BOSE-EINSTEIN CONDENSATES IN CURVED SPACE-TIME

From Concepts of General Relativity to Tidal Corrections for Quantum Gases in Local Frames

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Abstract

Measuring effects of General Relativity and beyond in the gravitational field of the Earth is a main goal of current research and cutting-edge technology. These effects, predicted by Einstein's theory, already play an important role in everyday life, for example in enabling for precise positioning and time keeping in global satellite navigation systems, such as GPS and GALILEO.

The breakthrough experimental realisation of Bose-Einstein condensation in 1995, some 70 years after Einstein's prediction, has since established matter-wave interferometers in laboratories worldwide, in which laser pulses are used to coherently split, reflect, and recombine a Bose-Einstein condensate. While already enabling highly accurate quantum sensors for technological applications as accelerometers, gyroscopes, and gravity gradiometers, matter-wave interferometers with their unprecedented potential sensitivity are highly anticipated to serve as formidable quantum probes of fundamental physics. For this reason, concentrated international efforts are currently under way to develop this promising technology into robust and sensitive instruments.

The German QUANTUS collaboration is at the forefront of this development, having demonstrated the first Bose-Einstein condensates and matter-wave interferometers in free fall, and having recently achieved the very first BEC in space on the sounding-rocket mission MAIUS-1 in early 2017. As its long-time goal, QUANTUS is aiming at a quantum test of Einstein's famous Equivalence Principle, which is at the heart of General Relativity as a geometric theory of gravity.

In this context, it is relevant to develop a precise description of free fall in Earth's gravity beyond the usual Newtonian approximation, and thus to take into account the full reality of curved space-time in terms Einstein's theory of General Relativity. In this thesis, we take a grand tour of the relevant concepts of Special and General Relativity and eventually apply these to the modelling of free falling quantum gases. We base our description on the experimentally relevant local inertial and non-inertial frames which we can think of as moving along with experiments in free fall, for example in a drop tower, or in satellites orbiting the Earth, such as the International Space Station ISS. Our main tool are Fermi normal coordinates attached to these frames, which provide a local curvature expansion around flat space-time that exhibits local tidal effects, and can thus be seen as an expansion around the Equivalence Principle. Being fairly under-represented in the literature, we extensively discuss these Fermi coordinates, as well as the so-called Riemann normal coordinates on which they are built. In particular, we provide a new combinatorial interpretation for the complicated polynomials in the Riemann tensor and its derivatives, which arise in the expansion for the tetrads and the metric in these coordinates.

We finally apply these methods to the mean-field description of free falling Bose-Einstein condensates in the gravitational field of the Earth. Modelling the space-time curvature around our planet in terms of the Schwarzschild metric, we explicitly calculate the metric in Fermi coordinates for local inertial frames in free fall along purely radial geodesics, which approximates the experimental situation in a drop tower, as well as along circular equatorial geodesics which can be used to model the situation on satellites, such as the ISS. We then use these metrics in the non-linear Klein-Gordon equation which can be seen to generalise the usual Gross-Pitaevskii equation to curved space-time. Performing the non-relativistic limit, we obtain the different local tidal-type Newtonian and relativistic corrections and discuss their orders of magnitude.

Zusammenfassung

Effekte der allgemeinen Relativitätstheorie und darüber hinaus im Schwerkraftfeld der Erde zu messen ist ein Hauptziel gegenwärtiger Forschung und Technologieentwicklung. Diese Effekte spielen bereits eine wichtige Rolle im alltäglichen Lebens, so zum Beispiel bei der hochgenauen Orts- und Zeitbestimmung mit globalen Navigationssatellitensystemen wie GPS und GALILEO, die heutzutage allgegenwärtig sind.

Die ebenfalls von Einstein vorhergesagte, und 1995 in einem technologischen Durchbruch erstmals experimentell realisierte Bose-Einstein-Kondensation, hat seither Materiewellen-Interferometer etabliert, in welchen Laser-Pulse genutzt werden um Bose-Einstein-Kondensate kohärent aufzuspalten, abzulenken und zu rekombinieren. Während diese bereits überaus erfolgreich als hochgenaue Sensoren zur Beschleunigungsmessung, als Gyroskope, sowie als Gravimeter zur Schwerefeldmessung eingesetzt werden, besitzen Materiewellen-Interferometer mit ihrer beispiellos hohen potentiellen Empfindlichkeit ein enormes Potential als hervorragende Quantensonden für die fundamentale Physik. Aus diesem Grund werden weltweit gegenwärtig erhebliche Anstrengungen unternommen um diese vielversprechende Technologie zu robusten und hochempfindlichen Instrumenten zu entwickeln.

Das deutsche QUANTUS-Projekt (QUANTengase Unter Schwerelosigkeit) steht an vorderster Front dieser Technologieentwicklung. QUANTUS konnte in den vergangenen Jahren sowohl die ersten Bose-Einstein-Kondensate, wie auch die ersten Materiewellen-Interferometer im freien Fall demonstrieren. Anfang 2017 gelang dann im Rahmen der MAIUS-1 Mission auf einer Höhenforschungsrakete die Erzeugung des weltweit ersten Bose-Einstein-Kondensats im Weltraum. Hierbei verfolgt QUANTUS das Ziel, mithilfe der gleichzeitigen Interferometrie an zwei verschiedenen atomaren Spezies, einen Quantentest des einsteinschen Äquivalenzprinzips durchzuführen, welches die Grundlage der allgemeinen Relativitätstheorie als geometrische Theorie der Gravitation darstellt.

Vor diesem Hintergrund ist es relevant eine Beschreibung von frei fallenden Experimenten im Gravitationsfeld der Erde zur Verfügung zu haben, die über die übliche newtonschen Näherung hinausgeht, und die die Realität der gekrümmten Raumzeit in einem vollständig kovarianten Ansatz innerhalb der allgemeinen Relativitätstheorie abbildet.

In dieser Dissertation arbeiten wir detailliert die für eine solche umfassende Beschreibung relevanten Konzepte der speziellen und allgemeinen Relativitätstheorie heraus. Dabei bauen wir unsere sehr allgemeine Beschreibung auf die experimentell relevanten und durch sogenannte Vierbeine repräsentierten inertialen, bzw. nicht-inertialen lokalen Koordinatenrahmen auf, die man sich als mit dem Experiment mitfallende Koordinatensysteme vorstellen kann, so z. B. im Fallturm, oder auf Satelliten in der Erdumlaufbahn, wie beispielsweise der Internationale Raumstation ISS. Unser Hauptwerkzeug sind dabei mit diesen lokalen Rahmen verbundene Fermi-Normalkoordinaten, die eine lokale Krümmungsentwicklung um die flache Raumzeit darstellen, und somit als eine Entwicklung um das Äquivalenzprinzip verstanden werden können. Diese in der Forschungsliteratur stark unterrepräsentierten Fermi-Koordinaten, sowie die ihnen zugrundeliegenden sogenannten

Riemannschen Normalkoordinaten, werden von uns ausgiebig diskutiert. Insbesondere können wir eine neue kombinatorische Interpretation für die bei der Entwicklung des Vierbeins und der Metrik in diesen Koordinaten auftretenden komplizierten Polynome im Riemann-Tensor und seinen Ableitungen angeben.

Abschließend wenden wir diese von uns erarbeiteten allgemein-relativistischen Entwicklungen auf die mean-field-Beschreibung von im Schwerkraftfeld der Erde frei fallende Bose-Einstein-Kondensaten an. Dabei modellieren wir die von der Erde erzeugte Raumzeit-Krümmung durch die Schwarzschild-Metrik, für die wir die lokale Metrik in Fermi-Koordinaten entlang von radialen und Kreisgeodäten ausrechnen. Mithilfe einer relativistischen Verallgemeinerung der Gross-Pitaevskii-Gleichung in Form der nichtlineare Klein-Gordon-Gleichung und der Metrik in Fermi-Koordinaten, erhalten wir im nichtrelativistischen Grenzfall die verschiedenen lokalen gezeitenartigen newtonschen und relativistischen Korrekturen und diskutieren ihre Größenordnungen.

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1 Introduction

Recently, gravity has been "making waves" with the celebrated first *direct* detection of gravitational waves from the in-spiral and violent merger of a binary black hole by the LIGO collaboration's two large laser interferometers in 2015 [1–3], and the subsequent first detection of gravitational waves from a binary neutron star merger in 2017, jointly with the newly-operational Virgo detector in Italy [4]. For their "decisive contributions to the LIGO detector and the observation of gravitational waves", LIGO's Rainer Weiss, Barry C. Barish, and Kip S. Thorne were awarded the Nobel Prize in Physics for 2017.

Another type of wave has also created much excitement in the last years. These are the macroscopic matter waves of Bose-Einstein condensates, a fragile collective quantum state of atoms, which arises when atom clouds suspended in ultra-high vacuum are cooled to near absolute zero temperature, where they can be made to condense into their quantum state of lowest energy. Matter waves can be coherently split, reflected, and recombined by lasers, thereby creating interferometers with potentially unprecedented sensitivity. A great world-wide effort is currently underway to turn these devices into practical quantum sensors.

Both these important phenomena of modern physics, the theory of general relativity which gives rise to gravitational waves, and Bose-Einstein condensates as giant matter waves, are connected with the name Albert Einstein, and both come together in the present thesis.

Einstein's General Relativity and the Equivalence Principle

Einstein's theory of General Relativity of 1915 is a mathematically beautiful geometric theory of gravity in which the flat space-time of special relativity becomes curved and dynamical, with an important part of its curvature being sourced by all forms of matter-energy-momentum in the universe. It describes our world on large scales where gravity dominates, from planets, stars, and galaxies, up to the universe itself.

Einstein's geometrisation of gravity is crucially based on the universality of free fall, i.e. on the theoretical assumption and experimental fact known already to Galileo Galilei that in vacuo, all bodies fall at the same rate, independent of their mass and composition. This means that gravitational mass and inertial mass are equal, which makes Newton's equations of motion for a test body subject to gravity completely independent of that body's mass. This is the (weak) *equivalence principle*, famously described by Einstein in terms of his elevator gedanken experiment.

General relativity, and in particular the equivalence principle, have stood the test of time, i.e. its predictions have been tested throughout its history, from tests on Earth with torsion pendula and the now classic solar-system tests like lunar laser ranging (see [5] for a comprehensive overview), to modern satellite-based free-fall tests, such as the recent MICROSCOPE mission [6]. Excitingly, it is anticipated that *quantum* tests of the equivalence principle employing interferometry with Bose-Einstein condensates will become available in the near future, and offer unprecedented precision. This will be motivated in the following two sections below.

Bose-Einstein Condensates and Matter-Wave Interferometry

The history of what we now call Bose-Einstein condensation (BEC) goes back to Indian self-taught scholar and polymath Satyendra Nath Bose. In 1924, while working as a lecturer at Dhaka University, Bose was able to derive Planck's law of black-body radiation with purely statistical arguments, treating photons as indistinguishable particles and "...only assuming that the ultimate elementary region in the phase-space has the content h^3..."[†], thereby creating the field of quantum statistics. However, his paper was turned down by some major European physics journals. It was not until he sent it to Albert Einstein who immediately recognised its importance, translating it to German and arranging for it to be published in the prestigious *Zeitschrift für Physik* on Bose's behalf [7]. Einstein then applied Bose's novel statistical method not to photons, but to the atoms of ideal gases in a subsequent two-part article [8, 9], published in 1924 and 1925. Therein, he made the surprising discovery that above a certain critical phase space density connected to a corresponding critical temperature, all atoms would "condense" into the the quantum state of lowest energy, usually the ground state.

From this theoretical prediction, it took years before interest in Bose-Einstein condensation arose again in connection with the discovery of superfluidity in ultracold liquid helium by F. London in 1938 [10, 11], for which Landau developed his two-fluid theory [12], and Bogoliubov applied his microscopic theory of interacting Bose gases some ten years later [13]. However, due to helium being a liquid at these low temperatures these predictions were found to agree only qualitatively with the corresponding experiments. Clearly, a better system was needed for experimentally testing Einstein's predictions. Again, it took years of refinement of experimental methods, such as the development of optical and magnetic trapping, and in particular that of evaporative cooling and laser-cooling techniques by Steven Chu, Claude Cohen-Tannoudji, William D. Phillips [14], and others in the 1970s (earning them the Nobel Prize in Physics of 1997), until three groups succeeded in experimentally achieving Bose-Einstein condensation in ultracold dilute alkali metal vapours in 1995. First realised by the group of Eric A. Cornell and Carl E. Wieman in a small sample of ^{87}Rb [15], a few months later, the group of Wolfgang Ketterle was able to create a condensate of ^{23}Na with massively more atoms [16], which enabled them to demonstrate and study coherence properties of a BEC for the first time [17], and thus exhibiting its wave nature. This breakthrough was recognised with the Nobel Prize in Physics in 2001, awarded in equal parts to Eric A. Cornell, Wolfgang Ketterle and Carl E. Wieman.

The concept of this wave nature of matter goes back to the very early days of quantum theory, when Louis de Broglie (Nobel Prize for Physics in 1929) suggested in his PhD thesis of 1924 [18] that like the photons that make up light, particles with mass such as electrons, should also be described in terms of waves, proposing his famous formula for

[†] S. N. Bose in his letter to A. Einstein dated 4$^{\text{th}}$ of June, 1924. Here, h is the Planck constant.

what is now called the de Broglie wavelength λ_{dB} of these *matter waves*,

$$\lambda_{\text{dB}} = \frac{h}{p} = \frac{h}{mv}\,, \tag{1.1}$$

where h is the Planck constant an p the particle's momentum in terms of its mass m and velocity v. When cold atom sources were becoming available, it was soon realised that – due to its inverse mass dependence – the de Broglie wavelength of atoms is typically orders of magnitude smaller than that of light and would thus make matter-wave interferometers enormously more sensitive as compared to the usual laser interferometers. After early interferometry experiments with electrons [19–21], and in particular with neutrons [22–25], the concept of an atom interferometer was in fact patented by Altschuler and Franz in 1973, and has been discussed ever since. Apart from their much larger mass, atoms offer many other advantages over electrons and neutrons (see, e.g. the review [26]), a crucial example being of course that their electron shell allows for their manipulation with lasers.

In particular the macroscopic matter waves of Bose-Einstein condensates are ideally suited for atom interferometry due to their small momentum spread [27], and nowadays, these matter-wave interferometers are considered to be a universal tool in physics [28], finding applications as accelerometers, gyroscopes, and gravity gradiometers, as well as in probing fundamental physics, for example in improving gravitational-wave detectors by coupling them to laser interferometers [29–31], or in exciting proposals to test Einstein's Equivalence Principle at the quantum level [32–34].

The QUANTUS Project – BEC and Atom-Interferometry in Free Fall

Since 2004, the German Aerospace Centre DLR has been funding the QUANTUS project (QUANTen Gase Unter Schwerelosigkeit – Quantum Gases in Weightlessness) and its sister projects, a collaboration of several research groups with the aim to establish ultracold atoms as high precision quantum sensors in the microgravity of free fall. Its first experimental apparatus, called QUANTUS-1, was built with the goal of demonstrating Bose-Einstein condensation in microgravity, for which the 146 m-tall drop tower at the University of Bremen's Centre of Applied Space Technology and Microgravity (ZARM) was chosen, offering up to 4.7 seconds of free fall in drop mode. This involved the development of a robust and compact experiment, able to fit inside a standardised ZARM drop capsule, and capable of withstanding the high accelerations that occur when the capsule plunges into the Styrofoam-filled recovery vessel at the bottom of the drop tower. Thus, a whole BEC experiment had to be ruggedised and dramatically scale down from the usual laboratory scales to the scale of a typical drop capsule 2 m tall and 80 cm wide, which includes lasers, electronics, and an imaging system, as well as the power supply in the form of batteries. With the first BEC of ^{87}Rb atoms in extended free fall created in 2007, later drop campaigns reached unprecedented expansion times of ≈ 1 s and truly macroscopic sizes of ≈ 2 mm [35, 36].

Starting already in 2008, a new and improved apparatus was conceived in the form of the QUANTUS-2 experiment, which was built to utilise the drop tower's catapult mode, offering an almost doubled free-fall time of 9.2 s. The goal of this second phase of the QUANTUS project now was to demonstrate the feasibility of a simultaneous, atom-

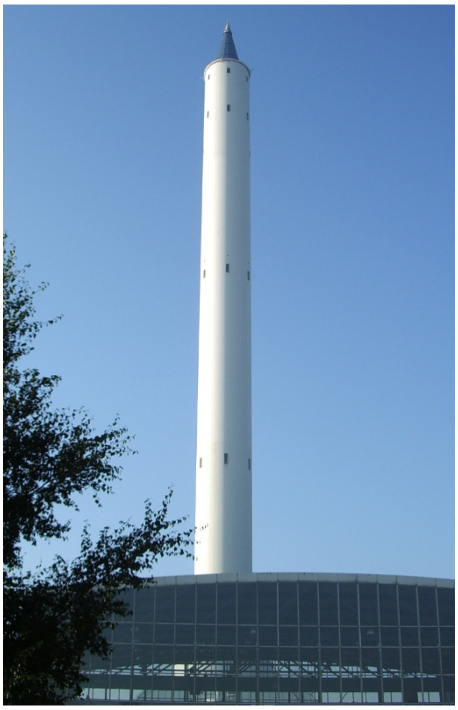

Figure 1.1.: The 146 m-tall drop tower at the University of Bremen's ZARM facility, in which most of the micro-gravity experiments of the QUANTUS collaboration are performed.

interferometric comparison of BEC's of two different atomic species, with the long-time goal of performing a quantum test of Einstein's equivalence principle with Bose-Einstein condensates [33]. After becoming operational, the first interferometry experiments with BECs in microgravity were reported in 2013 [37].

After the ground-based experiments QUANTUS-1 and -2, the next phase was a third evolution and further compactification of the experiment, which enabled for it to be installed in the payload section of a sounding rocket, this incarnation being given a new name MAIUS. After some delays, the MAIUS-1 sounding rocket mission was successfully launched on 23$^{\text{rd}}$ January 2017 from the space-port at Kiruna, Sweden, creating the first-ever Bose-Einstein condensates in space (defined to start above the Kármán line at 100 km altitude) [38]. On its sub-orbital trajectory reaching an apogee of 243 km altitude and covering a ground-track distance of just about 60 km in northern Sweden, the MAIUS-1 apparatus *autonomously* performed a total of 110 experiments related to matter-wave interferometry, a large majority of which occurring during the 6 minutes of exceptionally good microgravity conditions in space above 100 km altitude, but some of them even during the sounding rocket's boost phase under strong accelerations. An illustration of the flight phases and typical experiments performed at different stages is shown in Figure 1.2.

Setting and Approach

As motivated above, in the context of the promises of atom interferometry and the enormous technological progress that has been made in recent years in developing compact and

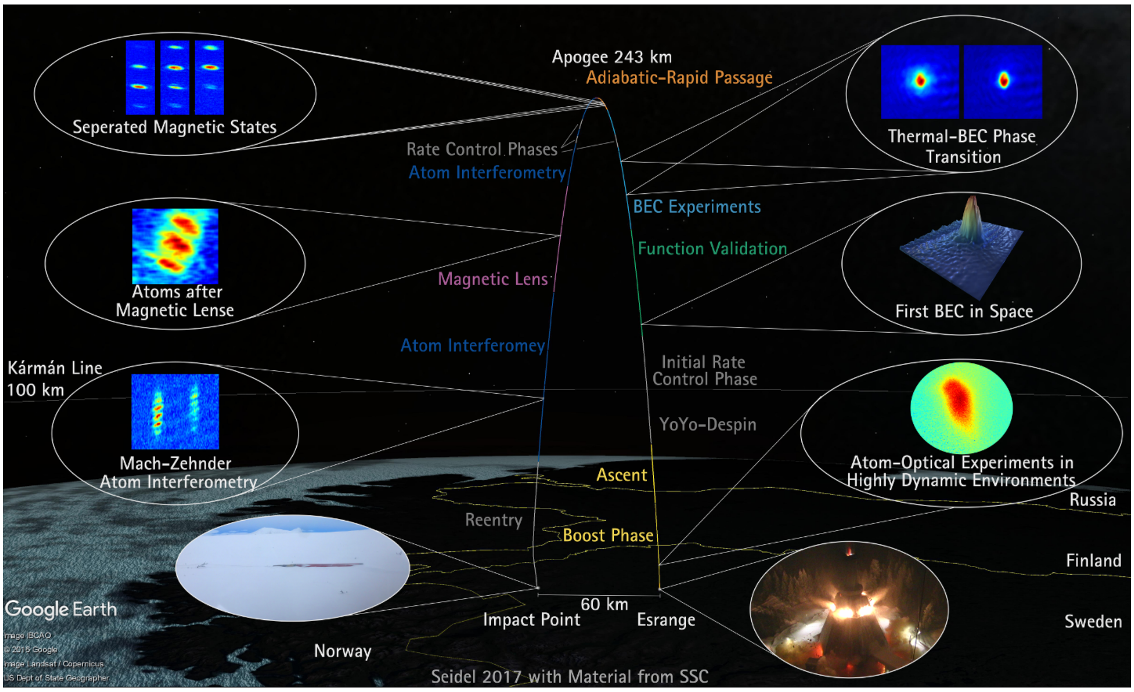

Figure 1.2.: The first BECs in space: Illustration of the different flight phases and corresponding BEC experiments performed in space in the microgravity of free fall on the QUANTUS collaboration's MAIUS-1 sounding rocket mission above northern Sweden on 23$^{\text{rd}}$ January 2017. The miniaturised, extremely robust, and autonomously operating apparatus achieved continuous laser lock an performed atom-optics experiments even during the initial boost phase under strong accelerations (image: S. Seidel, priv. comm.).

robust matter-wave interferometers which will soon be ready for extended measurement campaigns in space, it is relevant – on the theoretical side – to go beyond the usual Newtonian description of Bose-Einstein condensates and to generalise their description to the curved space-time of general relativity.

Thus, in this thesis we ask the following questions: How can local experiments on Earth or in space be described within general relativity, and how can this be applied consistently to a mean-field description of Bose-Einstein condensates in terms of a suitable relativistic generalisation of the Gross–Pitaevskii equation? Moreover, of what type are the arising corrections to Newtonian physics, and how can these be quantified?

In order to answer the above questions, and because of the local nature of these experiments, we base our description on local frames and their attached local coordinates, since these are the natural reference systems of observers (i.e. either scientists in person, or a readout device such as, e.g., a CCD camera) which move along with the experimental apparatus. These frames will be inertial if the attached experiment is in free fall, but can also be non-inertial, i.e. rotating and accelerating. In this description and at the origin of these local coordinates, the metric tensor that describes space-time curvature is locally the flat Minkowski metric of special relativity, but quadratic and higher-order curvature corrections of tidal nature appear at second and higher orders in the radial distance from the origin. In this sense, these so-called Fermi normal coordinates can be seen to provide an expansion around Einstein's equivalence principle.

Overview of this Thesis

After introducing the usual Newtonian mean-field description of Bose-Einstein condensates in terms of the Gross-Pitaevskii equation and motivating its generalisation to special relativity in chapter 2, we start in chapter 3 with a – hopefully pedagogical – introduction to the necessary mathematical basics and elements of differential geometry, such as manifolds, tensors, connections and covariant derivatives, etc.

Chapter 4 begins with a brief recapitulation of the basics of special relativity in its usual formulation in terms of global inertial frames, and of the Poincaré and Lorentz groups and transformations. We then turn our attention to the description of special relativity in general coordinates and non-inertial frames, which necessitates already the introduction of general coordinate transformations and a general metric tensor, as well as non-trivial tetrads along observers' world-lines, and their transport law in terms of acceleration and rotation. We then explicitly construct such non-inertial local coordinates and work out expressions for the tetrads and the metric, before we show how the tetrad's transport equation can be solved explicitly for time-independent inertial forces, which leads to different Lorentz-invariant types of motion, and in particular, to exact local frames for circular world-lines. We close that chapter with an expansion of the geodesic equation in these local coordinates, which is seen to describe the geodesic motion of test particles as seen from the fiducial non-inertial frame in which their motion occurs.

Building on chapter 4, the following chapter 5 introduces some elements of General Relativity, starting with its foundations, i.e. the geometric nature of gravity and the equivalence principles. We then move on to the discussion of space-time symmetries in terms of Killing vector fields that need to be imposed in order to find exact solutions, thereby motivating stationary and axisymmetric space-times as the most general family that can usually be solved exactly, and which is seen to describe idealised astrophysical objects such as planets, stars, and black holes. Subsequently, we come to the two well-known examples of these, namely the Schwarzschild and Kerr metrics. We continue with a motivation of the hierarchy of general equations of motion, exhibiting the Mathisson-Papapetrou-Dixon equation, before we restrict ourselves to the lowest order of the hierarchy in terms of the usual geodesic equation. As a simple example, we discuss the radial geodesics of the Schwarzschild metric, and then turn to the description of inertial and non-inertial observers in General Relativity, where we focus on frames along circular world-lines. The following section is then devoted to the curvature tensors, where we discuss, in particular, a convenient representation of the Weyl tensor in terms of two 3×3 Cartesian matrices and its connection to the famous Petrov classification of vacuum space-times. We close in the last section with a discussion of the limitations of exact metrics, giving a brief introduction to aspects of metric perturbation theory and providing an outlook to the post-Minkowskian and post-Newtonian expansions that are built on such a first-order perturbation approach.

Our chapter 6 then provides a pedagogical introduction to, as well as an extensive discussion of the two different types of local frames and their attached local coordinates, one being purely mathematical and one physical. At first we show how purely mathematical (but unphysical) local coordinates can be set up around a single space-time event in terms of a first-order Taylor expansion along all spacial geodesics emanating from that event, and then how these coordinates can be extended to arbitrary orders in radial distance from the origin in terms of so-called *Riemann normal coordinates*. Clearly, if the coordinate system is given by a Taylor expansion, then so are the components of all tensor fields. Therefore, our main focus is on calculating their tensorial expansion coefficients, most importantly those for the inverse tetrad and the metric, for which we derive closed formulas and relate these to a new combinatorial interpretation in terms of restricted non-commutative Bell polynomials.

In chapter 7 finally, we apply the methods introduced previously to properly describe a free falling Bose-Einstein condensate. After motivating the non-linear Klein-Gordon equation in curved space-time as the natural covariant generalisation of the Gross-Pitaevskii equation for a corresponding mean-field description of BECs in General Relativity, we first explicitely calculate the second-order metrics in Fermi normal coordinates for BECs in free fall along purely radial and circular equatorial geodesics of the Schwarzschild metric, which we take as our approximate model of the weak physical space-time metric of the Earth. We then expand the non-linear Klein-Gordon equation in terms of a convenient perturbation approach. Therein, we are able to compactly perform the non-relativistic limit, in which relativistic corrections appear in a systematic way, up to the order of our expansion. We exhibit these correction terms for the simpler case of a BEC in purely radial free fall, and then move on to a discussion of the interesting fully relativistic tidal potential that arises from the Fermi metric for BECs on circular orbits, providing the orders of magnitude of the main relativistic corrections. This establishes a consistent perturbative approach to freely falling BECs in the weak gravitational field of the Earth.

2 Ultracold Quantum Gases and Bose-Einstein Condensates

2.1. Bose-Einstein Condensation

Bose-Einstein condensation of massive particles is a consequence of the Pauli principle which requires that bosonic particles have symmetric quantum wave functions. Therefore, many of these particles can occupy the same quantum state at low temperature. Physically, there exists a critical temperature T_c at which one has a finite probability for two particles to occupy the same phase-space volume $\Delta x\,\Delta p = h$. A simple argument to derive this critical temperature equates the thermal de Broglie wavelength,

$$\lambda(T) = \frac{h}{\sqrt{3mk_\mathrm{B}T}}\,, \tag{2.1}$$

to the mean inter-particle distance, $d = \rho^{-1/3}$. Here we have introduced the density ρ and mass m of a particle ensemble, as well as the thermodynamic energy $k_\mathrm{B}T$, where k_B is Boltzmann's constant. The equality of both length scales, $d = \lambda(T_\mathrm{c})$, then defines this critical temperature, which is given by

$$T_\mathrm{c} = \frac{h^2\rho^{2/3}}{3mk_\mathrm{B}}\,. \tag{2.2}$$

This simple order-of-magnitude estimate applies to a three-dimensional homogeneous gas, whereas much more elaborate discussions, in particular for trapped systems, can be found in [39, 40].

2.2. Mean-Field Description and Gross-Pitaevskii Equation

In the simple discussion above, we have so far disregarded the effect of inter-particle interactions. However, these interactions are key to evaporative cooling and to thermalisation, without both of which no BEC could ever be achieved experimentally. There are now two standard approaches for introducing interactions into condensed many-particle systems, the first being the Hartree-Fock method, and the second one the so-called symmetry-breaking approach. We shall follow the latter, as we are only interested in deriving a classical field theory for the matter-wave field. We start by introducing a non-relativistic quantum field of interacting bosons in terms of (see, e.g. [41]), the field operators $\hat{\Psi}(\boldsymbol{x},t)$ and $\hat{\Psi}^\dagger(\boldsymbol{x},t)$ that annihilate and create a particle of mass m at position $\boldsymbol{x}$, respectively. These satisfy the bosonic equal-time commutation relations in the Heisenberg picture,

$$[\hat{\Psi}(\boldsymbol{x},t),\hat{\Psi}(\boldsymbol{x}',t)] = 0 = [\hat{\Psi}(\boldsymbol{x},t),\hat{\Psi}^\dagger(\boldsymbol{x}',t)]\,, \qquad [\hat{\Psi}(\boldsymbol{x},t),\hat{\Psi}^\dagger(\boldsymbol{x}',t)] = \delta(\boldsymbol{x}-\boldsymbol{x}')\,. \tag{2.3}$$

The energy of a dilute gas is a series expansion in terms of powers of the one-particle density. As the density of atomic quantum gases is extremely low (typically $\rho < 10^{14}\,\mathrm{cm}^{-3}$), one can truncate this series at the level of two-particle interactions (disregarding three-particle collisions). In this limit, the field Hamiltonian $\hat{H}$ reads,

$$\begin{aligned}\hat{H}[\hat{\Psi},\hat{\Psi}^\dagger] = \int d^3x\,\hat{\Psi}^\dagger(\boldsymbol{x})\Big[-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + V_{\text{ext}}(\boldsymbol{x},t)\Big]\hat{\Psi}(\boldsymbol{x})&\\ +\tfrac{1}{2}\int d^3x\int d^3x'\,\hat{\Psi}^\dagger(\boldsymbol{x})\hat{\Psi}^\dagger(\boldsymbol{x}')\,V_{\text{int}}(\boldsymbol{x}-\boldsymbol{x}')\,\hat{\Psi}(\boldsymbol{x}')\hat{\Psi}(\boldsymbol{x})\,,&\end{aligned} \tag{2.4}$$

where V_{int} models the inter-atomic van der Waals potential giving rise to the two-particle interactions (and is usually obtained from spectroscopy). The corresponding conserved particle-number operator is given by,

$$\hat{N}[\hat{\Psi},\hat{\Psi}^\dagger] = \int d^3x\,\hat{\Psi}^\dagger(\boldsymbol{x})\hat{\Psi}(\boldsymbol{x})\,. \tag{2.5}$$

The dynamics of the quantum field $\hat{\Psi}(\boldsymbol{x},t)$ can now be derived from the Heisenberg equation of motion for the many-body Hamiltonian (2.4),

$$\begin{aligned}i\hbar\frac{d}{dt}\hat{\Psi}(\boldsymbol{x},t) &= [\hat{\Psi}(\boldsymbol{x},t),\hat{H}]\\ &= \Big[-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + V_{\text{ext}}(\boldsymbol{x},t) + \int d^3x'\,\hat{\Psi}^\dagger(\boldsymbol{x}')V_{\text{int}}(\boldsymbol{x}-\boldsymbol{x}')\hat{\Psi}(\boldsymbol{x}')\Big]\hat{\Psi}(\boldsymbol{x},t)\,,\end{aligned} \tag{2.6}$$

resulting in a non-linear, operator-valued Schrödinger equation which is notoriously difficult to solve for N particles.

We now assume that we have many particles, $N \gg 1$, at temperatures $T \ll T_{\mathrm{c}}$. Then, the quantum field $\hat{\Psi}(\boldsymbol{x},t)$ acquires a large classical amplitude, given by the macroscopically populated expectation value $\psi(\boldsymbol{x},t) \coloneqq \langle\hat{\Psi}(\boldsymbol{x},t)\rangle$, which is called the condensate field. The complex function $\psi(\boldsymbol{x},t)$ thus represents the condensed part, i.e. the BEC, which is now a classical field having the meaning of an order parameter. Thus the field-operator can be written as,

$$\hat{\Psi}(\boldsymbol{x},t) = \psi(\boldsymbol{x},t) + \delta\hat{\Psi}(\boldsymbol{x},t)\,, \tag{2.7}$$

in terms of ψ and a small operator-valued residual perturbation with vanishing mean value, $\langle\delta\hat{\Psi}(\boldsymbol{x},t)\rangle = 0$, that is taken to represent the non-condensed atoms. Due to relation (2.5), and the assumption of vanishing thermal fraction, we have the normalisation $N = \int d^3x\,\psi(\boldsymbol{x},t)^*\psi(\boldsymbol{x},t)$. This interpretation of the mean field represents only the low-energy response to scattering theory. At low temperatures, this is characterised by the s-wave scattering length a_{s} and one can approximate the full van der Waals interaction potential by a so-called pseudo potential [42],

$$V_{\text{int}}(\boldsymbol{x}) = \frac{4\pi\hbar^2 a_{\mathrm{s}}}{m}\delta(\boldsymbol{x})\,. \tag{2.8}$$

Inserting now (2.7) into the Heisenberg equation (2.6) then yields the Gross-Pitaevskii equation which was derived independently by Gross [43, 44] and Pitaevskii [45],

$$\boxed{i\hbar \frac{d\psi(\boldsymbol{x},t)}{dt} = \left[-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + V_{\text{ext}}(\boldsymbol{x},t) + \frac{4\pi\hbar^2 a_{\text{s}}}{m}|\psi|^2\right]\psi(\boldsymbol{x},t)\,.} \tag{2.9}$$

2.3. Special Relativistic Generalisation of Gross-Pitaevskii

Since our aim is to find a relativistic generalisation of the Gross-Pitaevskii equation (2.9), it is helpful to summarise its relevant properties. For one, (2.9) is a *scalar* equation for the scalar condensate field $\psi(\boldsymbol{r},t)$. Secondly, it features a characteristic $|\psi|^2$ interaction term that models the particle-particle interaction in a BEC in the approximation of pure s-wave scattering.

Writing down a special relativistic generalisation Gross-Pitaevskii equation (2.9), is now straightforward: being a non-linear Schrödinger equation, and recalling that the relativistic generalisation of the Schrödinger equation is the Klein-Gordon equation for a complex scalar field ϕ with mass m, we only have to deal with the non-linear interaction term in (2.9). Since this term is a scalar field and thus invariant under all transformations (Lorentz and also general-coordinate, etc.), such a term can also be added without modification in the form $\xi|\phi|^2\phi$, where we have introduced a coupling constant ξ. Thus, the special-relativistic generalisation of (2.9) can be written as,

$$\boxed{\partial^\alpha\partial_\alpha\phi + \left(\frac{mc}{\hbar}\right)^2\phi + \xi|\phi|^2\phi = 0\,,} \tag{2.10}$$

where the kinetic operator is the usual d'Alembertian $\partial^\alpha\partial_\alpha = \eta^{\alpha\beta}\partial_\alpha\partial_\beta$ in Minkowski space. If ϕ is taken to be a relativistic *quantum* field, (2.10) is the equation of motion of the so-called $|\phi|^4$-theory that frequently serves as a toy model in textbooks on quantum field theory. Note that there is no trapping "potential" in (2.10), since the usual Newtonian or classical-physics potentials are not covariant, except if they are "dynamically created" by another relativistic field, e.g. as an interaction term. In this sense, the non-linear term $\xi|\phi|^2$ in (2.10) is usually referred to as a "self-interaction potential".

Equation (2.10) follows from the variation principle for an action that is determined by a corresponding Lagrangean density of $|\phi|^4$-theory ,

$$\mathcal{L}[\phi,\phi^*,\partial_\alpha\phi,\partial_\alpha\phi^*] = \tfrac{1}{2}\Big\{\eta^{\alpha\beta}(\partial_\alpha\phi^*)(\partial_\beta\phi) - \left(\frac{mc}{\hbar}\right)^2\phi^*\phi - \tfrac{1}{2}\xi(\phi^*\phi)^2\Big\}\,, \tag{2.11}$$

Being its relativistic generalisation, the Klein-Gordon equation reduces to the Schrödinger equation upon performing the non-relativistic limit, which is effected by separating off the phase associated with the rest-mass energy $E_0 = mc^2$, i.e.

$$\phi = \psi\,\exp\!\Big(-i\frac{mc^2}{\hbar}\tau\Big)\,. \tag{2.12}$$

In the same fashion, the non-linear Klein-Gordon equation (2.10) then reduces to the Gross-Pitaevskii equation (2.9), as long as ϕ is a *complex* scalar field: Separating the kinetic term of (2.10) into space and time components, we have,

$$\left(\tfrac{1}{c}\right)^2\partial_t^2\phi - \boldsymbol{\nabla}^2\phi + \left(\frac{mc}{\hbar}\right)^2\phi + \xi|\phi|^2\phi = 0\,, \tag{2.13}$$

so we need to calculate the second time derivative of ϕ in terms of the ansatz (2.12). We obtain,

$$\dot{\phi} = \left[\dot{\psi} - i\left(\frac{mc^2}{\hbar}\right)\psi\right]e^{-i\frac{mc^2}{\hbar}t}\,, \tag{2.14a}$$

$$\ddot{\phi} = \left[\ddot{\psi} - 2i\left(\frac{mc^2}{\hbar}\right)\dot{\psi} - \left(\frac{mc^2}{\hbar}\right)^2\psi\right]e^{-i\frac{mc^2}{\hbar}t}\,. \tag{2.14b}$$

Using these in (2.13), as as well as multiplying through by $-\frac{\hbar^2}{2m}$ yields,

$$i\hbar\dot{\psi} = -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\psi + \frac{\hbar^2\xi}{2m}|\psi|^2\psi + \frac{\hbar^2}{2mc^2}\ddot{\psi}\,, \tag{2.15}$$

which also gives $\xi = 8\pi a_\mathrm{s}$ upon comparing with (2.9). Here the $\ddot{\psi}$ term is a relativistic correction, which can be inferred from the prefactor $\propto \left(\frac{1}{c}\right)^2$. This second time derivative of ψ can be seen a coupling to the anti-particle aspect of the ψ field. Of course, upon neglecting this relativistic correction, we retain,

$$i\hbar\dot{\psi} = -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\psi + \frac{4\pi\hbar^2 a_\mathrm{s}}{m}|\psi|^2\psi\,, \tag{2.16}$$

which is just the Gross-Pitaevskii equation (2.9).

3 Elements of Differential Geometry

In this chapter, we set the mathematical stage for the physics of special relativity and of the curved space-time of general relativity that we discuss in the rest of the present thesis. On the one hand, our aim is to collect the necessary concepts and formulae for later reference, on the other hand, we also wish to provide a brief and minimal introduction to basic elements of differential geometry to readers who are not fully familiar with the field of general relativity and its mathematical foundations, hoping that in this way, this thesis will become more accessible.

We hope to do this by going step by step, starting from the most general physically acceptable concept of a differential manifold, and then introducing the additional physically relevant mathematical structures, such as parallel translation, norm (i.e., a metric), etc., in a somewhat informal way. Correspondingly, we introduce vectors and tensors in an index-free fashion as intrinsically geometrical objects, and then show how one obtains the corresponding component "tensors", that we shall use in the rest of this thesis.

Literature we have found useful in preparing this chapter includes Nakahara's textbook [46], as well as the book by Straumann [47, part III], some parts of Misner, Thorne and Wheeler's *Gravitation* [48] and also Wald [49] and Chandrasekhar [50, chapter 1].

3.1. Basics on Manifolds, Bases, Tensors, and Differential Forms

The natural setting for a geometric theory of gravity is a differentiable manifold $\mathcal{M}$, which can be thought of very broadly as a generalisation of the usual concept of a vector space to a new mathematical entity which may be curved and which consists of all the actual (tangent) vector spaces, one for each point.

3.1.1. Differentiable Manifolds

Heuristically, a manifold $\mathcal{M}$ is a topological space that locally resembles Euclidean space $\mathbb{R}^N$ in a neighbourhood of every point, although it may be different from $\mathbb{R}^N$ globally, in particular, it may be curved. Mathematically, an N-dimensional differentiable manifold can be defined by the following criteria (we follow Nakahara [46]):

1. $\mathcal{M}$ is provided with a family (called an *atlas*) $\{(U_i, \chi_i)\}$ of tuples (U_i, χ_i), which are called *charts* for obvious reasons.

2. The $\{U_i\}$ are a family of open sets which cover $\mathcal{M}$, i.e., $\bigcup_i U_i = \mathcal{M}$, called the coordinate neighbourhood. The map $\chi_i : U_i \to \tilde{U}_i$ is a homeomorphism from U_i onto a corresponding open subset $\tilde{U}_i$ of $\mathbb{R}^N$, called the *coordinate function*.

3. Given two non-intersecting open sets U_i and U_j, the map $\chi_{ij} = \chi_i \circ \chi_j^{-1}$ from $\chi_j(U_i \cap U_j)$ to $\chi_i(U_i \cap U_j)$ is infinitely differentiable.

The homeomorphism χ_i is represented by N functions $\{X^\mu\}$, which are also called *coordinates*. The significance of differentiable manifolds lies in the fact that we may use the usual calculus developed for $\mathbb{R}^n$, where smoothness of the coordinate transformations ensures the independence of the chosen coordinates.

To every point $\mathscr{P}$ of the manifold, there is attached a real vector space, called the *tangent space*, and denoted $\mathcal{T}_{\mathscr{P}}\mathcal{M}$. The tangent space consists of all the vectors which are tangent to those curves in $\mathcal{M}$ that pass through the point $\mathscr{P}$. Every vector space has an associated dual space of one-forms, or co-vectors, and thus the dual space to $\mathcal{T}_{\mathscr{P}}\mathcal{M}$ is called the co-tangent space and denoted $\mathcal{T}^*_{\mathscr{P}}\mathcal{M}$.

3.1.2. Bases – Coordinate and Non-Coordinate

For every point $\mathscr{P} \in \mathcal{M}$, a natural basis for tangent vector fields in $\mathcal{T}_{\mathscr{P}}\mathcal{M}$ is induced by the partial derivatives $\partial_\mu := \partial/\partial X^\mu$ along the coordinate lines of a given coordinate chart. This basis $\{\boldsymbol{\partial}_\mu\}$ is simply called the *coordinate basis* and the corresponding basis of $\mathcal{T}^*_{\mathscr{P}}\mathcal{M}$, or co-basis, is then given by $\{\boldsymbol{d}X^\mu\}$. The coordinate basis is the basis that is implicitly used in the usual component approach to tensor calculus ("Ricci calculus"), that should be most familiar to physicists, and that we will also use in most of this thesis.

However, the coordinate basis is only a special case of a general basis, and it is useful to discuss the arising properties and relations for the general case first and specialise later. The vectors $\boldsymbol{e}_\alpha$ of a general *non-coordinate* (or *non-holonomic*) basis $\{\boldsymbol{e}_\alpha\}$ can then be constructed in terms of a general coordinate transformation ${e_\alpha}^\mu$ as as a linear combination of the coordinate basis vectors,

$$\boldsymbol{e}_\alpha = {e_\alpha}^\mu \boldsymbol{\partial}_\mu \,, \qquad\qquad {e_\alpha}^\mu \in \mathrm{GL}(N, \mathbb{R}) \,. \tag{3.1}$$

A given basis of $\mathcal{T}_{\mathscr{P}}\mathcal{M}$ then induces a corresponding co-basis of the co-tangent space $\mathcal{T}^*_{\mathscr{P}}\mathcal{M}$ through the natural but basis-dependent definition of an inner product,

$$\boldsymbol{\theta}^\beta(\boldsymbol{e}_\alpha) := \delta^\beta{}_\alpha \,, \tag{3.2}$$

which associates with every $\boldsymbol{e}_\alpha$ the corresponding co-vector $\boldsymbol{\theta}^\alpha$ of the co-basis $\{\boldsymbol{\theta}^\alpha\}$. Just as with (3.1), co-basis vectors can be expanded in terms of the coordinate co-basis and the Jacobi matrix of a general coordinate transformation $\theta^\alpha{}_\mu$, as

$$\boldsymbol{\theta}^\beta = \theta^\beta{}_\nu \boldsymbol{d}X^\nu \,, \qquad\qquad \theta^\beta{}_\nu \in \mathrm{GL}(N, \mathbb{R}) \,, \tag{3.3}$$

where (3.2) leads to

$$\theta^\beta{}_\nu {e_\alpha}^\nu = \delta^\beta{}_\alpha \,. \tag{3.4}$$

In contrast to the coordinate-basis vectors introduced above, the non-coordinate basis vectors have a non-vanishing Lie bracket or commutator,

$$[\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta] = c^\delta{}_{\alpha\beta}\, \boldsymbol{e}_\delta \,, \qquad\qquad c^\gamma{}_{\alpha\beta} := \theta^\gamma{}_\nu \left({e_\alpha}^\mu \partial_\mu {e_\beta}^\nu - {e_\beta}^\mu \partial_\mu {e_\alpha}^\nu\right) , \tag{3.5}$$

which defines the anti-symmetric *structure coefficients* or *coefficients of anholonomicity* $c^\gamma{}_{\alpha\beta}$ of the basis.

3.1.3. Tensors and their Transformation Law

An (n,s) tensor $\boldsymbol{A}$ is a multi-linear map

$$\boldsymbol{A}:\underbrace{\mathcal{T}^*_{\mathscr{P}}\times\cdots\times\mathcal{T}^*_{\mathscr{P}}}_{n\text{ times}}\times\underbrace{\mathcal{T}_{\mathscr{P}}\times\cdots\times\mathcal{T}_{\mathscr{P}}}_{s\text{ times}}\to\mathbb{R}\,, \tag{3.6}$$

that maps n elements (i.e. co-vectors) $\boldsymbol{\chi}_1,\ldots,\boldsymbol{\chi}_n$ of $\mathcal{T}^*_{\mathscr{P}}\mathcal{M}$ and s elements (i.e. vectors) $\boldsymbol{X}_1,\ldots,\boldsymbol{X}_s$ of $\mathcal{T}_{\mathscr{P}}\mathcal{M}$ to a real number,

$$(\boldsymbol{\chi}_1,\ldots,\boldsymbol{\chi}_n,\,\boldsymbol{X}_1,\ldots,\boldsymbol{X}_s)\mapsto\boldsymbol{A}(\boldsymbol{\chi}_1,\ldots,\boldsymbol{\chi}_n,\,\boldsymbol{X}_1,\ldots,\boldsymbol{X}_s)\in\mathbb{R}\,. \tag{3.7}$$

The tensors themselves can be expanded naturally in a given basis in terms of a tensor product[†], of the respective basis vectors and co-vectors, and of their array of components in that basis. For our (n,s)-tensor $\boldsymbol{A}$, this reads

$$\boldsymbol{A}=A^{\sigma_1\cdots\sigma_n}{}_{\nu_1\cdots\nu_s}\,\boldsymbol{e}_{\sigma_1}\otimes\cdots\otimes\boldsymbol{e}_{\sigma_n}\otimes\boldsymbol{\theta}^{\nu_1}\otimes\cdots\otimes\boldsymbol{\theta}^{\nu_s}\,, \tag{3.8}$$

where $A^{\sigma_1\cdots\sigma_n}{}_{\nu_1\cdots\nu_s}$ is said component array in the (general) basis $\{\boldsymbol{e}_\sigma\}$. Note that the order of basis and co-basis vectors is reversed in the basis expansion (3.8) as opposed to the order of co-vectors $\boldsymbol{\chi}$ and vectors $\boldsymbol{X}$ in (3.7) above. This is necessary, since in this way we can insert the basis vectors and co-basis vectors in the argument "slots" $\_$ of $\boldsymbol{A}$, i.e.,

$$\boldsymbol{A}(\_,\ldots,\_,\_,\ldots,\_)=A^{\sigma_1\cdots\sigma_n}{}_{\nu_1\cdots\nu_s}\,\boldsymbol{e}_{\sigma_1}(\_)\otimes\cdots\otimes\boldsymbol{e}_{\sigma_n}(\_)\otimes\boldsymbol{\theta}^{\nu_1}(\_)\otimes\cdots\otimes\boldsymbol{\theta}^{\nu_s}(\_)\,, \tag{3.9}$$

which, by virtue of (3.2), yields its components in that basis as an array of real numbers, and thus makes contact with the above definitions (3.6) and (3.7), i.e.,

$$\boldsymbol{A}(\boldsymbol{\theta}^{\mu_1},\ldots,\boldsymbol{\theta}^{\mu_n},\,\boldsymbol{e}_{\rho_1},\ldots,\boldsymbol{e}_{\rho_s})=A^{\mu_1\cdots\mu_n}{}_{\rho_1\cdots\rho_s}\,. \tag{3.10}$$

Transformation of Tensors Under a Change of Basis

It is important to realise that vectors and tensors are invariant objects, i.e. they do not change under a transformation of coordinates that corresponds to a change of basis,

$$\tilde{\boldsymbol{e}}_\sigma=(\Lambda^{-1})^\nu{}_\sigma\boldsymbol{e}_\nu\,, \tag{3.11}$$

in terms of the Jacobian matrix of general coordinate transformation, $(\Lambda^{-1})^\nu{}_\sigma\in\mathrm{GL}(N,\mathbb{R})$. For a tangent vector $\boldsymbol{X}$, we then have

$$\boldsymbol{X}=X^\sigma\boldsymbol{e}_\sigma=X^\rho\Lambda^\sigma{}_\rho\,(\Lambda^{-1})^\nu{}_\sigma\boldsymbol{e}_\nu=\tilde{X}^\sigma\tilde{\boldsymbol{e}}_\sigma\,. \tag{3.12}$$

This means, that under a coordinate transformation, the components X^σ must *contravary* with respect to the basis vectors $\boldsymbol{e}_\sigma$, i.e. they must change with the inverse of the transformation matrix that effects the transformation of the basis vectors. Thus, the components with an upper index X^σ, being those of a vector, are referred to as

[†] We assume that the reader is already familiar with the notion of tensor product.

contravariant. Clearly, the inverse is then true for a co-vector $\boldsymbol{\chi}$,

$$\boldsymbol{\chi} = \chi_\nu \boldsymbol{\theta}^\nu = \chi_\nu (\Lambda^{-1})^\nu{}_\rho \, \Lambda^\rho{}_\sigma \boldsymbol{\theta}^\sigma = \tilde{\chi}_\rho \tilde{\boldsymbol{\theta}}^\rho \,, \tag{3.13}$$

i.e., the components of a co-vector must *co-vary* with respect to the basis vectors and, consequently, the lower-index components of a co-vector are referred to as *covariant.*

By multilinearity, the above transformation law for vectors extends to all higher-rank tensors. For our (n, s)-tensor $\boldsymbol{A}$, this reads explicitly,

$$\boldsymbol{A} = \tilde{A}^{\sigma'_1 \cdots \sigma'_n}{}_{\nu'_1 \cdots \nu'_s} \, \tilde{\boldsymbol{e}}_{\sigma'_1} \otimes \cdots \otimes \tilde{\boldsymbol{e}}_{\sigma'_n} \otimes \tilde{\boldsymbol{\theta}}^{\nu'_1} \otimes \cdots \otimes \tilde{\boldsymbol{\theta}}^{\nu'_s} \,, \tag{3.14}$$

where $\tilde{\boldsymbol{e}}_{\sigma'} = \Lambda_{\sigma'}{}^\sigma \boldsymbol{e}_\sigma$ and $\tilde{\boldsymbol{\theta}}^{\nu'} = \Lambda^{\nu'}{}_\nu \boldsymbol{\theta}^\nu$ are the transformed basis and co-basis vectors, and

$$\tilde{A}^{\sigma'_1 \cdots \sigma'_n}{}_{\nu'_1 \cdots \nu'_s} = A^{\sigma_1 \cdots \sigma_n}{}_{\nu_1 \cdots \nu_s} \Lambda_{\sigma_1}{}^{\sigma'_1} \cdots \Lambda_{\sigma_n}{}^{\sigma'_n} \Lambda^{\nu_1}{}_{\nu'_1} \cdots \Lambda^{\nu_n}{}_{\nu'_n} \tag{3.15}$$

are the transformed components.

3.1.4. Symmetrisation and Anti-Symmetrisation

For a component $(0, n)$-tensor A, symmetrisation and anti-symmetrisation of index slots is indicated by enclosing the indices in round and square brackets, respectively,

$$A_{(\nu_1 \nu_2 \cdots \nu_n)} := \frac{1}{n!} \sum_{\pi \in S_n} A_{\nu_{\pi(1)} \nu_{\pi(2)} \cdots \nu_{\pi(n)}} \,, \tag{3.16a}$$

$$A_{[\nu_1 \nu_2 \cdots \nu_n]} := \frac{1}{n!} \sum_{\pi \in S_n} \operatorname{sgn}(\pi) A_{\nu_{\pi(1)} \nu_{\pi(2)} \cdots \nu_{\pi(n)}} \,, \tag{3.16b}$$

with the sign function sgn, and where the sum runs over all permutations π of the indices $\nu_1, \nu_2, \ldots, \nu_n$, S_n being the symmetric group of degree n. Thus, for $n = 2$, we have

$$A_{(\mu\nu)} := \tfrac{1}{2}[A_{\mu\nu} + A_{\nu\mu}] \qquad \text{(symmetrisation)}, \tag{3.17}$$

$$A_{[\mu\nu]} := \tfrac{1}{2}[A_{\mu\nu} - A_{\nu\mu}] \qquad \text{(anti-symmetrisation)}\,. \tag{3.18}$$

In the case that only a non-adjacent subset of indices of $A_{\mu_1 \mu_2 \ldots \mu_n}$ are to be (anti-) symmetrised over, it is customary to set apart the ones which do not take part in the symmetrisation by enclosing them in vertical bars, as in

$$A_{(\nu_1 | \nu_2 | \nu_3)} = \tfrac{1}{2}[A_{\nu_1 \nu_2 \nu_3} + A_{\nu_3 \nu_2 \nu_1}] \,, \tag{3.19}$$

where the middle index ν_2 is excluded from symmetrisation.

3.1.5. Index Symmetries and Independent Components of Tensors

Consider a general tensor (or any indexed symbol) $A_{\nu_1 \nu_2 \cdots \nu_m}$ in N dimensions and with m index slots. Let A be symmetric in s and antisymmetric in a of these index slots, while the remaining $m - s - a$ slots (with $m \geq s + a$) carry no symmetry. For example, we could have the configuration

$$A_{\nu_1\nu_2\cdots\nu_m} = A_{(\nu_1\cdots\nu_s)\nu_{s+1}\cdots\nu_{s+a}[\nu_{s+a+1}\cdots\nu_m]} \,. \tag{3.20}$$

How many independent components does A have? Clearly, for $s = 0 = a$, the tensor is unsymmetrical, with N^m independent components in general. For $s, a \neq 0$, considering first the s symmetric index slots, there are a multi-set $\left(\!\binom{N}{s}\!\right) = \binom{N+s-1}{s}$ of inequivalent choices of index values including repetitions, which correspond to traces. For the a antisymmetric index slots, we have instead $\binom{N}{a}$ inequivalent choices, which yields a total of

$$N^{m-s-a} \binom{N+s-1}{s}\binom{N}{a} \tag{3.21}$$

inequivalent index configurations and thus degrees of freedom for $A_{\nu_1\nu_2\cdots\nu_m}$.

3.1.6. Differential Forms

A *differential form* of order n, or shorter, an n-form is a totally antisymmetric tensor field of type $(0, n)$, so that 0-forms are scalar fields and 1-forms are co-vector fields. In order to express differential forms in bases in a way that makes their antisymmetry transparent, one introduces the *exterior product* $\wedge$ as a totally antisymmetrised tensor product, so that a coordinate basis of n-forms is given in terms of the basis 1-forms by

$$\boldsymbol{d}X^{\nu_1} \wedge \boldsymbol{d}X^{\nu_2} \wedge \cdots \wedge \boldsymbol{d}X^{\nu_n} := \sum_{\pi\in S_n} \operatorname{sgn}(\pi)\, \boldsymbol{d}X^{\nu_{\pi(1)}} \otimes \boldsymbol{d}X^{\nu_{\pi(2)}} \otimes \cdots \otimes \boldsymbol{d}X^{\nu_{\pi(n)}} \,, \tag{3.22}$$

and thus a general n-form $\boldsymbol{\omega}$ can be expanded in terms of this coordinate-basis n-form as

$$\boldsymbol{\omega} = \frac{1}{n!}\omega_{\nu_1\nu_2\cdots\nu_n} \boldsymbol{d}X^{\nu_1} \wedge \boldsymbol{d}X^{\nu_2} \wedge \cdots \wedge \boldsymbol{d}X^{\nu_n} \,, \tag{3.23}$$

where the coefficients $\omega_{\nu_1\nu_2\cdots\nu_n}$ are totally antisymmetric, i.e. $\omega_{\nu_1\nu_2\cdots\nu_n} = \omega_{[\nu_1\nu_2\cdots\nu_n]}$.

The set of n-forms at every point $\mathscr{P}$ of a differentiable manifold $\mathcal{M}$ of dimension N forms a vector space, denoted $\bigwedge^N(\mathcal{T}^*_{\mathscr{P}}\mathcal{M})$. This space is of dimension $\binom{N}{n}$, since according to the discussion in subsection 3.1.5, this is the number of non-vanishing choices of $(\nu_1, \nu_2, \ldots, \nu_n)$ out of $(1, 2, \ldots, N)$ in (3.22). According to the well-known equality $\binom{N}{n} = \binom{N}{N-n}$, $\bigwedge^N(\mathcal{T}^*_{\mathscr{P}}\mathcal{M})$ is then isomorphic to the space $\bigwedge^{N-n}(\mathcal{T}^*_{\mathscr{P}}\mathcal{M})$ of $(N-n)$-forms at $\mathscr{P}$. On pseudo-Riemannian manifolds (subsection 3.3.4), this leads to the definition of a corresponding isomorphism called the *Hodge dual*, that we introduce in subsection 3.5.2.

In this sense, a 2-form, say $\boldsymbol{F}$ as for the Faraday tensor of electromagnetism,

$$\boldsymbol{F} = \tfrac{1}{2}F_{\mu\nu}\left(\boldsymbol{d}X^\mu \otimes \boldsymbol{d}X^\nu - \boldsymbol{d}X^\nu \otimes \boldsymbol{d}X^\mu\right) = \tfrac{1}{2}F_{\mu\nu}\, \boldsymbol{d}X^\mu \wedge \boldsymbol{d}X^\nu \,, \tag{3.24}$$

is also called a *bivector*, with $\binom{4}{2} = 6$ independent components in $N = 4$ dimensions, which, in components, we write in terms of a *bivector index* $\Sigma \in \{1, 2, 3, 4, 5, 6\}$ as

$$F_\Sigma = F_{\mu\nu} \,, \tag{3.25}$$

where the bivector index Σ corresponds to the antisymmetric combinations of the normal indices $\mu\nu$, i.e., $\Sigma \in \{01, 02, 03, 23, 31, 12\}$.

3.2. The Lie Derivative

In this and the next section, we come to the two notions of derivative that are important for general Relativity. We saw in the preceding section, that on a general differentiable manifold, vectors and tensors are defined only in the tangent space at each point. It is important to clarify that *per se*, there is no way of comparing two vectors (and thus tensors), that live in their respective tangent space at two different points, even when these points are only infinitesimally separated. However, the definition of derivative of a vector field as the limit of a difference quotient between two points requires exactly that.

The Lie derivative, which we are going to introduce first, achieves this by exploiting the Lie-group structure of a differentiable manifold, i.e., by using the flow (the integral curves) of another vector field to transport a vector from one tangent space to the other. It thus evaluates the change of a vector or tensor field along the integral curves of another vector field in a coordinate-invariant fashion. It commutes with contractions and the tensor product.

The second important notion of derivative, that we introduce in the next section then is the covariant derivative, which – in contrast to the Lie derivative – introduces an additional structure on the manifold, called an affine connection.

As mentioned above, the flow $\sigma(\lambda, x_0)$ of a vector field $\boldsymbol{X}$ represents its integral curve through the point x_0, parametrised by λ. In terms of its coordinates σ^μ and the components X^μ of $\boldsymbol{X}$, the flow then satisfies,

$$\dot{\sigma}^\mu(\lambda, x_0) = X^\mu(\sigma(\lambda, x_0))\,, \qquad \sigma^\mu(0, x_0) = {x_0}^\mu\,, \tag{3.26}$$

where the over-dot denotes total differentiation with respect to λ.

Thought of as a map $\sigma : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$, the flow generated by the vector field $\boldsymbol{X}$ can, in particular, be regarded as a one-parameter group of diffeomorphisms $\sigma_\lambda \equiv \sigma(\lambda, x_0)$ (invertible point-to-point maps between manifolds that represent active coordinate transformations). This, in turn, give rise to an induced map $(\sigma_\lambda)_*$ (called the *pushforward*) between the corresponding tangent spaces, which can now finally be used to "push forward" $[(\sigma_\epsilon)_*]$ or "pull back" [with the *pullback* $(\sigma_\epsilon)^* \coloneqq (\sigma_{-\epsilon})_* = (\sigma_\epsilon)^{-1}$] a vector an infinitesimal distance ϵ along the flow between two nearby points.

In order to have a well-defined derivative of a vector field $\boldsymbol{Y}$ at a point x along the flow in terms of the limit of its finite difference with $\boldsymbol{Y}$ at a neighbouring point $\sigma_\epsilon(x)$, we first pull back $\boldsymbol{Y}|_{\sigma_\epsilon(x)}$ from the neighbouring tangent space $\mathcal{T}_{\sigma_\epsilon(x)}\mathcal{M}$ to $\mathcal{T}_x\mathcal{M}$ by $(\sigma_{-\epsilon})_*$, after which we can take the difference between the two vectors, which now both live in $\mathcal{T}_x\mathcal{M}$. Thus, the Lie derivative of a vector field $\boldsymbol{Y}$ along the flow σ of another vector field $\boldsymbol{X}$ is defined as,

$$\mathscr{L}_{\boldsymbol{X}}\,\boldsymbol{Y} \coloneqq \lim_{\epsilon\to 0} \frac{1}{\epsilon}\Big[(\sigma_{-\epsilon})_*\boldsymbol{Y}|_{\sigma_\epsilon(x)} - \boldsymbol{Y}|_x\Big]\,. \tag{3.27}$$

In coordinates x^ν, $\sigma_\epsilon(x)$ is given to linear order in ϵ by $(\sigma_\epsilon)^\nu(x) = x^\nu + \epsilon X^\nu$, and similarly, $\boldsymbol{Y}$ at $\sigma_\epsilon(x)$ is expanded as

$$\boldsymbol{Y}|_{\sigma_\epsilon(x)} = \left[Y^\mu(x) + \epsilon X^\lambda(x)\partial_\lambda Y^\mu(x)\right]\boldsymbol{e}_\mu\big|_{x+\epsilon X} . \tag{3.28}$$

With $(\sigma_{-\epsilon})^\nu(x) = x^\nu - \epsilon X^\nu$, the pullback of $\boldsymbol{Y}|_{\sigma_\epsilon(x)}$ to $\mathcal{T}_x\mathcal{M}$ in (3.27) is then expanded as

$$\begin{aligned}(\sigma_{-\epsilon})_*\boldsymbol{Y}|_{\sigma_\epsilon(x)} &= \left[Y^\mu(x) + \epsilon X^\lambda(x)\partial_\lambda Y^\mu(x)\right]\partial_\mu\left[x^\nu - \epsilon X^\nu(x)\right]\boldsymbol{e}_\nu\big|_x \\ &= \left[Y^\mu(x) + \epsilon X^\lambda(x)\partial_\lambda Y^\mu(x)\right]\left[\delta^\nu{}_\mu - \epsilon\partial_\mu X^\nu(x)\right]\boldsymbol{e}_\nu\big|_x \\ &= \boldsymbol{Y}\big|_x + \epsilon\left[X^\lambda(x)\partial_\lambda Y^\nu(x) - Y^\lambda(x)\partial_\lambda X^\nu(x)\right]\boldsymbol{e}_\nu\big|_x .\end{aligned} \tag{3.29}$$

Thus, we can evaluate the limit in (3.27) and find that the Lie derivative of a vector field $\boldsymbol{Y}$ at the point x can be written in terms of a Lie bracket,

$$\mathscr{L}_{\boldsymbol{X}}\,\boldsymbol{Y} = (X^\lambda\partial_\lambda Y^\nu - Y^\lambda\partial_\lambda X^\nu)\boldsymbol{e}_\nu = [\boldsymbol{X}, \boldsymbol{Y}] . \tag{3.30}$$

The first term in the brackets of (3.30) comprises a partial derivative of the components of the field $\boldsymbol{Y}$ being acted upon, while the second term corrects for a change in $\boldsymbol{X}$ along the direction of $\boldsymbol{Y}$, coming from the linear change in the corresponding basis vector.

Thus, we have that when acting directly on basis vectors $\boldsymbol{e}_\nu$ instead of on vector fields $X^\nu\boldsymbol{e}_\nu$, this first term involving derivatives of their components X^ν in (3.30) is absent, i.e., we have

$$\mathscr{L}_{\boldsymbol{X}}\,\boldsymbol{e}_\rho = -(\partial_\rho X^\nu)\boldsymbol{e}_\nu . \tag{3.31}$$

The action of the Lie derivative on a one-form $\boldsymbol{\chi}$ is then obtained by making use of the fact that $\mathscr{L}_{\boldsymbol{X}}$ commutes with contractions, $\boldsymbol{\chi}(\boldsymbol{Y}) = \chi_\nu Y^\nu$, and that it reduces to an ordinary derivative when acting on these, since they are scalars. Therefore, we can write

$$\mathscr{L}_{\boldsymbol{X}}\,\boldsymbol{\chi}(\boldsymbol{Y}) = \mathscr{L}_{\boldsymbol{X}}\,(\chi_\nu Y^\nu) = X^\rho\partial_\rho(\chi_\nu Y^\nu) = X^\rho(\partial_\rho\chi_\nu)Y^\nu + \chi_\nu X^\rho(\partial_\rho Y^\nu) , \tag{3.32}$$

on the one hand, and by the product rule,

$$\begin{aligned}\mathscr{L}_{\boldsymbol{X}}\,\boldsymbol{\chi}(\boldsymbol{Y}) &= (\mathscr{L}_{\boldsymbol{X}}\,\boldsymbol{\chi})(\boldsymbol{Y}) + \boldsymbol{\chi}(\mathscr{L}_{\boldsymbol{X}}\,\boldsymbol{Y}) \\ &= (\mathscr{L}_{\boldsymbol{X}}\,\boldsymbol{\chi})_\nu Y^\nu + \chi_\nu[\boldsymbol{X}, \boldsymbol{Y}]^\nu \\ &= (\mathscr{L}_{\boldsymbol{X}}\,\boldsymbol{\chi})_\nu Y^\nu + \chi_\nu(X^\rho\partial_\rho Y^\nu - Y^\rho\partial_\rho X^\nu) ,\end{aligned} \tag{3.33}$$

on the other hand, where we have used (3.30). Equating the two above expressions for $\mathscr{L}_{\boldsymbol{X}}\,\boldsymbol{\chi}(\boldsymbol{Y})$ finally leaves us with,

$$\mathscr{L}_{\boldsymbol{X}}\,\boldsymbol{\chi} = (X^\rho\partial_\rho\chi_\nu + \chi_\rho\partial_\nu X^\rho)\boldsymbol{\theta}^\nu , \tag{3.34}$$

where the first term is again the partial derivative of the components of $\boldsymbol{\chi}$ along the vector field and the second term acts as a correction that measures the change in the vector field itself, this time with a plus sign. Correspondingly, for the action on the basis one-forms $\boldsymbol{\theta}^\nu$, we obtain

$$\mathscr{L}_{\boldsymbol{X}}\,\boldsymbol{\theta}^\nu = +(\partial_\rho X^\nu)\boldsymbol{\theta}^\rho . \tag{3.35}$$

We now collect the properties of the Lie derivative. Let $\boldsymbol{X}, \boldsymbol{Y}$ be vector fields, $\boldsymbol{T}$, $\boldsymbol{T}_1, \boldsymbol{T}_2$ tensor fields, and χ, χ_1, χ_2 scalar fields on $\mathcal{M}$ and α_1, α_2 real numbers, then the Lie derivative $\mathscr{L}_{\boldsymbol{X}}$ along $\boldsymbol{X}$ has the following properties,

$$\mathscr{L}_{\boldsymbol{X}}\, \chi = \partial_{\boldsymbol{X}} \chi\,, \tag{3.36a}$$
$$\mathscr{L}_{\boldsymbol{X}}\, \boldsymbol{Y} = [\boldsymbol{X}, \boldsymbol{Y}]\,, \tag{3.36b}$$
$$\mathscr{L}_{\boldsymbol{X}}\, \chi \boldsymbol{T} = (\partial_{\boldsymbol{X}} \chi)\boldsymbol{T} + \chi \mathscr{L}_{\boldsymbol{X}}\, \boldsymbol{T}\,, \tag{3.36c}$$
$$\mathscr{L}_{\boldsymbol{X}}\, (\boldsymbol{T}_1 \otimes \boldsymbol{T}_2) = (\mathscr{L}_{\boldsymbol{X}}\, \boldsymbol{T}_1) \otimes \boldsymbol{T}_2 + \boldsymbol{T}_1 \otimes (\mathscr{L}_{\boldsymbol{X}}\, \boldsymbol{T}_2)\,, \tag{3.36d}$$
$$\mathscr{L}_{\boldsymbol{X}}\, (\boldsymbol{T}_1 + \boldsymbol{T}_2) = \mathscr{L}_{\boldsymbol{X}}\, \boldsymbol{T}_1 + \mathscr{L}_{\boldsymbol{X}}\, \boldsymbol{T}_2\,, \tag{3.36e}$$
$$\mathscr{L}_{\chi_1 \boldsymbol{X} + \chi_2 \boldsymbol{Y}}\, \boldsymbol{Z} = \chi_1 \mathscr{L}_{\boldsymbol{X}}\, \boldsymbol{Z} + \chi_2 \mathscr{L}_{\boldsymbol{Y}}\, \boldsymbol{Z}\,, \tag{3.36f}$$

where $\partial_{\boldsymbol{X}} f = X^\nu \partial_\nu f$ in a coordinate basis.

The importance of the Lie derivative for general relativity lies in its connection with space-time symmetries, that is, a vanishing of the Lie derivative of the metric tensor $\boldsymbol{g}$ (which is introduced below) with respect to a vector field $\boldsymbol{X}$ means that $\boldsymbol{X}$ is the inifinitesimal generator of a symmetry of the corresponding space-time. In order to prepare for the corresponding discussion in later chapters, we therefore close this subsection by applying the Lie derivative to a $(0, 2)$-tensor $\boldsymbol{g}$, expanded in terms of a coordinate co-basis, $\boldsymbol{g} = g_{\mu\nu} \boldsymbol{d}X^\mu \otimes \boldsymbol{d}X^\nu$. Using (3.36c) with χ playing the part of the component functions $g_{\mu\nu}$, and (3.36d) for the product of (co-)basis vectors, we can deduce the action of $\mathscr{L}_{\boldsymbol{X}}$ on $\boldsymbol{g}$,

$$\begin{aligned} \mathscr{L}_{\boldsymbol{X}}\, \boldsymbol{g} &= (X^\rho \partial_\rho g_{\mu\nu}) \boldsymbol{d}X^\mu \otimes \boldsymbol{d}X^\nu + g_{\mu\nu} (\mathscr{L}_{\boldsymbol{X}}\, \boldsymbol{d}X^\mu) \otimes \boldsymbol{d}X^\nu + g_{\mu\nu} \boldsymbol{d}X^\mu \otimes (\mathscr{L}_{\boldsymbol{X}}\, \boldsymbol{d}X^\nu) \\ &= [(X^\rho \partial_\rho) g_{\mu\nu} + (\partial_\mu X^\rho) g_{\rho\nu} + (\partial_\nu X^\rho) g_{\mu\rho}] \boldsymbol{d}X^\mu \otimes \boldsymbol{d}X^\nu\,. \end{aligned} \tag{3.37}$$

3.3. Affine Connections and Covariant Derivatives

In this section we now come to the most important notion of derivative in general relativity, namely the *covariant derivative*. As already remarked in the last section on the Lie derivative, the covariant derivative requires the introduction of an additional structure on our differentiable manifold $\mathcal{M}$, called an *affine connection*, in order to "connect" the tangent spaces of two infinitesimally close points. Below, we first introduce connections in a somewhat formal manner, before we turn to their coefficients in different bases, followed by the covariant derivative and the equation of parallel transport. We then specialise to the (pseudo-) Riemannian case, i.e., to a manifold with metric tensor and its Levi-Cività connection, as required for general relativity.

3.3.1. Affine Connections

For vector fields $\boldsymbol{X}, \boldsymbol{Y}$ and $\boldsymbol{Z}$, scalar fields χ, χ_1, χ_2, and real numbers α_1 and α_2, an affine connection or covariant derivative $\boldsymbol{\nabla}$ is a map $(\boldsymbol{X}, \boldsymbol{Y}) \mapsto \boldsymbol{\nabla}_{\boldsymbol{X}} \boldsymbol{Y}$ with the following properties of a derivative,

$$\boldsymbol{\nabla}_{\boldsymbol{X}} \chi = \partial_{\boldsymbol{X}} \chi\,, \tag{3.38a}$$
$$\boldsymbol{\nabla}_{\boldsymbol{X}} (\alpha_1 \boldsymbol{Y} + \alpha_2 \boldsymbol{Z}) = \alpha_1 \boldsymbol{\nabla}_{\boldsymbol{X}} \boldsymbol{Y} + \alpha_2 \boldsymbol{\nabla}_{\boldsymbol{X}} \boldsymbol{Z}\,, \tag{3.38b}$$
$$\boldsymbol{\nabla}_{\chi_1 \boldsymbol{X} + \chi_2 \boldsymbol{Y}} \boldsymbol{Z} = \chi_1 \boldsymbol{\nabla}_{\boldsymbol{X}} \boldsymbol{Z} + \chi_2 \boldsymbol{\nabla}_{\boldsymbol{Y}} \boldsymbol{Z}\,, \tag{3.38c}$$

$$\boldsymbol{\nabla}_{\boldsymbol{Y}}(\chi \boldsymbol{X}) = (\partial_{\boldsymbol{Y}} \chi)\boldsymbol{X} + \chi \boldsymbol{\nabla}_{\boldsymbol{Y}} \boldsymbol{X} \,. \tag{3.38d}$$

As for the Lie derivative, we also demand compatibility with the multi-linear structure, i.e., with the tensor product. Thus, for two arbitrary tensors $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$, we have

$$\boldsymbol{\nabla}_{\boldsymbol{X}}(\boldsymbol{T}_1 \otimes \boldsymbol{T}_2) = (\boldsymbol{\nabla}_{\boldsymbol{X}} \boldsymbol{T}_1) \otimes \boldsymbol{T}_2 + \boldsymbol{T}_1 \otimes (\boldsymbol{\nabla}_{\boldsymbol{X}} \boldsymbol{T}_2) \,. \tag{3.38e}$$

Note however, that the connection itself is not a tensor since it is not linear with respect to functions (i.e. scalar fields) in its second argument, as evident from property (3.38d).

As the name suggests, an affine connection "connects" nearby tangent spaces, i.e. it provides a prescription to compare vectors and tensors that live on the different tangent spaces $\mathcal{T}_{\mathscr{P}}\mathcal{M}$ and $\mathcal{T}_{\mathscr{P}'}\mathcal{M}$ of two (infinitesimally) separated points $\mathscr{P}$ and $\mathscr{P}'$. Just as in Euclidean space vectors at different points can be compared by parallel transporting one to the location of the other, the existence of an affine connection defines (infinitesimal) parallel transport of vectors and tensors between $\mathscr{P}$ and $\mathscr{P}'$, where $\boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{Y}$ is the change in $\boldsymbol{Y}$ if it is transported from $\mathscr{P}$ along $\boldsymbol{X}$ to $\mathscr{P}'$, $\boldsymbol{Y}$ being a vector that lives in the tangent space at $\mathscr{P}$.

