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On the hydrodynamic behaviour of a particle system with nearest neighbour interactions

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Abstract

In this thesis we will study a system of Brownian particles on the real line, which are coupled through the nearest neighbours by an attractive potential. This model is related to the Ginzburg-Landau model. We will prove two results. The first result is the hydrodynamic equation for the particle density. More precisely, we show that the empirical measure of the particle positions converges in the hydrodynamic limit to a deterministic and absolutely continuous probability measure, where the density solves a nonlinear heat equation. The crucial idea will be the reduction of the particle model to the height model, in the literature also called Ginzburg-Landau interface model. We will obtain the claimed result by taking the limit in the height model and passing back to the particle model. Further, we will outline how this approach generalises to multiple dimensions. The second result is the characterisation of the equilibrium fluctuations in the case of quadratic potential. We will consider the fluctuation field, which is defined as the square root of the number of particles times the difference of the empirical measure of the particle positions and its expectation. Assuming the initial distribution of the particle system to be stationary, we will show that the fluctuation field converges in the hydrodynamic limit to an infinite-dimensional Ornstein-Uhlenbeck process. The proof will consist of characterising the accumulation points of the distributions of fluctuation fields by means of a martingale problem and showing tightness.

Zusammenfassung

In dieser Dissertation wird ein System von Brownschen Teilchen auf den reellen Zahlen studieren, wobei die Teilchen über die nächsten Nachbarn mit einem anziehenden Potential gekoppelt sind. Dieses Modell ist verwandt mit dem Ginzburg-Landau Modell. Wir zeigen zwei Resultate. Das erste Resultat ist die hydrodynamische Gleichung für die Teilchendichte. Genauer gesagt zeigen wir, dass das empirische Maß der Teilchenpositionen im hydrodynamischen Grenzwert gegen ein deterministisches und absolut stetiges Wahrscheinlichkeitsmaß konvergiert, wobei die Dichte eine nichtlineare Wärmeleitungsgleichung löst. Die wesentliche Idee wird es sein, das Teilchenmodell auf das Höhenmodell, in der Literatur Ginzburg-Landau Grenzflächenmodell genannt, zu reduzieren. Indem wir den Grenzwert im Höhenmodell bilden und dann zurück zum Teilchenmodell wechseln, erhalten wir das genannte Resultat. Wir skizzieren außerdem die Verallgemeinerung dieses Ansatzes auf den mehrdimensionalen Fall. Das zweite Resultat ist die Charakterisierung der Gleichgewichtsfluktuationen der Teilchendichte bei quadratischem Potential. Dazu betrachten wir das Fluktuationsfeld, das definiert ist als die Wurzel der Teilchenzahl mal die Differenz von dem empirischen Maß der Teilchenpositionen zu seinem Erwartungswert. Wir nehmen an, dass die Verteilung des Teilchensystem zu Beginn stationär ist. Dann zeigen wir, dass das Fluktuationsfeld im hydrodynamischen Grenzwert gegen einen unendlichdimensionalen Ornstein-Uhlenbeck Prozess konvergiert. Der Beweis wird darin bestehen, die Häufungspunkte der Verteilungen der Fluktuationsfelder durch ein Martingalproblem zu charakterisieren und Straffheit zu zeigen.

Preface

This thesis was written from April 2014 to July 2019 under the supervision of Prof. Dr. Volker Betz and Prof. Dr. Makiko Sasada.

I am thankful for the help of many people during the time of writing this thesis. First and foremost, I would like to express my deepest gratitude towards Prof. Dr. Volker Betz for his guidance and encouragement. To Prof. Dr. Makiko Sasada I am thankful for her supervision during my research stay in Tokyo. I am also thankful to Prof. Dr. Tadahisa Funaki for his hospitality during multiple stays in Tokyo. Many thanks go to the stochastics group and the International Research Training Group 1529, especially to Johannes Ehlert and Helge Schäfer for proofreading a draft of this thesis. Finally, I would like to thank my family and friends for their constant support.

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Alexander Dalinger

Contents

1 Introduction

Imagine a large particle system, e. g. a fluid. At microscopic scale the particles move seemingly chaotic, whereas at macroscopic scale the system behaves more organised, and macroscopic quantities like temperature, pressure and density can be observed. The approach of statistical mechanics builds upon this observation: First, the equilibrium states of the system are examined and characterised by a few macroscopic quantities $q = (q_1, \dots, q_m)$. Once the equilibrium states are known, the time evolution of the system can be investigated. Let the system be contained in a domain D. Since the interaction of particles is relatively strong for small distances, it can be assumed that the system is in a local equilibrium state, i. e. at time t and in a neighbourhood of $x \in D$ the system is in the equilibrium state with parameter $q(t, x)$. It can be expected that q is governed by a partial differential equation, the so called hydrodynamic equation. To make the above approach mathematically feasible, we need to make two modifications. When particles move deterministically according to the equation of motion, it is difficult to establish that the system is in a local equilibrium state. This problem can be overcome by adding randomness to the motion of the particles. Moreover, it does not suffices to have a large number of particles. We will work in the hydrodynamic limit, i. e. the number of particles and the volume of the domain containing the particles go to infinity such that their proportion remains constant. This guaranties the determinacy of the macroscopic quantities.

In this thesis we will study a system of Brownian particles $X_k(t)$, $k = 1, \dots, N$ on the real line, which are coupled through the nearest neighbours by an attractive potential. This model is related to the Ginzburg-Landau model. We will impose a boundary condition that spans an interval of length proportional to N. This prevents the particles from accumulating in one point, and we obtain the the hydrodynamic limit as $N \to \infty$. There are two main results. The first result is the hydrodynamic equation for the particle density. For this purpose, we regard $X_k(t)$ as the position of the k-th particle at time t, and consider the empirical measure

$$\mu_t^N(dx) = \frac{1}{N} \sum_{k=1}^{N} \delta_{N^{-1} X_k(N^2 t)}(dx).$$

This is a random probability measure on the real line. Let the space of probability measures on the real line be equipped with the topology of weak convergence of measures. We will prove that, for fixed t, μ_t^N converges as $N \to \infty$ in probability to a deterministic probability measure $m(t, x)dx$, where m solves a nonlinear heat equation. The crucial idea will be the reduction of the particle model to the height model, in the literature also called Ginzburg-Landau interface model. The height model describes an interface that separates two distinct phases. The second result is the characterisation of the equilibrium fluctuations for the particle density in the case of quadratic potential. We consider the fluctuation field $(F_t^N)_{t\in[0,T]}$ given by

$$F_t^N(f) = \sqrt{N}\left(\int_{-\infty}^{\infty} f d\mu_t^N - \mathbb{E}\left[\int_{-\infty}^{\infty} f d\mu_t^N\right]\right)$$

for suitable test functions f. Starting the particle system from an stationary distribution, we will prove that $(F_t^N)_{t\in[0,T]}$ converges as $N \to \infty$ in distribution to an infinite-dimensional Ornstein-Uhlenbeck process. In the proof we will follow the approach of [KL13, chapter 11], which consists of characterising the accumulation points of the distributions of fluctuation fields by means of a martingale problem and showing tightness.

Similar results for other models were already established. The hydrodynamic limit for a class of zero-range processes was studied in [KL13, chapter 5 and 11]. The hydrodynamic equation for the charge density was obtained, and its equilibrium fluctuations are characterised. More precisely, $X_k(t)$ is regarded as the charge of the k-th particle at time t, and

$$\mu_t^N(dx) = \frac{1}{N}\sum_{k=1}^{N} X_k(t)\delta_{N^{-1}k}(dx)$$

is considered. This is a random signed measure on the real line. Let the space of signed measures on the real line be equipped with the topology of weak convergence of measures. It is proven that, for fixed t, μ_t^N converges as $N \to \infty$ in probability to a deterministic measure $m(t,x)dx$, where m solves a nonlinear partial differential equation.

The hydrodynamic limit for the Ginzburg-Landau model on a one-dimensional lattice, where the dynamics is defined such that the total charge is preserved, was studied in [GPV88]. The hydrodynamic equation for the charge density is obtained by using estimates of the entropy and its rate of change. For the Ginzburg-Landau model on an infinite lattice the equilibrium fluctuations of the charge density where characterised in [Zhu90] and its non-equilibrium fluctuations in [CY92].

The hydrodynamic limit of the Ginzburg-Landau interface model was studied for different domains. For a torus it was shown in [FS97] that the height converges to the solution of a nonlinear partial differential equation. For bounded domains the same was shown in [Nis03] and the equilibrium fluctuations of the height differences were characterised in [GOS01].

In [Var91] the hydrodynamic limit for a system of Brownian particles on a circle was studied, where any two particles are coupled by a repulsive potential. The hydrodynamic equation for the particle density is obtained by using similar methods as in [GPV88]. The hydrodynamic limit for an infinitely extended system of Brownian particles, where any two particles are coupled by an attractive potential, was studied in [Spo86]. Using a cluster expansion and a compactness argument, the equilibrium fluctuations of the density are characterised. Another related model is Dyson's Brownian motion, known from random matrix theory. A treatment of this model can be found in [AGZ10].

2 Models

2.1 Particle model

2.1.1 General potential

Let a positive integer N, $\rho > 0$ and a function $V\colon \mathbb{R} \to \mathbb{R}$ be given. We assume $V \in C^2(\mathbb{R})$, V is even and $c_- \leq V'' \leq c_+$ for constants $c_-, c_+ > 0$. From the physical point of view, N is the number of particles, ρ the average density and V the potential. As state space we choose $\mathbb{R}^N$, i. e. we have N particles on the real line, and we denote a typical element in the state space as $x = (x_k)_{k=1}^N$. We define the Hamiltonian $H\colon \mathbb{R}^N \to \mathbb{R}$ by

$$H(x) = \sum_{k=0}^{N} V(x_k - x_{k+1}) \tag{2.1}$$

with the convention $x_0 = 0$, $x_{N+1} = \frac{N}{\rho}$. The derivative of $H(x)$ with respect to x_k is

$$\begin{aligned}\partial_{x_k} H(x) &= -V'(x_{k-1} - x_k) + V'(x_k - x_{k+1}) \\ &= V'(x_k - x_{k-1}) + V'(x_k - x_{k+1}).\end{aligned}$$

Corresponding to this, we introduce a random time evolution of the particles by means of the system of stochastic differential equations

$$\begin{cases} dX_k(t) = -\displaystyle\sum_{l\in\mathbb{Z}, |k-l|=1} V'(X_k(t) - X_l(t))dt + \sqrt{2}dW_k(t), \quad k = 1, \dots, N, \\ \quad X_0 = 0, \; X_{N+1} = \dfrac{N}{\rho}, \end{cases} \tag{2.2}$$

where W_k, $k = 1, \dots, N$ are independent one-dimensional Brownian motions.

We regard $X_k(t)$ as the position of the k-th particle at time t. Then we can expect the particles to spread out evenly between the boundary particles as time evolves. Indeed, let $X_{k-1}(t) < X_k(t) < X_{k+1}(t)$. We first assume $X_k(t)$ to be closer to $X_{k-1}(t)$ than to $X_{k+1}(t)$, i. e. $0 < X_k(t) - X_{k-1}(t) < X_{k+1}(t) - X_k(t)$. Then the drift of (2.2) is positive, which suggests that $X_k(t)$ increases within a short time interval. Conversely, we assume $X_k(t)$ to be closer to $X_{k+1}(t)$ than to $X_{k-1}(t)$, i. e. $0 < X_{k+1}(t) - X_k(t) < X_k(t) - X_{k-1}(t)$. Then the drift of (2.2) is negative, which suggests that $X_k(t)$ decreases within a short time interval. The term $\sqrt{2}dW_k(t)$ in (2.2) adds randomness to the motion of the particles.

The time evolution of the k-th particle depends only on the $(k-1)$-th and $(k+1)$-th particle. In this sense, the particle system (2.2) has nearest neighbour interactions. We

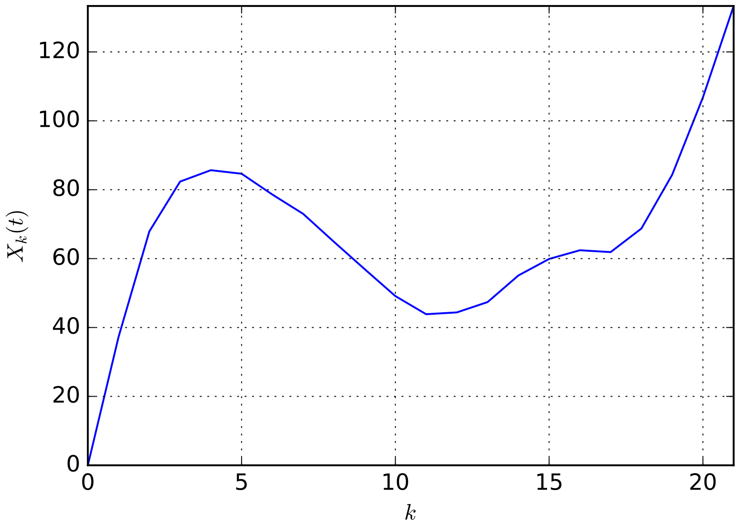

Figure 2.1: An example particle configuration.

can assume the particles to be initially enumerated in a nondecreasing order: $X_1(0) \leq \cdots \leq X_N(0)$. But at later times the particles positions can change so that between two interacting particles there are other particles. This behaviour is unnatural from the physical point of view. We will come back to this issue at the end of this section.

Remark 2.1. *1. The assumptions on V are needed for the height model, which will be introduced in the next section, but they can also be justified for the particle model. The assumption that V is even implies that the energy depends only on the absolute value of distance between two consecutive particles. Moreover, we assumed $V \in C^2(\mathbb{R})$ and $c_- \leq V'' \leq c_+$. The lower bound for V'' implies that H has a unique minimum. Indeed, $\nabla H(x) = 0$ implies*

$$V'(x_{k-1} - x_k) = V'(x_k - x_{k+1})$$

for every $k = 1, \ldots, N$. Since V'' is positive, V' is strictly increasing, and in particular, V' is injective. From this we conclude

$$x_{k-1} - x_k = x_k - x_{k+1}$$

for every $k = 1, \ldots, N$. The upper bound for V'' will be needed later to prove existence and uniqueness for solutions of (2.2)*.*

2. The boundary condition in (2.2) *can be regarded as a periodic boundary condition by identifying X_0 and X_{N+1} with each other.*

We set $X(t) = (X_k(t))_{k=1}^N$ and $W(t) = (W_k(t))_{k=1}^N$. Then (2.2) becomes in vector form

$$dX(t) = -\nabla H(X(t))dt + \sqrt{2}dW(t). \tag{2.3}$$

Lemma 2.2. *Let $T > 0$ and Y be a N-dimensional random vector that has finite second moment and is independent of the N-dimensional Brownian motion $W(t)$, $t \in [0,T]$. Then (2.3) has an almost surely unique solution $X(t)$, $t \in [0,T]$ with initial condition Y, i. e.*

$$X(t) = Y - \int_0^t \nabla H(X(s))ds + \sqrt{2}W(t) \text{ a. s.}, \quad t \in [0,T],$$

$X(t)$ is continuous in t and adapted to the filtration generated by Y and $W(t)$, $t \in [0,T]$.

Proof. It suffices to shown that the drift and diffusion coefficient of (2.3) are Lipschitz continuous and grow at most linearly. Then the claim follows from the existence and uniqueness result [Øks10, theorem 5.2.1]. This is obvious for the diffusion coefficient, so it remains to consider the drift coefficient.

Using Jensen's inequality, the mean value theorem and the boundedness of V'', we estimate

$$\begin{aligned}
&|\nabla H(x) - \nabla H(y)|^2 \\
&\leq 2\sum_{k=1}^N (|V'(x_k - x_{k-1}) - V'(y_k - y_{k-1})|^2 + |V'(x_k - x_{k+1}) - V'(y_k - y_{k+1})|^2) \\
&\leq 4c_+^2 \sum_{k=1}^N (|x_{k-1} - y_{k-1}|^2 + 2|x_k - y_k|^2 + |x_{k+1} - y_{k+1}|^2).
\end{aligned}$$

The last line is bounded above by a constant times $|x-y|^2$ since by convention $x_0 = y_0$, $x_{N+1} = y_{N+1}$. Similarly, it can be estimated

$$\begin{aligned}
|\nabla H(x)|^2 &= \sum_{k=1}^N |V'(x_{k-1} - x_k) - V'(x_k - x_{k+1})|^2 \\
&\leq 3c_+^2 \sum_{k=1}^N (|x_{k-1}|^2 + 4|x_k|^2 + |x_{k+1}|^2),
\end{aligned}$$

and the last line is bounded above by a constant times $|x|^2$ plus a constant depending on the boundary terms. This concludes the proof. □

We want to study the particle density. For this purpose, we consider the empirical measure

$$\mu_t^N(dx) = \frac{1}{N}\sum_{k=1}^N \delta_{N^{-1}X_k(N^2t)}(dx), \tag{2.4}$$

where $\delta_x(y) = \delta(x-y)$ is Kronecker's delta. Notice that μ_t^N is a random probability measure on the real line. By rescaling the particle positions in space with the factor N^{-1}, the bulk of the particles is in $[0, \rho^{-1}]$.

2.1.2 Quadratic potential

An important admissible potential is the quadratic potential $V(\eta) = \frac{\beta}{2}\eta^2$ with a constant $\beta > 0$. In this case, the particle system (2.2) becomes

$$\begin{cases} dX_k(t) = \beta\{X_{k-1}(t) - 2X_k(t) + X_{k+1}(t)\}dt + \sqrt{2}dW_k(t), \quad k = 1, \dots, N, \\ X_0 = 0, \; X_{N+1} = \dfrac{N}{\rho}. \end{cases} \tag{2.5}$$

As before, we set $X(t) = (X_k(t))_{k=1}^N$ and $W(t) = (W_k(t))_{k=1}^N$. Moreover, let L be the discrete Laplacian of size N, i. e. $L \in \mathbb{R}^{N\times N}$ is the matrix with entries $L_{kk} = 2$, $L_{kl} = -1$ if $|k-l| = 1$ and $L_{kl} = 0$ otherwise, and let $b \in \mathbb{R}^N$ be the vector with entries $b_N = N/\rho$ and $b_k = 0$ otherwise. Then (2.5) becomes in vector form

$$\begin{aligned} dX(t) &= \beta\{b - LX(t)\}dt + \sqrt{2}dW(t) \\ &= \beta L\{L^{-1}b - X(t)\}dt + \sqrt{2}dW(t). \end{aligned} \tag{2.6}$$

The solution $X(t)$, $t \in [0,T]$ of (2.6) is the N-dimensional Ornstein-Uhlenbeck process given by

$$X(t) = e^{-t\beta L}X(0) + (I - e^{-t\beta L})L^{-1}b + \int_0^t e^{-(t-s)\beta L}\sqrt{2}dW(s), \tag{2.7}$$

cf. appendix A.1. Taking the limit $t \to \infty$ we obtain a stationary distribution for (2.5), which is Gaussian with mean $L^{-1}b$ and covariance matrix $(\beta L)^{-1}$. The inverse of L can be explicitly calculated, it is the matrix $Q \in \mathbb{R}^{N\times N}$ with entries

$$Q_{kl} = \min(k,l) - \frac{kl}{N+1}.$$

Therefore, it holds with respect to the stationary distribution

$$\mathbb{E}[X_k] = Q_{kN}b_N = \frac{k}{\rho}\frac{N}{N+1}$$

and

$$\begin{aligned} \mathbb{E}[(X_{k+1} - X_k)^2] &= \mathrm{Var}[X_{k+1} - X_k] + \mathbb{E}[X_{k+1} - X_k]^2 \\ &= \frac{Q_{k+1,k+1} - 2Q_{k,k+1} + Q_{kk}}{\beta} + \left(\frac{1}{\rho}\frac{N}{N+1}\right)^2 \\ &= \frac{1}{\beta}\frac{N}{N+1} + \left(\frac{1}{\rho}\frac{N}{N+1}\right)^2. \end{aligned} \tag{2.8}$$

Remark 2.3. *The position of e. g. the middle particle has variance of order N with respect to the stationary distribution since $Q_{N/2,N/2} = \frac{N(N+2)}{4(N+1)}$.*

2.2 Height model

The height model describes an interface that separates two distinct phases. It is based on a particle system similar to (2.2). We introduce the height model by following [Nis03].

