{\psi_i\leqslant 0})\backslash\Omega_{i-1}}$ for $i = 1, \ldots, m$ with $\Omega_0 = \varnothing$ and $\Omega_{m+1} = \Omega\backslash(\Omega_m\backslash\mathcal{H}_{\psi_m})$.

Assume that the assertion of Theorem 3.1.3 holds for $\mathcal{G}_{\Omega_i}$ for $i = 1, \ldots, m+1$. Assume that there is a section $\gamma \in \mathcal{E}_{\Omega'}(\overline{R})$ and deformations $\gamma_{\Omega_i} \in \mathcal{E}_{\Omega_i}(R)$ of the image of γ in $\mathcal{E}_{\Omega_i}$ for all $1 \leqslant i \leqslant m+1$. Then there exists a deformation $\gamma_{\Omega'} \in \mathcal{E}_{\Omega'}(R)$ of γ.

We will prove Theorem 3.1.3 by induction on Ω and use this lemma in the inductive step. Hence, it is feasible to assume the validity of Theorem 3.1.3 for subsets of Ω here. Once we have established Theorem 3.1.3 in full (in particular for the application of the lemma in the proof of Proposition 3.1.10), these conditions of course are vacuous. Before we give the proof of the lemma, let us briefly discuss an example that nicely illustrates the main idea.

Example 3.1.7. We consider $G = \mathrm{GL}_2$ over $k = \mathbb{F}_q((\varpi))$ with T the split maximal diagonal torus. Then $X^*(T) \cong \mathbb{Z}^2$ with roots $\Phi = \{\pm(1,-1)\} \subseteq X^*(T)$, where the choice of the positive root $a = (1,-1)$ corresponds to the choice of the Borel subgroup given by upper triangular matrices. Let us consider the interval $\Omega = [0,2] \subseteq \mathbb{R} \cong \mathcal{A}(\mathrm{GL}_2, T)$ with $\Omega_1 = [0,1]$ and $\Omega_2 = [1,2]$.

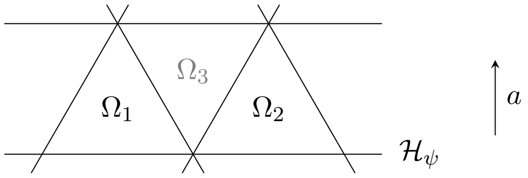


Let us consider the case $R = \mathbb{F}_q[\![\varpi]\!]/(\varpi^2)$ and $\overline{R} = R/(\varpi) = \mathbb{F}_q$. In this case, for a smooth affine group scheme $\mathcal{G}$ over $\mathcal{O}$, the module $\mathfrak{g} = e^*\omega^\vee_{\mathcal{G}/\mathcal{O}} \otimes_{\mathbb{F}_q} (\varpi)/(\varpi^2)$ is given by the tangent space of $\mathcal{G}$ at the identity section in its special fibre. Let us assume we are in the situation of Lemma 3.1.6 (1). We are given a section $\gamma \in \mathcal{E}_{[0,2]}(\mathbb{F}_q)$ and sections $\gamma_{[0,1]} \in \mathcal{E}_{[0,1]}(\mathbb{F}_q[\![\varpi]\!]/(\varpi^2))$ and $\gamma_{[1,2]} \in \mathcal{E}_{[1,2]}(\mathbb{F}_q[\![\varpi]\!]/(\varpi^2))$ that lift γ. Recall that by Lemma 3.1.4, for $\Omega' \prec \Omega$ the set of all lifts of γ in $\mathcal{E}_{\Omega'}$ is a torsor under $\mathfrak{g}_{\Omega'}$. Hence, after fixing a trivialisation of $\mathcal{E}_{\{1\}}$, the images of the lifts $\gamma_{[0,1]}, \gamma_{[1,2]}$ in $\mathcal{E}_{\{1\}}$ induce points in $\mathfrak{g}_{\{1\}}$. Thus, the question becomes if the intersection of the orbits $\mathfrak{g}_{[0,1]} \cdot \gamma_{[0,1]} \cap \mathfrak{g}_{[1,2]} \cdot \gamma_{[1,2]}$ in $\mathfrak{g}_{\{1\}}$ is non-empty, where $\mathfrak{g}_{[0,1]}$ acts via the natural map $\mathfrak{g}_{[0,1]} \to \mathfrak{g}_{\{1\}}$, similarly for $\mathfrak{g}_{[1,2]}$.

For $\Omega' \prec \Omega$, we decompose the Lie algebras into its root spaces $\mathfrak{g}_{\Omega'} = \mathfrak{u}_{a,\Omega'} \oplus \mathfrak{h} \oplus \mathfrak{u}_{-a,\Omega'}$, where $a = (1,-1)$ is the positive root. In this situation, the root spaces $\mathfrak{u}_{\pm a,\Omega'}$ are one-dimensional while the Cartan $\mathfrak{h}$ is two-dimensional. Then the induced map $\mathfrak{g}_{[0,1]} \to \mathfrak{g}_{\{1\}}$ is the identity on the Cartan algebra $\mathfrak{h}$ as well as on the positive root space $\mathfrak{u}_{a,[0,1]} = \mathfrak{u}_{a,\{1\}}$ by Lemma 3.1.5 while it is the zero map $\mathfrak{u}_{-a,[0,1]} \to \mathfrak{u}_{-a,\{1\}}$ on the negative root spaces. By a similar argument, for the second facet $\Omega_2 = [1,2]$ the map $\mathfrak{g}_{[1,2]} \to \mathfrak{g}_{\{1\}}$ is the identity on the Cartan and the negative root space, while it is the zero map on the positive root space.

Decomposing the lifts $\gamma_{[0,1]}$ and $\gamma_{[1,2]}$ in their components, this shows that by the action of $\mathfrak{g}_{[0,1]}$ we can guarantee that the $\mathfrak{u}_a$-components agree and by the action of $\mathfrak{g}_{[1,2]}$ we can get matching components in the $\mathfrak{u}_{-a}$-component. This shows the non-emptiness of the intersection of the orbits and hence the existence of a compatible set of lifts.

In order to guarantee the correct mapping property in the other directions, it is necessary to have the convexity assumption. This can be seen in the following example in the GL_3-case:



We are given two chambers Ω_1 and Ω_2 in the standard apartment in the Bruhat-Tits building of GL_3 that intersect in a single vertex. In particular, $\Omega_1 \cup \Omega_2$ is not convex. The base of both of the triangles lies in some affine root hyperplane $\mathcal{H}_\psi$ with $\dot{\psi} = a$ while both Ω_1 and Ω_2 are contained in the positive half space $\mathcal{H}_{\psi \geqslant 0}$. But this means that both $\mathfrak{u}_{a,\Omega_1} \to \mathfrak{u}_{a,\Omega_1 \cap \Omega_2}$ and $\mathfrak{u}_{a,\Omega_2} \to \mathfrak{u}_{a,\Omega_1 \cap \Omega_2}$ are the zero maps. Hence, it is in general not possible to lift sections in this situation.

The difference to the convex case is the following. We have $\mathrm{cl}(\Omega_1 \cup \Omega_2) = \Omega_1 \cup \Omega_2 \cup \Omega_3$, where Ω_3 is the triangle "between" Ω_1 and Ω_2. For a pair of $\mathcal{G}_{\Omega_1}$- (respectively $\mathcal{G}_{\Omega_2}$-) torsors $\mathcal{E}_{\Omega_1}$ and $\mathcal{E}_{\Omega_2}$ the existence of a compatible $\mathcal{G}_{\Omega_3}$-torsor $\mathcal{E}_{\Omega_3}$ (such a torsor does not exist in general!) can be interpreted as a compatibility condition on the a-root spaces, as it will guarantee by Lemma 3.1.6 (1) that for two given lifts $\gamma_{\Omega_1} \in \mathcal{E}_{\Omega_1}(\mathbb{F}_q \llbracket \varpi \rrbracket / (\varpi^2))$ and $\gamma_{\Omega_2} \in \mathcal{E}_{\Omega_2}(\mathbb{F}_q \llbracket \varpi \rrbracket / (\varpi^2))$ their image in $\mathfrak{u}_{a,\Omega_1 \cap \Omega_2}$ agrees.

Proof of Lemma 3.1.6. (1) Given some $\Omega' \prec \Omega$ (for which Theorem 3.1.3 holds), the set of all lifts of $\gamma \in \mathcal{E}_{\Omega'}(\overline{R})$ to $\mathcal{E}_{\Omega'}(R)$ is a torsor under $\mathfrak{g}_{\Omega'} = \mathfrak{g}_{\Omega',(R,I)}$ (if such lifts exist at all) by Lemma 3.1.4. Using the decomposition of the big open cell in $\mathcal{G}_{\Omega'}$, we can decompose $\mathfrak{g}_{\Omega'}$ into the root spaces as

$$\mathfrak{g}_{\Omega'} = \bigoplus_{a \in \Phi^-_{\mathrm{nd}}} \mathfrak{u}_{a,\Omega'} \oplus \mathfrak{h} \oplus \bigoplus_{a \in \Phi^+_{\mathrm{nd}}} \mathfrak{u}_{a,\Omega'}.$$

After fixing a trivialisation of $\mathcal{E}_{\Omega_1 \cap \Omega_2}$, the images of the lifts γ_{Ω_1} and γ_{Ω_2} in $\mathcal{E}_{\Omega_1 \cap \Omega_2}$ thus define elements of $\mathfrak{g}_{\Omega_1 \cap \Omega_2}$. The question whether there exists a lift $\gamma_{\Omega_1 \cup \Omega_2} \in \mathcal{E}_{\Omega_1 \cup \Omega_2}(R)$ of γ, or in other words, a compatible pair of lifts γ'_{Ω_1} and γ'_{Ω_2} in $\mathcal{E}_{\Omega_1}$ (respectively in $\mathcal{E}_{\Omega_2}$), is thus the question if the orbits in $\mathfrak{g}_{\Omega_1 \cap \Omega_2}$ have a non-empty intersection

$$\mathfrak{g}_{\Omega_1} \cdot \gamma_{\Omega_1} \cap \mathfrak{g}_{\Omega_2} \cdot \gamma_{\Omega_2} \neq \varnothing.$$

We treat this question componentwise with respect to the decomposition into root spaces. On the torus part this is clear as the maps $\mathfrak{g}_{\Omega_i} \to \mathfrak{g}_{\Omega_1 \cap \Omega_2}$ restrict to isomorphisms on $\mathfrak{h}$ by construction for $i = 1, 2$. It suffices to show that for all roots

$a \in \Phi_{\mathrm{nd}}$ at least one of $\mathfrak{g}_{\Omega_i} \to \mathfrak{g}_{\Omega_1 \cap \Omega_2}$ restricts to an isomorphism $\mathfrak{u}_{a,\Omega_i} \to \mathfrak{u}_{a,\Omega_1 \cap \Omega_2}$. For $a = \pm\dot{\psi}$ this directly follows from Lemma 3.1.5.

Let now $a \in \Phi \backslash \{\pm\dot{\psi}\}$, and let $\psi' \in \mathcal{A}$ the minimal affine functional with gradient $\dot{\psi}' = a$ such that $\Omega_1 \cap \Omega_2 \subseteq \mathcal{H}_{\psi' \leqslant 0}$. By the convexity assumption, at least one of the Ω_i is contained in $\mathcal{H}_{\psi' \leqslant 0}$ for $i = 1, 2$. But then $\mathfrak{u}_{a,\Omega_i} \xrightarrow{\cong} \mathfrak{u}_{a,\Omega_1 \cap \Omega_2}$ is an isomorphism by Lemma 3.1.5.

(2) For each $i = 1, \ldots, m$, the pair of subsets $\bigcup_{1 \leqslant j \leqslant i} \Omega_j, \Omega_{i+1}$ of Ω' satisfies the assumptions of (1) by construction (in particular, their intersection is contained in $\mathcal{H}_{\psi_i}$). Using induction on i, we construct lifts of γ for all $\mathcal{E}_{\bigcup_{1 \leqslant j \leqslant i} \Omega_i}$ using (1), and hence in particular for $\mathcal{E}_{\Omega'}$.

$\square$

Proof of Theorem 3.1.3. We first remark that the limit $\varprojlim_{\mathfrak{f} < \Omega} \mathcal{G}_{\mathfrak{f}}$ is a finite limit of affine $\mathcal{O}$-group schemes of finite type, hence is again an affine $\mathcal{O}$-group scheme of finite type. Moreover, as all transition maps are identities on the generic fibres, the generic fibre of the limit is isomorphic to G and ρ induces an isomorphism on the generic fibre.

By étale descent it suffices to work over $\breve{k}$, the completion of the maximal unramified extension of k. We may thus assume that $k = \breve{k}$, in which case G is quasi-split by assumption. Moreover, we have

$$(\varprojlim_{\mathfrak{f} < \Omega} \mathcal{G}_{\mathfrak{f}})(\mathcal{O}) = \varprojlim_{\mathfrak{f} < \Omega} (\mathcal{G}_{\mathfrak{f}}(\mathcal{O})) = \bigcap_{\mathfrak{f} < \Omega} G(k)^0_{\mathfrak{f}} = G(k)^0_{\Omega}.$$

It remains to show that $\varprojlim_{\mathfrak{f} < \Omega} \mathcal{G}_{\mathfrak{f}}$ is smooth, as smoothness implies by [BT84, § 1.7.3] that $\varprojlim_{\mathfrak{f} < \Omega} \mathcal{G}_{\mathfrak{f}}$ is étoffé in the sense of [BT84, Définition 1.7.1]. But this means that ρ is an isomorphism by the previous observations.

We use induction on Ω to show that $\varprojlim_{\mathfrak{f} < \Omega} \mathcal{G}_{\mathfrak{f}}$ is smooth. Let us fix some enumeration of the set of non-divisible positive roots $\Phi^+_{\mathrm{nd}} = \{a_1, \ldots, a_m\}$. We inductively cut down Ω into slices by hyperplanes with gradient a_i and in each step use Lemma 3.1.6 (2) to construct lifts of the section in the special fibre. For the start of the induction, note that the theorem clearly is satisfied when Ω is (the closure of) a facet. More concretely, in the last step of the induction we write $\Omega = \bigcup_{1 \leqslant i \leqslant m+1} \Omega_i$ using the notation from Lemma 3.1.6 (2) with $a = a_1$. By induction, we assume that the theorem holds for each Ω_i (that we got by cutting down each Ω_i using hyperplanes with gradient a_2).

We check that $\varprojlim_{\mathfrak{f} < \Omega} \mathcal{G}_{\mathfrak{f}}$ is formally smooth. Let R be an $\mathcal{O}$-algebra and let $I \subseteq R$ be an ideal of square zero. We denote by $\overline{R} = R/I$. Let us fix a section $\overline{g} \in \varprojlim_{\mathfrak{f} < \Omega} \mathcal{G}_{\mathfrak{f}}(\overline{R})$. Using the inductive hypothesis, there exist sections $g_i \in \varprojlim_{\mathfrak{f} \prec \Omega_i} \mathcal{G}_{\mathfrak{f}}(R) = \mathcal{G}_{\Omega_i}(R)$. By Lemma 3.1.6 (2), we then obtain a lift $g \in \varprojlim_{\mathfrak{f} \prec \Omega} \mathcal{G}_{\mathfrak{f}}(R)$. As $\varprojlim_{\mathfrak{f} \prec \Omega} \mathcal{G}_{\mathfrak{f}}$ is an affine scheme of finite presentation over $\mathcal{O}$, this shows that $\mathcal{G}_\Omega$ is smooth. This finishes the proof of the theorem. □

Corollary 3.1.8. *The Bruhat-Tits group scheme $\mathcal{G}_\Omega$ is isomorphic to the closure of the diagonal in the generic fibre*

$$G \xrightarrow{\Delta} \prod_{\mathfrak{f} \prec \Omega} \mathcal{G}_{\mathfrak{f}}.$$

Proof. The inclusion $\varprojlim_{\mathfrak{f} \prec \Omega} \mathcal{G}_{\mathfrak{f}} \to \prod_{\mathfrak{f} \prec \Omega} \mathcal{G}_{\mathfrak{f}}$ is a closed immersion since all $\mathcal{G}_{\mathfrak{f}}$ are affine and thus separated over $\mathcal{O}$. Since $\mathcal{G}_\Omega$ is in particular flat over $\mathcal{O}$, it is the closure of its generic fibre. The claim then follows from Theorem 3.1.3. □

Remark 3.1.9. Let $\Omega \subseteq \mathcal{B}(G, k)$ be a bounded subset that is not necessarily contained in a single apartment. Theorem 3.1.3 suggests a way to associate an $\mathcal{O}$-group scheme to Ω, namely to define

$$\mathcal{G}_\Omega = \varprojlim_{\mathfrak{f} \prec \Omega} \mathcal{G}_{\mathfrak{f}}.$$

It is however neither clear whether $\mathcal{G}_\Omega$ is smooth nor whether it has a connected special fibre.

3.1.3. Torsors for deep Bruhat-Tits group schemes

We consider torsors for the Bruhat-Tits group schemes above. Recall that a limit of $\mathcal{G}_{\mathfrak{f}}$-torsors for facets $\mathfrak{f} \prec \Omega$ is a $\mathcal{G}_\Omega$-pseudo torsor by Lemma 3.1.1, but may fail to be a $\mathcal{G}_\Omega$-torsor in general. We give a criterion when a limit of $\mathcal{G}_{\mathfrak{f}}$-torsors is already a $\mathcal{G}_\Omega$-torsor.