3.3.2. Connection Coefficients and their Transformation Law

Since the connection is linear, the result of evaluating $\boldsymbol{\nabla}$ in a certain basis can again be expressed in terms of a linear combination of the basis vectors. If we take our general basis $\{\boldsymbol{e}_\alpha\}$, we have

$$\boldsymbol{\nabla}_{\boldsymbol{e}_\alpha} \boldsymbol{e}_\beta = \tilde{\Gamma}^\delta{}_{\alpha\beta}\, \boldsymbol{e}_\delta \,, \qquad\qquad e_\alpha{}^\nu \nabla_\nu e_\beta{}^\mu = \tilde{\Gamma}^\delta{}_{\alpha\beta}\, e_\delta{}^\mu \,, \tag{3.39}$$

in terms of connection coefficients $\tilde{\Gamma}^\delta{}_{\alpha\beta}$ in that basis, i.e. the connection coefficients specify how the basis vectors change from point to point, or from one tangent space to the other. Acting instead on an element of the corresponding co-basis $\{\boldsymbol{\theta}^\alpha\}$, we find in view of the definition (3.2), that

$$\boldsymbol{\nabla}_{\boldsymbol{e}_\alpha} \boldsymbol{\theta}^\beta = -\tilde{\Gamma}^\beta{}_{\alpha\delta}\, \boldsymbol{\theta}^\delta \,, \qquad\qquad e_\alpha{}^\nu \nabla_\nu \theta^\beta{}_\mu = -\tilde{\Gamma}^\beta{}_{\alpha\delta}\, \theta^\delta{}_\mu \,, \tag{3.40}$$

We obtain the coefficients themselves by acting on (3.39) [on (3.40)] with an element of the corresponding co-basis (basis), respectively,

$$\tilde{\Gamma}^\kappa{}_{\alpha\beta} \coloneqq (\boldsymbol{\theta}^\kappa \boldsymbol{\nabla}_{\boldsymbol{e}_\alpha})(\boldsymbol{e}_\beta) = -(\boldsymbol{\nabla}_{\boldsymbol{e}_\alpha} \boldsymbol{\theta}^\kappa)(\boldsymbol{e}_\beta) \,. \tag{3.41}$$

The connection coefficients are not tensors, since in contrast to (3.15), they transform inhomogeneously, as we will now demonstrate. In fact, their transformation law is contained in the above equation (3.41). If we expand our general basis vectors in terms of the coordinate basis in the way that they were introduced in (3.1), we can write out the right-hand side of (3.41) in terms of the connection components in a coordinate basis, which we denote simply by $\Gamma^\sigma{}_{\mu\nu}$ (and which we shall call Christoffel symbols in the context of the Levi-Cività connection to be discussed below), obtaining step by step,

$$\begin{aligned}\tilde{\Gamma}^{\kappa}{}_{\alpha\beta} &= \boldsymbol{\theta}^{\kappa}\boldsymbol{\nabla}_{\boldsymbol{e}_\alpha}\boldsymbol{e}_\beta = \theta^{\kappa}{}_{\mu}\boldsymbol{d}X^{\mu}\, e_{\alpha}{}^{\nu}\boldsymbol{\nabla}_{\boldsymbol{\partial}_\nu} e_{\beta}{}^{\sigma}\boldsymbol{\partial}_\sigma \\ &= \theta^{\kappa}{}_{\mu}\boldsymbol{d}X^{\mu}\, e_{\alpha}{}^{\nu}\left[(\partial_\nu e_{\beta}{}^{\sigma})\boldsymbol{\partial}_\sigma + e_{\beta}{}^{\sigma}\Gamma^{\lambda}{}_{\nu\sigma}\boldsymbol{\partial}_\lambda\right] \\ &= \theta^{\kappa}{}_{\mu}\left[e_{\alpha}{}^{\nu}\partial_\nu e_{\beta}{}^{\mu} + e_{\alpha}{}^{\nu}e_{\beta}{}^{\sigma}\Gamma^{\mu}{}_{\nu\sigma}\right],\end{aligned} \tag{3.42}$$

where in the last line we have used $\boldsymbol{d}X^{\mu}(\boldsymbol{\partial}_\sigma) = \delta^{\mu}{}_{\sigma}$, etc.

We said above that the connection coefficients transform non-tensorially. In looking at the last line of the transformation law (3.42), we can actually be more specific: Evidently, the first and the last index of the connection coefficients in fact transform in a tensorial fashion, i.e., with an instance of a transformation matrix each [remembering that $\theta^{\kappa}{}_{\mu}, e_{\alpha}{}^{\nu} \in \mathrm{GL}(N,\mathbb{R})$], while it is only the second, or derivative index, that transforms non-tensorially.

3.3.3. Covariant Derivative

Clearly, if we can compare vectors at two points that are infinitesimally close, we can define derivatives and this is in fact our main motivation for introducing affine connections. Thus, the connection naturally provides a so-called covariant derivative that generalises the notion of directed derivative from vector calculus to differentiable manifolds. Here, *covariant* means that the derivative respects the tensorial structure [cf. (3.38e)], i.e. covariant derivatives of tensors are again tensors of increased rank, so that the covariant derivative of an (n,s)-tensor results in an $(n,s+1)$-tensor.

Taken in a coordinate basis $\boldsymbol{Y} = Y^{\mu}\boldsymbol{\partial}_\mu$ and $\boldsymbol{X} = X^{\nu}\boldsymbol{\partial}_\nu$, the scalar-vector Leibnitz rule (3.38d) yields the usual component version of the covariant derivative for vector fields, where the component functions X^{μ} play the role of the scalar field χ for every value of μ and the basis vectors $\boldsymbol{\partial}_\mu$ play the role of $\boldsymbol{X}$ in (3.38d). Thus, we can write

$$\begin{aligned}\boldsymbol{\nabla}_{\boldsymbol{Y}}\boldsymbol{X} &= Y^{\mu}(\partial_\mu X^{\nu})\boldsymbol{\partial}_\nu + Y^{\mu}X^{\nu}\,\boldsymbol{\nabla}_{\boldsymbol{\partial}_\mu}\boldsymbol{\partial}_\nu \\ &= Y^{\mu}(\partial_\mu X^{\sigma} + \Gamma^{\sigma}{}_{\mu\nu}X^{\nu})\boldsymbol{\partial}_\sigma\,,\end{aligned} \tag{3.43}$$

where in the second line we have used (3.39) for a coordinate basis.

The action of the covariant derivative on general tensors is then specified by its compatibility with tensor product, i.e. by the tensorial Leibniz rule (3.38e), since the space of (n,s)-tensors is spanned by the tensor product of n basis vectors and s elements of the co-basis, as in (3.8). Starting out in a general basis, the action of the connection on an (n,s)-tensor, i.e., its covariant derivative with respect to $\boldsymbol{X}$, reads

$$\begin{aligned}\boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{A} = {} & (\partial_{\boldsymbol{X}} A^{\sigma_1\cdots\sigma_n}{}_{\nu_1\cdots\nu_s})\,\boldsymbol{e}_{\sigma_1}\otimes\cdots\otimes\boldsymbol{e}_{\sigma_s}\otimes\boldsymbol{\theta}^{\nu_1}\otimes\cdots\otimes\boldsymbol{\theta}^{\nu_n} \\ & + A^{\sigma_1\cdots\sigma_n}{}_{\nu_1\cdots\nu_s}(\boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{e}_{\sigma_1})\otimes\boldsymbol{e}_{\sigma_2}\otimes\cdots\otimes\boldsymbol{e}_{\sigma_s}\otimes\boldsymbol{\theta}^{\nu_1}\otimes\cdots\otimes\boldsymbol{\theta}^{\nu_n} + \cdots + \\ & + A^{\sigma_1\cdots\sigma_n}{}_{\nu_1\cdots\nu_s}\boldsymbol{e}_{\sigma_1}\otimes\cdots\otimes\boldsymbol{e}_{\sigma_{s-1}}\otimes\left(\boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{e}_{\sigma_s}\right)\otimes\boldsymbol{\theta}^{\nu_1}\otimes\cdots\otimes\boldsymbol{\theta}^{\nu_n} + \cdots + \\ & + A^{\sigma_1\cdots\sigma_n}{}_{\nu_1\cdots\nu_s}\boldsymbol{e}_{\sigma_1}\otimes\cdots\otimes\boldsymbol{e}_{\sigma_s}\otimes\left(\boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{\theta}^{\nu_1}\right)\otimes\boldsymbol{\theta}^{\nu_2}\cdots\otimes\boldsymbol{\theta}^{\nu_n} + \cdots + \\ & + A^{\sigma_1\cdots\sigma_n}{}_{\nu_1\cdots\nu_s}\boldsymbol{e}_{\sigma_1}\otimes\cdots\otimes\boldsymbol{e}_{\sigma_s}\otimes\boldsymbol{\theta}^{\nu_1}\otimes\cdots\otimes\boldsymbol{\theta}^{\nu_{n-1}}\otimes\left(\boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{\theta}^{\nu_n}\right).\end{aligned} \tag{3.44}$$

Using now (3.39) and (3.40) and renaming dummy indices, we can factor out the tensor products. Specialising to a coordinate basis and co-basis for simplicity, i.e. setting $\boldsymbol{e}_\sigma = \boldsymbol{\partial}_\sigma$

and $\boldsymbol{\theta}^\nu = \boldsymbol{d}X^\nu$, we thus obtain

$$\boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{A} = X^\rho \, A^{\sigma_1\cdots\sigma_n}{}_{\nu_1\cdots\nu_s;\,\rho}\,\boldsymbol{\partial}_{\sigma_1} \otimes\cdots\otimes \boldsymbol{\partial}_{\sigma_s} \otimes \boldsymbol{d}X^{\nu_1} \otimes\cdots\otimes \boldsymbol{d}X^{\nu_n}\,, \tag{3.45}$$

where the components $A^{\sigma_1\cdots\sigma_n}{}_{\nu_1\cdots\nu_s;\,\rho}$ of the covariant derivative of $\boldsymbol{A}$ are given by

$$\begin{aligned} A^{\sigma_1\cdots\sigma_n}{}_{\nu_1\cdots\nu_s;\,\rho} = A^{\sigma_1\cdots\sigma_n}{}_{\nu_1\cdots\nu_s,\,\rho} & \\ + \Gamma^{\sigma_1}{}_{\rho\lambda} A^{\lambda\sigma_2\cdots\sigma_n}{}_{\nu_1\cdots\nu_s} + \cdots + \Gamma^{\sigma_n}{}_{\rho\lambda} A^{\sigma_1\cdots\sigma_{n-1}\lambda}{}_{\nu_1\cdots\nu_s} & \\ - \Gamma^{\lambda}{}_{\rho\nu_1} A^{\sigma_1\cdots\sigma_n}{}_{\lambda\nu_2\cdots\nu_s} - \cdots - \Gamma^{\lambda}{}_{\rho\nu_n} A^{\sigma_1\cdots\sigma_n}{}_{\nu_1\cdots\nu_{s-1}\lambda}\,, & \end{aligned} \tag{3.46}$$

in terms of their partial derivatives and the Christoffel symbols $\Gamma^\sigma{}_{\mu\nu}$. Thus in terms of components, we find that in addition to the partial derivative, for every contravariant (upper) index σ of $A^{\sigma_1\cdots\sigma_n}{}_{\nu_1\cdots\nu_s}$, there is a correction term $+\Gamma^\sigma{}_{\rho\lambda}A^{\cdots\lambda\cdots}{}_{\dots}$, and every covariant (lower) index ν gets corrected by a term $-\Gamma^\lambda{}_{\rho\nu}A^{\cdots}{}_{\dots\lambda\dots}$. Above we have also introduced the *comma notation* for the partial derivative, and the *semi-colon notation* for the covariant derivative in terms of components, so that

$$A^{\sigma_1\cdots\sigma_n}{}_{\nu_1\cdots\nu_s,\,\rho} \coloneqq \partial_\rho A^{\sigma_1\cdots\sigma_n}{}_{\nu_1\cdots\nu_s}\,, \tag{3.47}$$

$$A^{\sigma_1\cdots\sigma_n}{}_{\nu_1\cdots\nu_s;\,\rho} \coloneqq \nabla_\rho A^{\sigma_1\cdots\sigma_n}{}_{\nu_1\cdots\nu_s}\,, \tag{3.48}$$

where we note that there is only one comma or semi-colon and that all indices that come after a comma or semi-colon belong to a partial or covariant derivative, respectively.

On the one hand, writing the component derivatives in this way makes it clear that the (covariant) rank of the component tensor being acted on is increased by one; on the other hand, the above notation is very compact and thus well suited when we are dealing with a large number of derivatives. We shall make extensive use of the comma and semi-colon notation in chapter 6 for the coefficient tensors in Riemann and Fermi normal coordinate expansions.

Equation of Parallel Transport and Geodesics

Consider a vector field $\boldsymbol{w}$ that is transported along a curve $\mathscr{C}$, which is the integral curve of a second vector field $\boldsymbol{v}$. In the case that $\boldsymbol{w}$ remains parallel to itself along $\boldsymbol{v}$, we say that $\boldsymbol{w}$ is *parallel transported* along $\mathscr{C}$, which means that its covariant change along $\mathscr{C}$ as measured by the connection in terms of the covariant derivative, $\boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{w}$, vanishes, i.e.,

$$\boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{w} = \boldsymbol{0}\,. \tag{3.49}$$

Equation (3.49) is then called the *equation of parallel transport* for $\boldsymbol{w}$.

Given a coordinate chart $\{X^\mu\}$ and taking $\boldsymbol{w}$ and $\boldsymbol{v}$ in the corresponding coordinate basis for simplicity, i.e., $\boldsymbol{w} = w^\sigma\boldsymbol{\partial}_\sigma$ and $\boldsymbol{v} = v^\sigma\boldsymbol{\partial}_\sigma$, the components of the tangent vector $\boldsymbol{v}$ are given by

$$v^\sigma(\lambda) \coloneqq \frac{dX^\sigma_{\mathscr{C}}(\lambda)}{d\lambda}\,, \tag{3.50}$$

in terms of the curve's parametrisation $X^\sigma_{\mathscr{C}}(\lambda)$, where λ is an arbitrary affine parameter along $\mathscr{C}$. An affine parameter is a parameter that is determined up to constant affine transformations, i.e., two affine parameters λ and $\tilde{\lambda}$ are related by

$$\tilde{\lambda} = a\lambda + b\,, \qquad \text{where } a, b = \text{const., and } a, b \in \mathbb{R}\,. \tag{3.51}$$

Using (3.43), the component-version of the equation of parallel transport (3.49) reads

$$v^\nu \nabla_\nu w^\sigma = v^\nu \left[\partial_\nu w^\sigma + \Gamma^\sigma{}_{\nu\rho} w^\rho\right] = 0\,, \tag{3.52a}$$

or, in terms of the total parametric derivative along the curve $\mathscr{C}$,

$$v^\nu \nabla_\nu w^\sigma \equiv \frac{Dw^\sigma}{d\lambda} = \frac{dw^\sigma}{d\lambda} + \Gamma^\sigma{}_{\nu\rho} \frac{dX^\sigma_\mathscr{C}}{d\lambda} w^\rho = 0\,. \tag{3.52b}$$

Here, D is the so-called *covariant differential*,

$$Dw^\sigma := dw^\sigma + v^\mu \Gamma^\sigma{}_{\mu\nu}\, dw^\nu\,. \tag{3.53}$$

An important special case of (3.49), (3.52a) occurs if we take $\boldsymbol{w} = \boldsymbol{v}$, which means that the vector field $\boldsymbol{v}$ remains parallel to itself when transported along its own integral curve. Curves $\mathscr{C}$ that parallel-transport their own tangent vector in this sense are called autoparallels in general, and *geodesics* on (pseudo-)Riemannian manifolds (see below). Apart from being the straightest curves ("autoparallel") as measured by the connection, geodesics have the additional property of being extremal, i.e. the shortest curves connecting two points. Equation (3.49) then becomes the *geodesic equation*,

$$\boldsymbol{\nabla}_{\boldsymbol{v}} \boldsymbol{v} = \boldsymbol{0}\,, \tag{3.54}$$

with the well-known two component forms, again written once in terms of partial derivatives,

$$v^\nu \nabla_\nu v^\sigma = v^\nu \partial_\nu v^\sigma + \Gamma^\sigma{}_{\nu\rho} v^\nu v^\rho = 0\,, \tag{3.55a}$$

and once in terms of the total parametric derivative,

$$v^\nu \nabla_\nu v^\sigma \equiv \frac{Dv^\sigma(\lambda)}{d\lambda} = \frac{d^2 X^\sigma_\mathscr{C}(\lambda)}{d\lambda^2} + \Gamma^\sigma{}_{\nu\rho} \frac{dX^\mu_\mathscr{C}}{d\lambda} \frac{dX^\nu_\mathscr{C}}{d\lambda} = 0\,. \tag{3.55b}$$

3.3.4. Pseudo-Riemannian Manifold and Metric

Given a differentiable manifold $\mathcal{M}$ and two tangent vector fields $\boldsymbol{X}, \boldsymbol{Y} \in \mathcal{T}_\mathscr{P}\mathcal{M}$ (or co-vector fields $\boldsymbol{\chi}, \boldsymbol{\xi} \in \mathcal{T}^*_\mathscr{P}\mathcal{M}$) we can define an inner product on the tangent (co-tangent) space at every point $\mathscr{P} \in \mathcal{M}$,

$$\boldsymbol{g}(\boldsymbol{X}, \boldsymbol{Y}) =: \boldsymbol{X} \cdot \boldsymbol{Y}\,, \qquad \boldsymbol{g}^{-1}(\boldsymbol{\xi}, \boldsymbol{\omega}) =: \boldsymbol{\chi} \cdot \boldsymbol{\xi}\,, \tag{3.56}$$

respectively, if we also have a *pseudo-Riemannian metric*, i.e., a $(0,2)$ tensor field $\boldsymbol{g}$ that satisfies the two natural properties

1. $\boldsymbol{g}(\boldsymbol{X}, \boldsymbol{Y}) = \boldsymbol{g}(\boldsymbol{Y}, \boldsymbol{X})$ (symmetry),
2. $\boldsymbol{g}(\boldsymbol{X}, \boldsymbol{Y}) = 0$ for any $\boldsymbol{X}$ implies $\boldsymbol{Y} = \boldsymbol{0}$ (non-degeneracy).

Then the pair $(\mathcal{M}, \boldsymbol{g})$ is called a *(pseudo-)Riemannian manifold*. Property 2 is a generalisation of positive-definiteness $\boldsymbol{g}(\boldsymbol{X}, \boldsymbol{X}) \geq 0$ for a Riemannian manifold. Since it canonically induces a norm, the metric determines the physically important concepts of lengths,

$$\|\boldsymbol{X}\| = \sqrt{\boldsymbol{X} \cdot \boldsymbol{X}}\,, \tag{3.57}$$

and angles between vectors: For every tangent space, i.e. at every point of the manifold, the angle between two vector fields $\boldsymbol{X}$ and $\boldsymbol{Y}$ is given by

$$\cos\theta = \frac{\boldsymbol{X} \cdot \boldsymbol{Y}}{\|\boldsymbol{X}\|\,\|\boldsymbol{Y}\|}\,. \tag{3.58}$$

We can contract the metric with its inverse, which yields the unit tensor $\delta^{\mu}{}_{\nu}$,

$$g^{\mu\lambda} g_{\lambda\nu} = \delta^{\mu}{}_{\nu}\,, \tag{3.59}$$

which is obviously a covariant statement (i.e. $\delta^{\mu}{}_{\nu}$ is covariantly defined in this way, whereas $\delta_{\mu\nu}$ or $\delta^{\mu\nu}$ would not be covariant symbols).

3.3.5. Orthonormal Non-Coordinate Bases

We now return to the case of a non-coordinate basis, that was first encountered in subsection 3.1.2. With a scalar product between vectors at hand, we can introduce the most important case of a non-coordinate basis, probably being the only practically used one, namely the *orthonormal* non-coordinate basis $\{\boldsymbol{e}_{\hat{\alpha}}\}$, which we distinguish from the general basis above by putting hats on the basis indices.

Just as for general bases (3.1) and (3.3), the orthonormal basis vectors and co-vectors are written in terms of a coordinate basis and co-basis as

$$\boldsymbol{e}_{\hat{\alpha}} = e_{\hat{\alpha}}{}^{\nu}\,\boldsymbol{\partial}_{\nu}\,, \qquad \boldsymbol{e}^{\hat{\alpha}} = e^{\hat{\alpha}}{}_{\nu}\,\boldsymbol{d}X^{\nu}\,. \tag{3.60}$$

What sets an orthonormal non-coordinate basis apart from general bases are the components of the metric in that basis, as we shall now see. For comparison, we expand $\boldsymbol{g}$ in terms of a coordinate co-basis and its respective metric components $g_{\mu\nu}$ on the one hand, and in terms of the orthonormal non-coordinate co-basis $\{\boldsymbol{e}^{\hat{\alpha}}\}$ on the other hand. This reads,

$$\boldsymbol{g} = g_{\mu\nu}\,\boldsymbol{d}X^{\mu} \otimes \boldsymbol{d}X^{\nu} \qquad \text{in a coordinate basis,} \tag{3.61a}$$

$$\boldsymbol{g} = \eta_{\hat{\alpha}\hat{\beta}}\,\boldsymbol{e}^{\hat{\alpha}} \otimes \boldsymbol{e}^{\hat{\beta}} \qquad \text{in an orthonormal non-coordinate basis.} \tag{3.61b}$$

While the components $g_{\mu\nu}$ are functions of the coordinates, the coefficients of the metric in the orthonormal non-coordinate basis are defined to be constant and diagonal, given by the *Minkowski metric*,

$$\eta_{\hat{\alpha}\hat{\beta}} := \mathrm{diag}(\,1, -1, -1, -1\,)\,. \tag{3.62}$$

The action of $\boldsymbol{g}$ in (3.61b) on the orthonormal basis vectors $\boldsymbol{e}_{\hat{\alpha}}$ is then interpreted as their orthonormality condition. Thus, orthonormal non-coordinate bases are defined with respect to the metric by

$$\boldsymbol{g}(\boldsymbol{e}_{\hat{\alpha}}, \boldsymbol{e}_{\hat{\beta}}) = \boldsymbol{e}_{\hat{\alpha}} \cdot \boldsymbol{e}_{\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}}\,, \qquad g_{\mu\nu}\, e_{\hat{\alpha}}{}^{\mu} e_{\hat{\beta}}{}^{\nu} = \eta_{\hat{\alpha}\hat{\beta}}\,, \tag{3.63a}$$

$$\boldsymbol{g}^{-1}(\boldsymbol{e}^{\hat{\alpha}}, \boldsymbol{e}^{\hat{\beta}}) = \boldsymbol{e}^{\hat{\alpha}} \cdot \boldsymbol{e}^{\hat{\beta}} = \eta^{\hat{\alpha}\hat{\beta}}\,, \qquad g^{\mu\nu}\, e^{\hat{\alpha}}{}_{\mu} e^{\hat{\beta}}{}_{\nu} = \eta^{\hat{\alpha}\hat{\beta}}\,, \tag{3.63b}$$

where $\boldsymbol{g}^{-1}$ is the inverse metric tensor.

In the component approach, the coefficients $e_{\hat{\alpha}}{}^{\nu}$ of an orthonormal non-coordinate basis are called a *tetrad*. We shall discuss tetrads and their importance for special and general relativity, in particular for the description of observers and inertial frames extensively in section 4.2 and some of the following chapters.

3.3.6. The Levi-Cività Connection of General Relativity

In general, an affine connection has two tensorial invariants, namely curvature (which we will introduce in a separate section below) and torsion, which is given in terms of Cartan's *torsion tensor*

$$\boldsymbol{T}(\boldsymbol{X}, \boldsymbol{Y}) \coloneqq \boldsymbol{\nabla}_{\boldsymbol{X}} \boldsymbol{Y} - \boldsymbol{\nabla}_{\boldsymbol{Y}} \boldsymbol{X} - [\boldsymbol{X}, \boldsymbol{Y}]\,, \tag{3.64}$$

and taken to vanish in general relativity. Geometrically, non-vanishing torsion means that infinitesimal parallelogrammes generated by the right-hand side of (3.64) don't close, even in a coordinate basis where $[\boldsymbol{X}, \boldsymbol{Y}] = \boldsymbol{0}$, the amount of this kind of integrability obstruction of the affine connection being measured by the torsion tensor $\boldsymbol{T}$.

However, if our manifold is equipped with a metric tensor, i.e., if it is (pseudo-) Riemannian, it turns out that there is a privileged affine connection induced by the metric and uniquely specified by the following two conditions:

1. That the connection be symmetric, i.e., torsion-free, $\boldsymbol{T} = \boldsymbol{0}$, so that (3.64) becomes

$$\boldsymbol{\nabla}_{\boldsymbol{X}} \boldsymbol{Y} - \boldsymbol{\nabla}_{\boldsymbol{Y}} \boldsymbol{X} = [\boldsymbol{X}, \boldsymbol{Y}]\,. \tag{3.65}$$

2. That the connection be metric compatible,

$$\boldsymbol{\nabla}_{\boldsymbol{Z}}\, \boldsymbol{g}(\boldsymbol{X}, \boldsymbol{Y}) = \boldsymbol{0}\,, \qquad \nabla_{\sigma} g_{\mu\nu} = 0\,. \tag{3.66}$$

In view of (3.56), condition 2 means in particular that parallel transport preserves the scalar product, and this privileged connection is called the *Levi-Cività connection*.

From the above criteria, we can now embark on deriving the Levi-Cività connection in terms of its connection coefficients. If in (3.65) we use for $\boldsymbol{X}$ and $\boldsymbol{Y}$ our general basis vectors, i.e. $\boldsymbol{X} = \boldsymbol{e}_{\alpha}$ and $\boldsymbol{Y} = \boldsymbol{e}_{\beta}$, we can use (3.39) with (3.65) to arrive at

$$2\tilde{\Gamma}^{\delta}{}_{[\alpha\beta]}\, \boldsymbol{e}_{\delta} = \boldsymbol{\nabla}_{\boldsymbol{e}_{\alpha}} \boldsymbol{e}_{\beta} - \boldsymbol{\nabla}_{\boldsymbol{e}_{\beta}} \boldsymbol{e}_{\alpha} = [\boldsymbol{e}_{\alpha}, \boldsymbol{e}_{\beta}] = c^{\delta}{}_{\alpha\beta}\, \boldsymbol{e}_{\delta}\,, \tag{3.67}$$

where in the last equation, we have made use of the definition (3.5) of the structure coefficients of our general basis. After contracting away the basis vectors, with $\boldsymbol{e}^{\delta}$, we thus conclude, that in a general non-coordinate basis, the antisymmetric part of the connection coefficients (3.41) is determined by the structure coefficients of the basis vectors,

$$\tilde{\Gamma}^{\delta}{}_{[\alpha\beta]} = \tfrac{1}{2}\, c^{\delta}{}_{\alpha\beta}\,. \tag{3.68}$$

We note that plus sign on the right-hand side of (3.68) comes from our usage of the "del convention" for the placement of the derivative index as first lower index on the connection coefficients, as illustrated in equation (3.39).

In order to derive an expression for the Levi-Cività connection, we still have to implement condition 2., i.e. metric compatibility. This means that we have to evaluate (3.66),

$$\nabla_\sigma g_{\mu\nu} = g_{\mu\nu,\,\sigma} - \tilde{\Gamma}^{\lambda}{}_{\sigma\mu} g_{\lambda\nu} - \tilde{\Gamma}^{\lambda}{}_{\sigma\nu} g_{\mu\lambda} = 0\,, \tag{3.69}$$

which yields

$$g_{\mu\nu,\,\sigma} = 2\tilde{\Gamma}_{(\mu|\sigma|\nu)}\,. \tag{3.70}$$

With hindsight, we now add up three permutations of (3.70),

$$g_{\sigma\mu,\,\nu} + g_{\sigma\nu,\,\mu} - g_{\mu\nu,\,\sigma} = 2\big[\tilde{\Gamma}_{(\sigma|\mu|\nu)} + \tilde{\Gamma}_{(\sigma|\nu|\mu)} - \tilde{\Gamma}_{(\mu|\sigma|\nu)}\big]\,. \tag{3.71}$$

Extending the right-hand side with $0 = \tilde{\Gamma}_{\sigma\mu\nu} - \tilde{\Gamma}_{\sigma\mu\nu}$, we can combine the eight terms into two times $\tilde{\Gamma}_{\sigma\mu\nu}$ plus three combinations that are antisymmetric in the two last indices,

$$g_{\sigma\mu,\,\nu} + g_{\sigma\nu,\,\mu} - g_{\mu\nu,\,\sigma} = 2\big[\tilde{\Gamma}_{\sigma\mu\nu} + \tilde{\Gamma}_{\mu[\nu\sigma]} - \tilde{\Gamma}_{\nu[\sigma\mu]} - \tilde{\Gamma}_{\sigma[\mu\nu]}\big]\,. \tag{3.72}$$

The latter are given by the (index-lowered) structure coefficients $c_{\lambda\mu\nu}$ of the basis according to (3.68), and rearranging, we obtain the components of the Levi-Cività connection in a general basis,

$$\tilde{\Gamma}_{\sigma\mu\nu} = \tfrac{1}{2}\big[g_{\sigma\mu,\,\nu} + g_{\sigma\nu,\,\mu} - g_{\mu\nu,\,\sigma} + c_{\sigma\mu\nu} - c_{\nu\mu\sigma} - c_{\mu\nu\sigma}\big]\,. \tag{3.73}$$

in terms of partial derivatives of the metric on the one hand, and of the commutation or structure coefficients of the basis vectors (3.5) on the other hand. These are usually displayed with the first index raised, which we effect by contracting with the inverse metric, as usual,

$$\tilde{\Gamma}^{\sigma}{}_{\mu\nu} = \tfrac{1}{2} g^{\sigma\lambda}\big[g_{\sigma\mu,\,\nu} + g_{\sigma\nu,\,\mu} - g_{\mu\nu,\,\sigma} + c_{\sigma\mu\nu} - c_{\nu\mu\sigma} - c_{\mu\nu\sigma}\big]\,. \tag{3.74}$$

Usually however, one does not work in a general basis since it is not particularly convenient, but either in a coordinate basis $\{\boldsymbol{\partial}_\mu\}$, or in an orthonormal non-coordinate basis $\{\boldsymbol{e}_{\hat{\alpha}}\}$. With respect to the connection coefficients (3.73), (3.74) these two can be considered extreme cases: In the case of a coordinate basis, the basis vectors commute and thus the structure coefficients (3.5) in (3.74) vanish, so the connection coefficients are determined by partial derivatives of the metric alone. The Levi-Cività connection coefficients $\tilde{\Gamma}^{\sigma}{}_{\mu\nu}$ are then called *Christoffel symbols*, denoted $\Gamma^{\sigma}{}_{\mu\nu}$, and given by the well-known expression

$$\Gamma^{\sigma}{}_{\mu\nu} = \tfrac{1}{2}\, g^{\sigma\lambda}\big[g_{\sigma\mu,\,\nu} + g_{\sigma\nu,\,\mu} - g_{\mu\nu,\,\sigma}\big]\,. \tag{3.75}$$

While the components of the Levi-Cività connection in a general basis (3.74) have no particular index symmetry, they become symmetric in a coordinate basis, as can easily be seen from the right-hand side of (3.75), so we have $\Gamma^{\sigma}{}_{\mu\nu} = \Gamma^{\sigma}{}_{(\mu\nu)}$ for the Christoffel symbols.

The other extreme are the connection coefficients in an orthonormal non-coordinate basis. In this case, the metric is constant, $g_{\mu\nu} = \eta_{\mu\nu}$, and thus its partial derivatives in (3.74) vanish, so that we are left with the part that is composed of the commutation coefficients $c^{\delta}{}_{\mu\nu}$ alone, i.e.,

$$\tilde{\Gamma}^{\kappa}{}_{\alpha\beta} = \tfrac{1}{2}\,\eta^{\kappa\delta}\left[c_{\delta\alpha\beta} - c_{\beta\alpha\delta} - c_{\alpha\beta\delta}\right]. \tag{3.76}$$

When index-lowered, i.e., without the inverse metric, they become anti-symmetric in the first and the last index,

$$\tilde{\Gamma}_{\kappa\alpha\beta} = \tilde{\Gamma}_{[\kappa|\alpha|\beta]}\,, \tag{3.77}$$

which is also obvious from the symmetries of the right-hand side of (3.76). The (index-lowered) connection coefficients in an orthonormal basis are also called *Ricci rotation coefficients* or, in the context of the Newman-Penrose formalism, *spin coefficients*.

3.4. Riemann Curvature Tensor

The existence of an affine connection on a manifold (which need not be a Levi-Cività connection) allows us to define the important notion of curvature. Let $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ be vector fields, then the Riemann curvature tensor is an $(1,3)$-tensor field defined as the commutator of covariant derivatives of $\boldsymbol{X}$ and $\boldsymbol{Y}$, acting on the field $\boldsymbol{Z}$, as

$$\boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{Z} \coloneqq (\nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}} - \nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{X}})\boldsymbol{Z} - \nabla_{[\boldsymbol{X},\boldsymbol{Y}]}\boldsymbol{Z}\,. \tag{3.78}$$

The notation $\boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{Z}$ is conventional and clarifies that (at least in this index-free setting) the Riemann tensor can be thought of as a differential operator, acting on the vector field $\boldsymbol{Z}$. In comparison to the usual tensor notation introduced in (3.7)–(3.10) it would be written as $\boldsymbol{R}(\boldsymbol{\chi},\boldsymbol{Z},\boldsymbol{X},\boldsymbol{Y})$ in terms of a co-vector field $\boldsymbol{\chi}$. Expanding $\boldsymbol{R}$ in a coordinate basis as in (3.8) then reads

$$\boldsymbol{R} = R^{\sigma}{}_{\rho\mu\nu}\,\boldsymbol{\partial}_{\sigma} \otimes \boldsymbol{d}X^{\rho} \otimes \boldsymbol{d}X^{\mu} \otimes \boldsymbol{d}X^{\nu}\,, \tag{3.79}$$

in terms of its components $R^{\sigma}{}_{\rho\mu\nu}$ in this basis. Clearly, this usual component-form of the Riemann tensor is obtained upon taking the vector fields $\boldsymbol{X}$ and $\boldsymbol{Y}$ in (3.78) to be coordinate basis vectors, i.e., $\boldsymbol{X} = \boldsymbol{\partial}_{\mu}$ and $\boldsymbol{Y} = \boldsymbol{\partial}_{\nu}$, which commute, so we have $[\boldsymbol{X},\boldsymbol{Y}] = [\boldsymbol{\partial}_{\mu},\boldsymbol{\partial}_{\nu}] = \boldsymbol{0}$, and the third term on the right-hand side of (3.78) vanishes. Thus, the usual component form of the Riemann tensor is given in terms of partial derivatives and contractions of the Christoffel symbols (3.75), by

$$R^{\sigma}{}_{\rho\mu\nu} = \partial_{\mu}\Gamma^{\sigma}{}_{\rho\nu} - \partial_{\nu}\Gamma^{\sigma}{}_{\rho\mu} + \Gamma^{\sigma}{}_{\lambda\mu}\Gamma^{\lambda}{}_{\rho\nu} - \Gamma^{\sigma}{}_{\lambda\nu}\Gamma^{\lambda}{}_{\rho\mu}\,, \qquad \text{(coordinate basis)}. \tag{3.80}$$

From this definition it is clear that the Riemann tensor (3.80) is antisymmetric in the last pair of indices,

$$R^{\sigma}{}_{\rho\,\mu\nu} = R^{\sigma}{}_{\rho[\mu\nu]}\,. \tag{3.81}$$

It further satisfies the two so-called Bianchi identities, one algebraic and one differential one,

$$R^{\sigma}{}_{\rho\mu\nu} + R^{\sigma}{}_{\mu\nu\rho} + R^{\sigma}{}_{\nu\rho\mu} = 0 \qquad \text{(1st Bianchi identity)}, \tag{3.82a}$$

$$R^{\sigma}{}_{\rho\,\mu\nu;\lambda} + R^{\sigma}{}_{\rho\,\nu\lambda;\mu} + R^{\sigma}{}_{\rho\,\lambda\mu;\nu} = 0 \qquad \text{(2nd Bianchi identity)}. \tag{3.82b}$$

where the last three lower indices are permuted cyclically, to make the three resulting terms vanish, and we have used the semi-colon notation for covariant derivatives, as introduced above.

In a pseudo-Riemannian manifold, we can use the metric to lower the first index, $R_{\sigma\rho\mu\nu} = g_{\sigma\lambda} R^{\lambda}{}_{\rho\mu\nu}$; the resulting totally covariant Riemann tensor now displays the additional index symmetries

$$R_{\sigma\rho\,\mu\nu} = -R_{\rho\sigma\,\mu\nu}\,, \qquad \text{or} \quad R_{\sigma\rho\,\mu\nu} = R_{[\sigma\rho]\mu\nu}\,, \tag{3.83}$$

$$R_{\sigma\rho\,\mu\nu} = R_{\mu\nu\,\sigma\rho}\,, \tag{3.84}$$

so that the symmetries of the totally covariant Riemann tensor consist of the first and second Bianchi identities (3.82a) and (3.82b) (so-called *multi-term* symmetries), as well as the antisymmetry in every index pair and the exchange symmetry between both index pairs (so-called *mono-term* symmetries), the latter of which we can write compactly as

$$R_{\sigma\rho\mu\nu} = R_{([\sigma\rho][\mu\nu])}\,. \tag{3.85}$$

Due to its symmetries, the Riemann tensor only has two non-vanishing traces. The first, or partial trace defines the *Ricci tensor*,

$$R_{\mu\nu} \coloneqq R^{\lambda}{}_{\mu\lambda\nu}\,. \tag{3.86}$$

And the full trace yields the so-called *Ricci-* or *curvature scalar*†,

$$R_{\mathrm{Ric}} \coloneqq R^{\lambda}{}_{\lambda} = R^{\mu\nu}{}_{\mu\nu}\,. \tag{3.87}$$

The Ricci tensor and scalar make up the left-hand (or “geometry“) side of the Einstein field equations (5.5), which will be introduced in subsection 5.1.1.

†Usually one uses a capital ‘R’ for the Ricci scalar, i.e. the same symbol as for the Riemann and Ricci tensors. This would, however, collide with our usage of R for the radial coordinate in the Schwarzschild metric.

3.5. Quantities Derived from the Metric Tensor

3.5.1. Epsilon Tensor and Metric Determinant

In N dimensions, we define the totally antisymmetric Levi-Cività symbol* as usual, namely

$$\epsilon_{\nu_1\nu_2\cdots\nu_N} := \begin{cases} +1 & \text{for } \nu_1\nu_2\cdots\nu_N \text{ an even permutation of } 1,2,\ldots,N \\ -1 & \text{for } \nu_1\nu_2\cdots\nu_N \text{ an odd permutation of } 1,2,\ldots,N \\ \ \ 0 & \text{otherwise.} \end{cases} \tag{3.88}$$

Note that sometimes a convention is used in which the sign is reversed. Under general coordinate transformations, the Levi-Cività symbol (3.88) does not transform as a tensor, but acquires an additional factor of the Jacobian determinant of the transformation (i.e., it is a so-called tensor density of weight $+1$). In order to turn (3.88) into a tensor, one compensates for this with an inverse Jacobian which, for pseudo-Riemannian manifolds, is conventionally written as $\sqrt{-g}$, where‡

$$g := \det(g_{\mu\nu}) \tag{3.89}$$

is the metric determinant. The *epsilon tensor* $\boldsymbol{\varepsilon}$ is thus defined as

$$\boldsymbol{\varepsilon} := \varepsilon_{\nu_1\nu_2\cdots\nu_N} \boldsymbol{d}X^{\nu_1} \otimes \boldsymbol{d}X^{\nu_2} \otimes \cdots \otimes \boldsymbol{d}X^{\nu_N} = \frac{1}{N!}\varepsilon_{\nu_1\nu_2\cdots\nu_N} \boldsymbol{d}X^{\nu_1} \wedge \boldsymbol{d}X^{\nu_2} \wedge \cdots \wedge \boldsymbol{d}X^{\nu_N}\,, \tag{3.90}$$

i.e., it is the unique unit constant form that has the same rank as the underlying manifold has dimensions. In terms of its components, we have

$$\varepsilon_{\nu_1\nu_2\cdots\nu_N} := \sqrt{-g}\,\epsilon_{\nu_1\nu_2\cdots\nu_N}\,, \qquad \varepsilon^{\nu_1\nu_2\cdots\nu_N} := \frac{\epsilon^{\nu_1\nu_2\cdots\nu_N}}{\sqrt{-g}}\,. \tag{3.91}$$

Note that (3.91) is really a pseudo-tensor, since it changes sign under improper transformations, such as reflections. From this we have that a total contraction of the epsilon tensor with vectors or tensors then gives rise to a *pseudo-scalar*.

From (3.91), we then have the following formula for the determinant of the metric, and its inverse, $1/g = \det(g^{\mu\nu})$

$$\begin{aligned} g &= \det(g_{\mu\nu}) = \epsilon^{\nu_1\nu_2\cdots\nu_N} g_{1\nu_1}\, g_{2\nu_2} \cdots g_{N\nu_N} \\ 1/g &= \det(g^{\mu\nu}) = \epsilon_{\sigma_1\sigma_2\cdots\sigma_N} g^{1\sigma_1}\, g^{2\sigma_2} \cdots g^{N\sigma_N}\,. \end{aligned} \qquad \text{(Leibniz formula)} \tag{3.92}$$

We can write the Leibnitz formula (3.92) in a more symmetric form as follows: we start with $\varepsilon^{\nu_1\nu_2\cdots\nu_N}$ as defined in (3.91) and lower indices with the metric,

$$\varepsilon_{\nu_1\nu_2\cdots\nu_N} = \frac{\epsilon^{\sigma_1\sigma_2\cdots\sigma_N}}{\sqrt{-g}}\, g_{\nu_1\sigma_1}\, g_{\nu_2\sigma_2} \cdots g_{\nu_N\sigma_N}\,. \tag{3.93}$$

*A nice and compact discussion of the Levi-Cività symbol and epsilon tensor in general relativity can be found in the textbook [51, Sec. 3.1.1].

‡Here, we focus exclusively on Lorentzian manifolds, for which the metric determinant is negative, and thus one writes $\sqrt{-g}$. For the general case one should write $\sqrt{|g|}$ instead.

Now contracting both sides with $\varepsilon^{\nu_1\nu_2\cdots\nu_N}$ and taking into account the "normalisation condition"

$$\varepsilon^{\nu_1\nu_2\cdots\nu_N}\varepsilon_{\nu_1\nu_2\cdots\nu_N} = \epsilon^{\nu_1\nu_2\cdots\nu_N}\epsilon_{\nu_1\nu_2\cdots\nu_N} = N!\,, \tag{3.94}$$

gives $N!$ times the Leibnitz rule (3.92), which we rearrange to yield,

$$g = \frac{1}{N!}\,\epsilon^{\nu_1\nu_2\cdots\nu_N}\epsilon^{\sigma_1\sigma_2\cdots\sigma_N}\,g_{\nu_1\sigma_1}\,g_{\nu_2\sigma_2}\cdots g_{\nu_N\sigma_N}\,. \tag{3.95}$$

Products of Two Epsilon Tensors

In components, a product of two epsilon tensors can be expressed in terms of an anti-symmetrised product of N metric factors,

$$\varepsilon^{\sigma_1\sigma_2\cdots\sigma_N}\varepsilon_{\nu_1\nu_2\cdots\nu_N} = N!\,g^{\sigma_1}{}_{[\nu_1}\,g^{\sigma_2}{}_{\nu_2}\cdots g^{\sigma_N}{}_{\nu_N]} = N!\,\delta^{\sigma_1}{}_{[\nu_1}\,\delta^{\sigma_2}{}_{\nu_2}\cdots\delta^{\sigma_N}{}_{\nu_N]}\,, \tag{3.96}$$

where the anti-symmetrisation is understood to extend only over the lower indices $\nu_1\cdots\nu_N$, and the factor of $N!$ is necessary to compensate for the combinatorial factor in the definition (3.16b) of the anti-symmetrisation brackets. If we now contract on the first k of the σ and ν indices, we obtain

$$\varepsilon^{\lambda_1\cdots\lambda_k\,\sigma_{k+1}\cdots\sigma_N}\varepsilon_{\lambda_1\cdots\lambda_k\,\nu_{k+1}\cdots\nu_N} = k!(N-k)!\,g^{\sigma_{k+1}}{}_{[\nu_{k+1}}\,g^{\sigma_{k+2}}{}_{\nu_{k+2}}\cdots g^{\sigma_N}{}_{\nu_N]}\,. \tag{3.97}$$

3.5.2. The Hodge Dual

We now come back to the important correspondence between the spaces $\bigwedge^n(\mathcal{T}^*_{\mathscr{P}}\mathcal{M})$ of n-forms and $\bigwedge^{N-n}(\mathcal{T}^*_{\mathscr{P}}\mathcal{M})$ of $(N-n)$-forms, that was motivated in subsection 3.1.6. On a (pseudo-) Riemannian manifold, this correspondence can be used to define a natural isomorphism $*$,

$$* : \bigwedge^n(\mathcal{T}^*_{\mathscr{P}}\mathcal{M}) \to \bigwedge^{N-n}(\mathcal{T}^*_{\mathscr{P}}\mathcal{M})\,, \tag{3.98}$$

between them, called the Hodge dual or Hodge star operator.

Thus, if $\boldsymbol{\omega}$ is totally antisymmetric, i. e. an n-form as in (3.23), then it is dual to the $(N-n)$-form

$$*\boldsymbol{\omega} := \frac{\sqrt{-g}}{n!(N-n)!}\omega_{\nu_1\cdots\nu_n}\epsilon^{\nu_1\cdots\nu_n}{}_{\nu_{n+1}\cdots\nu_N}\boldsymbol{d}X^{\nu_{n+1}}\wedge\boldsymbol{d}X^{\nu_{n+2}}\wedge\cdots\wedge\boldsymbol{d}X^{\nu_N}\,, \tag{3.99}$$

and for the coordinate-basis n-form, this reads

$$*(\boldsymbol{d}X^{\nu_1}\wedge\cdots\wedge\boldsymbol{d}X^{\nu_n}) = \frac{\sqrt{-g}}{(N-n)!}\epsilon^{\nu_1\cdots\nu_n}{}_{\nu_{n+1}\cdots\nu_N}\boldsymbol{d}X^{\nu_{n+1}}\wedge\boldsymbol{d}X^{\nu_{n+2}}\wedge\cdots\wedge\boldsymbol{d}X^{\nu_N}\,. \tag{3.100}$$

On a Lorentzian manifold, applying the Hodge star to an n-form $\boldsymbol{\omega}$ twice yields $\boldsymbol{\omega}$ up to a sign,

$$**\boldsymbol{\omega} = (-1)^{n(N-n)+1}\boldsymbol{\omega}\,, \tag{3.101}$$

i.e., $(-1)^{n(N-n)+1} * *$ is an identity map on the space $\bigwedge^n(\mathcal{T}^*_{\mathscr{P}}\mathcal{M})$ of n-forms. Being a duality up to a sign, the inverse of the Hodge star operator can then be defined by

$$*^{-1} := (-1)^{n(N-n)+1} * \, . \tag{3.102}$$

If $\mathcal{M}$ is an orientable manifold, there exists a nowhere-vanishing top-dimensional, i.e. here, N-form called the *volume form*, which can be written as the Hodge dual of the constant 0-form 1,

$$*1 = \frac{1}{N!}\varepsilon_{\nu_1\nu_2\cdots\nu_N}\, \boldsymbol{d}X^{\nu_1} \wedge \boldsymbol{d}X^{\nu_2} \wedge \cdots \wedge \boldsymbol{d}X^{\nu_N} = \sqrt{-g}\, \boldsymbol{d}X^1 \wedge \boldsymbol{d}X^2 \wedge \cdots \wedge \boldsymbol{d}X^N \tag{3.103}$$

and which serves as a natural measure for integration on $\mathcal{M}$.

4 Elements of Special Relativity

We begin this chapter in section 4.1 with a brief review of the basics of special relativity in the way that it is usually introduced, namely in its inertial-frame form. We then turn to special relativity in general coordinates and, most importantly, in non-inertial frames in section 4.2, thereby introducing already the basic elements of the more general Riemannian description of space-time that forms the basis of the theory of general relativity. In particular, we discuss tetrads as representing the local Lorentz frame of general, idealised observers, as well as the geometric transport law along their world-lines (i.e. the frame's "equation of motion"), written in terms of the transport tensor $\Omega^{\kappa}{}_{\alpha}$, a two-form which can be decomposed into the frame's proper acceleration and rotation vectors. We briefly point out the close connections to covariant electrodynamics and the Faraday tensor.

In section section 4.3 we start by introducing the transformation to local coordinates for non-inertial observers and calculate the tetrads and metric in these. We then show how the non-inertial observers' transport equation can be solved approximately in terms of a time-ordered exponential, thereby introducing the important concept of the so-called parallel propagator, and then derive its exact solution for time-independent inertial forces. A particular case of this solution turns out to correspond to the interesting case of observers in circular orbit around the origin of coordinates. In preparation for the discussion of circular world-lines in general relativity, we subsequently show how these observers can be described more conveniently in terms of inertial Minkowski coordinates. Finally, section 4.4 concludes the present chapter with the discussion of geodesic motion of a particle as observed within a non-inertial frame in terms of the non-affinely-parametrised geodesic equation, written in the frame's local coordinates.

4.1. Special Relativity in Inertial Frames

The arena of special relativity is Minkowski space-time, a flat pseudo-Riemannian manifold with infinitesimal length given in terms of an affine parameter λ by the line element

$$d\lambda^2 = \eta_{\alpha\beta}\, dx^\alpha dx^\beta = (c\, dt)^2 - (dx)^2 - (dy)^2 - (dz)^2\,, \tag{4.1}$$

with Lorentzian coordinates $x^\alpha = (ct, x^a)$, and with the usual flat-space Minkowski metric (3.62). The line-element (4.1) measures four-dimensional distance between space-time events and is the fundamental invariant scalar in special relativity (as well as in general relativity), from which the theory is built.[†] Being a four-dimensional differential version of the Pythagorean theorem, albeit in a non-Euclidean space as a consequence of minus

[†]Concerning notation, we use letters from the beginning of the Greek alphabet, $\alpha, \beta, \gamma, \delta$ and κ, as indices for Minkowski-space coordinate functions, vectors, and tensors.

signs in (4.1), the fundamental line element $d\lambda$ can also be zero or negative. It is called,

$$\begin{array}{ll} \textit{time-like} \text{ for} & \eta_{\alpha\beta}\, dx^\alpha dx^\beta > 0 \\ \textit{light-like} \text{ or } \textit{null} \text{ for} & \eta_{\alpha\beta}\, dx^\alpha dx^\beta = 0 \\ \textit{space-like} \text{ for} & \eta_{\alpha\beta}\, dx^\alpha dx^\beta < 0\,, \end{array} \tag{4.2}$$

where the signs hold for our choice of metric signature $(+,-,-,-)$; for the other sign convention the signs are switched. Restricting (4.1) to a curve that is given in terms of an affine parameter λ by $x^\alpha(\lambda)$, with tangent vector $v^\alpha(\lambda) := \frac{dx^\alpha(\lambda)}{d\lambda}$, the curve's tangent v^α trivially inherits its causality class from (4.2). Upon dividing through in (4.1) with $d\tau^2$, we find that v^α is normalised to unity and is time-like for $v^\alpha v_\alpha = \eta_{\alpha\beta} v^\alpha v^\beta = 1$, null or light-like for $v^\alpha v_\alpha = 0$, and space-like for $v^\alpha v_\alpha = -1$. The tangents to (future-directed) time-like curves lie *inside* the forward light cone, null curves *on*, and space-like curves *outside* the forward or backward light cones.

4.1.1. Global Inertial Frames and Coordinates of Inertial Observers

Special relativity, in the way that it is usually discussed, is formulated in terms of idealised inertial observers whose world-lines are geodesics in Minkowski space, i.e. straight lines. Since Minkowski space is flat and he is inertial, a fiducial observer, moving along his world-line $\mathscr{W}$, carries with him a *global* inertial frame of coordinate basis vectors and the corresponding attached coordinates which cover the whole of Minkowski space, such that

$$x^0 = ct = c\tau\,, \tag{4.3}$$

i.e., his (inertial) time coordinate t is equal to his proper time τ, which in turn is the physical time that the observer reads off his clock. We note that this identification of proper time with coordinate time is one of the defining ingredients of the notion of inertial coordinates, and thus of *inertial frame* for an observer. The other ingredient that should be mentioned explicitly is that the spacial inertial coordinates are always taken to be Cartesian (so that all spacial connection coefficients vanish), i.e.,

$$x^a = (x^1, x^2, x^3) = (x, y, z) = \boldsymbol{x}\,. \tag{4.4}$$

Note also that in a non-Cartesian coordinate basis (e.g. in spherical coordinates), the Minkowski metric will take a different form than (3.62), i.e., it will be non-constant in general. We shall discuss non-Cartesian coordinates and non-inertial frames in section 4.2 below.

Using $d\lambda = c\, d\tau$ in the line element (4.1) and dividing through with $d\tau^2$, we obtain the normalisation condition,

$$u_\alpha u^\alpha = \eta_{\alpha\beta}\, u^\alpha u^\beta = c^2\,, \tag{4.5}$$

on a four-velocity, $u^\alpha = u^\alpha(x^\beta)$, the latter always being a time-like vector, defined as the tangent to the observer's world-line,

$$u^\alpha := \frac{dx^\alpha}{d\tau}\,. \tag{4.6}$$

Clearly, the above condition (4.5) ensures that the relative velocity of physical observers never exceed the speed of light. In the global inertial frame of our fiducial observer, the four-velocity (4.6) then takes the simple form

$$u^\alpha = (c, 0, 0, 0)\,. \tag{4.7}$$

The fact that our observer is inertial is embodied in the vanishing of his four-acceleration,

$$a^\kappa := \frac{du^\kappa}{d\tau} = 0\,, \tag{4.8}$$

which is the equation of motion for point particles with mass but without a gauge (such as electric) charge. Here we should emphasise that this definition of four-acceleration only holds in inertial frames, whereas in non-inertial frames we have to apply the correspondence principle; in particular, the total derivative above has to be promoted to a total covariant derivative, (4.36), so we obtain (4.37), see below.

By differentiating (4.5) with respect to τ, we find that the four-acceleration is always normal to its corresponding four-velocity, $a^\kappa u_\kappa = 0$. This is clearly a covariant statement that also holds in non-inertial frames, as well as in general relativity. With his four-velocity (4.7) being purely time-like, this means that in an observer's inertial frame, the four-acceleration (4.8) is purely space-like, so it has the decomposition

$$a^\kappa = (0, a^k)\,. \tag{4.9}$$

The equation of motion for a massive particle (4.8) is obtained in the standard way by extremising the point-particle action of special relativity,

$$S(\tau_1, \tau_2) = -m \int_{c\tau_1}^{c\tau_2} d\lambda = -mc \int_{\tau_1}^{\tau_2} \sqrt{\eta_{\alpha\beta} u^\alpha u^\beta}\, d\tau = -mc^2 \int_{\tau_1}^{\tau_2} \sqrt{1-\beta^2}\, d\tau\,, \tag{4.10}$$

with the normalised velocity parameter $\beta := v/c$, defined in terms of the particle's three-velocity v. In the first equation, we have used (4.6), and in the last equation (4.7) for an inertial frame. We also note that the minus sign is necessary, since geodesic motion *maximises* proper time.

4.1.2. Lorentz Transformations and Lorentz Group

The coordinate transformations ${\mathit\Pi_\delta}^\beta$ that leave the fundamental line element (4.1) invariant,

$${\mathit\Pi_\gamma}^\alpha {\mathit\Pi_\delta}^\beta \eta_{\alpha\beta} = \eta_{\gamma\delta}\,, \tag{4.11}$$

are the Poincaré transformations ${\mathit\Pi_\delta}^\beta$, i.e., inhomogeneous linear transformations of the form

$$x^{\alpha'} = {\mathit\Pi^{\alpha'}}_\alpha\, x^\alpha = {\Lambda^{\alpha'}}_\alpha\, x^\alpha + b^{\alpha'}\,, \tag{4.12}$$

where ${\Lambda^{\alpha'}}_\alpha$ is a Lorentz transformation matrix and the constant vector $b^{\alpha'}$ is related to space-time translations, which are not relevant for our purposes. Instead, we shall focus exclusively on the case that ${\mathit\Pi^{\alpha'}}_\alpha$ is a proper Lorentz transformation which is a combination of a velocity transformations (a boost), and a spacial rotation, so that (4.11) reduces to

$$\Lambda_\gamma{}^\alpha \Lambda_\delta{}^\beta \eta_{\alpha\beta} = \eta_{\gamma\delta} \,. \tag{4.13}$$

From the global inertial frame of our fiducial observer with four velocity (4.7), another observer who's (primed) inertial frame moves with relative velocity v^a with respect to our fiducial observer's, is then seen to have a four-velocity given by

$$u^{\alpha'} = \Lambda^{\alpha'}{}_\delta \, u^\delta = \gamma(\, c,\; v^{a'}\,) \,. \tag{4.14}$$

Here, the γ factor is a normalisation factor which ensures that (4.5) is satisfied. It is easily calculated by inserting (4.14) into the normalisation condition (4.5), which yields

$$\gamma = \frac{1}{\sqrt{\eta_{\alpha\beta}\, u^\alpha u^\beta}} = \frac{1}{\sqrt{1-\beta^2}} \,, \tag{4.15}$$

where β is the magnitude of $\beta^i = u^i/c$.

Proper Orthochronous Lorentz Group

The Lorentz transformations $\Lambda^{\alpha'}{}_\alpha$ introduced above are elements of the Lorentz group $\mathrm{O}(1,3)$, which in turn is a subgroup of the Poincaré group. However, we shall be concerned exclusively with the so-called *proper orthochronous* Lorentz group, comprising those Lorentz transformations that preserve the direction of time with $\Lambda^0{}_0 = \gamma \geq 1$ ("orthochronous"), and also preserve spacial orientation, with $\det \Lambda^{\alpha'}{}_\alpha = +1$ ("proper"); see, e. g., Weinberg, [52, Sec. 2.4]. These form a subgroup $\mathrm{SO}(1,3)$ of the Lorentz group, given by boosts and rotations, which are those elements that are infinitesimally connected to the identity. Thus, making use of this property, we can write

$$\Lambda^{\alpha'}{}_\alpha = \delta^{\alpha'}{}_\alpha + \omega^{\alpha'}{}_\alpha \,, \tag{4.16}$$

in terms of a first-order Taylor ansatz with an infinitesimal matrix $\omega^{\alpha'}{}_\alpha$.

Since the Lorentz group is a Lie group, it is *generated* by the matrix version of $\omega^{\alpha'}{}_\alpha$ and thus we can invoke the defining normalisation condition (4.13) for (inverse) Lorentz transformation matrices,

$$\eta_{\gamma\delta} = \Lambda_\gamma{}^\alpha \eta_{\alpha\beta} \Lambda_\delta{}^\beta = (\delta_\gamma{}^\alpha + \omega_\gamma{}^\alpha)\eta_{\alpha\beta}(\delta_\delta{}^\beta + \omega_\delta{}^\beta) = \eta_{\gamma\delta} + \omega_{\gamma\delta} + \omega_{\delta\gamma} \,, \tag{4.17}$$

to show that $\omega_{\gamma\delta} = -\omega_{\delta\gamma}$, i. e., $\omega^{\alpha'}{}_\alpha$ is antisymmetric (when index-lowered), where we have kept terms up to first order only. In $N = 4$ dimensions, an antisymmetric matrix has $\frac{1}{2}N(N-1) = 6$ independent components, corresponding to the three parameters of boosts (the rapidities ζ^i), and rotations (the rotation angles ϑ^i), respectively. Usually, one writes a proper Lorentz transformation $\Lambda^{\alpha'}{}_\alpha$ in terms of a matrix of parameters and of a matrix of matrix-valued generators. This decomposition is accomplished straightforwardly by pulling out a combination of Minkowski-metric factors from it, as we will now demonstrate.

Starting with the infinitesimal expression (4.16), we have

$$\begin{aligned}\Lambda^{\alpha}{}_{\beta} &= \delta^{\alpha}{}_{\beta} + \omega^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta} + \eta^{\alpha\kappa}\eta^{\delta}{}_{\beta}\,\omega_{\kappa\delta} \\ &= \delta^{\alpha}{}_{\beta} + \tfrac{1}{2}\,\omega_{\kappa\delta}\big(\eta^{\alpha\kappa}\eta^{\delta}{}_{\beta} - \eta^{\alpha\delta}\eta^{\kappa}{}_{\beta}\big) = \delta^{\alpha}{}_{\beta} + \tfrac{1}{2}\,\omega_{\kappa\delta}\big(J^{\alpha}{}_{\beta}\big)^{\kappa\delta}\,,\end{aligned} \tag{4.18}$$

where in the last line, we have defined the matrix of generators,

$$\big(J^{\alpha}{}_{\beta}\big)^{\kappa\delta} := \eta^{\alpha\kappa}\delta^{\delta}{}_{\beta} - \eta^{\alpha\delta}\delta^{\kappa}{}_{\beta}\,, \tag{4.19}$$

as a matrix of matrices, $\mathsf{J}^{\kappa\delta} := \big(J^{\alpha}{}_{\beta}\big)^{\kappa\delta}$. The matrix $\mathsf{J}^{\kappa\delta}$ collects the usual generators of boosts and rotations, whereas the coefficients $\omega_{\kappa\delta}$ collect the corresponding parameters, so that, in matrix form, the infinitesimal expression (4.18) becomes

$$\Lambda = 1 + \tfrac{1}{2}\omega_{\alpha\beta}\mathsf{J}^{\alpha\beta}\,. \tag{4.20}$$

Until now, all expressions were manifestly covariant. The usual parameters and generator matrices can be retrieved through (formally) breaking this covariance in going to an inertial frame that singles out a particular time direction, where the above covariant matrices split into time and space components. Thus,

$$\mathsf{K}_i = \mathsf{J}^{0i} = \big(J^{\alpha}{}_{\beta}\big)^{0i}\,, \qquad\qquad \mathsf{L}_i = (*\mathsf{J}^{jk})_i = \tfrac{1}{2}\epsilon_{ijk}\big(J^{\alpha}{}_{\beta}\big)^{jk}\,, \tag{4.21}$$

and equally, the parameters of boosts and rotations emerge from $\omega_{\kappa\delta}$ according to

$$\zeta^i = \omega_{0i}\,, \qquad\qquad \vartheta^i = (*\omega_{jk})^i = \tfrac{1}{2}\epsilon^{ijk}\omega_{jk}\,. \tag{4.22}$$

In the above, the matrices K_i are the usual generators of boosts, and L_i the generators of rotations, with

$$\mathsf{K}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathsf{K}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathsf{K}_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{4.23a}$$

$$\mathsf{L}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \mathsf{L}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad \mathsf{L}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.23b}$$

which obey the Lorentz algebra

$$[\mathsf{L}_i, \mathsf{L}_j] = \epsilon_{ijk}\,\mathsf{L}_k\,, \qquad [\mathsf{K}_i, \mathsf{K}_j] = -\epsilon_{ijk}\,\mathsf{L}_k\,, \qquad [\mathsf{L}_i, \mathsf{K}_j] = \epsilon_{ijk}\,\mathsf{K}_k\,. \tag{4.24}$$

Matrix Exponential Representation of Proper Lorentz Transformations

As for any Lie group, a finite Lorentz transformation can now be written in its familiar form, in terms of the matrix exponential of parameters and generator matrices as

$$\mathsf{\Lambda}(\omega_{\alpha\beta}) = \exp\left(\tfrac{1}{2}\omega_{\alpha\beta}\mathsf{J}^{\alpha\beta} \right) = \exp(\zeta^i \mathsf{K}_i + \vartheta^i \mathsf{L}_i)\,, \tag{4.25}$$

so that, in matrix form, the infinitesimal expansion around the identity in (4.16) now reads

$$\mathsf{\Lambda} = \mathsf{1} + \tfrac{1}{2}\omega_{\alpha\beta}\mathsf{J}^{\alpha\beta} = \mathsf{1} + \zeta^i \mathsf{K}_i + \vartheta^i \mathsf{L}_i\,. \tag{4.26}$$

It is also customary to compactly write the matrix exponential in equation (4.25) in terms of vectors of parameters $\boldsymbol{\zeta} \coloneqq (\zeta^1, \zeta^2, \zeta^3)$, and $\boldsymbol{\vartheta} \coloneqq (\vartheta^1, \vartheta^2, \vartheta^3)$, as well as vectors of generator matrices $\boldsymbol{K} \coloneqq (\mathsf{K}_1, \mathsf{K}_2, \mathsf{K}_3)$, and $\boldsymbol{L} \coloneqq (\mathsf{L}_1, \mathsf{L}_2, \mathsf{L}_3)$, so that one has the representation

$$\mathsf{\Lambda} = e^{\boldsymbol{\zeta}\cdot\boldsymbol{K} + \boldsymbol{\vartheta}\cdot\boldsymbol{L}}\,. \tag{4.27}$$

Note, that often the generators (4.23), as well as the exponent of their matrix-exponential form (4.25), (4.27) of a finite proper Lorentz transformation are defined to be purely imaginary, i.e., with factors of i. In order to compensate for this, one has to additionally make the exponent of (4.25), (4.27) negative. The proper Lorentz transformations decompose into boosts (i.e., velocity transformations), that mix time and one spacial direction, and rotations, that form the subgroup SO(3). In the following discussion, we shall focus exclusively on the boost, i.e. velocity transformations, since they are the most interesting for our purposes and we assume that the reader is already familiar with the rotation group.

As noted above, an inertial frame singles out a specific time direction and thereby affects a natural (1+3)-split of space-time into space and time. Thus, the Lorentz boost matrices $\Lambda^{\alpha'}{}_{\alpha}$ decomposes into Cartesian components irreducible under SO(3); these being one Cartesian scalar $\Lambda^{0'}{}_{0}$, two vectors, $\Lambda^{a'}{}_{0}$, $\Lambda^{0'}{}_{a}$, and one tensor, $\Lambda^{a'}{}_{a}$. A general Lorentz boost with magnitude β in the direction parametrised by the Cartesian unit vector $n^{a'}$ can then be written in (1+3)-form as

$$\Lambda^{\alpha'}{}_{\alpha} = \left(\begin{array}{c|c} \Lambda^{0'}{}_{0} & \Lambda^{0'}{}_{a} \\ \hline \Lambda^{a'}{}_{0} & \Lambda^{a'}{}_{a} \end{array}\right) = \left(\begin{array}{c|c} \gamma & -\gamma\beta\, n_a \\ \hline -\gamma\beta\, n^{a'} & \delta^{a'}{}_{a} + (\gamma-1)\, n^{a'} n_a \end{array}\right), \tag{4.28}$$

where, clearly, the inverse transformation must follow from inverting the velocity parameter by replacing $\beta \to -\beta$, which reads

$$\Lambda_{\alpha'}{}^{\alpha} = \eta_{\alpha'\gamma}\eta^{\alpha\delta}\Lambda^{\gamma}{}_{\delta} = \left(\begin{array}{c|c} \gamma & \gamma\beta\, n^a \\ \hline \gamma\beta\, n_{a'} & \delta_{a'}{}^{a} + (\gamma-1)\, n_{a'} n^a \end{array}\right). \tag{4.29}$$

Here we note that, in this Cartesian context, indices are raised and lowered with the Cartesian metric δ_{ij}, i.e., all signs coming from the Minkowski metric are made explicit.

It is convenient to write the boost-parameter vector in (4.27) in terms of its magnitude $\zeta = \|\boldsymbol{\zeta}\|$ and the spacial unit vector $\boldsymbol{n}$ as $\boldsymbol{\zeta} = \zeta\, \boldsymbol{n}$. With this notation, it is straightforward to check that the following identities hold,

$$\boldsymbol{n}\cdot\boldsymbol{K} = \left(\begin{array}{c|c} 0 & \boldsymbol{n}^\intercal \\ \hline \boldsymbol{n} & 0_{3\times 3}\end{array}\right), \qquad (\boldsymbol{n}\cdot\boldsymbol{K})^{2n} = \left(\begin{array}{c|c} 1 & \boldsymbol{0}^\intercal \\ \hline \boldsymbol{0} & \boldsymbol{n}\boldsymbol{n}^\intercal\end{array}\right), \qquad (\boldsymbol{n}\cdot\boldsymbol{K})^{2n+1} = \boldsymbol{n}\cdot\boldsymbol{K}, \tag{4.30}$$

with $n \in \mathbb{N}_+$ and where $\boldsymbol{n}\boldsymbol{n}^\intercal$ is a dyadic, i.e., the tensor product of $\boldsymbol{n}$ with itself. We can now evaluate the matrix exponential (4.27) in the usual way by separating its series definition into a sum of even and odd terms, and recognising the definition of $\cosh\zeta$ and $\sinh\zeta$, respectively,

$$\Lambda = e^{\zeta\boldsymbol{n}\cdot\boldsymbol{K}} = \sum_{n=0}^{\infty}\frac{\zeta^n}{n!}(\boldsymbol{n}\cdot\boldsymbol{K})^n = 1 + \left(\begin{array}{c|c} 1 & \boldsymbol{0}^\intercal \\ \hline \boldsymbol{0} & \boldsymbol{n}\boldsymbol{n}^\intercal\end{array}\right)[\cosh(\zeta)-1] + \left(\begin{array}{c|c} 0 & \boldsymbol{n}^\intercal \\ \hline \boldsymbol{n} & 0_{3\times 3}\end{array}\right)\sinh(\zeta). \tag{4.31}$$

Thus we obtain the following well-known representation in terms of the hyperbolic angle ζ, which illustrates that Lorentz boosts are hyperbolic rotations between space and time,

$$\Lambda = \left(\begin{array}{c|c} \cosh(\zeta) & \boldsymbol{n}^\intercal\sinh(\zeta) \\ \hline \boldsymbol{n}\sinh(\zeta) & 1_3 + (\cosh\zeta - 1)\boldsymbol{n}\boldsymbol{n}^\intercal\end{array}\right), \tag{4.32}$$

with 1_3 being the three-dimensional unit matrix. In components, this reads

$$\Lambda^{\alpha'}{}_{\alpha} = \left(\begin{array}{c|c} \cosh(\zeta) & -\sinh(\zeta)\, n_a \\ \hline -\sinh(\zeta)\, n^{a'} & \delta^{a'}{}_a + (\cosh\zeta - 1)\, n^{a'} n_a\end{array}\right), \tag{4.33}$$

where comparison with (4.28) yields $\gamma = \cosh(\zeta)$, $\gamma\beta = \sinh(\zeta)$ and, consequently, $\beta = \tanh(\zeta)$.

4.2. Special Relativity in General Coordinates and Non-Inertial Frames

In the preceding section we discussed inertial observers in Minkowski space. In the present section we will turn to the description of non-inertial observers in general coordinates, which includes the important case that the observers can be accelerating and rotating, but also allows for the choice of non-Cartesian spacial coordinates in the description of their motion. This step in fact necessitates a generalisation of all the special relativistic concepts that were introduced so far to the generally covariant case, which in turn will make the transition to a pseudo-Riemannian curved space-time and thus to general relativity in the next chapter more straightforward. A good reference for most of what we shall be discussing in the following is the recent and comprehensive textbook by Gourgoulhon [53]. We have also profited from [54], and [55, 56].

4.2.1. General Coordinates and Correspondence Principle

In order to describe physical events, we now introduce general coordinates, X^κ, for which we reserve a capital X, both in the present context of special relativity, as well as later in general relativity. Also in the present special relativistic context, we will continue to use letters from the beginning of the Greek alphabet as space-time indices.

Two different sets of general coordinates, X^κ and $\tilde{X}^{\kappa'}$, are then connected by the transformation

$$\tilde{X}^{\kappa'}(X^\kappa)\,, \qquad \text{with Jacobian } \frac{\partial X^\kappa}{\partial \tilde{X}^{\kappa'}} \in \mathrm{GL}(4,\mathbb{R})\,, \tag{4.34}$$

and their respective inverse. Given a quantity a Lorentz tensor in inertial coordinates, one can usually transition to the corresponding tensor in general coordinates by applying the following correspondence rules for metric, derivatives and volume element,

$$\eta_{\kappa\delta} \longrightarrow g_{\kappa\delta}\,, \qquad \partial_\alpha \longrightarrow \nabla_\alpha\,, \qquad \epsilon_{\alpha\beta\gamma\delta} \longrightarrow \varepsilon_{\alpha\beta\gamma\delta}\,, \qquad d^4x \longrightarrow \sqrt{-g}\,d^4X\,, \tag{4.35}$$

where g is the metric determinant, $\varepsilon_{\alpha\beta\gamma\delta}$ the epsilon tensor (3.91), and where the second rule "partial derivative becomes covariant derivative" leads in particular to the following replacement prescription for the directed derivative along a curve with tangent v^α,

$$v^\alpha \partial_\alpha \equiv \frac{d}{d\lambda} \longrightarrow \frac{D}{d\lambda} \equiv v^\alpha \nabla_\alpha\,. \tag{4.36}$$

Here, $Dw^\kappa = dw^\kappa + v^\alpha \Gamma^\kappa{}_{\alpha\beta} w^\beta$ is the covariant differential of a vector w^κ that was introduced in (3.53). In the context of general relativity, the relations in (4.35) are often referred to as the *correspondence principle*. In view of (4.36), in general coordinates, the definition (4.8) of the four-acceleration becomes

$$a^\kappa := u^\alpha \nabla_\alpha u^\kappa = \frac{Du^\kappa}{d\tau} = \frac{du^\kappa}{d\tau} + \Gamma^\kappa{}_{\alpha\beta} u^\alpha u^\beta\,, \tag{4.37}$$

which, for $a^\kappa = 0$, reduces to the geodesic equation (3.55a), respective (3.55b). Turning now to the fundamental line element, infinitesimal length $d\lambda$, being a scalar, is invariant under general coordinate transformations (4.34) and thus, when expressed in terms of the general coordinates, the special relativistic line element (4.1) becomes

$$d\lambda^2 = g_{\kappa\delta}\, dX^\kappa dX^\delta\,. \tag{4.38}$$

The metric tensor, $g_{\kappa\delta} \equiv g_{\kappa\delta}(X^\gamma)$, is related to the Minkowski metric (3.62) by a certain special kind of general coordinate transformation $X^\kappa(x^{\hat{\alpha}})$ with Jacobian $\frac{\partial X^\kappa}{\partial x^{\hat{\alpha}}}$, namely the one that transforms a vector in general coordinates back to the local inertial coordinates $x^{\hat{\alpha}}$, that is we have

$$g_{\kappa\delta} \frac{\partial X^\kappa}{\partial x^{\hat{\alpha}}} \frac{\partial X^\delta}{\partial x^{\hat{\beta}}} = \eta_{\hat{\alpha}\hat{\beta}}\,. \tag{4.39}$$

In equation (4.39) above we have introduced a new notation which helps to distinguish between indices that belong to general coordinates and indices of an inertial frame: from now on, we will use a hat as in $\hat{\alpha}$ to mark frame indices, while general-coordinate indices carry no hats.[†]

4.2.2. Tetrads and their Geometric Transport along time-like Curves

We shall now make contact with the orthonormal non-coordinate bases introduced in subsection 3.3.5. The particular kind of general coordinate transformation introduced in (4.39) above is called a *tetrad*,

$$e_{\hat{\alpha}}{}^{\kappa} := \frac{\partial X^{\kappa}}{\partial x^{\hat{\alpha}}}\,, \tag{4.40}$$

(or sometimes a *Vierbein*), meaning a "four-legged" quantity, in extension of the concept of a triad or *Dreibein* (of vectors). Its four four-vectors

$$e_{\hat{\alpha}}{}^{\kappa} = \left(e_{\hat{0}}{}^{\kappa},\, e_{\hat{a}}{}^{\kappa}\right), \tag{4.41}$$

with $e_{\hat{0}}{}^{\kappa}$ time-like and the three spatial ones, $e_{\hat{a}}{}^{\kappa}$, space-like, are nothing else but the components of the orthonormal non-coordinate basis vectors (3.60). In view of (3.63a), equation (4.39) and its inverse are then interpreted as the defining normalisation relations for the tetrads, i.e., we have

$$\eta_{\hat{\alpha}\hat{\beta}} = e_{\hat{\alpha}}{}^{\kappa}\, e_{\hat{\beta}}{}^{\delta}\, g_{\kappa\delta}\,, \qquad\qquad g_{\kappa\delta} = \eta_{\hat{\alpha}\hat{\beta}}\, e^{\hat{\alpha}}{}_{\kappa}\, e^{\hat{\beta}}{}_{\delta}\,, \tag{4.42}$$

as well as from (3.63b) the corresponding relations for the inverse metrics,

$$\eta^{\hat{\alpha}\hat{\beta}} = e^{\hat{\alpha}}{}_{\kappa}\, e^{\hat{\beta}}{}_{\gamma}\, g^{\kappa\gamma}\,, \qquad\qquad g^{\kappa\delta} = \eta^{\hat{\alpha}\hat{\beta}}\, e_{\hat{\alpha}}{}^{\kappa}\, e_{\hat{\beta}}{}^{\delta}\,. \tag{4.43}$$

Since tetrads are transformation matrices that locally relate two different spaces, they generally carry two indices of different types, one for each space. Thus, they can be interpreted as a vector in either of these spaces, and the two different types of vector indices are acted upon by the respective coordinate transformation of that space, with the other index being inert, i.e. transforming as a scalar.

More concretely, two tetrads $e_{\hat{\alpha}}{}^{\kappa}$ and $\tilde{e}_{\hat{\alpha}'}{}^{\kappa}$ defined by (4.42) in the same tangent space at a space-time point $\mathscr{P}$ are related by Lorentz transformations, which act exclusively on the (hatted) tetrad or frame index $\hat{\alpha}$, and "do not see" the "general-coordinate" (or later, in general relativity, "space-time") index κ,

$$\tilde{e}_{\hat{\alpha}'}{}^{\kappa}\big|_{\mathscr{P}} = \Lambda_{\hat{\alpha}'}{}^{\hat{\alpha}}\, e_{\hat{\alpha}}{}^{\kappa}\big|_{\mathscr{P}}\,, \tag{4.44}$$

[†] The hat notation for frame indices seems to be one of the two standard conventions in the literature (used by, e.g., MTW [48]). The other common one is to enclose frame indices in brackets, e.g., $e_{(\alpha)}{}^{\kappa}$ (as in [56], or [54]), which would collide with our notation of index symmetrisation (this also being a standard in the literature). Seldomly, one finds underlines (e.g., Marzlin [57]), or sans-serif fonts (e.g., Poisson, Pound and Vega [58]) being used. Sometimes the order of the indices is interchanged, or the spacing between them is collapsed altogether. Some authors also use the symbol λ to denote a tetrad.

Conversely, general coordinate transformations (4.34) act on the general-coordinate index κ with the tetrad index $\hat{\alpha}$ being inert,

$$\tilde{e}_{\hat{\alpha}}{}^{\kappa'} = \frac{\partial \tilde{X}^{\kappa'}}{\partial X^{\kappa}}\, e_{\hat{\alpha}}{}^{\kappa} \,. \tag{4.45}$$

In order to uniquely fix the freedom in the choice of Lorentz frame, i.e. of local time direction in (4.44), and in order for the tetrad to describe the (infinitesimal) reference frame of an idealised physical observer, the time-like vector is conventionally chosen parallel to the four-velocity u^{κ} along the world line of that fiducial observer, that is we set

$$e_{\hat{0}}{}^{\kappa} = \frac{u^{\kappa}}{c} \,, \tag{4.46}$$

and thus the tetrad's time-like vector $e_{\hat{0}}{}^{\kappa}$ points along the observer's local time direction.

4.2.3. Transport along an Observer's World-Line

We now turn to the question of how the tetrad changes between neighbouring points along the general world-line $\mathscr{W}$ that a fiducial idealised physical observer traces out as he moves through four-dimensional space-time. For this, we assume that we have installed along $\mathscr{W}$ a tetrad that is adapted to the observer's four-velocity $u^{\kappa}(\tau)$ according to (4.46).