Let a positive integer N and a potential V as in the particle model be given. Moreover, let $D \subset \mathbb{R}^d$ be a bounded domain with discretisation

$$D_N = \left\{ k \in \mathbb{Z}^d : B\left(\frac{k}{N}, \frac{5}{N}\right) \subset D \right\} \tag{2.9}$$

where $B(x,r) = \prod_{m=1}^{d} [x_m - r/2, x_m + r/2)$ is the half-open cube in $\mathbb{R}^d$ with side length r that is centered at x. As state space we choose $\mathbb{R}^{D_N}$, and we denote a typical element in the state space as $x = (x_k)_{k \in D_N}$. Let $|\cdot|$ be the Euclidean norm on $\mathbb{R}^d$, $C_0^2(\mathbb{R}^d)$ the space of two times continuously differentiable functions from $\mathbb{R}^d$ to $\mathbb{R}$ that vanish at infinity and $f \in C_0^2(\mathbb{R}^d)$. We define the Hamiltonian $H_{D_N} : \mathbb{R}^{D_N} \to \mathbb{R}$ by

$$H_{D_N}(x) = \frac{1}{2} \sum_{\substack{k,l \in D_N \\ |k-l|=1}} V(x_k - x_l) + \sum_{\substack{k \in D_N, l \in \mathbb{Z}^d \setminus D_N \\ |k-l|=1}} V(x_k - x_l)$$

with the convention

$$x_k = N^{d+1} \int_{B(N^{-1}k, N^{-1})} f(\theta) d\theta, \quad k \in \mathbb{Z}^d \setminus D_N. \tag{2.10}$$

The partial derivative of $H_{D_N}(x)$ with respect to x_k is

$$\partial_{x_k} H_{D_N}(x) = \sum_{l \in \mathbb{Z}^d, |k-l|=1} V'(x_k - x_l).$$

Corresponding to this, we introduce a random time evolution of the particles by means of the system of stochastic differential equations

$$\begin{cases} dX_k(t) = -\displaystyle\sum_{l \in \mathbb{Z}^d, |k-l|=1} V'(X_k(t) - X_l(t))dt + \sqrt{2} dW_k(t), & k \in D_N, \\ \quad X_k = N^{d+1} \displaystyle\int_{B(N^{-1}k, N^{-1})} f(\theta) d\theta, & k \in \mathbb{Z}^d \setminus D_N, \end{cases} \tag{2.11}$$

where W_k, $k \in D_N$ are independent one-dimensional Brownian motions. As in lemma 2.2, it can be shown that the particle system (2.11) has a unique solution $X(t) = (X_k(t))_{k \in \mathbb{Z}^d}$, $t \in [0,T]$ for every $T > 0$ if $X(0)$ has finite second moment and is independent of the $|D_N|$-dimensional Brownian motion $W(t) = (W_k(t))_{k \in D_N}$, $t \in [0,T]$.

We define the random function $h^N : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ by

$$h^N(t,\theta) = \sum_{k \in \mathbb{Z}^d} \frac{X_k(N^2 t)}{N} 1_{B(N^{-1}k, N^{-1})}(\theta).$$

Then we regard $h^N(t,\cdot)$ as the height of an interface at time t.

The hydrodynamic limit for heights was already established. Before we state it, we define the surface tension. Let $B=(0,1)^d$ with discretisation B_N given by (2.9). Then we set for $u\in\mathbb{R}^d$

$$Z^u_{B_N}=\int_{\mathbb{R}^{B_N}}\exp(-H_{B_N}(x))dx,$$

where we choose in (2.10) some $f\in C^2_0(\mathbb{R}^d)$ such that $f(\theta)=u^T\theta$ for $\theta\in B$.

Definition 2.4. *The finite volume surface tension $\sigma_{B_N}\colon\mathbb{R}^d\to\mathbb{R}$ is defined by*

$$\sigma_{B_N}(u)=-\frac{1}{|B_N|}\log\left(\frac{Z^u_{B_N}}{Z^0_{B_N}}\right),$$

and the (infinite volume) surface tension $\sigma\colon\mathbb{R}^d\to\mathbb{R}$ is defined as

$$\sigma(u)=\lim_{N\to\infty}\sigma_{B_N}(u). \tag{2.12}$$

The existence of the limit in (2.12) was proven in [FS97, proposition 1.1 in appendix II]. The surface tension measures the change of energy that comes from tilting a height by u.

Theorem 2.5 ([Nis03, theorem 2.1])**.** *If there exists a function $h_0\in L^2(D)$ satisfying*

$$\lim_{N\to\infty}\mathbb{E}[\|h^N(0,\cdot)-h_0\|^2_{L^2}]=0,$$

then it holds for every $t\in(0,T]$

$$\lim_{N\to\infty}\mathbb{E}[\|h^N(t,\cdot)-h(t,\cdot)\|^2_{L^2}]=0.$$

Here h is the unique weak solution of the nonlinear partial differential equation

$$\begin{cases}\partial_t h(t,\theta)=\operatorname{div}[\nabla\sigma(\nabla h(t,\theta))]\\ \qquad\qquad\;=\displaystyle\sum_{k=1}^d\partial_{\theta_k}\{\partial_{u_k}\sigma(\nabla h(t,\theta))\}, & \theta\in D,\ t\in(0,T),\\ h(t,\theta)=f(\theta), & \theta\in\partial D,\ t\in(0,T],\\ h(0,\theta)=h_0(\theta), & \theta\in D.\end{cases} \tag{2.13}$$

The hydrodynamic limit for heights states that the height $h^N(t,\cdot)$ converges as $N\to\infty$ to a deterministic function $h(t,\cdot)$, where h is the unique weak solution of a partial differential equation.

Remark 2.6. *1. Let $C^1_0(D)$ be the space of continuously differentiable functions from D to $\mathbb{R}$ that vanish on the boundary of D. By definition, h is a weak solution of* (2.13) *with initial condition $h_0\in L^2(D)$ if the following conditions are satisfied:*

a) $h \in L^2([0,T], H^1(D)) \cap C([0,T], L^2(D))$,

b) $h - f \in L^2([0,T], H_0^1(D))$,

c) *for every* $J \in C^1([0,T] \times D)$ *such that* $J(t,\cdot) \in C_0^1(D)$ *for every* $t \in [0,T]$, *it holds*

$$\int_D h(t,\theta)J(t,\theta)d\theta = \int_D h_0(\theta)J(0,\theta)d\theta + \int_0^t \int_D h(s,\theta)\partial_s J(s,\theta)d\theta ds \\ - \int_0^t \int_D \nabla\sigma(\nabla h(s,\theta))^T \nabla J(s,\theta)d\theta ds.$$

2. *The solution* h *of* (2.13) *defines a gradient flow. To see this, we consider the total surface tension* $\Sigma\colon C^1(D) \to \mathbb{R}$ *defined by*

$$\Sigma(g) = \int_D \sigma(\nabla g(\theta))d\theta.$$

The functional derivative of Σ *at* g *is*

$$\frac{\delta\Sigma}{\delta g(\theta)}(g) = -\operatorname{div}[\nabla\sigma(\nabla g)],$$

and therefore,

$$\partial_t h = -\frac{\delta\Sigma}{\delta h(\theta)}(h).$$

From this it follows that h *is the gradient flow that starts in* h_0 *and minimises the total surface tension as time evolves.*

In the following remark we will illustrate the connection between the height and particle model. We will use it later to prove the hydrodynamic equation for the particle density.

Remark 2.7. *In the definition of the height model we choose* $d = 1$, $D = (0,1)$ *and* $f \in C_0^2(\mathbb{R})$ *such that* $f(\theta) = \theta/\rho$ *on* D. *Then we have* $D_N = \{3,\dots,N-3\}$ *and the particle system* (2.11) *becomes*

$$\begin{cases} dX_k(t) = -\displaystyle\sum_{l\in\mathbb{Z},|k-l|=1} V'(X_k(t)-X_l(t))dt + \sqrt{2}dW_k(t), \\ \qquad\qquad\qquad\qquad\qquad\qquad k = 3,\dots,N-3, \\ X_k = N^2 \displaystyle\int_{B(N^{-1}k,N^{-1})} f(\theta)d\theta, \quad k \le 2 \text{ or } k \ge N-2. \end{cases} \tag{2.14}$$

Notice that $X_2 = 2/\rho$ *and* $X_{N-2} = (N-2)/\rho$. *Therefore, we expect* (2.14) *and* (2.2) *to exhibit the same macroscopic behaviour as* $N \to \infty$. *For fixed* t, *the height* $h^N(t,\cdot)\colon \mathbb{R} \to \mathbb{R}$ *is given by*

$$h^N(t,\theta) = \sum_{k=-\infty}^{\infty} \frac{X_k(N^2t)}{N} 1_{B(N^{-1}k,N^{-1})}(\theta),$$

and it converges in the sense of theorem 2.5, assuming it applies, to the unique weak solution of

$$\begin{cases} \partial_t h(t,\theta) = \partial_\theta \sigma'(\partial_\theta h(t,\theta)), & \theta \in (0,1),\ t \in (0,T), \\ h(t,0) = 0,\ h(t,1) = \dfrac{1}{\rho}, & t \in (0,T], \\ h(0,\theta) = h_0(\theta), & \theta \in (0,1). \end{cases} \tag{2.15}$$

Moreover, the empirical measure (2.4) *coincides with the pushforward measures* $\lambda \circ h^N(t, \cdot + (2N)^{-1})^{-1}$ *of* $h^N(t, \cdot + (2N)^{-1})$ *restricted to* $(0,1)$ *with respect to the Lebesgue measure* λ *on* $(0,1)$.

3 Hydrodynamic equation

3.1 Statement of the result

In this chapter we will derive the hydrodynamic equation for the particle density in the case of one dimension and general potential. For this purpose, we consider the empirical measures

$$\mu_t^N(dx) = \frac{1}{N}\sum_{k=1}^{N} \delta_{N^{-1}X_k(N^2t)}(dx)$$

and heights

$$h^N(t,\theta) = \sum_{k=-\infty}^{\infty} \frac{X_k(N^2t)}{N} 1_{B(N^{-1}k,N^{-1})}(\theta),$$

where $X(t)$, $t \in [0, N^2T]$ solves (2.14). Recall that the inverse distribution function $F^{-1}\colon (0,1) \to \mathbb{R}$ of a distribution function $F\colon \mathbb{R} \to [0,1]$ is given by

$$F^{-1}(\theta) = \inf\{x \in \mathbb{R} : F(x) > \theta\}.$$

The main result of this chapter is

Theorem 3.1. *Let F^N be the distribution function of μ_0^N and F the distribution function of a probability measure $m_0(x)dx$ supported in $[0,\rho^{-1}]$, where m_0 restricted to $(0,\rho^{-1})$ is two times continuously differentiable with Lipschitz continuous second derivative, $m_0'(0) = m_0'(\rho^{-1}) = 0$ and $\min_{[0,\rho^{-1}]} m_0 > 0$. We assume*

$$\lim_{N\to\infty} \mathbb{E}\left[\int_0^1 \left|h^N\left(0,\theta + \frac{1}{2N}\right) - (F^N)^{-1}(\theta)\right|^2 d\theta\right] = 0, \tag{3.1}$$

and

$$\lim_{N\to\infty} \mathbb{E}\left[\int_0^1 |(F^N)^{-1}(\theta) - F^{-1}(\theta)|^2 d\theta\right] = 0. \tag{3.2}$$

Then, for every $t \in (0,T]$, μ_t^N converges as $N \to \infty$ in probability to $m(t,x)dx$ in the space of probability measures on the real line equipped with the topology of weak convergence of measures. The function m is a classical solution of the nonlinear heat equation

$$\begin{cases} \partial_t m(t,x) = -\partial_x^2 \sigma'\left(\dfrac{1}{m(t,x)}\right), & t \in (0,T),\ x \in (0,\rho^{-1}), \\ m(0,x) = m_0(x), & x \in [0,\rho^{-1}]. \end{cases} \tag{3.3}$$

Let us consider the case of the quadratic potential $V(\eta) = \frac{\beta}{2}\eta^2$. Then we have $\sigma(u) = \frac{\beta}{2}u^2$, cf. remark A.6, and (3.3) becomes

$$\begin{cases} \partial_t m(t,x) = -\partial_x^2 \dfrac{\beta}{m(t,x)}, & t \in (0,T),\ x \in (0,\rho^{-1}), \\ m(0,x) = m_0(x), & x \in [0,\rho^{-1}]. \end{cases} \tag{3.4}$$

We have $-\partial_x^2 \frac{\beta}{m} = \partial_x \frac{\beta \partial_x m}{m^2}$, where we regard m as particle density and $\frac{\beta}{m^2}$ as the diffusion coefficient. In view of Fick's laws of diffusion, m diffuses over time and $\frac{\beta}{m^2}$ determines the speed of the diffusion. So the smaller the density of an area, the faster it is filled.

Remark 3.2. *We make some remarks on the assumptions of theorem 3.1:*

1. *Assumption* (3.1) *can be interpreted as the particles $X_k(0)$, $k = 1,\dots,N$ being enumerated in an nearly nondecreasing order. Indeed, we have*

$$F^N(x) = \frac{1}{N}\sum_{k=1}^{N} 1_{(-\infty,x]}(X_k(0)),$$

and from the definition of the inverse distribution function it follows $(F^N)^{-1} = \tilde{h}^N(0,\cdot + (2N)^{-1})$ on $(0,1)$, where $\tilde{h}^N(0,\cdot)$ is obtained from the height $h^N(0,\cdot)$ by enumerating the particles $X_k(0)$, $k = 1,\dots,N$ in a nondecreasing order.

2. *The integral in* (3.2) *can be written in terms of the quadratic Wasserstein metric W_2 as $W_2(\mu_0^N, m_0(x)dx)^2$, cf. [Vil03, theorem 2.18]. The quadratic Wasserstein metric is a metric on the space of probability measures on the real line with finite second moment, and it is known that convergence with respect to this metric is equivalent to weak convergence of measures together with convergence of the second moments.*

3. *Instead of* (3.2), *we can assume $0 \le X_k(0) \le N/\rho$ for $k = 3,\dots,N-3$ and*

$$\lim_{N\to\infty} \mathbb{E}\left[\int_0^{\rho^{-1}} |F^N(x) - F(x)|dx\right] = 0. \tag{3.5}$$

To see this, notice that μ_0^N and $m_0(x)dx$ are probability measures supported in $[0,\rho^{-1}]$, which implies $|(F^N)^{-1} - F^{-1}| \le \rho^{-1}$. From this it follows

$$\begin{aligned} \mathbb{E}\left[\int_0^{\rho^{-1}} |F^N(x) - F(x)|dx\right] &= \mathbb{E}\left[\int_0^1 |(F^N)^{-1}(\theta) - F^{-1}(\theta)|d\theta\right] \\ &\ge \rho\mathbb{E}\left[\int_0^1 |(F^N)^{-1}(\theta) - F^{-1}(\theta)|^2 d\theta\right], \end{aligned}$$

and the last line vanishes as $N \to \infty$ by assumption. Here we used that the L^1-difference does not change when passing to the inverse distribution functions.

4. *To show the existence of an initial condition satisfying the assumptions of theorem 3.1, let* $(Y_k)_{k=3}^{N-3}$ *be a sequence of independent and identically distributed random variables with distribution* $m_0(x)dx$. *Then* $X(0) = (X_k(0))_{k=3}^{N-3}$ *with* $X_k(0) = NY_k$ *satisfies the assumptions of the previous remark. Indeed, it is obvious that* $0 \leq X_k(0) \leq N/\rho$ *for* $k = 3, \dots, N-3$. *Glivenko-Cantelli's theorem implies*

$$\lim_{N\to\infty} \int_0^{\rho^{-1}} |F^N(x) - F(x)|dx = 0 \text{ a. s.}, \tag{3.6}$$

where the integral is bounded above by ρ^{-1}. *Taking the expectation of* (3.6) *and using dominated convergence, we get* (3.5). *To obtain* (3.1), *we change the particle enumeration, which does not affect* (3.5).

3.2 Proof of theorem 3.1

3.2.1 Formal derivation of the hydrodynamic equation

Before we give the rigorous proof of theorem 3.1, we will formally derive the hydrodynamic equation (3.3) for the particle system (2.5), i. e. in the case of the quadratic potential $V(\eta) = \frac{\beta}{2}\eta^2$. Let $C_c^2(\mathbb{R}^d)$ be the space of two times continuously differentiable functions from $\mathbb{R}^d$ to $\mathbb{R}$ with compact support. For $f \in C_c^2(\mathbb{R})$ we compute the time evolution of

$$\int_{-\infty}^{\infty} f d\mu_t^N = \frac{1}{N}\sum_{k=1}^{N} f\left(\frac{X_k(N^2t)}{N}\right).$$

Using Itô's lemma and

$$dX_k(N^2t) = N^2\beta\{X_{k-1}(N^2t) - 2X_k(N^2t) + X_{k+1}(N^2t)\}dt + \sqrt{2}dW_k(N^2t)$$

for $k = 1, \dots, N$, we get

$$\begin{aligned} d&\int_{-\infty}^{\infty} f d\mu_t^N \\ &= \frac{1}{N}\sum_{k=1}^{N}\left\{f'\left(\frac{X_k(N^2t)}{N}\right) d\frac{X_k(N^2t)}{N} + \frac{1}{2}f''\left(\frac{X_k(N^2t)}{N}\right)\left(d\frac{X_k(N^2t)}{N}\right)^2\right\} \\ &= \sum_{k=1}^{N} f'\left(\frac{X_k(N^2t)}{N}\right)\beta\{X_{k-1}(N^2t) - 2X_k(N^2t) + X_{k+1}(N^2t)\}dt \\ &\quad + \frac{1}{N^2}\sum_{k=1}^{N} f'\left(\frac{X_k(N^2t)}{N}\right)\sqrt{2}dW_k(N^2t) + \frac{1}{N}\sum_{k=1}^{N} f''\left(\frac{X_k(N^2t)}{N}\right)dt. \end{aligned} \tag{3.7}$$

The first sum in the last line of (3.7) is a martingale, and we omit it. Using summation by parts and Taylor's expansion, the sum in the third line of (3.7) can be written as

$$
\begin{aligned}
&-\sum_{k=1}^{N}\left\{f'\left(\frac{X_{k+1}(N^2t)}{N}\right)-f'\left(\frac{X_k(N^2t)}{N}\right)\right\}\beta\{X_{k+1}(N^2t)-X_k(N^2t)\}\\
&=-\frac{1}{N}\sum_{k=1}^{N}f''\left(\frac{X_k(N^2t)}{N}\right)\beta\{X_{k+1}(N^2t)-X_k(N^2t)\}^2,
\end{aligned}
\tag{3.8}
$$

where we omitted the boundary terms. We assume the particles to be in a local equilibrium, so $\{X_{k+1}(N^2t)-X_k(N^2t)\}^2$ in (3.8) can be replaced by its expectation with respect to the stationary distribution with parameter $\rho = m(t, N^{-1}X_k(N^2t))$:

$$
\{X_{k+1}(N^2t)-X_k(N^2t)\}^2=\frac{1}{\beta}\frac{N}{N+1}+\left(\frac{1}{m\left(t,\frac{X_k(N^2t)}{N}\right)}\frac{N}{N+1}\right)^2,
$$

cf. (2.8). Here we assumed m to be positive. Putting everything together, we get

$$
\begin{aligned}
d\int_{-\infty}^{\infty} f d\mu_t^N &= -\frac{1}{N}\sum_{k=1}^{N}f''\left(\frac{X_k(N^2t)}{N}\right)\frac{\beta}{m\left(t,\frac{X_k(N^2t)}{N}\right)^2}dt\\
&=-\int_{-\infty}^{\infty}f''(x)\frac{\beta}{m(t,x)^2}\mu_t^N(dx)dt,
\end{aligned}
\tag{3.9}
$$

up to an error of order N^{-1}.