Proposition 3.1.10. *Let $\Omega \subseteq \mathcal{A}$ be a bounded subset with $\Omega = \mathrm{cl}(\Omega)$ and let R be an $\mathcal{O}$-algebra. Let $(\mathcal{E}_{\mathfrak{f}})_{\mathfrak{f} \prec \Omega} \in \varprojlim_{\mathfrak{f} \prec \Omega} \mathfrak{B}(\mathcal{G}_{\mathfrak{f}})(R)$. Then*

$$\mathcal{E}_\Omega = \varprojlim_{\mathfrak{f} \prec \Omega} \mathcal{E}_{\mathfrak{f}}$$

is a smooth affine B-scheme. In particular, $\mathcal{E}_\Omega$ is a $\mathcal{G}_\Omega$-torsor if and only if $\mathcal{E}_\Omega \to \mathrm{Spec}(R)$ is surjective.

Proof. The second assertion follows from the first one using Lemma 3.1.1, Theorem 3.1.3 and [SGA1, Exposé XI, Proposition 4.2].

The first assertion is étale-local on $\operatorname{Spec}(R)$, so we may assume that G is quasi-split. It suffices to show that $\mathcal{E}_\Omega \to \operatorname{Spec}(R)$ is formally smooth, as $\mathcal{E}_\Omega$ is clearly representable by an affine R-scheme of finite presentation. But this follows from Lemma 3.1.6 (2) by induction on Ω as in the proof of Theorem 3.1.3. $\square$

The goal of this section is to show that the isomorphism of Bruhat-Tits group schemes of Theorem 3.1.3 induces an immersion on the level of the corresponding moduli stacks of $\mathcal{G}$-bundles on X. Therefore, let us now change perspective and consider (global) Bruhat-Tits group schemes in the following sense.

Definition 3.1.11. A smooth, affine group scheme $\mathcal{G} \to X$ is called a *(global) Bruhat-Tits group scheme* if it has geometrically connected fibres, its generic fibre $\mathcal{G}_K = G$ is a reductive group over K and if for all closed points x of X the pullback $\mathcal{G}_{\mathcal{O}_x} = \mathcal{G} \times_X \operatorname{Spec}(\mathcal{O}_x)$ is of the form $\mathcal{G}_\Omega$ for some bounded subset Ω contained in an apartment of the Bruhat-Tits building $B(G/K_x)$. The group scheme $\mathcal{G}$ is called a *parahoric (Bruhat-Tits) group scheme* if moreover all $\mathcal{G}_{\mathcal{O}_x}$ are parahoric group schemes.

Let G be a (connected) reductive group over the function field K of X. Bruhat-Tits group schemes with generic fibre G can be constructed as follows.

Construction 3.1.12. (1) There exists a reductive model $G \to U$ of G over some dense open subset $U \subseteq X$. For each of the finitely many points $x \in X \backslash U$ in the complement of U we choose a parahoric group scheme $\mathcal{G}^{(x)} \to \operatorname{Spec}(\mathcal{O}_x)$ with generic fibre $\mathcal{G}^{(x)}_{K_x} = G_{K_x}$. As $U \amalg \coprod_{x \in X \backslash U} \operatorname{Spec}(\mathcal{O}_x) \to X$ is an fpqc-cover, we can glue $G \to U$ with all $\mathcal{G}^{(x)}$ using fpqc-descent to obtain a smooth affine group scheme $\mathcal{G} \to X$, which is a parahoric group scheme by construction.

(2) Let us now fix a parahoric model $\mathcal{G} \to X$ and a closed point x_0 of X. For a connected bounded subset Ω in an apartment of the Bruhat-Tits building of $G_{K_{x_0}}$ as in the previous paragraph, we denote by $\mathcal{G}_\Omega \to \operatorname{Spec}(\mathcal{O}_{x_0})$ the corresponding (local) Bruhat-Tits group scheme. We glue $\mathcal{G}_\Omega$ with $\mathcal{G}$ along the identity over K_{x_0} and denote the resulting smooth affine group scheme over X by a slight abuse of notation again by $\mathcal{G}_\Omega$. Then $\mathcal{G}_\Omega$ is a Bruhat-Tits group scheme in the sense of the previous definition and parahoric if and only if Ω is contained in the closure of a facet.

The local homomorphisms $\rho_{\Omega',\Omega} \colon \mathcal{G}_\Omega \to \mathcal{G}_{\Omega'}$ over $\operatorname{Spec}(\mathcal{O}_{x_0})$ for $\Omega' \prec \Omega$ glue with the identity away from x_0 to morphisms of group schemes $\rho_{\Omega',\Omega} \colon \mathcal{G}_\Omega \to \mathcal{G}_{\Omega'}$ on X.

In particular, the isomorphism of Theorem 3.1.3 extends to an isomorphism

$$\mathcal{G}_\Omega \xrightarrow{\cong} \varprojlim_{\mathfrak{f} \prec \Omega} \mathcal{G}_{\mathfrak{f}}$$

of the corresponding global Bruhat-Tits group schemes.

For any smooth affine group scheme $\mathcal{H}$ on X, we denote by $\mathrm{Bun}_{\mathcal{H}}$ the moduli stack of $\mathcal{H}$-bundles on X. By the functoriality of Bun, the maps $\rho_{\mathfrak{f},\Omega}$ induce maps $\rho_{\mathfrak{f},\Omega,*}\colon \mathrm{Bun}_{\mathcal{G}_\Omega} \to \mathrm{Bun}_{\mathcal{G}_\mathfrak{f}}$ for all facets $\mathfrak{f} \prec \Omega$.

Theorem 3.1.13. *Let G be a reductive group over K, let x_0 be a closed point of X and let $\Omega = \mathrm{cl}(\Omega)$ be a bounded subset of an apartment in the Bruhat-Tits building $\mathcal{B}(G_{K_{x_0}}, K_{x_0})$. Let $\mathcal{G}_\Omega \to X$ be the corresponding Bruhat-Tits group scheme from Construction 3.1.12 (2). The map*

$$\rho_{\Omega,*} := \varprojlim_{\mathfrak{f}\prec\Omega} \rho_{\mathfrak{f},\Omega,*}\colon \mathrm{Bun}_{\mathcal{G}_\Omega} \to \varprojlim_{\mathfrak{f}\prec\Omega} \mathrm{Bun}_{\mathcal{G}_\mathfrak{f}}$$

induced by the $\rho_{\mathfrak{f},\Omega,*}$ for facets $\mathfrak{f} \prec \Omega$ is schematic and a quasi-compact open immersion.

Proof. By [Bre18, Proposition 3.19], the maps $\rho_{\mathfrak{f},\Omega,*}$ are schematic and quasi-projective for all facets $\mathfrak{f} \prec \Omega$. By Lemma A.0.2, the map $\rho_{\Omega,*}$ is schematic, separated and of finite type. Moreover, all $\mathrm{Bun}_{\mathcal{G}_\mathfrak{f}}$ are locally of finite type over $\mathbb{F}_q$ by [Hei10, Proposition 1]. By Lemma 3.1.1, the map $\rho_{\Omega,*}$ is a monomorphism.

We show that $\rho_{\Omega,*}$ is formally étale. Let R be a local artinian $\mathbb{F}_q$-algebra with maximal ideal $I \subseteq R$ of square zero. Let moreover $(\mathcal{E}_\mathfrak{f})_{\mathfrak{f}\prec\Omega} \in \varprojlim_{\mathfrak{f}\prec\Omega} \mathrm{Bun}_{\mathcal{G}_\mathfrak{f}}(R)$ such that $\varprojlim_{\mathfrak{f}\prec\Omega} \mathcal{E}_\mathfrak{f}$ is a $\mathcal{G}_\Omega$-torsor over $X_{\overline{R}}$, where $\overline{R} = R/I$. We claim that $\varprojlim_{\mathfrak{f}\prec\Omega} \mathcal{E}_\mathfrak{f}$ is already a $\mathcal{G}_\Omega$-torsor over X_R. The map $\widehat{(X_R)_{x_0}} \cup (X\backslash\{x_0\})_R \to X_R$ is a fpqc-cover, where $\widehat{(X_R)_{x_0}} = \mathrm{Spec}(\mathcal{O}_{x_0} \widehat{\otimes}_{\mathbb{F}_q} R)$, with $\mathcal{O}_{x_0} \widehat{\otimes}_{\mathbb{F}_q} V$ being the underlying $\mathbb{F}_q$-algebra of the completion of X_R along x_0. As all maps $\mathcal{G}_\Omega \to \mathcal{G}_\mathfrak{f}$ for $\mathfrak{f} \prec \Omega$ are the identity away from x_0, all transition maps $\mathcal{E}_{\mathfrak{f}',R} \times^{\mathcal{G}_{\mathfrak{f}'}} \mathcal{G}_\mathfrak{f} \to \mathcal{E}_{\mathfrak{f},R}$ are isomorphisms away from x_0. Using Proposition 3.1.10, it remains to check that the pullback to $\varprojlim_{\mathfrak{f}\prec\Omega} \mathcal{E}_\mathfrak{f} \to \widehat{(X_R)_{x_0}}$ is surjective, but the underlying topological spaces of $\widehat{(X_R)_{x_0}}$ and $\widehat{(X_{\overline{R}})_{x_0}}$ agree.

Hence, $\rho_{\Omega,*}$ is formally étale and thus a quasi-compact open immersion being a flat monomorphism of finite presentation. □

3.2. Bounds for shtukas

Global shtukas for GL_n were first introduced in [Dri87b] and generalised to split reductive groups (respectively to flat affine group schemes of finite type) by [Var04] and [AH21], respectively. In this section, we recall the definition and basic properties of moduli spaces of (iterated, global) shtukas. We use global bounds following [AH19] and introduce a new notion of local bounds in the style of [AH14] compatible with global bounds. For

Bruhat-Tits group schemes we construct (global and local) bounds given by cocharacters that recover the bounds from [Laf18] in the constant split reductive case.

Let $\mathcal{G} \to X$ be a smooth affine group scheme. Let I be a finite set and let $I = I_1 \cup \ldots \cup I_m$ be a partition of I. We write $I_\bullet = (I_1, \ldots, I_m)$.

Definition 3.2.1 ([AH21, Definition 3.3]). We denote by $\mathrm{Sht}_{\mathcal{G},X^I,I_\bullet}$ the stack fibered in groupoids over $\mathbb{F}_q$ whose S valued points are given by tuples

$$((x_i)_{i\in I}, (\mathcal{E}_j)_{j=0,\ldots,m}, (\varphi_j)_{j=1,\ldots,m}, \theta),$$

where

- $x_i \in X(S)$ are points on X called the *characteristic sections* (or *legs*) for $i \in I$,
- $\mathcal{E}_j \in \mathrm{Bun}_{\mathcal{G}}(S)$ are $\mathcal{G}$-bundles on X_S for $0 \leqslant j \leqslant m$,
- $\varphi_j \colon \mathcal{E}_{j-1}|_{X_S \setminus \bigcup_{i\in I_j} \Gamma_{x_i}} \xrightarrow{\cong} \mathcal{E}_j|_{X_S \setminus \bigcup_{i\in I_j} \Gamma_{x_i}}$ are isomorphisms of $\mathcal{G}$-bundles away from the graphs $\Gamma_{x_i} \subseteq X_S$ of the sections x_i, and
- $\theta \colon \sigma^* \mathcal{E}_m \xrightarrow{\cong} \mathcal{E}_0$ is an isomorphism of $\mathcal{G}$-bundles on X_S.

The projection to the characteristic sections defines a map $\mathrm{Sht}_{\mathcal{G},X^I,I_\bullet} \to X^I$. By [AH21, Theorem 3.15], $\mathrm{Sht}_{\mathcal{G},X^I,I_\bullet}$ is an ind-Deligne Mumford stack that is separated and locally of ind-finite type over X^I.

Let $I'_\bullet$ be a second partition of I that is finer than $I_\bullet$. The forgetful map

$$\mathrm{Sht}_{\mathcal{G},X^I,I'_\bullet} \to \mathrm{Sht}_{\mathcal{G},X^I,I_\bullet}$$

is an isomorphism over

$$U = \{\underline{x} = (x_i)_{i\in I} \in X^I \colon x_{i_1} \neq x_{i_2} \text{ for all } i_1, i_2 \in I_j \text{ and } 1 \leqslant j \leqslant m\} \subseteq X^I$$

by the argument in [Var04, Lemma A.8 a)].

When $I_\bullet = (I)$ is the trivial partition, we write $\mathrm{Sht}_{\mathcal{G},X^I} = \mathrm{Sht}_{\mathcal{G},X^I,(I)}$. Let us fix pairwise different closed points $y_i \in X$ for all $i \in I$. We denote by

$$\mathrm{Sht}^{\underline{y}}_{\mathcal{G},X^I} = \mathrm{Sht}_{\mathcal{G},X^I,I_\bullet} \times_{X^I} \mathrm{Spf}(\mathcal{O}_{\underline{y}}) = \mathrm{Sht}_{\mathcal{G},X^I} \times_{X^I} \mathrm{Spf}(\mathcal{O}_{\underline{y}})$$

the restriction of the moduli space of shtukas to the formal neighbourhood of $\mathcal{O}_{\underline{y}}$. By the previous observation, this stack does not depend on the choice of the partition $I_\bullet$ of I.

Assumption 3.2.2. In the following, we consider moduli spaces of shtukas in essentially three different situations.

(1) $\mathcal{G} \to X$ is a smooth affine group scheme. (The *smooth affine case*)

(2) G is a reductive group over K and $\mathcal{G} \to X$ is a smooth affine group scheme with generic fibre G. (The *generically reductive case*)

(3) G is a reductive group over K and $\mathcal{G}_\Omega \to X$ is a Bruhat-Tits group scheme for a bounded subset $\Omega = \mathrm{cl}(\Omega)$ of an apartment in the Bruhat-Tits building for $G_{K_{x_0}}$ for some fixed closed point x_0 of X as in Construction 3.1.12. (The *Bruhat-Tits case*)

3.2.1. Global bounds

We recall the notion of (global) bounds for shtukas following [AH19, Definition 3.1.3]. In the case where $\mathcal{G}$ is a Bruhat-Tits group scheme, we construct boundedness conditions given by cocharacters in the style of [Laf18].

We need the following iterated version of Beilinson-Drinfeld affine Grassmannians first introduced by [BD96] in the case of constant group schemes.

Definition 3.2.3. We denote by $\mathrm{Gr}_{\mathcal{G},X^I,I_\bullet}$ the functor on $\mathbb{F}_q$-schemes whose S valued points are given by tuples

$$((x_i)_{i\in I}, (\mathcal{E}_j)_{j=0,\dots,m}, (\varphi_j)_{j=1,\dots,m}, \varepsilon),$$

where

- $x_i \in X(S)$ are points on X called the *characteristic sections* (or *legs*) for $i \in I$,
- $\mathcal{E}_j \in \mathrm{Bun}_{\mathcal{G}}(S)$ are $\mathcal{G}$-bundles on X_S for $0 \leqslant j \leqslant m$,
- $\varphi_j : \mathcal{E}_{j-1}|_{X_S \setminus \bigcup_{i\in I_j} \Gamma_{x_i}} \xrightarrow{\cong} \mathcal{E}_j|_{X_S \setminus \bigcup_{i\in I_j} \Gamma_{x_i}}$ are isomorphisms of $\mathcal{G}$-bundles, and
- $\varepsilon : \mathcal{E}_m \xrightarrow{\cong} \mathcal{G} \times_X X_S$ is a trivialisation of $\mathcal{E}_m$.

Then $\mathrm{Gr}_{\mathcal{G},X^I,I_\bullet}$ is representable by an ind-scheme over X^I by [Hei10].

Let R be a $\mathbb{F}_q$-algebra. For a relative effective Cartier divisor $D \subseteq X_R$, the formal completion of X_R along D is a formal affine scheme. We denote by $\hat{\mathcal{O}}_D$ the underlying R-algebra and by $\hat{D} = \mathrm{Spec}(\hat{\mathcal{O}}_D)$ the corresponding affine scheme. Then D is a closed subscheme of $\hat{D}$ and we set $\hat{D}^0 = \hat{D} \setminus D$. We apply this construction in particular when $D = \Gamma_{\underline{x}} = \bigcup_{i\in I} \Gamma_{x_i}$ is the union of graphs of points $\underline{x} = (x_i)_{i\in I} \in X^I(R)$. In this case we write $\hat{\Gamma}_{\underline{x}} = \hat{D}$ and $\hat{\Gamma}^0_{\underline{x}} = \hat{D}^0$.