We recall that for an orthonormal non-coordinate basis, we have the general relation (3.39), i.e. (dropping the tilde on the connection coefficients),

$$e_{\hat{\alpha}}{}^{\delta} \nabla_{\delta} e_{\hat{\beta}}{}^{\kappa} = \Gamma^{\hat{\delta}}{}_{\hat{\alpha}\hat{\beta}} e_{\hat{\delta}}{}^{\kappa} \,, \tag{4.47}$$

so that the change of the basis vector $e_{\hat{\beta}}{}^{\kappa}$ in the direction of another basis vector $e_{\hat{\alpha}}{}^{\delta}$ is given as a linear combination of basis vectors in terms of the connection coefficients $\Gamma^{\hat{\kappa}}{}_{\hat{\alpha}\hat{\beta}}$. We now restrict (4.47) to a time-like curve, so that the directed covariant derivative becomes $e_{\hat{0}}{}^{\delta}\nabla_{\delta}$, which we write in terms of the four-velocity as

$$u^{\delta} \nabla_{\delta} e_{\hat{\alpha}}{}^{\kappa} = -\Omega^{\hat{\delta}}{}_{\hat{\alpha}}\, e_{\hat{\delta}}{}^{\kappa} \,, \tag{4.48}$$

see, e.g., [48, Sec. 6.5 and Exercise 6.8] and [53, Sec. 3.5]. Here, $\Omega^{\hat{\kappa}}{}_{\hat{\alpha}}$ is the so-called *transport matrix* [the sign on the right-hand side of (4.48) is conventional and is omitted by some authors]. Comparing (4.48) with the general expression in terms of the connection coefficients (4.47), we find

$$\Gamma^{\hat{\kappa}}{}_{\hat{0}\hat{\alpha}} = -\tfrac{1}{c}\Omega^{\hat{\kappa}}{}_{\hat{\alpha}} \,, \tag{4.49}$$

where we note that the transport matrix also inherits its antisymmetry from the orthonormal-frame connection coefficients,

$$\Omega_{\hat{\kappa}\hat{\alpha}} = -\Omega_{\hat{\alpha}\hat{\kappa}} \,, \tag{4.50}$$

and thus it has 6 independent components in general. The transport matrix expresses how the tetrad or frame $e_{\hat{\alpha}}{}^{\kappa}$ "twists" and "turns" in four-dimensional space-time as it is propagated (or, more geometrically: transported) along the world-line. More formally, it collects those connection coefficients that result from accelerations and rotations of the frame, as we shall see in the following.

In an inertial frame, e.g. in the global inertial coordinates of a fiducial Minkowski observer, we can express the directed derivative on the left-hand side of (4.48) in terms of the parametric derivative with respect to τ. This turns the general transport equation above into an ordinary differential equation,

$$\frac{de_{\hat{\alpha}}{}^{\kappa}(\tau)}{d\tau} = -\Omega^{\hat{\delta}}{}_{\hat{\alpha}}(\tau)\, e_{\hat{\delta}}{}^{\kappa}(\tau)\,, \tag{4.51}$$

cf. Misner, Thorne and Wheeler [48, chapter 6]. We note, that if the transport matrix vanishes altogether, i.e., for $\Omega^{\hat{\kappa}}{}_{\hat{\alpha}} = 0$, equation (4.48) reduces to the equation of parallel transport (3.52a) for the spacial basis vectors $e_{\hat{b}}{}^{\kappa}$, and to the geodesic equation for $e_{\hat{0}}{}^{\kappa}$, so that the whole tetrad is *parallel transported* along the world-line which is then a geodesic. In this case, the tetrad clearly provides an inertial frame.

In general however, the transport matrix will not vanish, i.e., the frame provided by the tetrad will be non-inertial. We will now investigate how the transport matrix relates to accelerations and rotations of the frame, starting with its time–space part, $\Omega_{\hat{a}\hat{0}}$. For that we compare the transport equation (4.48) with the definition of the four-acceleration (4.37), both taken in frame components, and with allowance for the requirement (4.46) that the time-like tetrad vector be parallel to the four-velocity. This yields

$$\Omega^{\hat{\kappa}}{}_{\hat{0}} = -\tfrac{1}{c}a^{\hat{\kappa}}\,, \qquad\qquad \Omega^{\hat{0}}{}_{\hat{\alpha}} = \tfrac{1}{c}a_{\hat{\alpha}}\,, \tag{4.52}$$

and equivalently, in terms of the connection coefficients, we have with (4.49) that

$$\Gamma^{\hat{\kappa}}{}_{\hat{0}\hat{0}} = \frac{a^{\hat{\kappa}}}{c^2}\,, \qquad\qquad \Gamma^{\hat{0}}{}_{\hat{0}\hat{\alpha}} = -\frac{a_{\hat{\alpha}}}{c^2}\,. \tag{4.53}$$

Clearly, since the above expressions (4.53) and (4.52) are written in terms of frame components, we can restrict $\hat{\kappa} = \hat{k}$ and $\hat{\alpha} = \hat{a}$ to purely spacial indices as a consequence of (4.9). The time–space part of the transport matrix is then uniquely determined by its antisymmetry (4.50) and the relation to accelerations (4.52) above, which provides 3 of its 6 independent components, with the other 3 residing in its purely spacial part. Assuming at first that these purely spacial components vanish, $\Omega^{\hat{k}}{}_{\hat{a}} = 0$, we can combine equations (4.52) into the manifestly antisymmetric expression,

$$\Omega^{\hat{\kappa}}{}_{\hat{\alpha}} = -\tfrac{1}{c}\big[a^{\hat{\kappa}}\eta_{\hat{0}\hat{\alpha}} - \eta^{\hat{0}\hat{\kappa}}a_{\hat{\alpha}}\big]\,. \tag{4.54}$$

However, more insight is gained if we make this equation generally covariant by applying the correspondence principle, i.e., by transforming the hatted inertial-frame indices with tetrads, which yields the transport tensor,

$$\Omega^{\kappa}{}_{\alpha} = e_{\hat{\kappa}}{}^{\kappa}\Omega^{\hat{\kappa}}{}_{\hat{\alpha}}e^{\hat{\alpha}}{}_{\alpha} = -\tfrac{1}{c}\big[a^{\kappa}e_{\hat{0}\alpha} - e_{\hat{0}}{}^{\kappa}a_{\alpha}\big]\,. \tag{4.55}$$

Using (4.46), we finally obtain the expression for the time–space, or acceleration-only part of the transport tensor in the form that it is usually displayed, i.e., in terms of the four-acceleration and four-velocity,

$$\Omega^{\kappa}{}_{\alpha} = -\tfrac{1}{c^2}\left[a^{\kappa}u_{\alpha} - u^{\kappa}a_{\alpha}\right], \qquad \text{(Fermi-Walker, or acceleration part).} \tag{4.56}$$

A tetrad undergoing acceleration-only transport according to (4.48) or (4.51) with a transport matrix given by (4.56) is said to be *Fermi-Walker transported* and thereby describes a non-inertial, accelerating frame.

So far, we have neglected the purely spacial part $\Omega^{\hat{k}}{}_{\hat{a}}$ of the transport matrix. Motivated by the assumption that this spacial part must be related to rotations of the frame, we can make the natural antisymmetric and tensorial ansatz

$$\Omega_{\kappa\alpha} = \tfrac{1}{c}u^{\delta}\omega^{\beta}\,\epsilon_{\delta\beta\kappa\alpha} = \tfrac{1}{c}u^{\delta}(*\omega)_{\delta\kappa\alpha}\,, \qquad \text{(rotation part),} \tag{4.57}$$

in terms of the Hodge dual of an angular-velocity four-vector ω^{δ}, which, like the four-acceleration, is then a purely spacial vector when expressed in the tetrad components of the corresponding observer, $\omega^{\hat{\alpha}} = (0, \omega^{\hat{a}})$. Just as in (4.53) with the acceleration, we can also write the spacial orthonormal-frame connection coefficients in terms of the rotation vector,

$$\Gamma^{\hat{k}}{}_{\hat{0}\hat{\jmath}} = -\tfrac{1}{c}\omega^{\hat{\imath}}\epsilon_{\hat{\imath}}{}^{\hat{k}}{}_{\hat{\jmath}}\,. \tag{4.58}$$

Combining (4.56) and (4.57), we can finally display the full transport matrix, consisting of an acceleration part, and a rotation part, i.e.,

$$\boxed{\Omega^{\kappa}{}_{\alpha} = -\tfrac{1}{c^2}\left[a^{\kappa}u_{\alpha} - u^{\kappa}a_{\alpha}\right] + \tfrac{1}{c}u^{\delta}\omega^{\beta}\,\epsilon_{\delta\beta}{}^{\kappa}{}_{\alpha}\,.} \tag{4.59}$$

There is a direct connection between the transport matrix in (4.48) and infinitesimal Lorentz transformations, as we shall now see (we follow [54, Appendix B 2.2]). We begin by recalling that the most general tetrad can be written as (4.44) in terms of another tetrad and a Lorentz transformation. Thus, we start very generally by considering two tetrads $\tilde{e}_{\hat{\alpha}'}{}^{\kappa}(\tau)$ and $e_{\hat{\alpha}}{}^{\kappa}(\tau)$ that exist in the same tangent space along a world-line. Without making any further assumptions yet, we can say that (4.44) must hold for every instant in time τ along the world-line, i.e., for every τ they will be related through a separate Lorentz transformation, so the relationship between the two tetrads along the world-line is given by a family of (inverse) Lorentz transformations $\Lambda_{\hat{\alpha}'}{}^{\hat{\alpha}}(\tau)$, parametrised by τ. In slight generalisation of (4.44), we thus have

$$\tilde{e}_{\hat{\alpha}'}{}^{\kappa}(\tau) = \Lambda_{\hat{\alpha}'}{}^{\hat{\alpha}}(\tau)\, e_{\hat{\alpha}}{}^{\kappa}(\tau)\,. \tag{4.60}$$

In a next step, we assume that $\tilde{e}_{\hat{\alpha}'}{}^{\kappa}$ undergoes general transport according to the transport equation (4.48) [which we use in its parameter form (4.51)]. We thus insert (4.60) into the transport law (4.51), which yields

$$\frac{d\Lambda_{\hat{\alpha}'}{}^{\hat{\alpha}}(\tau)}{d\tau}e_{\hat{\alpha}}{}^{\kappa}(\tau) + \Lambda_{\hat{\alpha}'}{}^{\hat{\alpha}}(\tau)\frac{de_{\hat{\alpha}}{}^{\kappa}(\tau)}{d\tau} = -\Omega^{\hat{\delta}'}{}_{\hat{\alpha}'}(\tau)\Lambda_{\hat{\delta}'}{}^{\hat{\delta}}(\tau)e_{\hat{\delta}}{}^{\kappa}(\tau)\,, \tag{4.61}$$

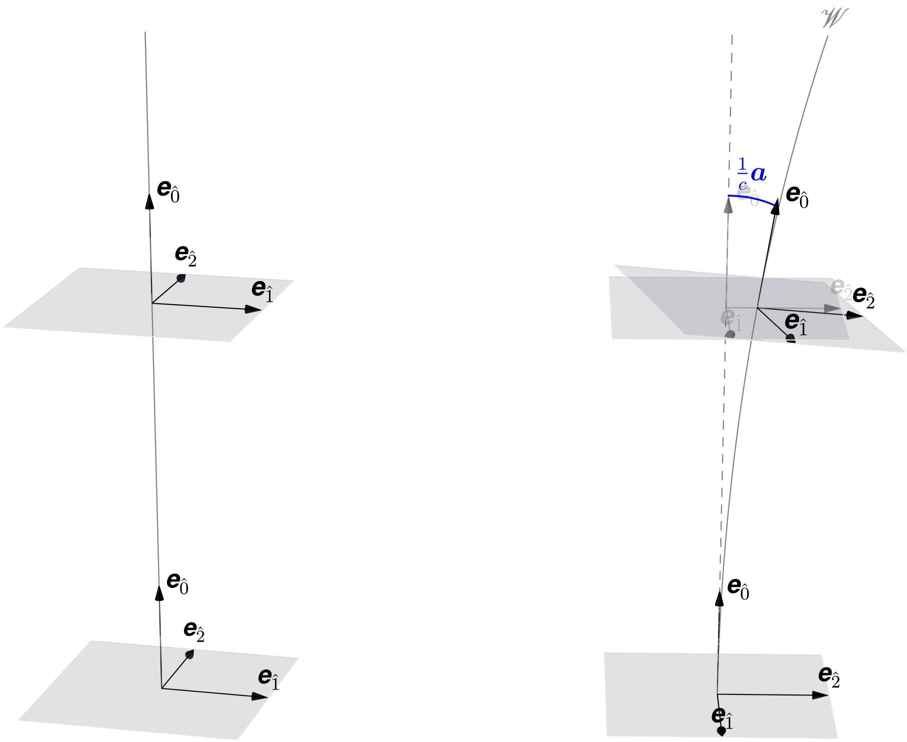

Figure 4.1.: Illustration of tetrad transport along different time-like world-lines. Left: an inertial tetrad undergoing parallel transport along its geodesic $\mathscr{G}$ which is a straight line in Minkowski space. Right: a non-inertial, accelerated frame undergoing Fermi-Walker transport along its world-line $\mathscr{W}$, with a momentarily co-moving inertial frame superimposed. The acceleration acting on $\mathbf{e}_{\hat{0}}$ effects an infinitesimal Lorentz boost of the frame, which leads to the accelerated frame tilting away from its momentarily co-moving inertial counterpart, by a corresponding (hyperbolic) angle determined by $\boldsymbol{a}$ (blue).

where we can rearrange and contract away the Lorentz transformation on the left-hand side with $\Lambda^{\hat{\alpha}'}{}_{\hat{\gamma}}$ to obtain a new transport equation

$$\frac{de_{\hat{\gamma}}{}^{\kappa}(\tau)}{d\tau} = -\widetilde{\Omega}^{\hat{\delta}}{}_{\hat{\gamma}}(\tau)\, e_{\hat{\delta}}{}^{\kappa}(\tau)\,, \tag{4.62}$$

in terms of the transformed transport matrix,

$$\widetilde{\Omega}_{\hat{\gamma}}{}^{\hat{\delta}}(\tau) := \Lambda^{\hat{\alpha}'}{}_{\hat{\gamma}}\Big[\frac{d\Lambda_{\hat{\alpha}'}{}^{\hat{\delta}}}{d\tau} + \Omega^{\hat{\delta}'}{}_{\hat{\alpha}'}\Lambda_{\hat{\delta}'}{}^{\hat{\delta}}\Big]\,. \tag{4.63}$$

In passing, we note that (4.63) has the form of the inhomogeneous transformation law between connections (3.42), here between the two types of connection coefficients $\Omega^{\hat{\delta}'}{}_{\hat{\alpha}'}$ and $\widetilde{\Omega}_{\hat{\gamma}}{}^{\hat{\delta}}$ that belong to the two different orthonormal bases $\tilde{e}_{\hat{\alpha}'}{}^{\kappa}$ and $e_{\hat{\alpha}}{}^{\kappa}$.

Up to this point, we have not yet specified how the tetrad $e_{\hat{\alpha}}{}^{\kappa}$ transports. If we now take $e_{\hat{\alpha}}{}^{\kappa}$ to be parallel transported, its time derivative and thus $\widetilde{\Omega}_{\hat{\gamma}}{}^{\hat{\delta}}$ in the transformed transport equation (4.62) vanishes and we find that (4.62) reduces to a linear ordinary differential equation for the family of (inverse) Lorentz transformations,

$$\frac{d\Lambda_{\hat{\alpha}}{}^{\hat{\delta}}(\tau)}{d\tau} = -\Omega^{\hat{\delta}'}{}_{\hat{\alpha}}(\tau)\Lambda_{\hat{\delta}'}{}^{\hat{\delta}}(\tau)\,, \tag{4.64}$$

where we have contracted away the tetrad and the Lorentz transformation on the right with their respective inverses. In a last step, we now assume that the two tetrads coincide at some initial time τ_0, i.e., that

$$e_{\hat{\alpha}}{}^{\kappa}(\tau_0) = \tilde{e}_{\hat{\alpha}'}{}^{\kappa}(\tau_0)\,, \tag{4.65}$$

which leads to the initial condition for the time-dependent family of Lorentz transformations being

$$\Lambda_{\hat{\delta}'}{}^{\hat{\delta}}(\tau_0) = \delta_{\hat{\delta}'}{}^{\hat{\delta}}\,. \tag{4.66}$$

We can now use the initial condition and the differential equation (4.64) for the above family of Lorentz transformations to write down its first-order solution $\Lambda_{\hat{\alpha}}{}^{\hat{\delta}}(\tau)$ around $\tau = \tau_0$. This yields

$$\Lambda_{\hat{\alpha}}{}^{\hat{\delta}}(\tau + \delta\tau) = \delta_{\hat{\alpha}}{}^{\hat{\delta}} - \Omega^{\hat{\delta}}{}_{\hat{\alpha}}(\tau_0)\,\delta\tau\,. \tag{4.67}$$

Comparing with (4.16), (4.18), this leads us to conclude that $\Omega^{\hat{\kappa}}{}_{\hat{\alpha}}$ generates infinitesimal Lorentz transformations, with

$$\omega^{\hat{\kappa}}{}_{\hat{\alpha}}(\tau) = \Omega^{\hat{\kappa}}{}_{\hat{\alpha}}\,\delta\tau\,. \tag{4.68}$$

In summary, we have found that the transport matrix in the general transport law (4.48) generates infinitesimal Lorentz transformations, i.e. boosts and rotations, that compensate for the acceleration and rotation of the frame.

A comparison between an inertial frame undergoing parallel transport and a non-inertial frame undergoing proper transport is displayed in the space-time diagram of Figure 4.1. The acceleration $\boldsymbol{a}$ acting on the non-inertial frame's time-like vector effects a continuous family of infinitesimal Lorentz boost that makes $e_{\hat{0}}{}^{\kappa}(\tau)$ tilt away from that of a momentarily co-moving inertial frame by a hyperbolic angle $|\boldsymbol{a}|/c$. The acceleration part of the transport matrix thus ensures that the non-inertial frame's time-like vector continues to be tangent to the world-line, as it "curves away" from the straight geodesic path of an inertial observer. This frame's spacial vectors $e_{\hat{a}}{}^{\kappa}(\tau)$ are then similarly transformed, so that they stay orthogonal to the changed time-like vector. Additionally, the spacial vectors are free to rotate with angular velocity $\omega^{\hat{\imath}}$ in the spacial subspace orthogonal to $e_{\hat{a}}{}^{\kappa}(\tau)$.

In the following, we close this section by commenting on the transport tensor and its analogue from electrodynamics. In the local 1+3-split of space-time into time and space affected by the tetrad, we can also write the transport tensor (4.59) in matrix form as,

$$\Omega^{\alpha}{}_{\beta} = \left(\begin{array}{c|c} 0 & \frac{1}{c}a_b \\ \hline -\frac{1}{c}a^a & \omega^{\hat{d}}\epsilon_{\hat{d}}{}^{\hat{a}}{}_{\hat{b}} \end{array}\right) = \left(\begin{array}{c|c} 0 & -\frac{1}{c}\boldsymbol{a}^{\mathsf{T}} \\ \hline -\frac{1}{c}\boldsymbol{a} & -(*\boldsymbol{\omega}) \end{array}\right) \tag{4.69}$$

where, in the second equation, we have expressed the matrix in terms of the Cartesian acceleration vector $\boldsymbol{a}$ and the spacial Hodge dual of the angular-velocity vector $\boldsymbol{\omega}$, making all signs explicit. Out of the two-form $\Omega_{\alpha\beta}$, we can construct two algebraic invariants, one by full contraction of $\Omega_{\alpha\beta}$ with itself, the other by full contraction with its Hodge dual $(*\Omega)_{\alpha\beta}$,

$$\mathcal{I}_1 := \tfrac{1}{2}\Omega_{\alpha\beta}\Omega^{\alpha\beta} = \boldsymbol{\omega}^2 - \left(\frac{\boldsymbol{a}}{c}\right)^2 \qquad \text{(Lorentz scalar)}, \tag{4.70a}$$

$$\mathcal{I}_2 := (*\Omega)_{\alpha\beta}\Omega^{\alpha\beta} = \tfrac{1}{c}\boldsymbol{a}\cdot\boldsymbol{\omega} \qquad \text{(Lorentz pseudo-scalar)}. \tag{4.70b}$$

While the first is a genuine Lorentz scalar, the second involves an epsilon (pseudo-) tensor from the definition of the Hodge dual (3.99) and thus acquires an additional sign under improper Lorentz transformations, which makes it a pseudo-scalar.

4.2.4. Analogy with Electrodynamics

It is interesting to note the close structural analogy between the transport matrix (4.59) and the electromagnetic field strength tensor $F_{\alpha\beta}$ on the one hand, and the transport equation (4.48) and the covariant Lorentz force law on the other hand. The Lorentz force law, giving the acceleration of a point charge q of mass m due to the electromagnetic field, reads

$$a^\kappa_{\text{em}} = \frac{q}{m} F^\kappa{}_\delta u^\delta\,, \tag{4.71}$$

where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the anti-symmetric electromagnetic field strength tensor and A_α the corresponding gauge field, which, in an inertial frame, decomposes into a scalar potential ϕ and three-vector potential A^a, i.e., $A^\alpha = (\phi/c, A^a)$. In said inertial frame, $F_{\alpha\beta}$ then decomposes into a spacial vector, which is the electric field E^i, and an antisymmetric spacial tensor, which is the (spacial, i.e. three-dimensional) Hodge dual of the magnetic field vector B^i,

$$E^i = -cF^{0i}\,, \tag{4.72a}$$

$$B^i = (*F)^{0i} = \tfrac{1}{2}F^{ab}\epsilon_{ab}{}^{0i}\,, \tag{4.72b}$$

so that $F_{\alpha\beta}$ takes the familiar form

$$F_{\alpha\beta} = \left(\begin{array}{c|c} 0 & -E_b/c \\ \hline E_a/c & -B^c\epsilon_{cab}\end{array}\right) = \left(\begin{array}{c|c} 0 & E^b/c \\ \hline -E^a/c & -B^c\epsilon_{cab}\end{array}\right) = \left(\begin{array}{c|c} 0 & \boldsymbol{E}/c \\ \hline -\boldsymbol{E}/c & *\boldsymbol{B}\end{array}\right) \tag{4.73}$$

[note that in contrast to (4.69) some signs are switched since we have displayed $F_{\alpha\beta}$ in index-lowered form].

Just as the two-form (4.59) above, $F_{\alpha\beta}$ too possesses two algebraic invariants, one Lorentz scalar and one pseudo-scalar,

$$\mathcal{I}^{\text{em}}_1 := \tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta} = \tfrac{1}{2}\left(\boldsymbol{B}^2 - \frac{\boldsymbol{E}^2}{c^2}\right), \qquad \text{(Lorentz scalar)} \tag{4.74a}$$

$$\mathcal{I}^{\text{em}}_2 := \tfrac{1}{4}(*F)_{\alpha\beta}F^{\alpha\beta} = \tfrac{1}{c}\boldsymbol{E}\cdot\boldsymbol{B}\,, \qquad \text{(Lorentz pseudo-scalar)} \tag{4.74b}$$

Note, that the scalar invariant (4.74a) is essentially the Lagrangian of electromagnetism.

Since $\mathcal{I}^{\text{em}}_1$ and $\mathcal{I}^{\text{em}}_2$ are Lorentz invariants, they give rise to a *local*, invariant classification of the electromagnetic field; see [53, Chapter 17] for a discussion and applications of this classification. Clearly, such an invariant classification can analogously be carried out for any two-form, and also for even-rank tensors that are pairwise antisymmetric in their index slots (i.e., "multi–two-forms"), an example being the Riemann tensor (3.80). In general relativity this leads to the important classification of the trace-free part of the Riemann tensor into so-called Petrov types, and the associated physical interpretation of the corresponding space-times; which we will address in subsection 5.6.4.

4.3. Local Frames and Coordinates for Accelerating and Rotating Observers in Special Relativity

While in the first part of the present chapter we introduced (besides the necessary formalism and notation) the central physical concept of inertial frame, as well as the corresponding mathematical concept of a tetrad and its differential equation, all of which govern non-inertial motion in special relativity, this section is devoted to the actual implementation, i.e. construction, of inertial frames and of the attached local inertial coordinate systems. We start with a discussion of the (formal) solution of the transport equation's initial value problem in terms of a time-ordered matrix exponential, thereby introducing the important concept of parallel propagator (or "translator"), and then turn to the general construction of local non-inertial frames. Subsequently, we show how the transport equation can be solved exactly for time-independent inertial forces and investigate a particular subclass of this solution, which turns out to describe the physically interesting case of an observer who is in circular orbital motion in Minkowski space. We conclude this section and thereby also this chapter by showing how the relative *geodesic* motion of another observer or particle can be described in the accelerating and rotating frame of our primary "spectator" observer.

4.3.1. Formal Solution of the Transport Equation and Parallel Propagator

We now return to the transport equation (4.48), in its inertial-frame form (4.51), with the intention of deriving its (formal) solution. At this point, however, it is worthwhile to discuss the solution in a more general setting, namely for the case of the equation of parallel transport (3.52a) in a (pseudo-) Riemannian space, of which the transport equation (4.48), or (4.51), is a special case, knowing that this more general case will become relevant when we discuss Riemann and Fermi normal coordinates and covariant expansions in chapter 6.

Thus, consider the equation of parallel transport (3.52a) along a curve with tangent $v^\kappa(\lambda)$, which we take to be normalised, $v^\lambda v_\lambda = 1$, without loss of generality, and which, in the present context, obviously stands for the u^κ in (4.48), written in terms of a general affine parameter λ (which, of course, stands for τ). The equation of parallel transport then reads,

$$\frac{de_{\hat\alpha}{}^\kappa(\lambda)}{d\lambda} = v^\lambda(\lambda)\,\Gamma^\kappa{}_{\lambda\nu}(\lambda)\,e_{\hat\alpha}{}^\nu(\lambda) = -\Omega^\kappa{}_\nu(\lambda)\,e_{\hat\alpha}{}^\nu(\lambda)\,, \tag{4.75}$$

where, in analogy to (4.49), we have defined the general (pseudo-) Riemannian "connection" or "transport" matrix as,

$$\Omega^\kappa{}_\nu := -v^\lambda\Gamma^\kappa{}_{\lambda\nu}\,. \tag{4.76}$$

Note that here the connection $\Gamma^\kappa{}_{\lambda\nu}$ is completely general, so it doesn't even have to derive from a metric (the notion of parallel transport is well defined on any affine manifold, i.e. any manifold with connection, so the manifold need not be Riemannian). Clearly, in special relativity it will always be possible to reduce this general "transport matrix" to (4.49) in an inertial frame with $\lambda = c\tau$ and $v^\mu = u^\mu/c$.

Parallel Propagator

We note that, somewhat formally, we can write the general solution of this matrix-valued, first-order ordinary differential equation in terms of a matrix, $g^\kappa{}_\alpha(\tau, \tau_0)$, which is generally known as the *parallel propagator*, as (see [59, Appendix I])

$$e_{\hat{\alpha}}{}^\kappa(\lambda) = g^\kappa{}_\nu(\lambda, \lambda_0)\, e_{\hat{\alpha}}{}^\nu(\lambda_0)\,. \tag{4.77}$$

The fact that one uses the same symbol g as for the metric is not an accident, as clearly, the initial condition for the parallel propagator must read

$$g^\kappa{}_\rho(\lambda_0, \lambda_0) = \delta^\kappa{}_\rho\,, \tag{4.78}$$

which is the mixed, i.e., $(1,1)$-form $\delta^\kappa{}_\rho = g^\kappa{}_\rho$ of the metric. In this sense, the parallel propagator can be thought of as a bi-local form of the metric, to which it must reduce for $\lambda = \lambda_0$.

More generally, the parallel propagator is the general solution to the equation of parallel transport along the unique geodesic that connects two points x' and x on a pseudo-Riemannian manifold: It takes a vector, say $v^{\sigma'}(x')$, at x' and parallel transports it to x along the unique geodesic that links these points, i.e.,

$$v^\sigma(x) = g^\sigma{}_{\sigma'}(x, x')\, v^{\sigma'}(x')\,, \tag{4.79}$$

[58, Sec. 5.2]. It can also be written in terms of a contraction of the tetrad at x with the inverse tetrad at x' on their tetrad index,

$$g^\sigma{}_{\sigma'}(x, x') = e_{\hat{\alpha}}{}^\sigma(x)\, e^{\hat{\alpha}}{}_{\sigma'}(x')\,. \tag{4.80}$$

Here one has made use of the fact that the frame components $v^{\hat{\alpha}}(x') = e^{\hat{\alpha}}{}_{\sigma'}(x')v^{\sigma'}(x')$ of a vector are constant under parallel transport, so it suffices to go to frame components at x via the tetrad there, and then go to space-time components again at x' via $e^{\hat{\alpha}}{}_{\sigma'}(x')$. The parallel propagator is an example of a bi-vector, in the sense of a bi-local vector. More precisely, a bi-vector is an object which transforms as a vector at two points x', and x, separately and which thus lives in the tangent spaces of these two points*. The previous

*Unfortunately, the term *bi-vector* has two completely different meanings in general relativity, the other one denoting a two-form seen as a vector in a six-dimensional space, as mentioned in subsection 3.1.6 and used in subsection 5.6.2. In contrast, a bi-vector in the present sense is the rank-one case of a bi-tensor. The reader may more familiar with well-known bi-scalars, such as the action $S(x', x)$ between two points, or the covariant Green's functions $G(x', x)$. Bi-tensors were apparently introduced by Ruse [60] and Synge [61] in the early 1930s, and independently by DeWitt and Brehme [62] in 1960, who fully developed the theory.

form of the parallel propagator in terms of a parameter λ is then recovered by inserting the parametrisation of the geodesic connecting x and x', say, $x = x(\lambda)$ and $x' = x'(\lambda_0)$.

With (4.75) and (4.77) we find, that the parallel propagator itself also satisfies the transport equation,

$$\frac{dg^\kappa{}_\rho}{d\tau}(\lambda, \lambda_0) = -\Omega^\kappa{}_\nu(\lambda)\, g^\nu{}_\rho(\lambda, \lambda_0)\,. \tag{4.81}$$

Formally integrating both sides of equation (4.81) then yields the Volterra-type integral equation,

$$g^\kappa{}_\rho(\lambda, \lambda_0) = \delta^\kappa{}_\rho - \int_{\lambda_0}^{\lambda} \Omega^\kappa{}_{\nu_1}(\lambda_1)\, g^{\nu_1}{}_\rho(\lambda_1, \lambda_0)\, d\lambda_1\,, \tag{4.82}$$

which can be formally solved by Picard iteration, i.e. by repeatedly re-inserting it back into itself, the first terms of which read

$$g^\kappa{}_\rho(\lambda, \lambda_0) = \delta^\kappa{}_\rho - \int_{\lambda_0}^{\lambda} \Omega^\kappa{}_\rho(\lambda_1)\, d\lambda_1 + \int_{\lambda_0}^{\lambda}\int_{\lambda_0}^{\lambda_1} \Omega^\kappa{}_{\nu_1}(\lambda_1)\, \Omega^{\nu_1}{}_\rho(\lambda_2)\, d\lambda_1\, d\lambda_2 + \cdots\,. \tag{4.83a}$$

The right-hand side of (4.83a) is actually a generalised exponential series, i.e. the series is of exponential type but with non-commuting factors. One can formally sum (4.83a) by introducing a non-linear path-ordering operator P which, when applied to a product of parameter-, i.e., λ-dependent functions or operators, indicates that the product be re-ordered in such a way, that the functions depending on smaller values of λ be to the left of those with larger values. Thus,

$$g^\kappa{}_\rho(\lambda, \lambda_0) = \Big(1 - \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\lambda_0}^{\lambda} \mathrm{P}_\lambda \big[\Omega(\lambda_1)\Omega(\lambda_2)\cdots\Omega(\lambda_{n-1})\Omega(\lambda_n)\big]\, d^n\lambda\Big)^\kappa{}_\rho\,, \tag{4.83b}$$

where, in the second line, we have written the $\Omega^\kappa{}_\nu$ in matrix form as Ω.

The expression on the right-hand side of (4.83b) now is the formal closed-form solution of the transport equation for the parallel propagator (4.77) in terms of a time-ordered matrix exponential, i.e.,

$$g^\kappa{}_\rho(\lambda, \lambda_0) = \Big(\mathrm{P}_\lambda \exp\Big\{-\int_{\lambda_0}^{\lambda} \Omega(\lambda')\, d\lambda'\Big\}\Big)^\kappa{}_\rho\,, \tag{4.84}$$

and thus the formal solution of the equation of parallel transport (4.75) follows by contracting the parallel propagator with the initial tetrad $e_{\hat\alpha}{}^\nu(\lambda_0)$ as in (4.77). Of course, this procedure is used in practice to obtain an approximate solution of the parallel-transport equation by truncating the resulting series at some finite order.

Specialising this general result to the case of the proper transport equation (4.51), with path ordering P_λ becoming time ordering P_τ, we conclude that its formal solution is given in terms of the inverse Minkowskian parallel propagator $g_{\hat\alpha}{}^{\hat\delta}(\tau, \tau_0)$ by

$$e_{\hat\alpha}{}^\kappa(\tau) = g_{\hat\alpha}{}^{\hat\delta}(\tau, \tau_0)\, e_{\hat\delta}{}^\kappa(\tau_0) = \Big(\mathrm{P}_\tau \exp\Big\{-\int_{\tau_0}^{\tau} \Omega(\tau')\, d\tau'\Big\}\Big)_{\hat\alpha}{}^{\hat\delta}\, e_{\hat\delta}{}^\kappa(\tau_0)\,. \tag{4.85}$$

Calculating the time-ordered matrix exponential in (4.85) for a general time-dependent transport matrix is generally believed to be impossible, so one has to resort to an approximate evaluation of (4.84), for example in terms of a Taylor expansion, or equivalently, in terms of a truncation at some finite n of the iterated solution (4.83b). However, in certain special cases, e.g. for time-independent $\Omega^{\kappa}{}_{\alpha}$ for which the time ordering in (4.84) becomes trivial, equations (4.84) and (4.85) reduce to a conventional matrix exponential, which can then be evaluated exactly, e.g. by means of the Cayley-Hamilton theorem. We shall discuss this case in subsection 4.3.3 below.

4.3.2. Local Coordinates for Non-Inertial Observers in Special Relativity

In the present subsection, we define local coordinates for inertial and non-inertial observers, based on their tetrad. From there, we work out the tetrad of the non-inertial observer in terms of the transport matrix, as well as the inverse tetrad, from which we obtain his metric. We assume that the world-line $\mathscr{W}$ of our non-inertial observer is given by $x^{\kappa}_{\mathscr{W}}(\tau)$ in terms of the global inertial coordinates x^{κ} of some fiducial Minkowski observer, together with his tetrad $\tilde{e}_{\hat{\alpha}}{}^{\kappa}(\tau)$, which represents his non-inertial frame. We then define the non-inertial observer's *local* coordinates,

$$\tilde{x}^{\hat{\alpha}} = (c\tau, \tilde{x}^{\hat{\imath}})\,, \tag{4.86}$$

where the $\tilde{x}^{\hat{\imath}}$ are Cartesian spacial coordinates, implicitly as linear extensions of the tetrad's spacial "legs" in terms of the following coordinate transformation [63],

$$x^{\kappa}(\tilde{x}^{\hat{\alpha}}) := x^{\kappa}_{\mathscr{W}}(\tau) + \tilde{x}^{\hat{\imath}}\tilde{e}_{\hat{\imath}}{}^{\kappa}(\tau)\,. \tag{4.87}$$

with inverse

$$\tilde{x}^{\hat{\imath}}(x^{\kappa}) = \big[x^{\kappa} - x^{\kappa}_{\mathscr{W}}(\tau)\big]\tilde{e}^{\hat{\imath}}{}_{\kappa}(\tau)\,. \tag{4.88a}$$

From the above coordinate transformation (4.87), we then obtain an expression for the corresponding Jacobian matrix in terms of the non-inertial observer's tetrad $\tilde{e}_{\hat{\alpha}}{}^{\kappa}(\tau)$. Only the time-like "leg" of the tetrad turns out to be non-trivial, for step by step, we obtain

$$\begin{aligned}\frac{\partial x^{\kappa}(\tilde{x}^{\hat{\alpha}})}{\partial \tilde{x}^{\hat{0}}} &= \frac{\partial x^{\kappa}_{\mathscr{W}}(\tau)}{\partial \tilde{x}^{\hat{0}}} + \tilde{x}^{\hat{\imath}}\frac{\partial \tilde{e}_{\hat{\imath}}{}^{\kappa}(\tau)}{\partial \tilde{x}^{\hat{0}}} \\ &= \tilde{e}_{\hat{0}}{}^{\kappa}(\tau) - \tfrac{1}{c}\Omega^{\hat{\delta}}{}_{\hat{\imath}}\,\tilde{x}^{\hat{\imath}}\tilde{e}_{\hat{\delta}}{}^{\kappa}(\tau) \\ &= \Big(1 - \tfrac{1}{c}\Omega^{\hat{0}}{}_{\hat{\imath}}\,\tilde{x}^{\hat{\imath}}\Big)\tilde{e}_{\hat{0}}{}^{\kappa}(\tau) - \tfrac{1}{c}\Omega^{\hat{\jmath}}{}_{\hat{\imath}}\,\tilde{x}^{\hat{\imath}}\,\tilde{e}_{\hat{\jmath}}{}^{\kappa}(\tau)\,,\end{aligned} \tag{4.89}$$

where, in the second and third lines, we have used the transport equation (4.51), writing

$$\frac{\partial \tilde{e}_{\hat{\imath}}{}^{\kappa}(\tau)}{\partial \tilde{x}^{\hat{0}}} = \frac{1}{c}\frac{d\tilde{e}_{\hat{\imath}}{}^{\kappa}(\tau)}{d\tau} = -\frac{1}{c}\Big[\Omega^{\hat{0}}{}_{\hat{\imath}}(\tau)\,\tilde{e}_{\hat{0}}{}^{\kappa}(\tau) + \Omega^{\hat{\jmath}}{}_{\hat{\imath}}(\tau)\,\tilde{e}_{\hat{\jmath}}{}^{\kappa}(\tau)\Big]\,. \tag{4.90}$$

The spacial part of the Jacobian matrix of (4.87) then just yields the non-inertial observer's tetrad,

$$\frac{\partial x^{\kappa}(\tilde{x}^{\hat{\alpha}})}{\partial \tilde{x}^{\hat{a}}} = \tilde{e}_{\hat{a}}{}^{\kappa}(\tau)\,. \tag{4.91}$$

We can simplify the left-hand sides of (4.89) and (4.91) by taking the reference coordinates x^κ to be attached to an instantaneously comoving inertial observer, whose frame momentarily coincides with $\tilde{e}_{\hat{\alpha}}{}^{\kappa}(\tau)$. This means that the above Jacobian matrix becomes trivial, i.e.,

$$\frac{\partial x^\kappa}{\partial \tilde{x}^{\hat{0}}} = \delta_{\hat{0}}{}^{\kappa}\,, \qquad \frac{\partial x^\kappa}{\partial \tilde{x}^{\hat{a}}} = \tilde{e}_{\hat{a}}{}^{\kappa} = \delta_{\hat{a}}{}^{\kappa}\,. \tag{4.92}$$

We can now rearrange (4.89) and solve for the non-trivial tetrad vector $\tilde{e}_{\hat{0}}{}^{\kappa}(\tau)$, obtaining

$$\tilde{e}_{\hat{0}}{}^{\kappa}(\tau,\tilde{x}) = \frac{1}{\left(1-\frac{1}{c}\Omega^{\hat{0}}{}_{\hat{\imath}}(\tau)\,\tilde{x}^{\hat{\imath}}\right)}\left[\delta_{\hat{0}}{}^{\kappa} + \tfrac{1}{c}\Omega^{\hat{k}}{}_{\hat{\imath}}(\tau)\tilde{x}^{\hat{\imath}}\,\delta_{\hat{k}}{}^{\kappa}\right]. \tag{4.93}$$

In the discussion of Fermi normal coordinates in the presence of inertial terms in section 6.3, we will encounter the exact result (4.93) in an expanded form. Thus, expanding out the denominator yields

$$\tilde{e}_{\hat{0}}{}^{\kappa}(\tau,\tilde{x}) = \delta_{\hat{0}}{}^{\kappa} + \tfrac{1}{c}\Omega^{\kappa}{}_{\hat{\imath}}\tilde{x}^{\hat{\imath}} + \left(\tfrac{1}{c}\right)^2\Omega^{\kappa}{}_{\hat{\imath}_1}\Omega^{\hat{0}}{}_{\hat{\imath}_2}\tilde{x}^{\hat{\imath}_1}\tilde{x}^{\hat{\imath}_2} + \left(\tfrac{1}{c}\right)^3\Omega^{\kappa}{}_{\hat{\imath}_1}\Omega^{\hat{0}}{}_{\hat{\imath}_2}\Omega^{\hat{0}}{}_{\hat{\imath}_3}\tilde{x}^{\hat{\imath}_1}\tilde{x}^{\hat{\imath}_2}\tilde{x}^{\hat{\imath}_3} + \cdots\,. \tag{4.94}$$

In order to calculate the metric, we also need the inverse tetrad of our non-inertial observer,

$$\tilde{e}^{\hat{0}}{}_{\kappa}(\tau) = \left(1-\tfrac{1}{c}\Omega_{\hat{0}\hat{\imath}}\,\tilde{x}^{\hat{\imath}}\right)\delta^{\hat{0}}{}_{\kappa}\,, \tag{4.95a}$$
$$\tilde{e}^{\hat{a}}{}_{\kappa}(\tau) = \delta^{\hat{a}}{}_{\kappa} - \tfrac{1}{c}\Omega_{\hat{a}\hat{\imath}}\tilde{x}^{\hat{\imath}}\,\delta^{\hat{a}}{}_{\kappa}\,. \tag{4.95b}$$

Metric for Non-Inertial Observers

The metric in the local non-inertial frame of a non-inertial, i.e., accelerating and rotating observer in special relativity, then follows from the orthonormality relation of the inverse tetrads (4.42),

$$g_{00}(\tilde{x}^{\hat{\alpha}}) = 1 - \tfrac{2}{c}\Omega_{\hat{0}\hat{\imath}}\,\tilde{x}^{\hat{\imath}} + \tfrac{1}{c^2}\Omega^{\delta}{}_{\hat{\imath}_1}\Omega_{\delta\hat{\imath}_2}\,\tilde{x}^{\hat{\imath}_1}\tilde{x}^{\hat{\imath}_2}\,, \tag{4.96a}$$
$$g_{0k}(\tilde{x}^{\hat{\alpha}}) = \quad -\tfrac{1}{c}\Omega_{k\hat{\imath}}\,\tilde{x}^{\hat{\imath}}\,, \tag{4.96b}$$
$$g_{jk}(\tilde{x}^{\hat{\alpha}}) = \eta_{jk}\,. \tag{4.96c}$$

We can also display the metric in terms of the usual Cartesian three-vectors $\boldsymbol{a}$, $\boldsymbol{\omega}$, and the position vector in the local coordinates $\tilde{\boldsymbol{x}}$ as

$$g_{00}(\tau,\tilde{\boldsymbol{x}}) = \left(1+\frac{\boldsymbol{a}\cdot\tilde{\boldsymbol{x}}}{c^2}\right)^2 - \left(\frac{\boldsymbol{\omega}\times\tilde{\boldsymbol{x}}}{c}\right)^2, \tag{4.97a}$$
$$g_{0k}(\tau,\tilde{\boldsymbol{x}}) = -\tfrac{1}{c}\left(\boldsymbol{\omega}\times\tilde{\boldsymbol{x}}\right)_k\,, \tag{4.97b}$$
$$g_{jk}(\tau,\tilde{\boldsymbol{x}}) = \eta_{jk}\,. \tag{4.97c}$$

Range of Validity of Local Non-Inertial Coordinates

Having worked out the transformation to local non-inertial coordinates, the question about their range of validity and applicability naturally emerges, since, e.g., from looking at (4.93) it is evident that the denominator vanishes and the whole expression becomes indefinite as the coordinate value tends to c^2/a. To this end one can define "critical" length scales $c/\sqrt{|\mathcal{I}_1|}$ and $c/\sqrt{|\mathcal{I}_2|}$ from the invariants (4.70), or equivalently, proper acceleration and rotation length scales $\mathscr{R}_a$, and $\mathscr{R}_\omega$ (cf. [56]),

$$\mathscr{R}_a := \frac{c^2}{a}\,, \qquad \text{and} \qquad \mathscr{R}_\omega := \frac{c}{\omega}\,, \tag{4.98}$$

where $a = \|\boldsymbol{a}\|$ and $\omega = \|\boldsymbol{\omega}\|$. At length scales of $\mathscr{R}_a$, the coordinate lines will generally start to intersect and at $\mathscr{R}_\omega$ we have reached the "light-cylinder". In order for the local coordinates to be non-degenerate, we must therefore have that our maximal coordinate values, or equivalently, the extent of any system that we are describing in terms of these, to be very much smaller than these scales,

$$\max|\tilde{x}^{\hat{\imath}}| \ll \min\{\mathscr{R}_a, \mathscr{R}_\omega\}\,. \tag{4.99}$$

As a concrete example for these lengths, we can take for $\mathscr{R}_a$ and $\mathscr{R}_\omega$ typical values of a, ω that an observer sitting on the surface of the rotating Earth would experience. We find,

$$\mathscr{R}_{a,\female} = \frac{c^2}{g_{\female}} = 9.16\times 10^{12}\,\mathrm{km} \approx 1\,\mathrm{ly}\,, \qquad \mathscr{R}_{\omega,\female} = \frac{c}{\Omega_{\female}} \approx 4.1\times 10^{9}\,\mathrm{km} \approx 27.5\,\mathrm{au}\,,$$

where $g_{\female} \approx 9.81\,\mathrm{m/s^2}$ is the usual (mean) gravitational acceleration at the surface of the Earth and $\Omega_{\female}$ is the rotation period with respect to to the so-called *stellar day*, i.e., the Earth's rotation period relative to the fixed stars. This means that such an observer's local coordinates would be valid well into the outer solar system (to roughly the orbit of Uranus, which has its Aphelion at about 20 au), with the rotation length $\mathscr{R}_\omega$ being – by far – the more restrictive in this case.

4.3.3. General Solution of Transport Equation for Time-Independent Inertial Forces

As mentioned at the end of subsection 4.3.1, the formal time-ordered-exponential solution (4.85) of the parallel propagator, and thus also of the tetrad, reduces to a conventional matrix exponential for time-independent inertial forces. In fact, Sartor [64] has shown how to evaluate the matrix-exponential representation (4.27) of the general (orthochronous) Lorentz transformation in closed form using the Cayley-Hamilton method [65], and since time-independent inertial forces are just the limit of an continuous family of Lorentz boosts and rotations, his calculation can be directly used also for our present purpose. This elegant approach uses the eigenvalues of the transport matrix and the Lorentz invariants (4.70), the general result thus enabling one to give a Lorentz-invariant classification of all possible time-independent inertial forces (essentially equivalent to the discussion of uniform acceleration and rotation by Friedman and Scarr in a recent series of papers [66–68]; see also [69, Sec. 7.9].†) We start with a quick recapitulation of the Cayley-Hamilton theorem and then follow Sartor* in deriving a closed-form solution of the matrix exponential (4.85)

†Note that Friedman and Scarr define the transport matrix (which they call A), and also its first invariant, $\mathcal{I}_1$, with the opposite sign from our convention.

*Sartor uses a different sign for the spacial sub-matrix [as evident from his equation (9)] of his Lorentz matrix L (corresponding to our Ω), so his spacial sub-matrix $\mathcal{R}_\omega$ is to be identified with the negative Hodge dual of the rotation vector, $-(*\boldsymbol{\omega})$, see below.

for this special case, which yields the accelerating and rotating tetrad and its attached local coordinates for time-independent inertial forces.

Cayley-Hamilton Theorem

The Cayley-Hamilton theorem states that every square matrix over the fields $\mathbb{R}$ or $\mathbb{C}$ satisfies its own characteristic equation. Let A be an $n \times n$ matrix with characteristic equation $\chi(\lambda) = 0$ given by

$$\chi(\lambda) = \det(\lambda \mathsf{1} - \mathsf{A}) = 0\,, \tag{4.100}$$

where $\mathsf{1}$ is the $n \times n$ unit matrix. Expanding the determinant yields the characteristic equation

$$\chi(\lambda) = \sum_{i=0}^{n} \chi_i \lambda^i = \lambda^n - \operatorname{tr}(\mathsf{A})\lambda^{n-1} + \cdots + (-1)^n \det(\mathsf{A})\,. \tag{4.101}$$

The Caley-Hamilton theorem now states that also

$$\chi(\mathsf{A}) = \sum_{i=0}^{n} \chi_i \mathsf{A}^i = \mathsf{A}^n - \operatorname{tr}(\mathsf{A})\mathsf{A}^{n-1} + \cdots + (-1)^n \det(\mathsf{A})\mathsf{1} \tag{4.102a}$$

$$= (\lambda_1 \mathsf{1} - \mathsf{A})^{m_1} (\lambda_2 \mathsf{1} - \mathsf{A})^{m_2} \cdots (\lambda_k \mathsf{1} - \mathsf{A})^{m_k} = 0\,, \tag{4.102b}$$

where $\lambda_1, \ldots, \lambda_k$ are the k distinct eigenvalues with their corresponding multiplicities $m_1, \ldots, m_k$. Since it provides a relation between the n^{th} power of an $n \times n$ matrix and its $n-1$ lower powers as well as the unit matrix, the Cayley-Hamilton theorem allows one to compute any matrix-valued function $f(\mathsf{A})$ – and thus in particular the matrix exponential – from the knowledge of these $n-1$ lower powers of A alone.

Reduction of the Matrix Exponential of $\boldsymbol{\Omega}$

In order to keep the notation during the following derivation as simple and concise as possible, we shall temporarily hide all factors of c and also pull the factor of proper time into the matrix Ω, as well as into the parameters $\boldsymbol{a}$ and $\boldsymbol{\omega}$, so that they become dimensionless. Since Ω is antisymmetric, $\operatorname{tr}\Omega = 0$, so the third-order term in the characteristic equation for Ω vanishes. In fact, there is also no linear term, so we are left with the bi-quadratic

$$\chi(\lambda) = \lambda^4 + \mathcal{I}_1 \lambda^2 - (\mathcal{I}_2)^2 = \lambda^4 + (\boldsymbol{\omega}^2 - \boldsymbol{a}^2)\lambda^2 - (\boldsymbol{a} \cdot \boldsymbol{\omega})^2 = 0\,, \tag{4.103}$$

with the two Lorentz invariants (4.70) of the transport matrix as coefficients. The characteristic equation (4.103) has negative discriminant and real coefficients, so it features two real roots and a non-real complex-conjugate pair. It is convenient to write these as $\pm\lambda_1$ and $\pm i\lambda_2$ in terms of their real and imaginary parts, $\lambda_1, \lambda_2 > 0$, which can then simply be expressed in terms of the invariants of Ω as

$$\lambda_{1/2} = \frac{1}{\sqrt{2}} \left[\sqrt{\mathcal{I}_1^2 + 4\mathcal{I}_2^2} \mp \mathcal{I}_1 \right]^{1/2}, \tag{4.104}$$

i.e., we have pulled out the minus sign from the inner expression, so the inner square root is always taken to be positive. Consider now the matrix exponential of the matrix Ω, defined in terms of its series,

$$e^{\pm\Omega} := \sum_{n=0}^{\infty} \frac{(\pm 1)^n}{n!} \Omega^n = 1 \pm \Omega + \tfrac{1}{2}\Omega^2 \pm \tfrac{1}{6}\Omega^3 + \cdots . \tag{4.105}$$

From the Cayley-Hamilton theorem we have that $\chi(\Omega) = 0$, i.e. the characteristic matrix polynomial,

$$\chi(\Omega) = \Omega^4 + (\boldsymbol{\omega}^2 - \boldsymbol{a}^2)\Omega^2 - (\boldsymbol{a}\cdot\boldsymbol{\omega})^2 1 = 0 , \tag{4.106}$$

which implies that the general matrix exponential (4.105) can be written in terms of the unit matrix 1 and the first three powers of Ω, with suitable coefficients c_0, c_1, c_2, c_3 to be determined below. It thus reduces to

$$e^{\pm\Omega} = c_0 1 \pm c_1 \Omega + c_2 \Omega^2 \pm c_3 \Omega^3 . \tag{4.107}$$

We first calculate the required powers of Ω. With (4.69) we obtain,

$$\Omega = \left(\begin{array}{c|c} 0 & -\boldsymbol{a}^\intercal \\ \hline -\boldsymbol{a} & *\boldsymbol{\omega} \end{array}\right) , \tag{4.108a}$$

$$\Omega^2 = \left(\begin{array}{c|c} a^2 & (\boldsymbol{\omega}\times\boldsymbol{a})^\intercal \\ \hline -\boldsymbol{\omega}\times\boldsymbol{a} & \boldsymbol{a}\boldsymbol{a}^\intercal + \boldsymbol{\omega}\boldsymbol{\omega}^\intercal - \omega^2 1_3 \end{array}\right) , \tag{4.108b}$$

with Ω^3 given in terms of Ω and its four-dimensional Hodge dual $*\Omega$, i.e.,

$$\Omega^3 = -(\omega^2 - a^2) \left(\begin{array}{c|c} 0 & -\boldsymbol{a}^\intercal \\ \hline -\boldsymbol{a} & *\boldsymbol{\omega} \end{array}\right) - (\boldsymbol{a}\cdot\boldsymbol{\omega}) \left(\begin{array}{c|c} 0 & \boldsymbol{\omega}^\intercal \\ \hline \boldsymbol{\omega} & *\boldsymbol{a} \end{array}\right) , \tag{4.108c}$$

and we have used that $*\boldsymbol{\omega}$ acts as the "cross-product operator" when applied to vectors, i.e. $*\boldsymbol{\omega}\boldsymbol{a} = \boldsymbol{\omega}\times\boldsymbol{a}$, as well as the relation $(*\boldsymbol{\omega})(*\boldsymbol{\omega}) = \boldsymbol{\omega}\boldsymbol{\omega}^\intercal - \omega^2 1_3$, with ω being the norm of $\boldsymbol{\omega}$.

In order to determine the coefficients in (4.107), we use the eigenvalue equation of Ω. If λ is an eigenvalue of Ω, we act on some eigenvector of Ω with the matrix exponential (4.107), which yields the corresponding equation for the eigenvalues,

$$e^{\lambda} = c_0 + c_1\lambda + c_2\lambda^2 + c_3\lambda^3 , \tag{4.109}$$

which specialises to the following relations for the real and the imaginary exponential,

$$e^{\pm\lambda_1} = c_0 \pm c_1\lambda_1 + c_2\lambda_1^2 \pm c_3\lambda_1^3 , \qquad e^{\pm i\lambda_2} = c_0 \pm i c_1\lambda_2 - c_2\lambda_2^2 \mp i c_3\lambda_2^3 . \tag{4.110}$$

Adding and subtracting both equations for different signs and the same eigenvalue, respectively, yields at first

$$\begin{aligned} \cosh\lambda_1 &= c_0 + c_2\lambda_1^2 , & \cos\lambda_2 &= c_0 - c_2\lambda_2^2 , \\ \sinh\lambda_1 &= c_1\lambda_1 + c_3\lambda_1^3 , & \sin\lambda_2 &= c_1\lambda_2 - c_3\lambda_2^3 , \end{aligned} \tag{4.111}$$

Type	Invariants	Eigenvalues	Physical example
null	$\mathcal{I}_1 = 0 = \mathcal{I}_2$	$\lambda_1 = 0 = \lambda_2$	—
(mostly) linear	$\mathcal{I}_1 < 0,\ \mathcal{I}_2 = 0$	$\lambda_1 = \sqrt{-\mathcal{I}_1} > 0$ $\lambda_2 = 0$	accelerating observer with rotating tetrad ("generalised Rindler observer")
(mostly) rotational	$\mathcal{I}_1 > 0,\ \mathcal{I}_2 = 0$	$\lambda_1 = 0$ $\lambda_2 = \sqrt{\mathcal{I}_1} > 0$	accelerated observer in circular orbit; equatorial observer on rotating earth
general	$\mathcal{I}_1 \neq 0,\ \mathcal{I}_2 \neq 0$	$\lambda_1 > 0$ $\lambda_2 > 0$	non-equatorial observer on rotating earth

Table 4.1.: Classification of inertial fields for uniform acceleration and rotation in terms of the Lorentz invariants $\mathcal{I}_1, \mathcal{I}_2$ and eigenvalues λ_1, λ_2 of the transport matrix $\Omega = (\Omega^\kappa{}_\alpha)$; cf. [66].

from which we obtain expressions for the coefficients c_i in terms of trigonometric and hyperbolic functions of the (real parts of) the eigenvalues,

$$\begin{aligned} c_0 &= \frac{\lambda_2^2 \cosh\lambda_1 + \lambda_1^2 \cos\lambda_2}{\lambda_1^2 + \lambda_2^2}\,, & c_1 &= \frac{\lambda_2^3 \sinh\lambda_1 + \lambda_1^3 \sin\lambda_2}{\lambda_1\lambda_2(\lambda_1^2 + \lambda_2^2)}\,, \\ c_2 &= \frac{\cosh\lambda_1 - \cos\lambda_2}{\lambda_1^2 + \lambda_2^2}\,, & c_3 &= \frac{\lambda_2 \sinh\lambda_1 - \lambda_1 \sin\lambda_2}{\lambda_1\lambda_2(\lambda_1^2 + \lambda_2^2)}\,. \end{aligned} \tag{4.112}$$

This leads to the following closed-form expression for the matrix exponential of Ω in (4.107),

$$\begin{aligned} e^{\pm\Omega} = \frac{\lambda_2^2 \cosh\lambda_1 + \lambda_1^2 \cos\lambda_2}{\lambda_1^2 + \lambda_2^2}\,1 &\pm \frac{\lambda_2^3 \sinh\lambda_1 + \lambda_1^3 \sin\lambda_2}{\lambda_1\lambda_2(\lambda_1^2 + \lambda_2^2)} \left(\begin{array}{c|c} 0 & -\boldsymbol{a}^\intercal \\ \hline -\boldsymbol{a} & *\boldsymbol{\omega} \end{array}\right) \\ &+ \frac{\cosh\lambda_1 - \cos\lambda_2}{\lambda_1^2 + \lambda_2^2} \left(\begin{array}{c|c} a^2 & -(\boldsymbol{\omega}\times\boldsymbol{a})^\intercal \\ \hline \boldsymbol{\omega}\times\boldsymbol{a} & \boldsymbol{a}\boldsymbol{a}^\intercal + \boldsymbol{\omega}\boldsymbol{\omega}^\intercal - \omega^2 1_3 \end{array}\right) \\ &\mp \frac{\lambda_2 \sinh\lambda_1 - \lambda_1 \sin\lambda_2}{\lambda_1\lambda_2(\lambda_1^2 + \lambda_2^2)} (\omega^2 - a^2)\Omega \\ &\mp \frac{\lambda_2 \sinh\lambda_1 - \lambda_1 \sin\lambda_2}{\lambda_1\lambda_2(\lambda_1^2 + \lambda_2^2)} (\boldsymbol{a}\cdot\boldsymbol{\omega}) \left(\begin{array}{c|c} 0 & \boldsymbol{\omega}^\intercal \\ \hline \boldsymbol{\omega} & *\boldsymbol{a} \end{array}\right)\,, \end{aligned} \tag{4.113}$$

and thus to the exact solution of the initial value problem associated with the transport equation (4.48) in the form (4.51) for time-independent acceleration and rotation.*

Classification of Inertial Fields and their Physical Interpretation

From their definition in (4.70), we see that the invariant $\mathcal{I}_1$ measures the difference between the magnitudes of the acceleration and rotation vectors squared, while $\mathcal{I}_2$ measures the magnitude of their projection. Since they are associated with different values of these two Lorentz invariants, the solution (4.113) decomposes into different Lorentz-invariant classes, depending on whether one, or both of the invariants are positive, negative, or vanish altogether. Thus we obtain, in essence, the Lorentz invariant classification of inertial fields into different types, depending on the relative magnitude and projection of $\boldsymbol{a}$ and $\boldsymbol{\omega}$. These classes are illustrated in Table 4.1, together with the corresponding values of the invariants and eigenvalues, as well as a physical interpretation of the type of motion (note that we have suppressed the two subtypes of the "null" case; the full classification was carried out by Synge [73]).

In the case where both invariants vanish, i.e. $\mathcal{I}_1 = 0 = \mathcal{I}_2$, the vectors $\boldsymbol{\omega}$ and $\boldsymbol{a}$ are perpendicular and of equal magnitude. This case is called "null acceleration", since all eigenvalues vanish. [Note that in the analogue situation in electromagnetism, where the classification of electromagnetic fields is achieved in terms of a classification of the field-strength tensor (4.73), this case represents electromagnetic radiation.] If $\mathcal{I}_2 = 0$ and $\mathcal{I}_1 > 0$, we have (mostly) rotational acceleration, i.e. the total magnitude of the general "acceleration" is dominated by the rotational part and can – in a certain frame – be reduced to *purely* rotational acceleration. In contrast, if instead $\mathcal{I}_1 < 0$, the usual linear acceleration dominates over the rotational contribution, which is called (mostly) linear acceleration. This type can equally be reduced to *purely* linear acceleration in a special frame. The last case is the general one, where both invariants are non-zero and consequently the eigenvalues λ_1, λ_2 are distinct and non-vanishing. This means that $\boldsymbol{\omega}$ and $\boldsymbol{a}$ are not perpendicular and of different magnitude.

The three Lorentz invariant special cases are extracted by performing the corresponding limits of (4.113), which yields,

$$\lim_{\substack{\lambda_1\to 0\\ \lambda_2\to 0}} e^{\pm\Omega} = 1 \pm \Omega + \tfrac{1}{2!}\Omega^2\,, \tag{4.114a}$$

$$\lim_{\lambda_2\to 0} e^{\pm\Omega} = 1 \pm \Omega + \frac{\cosh(\lambda_1)-1}{\lambda_1^2}\,\Omega^2 \pm \frac{\sinh(\lambda_1)-\lambda_1}{\lambda_1^3}\,\Omega^3\,, \qquad \lambda_1 = \sqrt{\left(\frac{a}{c}\right)^2 - \omega^2}\tau\,, \tag{4.114b}$$

$$\lim_{\lambda_1\to 0} e^{\pm\Omega} = 1 \pm \Omega + \frac{1-\cos(\lambda_2)}{\lambda_2^2}\,\Omega^2 \pm \frac{\lambda_2-\sin(\lambda_2)}{\lambda_2^3}\,\Omega^3\,, \qquad \lambda_2 = \sqrt{\omega^2 - \left(\frac{a}{c}\right)^2}\tau\,. \tag{4.114c}$$

As a check, we shall first specialise the last two to the expressions for a pure boost and a pure rotation, respectively, by additionally taking $\omega \to 0$ in (4.114b) so that $\lambda_1 = \frac{a}{c}\tau$, and $a \to 0$ in (4.114c), which makes $\lambda_2 = \omega\tau$. Firstly for the pure rotation case, we find that

*An equivalent formalism uses the elegant Frenet-Serret approach to the geometric properties of frames along curves, applied to tetrad transport in space-time, in which the role of $\Omega^{\kappa}{}_{\alpha}$ is played by the Frenet-Serret matrix, and that of a and ω by the world-line's Frenet-Serret curvatures and torsions. This is due to Iyer and Vishveshwara [70–72], who also use the Cayley-Hamilton method to solve the Frenet-Serret equations.

(4.114c) reduces to,

$$\lim_{\substack{\lambda_1\to 0\\ a\to 0}} e^{\pm\Omega\tau} = \left(\begin{array}{c|c} 1 & \mathbf{0}^\intercal \\ \hline \mathbf{0} & \cos(\omega\tau)1_3 + (1-\cos\omega\tau)\boldsymbol{n}_\omega\boldsymbol{n}_\omega^\intercal \pm \sin(\omega\tau) * \boldsymbol{n}_\omega \end{array}\right), \tag{4.115}$$

which is the axis-angle representation of a rotation, with $\boldsymbol{n}_\omega$ as the Euler axis. Secondly, the purely linear case (4.114b) is seen to reduce to a time-dependent Lorentz boost, i.e. to (4.33) with $\zeta = \lambda_1 = \frac{a}{c}\tau$. We note that we have first reinstalled the proper dimensionful units in all quantities of (4.114), pulling out the respective factors of c and proper time τ, which is achieved by the following substitutions:

$$\Omega \to \Omega\tau\,, \qquad a \to \tfrac{1}{c}a\tau\,, \qquad \omega \to \omega\tau\,, \qquad \mathcal{I}_{1/2} \to \mathcal{I}_{1/2}\,\tau^2\,, \qquad \lambda_{1/2} \to \lambda_{1/2}\,\tau\,. \tag{4.116}$$

In carrying out the simplifications for the mostly linear and mostly rotational cases (4.114b) and (4.114c) with full dependence on a and ω, one finds that it makes sense to pull out from the square root in λ_2 a factor of $\frac{a}{c}$, and correspondingly from that in λ_1 a factor of ω, i.e.

$$\lambda_1 = \frac{a}{c}\tau\sqrt{1-\left(\frac{\omega c}{a}\right)^2} = \frac{a\tau}{\gamma_a c}\,, \qquad \lambda_2 = \omega\tau\sqrt{1-\left(\frac{a}{c\omega}\right)^2} = \frac{\omega\tau}{\gamma_\omega}\,, \tag{4.117}$$

and thus to rewrite everything in terms of dimensionless quantities, thereby defining "accelerational" velocity factors β_a, β_ω and their associated gamma factors, by

$$\beta_\omega \coloneqq \frac{a}{c\omega}\,, \qquad \beta_a \coloneqq \frac{c\omega}{a}\,; \qquad\qquad \gamma_a \coloneqq \frac{1}{\sqrt{1-\beta_a^2}}\,, \qquad \gamma_\omega \coloneqq \frac{1}{\sqrt{1-\beta_\omega^2}}\,. \tag{4.118}$$

As before, it also proves convenient to decompose the vectors $\boldsymbol{a}$ and $\boldsymbol{\omega}$ into their magnitudes a and ω and directions given by Cartesian unit vectors $\boldsymbol{n}_a$ and $\boldsymbol{n}_\omega$, respectively, i.e.,

$$\boldsymbol{a} = a\,\boldsymbol{n}_a\,, \qquad\qquad \boldsymbol{\omega} = \omega\,\boldsymbol{n}_\omega\,. \tag{4.119}$$

Because of its direct physical significance for the description of frames along circular world-lines, in the following subsection 4.3.4, we shall exclusively focus on the rotation-dominated case.

4.3.4. Rotation-Dominated Case: Observers on Circular World-Lines in Minkowski Space

Now, let $e_{\hat\alpha}{}^\kappa(\tau)$ be an exact solution of the transport equation in terms of the matrix exponential (4.113), i.e.,

$$e_{\hat\alpha}{}^\kappa(\tau) = \left(e^{\pm\Omega\tau}\right)^\kappa{}_\delta\, e_{\hat\alpha}{}^\delta(0)\,. \tag{4.120}$$

We focus on the case of mostly rotational acceleration and take the initial tetrad there to be trivial, $e_{\hat\alpha}{}^\kappa(0) = \delta_{\hat\alpha}{}^\kappa$. With $\mathcal{I}_2 = \boldsymbol{\omega}\cdot\boldsymbol{a} = 0$, we have that $\Omega \propto \Omega^3$, so these terms combine, and equation (4.114c) simplifies to,

$$\begin{aligned}\lim_{\lambda_1\to 0} e^{\pm\Omega} = 1 \pm \gamma_\omega \sin\Big(\frac{\omega\tau}{\gamma_\omega}\Big) \left(\begin{array}{c|c} 0 & -\beta_\omega \boldsymbol{n}_a^\intercal \\ \hline -\beta_\omega \boldsymbol{n}_a & *\boldsymbol{n}_\omega \end{array}\right) \\ + \gamma_\omega^2 \Big[1 - \cos\Big(\frac{\omega\tau}{\gamma_\omega}\Big)\Big] \left(\begin{array}{c|c} \beta_\omega^2 & -\beta_\omega(\boldsymbol{n}_\omega \times \boldsymbol{n}_a)^\intercal \\ \hline \beta_\omega(\boldsymbol{n}_\omega \times \boldsymbol{n}_a) & \beta_\omega^2 \boldsymbol{n}_a \boldsymbol{n}_a^\intercal + \boldsymbol{n}_\omega \boldsymbol{n}_\omega^\intercal - 1_3 \end{array}\right).\end{aligned} \tag{4.121}$$

This still looks complicated and somewhat hard to interpret physically, in part because the orientations $\boldsymbol{n}_a$ and $\boldsymbol{n}_\omega$ of $\boldsymbol{a}$ and $\boldsymbol{\omega}$, respectively, are completely general. However, it turns out that the present class of time-independent (mostly) rotational acceleration corresponds physically to the important case of a circular orbit, i.e. the tetrad (4.120) represents the local frame of an observer who is orbiting around the coordinate origin with constant angular velocity Ω, which is related to his proper rotation ω. In order to show this, we will first specialise the general tetrad above by taking the non-inertial observer's proper acceleration to point in the positive X-direction by setting $\boldsymbol{n}_a = (1,0,0)^\intercal$, and at the same time taking his proper rotation vector $\boldsymbol{\omega}$ to point along the Z-axis with $\boldsymbol{n}_\omega = (0,0,1)^\intercal$, as usual, and which yields $*\boldsymbol{n}_\omega = \left(\begin{smallmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ for the Hodge dual of $\boldsymbol{n}_\omega$ that appears in (4.121). This reduces equation (4.121) to the more manageable form

$$e_{\hat\alpha}{}^{\kappa}(\tau) = \left(\begin{array}{c|ccc} \gamma_\omega^2\big[1-\beta_\omega^2\cos(\frac{\omega\tau}{\gamma_\omega})\big] & \mp\beta_\omega\gamma_\omega\sin(\frac{\omega\tau}{\gamma_\omega}) & -\beta_\omega\gamma_\omega^2\big[1-\cos(\frac{\omega\tau}{\gamma_\omega})\big] & 0 \\ \hline \mp\beta_\omega\gamma_\omega\sin(\frac{\omega\tau}{\gamma_\omega}) & \cos(\frac{\omega\tau}{\gamma_\omega}) & \pm\gamma_\omega\sin(\frac{\omega\tau}{\gamma_\omega}) & 0 \\ -\beta_\omega\gamma_\omega^2\big[1-\cos(\frac{\omega\tau}{\gamma_\omega})\big] & \mp\gamma_\omega\sin(\frac{\omega\tau}{\gamma_\omega}) & \gamma_\omega^2\big[\cos(\frac{\omega\tau}{\gamma_\omega})-\beta_\omega^2\big] & 0 \\ 0 & 0 & 0 & 1 \end{array}\right). \tag{4.122}$$

The appearance of β_ω in (4.122) is an indication that this frame is moving with respect to the inertial coordinates of a fiducial Minkowski observer, so the two frames must be related by a Lorentz boost. Since we are restricted to the (mostly) rotational Lorentz-invariant class by the conditions $\mathcal{I}_2 = \frac{1}{c}\boldsymbol{\omega}\cdot\boldsymbol{a} = 0$ and $\mathcal{I}_1 = \boldsymbol{\omega}^2 - \boldsymbol{a}^2/c^2 > 0$, we must certainly be able to boost ourselves to a Lorentz frame where the acceleration vanishes, $\boldsymbol{a} = \boldsymbol{0}$, thereby specialising our system to *purely* rotational acceleration by this choice of frame. This is accomplished by transforming $\Omega^\kappa{}_\alpha$ (being a rank-two Lorentz tensor) by an Lorentz boost with parameter β_ω,

$$\tilde\Omega^{\kappa'}{}_{\alpha'} = \Lambda^{\kappa'}{}_\delta \Omega^\delta{}_\beta \Lambda_{\alpha'}{}^\beta = \Lambda\,\Omega\,\Lambda^{-1}, \tag{4.123}$$

and so the full transport equation transforms as,

$$\tilde e_{\hat\alpha'}{}^{\kappa}(\tau) = \big(\Lambda\, e^{\pm\Omega\tau}\,\Lambda^{-1}\big)^{\hat\delta}{}_{\hat\alpha}\tilde e_{\hat\delta}{}^{\kappa}(0) = e^{\pm\Lambda\Omega\Lambda^{-1}\tau}\tilde e_{\hat\alpha}{}^{\kappa}(0), \qquad \tilde e_{\hat\alpha'}{}^{\kappa}(\tau) = (\Lambda)_{\hat\alpha'}{}^{\hat\delta} e_{\hat\delta}{}^{\kappa}(\tau). \tag{4.124}$$

Indeed, we have that equation (4.122) becomes a pure rotation (cf. Friedman and Scarr [74]),

$$\tilde e_{\hat\alpha'}{}^{\kappa'}(\tau) = \Lambda^\alpha{}_{\alpha'} e_{\hat\alpha}{}^{\kappa}(\tau)\Lambda_\kappa{}^{\kappa'} = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & \cos(\frac{\omega\tau}{\gamma_\omega}) & \pm\sin(\frac{\omega\tau}{\gamma_\omega}) & 0 \\ 0 & \mp\sin(\frac{\omega\tau}{\gamma_\omega}) & \cos(\frac{\omega\tau}{\gamma_\omega}) & 0 \\ 0 & 0 & 0 & 1 \end{array}\right). \tag{4.125}$$

Integrating the time-like vector $e_{\hat{0}}{}^{\kappa}(\tau)$ of (4.122), and noting that we had implicitly chosen the initial conditions of the motion in the local coordinates to be $\tau_0 = 0$, $x_0 = 1$, and $y_0 = 0 = z_0$, yields the corresponding world-line $\mathscr{W}$ in local coordinates $x^\kappa = (c\tau, x, y, z)$,

$$\begin{aligned} x^0_{\mathscr{W}}(\tau) &= \gamma_\omega \big[X^0 - \beta_\omega R_0 \sin\Big(\frac{\omega\tau}{\gamma_\omega}\Big)\big]\,, \qquad & x_{\mathscr{W}}(\tau) &= \pm R_0 \big[1 - \cos\Big(\frac{\omega\tau}{\gamma_\omega}\Big)\big]\,, \\ y_{\mathscr{W}}(\tau) &= -\gamma_\omega \big[\beta_\omega X^0 - R_0 \sin\Big(\frac{\omega\tau}{\gamma_\omega}\Big)\big]\,, \qquad & z_{\mathscr{W}}(\tau) &= Z_0\,. \end{aligned} \tag{4.126}$$

Here, we have used $X^0 = cT = \gamma_\omega c\tau$, being the time coordinate of the fiducial inertial Minkowski observer, and have interpreted the only combination of ω and a with dimension of length as the orbital radius, $R_0 \coloneqq \frac{a}{\omega^2}\gamma_\omega^2 = \frac{c}{\omega}\beta_\omega\gamma_\omega^2$. Equations (4.126) can be seen to describe a cycloid in the x–t plane, which is "distorted" by the Minkowski background coordinates $X^\kappa = \big(X^0, X, Y, Z\big)$ being Lorentz boosted in the Y-direction.

4.3.5. Circular World-Lines in Terms of Stationary Coordinates

Above we calculated the local frame of an (constantly) accelerating and rotating observer by direct solution of the transport equation, which was implicitly expressed in terms of his own local coordinates. We then specialised to the rotation-dominated case, which was seen to describe the physical situation of such an observer in circular orbital motion around the origin of the background inertial Minkowski coordinates, and which could be further specialised to the purely rotating coordinates of the Minkowski observer at the centre of inertial coordinates by a Lorentz boost. Since the above Cartesian coordinates are obviously not well adapted to the present cylindrically symmetric world-lines, we shall reformulate the problem in terms of stationary, i.e. uniformly rotating spherical Minkowski coordinates.

Rotating Metric Transformation

We start by writing the Minkowski metric (3.62) in spherical coordinates, $X^\kappa = \big(cT, R, \mathit{\Theta}, \mathit{\Phi}\big)$,

$$d\lambda^2 = (c\,dT)^2 - dR^2 - R^2\big[d\mathit{\Theta}^2 + \sin^2(\mathit{\Theta}) d\mathit{\Phi}^2\,\big]\,. \tag{4.127}$$

Now introducing uniformly rotating spherical coordinates, $X'^\kappa = \big(cT, R, \mathit{\Theta}, \mathit{\Phi}'\big)$, by setting

$$\mathit{\Phi}'(T) \coloneqq \mathit{\Phi} + \Omega T\,, \qquad \Omega \coloneqq \frac{d\mathit{\Phi}'}{dT} = \text{const.}, \tag{4.128}$$

which yields the differentials $d\mathit{\Phi}' = d\mathit{\Phi} + \Omega\, dT$ and $(d\mathit{\Phi}')^2 = d\mathit{\Phi}^2 + 2(\frac{\Omega}{c})d\mathit{\Phi}(c\,dT) + (\frac{\Omega}{c})^2(c\,dT)^2$, and thus, in terms of $d\mathit{\Phi}'$, the relation

$$d\mathit{\Phi}^2 = d\mathit{\Phi}'^2 - 2\Big(\frac{\Omega}{c}\Big)d\mathit{\Phi}'\,(c\,dT) + \Big(\frac{\Omega}{c}\Big)^2(c\,dT)^2\,. \tag{4.129}$$

On using this to replace $d\Phi^2$ in (4.127) above, we generate a correction term to the metric's time–time component, as well as a $d\Phi\, dT$ cross term, so that the metric becomes,

$$\begin{aligned} d\lambda^2 = \Big[1 - \Big(\frac{\Omega}{c}\Big)^2 (R\sin\Theta)^2\Big](c\,dT)^2 - dR^2 - R^2\big[d\Theta^2 + \sin^2(\Theta)d\Phi'^2\big] \\ + 2\Big(\frac{\Omega}{c}\Big)(R\sin\Theta)^2 d\Phi'\,(c\,dT)\,. \end{aligned} \tag{4.130}$$

This metric is adapted to observers which are at rest with respect to its rotating coordinates, i.e. to observers at fixed radial distance R and fixed Φ', rotating with constant angular velocity Ω around the coordinate origin, as seen by the fiducial inertial Minkowski observer with metric (4.127) there. In these coordinates, their world-lines, $X^{\kappa}_{\mathscr{W}}(\tau) = (cT_{\mathscr{W}}, R_{\mathscr{W}}, \Theta_{\mathscr{W}}, \Phi_{\mathscr{W}})$, with

$$T_{\mathscr{W}}(\tau) = \Gamma\tau\,, \qquad R_{\mathscr{W}}(\tau) = R_0\,, \qquad \Theta_{\mathscr{W}}(\tau) = \Theta_0\,, \qquad \Phi_{\mathscr{W}}(\tau) = \Phi_0 + \Gamma\Omega\tau\,, \tag{4.131}$$

are thus circular helices in space-time, the spacial projection of which is a circle. In this parametrisation in terms of inertial Minkowski coordinates, the orbiting observer has four-velocity,

$$\tilde{u}^{\kappa} = \Gamma(c, 0, 0, \Omega)\,, \tag{4.132}$$

where we have introduced the *coordinate gamma factor* or *coordinate red-shift factor*,

$$\Gamma := \frac{dT}{d\tau} = \frac{1}{\sqrt{g_{TT}}} = \Big[1 - \Big(\frac{\Omega}{c}\Big)^2(R\sin\Theta)^2\Big]^{-1/2}\,, \tag{4.133}$$

in terms of the time–time component of the above rotating metric (4.130). Note that, since we are in Minkowski space and they describe the same Lorentz-boost, we must have that $\Gamma = \gamma_{\omega}$.