We assume that $\mu_t^N(dx)$ converges as $N\to\infty$ in probability to $m(t,x)dx$ in the space of probability measures on the real line equipped with the topology of weak convergence of measures. Then (3.9) becomes in the limit $N\to\infty$, at least formally,

$$
\begin{aligned}
\int_0^t\int_{-\infty}^{\infty} f(x)\partial_s m(s,x)dxds &= -\int_0^t\int_{-\infty}^{\infty} f''(x)\frac{\beta}{m(s,x)}dxds\\
&= -\int_0^t\int_{-\infty}^{\infty} f(x)\partial_x^2\frac{\beta}{m(s,x)}dxds,
\end{aligned}
$$

where we integrated by parts two times and used that f has compact support. This concludes that m satisfies a weak formulation of (3.4).

Remark 3.3. *Another interesting particle system is given by means of the system of stochastic differential equations*

$$
\begin{cases}
dX_k(t)=\beta\displaystyle\sum_{k\in\mathbb{Z},|k-l|=1}\frac{1}{X_k(t)-X_l(t)}dt+dW_k(t), \quad k=1,\dots,N,\\
\qquad X_0=0,\ X_{N+1}=\dfrac{N}{\rho},
\end{cases}
\tag{3.10}
$$

which is related to Dyson's Brownian motion. The solutions of (3.10) *exhibit the same qualitative behaviour as the solution of* (2.2) *in the sense that we can expect the particles to spread out evenly between the boundary particles as time evolves. Indeed, let* $X_{k-1}(t) < X_k(t) < X_{k+1}(t)$. *We first assume* $X_k(t)$ *to be closer to* $X_{k-1}(t)$ *than to* $X_{k+1}(t)$, *i. e.* $0 < X_k(t) - X_{k-1}(t) < X_{k+1}(t) - X_k(t)$. *Then the drift of* (3.10) *is positive, which suggests that* $X_k(t)$ *increases within a short time interval. Conversely, we assume* $X_k(t)$ *to be closer to* $X_{k+1}(t)$ *than to* $X_{k-1}(t)$, *i. e.* $0 < X_{k+1}(t) - X_k(t) < X_k(t) - X_{k-1}(t)$. *Then the drift of* (3.10) *is negative, which suggests that* $X_k(t)$ *decreases within a short time interval.*

As for the particle system (2.14), *we can formally derive the hydrodynamic equation for* (3.10). *Using Itô's lemma, we get*

$$\begin{aligned} &d\int_{-\infty}^{\infty} f d\mu_t^N \\ &= \sum_{k=1}^{N} f'\left(\frac{X_k(N^2t)}{N}\right)\beta\left\{\frac{1}{X_k(N^2t)-X_{k-1}(N^2t)}+\frac{1}{X_k(N^2t)-X_{k+1}(N^2t)}\right\}dt \\ &\quad+\frac{1}{N^2}\sum_{k=1}^{N} f'\left(\frac{X_k(N^2t)}{N}\right)\sqrt{2}dW_k(N^2t)+\frac{1}{N}\sum_{k=1}^{N} f''\left(\frac{X_k(N^2t)}{N}\right)dt. \end{aligned}$$

We omit the martingale term and approximate the sum in the second line of the above equation by

$$\begin{aligned} &-\sum_{k=1}^{N}\left\{f'\left(\frac{X_{k+1}(N^2t)}{N}\right)-f'\left(\frac{X_k(N^2t)}{N}\right)\right\}\frac{\beta}{X_k(N^2t)-X_{k+1}(N^2t)} \\ &=\frac{\beta}{N}\sum_{k=1}^{N} f''\left(\frac{X_k(N^2t)}{N}\right). \end{aligned}$$

In contrast to the previous calculation, $\{X_{k+1}(N^2t) - X_k(N^2t)\}^2$ *did not need to be replaced by its expectation. The remaining arguments work without changes. Then it follows that* m *satisfies a weak formulation of the linear heat equation*

$$\partial_t m = (1+\beta)\partial_x^2 m.$$

To extend the above calculation to a proof of theorem 3.1, one could try to follow the approach in [KL13, chapter 4]. The idea of this approach is as follows: We regard $(\mu_t^N)_{t\in[0,T]}$ as an element in $C([0,T],\mathcal{M}_+(\mathbb{R}))$, where $\mathcal{M}_+(\mathbb{R})$ is the space of positive measures on the real line with the topology of vague convergence of measures. Since $\mathcal{M}_+(\mathbb{R})$ is a Polish space, $C([0,T],\mathcal{M}_+(\mathbb{R}))$ with the uniform topology is also a Polish space. Let Q^N be the distribution of $(\mu_t^N)_{t\in[0,T]}$, which is an element in the space of probability measure on $C([0,T],\mathcal{M}_+(\mathbb{R}))$ equipped with the topology of weak convergence of measures. Then the proof of theorem 3.1 can be divided into checking three statements:

1. Each accumulation point Q of $\{Q^N\}_{N\in\mathbb{N}}$ is a Dirac measure concentrated on $(m(t,x)dx)_{t\in[0,T]}$, where m is the unique solution of a partial differential equation.
2. The family $\{Q^N\}_{N\in\mathbb{N}}$ is tight.
3. For fixed t, μ_t^N converges as $N\to\infty$ in probability to $m(t,x)dx$.

The first statement implies uniqueness of the accumulation points Q, whereas the second statement and Prokhorov's theorem imply the relative compactness of $\{Q^N\}_{N\in\mathbb{N}}$. Together it follows the convergence of Q^N along the whole sequence. The third statement is exactly the convergence as stated in theorem 3.1. Since we encountered difficulties in the proof of tightness, we will use a different approach in the next section to prove theorem 3.1.

3.2.2 Rigorous proof

We will use the following strategy to prove theorem 3.1:

1. We will pass from empirical measures to heights.
2. Applying theorem 2.5 and passing back to empirical measures, we will recover the convergence of empirical measures.
3. The proof will be concluded by checking that the limit measure is absolutely continuous and its density satisfies the hydrodynamic equation.

We set $h_0 = F^{-1}$. Recall that we assumed m_0 restricted to $(0,\rho^{-1})$ to be two times continuously differentiable with Lipschitz continuous second derivative, $m_0'(0) = m_0'(\rho^{-1}) = 0$ and $\min_{[0,\rho^{-1}]} m_0 > 0$. Extending h_0 to $[0,1]$, it can be shown that $h_0 \in C^{(3)}([0,1])$ with Lipschitz continuous third derivative, $h_0''(0) = h_0''(1) = 0$, $h_0(0) = 0$ and $h_0(1) = \rho^{-1}$. From this it follows that the weak solution h of (2.15) with initial condition h_0 is even a classical solution and that the partial derivatives $\partial_\theta^3 h, \partial_\theta\partial_t h$ exist and are continuous, cf. corollary A.8.

Change from empirical measures to heights means that instead of μ_t^N, we consider $\tilde{h}^N(t,\cdot)$ from part one of remark 3.2. Next, we will show that theorem 2.5 applies. For this purpose, we need

Lemma 3.4. *Let h^N, h be as in theorem 2.5. Then it holds for every $t\in[0,T]$*

$$\lim_{N\to\infty} \mathbb{E}\left[\int_0^1 \left|h^N(t,\theta) - h(t,\theta)\right|^2 d\theta\right] = 0 \tag{3.11}$$

if and only if

$$\lim_{N\to\infty} \mathbb{E}\left[\int_0^1 \left|h^N\left(t,\theta+\frac{1}{2N}\right) - h(t,\theta)\right|^2 d\theta\right] = 0. \tag{3.12}$$

Proof. We only prove that (3.11) implies (3.12). The proof of the converse implication is similar. A change of variables yields

$$\begin{aligned}
&\mathbb{E}\left[\int_0^1 \left|h^N\left(t,\theta+\frac{1}{2N}\right)-h(t,\theta)\right|^2 d\theta\right]\\
&=\mathbb{E}\left[\int_{(2N)^{-1}}^{1+(2N)^{-1}} \left|h^N(t,\theta)-h\left(t,\theta-\frac{1}{2N}\right)\right|^2 d\theta\right]\\
&=\mathbb{E}\left[\int_0^1 \left|h^N(t,\theta)-h\left(t,\theta-\frac{1}{2N}\right)\right|^2 d\theta\right]+\mathcal{O}(N^{-1}),
\end{aligned}$$

where we used for the last line that $h^N(t,\theta)-h(t,\theta-(2N)^{-1})$ is bounded outside of $\left[\frac{3}{2N},1-\frac{3}{2N}\right)$ uniformly in N. The expectation in the last line can be bounded above with Jensen's inequality by

$$2\mathbb{E}\left[\int_0^1 \left|h^N(t,\theta)-h(t,\theta)\right|^2 d\theta\right]+2\int_0^1 \left|h(t,\theta)-h\left(t,\theta-\frac{1}{2N}\right)\right|^2 d\theta,$$

which vanishes as $N\to\infty$ by (3.11) and the continuity of the translation operator on $L^2(\mathbb{R})$. □

Combining (3.1) and (3.2) with Jensen's inequality, we get

$$\lim_{N\to\infty}\mathbb{E}\left[\int_0^1 \left|h^N\left(0,\theta+\frac{1}{2N}\right)-h_0(\theta)\right|^2 d\theta\right]=0,$$

and from lemma 3.4 it follows that theorem 2.5 applies. So we have for every $t\in[0,T]$

$$\lim_{N\to\infty}\mathbb{E}\left[\int_0^1 \left|h^N(t,\theta)-h(t,\theta)\right|^2 d\theta\right]=0. \tag{3.13}$$

Next, we will prove convergence of the empirical measure μ_t^N as $N\to\infty$. Recall from remark 2.7 that $\mu_t^N=\lambda\circ h^N(t,\cdot+(2N)^{-1})^{-1}$, where λ is the Lebesgue measure on $(0,1)$ and $h^N(t,\cdot+(2N)^{-1})$ is restricted to $(0,1)$.

Lemma 3.5. *For every $t\in(0,T]$, the empirical measure μ_t^N converges as $N\to\infty$ in probability to $\lambda\circ h(t,\cdot)^{-1}$ in the space of probability measures on the real line equipped with the topology of weak converges of measures.*

Proof. Applying lemma 3.4 to (3.13), we get

$$\lim_{N\to\infty}\mathbb{E}\left[\int_0^1 \left|h^N\left(t,\theta+\frac{1}{2N}\right)-h(t,\theta)\right|^2 d\theta\right]=0.$$

From this it follows with Markov's inequality that $h^N(t,\cdot+(2N)^{-1})$ converges in probability to $h(t,\cdot)$ in the space $L^2((0,1))$. Let $\mathcal{P}(\mathbb{R})$ be the space of probability measures on the real line equipped with the topology of weak converges of measures. Since

$L^2((0,1)) \to \mathcal{P}(\mathbb{R}),\ h \mapsto \lambda \circ h^{-1}$ is a continuous map between metric space, the continuous mapping theorem applies. This concludes that $\mu_t^N = \lambda \circ h^N(t, \cdot + (2N)^{-1})^{-1}$ converges in probability to $\lambda \circ h(t,\cdot)^{-1}$ in the space $\mathcal{P}(\mathbb{R})$. □

It remains to show that the limit measure $\lambda \circ h(t,\cdot)^{-1}$ is absolutely continuous with density $m(t,\cdot)$ and that m is a classical solution of (3.3). The next lemma implies that $h(t,\cdot)\colon (0,1) \to (0,\rho^{-1})$ is a C^1-diffeomorphism for every $t \in (0,T]$. This will enable us to prove the absolute continuity of $\lambda \circ h(t,\cdot)^{-1}$.

Lemma 3.6. *For every $t \in (0,T]$ we have*

$$\min_{\theta \in [0,1]} \partial_\theta h(t,\theta) > 0.$$

Proof. Setting $a(t,\theta) = \sigma''(\partial_\theta h(t,\theta))$, h solves the linear partial differential equation

$$\begin{cases} \partial_t h(t,\theta) = a(t,\theta)\partial_\theta^2 h(t,\theta), & t \in (0,T),\ \theta \in (0,1), \\ h(t,0) = 0,\ h(t,1) = \dfrac{1}{\rho}, & t \in (0,T], \\ h(0,\theta) = h_0(\theta), & \theta \in [0,1], \end{cases} \tag{3.14}$$

and $g = \partial_\theta h$ solves the linear partial differential equation

$$\begin{cases} \partial_t g(t,\theta) = \partial_\theta a(t,\theta)\partial_\theta g(t,\theta) + g(t,\theta)\partial_\theta^2 g(t,\theta), & t \in (0,T),\ \theta \in (0,1), \\ g(0,\theta) = h_0'(\theta), & \theta \in [0,1]. \end{cases}$$

So both h and $\partial_\theta h$ satisfy the maximum principle, cf. theorem A.9. In view of the maximum principle for $\partial_\theta h$ and the assumption $\min_{\theta\in[0,1]} \partial_\theta h(0,\theta) > 0$, it suffices to show $\partial_\theta h(t,\theta) > 0$ for every $t \in (0,T]$ and $\theta = 0,1$.

We will argue by contradiction. Let us first assume $\partial_\theta h(t,0) < 0$ for some $t \in (0,T]$. Since $\partial_\theta h(t,\theta)$ is continuous in θ, it holds $\partial_\theta h(t,\theta) < 0$ for every θ sufficiently close to 0, and Taylor's expansion yields $h(t,\theta) < h(t,0)$. Using that the boundary condition in (3.14) is independent of t, it follows that h assumes its minimum for $t \in (0,T]$ and $\theta > 0$, which contradicts the maximum principle for h.

If $\partial_\theta h(t,0) = 0$ for some $t \in (0,T]$, we tilt h by setting $\tilde{h}(t,\theta) = h(t,\theta) - \epsilon\theta$ for some $0 < \epsilon < \min_{\theta\in[0,1]} h_0'(\theta)$. Then $\tilde{h}$ solves the linear partial differential equation

$$\begin{cases} \partial_t h(t,\theta) = a(t,\theta)\partial_\theta^2 h(t,\theta), & t \in (0,T),\ \theta \in (0,1), \\ h(t,0) = 0,\ h(t,1) = \dfrac{1}{\rho} - \epsilon, & t \in (0,T], \\ h(0,\theta) = h_0(\theta) - \epsilon\theta, & \theta \in [0,1], \end{cases}$$

and in particular, $\tilde{h}$ satisfies the maximum principle. Since $\partial_\theta \tilde{h}(t,0) < 0$, we can argue as before to obtain a contradiction.

In the case $\theta = 1$, instead of the minimum of h, we need to consider its maximum. The remaining arguments work without changes. This concludes the proof. □

Let $t \in (0,T]$ be given. Then a change of variables shows for every $f \in C((0,\rho^{-1}))$

$$\int_0^1 f(h(t,\theta))d\theta = \int_0^{\rho^{-1}} f(x)\partial_x h(t,\cdot)^{-1}(x)dx, \tag{3.15}$$

so $\lambda \circ h(t,\cdot)^{-1}$ has density

$$m(t,x) = \partial_x h(t,\cdot)^{-1}(x) = \frac{1}{\partial_\theta h(t,h(t,\cdot)^{-1}(x))}. \tag{3.16}$$

Lemma 3.7. *The function m from* (3.16) *is a classical solution of* (3.3).

Proof. Differentiating the identity $h(t,h(t,\cdot)^{-1}(x)) = x$ with respect to t, we get

$$(\partial_t h)(t,h(t,\cdot)^{-1}(x)) + \partial_\theta h(t,h(t,\cdot)^{-1}(x))\partial_t h(t,\cdot)^{-1}(x) = 0,$$

and using $\partial_t h(t,\theta) = \partial_\theta \sigma'(\partial_\theta h(t,\theta))$, it follows

$$\begin{aligned} \partial_t h(t,\cdot)^{-1}(x) &= -\partial_\theta \sigma'(\partial_\theta h(t,h(t,\cdot)^{-1}(x)))\partial_x h(t,\cdot)^{-1}(x) \\ &= -\partial_x \sigma'(\partial_\theta h(t,h(t,\cdot)^{-1}(x))). \end{aligned} \tag{3.17}$$

Then we differentiate (3.17) with respect to x and obtain

$$\partial_t m(t,x) = -\partial_x^2 \sigma'\left(\frac{1}{m(t,x)}\right).$$

Since the initial condition in (3.3) holds by definition, this concludes the proof. □

This lemma concludes the proof of theorem 3.1.

As we pointed out before, our model exhibits an unnatural behaviour. Namely, the nearest neighbours are determined by the particle enumeration, which is fixed at the beginning. At later times the particles positions can change so that between two interacting particles there are other particles. We will show the following: Enumerating the particles in a nondecreasing order and restarting the dynamics at any given time does not affect the hydrodynamic equation for the particle density. This suggests that the physical model, which is obtained by enumerating the particles in a nondecreasing order every time two particles pass each other, and our model have the same hydrodynamic equation for the particle density.

Let the assumptions of theorem 3.1 be satisfied and fix $t \in [0,T]$. From the proof of theorem 3.1 we know that (3.13) holds and $h(t,\cdot)$ is nondecreasing. Let $\tilde{h}^N(0,\cdot)$ be the height obtained from $h^N(t,\cdot)$ by enumerating the particles at time $N^2 t$ in a nondecreasing order. More precisely, we have

$$h^N(t,\theta) = \sum_{k=-\infty}^{\infty} \frac{X_k(N^2 t)}{N} 1_{B(N^{-1}k,N^{-1})}(\theta)$$

and we put

$$\tilde{h}^N(0,\theta) = \sum_{k=-\infty}^{\infty} \frac{\tilde{X}_k(N^2t)}{N} 1_{B(N^{-1}k,N^{-1})}(\theta),$$

where $\tilde{X}_k(N^2t)$ is the k-th largest element of $\{X_l(N^2t) : l = 1,\dots,N\}$ if $k = 1,\dots,N$ and $\tilde{X}_k(N^2t) = X_k(N^2t)$ otherwise.