Remark 3.2.4. Using Beauville-Laszlo descent [BL95] (compare also [BD96, Remark 2.3.7 and Theorem 2.12.1] and [LS97]), the affine Grassmannian has the following alternative description, compare [Laf18, Construction 1.8]. Let R be a $\mathbb{F}_q$-algebra. Then an R-point of $\mathrm{Gr}_{\mathcal{G},X^I,I_\bullet}$ is given by a tuple

$$((x_i)_{i\in I}, (\mathcal{E}_j)_{j=0,\dots,m}, (\varphi_j)_{j=1,\dots,m}, \varepsilon),$$

where the $\mathcal{E}_j$ are now $\mathcal{G}$-torsors on $\hat{\Gamma}_{\underline{x}}$ and the φ_j are isomorphisms over $\hat{\Gamma}_{\underline{x}}\backslash\hat{\Gamma}_{\underline{x}_j}$, where $\underline{x}_j = (x_i)_{i\in I_j}$.

Let $U \subseteq X^I$ be the complement of all diagonals. Using this description of the affine Grassmannian, we find that $\mathrm{Gr}_{\mathcal{G},X^I,I_\bullet}|_U = (\prod_{i\in I}\mathrm{Gr}_{\mathcal{G},X})|_U$.

We also make use of a global version of the (positive) loop group.

Definition 3.2.5. The global loop group $\mathcal{L}_{X^I}\mathcal{G}$ is the functor on the category of $\mathbb{F}_q$-algebras

$$\mathcal{L}_{X^I}\mathcal{G} \colon R \mapsto \left\{(\underline{x}, g) \colon \underline{x} \in X^I(R), g \in \mathcal{G}(\hat{\Gamma}_{\underline{x}}{}^0)\right\}.$$

The positive global loop group $\mathcal{L}^+_{X^I}\mathcal{G}$ is the functor on the category of $\mathbb{F}_q$-algebras

$$\mathcal{L}^+_{X^I}\mathcal{G} \colon R \mapsto \left\{(\underline{x}, g) \colon \underline{x} \in X^I(R), g \in \mathcal{G}(\hat{\Gamma}_{\underline{x}})\right\}.$$

By [Hei10, Proposition 2], $\mathcal{L}_{X^I}\mathcal{G}$ is representable by an ind-group scheme over X^I and $\mathcal{L}^+_{X^I}\mathcal{G}$ is representable by an affine group scheme over X^I with geometrically connected fibres. Moreover, the projection $\mathcal{L}\mathcal{G} \to \mathrm{Gr}_{\mathcal{G},X^I}$ induces an isomorphism of fpqc-sheaves $\mathcal{L}_{X^I}\mathcal{G}/\mathcal{L}^+_{X^I}\mathcal{G} \to \mathrm{Gr}_{\mathcal{G},X^I}$. There is a natural left $\mathcal{L}^+_{X^I}\mathcal{G}$-action on $\mathrm{Gr}_{\mathcal{G},X^I,I_\bullet}$ given by changing the trivialisation ε.

Remark 3.2.6. It is well-known that there is a formally smooth map

$$\mathrm{Sht}_{\mathcal{G},X^I,I_\bullet} \to [\mathcal{L}^+_{X^I}\mathcal{G}\backslash \mathrm{Gr}_{\mathcal{G},X^I,I_\bullet}],$$

compare for example [AH19, Theorem 3.2.1] and [Laf18, Proposition 2.8]. In this sense, the affine Grassmannian is a local model for the moduli stack of shtukas.

We define (global) bounds for shtukas as certain subschemes of the affine Grassmannian following [AH19, Definition 3.1.3].

Definition 3.2.7. We fix an algebraic closure K^{alg} of the function field $K = K(X)$ of X. For a finite extension K' of K in K^{alg} we denote by $\widetilde{X}_{K'}$ the normalisation of X in K'. It is a smooth projective curve over $\mathbb{F}_q$ together with a finite morphism $\widetilde{X}_{K'} \to X$.

(1) Let K_1 and K_2 be two finite extensions of K. Two locally closed subschemes $Z_1 \subseteq \mathrm{Gr}_{\mathcal{G},X^I,I_\bullet} \times_{X^I} \widetilde{X}^I_{K_1}$ and $Z_2 \subseteq \mathrm{Gr}_{\mathcal{G},X^I,I_\bullet} \times_{X^I} \widetilde{X}^I_{K_2}$ are called *equivalent* if there is a finite extension $K_1.K_2 \subseteq K' \subseteq K^{\mathrm{alg}}$ of the composite $K_1.K_2$ of K_1 and K_2, such that $Z_1 \times_{\widetilde{X}^I_{K_1}} \widetilde{X}^I_{K'} = Z_2 \times_{\widetilde{X}^I_{K_2}} \widetilde{X}^I_{K'}$ in $\mathrm{Gr}_{\mathcal{G},X^I,I_\bullet} \times_{X^I} \widetilde{X}^I_{K'}$.

Let $\mathcal{Z}$ be an equivalence class of locally closed subschemes $Z_{K'} \subseteq \mathrm{Gr}_{\mathcal{G},X^I,I_\bullet} \times_{X^I} \widetilde{X}^I_{K'}$ and let $G_{\mathcal{Z}} := \{g \in \mathrm{Aut}(K^{\mathrm{alg}}/K) : g^*(\mathcal{Z}) = \mathcal{Z}\}$. The *field of definition* $K_{\mathcal{Z}}$ of $\mathcal{Z}$ is the intersection of the fixed field of $G_{\mathcal{Z}}$ in K^{alg} with all the finite extensions of K over which a representative of $\mathcal{Z}$ exists.

(2) A *bound* is an equivalence class $\mathcal{Z}$ of quasi-compact locally closed subschemes $Z_{K'} \subset \mathrm{Gr}_{\mathcal{G},X^I,I_\bullet} \times_{X^I} \widetilde{X}^I_{K'}$ that admits a representative $Z_{K_{\mathcal{Z}}}$ over its field of definition $K_{\mathcal{Z}}$ that is moreover stable under the left $\mathcal{L}^+_{X^I}\mathcal{G} \times_{X^I} \widetilde{X}^I_{K_{\mathcal{Z}}}$-action on $\mathrm{Gr}_{\mathcal{G},X^I,I_\bullet} \times_{X^I} \widetilde{X}^I_{K_{\mathcal{Z}}}$. The field of definition $K_{\mathcal{Z}}$ of $\mathcal{Z}$ is called the *reflex field* of $\mathcal{Z}$, and the corresponding curve $X_{\mathcal{Z}} := \widetilde{X}_{K_{\mathcal{Z}}}$ is called the *reflex curve* of $\mathcal{Z}$.

By abuse of notation we usually identify $\mathcal{Z}$ with its representative over the reflex curve. Such a representative is unique by Lemma 3.2.10 below.

(3) Let $\mathcal{Z}$ be a bound in the above sense and let

$$\underline{\mathcal{E}} = ((x_i)_{i\in I}, (\mathcal{E}_j)_{j=0,\dots,m}, (\varphi_j)_{j=1,\dots,m}, \theta) \in (\mathrm{Sht}_{\mathcal{G},X^I,I_\bullet} \times_{X^I} X^I_{\mathcal{Z}})(S).$$

By [HR20, Lemma 3.4], there exists an étale cover $T \to S$ such that $\hat{\Gamma}_{\underline{x}_T} \to \hat{\Gamma}_{\underline{x}}$ trivializes $\mathcal{E}_m|_{\hat{\Gamma}_{\underline{x}}}$. Fixing a trivialisation $\alpha\colon \mathcal{E}_m|_{\hat{\Gamma}_{\underline{x}_T}} \xrightarrow{\cong} \mathcal{G}|_{\hat{\Gamma}_{\underline{x}_T}}$ defines a point in $(\mathrm{Gr}_{\mathcal{G},X^I,I_\bullet} \times_{X^I} X^I_{\mathcal{Z}})(T)$, compare Remark 3.2.4. We say that $\underline{\mathcal{E}}$ is *bounded by* $\mathcal{Z}$ if this point factors through $\mathcal{Z}$. As $\mathcal{Z}$ is invariant under the left $\mathcal{L}^+_{X^I}\mathcal{G}$-action, the definition is independent of the choice of the trivialisation α.

We denote by $\mathrm{Sht}^{\mathcal{Z}}_{\mathcal{G},X^I,I_\bullet} \to X^I_{\mathcal{Z}}$ the moduli stack of $\mathcal{G}$-shtukas bounded by $\mathcal{Z}$ in this sense. As in the unbounded case, for a tuple $(y_i)_{i\in I}$ of pairwise distinct closed points of $X_{\mathcal{Z}}$ we write

$$\mathrm{Sht}^{\mathcal{Z},\underline{y}}_{\mathcal{G},X^I} = \mathrm{Sht}^{\mathcal{Z}}_{\mathcal{G},X^I} \times_{X^I_{\mathcal{Z}}} \mathrm{Spf}(\mathcal{O}_{\underline{y}}).$$

Let us recall some properties of this stack of bounded global $\mathcal{G}$-shtukas.

Remark 3.2.8. By [AH19, Theorem 3.1.6], the moduli stack of bounded $\mathcal{G}$-shtukas $\mathrm{Sht}^{\mathcal{Z}}_{\mathcal{G},X^I,I_\bullet}$ is a Deligne-Mumford stack locally of finite type and separated over X^I, and a locally closed substack of $\mathrm{Sht}_{\mathcal{G},X^I,I_\bullet}$. The diagonal of $\mathrm{Sht}^{\mathcal{Z}}_{\mathcal{G},X^I,I_\bullet}$ is schematic, finite and unramified by [AH21, Corollary 3.16].

Remark 3.2.9. There is a version of the local model theorem also for the moduli space of bounded shtukas. Let $\mathcal{Z}$ be a bound. By [AH19, Theorem 3.2.1], its representative $\mathcal{Z}$ inside the affine Grassmannian $\mathrm{Gr}_{\mathcal{G},X^I,I_\bullet} \times_{X^I} X^I_{\mathcal{Z}}$ is an étale local model for $\mathrm{Sht}^{\mathcal{Z}}_{\mathcal{G},X^I,I_\bullet}$. Moreover, the $\mathcal{L}^+_{X^I}\mathcal{G}$-action on $\mathcal{Z}$ factors through a finite-dimensional quotient $\mathcal{H}$ of $\mathcal{L}^+_{X^I}\mathcal{G}$ and we have a smooth map $\mathrm{Sht}^{\mathcal{Z}}_{\mathcal{G},X^I,I_\bullet} \to [\mathcal{H}\backslash\mathcal{Z}]$, compare [Laf18, Proposition 2.8].

The following lemma is a global analogue of [AH14, Remark 4.6] and shows in particular, that the representative of a bound $\mathcal{Z}$ over the reflex field is unique.

Lemma 3.2.10. *Let Z_{1,K_1} and Z_{2,K_2} be two closed subschemes of $\mathrm{Gr}_{\mathcal{G},X^I,I_\bullet} \times_{X^I} \widetilde{X}^I_{K_1}$ and $\mathrm{Gr}_{\mathcal{G},X^I,I_\bullet} \times_{X^I} \widetilde{X}^I_{K_2}$, respectively. Then Z_{1,K_1} and Z_{2,K_2} are equivalent if and only if $Z_{1,K'} = Z_{2,K'}$ for all finite extensions K' of K containing both K_1 and K_2.*

Proof. Let Z_{1,K_1} and Z_{2,K_2} be equivalent and let K'' be a common (finite) extension of K_1 and K_2 such that $Z_{1,K''} = Z_{2,K''}$. Let moreover K'/K be another finite extension of K containing both K_1 and K_2. The question if $Z_{1,K'} = Z_{2,K'}$ in $\mathrm{Gr}_{\mathcal{G},X^I,I_\bullet} \times_{X^I} \widetilde{X}^I_{K'}$ is fpqc-local and satisfied after the fpqc base change along $\widetilde{X}^I_{K'.K''} \to \widetilde{X}^I_{K'}$ by assumption. Note that the flatness of the map follows from the flatness of the normalisation map $\widetilde{X}_{K'.K''} \to \widetilde{X}_{K'}$. Hence, $Z_{1,K'} = Z_{2,K'}$. The other direction is clear. □

Remark 3.2.11. Our definition has a couple of subtle differences compared with [AH19, Definition 3.1.3]. We do not require our bounds to be closed but only locally closed subschemes of the affine Grassmannian. This allows us to also consider for example Schubert cells as bounds.

On the other hand, we require the bounds to have a representative over the reflex field. We do not know if such a representative always exists in this generality, as noted in [AH19, Remark 3.1.4]. However, this condition is certainly satisfied for bounds given by Schubert varieties, in which case the reflex field of the bound is the reflex field of the corresponding cocharacter. Moreover, we use the existence of a representative over the reflex field for example in the proof of Lemma 3.2.13.

By Lemma 3.2.10, a point $\underline{\mathcal{E}} \in (\mathrm{Sht}_{\mathcal{G},X^I,I_\bullet} \times_{X^I} X^I_{\mathcal{Z}})(S)$ is bounded by $\mathcal{Z}$ if and only if after the choice of some trivialisation of $\underline{\mathcal{E}}$ over some fppf-cover $T \to S$ the induced point $T \times_{X^I_{\mathcal{Z}}} \widetilde{X}^I_{K'} \to \mathrm{Gr}_{\mathcal{G},X^I,I_\bullet} \times_{X^I} \widetilde{X}^I_{K'}$ factors through $Z_{K'}$ for some (or equivalently for all) representative $Z_{K'}$ of $\mathcal{Z}$. In particular, the notion of bounded shtukas above agrees in this aspect with the defintion of [AH19].

In our setting, the notion of a shtuka datum (respectively a map of shtuka data) in the sense of [Bre18, Definitions 3.1 and 3.9] restricts to the following.

Definition 3.2.12. A *shtuka datum* $(\mathcal{G}, \mathcal{Z})$ is a pair of a smooth affine group scheme $\mathcal{G} \to X$ and a bound $\mathcal{Z}$ in $\mathrm{Gr}_{\mathcal{G},X^I,I_\bullet} \times_{X^I} X^I_{\mathcal{Z}}$, where $X_{\mathcal{Z}}$ is the reflex curve of $\mathcal{Z}$. A *map of shtuka data* $f\colon (\mathcal{G}, \mathcal{Z}) \to (\mathcal{G}', \mathcal{Z}')$ is a map of group schemes $f\colon \mathcal{G} \to \mathcal{G}'$ such that the map

$$\mathcal{Z} \times_{X^I_{\mathcal{Z}}} X^I_{\mathcal{Z}.\mathcal{Z}'} \hookrightarrow \mathrm{Gr}_{\mathcal{G},X^I,I_\bullet} \times_{X^I_{\mathcal{Z}}} X^I_{\mathcal{Z}.\mathcal{Z}'} \xrightarrow{f_*} \mathrm{Gr}_{\mathcal{G}',X^I,I_\bullet} \times_{X^I_{\mathcal{Z}}} X^I_{\mathcal{Z}.\mathcal{Z}'}$$

factors through $\mathcal{Z}' \times_{X^I_{\mathcal{Z}'}} X^I_{\mathcal{Z}.\mathcal{Z}'}$, where $X_{\mathcal{Z}.\mathcal{Z}'} = \widetilde{X}_{K_{\mathcal{Z}}.K_{\mathcal{Z}'}}$ is the normalisation of the compositum of the reflex fields of $\mathcal{Z}$ and $\mathcal{Z}'$, respectively.

A map of shtuka data $f\colon (\mathcal{G}, \mathcal{Z}) \to (\mathcal{G}', \mathcal{Z}')$ induces a map on the corresponding moduli stacks of shtukas

$$f_*\colon \mathrm{Sht}^{\mathcal{Z}}_{\mathcal{G},X^I,I_\bullet} \times_{X^I_{\mathcal{Z}}} X^I_{\mathcal{Z}.\mathcal{Z}'} \to \mathrm{Sht}^{\mathcal{Z}'}_{\mathcal{G}',X^I,I_\bullet} \times_{X^I_{\mathcal{Z}'}} X^I_{\mathcal{Z}.\mathcal{Z}'}$$

by the following lemma that is an analogue of [Bre18, Lemma 3.15].

Lemma 3.2.13. *Let $f\colon (\mathcal{G}, \mathcal{Z}) \to (\mathcal{G}', \mathcal{Z}')$ be a map of shtuka data. Let*

$$\underline{\mathcal{E}} \in (\mathrm{Sht}^{\mathcal{Z}}_{\mathcal{G},X^I,I_\bullet} \times_{X^I} X^I_{\mathcal{Z}.\mathcal{Z}'})(S).$$

Then $f_*\underline{\mathcal{E}} \in (\mathrm{Sht}_{\mathcal{G}',X^I,I_\bullet} \times_{X^I} X^I_{\mathcal{Z}.\mathcal{Z}'})(S)$ is bounded by $\mathcal{Z}'$.