Local Frame in Terms of Stationary Coordinates

While the tetrad that is adapted to the Minkowski metric in spherical coordinates can be read off from (4.127) as,

$$\tilde{e}_{\hat{\alpha}}{}^{\kappa} = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & & 0 \\ 0 & & \frac{1}{R} & 0 \\ 0 & 0 & 0 & \frac{1}{R\sin\Theta} \end{array}\right), \qquad \tilde{e}^{\hat{\alpha}}{}_{\kappa} = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & R\sin\Theta \end{array}\right), \tag{4.134}$$

the tetrad adapted to the metric rotating spherical coordinates (4.130), and thus also to any observer who is at rest with respect to these, reads

$$\tilde{e}_{\hat{\alpha}}{}^{\kappa} = \begin{pmatrix} \Gamma & 0 & 0 & \Gamma\big(\frac{\Omega}{c}\big) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{R} & 0 \\ \Gamma\frac{\Omega}{c}R\sin\Theta & 0 & 0 & \frac{\Gamma}{R\sin\Theta} \end{pmatrix}, \qquad \tilde{e}^{\hat{\alpha}}{}_{\kappa} = \begin{pmatrix} \frac{\Gamma}{R\sin\Theta} & 0 & 0 & -\Gamma\frac{\Omega}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & R & 0 \\ -\Gamma\frac{\Omega}{c}R\sin\Theta & 0 & 0 & \Gamma \end{pmatrix}. \tag{4.135}$$

The four-acceleration of the orbiting observer is calculated using (4.37), which happens to receive contributions only from the Christoffel symbols, since the four-velocity (4.132) is constant, yielding for the two non-vanishing components,

$$\begin{aligned} a^1_{\text{circ}} &= \Gamma^1{}_{00}\,(u^0_{\text{circ}})^2 + \Gamma^1{}_{33}\,(u^3_{\text{circ}})^2 &&= -\mathit{\Gamma}^2\Omega^2 R\sin^2\mathit{\Theta}\,,\\ a^2_{\text{circ}} &= \Gamma^2{}_{33}\,(u^3_{\text{circ}})^2 &&= -\mathit{\Gamma}^2\Omega^2\sin(\mathit{\Theta})\cos(\mathit{\Theta})\,. \end{aligned} \tag{4.136}$$

In the coordinates of our fiducial inertial observer, the non-inertial frame moving on an (equatorial, for simplicity) circular orbit thus has a coordinate velocity $v_{\text{coord}} = \Omega R$ and a corresponding inward centripetal coordinate acceleration of $a_{\text{coord}} = -\Omega^2 R$, needed to keep it on his orbit. These are obviously just the well-known expressions from Newtonian physics, yet they are coordinate quantities and do not directly correspond to the proper acceleration and rotation that an observer in such an orbiting frame will measure locally.

From (4.136), we obtain the *proper* acceleration by taking frame components in the rotating frame (4.135),

$$a^{\hat{1}}_{\text{circ}} = a^1_{\text{circ}} \qquad\qquad a^{\hat{2}}_{\text{circ}} = -\mathit{\Gamma}^2\Omega^2 R\sin(\mathit{\Theta})\cos(\mathit{\Theta})\,. \tag{4.137}$$

Analogously, one can get an expression for the proper rotation $\omega^{\hat{\imath}}$ by explicitly evaluating the left-hand side of (4.48), contracted with an inverse tetrad, in terms of the rotating Minkowski coordinates, i.e.,

$$e^{\hat{k}}{}_{\kappa}\, e_{\hat{0}}{}^{\alpha}\nabla_\alpha e_{\hat{a}}{}^{\kappa} = \tfrac{1}{c}\Omega^{\hat{k}}{}_{\hat{a}} = -\tfrac{1}{c}\epsilon^{\hat{k}}{}_{\hat{a}\hat{\imath}}\,\omega^{\hat{\imath}}\,. \tag{4.138}$$

Fortunately, this is not necessary in the present case since we know that $\boldsymbol{\omega}$ points in Z, i.e. $\mathit{\Theta}$ direction, and its magnitude can be read off from the trigonometric functions in (4.125), upon noting the correspondence $\mathit{\Omega}T = \omega\tau/\gamma_\omega$, together with $T = \mathit{\Gamma}\tau$. We thus have,

$$\omega = \mathit{\Gamma}^2\Omega\,, \qquad \text{and} \quad \beta_\omega = \frac{a}{c\omega} = \Big(\frac{\Omega}{c}\Big) R\sin\mathit{\Theta}\,, \tag{4.139}$$

(see e.g. Letaw and Pfautsch [75, Sec. IV]).

4.4. Geodesic Motion as Observed by Non-Inertial Observers

Before we move on to the relevant aspects of general relativity in the next chapter, we introduce an important application of local coordinates, namely the general description of motion in our local frame. Thus, in the present subsection, we are interested in describing the geodesic motion of a second, *inertial*, observer or particle, as seen from the accelerating and rotating frame of our primary *non-inertial* observer, and described in terms of his local coordinates x^α.[†] In order to achieve this, it is necessary to use a non-affine parametrisation of the inertial observer's geodesic equation (3.55b), i.e.,

$$\frac{dv^\kappa}{d\lambda} + \Gamma^\kappa{}_{\alpha_1\alpha_2}\, v^{\alpha_1} v^{\alpha_2} = 0\,, \tag{4.140}$$

[†] see, e.g., the textbook by Misner, Thorne and Wheeler [48, p. 174], or Straumann [47, Sec. 2.10].

where $\upsilon^\kappa = \frac{dx^\kappa}{d\lambda}$ is the tangent to the inertial observer's world-line, λ being the affine parameter proportional to his proper time that he uses to parametrise his motion.

As an aside, we note that we can in principle use any parameter we like and are not restricted to affine parameters (3.51). However, the reason that one usually restricts the parametrisation in the equation of motion to affine parameters is that only for these does the possible inhomogeneity have the meaning of a physical acceleration as in (4.37). For non-affine parametrisation, there is an additional non-physical coordinate-acceleration term which comes from the chain rule of differentiation, as we shall see below.

We will thus formulate the problem of describing the second, inertial observer/particle's motion in terms of the local coordinates (as discussed in the above subsection 4.3.2) adapted to our first, non-inertial observer, using *coordinate time* x^0 as our non-affine parameter, in terms of which the tangent to the observer's world-line can be expressed as

$$\upsilon^\kappa = \left(\frac{dx^0}{d\lambda}\right)\frac{dx^\kappa}{dx^0} = \gamma \mathrm{v}^\kappa \,, \tag{4.141}$$

where v^κ is the *coordinate four-velocity* (for which we write a slanted Roman v), and which is written in terms of the *coordinate velocity*,

$$\mathrm{v}^k := \frac{dx^k}{dx^0} \,, \tag{4.142}$$

and the *gamma factor* (or *redshift factor*), which is defined to be the derivative of the time coordinate with respect to the affine parameter,

$$\gamma(x) := \frac{dx^0}{d\lambda} \,. \tag{4.143}$$

We note that the coordinate four-velocity then decomposes as

$$\mathrm{v}^\kappa = \delta^\kappa{}_0 + \delta^\kappa{}_k \mathrm{v}^k \,, \tag{4.144}$$

and (4.141) thus becomes a Lorentz–boost-type ansatz for the four-velocity υ^κ of the second, inertial, observer or particle,

$$\upsilon^\kappa(x) = \gamma(x)\left[\delta^\kappa{}_0 + \delta^\kappa{}_k \mathrm{v}^k(x)\right] . \tag{4.145}$$

An expression for γ is obtained from the general line element $d\lambda^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ by separating it into space and time components, and subsequent division by dx^0,

$$\gamma = \frac{1}{\sqrt{g_{\alpha\beta}\mathrm{v}^\alpha \mathrm{v}^\beta}} = \frac{1}{\sqrt{g_{00} + 2g_{0a}\mathrm{v}^a + g_{ab}\mathrm{v}^a \mathrm{v}^b}} \,. \tag{4.146}$$

So parametrised in terms of coordinate time, the geodesic equation (4.140) is then equivalent to the system

$$\frac{d\gamma}{dx^0} = -\gamma\Gamma^0{}_{\alpha_1\alpha_2}\mathrm{v}^{\alpha_1}\mathrm{v}^{\alpha_2} \,, \tag{4.147a}$$

$$\frac{d(\gamma \mathrm{v}^k)}{dx^0} = \gamma \frac{d\mathrm{v}^k}{dx^0} + \left(\frac{d\gamma}{dx^0}\right)\mathrm{v}^k = -\gamma \Gamma^k{}_{\alpha_1\alpha_2}\, \mathrm{v}^{\alpha_1}\mathrm{v}^{\alpha_2}\,, \tag{4.147b}$$

which can be combined into a single equation for the purely spacial coordinate velocity,

$$\boxed{\frac{d\mathrm{v}^k}{dx^0} = -\left(\Gamma^k{}_{\alpha_1\alpha_2} - \Gamma^0{}_{\alpha_1\alpha_2}\mathrm{v}^k\right)\mathrm{v}^{\alpha_1}\mathrm{v}^{\alpha_2}\,.} \tag{4.148}$$

Inserting now the non-vanishing connection coefficients which encode the inertial forces, i.e., (4.53) for accelerations and (4.58) for rotations, we arrive at

$$\begin{aligned}\frac{d\mathrm{v}^k}{dx^0} &= -\Gamma^k{}_{00} - 2\Gamma^k{}_{0a}\mathrm{v}^a + 2\Gamma^0{}_{0a}\mathrm{v}^a\mathrm{v}^k \\ &= -\frac{a^k}{c^2} - \frac{2}{c}\epsilon^k{}_{ia}\omega^i\mathrm{v}^a + \frac{2}{c^2}(a_i\mathrm{v}^i)\mathrm{v}^k\,,\end{aligned} \tag{4.149a}$$

which can also be written in terms of Cartesian vectors, so that it takes the familiar form

$$\boxed{\frac{d\mathbf{v}}{dx^0} = -\frac{\boldsymbol{a}}{c^2} + \frac{2}{c^2}(\boldsymbol{a}\cdot\mathbf{v})\mathbf{v} - \frac{2}{c}\boldsymbol{\omega}\times\mathbf{v}\,,} \tag{4.149b}$$

where we recall that v^k, i.e. $\mathbf{v}$, is a Newtonian velocity divided by c. Here, the second term on the right is a special relativistic red-shift correction coming from the first-order term in the expansion of the relativistic gamma factor (4.143) and the third term is the usual Coriolis acceleration. Clearly, the minus sign in front of the acceleration term $\boldsymbol{a}/c^2$ in (4.149) comes from the fact that the frame of the observer is accelerating, while the particle that is being observed is inertial, so the observer *perceives* it to be accelerating away from him in the opposite direction.

Note that above we have only expanded the equation of motion, i.e. the Christoffel symbols, to zeroth order. This means that no terms quadratic in $\boldsymbol{a}$ and $\boldsymbol{\omega}$ (centrifugal acceleration) and no time derivatives thereof (Euler acceleration), appear in equation (4.149). These quadratic terms are obtained upon inserting into (4.148) the first-order expansion of the Christoffel symbols in terms of the local coordinates, the corresponding first-order derivatives of which can be calculated by using the definition of the Riemann tensor (3.80), which is taken to vanish in the present special relativistic context, $R^\kappa{}_{\gamma\alpha\beta} = 0$.

The above method of calculating the equation of motion is quite general and can also be used in general relativity, where it is applied in the context of an extension of the concept of local inertial coordinates that we have discussed in the present chapter, namely Fermi normal coordinates. We shall carry out the above-mentioned higher-order expansion of the Christoffel symbols in section 6.4, including the curvature terms that result from space-time curvature. The corresponding higher-order expansion of the equation of motion in curved space-time then follows in subsection 6.4.1.

5 Elements of General Relativity

After motivating general relativity as a natural consequence of the geometric nature of gravity, introducing the equivalence principles, and the Einstein field equations in section 5.1, we turn to the general setup for their exact solution in terms of space-time symmetries that must be imposed in section 5.2. Thereby, we briefly review the topic of Killing vector fields that represent these symmetries, introduce maximally symmetric space-times, in particular the de Sitter metric, and mention the connection to the Friedmann-Lemaître-Robertson-Walker metric of the cosmological standard model. Subsequently, we motivate how normal forms for the metric of stationary and static space-times follow from their single time-like Killing vector.

Continuing along these lines, in section 5.3 we introduce the physically important class of stationary and axisymmetric space-times and the normal form for their metrics in terms of their symmetry structure, comprising a time-like, as well as an additional azimuthal Killing vector field that represents their axial symmetry. We then come to the most important representatives of static and stationary metrics, being respectively the Schwarzschild and Kerr metrics.

The following section 5.4 is devoted to equations of motion. We start by discussing the general question of equations of motion for possibly extended, but (approximately) non-gravitating particles in general relativity, which follow from a general multipole expansion of the stress-energy-momentum conservation relation. Continuing with the geodesic equation as the zero-pole part of this expansion, we briefly discuss constants of motion and its general solution procedure in stationary axisymmetric space-times, before we explicitly solve the equation Schwarzschild space-time for the simple case of purely radial infall. The subsequent section 5.5 is then devoted to the discussion of frames for inertial and non-inertial observers in general relativity, where we focus on circular world-lines.

In section 5.6 we discuss aspects of curvature in terms of the Riemann and Weyl tensors, exhibiting an interesting and very convenient representation of the Weyl tensor in terms of two symmetric and trace-free Cartesian matrices, called its electric and magnetic parts, in an analogy with electrodynamics. This leads us to the famous Petrov classification of vacuum space-times and to the associated simple normal forms for the Weyl tensor's frame components in a particular curvature-adapted frame, which serves as a convenient starting point for the calculation of these components in arbitrary observer-adapted frames, appearing prominently in the Fermi normal coordinate metric that is introduced in the next chapter.

We conclude the present chapter with a discussion of the mathematical limitations of exact metrics in the realistic modelling of precision experiments, and an introduction to metric perturbation theory, as well as an outlook to the post-Minkowskian and post-Newtonian expansions which are certain implementations thereof.

5.1. Geometric Nature of Gravity and Equivalence Principle

Combining Newton's 2nd. law of motion $\boldsymbol{F} = m_\mathrm{i}\ddot{\boldsymbol{r}}$, where m_i is the *inertial mass* with Newton's law of universal gravitation using the principle of *actio = reactio*,

$$\boldsymbol{F}_\mathrm{g} = -\frac{\mathrm{G}Mm_\mathrm{g}}{|\boldsymbol{r}|^2}\boldsymbol{e}_r\,, \tag{5.1}$$

with the gravitational mass m_g, which is the mass that reacts to the gravitational field, leads to the usual field equation of Newtonian gravity,

$$\ddot{\boldsymbol{r}} = -\frac{m_\mathrm{g}}{m_\mathrm{i}}\frac{\mathrm{G}M}{|\boldsymbol{r}|^2}\boldsymbol{e}_r\,, \tag{5.2}$$

where G is the gravitational constant and M the mass of the attracting object, i.e. for our purposes, the Earth. It is now an experimental fact already known to Galileo Galilei, that all objects fall at the same speed, independent of their mass and internal composition, i.e. they feel the same gravitational acceleration, so that the prefactor in (5.2) is unity and one concludes that the inertial and gravitational masses are equal, $m_\mathrm{g} = m_\mathrm{i}$. This is the well-known *universality of free fall* (UFF).

Looking now at (5.2) in a slightly different way, one may notice that the equality of inertial and gravitation mass means that gravity can be transformed away by going to a free-falling coordinate system. Thus, gravity can be *geometrised*, i.e., elevated from the status of a force to a property of space-time itself. Since we already know that the theory of Special Relativity is the appropriate generalisation of non-gravitational Newtonian physics, this should be implemented as a generalisation of this metric description of flat space-time. One is thus forced to allow for *general coordinate transformations* (as introduced in subsection 4.2.1) on top of special relativity, which means that a general pseudo-Riemannian metric $g_{\mu\nu}$ should take the place of the Minkowski one, i.e. adapting from (4.39),

$$g_{\mu\nu}\frac{\partial X^\mu}{\partial x^{\hat{\alpha}}}\frac{\partial X^\nu}{\partial x^{\hat{\beta}}} = \eta_{\hat{\alpha}\hat{\beta}}\,, \qquad\qquad g_{\mu\nu} = \eta_{\hat{\alpha}\hat{\beta}}\frac{\partial x^{\hat{\alpha}}}{\partial X^\mu}\frac{\partial x^{\hat{\beta}}}{\partial X^\nu}\,. \tag{5.3}$$

This, in turn, then leads to the interpretation that the space-time of general relativity is curved, and thus that it is mathematically described as a pseudo-Riemannian manifold. Locally however, we can always recover flat Minkowski space-time with $g_{\mu\nu} = \eta_{\mu\nu}$, this is the first equation in (5.3). The physical content of this is embodied in the famous *equivalence principle* which is usually taken as the physical and philosophical "foundation" of the theory of general relativity. There are in fact three versions of the equivalence principle, the *weak*, *Einstein*, and *strong* forms, that we list in the following. The weak one is just the notion of universality of free fall:

Weak Equivalence Principle (WEP) The motion of any freely falling test particle is independent of its mass, composition and structure, that is $m_\mathrm{g} = m_\mathrm{i}$.

The so-called Einstein Equivalence Principle, then combines the universality of free fall with local Lorentz and position invariance, thereby makes Newtonian gravity compatible with special relativity (see, e.g. [76]). It can be stated as follows:

Einstein Equivalence Principle (EEP) The WEP holds and: "The outcome of any local non-gravitational experiment in a freely falling laboratory is independent of the velocity of the laboratory and its location in space-time".*

While the WEP was merely about the equivalence of inertial and gravitational mass for idealised test-particles, the EEP now paraphrases (5.3): i.e., that gravity can always be transformed away locally, in a small environment around every point where one then recovers the laws of special relativity. What "small" means in this context will generally depend on the radius of curvature of space-time at that point, essentially being proportional to the inverse square root of the Riemann tensor, as we shall see in chapter 6.

The Einstein equivalence principle is "at the heart" of General Relativity, since if the EEP is valid, then one can convincingly argue that gravity must be a curved–space-time phenomenon†, more precisely, it must satisfy the postulates of a *metric theory of gravity*. These are:

1. Space-time is a pseudo-Riemannian manifold with the metric having Lorentzian signature $(+,-,-,-)$
2. The world-lines of test bodies are geodesics of the pseudo-Riemannian metric
3. In freely falling – i.e. inertial – frames, the non-gravitational laws are those of Special Relativity

See, e.g., the standard reference [78, Sec. 2.3] by Will, and also his more recent review article [5]. An even stronger form of the EEP is usually conjectured to hold, the so-called

Strong Equivalence Principle (SEP) The WEP is valid for self-gravitating bodies as well as for test bodies. "The gravitational motion of a small test body depends only on its initial position in space-time and velocity, and not on its constitution." And also: "The outcome of *any* local experiment (gravitational or not) in a freely falling laboratory is independent of the velocity of the laboratory and its location in space-time."

5.1.1. Einstein Field Equations

The field equations of general relativity can be obtained from the Einstein-Hilbert action,

$$S_{\mathrm{EH}} := \frac{1}{2\kappa}\int \left[R_{\mathrm{Ric}} - 2\Lambda\right]\sqrt{-g}\,d^4X\,, \qquad \kappa = \frac{8\pi G}{c^4}\,, \tag{5.4}$$

where R_{Ric} is the Ricci scalar (3.86), Λ is the so-called cosmological constant, and G is Newton's gravitational constant, as usual. It is then a standard textbook exercise to

*The structure of the three equivalence principles is not hierarchical in the sense that there are naturally connections between the WEP and the EEP. Indeed, Schiff's conjecture states that any complete, self-consistent theory of gravity that embodies WEP necessarily embodies EEP [5].

†In general, one should be more careful here since General Relativity can be equivalently reformulated on the one hand in terms of torsion only, where curvature (and also non-metricity) is taken to vanish (this is called "Teleparallel Equivalent of General Relativity", TEGR), and on the other hand in terms of non-metricity only, where both curvature and torsion are taken to vanish ("Symmetric Teleparallel Equivalent of General Relativity", STEGR). Their action functionals differ from the Einstein-Hilbert action (5.4) only by a total divergence, so these formulations are dynamically equivalent to general relativity, see [77].

show that the vanishing variation of (5.4), i.e., $\delta S_{\mathrm{EH}} = 0$, (using e.g. the relations in subsection 5.7.2) leads to the Einstein field equations,

$$\boxed{R_{\mu\nu} - \tfrac{1}{2} R_{\mathrm{Ric}}\, g_{\mu\nu} + \Lambda\, g_{\mu\nu} = \kappa\, T_{\mu\nu}\ ,} \tag{5.5}$$

where $R_{\mu\nu}$ is the Ricci tensor (3.86) and $T_{\mu\nu}$ the stress-energy-momentum tensor. The left-hand side of (5.5) is the "geometry side". The so-called Einstein tensor, $G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R_{\mathrm{Ric}}\, g_{\mu\nu} + \Lambda\, g_{\mu\nu}$, is the only tensor that can be built from the metric and its first and second partial derivatives (the latter being contained in the tensors $R_{\mu\nu}$ and R_{Ric}), that also has a vanishing covariant gradient,

$$\nabla^{\sigma} \left[R_{\sigma\nu} - \tfrac{1}{2} R_{\mathrm{Ric}}\, g_{\sigma\nu} \right] = 0\,, \tag{5.6}$$

[being a consequence of the twice-contracted 2^{nd} Bianchi identity (3.82b)] so as to be compatible with the generalised energy momentum conservation relation,

$$\nabla_{\sigma} T^{\sigma\nu} = 0\,. \tag{5.7}$$

We shall see in section 5.4 that this covariant conservation law gives rise to generalised equations of motion and – in the test-particle limit – to the geodesic equation.

The right-hand side of (5.5), i.e. the "matter side" is determined by the stress-energy-momentum tensor $T_{\mu\nu}$, a symmetric tensor field which collects the densities and fluxes of energy and momentum, as well as shear stress and pressure of all matter fields into one tensorial quantity. Thus, the the physical essence of Einstein's field equations (5.5) can be summarised by a famous quote of J. A. Wheeler:

> 'Matter tells space-time how to curve, and curved space-time tells matter how to move'.

Solutions of the homogeneous Einstein equations, $R_{\mu\nu} = 0$, for which $T_{\mu\nu} = 0$, are called *vacuum space-times*. Most astrophysically relevant solutions are of this type, as they model the space-time exterior to an isolated mass, such as a planet, a star, or a black hole. The most well-known exact metrics in this category are probably the Schwarzschild and Kerr metrics, which are the only exact solutions that we will discuss in this thesis, focusing mostly on the Schwarzschild metric for simplicity and only comparing expression to their more general Kerr counterparts here and there, in order to provide a larger perspective. Concerning vacuum solutions in general, we will discuss their curvature tensor and its normal forms later in this chapter in subsection 5.6.2.

The Einstein equations (5.5) are initially a system of 10 coupled, highly non-linear second-order partial differential equations, the solution of which is among the most involved problems in classical theoretical physics. However of these 10, only 6 equations are independent, since the Einstein tensor is subject to the four differential constraints (5.6), which are of geometric nature. In order to solve (5.5), three approaches are available: exact solution, approximate solution employing metric perturbation theory, and since the early 2000s also numerical solution of the full Einstein equations [79, 80], which we do not discuss in this thesis. Firstly, in order to obtain an exact solution, it is necessary to impose symmetries and make use of symmetry-adapted coordinates in order to reduce their complexity to a manageable level, thereby reducing (5.5) to a number of (ideally linear, but often non-linear) *ordinary* differential equations. The most general result of this procedure are the stationary and axisymmetric space-times, that feature two so-called *Killing vector* fields, which represent symmetry directions of the metric, one time-like and one azimuthal one. The above-mentioned prototypical and well-known examples of these are the spherically symmetric Schwarzschild-Droste and the axisymmetric Kerr metrics, that we will discuss in subsection 5.3.2 and subsection 5.3.3, respectively.

5.2. Space-Time Symmetries: Killing Fields and Stationary Metrics

Since we saw above that exact solutions to the Einstein's field equations (5.5) can typically only obtained by imposing symmetry conditions, so the study of space-time symmetries is an important aspect of general relativity. It leads to the concept of Killing vector fields, i.e. to vector fields that are the infinitesimal generators of isometries of the space-time manifold.

5.2.1. Killing Vector Fields

A vector field ξ^σ is then called an *Killing vector field* if the Lie derivative of the metric with respect to ξ^σ vanishes,

$$\mathscr{L}_{\xi^\sigma}\, g_{\mu\nu} = 0\,. \tag{5.8}$$

We had already worked out the action of Lie derivative on the metric tensor in (3.37). This equation can be rewritten in a slightly modified form and in terms of covariant derivatives, as we shall briefly demonstrate. We first apply an inverse product rule with the two last terms on the right-hand side, i.e. $g_{\lambda\nu}\partial_\mu\xi^\lambda = \partial_\mu(\xi^\lambda g_{\lambda\nu}) - \xi^\lambda\partial_\mu g_{\lambda\nu}$, thereby moving all partial derivatives from the Killing vector field onto the metric and its contraction with ξ^σ, the latter of which becomes ξ_ν. Thus, we obtain from (3.37),

$$\begin{aligned} \mathscr{L}_{\xi^\sigma}\, g_{\mu\nu} &= \xi^\sigma\partial_\sigma g_{\mu\nu} + g_{\lambda\nu}\partial_\mu\xi^\lambda + g_{\mu\lambda}\partial_\nu\xi^\lambda \\ &= \partial_\mu\xi_\nu - \Gamma^\lambda{}_{\mu\nu}\xi_\lambda + \partial_\nu\xi_\mu - \Gamma^\lambda{}_{\nu\mu}\xi_\lambda\,, \end{aligned} \tag{5.9}$$

where in the last line we have extended with two Christoffel symbols, which result in two index-permuted covariant derivatives of the Killing vector field. Setting now the above result equal to zero, we obtain the *Killing equation*,

$$\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu = 0\,, \tag{5.10}$$

which is then nothing else but a manifestly covariant version of equation (5.8).

Each Killing vector corresponds to a quantity which is conserved along geodesics. If $X^\nu_{\mathscr{G}}(\lambda)$ parametrises a geodesic $\mathscr{G}$ and ξ^σ is a Killing vector of the underlying space-time, then the contraction of ξ_σ with its tangent $v^\sigma(\lambda) = \frac{dX^\nu_{\mathscr{G}}}{d\lambda}$ is a conserved quantity, since

$$\frac{D}{d\lambda}(K_\sigma v^\sigma) = (v^\nu\nabla_\nu\xi_\sigma)v^\sigma + \xi_\sigma(v^\nu\nabla_\nu v^\sigma) = \tfrac{1}{2}(\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu)v^\mu v^\nu = 0\,. \tag{5.11}$$

In the first equation we have used the geodesic equation for v^σ, i.e., $v^\nu \nabla_\nu v^\sigma = 0$, and in the remaining term, recognised the Killing equation (5.10). Using the Killing equation (5.10) and the definition of the Riemann tensor (3.78) for a coordinate basis, one obtains first $\nabla_\mu \nabla_\nu \xi_\sigma - \nabla_\mu \nabla_\sigma \xi_\nu = \xi_\lambda R^\lambda{}_{\sigma\mu\nu}$, and by adding and subtracting three index-permuted version of this and using the 1st Bianchi identity (3.82a) on the right-hand side, one obtains Killing's identity,

$$\nabla_\mu \nabla_\nu \xi_\sigma = \xi_\lambda R^\lambda{}_{\nu\mu\sigma} \,. \tag{5.12}$$

An important consequence of this is that any Killing vector field ξ^σ is completely determined by the values of ξ^σ and $\nabla_\mu \xi_\sigma$ at any point of the manifold. This can be seen by integrating the system of ordinary differential equation,

$$\begin{aligned} \frac{D\xi_\mu}{d\lambda} &= v^\lambda \nabla_\lambda \xi_\mu = v^\lambda K_{\lambda\mu} \,, \\ \frac{DK_{\mu\nu}}{d\lambda} &= v^\lambda \nabla_\lambda K_{\mu\nu} = \xi_\lambda R^\lambda{}_{\nu\mu\sigma} v^\sigma \,, \end{aligned} \tag{5.13}$$

where $K_{\mu\nu} := \nabla_\mu \xi_\nu$ is the Killing two-form, along any curve with tangent v^ν. In N dimensions, ξ^σ has N, and the antisymmetric tensor $K_{\mu\nu}$ has $N(N-1)/2$ components (we can interpret that ξ^σ corresponds to translations and $K_{\mu\nu}$ to generalised rotations), and thus, in an N-dimensional space-time, the maximal number of linearly independent Killing vectors is $\frac{1}{2}N(N+1)$, since this is the dimension of the space of initial data for $(\xi_\sigma, K_{\mu\nu})$. Clearly, this is also the number of independent components of the metric tensor.

5.2.2. Maximally Symmetric Space-Times: From Minkowski to Cosmology

Space-times in which this maximum is attained are called *maximally symmetric* and due to equations (5.13), the metric coefficients of maximally symmetric metrics are at most quadratic polynomials in the coordinates. Consequently, they are of constant curvature, that is, the curvature tensors are given by,

$$R_{\mathrm{Ric}} = \text{const.}, \qquad R_{\mu\nu} = \frac{R_{\mathrm{Ric}}}{N} g_{\mu\nu} \,, \qquad R_{\sigma\rho\mu\nu} = \frac{R_{\mathrm{Ric}}}{N(N-1)} (g_{\sigma\mu} g_{\rho\nu} - g_{\sigma\nu} g_{\rho\mu}) \,, \tag{5.14}$$

which leads to their characteristic property that the Riemann curvature tensor is covariantly constant (or "parallel"), $\nabla_\kappa R^\sigma{}_{\rho\mu\nu} = 0$. In general relativity, we have $N = 4$, so the maximal number of linearly independent Killing vectors is 10. For Minkowski space-time, this number can then be seen to correspond to Poincaré transformations (4.12), i.e., to the 3 independent directions of Lorentz boosts, plus 3 possible spacial rotation axes plus 4 directions of translation. Maximally symmetric space-times come in three kinds, depending on the *sign* of their curvature. For zero curvature one has flat Minkowski space, and for the two kinds with non-zero curvature, the positive-curvature de Sitter (dS) and negative-curvature anti de Sitter (AdS) space can be considered as Lorentzian analogues of an N-sphere and a hyperbolic N-space, respectively. They are maximally symmetric vacuum solutions of Einstein's field equations (5.5) for a positive, $\Lambda > 0$ (de Sitter space), and negative cosmological constant, $\Lambda < 0$ (anti de Sitter space).

These non-flat N-dimensional maximally symmetric space-times are then quadrics, i.e. quadratic hyper-surfaces embedded in an $(N+1)$-dimensional Minkowski space. Their metric can be constructed from the flat metric of the embedding space as follows. We let $N = 4$ for the case of General Relativity, and denote the coordinates of the 5-dimensional embedding space by $X^A = X^\mu + X^4$, so $A, B \in \{0, 1, 2, 3, 4\}$ are the 5-dimensional indices. The line element then reads,

$$d\lambda^2 = \eta_{AB} dX^A dX^B = \eta_{\mu\nu} dX^\mu dX^\nu - (dX^4)^2 \,, \tag{5.15}$$

the quadratic 4-dimensional hypersurface being given in terms of the quadrics' curvature radius R_0 by the equation

$$\eta_{AB} dX^A dX^B = -kR_0^2 \,, \tag{5.16}$$

where $k \in \{-1, 0, +1\}$ determines the sign of the hypersurface's scalar curvature, so that $R_{\mathrm{Ric}} = k/R_0^2$. Note that for $k = +1$, the additional dimension is spacial (de Sitter space), whereas for $k = -1$ it is time-like (anti de Sitter space). From equation (5.16), we obtain an expression for the differential of the additional spacial coordinate X^4,

$$dX^4 = \pm \frac{X_\nu dX^\nu}{\sqrt{X^\sigma X_\sigma + kR_0^2}} \,, \tag{5.17}$$

with which we can remove the additional dimension from the 5-dimensional metric (5.15), which then leads to the induced metric on the hypersurface taking the form,

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{X_\mu X_\nu}{X^\sigma X_\sigma + kR_0^2} \,. \tag{5.18}$$

de Sitter Space-Time

Focussing now on the positive-curvature de Sitter space-time, i.e. $k = +1$, the hypersurface is a hyperboloid of one sheet, which can be parametrised in terms of hyperbolic functions. In terms of the line element for the two-dimensional round sphere, $\left[d\Theta^2 + \sin^2(\Theta) d\Phi^2\right]$, the de Sitter line element can be written as,

$$d\lambda^2_{\mathrm{dS}_4} = (c\, dT)^2 - R_0^2 \cosh^2\!\left(\frac{cT}{R_0}\right) \left\{ \frac{dR^2}{1 - (R/R_0)^2} + R^2 \left[d\Theta^2 + \sin^2(\Theta) d\Phi^2\right] \right\} , \tag{5.19}$$

the dR^2 coefficient of which becomes singular at $R = R_0$. To remedy this, one introduces an angular radius coordinate χ by $R = R_0 \sin(\chi)$, in terms of which $d\chi = dR/\sqrt{1 - (R/R_0)^2}$, and leading to,

$$d\lambda^2 = (c\, dT)^2 - R_0^2 \cosh^2\!\left(\frac{cT}{R_0}\right) \left\{ d\chi^2 + \sin^2(\chi) \left[d\Theta^2 + \sin^2(\Theta) d\Phi^2\right] \right\} , \tag{5.20}$$

which shows that the spacial sections of (5.20) are three-spheres S^3. The metric of de Sitter space can be written in terms of yet another Cartesian-like set of coordinates, $x^\alpha = (ct, x^a)$, which we will encounter in the context of exact Riemann and Fermi coordinates in sections 6.2 and 6.3. The transformation reads (cf. [81] for an atlas of coordinate charts for de Sitter space),

$$X^0 = R_0 \cos\!\left(\frac{r}{R_0}\right) \sinh\!\left(\frac{cT}{R_0}\right) , \quad X^i = \frac{R_0}{r} \sin\!\left(\frac{r}{R_0}\right) x^i \,, \tag{5.21}$$

where $r = \sqrt{x^a x_a}$, with $r \in [0, \frac{1}{2}\pi R_0)$, leading to the following form of the line element,

$$d\lambda^2 = g_{\alpha\beta} dx^\alpha dx^\beta = \cos^2\Big(\frac{r}{R_0}\Big)(c\,dt)^2 - \Big[\frac{x_a x_b}{r^2} + \frac{\sin^2(r/R_0)}{(r/R_0)^2}\Big(g_{ab} - \frac{x_a x_b}{r^2}\Big)\Big]\,. \tag{5.22}$$

Cosmology: The Friedmann-Lemaître-Robertson-Walker Metric

In order to round up the line of reasoning of this subsection, we briefly point out the connection between the maximally symmetric solutions mentioned above and the standard cosmological space-time model, referred to as Friedmann-Lemaître-Robertson-Walker (FLRW) metric. Built on the observed large-scale *spacial* homogeneity and isotropy of the universe, one is led to consider metrics with homogeneous and isotropic spacial sections. This gives such metrics a "warped-product" form, i.e. the metric is almost block-diagonal between the time and space sub-manifolds, up to a time-dependent ("warp") function multiplying the spacial part, which is then their only degree of freedom.

In extension of (5.19), the FLRW metric for spatially open ($k = -1$), flat ($k = 0$), or closed ($k = +1$) expanding or contracting universes then reads,

$$d\lambda^2 = (c\,dT)^2 - a^2(T)\Big[\frac{dR^2}{1 - kR^2} + R^2\big(d\Theta^2 + \sin^2(\Theta)d\Phi^2\big)\Big]\,. \tag{5.23}$$

The FLRW metrics can thus be seen as slight generalisations of the maximally symmetric ones, where the prefactor of the spacial part in (5.19), (5.20) is generalised to $a^2(T)$, called the *scale factor*, and being a generic function of coordinate time, where in the de Sitter case, $a^2(T) = R_0^2 \cosh^2(cT/R_0)$.

5.2.3. Stationary and Static Space-Times

A space-time determined by a solution $g_{\mu\nu}$ of the Einstein field equations (5.5) is called *stationary*, if the metric possesses a time-like Killing vector. Thus, in a sense, the space-time is then "time-translation invariant", and choosing the Killing vector's integral curves as time coordinate lines, all metric coefficients must be time independent.[‡]

Geometrically, this means that we can think of the space-time as being foliated with a (three-dimensional) space-like "hypersurface" Σ, the three-geometry of which is time-independent.[§] However, in following the Killing trajectories, i.e. the time-coordinate lines, from one hypersurface at T to the infinitesimally neighbouring one at $T + dT$, this next "hypersurface" can be locally "shifted", or "rotated" infinitesimally with respect to the previous one in a manner that will depend on spacial position on Σ in general, but *not* on coordinate time T. This is quantified by the three-form $\xi_{[\rho}\nabla_\mu \xi_{\nu]}$, which is dual to a spacial vector, the *shift vector* field N^i, i.e.,

$$N_\sigma = \varepsilon_{\sigma\rho\mu\nu}\xi^\rho \nabla^\mu \xi^\nu\,, \tag{5.24}$$

[‡] A more "rigorous" discussion can be found in Sec. 7.1 of Wald's textbook [49].

[§] Technically, Σ is only a true hypersurface in the static case. In a general stationary space-time it is a quotient space, i.e., the image of a map given by the flow (i.e. the integral curves) of the time-like Killing vector ξ^σ (cf. the discussion in section 3.2); see [82, chapter 18], [83].

which then carries points from one constant-time hypersurface to the next. Equivalently, the shift vector field measures the failure of ξ^σ to be orthogonal to the spacial hypersurfaces Σ.

Given an initial metric $g_{\mu\nu}$, and writing the squared norm of the time-like Killing vector field as $N^2 := \xi_\sigma \xi^\sigma$, the decomposition¶ with respect to ξ^σ is achieved via the projector $\delta^\sigma{}_\nu - N^{-2}\xi^\sigma\xi_\nu$, which projects onto Σ, as well as $N^{-2}\xi^\sigma\xi_\nu$, which projects parallel to ξ^σ, i.e.

$$d\lambda^2 = N^{-2}\xi_\mu\xi_\nu \, dX^\mu dX^\nu - N^{-2}(N^2 g_{\mu\nu} - \xi_\mu\xi_\nu) dX^\mu dX^\nu \,. \tag{5.25}$$

The infinitesimal distance between two adjacent spacial hypersurfaces must then be determined by the squared norm of the time-like Killing vector field, which is written as $N^2 := \xi_\sigma\xi^\sigma$, in terms of the so-called *lapse function* $N = N(X^i)$. This means that the time–time component of the stationary metric is given by $g_{TT} = N^2$, there must be a $dT\, dX^i$ cross-term, and the metric of the spacial "hypersurfaces", $\gamma_{ij} = \gamma_{ij}(X^i)$, reads, $\gamma_{\mu\nu} = N^2 g_{\mu\nu} - \xi_\mu\xi_\mu$. Consequently, in a coordinate system where $\xi^\sigma = (1, 0, 0, 0)$, the normal form of the stationary metric reads,

$$\boxed{d\lambda^2 = N^2\big[c\, dT - N_i\, dX^i\big]^2 - N^{-2}\gamma_{ij} dX^i dX^j \,.} \tag{5.26}$$

Static space-times are special cases of stationary ones, with the additional property, that the time-like Killing vector field is everywhere *orthogonal* to the spacial hypersurfaces Σ (which are now true hypersurfaces), so that the shift vector fields vanishes, $N_\sigma = 0$. Consequently, the normal form of its metric reduces from (5.26) to

$$\boxed{d\lambda^2 = N^2(c\, dT)^2 - N^{-2}\gamma_{ij} dX^i dX^j \,,} \tag{5.27}$$

i.e., it becomes block diagonal between time and the spacial directions in these "Killing-adapted" coordinates.

5.3. Exact Metrics: Schwarzschild, Kerr, and Beyond

For physical reasons, we are interested in the metric outside of an isolated object (i.e. of a planet or a star), which is possibly rotating with constant angular velocity. According to our considerations above, this means that symmetry-wise the metric will be stationary and axisymmetric, and furthermore that it must be a vacuum solution of the Einstein field equations that is asymptotically flat (the metric approaches the flat Minkowski metric far away from the isolated object). The class of stationary and axisymmetric space-times is the class of exact solutions of Einstein's field equations (5.5) with the least symmetry, i.e., smallest number of Killing vectors (namely two), that are generally thought to be analytically treatable. As motivated above, they are also the ones that are of direct physical relevance as an idealised (stationary, isolated, asymptotically flat, . . .) description of astrophysical objects like (neutron-) stars, black holes, etc. Therefore, we shall begin this section by motivating the general form of a stationary and axisymmetric metric, and then introduce the Schwarzschild and Kerr metrics.

¶This is called a Geroch decomposition [84]

5.3.1. The Stationary and Axisymmetric Metric

We saw above that stationary metrics are characterised by a time-like Killing vector field ξ^σ_T. If additionally, the space-time is also axisymmetric, we have an additional space-like Killing vector, ξ^σ_Φ, with closed space-like integral curves. These two Killing vectors commute*, $[\xi^\sigma_T, \xi^\sigma_\Phi] = 0$, so that one can choose as symmetry-adapted coordinates their integral curves $X^0 \equiv cT$ and $X^3 \equiv \Phi$, i.e. in these coordinates, the azimuthal Killing vector is given by $\boldsymbol{\partial}_\Phi$ and the time-like one by $\boldsymbol{\partial}_T$. The existence of an additional Killing vector now means that the normal form (5.26) of the stationary metric can be further simplified.

Normal Form of the Metric

As a consequence of the inevitable presence of a $dT\,d\Phi$ cross-term in the line element, the metric must be simultaneously invariant under time inversion, $T \to -T$, and inversion of azimuthal angle, $\Phi \to -\Phi$ (i.e, inverting the time direction means that the isolated object will rotate backwards), which immediately leads to the conditions that any other cross-terms, i.e. those mixed components of $g_{\mu\nu}$ that involve either T or Φ must vanish,

$$g_{01} = 0 = g_{02}\,, \qquad \text{and} \qquad g_{13} = 0 = g_{23}\,, \tag{5.28}$$

but initially leaves components along the anti-diagonal of $g_{\mu\nu}$, i.e. g_{12} and g_{03}, unconstrained. In addition, the metric components can now only depend on the radial and polar variables, i.e. on $X^1 \equiv R$ and $X^2 \equiv \Theta$. As a consequence of these restrictions, the metric is block-diagonal, and the R–Θ-subspace in the metric can be locally considered as a separate two-dimensional Riemannian manifold. Since two-dimensional manifolds are known to be conformally flat (i.e their Weyl tensor vanishes, see Table 5.1), their metric being locally proportional to the Minkowski metric, we conclude that the metric of this sub-manifold must be diagonal, so g_{12} can be taken to vanish too [86, Sec. 13.1]. Although conformal flatness of this subspace means that these two metric components can be chosen to be equal, $g_{\Theta\Theta} = g_{RR}$, we shall not implement this last possible simplification, since we want to write the metric in spherical coordinates which necessarily leaves these distinct.

These considerations finally results in a considerable simplification of the metric, which can then be brought into the block diagonal standard form,

$$g_{\mu\nu} = \left(\begin{array}{c|cc|c} g_{TT} & & & g_{T\Phi} \\ \hline & g_{RR} & & \\ & & g_{\Theta\Theta} & \\ \hline g_{\Phi T} & & & g_{\Phi\Phi} \end{array}\right), \quad g^{\mu\nu} = \left(\begin{array}{c|cc|c} \frac{g_{\Phi\Phi}}{g_{TT}g_{\Phi\Phi}-(g_{T\Phi})^2} & & & \frac{-g_{T\Phi}}{g_{TT}g_{\Phi\Phi}-(g_{T\Phi})^2} \\ \hline & \frac{1}{g_{RR}} & & \\ & & \frac{1}{g_{\Theta\Theta}} & \\ \hline \frac{-g_{\Phi T}}{g_{TT}g_{\Phi\Phi}-(g_{T\Phi})^2} & & & \frac{g_{TT}}{g_{TT}g_{\Phi\Phi}-(g_{T\Phi})^2} \end{array}\right). \tag{5.29}$$

A particular form of (5.26) is achieved in the corresponding line element,

$$d\lambda^2 = g_{TT}(c\,dT)^2 + g_{RR}dR^2 + g_{\Theta\Theta}d\Theta^2 + g_{\Phi\Phi}d\Phi^2 + 2g_{T\Phi}\,(c\,dT)d\Phi\,, \tag{5.30}$$

*Carter showed in [85] that no generality is lost when considering only commuting Killing vector fields.

upon rearranging it and completing the square,

$$d\lambda^2 = N^2(c\,dT)^2 + g_{\Phi\Phi}\big[d\Phi + N^\Phi(c\,dT)\big]^2 + g_{RR}dR^2 + g_{\Theta\Theta}d\Theta^2\,, \tag{5.31}$$

so that the lapse function introduced above is written in terms of the determinant of the T–Φ sub-space, as

$$N := \sqrt{g_{TT} - (g_{T\Phi})^2/g_{\Phi\Phi}}\,, \tag{5.32}$$

and the (only non-vanishing component of) the *shift vector field* N^i reads,

$$N^\Phi := \frac{g_{T\Phi}}{g_{\Phi\Phi}}\,. \tag{5.33}$$

5.3.2. Spherically Symmetry: Schwarzschild-Droste Space-Time

The "Schwarzschild" metric was the very first exact solution of Einstein's (vacuum) field equations, derived under the assumption of spherical symmetry by Karl Schwarzschild in 1916 [87], and around the same time independently by Dutch physicist Johannes Droste [88, 89]*. According to a collection of results by several people commonly referred to as *Birkhoff's theorem*, spherical symmetry is a very strong assumption in that every spherically symmetric vacuum solution of the Einstein equations (5.5) is necessarily static and asymptotically flat.

Geometrically, spherical symmetry means that the spacial hypersurfaces of the foliation of space-time discussed in subsection 5.2.3 must itself consist a family of concentric spheres. This results in that we can view a spherically symmetric metric as an almost direct product, which leads to it assuming the form of a covariant 2+2 split, which we will explore in Appendix A in order to derive a closed-form expression for its Weyl tensor. The Schwarzschild metric is best known in the pseudo-spherical Schwarzschild coordinates,

$$X^\mu = (cT, R, \Theta, \Phi)\,, \tag{5.34}$$

where it takes the well-known and simple form

$$\boxed{d\lambda^2 = B(R)(c\,dT)^2 - \frac{dR^2}{B(R)} - R^2\big[d\Theta^2 + \sin^2(\Theta)d\Phi^2\big].} \tag{5.35}$$

Here, $B(R)$ is the Schwarzschild function given by the simple expression,

$$B(R) = \Big(1 - \frac{R_\mathrm{S}}{R}\Big)\,, \qquad R_\mathrm{S} = \frac{2GM}{c^2}\,, \tag{5.36}$$

*What we today call the Schwarzschild solution was in fact independently derived by Johannes Droste around the same time in his PhD thesis [88], that he defended on December 8th 1916. He published his results in English language in a subsequent paper [89, 90] just four months after that of Schwarzschild. They contain (among other) a clearer and much more modern-looking presentation of the derivation, including a solution $R(\Phi)$ of the geodesic equation in terms of Weierstraß' elliptic $\wp$ function, a complete discussion of geodesic motion, and the line element in isotropic coordinates [our equation (5.39)]. Therefore, it appears appropriate to speak instead of the "Schwarzschild-Droste solution". This is also the view recently taken by Rothman in an Editor's Note of *General Relativity and Gravitation* [91].

and R_S denotes the Schwarzschild radius, i.e. the geometrised mass parameter. The length R_S is the single characteristic scale in Schwarzschild space-time and thus the ratio R_S/R in the Schwarzschild function (5.36) is the dimensionless scale for curvature effects, i.e. curvature is strong when R_S/R is of order 1, and weak when $R_S/R \ll 1$.

Taken by itself, this space-time describes a static black hole, i.e. it features a true curvature singularity at the origin at $R = 0$, whereas the apparent singularity in the line element (5.35) at $R = R_S$ is a coordinate singularity that can be removed by a different choice of coordinates. Physically, the two-sphere with coordinate radius $R = R_S$ marks the location of the black hole's event horizon. As always, these two types of singularity can be distinguished by calculating a curvature invariant like the Kretschmann scalar, $\mathcal{K} = R_{\sigma\rho\mu\nu}R^{\sigma\rho\mu\nu} \sim \frac{1}{R^6}$. The Schwarzschild metric is asymptotically flat, i.e., we have $g_{\mu\nu} \to \eta_{\mu\nu}$ for $R \to \infty$, and reduces to flat Minkowski space-time in the limit $R_S \to 0$, i.e. for vanishing mass.

By comparing the Schwarzschild line-element (5.35) to the one of Minkowski space-time in spherical coordinates (4.127), one might be led to infer that R is a "normal" radial coordinate, at least when R_S is small and the metric closed to Minkowskian, however this is deceptive. It turns out that R is not even always a spacial coordinate, since below the event horizon, i.e. for $0 < R < R_S$, T and R change roles, as the metric coefficient $B(R)$ is negative there, so T becomes a spacial and R a time coordinate. Thus, R is *not* the *distance* to the coordinate origin at $R = 0$. Instead, the Schwarzschild radial coordinate R in the line-element (5.35) is a so-called *areal radius*, i.e. it has a geometric meaning as the familiar radius associated with the Euclidean surface area $4\pi R^2$ of the concentric spheres where T and R are constant (see e.g. [92]). It is thus only asymptotically, i.e. for $R \gg R_S$, and in weak fields, $R_S/R \ll 1$, that one can approximately identify R with the Cartesian radial coordinate in Newtonian physics.

Although the problem with the interpretation of coordinates is fundamentally associated with General Relativity and cannot be overcome (instead we have to give up our simple Newtonian picture of the world when dealing with non-negligible curvature), we can at least simplify the spacial sector of the Schwarzschild metric by introducing the following transformation to an isotropic radial coordinate R_{iso},

$$R = R_{\mathrm{iso}}\Big(1 + \frac{R_S}{4R_{\mathrm{iso}}}\Big)\,, \qquad R_{\mathrm{iso}} = \frac{R}{2}\Big[1 + \sqrt{B(R_{\mathrm{iso}})}\,\Big] - \frac{R_S}{4}\,, \tag{5.37}$$

which brings (5.35) into the *isotropic* form, where the speed of light is equal in all directions, in that all spacial directions now share the same prefactor,

$$d\lambda^2 = B_1(R_{\mathrm{iso}})(c\,dT)^2 - B_2(R_{\mathrm{iso}})\big[dR_{\mathrm{iso}}^2 + R_{\mathrm{iso}}^2(d\Theta^2 + \sin^2(\Theta)d\Phi^2)\big]\,. \tag{5.38}$$

Here, the terms in the square brackets are just the spacial part of the Minkowski metric in spherical coordinates $(R_{\mathrm{iso}}, \Theta, \Phi)$, so that (5.38) can now also be expressed in a Cartesian form, which just reads,

$$d\lambda^2 = B_1(R_{\mathrm{iso}})(c\,dT)^2 - B_2(R_{\mathrm{iso}})\big[dX^2 + dY^2 + dZ^2\big]\,, \tag{5.39}$$

where the two transformed coefficient functions are given by,

$$B_1(R_{\rm iso}) = \frac{\left(1 - \frac{R_{\rm S}}{4R_{\rm iso}}\right)^2}{\left(1 + \frac{R_{\rm S}}{4R_{\rm iso}}\right)^2}\,, \qquad B_2(R_{\rm iso}) = \left(1 + \frac{R_{\rm S}}{4R_{\rm iso}}\right)^4 . \tag{5.40}$$

These isotropic coordinates are thus the starting point for performing a Newtonian limit in the context of Schwarzschild-like static space-times.

Birkhoff's theorem implies that the (exterior) Schwarzschild metric in (5.35) is the *unique* spherically symmetric, static and asymptotically flat solution of the vacuum field equations (5.5) for vanishing cosmological constant, $\Lambda = 0$. This means that the mass monopole is thought to always be the dominant contribution to space-time curvature for all astrophysically relevant objects [planets, (neutron-) stars, black holes, etc.]. Approximating the space-time curvature around the Earth by a Schwarzschild metric, the Schwarzschild radius of the Earth is found to be $R_{\rm S} \approx 9\,\mathrm{mm}$, so $R_{\rm S}/R$ is very small. In fact, for the surface of the Earth at $R_{♁}$, the scale of curvature effects is

$$\frac{R_{\rm S}}{R_{♁}} \approx \frac{9\,\mathrm{mm}}{6378\,\mathrm{km}} \approx 10^{-9}\,, \tag{5.41}$$

which is obviously why we usually don't notice that we live in a curved space-time, and developed Newtonian mechanics.

5.3.3. Kerr Space-Time

The Kerr metric is the simplest example of the stationary and axisymmetric metric (5.26), discovered in 1963 by New Zealand mathematician Roy Kerr [93]. As a vacuum solution of Einstein's field equations it depends on two parameters, the usual geometrised "mass" parameter $R_{\rm S}$ analogous to the Schwarzschild case, and additionally the "angular-momentum" parameter a, which governs the single non-vanishing, off-diagonal metric coefficient in (5.29) and is physically related to the dragging of inertial frames caused by the rotation of the space-time itself. It is usually expressed in the Schwarzschild-like Boyer-Lindquist coordinates $X^\mu = (cT, R, \varTheta, \varPhi)$, in which the number of non-diagonal metric coefficients is minimised to a single one, namely $g_{T\varPhi}$, and the metric tensor thus takes on the standard form (5.29), as discussed in subsection 5.3.1. In "expanded" form, it reads,

$$\boxed{\begin{aligned} d\lambda^2 = \left(1 - \frac{R_{\rm S}R}{\rho^2}\right)(c\,dT)^2 + \frac{2R_{\rm S}Ra\sin^2\varTheta}{\rho^2}(c\,dT)d\varPhi - \rho^2\left(\frac{dR^2}{\varDelta} + d\varTheta^2\right) \\ - \left(R^2 + a^2 - \frac{R_{\rm S}Ra^2}{\rho^2}\sin^2\varTheta\right)\sin^2(\varTheta)d\varPhi^2, \end{aligned}} \tag{5.42}$$

where the coefficient functions $\rho \equiv \rho(R,\varTheta)$, $\varDelta \equiv \varDelta(R)$ and $\varSigma \equiv \varSigma(R,\varTheta)$ are given respectively by,

$$\rho^2 := R^2 + (a\cos\varTheta)^2 = R^2\left[1 + \left(\frac{a}{R}\right)^2\cos^2\varTheta\right], \tag{5.43a}$$

$$\Delta := R^2 - R_S R + a^2 = R^2\Big[B(R) + \Big(\frac{a}{R}\Big)^2\Big]\,, \tag{5.43b}$$

$$\Sigma^2 = (R^2 + a^2)^2 - \Delta(a\sin\Theta)^2\,, \quad \text{and where} \tag{5.43c}$$

$$a = \frac{L_z}{Mc} \quad \text{is the Kerr angular momentum parameter.} \tag{5.43d}$$

In the second equation respectively, we have made contact with the corresponding quantities in Schwarzschild space-time by expressing them in terms of the Schwarzschild function (5.36).* This shows that ρ is essentially the deformed version of the radial coordinate R, and Δ looks like a deformation of the Schwarzschild metric's coefficient function $B(R)$. Taken as describing a rotating black hole, Kerr space-time has a more complex structure, including a ring singularity at the centre, an inner and an outer horizon, and the so-called ergosurface around these, with the region between the ergosurface and the outer horizon, called the ergo-sphere, being a region where the frame-dragging of space-time is so strong that test-particles are constrained to co-rotate with th black hole. For details we refer the reader to any good textbook on general relativity, e.g. to the introductory but complete [86, Ch. 13], or the dedicated [94].

By completing the square twice, one obtains a more compact form of the Boyer-Lindquist line element (5.42), making it straightforward to read off the metric-adapted *inverse* tetrad,

$$\boxed{d\lambda^2 = \frac{\Delta}{\rho^2}\big[c\,dT - a\sin^2(\Theta)d\Phi\big]^2 - \frac{\rho^2}{\Delta}dR^2 - \rho^2 d\Theta^2 - \frac{\sin^2\Theta}{\rho^2}\big[(R^2+a^2)d\Phi - a\,cdT\big]^2\,.} \tag{5.44}$$

The fact that the space-time itself described by Kerr metric (5.42) above is actually rotating with "angular momentum" a can be inferred by writing (5.42) in a slightly different form which is more suggestive of a rotating object,

$$d\lambda^2 = \frac{\rho^2\Delta}{\Sigma^2}(c\,dT)^2 - \Big(\frac{\Sigma}{\rho}\Big)^2\sin^2(\Theta)\big(d\Phi - N^\Phi\,cdT\big)^2 - \rho^2\Big[\frac{dR^2}{\Delta} + d\Theta^2\Big]\,, \tag{5.45}$$

where the shift vector is given by

$$N^\Phi = \frac{cR_S Ra}{\Sigma^2}\,. \tag{5.46}$$

Limits of the Kerr Metric

For a slowly rotating source, the Kerr metric (5.42) is straightforwardly linearised in the Kerr parameter a to obtain the Schwarzschild metric, augmented with a non-diagonal term linear in a,

$$d\lambda^2 = B(R)(c\,dT)^2 - \frac{dR^2}{B(R)} - R^2\big[d\Theta^2 + \sin^2(\Theta)d\Phi^2\big] + 2\frac{R_S}{R}a\sin^2(\Theta)(c\,dT)d\Phi. \tag{5.47}$$

*We should caution the reader that in the literature, the definitions of ρ, Δ, and Σ are often swapped, i.e. there are several different conventions. We use the convention adopted by Misner, Thorne and Wheeler [48].

In the static limit, obtained by taking $a \to 0$, it is then obvious that we recover from (5.47) the Schwarzschild metric (5.35), i.e. the Kerr space-time also becomes spherically symmetric in this limit. Taking instead the zero-mass limit of, $R_\mathrm{S} \to 0$ in (5.42), we obtain,

$$d\lambda^2 = (c\,dT)^2 - \frac{\rho^2 dR^2}{R^2 + a^2} - \rho^2 d\Theta^2 - (R^2 + a^2)\sin^2(\Theta)d\Phi^2, \tag{5.48}$$

which turns out to be the Minkowski metric, $d\lambda^2 = (c\,dT)^2 - dX^2 - dY^2 - dZ^2$, expressed in a certain type of oblate spheroidal coordinates given by,

$$X = \sqrt{R^2 + a^2}\sin(\Theta)\cos(\Phi)\,, \quad Y = \sqrt{R^2 + a^2}\sin(\Theta)\sin(\Phi)\,, \quad Z = R\cos(\Theta)\,, \tag{5.49}$$

i.e., the surfaces $R =$ const. are ellipsoids of revolution, and the surfaces $\Theta =$ const. hyperbolae of revolution around the Z-axis, respectively. So, just as in the Schwarzschild case, the Kerr space-time – being its rotating generalisation – *locally* also becomes flat Minkowski space-time in the limit of vanishing mass.[†]

5.4. General Equations of Motion: From Extended Bodies to Point-Particles and their Geodesics

Unlike in the Newtonian case, where the solution of the Kepler problem is a standard exercise in classical mechanics and where the orbiting bodies can have arbitrary masses, the general, unrestricted two-body problem in full general relativity is unsolved, i.e., no exact solution of the Einstein equations (5.5) for two dynamical gravitating bodies with masses M_1, M_2 is known. Instead, one usually has to make use of a separation of mass scales, e.g. $m \ll M$ for $m \coloneqq M_1$, $M \coloneqq M_1$. In the extreme case of the so-called *test-particle* limit of m, it is then assumed that the gravitation of the fiducial test mass m can be neglected altogether, and that it is taken to be *point-like*, i.e., it has *no internal structure* that would lead to multipole moments, besides the zeroth-order mass moment m.

In the following two subsections, we would like to start by motivating the area between these two extremes, i.e. between the full dynamical (but as-yet unsolved) two-body problem on the one hand, and the hypothetical bare point-like test particle. While (just as in electrodynamics) the ubiquitous self-fields are always present in a fully relativistic, field-theoretic description of motion and difficult to deal with theoretically, this is neglected in the approach to *multipolar equations of motion* that we introduce in the following, where the body is still assumed to be only negligibly gravitating, but one allows for a full set of multipole moments. We then turn back to geodesics, since these multipolar effects are extremely small within the solar system and on Earth orbits. After briefly discussing the constants of motion for the general geodesic problem in stationary axisymmetric space-times, we turn to the Schwarzschild case, where we explicitly solve the case of purely radial infall.

[†]This holds locally. Globally, the zero-mass limit of Kerr space-time is a Minkowski wormhole, since taking the limit of vanishing mass, $M \to 0$, can not change the space-time's global topology, cf. [95].

5.4.1. Multipolar Equations of Motion

The multipolar expansion scheme for equations of motion that we briefly outline below was initiated by Mathisson in 1937 [96] (see [97] for a republication of the English translation), improved by Papapetrou [98] and Tulczyjew [99] in the 1950s, and subsequently extended and covariantised by Dixon in the 1970s [100–103]. (For details, see the recent comprehensive historical account [104] by Dixon himself)

The general equation of motion for any body in general relativity must follow from the covariant conservation of its stress-energy-momentum tensor (5.7). An extended body traces out a world-tube as it moves through space-time, and one generally assumes that the body's stress-energy-momentum tensor $T^{\mu\nu}$ is non-vanishing only within this world-tube. One then replaces the differential energy-momentum conservation relation (5.7) by its equivalent integral form, and rewrites the continuous stress-energy-momentum density of the extended body in terms of a multi-polar expansion around some initially arbitrary, but suitably defined fiducial world-line within its world-tube, in terms of suitably defined multipole moments. In a somewhat involved and technical procedure (which has actually only ever been carried up to quadrupolar order), this then gives rise to a hierarchy of coupled equations of motion for the different multipole moments of the body. Truncating this hierarchy at zeroth order, one thereby obtains the equation of motion for the body's *kinematical* momentum $p^\sigma := mv^\sigma$,

$$p^\sigma \nabla_\sigma p^\mu = 0\,, \tag{5.50}$$

which is just the geodesic equation (4.37), that represents the trivial monopole member in this hierarchy. While its solution yields the first approximation to a general extended body's physical world-line, it will generally be modified by the coupling of the body's multipole moments to those of space-time curvature, evaluated at the fiducial reference world-line, as we shall for the next member in said hierarchy of equations of motion below. Since for the Earth these corrections are very small, we shall neglect them after the following subsection, where we exhibit the next and first non-trivial order in the hierarchy of coupled equations of motion for the extended particle's multipole moments, namely that of the particle's classical spin. Besides serving as an example, this so-called Mathisson-Papapetrou-Tulczyjew-Dixon equation is a convenient starting point, e.g. for the discussion of idealised gyroscopes.

5.4.2. Pole-Dipole Approximation: The Mathisson-Papapetrou-Dixon Equation

Dixon defines the above-mentioned multipole moments in terms of integrals over the leaves of a spacial foliation of the body's world-tube [100]. At the so-called pole–dipole order, these are: the body's *canonical*, i.e. total momentum P^σ ("pole"), and its antisymmetric spin tensor $S^{\mu\nu}$ ("dipole"); their equations of motion being the Mathisson-Papapetrou-Dixon (MPD) equation,

$$\frac{DP^\sigma}{d\tau} = -\tfrac{1}{2} R^\sigma{}_{\rho\mu\nu} v^\rho S^{\mu\nu} := F^\sigma_{\mathrm{M}}\,, \tag{5.51a}$$

$$\frac{DS^{\mu\nu}}{d\tau} = P^\mu v^\nu - v^\mu P^\nu\,, \tag{5.51b}$$

where P^σ is defined in terms of p^σ and $S^{\mu\nu}$ as,

$$P^\sigma = p^\sigma + \frac{DS^{\sigma\lambda}}{d\tau} v_\lambda \,. \tag{5.52}$$

The right-hand-side of (5.51a), here denoted F^σ_{M}, is the spin- or *Mathisson force*, i.e. in a local Lorentz frame it has the interpretation of an additional force on the particle due to lowest-order coupling of the particle's multipole moments with those of space-time curvature (here $S^{\mu\nu}$ with $R^\sigma{}_{\rho\mu\nu}$)*. The above system of equations (5.51) is not closed, since there are only 10 equations (4 for P^σ, 6 for $S_{\mu\nu}$), but 13 unknown functions (3 additional degrees of freedom of the tangent v^σ to the fiducial world-line), so one has to impose an additional *spin supplementary condition*, e.g. $S_{\mu\nu}v^\nu = 0$ (Pirani), or $S_{\mu\nu}P^\nu = 0$ (Tulczyjew), etc. (See [105] for a survey of the various spin supplementary conditions and their relationship, and [106] for an analysis of their respective connection with the corresponding centre of mass†.)

This above-mentioned coupling of the body's spin moment with local space-time curvature means that it's centre of mass (which itself also becomes ambiguous in general) deviates from its usual geodesic trajectory. The magnitude of the spin tensor is determined by the spin scalar $S^2 \coloneqq \frac{1}{2}S^{\mu\nu}S_{\mu\nu}$, and since typical classical spinning systems, such as gyroscopes, have a comparatively small S (cf. [108]), one will usually linearise the MPD equations. This so-called linear-in-spin approximation of (5.51a) then reads,

$$m\frac{Du^\sigma}{d\tau} = -\tfrac{1}{2}R^\sigma{}_{\rho\mu\nu}v^\rho S^{\mu\nu} \,, \tag{5.53}$$

which is utilised, e.g., in [108] to calculate the above-mentioned spin-geodesic deviation in Schwarzschild space-time. Let us also note, that it has been shown that the generally covariant Dirac equation essentially reduces to the Mathisson-Papapetrou-Dixon equations in non-relativistic limit, as shown by [109, 110] and others.

5.4.3. Constants of Motion and Geodesic Equations

Before we come to concrete examples for geodesics in Schwarzschild space-time, we find it instructive to start out more generally by outlining the Lagrangean and Hamiltonian analysis that leads to the equations of motion for general metrics and then for the axially symmetric metric (5.30) in terms of the particular constants of motion in that space-time.

From the general expression for the line element, we obtain the normalisation of the four-velocity by writing $d\lambda^2 = c^2 d\tau^2$ and dividing through with $d\tau^2$,

$$g_{\mu\nu}\dot{X}^\mu\dot{X}^\nu = c^2 \,, \qquad \text{with } \dot{X}^\mu = \frac{dX^\mu}{d\tau} \,. \tag{5.54}$$

*In fact, the Mathisson force can be interpreted as the classical gravitational analogue of the Stern-Gerlach force in electrodynamics.

† To the reader interested in the subject of general equations of motion, we highly recommend the recent and comprehensive textbook *Equations of Motion in Relativistic Gravity* edited by Pützfeld, Lämmerzahl and Schutz [107].

The Lagrangean for test-particle motion then follows by multiplication with $\frac{1}{2}m$, where m is the mass of the test particle. We thus have

$$L = \tfrac{1}{2}m\, g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu = \tfrac{1}{2}mc^2\,, \tag{5.55}$$

where we would like to note two things: First, the Lagrangean is identical with the Hamiltonian, $H = L$, since general relativity is a purely kinetic theory in the sense that there are no potentials. Secondly, the Lagrangean and Hamiltonian are themselves constants of the motion. Instead of with the proper Lagrangean (5.55), it is customary to drop the factor $m/2$ and work directly with the four-velocity normalisation (5.54), taking this as a "modified Lagrangean" $\tilde{L}$ having dimensions of velocity squared, which makes its conjugate momenta $\tilde{P}$ to momenta per unit mass m, i.e., they have dimensions of velocity. Thus, these "modified" conjugate momenta $\tilde{P}$ of (5.54) are then obtained in the usual way,

$$\tilde{P}_\mu = \frac{\partial \tilde{L}}{\partial \dot{X}^\mu} = g_{\mu\nu}\dot{X}^\nu\,. \tag{5.56}$$

Thus, for the axially symmetric metric in (5.30), we have the Lagrangean

$$\tilde{L} = \left[g_{TT}(c\dot{T})^2 + g_{RR}\dot{R}^2 + g_{\Theta\Theta}\dot{\Theta}^2 + g_{\Phi\Phi}\dot{\Phi}^2 + 2g_{T\Phi}(c\dot{T})\dot{\Phi}\right] = c^2\,. \tag{5.57}$$

Since the space-time is stationary and axisymmetric, this Lagrangean does not depend on T and Φ, which also follows from inspection of the Euler-Lagrange equations,

$$\frac{\partial \tilde{L}}{\partial X^\mu} = \frac{d}{d\tau}\left(\frac{\partial \tilde{L}}{\partial \dot{X}^\mu}\right) = \frac{dX^\mu}{d\tau}\,, \tag{5.58}$$

and thus we have that these coordinates are cyclic and the corresponding canonical momenta (5.56) are conserved, i.e. they are constants of motion. The lack of T-dependence of the metric functions thus implies that total energy $\mathcal{E}$, given by

$$P_T = \frac{\mathcal{E}}{mc} = g_{TT}\left(\frac{c\,dT}{d\tau}\right) + g_{T\Phi}\left(\frac{d\Phi}{d\tau}\right), \tag{5.59a}$$

is a conserved quantity, and finally, the missing Φ-dependence leads to angular momentum around the axis of symmetry, L_Z per unit mass, being conserved, i.e.,

$$P_\Phi = l \coloneqq \frac{L_Z}{m} = g_{\Phi\Phi}\dot{\Phi} + g_{T\Phi}\,(c\dot{T})\,. \tag{5.59b}$$

Since the axisymmetric space-time we are dealing with has the important property of being asymptotically flat, i.e. it reduces to flat Minkowski space-time, $g_{\mu\nu} \to \eta_{\mu\nu}$ for $R \to \infty$, the quantities $\mathcal{E}$ and L_Z can be straightforwardly interpreted respectively as the test particle's energy at infinity which is related to the usual Keplerian energy, and axial angular momentum at infinity.

From (5.57), we now have four equations of motion, which can generally be written as

$$\left(\frac{dT}{d\tau}\right)^2 = F_T(R,\Theta), \qquad \left(\frac{dR}{d\tau}\right)^2 = F_R(R,\Theta), \qquad \left(\frac{d\Theta}{d\tau}\right)^2 = F_\Theta(R,\Theta), \qquad \left(\frac{d\Phi}{d\tau}\right)^2 = F_\Phi(R,\Theta), \tag{5.60}$$

where the right-hand sides F_T, F_R, F_Θ, and F_Φ are written in terms of the canonical momenta (5.59) and the constants of motion in (5.59a) and (5.59b). This is a coupled system of four ordinary differential equations. In order to separate it, one generally needs four conserved quantities, of which three are already given by the normalisation of the four-velocity (5.54), together with P_T and P_Φ, which result from the two Killing vectors. Fortunately, there exists a fourth conserved quantity, called the *Carter constant* [111, 112], due to the existence of a higher-order symmetry (a rank-two Killing-Yano tensor) in all space-times of Petrov type D (cf. subsection 5.6.4), which essentially includes the present class of stationary and axisymmetric ones.

Their solutions are then written in terms of integrals, most of which can actually be solved exactly [113, 114] in terms of *coordinate time* T and with the help of elliptic and hyper-elliptic functions and integrals, using algebro-geometric methods [115–117]. Unfortunately for our purposes, it turns out that nobody knows how to solve the above integrals for $T(\tau)$, $R(\tau)$, $\Theta(\tau)$, and $\Phi(\tau)$ in terms of *proper time* τ.* For this reason, we will resort to briefly discussing only radial Schwarzschild geodesics in subsection 5.4.5, where the relation $T(\tau)$ is straightforward to work out, but can only be given in terms of a series, and later in subsection 5.5.5 focus on the physically interesting situation of circular geodesics, where this relationship is trivial.

5.4.4. Geodesics of the Schwarzschild Metric

In Schwarzschild space-time we have from (5.59a), as well as (5.59b), the following expressions for the conserved quantities

$$\frac{\mathcal{E}}{mc^2} = B(R)\frac{dT}{d\tau}\,, \qquad \text{and} \qquad l = -R^2\sin^2(\Theta)\frac{d\Phi}{d\tau}\,, \tag{5.61}$$

Dividing the Schwarzschild line element (5.35) by $d\tau^2$, we may substitute the two constants of motion (5.61) to obtain

$$c^2 = B(R)^{-1}\frac{\mathcal{E}^2}{m^2c^2} - B(R)^{-1}\left(\frac{dR}{d\tau}\right)^2 - \left(\frac{l}{R\sin\Theta}\right)^2, \tag{5.62}$$

where we can divide by $B(R)$ and rearrange to obtain the radial equation for test-particle motion in Schwarzschild space-time,

$$\left(\frac{dR}{d\tau}\right)^2 = \frac{\mathcal{E}^2}{m^2c^2} - B(R)\Big[c^2 + \Big(\frac{l}{R\sin\Theta}\Big)^2\Big]\,. \tag{5.63}$$

*Private communication with E. Hackmann

Expanding out the right-hand side, we then have the more familiar form with

$$\left(\frac{dR}{d\tau}\right)^2 = \frac{\mathcal{E}^2 - m^2c^4}{m^2c^2} + \frac{2GM}{R} - \frac{l^2}{R^2} + \frac{R_\mathrm{S}}{R}\left(\frac{l}{R\sin\Theta}\right)^2, \tag{5.64}$$

where the correspondence with the radial equation of the Kepler problem of Newtonian physics is exhibited most clearly. Indeed, if we restrict the motion to the equatorial plane by setting $\Theta = \pi/2$ and put

$$m = \mu\,, \qquad \frac{2E}{\mu} = \frac{\mathcal{E}^2 - m^2c^4}{m^2c^2}\,, \qquad R_\mathrm{S} \to \infty\,, \tag{5.65}$$

where μ is the usual reduced mass in the two-body problem, we obtain the Newtonian radial equation

$$\dot{R}^2 = \frac{2E}{\mu} + \frac{2GM}{R} - \left(\frac{l}{R}\right)^2, \tag{5.66}$$

(see, e.g. the comprehensive and recent textbook [118, Sec. 3.2.3] for a nice discussion), so we find that for the Schwarzschild case, the radial equation of the Kepler problem is modified by a dimensionless curvature correction R_S/R containing the Schwarzschild radius. It is this additional $1/R^3$ term that turns the integral corresponding to (5.64) into an elliptic integral. Apart from this term, the discussion of test-particle motion in Schwarzschild space-time closely parallels that of the Kepler problem.

5.4.5. Radial Geodesics

For purely radial motion, we have Φ = const. and may also restrict ourselves to the equatorial plane, i.e. we set $\Theta = \pi/2$ in (5.35), or equivalently, $l = 0$ in the radial equation (5.63), which leaves us with

$$\left(\frac{dR}{d\tau}\right)^2 = \left(\frac{\mathcal{E}}{mc}\right)^2 - B(R)c^2\,, \tag{5.67}$$

where we can express the energy $\mathcal{E}$ through the (constant) drop height R_0, by using the initial condition $\dot{R}|_{R=R_0} = 0$,

$$\frac{\mathcal{E}}{mc^2} = \sqrt{B(R_0)}\,. \tag{5.68}$$

The first equation in (5.61) for the time-like Killing vector then becomes

$$\frac{dT}{d\tau} = \frac{\sqrt{B(R_0)}}{B(R)}\,, \tag{5.69}$$

and the radial equation becomes

$$\left(\frac{dR}{d\tau}\right)^2 = c^2\left[B(R_0) - B(R)\right] = \frac{2GM}{R_0}\left(\frac{R_0}{R} - 1\right). \tag{5.70}$$

Equation (5.70), in turn, has the form of the differential equation of a cycloid, the solution of which is given in terms of a cycloid parameter ξ by

$$\tau(\xi) = \frac{R_0}{2}\sqrt{\frac{R_0}{2GM}}(\xi + \sin\xi)\,, \tag{5.71}$$

$$R(\xi) = \frac{R_0}{2}(1+\cos\xi) = R_0\cos^2(\xi/2)\,. \tag{5.72}$$

Inverting the series representation of (5.71) and inserting into (5.72) yields the solution $R(\tau)$

$$\begin{aligned} R(\tau) &= R_0 - \frac{GM}{2R_0^2}\tau^2 - \frac{(GM)^2}{12\,R_0^5}\tau^4 + \mathscr{O}(\tau^6) \\ &= R_0 - \frac{g_\mathrm{N}}{2}\tau^2 - \frac{g_\mathrm{N}^2}{12\,R_0}\tau^4 + \mathscr{O}(\tau^6)\,, \end{aligned} \tag{5.73}$$

with $g_\mathrm{N} \equiv g_\mathrm{N}(R_0) = GM/R_0^2$ denoting the local gravitational acceleration at R_0.