Lemma 3.8. *It holds*

$$\int_0^1 \left|\tilde{h}^N(0,\theta) - h(t,\theta)\right|^2 d\theta \le \int_0^1 \left|h^N(t,\theta) - h(t,\theta)\right|^2 d\theta.$$

Proof. It suffices to show that, if $X_k(N^2t) > X_{k+1}(N^2t)$ for some $k = 1,\dots,N-1$, then interchanging the enumeration of these particles decreases the integral

$$\int_0^1 |h^N(t,\theta) - h(t,\theta)|^2 d\theta.$$

Let $\theta_1 \in B(N^{-1}k, N^{-1})$ and $\theta_2 \in B(N^{-1}(k+1), N^{-1})$. Then this follows from

$$\begin{aligned}&|h^N(t,\theta_1) - h(t,\theta_1)|^2 + |h^N(t,\theta_2) - h(t,\theta_2)|^2 \\ &> |h^N(t,\theta_2) - h(t,\theta_1)|^2 + |h^N(t,\theta_1) - h(t,\theta_2)|^2,\end{aligned}$$

which can be shown by expanding the squares. □

Combining lemma 3.8 and (3.13), it follows

$$\lim_{N\to\infty} \mathbb{E}\left[\int_0^1 \left|\tilde{h}^N(0,\theta) - h(t,\theta)\right|^2 d\theta\right] = 0.$$

Let $\tilde{h}^N(s,\cdot)$ be the height at time $s \in [0,T]$ started from $\tilde{h}^N(0,\cdot)$. Then theorem 2.5 implies

$$\lim_{N\to\infty} \mathbb{E}\left[\int_0^1 \left|\tilde{h}^N(s,\theta) - \tilde{h}(s,\theta)\right|^2 d\theta\right] = 0,$$

where $\tilde{h}$ solves (2.15) with initial condition $h(t,\cdot)$. Since $h(t+\cdot,\cdot)$ also solves (2.15) with initial condition $h(t,\cdot)$, and the solution of (2.15) is unique, it follows $\tilde{h} = h(t+\cdot,\cdot)$. In conclusion, we can change the particle enumeration to be in a nondecreasing order for any given time without changing the hydrodynamic equation for the particle density, and in particular, we can do this for finitely many times.

Remark 3.9. *Two open problems remain: 1) The uniqueness of the solution m of the hydrodynamic equation* (3.3) *needs to be proven. 2) The conjecture that the particle model* (2.14) *and the physical model, which is obtained by enumerating the particles in a non-decreasing order every time two particles pass each other, have the same hydrodynamic equation for the particle density needs to be proven.*

3.3 Generalisation to multiple dimensions

In this section we will generalise theorem 3.1 to $d \geq 2$ dimensions. First, we will generalise the particle model. For this purpose, let $D \subset \mathbb{R}^d$ be a bounded domain with discretisation D_N given by (2.9). As state space we choose $(\mathbb{R}^d)^{D_N}$, and we denote a typical element in the state space as $x = (x_k)_{k \in D_N}$. Let V_m, $m = 1, \ldots, d$ be potentials as in the one-dimensional case, and define the potential $V \colon \mathbb{R}^d \to \mathbb{R}$ by

$$V(\eta) = \sum_{m=1}^{d} V_m(\eta^{(m)}),$$

where $\eta = (\eta^{(m)})_{m=1}^d$. An important admissible potential is the quadratic potential $V(\eta) = \frac{\beta}{2}|\eta|^2$ for a constant $\beta > 0$. Let $f_m \in C_0^2(\mathbb{R}^d)$ for $m = 1, \ldots, d$. We define the Hamiltonian $H_{D_N} \colon (\mathbb{R}^d)^{D_N} \to \mathbb{R}$ by

$$H_{D_N}(x) = \frac{1}{2} \sum_{\substack{k,l \in D_N \\ |k-l|=1}} V(x_k - x_l) + \sum_{\substack{k \in D_N, l \in \mathbb{Z}^d \setminus D_N \\ |k-l|=1}} V(x_k - x_l) \tag{3.18}$$

with the convention

$$x_k = \left(N^{d+1} \int_{B(N^{-1}k, N^{-1})} f_m(\theta) d\theta \right)_{m=1}^d, \quad k \in \mathbb{Z}^d \setminus D_N.$$

The gradient of $H_{D_N}(x)$ with respect to the vector x_k is

$$\nabla_{x_k} H_{D_N}(x) = \sum_{l \in \mathbb{Z}^d, |k-l|=1} \left\{ V_m' \left(x_k^{(m)} - x_l^{(m)} \right) \right\}_{m=1}^d.$$

Corresponding to this, we introduce a random time evolution of the particles by means of the system of stochastic differential equations

$$\begin{cases} dX_k(t) = - \sum_{l \in \mathbb{Z}^d, |k-l|=1} \left\{ V_m' \left(X_k^{(m)}(t) - X_l^{(m)}(t) \right) \right\}_{m=1}^d dt + \sqrt{2} dW_k(t), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k \in D_N, \\ X_k = \left(N^{d+1} \int_{B(N^{-1}k, N^{-1})} f_m(\theta) d\theta \right)_{m=1}^d, \quad k \in \mathbb{Z}^d \setminus D_N, \end{cases} \tag{3.19}$$

where W_k, $k \in D_N$ are independent d-dimensional Brownian motions. Notice that (3.19) consists of d independent one-dimensional particles systems. As before, we define the random function $h^N \colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ by

$$h^N(t, \theta) = \sum_{k \in \mathbb{Z}^d} \frac{X_k(N^2 t)}{N} 1_{B(N^{-1}k, N^{-1})}(\theta).$$

Let $h^{N,(m)}$ be the m-th component of h^N. If theorem 2.5 applies, then $h^{N,(m)}$ converges to the unique weak solution $h^{(m)}$ of

$$\begin{cases} \partial_t h(t,\theta) = \operatorname{div}[\nabla\sigma_m(\nabla h(t,\theta))] \\ \qquad\qquad = \sum_{n=1}^{d} \partial_{\theta_n}\{\partial_{u_n}\sigma_m(\nabla h(t,\theta))\}, & t \in (0,T),\ \theta \in D, \\ h(t,\theta) = f_m(\theta), & t \in (0,T),\ \theta \in \partial D, \\ h(0,\theta) = h_0^{(m)}(\theta), & \theta \in D, \end{cases} \tag{3.20}$$

where σ_m is the surface tension associated to the potential V_m.

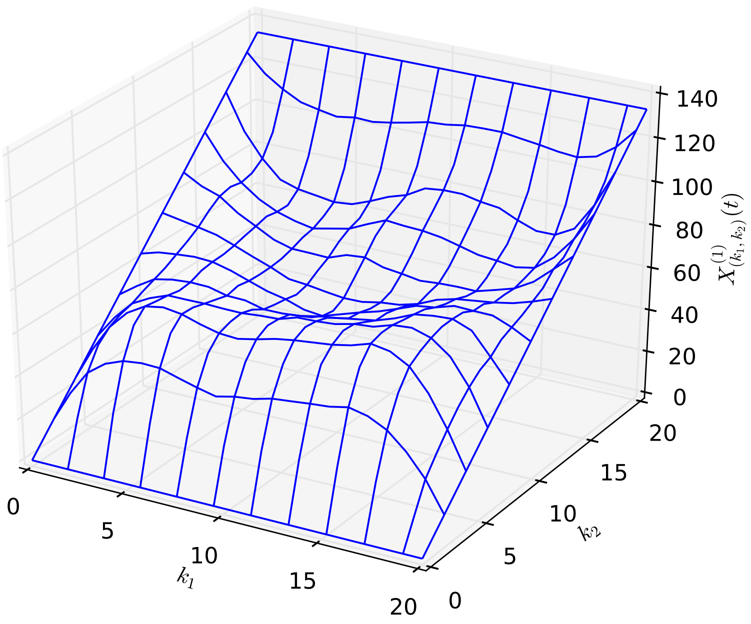

Figure 3.1: The first components of an example particle configuration.

We choose $D = (0,1)^d$ and $f_m \in C_0^2(\mathbb{R}^d)$ such that $f_m(\theta) = \theta_m/\rho$ on D for $m = 1,\dots,d$. Then we have $D_N = \{3,\dots,N-3\}^d$, the particle system (3.19) becomes

$$\begin{cases} dX_k(t) = -\sum_{l\in\mathbb{Z}^d, |k-l|=1} \left\{V'_m\left(X_k^{(m)}(t) - X_l^{(m)}(t)\right)\right\}_{m=1}^{d} dt + \sqrt{2}dW_i(t), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k \in \{3,\dots,N-3\}^d, \\ X_k = \left(N^{d+1}\int_{B(N^{-1}k,N^{-1})} f_m(\theta)d\theta\right)_{m=1}^{d}, \quad k \in \mathbb{Z}^d \setminus \{3,\dots,N-3\}^d, \end{cases} \tag{3.21}$$

where $X_k = k/\rho$ for every $k \in \{1, \dots, N-1\}^d \setminus \{3, \dots, N-3\}^d$, and (3.20) becomes

$$\begin{cases} \partial_t h(t,\theta) = \operatorname{div}[\nabla\sigma_m(\nabla h(t,\theta))], & t \in (0,T),\ \theta \in (0,1)^d, \\ h(t,\theta) = \dfrac{\theta_m}{\rho}, & t \in (0,T),\ \theta \in \partial(0,1)^d, \\ h(0,\theta) = h_0^{(m)}(\theta), & \theta \in (0,1)^d. \end{cases} \tag{3.22}$$

We assume $h^{(m)}$ to be even a classical solution.

Next, we will generalise the assumptions of theorem 3.1. In (3.1) we compare the height $h^N(0,\cdot)$ with the inverse distribution function $(F^N)^{-1}$. We know from part one of remark 3.2 that $(F^N)^{-1} = \tilde{h}^N(0,\cdot+(2N)^{-1})$ on $(0,1)$, where $\tilde{h}^N(0,\cdot)$ is obtained from the height $h^N(0,\cdot)$ by enumerating the particles $X_k(0)$, $k=1,\dots,N$ in a nondecreasing order. More precisely,

$$\tilde{h}^N(0,\theta) = \sum_{k=-\infty}^{\infty} \frac{\tilde{X}_k(0)}{N} 1_{B(N^{-1}k,N^{-1})}(\theta),$$

where $\tilde{X}_k(0)$ is the k-th largest element of $\{X_l(0) : l=1,\dots,N\}$ if $k=1,\dots,N$ and $\tilde{X}_k(0) = X_k(0)$ otherwise. Notice that the measurability of $\tilde{X}_k(0)$ for $k=1,\dots,N$ follows from the fact that $\tilde{X}_k(0) \geq a$ is equivalent to $\sum_{l=1}^N 1_{[a,\infty)}(X_l(0)) \geq k$. It can be shown that the height $\tilde{h}^N(0,\cdot)$ is optimal in the sense that the energy

$$\sum_{k=0}^{N} V(\tilde{X}_k(0) - \tilde{X}_{k+1}(0)),$$

cf. (2.1), is minimal among all enumerations of the particles $X_k(0)$, $k=1,\dots,N$. Adapting this observation to the d-dimensional case, we set $\tilde{X}_k(0) = X_{\sigma(k)}(0)$ if $k \in \{1,\dots,N\}^d$ and $\tilde{X}_k(0) = X_k(0)$ otherwise, where $\sigma = \sigma((X_k(0)_{k\in\{1,\dots,N\}^d})$ is a random permutation of $\{1,\dots,N\}^d$ minimising the energy

$$\frac{1}{2} \sum_{\substack{k,l\in\{1,\dots,N\}^d \\ |k-l|=1}} V(\tilde{X}_k(0) - \tilde{X}_l(0)) + \sum_{\substack{k\in\{1,\dots,N\}^d \\ l\in\mathbb{Z}^d\setminus\{1,\dots,N\}^d \\ |k-l|=1}} V(\tilde{X}_k(0) - \tilde{X}_l(0)), \tag{3.23}$$

cf. (3.18). Notice that $\tilde{X}_k(0)$ is measurable for $k \in \{1,\dots,N\}^d$ since it can be proven that the map $S_{N^d} \times \mathbb{R}^{N^d} \to \mathbb{R}^{N^d}$, $(\pi, x) \mapsto (x_{\pi(k)})_{k\in\{1,\dots,N\}^d}$ is measurable, where the symmetric group S_{N^d} of the set $\{1,\dots,N\}^d$ is equipped with the discrete σ-algebra, $\mathbb{R}^{N^d}$ with the Borel σ-algebra and $S_{N^d} \times \mathbb{R}^{N^d}$ with the associated product σ-algebra. We define the height $T^N \colon \mathbb{R}^d \to \mathbb{R}^d$ by

$$T^N(\theta) = \sum_{k\in\mathbb{Z}^d} \frac{\tilde{X}_k(0)}{N} 1_{B(N^{-1}k,N^{-1})}(\theta). \tag{3.24}$$

Let $L^2((0,1)^d, \mathbb{R}^d)$ be the space of square-integrable functions from $(0,1)^d$ to $\mathbb{R}^d$. Then (3.1) generalises to

$$\lim_{N\to\infty} \mathbb{E}\left[\int_{(0,1)^d} \left|h^N(0,\theta) - T^N(\theta)\right|^2 d\theta\right] = 0 \tag{3.25}$$

and (3.2) to

$$\lim_{N\to\infty} \mathbb{E}\left[\int_{(0,1)^d} |T^N(\theta) - T(\theta)|^2 d\theta\right] = 0, \tag{3.26}$$

where $T \in L^2((0,1)^d, \mathbb{R}^d)$ and $\lambda \circ T^{-1} = m_0(x)dx$ with support in $[0, \rho^{-1}]^d$.

We set $h_0 = T$. Combining (3.25) and (3.26) with Jensen's inequality, we get

$$\lim_{N\to\infty} \mathbb{E}\left[\int_{(0,1)^d} \left|h^N(0,\theta) - h_0(\theta)\right|^2 d\theta\right] = 0.$$

So theorem 2.5 applies, and it follows

$$\lim_{N\to\infty} \mathbb{E}\left[\int_{(0,1)^d} \left|h^N(t,\theta) - h(t,\theta)\right|^2 d\theta\right] = 0$$

for every $t \in (0,T]$, where the k-th component $h^{(k)}$ of h solves (3.22) with initial condition $h_0^{(k)}$.

Let us consider the empirical measure

$$\mu_t^N(dx) = \frac{1}{N^d} \sum_{k\in\{1,\dots,N\}^d} \delta_{N^{-1}X_k(N^2t)}(dx),$$

where $X(t)$, $t \in [0, N^2T]$ solves (3.21). It is straight forward to generalise lemma 3.4 and 3.5. So μ_t^N converges in probability to $\lambda \circ h(t,\cdot)^{-1}$ in the space of probability measures on $\mathbb{R}^d$ equipped with the topology of weak convergence of measures. We were not able generalise lemma 3.6 to prove the absolutely continuity of the limit measure $\lambda \circ h(t,\cdot)^{-1}$. For this reason, we assume $h(t,\cdot)\colon (0,1)^d \to (0,\rho^{-1})^d$ to be a C^1-diffeomorphism for every $t \in [0,T]$. Then a calculation analogous to (3.15) shows that $\lambda \circ h(t,\cdot)^{-1}$ has density

$$m(t,x) = |\det[D_x h(t,\cdot)^{-1}(x)]| = \frac{1}{|\det[D_\theta h(t, h(t,\cdot)^{-1}(x))]|}. \tag{3.27}$$

We define the trajectory field $T\colon (0,T) \times (0,\rho^{-1})^d \to (0,\rho^{-1})^d$ by

$$T(t,x) = h(t, h(0,\cdot)^{-1}(x)),$$

and its velocity field $v\colon (0,T) \times (0,\rho^{-1})^d \to \mathbb{R}^d$ by

$$v(t, T(t,x)) = \partial_t T(t,x). \tag{3.28}$$

Lemma 3.10. *The function* m *from* (3.27) *is a classical solution of the linear transport equation*

$$\begin{cases} \partial_t m(t,x) = -\operatorname{div}[m(t,x)v(t,x)], \\ \qquad\qquad = -\sum_{n=1}^{d} \partial_{x_n}\{m(t,x)v(t,x)\}_n, & t \in (0,T),\ x \in (0,\rho^{-1})^d, \\ m(0,x) = m_0(x), & x \in [0,\rho^{-1}]^d, \end{cases} \tag{3.29}$$

where v *is given by* (3.28)*.*

Proof. Notice that $\lambda \circ h(0,\cdot)^{-1} = (\lambda \circ h(t,\cdot)^{-1}) \circ T(t,\cdot)$. Then a change of variables shows for every $f \in C((0,\rho^{-1})^d)$

$$\begin{aligned} \int f(x)m_0(x)dx &= \int f(T(t,\cdot)^{-1}(y))m(t,y)dy \\ &= \int f(x)m(t,T(t,x))|\det[D_xT(t,x)]|dx, \end{aligned}$$

and from this it follows

$$m_0(x) = m(t,T(t,x))|\det[D_xT(t,x)]|. \tag{3.30}$$

Differentiating (3.30) with respect to t, we get

$$\begin{aligned} 0 &= (\partial_t m)(t,T(t,x))|\det[D_xT(t,x)]| \\ &\quad + (D_x m)(t,T(t,x))\partial_t T(t,x)|\det[D_xT(t,x)]| \\ &\quad + m(t,T(t,x))\frac{\det[D_xT(t,x)]^2}{|\det[D_xT(t,x)]|}\operatorname{tr}[D_xT(t,x)^{-1}\partial_t D_xT(t,x)], \end{aligned}$$

or equivalently,

$$\begin{aligned} &(\partial_t m)(t,T(t,x)) \\ &= -(D_x m)(t,T(t,x))\partial_t T(t,x) - m(t,T(t,x))\operatorname{tr}[D_xT(t,x)^{-1}\partial_t D_xT(t,x)]. \end{aligned} \tag{3.31}$$

Then we choose $x = T(t,\cdot)^{-1}(y)$ in (3.31) and obtain

$$\begin{aligned} \partial_t m(t,y) &= -D_y m(t,y)(\partial_t T)(t,T(t,\cdot)^{-1}(y)) \\ &\quad - m(t,y)\operatorname{tr}[D_xT(t,T(t,\cdot)^{-1}(y))^{-1}(\partial_t D_xT)(t,T(t,\cdot)^{-1}(y))] \\ &= -D_y m(t,y)v(t,y) - m(t,y)\operatorname{tr}[D_y v(t,y)], \end{aligned}$$

where the last line is equal to $-\operatorname{div}[m(t,x)v(t,x)]$. Since the initial condition in (3.29) holds by definition, this concludes the proof. □

Putting everything together, we have proven

Theorem 3.11. *Let T^N be given by (3.24), $T \in L^2((0,1)^d, \mathbb{R}^d)$ and $\lambda \circ T^{-1} = m_0(x)dx$ with support in $[0, \rho^{-1}]^d$. We assume*

$$\lim_{N \to \infty} \mathbb{E} \left[\int_{(0,1)^d} \left| h^N(0,\theta) - T^N(\theta) \right|^2 d\theta \right] = 0,$$

and

$$\lim_{N \to \infty} \mathbb{E} \left[\int_{(0,1)^d} |T^N(\theta) - T(\theta)|^2 d\theta \right] = 0,$$

Then, for every $t \in (0, T]$, μ_t^N converges as $N \to \infty$ in probability to $m(t,x)dx$ in the space of probability measures on $\mathbb{R}^d$ equipped with the topology of weak convergence of measures. The function m is a classical solution of (3.29).