Proof. Let $\underline{\mathcal{E}} = ((x_i)_{i\in I}, (\mathcal{E}_j)_{j=0,\dots,m}, (\varphi_j)_{j=1,\dots,m}, \theta) \in (\mathrm{Sht}^{\mathcal{Z}}_{\mathcal{G},X^I,I_\bullet} \times_{X^I} X^I_{\mathcal{Z}.\mathcal{Z}'})(S)$. Let $T \to S$ be a fppf-cover that trivialises $\mathcal{E}_m|_{\hat{\Gamma}_{\underline{x}}}$ and choose a trivialisation $\alpha\colon \mathcal{E}_m|_{\hat{\Gamma}_{\underline{x}_T}} \xrightarrow{\cong} \mathcal{G}|_{\hat{\Gamma}_{\underline{x}_T}}$. Then $(\underline{\mathcal{E}}_T, \alpha)$ defines an T-valued point in $\mathrm{Gr}_{\mathcal{G},X^I,I_\bullet} \times_{X^I} X^I_{\mathcal{Z}.\mathcal{Z}'}$. As $\underline{\mathcal{E}}$ is bounded by $\mathcal{Z}$, the induced point $T \times_{X^I_{\mathcal{Z}}} X^I_{\mathcal{Z}.\mathcal{Z}'} \to \mathrm{Gr}_{\mathcal{G},X^I,I_\bullet} \times_{X^I} X^I_{\mathcal{Z}.\mathcal{Z}'}$ factors through $\mathcal{Z} \times_{X^I_{\mathcal{Z}'}} X^I_{\mathcal{Z}.\mathcal{Z}'}$.

Then the map

$$T \hookrightarrow T \times_{X^I_{\mathcal{Z}}} X^I_{\mathcal{Z}.\mathcal{Z}'} \to \mathrm{Gr}_{\mathcal{G},X^I,I_\bullet} \times_{X^I} X^I_{\mathcal{Z}.\mathcal{Z}'}$$

factors through $\mathcal{Z} \times_{X_{\mathcal{Z}}} X^I_{\mathcal{Z}.\mathcal{Z}'}$, hence its image under f_* lies in $\mathcal{Z}' \times_{X_{\mathcal{Z}'}} X^I_{\mathcal{Z}.\mathcal{Z}'}$ by assumption. Thus, the map $T \times_{X^I_{\mathcal{Z}'}} X^I_{\mathcal{Z}.\mathcal{Z}'} \to \mathrm{Gr}_{\mathcal{G}',X^I,I_\bullet} \times_{X^I} X^I_{\mathcal{Z}.\mathcal{Z}'}$ factors through $\mathcal{Z}' \times_{X^I_{\mathcal{Z}'}} X^I_{\mathcal{Z}.\mathcal{Z}'}$, too. □

Note that we used the existence of a representative of the bounds over their respective reflex fields. We do not know how to prove the lemma without this assumption.

Construction 3.2.14 (Bounds from cocharacters in the generically reductive case)**.** Let us now construct bounds given by cocharacters in the generically reductive case (compare Assumption 3.2.2 (2)).

Let G be a reductive group over K and let μ be a conjugacy class of geometric cocharacters of G with reflex field K_μ. Let K'/K be a finite separable extension that splits G. We denote by $\mathrm{Gr}_{G_{K'}}^{\leqslant\mu} \subseteq \mathrm{Gr}_{G_{K'}} = \mathrm{Gr}_G \times_K K'$ the Schubert variety inside the (classical) affine Grassmannian for $G_{K'}$. The Schubert variety is already defined over the reflex field of μ and hence descends to a closed subscheme $\mathrm{Gr}_G^{\leqslant\mu} \subseteq \mathrm{Gr}_G \times_K K_\mu$.

Let now $\mathcal{G} \to X$ be a smooth affine group scheme with generic fibre $\mathcal{G}_K = G$. By [RS21], the generic fibre of Beilinson-Drinfeld Grassmannian for $\mathcal{G}$ can be identified (non canonically) with the affine Grassmannian for G, $\mathrm{Gr}_{\mathcal{G},X} \times_X \mathrm{Spec}(K) \cong \mathrm{Gr}_G$. We use this observation to define $\mathrm{Gr}_{\mathcal{G},X}^{\leqslant\mu}$ as the scheme-theoretic image

$$\mathrm{Gr}_{\mathcal{G},X}^{\leqslant\mu} = \mathrm{image}\left(\mathrm{Gr}_G^{\leqslant\mu} \hookrightarrow \mathrm{Gr}_{\mathcal{G},X} \times_X X_\mu\right)$$

where we denote by $X_\mu = \widetilde{X}_{K_\mu}$ the reflex curve of μ. Note that this definition is independent of the choice of the identification of the generic fibre.

Let $\underline{\mu} = (\mu_i)_{i\in I}$ be a tuple of conjugacy classes of cocharacters μ_i of G. We denote by $K_{\underline{\mu}}$ the compositum of all reflex fields of the μ_i and by $X_{\underline{\mu}} = \widetilde{X}_{K_{\underline{\mu}}}$. We denote by $\mathrm{Gr}_{\mathcal{G},X^I,I_\bullet}^{\leqslant\underline{\mu}} \subseteq \mathrm{Gr}_{\mathcal{G},X^I,I_\bullet} \times_{X^I} X_{\underline{\mu}}^I$ the Zariski closure of the preimage of $\prod_{i\in I}\left(\mathrm{Gr}_{\mathcal{G},X}^{\leqslant\mu_i} \times_{X_{\mu_i}} X_{\underline{\mu}}\right)$ under the isomorphism $\mathrm{Gr}_{\mathcal{G},X^I,I_\bullet}|_U \xrightarrow{\cong} (\prod_{i\in I}\mathrm{Gr}_{\mathcal{G},X})|_U$, where $U \subseteq X^I$ is the complement of all diagonals in X^I.

By construction, the equivalence class of $\mathrm{Gr}_{\mathcal{G},X^I,I_\bullet}^{\leqslant\underline{\mu}}$ defines a bound for $\mathcal{G}$ with reflex curve $X_{\underline{\mu}}$ and $\mathrm{Gr}_{\mathcal{G},X^I,I_\bullet}^{\leqslant\underline{\mu}}$ is a representative of this bound over $X_{\underline{\mu}}$. We say that a global $\mathcal{G}$-shtuka is *bounded by* $\underline{\mu}$ if it is bounded by $\mathrm{Gr}_{\mathcal{G},X^I,I_\bullet}^{\leqslant\underline{\mu}}$ and denote by $\mathrm{Sht}_{\mathcal{G},X^I,I_\bullet}^{\leqslant\underline{\mu}} \subseteq \mathrm{Sht}_{\mathcal{G},X^I,I_\bullet} \times_{X^I} X_{\underline{\mu}}^I$ the corresponding moduli stack of global $\mathcal{G}$-shtukas bounded by $\mathrm{Gr}_{\mathcal{G},X^I,I_\bullet}^{\leqslant\underline{\mu}}$.

Lemma 3.2.15. *Let G be a reductive group and let $f\colon \mathcal{G} \to \mathcal{G}'$ be a map of smooth affine group schemes with generic fibres G such that f is an isomorphism over a dense open subset U of X. Let $\underline{\mu} = (\mu_i)_{i\in I}$ be a tuple of conjugacy classes of geometric cocharacters of G. Then f induces a map $f_*\colon \mathrm{Gr}_{\mathcal{G},X^I,I_\bullet}^{\leqslant\underline{\mu}} \to \mathrm{Gr}_{\mathcal{G}',X^I,I_\bullet}^{\leqslant\underline{\mu}}$ that is an isomorphism over U^I.*

Proof. That f_* is defined and an isomorphism over U^I is clear. That f_* extends to a map over X^I follows by the construction of $\mathrm{Gr}_{\mathcal{G},X^I,I_\bullet}^{\leqslant\underline{\mu}}$ as a schematic closure. □

Remark 3.2.16. Let us comment on how the bounds constructed above compare to other notions of bounds given by cocharacters in the literature.

(1) When $\mathcal{G}$ is constant split reductive, our bounds agree with the bounds of [Laf18, Définition 1.12]. This in particular includes the case of Drinfeld shtukas in [Dri87b], that means shtukas for $\mathcal{G} = \mathrm{GL}_n$ and $\underline{\mu} = ((1, 0, \ldots, 0), (0, \ldots, 0, -1))$. In a similar fashion, the bounds used in the unitary case in [FYZ21b; FYZ21a] can be realised in this way.

(2) Already in the split case, there are several other ways to define bounds given by cocharacters, compare [Var04] and [AH21]. In general, these definitions do not agree, see for example [Laf18, Remarque 1.8]. The proof of our main Theorem 3.3.8 does not rely on the concrete construction of the bounds, but only on the fact that the bounds constructed above satisfy Lemma 3.2.15 and the conditions of Theorem 3.3.3.

(3) In the non-split case, [Laf18, § 12.3.1] constructs bounds for parahoric group schemes $\mathcal{G}$ that are given by representations of the L-group of G. Starting from a cocharacter μ of a split maximal torus T of G (defined over some finite extension of K), we can take the direct sum W of all Galois translates of μ. We can then (at least in the generic fibre) recover $\mathrm{Gr}_{\mathcal{G},X^I,I_\bullet}^{\leqslant\underline{\mu}}$ as a component in the base change $\mathrm{Gr}_{\mathcal{G},X^I,I_\bullet}^{W} \times_{X^I} X_{\underline{\mu}}^I$, where $\mathrm{Gr}_{\mathcal{G},X^I,I_\bullet}^{W}$ denotes the bound given by W from [Laf18]. However, in order to study the geometry of the special fibre of our moduli spaces of shtukas it seems to be necessary to use the finer bounds.

3.2.2. Local bounds

We define similar bounds for local shtukas. [AH14] define a notion of local boundedness conditions. However, using their definition the local and global notions are not compatible in a natural way in general, compare Remark 3.2.22 below. We introduce a variant of their notion of local bounds that are naturally compatible with the global bounds defined above.

We start by giving the definition of local shtukas. We continue to use the notation in the local setting from above. Let $k = \mathbb{F}(\!(t)\!)$ be a local field in characteristic p with ring of integers $\mathcal{O} = \mathbb{F}[\![t]\!]$ and finite residue field $\mathbb{F}$. Let $\mathcal{G} \to \mathcal{O}$ be a smooth affine group scheme. We denote by $L\mathcal{G}$ (respectively $L^+\mathcal{G}$) the (positive) loop group of $\mathcal{G}$ defined as functors on the category of $\mathbb{F}$-algebras as

$$R \mapsto L\mathcal{G}(R) = \mathcal{G}(R(\!(t)\!)) \qquad \text{and} \qquad R \mapsto L^+\mathcal{G}(R) = \mathcal{G}(R[\![t]\!]),$$

respectively. The loop group $L\mathcal{G}$ is representable by an ind-group scheme of ind-finite type over $\mathbb{F}$, the positive loop group is representable by an affine (infinite dimensional) group scheme over $\mathbb{F}$. Recall that the (classical) affine Grassmannian $\mathrm{Gr}_{\mathcal{G}}$ for $\mathcal{G}$ is given by the fpqc-sheafification of the quotient $\mathrm{Gr}_{\mathcal{G}} = (L\mathcal{G}/L^+\mathcal{G})_{\text{fpqc}}$. Moreover, using the inclusion $L^+\mathcal{G} \to L\mathcal{G}$, there is a natural way to associate to a $L^+\mathcal{G}$-torsor $\mathcal{E}^+$ its corresponding $L\mathcal{G}$-torsor $\mathcal{E}$.

For an $\mathbb{F}[\![t]\!]$-algebra R we denote by $\zeta \in R$ the image of t. We denote by $\mathcal{N}ilp_{\mathbb{F}[\![\zeta]\!]}$ the category of $\mathbb{F}[\![t]\!]$-algebras where ζ is nilpotent.

Definition 3.2.17. Let $R \in \mathcal{N}ilp_{\mathbb{F}[\![\zeta]\!]}$. A *local $\mathcal{G}$-shtuka* over R is a pair $\underline{\mathcal{E}} = (\mathcal{E}^+, \varphi)$ consisting of a $L^+\mathcal{G}$-torsor $\mathcal{E}^+$ on R and an isomorphism of $L\mathcal{G}$-torsors $\varphi\colon \sigma^*\mathcal{E} \to \mathcal{E}$.

Instead of defining bounds as certain subschemes in $\mathrm{Gr}_{\mathcal{G}} \hat{\times} \mathrm{Spf}(\mathbb{F}[\![t]\!])$ as in [AH14], we use the following local variant of Beilinson-Drinfeld affine Grassmannians following [Ric21] to define local bounds.

Definition 3.2.18. The Beilinson-Drinfeld affine Grassmannian $\mathrm{Gr}_{\mathcal{G},\mathcal{O}}$ for $\mathcal{G}$ is the functor on $\mathcal{O}$-algebras defined by

$$R \mapsto \left\{ (\mathcal{E}, \alpha)\colon \begin{array}{l} \mathcal{E} \text{ a } \mathcal{G}\text{-torsor on } \mathrm{Spec}(R[\![t-\zeta]\!]), \\ \alpha\colon \mathcal{E}|_{R(\!(t-\zeta)\!)} \xrightarrow{\cong} \mathcal{G}_{R(\!(t-\zeta)\!)} \text{ a trivialisation over} R(\!(t-\zeta)\!) \end{array} \right\}.$$

By [Ric21], $\mathrm{Gr}_{\mathcal{G},\mathcal{O}}$ is representable by an ind-scheme over $\mathcal{O}$. Moreover, for a smooth, affine group scheme $\mathcal{G} \to X$ and a closed point $x \in X$ we have a canonical isomorphism $\mathrm{Gr}_{\mathcal{G}_{\mathcal{O}_x},\mathcal{O}_x} = \mathrm{Gr}_{\mathcal{G},X} \times_X \mathrm{Spec}(\mathcal{O}_x)$.

The affine Grassmannian $\mathrm{Gr}_{\mathcal{G},\mathcal{O}}$ carries an action of the positive loop group $\mathcal{L}^+_{\mathcal{O}}\mathcal{G}$ defined as the functor on $\mathcal{O}$-algebras by

$$R \mapsto (\mathcal{L}^+_{\mathcal{O}}\mathcal{G})(R) = \mathcal{G}(R[\![t-\zeta]\!]).$$

Note that the special fibre of $\mathrm{Gr}_{\mathcal{G},\mathcal{O}}$ is the classical affine Grassmannian for $\mathcal{G}$, while the generic fibre of $\mathrm{Gr}_{\mathcal{G},\mathcal{O}}$ is the B_{dR}-affine Grassmannian for $G = \mathcal{G}_k$.

In order to define bounded local shtukas, we need to construct points in (the formal completion of) $\mathrm{Gr}_{\mathcal{G},\mathcal{O}}$ from a local shtuka. This is done as follows. Let $\underline{\mathcal{E}} = (\mathcal{E}, \varphi)$ be a local shtuka over $R \in \mathcal{N}ilp_{\mathbb{F}[\![\zeta]\!]}$. Let $R \to R'$ be an fppf-cover that trivialises $\mathcal{E}$. As $\zeta \in R$ is nilpotent by assumption, we have $R[\![t-\zeta]\!] = R[\![t]\!]$. Using the equivalence of $L^+\mathcal{G}$-torsors over R with formal $\hat{\mathcal{G}} = \mathcal{G} \times_{\mathbb{F}[\![t]\!]} \mathrm{Spf}(\mathbb{F}[\![t]\!])$-torsors over $\mathrm{Spf}(R[\![t]\!]) = \mathrm{Spf}(R[\![t-\zeta]\!])$ from [AH14, Proposition 2.4], a trivialisation $\alpha\colon \mathcal{E}_{R'} \xrightarrow{\cong} \hat{\mathcal{G}}_{\mathrm{Spf}(R'[\![t-\zeta]\!])}$ defines a R'-rational point in $\widehat{\mathrm{Gr}}_{\mathcal{G},\mathbb{F}[\![t]\!]} := \mathrm{Gr}_{\mathcal{G},\mathbb{F}[\![t]\!]} \times_{\mathrm{Spec}(\mathbb{F}[\![t]\!])} \mathrm{Spf}(\mathbb{F}[\![t]\!])$ given by $(\sigma^*\mathcal{E}, \alpha \circ \varphi)$.

Using this version of affine Grassmannians, we define local bounds in the style of [AH14, Definitions 4.5 and 4.8].

Definition 3.2.19. Let us fix an algebraic closure k^{alg} of k.

(1) Let $\mathcal{O} \subseteq \mathcal{O}_1, \mathcal{O}_2$ be two finite extensions of discrete valuation rings in k^{alg}. We call two locally closed subschemes

$$Z_1 \subseteq \mathrm{Gr}_{\mathcal{G},\mathcal{O}} \times_{\mathrm{Spec}(\mathcal{O})} \mathrm{Spec}(\mathcal{O}_1) \qquad \text{and} \qquad Z_2 \subseteq \mathrm{Gr}_{\mathcal{G},\mathcal{O}} \times_{\mathrm{Spec}(\mathcal{O})} \mathrm{Spec}(\mathcal{O}_2)$$

equivalent if there is a common finite extension $\mathcal{O}_1, \mathcal{O}_2 \subseteq \mathcal{O}'$ of discrete valuation rings in k^{alg} such that $Z_1 \times_{\mathrm{Spec}(\mathcal{O}_1)} \mathrm{Spec}(\mathcal{O}') = Z_2 \times_{\mathrm{Spec}(\mathcal{O}_2)} \mathrm{Spec}(\mathcal{O}')$ in $\mathrm{Gr}_{\mathcal{G},\mathcal{O}} \times_{\mathrm{Spec}(\mathcal{O})} \mathrm{Spec}(\mathcal{O}')$.