The similarity between the Newtonian expression $R_\mathrm{Newton}(t)$ and the general relativistic one (5.73) allows us to compare the two cases for an observer in radial free fall in Newtonian gravity and in the Schwarzschild field, respectively. In order to do this, we use the asymptotic flatness of Schwarzschild space-time, and thus the fact that the time coordinate T becomes Minkowskian (and thus equivalent to Newtonian) time for $R \to \infty$, so the comparison to should be done with (5.73) expressed in terms of Schwarzschild coordinate time T instead. In order to derive an expression for $R(T)$, we start by differentiating (5.71),

$$\frac{d\tau}{d\xi} = \frac{R_0}{2}\sqrt{\frac{R_0}{2GM}}(1+\cos\xi)\,, \tag{5.74}$$

and this, together with (5.69), yields a differential equation for the coordinate time $T(\xi)$,

$$\frac{dT}{d\xi} = \frac{R_0}{2}\sqrt{\frac{R_0}{2GM}}\frac{\sqrt{B(R_0)}}{B(R)}(1+\cos\xi)\,, \tag{5.75}$$

which, upon substitution of (5.72), leads to the following integral,

$$T(\xi) = \frac{R_0}{2}\sqrt{\frac{R_0}{2GM}}\sqrt{B(R_0)}\int_0^\xi d\xi'\,\frac{[1+\cos\xi']^2}{\alpha+\cos\xi'}\,, \tag{5.76}$$

where we have abbreviated $\alpha = 1 - 2R_\mathrm{S}/R_0$. This integral can be solved, yielding

$$\begin{aligned} T(\xi) = \frac{R_0}{2}\sqrt{\frac{R_0}{2GM}}\sqrt{B(R_0)}\Bigg[&\Big(1-\frac{2R_\mathrm{S}}{R_0}\Big)\xi + \sin(\xi) \\ &+ \frac{\Big(1-\frac{2R_\mathrm{S}}{R_0}\Big)^2}{2\sqrt{\frac{R_\mathrm{S}}{R_0}}\sqrt{1-\frac{R_\mathrm{S}}{R_0}}}\ln\left(\frac{\sqrt{R_0/R_\mathrm{S}-1}+\tan(\xi/2)}{\sqrt{R_0/R_\mathrm{S}-1}-\tan(\xi/2)}\right)\Bigg]\,, \end{aligned} \tag{5.77}$$

so what remains is to express this solution in terms of R, for which we can obtain an expression for the cycloid parameter $\xi(R)$ from (5.72), i.e.,

$$\xi(R) = 2\arccos\Bigl(\sqrt{R/R_0}\Bigr)\,. \tag{5.78}$$

With this, equation (5.77) is finally found to read,

$$\begin{aligned} T(R) = \frac{R_0}{2}\sqrt{\frac{R_0}{2GM}}\sqrt{B(R_0)}\Biggl[\Bigl(1-\frac{2R_\mathrm{S}}{R_0}\Bigr)2\arccos\Biggl(\sqrt{\frac{R}{R_0}}\Biggr) + 2\sqrt{\frac{R}{R_0}\Bigl(1-\frac{R}{R_0}\Bigr)} \\ + \frac{\Bigl(1-\frac{2R_\mathrm{S}}{R_0}\Bigr)^2}{2\sqrt{\frac{R_\mathrm{S}}{R_0}}\sqrt{1-\frac{R_\mathrm{S}}{R_0}}}\ln\Biggl(\frac{\sqrt{R_0/R_\mathrm{S}-1}+\sqrt{R_0/R-1}}{\sqrt{R_0/R_\mathrm{S}-1}-\sqrt{R_0/R-1}}\Biggr)\Biggr]\,. \end{aligned} \tag{5.79}$$

The best we can do now to invert this expression is inverting in terms of series. This yields the solution of the radial equation for radial free fall in Schwarzschild space-time (5.73), but expressed in terms of coordinate time T, which reads

$$R(T) = R_0 - \tfrac{1}{2}g_\mathrm{N}T^2 - \tfrac{1}{12}\frac{g_\mathrm{N}^2(1-3R_\mathrm{S}/R_0)}{B(R_0)R_0}T^4 + \mathscr{O}(T^6)\,. \tag{5.80}$$

5.5. Inertial and Non-Inertial Observers in General Relativity

After having introduced some of the basics of general relativity and the relevant exact metrics in the beginning of the present chapter, we now come to the question of how to represent physical observers, more specifically, the frames of idealised inertial and non-inertial observers in the curved space-time of general relativity in terms of tetrads. In the context of special relativity in general frames as discussed in subsection 4.2.2, we had introduced the tetrad (4.40) in terms of the Jacobian matrix of the particular coordinate transformation that yields the flat Minkowski metric when applied to $g_{\mu\nu}$, which we interpreted as the defining orthonormality relations (4.42) for the tetrad. Since the discussion in subsection 4.2.2 was covariant and done in terms of general coordinates, it is clear that all formulae of that subsection will be valid also in the present general relativistic context. Following up on the discussion in section 5.2, we shall always assume that the space-time metric $g_{\mu\nu}$ is given in some a-priori and (ideally and typically) symmetry-adapted coordinates X^σ. In the context of local inertial frames and Riemann and Fermi coordinates, i.e. in much of the rest of this thesis, this given metric will be referred to as *background metric*, and similarly, the a-priori coordinates X^σ will be called *background coordinates*.

As before, we will be concerned with a tetrad that is defined along a time-like world-line $\mathscr{W}$, which includes the possibility of a rotation of the spacial tetrad vectors on the one hand, and of the whole frame being accelerated on the other, where – of course – both can occur simultaneously. We will always assume that the tetrad's time-like vector is chosen parallel to the four velocity u^μ along $\mathscr{W}$, according to (4.46).

5.5.1. Metric-Adapted Tetrad

With our background metric given, we can then employ the tetrad orthonormality condition (4.42) to obtain an initial *metric-adapted* tetrad ${e_{\hat{\alpha}}}^{\mu}(X^{\sigma})$. Since the background metric is written in the a-priori coordinates X^{σ}, the metric-adapted tetrad's time-like vector ${e_{\hat{0}}}^{\mu}$ points, by construction, along the time coordinate lines X^0 of the background space-time, i.e. it is parallel to the time vector $\partial_0 = \partial/\partial X^0$ of the metric's coordinate basis. Physically, this means that the so-constructed tetrad is adapted to the specific observer that, in turn, the a-priori coordinate basis of the background metric is adapted to implicitly. For example, the Schwarzschild metric (5.35) in standard Schwarzschild coordinates (5.34) represents an observer who is static, i.e., at rest with respect to the source of the Schwarzschild field (the Earth, for our purposes), sitting at a fixed value of the Schwarzschild radial coordinate R above the Earth's surface, with his local time direction parallel to the corresponding time coordinate $X^0 = cT$.

In the context of exact metrics, the metric-adapted tetrad is usually comparatively simple, as long as the background metric is given in symmetry-adapted coordinates, which will always be the case in practice and which is certainly true for the class of stationary axisymmetric metrics of relevance for the description of astrophysical bodies as elaborated in section 5.3. Because of this comparative simplicity, the metric-adapted tetrad thus serves as a convenient starting point for all calculations, the simplest example being [apart from the Minkowski tetrad in spherical coordinates (4.134), which it generalises] the case of a diagonal metric such as the Schwarzschild one, where the adapted tetrad is diagonal and can thus be directly read off the metric in terms of inverse square roots of its components. We will treat the Schwarzschild metric-adapted tetrad or "natural static frame of the Schwarzschild observer" in Equation 5.82 below. If the metric is nearly diagonal, such as for the Kerr metric, some more labour is needed, however we note that for the Kerr metric (and similarly for all "non-accelerating" vacuum metrics of Petrov type D [119]) there exists a standard, symmetric choice of metric-adapted tetrad due to Carter [120].

5.5.2. Tetrads Adapted to General Observers

Although so far we have typically obtained a simple initial tetrad, we are usually not interested in an inertial frame for the corresponding metric-adapted observer himself, but for a different *target observer*, who will then be adapted to a certain target world-line, such as, e.g., a circular geodesic. Since according to (4.44), two tetrads at the same space-time point are connected by Lorentz transformations that act on the tetrad index, all we have to do for this step is to adapt the background tetrad's Lorentz frame to that of the target observer. This is then accomplished by Lorentz-boosting it appropriately, so that – as visualised in a space-time diagram – its time-like vector ${e_{\hat{0}}}^{\mu}$ comes to lie in the direction of the target observer's four-velocity.

While generally, the target observer can be made inertial by choosing the boost parameter β, and therefore the target observer's velocity such as to make his four-acceleration vanish, the boost will generally also result in a rotating tetrad, i.e. the boosted tetrad is not initially parallel transported In order to determine the transport properties of a tetrad, we then have to calculate its transport equation (4.48). Although this might be very difficult for general space-times, Marck [121] has shown that at least in the Schwarzschild and

Kerr space-times (and also in other Petrov-type-D space-times), a parallel transported, i.e. non-rotating tetrad can always be constructed from the metric-adapted one, owing to the special separability property of these space-times discovered by Carter in 1968 [111].

5.5.3. Metric-Adapted Frames for Schwarzschild and Kerr Space-Time

We start the calculation of tetrads that are adapted to more general observers, by first exhibiting the respective metric-adapted frames in Schwarzschild and Kerr space-times, and briefly discussing the properties of the attached observers.

Kerr

From the second, more compact form of the Kerr line element in Boyer-Lindquist coordinates (5.44), one easily reads off the *inverse* tetrad,

$$e^{\hat{\alpha}}{}_{\sigma} = \begin{pmatrix} \frac{\sqrt{\varDelta}}{\rho} & 0 & 0 & -\frac{\sqrt{\varDelta}}{\rho} a \sin^2\Theta \\ 0 & \frac{\rho}{\sqrt{\varDelta}} & 0 & 0 \\ 0 & 0 & \rho & 0 \\ -\frac{a\sin\Theta}{\rho} & 0 & 0 & \frac{\sin\Theta}{\rho}(R^2+a^2) \end{pmatrix}, \tag{5.81a}$$

which is block-diagonal and thus easily inverted, resulting in the following metric-adapted tetrad,

$$e_{\hat{\alpha}}{}^{\sigma} = \begin{pmatrix} \frac{(R^2+a^2)}{\rho\sqrt{\varDelta}} & 0 & 0 & \frac{a\sin\Theta}{\rho} \\ 0 & \frac{\sqrt{\varDelta}}{\rho} & 0 & 0 \\ 0 & 0 & 1/\rho & 0 \\ \frac{a}{\rho\sqrt{\varDelta}} & 0 & 0 & \frac{1}{\rho\sin\Theta} \end{pmatrix}. \tag{5.81b}$$

The above tetrads (5.81b), (5.81) are known as the *Carter frame* [111] already mentioned in subsection 5.5.2 above, i.e., the unique frame that is adapted to the curvature (aligned with the in- and outgoing principal null directions) in Kerr space-time, that makes the Weyl tensor diagonal in a sense discussed in subsection 5.6.4 below.

Schwarzschild

The metric-adapted tetrad that can be read off the Schwarzschild metric corresponds physically to the natural static frame of the Schwarzschild observer,

$$e_{\hat{\alpha}}{}^{\mu} = \begin{pmatrix} \frac{1}{\sqrt{B(R)}} & 0 & 0 & 0 \\ 0 & \sqrt{B(R)} & 0 & 0 \\ 0 & 0 & \frac{1}{R} & 0 \\ 0 & 0 & 0 & \frac{1}{R\sin\Theta} \end{pmatrix}, \tag{5.82}$$

which can also be obtained by taking the Schwarzschild limit, $a \to 0$, of the Kerr metric's adapted tetrad (5.81b) upon using the definitions of ρ and $\varDelta$ in (5.43). The natural static Schwarzschild observer's world-line is just $X^\kappa(\tau) = (\varGamma_{\text{stat}}\tau, R_0, \Theta_0, \varPhi_0)$, his four velocity being given simply by,

$$u^{\mu}_{\text{stat}} = \mathit{\Gamma}_{\text{stat}}(c,\, 0,\, 0,\, 0)\,, \qquad \text{and where} \quad \mathit{\Gamma}_{\text{stat}} := \frac{1}{\sqrt{B(R)}} \tag{5.83}$$

is his coordinate red-shift factor. This observer is accelerated, as can be seen by computing his (coordinate) four-acceleration (4.37), the only non-vanishing component of which is the radial one,

$$a^1 = \Gamma^1{}_{00}\big(u^0_{\text{stat}}\big)^2 = \frac{GM}{R^2}\,, \tag{5.84}$$

being just the Newtonian value. The sign on the right-hand side is positive, consistent with the fact that the static observer must accelerate radially away from the Earth, countering the "pull of gravity", in order to remain hovering at a fixed position, e.g. by "firing his rocket engines". The static observer's *proper* acceleration then follows by taking frame components with respect to the inverse of (5.82),

$$a^{\hat{1}} = e^{\hat{1}}{}_{\sigma}a^{\sigma} = \frac{GM}{R^2\sqrt{B(R)}}\,. \tag{5.85}$$

5.5.4. Schwarzschild Radial Free-Fall Frame

We now come to the frame adapted to the simplest geodesics of the Schwarzschild metric, namely the (infalling) radial world-lines of the radial free-fall observer discussed in subsection 5.4.5. They are constructed by Lorentz-boosting the static metric-adapted tetrad (5.82) in negative radial direction, i.e,

$$e_{\hat{\alpha}'}{}^{\mu} = \Lambda_{\hat{\alpha}'}{}^{\hat{\alpha}}e_{\hat{\alpha}}{}^{\mu}\,, \tag{5.86}$$

which yields the following tetrad of the radial-free-fall frame,

$$e_{\hat{\alpha}'}{}^{\mu} = \begin{pmatrix} \frac{\gamma}{\sqrt{B(R)}} & \beta\gamma\sqrt{B(R)} & 0 & 0 \\ \beta\gamma\sqrt{B(R)} & \frac{\gamma}{\sqrt{B(R)}} & 0 & 0 \\ 0 & 0 & \frac{1}{R} & 0 \\ 0 & 0 & 0 & \frac{1}{R\sin\mathit{\Theta}} \end{pmatrix}, \tag{5.87}$$

where the radial boost velocity and associated gamma factor are given in terms of the radial velocity (5.70) by,

$$\beta_{\text{rad}} = -\sqrt{R_{\text{S}}\Big(\frac{1}{R} - \frac{1}{R_0}\Big)}\,, \qquad \gamma_{\text{rad}} = \frac{1}{\sqrt{1 - \frac{R_{\text{S}}}{R} + \frac{R_{\text{S}}}{R_0}}}\,, \tag{5.88}$$

and where $R(\tau)$ is the solution (5.73) of the radial equation (5.70).

5.5.5. Uniformly Rotating Circular World-Lines and their Adapted Frames

Apart from the purely radial geodesics that we discussed in subsection 5.4.5, there is another class of world-lines with considerable physical significance where the equations of motion are actually trivial to solve, namely that of uniformly rotating circular world-lines that we already encountered in the context of Minkowski space-time in subsection 4.3.5.

Circular World-Lines as Motion along Killing Trajectories

Observers in uniform circular motion around a (possibly rotating) body who's gravitational field is modelled in terms of a stationary axially symmetric metric, are moving on the symmetry surface spanned by the two Killing vectors $\boldsymbol{\xi}_{(T)}$ and $\boldsymbol{\xi}_{(\Phi)}$, i.e. in adapted coordinates, on the coordinate cylinders spanned by the two symmetry-adapted basis vectors $\boldsymbol{\partial}_T$ and $\boldsymbol{\partial}_\Phi$, so that their four-velocity $\boldsymbol{u}$ can be written as a constant linear combination of these, which is then a Killing vector too*. Since the motion is along symmetry orbits of the metric, all parameters that describe these world-lines, such as the velocity parameter, the four-acceleration, and the rotation of local frames, are constants. On the one hand, this makes circular world-lines especially easy to work with. On the other, they still exhibit all the features of bound accelerated, or geodesic orbital motion, such as precession of the local frame vectors with respect to the fixed stars, and most importantly for our purposes, they can readily be used to (approximately) model the orbits of some of the most important Earth satellites such as the International Space Station (ISS) that is on an almost circular orbit.

Parametrisations of Circular World-Lines

The angular motion of observers on these circular world-lines may be parametrised either in terms of the time coordinate of the co-moving local frame, i.e, proper time τ, or in terms of coordinate time T. Their four-velocity vector is then written in the first case in terms of a Lorentz boost of the static or stationary metric-adapted tetrad in the Φ-direction, with boost parameter $\beta_{\text{circ}} = \frac{d\Phi}{d\tau}$, and in the second case, in terms of the constant angular velocity $\Omega = \frac{d\Phi}{dT}$ with respect to infinity (i.e., to the fixed stars) as,

$$\boldsymbol{u} = \Gamma\big[\boldsymbol{\partial}_T + \Omega\,\boldsymbol{\partial}_\Phi\big] = \gamma\big[\boldsymbol{e}_{\hat{0}} + \beta\,\boldsymbol{e}_{\hat{\Phi}}\big]\,, \tag{5.89}$$

(cf., e.g. Bini, Cherubini, Geralico and Jantzen [122]). Just as in (4.130) in subsection 4.3.5, we shall begin by transforming the line element to rotating coordinates, however here we start with the general stationary and axisymmetric line element (5.30). Defining $\Phi'(T) = \Phi + \Omega T$ for Ω a constant angular coordinate-velocity, we obtain the differentials,

$$\begin{aligned} d\Phi &= d\Phi' - \Big(\frac{\Omega}{c}\Big)(c\,dT)\,,\\ d\Phi^2 &= d\Phi'^2 - 2\Big(\frac{\Omega}{c}\Big)d\Phi'\,(c\,dT) + \Big(\frac{\Omega}{c}\Big)^2(c\,dT)^2\,. \end{aligned} \tag{5.90}$$

Inserting this into the general stationary stationary axisymmetric line element (5.30), yields the metric in rotating coordinates adapted to observers on circular world-lines, which then orbit the origin of background coordinates,

*In fact, this holds also for so-called *quasi-Killing vectors* (cf. Iyer and Vishveshwara [72]), where the angular-velocity parameter Ω is not constant, but only Lie dragged, i.e. $\mathscr{L}_{\boldsymbol{\xi}_{(T)}}\,\Omega = 0$.

$$d\lambda^2 = \varGamma^{-2}(c\,dT)^2 + 2\Big[g_{T\varPhi} - g_{\varPhi\varPhi}\Big(\frac{\Omega}{c}\Big)\Big](c\,dT)d\varPhi' + g_{RR}dR^2 + g_{\varTheta\varTheta}d\varTheta^2 + g_{\varPhi\varPhi}(d\varPhi')^2\,, \tag{5.91}$$

where the *coordinate gamma factor* or *coordinate red-shift factor* $\varGamma_{\mathrm{circ}}$ is the normalisation factor of a four-velocity that is adapted to circular world-lines in the general stationary axisymmetric metric (5.30), which can also be determined from the normalisation relation of the four-velocity as usual, yielding,

$$\varGamma := \Big[g_{TT} + 2g_{T\varPhi}\Big(\frac{\Omega}{c}\Big) + g_{\varPhi\varPhi}\Big(\frac{\Omega}{c}\Big)^2\Big]^{-1/2}. \tag{5.92}$$

In Schwarzschild space-time, the line element in rotating coordinates (5.91) reduces to,

$$d\lambda^2 = \Big[B(R) - \Big(\frac{\Omega}{c}\Big)^2(R\sin\varTheta)^2\Big](c\,dT)^2 + 2\Big(\frac{\Omega}{c}\Big)(R\sin\varTheta)^2 d\varPhi'\,(cdT) \\ - \frac{dR^2}{B(R)} - R^2\left[d\varTheta^2 + \sin^2(\varTheta)d\varPhi^2\right]. \tag{5.93}$$

and the ansatz (5.93) for the four-velocity along circular world-lines reads,

$$u^{\mu}_{\mathrm{circ}} = c\gamma\left(\frac{1}{\sqrt{B(R)}}, 0, 0, \frac{\beta_{\mathrm{circ}}}{R\sin\varTheta}\right) = \varGamma\,(c, 0, 0, \Omega)\,, \tag{5.94}$$

respectively in the two parametrisations, where the general coordinate red-shift factor in (5.92) reduces to

$$\varGamma := \frac{dT}{d\tau} = \Big[B(R) - \Big(\frac{\Omega}{c}\Big)^2(R\sin\varTheta)^2\Big]^{-1/2}. \tag{5.95}$$

Note, that the velocity parameters in these two parametrisations are related by,

$$\beta_{\mathrm{circ}} = \Big(\frac{\Omega}{c}\Big)\frac{R\sin(\varTheta)}{\sqrt{B(R)}}\,, \tag{5.96}$$

which leads to the following relation between the corresponding red-shift factors,

$$\gamma = \varGamma\sqrt{B(R)} = \frac{\sqrt{B(R)}}{\sqrt{1 - (\frac{\Omega}{c})^2(R\sin\varTheta)^2}} = \frac{1}{\sqrt{1-\beta^2_{\mathrm{circ}}}}\,. \tag{5.97}$$

Having established the different natural parametrisations of the motion, we turn to the tetrad that is adapted to the observers following the circular world-lines. Like the four-velocity, this tetrad too is obtained by Lorentz boosting the natural static Schwarzschild observer's tetrad (5.82) along the $\varPhi$-direction of the Schwarzschild coordinates. This yields,

$$\bar{e}_{\hat{\alpha}}{}^{\mu} = \begin{pmatrix} \varGamma & 0 & 0 & \varGamma(\frac{\Omega}{c}) \\ 0 & \sqrt{B(R)} & 0 & 0 \\ 0 & 0 & \frac{1}{R} & 0 \\ \varGamma(\frac{\Omega}{c})\frac{R\sin(\varTheta)}{\sqrt{B(R)}} & 0 & 0 & \varGamma\frac{\sqrt{B(R)}}{R\sin(\varTheta)} \end{pmatrix}, \tag{5.98}$$

with inverse

$$\bar{e}^{\hat{\alpha}}{}_{\mu} = \begin{pmatrix} \mathit{\Gamma}\frac{\sqrt{B(R)}}{R\sin(\mathit{\Theta})} & 0 & 0 & -\mathit{\Gamma}\left(\frac{\Omega}{c}\right) \\ 0 & \frac{1}{\sqrt{B(R)}} & 0 & 0 \\ 0 & 0 & R & 0 \\ -\mathit{\Gamma}\left(\frac{\Omega}{c}\right)\frac{R\sin(\mathit{\Theta})}{\sqrt{B(R)}} & 0 & 0 & \mathit{\Gamma} \end{pmatrix} . \tag{5.99}$$

Observers following circular world-lines are accelerated in general. Their four-acceleration is easily calculated to be,

$$\begin{aligned} a^1_{\text{circ}} &= \Gamma^1{}_{00}\left(u^0_{\text{circ}}\right)^2 + \Gamma^1{}_{33}\left(u^3_{\text{circ}}\right)^2 &&= \frac{GM}{R^2}\mathit{\Gamma}^2 B(R)\Big[1 - 2\frac{R^3}{R_{\text{S}}}\left(\frac{\Omega}{c}\right)^2\sin^2\mathit{\Theta}\Big] , \\ a^2_{\text{circ}} &= \Gamma^2{}_{33}\left(u^3_{\text{circ}}\right)^2 &&= -\mathit{\Gamma}^2\Omega^2\sin(\mathit{\Theta})\cos(\mathit{\Theta}) . \end{aligned} \tag{5.100}$$

Here, the radial component consists of two terms, the first one of which can be seen to correspond to the usual radial acceleration that the natural static Schwarzschild observer is subjected to (5.84) (for $\Omega \to 0$, we have $\mathit{\Gamma}^2 B(R) \to 1$), while the second term is a kind of centrifugal correction which is largest at the equator and vanishes at the poles. In contrast, the $\mathit{\Theta}$-component of the four-acceleration vanishes on the equator and at the poles and is largest half-way between the latter.

Circular Geodesics

Requiring now that the world-line be a geodesic, i.e. that $a^\mu_{\text{circ}} = 0$ in (5.100), we see from the second equation there that the orbit must be equatorial (i.e., we must have $\mathit{\Theta} = \pi/2$) and find from the first equation that the geodesic orbital frequency is determined as,

$$\Omega = \pm\sqrt{\frac{GM}{R_0^3}} , \tag{5.101}$$

for counter-clockwise (+) and clockwise (−) orbits, respectively, which is just the Keplerian value for the angular frequency. In the geodesic case, the red-shift factor (5.95) in turn reduces to

$$\mathit{\Gamma}(R) = \left(1 - \frac{R_{\text{S}}}{R} - \frac{R_{\text{S}}}{2R}\right)^{-1/2} = \left(1 - \frac{3R_{\text{S}}}{2R_0}\right)^{-1/2} \qquad \text{(circular equatorial geodesic)}, \tag{5.102}$$

and thus with (5.36), we find that two thirds of the total relativistic precession experienced by a geodesic frame in equatorial circular orbit is due to gravitational redshift, while one third is special-relativistic time dilation due to the orbital velocity, a term which, in Minkowski space, is responsible for Thomas precession. In the geodesic case we then have in terms of the different parametrisations,

$$\left(\frac{\Omega}{c}\right) = \frac{\sqrt{B(R)}}{R\sin(\mathit{\Theta})}\beta = \sqrt{\frac{R_{\text{S}}}{2R^3}}\frac{1}{\sin(\mathit{\Theta})} . \tag{5.103}$$

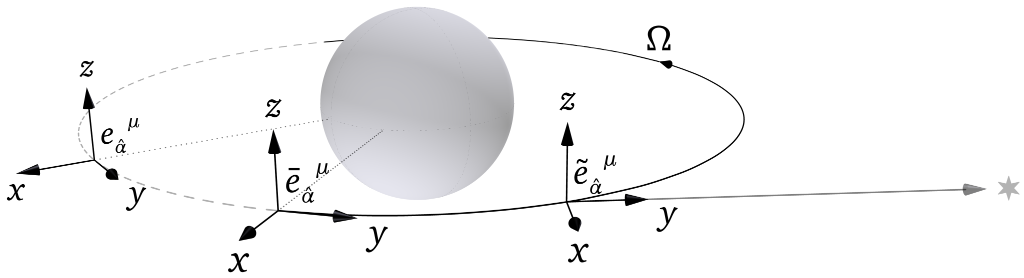

Figure 5.1.: Construction of a parallel transported frame adapted to circular world-lines, starting from the metric-adapted frame $e_{\hat{\alpha}}{}^{\mu}$ which is static, accelerated radially inwards, and also radially pointing. It can be made inertial via a Lorentz boost in Φ direction with appropriate velocity, resulting in a frame $\bar{e}_{\hat{\alpha}}{}^{\mu}$ adapted to inertial observers on a circular orbit. However, in our parametrisation in terms of coordinate quantities, this frame rotates once per orbit with respect to distant fixed stars, since it is still radially oriented. Undoing this rotation finally yields a frame $\tilde{e}_{\hat{\alpha}}{}^{\mu}$, that is parallel transported and non-rotating, i.e. inertially pointing with respect to distant fixed stars.

While the tetrad (5.98) is adapted to the symmetry of the Schwarzschild space-time, its spatial unit vectors are not parallel transported along the circular geodesic, since by construction, the first unit vector $e_{\hat{1}}{}^{\mu}$ always points radially outwards and thus the spatial part of the tetrad rotates once per orbit, i. e. with angular frequency Ω, with respect to the asymptotic Schwarzschild observer (to the fixed stars). This can be seen from the spatial part of the equation of parallel transport for the tetrad (5.98), which is calculated to be

$$u^{\nu}\nabla_{\nu}e_{\hat{\imath}}{}^{\mu} = -\epsilon_{\hat{0}\hat{\imath}\hat{\jmath}}{}^{\hat{k}}\,\omega^{\hat{\jmath}}\,e_{\hat{k}}{}^{\mu}\,, \tag{5.104}$$

the right-hand side of which constitutes an infinitesimal rotation with $(\omega^{\hat{k}}) = (0, \Omega, 0)$. In order to obtain a parallel transported tetrad, we must undo this rotation by forming linear combinations of the old basis vectors. A tetrad $\tilde{e}_{\hat{0}}{}^{\mu}$ which is parallel transported along the whole circular world-line is then given simply by rotating the tetrad backwards, i.e.,

$$\begin{aligned} \tilde{e}_{\hat{0}}{}^{\mu} &= \bar{e}_{\hat{0}}{}^{\mu}\,, \quad \tilde{e}_{\hat{1}}{}^{\mu} = \bar{e}_{\hat{1}}{}^{\mu}\cos\Phi(\tau) - \bar{e}_{\hat{3}}{}^{\mu}\sin\Phi(\tau)\,, \\ \tilde{e}_{\hat{2}}{}^{\mu} &= \bar{e}_{\hat{2}}{}^{\mu}\,, \quad \tilde{e}_{\hat{3}}{}^{\mu} = \bar{e}_{\hat{1}}{}^{\mu}\sin\Phi(\tau) + \bar{e}_{\hat{3}}{}^{\mu}\cos\Phi(\tau)\,. \end{aligned} \tag{5.105}$$

This parallel-transported and *non-rotating* circular-geodesic frame then displays explicitly as,

$$\tilde{e}_{\hat{\alpha}}{}^{\mu} = \begin{pmatrix} \Gamma & 0 & 0 & \Gamma(\Omega/c) \\ -\Gamma\frac{(\Omega/c)R\sin(\Theta)}{\sqrt{B(R)}}\sin\Phi(\tau) & \sqrt{B(R)}\cos\Phi(\tau) & 0 & -\Gamma\frac{\sqrt{B(R)}}{R\sin(\Theta)}\sin\Phi(\tau) \\ 0 & 0 & 1/R & 0 \\ \Gamma\frac{(\Omega/c)R\sin(\Theta)}{\sqrt{B(R)}}\cos\Phi(\tau) & \sqrt{B(R)}\sin\Phi(\tau) & 0 & \Gamma\frac{\sqrt{B(R)}}{R\sin(\Theta)}\cos\Phi(\tau) \end{pmatrix}, \tag{5.106}$$

where we have written the arguments of the trigonometric functions in terms of proper time τ, introducing $\Phi(\tau) = \Gamma\Omega\tau$. The construction of this parallel-transported, non-rotating frame along circular world-lines, starting with the metric-adapted one, is displayed in Figure 5.1.

5.6. Aspects of Curvature: Riemann and Weyl Tensors

This section is devoted to a discussion of the curvature tensors, especially the Riemann tensor and its completely trace-free part the Weyl tensor, to which it reduces in vacuum space-times such as those of Schwarzschild and Kerr. What we discuss in the following is to be seen as an important preparation for calculating the Fermi normal coordinate expansions that we introduce in chapter 6, since in the tensorial coefficients of these expansions, the frame components of the Riemann (Weyl) tensor appear "beyond the equivalence principle" at quadratic order, and sums of complicated combinations of its partial or covariant derivatives at higher orders.

We start with introducing the geodesic deviation equation which governs the time evolution of a "deviation" vector that connects two infinitesimally separated geodesics, and which also appears as the basis for covariant recursions that give rise to the above-mentioned coefficients of the expansion of tensor fields in Riemann and Fermi normal coordinates. We then show how the frame components of the Weyl tensor can be conveniently represented in six-dimensional bi-vector space in terms of two trace-free 3×3 Cartesian matrices, called its "electric" and "magnetic" parts, and how this is connected with the famous Petrov classification of vacuum space-times into six algebraic types. Their simple normal forms lead to a convenient starting point for calculating the Weyl tensor's frame components in a target frame of interest by applying to these matrices a Lorentz boost in bi-vector space. This method is ideally suited to make manual calculations of the Weyl tensor's components in arbitrary frames straightforward and directly applies to the important case of Schwarzschild and Kerr.

5.6.1. The Geodesic Deviation Equation

Consider a reference geodesic $\mathscr{G}$ given by $\bar{X}^\sigma_{\mathscr{G}}(\lambda)$ and parametrised in terms of an affine parameter λ. We are interested in the relation to an infinitesimally neighbouring geodesic $\mathscr{G}'$, parametrised in terms of another affine parameter, say η, and given by $\tilde{X}^\sigma_{\mathscr{G}'}(\eta)$. If the two geodesics are infinitesimally close, we can describe the neighbouring geodesic in terms of a first-order *deviation vector* $\delta X^\sigma(\lambda)$ from the reference geodesic, i.e. we have,

$$\tilde{X}^\sigma(\eta) \coloneqq \bar{X}^\sigma(\lambda) + \delta X^\sigma(\lambda)\,. \tag{5.107}$$

The deviation vector points from $\mathscr{G}$ to $\mathscr{G}'$ and connects points of equal parameter value λ. Using the two geodesic equations for $\mathscr{G}$ and $\mathscr{G}'$ in terms of the above deviation ansatz, one can derive a second-order ODE for $\delta X^\sigma(\lambda)$, called the *geodesic deviation equation* (or, in a mathematical context, *Jacobi equation*). This equation is in fact the "equation of motion" that follows from the *second variation* of the geodesic action, the first variation being of course the geodesic equation itself. For the actual derivation using the component approach, we refer the reader to the detailed discussion in the very nice textbook by Ciufolini and Wheeler [123, Sec. 2.5]. Instead, we just display the result and provide a concise index-free derivation below. The geodesic deviation equation reads,

$$\boxed{\frac{D^2 w^\sigma}{d\lambda^2} = v^{\nu_1}\nabla_{\nu_1} v^{\nu_2}\nabla_{\nu_2} w^\sigma = -R^\sigma{}_{\rho\mu\nu} v^\rho w^\mu v^\nu\,,} \tag{5.108}$$

where we have denoted the tangent to the reference geodesic by v^ρ and where $w^\mu := \delta X^\mu$ is the corresponding deviation vector field.

The very short and elegant index-free derivation of the geodesic deviation equation [124] for coordinate vector fields goes as follows. As above, the tangent to the reference geodesic will be denoted $\boldsymbol{v}$ and the the deviation vector by $\boldsymbol{w}$. The geodesic equation for the vector field $\boldsymbol{v}$ is then given by (3.54), i.e. $\boldsymbol{\nabla_v v} = \boldsymbol{0}$. For coordinate vector fields (i. e. vector fields defined with respect to a coordinate basis) $\boldsymbol{v} := v^\alpha \boldsymbol{\partial}_\alpha$ and $\boldsymbol{w} := w^\gamma \boldsymbol{\partial}_\gamma$ and in the absence of torsion, we have from (3.64) that

$$\boldsymbol{\nabla_v w} = \boldsymbol{\nabla_w v}\,, \tag{5.109}$$

which is the index-free version of the statement that the Christoffel symbols are symmetric, $\Gamma^\kappa{}_{\alpha\delta} = \Gamma^\kappa{}_{(\alpha\delta)}$, as can be seen from writing out the covariant derivatives with the above definitions of $\boldsymbol{v}$ and $\boldsymbol{w}$ and noting that the partial derivatives vanish due to $[\boldsymbol{v},\boldsymbol{w}] = (v^\alpha\partial_\alpha w^\kappa - w^\alpha\partial_\alpha v^\kappa)\boldsymbol{\partial}_\kappa = 0$. For coordinate vector fields, the index-free definition of the Riemann tensor (3.78) then reduces to

$$\boldsymbol{\nabla_v \nabla_w v} - \boldsymbol{\nabla_w \nabla_v v} = \boldsymbol{R}(\boldsymbol{v},\boldsymbol{w})\boldsymbol{v}\,, \tag{5.110}$$

where we find that the second term on the left-hand side of (5.110) now vanishes in view of the geodesic equation (3.54) for $\boldsymbol{v}$. Using (3.65) with the first term then yields the index-free version of the geodesic deviation equation (5.108),

$$\boxed{\boldsymbol{\nabla_v \nabla_v w} = -\boldsymbol{R}(\boldsymbol{w},\boldsymbol{v})\boldsymbol{v}\,.} \tag{5.111}$$

where we have also swapped the vector fields in the argument of $\boldsymbol{R}(\boldsymbol{v},\boldsymbol{w})$, in order to make their ordering correspond to that in equation (5.108), which gives a minus sign. The importance of the geodesic deviation equation for our purposes lies in the fact that the coefficients of the Riemann and Fermi normal expansions that we shall discuss in chapter 6 essentially follow from it and its higher covariant derivatives.

5.6.2. The Weyl Tensor

Associated with the name Ricci (after the Italian mathematician Gregorio Ricci-Curbastro) is also the following important invariant decomposition of the (index-lowered) Riemann tensor into its completely trace-free part $C_{\sigma\rho\mu\nu}$, the partial trace $R_{\mu\nu}$, and the total trace R_{Ric}. For an N-dimensional (pseudo-)Riemannian manifold, the Ricci decomposition reads:

$$R_{\sigma\rho\mu\nu} = C_{\sigma\rho\mu\nu} + \frac{2}{N-2}\big(g_{\sigma[\mu|}R_{\rho|\nu]} - g_{\rho[\mu|}R_{\sigma|\nu]}\big) - \frac{2}{(N-1)(N-2)} R_{\mathrm{Ric}}\, g_{\sigma[\mu|}g_{\rho|\nu]}\,, \tag{5.112}$$

where the factors of 2 in the numerators of the numerical fractions compensate the 1/2! coming from the definition (3.16b) of antisymmetrisations. The completely trace-free part $C_{\sigma\rho\mu\nu}$ defined by (5.112) is called the *Weyl tensor*. In general relativity, $N = 4$, so we have

$$C_{\sigma\rho\mu\nu} = R_{\sigma\rho\mu\nu} - \big(g_{\sigma[\mu|}R_{\rho|\nu]} - g_{\rho[\mu|}R_{\sigma|\nu]}\big) + \tfrac{1}{3}R_{\text{Ric}}\, g_{\sigma[\mu|}g_{\rho|\nu]}\,. \tag{5.113}$$

As a consequence of (5.112), the Weyl tensor has the same symmetries (3.85), (3.82a) and (3.82b) as the Riemann tensor, but is completely trace-free,

$$C^{\lambda}{}_{\mu\lambda\nu} = 0\,, \qquad\qquad C^{\mu\nu}{}_{\mu\nu} = 0\,. \tag{5.114}$$

In the following we shall discuss the number of independent components of the Riemann tensor in terms of the number of independent components of its irreducible parts $C_{\sigma\rho\mu\nu}$, $R_{\mu\nu}$ and R_{Ric}. As a consequence of the pair-exchange symmetry (3.84), the Riemann tensor can be thought of Symmetric 2×2 matrix $R_{\Sigma\Pi}$ with the compound indices $\Sigma = (\sigma\rho)$, and $\Pi = (\mu\nu)$ taking only the antisymmetric combinations. Its constituent 4×4 matrices are antisymmetric. According to the discussion in subsection 3.1.5, a symmetric $M\times M$ matrix has $\binom{M+1}{2} = \frac{1}{2}M(M+1)$ independent components, and an antisymmetric $N\times N$ matrix has $\binom{N}{2} = \frac{1}{2}N(N-1)$, we have with $M = \frac{1}{2}N(N-1)$, that

$$\tfrac{1}{2}M(M+1) = \tfrac{1}{2}\big[\tfrac{1}{2}N(N-1)\big]\big[\tfrac{1}{2}N(N-1)+1\big] = \tfrac{1}{8}\big[N^4 - 2N^3 + 3N^2 - 2N\big] \tag{5.115}$$

independent components which yields 21 for $N = 4$. We now have to implement the additional algebraic constraints coming from the 1st Bianchi identity (3.82a). We start by noting that, with the first index lowered, (3.82a) is equivalent to

$$R_{\sigma[\rho\mu\nu]} = 0\,, \tag{5.116}$$

which, in turn, is equivalent to the vanishing of the totally antisymmetric part of the Riemann tensor,

$$R_{[\sigma\rho\mu\nu]} = 0\,, \tag{5.117}$$

if the pair (anti-)symmetries (3.85) are already imposed. Both of the above relations can be seen by writing out the respective index permutations and using (3.85). We have thus written the condition (3.82a) in a form that allows us to incorporate it into our counting scheme above. According to what we saw above for antisymmetric indices, this additional constraint further reduces the number of independent index configurations and thus the number of of independent components by $\binom{N}{4} = \frac{1}{24}N(N-1)(N-2)(N-3)$.

Thus, we have that in N dimensions, the number of independent components C_N of the Riemann tensor is given by

$$C_N = \tfrac{1}{4}N(N-1)\big[\tfrac{1}{2}N(N-1)+1\big] - \frac{N(N-1)(N-2)(N-3)}{24} = \tfrac{1}{12}N^2(N^2-1)\,, \tag{5.118}$$

which yields $C_4 = 20$ in the four-dimensional space-time of general relativity. The first few numbers C_N for dimensions 1 to 5, together with the respective numbers of independent components of the Weyl and Ricci tensors are summarised in Table 5.1.

dimension N	1	2	3	**4**	5
Riemann tensor C_N	—	1	6	**20**	50
Weyl tensor	—	—	—	**10**	35
Ricci tensor	1	3	6	**10**	15

Table 5.1.: The number of independent components of the curvature tensors in dimensions one to five. In general relativity, the Riemann tensor has 20 independent components that are evenly distributed between the Weyl and the Ricci tensors, with 10 independent components each.

We note that on a two-dimensional manifold, i.e. for a surface, the Riemann tensor has only one independent component and is thus determined entirely by the Ricci scalar,

$$R_{\sigma\rho\mu\nu} = \tfrac{1}{2} R_{\mathrm{Ric}} (g_{\sigma\mu} g_{\rho\nu} - g_{\sigma\nu} g_{\rho\mu}) \,, \qquad \text{in 2 dimensions} \,. \tag{5.119}$$

The antisymmetry of the two index pairs (3.85) leads to a very useful representation of the Riemann tensor in terms of its non-vanishing components, as we shall see in the following.

5.6.3. Bi-Vector Representation of Weyl Tensor in Terms of Cartesian Matrices

By defining the above-mentioned compound indices Σ, Π, which take their values in the antisymmetric combinations of single indices, $\Sigma, \Pi \in \{01, 02, 03, 23, 31, 12\}$, the Riemann tensor can be written as a 6×6 matrix in terms of three different 3×3 constituent matrices, E_{ij}, B_{ij}, and Q_{ij}, where i, j are spacial indices. Explicitly this reads,

$$\mathcal{R}_{\Sigma\Pi} = \left(\begin{array}{ccc|ccc} R_{0101} & R_{0102} & R_{0103} & R_{0123} & R_{0131} & R_{0112} \\ R_{0201} & R_{0202} & R_{0203} & R_{0223} & R_{0231} & R_{0212} \\ R_{0301} & R_{0302} & R_{0303} & R_{0323} & R_{0331} & R_{0312} \\ \hline R_{2301} & R_{2302} & R_{2303} & R_{2323} & R_{2331} & R_{2312} \\ R_{3101} & R_{3102} & R_{3103} & R_{3123} & R_{3131} & R_{3112} \\ R_{1201} & R_{1202} & R_{1203} & R_{1223} & R_{1231} & R_{1212} \end{array}\right) = \left(\begin{array}{c|c} \mathsf{E}_{ij} & \mathsf{B}_{ij} \\ \hline \mathsf{B}_{ji} & \mathsf{Q}_{ij} \end{array}\right) \tag{5.120}$$

This "Riemann matrix" is symmetric, $\mathcal{R}_{\Sigma\Pi} = \mathcal{R}_{\Pi\Sigma}$, on account of the index-pair exchange symmetry (3.84). This also means that the matrices E_{ij} and Q_{ij} along the diagonal of (5.120) are symmetric. The 1st Bianchi identity (3.82a), on the other hand, yields

$$R_{0ijk} + R_{0jki} + R_{0kij} = 0 \,, \qquad i, j, k \text{ different} \,, \tag{5.121}$$

so that we find that B_{ij} is trace-free. In summary, E_{ij}, B_{ij} and Q_{ij} have the following properties:

$$\mathsf{E}_{ij} = \mathsf{E}_{(ij)} \,, \qquad \mathsf{Q}_{ij} = \mathsf{Q}_{(ij)} \,, \qquad \operatorname{tr} \mathsf{B} = 0 \,. \tag{5.122}$$

Geometrically, the four 3×3-matrices introduced above arise from a $1+3$ split of space-time that singles out a certain time direction, which is given by the time-like vector of a tetrad. More specifically, they result from projecting the second and the last indices of: the Riemann tensor $R_{\sigma\rho\mu\nu}$ itself, of the left dual $(*R)_{\sigma\rho\mu\nu}$ of the Riemann tensor, and of its double dual $(*R*)_{\sigma\rho\mu\nu}$, with the time-like tetrad vector $e_{\hat{0}}{}^{\sigma}$. Thus, these matrices are given by,

$$\mathsf{E}_{ij} = R_{\hat{0}\hat{\imath}\hat{0}\hat{\jmath}} = R_{\sigma\mu\rho\nu}\, e_{\hat{0}}{}^{\sigma} e_{\hat{\imath}}{}^{\mu} e_{\hat{0}}{}^{\rho} e_{\hat{\jmath}}{}^{\nu} \,, \tag{5.123a}$$

$$\mathsf{B}_{ij} = (*R)_{\hat{0}\hat{\imath}\hat{0}\hat{\jmath}} = \tfrac{1}{2}\varepsilon_{\sigma\mu}{}^{\lambda_1\lambda_2} R_{\lambda_1\lambda_2\rho\nu}\, e_{\hat{0}}{}^{\sigma} e_{\hat{\imath}}{}^{\mu} e_{\hat{0}}{}^{\rho} e_{\hat{\jmath}}{}^{\nu} \,, \tag{5.123b}$$

$$\mathsf{Q}_{ij} = (*R*)_{\hat{0}\hat{\imath}\hat{0}\hat{\jmath}} = \tfrac{1}{4}\varepsilon_{\sigma\mu}{}^{\lambda_1\lambda_2} R_{\lambda_1\lambda_2\lambda_3\lambda_4} \varepsilon_{\rho\nu}{}^{\lambda_3\lambda_4}\, e_{\hat{0}}{}^{\sigma} e_{\hat{\imath}}{}^{\mu} e_{\hat{0}}{}^{\rho} e_{\hat{\jmath}}{}^{\nu} \,, \tag{5.123c}$$

with inverse relations given in terms of frame components of the Riemann tensor by,

$$R_{\hat{a}\hat{b}\hat{c}\hat{0}} = -\epsilon_{\hat{a}\hat{b}}{}^{\hat{\imath}}\mathsf{B}_{\hat{\imath}\hat{c}} \,, \qquad R_{\hat{a}\hat{b}\hat{c}\hat{d}} = \epsilon_{\hat{a}\hat{b}}{}^{\hat{\imath}}\mathsf{Q}_{\hat{\imath}\hat{\jmath}}\epsilon^{\hat{\imath}}{}_{\hat{c}\hat{d}} \,. \tag{5.124}$$

Note how equations (5.123) are the analogue of the definition (4.72a) of the electric and magnetic field in terms of a projection of the first index of the electromagnetic field-strength tensor $F_{\mu\nu}$ for $\boldsymbol{E}$ and of the first index of its dual, $(*F)_{\mu\nu}$, for $\boldsymbol{B}$. Consequently, the matrix form (5.120) of the Riemann tensor directly corresponds to the decomposition (4.73) of the electromagnetic field-strength tensor $F_{\mu\nu}$ in terms of the electric and magnetic field three-vectors $\boldsymbol{E}$ and $\boldsymbol{B}$. The physical interpretation of this is clear: Since they are described by different frames, different observers in an "electromagnetic" field given by $F_{\mu\nu}$ will measure different electric and magnetic fields E^i and B^i. Because of this close analogy, the Cartesian tensor E_{ij} is known as the *electric part* of the Riemann tensor and sometimes also as the *electrogravitic tensor*, while B_{ij} is its *magnetic part*, and Q_{ij} simply as the *spacial part.*

5.6.4. Petrov Classification of Space-Times and Normal Forms of the Weyl Tensor

If the metric, from which the Riemann tensor is derived, is a *vacuum* solution of the Einstein field equations (5.5), then according to the Ricci decomposition (5.112), the Riemann tensor is given by the Weyl tensor alone, i. e., we have $R_{\sigma\rho\mu\nu} \equiv C_{\sigma\rho\mu\nu}$ and an important simplification occurs in its above bi-vector decomposition (5.120). This leads to an invariant local classification, as well as to the associated physical interpretation, of vacuum space-times according to the algebraic types of the Weyl matrix*,

$$\mathcal{C}_{\Sigma\Pi} = C_{\sigma\rho\mu\nu} \,, \tag{5.125}$$

via its eigenvalue problem in six-dimensional bi-vector space, written alternatively as,

$$\tfrac{1}{2}C^{\sigma\rho}{}_{\mu\nu} f^{\mu\nu} = \lambda f^{\sigma\rho} \,, \qquad \text{or} \qquad \mathcal{C}^{A}{}_{B} f^{B} = \lambda f^{A} \,. \tag{5.126}$$

*This is the famous Petrov, or "Petrov–Pirani–Penrose" classification, which was obtained first by Petrov in 1954 [125, 126], later independently by Pirani in 1957 [127], and elaborated and unified by Penrose in his spinor approach to General Relativity [128]. Cf. also the corresponding Editor's Note by MacCallum [129]. A standard reference for this and the next subsection on the Petrov classification is Chapter 4 of the comprehensive textbook on exact solutions in general relativity by Stephani et al. [82]; see also the textbooks [130, Sec. 9.3], and [131, Chapter 8], as well as the paper [132].

For a vacuum space-time, we have the condition that both the Ricci tensor (3.86) and the Ricci scalar (3.87) vanish by virtue of the Einstein field equations (5.5). In passing from the general Riemann tensor and its bi-vector representation (5.120) to the corresponding representation of the Weyl tensor, we have to implement the trace-free conditions (5.114) in bi-vector space, i.e., in terms of the above matrices E, B, and Q. Thus, we have to raise a bi-vector index on $\mathcal{C}_{\Sigma\Pi}$, which is accomplished with the bi-vector metric,

$$g_{\Sigma\Pi} = g_{\sigma\rho\mu\nu} = g_{\sigma\mu}g_{\rho\nu} - g_{\sigma\nu}g_{\rho\mu} \,. \tag{5.127}$$

This would, however, change its constituent matrices E_{ij}, B_{ij} and Q_{ij} in an unpredictable, metric-dependent way. Instead, it is better to consider the whole construction of (5.120) in an inertial frame, where the metric is given by the Minkowski metric and one uses its corresponding bi-vector representation,

$$\eta_{AB} = \eta_{\gamma\delta\alpha\beta} = \eta_{\gamma\alpha}\eta_{\delta\beta} - \eta_{\gamma\beta}\eta_{\delta\alpha} = \left(\begin{array}{c|c} -1_3 & \\ \hline & 1_3 \end{array}\right) = \eta^{AB} \,, \tag{5.128}$$

for raising and lowering indices, where 1_3 denotes the 3×3 unit matrix, as always in this context. This yields

$$\mathcal{R}^A{}_B = \eta^{AC}\mathcal{R}_{CB} = \left(\begin{array}{c|c} -\mathsf{E}_{ij} & -\mathsf{B}_{ij} \\ \hline \mathsf{B}_{ji} & \mathsf{Q}_{ij} \end{array}\right) , \tag{5.129}$$

and thus, for a vacuum space-time, the vanishing of the Ricci scalar leads to the condition

$$\mathcal{R}^A{}_A = -\mathsf{E}_{ij} + \mathsf{Q}_{ij} = 0 \,, \tag{5.130}$$

which means that in vacuum, Q is just given by the negative of E. Additionally, from the time-time component of the Ricci tensor we obtain $R_{00} = R^i{}_{0i0} = \mathsf{E}^i{}_i = 0$, i.e. E_{ij} becomes trace-free. As a result, in the bi-vector representation, the Weyl tensor (5.113) takes on the following simple form in terms of the symmetric, trace-free matrices E_{ij} and B_{ij},

$$\mathcal{C}_{AB} = \left(\begin{array}{c|c} \mathsf{E}_{ij} & \mathsf{B}_{ij} \\ \hline \mathsf{B}_{ij} & -\mathsf{E}_{ij} \end{array}\right) , \qquad \text{with} \qquad \begin{array}{ll} \mathsf{E}_{ij} = \mathsf{E}_{(ij)} \,, & \operatorname{tr}\mathsf{E} = 0 \,, \\ \mathsf{B}_{ij} = \mathsf{B}_{(ij)} \,, & \operatorname{tr}\mathsf{B} = 0 \,. \end{array} \tag{5.131}$$

Eigenvalue Problem, Characteristic Equation, and Invariants of the Weyl Tensor

Rather than directly evaluating the eigenvalue problem (5.126) and carrying out the classification of the six-dimensional real Weyl operator

$$\mathcal{C}^A{}_B = \eta^{AC}\mathcal{C}_{CB} = \left(\begin{array}{c|c} -\mathsf{E} & -\mathsf{B} \\ \hline \mathsf{B} & -\mathsf{E} \end{array}\right) , \tag{5.132}$$

in terms of the real matrices E and B, it is much more convenient to cast the problem into the form of a complex three-dimensional eigenvalue equation (exploiting thereby the closure of the field of complex numbers and allowing for a straightforward application of the Jordan decomposition below). This is achieved as follows [132]. In analogy to the case of the electromagnetic field, and borrowing the corresponding notation for lack of imagination, we decompose the eigenbivector f^A in terms of two three-dimensional vectors as $f^A = (\boldsymbol{E}, \boldsymbol{B})^\intercal$. The eigenvalue problem (5.126) then splits up into a pair of three-dimensional equations,

$$\mathsf{E}\boldsymbol{E} + \mathsf{B}\boldsymbol{B} = -\lambda \boldsymbol{E}\,, \qquad \mathsf{B}\boldsymbol{E} - \mathsf{E}\boldsymbol{B} = \lambda \boldsymbol{B}\,. \tag{5.133}$$

We can now take appropriate complex linear combinations in order to build up a combined complex eigenvalue problem. Multiplying the second equation by $(\pm i)$ and adding it to the first yields the two equations,

$$(\mathsf{E} \pm i\mathsf{B})(\boldsymbol{E} \mp i\boldsymbol{B}) = -\lambda(\boldsymbol{E} \mp i\boldsymbol{B})\,. \tag{5.134}$$

We can thus define the complex, trace-free, 3×3 "Petrov matrix" P as their complex linear combination,

$$\mathsf{P} \coloneqq \mathsf{E} + i\mathsf{B}\,, \qquad \operatorname{tr}\mathsf{P} = 0\,, \tag{5.135}$$

which leads to the complex eigenvalue problem for P,

$$\mathsf{P}\boldsymbol{e} = -\lambda \boldsymbol{e}\,, \qquad \chi_{\mathsf{P}}(\lambda) = \det(\lambda \mathsf{1} - \mathsf{P}) = 0\,, \tag{5.136}$$

with three complex eigenvalues $\lambda_1, \lambda_2, \lambda_3$. Clearly, the trace-free condition for P now means,

$$\lambda_1 + \lambda_2 + \lambda_3 = 0\,, \tag{5.137}$$

which in turn means that the characteristic equation for P reduces to a depressed cubic,

$$\lambda^3 - \mathcal{I}\lambda - \mathcal{J} = 0\,. \tag{5.138}$$

Its two coefficients, $\mathcal{I}$ and $\mathcal{J}$, being the complex invariants of P, are then related to the eigenvalues and to the four real invariants of the Weyl tensor,

$$\begin{aligned} \mathcal{I} &= \tfrac{1}{2}\operatorname{tr}(\mathsf{P}^2) = \tfrac{1}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) = \mathcal{I}_1^{\mathrm{W}} + i\mathcal{I}_2^{\mathrm{W}} \\ \mathcal{J} &= \det \mathsf{P} = \lambda_1\lambda_2\lambda_3 = \tfrac{1}{3}\operatorname{tr}(\mathsf{P}^3) = \tfrac{1}{3}(\lambda_1^3 + \lambda_2^3 + \lambda_3^3)\,. \end{aligned} \tag{5.139}$$

Here, the real and imaginary parts of $\mathcal{I}$ are the two quadratic invariants of the Weyl tensor,

$$\mathcal{I}_1^{\mathrm{W}} = C_{\gamma\delta\alpha\beta}C^{\gamma\delta\alpha\beta} = 4\mathcal{C}_{AB}\mathcal{C}^{AB} = 4\operatorname{tr}\left(\begin{array}{c|c} \mathsf{E}^2 - \mathsf{B}^2 & -(\mathsf{EB} + \mathsf{BE}) \\ \hline \mathsf{EB} + \mathsf{BE} & \mathsf{E}^2 - \mathsf{B}^2 \end{array}\right) = 8\operatorname{tr}(\mathsf{E}^2 - \mathsf{B}^2), \tag{5.140a}$$

$$\mathcal{I}_2^{\mathrm{W}} = (*C)_{\sigma\rho\mu\nu}C^{\sigma\rho\mu\nu} = 4\epsilon_A{}^C\mathcal{C}_{CB}\mathcal{C}^{AB} = 4\operatorname{tr}\left(\begin{array}{c|c} \mathsf{EB} + \mathsf{BE} & \mathsf{E}^2 - \mathsf{B}^2 \\ \hline -(\mathsf{E}^2 - \mathsf{B}^2) & \mathsf{EB} + \mathsf{BE} \end{array}\right) = 16\operatorname{tr}(\mathsf{EB}), \tag{5.140b}$$

where we note that the epsilon tensor in bi-vector space reads,

$$\epsilon_{AB} = \left(\begin{array}{c|c} & \mathsf{1}_3 \\ \hline \mathsf{1}_3 & \end{array}\right)\,, \tag{5.141}$$

with the normalisation factor of $\frac{1}{2}$ in the Hodge dual not being necessary here, since bi-vector indices $A, B, C, \ldots$ represent only one of the two orderings of their constituent normal indices that cause double counting in contractions. This is also the cause for the additional numerical factors of 4 in the expressions (5.140a) for the invariants above.

Jordan Decomposition

We recall (see e.g. [133, Sec. 3.1]) that every complex $n \times n$ matrix M is similar to a block diagonal matrix J, its so-called *Jordan normal form*, i.e.,

$$\mathsf{M} = \mathsf{S}\,\mathsf{J}\,\mathsf{S}^{-1}\,, \qquad \mathsf{J} = \mathsf{J}_{n_1} \oplus \mathsf{J}_{n_2} \oplus \cdots \oplus \mathsf{J}_{n_k}\,, \tag{5.142}$$

where every *Jordan block* $\mathsf{J}_{n_i}(\lambda_i)$ is an $n_i \times n_i$ upper triangular matrix, given in terms of the eigenvalue $\lambda \in \mathbb{C}$ of the associated eigenvalue problem by

$$\mathsf{J} = \begin{pmatrix} \boxed{\mathsf{J}_{n_1}} & & \\ & \ddots & \\ & & \boxed{\mathsf{J}_{n_k}} \end{pmatrix}, \qquad \mathsf{J}_{n_i}(\lambda_i) = \begin{pmatrix} \lambda_k & 1 & & \\ & \lambda_k & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{pmatrix}, \tag{5.143}$$

and where $n_1 + n_2 + \cdots + n_k = n$. The orders n_i and the eigenvalues λ_i need not be distinct. The number k of Jordan blocks (counting possible multiple occurrences of the same block) corresponds to the number of linearly independent eigenvectors of J. The number of Jordan blocks corresponding to a given eigenvalue λ is the *geometric multiplicity* of that eigenvalue (the dimension of the associated eigenspace). Clearly, an eigenvalue's geometric multiplicity must then be less or equal to its algebraic multiplicity. Since the Jordan decomposition is unique up to a reordering of Jordan blocks, a complex $n \times n$ matrix can now be classified according to the different combinations of the geometric and algebraic multiplicities of its eigenvalues.

Normal Forms of the Weyl Tensor

Applying this to our complex 3×3 Petrov matrix P, we find that there can be six different types in general. We start with the 3 main types with respect to the number of independent eigenvectors: type I possesses 3 independent eigenvectors, type II has 2 and finally, type III has only one. This means that type I is diagonalisable, i.e., the sum of the geometric multiplicities is equal to 3, while for type II it is equal to 2 and for type III equal to 1.

The geometric multiplicities now translate directly into the number of Jordan blocks in the Jordan normal form. Thus, the normal forms of P for these initial main types read,

$$\mathsf{P}_{\mathrm{I}} = \begin{pmatrix} \boxed{\lambda_1} & & \\ & \boxed{\lambda_2} & \\ & & \boxed{\lambda_3} \end{pmatrix}, \qquad \mathsf{P}_{\mathrm{II}} = \begin{pmatrix} \boxed{\lambda_1} & \\ & \boxed{\begin{matrix} \lambda_2 & 1 \\ 0 & \lambda_2 \end{matrix}} \end{pmatrix}, \qquad \mathsf{P}_{\mathrm{III}} = \begin{pmatrix} \boxed{\begin{matrix} \lambda_1 & 1 & \\ & \lambda_1 & 1 \\ & & \lambda_1 \end{matrix}} \end{pmatrix}. \tag{5.144}$$

The first two types I and II of these now feature two and one degenerate subtypes, respectively, depending on the algebraic degeneracies of their eigenvalues, and we thus have three possibilities for their algebraic multiplicity:

$$\begin{aligned} \text{case i:} \quad & \lambda_1 \neq \lambda_2 \neq \lambda_3 \\ \text{case ii:} \quad & \lambda_1 \neq \lambda_2 = \lambda_3 \\ \text{case iii:} \quad & \lambda_1 = \lambda_2 = \lambda_3 = 0\,. \end{aligned} \tag{5.145}$$

Type I degenerates first into the subtype D, if two of the eigenvalues are equal. In this case we write $\lambda \coloneqq -\lambda_2 = -\lambda_3$, so the trace-free condition (5.137) yields $\lambda_1 = 2\lambda$. Subtype D in turn can degenerate further into type O (a stylised 0) where all three eigenvalues are equal and vanishing, so P_O is the zero matrix and correspondingly, the Weyl tensor vanishes for Petrov type O. Type II initially features two independent eigenvalues, one for each of the two Jordan blocks of dimensions one and two in P_II, respectively. It can then degenerate into its single subtype N (for "null") if the two eigenvalues coincide and thus vanish, $\lambda_1 = \lambda_2 = 0$.

Finally, type III already features only a single vanishing eigenvalue, so its associated canonical form P_III consists of a single Jordan block of dimension 3 with vanishing diagonal. The normal forms of the Petrov matrix P for the six Petrov types thus read,

$$\mathsf{P}_\mathrm{I} = \begin{pmatrix} \boxed{\lambda_1} & & \\ & \boxed{\lambda_2} & \\ & & \boxed{-(\lambda_1+\lambda_2)} \end{pmatrix}, \quad \mathsf{P}_\mathrm{D} = \begin{pmatrix} \boxed{2\lambda} & & \\ & \boxed{-\lambda} & \\ & & \boxed{-\lambda} \end{pmatrix}, \quad \mathsf{P}_\mathrm{O} = \begin{pmatrix} \boxed{0} & & \\ & \boxed{0} & \\ & & \boxed{0} \end{pmatrix},$$
$$\mathsf{P}_\mathrm{II} = \begin{pmatrix} \boxed{2\lambda} & \\ & \boxed{\begin{matrix} -\lambda & 1 \\ 0 & -\lambda \end{matrix}} \end{pmatrix}, \quad \mathsf{P}_\mathrm{N} = \begin{pmatrix} \boxed{\lambda_1} & \\ & \boxed{\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}} \end{pmatrix}, \quad \mathsf{P}_\mathrm{III} = \begin{pmatrix} \boxed{\begin{matrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{matrix}} \end{pmatrix}. \tag{5.146}$$

In Table 5.2 we display a summary of these Petrov types, together with the standard physical interpretation of the corresponding space-times due to Szekeres [134], as well as an example, where applicable.

Weyl Tensor of Petrov Type D: Schwarzschild and Kerr

From now on, we shall exclusively focus on Weyl tensors of Petrov type D because of their direct astrophysical relevance as describing the Riemann curvature outside of planets, stars, black holes, etc. Remembering the definition of P in equation (5.135) and defining the electric and magnetic "Weyl scalars" $\mathcal{E} \coloneqq \mathrm{Re}(\lambda)$ and $\mathcal{B} \coloneqq \mathrm{Im}(\lambda)$, we immediately obtain

*This is short for "plane-fronted waves with parallel propagation", e.g. the gravitational radiation detected by LIGO, but can be any other type of plane-fronted radiation that locally propagates parallely with the speed of light, such as electromagnetic radiation.

Petrov type	eigenvalues	physical interpretation	example
I	$\lambda_1 \neq \lambda_2 \neq \lambda_3$	(general space-time)	—
D	$\lambda_1 \neq \lambda_2 = \lambda_3$	Newton-Coulomb-like central field of isolated source (planet, star, black hole)	Schwarzschild, Kerr
O	$\lambda_1 = \lambda_2 = \lambda_3 = 0$	space-time (conformally) flat	FLRW standard cosmological model
II	$\lambda_1 \neq \lambda_2$	"mixture" of type D and N effects	
N	$\lambda_1 = 0 = \lambda_2$	transverse gravitational radiation	pp-wave space-time*
III	$\lambda_1 = 0$	longitudinal gravitational radiation	—

Table 5.2.: Local classification of vacuum space-times: The six Petrov types (corresponding to the possible algebraic types of the Weyl tensor) with the main types (bold) and the corresponding degenerate types (non-bold) indicated, together with their eigenvalue structure, the physical interpretation of the type of space-time, as well as an example thereof.

the diagonal normal forms of the electric and magnetic parts of the Weyl matrix (5.131),

$$\mathsf{E} = \begin{pmatrix} 2\mathcal{E} & & \\ & -\mathcal{E} & \\ & & -\mathcal{E} \end{pmatrix}, \qquad \mathsf{B} = \begin{pmatrix} -2\mathcal{B} & & \\ & \mathcal{B} & \\ & & \mathcal{B} \end{pmatrix}, \tag{5.147}$$

and this means that we only have to compute λ, i.e., the real quantities $\mathcal{E}$ and $\mathcal{B}$, which in turn means that we only have to compute two frame-components of the Riemann tensor, say $R_{\hat{0}\hat{1}\hat{0}\hat{1}} = 2\mathcal{E}$ for the electric part and $R_{\hat{0}\hat{1}\hat{2}\hat{3}} = 2\mathcal{B}$ for the magnetic part, provided we are given the unique adapted tetrad that makes the Weyl tensor "diagonal" in this sense.

For Schwarzschild space-time (5.35), this unique diagonal form of E and B is realised in the metric-adapted static frame (5.82) of the standard Schwarzschild metric (5.35), which demonstrates that the standard Schwarzschild coordinates are optimally adapted to the space-time's curvature structure. As the closed forms (A.30) and (A.35) of our coordinate-invariant calculation in Appendix A show, in standard Schwarzschild coordinates we have,

$$\mathcal{E} = \frac{R_S}{2R^3}\,, \quad \mathcal{B} = 0\,, \qquad \text{so} \qquad \mathsf{E} = \frac{R_S}{2R^3}\,\mathrm{diag}(2,-1,-1)\,, \qquad \mathsf{B} = 0\,, \tag{5.148}$$

while for the isotropic Schwarzschild metric (5.38), written in terms of the isotropic radial coordinate R_{iso}, only the radial coordinate in the prefactor is modified according to $R \to R_{\text{iso}}\sqrt{B_2(R_{\text{iso}})}$ and we thus have

$$\mathcal{E}_{\text{iso}} = \frac{R_S}{2R_{\text{iso}}^3 B_2(R_{\text{iso}})^{3/2}}\,, \qquad \text{so} \qquad \mathsf{E}_{\text{iso}} = \frac{R_S}{2R_{\text{iso}}^3 B_2(R_{\text{iso}})^{3/2}}\,\mathrm{diag}(2,-1,-1)\,, \tag{5.149}$$

with $\mathcal{B}_{\text{iso}} = 0$ and $\mathsf{B}_{\text{iso}} = \mathsf{0}$. Physically, the existence of a (static) frame in which the "magnetic" part B of the Weyl tensor vanishes means that there is no intrinsic gravitomagnetic effect (i.e., no frame-dragging or "rotation" of space-time itself as seen from asymptotic Minkowski space-time) in Schwarzschild space-time, which should also be clear from the static nature of the metric. However, such effects coming from a non-zero B can be locally "generated" *for certain observers* (i.e. those, which are in non-radial motion with respect to the coordinates and thus to the source) by projecting the Weyl tensor onto a non-static tetrad instead, the simplest case being a stationary frame that follows a circular orbit. This will be investigated below.

For the Kerr metric in Boyer-Lindquist coordinates (5.42), the Weyl tensor tensor takes on the normal form (5.147) in the Carter frame (5.81b). The electric and magnetic curvature scalars $\mathcal{E}$ and $\mathcal{B}$ are given, respectively, by

$$\mathcal{E} = \frac{R_{\mathrm{S}} R}{2\rho^6}\left[R^2 - 3(a\cos\Theta)^2\right] = \frac{R_{\mathrm{S}}}{2R^3}\left[\frac{1 - 3(\frac{a}{R})^2\cos^2\Theta}{1 - (\frac{a}{R})^2\cos^2\Theta}\right], \tag{5.150a}$$

$$\mathcal{B} = \frac{R_{\mathrm{S}}\, a\cos\Theta}{2\rho^6}\left[3R^2 - (a\cos\Theta)^2\right] = \frac{R_{\mathrm{S}}}{2R^3}\left(\frac{a\cos\Theta}{R}\right)\left[\frac{3 - (\frac{a}{R})^2\cos^2\Theta}{1 - (\frac{a}{R})^2\cos^2\Theta}\right], \tag{5.150b}$$

which can be seen to reduce to the Schwarzschild expression (5.148) in the limit $a \to 0$. The non-vanishing of the magnetic Weyl scalar $\mathcal{B}$, respectively the magnetic part of the Weyl tensor B, in this "best-possible" frame indicates once more that there is "something magnetic" about the Kerr metric, i.e. space-time itself is rotating around the centre of symmetry.

5.6.5. Lorentz Transformations in Bi-Vector Space and Boosted Weyl Matrix

Above, we have worked out the simple and invariant representation of the Weyl tensor in a certain orthonormal frame in terms of the Cartesian matrices E and B, and noted that this simple normal form is attained in a single specific frame that is aligned to the Weyl curvature. However, we had previously argued in section 5.5, that we are usually interested in the frame components of the Riemann/Weyl tensor with respect to a different target frame. Since tetrads at the same space-time point are connected through Lorentz transformations, we shall now work out how Lorentz transformations, in particular Lorentz boosts, look in six dimensional bivector space, i.e., how they act on the bivector-representation of the Weyl tensor and thus on E and B (cf. [135, 136]). As in the case of the bivector metric (5.127), a Lorentz transformation matrix in bivector space takes the following form,

$$\Lambda^{A'}{}_{A} = \Lambda^{\alpha'\beta'}{}_{\alpha\beta} = \Lambda^{\alpha'}{}_{\alpha}\Lambda^{\beta'}{}_{\beta} - \Lambda^{\alpha'}{}_{\beta}\Lambda^{\beta'}{}_{\alpha}\,, \tag{5.151}$$

featuring the proper double two-form or Riemann index symmetries appropriate for an operator in bivector space. Just as the Riemann tensor in (5.123) itself, the "bivectorised" Lorentz transformations (5.151) too split into three parts: an "electric" part, $\Lambda^{0b'}{}_{0b}$, a "magnetic" part given by the left dual, $(*\Lambda)^{0b'}{}_{0b}$, and a purely spacial part given by the double dual, $(*\Lambda*)^{0b'}{}_{0b}$. For the boost part of these we have specifically,

$$\Lambda^{0'b'}{}_{0b} = \Lambda^{0'}{}_{0}\Lambda^{b'}{}_{b} - \Lambda^{0'}{}_{b}\Lambda^{b'}{}_{0} = \gamma\delta^{b'}{}_{b} + (1-\gamma)n^{b'}n_b\,, \tag{5.152a}$$

$$(*\Lambda)^{0'b'}{}_{0b} = \tfrac{1}{2}\epsilon^{0'b'}{}_{k'l'}\Lambda^{k'l'}{}_{0b} = \tfrac{1}{2}\epsilon^{0'b'}{}_{k'l'}(\Lambda^{k'}{}_{0}\Lambda^{l'}{}_{b} - \Lambda^{k'}{}_{b}\Lambda^{l'}{}_{0}) = -\gamma\beta\epsilon_{b'bk}n^k\,, \tag{5.152b}$$

$$(\Lambda*)^{0'b'}{}_{0b} = \tfrac{1}{2}\Lambda^{0'b'}{}_{kl}\epsilon^{kl}{}_{0b} = \tfrac{1}{2}(\Lambda^{0'}{}_{k}\Lambda^{b'}{}_{l} - \Lambda^{0'}{}_{l}\Lambda^{b'}{}_{k})\epsilon^{kl}{}_{0b} = -\gamma\beta\epsilon_{bb'k}n^k\,, \tag{5.152c}$$

$$(*\Lambda*)^{0'b'}{}_{0b} = \tfrac{1}{4}\epsilon^{0b'}{}_{k'l'}\Lambda^{k'l'}{}_{mn}\epsilon^{mn}{}_{0b} = -\left[\gamma\delta^{b'}{}_{b} + (1-\gamma)n_{b'}n_b\right], \tag{5.152d}$$

where the right-hand side of the equations is Cartesian in the sense that we have made all signs from the Minkowski metric explicit, and have taken the canonical Cartesian three-dimensional epsilon symbol to naturally have all lower indices, ϵ_{ijk}. As above for the Riemann tensor, we can thus write the bi-vector representation $\Lambda^{\alpha'\beta'}{}_{\alpha\beta}$ of a general Lorentz boost $\Lambda^{\alpha'}{}_{\alpha'}$ in terms of two Cartesian 3×3 matrices K and H, as

$$\Lambda^{\hat{A}'}{}_{\hat{A}} = \Lambda^{\alpha'\beta'}{}_{\alpha\beta} = \left(\begin{array}{c|c} \mathsf{K} & \mathsf{H} \\ \hline \mathsf{H}^\mathsf{T} & \mathsf{K}\end{array}\right) = \left(\begin{array}{c|c} \mathsf{K} & \mathsf{H} \\ \hline -\mathsf{H} & \mathsf{K}\end{array}\right), \qquad \text{with} \qquad \begin{aligned}\mathsf{K} &= \mathsf{K}^\mathsf{T}\,,\\ \mathsf{H} &= -\mathsf{H}^\mathsf{T}\,,\end{aligned} \tag{5.153}$$

where these boost matrices are given by

$$\mathsf{K}^{a'}{}_{a} \coloneqq \Lambda^{0a'}{}_{0a} \quad = \gamma\delta^{a'}{}_{a} + (1-\gamma)n^{a'}n_a = \delta^{a'}{}_{a}\cosh(\zeta) + n^{a'}n_a(1-\cosh\zeta)\,, \tag{5.154a}$$

$$\mathsf{H}^{a'}{}_{a} \coloneqq (\Lambda*)^{0a'}{}_{0a} = \gamma\beta\,\epsilon^{a'}{}_{ak}n^k \qquad\qquad = \epsilon^{a'}{}_{ak}n^k\sinh(\zeta)\,, \tag{5.154b}$$

from which it is evident that $\mathsf{K}^{a'}{}_{a}$ is symmetric, while $\mathsf{H}^{a'}{}_{a}$ is antisymmetric. An inverse Lorentz boost is then simply given by taking $\beta \to -\beta$ in (5.154), which makes $\mathsf{K} \to -\mathsf{K}$ in (5.153), so that we have

$$(\Lambda^{-1})^{\hat{A}}{}_{\hat{A}'} = (\Lambda^{-1})^{\alpha\beta}{}_{\alpha'\beta'} = \left(\begin{array}{c|c} \mathsf{K} & -\mathsf{H} \\ \hline \mathsf{H} & \mathsf{K}\end{array}\right). \tag{5.155}$$

We note that, writing the defining relation for Lorentz transformations in bi-vector space as follows,

$$\eta_{AB}(\Lambda^{-1})^{\hat{A}}{}_{\hat{A}'}(\Lambda^{-1})^{\hat{B}}{}_{\hat{B}'} = \eta_{\hat{A}'\hat{B}'}\,, \tag{5.156}$$

leads to the following constraints on the matrices K and H,

$$\mathsf{K}^2 + \mathsf{H}^2 = \mathsf{1}\,, \qquad \text{and} \qquad \mathsf{HK} - \mathsf{KH} = \mathsf{0}\,. \tag{5.157}$$

With these preparations at hand, we now turn to the transformation properties of the Weyl tensor in bivector space. Under a Lorentz boost $\Lambda^{\hat{\alpha}}{}_{\hat{\alpha}'}$, its frame components transform according to the general relation, i.e. as

$$\tilde{C}_{\hat{\gamma}'\hat{\delta}'\hat{\alpha}'\hat{\beta}'} = C_{\hat{\gamma}\hat{\delta}\hat{\alpha}\hat{\beta}}(\Lambda^{-1})^{\hat{\gamma}}{}_{\hat{\gamma}'}(\Lambda^{-1})^{\hat{\delta}}{}_{\hat{\delta}'}(\Lambda^{-1})^{\hat{\alpha}}{}_{\hat{\alpha}'}(\Lambda^{-1})^{\hat{\beta}}{}_{\hat{\beta}'}\,. \tag{5.158}$$

With the help of the bivector representations (5.153) and (5.131), this now becomes

$$\tilde{C}_{\hat{A}'\hat{B}'} = (\Lambda^{-1})^{\hat{A}}{}_{\hat{A}'} C_{\hat{A}\hat{B}} (\Lambda^{-1})^{\hat{B}}{}_{\hat{B}'} = \left[\begin{pmatrix} \mathsf{K} & \mathsf{H} \\ -\mathsf{H} & \mathsf{K} \end{pmatrix} \begin{pmatrix} \mathsf{E} & \mathsf{B} \\ \mathsf{B} & -\mathsf{E} \end{pmatrix} \begin{pmatrix} \mathsf{K} & \mathsf{H} \\ -\mathsf{H} & \mathsf{K} \end{pmatrix} \right] = \begin{pmatrix} \tilde{\mathsf{E}} & \tilde{\mathsf{B}} \\ \tilde{\mathsf{B}} & -\tilde{\mathsf{E}} \end{pmatrix}, \tag{5.159}$$

with the Lorentz-boosted matrices $\tilde{\mathsf{E}}$ and $\tilde{\mathsf{B}}$ being given explicitly by the following relations, which we prefer to display in terms of components,

$$\tilde{\mathsf{E}}_{a'b'} = \mathsf{K}^a{}_{a'} \mathsf{E}_{ab} \mathsf{K}^b{}_{b'} + \mathsf{K}^a{}_{a'} \mathsf{B}_{ab} \mathsf{H}^b{}_{b'} - \mathsf{H}^a{}_{a'} \mathsf{B}_{ab} \mathsf{K}^b{}_{b'} + \mathsf{H}^a{}_{a'} \mathsf{E}_{ab} \mathsf{H}^b{}_{b'} \tag{5.160a}$$

$$\tilde{\mathsf{B}}_{a'b'} = \mathsf{K}^a{}_{a'} \mathsf{B}_{ab} \mathsf{K}^b{}_{b'} - \mathsf{K}^a{}_{a'} \mathsf{E}_{ab} \mathsf{H}^b{}_{b'} + \mathsf{H}^a{}_{a'} \mathsf{E}_{ab} \mathsf{K}^b{}_{b'} + \mathsf{H}^a{}_{a'} \mathsf{B}_{ab} \mathsf{H}^b{}_{b'} \,. \tag{5.160b}$$

Using equations (5.154) while keeping E and B general, this expands to,

$$\begin{aligned} \tilde{\mathsf{E}}_{a'b'} = \gamma^2 \mathsf{E}_{a'b'} &+ 2\gamma(1-\gamma)(\mathsf{E}_{a'b} n^b) n_{b'} + (1-\gamma)^2 n_{a'} (n^a \mathsf{E}_{ab} n^b) n_{b'} \\ &+ \gamma^2 \beta^2 (\epsilon^a{}_{a'k} \mathsf{E}_{ab} \epsilon^b{}_{b'l}) n_k n_l + \gamma^2 \beta [\mathsf{B}_{a'b} \epsilon^b{}_{b'k} - \epsilon^a{}_{a'k} \mathsf{B}_{ab'}] n_k \\ &+ \gamma\beta(1-\gamma)[n_{a'}(n^a \mathsf{B}_{ab}) \epsilon^b{}_{b'k} - \epsilon^a{}_{a'k} (\mathsf{B}_{ab} n^b) n_{b'}] n_k \end{aligned} \tag{5.161a}$$

$$\begin{aligned} \tilde{\mathsf{B}}_{a'b'} = \gamma^2 \mathsf{B}_{a'b'} &+ 2\gamma(1-\gamma)(\mathsf{B}_{a'b} n^b) n_{b'} + (1-\gamma)^2 n_{a'} (n^a \mathsf{B}_{ab} n^b) n_{b'} \\ &+ \gamma^2 \beta^2 (\epsilon^a{}_{a'k} \mathsf{B}_{ab} \epsilon^b{}_{b'l}) n_k n_l - \gamma^2 \beta [\mathsf{E}_{a'b} \epsilon^b{}_{b'k} - \epsilon^a{}_{a'k} \mathsf{E}_{ab'}] n_k \\ &- \gamma\beta(1-\gamma)[n_{a'}(n^a \mathsf{E}_{ab}) \epsilon^b{}_{b'k} - \epsilon^a{}_{a'k} (\mathsf{E}_{ab} n^b) n_{b'}] n_k \,, \end{aligned} \tag{5.161b}$$

where we find that the expression for $\tilde{\mathsf{B}}_{a'b'}$ follows from replacing $\mathsf{E} \to \mathsf{B}$ and $\beta \to -\beta$ in that for $\tilde{\mathsf{E}}_{a'b'}$. These transformation relations for the electric and magnetic parts of the Weyl tensor under Lorentz boosts are the result of this subsection.