Remark 3.12.

1. *Notice that we proved theorem 3.11 under the assumption that the solution $h^{(m)}$ of (3.20) is not only a weak solution, but a classical solution, and that $h(t,\cdot)\colon (0,1)^d \to (0,\rho^{-1})^d$ is a C^1-diffeomorphism for every $t \in [0,T]$. Maybe a weak formulation of the hydrodynamic equation (3.29) can be proven by using the weak solution $h^{(m)}$.*

2. *Two open problems remain: 1) The uniqueness of the solution m of the hydrodynamic equation (3.3) needs to be proven. 2) The conjecture that the particle model (3.21) and the physical model, which is obtained by enumerating the particles at all times such that the energy (3.23) is minimal, have the same hydrodynamic equation for the particle density needs to be proven.*

4 Equilibrium fluctuations

4.1 Statement of the result

In this chapter we will characterise the equilibrium fluctuations of the particle density in the case of one dimension and quadratic potential. We consider the fluctuation field $(F_t^N)_{t\in[0,T]}$ given by

$$F_t^N(f) = \sqrt{N}\left(\int_{-\infty}^{\infty} f d\mu_t^N - \mathbb{E}\left[\int_{-\infty}^{\infty} f d\mu_t^N\right]\right)$$
$$= \frac{1}{\sqrt{N}}\sum_{k=1}^{N}\left(f\left(\frac{X_k(N^2t)}{N}\right) - \mathbb{E}\left[f\left(\frac{X_k(N^2t)}{N}\right)\right]\right)$$

for suitable test functions f. Here μ_t^N is the empirical measure (2.4) and $X(t)$, $t \in [0, N^2T]$ solves the particle system (2.5). The initial distribution is Gaussian with mean $L^{-1}b$ and covariance matrix $(\beta L)^{-1}$, where L is the discrete Laplacian of size N and $b \in \mathbb{R}^N$ is the vector with entries $b_N = N/\rho$ and $b_k = 0$ otherwise. From section 2.1.2 we know that this initial distribution is stationary for (2.5).

As test functions we choose $f \in C^\infty([0, \rho^{-1}])$ satisfying $f^{(2k+1)}(0) = f^{(2k+1)}(\rho^{-1}) = 0$ for every non-negative integer k, and these functions are extended to the real line such that they are even and $2\rho^{-1}$-periodic. Let S be the space of test functions equipped with the topology generated by the seminorms

$$|f|_k = \sum_{l=0}^{k} \|f^{(l)}\|_\infty \tag{4.1}$$

for every non-negative integer k. Since S is a closed subspace of the nuclear Fréchet space of $2\rho^{-1}$-periodic functions $f \in C^\infty(\mathbb{R})$ equipped with the topology generated the seminorms (4.1), S is a nuclear Fréchet space and there exists a family of seminorms induced by inner products that is equivalent to (4.1), cf. [AS99, chapter II and III] and [HKPS93, appendix A.5]. Further, let S' be the topological dual space of S equipped with the strong topology, i. e. the topology generated by the seminorms

$$\|F\|_B = \sup_{f\in B} |F(f)|$$

for every bounded subset $B \subset S$, and $C([0, T], S')$ the space of continuous functions from $[0, T]$ to S' equipped with the topology that is generated by the seminorms

$$\left|\left|\left|(F_t)_{t\in[0,T]}\right|\right|\right|_B = \sup_{t\in[0,T]} \|F_t\|_B$$

for every bounded subset $B \subset S$.

Notice that a subset $B \subset S$ is bounded if and only if $\sup_{f \in B} |f|_k$ is bounded for every non-negative integer k, and that $\{F_t : t \in [0,T]\}$ is equicontinuous for every $(F_t)_{t\in[0,T]} \in C([0,T], S')$ by the uniform boundedness principle.

Remark 4.1. *Let us make some remarks on the space S:*

1. *The choice of S will be crucial in the proof of theorem 4.4. More precisely, in* (4.8) *we will need the invariance of S under the Neumann Laplacian, in the proof of lemma 4.11 we will need the boundary condition and in the proof of proposition 4.7 we will need the invariance of S under the semigroup generated by the Neumann Laplacian.*
2. *The Borel σ-algebra on S' and the cylindrical σ-algebra on S' coincide, cf. [Bad70, Exposé No 8].*

Lemma 4.2. *The fluctuation field $(F_t^N)_{t\in[0,T]}$ is supported in $C([0,T], S')$.*

Proof. Since $F_t^N(f)$ is linear in f and $|F_t^N(f)| \leq C|f|_0$, it holds $F_t^N \in S'$. Let $B \subset S$ be bounded. Using the stationarity of the initial distribution and Taylor's expansion, we get

$$\begin{aligned}\sup_{f\in B} |F_s^N(f) - F_t^N(f)| &\leq \sup_{f\in B} \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \left| f\left(\frac{X_k(N^2 s)}{N}\right) - f\left(\frac{X_k(N^2 t)}{N}\right)\right| \\ &\leq \frac{C}{\sqrt{N}} \sum_{k=1}^{N} \left| \frac{X_k(N^2 s) - X_k(N^2 t)}{N} \right|\end{aligned}$$

From this it follows that $\|F_s^N - F_t^N\|_B$ vanishes as $s \to t$. □

Let $f \in L^2([0,\rho^{-1}])$. Since the functions $h_0 = \sqrt{\rho}$ and $h_z(u) = \sqrt{2\rho}\cos(\rho\pi z u)$ for positive integers z form an orthonormal basis of $L^2([0,\rho^{-1}])$ with the usual inner product $\langle\cdot,\cdot\rangle_{L^2}$, we have

$$f = \sum_{z=0}^{\infty} \langle f, h_z\rangle_{L^2} h_z.$$

Let Δ be the Neumann Laplacian on $L^2([0,\rho^{-1}])$, $(T_t)_{t\in[0,\infty)}$ the semigroup generated by $\mathfrak{A} = \frac{\beta}{\rho^2}\Delta$ on $L^2([0,\rho^{-1}])$ and $\mathfrak{B} = \sqrt{2\rho}\nabla$. Solving the heat equation in $[0,\rho^{-1}]$ with Neumann boundary condition and initial condition f, we get for every $t \geq 0$

$$T_t f = \sum_{z=0}^{\infty} \langle f, h_z\rangle_{L^2} e^{-(\sqrt{\beta}\pi z)^2 t} h_z,$$

cf. [Str07, chapter 4]. We can regard S as a subset of $L^2([0,\rho^{-1}])$. Then we define $T_t f$ for $f \in S$ by applying T_t to the restriction of f to $[0,\rho^{-1}]$ and extending the resulting function to the real line such that it is even and $2\rho^{-1}$-periodic. We will use the notation $f_t = T_t f$.

Lemma 4.3. *Let $f \in S$ and $t > 0$. Then it holds $f_t \in S$.*

Proof. We will first prove that the restriction of f_t to $[0, \rho^{-1}]$ is infinitely differentiable and can be differentiated under the sum sign, i. e.

$$(f_t)^{(k)} = \sum_{z=0}^{\infty} \langle f, h_z \rangle_{L^2} e^{-(\sqrt{\beta}\pi z)^2 t} h_z^{(k)} \tag{4.2}$$

for every positive integer k. It suffices to show the uniform convergence of the right-hand side of (4.2) on $[0, \rho^{-1}]$ for every non-negative integer k. We have

$$\left| \langle f, h_z \rangle_{L^2} e^{-(\sqrt{\beta}\pi z)^2 t} h_z^{(k)}(u) \right| \leq C |\langle f, h_z \rangle_{L^2} (\rho \pi z)^k|$$

for every $u \in [0, \rho^{-1}]$ and

$$2 \sum_{z=0}^{\infty} |\langle f, h_z \rangle_{L^2} (\rho \pi z)^k| \leq \sum_{z=0}^{\infty} \langle f, h_z \rangle_{L^2}^2 (1 + (\rho \pi z)^2)^{k+1} + \sum_{z=0}^{\infty} (1 + (\rho \pi z)^2)^{-1}. \tag{4.3}$$

Since h_z is an eigenfunction of $I - \Delta$ with eigenvalue $1 + (\rho \pi z)^2$ and

$$\begin{aligned} \sum_{z=0}^{\infty} \langle f, h_z \rangle_{L^2} \langle f, (I - \Delta)^{k+1} h_z \rangle_{L^2} &= \langle f, (I - \Delta)^{k+1} f \rangle_{L^2} \\ &\leq \|f\|_{L^2} \|(I - \Delta)^{k+1} f\|_{L^2}, \end{aligned}$$

we see that the right-hand side of (4.3) is finite. Applying Weierstrass' M-test, we obtain the uniform convergence of the right-hand side of (4.2) on $[0, \rho^{-1}]$.

Using (4.2), we see that $(f_t)^{(2k+1)}(0) = (f_t)^{(2k+1)}(\rho^{-1}) = 0$ for every non-negative integer k. This concludes $f_t \in S$. □

The main result of this chapter is

Theorem 4.4. *The fluctuation field $(F_t^N)_{t \in [0,T]}$ converges as $N \to \infty$ in distribution to the unique solution $(F_t)_{t \in [0,T]}$ of the martingale problem in proposition 4.7 below such that F_0 is a Gaussian random field with mean 0 and covariance*

$$\mathbb{E}[F_0(f) F_0(g)] = \frac{\rho^2}{\beta} \int_0^{\rho^{-1}} \left(f(u) - \int_0^{\rho^{-1}} f(v) \rho dv \right) g(u) \rho du \tag{4.4}$$

for every $f, g \in S$. Moreover, the space-time covariance of $(F_t)_{t \in [0,T]}$ is given by

$$\mathbb{E}[F_s(f) F_t(g)] = \frac{\rho^2}{\beta} \int_0^{\rho^{-1}} \left(f(u) - \int_0^{\rho^{-1}} f(v) \rho dv \right) g_{t-s}(u) \rho du \tag{4.5}$$

for every $0 \leq s \leq t \leq T$ and $f, g \in S$.

Before we state proposition 4.7, we state two lemmas that are needed in its proof.

Lemma 4.5. *Let $f \in S$, $t \geq 0$ and $\epsilon > 0$. Then it holds*

$$f_{t+\epsilon} = f_t + \epsilon \mathfrak{A} f_t + h(t,\epsilon), \tag{4.6}$$

where $h(t,\epsilon) \in S$ and $\epsilon^{-1}h(t,\epsilon)$ vanishes as $\epsilon \to 0$ uniformly in t. In particular, the map

$$[0,\infty) \to S,\ t \mapsto f_t$$

is continuous.

Proof. Using the Taylor expansion

$$e^{-(\sqrt{\beta}\pi z)^2(t+\epsilon)} = e^{-(\sqrt{\beta}\pi z)^2 t} - (\sqrt{\beta}\pi z)^2 e^{-(\sqrt{\beta}\pi z)^2 t}\epsilon + \frac{(\sqrt{\beta}\pi z)^4 e^{-(\sqrt{\beta}\pi z)^2 \xi}}{2}\epsilon^2$$

with some $\xi \in [t, t+\epsilon]$, we obtain that (4.6) holds and that

$$h(t,\epsilon) = \sum_{z=0}^{\infty} \langle f, h_z \rangle_{L^2} \frac{(\sqrt{\beta}\pi z)^4 e^{-(\sqrt{\beta}\pi z)^2 \xi}}{2}\epsilon^2 h_z.$$

As in the proof of lemma 4.3, it can be shown that $h(t,\epsilon) \in S$ and $|\epsilon^{-1}h(t,\epsilon)|_k$ vanishes as $\epsilon \to 0$ uniformly in t for every non-negative integer k. This concludes the proof. □

Lemma 4.6. *Let $(F_t)_{t\in[0,T]} \in C([0,T], S')$ and $f \in S$. Then the map*

$$[0,T]^2 \to \mathbb{R},\ (s,t) \mapsto F_s(\mathfrak{A} f_t)$$

is continuous.

Proof. We have

$$|F_s(\mathfrak{A} f_t) - F_u(\mathfrak{A} f_v)| \leq |F_s(\mathfrak{A} f_t) - F_u(\mathfrak{A} f_t)| + |F_u(\mathfrak{A} f_t) - F_u(\mathfrak{A} f_v)|, \tag{4.7}$$

and we will show that each term on the right-hand side of (4.7) vanishes as $(v,u) \to (s,t)$. The first term vanishes as $u \to s$ since $(F_t)_{t\in[0,T]} \in C([0,T], S')$. From lemma 4.5 we obtain that $\mathfrak{A} f_v$ is continuous in v. Together with the equicontinuity of $\{F_u : u \in [0,T]\}$ it follows that the second term on the right-hand side of (4.7) vanishes as $(u,v) \to (s,t)$. □

Proposition 4.7. *Let $(F_t)_{t\in[0,T]}$ a random element with values in $C([0,T], S')$ such that for every $f \in S$*

$$\begin{aligned} M_t(f) &= F_t(f) - F_0(f) - \int_0^t F_s(\mathfrak{A} f) ds, \\ V_t(f) &= M_t(f)^2 - \|\mathfrak{B} f\|_{L^2}^2 t \end{aligned} \tag{4.8}$$

define martingales with respect to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ given by $\mathcal{F}_t = \sigma(F_s(f) : s \in [0,t],\ f \in S)$. Then, for every $0 \leq s < t \leq T$ and $f \in S$, the distribution of $F_t(f)$ conditioned on $\mathcal{F}_s$ is Gaussian with mean $F_s(f_{t-s})$ and variance $\int_s^t \|\mathfrak{B} f_{t-r}\|_{L^2}^2 dr$. In particular, the distribution of $(F_t)_{t\in[0,T]}$ is uniquely determined by the distribution of F_0.

Proof. Levy's characterisation theorem shows that $(\|\mathfrak{B}f\|_{L^2}^{-1}M_t(f))_{t\in[0,T]}$ is a Brownian motion with respect to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$. In particular, for every $s \in [0,T]$, the stochastic process $(Y_t^s(f))_{t\in[s,T]}$ defined by

$$Y_t^s(f) = \exp\left(i\left(F_t(f) - F_s(f) - \int_s^t F_r(\mathfrak{A}f)dr\right) + \frac{1}{2}\|\mathfrak{B}f\|_{L^2}^2(t-s)\right)$$

is a complex martingale with respect to the filtration $(\mathcal{F}_t)_{t\in[s,T]}$.

Next, we will show that, for every $s \in [0,T]$, the stochastic process $(Z_t^s(f))_{t\in[0,s]}$ defined by

$$Z_t^s(f) = \exp\left(iF_t(f_{s-t}) + \frac{1}{2}\int_0^t \|\mathfrak{B}f_{s-r}\|_{L^2}^2 dr\right)$$

is a complex martingale with respect to the filtration $(\mathcal{F}_t)_{t\in[0,s]}$. It is obvious that $Z_t^s(f)$ is $\mathcal{F}_t$-measurable and integrable. To prove the martingale property, we fix $0 \le t_1 < t_2 \le s$ and define the partition $s_k = t_1 + \frac{k}{N}(t_2 - t_1)$, $k = 0, \ldots, N$ of $[t_1, t_2]$. Then it holds

$$\begin{aligned} &\prod_{k=0}^{N-1} Y_{s_{k+1}}^{s_k}(f_{s-s_k}) \\ &= \exp\left(i\sum_{k=0}^{N-1}\left\{F_{s_{k+1}}(f_{s-s_k}) - F_{s_k}(f_{s-s_k}) - \int_{s_k}^{s_{k+1}} F_r(\mathfrak{A}f_{s-s_k})dr\right\}\right. \\ &\quad \left. + \frac{1}{2}\sum_{k=0}^{N-1}\|\mathfrak{B}f_{s-s_k}\|_{L^2}^2(s_{k+1}-s_k)\right). \end{aligned} \tag{4.9}$$

Applying lemma 4.5, the first sum on the right-hand side of (4.9) becomes

$$\begin{aligned} \sum_{k=0}^{N-1}\Bigg\{&F_{s_{k+1}}(f_{s-s_{k+1}}) - F_{s_k}(f_{s-s_k}) \\ &+ (s_{k+1}-s_k)F_{s_{k+1}}(\mathfrak{A}f_{s-s_{k+1}}) - \int_{s_k}^{s_{k+1}} F_r(\mathfrak{A}f_{s-s_k})dr \\ &+ F_{s_{k+1}}(h(s-s_{k+1}, s_{k+1}-s_k))\Bigg\}, \end{aligned}$$

which converges as $N \to \infty$ to $F_{t_2}(f_{s-t_2}) - F_{t_1}(f_{s-t_1})$ by the mean value theorem, lemma 4.6 and the equicontinuity of $\{F_t : t \in [t_1, t_2]\}$. The second sum on the right-hand side of (4.9) converges to $\int_{t_1}^{t_2}\|\mathfrak{B}f_{s-r}\|_{L^2}^2 dr$ since lemma 4.5 implies that $\mathfrak{B}f_{s-r}$ is continuous in r and uniform convergence on $[0,\rho^{-1}]$ implies convergence in the space $L^2([0,\rho^{-1}])$. So (4.9) converges as $N \to \infty$ to

$$\exp\left(i(F_{t_2}(f_{s-t_2}) - F_{t_1}(f_{s-t_1})) + \frac{1}{2}\int_{t_1}^{t_2}\|\mathfrak{B}f_{s-r}\|_{L^2}^2 dr\right) = \frac{Z_{t_2}^s(f)}{Z_{t_1}^s(f)},$$

and the dominated convergence theorem implies for every bounded and $\mathcal{F}_{t_1}$-measurable function g

$$\lim_{N\to\infty} \mathbb{E}\left[\prod_{k=0}^{N-1} Y_{s_{k+1}}^{s_k}(f_{s-s_k})g\right] = \mathbb{E}\left[\frac{Z_{t_2}^s(f)}{Z_{t_1}^s(f)}g\right]. \tag{4.10}$$

Conditioning on $\mathcal{F}_{s_{N-1}}$ and using that $(Y_t^s(f))_{t\in[s,T]}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\in[s,T]}$, the expectation on the left-hand side of (4.10) becomes

$$\mathbb{E}\left[\prod_{k=0}^{N-2} Y_{s_{k+1}}^{s_k}(f_{s-s_k})E[Y_{s_N}^{s_{N-1}}(f_{s-s_{N-1}})|\mathcal{F}_{s_{N-1}}]g\right] = \mathbb{E}\left[\prod_{k=0}^{N-2} Y_{s_{k+1}}^{s_k}(f_{s-s_k})g\right].$$

Repeating this argument by successively conditioning on $\mathcal{F}_{s_{N-2}}, \dots, \mathcal{F}_{s_1}$, we get

$$\mathbb{E}\left[\frac{Z_{t_2}^s(f)}{Z_{t_1}^s(f)}g\right] = \mathbb{E}[g],$$

and replacing g by $Z_{t_1}^s(f)g$, we conclude that $(Z_t^s(f))_{t\in[0,s]}$ satisfies the martingale property.