(2) A *local bound* is an equivalence class $\mathcal{Z}$ of quasi-compact locally closed subschemes of $\mathrm{Gr}_{\mathcal{G},\mathcal{O}}$ such that all representatives are stable under the $\mathcal{L}^+_{\mathcal{O}}\mathcal{G}$-action and such that $\mathcal{Z}$ admits a representative over its field of definition (also called its *reflex field*) as defined in [AH14, Definition 4.5].

(3) Let $\mathcal{Z}$ be a bound in the above sense and let $\underline{\mathcal{E}} = (\mathcal{E}, \varphi)$ be a local shtuka over $R \in \mathcal{N}ilp_{\mathbb{F}[\![\zeta]\!]}$. Let $R \to R'$ be an fppf-cover that trivialises $\mathcal{E}$ and choose a trivialisation α of $\mathcal{E}$ over R'. We say that $\underline{\mathcal{E}}$ *is bounded* by $\mathcal{Z}$ if for all representatives $Z_{\mathcal{O}'}$ of $\mathcal{Z}$ over $\mathcal{O}'$, the point in $\widehat{\mathrm{Gr}}_{\mathcal{G},\mathcal{O}}(R')$ induced by α factors through $Z_{\mathcal{O}'}$. As $Z_{\mathcal{O}'}$ is invariant under the left $\mathcal{L}^+_{\mathcal{O}'}\mathcal{G}$-action, the definition is independent of the choice of the trivialisation α.

Remark 3.2.20. The discussion of [AH14, Remarks 4.6, 4.7 and 4.9] (respectively their global analogues in Lemma 3.2.10 and Remark 3.2.11) also applies in this setting. In particular, the representative of a bound over its reflex field is unique and it suffices to check boundedness of a local shtuka for a single representative. By a slight abuse of notation we may thus identify a bound with its representative over its reflex field. Note that it is not known if an equivalence class of $\mathcal{L}^+_{\mathcal{O}}\mathcal{G}$-stable subschemes in $\mathrm{Gr}_{\mathcal{G},\mathcal{O}}$ always admits a representative over its reflex field.

As in the global case (compare Construction 3.2.14) we define bounds given by cocharacters when the generic fibre of $\mathcal{G}$ is reductive. When $\mathcal{G}$ is parahoric, these bounds coincide with the global Schubert varieties defined in [Ric16, Definition 2.3].

Definition 3.2.21. Assume that the generic fibre $G = \mathcal{G}_k$ of $\mathcal{G}$ is reductive. Let μ be a conjugacy class of geometric cochcaracters of G with reflex field k_μ. Let $\mathcal{O}_\mu$ be the ring of integers in k_μ. Then $\mathrm{Gr}^{\leqslant\mu}_{\mathcal{G},\mathcal{O}}$ is defined to be the scheme-theoretic closure of $\mathrm{Gr}^{\leqslant\mu}_{G}$ inside $\mathrm{Gr}_{\mathcal{G},\mathcal{O}} \times_{\mathrm{Spec}(\mathcal{O})} \mathrm{Spec}(\mathcal{O}_\mu)$.

Clearly, $\mathrm{Gr}^{\leqslant\mu}_{\mathcal{G},\mathcal{O}}$ defines a local bound with reflex ring $\mathcal{O}_\mu$. Note that when $\mathcal{G}$ is constant split reductive, the bounds defined here may differ from the bound given by μ in [HV11, Definition 3.5], compare [Zhu17, Remark 2.1.7] and [Laf18, Remark 1.18]. However, they do agree when μ is minuscule and G^{der} is simply connected.

Remark 3.2.22. Morally, the difference between bounds defined as (locally) closed subschemes of $\mathrm{Gr}_{\mathcal{G}} \widehat{\times} \mathrm{Spf}(\mathcal{O})$ (as in [AH14]) and $\mathrm{Gr}_{\mathcal{G},\mathcal{O}}$ as defined above is the following. As noted in [AH14, Example 4.13], the first kind of subschemes naturally gives rise to bounds along $\tilde{t} = t$ (in the notation of [HV11]), while our bounds give rise to bounds along $\tilde{t} = t - \zeta$. In this sense, it seems more natural to define bounds for local shtukas inside $\mathrm{Gr}_{\mathcal{G},\mathcal{O}}$, compare [HV11, Remark 3.6]. When $\mathcal{G}$ is constant split reductive, the bounds given by $(\mu, t-\zeta)$ of [HV11] can be represented inside $\mathrm{Gr}_{\mathcal{G}} \widehat{\times} \mathrm{Spf}(\mathcal{O})$ by [AH14, Example 4.13]. However, this may fail to be the case in general.

3.2.3. Local-global compatibility.

We explain how to construct local bounds from global ones. We recall the global-to-local functor for shtukas from [AH14, Section 5] and show that our notions of global and local bounds are compatible in the sense that a global shtuka is bounded if and only if its corresponding local shtukas are bounded by the associated local bounds. This observation gives rise to a bounded version of the Serre-Tate Theorem [AH14, Theorem 5.13].

We use the following notation following [AH14, Remark 5.2]. Let $y \in X$ be a closed point. We denote by $\mathcal{O}_y$ the completed local ring at y, and by $\mathfrak{m}_y \subseteq \mathcal{O}_y$ and $\mathbb{F}_y = \mathcal{O}_y/\mathfrak{m}_y$ its maximal ideal with uniformiser ϖ_y and residue field, respectively. Let $x \in X(R)$ be a section of X such that x factors through $\mathrm{Spf}(\mathcal{O}_y)$, in other words, the image of the uniformiser ϖ_y in R is nilpotent. Then the $\mathfrak{m}$-adic completion of $\mathcal{O}_y \otimes_{\mathbb{F}_q} R$ factors as

$$\mathcal{O}_y \widehat{\otimes}_{\mathbb{F}_q} R = (\mathbb{F}_y \otimes_{\mathbb{F}_q} R)\llbracket \varpi_y \rrbracket = \prod_{1 \leqslant \ell \leqslant [\mathbb{F}_y : \mathbb{F}_q]} \mathcal{O}_y \widehat{\otimes}_{\mathbb{F}_y} R = \prod_{1 \leqslant \ell \leqslant [\mathbb{F}_y : \mathbb{F}_q]} R\llbracket \varpi_y \rrbracket .$$

The ℓ-th factor is defined by the ideal $\mathfrak{a}_\ell = \langle a \times 1 - 1 \otimes x(a)^{q^\ell} : a \in \kappa_y \rangle$ in $\mathcal{O}_y \widehat{\otimes}_{\mathbb{F}_q} R$ and the Frobenius σ cyclically permutes the factors.

Remark 3.2.23. We explain how global bounds give rise to local bounds following [AH19, Proposition 4.3.3]. Let $\mathcal{G} \to X$ be a smooth affine group scheme and let $\mathcal{Z}$ be a global bound for $\mathcal{G}$. Let us fix a tuple $\underline{y} = (y_i)_{i \in I} \in X^I$ of pairwise distinct closed points in X. Using the isomorphism $\mathrm{Gr}_{\mathcal{G},X^I,I_\bullet}|_U = (\prod_{i \in I} \mathrm{Gr}_{\mathcal{G},X})|_U$ over the complement of all diagonals U in X^I, we denote by $\mathcal{Z}_i$ the image of $\mathcal{Z}$ under the projection to the i-th component. Then $\mathcal{Z}_i \subseteq \mathrm{Gr}_{\mathcal{G},X} \times_X X_{\mathcal{Z}}$ is a quasi-compact locally closed subscheme stable under the action of $\mathcal{L}_X^+\mathcal{G}$.

Let y_i' be a closed point of $\widetilde{X}_{\mathcal{Z}}$ lying over y_i. We denote by $\mathcal{Z}_{y_i'} = \mathcal{Z}_i \times_{\tilde{X}_{K'}} \mathrm{Spec}(\mathcal{O}_{y_i'})$. Then $\mathcal{Z}_{y_i'} \subseteq \mathrm{Gr}_{\mathcal{G},\mathcal{O}_y} \times_{\mathrm{Spec}(\mathcal{O}_y)} \mathrm{Spec}(\mathcal{O}_{y_i'})$ is a locally closed subscheme stable under the loop group action. In particular, for a tuple of points $\underline{y}' = (y_i')_{i\in I}$ of $\mathcal{X}_{\mathcal{Z}}^I$ lying over $\underline{y}$, we can associate to a global bound $\mathcal{Z}$ an I-tuple of equivalence classes of $\mathcal{L}_{\mathcal{O}}^+\mathcal{G}$-stable subschemes $(\mathcal{Z}_{y_i'})_{i\in I}$. Note that it is not clear in general that the equivalence class of subschemes defined by $\mathcal{Z}_{y_i'}$ does indeed admit a representative over its reflex ring (which will in general be different from $\mathcal{O}_{y_i'}$).

However, in the generically reductive case and $\mathcal{Z} = \mathrm{Gr}_{\mathcal{G},X^I,I_\bullet}^{\leqslant\mu}$ for an I-tuple of conjugacy classes of geometric cocharacters of $G = \mathcal{G}_K$ we get $\mathcal{Z}_{y_i'} = \mathrm{Gr}_{\mathcal{G}_{\mathcal{O}_{y_i}},\mathcal{O}_{y_i}}^{\leqslant\mu_i} \times_{\mathrm{Spec}(\mathcal{O}_{\mu_i})} \mathrm{Spec}(\mathcal{O}_{y_i'})$ by construction, so in this case the $\mathcal{Z}_{y_i'}$ do indeed define local bounds.

Remark 3.2.24. More precisely, [AH19, Proposition 4.3.3] construct local bounds in the sense of [AH14] by further pulling back the global bound to a subscheme in $\mathrm{Gr}_{\mathcal{G}} \hat{\times}_{\mathbb{F}_q} \mathrm{Spf}(\mathcal{O})$. In particular, the local bounds associated to $\mathrm{Gr}_{G,X^I,I_\bullet}^{\leqslant\mu}$ in the split reductive case are $\mathrm{Gr}_G^{\leqslant\mu_i} \hat{\times}_{\mathbb{F}_q} \mathrm{Spf}(\mathcal{O})$ rather than $\mathrm{Gr}_{\mathcal{G}_{\mathcal{O}_{y_i}},\mathcal{O}_{y_i}}^{\leqslant\mu_i}$, compare Remark 3.2.22.

Global-to-local functor

We explain how to associate local shtukas to global shtukas following [AH14, Section 5]. Let us fix a tuple $\underline{y} = (y_i)_{i\in I}$ of pairwise distinct closed points of X. Let $\underline{\mathcal{E}} = ((x_i)_{i\in I}, \mathcal{E}, \varphi) \in \mathrm{Sht}_{\mathcal{G},X^I}^{\underline{y}}(R)$. By the observation above, the y_i-adic completion of $\mathcal{E}$ decomposes as

$$\mathcal{E} \hat{\times}_{X_R} \mathrm{Spf}(\mathcal{O}_{y_i} \hat{\otimes}_{\mathbb{F}_q} R) = \coprod_{1\leqslant \ell \leqslant [\mathbb{F}_{y_i}\,:\,\mathbb{F}_q]} \mathcal{E} \hat{\times}_{X_R} \mathrm{Spf}(R[\![\varpi_{y_i}]\!]),$$

and each component is a formal $\hat{\mathcal{G}}_{y_i} = \mathcal{G} \times_X \mathrm{Spf}(\mathcal{O}_{y_i})$-torsor over R. Hence, $\widehat{\underline{\mathcal{E}}_{y_i}} = \big(\mathcal{E} \hat{\times}_{X_R} V(\mathfrak{a}_0), \varphi^{\deg(y_i)}\big)$ is a local $\mathcal{G}_{\mathcal{O}_{y_i}}$-shtuka over R.

Definition 3.2.25. The *global-to-local functor* associates to a global shtuka $\underline{\mathcal{E}} \in \mathrm{Sht}_{\mathcal{G},X^I}^{\underline{y}}(R)$ a tuple of local $\mathcal{G}_{y_i}$-shtukas for $i \in I$ given by

$$\widehat{\underline{\mathcal{E}}_{\underline{y}}} = (\widehat{\underline{\mathcal{E}}_{y_i}})_{i\in I}.$$

Then, $\widehat{\underline{\mathcal{E}}_{y_i}}$ is called the *local shtuka* of $\underline{\mathcal{E}}$ at y_i.

Remark 3.2.26. In a similar fashion, for a closed point y of X we can associate to a global shtuka $\underline{\mathcal{E}} = ((x_i), (\mathcal{E}_j), (\varphi_j), \theta) \in \mathrm{Sht}_{\mathcal{G},X^I,I_\bullet}|_{(X\setminus\{y\})^I}(R)$ with characteristic sections away from y an étale local shtuka at y by [AH14, Remark 5.6] as follows. We denote by $\widetilde{\mathcal{G}}_y =$

$\mathrm{Res}_{\mathbb{F}_y/\mathbb{F}_q}\mathcal{G}_{\mathcal{O}_y}$. Then $\widetilde{\mathcal{G}}_y$ is a smooth affine group scheme over $\mathbb{F}_q[\![\varpi_y]\!]$. The *étale local* $\widetilde{\mathcal{G}}_y$-*shtuka* associated to $\mathcal{E}$ is then given by $\underline{\widetilde{\mathcal{E}}}_y = (\widetilde{\mathcal{E}}_y, \varphi)$ with $\widetilde{\mathcal{E}}_y = \mathrm{Res}_{\mathbb{F}_y/\mathbb{F}_q}(\mathcal{E}_m\widehat{X_R}(\mathcal{O}_y\widehat{\otimes}_{\mathbb{F}_q}R))$ and $\varphi = \varphi_m \circ \ldots \circ \varphi_0 \circ \theta$. Note that $\underline{\widetilde{\mathcal{E}}}_y$ is called étale as φ is an isomorphism by assumption.

The global-to-local functor is compatible with our notion of bounds in the following sense. Let us fix a global bound $\mathcal{Z}$ for $\mathcal{G}$ and a tuple of closed points $\underline{y}' = (y_i')_{i\in I \in X^I_{\mathcal{Z}}}$ such that y_i' lies over y_i. We denote by $\mathrm{Sht}^{\underline{y}'}_{\mathcal{G},X^I} = \mathrm{Sht}_{\mathcal{G},X^I} \times_{X^I} \mathrm{Spf}(\mathcal{O}_{\underline{y}'})$.

Proposition 3.2.27. *Assume that the associated local equivalence classes $\mathcal{Z}_{y_i'}$ constructed in Remark 3.2.23 are local bounds. A global shtuka $\underline{\mathcal{E}} \in \mathrm{Sht}^{\underline{y}'}_{\mathcal{G},X^I}(R)$ is bounded by $\mathcal{Z}$ if and only if for all $i \in I$ its associated local shtuka $\widehat{\underline{\mathcal{E}}_{y_i}}$ at y_i is bounded by $\mathcal{Z}_{y_i'}$.*

Proof. Let us fix an fppf-cover $R' \to R$ and a trivialisation $\alpha\colon \mathcal{E}|_{\hat{\Gamma}_{\underline{x}_{R'}}} \xrightarrow{\cong} \mathcal{G}|_{\hat{\Gamma}_{\underline{x}_{R'}}}$. As the $(y_i)_{i\in I}$ were assumed to be pairwise distinct, we have $\hat{\Gamma}_{\underline{x}_{R'}} = \bigcup_{i\in I}\hat{\Gamma}_{x_{i,R'}}$. Moreover, by [AH14, Lemma 5.3] we have $\hat{\Gamma}_{x_{i,R'}} = V(\mathfrak{a}_0)$. By construction, the induced point $(\underline{\mathcal{E}}_{R'}, \alpha) \in \mathrm{Gr}_{\mathcal{G},X^I}(R')$ factors through $\mathcal{Z}$ if and only if the restriction of α to $\hat{\Gamma}_{x_{i,R'}}$ factors through $\mathcal{Z}_{y_i'}$ for all $i \in I$, or equivalently the corresponding point $R' \times_{\mathcal{O}_{\mathcal{Z}_{y_i'}}} \mathcal{O}_{y_i'} \to \mathrm{Gr}_{\mathcal{G},\mathcal{O}_{y_i}} \times_{\mathrm{Spec}(\mathcal{O}_{y_i})} \mathrm{Spec}(\mathcal{O}_{y_i'})$ factors through $\mathcal{Z}_{y_i'}$. But this is the case if and only if the local shtuka $\widehat{\underline{\mathcal{E}}_{y_i}}$ at y_i is bounded by $\mathcal{Z}_{y_i'}$ by definition. □

Remark 3.2.28. Let $\underline{y} = (y_i)_{i\in I}$ be a tuple of pairwise distinct closed points of X. Let $(\mathcal{Z}_i)_{i\in I}$ be a tuple of local bounds at $\underline{y}$. We denote by $\mathcal{O}_{(\mathcal{Z}_i)_{i\in I}} = \widehat{\bigotimes}_{i\in I}\mathcal{O}_{\mathcal{Z}_i}$. As in [AH19, Definition 4.3.2], we say a global shtuka $\underline{\mathcal{E}} \in \mathrm{Sht}^{\underline{y}}_{\mathcal{G},X^I} \times_{\mathrm{Spf}(\mathcal{O}_{\underline{y}})} \mathrm{Spf}(\mathcal{O}_{(\mathcal{Z}_i)_{i\in I}})$ is bounded by $(\mathcal{Z}_i)_{i\in I}$ if its associated local shtuka at y_i is bounded by $\mathcal{Z}_i$ for all $i \in I$. When the local bounds come from a global bound, the previous proposition shows that this notion of local boundedness conditions agrees with the global one. We do not explore these local boundedness conditions for global shtukas further here as the bounds we are later interested in, namely the ones given by cocharacters, arise from global bounds.