5.7. Approximate Space-Times: Metric Perturbation Theory, Post-Minkowskian, and Post-Newtonian Expansions

In the last sections we have mostly discussed exact space-times and their general properties, focussing on axisymmetric vacuum space-times such as the well-known Schwarzschild and Kerr solutions, because of their astrophysical relevance on the one hand, and practical relevance as backgrounds for local coordinate expansions around the (non-)inertial frames on the other. Exact metrics are, in a certain sense, nice to work with since most quantities of interest can be calculated exactly in terms of fairly simple and compact closed-form expressions. They also allow one to develop an intuition of the different types effects that occur in general relativity with respect to our usual Newtonian view of the world. These are, e.g., the different contributions to observed red-shift of orbiting observers (as corrected for in satellite navigation systems such as GPS and Galileo), or the different types of curvature such as the electric-type Schwarzschild-like one, and the magnetic-type frame-dragging one that is additionally present e.g. in Kerr family of space-times.

In the last two decades, or so, enormous progress has been made in terms of our knowledge of the gravitational near-Earth environment. This is perhaps best illustrated by high-precision gravity satellite missions such as GRACE and GOCE: In 2002, the Gravity Recovery and Climate Experiment (GRACE), a cooperation between NASA and the

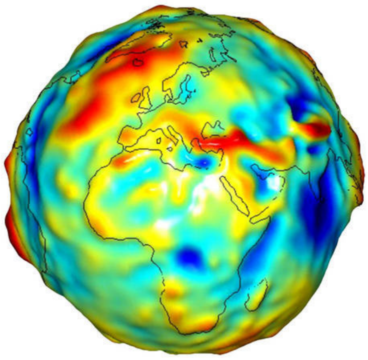

Figure 5.2.: A (greatly exaggerated in amplitude) rendering of the Earth's (Newtonian) geoid from gravity data obtained by the GRACE mission, showing the complicated multipolar nature of the shape of our planet, and correspondingly also of its gravitational field (image: `www.nasa.gov/mission_pages/Grace`).

German Aerospace Centre DLR, launched a pair of identical small satellites into the same, very nearly circular, polar orbit. of 6,700 km. In their orbit, the satellites are separated by approximately 200 km along which they share a microwave link that allows to track the distance between the two spacecraft to a precision on the order of millimetres, and thus to measure the multipolar structure of the Earth's gravitational field to a precision three orders of magnitude better than previously possible, ushering in a new era of high-precision global gravity models. In Figure 5.2 we illustrate this complicated multi-polar structure of the Earth's gravitational field with a (greatly exaggerated in amplitude) rendering of the Earth's geoid, in terms of gravity-field data obtained by the GRACE mission. Between 2009 and 2013, ESA's Gravity Field and Steady-State Ocean Circulation Explorer (GOCE) then provided complementary measurements of higher resolution due to its much lower orbit (which necessitated permanent active propulsion in form of an ion thruster). Mission such as GRACE and GOCE can be seen to perform covariant measurements of the tidal gravitational field of the Earth, i.e. they essentially measure frame components of the Riemann (i.e. Weyl) tensor and its higher covariant derivatives, either in the spacecraft's local frame for single-satellite missions via accelerometers etc. (GOCE), or via geodesic deviation along their common geodesic orbit for two-satellite missions (GRACE)*.

5.7.1. Limitations of Exact Metrics

Solving Einstein's equations exactly naturally comes with many idealisations and simplifying assumptions. In particular, we are fundamentally forced to rely on symmetries in order to obtain an exact solution as discussed in, e.g., section 5.2. Inevitably, however, as we attempt to make our exact solution general enough for physical reality in the sense

*We note that, unfortunately from the relativist's point of view, the extensive data processing that is subsequently performed on ground typically involves Newtonian assumptions, since after all, the main customers are not interested in any minute relativistic corrections (yet). What one needs from the relativistic point of view are not the standard published data products of these missions, but the (locally) calibrated raw data, which is usually available, but less convenient to work with.

metric	symmetry	parameters	rotating source?	multiple sources?	multipolar source?
Schwarzschild	spherical, static	R_{S}	no	no	no
Kerr	axisymmetric, stationary	R_{S}, a	yes	no	no
Quevedo-Mashhoon	axisymmetric, stationary	$R_{\mathrm{S}}, a, \ldots$	yes	no	only axisymmetric

Table 5.3.: A (non-exhaustive) comparison of different exact metrics which one might consider for modelling of the exterior gravitation field in the vicinity of the Earth or another planet, versus the requirements from the point of view of an accurate relativistic description for planned an future high-precision experiments (weakly rotating main source, freely prescribable multipole moments with some of these weakly time-dependent, a number of secondary sources that can possibly be treated as point particles, the ability to include local nearby sources; cf. [137]).

discussed above, we will lose all symmetries. In the present subsection, we would like to briefly touch upon these limitations of exact metrics.

As motivated above, an important part of modelling physical reality within general relativity thus consists of having a metric that allows for the description of slowly rotating isolated gravitating bodies in terms of a complete set of multipole moments, which must also be able to exhibit weak time-dependence. Since the solar system consists of many such bodies which influence the gravitational field near the Earth, most notably the Sun and the Moon (experienced e.g. a tides on Earth), this is another requirement for such a metric: it must be able to jointly describe several bodies, some in terms of multipole moments, others (which might be too small and too far away for their multipoles to be of experimental relevance) as Schwarzschild-like point particles. Lastly, it may also be necessary to include local nearby masses, e.g. propellant tanks in satellite experiments.

Unfortunately, when working with exact solutions, we are generally “stuck” with a single gravitating source; which is clearly not how the universe (other stars, galaxies), or solar system (Sun, planets, asteroids), or even the environment of the Earth (Moon) looks. While there are a few special exact “multi-body” solutions, these are generally static and somewhat unphysical, since any realistic collection of sources will not remain static, with its different parts starting to move relative to one another and thus generate time-varying curvature and thus couple to gravitational waves (See [138] for a discussion of the gravitational field of compact objects in general relativity). In Table 5.3 we compare some exact metrics in terms of their (in)ability to describe aspects of space-time curvature in the solar system and near the Earth, starting from the Schwarzschild one representing a point source and then moving on to the Kerr metric which can represent the field of a rotating source, and to the so-called Quevedo-Mashhoon metric [139–142], which, being a generalisation of Kerr, includes arbitrary but only axisymmetric multipole moments. See [137] for a more detailed discussion of the issues discussed in this subsection.

5.7.2. Metric Perturbation Theory

The above-mentioned general and practical issues with using exact solutions to describe the physical gravitational field near the Earth or in the solar system, naturally lead us to consider *approximate* solutions of the Einstein equations, i.e. to metric perturbation theory. In the following subsections, we shall give a brief introduction to this subject, mentioning two important applications as far as solar-system experiments are concerned, namely the post-Minkowskian and post-Newtonian expansions. Although this is of course an interesting subject of its own, it is at the same time fairly complex, and thus a full discussion is beyond the scope of this thesis.

In metric perturbation theory, the aim is to approximate the complicated *physical* metric $g_{\mu\nu}$ by a one-parameter sequence of approximate solutions $g_{\mu\nu}(\epsilon)$ around a *background metric* $\bar{g}_{\mu\nu}$, which can be an exact solution like the Schwarzschild metric, or $\eta_{\mu\nu}$, if we want to perturb around flat space-time. Such a perturbation expansion of the metric generally reads,

$$g_{\mu\nu}(\epsilon) = \bar{g}_{\mu\nu} + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} h^{(n)\mu\nu} = \bar{g}_{\mu\nu} + \epsilon\, h^{(1)\mu\nu} + \frac{\epsilon^2}{2!} h^{(2)\mu\nu} + \frac{\epsilon^3}{3!} h^{(3)\mu\nu} + \mathscr{O}(\epsilon^4)\,, \qquad (5.162)$$

where ϵ is a formal expansion parameter that helps to identify the terms belonging to a certain order.

Before we go on, we should note that as such, this formalism is *not* (yet) tied to any specific interpretation of the $h^{(n)\mu\nu}$, e.g. as representing a perturbation of background space-time in metric perturbation theory. What we have written in (5.162) above is nothing more than a general n^{th} order expansion approach for the metric. We stress this point since as such, (5.162) can also be employed with a different interpretation, namely that of representing, e.g., the terms in the expansion of the metric in Riemann or Fermi normal coordinates as developed in our chapter 6 below. In this setting of (5.162) representing, e.g. Fermi coordinate expansions, the above formal expansion parameter ϵ would become proper length s along all space-like geodesics intersecting the origin of coordinates, and the background metric would be chosen Minkowskian, $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$, the "perturbation" actually representing a Taylor expansion of the metric.

In the specific context of metric perturbation theory then, one writes the general expansion (5.162) also as,

$$g_{\mu\nu}(\epsilon) = \bar{g}_{\mu\nu} + \sum_{n=1}^{\infty} \frac{1}{n!} \delta^n g_{\mu\nu}\,, \qquad \delta^n g_{\mu\nu} = \epsilon^n h^{(n)}_{\mu\nu}\,, \qquad (5.163)$$

where $\delta^n g_{\mu\nu}$ is the n^{th} variation of the metric. Metric perturbation theory is regularly applied in several sub-fields of general relativity. Among these are: the usual (perturbative) description of gravitational waves propagating far from their sources (mostly restricted to first order around Minkowski space-time), in investigations of the stability of space-times (historically in investigations of the stability of black-hole solutions, see e.g., the pioneering work of Regge and Wheeler in the perturbations of Schwarzschild space-time [143]), and in modern times in modelling the post-merger phase of coalescing compact astrophysical binary systems (binary black holes and neutron stars), where typically, a highly excited single black hole has formed, which can be well described by black hole perturbation theory around, e.g. a Kerr background metric, as it rapidly radiates off its event horizon's multipole moments in gravitational radiation (this is called the "ring-down" phase). Lastly, an important application of metric perturbation theory (usually also first order around Minkowski space-time) is in deriving the post-Minkowskian and post-Newtonian expansion of the metric around flat space-time, which we briefly motive in subsection 5.7.3 below.

From First-Order Metric Perturbation to Linearised Einstein Equations

In order to get a feeling for how metric perturbation theory works, we shall truncate (5.162), (5.163) to first order, and explicitly derive all relations for the perturbations of the inverse metric, the Christoffel symbols, the Riemann and Ricci tensors, as well as the Ricci scalar. This leads us to the Einstein equations in first-order perturbation theory around a general background $\bar{g}_{\mu\nu}$. We thus start with the linear perturbation ansatz,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}\,, \qquad \text{where} \quad h_{\mu\nu} \coloneqq \delta g_{\mu\nu}\,, \tag{5.164}$$

as introduced above. To begin with, we need to calculate the first-order expansion of the inverse physical metric, which can also be written the general form,

$$g^{\mu\nu} = \bar{g}^{\mu\nu} + \delta g^{\mu\nu}\,. \tag{5.165}$$

Its variation $\delta g^{\mu\nu}$ can then be obtained from the vanishing variation of the orthonormality condition of the physical metric with itself, i.e. from $\delta(g_{\mu\sigma}g^{\sigma\nu}) = (\delta g_{\mu\sigma})\bar{g}^{\sigma\nu} + \bar{g}_{\mu\sigma}(\delta g^{\sigma\nu}) = 0$, where we have silently linearised, using (5.164) and (5.165). This leads to,

$$\delta g^{\mu\nu} = -\bar{g}^{\mu\sigma}\bar{g}^{\nu\rho}\,\delta g_{\rho\sigma}\,, \qquad \text{which yields} \qquad h^{\mu\nu} = \bar{g}^{\mu\rho}\bar{g}^{\nu\sigma}h_{\rho\sigma}\,, \tag{5.166}$$

so that the first-order perturbation expansion (5.165) is written in terms of $h_{\rho\sigma}$ as

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - \bar{g}^{\mu\rho}\bar{g}^{\nu\sigma}h_{\rho\sigma}\,. \tag{5.167}$$

This tells us that indices on all quantities are to be raised and lowered with the background metric, which has the practical consequence that no factors of the perturbation $h_{\mu\nu}$ can be hidden in differently placed indices.

Having the physical metric and its inverse in terms of the perturbation $h_{\mu\nu}$ at hand, the next steps are to calculate all the derived quantities, i.e. the perturbation expansions of the Christoffel symbols and of the Riemann tensor, and most importantly of its contractions, being the Ricci tensor and scalar, which enter the Einstein field equations. The corresponding first-order expansion of the physical Christoffel symbols is then written as

$$\Gamma^{\sigma}{}_{\mu\nu} = \bar{\Gamma}^{\sigma}{}_{\mu\nu} + \delta\Gamma^{\sigma}{}_{\mu\nu}\,, \tag{5.168}$$

with its perturbation $\delta\Gamma^{\sigma}{}_{\mu\nu} = \delta(g^{\sigma\lambda}\Gamma_{\lambda\mu\nu})$ given by,

$$\delta\Gamma^{\sigma}{}_{\mu\nu} = (\delta g^{\sigma\lambda})\,\bar{\Gamma}_{\lambda\mu\nu} + \bar{g}^{\sigma\lambda}\,\delta\Gamma_{\lambda\mu\nu}\,, \tag{5.169}$$

in terms of the index-lowered perturbation,

$$\delta\Gamma_{\lambda\mu\nu} = \tfrac{1}{2}\left[\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}\right] , \tag{5.170}$$

and the background Christoffel symbols $\bar{\Gamma}^\sigma{}_{\mu\nu}$. Together with (5.169), this yields,

$$\delta\Gamma^\sigma{}_{\mu\nu} = \tfrac{1}{2}\,\bar{g}^{\sigma\lambda}\left[\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu} - 2\bar{\Gamma}^\rho{}_{\mu\nu} h_{\lambda\rho}\right] , \tag{5.171}$$

which can be written as a sum of three covariant derivatives of the metric perturbation by adding zero in the form $\bar{\Gamma}^\rho{}_{\lambda\mu} h_{\rho\nu} - \bar{\Gamma}^\rho{}_{\mu\lambda} h_{\rho\nu} = 0 = \bar{\Gamma}^\rho{}_{\lambda\nu} h_{\mu\rho} - \bar{\Gamma}^\rho{}_{\nu\lambda} h_{\mu\rho}$, i.e. we are left with,

$$\delta\Gamma^\sigma{}_{\mu\nu} = \tfrac{1}{2}\,\bar{g}^{\sigma\lambda}\left[\nabla_\mu h_{\lambda\nu} + \nabla_\nu h_{\mu\lambda} - \nabla_\lambda h_{\mu\nu}\right] . \tag{5.172}$$

This shows that $\delta\Gamma^\sigma{}_{\mu\nu}$ is a tensor, in contrast to $\bar{\Gamma}^\sigma{}_{\mu\nu}$ and $\Gamma^\sigma{}_{\mu\nu}$. Analogously, the physical Riemann tensor is written in terms of a perturbation of the background one, $\bar{R}^\sigma{}_{\rho\mu\nu}$, as

$$R^\sigma{}_{\rho\mu\nu} = \bar{R}^\sigma{}_{\rho\mu\nu} + \delta R^\sigma{}_{\rho\mu\nu} , \tag{5.173}$$

and its perturbation $\delta R^\sigma{}_{\rho\mu\nu}$ follows immediately from inserting the $\delta\Gamma^\sigma{}_{\mu\nu}$ into the definition (3.80) of $R^\sigma{}_{\rho\mu\nu}$, i.e.

$$\delta R^\sigma{}_{\rho\mu\nu} = 2\nabla_{[\mu|}\delta\Gamma^\sigma{}_{\rho|\nu]} , \tag{5.174}$$

The physical Ricci tensor is then given by the contraction $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$. We thus have

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + \delta R_{\mu\nu} , \tag{5.175}$$

with $\delta R_{\mu\nu} = \delta R^\lambda{}_{\mu\lambda\nu}$. The perturbation of the Ricci tensor can be expanded in terms of the metric perturbation, yielding

$$\begin{aligned} \delta R_{\mu\nu} &= \nabla_\lambda\, \delta\Gamma^\lambda{}_{\mu\nu} - \nabla_\nu\, \delta\Gamma^\lambda{}_{\mu\lambda} \\ &= \tfrac{1}{2}\left\{2\nabla^\lambda\nabla_{(\mu|}h_{\nu)\lambda} - \nabla_\nu\nabla_\mu h - \Box h_{\mu\nu}\right\} , \end{aligned} \tag{5.176}$$

where $h := h^\lambda{}_\lambda$ is the trace of the metric perturbation and we have introduced the usual shorthand $\Box := \nabla^\lambda\nabla_\lambda$ to denote the d'Alembertian in curved space-time. A further contraction of the physical Ricci tensor yields the Ricci scalar, $R_{\mathrm{Ric}} = R^\lambda{}_\lambda$, which we can also decompose as $R_{\mathrm{Ric}} = \bar{R}_{\mathrm{Ric}} + \delta R_{\mathrm{Ric}}$. We have

$$\begin{aligned} R_{\mathrm{Ric}} &= g^{\sigma\lambda} R_{\sigma\lambda} = (\bar{g}^{\sigma\lambda} - h^{\sigma\lambda})(\bar{R}_{\sigma\lambda} + \delta R_{\sigma\lambda}) \\ &= \bar{R}_{\mathrm{Ric}} + \bar{g}^{\sigma\lambda}\,\delta R_{\sigma\lambda} - h_{\sigma\lambda}\bar{R}^{\sigma\lambda} , \end{aligned} \tag{5.177}$$

thus, the perturbation δR_{Ric} of the Ricci scalar then reads

$$\delta R_{\mathrm{Ric}} = \bar{g}^{\sigma\lambda}\,\delta R_{\sigma\lambda} - h_{\sigma\lambda}\bar{R}^{\sigma\lambda} = \nabla^\sigma\nabla^\lambda h_{\sigma\lambda} - \Box h - h_{\sigma\lambda}\bar{R}^{\sigma\lambda} . \tag{5.178}$$

So that we can finally write down the perturbation of the Einstein tensor, being

$$\begin{aligned}\delta G_{\mu\nu} &= \delta R_{\mu\nu} - \tfrac{1}{2}(\bar{g}_{\mu\nu}\,\delta R_{\text{Ric}} + h_{\mu\nu}\bar{R}_{\text{Ric}}) \\ &= \tfrac{1}{2}\Big\{2\nabla^\lambda\nabla_{(\mu}h_{\nu)\lambda} - \nabla_\nu\nabla_\mu h - \Box h_{\mu\nu} - h_{\mu\nu}\bar{R}_{\text{Ric}} - \bar{g}_{\mu\nu}(\nabla^\lambda\nabla^\sigma h_{\lambda\sigma} - \Box h - h_{\lambda\sigma}\bar{R}^{\lambda\sigma})\Big\}\,.\end{aligned} \tag{5.179}$$

This can now be somewhat simplified by writing (5.179) in terms of the *trace-reversed* perturbation

$$\tilde{h}_{\mu\nu} := h_{\mu\nu} - \tfrac{1}{2}\bar{g}_{\mu\nu}h\,. \tag{5.180}$$

However, at this point we abandon the present discussion. We just note, that now one could work out the first-order perturbations around, e.g., the Schwarzschild metric, for which there exists a very nice formalism due to Gerlach and Sengupta [144–146] (see also references [147–150] for some more recent higher-order and gauge-invariant applications of this approach), using the symmetry-adapted 2+2 split for a spherical space-time that we employ in Appendix A below in order to calculate a coordinate-independent, closed-form expression for the Riemann tensor.

General n^{th}-Order Formulae

In the previous sub-subsection, we have worked out the first-order perturbations of the tensors and derived quantities of interest by hand in order to get a feeling for metric perturbation theory. Here, we briefly want show the structure of these expansions for arbitrary perturbation order n, in particular for the inverse metric, and how general closed-form expressions for these can be obtained (cf. [151]). The perturbation expansions of the various derived quantities follow from the fact that the perturbation operator δ is a derivation, i.e. it obeys the Leibnitz rule, so that δ^n acting on a product of l tensors $\boldsymbol{T}_1\boldsymbol{T}_2\cdots\boldsymbol{T}_l$ is expanded in terms of a multinomial as

$$\delta^n(\boldsymbol{T}_1\boldsymbol{T}_2\cdots\boldsymbol{T}_l) = \sum_{l=1}^{\infty}\sum_{\substack{k_1,\ldots,k_l\geq 1\\ k_1+\cdots+k_l=n}}\binom{n}{k_1,k_1,\ldots,k_l}(\delta^{k_1}\boldsymbol{T}_1)(\delta^{k_2}\boldsymbol{T}_2)\cdots(\delta^{k_l}\boldsymbol{T}_l)\,, \tag{5.181}$$

where the sum ranges over the 2^{n-1} integer compositions (i.e. “sorted” partitions) of n into $l \leq n$ positive integers k_i, i.e. over all l-tuples $(k_1,\ldots,k_l)$ with $k_1+\cdots+k_l = n$ (compare for the related discussion in subsection 6.2.8). The inverse metric then follows from inverting the series (5.162). A closed formula for its n^{th}-order perturbation can be found as follows. We act with δ^n on the identity $g^{\mu\lambda}g_{\lambda\nu} = \delta^\mu{}_\nu$, use the Leibnitz rule,

$$\sum_{k=0}^{n}\binom{n}{k}(\delta^{n-k}g^{\mu\lambda})(\delta^k g_{\lambda\nu}) = 0\,, \tag{5.182}$$

and pull out the $k=0$ term, which yields the following binomial-type recursion for $\delta^n g^{\mu\nu}$,

$$\delta^n g^{\mu\nu} = -\sum_{k=1}^{n}\binom{n}{k}(\delta^{n-k}g^{\mu\lambda})(\delta^k g_{\lambda\sigma})\bar{g}^{\sigma\nu}\,. \tag{5.183}$$

This recursion can be solved by iteration. We iterate $l-1$ times, where the possible range of l is $1 \leq l \leq n$, yielding a string of l factors of $(\delta^{k_i} g_{\lambda\sigma})\bar{g}^{\sigma\nu}$. The l sums running over indices $k_1, k_2, \ldots, k_l$, from 1 to $n, n-k_1, \ldots, n-(k_1+\cdots+k_{l-1})$, respectively, can again be combined into a sum over all compositions $k_1, \ldots, k_l$ of the integer n, i.e. the summation is subject to the condition $k_1 + \cdots + k_l = n$. The resulting numerical coefficient can be written as,

$$\binom{n}{k_1} \prod_{i=1}^{l-1} \binom{n-(k_1+\cdots+k_i)}{k_{i+1}} = \binom{n}{k_1, \ldots, k_l} := \frac{n!}{k_1! k_2! \cdots k_l!}\,, \tag{5.184}$$

where we have used the well-known decomposition of a multinomial into binomials. Thus, we obtain,

$$\delta^n g^{\mu\sigma} = \sum_{l=1}^{n} (-1)^l \sum_{\substack{k_1,\ldots,k_l \geq 1 \\ k_1+\cdots+k_l=n}} \binom{n}{k_1, \cdots, k_l} \bar{g}^{\mu\nu_1} (\delta^{k_1} g_{\nu_1\nu_2}) \bar{g}^{\nu_2\nu_3} (\delta^{k_2} g_{\nu_3\nu_4}) \cdots \bar{g}^{\nu_{l-1}\nu_l} (\delta^{k_l} g_{\nu_l\nu_{l+1}}) \bar{g}^{\nu_{l+1}\sigma}\,, \tag{5.185}$$

which leads to the following closed-form expression for the perturbation expansion of the inverse metric,

$$g^{\mu\nu}(\epsilon) = \bar{g}^{\mu\nu} + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} \sum_{l=1}^{n} (-1)^l \sum_{\substack{k_1,\ldots,k_l \geq 1 \\ k_1+\cdots+k_l=n}} h^{(\boldsymbol{k_1})\mu\lambda_1} h^{(\boldsymbol{k_2})}{}_{\lambda_1}{}^{\lambda_2} \cdots h^{(\boldsymbol{k_{l-1}})\nu\lambda_{k_1}} h^{(\boldsymbol{k_l})}{}_{\lambda_1}{}^{\lambda_2}\,, \tag{5.186}$$

and up to 3$^{\text{rd}}$ order, this reads explicitly,

$$\begin{aligned} g^{\mu\nu}(\epsilon) &= \bar{g}^{\mu\nu} - \epsilon\, h^{(\mathbf{1})\mu\nu} - \frac{\epsilon^2}{2}\left[h^{(\mathbf{2})\mu\nu} - 2h^{(\mathbf{1})\mu\lambda} h^{(\mathbf{1})}{}_{\lambda}{}^{\nu}\right] \\ &\quad - \frac{\epsilon^3}{6}\Big[h^{(\mathbf{3})\mu\nu} - 3\big(h^{(\mathbf{2})\mu\lambda} h^{(\mathbf{1})}{}_{\lambda}{}^{\nu} + h^{(\mathbf{1})\mu\lambda} h^{(\mathbf{2})}{}_{\lambda}{}^{\nu}\big) + 6h^{(\mathbf{1})\mu\lambda_1} h^{(\mathbf{1})}{}_{\lambda_1\lambda_2} h^{(\mathbf{1})\lambda_2\nu}\Big] + \mathscr{O}(\epsilon^4). \end{aligned} \tag{5.187}$$

Perturbation expansions for other quantities then follow along the same lines, for example, a closed expression for the n^{th}-order perturbation of the metric determinant can be obtained from (3.95), i.e. by iterating on,

$$g(\epsilon) = \det g_{\mu\nu}(\epsilon) = \frac{1}{4!} \epsilon^{\sigma_1\sigma_2\sigma_3\sigma_4} \epsilon^{\nu_1\nu_2\nu_3\nu_4} g_{\sigma_1\nu_1}(\epsilon) \cdots g_{\sigma_4\nu_4}(\epsilon)\,, \tag{5.188}$$

the lowest-order terms of which are given by,

$$g(\epsilon) = \bar{g}\Big\{1 + \epsilon\, h^{(\mathbf{1})\lambda}{}_{\lambda} + \frac{\epsilon^2}{2}\big[h^{(\mathbf{2})\lambda}{}_{\lambda} + (h^{(\mathbf{1})\lambda}{}_{\lambda})^2 - h^{(\mathbf{1})\mu\nu} h^{(\mathbf{1})}{}_{\mu\nu}\big] + \mathscr{O}(\epsilon^3)\Big\}\,. \tag{5.189}$$

The related square-root of the metric determinant, being the coefficient of the volume element, is then found to have the following low-order perturbation expansion,

$$\sqrt{-g(\epsilon)} = \sqrt{-\bar{g}}\,\Big\{1 - \frac{\epsilon}{2} h^{(\mathbf{1})\lambda}{}_{\lambda} + \frac{\epsilon^2}{4}\big[h^{(\mathbf{2})\lambda}{}_{\lambda} + \tfrac{1}{2}(h^{(\mathbf{1})\lambda}{}_{\lambda})^2 - h^{(\mathbf{1})\mu\nu} h^{(\mathbf{1})}{}_{\mu\nu}\big] + \mathscr{O}(\epsilon^3)\Big\}\,, \tag{5.190}$$

where $\bar{g}$ is the determinant of the background metric. Let us note that these formulae are nowadays implemented in computer algebra systems, such as the `xPert` companion package [151] to the excellent open source system `xAct` [152] on top of Mathematica.

5.7.3. Weak Fields and slow Motion: Post-Minkowskian and Post-Newtonian Theory

Before we close this chapter, we would like to motivate an important application of first-order perturbation theory in general relativity, namely the post-Minkowskian and post-Newtonian expansions, which should ultimately be used to describe the gravitational field around the earth or in the solar system, where space-time is weakly curved and velocities are small compared to the speed of light. As argued in subsection 5.7.1, only a perturbative description is ultimately suited overcome the limitations of exact metrics, and to *consistently* describe space-time within the solar system, being an n-body system of weakly gravitating and slowly rotating, approximately spherically objects (sun, planets, asteroids), each of which can be modelled in terms of relativistic multipole moments to a certain desired order. The post-Minkowskian expansion potentially offers such a description. Being a weak-field expansion in terms of Newton's gravitational constant G_{N}, it keeps all velocities fully general, in terms of retarded integrals of the type,

$$\tilde{h}_{\mu\nu}(t,\boldsymbol{x}) \propto \int \frac{T_{\mu\nu}\left(t-\frac{1}{c}|\boldsymbol{x}-\boldsymbol{x}'|,\boldsymbol{x}'\right)}{|\boldsymbol{x}-\boldsymbol{x}'|}\, c\,dt'\,d^3x'\,. \tag{5.191}$$

As such it is then the starting point of the post-Newtonian expansion, where all retardations are expanded out, so that the resulting post-Newtonian gravitational "potentials" are then *instantaneous*, just as in Newtonian mechanics.

The post-Minkowskian and post-Newtonian formalism is nowadays an extensive subject of its own, on which whole books are written, see e.g. the comprehensive and recent textbook by Poisson and Will [118] which has a slight focus on gravitational-wave emission from binary sources, and also Kopeikin, Efroimsky and Kaplan [153] and Soffel [154], which focuses more on relativistic celestial mechanics and astrometry from the solar-system perspective. Generally, there are two slightly different approaches to post-Minkowskian and post-Newtonian theory, the Will-Wiseman formalism exhibited in [118] on the one hand, and the more systematic Blanchet-Darmour-Iyer formalism [155] on the other. While both start from the so-called Landau-Lifshitz formulation of general relativity (cf. Poisson and Will [118, Sec. 6.1]), the approach by Blanchet, Darmour, and Iyer seems to be more elegant, since the the metric perturbation $h_{\mu\nu}$ for vacuum space-times exterior to the source is directly decomposed into two sets of symmetric and trace-free (STF) Cartesian tensors, the *mass moments*, and *current moments* (the lowest-order incarnations of which are essentially the electric and magnetic parts E and B of the Weyl tensor from subsection 5.6.2, which were also seen to be Cartesian STF tensors). A recent review on this approach can be found in the Living Review by Blanchet [156].

6 Inertial Frames and Fermi Coordinates

In the present chapter we introduce the main tools that we will employ to locally model Bose-Einstein coordinates in general relativity, namely Fermi normal coordinates. Since Fermi coordinates are in essence an extension of local inertial coordinates, and thus an extension of a certain kind of inertial frame, we first come back to discuss in some detail the construction of local inertial frames, more specifically of their attached local inertial coordinates in a general pseudo-Riemannian manifold. Thereby we hope to show, how these two different kinds of local inertial frame each leads to a specific set of normal coordinates, these being Riemann normal coordinates in the first case, and Fermi normal coordinates in the second.

Since the Riemann normal coordinate expansions serve as the basis for the construction of Fermi normal coordinates, we begin by discussing Riemann coordinates in some detail. The important objective here is, to work out the expansions of tensor fields in Riemann and later also in Fermi coordinates, most importantly those of the tetrads and of the metric. For this, we initially introduce and pursue what we call the *traditional* approach that goes back to the 1920s, in terms of so-called generalised connection coefficients that follow from partial derivatives of the Christoffel symbols in the equation of parallel transport.

We then move on to the more modern approach for deriving tensorial expansion coefficients, in terms of manifestly covariant recursions, which can be seen to essentially follow from the equation of geodesic deviation. These recursions can be solved in closed form, which we demonstrate extensively. We also provide a new combinatorial interpretation for the above-mentioned complicated tensorial expansion coefficients of the tetrads and metric that are generated by these covariant recursions and their closed solutions.

6.1. Inertial Frames and Local Lorentz Coordinates

Physically speaking, by the term *local inertial frame* we generally understand a mathematical realisation of the equivalence principle, i.e., that space-time curvature can be transformed away locally, as discussed in chapter 5. Thus, it is a local coordinate frame or basis in which

1. space-time as represented by the metric is flat, $g_{\mu\nu} = \eta_{\mu\nu}$, and
2. gravitational forces (in Newtonian language) or, correspondingly, the Christoffel symbols vanish, ${\Gamma^\kappa}_{\alpha\beta} = 0$.

Clearly, in modelling space-time as a pseudo-Riemannian manifold, it is natural to base such a local inertial frame on a non-coordinate basis or tetrad as discussed for the case of special relativity in subsection 4.2.2, and thus the first of the two above requirements will be naturally fulfilled in view of the tetrad normalisation condition (4.42).

It thus remains to be shown that, additionally, the Christoffel symbols can be made to vanish by a suitable choice of coordinates. That this must be possible in general can be inferred from a simple counting of the number of independent components in a general ansatz for the metric and the coordinate transformation in terms of a Taylor expansion, as we will now show, before we explicitly construct the corresponding coordinate transformation, first around a single point in space-time, and subsequently along the whole world-line of a physical observer.

Counting Independent Components

For the present counting argument, we assume that the metric in the sought-for coordinates is given as a Taylor expansion around some point $\mathscr{P}$ in space-time. At zeroth order, the metric tensor at $\mathscr{P}$ will then transform with (two instances) of the Jacobian matrix of the transformation, as in (4.39). This Jacobian matrix has $4 \times 4 = 16$ independent components in general, 6 of which correspond to the freedom of performing (proper) Lorentz transformations, i.e., boosts and rotations, which leaves 10 components to encode gravity. This is seen to match with the number of independent components of the metric tensor, that – being a symmetric 4×4 matrix – also has 10 independent components. Clearly, the metric can then always be transformed to the flat Minkowski metric, and this statement is nothing else than (4.42), with the Jacobian matrix given by the tetrad (4.40), that we encountered in subsection 4.2.2.

At first order of the Taylor expansion, the first partial derivatives of the metric transform with the second partial derivatives of the coordinate transformation, or equivalently, with the first partial derivatives of the Jacobian matrix, which features $10 \times 4 = 40$ independent components (in a coordinate basis), which matches precisely with the number of independent components of the Christoffel symbols. Thus, by choosing the coordinate transformation appropriately, it is always possible to make the Christoffel symbols vanish.

At third order, however, we find that the second partial derivatives of the metric possess $10 \times 10 = 100$ independent components in general, while the second partial derivatives of the Jacobian matrix have $4\binom{4+3-1}{3} = 80$ independent components. This leaves 20 components that can't be transformed to zero by a coordinate transformation, and according to the discussion in subsection 5.6.2 (as summarised in Table 5.1), this is exactly the number of independent components of the Riemann curvature tensor.

There are actually two different kinds of inertial frames, both are incarnations of the equivalence principle. The first one is based on a single point (or "event") in space-time and is thus somewhat unphysical in the sense that it cannot be used to describe physical observers, who are always in "motion" through four-dimensional space-time, at least along the time coordinate lines. This inertial frame can be extended radially away from the original point and leads to Riemann normal coordinates. The other kind of inertial frame is constructed to be valid along a whole world-line, thus, in contrast to the previous one, it is well suited for the description of physical observers. This physical inertial frame involves a 1+3-split (or "threading") of space-time into time (the observer's local time direction) and space (the observer's local rest space).

Inertial Frame at a Single Point

In the following, we consider a (pseudo-) Riemannian *background* space-time manifold $\mathscr{M}$ given in some a priori coordinates X^μ, for which we shall use capital letters as before. Let the metric tensor of this space-time be $g_{\mu\nu}(X^\sigma)$ and let a tetrad basis field $e_{\hat\alpha}{}^\mu(X^\sigma)$ with $g_{\mu\nu}e_{\hat\alpha}{}^\mu e_{\hat\beta}{}^\nu = \eta_{\hat\alpha\hat\beta}$ be defined on $\mathscr{M}$ as usual.

Now let $\mathscr{P}$ be a point in space-time with coordinates $X^\mu_{\mathscr{P}}$, where we want to set up our inertial frame. In order to construct a system of local *inertial* coordinates were all connection coefficients vanish, we shall, in the following, investigate the second-order solution of the geodesic equation (3.52a) along all geodesics $\mathscr{C}$ with coordinate representation $X^\sigma_{\mathscr{C}}(\lambda)$, that intersect our point $\mathscr{P}$, i.e.,

$$\frac{d^2X^\sigma_{\mathscr{C}}(\lambda)}{d\lambda^2} + \Gamma^\sigma{}_{\nu_1\nu_2}(X^\mu_{\mathscr{C}}(\lambda))\frac{dX^{\nu_1}_{\mathscr{C}}(\lambda)}{d\lambda}\frac{dX^{\nu_2}_{\mathscr{C}}(\lambda)}{d\lambda} = 0\,, \tag{6.1}$$

in terms of a Taylor expansion (we shall drop the clumsy argument $X^\mu_{\mathscr{C}}(\lambda)$ of the Christoffel symbols and other tensors from now on and simply abbreviate with λ where necessary). At $\mathscr{P}$ we shall choose, without loss of generality, $\lambda = 0$ for the affine parameter. Equation (6.1) is a second-order, ordinary differential equation, which requires two initial conditions in order to uniquely specify the solution of the associated initial-value problem, i.e.,

$$X^\sigma_{\mathscr{P}} = X^\sigma_{\mathscr{C}}(0) \tag{6.2a}$$

$$v^\sigma = \left.\frac{dX^\sigma_{\mathscr{C}}(\lambda)}{d\lambda}\right|_{\mathscr{P}} = \left.\frac{dX^\sigma_{\mathscr{C}}(\lambda)}{d\lambda}\right|_{\lambda=0}, \tag{6.2b}$$

To second order in λ, this Taylor-series solution is given in terms of these two initial conditions and takes the form

$$X^\sigma(\lambda) = X^\sigma_{\mathscr{P}} + \left.\frac{dX^\sigma}{d\lambda}\right|_{\mathscr{P}}\lambda + \frac{1}{2}\left.\frac{d^2X^\sigma}{d\lambda^2}\right|_{\mathscr{P}}\lambda^2 + \mathscr{O}(\lambda^3)\,, \tag{6.3}$$

where the first-order coefficient is just the initial tangent $v^\sigma|_{\mathscr{P}}$ and the second-order coefficient is determined by the geodesic equation (6.1), so that we have

$$X^\sigma(\lambda) = X^\sigma_{\mathscr{P}} + v^\sigma\lambda - \tfrac{1}{2}\Gamma^\sigma{}_{\nu_1\nu_2}v^{\nu_1}v^{\nu_2}\big|_{\mathscr{P}}\lambda^2 + \mathscr{O}(\lambda^3)\,. \tag{6.4}$$

We may also assume, without loss of generality, that the initial tangent is normalised to unity, i.e., that $v^\mu v_\mu \equiv g_{\mu\nu}v^\mu v^\nu = 1$. We can now introduce local inertial coordinates in terms of the constant tetrad components $v^{\hat\alpha} = e^{\hat\alpha}{}_\sigma v^\sigma|_{\mathscr{P}}$ of the tangent vector v^σ at $\mathscr{P}$ by scaling with the affine parameter along the geodesics,

$$x^{\hat\alpha} = v^{\hat\alpha}\lambda\,, \tag{6.5}$$

by replacing $v^\nu|_{\mathscr{P}}\lambda = e_{\hat\alpha}{}^\nu|_{\mathscr{P}}x^{\hat\alpha}$ in the expansion (6.4) above. We have thus chosen for coordinate lines the straight lines of length λ along the normalised initial tangent directions in Minkowski space $v^{\hat\alpha}$. This turns the second-order Taylor-series solution (6.3) of the geodesic equation into a coordinate transformation from the local inertial coordinates (6.5) to the a priori coordinates X^σ of the background space-time,

$$X^\sigma(x^\alpha) = X^\sigma_{\mathscr{P}} + e_{\hat{\alpha}}{}^\sigma\big|_{\mathscr{P}} x^{\hat{\alpha}} - \tfrac{1}{2}\Gamma^\sigma{}_{\nu_1\nu_2} e_{\hat{\alpha}_1}{}^{\nu_1} e_{\hat{\alpha}_2}{}^{\nu_2}\Big|_{\mathscr{P}} x^{\hat{\alpha}_1} x^{\hat{\alpha}_2} + \mathscr{O}(x^3)\,. \tag{6.6}$$

Equation (6.6) is straightforward to invert, and yields the transformation from the a priori to the local inertial coordinates,

$$x^\beta(X^\nu) = e^{\hat{\beta}}{}_\nu\big|_{\mathscr{P}}(X^\nu - X^\nu_{\mathscr{P}}) + \tfrac{1}{2} e^{\hat{\beta}}{}_\sigma \Gamma^\sigma{}_{\nu_1\nu_2}\Big|_{\mathscr{P}}(X^{\nu_1} - X^{\nu_1}_{\mathscr{P}})(X^{\nu_2} - X^{\nu_2}_{\mathscr{P}}) + \mathscr{O}(X^3)\,. \tag{6.7}$$

The Jacobian of the coordinate transformation (6.6) and its inverse (6.7) are then given to first order by (we have started abbreviating the coordinate arguments X^σ and $x^{\hat{\alpha}}$ in the $\mathscr{O}$-symbols as X and x, respectively)

$$\frac{\partial X^\sigma(x^\alpha)}{\partial x^{\hat{\beta}}} = e_{\hat{\beta}}{}^\sigma\big|_{\mathscr{P}} - \Gamma^\sigma{}_{\nu_1\nu_2} e_{\hat{\alpha}}{}^{\nu_1} e_{\hat{\beta}}{}^{\nu_2}\Big|_{\mathscr{P}} x^{\hat{\alpha}} + \mathscr{O}(x^2)\,, \tag{6.8a}$$

$$\frac{\partial x^{\hat{\beta}}(X^\nu)}{\partial X^\sigma} = e^{\hat{\beta}}{}_\sigma\big|_{\mathscr{P}} + e^{\hat{\beta}}{}_\lambda \Gamma^\lambda{}_{\nu_1\sigma}\Big|_{\mathscr{P}}(X^{\nu_1} - X^{\nu_1}_{\mathscr{P}}) + \mathscr{O}(X^2)\,, \tag{6.8b}$$

respectively, where we note that the tetrad is recovered by restricting the Jacobian to $\mathscr{P}$, i.e.,

$$e_{\hat{\beta}}{}^\sigma\big|_{\mathscr{P}} = \frac{\partial X^\sigma(x^\alpha)}{\partial x^{\hat{\beta}}}\bigg|_{\mathscr{P}}\,, \qquad e^{\hat{\beta}}{}_\sigma\big|_{\mathscr{P}} = \frac{\partial x^{\hat{\beta}}(X^\nu)}{\partial X^\sigma}\bigg|_{\mathscr{P}}\,. \tag{6.9}$$

In order to calculate the connection coefficients in local inertial coordinates, we also need the first partial derivatives of the Jacobian, which read

$$\frac{\partial^2 X^\sigma(x^\alpha)}{\partial x^\alpha \partial x^\beta} = -\Gamma^\sigma{}_{\nu_1\nu_2} e_{\hat{\alpha}}{}^{\nu_1} e_{\hat{\beta}}{}^{\nu_2}\Big|_{\mathscr{P}} + \mathscr{O}(x)\,, \tag{6.10}$$

The general inhomogeneous transformation law for connection coefficients (3.42) then yields

$$\tilde{\Gamma}^\kappa{}_{\alpha\beta}\big|_{\mathscr{P}} = e^{\hat{\kappa}}{}_\sigma\big(-\Gamma^\sigma{}_{\nu_1\nu_2} e_{\hat{\alpha}}{}^{\nu_1} e_{\hat{\beta}}{}^{\nu_2} + e_{\hat{\alpha}}{}^\mu e_{\hat{\beta}}{}^\nu \Gamma^\sigma{}_{\mu\nu}\big)\Big|_{\mathscr{P}} + \mathscr{O}(x) = 0 + \mathscr{O}(x)\,. \tag{6.11}$$

We have thus explicitly constructed a coordinate transformation that transforms away the connection coefficients to linear order around the point $\mathscr{P}$.

Inertial and Non-Inertial Frames along a World-Line

As remarked above, the inertial frame based on a single point is not suited for the description of physical observers, since these necessarily move along time-like world-lines $\mathscr{W}$ and are therefore *always in motion* through space-time, at least along their local time direction given by u^μ. The definition of local inertial coordinates around a point (6.5) is now generalised to apply to *every point* along the observer's world-line,

$$x^{\hat{0}} = c\tau\,, \qquad \text{proper time along } \mathscr{W} \tag{6.12a}$$

$$x^{\hat{\imath}} = v^{\hat{\imath}} s\,, \qquad v^{\hat{\imath}} = e^{\hat{\imath}}{}_{\sigma} v^{\sigma}\,, \qquad \text{proper distance normal to } \mathscr{W}\,. \tag{6.12b}$$

This means that we apply the inertial-coordinate construction of section 6.1 now exclusively to the spacial sub-space $\Sigma_{\boldsymbol{u}}(\tau)$ orthogonal to the world-line $\mathscr{W}$, i.e., to the observer's rest space, which yields the spacial inertial coordinates, the time coordinate being given by the observer's proper time τ (see the illustration in Figure 6.1). Thus the tetrad effects a local 1+3 split (or *threading*) of space-time into time and the observer's purely space-like rest space orthogonal to $e_{\hat{0}}{}^{\mu}$.

In general, we require that the tetrad and metric in these local (non-)inertial coordinates be those of an accelerating and rotating observer in special relativity (4.96), as derived in subsection 4.3.2. The coordinate transformation from background to local inertial coordinates along a world-line is again taken to be,

$$X^{\sigma}(x^{\hat{\imath}}, x^{\hat{0}}) = X^{\sigma}_{\mathscr{W}}(x^{\hat{0}}) + e_{\hat{\imath}}{}^{\sigma}(x^{\hat{0}})\big|_{\mathscr{W}} x^{\hat{\imath}} - \tfrac{1}{2}\Gamma^{\sigma}{}_{\nu_1\nu_2}(x^{\hat{0}})\, e_{\hat{\imath}_1}{}^{\nu_1}(x^{\hat{0}})\, e_{\hat{\imath}_2}{}^{\nu_2}(x^{\hat{0}})\Big|_{\mathscr{W}} x^{\hat{\imath}_1} x^{\hat{\imath}_2} + \mathscr{O}(x^3)\,, \tag{6.13}$$

which differs from (6.6) only in its split coordinate dependence on $x^{\hat{0}}$ and $x^{\hat{\imath}}$, as well as in that all coefficients of the first-order Taylor expansion now depend explicitly on $x^{\hat{0}}$. The inverse transformation is,

$$x^{a}(X^{\nu}) = e^{\hat{a}}{}_{\nu}(x^{\hat{0}})\big|_{\mathscr{W}}(X^{\nu} - X^{\nu}_{\mathscr{W}}) + \tfrac{1}{2} e^{\hat{a}}{}_{\sigma}\Gamma^{\sigma}{}_{\nu_1\nu_2}\Big|_{\mathscr{W}}(X^{\nu_1} - X^{\nu_1}_{\mathscr{W}})(X^{\nu_2} - X^{\nu_2}_{\mathscr{W}}) + \mathscr{O}(X^3)\,, \tag{6.14a}$$

$$x^{0}(X^{\nu}) = e^{\hat{0}}{}_{\nu}(x^{\hat{0}})\big|_{\mathscr{W}}(X^{\nu} - X^{\nu}_{\mathscr{W}}) + \tfrac{1}{2} e^{\hat{0}}{}_{\sigma}\Gamma^{\sigma}{}_{\nu_1\nu_2}\Big|_{\mathscr{W}}(X^{\nu_1} - X^{\nu_1}_{\mathscr{W}})(X^{\nu_2} - X^{\nu_2}_{\mathscr{W}}) + \mathscr{O}(X^3)\,. \tag{6.14b}$$

In contrast to the construction in section 6.1, where the connection components could be completely transformed away, we now find that in the present context of a non-inertial observer's frame this is only partially possible since some of the connection coefficients on the world-line are determined by the observer's transport law (4.48), (4.51). As in section 6.1, we calculate the Jacobian matrices of (6.13) and (6.7). Its spacial vector is just the inertial-coordinate-around-a-point expression (6.8), restricted to the spacial sub-space, as anticipated,

$$\frac{\partial X^{\sigma}(x^{\hat{\imath}}, x^{\hat{0}})}{\partial x^{\hat{a}}} = e_{\hat{a}}{}^{\sigma}(x^{\hat{0}})\big|_{\mathscr{W}} - \Gamma^{\sigma}{}_{\nu_1\nu_2}(x^{\hat{0}}) e_{\hat{a}}{}^{\nu_1}(x^{\hat{0}}) e_{\hat{\imath}}{}^{\nu_2}(x^{\hat{0}})\Big|_{\mathscr{W}} x^{\hat{\imath}} + \mathscr{O}(x^2)\,, \tag{6.15}$$

with the all-spacial second derivative given by,

$$\frac{\partial^2 X^{\sigma}(x^{\hat{\imath}}, x^{\hat{0}})}{\partial x^{\hat{\imath}_1}\partial x^{\hat{\imath}_2}} = -\Gamma^{\sigma}{}_{\nu_1\nu_2}(x^{\hat{0}}) e_{\hat{\imath}_1}{}^{\nu_1}(x^{\hat{0}}) e_{\hat{\imath}_2}{}^{\nu_2}(x^{\hat{0}})\Big|_{\mathscr{W}} + \mathscr{O}(x)\,. \tag{6.16}$$

The time-like vector, however, yields a contribution from acceleration and rotation, given by the transport matrix,

$$\frac{\partial X^{\sigma}(x^{\hat{\imath}}, x^{\hat{0}})}{\partial x^{\hat{0}}} = \Big[1 - \tfrac{1}{c}\Omega^{\hat{0}}{}_{\hat{\imath}}(x^{\hat{0}}) x^{\hat{\imath}}\Big] e_{\hat{0}}{}^{\sigma}(x^{\hat{0}})\big|_{\mathscr{W}} - \tfrac{1}{c}\Omega^{\hat{d}}{}_{\hat{\imath}}(x^{\hat{0}}) x^{\hat{\imath}} e_{\hat{d}}{}^{\sigma}(x^{\hat{0}})\big|_{\mathscr{W}} + \mathscr{O}(x^2)\,, \tag{6.17}$$

which is just the special-relativistic expression (4.89) from subsection 4.3.2. The mixed time–time and time–space second partial derivatives of the coordinate transformation are then given by the transport matrix and can be combined into,

$$\frac{\partial^2 X^\sigma(x^{\hat\imath}, x^{\hat 0})}{\partial x^{\hat 0}\partial x^{\hat\alpha}} = -\tfrac{1}{c}\Omega^{\hat\delta}{}_{\hat\alpha}(x^{\hat 0}) e_{\hat\delta}{}^\sigma(x^{\hat 0})\big|_{\mathscr{W}} + \mathscr{O}(x)\,. \tag{6.18}$$

With the inverse Jacobian being given by (6.8), and as above in (6.11), we calculate the zeroth-order expression for the Christoffel symbols in the present, physical local inertial coordinates. The purely spacial Christoffel symbols are then given by the restriction of (6.11) to the spacial subspace orthogonal to $\mathscr{W}$. The mixed-index Christoffel symbols then transform according to,

$$\begin{aligned}
\tilde\Gamma^{\hat\kappa}{}_{\hat 0\hat\alpha}(x) &= \frac{\partial x^{\hat\kappa}(X^\nu)}{\partial X^\sigma}\left[\frac{\partial^2 X^\sigma(x^{\hat\imath}, x^{\hat 0})}{\partial x^{\hat 0}\partial x^{\hat\alpha}} + \Gamma^\sigma{}_{\nu_1\nu_2}\frac{\partial X^{\nu_1}(x^{\hat\imath}, x^{\hat 0})}{\partial x^{\hat 0}}\frac{\partial X^\sigma(x^{\hat\imath}, x^{\hat 0})}{\partial x^{\hat\alpha}}\right] \\
&= e^{\hat\kappa}{}_\sigma(x^{\hat 0})\left[\partial_{\hat 0} e_{\hat\alpha}{}^\sigma(x^{\hat 0}) + \Gamma^\sigma{}_{\nu_1\nu_2} e_{\hat 0}{}^{\nu_1} e_{\hat\alpha}{}^{\nu_2}\right] + \mathscr{O}(x) \\
&= -\tfrac{1}{c}\Omega^{\hat\kappa}{}_{\hat\alpha}(x^{\hat 0}) + \mathscr{O}(x)\,.
\end{aligned} \tag{6.19}$$

where, in the second equation, we have used the general transformation relation (3.39) between the Christoffel symbols and the orthonormal-frame connection coefficients, i.e.,

$$e^{\hat\kappa}{}_\sigma\left[e_{\hat 0}{}^\nu\partial_\nu e_{\hat\alpha}{}^\sigma + \Gamma^\sigma{}_{\nu\lambda} e_{\hat 0}{}^\nu e_{\hat\alpha}{}^\lambda\right] = -\tfrac{1}{c}\Omega^{\hat\kappa}{}_{\hat\alpha}\,. \tag{6.20}$$

In summary, the purely spacial part of the Christoffel symbols can be made to vanish at $\mathscr{W}$, while the mixed time–space Christoffels are given by a transport matrix in the non-inertial case, i.e.,

$$\tilde\Gamma^{\hat\kappa}{}_{\hat\imath_1\hat\imath_2}(x) = 0 + \mathscr{O}(x)\,, \qquad \tilde\Gamma^{\hat\kappa}{}_{\hat 0\hat\alpha}(x) = -\tfrac{1}{c}\Omega^{\hat\kappa}{}_{\hat\alpha}\,. \tag{6.21}$$

6.2. Riemann Normal Coordinates

Before we continue the above discussion of an observer's physical local inertial frame and coordinates in the next section with the introduction of Fermi normal coordinates, we first introduce and discuss Riemann coordinates in a somewhat extensive fashion. This is essential, since the Riemann normal coordinate expansion is actually used in the spacial sector of Fermi coordinates, and all the expansion techniques developed for the former, i.e. essentially all results from the present section, can be used with only minor adaptions also in the discussion of Fermi coordinates.

The concept of what are today called Riemann normal coordinates was originally introduced by Bernhard Riemann in his habilitation colloquium in front of the philosophical faculty at the university of Göttingen in 1851, being published only posthumously, see e.g. [157, p. 254]. These concepts were only worked out into a coherent formalism much later by Veblen and Thomas and others [158, 159] beginning in the early 1920s. Nowadays, Riemann normal coordinates are an important mathematical tool in differential geometry and general relativity, having been used, for example, in the proof of the Atiyah-Singer index theorem [160], and in approaches to numerical relativity [161], to name just a few. As such they are a classic subject and treated in many of the classic texts on differential geometry and general relativity [162–164], at least in the traditional approach due to Veblen and Thomas. More recent literature references, some of which employ the more modern second-order recursion approach are [165–170].

6.2.1. Power-Series Solution of the Geodesic Equation

We begin, along the same lines as in section 6.1, by investigating the power-series solution of the geodesic equation, this time more generally and to arbitrary order n. As an ordinary differential equation, (6.1) admits, in particular, an power-series solution of the general form,

$$X^\sigma_{\mathscr{C}}(\lambda) = X^\sigma_0 + v^\sigma \lambda + \sum_{n=2}^{\infty} \frac{1}{n!} \frac{d^n X^\sigma_{\mathscr{C}}(\lambda)}{d\lambda^n}\bigg|_{\mathscr{P}} \lambda^n \,. \tag{6.22}$$

In order to evaluate (6.22) further, we clearly need to find an expression for the higher parametric derivatives of $X^\mu_{\mathscr{C}}(\lambda)$. Differentiating the geodesic equation (6.1) with respect to λ yields

$$\begin{aligned}\frac{d^3 X^\sigma_{\mathscr{C}}(\lambda)}{d\lambda^3} &+ \Big(\frac{d}{d\lambda}\Gamma^\sigma{}_{\nu_1\nu_2}(\lambda)\Big)\frac{dX^{\nu_1}_{\mathscr{C}}(\lambda)}{d\lambda}\frac{dX^{\nu_2}_{\mathscr{C}}(\lambda)}{d\lambda} \\ &+ \Gamma^\sigma{}_{\nu_1\nu_2}(\lambda)\frac{d^2 X^{\nu_1}_{\mathscr{C}}(\lambda)}{d\lambda^2}\frac{dX^{\nu_2}_{\mathscr{C}}(\lambda)}{d\lambda} + \Gamma^\sigma{}_{\nu_1\nu_2}(\lambda)\frac{dX^{\nu_1}_{\mathscr{C}}(\lambda)}{d\lambda}\frac{d^2 X^{\nu_2}_{\mathscr{C}}(\lambda)}{d\lambda^2} = 0\,,\end{aligned} \tag{6.23}$$

and using the geodesic equation again for the second derivatives in the last terms, we obtain

$$\frac{d^3 X^\sigma_{\mathscr{C}}(\lambda)}{d\lambda^3} + (\Gamma^\sigma{}_{\nu_1\nu_2,\nu_3} - \Gamma^\lambda{}_{\nu_3\nu_1}\Gamma^\sigma{}_{\lambda\nu_2} - \Gamma^\lambda{}_{\nu_3\nu_2}\Gamma^\sigma{}_{\nu_1\lambda})\frac{dX^{\nu_1}_{\mathscr{C}}(\lambda)}{d\lambda}\frac{dX^{\nu_2}_{\mathscr{C}}(\lambda)}{d\lambda}\frac{dX^{\nu_3}_{\mathscr{C}}(\lambda)}{d\lambda} = 0\,. \tag{6.24}$$

It will prove convenient to write (6.24) and its higher-order generalisation compactly as

$$\frac{d^3 X^\sigma_{\mathscr{C}}(\lambda)}{d\lambda^3} + \Gamma^\sigma{}_{\nu_1\nu_2\nu_3}(\lambda)\frac{dX^{\nu_1}_{\mathscr{C}}(\lambda)}{d\lambda}\frac{dX^{\nu_2}_{\mathscr{C}}(\lambda)}{d\lambda}\frac{dX^{\nu_3}_{\mathscr{C}}(\lambda)}{d\lambda} = 0\,, \tag{6.25}$$

where we have introduced the very convenient abbreviation,

$$\Gamma^\sigma{}_{\nu_1\nu_2\nu_3}(\lambda) := \Gamma^\sigma{}_{\nu_1\nu_2,\nu_3} - \Gamma^\lambda{}_{\nu_3\nu_1}\Gamma^\sigma{}_{\lambda\nu_2} - \Gamma^\lambda{}_{\nu_3\nu_2}\Gamma^\sigma{}_{\nu_1\lambda}\,. \tag{6.26}$$

Here, the terms on the right-hand side also depend on the parameter λ through $\Gamma^\sigma{}_{\nu_1\nu_2}(\lambda)$ (and we use the same symbol for parameter and contraction index, which should not cause confusion). Continuing this process with higher derivatives of the geodesic equation (6.1), we obtain at n^{th} order,

$$\frac{d^n X^\sigma_{\mathscr{C}}(\lambda)}{d\lambda^n} + \Gamma^\sigma{}_{\nu_1\nu_2\cdots\nu_n}(\lambda)\frac{dX^{\nu_1}_{\mathscr{C}}(\lambda)}{d\lambda}\frac{dX^{\nu_2}_{\mathscr{C}}(\lambda)}{d\lambda}\cdots\frac{dX^{\nu_n}_{\mathscr{C}}(\lambda)}{d\lambda} = 0\,, \qquad n \geq 2\,, \tag{6.27}$$

where we have recursively defined the *generalised connection coefficients* (hiding from now on the dependence on the affine parameter λ) as

$$\Gamma^{\sigma}{}_{\nu_1\cdots\nu_n} := \Gamma^{\sigma}{}_{\nu_1\cdots\nu_{n-1},\,\nu_n} - \sum_{i=1}^{n-1} \Gamma^{\lambda}{}_{\nu_n\nu_i}\Gamma^{\sigma}{}_{\nu_1\cdots\nu_{i-1}\,\lambda\,\nu_{i+1}\cdots\nu_{n-1}}\,, \qquad n \geq 2\,. \tag{6.28}$$

Using (6.27), we are now in the position to replace the partial derivatives in (6.22) by writing it as

$$X^{\sigma}_{\mathscr{C}}(\lambda) = X^{\sigma}_{\mathscr{P}} + v^{\sigma}\lambda - \sum_{n=2}^{\infty}\frac{\lambda^n}{n!}\Gamma^{\sigma}{}_{\nu_1\nu_2\cdots\nu_n}v^{\nu_1}v^{\nu_2}\cdots v^{\nu_n}\big|_{\mathscr{P}}\,. \tag{6.29}$$

Here, we make the important observation, that due to the totally symmetric nature of the product $v^{\nu_1}v^{\nu_2}\cdots v^{\nu_n} \equiv v^{(\nu_1}v^{\nu_2}\cdots v^{\nu_n)}$ multiplying the generalised connection coefficient in (6.29), the latter must also be totally symmetric in its lower indices, i.e.,

$$\Gamma^{\sigma}{}_{\nu_1\nu_2\cdots\nu_n} = \Gamma^{\sigma}{}_{(\nu_1\nu_2\cdots\nu_n)}\,. \tag{6.30}$$

As a consequence of this total symmetrisation, the summation in the above definition of the generalised connection coefficients (6.28) simplifies somewhat [171, Chapter 6],

$$\Gamma^{\sigma}{}_{(\nu_1\nu_2\cdots\nu_n)} = \Gamma^{\sigma}{}_{(\nu_1\nu_2\cdots\nu_{n-1},\,\nu_n)} - (n-1)\,\Gamma^{\lambda}{}_{(\nu_1\nu_n|}\Gamma^{\sigma}{}_{\lambda|\nu_2\cdots\nu_{n-1})}\,. \tag{6.31}$$

6.2.2. Introduction of Local Inertial Coordinates

We now introduce the local inertial coordinates (6.5) of section 6.1, as straight lines tangent to all the geodesics that intersect our expansion point $\mathscr{P}$, where from now on, we shall refer to the local inertial coordinates (6.5) as *Riemann normal coordinates*. In generalisation of (6.6), we then interpret equation (6.29) as the coordinate transformation $X^{\sigma}(x^{\hat{\alpha}})$ from the geodesic Riemann normal coordinates $x^{\hat{\alpha}}$ to the background coordinates X^{σ}, i.e.,

$$X^{\sigma}(x^{\hat{\alpha}}) = X^{\sigma}_{\mathscr{P}} + e_{\hat{\alpha}}{}^{\sigma}\big|_{\mathscr{P}}x^{\hat{\alpha}} - \sum_{n=2}^{\infty}\frac{1}{n!}\Gamma^{\sigma}{}_{(\nu_1\nu_2\cdots\nu_n)}e_{\hat{\alpha}_1}{}^{\nu_1}\cdots e_{\hat{\alpha}_n}{}^{\nu_n}\Big|_{\mathscr{P}}x^{\hat{\alpha}_1}\cdots x^{\hat{\alpha}_n}\,, \tag{6.32}$$

where we have dropped the label $\mathscr{C}$ in accordance with this reinterpretation. This then defines a valid coordinate chart as long as the coordinate transformation is one-to-one, i.e., as long as the Jacobian of (6.32) is non-singular,

$$\left|\frac{\partial X^{\sigma}(x^{\alpha})}{\partial x^{\hat{\beta}}}\right| \neq 0\,. \tag{6.33}$$

In practice, this condition can usually be fulfilled – even in strong fields – by decreasing the range of validity and increasing the order of the expansion, and we shall discuss these issues in subsection 6.3.7 below.

Considerable simplifications occur in the coefficients, if we express the general expansion (6.32) in Riemann normal coordinates, where it must trivially reduce to (6.5). Thus, we have that

$$x^{\kappa}(x^{\hat{\alpha}}) = x^{\hat{\kappa}} - \sum_{n=2}^{\infty}\frac{1}{n!}\Gamma^{\kappa}{}_{(\hat{\alpha}_1\hat{\alpha}_2\cdots\hat{\alpha}_n)}\Big|_{\mathscr{P}}x^{\hat{\alpha}_1}\cdots x^{\hat{\alpha}_n}\,, \tag{6.34}$$

where we have used that $x^{\hat{\kappa}}_{\mathscr{P}} = 0$ is the origin of the expansion, as well as the fact that the tetrad in Riemann normal coordinates become trivial at the origin, $e_{\hat{\alpha}}{}^{\kappa}\big|_{\mathscr{P}} = \delta_{\hat{\alpha}}{}^{\kappa}$. Since (6.34) must reduce to the definition of Riemann normal coordinates (6.5) for every power of $x^{\hat{\alpha}}$ separately, one concludes that in those coordinates, and at the origin $\mathscr{P}$ of the expansion, the totally symmetric generalised connection coefficients vanish, i. e. we have, together with the vanishing of the Christoffel symbols at $\mathscr{P}$, the conditions (cf. [172])

$$\Gamma^{\kappa}{}_{\gamma\delta}\big|_{\mathscr{P}} = \Gamma^{\kappa}{}_{(\gamma\delta)}\big|_{\mathscr{P}} = 0\,, \tag{6.35a}$$

$$\Gamma^{\kappa}{}_{(\alpha_1\alpha_2\cdots\alpha_n)}\big|_{\mathscr{P}} = 0\,, \qquad\qquad n \geq 2\,. \tag{6.35b}$$

The second important observation now is, that from their definition, we see that the second term in (6.28), (6.31) containing the symmetrised connection coefficients vanishes, so that by induction – at the origin of Riemann normal coordinates – the n^{th}-order generalised connection coefficient is given by the $(n-2)$nd partial derivative of the Christoffel symbols, i.e., we have,

$$\Gamma^{\kappa}{}_{(\alpha_1\alpha_2\cdots\alpha_n)}\big|_{\mathscr{P}} = \Gamma^{\kappa}{}_{(\alpha_1\alpha_2,\,\alpha_3\cdots\alpha_n)}\big|_{\mathscr{P}} = 0\,. \tag{6.36}$$

Since partial derivatives commute and since the Christoffel symbols are symmetric in their last two indices, we also find that, at the origin of Riemann normal coordinates, the generalised connection coefficients have the symmetry ,

$$\Gamma^{\kappa}{}_{\alpha_1\alpha_2,\,\alpha_3\cdots\alpha_n}\big|_{\mathscr{P}} = \Gamma^{\kappa}{}_{(\alpha_1\alpha_2),\,(\alpha_3\cdots\alpha_n)}\big|_{\mathscr{P}}\,. \tag{6.37}$$

The generalised connection coefficients are actually tensors when evaluated *at the origin* of a Riemann normal coordinate system, i.e. at $\mathscr{P}$, termed *normal tensors*. These were introduced in the 1920s by Veblen and Thomas [158] (see also the recent review by Dixon [104, Sec. 17] and the classic textbook by Schouten [163, Sec. III.7]).

6.2.3. Expansion of Tensor Fields in Terms of Generalised Connection Coefficients

Having set up the coordinate transformation to Riemann normal coordinates, the next important task is to find the expansions of vector and tensor fields, most importantly of the metric. They will be constructed in a similar fashion as the coordinate transformation above in terms of a Taylor expansion along the Riemann coordinate lines. Geometrically, expansion of a vector (or tensor) field $w^{\kappa}(x)$ in Riemann coordinates then means parallel transporting it outwards along the geodesics intersecting our expansion point $\mathscr{P}$, which in turn means solving its equation of parallel transport. From a slightly more general point of view, in Riemann normal coordinates, the tetrad itself represents the parallel propagator (cf. section 4.3.1), since taking x' to refer to the origin $\mathscr{P}$, i.e. $x' = 0$, where the tetrad is trivial, we have from (4.80) that

$$g^{\kappa}{}_{\kappa'}(x, 0) = e_{\hat{\delta}}{}^{\kappa}(x)\,\delta^{\hat{\delta}}{}_{\kappa'}\,. \tag{6.38}$$

As a consequence, having the expansion of the tetrad, allows us to write down the expansion for *any* tensor by similarly contracting its frame indices at $\mathscr{P}$ with the the frame indices of $e_{\hat{\delta}}{}^{\kappa}(x)$ and $e^{\hat{\delta}}{}_{\kappa}(x)$, one for every index. As an example, for a tensor $W_{\kappa_1\cdots\kappa_n}(x)$, this then reads

$$W_{\kappa_1\cdots\kappa_n}(x) = W_{\hat{\kappa}_1\cdots\hat{\kappa}_n}(0)\, e^{\hat{\kappa}_1}{}_{\kappa_1}(x)\cdots e^{\hat{\gamma}_n}{}_{\kappa_n}(x)\,. \tag{6.39}$$

The problem of expanding tensor fields in Riemann coordinates thus reduces to expanding the tetrad and its inverse.

Expansion of the Tetrads

The equation of parallel transport for the tetrad and its inverse, $v^\alpha\nabla_\alpha e_{\hat{\beta}}{}^{\kappa}(x) = 0$ and $v^\alpha\nabla_\alpha e^{\hat{\beta}}{}_{\kappa}(x) = 0$, respectively, are written out in terms of partial derivatives and Christoffel symbols (and omitting the instances of v^α, see below) as,

$$\partial_\alpha e^{\hat{\beta}}{}_{\kappa}(x) = e^{\hat{\beta}}{}_{\delta}(x)\,\Gamma^{\delta}{}_{\alpha\kappa}(x)\,, \tag{6.40a}$$

$$\partial_\alpha e_{\hat{\beta}}{}^{\kappa}(x) = -\Gamma^{\kappa}{}_{\alpha\delta}(x)\, e_{\hat{\beta}}{}^{\delta}(x)\,, \tag{6.40b}$$

where we have paid attention to the "natural order" of the factors with respect to the Christoffel's indices for later convenience, and we note that the equations of parallel transport are linear homogeneous ordinary differential equations. Focussing on the inverse tetrad $e^{\hat{\beta}}{}_{\kappa}(x)$, the Taylor-series solution of (6.40a) generally reads,

$$e^{\hat{\beta}}{}_{\kappa}(x) = e^{\hat{\beta}}{}_{\kappa}\big|_{\mathscr{P}} + \sum_{n=1}^{\infty}\frac{1}{n!}\frac{d^n e^{\hat{\beta}}{}_{\kappa}(x)}{d\lambda^n} v^{\alpha_1}v^{\alpha_2}\cdots v^{\alpha_n}\big|_{\mathscr{P}}\lambda^n\,, \tag{6.41}$$

[and equivalently for (6.40a)] where the initial condition (meaning no parallel transport) must clearly be unity, i.e.,

$$e^{\hat{\beta}}{}_{\kappa}(0) = \delta^{\hat{\beta}}{}_{\kappa}\,, \qquad e_{\hat{\beta}}{}^{\kappa}(0) = \delta_{\hat{\beta}}{}^{\kappa}\,, \tag{6.42a}$$

and the first-order coefficient, comprising a Christoffel symbol, vanishes in Riemann normal coordinates due to (6.35a),

$$\partial_\alpha e^{\hat{\beta}}{}_{\kappa}(x)\big|_{\mathscr{P}} = 0\,, \qquad \partial_\alpha e_{\hat{\beta}}{}^{\kappa}(x)\big|_{\mathscr{P}} = 0\,. \tag{6.42b}$$

Just as in the derivation of the coordinate transformation (6.24), we obtain the higher-order coefficients in (6.41) by acting on the parallel-transport equation (6.40a) with derivatives of the required order. In doing so, we prefer to replace the total λ-derivatives with partial derivatives from now on, since this allows us to use the *comma notation* for the latter, thus uncluttering the notation considerably.

Acting now with $v^{\alpha_1}\partial_{\alpha_1}\cdots v^{\alpha_{n-1}}\partial_{\alpha_{n-1}}$ on equation (6.40a) above and symmetrising on the α indices, we first note that all instances of the Riemann tangent vector field v^{α_i} therein can be moved through the partial derivatives to their left. Upon using the parallel-transport equation (6.40a) repeatedly, they only lead to terms with (6.35b) which vanish at the origin of Riemann normal coordinates. Thus, we can effectively omit all v^{α_i}s from the discussion, which simplifies the notation. We just need to remember that the α-indices are not free indices, but are contracted with instances of the vector field v^α so that the partial derivatives are really directed derivatives along the Riemann normal coordinate lines, i.e. along the geodesics through the origin at $\mathscr{P}$. Of course, in evaluating the coefficients at $\mathscr{P}$, in the end each factor of $v^{\alpha}|_{\mathscr{P}} = v^{\hat{\alpha}}$ combines with a factor of the expansion parameter λ to become a Riemann normal coordinate, in accordance with definition (6.5).

Thus, in this simplified notation, the expansion of the inverse tetrad in Riemann normal coordinates (6.41) reads,

$$e^{\hat{\beta}}{}_{\kappa}(x) = e^{\hat{\beta}}{}_{\kappa}(0) + \partial_{\alpha} e^{\hat{\beta}}{}_{\kappa}(x)|_{\mathscr{P}} x^{\hat{\alpha}} + \tfrac{1}{2!}\partial_{(\alpha_1}\partial_{\alpha_1)} e^{\hat{\beta}}{}_{\kappa}(x)|_{\mathscr{P}} x^{\hat{\alpha}_1} x^{\hat{\alpha}_2} + \mathscr{O}(x^3)\,, \tag{6.43}$$

with initial values, corresponding to the zeroth and first coefficients, given by (6.42a) and (6.42b) above. The non-trivial coefficients for $n \geq 2$ are then found by acting with $n-1$ partial derivatives on the parallel-transport equations (6.40), which results in the following recursions for the tensorial coefficients in the expansion of the tetrad and its inverse,

$$\partial_{(\alpha_1} \cdots \partial_{\alpha_n)} e^{\hat{\beta}}{}_{\kappa}(x)\Big|_{\mathscr{P}} = \partial_{(\alpha_1} \cdots \partial_{\alpha_{n-1}|} \Big(e^{\hat{\beta}}{}_{\delta}(x)\, \Gamma^{\delta}{}_{\kappa|\alpha_n)}(x) \Big)\Big|_{\mathscr{P}}\,, \tag{6.44a}$$

$$\partial_{(\alpha_1} \cdots \partial_{\alpha_n)} e_{\hat{\beta}}{}^{\kappa}(x)\Big|_{\mathscr{P}} = -\partial_{(\alpha_1} \cdots \partial_{\alpha_{n-1}|} \Big(\Gamma^{\kappa}{}_{\delta|\alpha_n)}(x)\, e_{\hat{\beta}}{}^{\delta}(x) \Big)\Big|_{\mathscr{P}}\,. \tag{6.44b}$$

We can now use the generalised Leibniz rule on the right-hand sides to write these in a more explicit form, obtaining

$$\partial_{(\alpha_1} \cdots \partial_{\alpha_n)} e^{\hat{\beta}}{}_{\kappa}(x)\Big|_{\mathscr{P}} = \sum_{k=2}^{n} \binom{n-1}{k-1} \Big(\partial_{(\alpha_n} \cdots \partial_{\alpha_{n-k+1}|} e^{\hat{\beta}}{}_{\delta}(x) \Big) \Gamma^{\delta}{}_{\kappa|\alpha_k,\, \alpha_{k-1} \cdots \alpha_1)}\Big|_{\mathscr{P}}, \tag{6.45a}$$

$$\partial_{(\alpha_1} \cdots \partial_{\alpha_n)} e_{\hat{\beta}}{}^{\kappa}(x)\Big|_{\mathscr{P}} = -\sum_{k=2}^{n} \binom{n-1}{k-1} \Gamma^{\kappa}{}_{\delta(\alpha_1,\, \alpha_2 \cdots \alpha_k} \partial_{\alpha_{k+1}} \cdots \partial_{\alpha_n)} e_{\hat{\beta}}{}^{\delta}(x)\Big|_{\mathscr{P}}, \tag{6.45b}$$

where the sums starts with $k = 2$ respectively, since the terms with $k = 1$ vanish as mentioned above. The above recursions are written in terms of generalised connection coefficients $\Gamma^{\delta}{}_{\kappa(\alpha_1,\, \alpha_2 \cdots \alpha_n)}$ with one lower index *not* symmetrised over and different from α. We shall derive an (albeit complicated and recursive) general expression for these in terms of the Riemann tensor and its partial derivatives in the following subsection.