Next, we will identify the distribution of $F_t(f)$ conditioned on $\mathcal{F}_s$ by means of the characteristic function. Using that $(Z_r^t(f))_{r\in[0,t]}$ is a martingale with respect to the filtration $(\mathcal{F}_r)_{r\in[0,t]}$ and

$$Z_t^t(f) = \exp\left(iF_t(f) + \frac{1}{2}\int_0^t \|\mathfrak{B}f_{t-r}\|_{L^2}^2 dr\right),$$

we obtain

$$\begin{aligned}
&\mathbb{E}[\exp(iF_t(f))|\mathcal{F}_s] \\
&= \mathbb{E}\left[\exp\left(iF_t(f) + \frac{1}{2}\int_0^t \|\mathfrak{B}f_{t-r}\|_{L^2}^2 dr\right)\middle|\mathcal{F}_s\right]\exp\left(-\frac{1}{2}\int_0^t \|\mathfrak{B}f_{t-r}\|_{L^2}^2 dr\right) \\
&= \exp\left(iF_s(f_{t-s}) - \frac{1}{2}\int_s^t \|\mathfrak{B}f_{t-r}\|_{L^2}^2 dr\right).
\end{aligned}$$

Replacing f by θf, where $\theta \in \mathbb{R}$, this implies that the distribution of $F_t(f)$ conditioned on $\mathcal{F}_s$ is Gaussian with mean $F_s(f_{t-s})$ and variance $\int_s^t \|\mathfrak{B}f_{t-r}\|_{L^2}^2 dr$.

It remains to show that the distribution Q of $(F_t)_{t\in[0,T]}$ is uniquely determined by the distribution Q_0 of F_0. Part two of remark 4.1 implies that Q is uniquely determined by the probabilities $\mathbb{P}[F_{t_k}(f_k) \in A_k : k = 1, \dots, m]$, where m is a positive integer, $0 \le t_1 \le \dots \le t_m \le T$, $f_k \in S$ and $A_k \subset \mathbb{R}$ are Borel sets, and these probabilities are uniquely determined by Q_0 and the conditional probabilities $\mathbb{P}[F_{t_k}(f_k) \in A_k|\mathcal{F}_{t_{k-1}}]$. This concludes the proof. □

In the proof of proposition 4.7 we saw that $(\|\mathfrak{B}f\|_{L^2}^{-1}M_t(f))_{t\in[0,T]}$ is a Brownian motion. From this it follows that $(F_t)_{t\in[0,T]}$ satisfies

$$F_t(f) = F_0(f) + \int_0^t F_s(\mathfrak{A}f)ds + \sqrt{2\rho}W_t(f) \tag{4.11}$$

for every $f \in S$, where $(W_t)_{t\in[0,T]}$ is the cylindrical Brownian motion with space-time covariance

$$E[W_s(f)W_t(g)] = \min(s,t)\left\langle f', g'\right\rangle_{L^2}$$

for every $s, t \in [0,T]$ and $f, g \in S$. We can write (4.11) formally as

$$dF_t = \frac{\beta}{\rho^2}\Delta F_t dt + \sqrt{2\rho}dW_t,$$

which is the linearisation of the hydrodynamic equation (3.4) around ρ plus randomness. For this reason, $(F_t)_{t\in[0,T]}$ is called infinite-dimensional Ornstein-Uhlenbeck process.

To prove theorem 4.4, we will follow the approach in [KL13, chapter 11] (see also [HS78]). Let Q^N be the distribution of $(F_t^N)_{t\in[0,T]}$. Then Q^N is an element in the space of probability measures on $C([0,T], S')$ equipped with the topology of weak convergence of measures. The proof of theorem 4.4 is divided into checking two statements:

1. Let Q be an accumulation point of $\{Q^N\}_{N\in\mathbb{N}}$. Then Q satisfies the martingale problem in proposition 4.7 such that Q restricted to $\mathcal{F}_0$ is the distribution of a Gaussian random field with mean 0 and covariance (4.4).
2. The family $\{Q^N\}_{N\in\mathbb{N}}$ is tight.

The first statement implies uniqueness of the accumulation points Q, whereas the second statement and the Prokhorov-type theorem [Mit83, theorem 5.1] imply relative compactness of $\{Q^N\}_{N\in\mathbb{N}}$. Together it follows the convergence of Q^N along the whole sequence.

4.2 Proof of theorem 4.4

4.2.1 Martingale problem

We will start by showing that $F_t^N(f)$ is equal to a Gaussian random variable plus an error, and $(F_t^N)_{t\in[0,T]}$ satisfies an approximation of (4.8). This is the content of the following two lemmas.

Lemma 4.8. *Let $f \in S$ and $t \in (0,T]$. Then it holds*

$$F_t^N(f) = G_t^N(f) + R_t^N(f) \tag{4.12}$$

with

$$G_t^N(f) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} f'\left(\frac{m_k}{N}\right) \frac{X_k(N^2t) - m_k}{N},$$

$$R_t^N(f) = \frac{1}{2\sqrt{N}} \sum_{k=1}^{N} \left\{ f''(\xi_k(N^2t)) \left(\frac{X_k(N^2t) - m_k}{N}\right)^2 - \mathbb{E}\left[f''(\xi_k(N^2t)) \left(\frac{X_k(N^2t) - m_k}{N}\right)^2 \right] \right\},$$

where $m_k = \mathbb{E}[X_k(0)]$ *and* $\xi_k(N^2t)$ *takes values between* $N^{-1}m_k$ *and* $N^{-1}X_k(N^2t)$. *Moreover,* $G_t^N(f)$ *is a Gaussian random variable with mean* 0 *and variance*

$$\mathbb{E}[G_t^N(f)^2] = \frac{1}{N^2} \sum_{k,l=1}^{N} f'\left(\frac{m_k}{N}\right) f'\left(\frac{m_l}{N}\right) \frac{1}{\beta} \left(\frac{\min(k,l)}{N} - \frac{kl}{N(N+1)} \right),$$

and $\mathbb{E}[R_t^N(f)^2]$ *is of order* N^{-1}.

Proof. Using Taylor's expansion, we get

$$F_t^N(f) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \left\{ f'\left(\frac{m_k}{N}\right) \frac{X_k(N^2t) - m_k}{N} + \frac{1}{2} f''(\xi_k(N^2t)) \left(\frac{X_k(N^2t) - m_k}{N}\right)^2 - \mathbb{E}\left[f'\left(\frac{m_k}{N}\right) \frac{X_k(N^2t) - m_k}{N} + \frac{1}{2} f''(\xi_k(N^2t)) \left(\frac{X_k(N^2t) - m_k}{N}\right)^2 \right] \right\},$$

where $\xi_k(N^2t)$ takes values between m_k/N and $N^{-1}X_k(N^2t)$. Using the stationarity of the initial distribution, we get $\mathbb{E}[X_k(N^2t)] = m_k$. This concludes (4.12).

The stationarity of the initial distribution implies that $(X_k(N^2t))_{k=1}^N$ is a Gaussian random vector with mean $(m_k)_{k=1}^N$ and covariance matrix $(\beta L)^{-1}$. From this it follows that $G_t^N(f)$ is a Gaussian random variable with mean 0 and variance

$$\begin{aligned} &\mathbb{E}[G_t^N(f)^2] \\ &= \frac{1}{N^2} \sum_{k,l=1}^{N} f'\left(\frac{m_k}{N}\right) f'\left(\frac{m_l}{N}\right) \frac{1}{N} \mathbb{E}[(X_k(N^2t) - m_k)(X_l(N^2t) - m_l)] \\ &= \frac{1}{N^2} \sum_{k,l=1}^{N} f'\left(\frac{m_k}{N}\right) f'\left(\frac{m_l}{N}\right) \frac{1}{\beta} \left(\frac{\min(k,l)}{N} - \frac{kl}{N(N+1)} \right). \end{aligned}$$

It remains to prove that $\mathbb{E}[R_t^N(f)^2]$ is of order N^{-1}. Using Jensen's inequality, we estimate

$$\mathbb{E}[R_t^N(f)^2] \le \mathbb{E}\left[\left(\frac{1}{2\sqrt{N}}\sum_{k=1}^N f''(\xi_k(N^2t))\left(\frac{X_k(N^2t)-m_k}{N}\right)^2\right)^2\right]$$
$$\le \frac{\|f''\|_\infty^2}{4N^4}\sum_{k=1}^N \mathbb{E}[(X_k(N^2t)-m_k)^4].$$

Since $X_k(N^2t)$ is a Gaussian random variable with mean m_k and variance of order N, $\mathbb{E}[(X_k(N^2t)-m_k)^4] = 3\mathbb{E}[(X_k(N^2t)-m_k)^2]^2$ is of order N^2. This concludes that $\mathbb{E}[R_t^N(f)^2]$ is of order N^{-1}. □

Lemma 4.9. *Let $f \in S$ and $t \in (0,T]$. Then it holds*

$$F_t^N(f) = F_0^N(f) + \int_0^t F_s^N(\mathfrak{A}f)ds + M_t^N(f) + S_t^N(f) \tag{4.13}$$

with

$$M_t^N(f) = \frac{1}{N^{3/2}}\sum_{k=1}^N f'\left(\frac{m_k}{N}\right)\sqrt{2}W_k(N^2t),$$
$$S_t^N(f) = -\int_0^t R_s^N(\mathfrak{A}f)ds + \mathcal{O}(N^{-7/2})\int_0^t\sum_{k=1}^N(X_k(N^2s)-m_k)ds$$
$$+ R_t^N(f) - R_0^N(f).$$

Moreover, $\mathbb{E}[S_t^N(f)^2]$ and $\mathbb{E}[\sup_{t\in[0,T]}|S_t^N(f)|]$ vanish as $N\to\infty$.

Proof. Let $\overline{X}_k(N^2t) = X_k(N^2t)-m_k$. Using lemma 4.8, (2.5) and $m_{k-1}-2m_k+m_{k+1} = 0$ for every $k = 1,\dots,N$, we get

$$\begin{aligned} dF_t^N(f) &= \beta\sqrt{N}\sum_{k=1}^N f'\left(\frac{m_k}{N}\right)\{\overline{X}_{k-1}(N^2t) - 2\overline{X}_k(N^2t) + \overline{X}_{k+1}(N^2t)\}dt \\ &\quad + \frac{1}{N^{3/2}}\sum_{k=1}^N f'\left(\frac{m_k}{N}\right)\sqrt{2}dW_k(N^2t) + dR_t^N(f). \end{aligned} \tag{4.14}$$

Then summation by parts shows that the first sum on the right-hand side of (4.14) is equal to

$$\sum_{k=1}^N\left\{f'\left(\frac{m_{k-1}}{N}\right) - 2f'\left(\frac{m_k}{N}\right) + f'\left(\frac{m_{k+1}}{N}\right)\right\}\overline{X}_k(N^2t)$$
$$+ f'\left(\frac{m_1}{N}\right)\overline{X}_0(N^2t) - f'\left(\frac{m_{N+1}}{N}\right)\overline{X}_N(N^2t)$$
$$- f'\left(\frac{m_0}{N}\right)\overline{X}_1(N^2t) + f'\left(\frac{m_N}{N}\right)\overline{X}_{N+1}(N^2t),$$

where the last two lines vanish since $f \in S$, and Taylor's expansion shows

$$f'\left(\frac{m_{k-1}}{N}\right) - 2f'\left(\frac{m_k}{N}\right) + f'\left(\frac{m_{k+1}}{N}\right) = \frac{1}{(\rho N)^2} f^{(3)}\left(\frac{m_k}{N}\right) + \mathcal{O}(N^{-4}).$$

From this and lemma 4.8, it follows that the right-hand side of (4.14) is equal to

$$\begin{aligned}
&\beta\sqrt{N}\sum_{k=1}^{N}\left\{\frac{1}{(\rho N)^2} f^{(3)}\left(\frac{m_k}{N}\right) + \mathcal{O}(N^{-4})\right\}\overline{X}_k(N^2t)dt \\
&+ \frac{1}{N^{3/2}}\sum_{k=1}^{N} f'\left(\frac{m_k}{N}\right)\sqrt{2}dW_k(N^2t) + dR_t^N(f) \\
&= F_t^N(\mathfrak{A}f)dt + \frac{1}{N^{3/2}}\sum_{k=1}^{N} f'\left(\frac{m_k}{N}\right)\sqrt{2}dW_k(N^2t) \\
&- R_t^N(\mathfrak{A}f)dt + \mathcal{O}(N^{-7/2})\sum_{k=1}^{N}\overline{X}_k(N^2t)dt + dR_t^N(f).
\end{aligned}$$

This concludes (4.13).

We will next prove that $\mathbb{E}[S_t^N(f)^2]$ vanishes as $N \to \infty$. It suffices to prove this statement for each term constituting $S_t^N(f)$ separately. For the first term Jensen's inequality and the stationarity of the initial distribution show

$$\mathbb{E}\left[\left(\int_0^t R_s^N(\mathfrak{A}f)ds\right)^2\right] \leq \mathbb{E}\left[t\int_0^t R_s^N(\mathfrak{A}f)^2ds\right] = t^2\mathbb{E}[R_0^N(\mathfrak{A}f)^2],$$

which vanishes as $N \to \infty$ by lemma 4.8. For the second term we analogously get

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^t \frac{1}{N^{7/2}}\sum_{k=1}^{N}\overline{X}_k(N^2s)ds\right)^2\right] &\leq t^2\mathbb{E}\left[\left(\frac{1}{N^{7/2}}\sum_{k=1}^{N}\overline{X}_k(0)\right)^2\right] \\
&= \frac{t^2}{N^7}\sum_{k,l=1}^{N}\mathbb{E}[\overline{X}_k(0)\overline{X}_l(0)],
\end{aligned}$$

which vanishes as $N \to \infty$ since $\mathbb{E}[\overline{X}_k(0)\overline{X}_l(0)]$ is of order N. For the third and fourth term we know from lemma 4.8 that $\mathbb{E}[(R_t^N(f) - R_0^N(f))^2]$ vanishes as $N \to \infty$.

Finally, we will prove that $\mathbb{E}[\sup_{t\in[0,T]}|S_t^N(f)|]$ vanishes as $N \to \infty$. It suffices to prove this statement for each term constituting $S_t^N(f)$ separately. From the previous considerations it follows that for the first term

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t R_s^N(\mathfrak{A}f)ds\right|\right] \leq T\mathbb{E}\left[|R_0^N(\mathfrak{A}f)|\right]$$

and for the second term

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t \frac{1}{N^{7/2}}\sum_{k=1}^{N}\overline{X}_k(N^2 s)ds\right|\right] \le T\mathbb{E}\left[\left|\frac{1}{N^{7/2}}\sum_{k=1}^{N}\overline{X}_k(0)\right|\right]$$

vanish as $N\to\infty$. The third and fourth term are more involved. Elementary estimates show

$$\mathbb{E}\left[\sup_{t\in[0,T]}|R_t^N(f)-R_0^N(f)|\right] \le \|f''\|_\infty \mathbb{E}\left[\sup_{t\in[0,T]}\frac{1}{N^{5/2}}\sum_{k=1}^{N}\overline{X}_k(N^2 t)^2\right], \tag{4.15}$$

where the expectation on the right-hand side becomes with (2.7)

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\frac{1}{N^{5/2}}\sum_{k=1}^{N}\left\{e^{-N^2 t\beta L}(X(0)-L^{-1}b)+\int_0^{N^2 t} e^{-(N^2 t-s)\beta L}\sqrt{2}dW(s)\right\}_k^2\right]\\
&\le 2\mathbb{E}\left[\sup_{t\in[0,T]}\frac{1}{N^{5/2}}\sum_{k=1}^{N}\{e^{-N^2 t\beta L}(X(0)-L^{-1}b)\}_k^2\right] \\
&\quad+\frac{2}{N^{5/2}}\sum_{k=1}^{N}\mathbb{E}\left[\sup_{t\in[0,T]}\left\{\int_0^{N^2 t} e^{-(N^2 t-s)\beta L}\sqrt{2}dW(s)\right\}_k^2\right].
\end{aligned}\tag{4.16}$$

Notice that the sum in the second line of (4.16) is the squared Euclidean norm of $e^{-N^2 t\beta L}(X(0)-L^{-1}b)$, which is bounded above by the squared Euclidean norm of $X(0)-L^{-1}b$ since the eigenvalues of L are non-negative, cf. lemma A.1. So the expectation in the second line of (4.16) is bounded above by

$$\mathbb{E}\left[\frac{1}{N^{5/2}}\sum_{k=1}^{N}\overline{X}_k(0)^2\right] = \frac{1}{N^{5/2}}\sum_{k=1}^{N}\mathbb{E}[\overline{X}_k(0)^2],$$

which vanishes as $N\to\infty$ since $\mathbb{E}[\overline{X}_k(0)^2]$ is of order N. Since

$$\left\{\int_0^{N^2 t} e^{-(N^2 t-s)\beta L}\sqrt{2}dW(s)\right\}_k = \sum_{l=1}^{N}\int_0^{N^2 t}\{e^{-(N^2 t-s)\beta L}\}_{kl}\sqrt{2}dW_l(s)$$

defines a martingale, Doob's martingale inequality shows that the sum in the last line of (4.16) is bounded above by

$$\begin{aligned}
&4\sum_{k=1}^{N}\mathbb{E}\left[\left(\sum_{l=1}^{N}\int_0^{N^2 T}\{e^{-(N^2 T-s)\beta L}\}_{kl}\sqrt{2}dW_l(s)\right)^2\right]\\
&=8\sum_{k,l=1}^{N}\int_0^{N^2 T}\{e^{-(N^2 T-s)\beta L}\}_{kl}^2 ds.
\end{aligned}\tag{4.17}$$

The double sum in the last line of (4.17) is equal to

$$\int_0^{N^2T} \operatorname{tr}[e^{-2(N^2T-s)\beta L}]ds = \int_0^{N^2T} \sum_{k=1}^{N} e^{-2(N^2T-s)\beta\lambda_k} ds$$
$$= \sum_{k=1}^{N} \frac{1-e^{-2N^2T\beta\lambda_k}}{2\beta\lambda_k},$$

where λ_k, $k = 1, \ldots, N$ are the eigenvalues of L. From this and lemma A.1 it follows that the last line of (4.16) vanishes as $N \to \infty$. Putting everything together, we obtain that the right-hand side of (4.15) vanishes as $N \to \infty$. This concludes the proof. □

Let Q be an accumulation point of $\{Q^N\}_{N\in\mathbb{N}}$. We pass to a subsequence so that Q^N converges to Q, or equivalently, $(F_t^N)_{t\in[0,T]}$ converges in distribution to $(F_t)_{t\in[0,T]}$. We will show that F_0 is a Gaussian random field with mean 0 and covariance (4.4), and $(F_t)_{t\in[0,T]}$ satisfies the martingale problem in proposition 4.7. This is the content of the following two lemmas. Then proposition 4.7 will imply the uniqueness of the accumulation points Q.