The global-to-local functor also gives rise to a Serre-Tate theorem relating the deformation theory of global shtukas with the deformation theory of their associated local shtukas, compare [AH14, Theorem 5.10]. Let $S = \mathrm{Spec}(R) \in \mathcal{N}ilp_{\mathcal{O}_{\underline{y}}}$ and let $i\colon \overline{S} = \mathrm{Spec}(R/I) \hookrightarrow S$ be a closed subscheme defined by a nilpotent ideals I. Let $\underline{\bar{\mathcal{E}}} \in \mathrm{Sht}^{\mathcal{Z},\underline{y}}_{\mathcal{G},X^I}(\overline{S})$. We denote by $\mathrm{Def}^{\mathcal{Z}}_{\underline{\bar{\mathcal{E}}}}(S)$ the category of bounded deformations of $\underline{\mathcal{E}}$ to S, in other words, the category of pairs $(\underline{\mathcal{E}}, \beta\colon i^*\underline{\mathcal{E}} \tilde{\to} \underline{\bar{\mathcal{E}}})$ where $\underline{\mathcal{E}} \in \mathrm{Sht}^{\mathcal{Z},\underline{y}}_{\mathcal{G},X^I}(S)$ and β is an isomorphism of $\mathcal{G}$-shtukas over $\overline{S}$.

Similarly, for a local $\mathcal{G}_{y_i}$-shtuka $\underline{\bar{\mathcal{E}}}$ bounded by $\mathcal{Z}_{y_i}$ we define $\mathrm{Def}_{\underline{\bar{\mathcal{E}}}}^{\mathcal{Z}_{y_i}}(S)$ as the category of bounded deformations of $\underline{\bar{\mathcal{E}}}$ to S, that is, the category of pairs $(\underline{\mathcal{E}}, \beta\colon i^*\underline{\mathcal{E}} \xrightarrow{\sim} \underline{\bar{\mathcal{E}}})$ where $\underline{\mathcal{E}}$ is a local $\mathcal{G}_{y_i}$-shtuka on S bounded by $\mathcal{Z}_{y_i}$ and β is an isomorphism of local $\mathcal{G}_{y_i}$-shtukas over $\overline{S}$.

Corollary 3.2.29 (Bounded Serre-Tate Theorem for shutkas)**.** *Let $\underline{\bar{\mathcal{E}}} \in \mathrm{Sht}_{\mathcal{G},X^I}^{\mathcal{Z},\underline{y}}(\overline{S})$. The restriction of the global-to-local functor*

$$\widehat{(-)_{\underline{y}}}\colon \mathrm{Def}_{\underline{\bar{\mathcal{E}}}}^{\mathcal{Z}}(S) \to \prod_{i\in I} \mathrm{Def}_{\widehat{\underline{\bar{\mathcal{E}}}_{y_i}}}^{\mathcal{Z}_i}(S), \qquad (\underline{\mathcal{E}}, \beta) \mapsto (\widehat{\underline{\mathcal{E}}_{y_i}}, \widehat{\beta_{y_i}})_{i\in I}$$

is an equivalence of categories.

Proof. This follows directly from the unbounded case in [AH14, Theorem 5.10.] together with Proposition 3.2.27 □

3.3. Level maps and integral models with deep Bruhat-Tits level

We construct integral models for moduli spaces of shtukas with deep Bruhat-Tits level structures and show that these integral models admit proper, surjective and generically étale level maps. In order to do so, we first study the morphism on shtuka spaces induced by a generic isomorphism of group schemes extending a result of [Bre18].

3.3.1. Functoriality of shtuka spaces under generic isomorphisms

We study functoriality of shtuka spaces under homomorphisms of group schemes that are generic isomorphisms. We prove an analogue of [Bre18, Theorem 3.20] in our setting of shtukas with global bounds. In particular, we get the result on the whole curve and need not restrict the legs to a formal neighbourhood of fixed sections as in [Bre18]. Moreover, we show that the level maps in our setting are generically finite étale, which is not part of [Bre18]. This already shows that we have nice level maps in the parahoric case.

Remark 3.3.1. Let us first note the following functoriality properties of the affine Grassmannian in this setting.

(1) Let $f\colon \mathcal{G} \to \mathcal{G}'$ be a homomorphism of group schemes over X such that f is an isomorphism over a dense open subset $U \subseteq X$. The induced map

$$f_*\colon \mathrm{Gr}_{\mathcal{G},X^I,I_\bullet} \to \mathrm{Gr}_{\mathcal{G}',X^I,I_\bullet}$$

is then an isomorphism over U^I using the moduli description from Remark 3.2.4.

(2) In the Bruhat-Tits case (compare Assumption 3.2.2 (3)) it follows that the map

$$\rho_{\Omega,*}\colon\ \mathrm{Gr}_{\mathcal{G}_\Omega,X^I,I_\bullet} \to \varprojlim_{\mathfrak{f}<\Omega} \mathrm{Gr}_{\mathcal{G}_{\mathfrak{f}},X^I,I_\bullet}$$

is an open immersion by Theorem 3.1.13 and an isomorphism over $(X\backslash\{x_0\})^I$ using the previous observation.

(3) Moreover, using Lemma 3.2.15 we obtain a map

$$\rho_{\Omega,*}\colon\ \mathrm{Gr}^{\leqslant\mu}_{\mathcal{G}_\Omega,X^I,I_\bullet} \to \varprojlim_{\mathfrak{f}<\Omega} \mathrm{Gr}^{\leqslant\mu}_{\mathcal{G}_{\mathfrak{f}},X^I,I_\bullet}$$

that factors as a closed immersion followed by an open immersion

$$\mathrm{Gr}^{\leqslant\mu}_{\mathcal{G}_\Omega,X^I,I_\bullet} \to \mathrm{Gr}_{\mathcal{G}_\Omega,X^I,I_\bullet} \times_{\varprojlim_{\mathfrak{f}<\Omega}\mathrm{Gr}_{\mathcal{G}_{\mathfrak{f}},X^I,I_\bullet}} \varprojlim_{\mathfrak{f}<\Omega} \mathrm{Gr}^{\leqslant\mu}_{\mathcal{G}_{\mathfrak{f}},X^I,I_\bullet} \to \varprojlim_{\mathfrak{f}<\Omega} \mathrm{Gr}^{\leqslant\mu}_{\mathcal{G}_{\mathfrak{f}},X^I,I_\bullet}$$

and is hence locally closed immersion and an isomorphism over $(X\backslash\{x_0\})^I$.

We need the following lemma on twisted flag varieties in the local setting.

Lemma 3.3.2. *Let $k = \mathbb{F}((t))$ be the field of formal Laurent series over an arbitrary field $\mathbb{F}$ and let $\mathfrak{o} = \mathbb{F}[\![t]\!]$ the subring of formal power series. Let G be a smooth affine group scheme over k and let $\mathcal{G}$ and $\mathcal{G}'$ be two smooth integral models of G with geometrically connected fibres. Let $f\colon \mathcal{G} \to \mathcal{G}'$ be a homomorphism of $\mathfrak{o}$-group schemes that is the identity on G over k.*

(1) The corresponding twisted flag variety $L^+\mathcal{G}'/L^+\mathcal{G}$ is representable by a smooth and separated scheme of finite type over $\mathbb{F}$. If $\mathbb{F}$ is finite or separably closed, then

$$\left(L^+\mathcal{G}'/L^+\mathcal{G}\right)(\mathbb{F}) = \mathcal{G}'(\mathfrak{o})/\mathcal{G}(\mathfrak{o}).$$

(2) Assume that $\mathbb{F}$ is finite. We equip $G(k)$ with the analytic topology induced by the natural topology on k (note that k is locally compact in this case). Then $\mathcal{G}(\mathfrak{o})$ is a compact open subgroup of $\mathcal{G}'(\mathfrak{o})$. In particular, the quotient $\mathcal{G}'(\mathfrak{o})/\mathcal{G}(\mathfrak{o})$ is discrete and finite.

(3) Let S be an $\mathbb{F}$-scheme. Giving a $\mathcal{L}^+\mathcal{G}$-torsor over S is equivalent to giving a $\mathcal{L}^+\mathcal{G}'$-torsor $\mathcal{E}'$ over S together with an isomorphism $\mathcal{E}'/\mathcal{L}^+\mathcal{G} \xrightarrow{\cong} \mathcal{L}^+\mathcal{G}'/\mathcal{L}^+\mathcal{G}$.

Note that giving an isomorphism $\mathcal{E}'/\mathcal{L}^+\mathcal{G} \xrightarrow{\cong} \mathcal{L}^+\mathcal{G}'/\mathcal{L}^+\mathcal{G}$ in (3) is also clearly equivalent to giving a section in $\left(\mathcal{E}'/\mathcal{L}^+\mathcal{G}\right)(S)$.

Proof. (1) By the argument in the proof of [Bre18, Lemma 3.17], the quotient stack $L^+\mathcal{G}'/L^+\mathcal{G}$ is representable by a separated scheme of finite type over $\mathbb{F}$ that is moreover a closed subscheme of the affine Grassmannian $\mathrm{Gr}_{\mathcal{G}}$. As both $\mathcal{L}^+\mathcal{G}$ and $\mathcal{L}^+\mathcal{G}'$ are formally smooth over $\mathbb{F}$, the quotient $L^+\mathcal{G}'/L^+\mathcal{G}$ is hence formally smooth as well.

For the second claim, it suffices to show that $H^1(\mathbb{F}, L^+\mathcal{G})$ is trivial by the moduli description of the quotient stack. But this is shown in the proof of [Ric20, Corollary 3.22].

(2) Clearly, both $\mathcal{G}(\mathfrak{o})$ and $\mathcal{G}'(\mathfrak{o})$ are compact open subgroups of $G(k)$ by construction. The existence of the map f then means that $\mathcal{G}(\mathfrak{o})$ is a subgroup of $\mathcal{G}'(\mathfrak{o})$. The assertion on the quotient then directly follows from basic facts from topology.

(3) Given a $\mathcal{L}^+\mathcal{G}$-torsor $\mathcal{E}$ on S, its associated $\mathcal{L}^+\mathcal{G}'$-torsor is given by $\mathcal{E} \times^{\mathcal{L}^+\mathcal{G}} \mathcal{L}^+\mathcal{G}'$. The map on sections given by $(e, g) \mapsto g$ then induces an isomorphism

$$\mathcal{E}'/\mathcal{L}^+\mathcal{G} \xrightarrow{\cong} \mathcal{L}^+\mathcal{G}'/\mathcal{L}^+\mathcal{G}.$$

This construction is an equivalence.

□

Theorem 3.3.3. *Let $\mathcal{G}$ and $\mathcal{G}'$ be two smooth affine group schemes over X with geometrically connected fibres. Let $f\colon (\mathcal{G}, \mathcal{Z}) \to (\mathcal{G}', \mathcal{Z}')$ be a map of shtuka data such that the map $f\colon \mathcal{G} \to \mathcal{G}'$ is an isomorphism over $U = X \backslash \{y_1, \ldots, y_n\}$ for a finite set of closed points $\{y_1, \ldots, y_n\}$ of X.*

(1) The induced map

$$f_*\colon \mathrm{Sht}^{\mathcal{Z}}_{\mathcal{G},X^I,I_\bullet} \times_{X^I_{\mathcal{Z}}} X^I_{\mathcal{Z}.\mathcal{Z}'} \to \mathrm{Sht}^{\mathcal{Z}'}_{\mathcal{G}',X^I,I_\bullet} \times_{X^I_{\mathcal{Z}'}} X^I_{\mathcal{Z}.\mathcal{Z}'}$$

is schematic, separated and of finite type.

(2) Assume that $\mathcal{G}$ is a parahoric Bruhat-Tits group scheme and that $\mathcal{Z} \subseteq \mathrm{Gr}_{\mathcal{G},X^I,I_\bullet} \times_{X^I} X^I_{\mathcal{Z}}$ is a closed subscheme. Then the map f_* is moreover proper.

(3) Assume that $\mathcal{Z} \times_{X^I_{\mathcal{Z}}} X_{\mathcal{Z}.\mathcal{Z}'} \to \mathcal{Z}' \times_{X^I_{\mathcal{Z}}} X_{\mathcal{Z}.\mathcal{Z}'}$ is an isomorphism over $(U \times_X X_{\mathcal{Z}.\mathcal{Z}'})^I$. Then the map f_* is étale locally representable by the constant scheme

$$\prod_{i=1}^{n} \underline{\mathcal{G}'(\mathcal{O}_{y_i})/\mathcal{G}(\mathcal{O}_{y_i})}.$$

In particular, f_* is finite étale and surjective over $(U \times_X X_{\mathcal{Z}.\mathcal{Z}'})^I$.

(4) *Under the assumptions of (2) and (3) assume additionally that $\mathcal{Z}'$ is the schematic closure of $\mathcal{Z}'|_{(U\times_X X_{\mathcal{Z}'})^I}$ in $\mathrm{Gr}_{\mathcal{G}',X^I_{\mathcal{Z}'},I_\bullet}$. Then f_* is surjective.*

Remark 3.3.4. The first two statements are direct analogues of the corresponding statements in [Bre18, Theorem 3.20], while there is no analogue of the third assertion in [Bre18, Theorem 3.20]. In order to get surjectivity of the map f_*, in [Bre18] it is assumed that the bound $\mathcal{Z}$ arises as the base change of $\mathcal{Z}'$ under the map f_* on affine Grassmannians. This assumption does not seem adequate in our setting, in particular, it is not satisfied for the bounds given by cocharacters in the Bruhat-Tits case. We thus replace the assumption by the condition that the map on bounds is a generic isomorphism and that the bounds arise as schematic closures from their generic part, both of which are satisfied in our setting. Note that when $\mathcal{Z}$ arises as a base change, the map $\mathcal{Z}\to\mathcal{Z}'$ is clearly an isomorphism over U^I.

Note moreover that a similar statement also holds for moduli spaces of shtukas with local boundedness conditions as in Remark 3.2.28. In fact, the proof of [Bre18] for (1) and (2) directly translates to this setting.

Proof. (1) We proceed as in the proof of [Bre18, Theorem 3.20]. We consider the projection $\mathrm{Sht}^{\mathcal{Z}}_{\mathcal{G},X^I,I_\bullet}\to\prod_{j=1,\dots,m}\mathrm{Bun}_{\mathcal{G}}$ given by $\underline{\mathcal{E}}\mapsto(\mathcal{E}_j)_{j=1,\dots,m}$. Let us fix

$$\underline{\mathcal{E}}'=((x_i)_{i\in I},(\mathcal{E}'_j)_{j=0,\dots,m},(\varphi_j)_{j=1,\dots,m},\theta)\in\left(\mathrm{Sht}^{\mathcal{Z}'}_{\mathcal{G}',X^I,I_\bullet}\times_{X^I_{\mathcal{Z}'}}X^I_{\mathcal{Z}.\mathcal{Z}'}\right)(S).$$

We claim that the induced map

$$S\times_{\mathrm{Sht}^{\mathcal{Z}'}_{\mathcal{G}',X^I,I_\bullet}\times_{X^I_{\mathcal{Z}'}}X^I_{\mathcal{Z}.\mathcal{Z}'}}\left(\mathrm{Sht}^{\mathcal{Z}}_{\mathcal{G},X^I,I_\bullet}\times_{X^I_{\mathcal{Z}}}X^I_{\mathcal{Z}.\mathcal{Z}'}\right)\to S\times_{\prod_{j=1}^m\mathrm{Bun}_{\mathcal{G}'}}\prod_{j=1}^m\mathrm{Bun}_{\mathcal{G}}$$

is a quasi-compact locally closed immersion. This shows the assertion (1) using that $\mathrm{Bun}_{\mathcal{G}}\to\mathrm{Bun}_{\mathcal{G}'}$ is schematic and quasi-projective by [Bre18, Proposition 3.18].

In order to show the claim, let us fix a point

$$(s,(\mathcal{E}_j)_{j=1,\dots m},(\psi_j)_{j=1,\dots,m})\in(S\times_{\prod_{j=1}^m\mathrm{Bun}_{\mathcal{G}'}}\prod_{j=1}^m\mathrm{Bun}_{\mathcal{G}})(T),$$

where $s\colon T\to S$ is a map of schemes, the $\mathcal{E}_j$ are $\mathcal{G}$-bundles and $\psi_j\colon s^*\mathcal{E}'\xrightarrow{\cong}f_*\mathcal{E}$ is an isomorphism of $\mathcal{G}'$-bundles over X_T. As in the proof of [Bre18, Theorem 3.20], there is at most one T-valued point $(s,\underline{\mathcal{E}},\psi)$ of $S\times_{\mathrm{Sht}^{\mathcal{Z}'}_{\mathcal{G}',X^I,I_\bullet}}\mathrm{Sht}^{\mathcal{Z}}_{\mathcal{G},X^I,I_\bullet}$ mapping to

$(s, (\mathcal{E}_j)_{j=1,\dots m}, (\psi_j)_{j=1,\dots,m})$ as the maps φ_j of $\underline{\mathcal{E}}$ are already uniquely determined over an open dense subset by the φ'_j.