Expansion of the Metric

Although we can calculate the expansion of the metric in terms of that of the inverse tetrad by using (4.42) [this is then a special case of (6.39)] with $W_{\kappa_1\kappa_2}(x) \equiv g_{\kappa_1\kappa_2}(x)$], it is actually more convenient for a manual calculation to derive a recursion for its partial derivatives, just as in (6.44a) above. To this end, one starts from the metricity condition of the covariant derivative, $\nabla_{\alpha} g_{\kappa_1\kappa_2} = 0$, from which on obtains the expression for the metric's first partial derivative,

$$\partial_{\alpha} g_{\kappa_1\kappa_2}(x) = g_{\kappa_1\delta}(x)\, \Gamma^{\delta}{}_{\kappa_2\alpha}(x) + g_{\delta\kappa_2}(x)\, \Gamma^{\delta}{}_{\kappa_1\alpha}(x) = 2 g_{(\kappa_1|\delta}(x)\, \Gamma^{\delta}{}_{|\kappa_2)\alpha}(x)\,. \tag{6.46}$$

Again acting with $n-1$ partial α-derivatives yields a recursion which has the same structure as that for the inverse tetrad in (6.44a), i.e.,

$$\partial_{(\alpha_1}\cdots\partial_{\alpha_n)}g_{\kappa_1\kappa_2}(x)\Big|_{\mathscr{P}} = 2\partial_{(\alpha_1}\cdots\partial_{\alpha_{n-1}|}\Big(g_{(\kappa_1|\delta}(x)\,\Gamma^{\delta}{}_{|\kappa_2)|\alpha_n)}(x)\Big)\Big|_{\mathscr{P}}\,, \tag{6.47}$$

where the symmetrisations in the α-indices and in the two κ-indices act independently. Since this recursion for the metric and that for inverse tetrad have the same form, they also have the same solution (up to the additional factor of 2 and the corresponding symmetrisation in the two κ indices), so knowledge of the inverse tetrad's expansion makes it an almost trivial exercise to write down that of the metric.

The recursion (6.44a), (6.45) [and also (6.2.5)], leads to a sum consisting of one single generalised connection coefficient of the respective order n, and a chained contraction of lower-order generalised connection coefficients. Although it can be solved by iteration, yielding a multinomial-type closed-form expression (cf. subsection 6.2.8), below we will only calculate the first five of these, since there is a much better way of calculating the expansion of the tetrads in Riemann coordinates, which we shall introduce later on. Thus, with the initial conditions (6.42a) and (6.42b), the first few instances of (6.45) for the inverse tetrad read,

$$\partial_{(\alpha_1}\partial_{\alpha_2)}e^{\hat\beta}{}_{\kappa}(x)|_{\mathscr{P}} = \Gamma^{\hat\beta}{}_{\kappa(\alpha_1,\,\alpha_2)}\,, \tag{6.48a}$$

$$\partial_{(\alpha_1}\partial_{\alpha_2}\partial_{\alpha_3)}e^{\hat\beta}{}_{\kappa}(x)|_{\mathscr{P}} = \Gamma^{\hat\beta}{}_{\kappa(\alpha_1,\,\alpha_2\alpha_3)}\,, \tag{6.48b}$$

$$\partial_{(\alpha_1}\cdots\partial_{\alpha_4)}e^{\hat\beta}{}_{\kappa}(x)|_{\mathscr{P}} = \Gamma^{\hat\beta}{}_{\kappa(\alpha_1,\,\alpha_2\alpha_3\alpha_4)} + 3\Gamma^{\hat\beta}{}_{\delta(\alpha_1,\,\alpha_2|}\Gamma^{\delta}{}_{\kappa|\alpha_3,\,\alpha_4)}\,, \tag{6.48c}$$

$$\begin{aligned}\partial_{(\alpha_1}\cdots\partial_{\alpha_5)}e^{\hat\beta}{}_{\kappa}(x)|_{\mathscr{P}} = \Gamma^{\hat\beta}{}_{\kappa(\alpha_1,\,\alpha_2\alpha_3\alpha_4\alpha_5)} &+ 4\Gamma^{\hat\beta}{}_{\delta(\alpha_1,\,\alpha_2\alpha_3|}\Gamma^{\delta}{}_{\kappa|\alpha_4,\,\alpha_5)}\\ &+ 6\Gamma^{\hat\beta}{}_{\delta(\alpha_1,\,\alpha_2|}\Gamma^{\delta}{}_{\kappa|\alpha_3,\,\alpha_4\alpha_5)}\,,\end{aligned} \tag{6.48d}$$

where we note that by iterating, the "strings" of generalised connections grow to the left and that those strings where the order of the occuring generalised connections increases to the right carry larger numerical factors. For the tetrad itself, the first five terms are,

$$\partial_{(\alpha_1}\partial_{\alpha_2)}e_{\hat\beta}{}^{\kappa}(x)|_{\mathscr{P}} = \Gamma^{\kappa}{}_{\hat\beta(\alpha_1,\,\alpha_2)}\,, \tag{6.49a}$$

$$\partial_{(\alpha_1}\partial_{\alpha_2}\partial_{\alpha_3)}e_{\hat\beta}{}^{\kappa}(x)|_{\mathscr{P}} = \Gamma^{\kappa}{}_{\hat\beta(\alpha_1,\,\alpha_2\alpha_3)}\,, \tag{6.49b}$$

$$\partial_{(\alpha_1}\cdots\partial_{\alpha_4)}e_{\hat\beta}{}^{\kappa}(x)|_{\mathscr{P}} = \Gamma^{\kappa}{}_{\hat\beta(\alpha_1,\,\alpha_2\alpha_3\alpha_4)} + 3\Gamma^{\kappa}{}_{\delta(\alpha_1,\,\alpha_2|}\Gamma^{\delta}{}_{\hat\beta|\alpha_3,\,\alpha_4)}\,, \tag{6.49c}$$

$$\begin{aligned}\partial_{(\alpha_1}\cdots\partial_{\alpha_5)}e_{\hat\beta}{}^{\kappa}(x)|_{\mathscr{P}} = \Gamma^{\kappa}{}_{\hat\beta(\alpha_1,\,\alpha_2\alpha_3\alpha_4\alpha_5)} &+ 6\Gamma^{\kappa}{}_{\delta(\alpha_1,\,\alpha_2\alpha_3|}\Gamma^{\delta}{}_{\hat\beta|\alpha_4,\,\alpha_5)}\\ &+ 4\Gamma^{\kappa}{}_{\delta(\alpha_1,\,\alpha_2|}\Gamma^{\delta}{}_{\hat\beta|\alpha_3,\,\alpha_4\alpha_5)}\,.\end{aligned} \tag{6.49d}$$

Here, the iteration lets the strings of generalised connections grow to the right and those strings where the order of the terms decreases carry the higher numerical factors.

6.2.4. Calculation of the Generalised Connection Coefficients

In the preceding subsection 6.2.3 above, we saw that the expansion of tensors in Riemann normal coordinates leads to recursions in terms of the generalised connection coefficients with one index different from α and not taking part in the symmetrisation. In the present subsection 6.2.4, we will use the vanishing of the generalised connection coefficients in Riemann coordinates (6.36), and their special index symmetry (6.37) to derive a recursion for these in terms of the Riemann tensor and its higher *partial* derivatives.

Writing the generalised connection coefficients of order $n+1$ at the origin of Riemann normal coordinates as $\Gamma^{\kappa}{}_{(\alpha_1\alpha_2\cdots\alpha_n\beta)}$, we see there are two equivalent ways of pulling out the index β from the two first index slots, and $(n+1)-2 = n-1$ equivalent ways of pulling out β from the remaining $n-1$ index slots [167], so we have

$$\Gamma^{\kappa}{}_{(\alpha_1\alpha_2\cdots\alpha_n\beta)}\big|_{\mathscr{P}} = \frac{2}{n+1}\Gamma^{\kappa}{}_{\beta(\alpha_1,\,\alpha_2\cdots\alpha_n)} + \frac{n-1}{n+1}\Gamma^{\kappa}{}_{(\alpha_1\alpha_2,\,\alpha_3\cdots\alpha_n)\beta} = 0\,. \tag{6.50}$$

The factor $n+1$ in the denominator comes from adjusting the normalisation in the definition of symmetrisation; however, continuing with the second equation in (6.50), it clearly plays no further role. We now make the connection to the Riemann tensor, reprinting its definition (3.80) for the benefit of the reader with appropriately renamed indices,

$$R^{\kappa}{}_{\alpha_1\alpha_2\beta} = \Gamma^{\kappa}{}_{\alpha_1\beta,\,\alpha_2} - \Gamma^{\kappa}{}_{\alpha_1\alpha_2,\,\beta} + \Gamma^{\kappa}{}_{\delta\alpha_2}\Gamma^{\delta}{}_{\alpha_1\beta} - \Gamma^{\kappa}{}_{\delta\beta}\Gamma^{\delta}{}_{\alpha_1\alpha_2}\,. \tag{6.51}$$

Acting now with $n-1$ partial α-derivatives, and symmetrising on all α-indices, we obtain

$$\begin{aligned} R^{\kappa}{}_{(\alpha_1\alpha_2|\beta,\,|\alpha_3\cdots\alpha_n)} = \Gamma^{\kappa}{}_{\beta(\alpha_1,\,\alpha_2\cdots\alpha_n)} - \Gamma^{\kappa}{}_{(\alpha_1\alpha_2,\,\alpha_3\cdots\alpha_n)\beta} \\ + \left[\Gamma^{\kappa}{}_{\delta(\alpha_2|}\Gamma^{\delta}{}_{|\alpha_1|\beta}\right]_{,\,|\alpha_3\cdots\alpha_n)} - \left[\Gamma^{\kappa}{}_{\delta\beta}\Gamma^{\delta}{}_{(\alpha_1\alpha_2}\right]_{,\,\alpha_3\cdots\alpha_n)}\,, \end{aligned} \tag{6.52}$$

where we note that the last term on the right-hand side of (6.52) vanishes in view of (6.35a) and (6.35b). Using now (6.52) to replace the $\Gamma^{\kappa}{}_{(\alpha_1\alpha_2,\alpha_3\cdots\alpha_n)\beta}$ in (6.50), we obtain a non-linear recursion for the generalised connection coefficients with one index different from α at the origin of Riemann normal coordinates,

$$\boxed{\Gamma^{\kappa}{}_{\beta(\alpha_1,\,\alpha_2\cdots\alpha_n)} = \frac{n-1}{n+1}\Big[R^{\kappa}{}_{(\alpha_1\alpha_2|\beta,\,|\alpha_3\cdots\alpha_n)} - \big(\Gamma^{\kappa}{}_{\delta(\alpha_2|}\Gamma^{\delta}{}_{|\alpha_1|\beta}\big)_{,|\alpha_3\cdots\alpha_n)}\Big]\,, \qquad n \geq 1\,.} \tag{6.53}$$

Note, that this also holds for $n=1$ since the pre-factor vanishes in this case, so that equation (6.35a) is recovered.

Using again the generalised Leibniz rule, we could rewrite the term with the contracted Christoffel symbols explicitly in terms of lower-order generalised connection coefficients,

$$\left[\Gamma^{\kappa}{}_{\delta(\alpha_2|}\Gamma^{\delta}{}_{|\alpha_1|\beta}\right]_{,|\alpha_3\cdots\alpha_n)} = \sum_{k=2}^{n}\binom{n-2}{k-2}\Gamma^{\kappa}{}_{\delta(\alpha_1,\,\alpha_3\cdots\alpha_k|}\Gamma^{\delta}{}_{\beta|\alpha_2,\,\alpha_{k+1}\cdots\alpha_n)}\,, \tag{6.54}$$

however, this non-linear term clearly makes the general solution (i.e. an iteration in closed form) of the recursion (6.53) difficult, if not intractable. Furthermore, the right-hand side of (6.53) is not manifestly covariant, since it is written in terms of *partial* derivatives of the Riemann tensor instead of covariant derivatives. For the moment, we give the first few instances of (6.53), which can be straightforwardly calculated by hand, including the conversion of partial derivatives of the appearing Riemann tensors to their covariant derivatives, if this is desired. For $n = 2$ to $n = 5$, these read

$$\begin{aligned}
{\Gamma^\kappa}_{\beta(\alpha_1,\alpha_2)} &= \tfrac{1}{3}{R^\kappa}_{(\alpha_1\alpha_2)\beta}\,, && \text{(6.55a)}\\
{\Gamma^\kappa}_{\beta(\alpha_1,\alpha_2\alpha_3)} &= \tfrac{1}{2}{R^\kappa}_{(\alpha_1\alpha_2|\beta;|\alpha_3)} = \tfrac{1}{2}{R^\kappa}_{(\alpha_1\alpha_2|\beta,|\alpha_3)}\,, && \text{(6.55b)}\\
{\Gamma^\kappa}_{\beta(\alpha_1,\alpha_2\alpha_3\alpha_4)} &= \tfrac{3}{5}\big[{R^\kappa}_{(\alpha_1\alpha_2|\beta;|\alpha_3\alpha_4)} - \tfrac{2}{9}{R^\kappa}_{(\alpha_1\alpha_2|\delta}{R^\delta}_{|\alpha_3\alpha_4)\beta}\big]\,, && \text{(6.55c)}\\
{\Gamma^\kappa}_{\beta(\alpha_1,\alpha_2\cdots\alpha_5)} &= \tfrac{2}{3}\big[{R^\kappa}_{(\alpha_1\alpha_2|\beta;|\alpha_3\alpha_4\alpha_5)} - {R^\kappa}_{(\alpha_1\alpha_2|\delta}{R^\delta}_{|\alpha_3\alpha_4|\beta;|\alpha_5)}\big]\,, && \text{(6.55d)}\\
&= \tfrac{2}{3}\big[{R^\kappa}_{(\alpha_1\alpha_2|\beta,|\alpha_3\alpha_4\alpha_5)} - \tfrac{1}{2}\big({R^\kappa}_{(\alpha_1\alpha_2|\delta,|\alpha_3|}{R^\delta}_{|\alpha_4\alpha_5)\beta} && \text{(6.55e)}\\
&\qquad + {R^\kappa}_{(\alpha_1\alpha_2|\delta}{R^\delta}_{|\alpha_3\alpha_4|\beta,|\alpha_5)}\big)\big]\,,
\end{aligned}$$

see Hatzinikitas [166], who calculates (6.53) to $n = 6$.[†] Note that for the first three generalised connections above it makes no difference wether we write them in terms of (symmetrised) covariant or partial derivatives of Riemann tensors. This is obvious for its first (symmetrised) covariant derivative, which reads,

$$\begin{aligned}
{R^\kappa}_{(\alpha_1\alpha_2|\beta;|\alpha_3)} &= {R^\kappa}_{(\alpha_1\alpha_2|\beta,|\alpha_3)} + {\Gamma^\kappa}_{\delta(\alpha_1|}{R^\delta}_{|\alpha_2\alpha_3)\beta} - {R^\kappa}_{(\delta\alpha_2|\beta}{\Gamma^\delta}_{|\alpha_1\alpha_3)}\\
&\qquad - {R^\kappa}_{(\alpha_1\delta|\beta}{\Gamma^\delta}_{|\alpha_2\alpha_3)} - {R^\kappa}_{(\alpha_1\alpha_2|\delta}{\Gamma^\delta}_{\beta|\alpha_3)}\\
&= {R^\kappa}_{(\alpha_1\alpha_2|\beta,|\alpha_3)} + {\Gamma^\kappa}_{\delta(\alpha_1|}{R^\delta}_{|\alpha_2\alpha_3)\beta} - {R^\kappa}_{(\alpha_1\alpha_2|\delta}{\Gamma^\delta}_{\beta|\alpha_3)}\,,
\end{aligned} \tag{6.56}$$

reducing to its partial derivative, i.e. to the first term, at the origin at $\mathscr{P}$ due to (6.35a). For the second covariant derivative of the Riemann tensor, there is an accidental cancelation of second-order generalised connections, which also renders this equal to the second partial Riemann derivative. This fails to be true starting with ${\Gamma^\kappa}_{\beta(\alpha_1,\alpha_2\cdots\alpha_5)}$, for which we have displayed both versions for later convenience.

In summary, we have used the following results for the covariant derivatives of the Riemann tensor in Riemann normal coordinates, for turning its partial into covariant derivatives,

$$\begin{aligned}
{R^\kappa}_{(\alpha_1\alpha_2|\beta;|\alpha_3)}\big|_{\mathscr{P}} &= {R^\kappa}_{(\alpha_1\alpha_2|\beta,|\alpha_3)}\,,\\
{R^\kappa}_{(\alpha_1\alpha_2|\beta;|\alpha_3\alpha_4)}\big|_{\mathscr{P}} &= {R^\kappa}_{(\alpha_1\alpha_2|\beta,|\alpha_3\alpha_4)}\,,\\
{R^\kappa}_{(\alpha_1\alpha_2|\beta;|\alpha_3\alpha_4\alpha_5)}\big|_{\mathscr{P}} &= {R^\kappa}_{(\alpha_1\alpha_2|\beta,|\alpha_3\alpha_4\alpha_5)} - \tfrac{1}{2}{R^\kappa}_{(\alpha_1\alpha_2|\delta;|\alpha_3|}{R^\delta}_{|\alpha_4\alpha_5)\beta}\\
&\qquad + \tfrac{1}{2}{R^\kappa}_{(\alpha_1\alpha_2|\delta}{R^\delta}_{|\alpha_3\alpha_4|\beta;|\alpha_5)}\,.
\end{aligned} \tag{6.57}$$

6.2.5. Results for Expansions of Tetrad and Metric to 6th Order

With our manual calculations of the partial derivatives of the tetrad, the metric, and their respective inverses in terms of generalised connections (6.48), as well as of the generalised connections themselves in terms of Riemann tensors and its *covariant* derivatives (6.55) at hand, we can finally assemble their expansion coefficients and display the corresponding Taylor expansions to the calculated order. We first determine the coefficients, with the outer numerical factor being $\frac{n-1}{n+1}$, if n is the corresponding order of the coefficient.

[†] Hatzinikitas uses a different ordering of the lower indices on the Riemann tensor factors (strangely only for $n \geq 3$) which he compensates with an overall minus sign. With this in mind, our results agree.

Since it turns out to be interesting, we shall compare the expansion coefficients of the inverse tetrad in their *manifestly* covariant form, i.e. written in terms of covariant derivative of Riemann tensors, with their version in terms of *partial* Riemann derivatives. In the manifestly covariant case, the expansion coefficients of the inverse tetrad read,

$$
\begin{aligned}
\partial_{(\alpha_1}\partial_{\alpha_2)}e^{\hat\beta}{}_{\kappa}(x)\big|_{\mathscr{P}} &= \tfrac{1}{3}R^{\hat\beta}{}_{(\hat\alpha_1\hat\alpha_2)\hat\kappa}\,, && (6.58a)\\
\partial_{(\alpha_1}\partial_{\alpha_2}\partial_{\alpha_3)}e^{\hat\beta}{}_{\kappa}(x)\big|_{\mathscr{P}} &= \tfrac{1}{2}R^{\hat\beta}{}_{(\hat\alpha_1\hat\alpha_2|\hat\kappa;|\hat\alpha_3)}\,, && (6.58b)\\
\partial_{(\alpha_1}\cdots\partial_{\alpha_4)}e^{\hat\beta}{}_{\kappa}(x)\big|_{\mathscr{P}} &= \tfrac{3}{5}\big[R^{\hat\beta}{}_{(\hat\alpha_1\hat\alpha_2|\hat\kappa;|\hat\alpha_3\hat\alpha_4)} + \tfrac{1}{3}R^{\hat\beta}{}_{(\hat\alpha_1\hat\alpha_2|\hat\delta}R^{\hat\delta}{}_{|\hat\alpha_3\hat\alpha_4)\hat\kappa}\big]\,, && (6.58c)\\
\partial_{(\alpha_1}\cdots\partial_{\alpha_5)}e^{\hat\beta}{}_{\kappa}(x)\big|_{\mathscr{P}} &= \tfrac{2}{3}\big[R^{\hat\beta}{}_{(\hat\alpha_1\hat\alpha_2|\hat\kappa;|\hat\alpha_3\hat\alpha_4\hat\alpha_5)} + R^{\hat\beta}{}_{(\hat\alpha_1\hat\alpha_2|\hat\delta;|\hat\alpha_3|}R^{\hat\delta}{}_{|\hat\alpha_4\hat\alpha_5)\hat\kappa} && (6.58d)\\
&\qquad + \tfrac{1}{2}R^{\hat\beta}{}_{(\hat\alpha_1\hat\alpha_2|\hat\delta}R^{\hat\delta}{}_{|\hat\alpha_3\hat\alpha_4|\hat\kappa;|\hat\alpha_5)}\big]\,.
\end{aligned}
$$

As with the generalised connection coefficients in (6.55), we can freely exchange semicolons and commas in the second- to fourth-order terms, since it makes no difference at these orders. In contrast, the inner numerical coefficients in the fifth-order term are different when written in terms of partial derivatives, namely,

$$
\begin{aligned}
\partial_{(\alpha_1}\cdots\partial_{\alpha_5)}e^{\hat\beta}{}_{\kappa}(x)\big|_{\mathscr{P}} &= \tfrac{2}{3}\big[R^{\hat\beta}{}_{(\hat\alpha_1\hat\alpha_2|\hat\kappa,|\hat\alpha_3\hat\alpha_4\hat\alpha_5)} + \tfrac{1}{2}R^{\hat\beta}{}_{(\hat\alpha_1\hat\alpha_2|\hat\delta,|\hat\alpha_3|}R^{\hat\delta}{}_{|\hat\alpha_4\hat\alpha_5)\hat\kappa} && (6.59)\\
&\qquad + R^{\hat\beta}{}_{(\hat\alpha_1\hat\alpha_2|\hat\delta}R^{\hat\delta}{}_{|\hat\alpha_3\hat\alpha_4|\hat\kappa,|\hat\alpha_5)}\big]\,;
\end{aligned}
$$

in other words, the ordering of the inner numerical coefficients seems to be reversed as soon as the expansion order is high enough for differences between partial and covariant Riemann derivatives to appear. This fact is no coincidence and carries through to all orders, as we shall see below. Continuing with the other expansions, for the tetrad itself we have,

$$
\begin{aligned}
\partial_{(\alpha_1}\partial_{\alpha_2)}e_{\hat\beta}{}^{\kappa}(x)\big|_{\mathscr{P}} &= -\tfrac{1}{3}R^{\hat\kappa}{}_{(\hat\alpha_1\hat\alpha_2)\hat\beta}\,, && (6.60a)\\
\partial_{(\alpha_1}\partial_{\alpha_2}\partial_{\alpha_3)}e_{\hat\beta}{}^{\kappa}(x)\big|_{\mathscr{P}} &= -\tfrac{1}{2}R^{\hat\kappa}{}_{(\hat\alpha_1\hat\alpha_2|\hat\beta;|\hat\alpha_3)}\,, && (6.60b)\\
\partial_{(\alpha_1}\cdots\partial_{\alpha_4)}e_{\hat\beta}{}^{\kappa}(x)\big|_{\mathscr{P}} &= -\tfrac{3}{5}\big[R^{\hat\beta}{}_{(\hat\alpha_1\hat\alpha_2|\hat\kappa;|\hat\alpha_3\hat\alpha_4)} - \tfrac{7}{9}R^{\hat\beta}{}_{(\hat\alpha_1\hat\alpha_2|\hat\delta}R^{\hat\delta}{}_{|\hat\alpha_3\hat\alpha_4)\hat\kappa}\big]\,, && (6.60c)\\
\partial_{(\alpha_1}\cdots\partial_{\alpha_5)}e_{\hat\beta}{}^{\kappa}(x)\big|_{\mathscr{P}} &= -\tfrac{2}{3}\big[R^{\hat\kappa}{}_{(\hat\alpha_1\hat\alpha_2|\hat\beta;|\hat\alpha_3\hat\alpha_4\hat\alpha_5)} - \tfrac{3}{2}R^{\hat\kappa}{}_{(\hat\alpha_1\hat\alpha_2|\hat\delta;|\hat\alpha_3|}R^{\hat\delta}{}_{|\hat\alpha_4\hat\alpha_5)\hat\beta} && (6.60d)\\
&\qquad - 2R^{\hat\kappa}{}_{(\hat\alpha_1\hat\alpha_2|\hat\delta}R^{\hat\delta}{}_{|\hat\alpha_3\hat\alpha_4|\hat\beta;|\hat\alpha_5)}\big]\,.
\end{aligned}
$$

From its recursion , the expansion coefficients of the metric are then found to read

$$
\begin{aligned}
\partial_{\alpha}g_{\kappa_1\kappa_2}(x)\big|_{\mathscr{P}} &= 0\,, && (6.61a)\\
\partial_{(\alpha_1}\partial_{\alpha_2)}g_{\kappa_1\kappa_2}(x)\big|_{\mathscr{P}} &= \tfrac{2}{3}R_{\hat\kappa_1(\hat\alpha_1\hat\alpha_2)\hat\kappa_2}\,, && (6.61b)\\
\partial_{(\alpha_1}\partial_{\alpha_2}\partial_{\alpha_3)}g_{\kappa_1\kappa_2}(x)\big|_{\mathscr{P}} &= R_{\hat\kappa_1(\hat\alpha_1\hat\alpha_2|\hat\kappa_2;|\hat\alpha_3)}\,, && (6.61c)\\
\partial_{(\alpha_1}\cdots\partial_{\alpha_4)}g_{\kappa_1\kappa_2}(x)\big|_{\mathscr{P}} &= \tfrac{6}{5}\big[R_{\hat\kappa_1(\hat\alpha_1\hat\alpha_2|\hat\kappa_2;|\hat\alpha_3\hat\alpha_4)} + \tfrac{8}{9}R_{\hat\kappa_1(\hat\alpha_1\hat\alpha_2|\hat\delta}R^{\hat\delta}{}_{|\hat\alpha_3\hat\alpha_4)\hat\kappa_2}\big]\,, && (6.61d)
\end{aligned}
$$

$$\partial_{(\alpha_1}\cdots\partial_{\alpha_5)}g_{\kappa_1\kappa_2}(x)\big|_{\mathscr{P}} = \tfrac{4}{3}\big[R_{\hat\kappa_1(\hat\alpha_1\hat\alpha_2|\hat\kappa_2;\,|\hat\alpha_3\hat\alpha_4\hat\alpha_5)} + 2\big(R_{\hat\kappa_1(\hat\alpha_1\hat\alpha_2|\hat\delta;\,|\hat\alpha_3|}R^{\hat\delta}{}_{|\hat\alpha_4\hat\alpha_5)\hat\kappa_2} + R_{\hat\kappa_1(\hat\alpha_1\hat\alpha_2|\hat\delta}R^{\hat\delta}{}_{|\hat\alpha_3\hat\alpha_4|\hat\kappa_2;\,|\hat\alpha_5)}\big)\big]\,. \tag{6.61e}$$

Finally putting everything together, we display the full expansion of the tetrad to 5^{th} order,

$$\begin{aligned} e_{\hat\beta}{}^{\kappa}(x) = \delta_{\hat\beta}{}^{\hat\kappa} &- \tfrac{1}{6}R^{\hat\kappa}{}_{\hat\alpha_1\hat\alpha_2\hat\beta}x^{\hat\alpha_1}x^{\hat\alpha_2} - \tfrac{1}{12}R^{\hat\kappa}{}_{(\hat\alpha_1\hat\alpha_2|\hat\beta;\,|\hat\alpha_3)}x^{\hat\alpha_1}x^{\hat\alpha_2}x^{\hat\alpha_3} \\ &- \frac{1}{40}\Big[R^{\hat\kappa}{}_{(\hat\alpha_1\hat\alpha_2|\hat\beta;\,|\hat\alpha_3\hat\alpha_4)} - \tfrac{7}{9}R^{\hat\kappa}{}_{(\hat\alpha_1\hat\alpha_2|\hat\delta}R^{\hat\delta}{}_{|\hat\alpha_3\hat\alpha_4)\hat\beta}\Big]x^{\hat\alpha_1}x^{\hat\alpha_2}x^{\hat\alpha_3}x^{\hat\alpha_4} \\ &- \frac{1}{180}\Big[R^{\hat\kappa}{}_{(\hat\alpha_1\hat\alpha_2|\hat\beta;\,|\hat\alpha_3\hat\alpha_4\hat\alpha_5)} - \tfrac{3}{2}R^{\hat\kappa}{}_{(\hat\alpha_1\hat\alpha_2|\hat\delta;\,|\hat\alpha_3|}R^{\hat\delta}{}_{|\hat\alpha_4\hat\alpha_5)\hat\beta} \\ &\qquad - 2R^{\hat\kappa}{}_{(\hat\alpha_1\hat\alpha_2|\hat\delta}R^{\hat\delta}{}_{|\hat\alpha_3\hat\alpha_4|\hat\beta;\,|\hat\alpha_5)}\Big]x^{\hat\alpha_1}x^{\hat\alpha_2}x^{\hat\alpha_3}x^{\hat\alpha_4}x^{\hat\alpha_5} + \mathscr{O}(x^6)\,, \end{aligned} \tag{6.62a}$$

and similarly, for the inverse tetrad we have

$$\begin{aligned} e^{\hat\beta}{}_{\kappa}(x) = \delta^{\hat\beta}{}_{\hat\kappa} &+ \tfrac{1}{6}R^{\hat\beta}{}_{\hat\alpha_1\hat\alpha_2\hat\kappa}x^{\hat\alpha_1}x^{\hat\alpha_2} + \tfrac{1}{12}R^{\hat\beta}{}_{(\hat\alpha_1\hat\alpha_2|\hat\kappa;\,|\hat\alpha_3)}x^{\hat\alpha_1}x^{\hat\alpha_2}x^{\hat\alpha_3} \\ &+ \frac{1}{40}\big[R^{\hat\beta}{}_{(\hat\alpha_1\hat\alpha_2|\hat\kappa;|\hat\alpha_3\hat\alpha_4)} + \tfrac{1}{3}R^{\hat\beta}{}_{(\hat\alpha_1\hat\alpha_2|\hat\delta}R^{\hat\delta}{}_{|\hat\alpha_3\hat\alpha_4)\hat\kappa}\big]x^{\hat\alpha_1}x^{\hat\alpha_2}x^{\hat\alpha_3}x^{\hat\alpha_4} \\ &+ \frac{1}{180}\big[R^{\hat\beta}{}_{(\hat\alpha_1\hat\alpha_2|\hat\kappa;\,|\hat\alpha_3\hat\alpha_4\hat\alpha_5)} + R^{\hat\beta}{}_{(\hat\alpha_1\hat\alpha_2|\hat\delta;\,|\hat\alpha_3|}R^{\hat\delta}{}_{|\hat\alpha_4\hat\alpha_5)\hat\kappa} \\ &\qquad + \tfrac{1}{2}R^{\hat\beta}{}_{(\hat\alpha_1\hat\alpha_2|\hat\delta}R^{\hat\delta}{}_{|\hat\alpha_3\hat\alpha_4|\hat\kappa;\,|\hat\alpha_5)}\big]x^{\hat\alpha_1}x^{\hat\alpha_2}x^{\hat\alpha_3}x^{\hat\alpha_4}x^{\hat\alpha_5} + \mathscr{O}(x^6)\,. \end{aligned} \tag{6.62b}$$

We note that the outermost numerical coefficient at each order in the above results then reads $\frac{1}{n!}\frac{n-1}{n+1}$, and includes the minus sign for the tetrad, as discussed above. For the 5^{th}-order expansion of the metric, the outermost numerical coefficient reads $2\frac{1}{n!}\frac{n-1}{n+1}$, and we have

$$\begin{aligned} g_{\kappa_1\kappa_2}(x) = \eta_{\hat\kappa_1\hat\kappa_2} &+ \tfrac{1}{3}R_{\hat\kappa_1(\hat\alpha_1\hat\alpha_2)\hat\kappa_2}\,x^{\hat\alpha_1}x^{\hat\alpha_2} + \tfrac{1}{6}R_{\hat\kappa_1(\hat\alpha_1\hat\alpha_2|\hat\kappa_2;\,|\hat\alpha_3)}\,x^{\hat\alpha_1}x^{\hat\alpha_2}x^{\hat\alpha_3} \\ &+ \frac{1}{20}\Big[R_{\hat\kappa_1(\hat\alpha_1\hat\alpha_2|\hat\kappa_2;\,|\hat\alpha_3\hat\alpha_4)} + \tfrac{8}{9}R_{\hat\kappa_1(\hat\alpha_1\hat\alpha_2|\hat\delta}R^{\hat\delta}{}_{|\hat\alpha_3\hat\alpha_4)\hat\kappa_2}\Big]\,x^{\hat\alpha_1}x^{\hat\alpha_2}x^{\hat\alpha_3}x^{\hat\alpha_4} \\ &+ \frac{1}{90}\Big[R_{\hat\kappa_1(\hat\alpha_1\hat\alpha_2|\hat\kappa_2;\,|\hat\alpha_3\hat\alpha_4\hat\alpha_5)} + 2\Big(R_{\hat\kappa_1(\hat\alpha_1\hat\alpha_2|\hat\delta;\,|\hat\alpha_3|}R^{\hat\delta}{}_{|\hat\alpha_4\hat\alpha_5)\hat\kappa_2} \\ &\qquad + R_{\hat\kappa_1(\hat\alpha_1\hat\alpha_2|\hat\delta}R^{\hat\delta}{}_{|\hat\alpha_3\hat\alpha_4|\hat\kappa_2;\,|\hat\alpha_5)}\Big)\Big]\,x^{\hat\alpha_1}x^{\hat\alpha_2}x^{\hat\alpha_3}x^{\hat\alpha_4}x^{\hat\alpha_5} \\ &+ \mathscr{O}(x^5)\,. \end{aligned} \tag{6.63}$$

6.2.6. A Covariant Recursion for the Tetrad Expansion Coefficients

In the subsections above we have presented what one could call the "traditional approach" to Riemann normal-coordinate expansions in terms of generalised connection coefficients that is due to Veblen and Thomas, since in our view, this is the most transparent one to get started with. Although the formulae were seen to hold for any expansion order n in principle, this approach clearly suffers from the complications of a non-linear recursion for the generalised connection coefficients, its non-covariant character with respect to the derivatives, and its double-recursion nature. It is because of these inconveniences, that practically, the important expansion coefficients of the tetrad could only be obtained to a fairly low order, with considerable manual labour being necessary in going to higher n.

The more modern approach to the expanion coefficients of the inverse tetrad employs a second-order recursion on top of the geodesic deviation equation (5.108), with an additional numerical factor [the same one as in front of the expression for the generalised connections in terms of the Riemann tensor (6.53)] which arises from the Riemann coordinate conditions (6.35). For the inverse tetrad, it reads,

$$\boxed{\partial_{(\alpha_1} \cdots \partial_{\alpha_n)} e^{\hat{\beta}}{}_{\kappa}(x)\big|_{\mathscr{P}} = \frac{n-1}{n+1} \partial_{(\alpha_1} \cdots \partial_{\alpha_{n-2}|} \left(e^{\hat{\beta}}{}_{\delta}(x) R^{\delta}{}_{|\alpha_{n-1}\alpha_n)\kappa} \right)\Big|_{\mathscr{P}}, \quad n \geq 1 .} \tag{6.64}$$

This manifestly covariant recursion can actually be solved by iteration, yielding a closed-form, multinomial-type expression, which can then be employed to generate the expansion coefficients of the inverse tetrad and the metric in an efficient manner. Unfortunately, the corresponding closed-form solution for the tetrad itself seems much more difficult, as we shall see below. This type of recursion can be derived in at least two ways [165, 170], and probably also directly from the first-order recursion (6.44), if one makes use of the full original coordinate condition, i.e. of the vanishing of (6.31), which include all Christoffel terms. Note that the key point is that (6.64) is manifestly covariant, i.e in contrast to (6.44), there are no Christoffel symbols inside the partial derivatives, and thus we essentially bypass the second, nonlinear recursion (6.53) for the generalised connection coefficients, which presents the main difficulty in the traditional approach.

The initial conditions for these covariant recursions are given by the first equations of (6.42a) and (6.42b) respectively. Using (6.64), the first few terms in the expansion of the inverse tetrad can now easily be computed by hand. In contrast to the recursions of the "traditional approach", we now only have to iterate a single recursion, the result being already tensorial, i.e., given in terms of Riemann tensors and their partial derivative. However, the above recursion can actually be solved in closed form, which we will carry out in the following subsection.

In order to briefly motivate how (6.64) could probably be derived from (6.44), let us take a closer look at the first-order recursion for the inverse tetrad (6.44a). Acting the innermost partial derivative on the product yields,

$$\partial_{(\alpha_1} \cdots \partial_{\alpha_n)} e^{\hat{\beta}}{}_{\kappa}(x)\big|_{\mathscr{P}} = \partial_{(\alpha_1} \cdots \partial_{\alpha_{n-2}|} e^{\hat{\beta}}{}_{\delta}(x) \left[\Gamma^{\delta}{}_{\kappa(\alpha_1,\,\alpha_2)} + \Gamma^{\delta}{}_{(\alpha_1|\delta_1} \Gamma^{\delta_1}{}_{|\alpha_2)\kappa} \right], \tag{6.65}$$

and using now the vanishing of the "full" generalised connection coefficients, i.e. $\Gamma^{\kappa}{}_{(\alpha_1\alpha_2\cdots\alpha_n)} = 0$ instead of (6.50), for deriving the expression for $\Gamma^{\delta}{}_{\kappa(\alpha_1,\,\alpha_2)}$, we have,

$$\Gamma^{\kappa}{}_{(\alpha_1\alpha_2\alpha_3)}\Big|_{\mathscr{P}} = \left[\Gamma^{\kappa}{}_{(\alpha_1\alpha_2,\,\alpha_3)} - 2\Gamma^{\kappa}{}_{(\alpha_1|\delta} \Gamma^{\delta}{}_{|\alpha_1\alpha_2)} \right]\Big|_{\mathscr{P}} = 0 . \tag{6.66}$$

Pulling out one of the lower indices from the symmetrisation just as in (6.50), we obtain,

$$\left[2\Gamma^{\kappa}{}_{\beta(\alpha_1,\,\alpha_2)} + \Gamma^{\kappa}{}_{(\alpha_1\alpha_2),\,\beta} - 2\big(\Gamma^{\kappa}{}_{\beta\delta} \Gamma^{\delta}{}_{(\alpha_1\alpha_2)} + 2\Gamma^{\kappa}{}_{(\alpha_1|\delta} \Gamma^{\delta}{}_{|\alpha_2)\beta} \big) \right]\Big|_{\mathscr{P}} = 0 , \tag{6.67}$$

and discarding the vanishing term quadratic in the Christoffels with $\Gamma^{\delta}{}_{(\alpha_1\alpha_2)}$, and using the definition (6.51) of the Riemann tensor again, then leaves us with the $n=2$ analog of (6.53),

$$\Gamma^{\kappa}{}_{\beta(\alpha_1,\,\alpha_2)}\big|_{\mathscr{P}} = \tfrac{1}{3}R^{\kappa}{}_{(\alpha_1\alpha_2)\beta} + \Gamma^{\kappa}{}_{(\alpha_1|\delta}\Gamma^{\delta}{}_{|\alpha_2)\beta}\,. \tag{6.68}$$

Here, the important difference to (6.53) is, that the numerical factor $\frac{1}{3}$ now multiplies *only* the Riemann tensor [as exhibited in (6.64)].

6.2.7. Solution of Recursion for Inverse Tetrad in Riemann Normal Coordinates

The recursions (6.64) and (6.82), now lend themselves well to a solution by iteration in much the same way as that was encountered for the non-covariant recursions with generalised connections in subsections 6.2.4 and 6.2.3. While our derivation of the solution of these is based on the straightforward iteration approach by Gray in [165], it appears that this solution was rediscovered several times. In the physics literature, a solution that is equivalent to the one below was also later derived independently by van de Ven [169, p. 2326], and also by Müller, Schubert and van de Ven in [170], who work in an orthonormal basis and use a different method based on integral and differential equations for the tetrad. Apparently, Avramidi also derived a very similar recursion [174], in the context of heat kernel expansions in the bi-tensor formalism in his doctoral thesis [173], most of which seems to have been re-published in his textbook [175, Sec. 2.2].

In order to solve the recursion for the inverse tetrad, we first rewrite the right-hand sides of (6.64), (6.82) in terms of the generalised Leibniz rule, just as in (6.45); thus,

$$\partial_{(\alpha_1}\cdots\partial_{\alpha_n)}e^{\hat\beta}{}_{\kappa}(x)\Big|_{\mathscr{P}} = \frac{n-1}{n+1}\sum_{k=2}^{n}\binom{n-2}{k-2}R^{\delta}{}_{(\alpha_1\alpha_2|\kappa;\,|\alpha_3\cdots\alpha_k|}\partial_{|\alpha_{k+1}}\cdots\partial_{\alpha_n)}e^{\hat\beta}{}_{\delta}(x)\Big|_{\mathscr{P}}\,, \tag{6.69}$$

where k is the number of covariant-derivative indices α from the left-hand side of the recursion that are part of each Riemann factor in the sum; with $k-2$ of these derivatives actually acting on a particular Riemann factor. Now iterating (6.69) $l-1$ times and evaluating the result at $\mathscr{P}$ using the initial condition (6.42a) for the tetrad, we obtain

$$\begin{aligned}\partial_{(\alpha_1}\cdots\partial_{\alpha_n)}e^{\hat\beta}{}_{\kappa}(x)\Big|_{\mathscr{P}} = \sum_{l=1}^{\lfloor\frac{n}{2}\rfloor}\sum_{\substack{k_1,\dots,k_l=2\\k_1+\cdots+k_l=n}}\mathcal{C}_{\perp}(k_1,k_2,\dots,k_l)\,R^{\hat\beta}{}_{(\alpha_1\alpha_2|\delta_{l-1};\,|\alpha_3\cdots\alpha_{k_1}|}\\ \times R^{\delta_{l-1}}{}_{|\alpha_{k_1+1}\alpha_{k_1+2}|\delta_2;\,|\alpha_{k_1+3}\cdots\alpha_{k_2}|}\cdots R^{\delta_{l-1}}{}_{|\alpha_{k_{l-1}+1}\alpha_{k_{l-1}+2}|\kappa;\,|\alpha_{k_l-1}\cdots\alpha_{k_l})}\,,\end{aligned} \tag{6.70}$$

where the inner sum runs over all compositions (i.e. ordered partitions) $(k_1,k_2,\dots,k_l)$ of the integer n into l parts, corresponding to a chained contraction of l Riemann factors with k_i α-indices on the i^{th} factor. Since a single of these Riemann factors uses up at least two such indices and at most all n, and since – as a consequence of (6.42b) – all terms in the l-sum with an odd upper limit (i.e. $\frac{n}{2}+1$) vanish, l can be taken to run from 1 to $\lfloor\frac{n}{2}\rfloor$ (the integer part of $n/2$). The former case corresponds to a single Riemann tensor with

$n-2$ covariant derivatives, i.e.,

$$R^{\hat{\beta}}{}_{(\alpha_1\alpha_2|\kappa;|\alpha_3\cdots\alpha_n)}\,, \qquad\qquad l=1,\ k_1=n\,,$$

and the latter to a "maximal" product of $\lfloor\frac{n}{2}\rfloor$ Riemann tensors without derivatives, i.e.,

$$R^{\hat{\beta}}{}_{(\alpha_1\alpha_2|\delta_1}R^{\delta_1}{}_{|\alpha_3\alpha_4|\delta_2}R^{\delta_2}{}_{|\alpha_5\alpha_6|\delta_3}\cdots R^{\delta_{n/2-1}}{}_{|\alpha_{n-1}\alpha_n)\kappa}\,, \qquad\qquad l=\lfloor\tfrac{n}{2}\rfloor,\ k_i=2 \text{ for } 1\le i\le l\,.$$

Below, we shall provide a simple and instructive example in which this latter case is realised.

Numerical Coefficient of the Solution

The numerical coefficient $\mathcal{C}_\perp(k_1,k_2,\ldots,k_l)$ in (6.70) contains a product of all the l factors of the type $\frac{n-1}{n+1}$ and all the binomial coefficients that are accumulated in the $l-1$ iterations of the above recursion (6.69). The iteration procedure can be seen to entail,

$$\frac{n-1}{n+1}\binom{n-2}{k_1-2}\frac{n-k_1-1}{n-k_1+1}\binom{n-k_1-2}{k_2-2}\cdots\frac{n-(k_1+\cdots+k_l)-1}{n-(k_1+\cdots+k_l)+1}\binom{n-(k_1+\cdots+k_l)-2}{n-(k_1+\cdots+k_l)-2}\,,$$

which is readily condensed into the product

$$\mathcal{C}_\perp(k_1,\ldots,k_l) \coloneqq \prod_{i=0}^{l}\frac{n-(k_1+\cdots+k_{i-1})-1}{n-(k_1+\cdots+k_{i-1})+1}\binom{n-(k_1+\cdots+k_{i-1})-2}{k_i-2}\,. \tag{6.71}$$

We shall call the product of binomial coefficients in (6.71) the "parallel" numerical coefficient, $\mathcal{C}_\parallel(k_1,\ldots,k_l)$, since it will appear on its own in the case of Fermi normal coordinates in subsection 6.3.6. This factor can in fact be rewritten in terms of a multinomial coefficient: Using $k_1+\cdots+k_l=n$, and the well-known decomposition of a multinomial coefficient into products of binomial coefficients, $\binom{n}{n_1,\ldots,n_l}=\binom{n_1}{n_1}\binom{n_1+n_2}{n_2}\cdots\binom{n_1+\cdots+n_l}{n_l}$, we have the forms,

$$\begin{aligned}\mathcal{C}_\parallel(k_1,\ldots,k_l) \coloneqq \prod_{i=0}^{l}\binom{n-(k_1+\cdots+k_{i-1})-2}{k_i-2} &= \prod_{i=1}^{l}\binom{k_i+k_{i+1}+\cdots+k_l-2}{k_i-2}\\ &= \binom{n}{k_1,\ldots,k_l}\prod_{i=1}^{l}\frac{k_i(k_i-1)}{(k_i+k_{i+1}+\cdots+k_l)(k_i+k_{i+1}+\cdots+k_l-1)}\,,\end{aligned} \tag{6.72}$$

so that, taking into account a cancellation with the numerator of the first factor in (6.71), the full expression for $\mathcal{C}_\perp(k_1,\ldots,k_l)$ is written as,

$$\mathcal{C}_\perp(k_1,\ldots,k_l) = \binom{n}{k_1,\ldots,k_l}\prod_{i=1}^{l}\frac{k_i(k_i-1)}{(k_i+k_{i+1}+\cdots+k_l)(k_i+k_{i+1}+\cdots+k_l+1)} \tag{6.73}$$

(note the "+" sign in the last factor of the denominator of $\mathcal{C}_\perp$, versus a "$-$" for $\mathcal{C}_\parallel$). Here, the numerator $k_i(k_i-1)$ under the product comes simply from rewriting the original multinomial $\binom{n}{k_1-2,\ldots,k_l-2}$. The denominator, in contrast, is more interesting. It turns out that the coefficients

$$\frac{k_1 \cdots k_l}{k_1(k_1+k_2)\cdots(k_1+\cdots+k_l)} \tag{6.74}$$

form a partition of unity on the symmetric group S_l of order l, as we shall see in subsection 6.2.8, and are thus responsible for the non-commutative nature of the solution (6.70).

Solution in Terms of Partial Riemann Derivatives

In order to write the closed-form solution (6.70) in a more concise form, it is useful to define the following Riemann matrix

$$\tilde{\mathsf{R}}_n \coloneqq R^{\hat\kappa}{}_{(\hat\alpha_1\hat\alpha_2|\hat\beta,|\hat\alpha_3\cdots\hat\alpha_n)} x^{\hat\alpha_1} x^{\hat\alpha_2}\cdots x^{\hat\alpha_n}\,, \tag{6.75}$$

where the overtilde means that we are dealing with partial derivatives of the Riemann tensor. The Taylor expansion of the inverse tetrad,

$$e^{\hat\beta}{}_\kappa(x) = \delta^{\hat\beta}{}_\kappa + \sum_{n=2}^{\infty}\frac{1}{n!}\partial_{(\alpha_1}\cdots\partial_{\alpha_n)} e^{\hat\beta}{}_\kappa(x)\Big|_{\mathscr{P}} x^{\alpha_1}\cdots x^{\alpha_n}\,, \tag{6.76}$$

is then assembled as,

$$\boxed{e^{\hat\beta}{}_\kappa(x) = \delta^{\hat\beta}{}_\kappa + \sum_{n=2}^{\infty}\frac{1}{n!}\sum_{l=1}^{\lfloor\frac{n}{2}\rfloor}\sum_{\substack{k_1,\ldots,k_l\geq 2\\ k_1+\cdots+k_l=n}} \mathcal{C}_\perp(k_1,\ldots,k_l)(\tilde{\mathsf{R}}_{k_l}\tilde{\mathsf{R}}_{k_{l-1}}\cdots\tilde{\mathsf{R}}_{k_1})^{\hat\beta}{}_\kappa\,.} \tag{6.77}$$

The above closed-form solution (6.84) then yields the following result for the expansion of the inverse tetrad in terms of partial derivatives of the Riemann tensor, of which we display terms up to 9th order. We have,

$$\begin{aligned}
e^{\hat\beta}{}_\kappa(x) = \big[1 &+ \tfrac{1}{2!}\tfrac{1}{3}\tilde{\mathsf{R}}_2 + \tfrac{1}{3!}\tfrac{1}{2}\tilde{\mathsf{R}}_3 + \tfrac{1}{4!}\tfrac{3}{5}(\tilde{\mathsf{R}}_4 + \tfrac{1}{3}\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2) + \tfrac{1}{5!}\tfrac{2}{3}(\tilde{\mathsf{R}}_5 + \tfrac{1}{2}\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_2 + \tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_3)\\
&+ \tfrac{1}{6!}\tfrac{5}{7}(\tilde{\mathsf{R}}_6 + \tfrac{3}{5}\tilde{\mathsf{R}}_4\tilde{\mathsf{R}}_2 + 2\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_4 + 2\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_3 + \tfrac{1}{5}\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2)\\
&+ \tfrac{1}{7!}\tfrac{3}{4}(\tilde{\mathsf{R}}_7 + \tfrac{2}{3}\tilde{\mathsf{R}}_5\tilde{\mathsf{R}}_2 + \tfrac{10}{3}\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_5 + 3\tilde{\mathsf{R}}_4\tilde{\mathsf{R}}_3 + 5\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_4\\
&\qquad + \tfrac{1}{3}\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2 + \tfrac{2}{3}\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_2 + \tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_3)\\
&+ \tfrac{1}{8!}\tfrac{7}{9}(\tilde{\mathsf{R}}_8 + \tfrac{5}{7}\tilde{\mathsf{R}}_6\tilde{\mathsf{R}}_2 + 5\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_6 + 4\tilde{\mathsf{R}}_5\tilde{\mathsf{R}}_3 + 10\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_5 + 9\tilde{\mathsf{R}}_4\tilde{\mathsf{R}}_4 + \tfrac{3}{7}\tilde{\mathsf{R}}_4\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2\\
&\qquad + \tfrac{10}{7}\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_4\tilde{\mathsf{R}}_2 + 3\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_4 + \tfrac{10}{7}\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_2 + 2\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_3\\
&\qquad + 4\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_3 + \tfrac{1}{7}\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2)\\
&+ \tfrac{1}{9!}\tfrac{4}{5}(\tilde{\mathsf{R}}_9 - 21\tilde{\mathsf{R}}_7\tilde{\mathsf{R}}_2 - \tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_7 - 35\tilde{\mathsf{R}}_6\tilde{\mathsf{R}}_3 - 7\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_6 - 35\tilde{\mathsf{R}}_5\tilde{\mathsf{R}}_4 - 21\tilde{\mathsf{R}}_4\tilde{\mathsf{R}}_5\\
&\qquad + 35\tilde{\mathsf{R}}_5\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2 + 10\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_5\tilde{\mathsf{R}}_2 + \tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_5 + 63\tilde{\mathsf{R}}_4\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_2 + 21\tilde{\mathsf{R}}_4\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_3\\
&\qquad + 42\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_4\tilde{\mathsf{R}}_2 + 7\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_4 + 10\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_4\tilde{\mathsf{R}}_3 + 5\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_4 + 28\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_3)\,\big]^{\hat\beta}{}_\kappa\\
&+ \mathscr{O}(x^{10})\,.
\end{aligned} \tag{6.78}$$

Manifestly Covariant form of the Recursion and its Closed-Form Solution

We saw above that the closed-form solution (6.84) of the recursion (6.64) is written in terms of partial derivatives of Riemann tensors. In order to turn said recursion into one which produces strings of *covariant derivatives* of Riemann tensors instead, we shall make use of a neat trick. We first note that for $n = 3$, our recursion (6.64), i.e. the product $e^{\hat{\beta}}{}_{\delta}(x)R^{\delta}{}_{(\alpha_1\alpha_2)\kappa}(x)$ generates "one half" of the covariant derivative (6.56) of the Riemann tensor, i.e.,

$$\partial_{(\alpha_1|}\Big[e^{\hat{\beta}}{}_{\delta}(x)\, R^{\delta}{}_{|\alpha_2\alpha_3)\kappa}\Big]\Big|_{\mathscr{P}} = e^{\hat{\beta}}{}_{\delta}(x)\Big[R^{\delta}{}_{(\alpha_1\alpha_2|\kappa;\,|\alpha_3)} + \Gamma^{\delta}{}_{(\alpha_1|\delta_1}(x)R^{\delta_1}{}_{|\alpha_2\alpha_3)\kappa}\Big]\Big|_{\mathscr{P}}, \tag{6.79}$$

via the Leibnitz rule, with the Christoffel term coming from the parallel-transport equation of $e^{\hat{\beta}}{}_{\delta}(x)$. It is then clear, that we can similarly generate its full covariant derivative, as displayed above, by acting with a *partial* derivative on a Riemann tensor, "dressed", as before, with an inverse tetrad on the left and, in addition, a tetrad on the right. This reads,

$$\begin{aligned}\partial_{(\alpha_1|}\Big[e^{\hat{\beta}}{}_{\delta}(x)\, R^{\delta}{}_{|\alpha_2\alpha_3)\gamma}(x)\, e_{\hat{\gamma}}{}^{\gamma}(x)\Big] &= e^{\hat{\beta}}{}_{\delta}(x)\Big[R^{\delta}{}_{(\alpha_1\alpha_2|\gamma,\,|\alpha_3)} + \Gamma^{\delta}{}_{(\alpha_1|\delta_1}R^{\delta_1}{}_{|\alpha_2\alpha_3)\gamma} \\ &\qquad - R^{\delta}{}_{(\alpha_1\alpha_2|\delta_1}\Gamma^{\delta_1}{}_{|\alpha_3)\gamma}\Big]e_{\hat{\gamma}}{}^{\gamma}(x) \\ &= e^{\hat{\beta}}{}_{\delta}(x)R^{\delta}{}_{(\alpha_1\alpha_2|\gamma;\,|\alpha_3)}e_{\hat{\gamma}}{}^{\gamma}(x)\,,\end{aligned} \tag{6.80}$$

with the expression in brackets on the right-hand side of the first equation being the familiar expression (6.56) of the covariant derivative of the Riemann tensor in Riemann coordinates. This works recursively, i.e. we can also generate the second and higher covariant derivatives of $R^{\delta}{}_{(\alpha_1\alpha_2)\kappa}$ by acting with more partial derivatives. We note that here, the tetrad again appears in its role as parallel propagator, cf. (6.38).

We can now put this to use with our recursion (6.64) in the following way: we simply extend with unity in the form of a Kronecker delta on the right-hand side of the Riemann tensor inside the recursion, inserting the tetrad's orthonormality relation in the form, $\delta^{\gamma}{}_{\kappa} = e_{\hat{\gamma}}{}^{\gamma}(x)e^{\hat{\gamma}}{}_{\kappa}(x)$, so that we have,

$$\partial_{(\alpha_1}\cdots\partial_{\alpha_n)}e^{\hat{\beta}}{}_{\kappa}(x)\big|_{\mathscr{P}} = \frac{n-1}{n+1}\partial_{(\alpha_1}\cdots\partial_{\alpha_{n-2}|}\Big[\underbrace{e^{\hat{\beta}}{}_{\delta}(x)R^{\delta}{}_{|\alpha_{n-1}\alpha_n)\gamma}e_{\hat{\gamma}}{}^{\gamma}(x)}_{\text{generates } R^{\delta}{}_{(\alpha_1\alpha_2|\gamma;\,|\alpha_3\cdots\alpha_n)}}\, e^{\hat{\gamma}}{}_{\kappa}(x)\Big]\Big|_{\mathscr{P}}. \tag{6.81}$$

Contracting the Riemann tensor with $e^{\hat{\beta}}{}_{\delta}(x)$ and $e_{\hat{\gamma}}{}^{\gamma}(x)$, respectively, we thus obtain the manifestly covariant version of the recursion for the inverse tetrad's n^{th}-order Taylor coefficients in terms of *covariant derivatives* of Riemann tensors,

$$\boxed{\partial_{(\alpha_1}\cdots\partial_{\alpha_n)}e^{\hat{\beta}}{}_{\kappa}(x)\big|_{\mathscr{P}} = \frac{n-1}{n+1}\partial_{(\alpha_1}\cdots\partial_{\alpha_{n-2}|}\Big(R^{\hat{\beta}}{}_{|\alpha_{n-1}\alpha_n)\hat{\delta}}\, e^{\hat{\delta}}{}_{\kappa}(x)\Big)\Big|_{\mathscr{P}}. \quad n \geq 1\,.} \tag{6.82}$$

Note that, in contrast to (6.64), the tetrad now contracts with the Riemann tensor on a hatted tetrad index at its right-hand side. This recursion for the inverse tetrad was first derived by Müller, Schubert and van de Ven using a different but equivalent approach in a non-coordinate basis in [170] (they derive a differential equation and its corresponding integral form, before Taylor-expanding these to obtain the recursion). It is now easy to see, that (6.82) must have the same closed-form solution as (6.64), the only difference being, that iterating (6.82) produces strings of Riemann factors that grow to the right. Thus, defining the *covariant Riemann matrix*,

$$\mathsf{R}_n \coloneqq R^{\hat{\kappa}}{}_{(\hat{\alpha}_1\hat{\alpha}_2|\hat{\beta};|\hat{\alpha}_3\cdots\hat{\alpha}_n)} x^{\hat{\alpha}_1} x^{\hat{\alpha}_2} \cdots x^{\hat{\alpha}_n} \,, \tag{6.83}$$

in analogy to (6.75), we can write its closed-form solution as,

$$\boxed{e^{\hat{\beta}}{}_{\kappa}(x) = \delta^{\hat{\beta}}{}_{\kappa} + \sum_{n=2}^{\infty} \frac{1}{n!} \sum_{l=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{\substack{k_1,\ldots,k_l \geq 2 \\ k_1+\cdots+k_l=n}} \mathcal{C}_{\perp}(k_1,\ldots,k_l) \big(\mathsf{R}_{k_1}\mathsf{R}_{k_2}\cdots\mathsf{R}_{k_l}\big)^{\hat{\beta}}{}_{\kappa} \,,} \tag{6.84}$$

where the Riemann factors on the right-hand side are now in canonical order, i.e. $\mathsf{R}_{k_1}\mathsf{R}_{k_2}\cdots\mathsf{R}_{k_l}$, instead of $\tilde{\mathsf{R}}_{k_l}\tilde{\mathsf{R}}_{k_{l-1}}\cdots\tilde{\mathsf{R}}_{k_1}$ in (6.77). This then explains the previously-observed curious reversion of ordering in the numerical coefficients between the expansions as expressed in partial [equation (6.58d)] versus covariant Riemann derivatives [equation (6.59)]. Extending our manually calculated previous results (6.62b), the closed-form solution (6.84) of (6.82) yields the following *manifestly covariant* expansion of the inverse tetrad to 10^{th} order, written in terms of the covariant Riemann matrix R_n,

$$
\begin{aligned}
e^{\hat\beta}{}_{\kappa}(x) = \big[\, 1 &+ \tfrac{1}{2!}\tfrac{1}{3}\mathsf{R}_2 + \tfrac{1}{3!}\tfrac{1}{2}\mathsf{R}_3 + \tfrac{1}{4!}\tfrac{3}{5}(\mathsf{R}_4 + \tfrac{1}{3}\mathsf{R}_2\mathsf{R}_2) + \tfrac{1}{5!}\tfrac{2}{3}(\mathsf{R}_5 + \mathsf{R}_3\mathsf{R}_2 + \tfrac{1}{2}\mathsf{R}_2\mathsf{R}_3) \\
&+ \tfrac{1}{6!}\tfrac{5}{7}(\mathsf{R}_6 + 2\mathsf{R}_4\mathsf{R}_2 + \tfrac{3}{5}\mathsf{R}_2\mathsf{R}_4 + 2\mathsf{R}_3\mathsf{R}_3 + \tfrac{1}{5}\mathsf{R}_2\mathsf{R}_2\mathsf{R}_2) \\
&+ \tfrac{1}{7!}\tfrac{3}{4}(\mathsf{R}_7 + \tfrac{10}{3}\mathsf{R}_5\mathsf{R}_2 + \tfrac{2}{3}\mathsf{R}_2\mathsf{R}_5 + 5\mathsf{R}_4\mathsf{R}_3 + 3\mathsf{R}_3\mathsf{R}_4 + \mathsf{R}_3\mathsf{R}_2\mathsf{R}_2 \\
&\qquad + \tfrac{2}{3}\mathsf{R}_2\mathsf{R}_3\mathsf{R}_2 + \tfrac{1}{3}\mathsf{R}_2\mathsf{R}_2\mathsf{R}_3) \\
&+ \tfrac{1}{8!}\tfrac{7}{9}(\mathsf{R}_8 + 5\mathsf{R}_6\mathsf{R}_2 + \tfrac{5}{7}\mathsf{R}_2\mathsf{R}_6 + 10\mathsf{R}_5\mathsf{R}_3 + 4\mathsf{R}_3\mathsf{R}_5 + 9\mathsf{R}_4\mathsf{R}_4 \\
&\qquad + 3\mathsf{R}_4\mathsf{R}_2\mathsf{R}_2 + \tfrac{10}{7}\mathsf{R}_2\mathsf{R}_4\mathsf{R}_2 + \tfrac{3}{7}\mathsf{R}_2\mathsf{R}_2\mathsf{R}_4 + 4\mathsf{R}_3\mathsf{R}_3\mathsf{R}_2 \\
&\qquad + 2\mathsf{R}_3\mathsf{R}_2\mathsf{R}_3 + \tfrac{10}{7}\mathsf{R}_2\mathsf{R}_3\mathsf{R}_3 + \tfrac{1}{7}\mathsf{R}_2\mathsf{R}_2\mathsf{R}_2\mathsf{R}_2) \\
&+ \tfrac{1}{9!}\tfrac{4}{5}(\mathsf{R}_9 + 7\mathsf{R}_7\mathsf{R}_2 + \tfrac{2}{3}\mathsf{R}_2\mathsf{R}_7 + \tfrac{35}{2}\mathsf{R}_6\mathsf{R}_3 + 5\mathsf{R}_3\mathsf{R}_6 + 21\mathsf{R}_5\mathsf{R}_4 \\
&\qquad + 14\mathsf{R}_4\mathsf{R}_5 + 7\mathsf{R}_5\mathsf{R}_2\mathsf{R}_2 + \tfrac{5}{2}\mathsf{R}_2\mathsf{R}_5\mathsf{R}_2 + \tfrac{1}{2}\mathsf{R}_2\mathsf{R}_2\mathsf{R}_5 \\
&\qquad + 14\mathsf{R}_4\mathsf{R}_3\mathsf{R}_2 + 10\mathsf{R}_3\mathsf{R}_4\mathsf{R}_2 + 7\mathsf{R}_4\mathsf{R}_2\mathsf{R}_3 + 3\mathsf{R}_3\mathsf{R}_2\mathsf{R}_4 \\
&\qquad + \tfrac{15}{4}\mathsf{R}_2\mathsf{R}_4\mathsf{R}_3 + \tfrac{9}{4}\mathsf{R}_2\mathsf{R}_3\mathsf{R}_4 + 10\mathsf{R}_3\mathsf{R}_3\mathsf{R}_3 + \mathsf{R}_3\mathsf{R}_2\mathsf{R}_2\mathsf{R}_2 \\
&\qquad + \tfrac{3}{4}\mathsf{R}_2\mathsf{R}_3\mathsf{R}_2\mathsf{R}_2 + \tfrac{1}{2}\mathsf{R}_2\mathsf{R}_2\mathsf{R}_3\mathsf{R}_2 + \tfrac{1}{4}\mathsf{R}_2\mathsf{R}_2\mathsf{R}_2\mathsf{R}_3) \\
&+ \tfrac{1}{10!}\tfrac{9}{11}(\mathsf{R}_{10} + \tfrac{28}{3}\mathsf{R}_8\mathsf{R}_2 + \tfrac{7}{9}\mathsf{R}_2\mathsf{R}_8 + 28\mathsf{R}_7\mathsf{R}_3 + 6\mathsf{R}_3\mathsf{R}_7 + 42\mathsf{R}_6\mathsf{R}_4 \\
&\qquad + 20\mathsf{R}_4\mathsf{R}_6 + \tfrac{112}{3}\mathsf{R}_5\mathsf{R}_5 + 14\mathsf{R}_6\mathsf{R}_2\mathsf{R}_2 + \tfrac{35}{9}\mathsf{R}_2\mathsf{R}_6\mathsf{R}_2 \\
&\qquad + \tfrac{5}{9}\mathsf{R}_2\mathsf{R}_2\mathsf{R}_6 + \tfrac{112}{3}\mathsf{R}_5\mathsf{R}_3\mathsf{R}_2 + 20\mathsf{R}_3\mathsf{R}_5\mathsf{R}_2 + \tfrac{56}{3}\mathsf{R}_5\mathsf{R}_2\mathsf{R}_3 \\
&\qquad + \tfrac{70}{9}\mathsf{R}_2\mathsf{R}_5\mathsf{R}_3 + 4\mathsf{R}_3\mathsf{R}_2\mathsf{R}_5 + \tfrac{28}{9}\mathsf{R}_2\mathsf{R}_3\mathsf{R}_5 + 40\mathsf{R}_4\mathsf{R}_4\mathsf{R}_2 \\
&\qquad + 12\mathsf{R}_4\mathsf{R}_2\mathsf{R}_4 + 7\mathsf{R}_2\mathsf{R}_4\mathsf{R}_4 + 40\mathsf{R}_4\mathsf{R}_3\mathsf{R}_3 + 30\mathsf{R}_3\mathsf{R}_4\mathsf{R}_3 \\
&\qquad + 18\mathsf{R}_3\mathsf{R}_3\mathsf{R}_4 + 4\mathsf{R}_4\mathsf{R}_2\mathsf{R}_2\mathsf{R}_2 + \tfrac{7}{3}\mathsf{R}_2\mathsf{R}_4\mathsf{R}_2\mathsf{R}_2 + \tfrac{10}{9}\mathsf{R}_2\mathsf{R}_2\mathsf{R}_4\mathsf{R}_2 \\
&\qquad + \tfrac{1}{3}\mathsf{R}_2\mathsf{R}_2\mathsf{R}_2\mathsf{R}_4 + 6\mathsf{R}_3\mathsf{R}_3\mathsf{R}_2\mathsf{R}_2 + 4\mathsf{R}_3\mathsf{R}_2\mathsf{R}_3\mathsf{R}_2 + 2\mathsf{R}_3\mathsf{R}_2\mathsf{R}_2\mathsf{R}_3 \\
&\qquad + \tfrac{28}{9}\mathsf{R}_2\mathsf{R}_3\mathsf{R}_3\mathsf{R}_2 + \tfrac{14}{9}\mathsf{R}_2\mathsf{R}_3\mathsf{R}_2\mathsf{R}_3 + \tfrac{10}{9}\mathsf{R}_2\mathsf{R}_2\mathsf{R}_3\mathsf{R}_3 \\
&\qquad + \tfrac{1}{9}\mathsf{R}_2\mathsf{R}_2\mathsf{R}_2\mathsf{R}_2\mathsf{R}_2)\,\big]^{\hat\beta}{}_{\kappa} + \mathscr{O}(x^{11})\,.
\end{aligned}
\tag{6.85}
$$

Note that the number of terms at order $n \geq 1$ is given by F_{n-1}, where $\mathrm{F}_n = 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$ is the sequence of Fibonacci numbers, both for the expansion in terms of partial Riemann derivatives in (6.78), as well as for the manifestly covariant expansion (6.85) above. This is explained in subsection 6.2.8 below.

6.2.8. A Combinatorial Perspective on the Riemann and Fermi Expansions: Non-Commutative Composition Polynomials

From the above discussion of the closed-form solution for the Riemann expansion coefficients of the inverse tetrad, and the explicit structure of these in (6.78) and (6.85), the reader too has probably developed the suspicion that the structure of these expansions must have a combinatorial origin. Indeed, this is the case, as we shall motivate in the present section. To the best of our knowledge, these connections have not been noticed before.

Partitions and Restricted Compositions

We start with a reminder of some basic elements of partitions and compositions. An integer l-composition of n is an *ordered* partition of n into l summands, i.e. a decomposition of n

into l parts, $k_1, k_2, \ldots, k_l$, such that,

$$k_1 + k_2 + \cdots + k_l = n\,, \tag{6.86}$$

i.e., all sequences of integers $k_1, k_2, \ldots, k_l$ (we assume $k_i > 0$ throughout) which fulfill this linear Diophantine equation are an integer composition of n. For example, the 4 integer compositions of 3 are: 1|1|1, 1|2, 2|1, 3. The difference between partitions and compositions is that in the former, order doesn't matter, so all compositions with the same parts are considered to be the same partition: There are only 3 *partitions* of the integer 3, namely 1|1|1, 1|2, 3, i.e. the two compositions 1|2 and 2|1 belong to the same partition. In a certain sense, we can thus think of compositions as "non-commutative" partitions.

How many l-composition of n are there? The standard combinatorial argument goes as follows: we alternatingly place n 1s and $n-1$ boxes in a row, starting and ending with a 1, i.e.

$$1\square 1\square 1\square \cdots \square 1\square 1$$

Putting now either a "+" or a "," into the boxes, produces a unique composition of n. This amounts to $n-1$ binary choices, so having placed $l-1$ commas corresponding to composition with l parts, we obtain the well-known result that there are $\binom{n-1}{l-1}$ l-composition of n, and thus

$$\sum_{l=1}^{n} \binom{n-1}{l-1} = (1+1)^{n-1} = 2^{n-1} \tag{6.87}$$

compositions of n in total.

There are now different ways to restrict integer compositions, e.g., restricting the way the parts are arranged within the composition, or restricting the set from which the parts of the composition are taken (see, e.g. [176]). In the following, we shall be interested in the latter case. We call a composition a q-restricted composition, if its parts k_i are restricted to $q < k_i$ for a positive integer q. How many q-restricted l-compositions are there? This is fairly easy to see, since all compositions are solutions to (6.86): The general strategy is to start with that condition and manipulate the parts, to yield another composition, e.g. $k_1' + k_2' + \cdots + k_l' = n'$. Thus, starting with an unrestricted l-composition, in this simple case we just have to add q to every part, i.e.,

$$\begin{gathered} k_1' + \cdots + k_l' = (q + k_1) + \cdots + (q + k_l) = n\,, \quad \text{or,} \\ k_1 + \cdots + k_l = n - la = n'\,, \end{gathered}$$

where the new part variables $q < k_i'$ if $k_i \in \{1, \ldots, n\}$, as before. We thus end up with a total $\binom{n'-1}{l-1} = \binom{n-lq-1}{l-1}$ of q-restricted l-compositions (see, e.g. [177]), which yields, in particular,

$$\binom{n-l-1}{l-1} = \frac{(n-l-1)!}{(l-1)!(n-2l)!}\,. \tag{6.88}$$

for the number of 1-restricted l-compositions, that we shall meet below.

Non-Commutative Bell Polynomials

A short account of the (very limited) literature on non-commutative Bell polynomials that we are about to discuss can be found in the recent textbook by Mansour and Schork [178, Sec. 9.9.3], see also Lundervold and Munthe-Kaas [179, Sec. 4.2.1].

Let $\mathsf{X}_1, \mathsf{X}_2, \ldots, \mathsf{X}_n$ be formal non-commuting variables of some kind (our notation is suggestive of the X_i being matrices; below we will make the identification $\mathsf{X}_i = \Gamma_i$ and in a closely related context, $\mathsf{X}_i = \mathsf{R}_i$ for $i \geq 2$). We further define a derivation ∂ that acts on the X_i by increasing their order by one, i.e. as $\partial \mathsf{X}_i \coloneqq \mathsf{X}_{i+1}$, with the usual property $\partial \mathsf{1} = \mathsf{0}$. We now investigate the expansion of the n^{th} power,

$$(\mathsf{X}_1 + \partial)^n \,, \tag{6.89}$$

of the compound symbolic operator built from a linear combination of X_1 and ∂. Note, that this operator has the symbolic form of a covariant derivative, if we interpret X_1 as some kind of connection and the derivation ∂ as a directional derivative.

Acting (6.89) recursively on $\mathsf{1}$ can be seen to generate non-commutative polynomials in the $\mathsf{X}_1, \ldots, \mathsf{X}_n$ with the first ones being, $B_0 \coloneqq \mathsf{1}$ and $B_1 \coloneqq \mathsf{X}_1$, i.e.,

$$B_n = (\mathsf{X}_1 + \partial)^n B_0 = (\mathsf{X}_1 + \partial)^n \mathsf{1} = (\mathsf{X}_1 + \partial)^{n-1} \mathsf{X}_1 \,. \tag{6.90}$$

These are called *non-commutative Bell polynomials*, $B_n(\mathsf{X}_1, \ldots, \mathsf{X}_n)$, since they are a direct non-commutative generalisation of the exponential Bell polynomials first introduced as the prototypical partition polynomials by Bell in 1927 [180], and subsequently named in his honour. Non-commutative Bell polynomials were introduced in 1995 by Munthe-Kaas in the context of defining Runge-Kutta methods in the context of manifolds [181, 182]. Non-commutative (and commutative) Bell polynomials are defined by the recursion,

$$\boxed{\begin{aligned} &B_0 \coloneqq 1\,, && \\ &B_n \coloneqq (\mathsf{X}_1 + \partial) B_{n-1}\,, && n \geq 1\,, \end{aligned}} \tag{6.91}$$

with the first instances of (6.91) being:

$$\begin{aligned} &B_0 = 1\,, \quad B_1 = \mathsf{X}_1\,, \quad B_2 = \mathsf{X}_2 + \mathsf{X}_1^2\,, \quad B_3 = \mathsf{X}_3 + \mathsf{X}_2\mathsf{X}_1 + 2\mathsf{X}_1\mathsf{X}_2 + \mathsf{X}_1^3\,, \\ &B_4 = \mathsf{X}_4 + \mathsf{X}_3\mathsf{X}_1 + 3\mathsf{X}_1\mathsf{X}_3 + 3\mathsf{X}_2\mathsf{X}_2 + \mathsf{X}_2\mathsf{X}_1\mathsf{X}_1 + 2\mathsf{X}_1\mathsf{X}_2\mathsf{X}_1 + 3\mathsf{X}_1\mathsf{X}_1\mathsf{X}_2 + \mathsf{X}_1^4\,. \end{aligned}$$

By induction, one finds that they can also be written as a binomial recursion relation,

$$\boxed{B_n(\mathsf{X}_1, \ldots, \mathsf{X}_n) = \sum_{k=1}^{n} \binom{n-1}{k-1} B_{n-k}(\mathsf{X}_1, \ldots, \mathsf{X}_{n-k})\, \mathsf{X}_k\,,} \tag{6.92}$$

which is seen to be nothing else but a generalisation of the above sum over all compositions (6.87) to the polynomial case. Since the binomial coefficient there recursively counts the number of k-part compositions of n, these are thus *composition polynomials.*

The above recursion (6.92) can be iterated to yield a closed-form solution, which is conventionally written in terms of the so-called non-commutative *partial* Bell polynomials,

$$B_{n,l}(\mathsf{X}_1,\dots,\mathsf{X}_{n-l+1}) = \sum_{\substack{k_1,\dots,k_l\geq 1\\ k_1+\cdots+k_l=n}} \binom{n}{k_1,\dots,k_l} \frac{k_1\cdots k_l}{k_1(k_1+k_2)\cdots(k_1+\cdots+k_l)} \mathsf{X}_{k_1}\cdots\mathsf{X}_{k_l}\,, \tag{6.93}$$

which represent that part of B_n containing the words (or *strings*) of length l and which correspond directly to the l-part compositions of n. For example, $B_{3,2} = \mathsf{X}_2\mathsf{X}_1 + 2\mathsf{X}_1\mathsf{X}_2$, or $B_{4,3} = \mathsf{X}_2\mathsf{X}_1^2 + 2\mathsf{X}_1\mathsf{X}_2\mathsf{X}_1 + 3\mathsf{X}_1^2\mathsf{X}_2$, etc. The B_n then follow from summing (6.93) over all parts,

$$B_n(\mathsf{X}_1,\dots,\mathsf{X}_n) = \sum_{l=1}^{n} B_{n,l}(\mathsf{X}_1,\dots,\mathsf{X}_{n-l+1})\,, \tag{6.94}$$

(cf. [183]. The numerical coefficient in $B_{n,l}$ can be written in one of two forms [183]. Either, as above, in terms of a multinomial coefficient times a product of factors that we shall discuss in a moment, or in terms of a product of binomials, i.e.,

$$\binom{n}{k_1,\dots,k_l} \frac{k_1\cdots k_l}{k_1(k_1+k_2)\cdots(k_1+\cdots+k_l)} = \prod_{i=1}^{l-1} \binom{k_1+\ldots+k_{i+1}-1}{k_{i+1}-1}\,, \tag{6.95}$$

where we could also write $\binom{k_1+\cdots+k_{i+1}-1}{k_{i+1}-1} = \binom{k_1+\cdots+k_{i+1}-1}{k_1+\cdots+k_i}$, due to the well-known symmetry of the binomial coefficients. The product of factors $\frac{k_1\cdots k_l}{k_1(k_1+k_2)\cdots(k_1+\cdots+k_l)}$ in the first form on the left-hand-side forms a partition of unity on the symmetric group S_l of order l, i.e.,

$$\sum_{\pi\in\mathrm{S}_l} \frac{k_{\pi(1)}\cdots k_{\pi(l)}}{k_{\pi(1)}\big(k_{\pi(1)}+k_{\pi(2)}\big)\cdots\big(k_{\pi(1)}+\cdots+k_{\pi(l)}\big)} = 1\,, \tag{6.96}$$

where π denotes a permutation of the symbols $k_1,\dots,k_l$. Taking the above example of the compositions of $n=3$ again, we have,

$$\sum_{\substack{k_1,k_2\geq 1\\ k_1+k_2=3}} \frac{k_1k_2}{k_1(k_1+k_2)} = \tfrac{2}{3}+\tfrac{1}{3} = 1\,,$$

for the two 2-part compositions $1|2$ and $2|1$, respectively. We now make the important observation, that the recursion (6.92) for the Bell polynomials is just the recursion (6.45) for the inverse tetrad, $e^{\hat\beta}{}_{\kappa}(x)$, in terms of the generalised connection coefficients $\mathsf{\Gamma}_n$. Thus, in the context of Riemann normal coordinates [and also Fermi coordinates, compare for (6.119) and (6.119), where also the $\mathsf{\Gamma}_1$ are retained], we can identify $\mathsf{\Gamma}_n \equiv \mathsf{X}_n$, and the n^{th}-order expansion coefficient of the inverse tetrad can be written as,

$$\partial_{(\alpha_1}\cdots\partial_{\alpha_n)} e^{\hat\beta}{}_{\kappa}(x)\big|_{\mathscr{P}} = \sum_{l=1}^{n} \big[B_{n,l}(\mathsf{\Gamma}_1,\dots,\mathsf{\Gamma}_n)\big]^{\hat\beta}{}_{\kappa}\,, \tag{6.97}$$

in terms of the partial non-commutative Bell polynomials (6.93).