Lemma 4.10. *F_0 is a Gaussian random field with mean 0 and covariance*

$$\mathbb{E}[F_0(f)F_0(g)] = \frac{\rho^2}{\beta}\int_0^{\rho^{-1}} \left(f(u) - \int_0^{\rho^{-1}} f(v)\rho dv\right) g(u)\rho du \tag{4.18}$$

for every $f, g \in S$.

Proof. Since $(F_t^N)_{t\in[0,T]}$ converges in distribution to $(F_t)_{t\in[0,T]}$ by assumption and the map $C([0,T], S') \to \mathbb{R}$, $(F_t)_{t\in[0,T]} \mapsto F_0(f)$ is continuous, $F_0^N(f)$ converges in distribution to $F_0(f)$. Using lemma 4.8 and Slutsky's theorem, we see that $G_0^N(f)$ also converges in distribution to $F_0(f)$. From this it follows that $F_0(f)$ is a Gaussian random variable with mean 0 and variance

$$\begin{aligned}
&\lim_{N\to\infty} \mathbb{E}[G_0^N(f)^2] \\
&= \lim_{N\to\infty} \frac{1}{N^2}\sum_{k,l=1}^{N} f'\left(\frac{m_k}{N}\right) f'\left(\frac{m_l}{N}\right) \frac{1}{\beta}\left(\frac{\min(k,l)}{N} - \frac{kl}{N(N+1)}\right) \\
&= \int_0^1\int_0^1 f'\left(\frac{x}{\rho}\right) f'\left(\frac{y}{\rho}\right)\frac{1}{\beta}(\min(x,y) - xy)dxdy.
\end{aligned} \tag{4.19}$$

Dividing the integral with respect to x in the last line of (4.19) into the integral from 0 to y and the integral from y to 1, integrating by parts and changing variables, we see that the last line of (4.19) agrees with (4.18).

Let m be a positive integer and $f_k \in S$ for $k = 1, \ldots, m$. We consider the characteristic function of $(F_0(f_k))_{k=1}^m$:

$$\mathbb{E}\left[\exp\left(i\sum_{k=1}^{m}\theta_k F_0(f_k)\right)\right] = \mathbb{E}\left[\exp\left(iF_0\left(\sum_{k=1}^{m}\theta_k f_k\right)\right)\right], \tag{4.20}$$

where $\theta_k \in \mathbb{R}$. Then the previous considerations show that (4.20) is equal to

$$\exp\left(-\frac{\rho^2}{2\beta}\sum_{k,l=1}^{m}\theta_k\theta_l\int_0^{\rho^{-1}}\left\{f_k(u)-\int_0^{\rho^{-1}}f_k(v)\rho dv\right\}f_l(u)\rho du\right),$$

which implies that $(F_0(f_k))_{k=1}^m$ is a Gaussian random vector with mean 0 and covariance matrix given by (4.18). This concludes the proof. □

Lemma 4.11. $(F_t)_{t\in[0,T]}$ *satisfies the martingale problem in proposition 4.7.*

Proof. Let $f \in S$. We will show first that

$$M_t(f) = F_t(f) - F_0(f) - \int_0^t F_s(\mathfrak{A}f)ds$$

defines a martingale with respect to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$. The $\mathcal{F}_t$-measurability of $M_t(f)$ is obvious. To see the integrability of $M_t(f)$, we consider $M_t^N(f)$ from lemma 4.9. Using the continuity of the map $C([0,T],S') \to \mathbb{R}$, $(F_t)_{t\in[0,T]} \mapsto F_t(f) - F_0(f) - \int_0^t F_s(\mathfrak{A}f)ds$ and that $\mathbb{E}[S_t^N(f)^2]$ vanishes as $N \to \infty$, Slutsky's theorem implies that $M_t^N(f)$ converges as $N \to \infty$ in distribution to $M_t(f)$. Further, $(M_t^N(f))_{N\in\mathbb{N}}$ is uniformly integrable since $\mathbb{E}[M_t^N(f)^4]$ is bounded above uniformly in N:

$$\begin{aligned}\mathbb{E}[M_t^N(f)^4] &= \frac{4}{N^6}\sum_{k,l=1}^{N} f'\left(\frac{m_k}{N}\right)^2 f'\left(\frac{m_l}{N}\right)^2 \mathbb{E}\left[W_k(N^2t)^2W_l(N^2t)^2\right]\\ &\le \frac{12t^2}{N^2}\sum_{k,l=1}^{N} f'\left(\frac{m_k}{N}\right)^2 f'\left(\frac{m_l}{N}\right)^2 \le 12t^2\|f'\|_\infty^4,\end{aligned}$$

where we used the independence of the Brownian motions W_k, $k = 1,\dots,N$ and

$$\mathbb{E}[W_k(t)^4] = 3\mathbb{E}[W_k(t)^2]^2 = 3t^2.$$

From this it follows with Skorohod's representation theorem that $\mathbb{E}[M_t^N(f)^4]$ converges as $N \to \infty$ to $\mathbb{E}[M_t(f)^4]$ and $M_t(f)^4$ is integrable. In particular, $M_t(f)$ is integrable.

To prove the martingale property for $(M_t(f))_{t\in[0,T]}$, by the monotone class theorem, it suffices to show

$$\mathbb{E}[(M_t(f) - M_s(f))1_U] = 0 \tag{4.21}$$

for every $0 \le s \le t \le T$ and $U = \{F_{s_k}(f_k) \in A_k : k = 1,\dots,m\}$, where m is a positive integer, $0 \le s_1 \le \dots \le s_m \le s$, $f_k \in S$ and $A_k \subset \mathbb{R}$ are Borel sets. Let $U^N = \{F_{s_k}^N(f_k) \in A_k : k = 1,\dots,m\}$. Then it holds

$$\lim_{N\to\infty}\mathbb{E}[(M_t^N(f) - M_s^N(f))1_{U^N}] = \lim_{N\to\infty}\mathbb{E}[(M_t^N(f) - M_s^N(f))1_U] \tag{4.22}$$

in the path space representation of $(F_t^N)_{t\in[0,T]}$ and $(F_t)_{t\in[0,T]}$, where we used that $M_t^N(f) - S_t^N(f)$ is a function of $(F_t^N)_{t\in[0,T]}$ and $\mathbb{E}[S_t^N(f)^2]$ vanishes as $N \to \infty$. The left-hand side of (4.22) is equal to 0 since $(M_t^N(f))_{t\in[0,T]}$ is a martingale with respect to the filtration $(\mathcal{G}_t^N)_{t\in[0,T]}$ given by $\mathcal{G}_t^N = \sigma(W_k(s),\ X_k(0) : k = 1,\dots,N,\ s \in [0, N^2 t])$ and $U^N \in \mathcal{G}_s^N$. We know that $M_t^N(f)$ converges in distribution to $M_t(f)$ and $(M_t^N(f)1_U)_{N\in\mathbb{N}}$ is uniformly integrable. Together with Skorohod's representation theorem it follows that the right-hand side of (4.22) is equal to $\mathbb{E}[(M_t(f) - M_s(f))1_U]$. This concludes (4.21).

We will show next that

$$V_t(f) = M_t(f)^2 - \|\mathfrak{B}f\|_{L^2}^2 t$$

defines a martingale with respect to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$. The $\mathcal{F}_t$-measurability of $V_t(f)$ is obvious, and the integrability of $V_t(f)$ follows from the integrability of $M_t(f)^4$.

As before, we will prove the martingale property for $(V_t(f))_{t\in[0,T]}$ by showing

$$\mathbb{E}[(V_t(f) - V_s(f))1_U] = 0 \tag{4.23}$$

for every $0 \le s \le t \le T$ and $U = \{F_{s_k}(f_k) \in A_k : k = 1,\dots,m\}$. For this purpose, we consider

$$V_t^N(f) = M_t^N(f)^2 - \langle M_t^N(f)\rangle,$$

where $\langle\cdot\rangle$ denotes the quadratic variation and

$$\begin{aligned}\langle M_t^N(f)\rangle &= \frac{2}{N^3}\sum_{k,l=1}^N f'\left(\frac{m_k}{N}\right) f'\left(\frac{m_l}{N}\right)\langle W_k(N^2t), W_l(N^2t)\rangle \\ &= \frac{2t}{N}\sum_{k=1}^N f'\left(\frac{m_k}{N}\right)^2.\end{aligned}$$

Then it holds

$$\lim_{N\to\infty}\mathbb{E}\left[(V_t^N(f) - V_s^N(f))1_{U^N}\right] = \lim_{N\to\infty}\mathbb{E}\left[(V_t^N(f) - V_s^N(f))1_U\right] \tag{4.24}$$

in the path space representation of $(F_t^N)_{t\in[0,T]}$ and $(F_t)_{t\in[0,T]}$, where we used that $(M_t^N(f) - S_t^N(f))^2 - \langle M_t^N(f)\rangle$ is a function of $(F_t^N)_{t\in[0,T]}$, $\mathbb{E}[S_t^N(f)^2]$ vanishes as $N \to \infty$ and $\mathbb{E}[M_t^N(f)^4]$ is bounded above uniformly in N. The left-hand side of (4.24) vanishes as $N \to \infty$ since $(V_t^N(f))_{t\in[0,T]}$ is a martingale with respect to the filtration $(\mathcal{G}_t^N)_{t\in[0,T]}$ and $U^N = \{F_{s_k}^N(f_k) \in A_k : k = 1,\dots,m\} \in \mathcal{G}_s^N$. Since $M_t^N(f)$ converges as $N \to \infty$ to $M_t(f)$ and $\langle M_t^N(f)\rangle$ converges to $\|\mathfrak{B}f\|_{L^2}^2 t$, Slutsky's theorem implies that $V_t^N(f)$ converges as $N \to \infty$ to $V_t(f)$. Moreover, $(V_t^N(f))_{N\in\mathbb{N}}$ is uniformly integrable since $\mathbb{E}[M_t^N(f)^4]$ is bounded above uniformly in N. Together with Skorohod's representation theorem it follows that the right-hand side of (4.24) is equal to $\mathbb{E}[(V_t(f) - V_s(f))1_U]$. This concludes (4.23). □

To complete the proof of theorem 4.4, it remains to show (4.5).

Corollary 4.12. *The space-time covariance of* $(F_t^N)_{t\in[0,T]}$ *is given by*

$$\mathbb{E}[F_s(f)F_t(g)] = \frac{\rho^2}{\beta}\int_0^{\rho^{-1}}\left(f(u) - \int_0^{\rho^{-1}} f(v)\rho dv\right) g_{t-s}(u)\rho du$$

for every $0 \le s \le t \le T$ *and* $f, g \in S$.

Proof. Conditioning on $\mathcal{F}_s$ and using proposition 4.7, we get

$$\mathbb{E}[F_s(f)F_t(g)] = \mathbb{E}[F_s(f)\mathbb{E}[F_t(g)|\mathcal{F}_s]] = \mathbb{E}[F_s(f)F_s(g_{t-s})].$$

The proof is concludes by using the stationarity of the initial distribution and lemma 4.10. □

4.2.2 Tightness

We will prove the tightness of $\{Q_f^N\}_{N\in\mathbb{N}}$ for every $f \in S$, where Q_f^N is the distribution of $(F_t^N(f))_{t\in[0,T]}$. Then [Mit83, theorem 3.1] will show the tightness of $\{Q^N\}_{N\in\mathbb{N}}$. For this purpose, we will check the assumptions of

Proposition 4.13. *[Bil13, theorem 7.1] Let* $\{P_N\}_{N\in\mathbb{N}}$ *be a family of probability measures on* $C([0,T],\mathbb{R})$ *such that*

$$\lim_{a\to\infty}\limsup_{N\to\infty} P_N\left[(x_t)_{t\in[0,T]} : |x_0| \ge a\right] = 0,$$

and for every $\epsilon > 0$

$$\lim_{\delta\to\infty}\limsup_{N\to\infty} P_N\left[(x_t)_{t\in[0,T]} : \sup_{\substack{s,t\in[0,T]\\|s-t|\le\delta}} |x_s - x_t| \ge \epsilon\right] = 0.$$

Then $\{P_N\}_{N\in\mathbb{N}}$ *is tight.*

Lemma 4.14. *It holds for every* $f \in S$

$$\lim_{a\to\infty}\limsup_{N\to\infty} \mathbb{P}[|F_0^N(f)| \ge a] = 0.$$

Proof. Lemma 4.8 shows

$$\mathbb{E}[F_0^N(f)^2] \le 2\mathbb{E}[G_0^N(f)^2] + 2\mathbb{E}[R_0^N(f)^2],$$

where the first term on the right-hand side is bounded above uniformly in N and the second term vanishes as $N \to \infty$. From this it follows $\limsup_{N\to\infty}\mathbb{E}[F_0^N(f)^2] < \infty$, and Markov's inequality concludes the proof. □

Lemma 4.15. *It holds for every $f \in S$ and $\epsilon > 0$*

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\mathbb{P}\left[\sup_{\substack{s,t\in[0,T]\\|s-t|\le\delta}}|F_s^N(f)-F_t^N(f)|\ge\epsilon\right]=0. \tag{4.25}$$

Proof. From lemma 4.9 we know

$$F_t^N(f)-F_s^N(f)=\int_s^t F_s^N(\mathfrak{A}f)ds+M_t^N(f)-M_s^N(f)+S_t^N(f)-S_s^N(f). \tag{4.26}$$

It suffices to prove (4.25) for each term constituting $F_t^N(f)-F_s^N(f)$.

For the first term on the right-hand side of (4.26) we use Jensen's inequality and the stationarity of the initial distribution to estimate

$$\mathbb{E}\left[\sup_{\substack{0\le s\le t\le T\\t-s\le\delta}}\left(\int_s^t F_r^N(\mathfrak{A}f)dr\right)^2\right]\le\mathbb{E}\left[\sup_{\substack{0\le s\le t\le T\\t-s\le\delta}}(t-s)\int_s^t F_r^N(\mathfrak{A}f)^2dr\right]$$
$$\le\mathbb{E}\left[\delta\int_0^T F_r^N(\mathfrak{A}f)^2dr\right]=\delta T\mathbb{E}[F_0^N(\mathfrak{A}f)^2].$$

Then Markov's inequality shows

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\mathbb{P}\left[\sup_{\substack{0\le s\le t\le T\\t-s\le\delta}}\left|\int_s^t F_r^N(\mathfrak{A}f)dr\right|\ge\epsilon\right]=0.$$

We will next consider the second and third term on the right-hand side of (4.26). Since $M_t^N(f)$ is a continuous martingale, [RY99, theorem 1.6 in chapter V] shows $M_t^N(f)=W(\langle M_t^N(f)\rangle t)$, where W is a one-dimensional Brownian motion. So

$$\mathbb{P}\left[\sup_{\substack{s,t\in[0,T]\\|s-t|\le\delta}}|M_t^N(f)-M_s^N(f)|\ge\epsilon\right]$$
$$=\mathbb{P}\left[\sup_{\substack{s,t\in[0,T]\\|s-t|\le\delta}}\left|W(\langle M_t^N(f)\rangle t)-W(\langle M_s^N(f)\rangle s)\right|>\epsilon\right],$$

which vanishes as $\delta\to 0$ uniformly in N by Levy's modulus of continuity theorem and the estimate

$$|\langle M_t^N(f)\rangle t-\langle M_s^N(f)\rangle s|=\frac{2|t^2-s^2|}{N}\sum_{k=1}^N f'\left(\frac{m_k}{N}\right)\le 4T\|f'\|_\infty|t-s|.$$

For the two last terms on the right-hand side of (4.26) we know from lemma 4.9 that

$$\lim_{N\to\infty} \mathbb{E}\left[\sup_{\substack{s,t\in[0,T]\\|s-t|\leq\delta}} |S_t^N(f) - S_s^N|\right] = 0.$$

Then Markov's inequality shows

$$\lim_{\delta\to 0}\limsup_{N\to\infty} \mathbb{P}\left[\sup_{\substack{s,t\in[0,T]\\|s-t|\leq\delta}} |S_t^N(f) - S_s^N(f)| \geq \epsilon\right] = 0.$$

This concludes the proof. □

A Appendix

A.1 Finite-dimensional Ornstein-Uhlenbeck processes

The particle system (2.6) in the case of quadratic potential is a finite-dimensional Ornstein-Uhlenbeck process. In this section we will calculate its mean and covariance matrix and identify its stationary distribution. Let N be a positive integer. A $\mathbb{R}^N$-valued stochastic process $(X(t))_{t\in[0,\infty)}$ is called N-dimensional Ornstein-Uhlenbeck process if it solves the stochastic differential equation

$$dX(t) = A(m - X(t))dt + BdW(t) \tag{A.1}$$

where $m \in \mathbb{R}^N$, $A, B \in \mathbb{R}^{N\times N}$ and W is a N-dimensional Brownian motion. This Ornstein-Uhlenbeck process describes N Brownian particles attracted towards m, where A determines the strength of the attraction force and B the strength of the noise.

We can solve (A.1) explicitly. The solution $e^{-At}c$ for some $c \in \mathbb{R}^N$ of the homogeneous differential equation $dX(t) = -AX(t)dt$ and the variation of constants method lead to the approach $X(t) = e^{-At}Y(t)$. Applying Itô's lemma to $Y(t) = e^{At}X(t)$, we get

$$\begin{aligned} dY(t) &= e^{tA}AX(t)dt + e^{tA}dX(t) \\ &= e^{tA}AX(t)dt + e^{tA}A(m - X(t))dt + e^{tA}BdW(t) \\ &= e^{tA}Amdt + e^{tA}BdW(t), \end{aligned}$$

or equivalently,

$$\begin{aligned} e^{tA}X(t) &= X(0) + \int_0^t e^{sA}Amds + \int_0^t e^{sA}BdW(s) \\ &= X(0) + (e^{tA} - I)m + \int_0^t e^{sA}BdW(s). \end{aligned}$$

We conclude

$$X(t) = e^{-tA}X(0) + (I - e^{-tA})m + \int_0^t e^{-(t-s)A}BdW(s). \tag{A.2}$$

We assume the initial condition to be deterministic: $X(0) = x$. Then it follows from (A.2) that the Ornstein-Uhlenbeck process is a Gaussian process. In particular, $(X(t))_{t\in[0,\infty)}$ is characterised by its mean function

$$\mathbb{E}[X(t)] = e^{-tA}x + (I - e^{-tA})m$$

and covariance function

$$\begin{aligned}&\mathbb{E}[(X(s)-\mathbb{E}[X(s)])^T(X(t)-\mathbb{E}[X(t)])]\\&=\mathbb{E}\left[\left(\int_0^s e^{-(s-r)A}BdW(r)\right)^T\int_0^t e^{-(t-r)A}BdW(r)\right]\\&=\int_0^s (e^{-(s-r)A}B)^T e^{-(t-r)A}Bdr,\end{aligned}$$

for $0 \le s \le t < \infty$, where we used Itô's isometry for the last equation.