It remains to check that the locus where such an extension exists is closed in T. Let $D = X\backslash U$ be the effective Cartier divisor in X given by $\underline{y}$. Let $1 \leqslant j \leqslant m$. The map $\varphi'_{j,T} \colon \mathcal{E}'_{j-1}|_{X_T\backslash\bigcup_{i\in I_j}\Gamma_{\underline{x}_j}} \to \mathcal{E}'_j|_{X_T\backslash\bigcup_{i\in I_j}\Gamma_{\underline{x}_j}}$ defines a map $\varphi_j \colon \mathcal{E}_{j-1}|_{X_T\backslash(D\cup\bigcup_{i\in I_j})\Gamma_{\underline{x}_j}} \to \mathcal{E}_j|_{X_T\backslash(D\cup\bigcup_{i\in I_j}\Gamma_{\underline{x}_j})}$. Trivialising both $\mathcal{E}_{j-1}$ and $\mathcal{E}$ over $\hat{D} \cup \hat{\Gamma}_{\underline{x}_j}$ defines an element $\varphi_j \in \mathcal{G}(\hat{D}^0 \cup \Gamma_{\underline{x}_j})$. By the argument that the positive loop group is a closed subscheme of the loop group, the locus where φ_j can be extended to $\hat{D}\backslash\Gamma_{\underline{x}}$ is closed. Finally, the locus where this is bounded by $\mathcal{Z}$ is reprsentable by a quasi-compact immmersion.

(2) This follows from the argument in (1) as in the parahoric case the map $\mathrm{Bun}_{\mathcal{G}} \to \mathrm{Bun}_{\mathcal{G}'}$ is projective by [Bre18, Proposition 3.18].

(3) It suffices to show the first claim that the map f_* is étale locally representable by the constant scheme $\prod_{\ell=1}^n \underline{\mathcal{G}'(\mathcal{O}_{y_\ell})/\mathcal{G}(\mathcal{O}_{y_\ell})}$. We follow the proof of [Var04, Proposition 2.16]. Let

$$\underline{\mathcal{E}'} = ((x_i), (\mathcal{E}'_i), (\varphi'_i), \theta) \in \mathrm{Sht}^{\mathcal{Z}'}_{\mathcal{G}', X^I, I_\bullet}|_{U^I_{\mathcal{Z},\mathcal{Z}'}}(S).$$

For $\ell = 1, \dots, n$, we denote by $\widetilde{\underline{\mathcal{E}'}_{y_k}} = (\widetilde{\mathcal{E}'}_{y_\ell}, \varphi)$ the associated étale local shtuka of $\underline{\mathcal{E}'}$ at y_ℓ as defined in Remark 3.2.26. The fibre product

$$S' = S \times_{\underline{\mathcal{E}'}, \mathrm{Sht}^{\mathcal{Z}'}_{\mathcal{G}', X^I, I_\bullet}|_{U^I_{\mathcal{Z},\mathcal{Z}'}}, f_*} \mathrm{Sht}^{\mathcal{Z}}_{\mathcal{G}, X^I, I_\bullet}|_{U^I_{\mathcal{Z},\mathcal{Z}'}}$$

is then given by the set of tuples $(\widetilde{\underline{\mathcal{E}'}_{y_\ell}})_{\ell=1,\dots,n}$ of étale local $\widetilde{\mathcal{G}_{\mathcal{O}_{y_\ell}}}$-shtukas such that $f_*\widetilde{\underline{\mathcal{E}'}_{y_\ell}} = \widetilde{\underline{\mathcal{E}'}_{y_\ell}}$. As the claim is étale-local on S, we may assume that all $\widetilde{\mathcal{E}'}_{y_\ell}$ are trivial $\mathcal{L}^+\widetilde{\mathcal{G}'_{\mathcal{O}_{y_\ell}}}$-torsors. By Lemma 3.3.2 (3), the fibre product S' is then representable by the scheme of Frobenius fixed points of $\prod_{\ell=1}^n \widetilde{\mathcal{G}'_{\mathcal{O}_{y_\ell}}}/\mathcal{L}^+\widetilde{\mathcal{G}_{\mathcal{O}_{y_\ell}}}$, which is given by the constant scheme $\prod_{\ell=1}^n \underline{\left(\mathcal{L}^+\widetilde{\mathcal{G}'_{\mathcal{O}_{y_\ell}}}/\mathcal{L}^+\widetilde{\mathcal{G}_{\mathcal{O}_{y_\ell}}}\right)(\mathbb{F}_q)}$ by [Var04, Lemma 3.3]. By Lemma 3.3.2 (1), this scheme can be identified with $\prod_{\ell=1}^n \underline{\mathcal{G}'(\mathcal{O}_{y_\ell})/\mathcal{G}(\mathcal{O}_{y_\ell})}$, and by Lemma 3.3.2 (2) it is finite over $\mathbb{F}_q$.

(4) Let us fix a point $s \in \mathrm{Sht}^{\mathcal{Z}'}_{\mathcal{G}', X^I, I_\bullet}$. If s lies over U, it is in the image of $\mathrm{Sht}^{\mathcal{Z}'}_{\mathcal{G}', X^I, I_\bullet}$ by (3). Let us thus assume that s maps to $X^I\backslash U$. By the local model theorem (compare Remark 3.2.9), we have a smooth map $\mathrm{Sht}^{\mathcal{Z}'}_{\mathcal{G}', X^I, I_\bullet} \to [\mathcal{H}\backslash\mathcal{Z}']$, where $\mathcal{H}$ is a finite-dimensional quotient of $\mathcal{L}^+_{X^I}\mathcal{G}$. By assumption on $\mathcal{Z}'$, the image of s in

$[\mathcal{H}\backslash\mathcal{Z}']$ has a generalisation s' over U. As the local model map is smooth, s' lifts to a generalisation s'' of s in $\mathrm{Sht}^{\mathcal{Z}'}_{\mathcal{G}',X^I,I_\bullet}$. As f_* is generically surjective by (3), there is a point $t \in \mathrm{Sht}^{\mathcal{Z}}_{\mathcal{G},X^I,I_\bullet}$ mapping to s''. As f_* is proper by (2), specialisations lift along f_*. Hence, s is in the image of f_*.

□

Let us also state the result in the generically reductive case with bounds given by cocharacters.

Corollary 3.3.5. *Let G be a reductive group over K and let $f\colon \mathcal{G} \to \mathcal{G}'$ be a map of two smooth affine models of G that is an isomorphism over some dense open subset U of X. Let $\underline{\mu} = (\mu_i)_{i\in I}$ be an I-tuple of conjugacy classes of cocharacters for G. The induced map*

$$f_*\colon\ \mathrm{Sht}^{\leqslant\underline{\mu}}_{\mathcal{G},X^I,I_\bullet} \to \mathrm{Sht}^{\leqslant\underline{\mu}}_{\mathcal{G}',X^I,I_\bullet}$$

is schematic, separated and of finite type. Moreover, it is finite étale and surjective over $(U \times_X X_{\underline{\mu}})^I$. When $\mathcal{G}$ is a parahoric Bruhat-Tits group scheme, f_* is proper and surjective.

Proof. The bounds given by $\underline{\mu}$ for $\mathcal{G}$ and $\mathcal{G}'$ clearly satisfy the conditions of Theorem 3.3.3. □

3.3.2. Moduli spaces of shtukas with deep Bruhat-Tits level structure

In this section, we define the integral model of the moduli space of shtukas with deep Bruhat-Tits level structure as the schematic image of the moduli space of shtukas for the Bruhat-Tits group scheme inside the limit of all the corresponding spaces for parahoric level.

Proposition 3.3.6. *In the situation of Assumption 3.2.2 (3), the map*

$$\rho_{\Omega,*}\colon\ \mathrm{Sht}^{\leqslant\underline{\mu}}_{\mathcal{G}_\Omega,X^I,I_\bullet} \to \varprojlim_{\mathfrak{f}\prec\Omega} \mathrm{Sht}^{\leqslant\underline{\mu}}_{\mathcal{G}_\mathfrak{f},X^I,I_\bullet}$$

is schematic and representable by a quasi-compact locally closed immersion. Moreover, $\rho_{\Omega,*}$ is an open and closed immersion over $(X\backslash x_0)^I$. When Ω is (the closure of) a facet, $\rho_{\Omega,*}$ is an isomorphism.

Proof. The assertion in the case that Ω is a facet is clear. By Corollary 3.3.5 and Lemma A.0.2, the map is schematic, separated and of finite type. By Theorem 3.1.13, the corresponding map on the unbounded moduli stacks of shtukas is an open immersion.

Hence, $\rho_{\Omega,*}$ is certainly a locally closed immersion as being bounded by $\underline{\mu}$ is a closed condition.

Over $(X\backslash x_0)^I$, an element of $\mathrm{Sht}_{\mathcal{G}_\Omega,X^I,I_\bullet}$ is bounded by $\underline{\mu}$ if and only if its image under $\rho_{\mathfrak{f},\Omega,*}$ for one (or equivalently all) facet $\mathfrak{f} \prec \Omega$ is bounded by $\underline{\mu}$ by Lemma 3.2.15. Thus, $\rho_{\Omega,*}$ is an open immersion over $(X\backslash x_0)^I$. Moreover, the map $\rho_{\Omega,*}$ is finite away from x_0 by Lemma A.0.2, hence also a closed immersion.

□

Definition 3.3.7. The integral model $\overline{\mathrm{Sht}}^{\leqslant\underline{\mu}}_{\mathcal{G}_\Omega,X^I,I_\bullet}$ of the moduli space of shtukas with $\mathcal{G}_\Omega$-level is defined to be the schematic image in the sense of [EG21] of the map

$$\rho_{\Omega,*}\colon \mathrm{Sht}^{\leqslant\underline{\mu}}_{\mathcal{G}_\Omega,X^I,I_\bullet} \to \varprojlim_{\mathfrak{f}\prec\Omega} \mathrm{Sht}^{\leqslant\underline{\mu}}_{\mathcal{G}_\mathfrak{f},X^I,I_\bullet}.$$

By Proposition 3.3.6, we have $\overline{\mathrm{Sht}}^{\leqslant\underline{\mu}}_{\mathcal{G}_\mathfrak{f},X^I,I_\bullet} = \mathrm{Sht}^{\leqslant\underline{\mu}}_{\mathcal{G}_\mathfrak{f},X^I,I_\bullet}$ in the parahoric case. Moreover, the inclusion $\mathrm{Sht}^{\leqslant\underline{\mu}}_{\mathcal{G}_\Omega,X^I,I_\bullet} \to \overline{\mathrm{Sht}}^{\leqslant\underline{\mu}}_{\mathcal{G}_\Omega,X^I,I_\bullet}$ is an isomorphism away from x_0 by Proposition 3.3.6 together with the fact that the schematic closure commutes with flat base change.

By construction, we have level maps $\overline{\rho}_{\mathfrak{f},\Omega}\colon \overline{\mathrm{Sht}}^{\leqslant\underline{\mu}}_{\mathcal{G}_\Omega,X^I,I_\bullet} \to \mathrm{Sht}^{\leqslant\underline{\mu}}_{\mathcal{G}_\mathfrak{f},X^I,I_\bullet}$ for all facets $\mathfrak{f} \prec \Omega$. In particular, for $\Omega' \prec \Omega$ we obtain a map $\overline{\rho}_{\Omega',\Omega}\colon \overline{\mathrm{Sht}}^{\leqslant\underline{\mu}}_{\mathcal{G}_\Omega,X^I,I_\bullet} \to \varprojlim_{\mathfrak{f}\prec\Omega'} \mathrm{Sht}^{\leqslant\underline{\mu}}_{\mathcal{G}_\mathfrak{f},X^I,I_\bullet}$ that factors through $\overline{\mathrm{Sht}}^{\leqslant\underline{\mu}}_{\mathcal{G}_{\Omega'},X^I,I_\bullet}$ by construction.

Theorem 3.3.8. *Let Ω, Ω' be two bounded connected subsets of an appartment in the Bruhat-Tits building of $G_{K_{x_0}}$ such that $\Omega' \prec \Omega$. Then, the level map*

$$\overline{\rho}_{\Omega',\Omega}\colon \overline{\mathrm{Sht}}^{\leqslant\underline{\mu}}_{\mathcal{G}_\Omega,X^I,I_\bullet} \to \overline{\mathrm{Sht}}^{\leqslant\underline{\mu}}_{\mathcal{G}_{\Omega'},X^I,I_\bullet}$$

is schematic, proper, surjective and finite étale away from x_0.

Proof. As a first step, we show that $\overline{\rho}_{\Omega',\Omega}$ is schematic. By Lemmas A.0.1 and A.0.2, the map

$$\varprojlim_{\mathfrak{f}\prec\Omega} \mathrm{Sht}^{\leqslant\underline{\mu}}_{\mathcal{G}_\mathfrak{f},X^I,I_\bullet} \to \varprojlim_{\mathfrak{f}'\prec\Omega'} \mathrm{Sht}^{\leqslant\underline{\mu}}_{\mathcal{G}_{\mathfrak{f}'},X^I,I_\bullet}$$

is schematic. The claim for $\overline{\rho}_{\Omega',\Omega}$ then follows from Lemma A.0.3.

That the map is finite étale away from x_0 follows from the fact that the map $\mathrm{Sht}^{\leqslant\underline{\mu}}_{\mathcal{G}_\Omega,X^I,I_\bullet} \to \overline{\mathrm{Sht}}^{\leqslant\underline{\mu}}_{\mathcal{G}_\Omega,X^I,I_\bullet}$ is an isomorphism away from x_0 by the observation above together with Corollary 3.3.5.

Moreover, the map $\overline{\mathrm{Sht}}^{\leqslant\mu}_{\mathcal{G}_{\Omega'},X^I,I_\bullet} \to \varprojlim_{\mathfrak{f}\prec\Omega'} \mathrm{Sht}^{\leqslant\mu}_{\mathcal{G}_{\mathfrak{f}},X^I,I_\bullet}$ is a closed immersion by definition. Thus, by Lemma A.0.2, it suffices to consider the level maps

$$\overline{\mathrm{Sht}}^{\leqslant\mu}_{\mathcal{G}_\Omega,X^I,I_\bullet} \to \mathrm{Sht}^{\leqslant\mu}_{\mathcal{G}_{\mathfrak{f}},X^I,I_\bullet}$$

for facets $\mathfrak{f} \prec \Omega$ to show the properness. Similarly, by construction of $\overline{\mathrm{Sht}}^{\leqslant\mu}_{\mathcal{G}_\Omega}$, it suffices to show the claim for the projections

$$\varprojlim_{\mathfrak{f}\prec\Omega} \mathrm{Sht}^{\leqslant\mu}_{\mathcal{G}_{\mathfrak{f}},X^I,I_\bullet} \to \mathrm{Sht}^{\leqslant\mu}_{\mathcal{G}_{\mathfrak{f}},X^I,I_\bullet}.$$

But for the projections the claim follows from Lemma A.0.1. The surjectivity follows as in the parahoric case in the proof of Theorem 3.3.3. $\square$

3.4. Newton stratification

We recall the Newton stratification on stacks of global shtukas and define a Newton stratification on our integral models with deep level. We show that the expected closure relations of Newton strata are satisfied in the hyperspecial case.

Let $k \cong \mathbb{F}((t))$ be a local field in characteristic p with ring of integers $\mathcal{O} \cong \mathbb{F}[\![t]\!]$ and finite residue field $\mathbb{F}$. We denote by $\bar{k} = k^{\mathrm{sep}}$ a fixed separable closure and by $\breve{k} \cong \mathbb{F}^{\mathrm{alg}}((t))$ the completion of the maximal unramified extension of k. Let G/k be a reductive group and let us fix $T \subseteq G$ be a maximal torus defined over k. As $G_{\breve{k}} = G \times_k \breve{k}$ is quasi-split by a theorem of Steinberg, we can choose a Borel $B \subseteq G_{\breve{k}}$ containing $T_{\breve{k}}$. We denote by $X_*(T)$ its group of geometric cocharacters and by $\pi_1(G)$ the algebraic fundamental group of G given by the quotient of the cocharacter lattice by the coroot lattice.

We denote by $B(G)$ the set of σ-conjugacy classes in $G(\breve{k}) = LG(\mathbb{F}^{\mathrm{alg}})$. By [Kot85; Kot97; RR96], the elements of $B(G)$ are classified by two invariants: the *Kottwitz map* denoted by

$$\kappa\colon B(G) \to \pi_1(G)_{\mathrm{Gal}(\bar{k}/k)}$$

and the *Newton map* denoted by

$$\nu\colon B(G) \to (\mathrm{Hom}(\mathbb{D}_{\bar{k}}, G_{\bar{k}})/G(\bar{k}))^{\mathrm{Gal}(\bar{k}/k)},$$

where $\mathbb{D}$ denotes the pro-torus with character group $\mathbb{Q}$ and $G(\bar{k})$ acts by conjugation. Note that we can identify

$$(\mathrm{Hom}(\mathbb{D}_{\bar{k}}, G_{\bar{k}})/G(\bar{k}))^{\mathrm{Gal}(\bar{k}/k)} = X_*(T)^{+,\mathrm{Gal}(\bar{k}/k)}_{\mathbb{Q}} = X_*(T)^{+}_{\mathbb{Q},\mathrm{Gal}(\bar{k}/k)}$$

with the set of rational dominant (with respect to the choice of B) Galois-invariant cocharacters, and that $\kappa(b) = \nu(b)$ in $\pi_1(G)_{\mathbb{Q},\mathrm{Gal}(\bar{k}/k)}$.