Let $L \in \mathbb{R}^{N\times N}$ be the discrete Laplacian of size N, which is positive definite, and $b \in \mathbb{R}^N$ the vector with entries $b_N = N/\rho$ and $b_k = 0$ otherwise. Choosing $m = L^{-1}b$, $A = \beta L$ and $B = \sqrt{2}I$, we see that (A.1) coincides with (2.6). The mean function of this Ornstein-Uhlenbeck is

$$\mathbb{E}[X(t)] = e^{-t\beta L}x + (I - e^{-t\beta L})L^{-1}b \tag{A.3}$$

and it covariance function

$$\begin{aligned}\mathbb{E}[(X(s)-\mathbb{E}[X(s)])^T(X(t)-\mathbb{E}[X(t)])] &= 2\int_0^s e^{-(s+t-2r)\beta L}dr\\&=(\beta L)^{-1}(e^{-(t-s)\beta L}-e^{-(t+s)\beta L}).\end{aligned}$$

Setting $s = t$, we get the covariance matrix of $X(t)$:

$$\mathbb{E}[(X(t)-\mathbb{E}[X(t)])^T(X(t)-\mathbb{E}[X(t)])] = (\beta L)^{-1}(I - e^{-2t\beta L}). \tag{A.4}$$

Recall that weak convergence of Gaussian measures μ_k on $\mathbb{R}^N$ is equivalent to convergence of their means m_k and covariance matrices C_k, and the weak limit μ is again a Gaussian measure with mean $\lim_{k\to\infty} m_k$ and covariance matrix $\lim_{k\to\infty} C_k$. Since (A.3) converges as $t \to \infty$ to $L^{-1}b$ and (A.4) converges to $(\beta L)^{-1}$, the distribution of $X(t)$ converges as $t \to \infty$ weakly to a Gaussian measure with mean $L^{-1}b$ and covariance matrix $(\beta L)^{-1}$, and the latter is a stationary distribution for (A.1).

The following lemma states that the rescaled trace of the covariance matrix $(\beta L)^{-1}$ is bounded uniformly in N.

Lemma A.1. *Let λ_k, $k = 1, \dots, N$ be the eigenvalues of the discrete Laplacian L of size N. Then it holds*

$$\lambda_k = 2 - 2\cos\left(\frac{k\pi}{N+1}\right) \tag{A.5}$$

and

$$\sup_{N\in\mathbb{N}} \frac{1}{N^2}\sum_{k=1}^N \lambda_k^{-1} < \infty. \tag{A.6}$$

Proof. Let v be an eigenvector of L with eigenvalue λ, i. e. $Lv = \lambda v$. Then it holds

$$-v_{l-1} + 2v_l - v_{l+1} = \lambda v_l$$

for every $l = 1, \dots, N$ with the convention $v_0 = v_{N+1} = 0$, or equivalently,

$$v_{l+1} = 2\alpha v_l - v_{l-1}$$

with $2\alpha = 2 - \lambda$. Rescaling v such that $v_1 = 1$, we get

$$v_{l+1} = U_l(\alpha),$$

where U_l is the l-th Chebyshev polynomial of second kind. Since $U_N(\alpha) = v_{N+1} = 0$, α is a root of the N-th Chebyshev polynomial. From this it follows

$$\lambda = 2 - 2\alpha = 2 - 2\cos\left(\frac{k\pi}{N+1}\right)$$

for some $k = 1, \dots, N$. This concludes (A.5).

It remain to prove (A.6). For this purpose, we expand the cosine function in a Taylor series around 0. Then we get

$$2 - 2\cos(x) = 2 - 2\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \geq x^2 - \frac{x^4}{12}$$

for every $0 \leq x \leq \pi$. From this it follows

$$\lambda_k \geq \left(\frac{k\pi}{N+1}\right)^2 - \frac{1}{12}\left(\frac{k\pi}{N+1}\right)^4. \tag{A.7}$$

Using (A.7), we estimate

$$\begin{aligned}\frac{1}{N^2}\sum_{k=1}^{N} \lambda_k^{-1} &\leq \frac{1}{N^2}\sum_{k=1}^{N} \frac{1}{\left(\frac{k\pi}{N+1}\right)^2 - \frac{1}{12}\left(\frac{k\pi}{N+1}\right)^4} \\ &\leq \sum_{k=1}^{N} \frac{1}{(k\pi)^2} \frac{2}{1 - \frac{1}{12}\left(\frac{k\pi}{N+1}\right)^2} \leq \sum_{k=1}^{N} \frac{1}{(k\pi)^2} \frac{2}{1 - \frac{\pi^2}{12}},\end{aligned}$$

which converges as $N \to \infty$. This yields an upper bound for (A.6) that is independent of N. □

A.2 Surface tension

The surface tension (2.12) is an important quantity as it appears in the hydrodynamic equation (2.13) and (3.3). In this section we review properties of the surface tension and explicitly compute it in the case of one dimension and quadratic potential.

Proposition A.2 ([Fun05, theorem 5.3]). *The d-dimensional surface tension* $\sigma\colon \mathbb{R}^d \to \mathbb{R}$ *has the following properties:*

1. $\sigma \in C^1(\mathbb{R}^d)$ *and* $\nabla\sigma$ *is Lipschitz continuous.*
2. σ *is even: It holds* $\sigma(u) = \sigma(-u)$ *for every* $u \in \mathbb{R}^d$.
3. *For every* $u, v \in \mathbb{R}^d$ *it holds*
$$\frac{c_-}{2}|u-v|^2 \leq \sigma(v) - \sigma(u) - (v-u)\nabla\sigma(u) \leq \frac{c_+}{2}|u-v|^2,$$
where $c_-, c_+ > 0$ *are the constants from the assumptions on the potential* V*. In particular,* σ *is strictly convex.*
4. *For every* $u, v \in \mathbb{R}^d$ *it holds*
$$c_-|u-v|^2 \leq (u-v)(\nabla\sigma(u) - \nabla\sigma(v)) \leq c_+|u-v|^2$$

In the following we consider the one-dimensional surface tension. For $\lambda \in \mathbb{R}$ we define the probability measure ν_λ
$$\nu_\lambda(dx) = \frac{e^{-V(x)+\lambda x}}{Z_\lambda}dx,$$
where Z_λ is the normalisation constant. Further, we define the function $u\colon \mathbb{R} \to \mathbb{R}$ by
$$u(\lambda) = \int_{-\infty}^{\infty} x\nu_\lambda(dx).$$
The next lemma implies that u is infinitely differentiable.

Lemma A.3. *Let* k *be a non-negative integer. Then the function*
$$\mathbb{R} \to \mathbb{R},\ \lambda \mapsto \int_{-\infty}^{\infty} x^k e^{-V(x)+\lambda x}dx$$
is differentiable and it can be differentiated under the integral sign.

Proof. First, we will show the integrability of $x^k e^{-V(x)+\lambda x}$ with respect to x. This follows by expanding the potential V in a Taylor series around 0 and applying the assumptions on V:
$$\begin{aligned}\int_{-\infty}^{\infty} |x|^k e^{-V(x)+\lambda x}dx &\leq \int_{-\infty}^{\infty} |x|^k e^{-V(0)-\frac{c_-}{2}x^2+\lambda x}dx \\ &= \int_{-\infty}^{\infty} |x|^k e^{-V(0)-\frac{c_-}{2}\left(x-\frac{\lambda}{c_-}\right)^2+\frac{\lambda^2}{2c_-}}dx < \infty.\end{aligned}$$

To show the differentiability of u_k, we consider the difference quotient

$$\begin{aligned}&\frac{1}{h}\left(\int_{-\infty}^{\infty} x^k e^{-V(x)+(\lambda+h)x}dx - \int_{-\infty}^{\infty} x^k e^{-V(x)+\lambda x}dx\right)\\ &= \int_{-\infty}^{\infty} x^k \frac{1}{h}(e^{-V(x)+(\lambda+h)x} - e^{-V(x)+\lambda x})dx\\ &= \int_{-\infty}^{\infty} x^{k+1}e^{-V(x)+\xi(x)x}dx\end{aligned}$$

for $h \in [0, h_0]$, where we used the mean value theorem with some $\xi(x) \in [\lambda, \lambda + h]$. The integrand in the last line is bounded above uniformly in h by

$$|x|^{k+1}(e^{-V(x)+\lambda x} + e^{-V(x)+(\lambda+h_0)x}),$$

which is integrable. Applying the dominated convergence theorem, we can take the limit $h \to 0$ under the integral sign. This concludes the proof. □

Lemma A.3 shows that u can be written in terms of the normalisation constant Z_λ:

$$\begin{aligned}u(\lambda) &= \frac{1}{Z_\lambda}\frac{d}{d\lambda}\int_{-\infty}^{\infty} e^{-V(x)+\lambda x}dx\\ &= \frac{1}{Z_\lambda}\frac{d}{d\lambda}Z_\lambda = \frac{d}{d\lambda}\log(Z_\lambda).\end{aligned}$$

Since

$$\begin{aligned}u'(\lambda) &= \frac{\frac{d^2}{d\lambda^2}Z_\lambda}{Z_\lambda} - \left(\frac{\frac{d}{d\lambda}Z_\lambda}{Z_\lambda}\right)^2\\ &= \frac{1}{Z_\lambda}\frac{d^2}{d\lambda^2}\int_{-\infty}^{\infty} e^{-V(x)+\lambda x}dx - \left(\frac{1}{Z_\lambda}\frac{d}{d\lambda}\int_{-\infty}^{\infty} e^{-V(x)+\lambda x}dx\right)^2\\ &= \frac{1}{Z_\lambda}\int_{-\infty}^{\infty} x^2 e^{-V(x)+\lambda x}dx - \left(\frac{1}{Z_\lambda}\int_{-\infty}^{\infty} xe^{-V(x)+\lambda x}dx\right)^2 > 0,\end{aligned}$$

u is strictly increasing and admits an inverse.

Lemma A.4. *The the one-dimensional surface tension σ satisfies*

$$\sigma'(v) = \int_{-\infty}^{\infty} V'(x)\frac{1}{Z_{u^{-1}(v)}}e^{-V(x)+u^{-1}(v)x}dx = u^{-1}(v)$$

for every $v \in \mathbb{R}$.

Proof. The first equation is a consequence of [Fun05, theorem 5.5 and remark 4.5]. The second equation follows from

$$\begin{aligned}&\int_{-\infty}^{\infty} V'(x)\frac{1}{Z_{u^{-1}(v)}}e^{-V(x)+u^{-1}(v)x}dx\\ &= \int_{-\infty}^{\infty}(V'(x) - u^{-1}(v))\frac{1}{Z_{u^{-1}(v)}}e^{-V(x)+u^{-1}(v)x}dx + u^{-1}(v)\end{aligned}$$

and the fact that the integrand in the last line is the derivative of $-e^{-V(x)+u^{-1}(v)x}$ with respect to x, which vanishes as $x \to \pm\infty$ by the assumptions on the potential V. □

Corollary A.5. *The one-dimensional surface tension σ is infinitely differentiable.*

Proof. By lemma A.4, we need to prove that u^{-1} is infinitely differentiable. We know that u is infinitely differentiable and positive. Successively differentiating the identity $u(u^{-1}(v)) = v$, we observe that $(u^{-1})^{(k)}(v)$ is a fraction: The numerator is a polynomial of the first k derivatives of u and the first $k-1$ derivatives of u^{-1}, and the denominator is $u'(u^{-1}(v))$. This concludes the proof. □

Remark A.6. *Lemma A.4 can be used to explicitly compute the surface tension. We demonstrate this in the case of the quadratic potential $V(\eta) = \beta\eta^2$ with a constant $\beta > 0$. We first compute the normalisation constant Z_λ:*

$$\int_{-\infty}^{\infty} e^{-\beta x^2+\lambda x} dx = \int_{-\infty}^{\infty} e^{-\beta(x-\frac{\lambda}{2\beta})^2+\frac{\lambda^2}{4\beta}} dx = e^{\frac{\lambda^2}{4\beta}} \sqrt{\frac{\pi}{\beta}}.$$

From this it follows

$$u(\lambda) = \frac{d}{d\lambda} \log(Z_\lambda) = \frac{\lambda}{2\beta}.$$

Since $u^{-1}(v) = 2\beta v$, we conclude

$$\sigma(v) = \int_0^v u^{-1}(w)dw = \beta v^2.$$

Analogously, the surface tension in the case of multiple dimensions and quadratic potential $V(\eta) = \beta\eta^2$ is given by

$$\sigma(v) = \beta|v|^2,$$

cf. [Fun05, proposition 5.2].

A.3 Partial differential equations

Let $T > 0$, an open interval $D \subset \mathbb{R}$ and a function $\psi\colon \mathbb{R} \to \mathbb{R}$ be given. In this section we consider the nonlinear heat equation

$$\begin{cases} \partial_t h(t,\theta) = \partial_\theta \psi(\partial_\theta h(t,\theta)), & t \in (0,T),\ \theta \in D, \\ h(t,\theta) = f(\theta), & t \in (0,T],\ \theta \in \partial D, \\ h(0,\theta) = h_0(\theta), & \theta \in \overline{D}, \end{cases} \tag{A.8}$$

which is a generalisation of the nonlinear partial differential equation (2.13) in the one-dimensional case. The next theorem states that (A.8) has a unique classical solution if ψ, the boundary condition and initial condition are sufficiently smooth and the boundary condition is compatible with the initial condition.

Theorem A.7. *We assume $\psi \in C^3(\mathbb{R})$ and the existence of constants $\nu, \mu > 0$ such that*

$$\nu \leq \psi' \leq \mu. \tag{A.9}$$

Moreover, we assume $h_0 \in C^3(\overline{D})$, $h_0^{(3)}$ to be Lipschitz continuous and the compatibility of the boundary condition and initial condition in the sense that

$$h_0(\theta) = f(\theta), \qquad \partial_\theta \psi(h_0'(\theta)) = 0$$

for every $\theta \in \partial D$. Then equation (A.8) *has a unique classical solution h and the partial derivatives $\partial_\theta^3 h, \partial_\theta \partial_t h$ exist and are continuous.*

Proof. From [LSU88, theorem 6.1 in chapter V] it follows that (A.8) a unique classical solution. Then [LSU88, theorem 5.2 in chapter IV] can be applied to the corresponding linear partial differential equation, and we obtain the existence and continuity of the partial derivatives $\partial_\theta^3 h$ and $\partial_\theta \partial_t h$. □

Theorem A.7 applies for $\psi = \sigma'$ and $D = (0, 1)$, in which case (A.8) coincides with (2.15). This is the content of the following corollary.

Corollary A.8. *Let $\psi = \sigma'$ and $D = (0, 1)$. We assume $h_0 \in C^3(\overline{D})$, $h_0^{(3)}$ to be Lipschitz continuous and the boundary condition and initial condition to be compatible in the sense that*

$$h_0(\theta) = f(\theta), \qquad \partial_\theta \psi(h_0'(\theta)) = 0$$

for every $\theta \in \partial D$. Then equation (A.8) *has a unique classical solution h and the partial derivatives $\partial_\theta^3 h, \partial_\theta \partial_t h$ exist and are continuous.*

Proof. We check the assumptions of theorem A.7. From corollary A.5 we know that σ is infinitely differentiable. Using the forth statement of proposition A.2 with $u > v$, we get

$$c_- \leq \frac{\sigma'(u) - \sigma'(v)}{u - v} \leq c_+,$$

and we obtain (A.9) by taking the limit $u \to v$. This concludes the proof. □

Next, we state the maximum principle for linear partial differential equations.

Theorem A.9 ([LSU88, corollary 2.1 in chapter I])**.** *We assume h to be a classical solution of the linear partial differential equation*

$$\partial_t h(t, \theta) = a(t, \theta) \partial_\theta^2 h(t, \theta) + b(t, \theta) \partial_\theta h(t, \theta) + c(t, \theta) h(t, \theta)$$

for $t \in (0, T)$ and $\theta \in D$, where the coefficients a, b, c are bounded functions and $a \geq 0$. Let $D_T = D \times (0, T]$ be the parabolic cylinder and $\Gamma_T = \overline{D}_T \setminus D_T$ its boundary, then it holds

$$\max_{\overline{D}_T} h = \max_{\Gamma_T} h,$$
$$\min_{\overline{D}_T} h = \min_{\Gamma_T} h.$$

A.4 Simulations

In this section we provide the Python code to simulate the evolution of the particle system in the case of one and two dimensions. In the one-dimensional case we use the quadratic potential $V(\eta) = \beta\eta^2$, the parameters $N = 20$, $\beta = 1$ and $\rho = 0.15$ and we simulate up to time $t = 10$:

```
# import the necessary packages
import sys
import numpy as np
import matplotlib.pyplot as plt
import matplotlib.animation as animation

# set the parameters
n = 20
beta = 1
rho = 0.15
t = 100
dt = 0.1
dW = lambda dt: np.random.normal(loc = 0, scale = np.sqrt(dt))

# initiate the particle system
x = np.zeros((t, n+2))
x[0, 1:n+1] = np.random.uniform(low = 0, high = n/rho,\
                size = n)
x[:, n+1] = n/rho

# run the dynamics
for s in xrange(1, t):
   for k in xrange(1, n+1):
      x[s, k] = x[s-1, k] + beta*(x[s-1, k+1] + x[s-1, k-1] -\
                2*x[s-1, k])*dt +\
                np.sqrt(2) * dW(dt)

# plot the result
fig, ax = plt.subplots()
ax.set_xlabel(’$k$’)
ax.set_ylabel(’$X_k(t)$’)
ax.grid(True)
ax.autoscale(enable=True, axis=’both’, tight=True)
line, = ax.plot(x[0, :])
def update(i):
   if i < t:
      line.set_ydata(x[i, :])
```

```
    return line,
ani = animation.FuncAnimation(fig, update, interval=200)
plt.show()
```

In the two-dimensional case we simulate only the first component of the particle positions. We use the quadratic potential $V(\eta) = \beta\eta^2$, the parameters $N = 400$, $\beta = 1$ and $\rho = 0.15$ and we simulate up to time $t = 10$:

```
# import the necessary packages
import sys
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import axes3d

# set the parameters
n = 20
beta = 1
rho = 0.15
t = 100
dt = 0.1
dW = lambda dt: np.random.normal(loc = 0, scale = np.sqrt(dt))

# initiate the particle system
x = np.zeros((t, n+2, n+2))
x[0, 1:n+1, 1:n+1] = np.random.uniform(low = 0,\
                         high = n/rho, size = (n, n))
x[:, n+1, :] = n/rho
for k in xrange(1, n+1):
   x[:, k, 0] = x[:, k, n+1] = k/rho

# run the dynamics
for s in xrange(1, t):
   for k in xrange(1, n+1):
      for l in xrange(1, n+1):
         x[s, k, l] = x[s-1, k, l] +\
                      beta*(x[s-1, k+1, l] + x[s-1, k-1, l] +\
                      x[s-1, k, l+1] + x[s-1, k, l-1] -\
                      4*x[s-1, k, l]) * dt +\
                      np.sqrt(2) * dW(dt)

# plot the result
fig = plt.figure()
ax = fig.add_subplot(111, projection='3d')
ax.set_xlabel('$k_1$')
ax.set_ylabel('$k_2$')
```

```
ax.set_zlabel('$X_{(k_1,k_2)}^{(1)}(t)$')
xs = np.linspace(0, 20, n+2)
X, Y = np.meshgrid(xs, xs)
wframe = None
for s in xrange(0, t):
   oldcol = wframe
   Z = x[s, :, :]
   wframe = ax.plot_wireframe(X, Y, Z, rstride=2, cstride=2)
   if oldcol is not None:
      ax.collections.remove(oldcol)
   plt.pause(0.2)
plt.show()
```

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