The choice of Borel determines a set of simple positive roots and consequently defines the dominance order on $X_*(T)_{\mathbb{Q}}$ by $\mu_1 \leqslant \mu_2$ if $\mu_2 - \mu_1$ is a $\mathbb{Q}$-linear combination of positive simple roots with non-negative coefficients. Via κ and ν the dominance order induces a partial order on $B(G)$ by $b_1 \leqslant b_2$ if and only if $\kappa(b_1) = \kappa(b_2)$ and $\nu(b_1) \leqslant \nu(b_2)$.

Let $\mathcal{G} \to \mathrm{Spec}(\mathcal{O})$ be a smooth affine group scheme such that $\mathcal{G}_k = G$. Note that for an algebraically closed extension ℓ of $\mathbb{F}$ the set of σ-conjugacy classes in $LG(\ell)$ does not depend on the choice of ℓ by [RR96, Lemma 1.3]. It classifies quasi-isogeny classes of local $\mathcal{G}$-shtukas by associating to $(L^+\mathcal{G}, b)$ the class $[b] \in B(G)$. For a local $\mathcal{G}$-shtuka $\underline{\mathcal{E}}$ over $S = \mathrm{Spec}(R)$ and a point $s \in S$ we denote by $[\underline{\mathcal{E}}_s] \in B(G)$ the corresponding element. This does not depend on the choice of an algebraic closure of the residue field at s.

Let us shift perspective back to the global setting again and consider a smooth affine group scheme $\mathcal{G} \to X$ with generic fibre $\mathcal{G}_K = G$ a reductive group. Let us moreover fix a tuple $\underline{y} = (y_i)_{i \in I}$ of pairwise distinct closed points of X. Let us fix a bound $\mathcal{Z}$ and points $\underline{y}' = (y_i')_{i \in I} \in X^I_{\mathcal{Z}}$ lying over $\underline{y}$. We denote by $\mathrm{Sht}^{\mathcal{Z}}_{\mathcal{G},X^I,\mathbb{F}_{\underline{y}'}} = \mathrm{Sht}^{\mathcal{Z},\underline{y}'}_{\mathcal{G},X^I} \times_{\mathrm{Spf}(\mathcal{O}_{\underline{y}'})} \mathrm{Spec}(\mathbb{F}_{\underline{y}'})$ the special fibre of the moduli space of shtukas at $\underline{y}$.

Definition 3.4.1 ([Bre18, Definition 4.12])**.** Let ℓ be an algebraically closed extension of $\mathbb{F}_{\underline{y}'}$. The global-to-local functor induces maps

$$\begin{aligned} \delta_{\mathcal{G},y_i,\ell} \colon \mathrm{Sht}^{\mathcal{Z}}_{\mathcal{G},X^I,\mathbb{F}_{\underline{y}'}}(\ell) &\to B(G_{y_i}) \\ \underline{\mathcal{E}} &\mapsto [\widehat{\underline{\mathcal{E}}_{y_i}}] \end{aligned}$$

for all $i \in I$ and

$$\delta_{\mathcal{G},\underline{y},\ell} = \prod_{i \in I} \delta_{\mathcal{G},y_i,\ell} \colon \mathrm{Sht}^{\mathcal{Z}}_{\mathcal{G},X^I,\mathbb{F}_{\underline{y}'}}(\ell) \to \prod_{i \in I} B(G_{y_i}).$$

Let $\underline{b} = (b_i)_{i \in I} \in \prod_{i \in I} B(G_{y_i})$. The locus in $\mathrm{Sht}^{\mathcal{Z}}_{\mathcal{G},X^I,\mathbb{F}_{\underline{y}'}}$ where $\delta_{\mathcal{G},\underline{y}}$ maps to $\underline{b}$ is locally closed by [HV11, Theorem 7.11], compare also [RR96]. The reduced substack on this locally closed subset is denoted by $\mathrm{Sht}^{\mathcal{Z},\underline{b}}_{\mathcal{G},X^I,\mathbb{F}_{\underline{y}'}}$ and called the *Newton stratum* associated to $\underline{b}$.

The Newton map is compatible with changing the group scheme in the following sense.

Lemma 3.4.2 (compare [Bre18, Section 5.2])**.** *Let G/K be a reductive group and let $\mathcal{G}$ and $\mathcal{G}'$ be two smooth affine models of G over X. Let $f \colon (\mathcal{G}, \mathcal{Z}) \to (\mathcal{G}', \mathcal{Z}')$ be a map of shtuka* *data such that $f\colon \mathcal{G} \to \mathcal{G}'$ is given by the identity on G in the generic fibre. Recall that f induces a map*

$$f_*\colon \operatorname{Sht}^{\mathcal{Z}}_{\mathcal{G},X^I,I_\bullet} \times_{X^I_{\mathcal{Z}}} X^I_{\mathcal{Z}.\mathcal{Z}'} \to \operatorname{Sht}^{\mathcal{Z}'}_{\mathcal{G}',X^I,I_\bullet} \times_{X^I_{\mathcal{Z}'}} X^I_{\mathcal{Z}.\mathcal{Z}'}.$$

Then

$$\delta_{\mathcal{G}',\underline{y}} \circ f_* = \delta_{\mathcal{G},\underline{y}}.$$

Proof. The proof of [Bre18, Section 5.2] carries over to this situation. □

Let us now consider the Bruhat-Tits case, compare Assumption 3.2.2 (3). Thus, let $\Omega = \mathrm{cl}(\Omega)$ be a subset of an appartment of the Bruhat-Tits building of $G_{K_{x_0}}$ for a fixed closed point x_0 of X. Let $\mathcal{G}_\Omega$ be the corresponding Bruhat-Tits group scheme. Let $\underline{\mu} = (\mu_i)_{i\in I}$ be a conjugacy class of geometric cocharacters of G. Let moreover $\underline{y}' = (y_i')$ be a tuple of closed points of $X_{\underline{\mu}}$ lying over $\underline{y}$. In order to define a Newton stratification on $\overline{\operatorname{Sht}}^{\leqslant\underline{\mu}}_{\mathcal{G}_\Omega,X^I,\mathbb{F}_{\underline{y}'}}$, we note that by construction and by the previous lemma, we have that the map

$$\delta_{\mathcal{G}_{\mathfrak{f}},\underline{y}} \circ \rho_{\mathfrak{f},\Omega}\colon \overline{\operatorname{Sht}}^{\leqslant\underline{\mu},}_{\mathcal{G}_\Omega,X^I,\mathbb{F}_{\underline{y}'}} \to \operatorname{Sht}^{\leqslant\underline{\mu}}_{\mathcal{G}_{\mathfrak{f}},X^I,\mathbb{F}_{\underline{y}'}} \to \prod_{i\in I} B(G_{y_i})$$

does not depend on the choice of the facet $\mathfrak{f} \prec \Omega$. Hence, we obtain a well-defined map

$$\bar{\delta}_{\mathcal{G}_\Omega,\underline{y}}\colon \overline{\operatorname{Sht}}^{\leqslant\underline{\mu}}_{\mathcal{G}_\Omega,X^I,\mathbb{F}_{\underline{y}'}} \to \prod_{i\in I} B(G_{y_i}).$$

Let $\underline{b} = (b_i)_{i\in I} \in \prod_{i\in I} B(G_{y_i})$. The locus in $\overline{\operatorname{Sht}}^{\leqslant\underline{\mu}}_{\mathcal{G}_\Omega,X^I,\mathbb{F}_{\underline{y}'}}$ where $\bar{\delta}_{\mathcal{G}_\Omega,\underline{y}}$ maps to $\underline{b}$ is again locally closed by the result in the parahoric case together with Lemma 3.4.2.

Definition 3.4.3. Let $\underline{b} = (b_i)_{i\in I} \in \prod_{i\in I} B(G_{y_i})$. The *Newton stratum* in $\overline{\operatorname{Sht}}^{\leqslant\underline{\mu}}_{\mathcal{G}_\Omega,X^I,\mathbb{F}_{\underline{y}'}}$ associated to $\underline{b}$ is the reduced locally closed substack on the set of points where $\bar{\delta}_{\mathcal{G},\underline{y}}$ maps to $\underline{b}$. It is denoted by $\overline{\operatorname{Sht}}^{\leqslant\underline{\mu},\underline{b}}_{\mathcal{G}_\Omega,X^I,\mathbb{F}_{\underline{y}'}}$.

We have the obvious analogue of Lemma 3.4.2 for deep level, in other words, the Newton stratification for deep levels is still compatible with the level maps.

Corollary 3.4.4. *Let $\Omega' \prec \Omega$ be two connected bounded subsets of the Bruhat-Tits building. Then*

$$\bar{\delta}_{\mathcal{G}_{\Omega'},\underline{y}} \circ \bar{\rho}_{\Omega',\Omega} = \bar{\delta}_{\mathcal{G}_\Omega,\underline{y}}.$$

In particular, $\overline{\operatorname{Sht}}^{\leqslant\underline{\mu},\underline{b}'}_{\mathcal{G}_\Omega,X^I,\mathbb{F}_{\underline{y}'}} \cap \overline{\overline{\operatorname{Sht}}^{\leqslant\underline{\mu},\underline{b}}_{\mathcal{G}_\Omega,X^I,\mathbb{F}_{\underline{y}'}}} \neq \emptyset$ *only if* $\underline{b}' \leqslant \underline{b}$.

Proof. This follows from the construction and Lemma 3.4.2. The second statement then follows directly from the parahoric case in [Bre18, Proposition 4.11, Section 5], compare also [HV11, Theorem 7.11]. □

We conclude by showing the strong stratification property of the Newton stratification in the hyperspecial case.

Theorem 3.4.5. *Let $\mathcal{G} \to X$ be a parahoric group scheme that is hyperspecial at y_i for all $i \in I$. Let $\underline{\mu} = (\mu_i)_{i \in I}$ be a conjugacy class of geometric cocharacters of G. Then the Newton stratification at $\underline{y}'$ satisfies the strong stratification property in the sense that*

$$\overline{\mathrm{Sht}^{\leqslant \underline{\mu}, \underline{b}}_{\mathcal{G}, X^I, \mathbb{F}_{\underline{y}'}}} = \bigcup_{\underline{b}' \leqslant \underline{b}} \mathrm{Sht}^{\leqslant \underline{\mu}, \underline{b}'}_{\mathcal{G}, X^I, \mathbb{F}_{\underline{y}'}}$$

for all $\underline{b} \in \prod_{i \in I} B(G_{y_i})$.

Proof. Let $\underline{b}, \underline{b}' \in \prod_{i \in I} B(G_{y_i})$ with $\underline{b}' \leqslant \underline{b}$. It suffices to show that every closed point $\bar{s} = \underline{\mathcal{E}} \in \mathrm{Sht}^{\leqslant \underline{\mu}, \underline{b}'}_{\mathcal{G}, X^I, \mathbb{F}_{\underline{y}'}}(\mathbb{F}^{\mathrm{alg}}_{\underline{y}'})$ lies in the closure of $\mathrm{Sht}^{\leqslant \underline{\mu}, \underline{b}}_{\mathcal{G}, X^I, \mathbb{F}_{\underline{y}'}}$. Let R be the $\mathcal{O}_{\underline{y}'}$-algebra pro-representing the deformation functor of $\bar{s}$. Then $\bar{s}$ lies in the closure of $\overline{\mathrm{Sht}^{\leqslant \underline{\mu}, \underline{b}}_{\mathcal{G}, X^I, \mathbb{F}_{\underline{y}'}}}$ if and only if the same is true in the Newton stratification on Spec R. By the bounded Serre-Tate Theorem (Corollary 3.2.29) the universal deformation ring factors as Spec $R = \prod_{i \in I}$ Spec R_i, where R_i is the universal deformation ring of the corresponding bounded local shtuka at y_i. Under this isomorphism we have $\mathrm{Spec}(R)_{\underline{b}} = \prod_{i \in I} \mathrm{Spec}(R_i)_{b_i}$, where we denote by $\mathrm{Spec}(R_i)_{b_i}$ the corresponding Newton strata in Spec R_i for $i \in I$. On Spec R_i the closure properties hold by [Vie13, Theorem 2, Lemma 21 (2)], and thus they hold on Spec R. This proves the assertion. □

A. Some lemmata on algebraic stacks

We collect some results on finite connected limits of algebraic stacks we use below for which we could not find a reference in the literature.

In this section, I will always denote a connected index category and $(\mathcal{X}_i)_{i\in I}$ denotes a diagram over I of (fppf-) Artin stacks over some base scheme S.

Lemma A.0.1. *Assume that all algebraic stacks $\mathcal{X}_i$ have a diagonal that is schematic. Let all transition maps in $(\mathcal{X}_i)_{i\in I}$ be schematic. Then the projections $\varprojlim_{i\in I} \mathcal{X}_i \to \mathcal{X}_j$ are schematic for all $j \in I$.*

Moreover, assume that all $\mathcal{X}_i$ are separated over S and that all transition maps have a property $\mathbf{P}$ of morphisms of schemes that is stable under base change and composition and is smooth local on the target such that all proper maps have $\mathbf{P}$. Then the projections $\varprojlim_{i\in I} \mathcal{X}_i \to \mathcal{X}_j$ have property $\mathbf{P}$ for all $j \in I$.

Proof. It suffices to show the claim for fibre products and equalisers. For fibre products this is clear. Let us thus consider the equaliser diagram

$$\mathcal{X}_1 \underset{g}{\overset{f}{\rightrightarrows}} \mathcal{X}_2.$$

The equaliser of this diagram is given by the fibre product $\mathcal{X} = \mathcal{X}_2 \times_{\Delta,\mathcal{X}_2\times_S\mathcal{X}_2,(f,g)} \mathcal{X}_1$. Thus, the projection $\mathcal{X} \to \mathcal{X}_1$ arises as the base change of the diagonal of $\mathcal{X}_1$ and is thus schematic in the first case and moreover proper in the second case (as we assumed $\mathcal{X}_1$ to be separated). The projection $\mathcal{X} \to \mathcal{X}_2$ has the required properties as it is the composition $\mathcal{X} \to \mathcal{X}_1 \to \mathcal{X}_2$. □

Lemma A.0.2. *Let $(f_i \colon \mathcal{X} \to \mathcal{X}_i)_{i\in I}$ be a cone over the diagram $(\mathcal{X}_i)_{i\in I}$ such that all maps f_i are schematic. Then the limit $f \colon \mathcal{X} \to \varprojlim_{i\in I} \mathcal{X}_i$ is schematic as well.*

Assume moreover that all f_i are separated and have a property $\mathbf{P}$ of morphisms of schemes that is stable under base change and composition and is smooth local on the target such that all closed immersions have $\mathbf{P}$. Then f has $\mathbf{P}$.

Proof. Let Let T be an S-scheme. Let us fix a map $T \to \varprojlim_{i\in I} \mathcal{X}_i$. As different limits commute, we get that

$$T \times_{\varprojlim_{i\in I} \mathcal{X}_i} \mathcal{X} = \varprojlim_{i\in I}(T \times_{\mathcal{X}_i} \mathcal{X}),$$

which is representable by a scheme by assumption. For the second part, let us denote by $T_i = T \times_{\mathcal{X}_i} \mathcal{X}$. Then T_i is a separated T-scheme by assumption. As I is connected, we may take the limit on the right hand side in the category of T-schemes (as opposed to the category of S-schemes). We represent the limit as an equaliser between products

$$\varprojlim_{i\in I} T_i = \mathrm{eq}\left(\prod_{i\in I} T_i \rightrightarrows \prod_{i\in I} T_i\right),$$

where the products are taken in the category of T-schemes. As all T_i are separated over T, the inclusion of $\varprojlim_{i\in I} T_i \hookrightarrow \prod_{i\in I} T_i$ is a closed immersion. Moreover, as all $T_i \to T$ have property **P**, so does their product. Hence, $\varprojlim_{i\in I} T_i \to T$ has property **P**. □

Lemma A.0.3. *Let $f\colon \mathcal{X} \to \mathcal{X}'$ be a schematic map of algebraic stacks and let $\mathcal{Y} \subseteq \mathcal{X}$ and $\mathcal{Y}' \subseteq \mathcal{X}'$ be two closed substacks such that $f|_{\mathcal{Y}}$ factors through $\mathcal{Y}'$. Then $f|_{\mathcal{Y}}\colon \mathcal{Y} \to \mathcal{Y}'$ is schematic.*

Proof. Let S be a scheme and let us fix a map $y'\colon S \to \mathcal{Y}'$. As f is schematic, the fibre product $T = S \times_{y,\mathcal{X}',f} \mathcal{Y}$ is representable by a scheme. Then $T = S \times_{\mathcal{Y}'} \mathcal{Y}$. □

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