Expansion Coefficients as Restricted Non-Commutative Composition Polynomials

With the above considerations, it is now clear that the manifestly covariant recursion for the inverse tetrad (6.69) must also be related to composition polynomials. Indeed, one sees that the inverse tetrad's expansion coefficients in terms of the Riemann matrices R_k are (up to the additional numerical factor $\frac{n-1}{n+1}$) what we shall call 1-restricted non-commutative Bell polynomials $B_n^{\bar{1}}$, i.e. they are based on 1-restricted integer compositions, and given in this context in terms of the non-commuting variables $\mathsf{X}_k \equiv \mathsf{R}_k$ for $k \geq 2$, since no first-order terms appear there. They are then generated by the same recursion (6.92) as the standard non-commutative Bell polynomials by simply starting the summation with index $k = 2$, i.e.

$$\boxed{B_n^{\bar{1}}(\mathsf{X}_2, \ldots, \mathsf{X}_n) = \sum_{k=2}^{n} \binom{n-2}{k-2} B_{n-k}^{\bar{1}}(\mathsf{X}_2, \ldots, \mathsf{X}_{n-k})\, \mathsf{X}_k \,.} \tag{6.98}$$

We can now also explain why the number of terms at n^{th} order in the above expansions (6.85), (6.78) is given by a Fibonacci number F_{n-1}. This follows simply from the fact that the coefficients are in one-to-one correspondence with the 1-restricted compositions of n, of which there are a total of $\binom{n-l-1}{l-1}$, so summing over the number l of parts yields,

$$\sum_{l=1}^{n} \binom{n-l-1}{l-1} = \mathrm{F}_{n-1} \,, \tag{6.99}$$

which corresponds to a well-known definition of the Fibonacci numbers. It turns out that the connection between restricted compositions and Fibonacci numbers is well known in the combinatorics community [184] (see, e.g. [176, Sec. 3.3]).

6.2.9. Recursion and its Solution for the Metric

Since the metric is calculated by "squaring" the inverse tetrad, its expansion coefficients are obtained from a convolution of those of the tetrad. To this end, it is convenient to also write the n^{th}-order coefficients of the tetrad, inverse tetrad, and metric at $\mathscr{P}$ in matrix form as, e_n, $\bar{\mathsf{e}}_n$, and g_n, respectively, i.e.

$$\begin{aligned} (\mathsf{e}_n)_{\hat{\beta}}{}^{\kappa} &:= e_{\hat{\beta}}{}^{\kappa}{}_{,\hat{\alpha}_1\cdots\hat{\alpha}_n}\big|_{\mathscr{P}} x^{\hat{\alpha}_1}\cdots x^{\hat{\alpha}_n} \\ (\bar{\mathsf{e}}_n)^{\hat{\beta}}{}_{\kappa} &:= e^{\hat{\beta}}{}_{\kappa,\hat{\alpha}_1\cdots\hat{\alpha}_n}\big|_{\mathscr{P}} x^{\hat{\alpha}_1}\cdots x^{\hat{\alpha}_n} \end{aligned} \qquad (\mathsf{g}_n)_{\kappa_1\kappa_2} := g_{\kappa_1\kappa_2,\hat{\alpha}_1\cdots\hat{\alpha}_n}\big|_{\mathscr{P}} x^{\hat{\alpha}_1}\cdots x^{\hat{\alpha}_n} \,. \tag{6.100}$$

In terms of $\bar{\mathsf{e}}_n$, the expression for g_n then follows by acting with n partial derivatives on $g_{\kappa_1\kappa_2} = \eta_{\hat{\beta}\hat{\delta}} e^{\hat{\beta}}{}_{\kappa_1} e^{\hat{\delta}}{}_{\kappa_2}$, which we write as $\mathsf{g} = \bar{\mathsf{e}}^{\mathsf{T}} \eta\, \bar{\mathsf{e}}$, and using the Leibniz rule,

$$\mathsf{g}_n = \partial^n(\bar{\mathsf{e}}^{\mathsf{T}}\eta\,\bar{\mathsf{e}}) = \sum_{\substack{n_1,n_2\geq 0 \\ n_1+n_2=n}} \binom{n}{n_1, n_2} \bar{\mathsf{e}}_{n_1}^{\mathsf{T}}\, \eta\, \bar{\mathsf{e}}_{n_2} \,. \tag{6.101}$$

The first two instances are clearly $\mathsf{g}_0 = 1$ and $\mathsf{g}_1 = 0$, so we can start the sum with $n_1, n_2 \geq 2$. However we shall not work out the complete solution for the metric at this point.

6.2.10. Tetrad and Inverse Metric

For the tetrad itself we shall also restrict ourselves to providing an initial ansatz. Using the orthonormality condition (4.42), (4.43) in the first of its two mixed forms, which we write as $\mathsf{e}\,\bar{\mathsf{e}} = 1$, with e being the matrix of the tetrad, we simply act with n partial derivatives, which yields,

$$\partial^n(\mathsf{e}\,\bar{\mathsf{e}}) = \sum_{k=0}^{n} \binom{n}{k} \mathsf{e}_{n-k}\bar{\mathsf{e}}_k = 0\,, \tag{6.102}$$

in terms of the k^{th}-order derivatives e_k and e_k of e and e, respectively. Pulling out the term with $k = 0$, we obtain a binomial-type recursion for e_n,

$$\mathsf{e}_n = -\sum_{k=1}^{n} \binom{n}{k} \mathsf{e}_{n-k}\bar{\mathsf{e}}_k\,, \qquad n \geq 1\,, \tag{6.103}$$

where $\mathsf{e}_0 = 1$. This can be iterated $l-1$ times as before, yielding,

$$\mathsf{e}_n = \sum_{l=1}^{n} (-1)^l \sum_{\substack{n_1,\ldots,n_l \geq 1 \\ n_1+\cdots+n_l = n}} \binom{n}{n_1,\ldots,n_l} \bar{\mathsf{e}}_{n_1}\bar{\mathsf{e}}_{n_2}\cdots\bar{\mathsf{e}}_{n_l}\,, \tag{6.104}$$

which is an expression for the coefficients in the expansion of $e_{\hat{\beta}}{}^{\kappa}(x)$, in terms of those of the inverse tetrad, $e^{\hat{\beta}}{}_{\kappa}(x)$. In principle, we can now insert the corresponding n_i^{th}-order terms of the closed-form solution (6.84) of the inverse tetrad, i.e.,

$$\bar{\mathsf{e}}_{n_i} = \sum_{l_i=1}^{\lfloor \frac{n_i}{2} \rfloor} \sum_{\substack{k_{i1},\ldots,k_{il_i} \geq 2 \\ k_{i1}+\cdots+k_{il_i} = n_i}} \mathcal{C}_{\perp}(k_{i1},\ldots,k_{il_i})\, \mathsf{R}_{k_{i1}}\mathsf{R}_{k_{i2}}\cdots\mathsf{R}_{k_{il_i}}\,. \tag{6.105}$$

However, the result is extremely messy, and for now, we shall refrain from pursuing this direction any further. Instead, we content ourselves with displaying the lowest order terms: To 8^{th} order, the expansion of tetrad itself in Riemann coordinates reads,

$$\begin{aligned} e_{\hat{\beta}}{}^{\kappa}(x) = \big[\,1 &- \tfrac{1}{2!}\tfrac{1}{3}\mathsf{R}_2 - \tfrac{1}{3!}\tfrac{1}{2}\mathsf{R}_3 - \tfrac{1}{4!}\tfrac{3}{5}(\mathsf{R}_4 - \tfrac{7}{9}\mathsf{R}_2\mathsf{R}_2) - \tfrac{1}{5!}\tfrac{2}{3}(\mathsf{R}_5 - \tfrac{3}{2}\mathsf{R}_3\mathsf{R}_2 - 2\mathsf{R}_2\mathsf{R}_3) \\ &- \tfrac{1}{6!}\tfrac{5}{7}(\mathsf{R}_6 - 2\mathsf{R}_4\mathsf{R}_2 - \tfrac{3}{5}\mathsf{R}_2\mathsf{R}_4 - 2\mathsf{R}_3\mathsf{R}_3 + \tfrac{1}{5}\mathsf{R}_2\mathsf{R}_2\mathsf{R}_2) \\ &- \tfrac{1}{7!}\tfrac{3}{4}(\mathsf{R}_7 - \tfrac{10}{3}\mathsf{R}_5\mathsf{R}_2 - \tfrac{2}{3}\mathsf{R}_2\mathsf{R}_5 - 5\mathsf{R}_4\mathsf{R}_3 - 3\mathsf{R}_3\mathsf{R}_4 + \mathsf{R}_3\mathsf{R}_2\mathsf{R}_2 \\ &\qquad + \tfrac{2}{3}\mathsf{R}_2\mathsf{R}_3\mathsf{R}_2 + \tfrac{1}{3}\mathsf{R}_2\mathsf{R}_2\mathsf{R}_3) \\ &- \tfrac{1}{8!}\tfrac{7}{9}(\mathsf{R}_8 + 5\mathsf{R}_6\mathsf{R}_2 - \tfrac{5}{7}\mathsf{R}_2\mathsf{R}_6 - 10\mathsf{R}_5\mathsf{R}_3 - 4\mathsf{R}_3\mathsf{R}_5 - 9\mathsf{R}_4\mathsf{R}_4 \\ &\qquad + 3\mathsf{R}_4\mathsf{R}_2\mathsf{R}_2 + \tfrac{10}{7}\mathsf{R}_2\mathsf{R}_4\mathsf{R}_2 + \tfrac{3}{7}\mathsf{R}_2\mathsf{R}_2\mathsf{R}_4 + 4\mathsf{R}_3\mathsf{R}_3\mathsf{R}_2 \\ &\qquad + 2\mathsf{R}_3\mathsf{R}_2\mathsf{R}_3 + \tfrac{10}{7}\mathsf{R}_2\mathsf{R}_3\mathsf{R}_3 - \tfrac{1}{7}\mathsf{R}_2\mathsf{R}_2\mathsf{R}_2\mathsf{R}_2)\,\big]^{\kappa}{}_{\hat{\beta}} \\ &+ \mathscr{O}(x^{11})\,. \end{aligned} \tag{6.106}$$

6.2.11. An Exactly Soluble Example: de Sitter Space-Time and the Two-Sphere

As a check of the recursion (6.82), and more importantly of its solution (6.84), we are going to take the background space as a maximally symmetric and thus constant-curvature *Riemannian* manifold. In this case, the curvature tensors are given by (5.14), i.e. all the covariant derivatives of the Riemann tensor vanish. For simplicity, we are further going to assume that $R_{\mathrm{Ric}} = 2$ and that our Riemannian manifold is two-dimensional, thus it is just the two-sphere, which has constant unit Gaussian curvature. Since on a Riemannian manifold, the role of the flat Minkowski metric is played by the Kronecker delta $\delta_{\hat{a}\hat{b}}$, the frame components of the Riemann tensor are given by

$$R^{\hat{c}}{}_{\hat{a}_1\hat{a}_2\hat{b}} = \left(\delta^{\hat{c}}{}_{\hat{a}_2}\delta_{\hat{a}_1\hat{b}} - \delta^{\hat{c}}{}_{\hat{b}}\delta_{\hat{a}_2\hat{a}_2}\right). \tag{6.107}$$

The Riemann tensor being constant, it is then easy to see that the tetrad recursion (6.82) becomes trivial since the covariant derivatives "go through" the Riemann factor, i.e. the binomial in (6.69) reduces to a single term. Correspondingly, in the closed-form solution (6.84), the binomial coefficients become unity, so the numerical factor $\mathcal{C}_1$ reduces to

$$\mathcal{C}_1(2,2,\ldots,2) = \prod_{i=0}^{l}\frac{n-2i-1}{n-2i+1} = \frac{1}{n+1} = \frac{1}{2l+1}. \tag{6.108}$$

Defining a radial Riemann coordinate, $r^2 = x^a x_a$, the maximal product $(\mathsf{R}_2\mathsf{R}_2\cdots\mathsf{R}_2)$ of n Riemann tensors contracted with $x^{\hat{\alpha}_1}\cdots x^{\hat{\alpha}_n}$ reduces in this case to a single one decorated with a factor of r^n, since with (5.14) we find, that

$$\frac{R^{\hat{\beta}}{}_{\hat{\alpha}_i\hat{\alpha}_j\hat{\kappa}}}{r^2} = \left(\frac{x^{\hat{\beta}}x_{\hat{\kappa}}}{r^2} - \delta^{\hat{\beta}}{}_{\hat{\kappa}}\right) \tag{6.109}$$

is a projector and thus idempotent. So, using these simplifications in (6.84), as well as $n = 2l$, we initially obtain,

$$e^{\hat{b}}{}_k(x) = \delta^{\hat{b}}{}_k + \left[x^{\hat{b}}x_k - \delta^{\hat{b}}{}_k(x^a x_a)\right]\sum_{l=1}^{\infty}\frac{(-1)^{l-1}}{(2l+1)!}(x^a x_a)^{l-1}, \tag{6.110}$$

where we recognise the series definition of the sinc function, which finally yields,

$$e^{\hat{b}}{}_k(x) = \delta^{\hat{b}}{}_k + \sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{(2l+1)!}r^{2l-2}\left[x^{\hat{b}}x_k - \delta^{\hat{b}}{}_k r^2\right] = \frac{x^{\hat{b}}x_k}{r^2} + \frac{\sin(|r|)}{r}\left[\delta^{\hat{b}}{}_k - \frac{x^{\hat{b}}x_k}{r^2}\right]. \tag{6.111}$$

The metric turns out to have the same appearance, but with the prefactor $\sin(|r|)/r$ squared, i.e.,

$$g_{ij}(x) = \frac{x_i x_j}{r^2} + \frac{\sin^2(|r|)}{r^2}\left[\delta_{ij} - \frac{x_i x_j}{r^2}\right], \tag{6.112}$$

which we already recognised as the (spacial part of the) de Sitter metric (5.22) in a peculiar coordinate chart that we encountered in section 5.2. We thus find that this is the metric of de Sitter space-time in Riemann (and later Fermi) coordinates.

In our particular example, this is in fact the metric of the two-sphere in Cartesian coordinates: Upon introducing spherical Riemann coordinates, (r, ϕ), in the usual way as $x^{\hat{1}} \equiv x = r\cos(\phi)$ and $x^{\hat{2}} \equiv y = r\sin(\phi)$, we obtain

$$g_{ij}(x) = dr^2 + \sin^2(|r|)d\phi^2\,, \tag{6.113}$$

which takes the familiar form if we rename $r \to \theta$, so we see that the radial Riemann normal coordinate becomes an angle in this compact space.

6.3. Fermi Normal Coordinates

In the last section 6.2, we saw how the inertial frame at a point, being a purely mathematical realisation of the equivalence principle, can be extended outwards along all geodesics that intersect its origin, thereby defining a geodesic coordinate system called Riemann normal coordinates. In this section, we will use the Riemann coordinate expansion to extend the *physical* inertial frame along a world-line. This is done by using the tetrad and the local 1+3 split of space-time into time and space that it induces to set up a *spacial-only* Riemann normal coordinate system *orthogonal* to the central world-line $\mathscr{W}$. The coordinate system $x^\alpha = (x^0, x^i)$ that results from taking $x^0 = c\tau$ in terms of proper time on $\mathscr{W}$, together with the spacial Riemann coordinates x^i, is then called a *Fermi normal coordinate* system. This is in direct extension of the local inertial coordinates in (6.12).

More geometrically, Fermi normal coordinates can be considered a natural generalisation of Riemann normal coordinates that arises upon replacing the point $\mathscr{P}$ (being a zero-dimensional sub-manifold) at the origin of the Riemann normal coordinate system by a higher-dimensional sub-manifold $\mathscr{W}$, which – in the context of general relativity – is usually taken to be the one-dimensional world-line of an observer. The remaining Riemann normal systems then live in the three-dimensional spacial sub-manifold orthogonal to that world-line $\mathscr{W}$, so this construction amounts to a continuous family of Riemann normal coordinate systems along $\mathscr{W}$, parametrised by proper time τ.

The construction of Fermi normal coordinates (or *Fermi coordinates* for short) is illustrated in Figure 6.1. The purely spacial geodesics are the dashed red curves lying in the (curved) spacial rest-space $\Sigma_{\boldsymbol{u}}(\tau)$ of $\mathscr{W}$ at time τ. The coordinate expansion in terms of a Taylor-series solution of the geodesic equation along these has its coefficients evaluated at $\mathscr{W}$, so we can think of the red-dashed geodesics as being projected onto (flat) tangent space at $\mathscr{W}$. The geodesic coordinate lines thus become the straight lines (red).

6.3.1. Historical Development and Overview of Literature on Fermi Coordinates

The notion of Fermi coordinates, or more precisely that of what we have called a *physical inertial frame*, go back to a paper with the (translated) title "On the Phenomena That Occur in the Neighbourhood of a World Line" [185] by Enrico Fermi, published in three parts in 1922, in which he also introduces what nowadays is called Fermi-Walker transport (see [186] for an English translation). In his well-known textbook [187], Synge studied these coordinates in some depth and named the corresponding normal coordinate construction after Fermi. A few years later, Manasse and Misner then popularised Fermi coordinates with their influential paper [188] of 1964. While an introductory exposition of Fermi

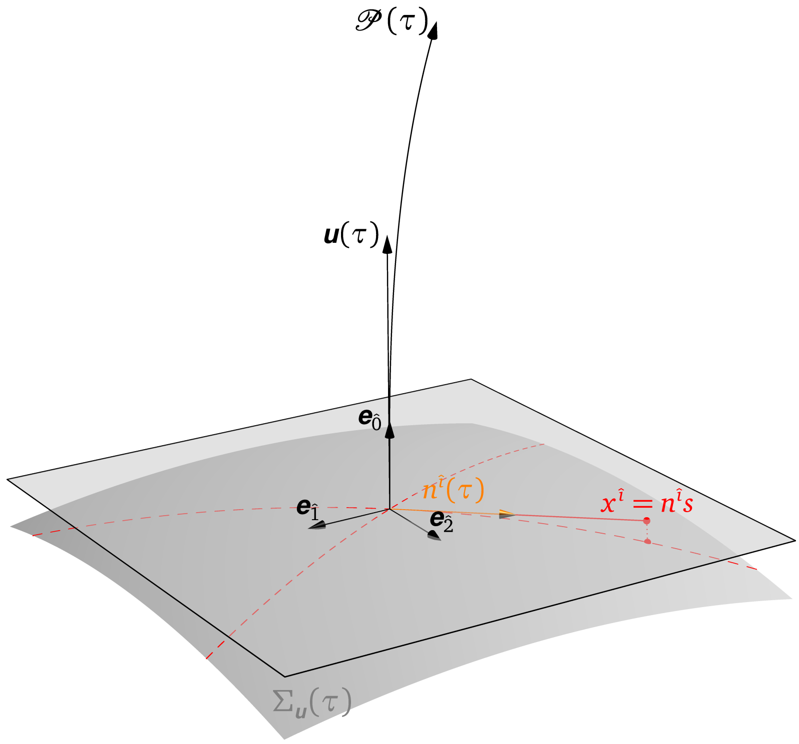

Figure 6.1.: Illustration of the geometrical construction of Fermi normal coordinates in terms of a 1+3 split of space-time along a time-like world-line that is effected by the tetrad.

coordinates appears in the famous textbook by Misner, Thorne and Wheeler [48, Sec. 13.6], together with a nice discussion of what they have termed *proper reference frame*, they do not embark on a calculation of the metric other that quoting the previous results of Manasse and Misner.

This gap was filled in the late 1970s by Ni and co-workers. Ni and Zimmermann calculated the metric (as well as the expansion of the equation of motion for general geodesics, cf. our section 6.4) to second order, with inertial (i.e., acceleration and rotation) terms included [189]. This work was extended by Li and Ni in [190], where they carry these calculations to 3$^{\text{rd}}$ order. In [191] Li and Ni then focused on Fermi coordinates around a geodesic, i.e., without the inertial terms, but carrying the expansions to 4$^{\text{th}}$ order. More recent works include [192], and also [193]. Halpern and Malin extensively discuss Riemann and Fermi normal coordinates in [172]. A more recent general reference for the topic of normal coordinates is the somewhat mathematical textbook by Iliev [171], see also the textbook by Gray [194], especially sections 2 and 9.3. We should also mention the three PhD theses by Marzlin [195], by Delva [196], and by Kajari [197] (see also [54] for the latter's main results).

On the side of concrete applications of Fermi coordinates, Parker expanded the covariant Dirac equation in Fermi coordinates to investigate the local influence of space-time curvature on the energy levels of a one-electron atom [198, 199]. This work was subsequently extended by Parker and Pimentel, who studied the perturbations to the hydrogen spectrum for a hydrogen atom freely falling on radial and circular geodesics in Schwarzschild space-time [200]. Gill, Wunner, Soffel and Ruder later considered the perturbations to energy levels and wave functions of hydrogen-like atoms with the same basic setup in [201], see also the more recent work [202]. In [203], Pinto employs the same basic approach as the previous works but considering only the non-relativistic limit, to investigate if Rydberg atoms could be used as probes for gravimetric measurements. Finally, in the context of quantum optics, Audretsch and Marzlin Atom interferometer, Ramsey fringes [204],

In an astrophysical context, Fermi coordinates were recently used by Ishii, Shibata and Mino to obtain relativistically accurate tidal potentials for the study of tidally disrupted stars in orbit around a Kerr black hole [205], and to study relativistic effects in the tidal interaction between a white dwarf and a massive Schwarzschild black hole in [206]. Let us note, that the paper by Puetzfeld, Obukhov and Lämmerzahl contains a timeline of works on Fermi-type coordinates [207, Table 1], which the interested reader may consult for further information and references.

Before we start, we should also clarify, that Fermi coordinates as such usually refer to an extension of an *inertial* frame, i.e., without the acceleration and rotation terms in the transport matrix $\Omega^{\hat{\kappa}}{}_{\hat{\alpha}}$. As mentioned above, Misner, Thorne and Wheeler [48] have coined the expression *proper reference frame coordinates* for the Fermi coordinates of an accelerating and rotating observer, which has since then been adopted by some authors. Some authors also use the term *Fermi-Walker coordinates* for the Fermi coordinates of an accelerated but non-rotating observer, i.e., Fermi coordinates based on a Fermi–Walker-transported tetrad. However, we feel that it is not very useful to distinguish the different versions by different names. Instead we shall refer to all of these just as *Fermi coordinates*, be they coordinates of an inertial, accelerating, or rotating observer, or both. In our view, all of these contain the defining key ingredients of Fermi coordinates, which are: (1) that they are based on a tetrad adapted to a world-line $\mathscr{W}$, and (2) that the spacial coordinate lines are taken to be geodesics orthogonal to the observer's four-velocity at $\mathscr{W}$ (although this last point is easily generalised).

6.3.2. Expansion of Tensors in Fermi Normal Coordinates

As in the case of Riemann coordinates in subsection 6.2.3 and subsection 6.2.4, we start by applying the "traditional approach" in terms of generalised connection coefficients to the expansion of the tetrad and its inverse in Fermi normal coordinates. Since the tetrad and its inverse act as parallel propagators, it is sufficient to treat only the expansions of these. The Taylor expansion of the inverse tetrad essentially parallels that of (6.41),

$$e^{\hat{\beta}}{}_{\kappa}(x) = e^{\hat{\beta}}{}_{\kappa}(0) + \sum_{n=1}^{\infty} \frac{1}{n!}\Big(\partial_{(i_1}\cdots\partial_{i_n)}e^{\hat{\beta}}{}_{\kappa}\Big)\big|_{\mathscr{W}} x^{i_1}x^{i_2}\cdots x^{i_n}\,, \tag{6.114}$$

and – as above – an expression for the coefficients is obtained from the parallel-transport equations for the tetrad and its inverse,

$$\partial_i e^{\hat{\alpha}}{}_{\kappa}(x) = e^{\hat{\alpha}}{}_{\delta}(x)\Gamma^{\delta}{}_{i\kappa}(x)\,, \tag{6.115a}$$

$$\partial_i e_{\hat{\alpha}}{}^{\kappa}(x) = -\Gamma^{\kappa}{}_{i\delta}(x)e_{\hat{\alpha}}{}^{\delta}(x)\,, \tag{6.115b}$$

and acting with $n-1$ partial derivatives,

$$\partial_{(i_1}\cdots\partial_{i_n)}e_{\hat{\alpha}}{}^{\kappa}(x) = \partial_{(i_1}\cdots\partial_{i_{n-1}|}\Big(\Gamma^{\kappa}{}_{\delta|i_n)}e_{\hat{\alpha}}{}^{\delta}(x)\Big)\,. \tag{6.116a}$$

$$\partial_{(i_1}\cdots\partial_{i_n)}e^{\hat{\alpha}}{}_{\kappa}(x) = \partial_{(i_1}\cdots\partial_{i_{n-1}|}\Big(e^{\hat{\alpha}}{}_{\delta}(x)\Gamma^{\delta}{}_{\kappa|i_n)}\Big)\,. \tag{6.116b}$$

where it is understood that the Fermi indices $i, i_1, \ldots, i_n$, etc. are all contracted with Fermi coordinates $x^i, x^{i_1}, \ldots, x^{i_n}$. As before, the initial condition (meaning no parallel transport) must clearly be unity, i.e.

$$e^{\hat{\alpha}}{}_{\kappa}(0) = \delta^{\hat{\alpha}}{}_{\kappa}\,, \qquad\qquad e_{\hat{\alpha}}{}^{\kappa}(0) = \delta_{\hat{\alpha}}{}^{\kappa}\,. \tag{6.117}$$

One complication with respect to the corresponding calculation in Riemann coordinates in (6.48) now is, that – at least for the general case of “non-inertial” Fermi coordinates (or “proper-reference-frame coordinates”) which are based on an accelerating and rotating tetrad – we now have to retain all the terms in the expansion of the right-hand sides of (6.116a), (6.116b), including those containing one or more Christoffel symbols, which considerably increases the size of the expressions. Applying the generalised Leibniz rule to the first-order recursions (6.116b), (6.116a) yields,

$$\partial_{(i_1}\cdots\partial_{i_n)}e^{\hat{\alpha}}{}_{\kappa}(x) = \sum_{k=0}^{n-1}\binom{n-1}{k}\Big(\partial_{(i_1}\cdots\partial_{i_k|}e^{\hat{\alpha}}{}_{\delta}(x)\Big)\Gamma^{\delta}{}_{\kappa|i_{k+1},\,i_{k+2}\cdots i_n)}\,. \tag{6.118a}$$

$$\partial_{(i_1}\cdots\partial_{i_n)}e_{\hat{\alpha}}{}^{\kappa}(x) = -\sum_{k=1}^{n}\binom{n-1}{k-1}\Gamma^{\kappa}{}_{\delta(i_1,\,i_2\cdots i_k}\partial_{i_{k+1}}\cdots\partial_{i_n)}e_{\hat{\alpha}}{}^{\delta}(x)\,. \tag{6.118b}$$

The first few lower-order iterations of the inverse tetrad, the three cases $n=2$ to $n=4$ expand to,

$$\partial_{(i_1}\partial_{i_2)}e^{\hat{\alpha}}{}_{\kappa}(x)|_{\mathscr{W}} = e^{\hat{\alpha}}{}_{\delta}(0)\Big[\Gamma^{\delta}{}_{\kappa(i_1,\,i_2)} + \Gamma^{\delta}{}_{\delta_1(i_1|}\Gamma^{\delta_1}{}_{\kappa|i_2)}\Big]\,, \tag{6.119a}$$

$$\begin{aligned}\partial_{(i_1}\partial_{i_2}\partial_{i_3)}e^{\hat{\alpha}}{}_{\kappa}(x)|_{\mathscr{W}} = e^{\hat{\alpha}}{}_{\delta}(0)\Big[&\Gamma^{\delta}{}_{\kappa(i_1,\,i_2 i_3)} + 2\Gamma^{\delta}{}_{\delta_1(i_1|}\Gamma^{\delta_1}{}_{\kappa|i_2,\,i_3)} + \Gamma^{\delta}{}_{\delta_1(i_1,\,i_2|}\Gamma^{\delta_1}{}_{\kappa|i_3)}\\ &+ \Gamma^{\delta}{}_{\delta_1(i_1|}\Gamma^{\delta_1}{}_{\delta_2|i_2|}\Gamma^{\delta_2}{}_{\kappa|i_3)}\Big]\,,\end{aligned} \tag{6.119b}$$

$$\begin{aligned}\partial_{(i_1}\cdots\partial_{i_4)}e^{\hat{\alpha}}{}_{\kappa}(x)|_{\mathscr{W}} = e^{\hat{\alpha}}{}_{\delta}(0)\Big[&\Gamma^{\delta}{}_{\kappa(i_1,\,i_2 i_3 i_4)} + 3\Gamma^{\delta}{}_{\delta_1(i_1|}\Gamma^{\delta_1}{}_{\kappa|i_2,\,i_3 i_4)} + 3\Gamma^{\delta}{}_{\delta_1(i_1,\,i_2|}\Gamma^{\delta_1}{}_{\kappa|i_3,\,i_4)}\\ &+ \Gamma^{\delta}{}_{\delta_1(i_1,\,i_2 i_3|}\Gamma^{\delta_1}{}_{\kappa|i_4)} + 3\Gamma^{\delta}{}_{\delta_1(i_1|}\Gamma^{\delta_1}{}_{\delta_2|i_2|}\Gamma^{\delta_2}{}_{\kappa|i_3,\,i_4)}\\ &+ 2\Gamma^{\delta}{}_{\delta_1(i_1|}\Gamma^{\delta_1}{}_{\delta_2|i_2,\,i_3|}\Gamma^{\delta_2}{}_{\kappa|i_4)} + \Gamma^{\delta}{}_{\delta_1(i_1,\,i_2|}\Gamma^{\delta_1}{}_{\delta_2|i_3|}\Gamma^{\delta_2}{}_{\kappa|i_4)}\\ &+ \Gamma^{\delta}{}_{\delta_1(i_1|}\Gamma^{\delta_1}{}_{\delta_2|i_2|}\Gamma^{\delta_2}{}_{\delta_3|i_3|}\Gamma^{\delta_3}{}_{\kappa|i_4)}\Big]\,,\end{aligned} \tag{6.119c}$$

and for the tetrad we obtain,

$$\partial_{(i_1}\partial_{i_2)}e_{\hat{\alpha}}{}^{\kappa}(x)|_{\mathscr{W}} = -\Big[\Gamma^{\kappa}{}_{\hat{\beta}(i_1,\,i_2)} - \Gamma^{\kappa}{}_{\delta(i_1|}\Gamma^{\delta}{}_{\hat{\beta}|i_2)}\Big]\,, \tag{6.120a}$$

$$\partial_{(i_1}\partial_{i_2}\partial_{i_3)}e_{\hat\beta}{}^{\kappa}(x)\big|_{\mathscr{W}} = -\Big[\Gamma^{\kappa}{}_{\hat\beta(i_1,\,i_2i_3)} - 2\Gamma^{\kappa}{}_{\delta(i_1,\,i_2|}\Gamma^{\delta}{}_{\hat\beta|i_3)} - \Gamma^{\kappa}{}_{\delta(i_1|}\Gamma^{\delta}{}_{\hat\beta|i_2,\,i_3)} + \Gamma^{\kappa}{}_{\delta(i_1|}\Gamma^{\delta}{}_{\delta_1|i_2|}\Gamma^{\delta_1}{}_{\hat\beta|i_3)}\Big]\,, \tag{6.120b}$$

$$\begin{aligned}\partial_{(i_1}\cdots\partial_{i_4)}e_{\hat\beta}{}^{\kappa}(x)\big|_{\mathscr{W}} = -\Big[&\Gamma^{\kappa}{}_{\hat\beta(i_1,\,i_2i_3i_4)} - 3\Gamma^{\kappa}{}_{\delta(i_1,\,i_2i_3|}\Gamma^{\delta}{}_{\hat\beta|i_4)} - 3\Gamma^{\kappa}{}_{\delta(i_1,\,i_2|}\Gamma^{\delta}{}_{\hat\beta|i_3,\,i_4)}\\ &- \Gamma^{\kappa}{}_{\delta(i_1|}\Gamma^{\delta}{}_{\hat\beta|i_2,\,i_3i_4)} + 3\Gamma^{\kappa}{}_{\delta(i_1,\,i_2|}\Gamma^{\delta}{}_{\delta_1|i_3|}\Gamma^{\delta_1}{}_{\hat\beta|i_4)}\\ &+ 2\Gamma^{\kappa}{}_{\delta(i_1|}\Gamma^{\delta}{}_{\delta_1|i_2,\,i_3|}\Gamma^{\delta_1}{}_{\hat\beta|i_4)} + \Gamma^{\kappa}{}_{\delta(i_1|}\Gamma^{\delta}{}_{\delta_1|i_2|}\Gamma^{\delta_1}{}_{\hat\beta|i_3,\,i_4)}\\ &- \Gamma^{\kappa}{}_{\delta(i_1|}\Gamma^{\delta}{}_{\delta_1|i_2|}\Gamma^{\delta_1}{}_{\delta_2|i_3|}\Gamma^{\delta_2}{}_{\hat\beta|i_4)}\Big]\,.\end{aligned} \tag{6.120c}$$

In the following, our task will be to calculate the generalised connection coefficients that appear in the expansions above.

6.3.3. Generalised Connection Coefficients in Fermi Coordinates

Since in Fermi coordinates we are dealing with a Riemann expansion that is confined to the local spacial sub-space orthogonal to the central world-line $\mathscr{W}$, spacial indices can be considered to be "Riemann normal" indices. In particular, the expansion indices i that are contracted with the tangent vectors v^i to the spacial geodesics emanating from $\mathscr{W}$ replace the α-indices that we used in the context of the pure Riemann normal coordinate expansion of section 6.2. Thus also in Fermi coordinates, for those generalised connection coefficients with all-spacial indices we have Equations (6.35a), (6.35b) and the purely spacial part of the connection, i.e. the Christoffel symbols with all-spacial indices vanish. The connection coefficients with at least one index from the time-like sub-manifold $\mathscr{W}$, however, are determined by the transport law of the tetrad along $\mathscr{W}$, i.e. by equation (4.49),

$$\nabla_i e_{\hat 0}{}^{\kappa}(x) \equiv \partial_i e_{\hat 0}{}^{\kappa}(x) + \Gamma^{\kappa}{}_{i\delta}e_{\hat 0}{}^{\delta}(x) = \widetilde\Gamma^{\hat\delta}{}_{i\hat 0}e_{\hat\delta}{}^{\kappa}(x)\,, \tag{6.121a}$$

$$\nabla_{\hat\imath} e^{\hat 0}{}_{\kappa}(x) \equiv \partial_{\hat\imath} e^{\hat 0}{}_{\kappa}(x) - \Gamma^{\delta}{}_{\hat\imath\kappa}e^{\hat 0}{}_{\delta}(x) = -\widetilde\Gamma^{\hat 0}{}_{i\hat\alpha}e^{\hat\alpha}{}_{\kappa}(x)\,. \tag{6.121b}$$

Specialising to the origin of Fermi coordinates at $\mathscr{W}$, the first equation yields

$$\Gamma^{\delta}{}_{\hat\imath\kappa}(x)e^{\hat\beta}{}_{\delta}(x)\big|_{\mathscr{W}} = \widetilde\Gamma^{\hat\beta}{}_{\hat\imath\hat 0}\,\delta^{\hat 0}{}_{\kappa} = -\tfrac{1}{c}\Omega^{\hat\beta}{}_{\hat\alpha}\,, \tag{6.122}$$

Fermi Coordinate Conditions

This leads to the following coordinate conditions for Fermi normal coordinates based on a "non-inertial" tetrad, i.e. one that undergoes proper transport along $\mathscr{W}$,

$$\Gamma^{\kappa}{}_{0\alpha}\big|_{\mathscr{W}} = \widetilde\Gamma^{\hat\kappa}{}_{\hat 0\hat\alpha}\big|_{\mathscr{W}} = -\tfrac{1}{c}\Omega^{\hat\kappa}{}_{\hat\alpha}\,, \tag{6.123a}$$

$$\Gamma^{\kappa}{}_{i_1i_2}\big|_{\mathscr{W}} = \Gamma^{\kappa}{}_{(i_1i_2)}\big|_{\mathscr{W}} = 0\,, \tag{6.123b}$$

$$\Gamma^{\kappa}{}_{(i_1i_2,\,i_3\cdots i_n)}\big|_{\mathscr{W}} = 0\,, \qquad n\geq 2\,. \tag{6.123c}$$

The coordinate condition (6.123b), (6.123c) must be preserved along $\mathscr{W}$, i.e. their time derivatives must vanish. Therefore, we also require the following *compatibility condition* to hold in Fermi coordinates,

$$\Gamma^{\kappa}{}_{(i_1 i_2,\, i_3 \cdots i_k\, 0 \cdots 0)}\big|_{\mathscr{W}} = 0\,, \qquad\qquad k \geq 2\,. \tag{6.123d}$$

For $\Omega^{\hat{\kappa}}{}_{\hat{\alpha}} = 0$, these reduce to the corresponding conditions for the usual Fermi coordinates based on a parallel-transported tetrad (see, e.g. Sec. 5 of [172]). Our task now is to calculate the generalised connection coefficients with one free lower index, since these appear in the n^{th} order derivatives of the equation of parallel transport, as explained in subsection 6.3.2 above. As a consequence of the local $(1+3)$-split at $\mathscr{W}$, there will be two kinds of these, depending on which submanifold the vector being parallel transported lies in, i.e. whether the vector is time-like or space-like.

The Spacial Sub-Space

For the spacial sub-space of Fermi coordinates, acting with n partial derivatives on the definition of the Riemann tensor just yields equation (6.52) restricted to the spacial sub-space, i.e., we can simply replace the indices there according to $\alpha \to i$ and $\beta \to b$,

$$R^{\kappa}{}_{(i_1 i_2 | b,\, | i_3 \cdots i_n)} = \Gamma^{\kappa}{}_{b (i_1,\, i_2 \cdots i_n)} - \Gamma^{\kappa}{}_{(i_1 i_2 |,\, b | i_3 \cdots i_n)} + \left[\Gamma^{\kappa}{}_{\delta (i_1 |} \Gamma^{\delta}{}_{b | i_2}\,\right]_{, | i_3 \cdots i_n)}\,; \tag{6.124}$$

from which we we obtain the Fermi-coordinate version of the non-linear recursion (6.53) for the generalised connection coefficients with a spacial lower index pulled out, evaluated at the central world-line $\mathscr{W}$,

$$\perp\!\Gamma^{\kappa}{}_{\beta (i_1,\, i_2 \cdots i_n)}\big|_{\mathscr{W}} = \frac{n-1}{n+1}\Big[R^{\kappa}{}_{(i_1 i_2 | \beta,\, | i_3 \cdots i_n)} - \big(\Gamma^{\kappa}{}_{\delta (i_1 |} \Gamma^{\delta}{}_{\beta | i_2}\,\big)_{, i_3 \cdots i_n)}\Big]\Big|_{\mathscr{W}}\,, \qquad n \geq 1\,. \tag{6.125}$$

As with Riemann normal coordinates in equations (6.55), we will calculate the first few generalised connection coefficients manually. In order to promote the appearing partial derivatives of the Riemann tensor to covariant derivatives, we start by calculating its first two covariant derivatives. In a first step, it is convenient to do that part of the calculation which is common to both the spacial and the time-like sub-space, in terms of a lower index β, before we specialise to the two different cases $\beta = b$ and $\beta = 0$. We have,

$$\begin{aligned}
R^{\kappa}{}_{(i_1 i_2 | \beta ;\, | i_3)} &= R^{\kappa}{}_{(i_1 i_2 | \beta ,\, | i_3)} + \Gamma^{\kappa}{}_{\delta (i_1 |} R^{\delta}{}_{| i_2 i_3) \beta} - R^{\kappa}{}_{(i_1 i_2 | \delta} \Gamma^{\delta}{}_{\beta | i_4)} && \text{(6.126a)}\\
&= R^{\kappa}{}_{(i_1 i_2 | \beta ,\, | i_3)} - \tfrac{1}{c}\Omega^{\kappa}{}_{(i_1 |} R^{\hat{0}}{}_{| i_2 i_3) \beta} + \tfrac{1}{c} R^{\kappa}{}_{(i_1 i_2 | \delta} \Omega^{\delta}{}_{| i_3)} \delta^{\hat{0}}{}_{\beta}\,, \\
R^{\kappa}{}_{(i_1 i_2 | \beta ;\, | i_3 i_4)} &= R^{\kappa}{}_{(i_1 i_2 | \beta ;\, | i_3 ,\, i_4)} + \Gamma^{\kappa}{}_{\delta (i_1 |} R^{\delta}{}_{| i_2 i_3 | \beta ;\, | i_4)} - R^{\kappa}{}_{(i_1 i_2 | \delta ;\, | i_3 |} \Gamma^{\delta}{}_{\beta | i_4)} \\
&= R^{\kappa}{}_{(i_1 i_2 | \beta ,\, | i_3 i_4)} + \Gamma^{\kappa}{}_{\delta (i_1 ,\, i_2 |} R^{\delta}{}_{| i_3 i_4) \beta} - R^{\kappa}{}_{(i_1 i_2 | \delta} \Gamma^{\delta}{}_{\beta | i_3 ,\, i_4)} && \text{(6.126b)}\\
&\quad - \big(\tfrac{1}{c}\big)\Omega^{\kappa}{}_{(i_1 |}\Big(R^{\hat{0}}{}_{| i_2 i_3 | \beta ;\, | i_4)} + \big(\tfrac{1}{c}\big)\Omega^{\hat{0}}{}_{| i_2 |} R^{\hat{0}}{}_{| i_3 i_4) \beta}\Big) \\
&\quad + \big(\tfrac{1}{c}\big)\Big(R^{\kappa}{}_{(i_1 i_2 | \delta ;\, | i_3 |} + \big(\tfrac{1}{c}\big)\Omega^{\kappa}{}_{(i_1 |} R^{\hat{0}}{}_{| i_2 i_3 | \delta}\Big)\Omega^{\delta}{}_{| i_4)} \delta^{\hat{0}}{}_{\beta} \\
&\quad - \tfrac{1}{c}\Omega^{\kappa}{}_{(i_1 |} R^{\hat{0}}{}_{| i_2 i_3 | \beta ;\, | i_4)} + \tfrac{1}{c} R^{\kappa}{}_{(i_1 i_2 | \delta ;\, | i_3 |} \Omega^{\delta}{}_{| i_4)} \delta^{\hat{0}}{}_{\beta}
\end{aligned}$$

where the terms in the two pairs of parentheses come from replacing first partial derivatives of the Riemann tensor with their first covariant derivatives using (6.126a). In the expansion of the second covariant derivative (6.126b), lower-order generalised connections appear, which must be specialised to either the generalised connections of the spacial sub-space, ${}_{\perp}\Gamma^{\kappa}{}_{\beta(i_1,\,i_2)}$, if (6.126b) is used in the expansion of the higher-order versions of these (i.e. here in the expansion of ${}_{\perp}\Gamma^{\kappa}{}_{\beta(i_1,\,i_2i_3i_4)}$), or conversely to the generalised connections of the time-like sub-space, ${}_{\parallel}\Gamma^{\kappa}{}_{0(i_1,\,i_2)}$.

Thus, specialising equations (6.126) first to the spacial sub-space, we find,

$$R^{\kappa}{}_{(i_1i_2|\beta;\,|i_3)} = R^{\kappa}{}_{(i_1i_2|\beta,\,|i_3)} - \tfrac{1}{c}\Omega^{\kappa}{}_{(i_1|}R^{\hat{0}}{}_{|i_2i_3)\beta} + \tfrac{1}{c}R^{\kappa}{}_{(i_1i_2|\delta}\Omega^{\delta}{}_{|i_3)}\delta^{\hat{0}}{}_{\beta}, \tag{6.127a}$$

$$\begin{aligned} R^{\kappa}{}_{(i_1i_2|\beta;\,|i_3i_4)} = R^{\kappa}{}_{(i_1i_2|\beta,\,|i_3i_4)} &+ 2\tfrac{1}{c}\Big[R^{\kappa}{}_{(i_1i_2|\delta;\,|i_3|}\Omega^{\delta}{}_{|i_4)}\delta^{\hat{0}}{}_{\beta} - \Omega^{\kappa}{}_{(i_1|}R^{\hat{0}}{}_{|i_2i_3|\beta;\,|i_4)}\Big] \\ &+ \tfrac{1}{3}\big(\tfrac{1}{c}\big)^2\Big[R^{\kappa}{}_{(i_1i_2|\delta}\Omega^{\delta}{}_{|i_3|}\Omega^{\hat{0}}{}_{|i_4)}\delta^{\hat{0}}{}_{\beta} - 4\Omega^{\kappa}{}_{(i_1|}\Omega^{\hat{0}}{}_{|i_2|}R^{\hat{0}}{}_{|i_3i_4)\beta} \\ &\qquad + \Omega^{\kappa}{}_{(i_1|}R^{\hat{0}}{}_{|i_2i_3|\delta}\Omega^{\delta}{}_{|i_4)}\delta^{\hat{0}}{}_{\beta}\Big], \end{aligned} \tag{6.127b}$$

which reduces to the corresponding expression (6.57) in Riemann coordinates as it should if inertial forces are absent, i.e. for $\Omega^{\kappa}{}_{\hat{\imath}} = 0$. The last expression simplifies in our present case of the calculation of ${}_{\perp}\Gamma^{\kappa}{}_{\beta(i_1,\,i_2i_3i_4)}$ since the index β becomes the purely spacial index $\hat{b}$ in this case, so the terms in (6.127b) with a trailing $\delta^{\hat{0}}{}_{\beta}$ vanish, yielding

$$R^{\kappa}{}_{(i_1i_2|\beta,\,|i_3i_4)} = R^{\kappa}{}_{(i_1i_2|\beta;\,|i_3i_4)} - 2\big(\tfrac{1}{c}\big)\Omega^{\kappa}{}_{(i_1|}R^{\hat{0}}{}_{|i_2i_3|\beta;\,|i_4)} - \tfrac{4}{3}\big(\tfrac{1}{c}\big)^2\Omega^{\kappa}{}_{(i_1|}\Omega^{\hat{0}}{}_{|i_2|}R^{\hat{0}}{}_{|i_3i_4)\beta}\,. \tag{6.128}$$

The first few generalised connection coefficients for the spacial sub-space can now be calculated in terms of the Riemann tensor, its covariant derivatives, and the transport matrix (6.123a). They read: $\Gamma^{\kappa}{}_{bi}\big|_{\mathscr{W}} = 0$ from (6.123b) for $n = 1$, and

$${}_{\perp}\Gamma^{\kappa}{}_{\beta(i_1,\,i_2)}\big|_{\mathscr{W}} = \tfrac{1}{3}\big[R^{\kappa}{}_{(i_1i_2)\beta} - \big(\tfrac{1}{c}\big)^2\Omega^{\kappa}{}_{(i_1|}\Omega^{\hat{0}}{}_{|i_2)}\delta^{\hat{0}}{}_{\beta}\big], \tag{6.129a}$$

$$\begin{aligned} {}_{\perp}\Gamma^{\kappa}{}_{\beta(i_1,\,i_2i_3)}\big|_{\mathscr{W}} = \frac{1}{2}\Big[&R^{\kappa}{}_{(i_1i_2|\beta;\,|i_3)} + \tfrac{4}{3}\big(\tfrac{1}{c}\big)\Omega^{\kappa}{}_{(i_1|}R^{\hat{0}}{}_{|i_2i_3)\beta} \\ &- \tfrac{2}{3}\big(\tfrac{1}{c}\big)R^{\kappa}{}_{(i_1i_2|\delta}\Omega^{\delta}{}_{|i_3)}\delta^{\hat{0}}{}_{\beta} - \tfrac{2}{3}\big(\tfrac{1}{c}\big)^3\Omega^{\kappa}{}_{(i_1|}\Omega^{\hat{0}}{}_{|i_2|}\Omega^{\hat{0}}{}_{|i_3)}\delta^{\hat{0}}{}_{\beta}\Big], \end{aligned} \tag{6.129b}$$

$$\begin{aligned} {}_{\perp}\Gamma^{\kappa}{}_{\beta(i_1,\,i_2i_3i_4)}\big|_{\mathscr{W}} = \frac{3}{5}\Big[&R^{\kappa}{}_{(i_1i_2|\beta;\,|i_3i_4)} - \tfrac{2}{9}R^{\kappa}{}_{(i_1i_2|\delta}R^{\delta}{}_{|i_3i_4)\beta} + \tfrac{5}{2}\big(\tfrac{1}{c}\big)\Omega^{\kappa}{}_{(i_1|}R^{\hat{0}}{}_{|i_2i_3|\beta;\,|i_4)} \\ &+ \tfrac{20}{9}\big(\tfrac{1}{c}\big)^2\Omega^{\kappa}{}_{(\hat{\imath}_1|}\Omega^{\hat{0}}{}_{|\hat{\imath}_2|}R^{\hat{0}}{}_{|\hat{\imath}_3\hat{\imath}_4)\beta}\Big]. \end{aligned} \tag{6.129c}$$

Note, how in contrast to the case of pure Riemann coordinates in equations (6.55), there are additional terms resulting from the non-linear coupling to the inertial forces. If we take the central world-line to be a geodesic, we have $\Omega^{\kappa}{}_{\alpha} = 0$ and equations (6.129) reduce to the result (6.55) for pure Riemann coordinates.

The Time-Like Sub-Space of the Central World-Line

Things are different for a time-like vector which does not lie in the spacial sub-space of the Riemann expansion. The first partial derivatives of the Christoffel symbols with one index zero are obtained from the definition of the Riemann tensor,

$$\Gamma^{\kappa}{}_{(i_1|\beta,\,|i_2)} = R^{\kappa}{}_{(i_1 i_2)\beta} + \Gamma^{\kappa}{}_{(i_1 i_2),\,\beta} - \Gamma^{\kappa}{}_{\delta(i_1|}\Gamma^{\delta}{}_{|i_2)\beta} + \Gamma^{\kappa}{}_{\delta\beta}\Gamma^{\delta}{}_{(i_1 i_2)}\,, \tag{6.130}$$

where the last term vanishes in view of (6.123b). Acting with $n-2$ partial i-derivatives, as usual, we obtain

$$R^{\kappa}{}_{(i_1 i_2|\beta,\,|i_3\cdots i_n)} = \Gamma^{\kappa}{}_{\beta(i_1,\,i_2\cdots i_n)} - \Gamma^{\kappa}{}_{(i_1 i_2|,\,\beta|i_3\cdots i_n)} + \left[\Gamma^{\kappa}{}_{\delta(i_1|}\Gamma^{\delta}{}_{\beta|i_2}\,\right]_{,|i_3\cdots i_n)}\,, \tag{6.131}$$

where the term $\Gamma^{\kappa}{}_{(i_1 i_2|,\,0|i_3\cdots i_n)}$ vanishes as a consequence of (6.123d). This directly leads us to the non-linear recursion for the generalised connection coefficients with one lower zero index that come from the Taylor expansion of vectors from the time-like subspace,

$$_{\parallel}\Gamma^{\kappa}{}_{0(i_1,\,i_2\cdots i_n)}\big|_{\mathscr{W}} = \Big[R^{\kappa}{}_{(i_1 i_2|0,\,|i_3\cdots i_n)} - \big(\Gamma^{\kappa}{}_{\delta(i_1|}\Gamma^{\delta}{}_{0|i_2}\,\big)_{,i_3\cdots i_n)}\Big]\Big|_{\mathscr{W}}\,, \qquad n\geq 2\,, \tag{6.132}$$

which, in contrast to (6.125), is now only valid for $n\geq 2$.

In order to turn the arising partial derivatives of the Riemann tensor into covariant derivatives, we evaluate the general expansions (6.126) in terms of the time-like generalised connection coefficients (6.132). This yields,

$$R^{\kappa}{}_{(\hat\imath_1\hat\imath_2|\beta;\,|\hat\imath_3)} = R^{\kappa}{}_{(\hat\imath_1\hat\imath_2|\beta,\,|\hat\imath_3)} - \tfrac{1}{c}\Omega^{\kappa}{}_{(\hat\imath_1|}R^{\hat 0}{}_{|\hat\imath_2\hat\imath_3)\beta} + \tfrac{1}{c}R^{\kappa}{}_{(\hat\imath_1\hat\imath_2|\delta}\Omega^{\delta}{}_{|\hat\imath_3)}\delta^{\hat 0}{}_{\beta}\,, \tag{6.133a}$$

$$\begin{aligned}R^{\kappa}{}_{(\hat\imath_1\hat\imath_2|\beta;\,|\hat\imath_3\hat\imath_4)} = R^{\kappa}{}_{(\hat\imath_1\hat\imath_2|\beta,\,|\hat\imath_3\hat\imath_4)} &+ 2\big(\tfrac{1}{c}\big)\Big[R^{\kappa}{}_{(\hat\imath_1\hat\imath_2|\delta;\,|\hat\imath_3|}\Omega^{\delta}{}_{|\hat\imath_4)}\delta^{\hat 0}{}_{\beta} - \Omega^{\kappa}{}_{(\hat\imath_1|}R^{\hat 0}{}_{|\hat\imath_2\hat\imath_3|\beta;\,|\hat\imath_4)}\Big]\\ &+ 2\big(\tfrac{1}{c}\big)^2\Big[\Omega^{\kappa}{}_{(\hat\imath_1|}R^{\hat 0}{}_{|\hat\imath_2\hat\imath_3|\hat\delta}\Omega^{\delta}{}_{|\hat\imath_4)}\delta^{\hat 0}{}_{\beta} - \Omega^{\kappa}{}_{(\hat\imath_1|}\Omega^{\hat 0}{}_{|\hat\imath_2|}R^{\hat 0}{}_{|\hat\imath_3\hat\imath_4)\beta}\Big],\end{aligned} \tag{6.133b}$$

Manual evaluation of equation (6.132) yields the following expressions for the first few instances of the "time-like" generalised connection coefficients $_{\parallel}\Gamma^{\kappa}{}_{\beta(i_1,\,i_2\cdots i_n)}$:

$$_{\parallel}\Gamma^{\kappa}{}_{\beta(i_1,\,i_2)}\big|_{\mathscr{W}} = \big[R^{\kappa}{}_{(i_1 i_2)\beta} - \Gamma^{\kappa}{}_{\delta(i_1|}\Gamma^{\delta}{}_{\beta|i_2)}\big]\big|_{\mathscr{W}} = R^{\kappa}{}_{(\hat\imath_1\hat\imath_2)\hat\beta} - \big(\tfrac{1}{c}\big)^2\Omega^{\kappa}{}_{(\hat\imath_1|}\Omega^{\hat 0}{}_{|\hat\imath_2)}\delta^{\hat 0}{}_{\beta}\,, \tag{6.134a}$$

$$\begin{aligned}_{\parallel}\Gamma^{\kappa}{}_{\beta(i_1,\,i_2 i_3)}\big|_{\mathscr{W}} &= \big[R^{\kappa}{}_{(i_1 i_2|\beta,\,|i_3)} - {}_{\parallel}\Gamma^{\kappa}{}_{\delta(i_1,\,i_2|}\Gamma^{\delta}{}_{\beta|i_3)} - \Gamma^{\kappa}{}_{\delta(i_1|}\,{}_{\parallel}\Gamma^{\delta}{}_{\beta|i_2,\,i_3)}\big]\big|_{\mathscr{W}}\\ &= R^{\kappa}{}_{(\hat\imath_1\hat\imath_2|\hat\beta;\,|\hat\imath_3)} + 2\big(\tfrac{1}{c}\big)\Omega^{\kappa}{}_{(\hat\imath_1|}R^{\hat 0}{}_{|\hat\imath_2\hat\imath_3)\hat\beta} - 2\big(\tfrac{1}{c}\big)^3\Omega^{\kappa}{}_{(\hat\imath_1|}\Omega^{\hat 0}{}_{|\hat\imath_2|}\Omega^{\hat 0}{}_{|\hat\imath_3)}\delta^{\hat 0}{}_{\beta}\,,\end{aligned} \tag{6.134b}$$

$$
\begin{aligned}
{}_{\Vert}\Gamma^{\kappa}{}_{\beta(i_1,\, i_2 i_3 i_4)}\big|_{\mathscr{W}} &= \big[R^{\kappa}{}_{(i_1 i_2|\beta,\, |i_3 i_4)} - {}_{\Vert}\Gamma^{\kappa}{}_{\delta(i_1,\, i_2 i_3|}\Gamma^{\delta}{}_{\beta|i_4)} - 2\,{}_{\Vert}\Gamma^{\kappa}{}_{\delta(i_1,\, i_2|}\,{}_{\Vert}\Gamma^{\delta}{}_{\beta|i_3,\, i_4)} \\
&\qquad\qquad - \Gamma^{\kappa}{}_{\hat{0}(i_1|}\,{}_{\Vert}\Gamma^{\hat{0}}{}_{\beta|i_2,\, i_3 i_4)}\big]\big|_{\mathscr{W}}\,, \\
&= R^{\kappa}{}_{(\hat{\imath}_1\hat{\imath}_2|\hat{\beta};\, |\hat{\imath}_3\hat{\imath}_4)} - 2R^{\kappa}{}_{(\hat{\imath}_1\hat{\imath}_2|\hat{\delta}}R^{\hat{\delta}}{}_{|\hat{\imath}_3\hat{\imath}_4)\hat{\beta}} \\
&\quad + \tfrac{1}{c}\big[3\Omega^{\kappa}{}_{(\hat{\imath}_1|}R^{\hat{0}}{}_{|\hat{\imath}_2\hat{\imath}_3|\hat{\beta};\, |\hat{\imath}_4)} - R^{\kappa}{}_{(\hat{\imath}_1\hat{\imath}_2|\hat{\delta};\, |\hat{\imath}_3|}\Omega^{\hat{\delta}}{}_{|\hat{\imath}_4)}\delta^{\hat{0}}{}_{\beta}\,\big] \\
&\quad + \big(\tfrac{1}{c}\big)^2\big[6\Omega^{\kappa}{}_{(\hat{\imath}_1|}\Omega^{\hat{0}}{}_{|\hat{\imath}_2|}R^{\hat{0}}{}_{|\hat{\imath}_3\hat{\imath}_4)\hat{0}} + 2R^{\kappa}{}_{(\hat{\imath}_1\hat{\imath}_2|\hat{\delta}}\Omega^{\hat{\delta}}{}_{|\hat{\imath}_3|}\Omega^{\hat{0}}{}_{|\hat{\imath}_4)}\delta^{\hat{0}}{}_{\beta}\,\big] \\
&\quad - 6\big(\tfrac{1}{c}\big)^4\Omega^{\kappa}{}_{(\hat{\imath}_1|}\Omega^{\hat{0}}{}_{|\hat{\imath}_2|}\Omega^{\hat{0}}{}_{|\hat{\imath}_3|}\Omega^{\hat{0}}{}_{|\hat{\imath}_4)}\delta^{\hat{0}}{}_{\beta}\,,
\end{aligned}
\tag{6.134c}
$$

6.3.4. Expansion Coefficients for Tetrads and Metric to 5th Order

Using the above expressions (6.129) and (6.134) for the generalised connections with the expansions (6.118a), (6.118b), we can finally write the coefficients of the expansion of the tetrads and metric (excluding the combinatorial $1/n!$, as before). For the spacial part we obtain,

$$\partial_i e_{\hat{a}}{}^{\kappa}(x)\big|_{\mathscr{W}} = 0\,, \tag{6.135a}$$

$$\partial_{(i_1}\partial_{i_2)} e_{\hat{a}}{}^{\kappa}(x)\big|_{\mathscr{W}} = -\tfrac{1}{3}R^{\kappa}{}_{(\hat{\imath}_1\hat{\imath}_2)\hat{a}}\,, \tag{6.135b}$$

$$\partial_{(i_1}\partial_{i_2}\partial_{i_3)} e_{\hat{a}}{}^{\kappa}(x)\big|_{\mathscr{W}} = -\tfrac{1}{2}\big[R^{\kappa}{}_{(\hat{\imath}_1\hat{\imath}_2|\hat{a};\, |\hat{\imath}_3)} + 2\big(\tfrac{1}{c}\big)\Omega^{\kappa}{}_{(\hat{\imath}_1|}R^{\hat{0}}{}_{|\hat{\imath}_2\hat{\imath}_3)\hat{a}}\,\big]\,, \tag{6.135c}$$

$$
\begin{aligned}
\partial_{(i_1}\cdots\partial_{i_4)} e_{\hat{a}}{}^{\kappa}(x)\big|_{\mathscr{W}} = -\tfrac{3}{5}\Big[&R^{\kappa}{}_{(\hat{\imath}_1\hat{\imath}_2|\hat{a};\, |\hat{\imath}_3\hat{\imath}_4)} + \tfrac{10}{3}\big(\tfrac{1}{c}\big)\Omega^{\kappa}{}_{(\hat{\imath}_1|}R^{\hat{0}}{}_{|\hat{\imath}_2\hat{\imath}_3|\hat{a};\, |\hat{\imath}_4)} \\
&- \tfrac{7}{9}R^{\kappa}{}_{(\hat{\imath}_1\hat{\imath}_2|\hat{\delta}}R^{\hat{\delta}}{}_{|\hat{\imath}_3\hat{\imath}_4)\hat{a}} + \tfrac{40}{9}\big(\tfrac{1}{c}\big)^2\Omega^{\kappa}{}_{(\hat{\imath}_1|}\Omega^{\hat{0}}{}_{|\hat{\imath}_2|}R^{\hat{0}}{}_{|\hat{\imath}_3\hat{\imath}_4)\hat{a}}\Big].
\end{aligned}
\tag{6.135d}
$$

Note how the coefficients in this Riemann normal expansion in the spacial subspace mix with inertial terms and Riemann factors possessing $\hat{0}$-indices from the orthogonal time-like submanifold, appearing through the splitting of the contraction indices. If the latter terms are absent in the expansion of the spacial tetrad vectors, $e_{\hat{b}}{}^{\kappa}(x)$, and their inverse, $e^{\hat{b}}{}_{\kappa}(x)$, below, we recover the (albeit spacial) Riemann normal terms in (6.60), and this represents a first simple check of our calculations. The second check involves the purely inertial terms of the form $\Omega^{\kappa}{}_{(\hat{\imath}_1|}\Omega^{\hat{0}}{}_{|\hat{\imath}_2|}\cdots\Omega^{\hat{0}}{}_{|\hat{\imath}_n)}$, and indirectly also the mixed inertial–curvature terms. It consists of verifying the correct special relativistic limit in which all curvature terms, i.e. Riemann factors, vanish. In calculating the expansion coefficients of the spacial tetrad vectors (6.135) above from the generalised connection coefficients (6.129) and (6.134), we find that all the purely inertial terms present in the latter two actually cancel, so that (6.135) correctly reproduce the trivial spacial tetrad (4.92) in the special relativistic limit.

For the expansion coefficients of the time-like tetrad vector, we have the following expressions,

$$\partial_i e_{\hat{0}}{}^{\kappa}(x)\big|_{\mathscr{W}} = \tfrac{1}{c}\Omega^{\kappa}{}_{\hat{\imath}}\,, \tag{6.136a}$$

$$\partial_{(i_1}\partial_{i_2)} e_{\hat{0}}{}^{\kappa}(x)\big|_{\mathscr{W}} = -\Big[R^{\kappa}{}_{(\hat{\imath}_1\hat{\imath}_2)\hat{0}} - 2\big(\tfrac{1}{c}\big)^2\Omega^{\kappa}{}_{(\hat{\imath}_1|}\Omega^{\hat{0}}{}_{|\hat{\imath}_2)}\Big]\,, \tag{6.136b}$$

$$
\begin{aligned}
\partial_{(i_1}\partial_{i_2}\partial_{i_3)} e_{\hat{0}}{}^{\kappa}(x)\big|_{\mathscr{W}} = -\,\Big[&R^{\kappa}{}_{(\hat{\imath}_1\hat{\imath}_2|\hat{0};\, |\hat{\imath}_3)} + \tfrac{1}{c}\Big(2R^{\kappa}{}_{(\hat{\imath}_1\hat{\imath}_2|\hat{\delta}}\Omega^{\hat{\delta}}{}_{|\hat{\imath}_3)} + 3\Omega^{\kappa}{}_{(\hat{\imath}_1|}R^{\hat{0}}{}_{|\hat{\imath}_2\hat{\imath}_3)\hat{0}}\Big) \\
&- 3!\big(\tfrac{1}{c}\big)^3\Omega^{\kappa}{}_{(\hat{\imath}_1|}\Omega^{\hat{0}}{}_{|\hat{\imath}_2|}\Omega^{\hat{0}}{}_{|\hat{\imath}_3)}\Big],
\end{aligned}
\tag{6.136c}
$$

$$\begin{aligned}\partial_{(i_1}\cdots\partial_{i_4)}e_{\hat{0}}{}^{\kappa}(x)\big|_{\mathscr{W}} = -\Big[&R^{\kappa}{}_{(\hat{\imath}_1\hat{\imath}_2|\hat{0};\,|\hat{\imath}_3\hat{\imath}_4)} - 5R^{\kappa}{}_{(\hat{\imath}_1\hat{\imath}_2|\hat{\delta}}R^{\hat{\delta}}{}_{|\hat{\imath}_3\hat{\imath}_4)\hat{0}}\\ &+\big(\tfrac{1}{c}\big)\Big(2R^{\kappa}{}_{(\hat{\imath}_1\hat{\imath}_2|\hat{\delta};\,|\hat{\imath}_3|}\Omega^{\hat{\delta}}{}_{|\hat{\imath}_4)} + 4\Omega^{\kappa}{}_{(\hat{\imath}_1|}R^{\hat{0}}{}_{|\hat{\imath}_2\hat{\imath}_3|\hat{0};\,|\hat{\imath}_4)}\Big)\\ &+\big(\tfrac{1}{c}\big)^2\Big(8R^{\kappa}{}_{(\hat{\imath}_1\hat{\imath}_2|\hat{\delta}}\Omega^{\hat{\delta}}{}_{|\hat{\imath}_3|}\Omega^{\hat{0}}{}_{|\hat{\imath}_4)} + 8\Omega^{\kappa}{}_{(\hat{\imath}_1|}R^{\hat{0}}{}_{|\hat{\imath}_2\hat{\imath}_3|\hat{\delta}}\Omega^{\hat{\delta}}{}_{|\hat{\imath}_4)}\\ &\qquad + 12\Omega^{\kappa}{}_{(\hat{\imath}_1|}\Omega^{\hat{0}}{}_{|\hat{\imath}_2|}R^{\hat{0}}{}_{|\hat{\imath}_3\hat{\imath}_4|\hat{0})}\Big)\\ &- 4!\big(\tfrac{1}{c}\big)^4\Omega^{\kappa}{}_{(\hat{\imath}_1|}\Omega^{\hat{0}}{}_{|\hat{\imath}_2|}\Omega^{\hat{0}}{}_{|\hat{\imath}_3|}\Omega^{\hat{0}}{}_{|\hat{\imath}_4)}\Big].\end{aligned} \tag{6.136d}$$

Here in contrast, the purely inertial terms from the generalised connections have all added up, resulting in a prefactor of $n!$ at order n. These prefactors then cancel the usual combinatoric $1/n!$ that occurs upon inserting the tensorial coefficients (6.136) into the Taylor expansion for $e_{\hat{0}}{}^{\kappa}(x)$, so that the expanded form (4.94) of the exact special relativistic expression for the time-like tetrad vector (4.93) is recovered in that limit.

For the calculation of the inverse tetrad, we perform the $(1+3)$-split in the inverse tetrad's "Fermi" index κ, which is split into $(0,k)$. Since both the time-like and the spacial co-vector of the inverse tetrad in special relativity (4.95a) and (4.95b) are exactly linear in $\Omega^{\hat{\beta}}{}_{\hat{\imath}}$ (and thus also in $x^{\hat{\imath}}$), we expect only this term at linear order in the corresponding Fermi expansion, and thus no higher-order purely inertial terms. Indeed, all higher-order, purely inertial contributions from the generalised connections cancel, and we obtain the following expansion coefficients for the time-part of the inverse tetrad's co-vector,

$$\partial_i e^{\hat{\alpha}}{}_0(x)\big|_{\mathscr{W}} = -\tfrac{1}{c}\Omega^{\hat{\alpha}}{}_{\hat{\imath}}\,, \tag{6.137a}$$

$$\partial_{(i_1}\partial_{i_2)}e^{\hat{\alpha}}{}_0(x)\big|_{\mathscr{W}} = R^{\hat{\alpha}}{}_{(\hat{\imath}_1\hat{\imath}_2)0}\,, \tag{6.137b}$$

$$\partial_{(i_1}\partial_{i_2}\partial_{i_3)}e^{\hat{\alpha}}{}_0(x)\big|_{\mathscr{W}} = R^{\hat{\alpha}}{}_{(\hat{\imath}_1\hat{\imath}_2|0;\,|\hat{\imath}_3)} - \big(\tfrac{1}{c}\big)R^{\hat{\alpha}}{}_{(\hat{\imath}_1\hat{\imath}_2|\hat{\delta}}\Omega^{\hat{\delta}}{}_{|\hat{\imath}_3)}\,, \tag{6.137c}$$

$$\partial_{(i_1}\cdots\partial_{i_4)}e^{\hat{\alpha}}{}_0(x)\big|_{\mathscr{W}} = R^{\hat{\alpha}}{}_{(\hat{\imath}_1\hat{\imath}_2|0;\,|\hat{\imath}_3\hat{\imath}_4)} + R^{\hat{\alpha}}{}_{(\hat{\imath}_1\hat{\imath}_2|\hat{\delta}}R^{\hat{\delta}}{}_{|\hat{\imath}_3\hat{\imath}_4)0} - 2\big(\tfrac{1}{c}\big)R^{\hat{\alpha}}{}_{(\hat{\imath}_1\hat{\imath}_2|\hat{\delta};\,|\hat{\imath}_3|}\Omega^{\hat{\delta}}{}_{|\hat{\imath}_4)}, \tag{6.137d}$$

whereas the coefficients for the spacial co-vector of the inverse tetrad are found to be those of Riemann normal coordinates with the free space-time indices spacial,

$$\partial_i e^{\hat{\alpha}}{}_k(x)\big|_{\mathscr{W}} = 0\,, \tag{6.138a}$$

$$\partial_{(i_1}\partial_{i_2)}e^{\hat{\alpha}}{}_k(x)\big|_{\mathscr{W}} = \tfrac{1}{3}R^{\hat{\alpha}}{}_{(\hat{\imath}_1\hat{\imath}_2)k}\,, \tag{6.138b}$$

$$\partial_{(i_1}\partial_{i_2}\partial_{i_3)}e^{\hat{\alpha}}{}_k(x)\big|_{\mathscr{W}} = \tfrac{1}{2}R^{\hat{\alpha}}{}_{(\hat{\imath}_1\hat{\imath}_2|k;\,|\hat{\imath}_3)}, \tag{6.138c}$$

$$\partial_{(i_1}\cdots\partial_{i_4)}e^{\hat{\alpha}}{}_k(x)\big|_{\mathscr{W}} = \tfrac{3}{5}\Big[R^{\hat{\alpha}}{}_{(i_1i_2|k;\,|i_3i_4)} + \tfrac{1}{3}R^{\hat{\alpha}}{}_{(i_1i_2|\hat{\delta}}R^{\hat{\delta}}{}_{|i_3i_4)k}\Big]. \tag{6.138d}$$

These result from a lengthy but straightforward calculation, especially for the two highest orders $n=3$ and $n=4$ of the time-like vectors of the tetrad and its inverse.

6.3.5. Expansions of Tetrads and Metric in Fermi Coordinates to 5th Order

Here we exhibit the results of our manual calculation in the previous subsections of the expansion of the tetrad and inverse, as well as of the metric and its inverse in terms of the above-calculated coefficients. To the best of our knowledge, the fourth-order terms in the below expression for the tetrad and metric are already new result. Thus, the expansions of the tetrads to 5^{th} order in Fermi coordinates is then assembled to read,

$$\begin{aligned}
e_{\hat{0}}{}^{\kappa}(x) = \delta_{\hat{0}}{}^{\kappa} &+ \tfrac{1}{c}\Omega^{\kappa}{}_{\hat{\imath}}x^{\hat{\imath}} - \tfrac{1}{2}\big[R^{\kappa}{}_{\hat{\imath}_1\hat{\imath}_2\hat{0}} - 2\big(\tfrac{1}{c}\big)^2\Omega^{\kappa}{}_{(\hat{\imath}_1|}\Omega^{\hat{0}}{}_{|\hat{\imath}_2)}\big]x^{\hat{\imath}_1}x^{\hat{\imath}_2} \\
&- \frac{1}{3!}\Big[R^{\kappa}{}_{(\hat{\imath}_1\hat{\imath}_2|\hat{0};|\hat{\imath}_3)} + \tfrac{1}{c}\Big(2R^{\kappa}{}_{(\hat{\imath}_1\hat{\imath}_2|\hat{\delta}}\Omega^{\hat{\delta}}{}_{|\hat{\imath}_3)} + 3\Omega^{\kappa}{}_{(\hat{\imath}_1|}R^{\hat{0}}{}_{|\hat{\imath}_2\hat{\imath}_3)\hat{0}}\Big) \\
&\qquad - 6\big(\tfrac{1}{c}\big)^3\Omega^{\kappa}{}_{(\hat{\imath}_1|}\Omega^{\hat{0}}{}_{|\hat{\imath}_2|}\Omega^{\hat{0}}{}_{|\hat{\imath}_3)}\Big]x^{\hat{\imath}_1}x^{\hat{\imath}_2}x^{\hat{\imath}_3} \\
&- \frac{1}{4!}\Big[R^{\kappa}{}_{(\hat{\imath}_1\hat{\imath}_2|\hat{0};|\hat{\imath}_3\hat{\imath}_4)} - 5R^{\kappa}{}_{(\hat{\imath}_1\hat{\imath}_2|\hat{\delta}}R^{\hat{\delta}}{}_{|\hat{\imath}_3\hat{\imath}_4)\hat{0}} \\
&\qquad + 2\big(\tfrac{1}{c}\big)R^{\kappa}{}_{(\hat{\imath}_1\hat{\imath}_2|\hat{\delta};|\hat{\imath}_3|}\Omega^{\hat{\delta}}{}_{|\hat{\imath}_4)} + 4\big(\tfrac{1}{c}\big)\Omega^{\kappa}{}_{(\hat{\imath}_1|}R^{\hat{0}}{}_{|\hat{\imath}_2\hat{\imath}_3|\hat{0};|\hat{\imath}_4)} \\
&\qquad + \big(\tfrac{1}{c}\big)^2\Big(8R^{\kappa}{}_{(\hat{\imath}_1\hat{\imath}_2|\hat{\delta}}\Omega^{\hat{\delta}}{}_{|\hat{\imath}_3|}\Omega^{\hat{0}}{}_{|\hat{\imath}_4)} + 8\Omega^{\kappa}{}_{(\hat{\imath}_1|}R^{\hat{0}}{}_{|\hat{\imath}_2\hat{\imath}_3|\hat{\delta}}\Omega^{\hat{\delta}}{}_{|\hat{\imath}_4)} \\
&\qquad + 12\Omega^{\kappa}{}_{(\hat{\imath}_1|}\Omega^{\hat{0}}{}_{|\hat{\imath}_2)}R^{\hat{0}}{}_{|\hat{\imath}_3\hat{\imath}_4|\hat{0})}\Big) \\
&\qquad - 4!\big(\tfrac{1}{c}\big)^4\Omega^{\kappa}{}_{(\hat{\imath}_1|}\Omega^{\hat{0}}{}_{|\hat{\imath}_2|}\Omega^{\hat{0}}{}_{|\hat{\imath}_3|}\Omega^{\hat{0}}{}_{|\hat{\imath}_4)}\Big]x^{\hat{\imath}_1}x^{\hat{\imath}_2}x^{\hat{\imath}_3}x^{\hat{\imath}_4} \\
&+ \mathscr{O}(x^5)\,,
\end{aligned} \tag{6.139a}$$

for the time-like vector, and the somewhat shorter expansion of the space-like vector reads,

$$\begin{aligned}
e_{\hat{a}}{}^{\kappa}(x) = \delta_{\hat{a}}{}^{\kappa} &- \tfrac{1}{6}R^{\kappa}{}_{\hat{\imath}_1\hat{\imath}_2\hat{a}}\,x^{\hat{\imath}_1}x^{\hat{\imath}_2} - \tfrac{1}{12}\big[R^{\kappa}{}_{(\hat{\imath}_1\hat{\imath}_2|\hat{a};|\hat{\imath}_3)} + 2\big(\tfrac{1}{c}\big)\Omega^{\kappa}{}_{(\hat{\imath}_1|}R^{\hat{0}}{}_{|\hat{\imath}_2\hat{\imath}_3)\hat{a}}\big]\,x^{\hat{\imath}_1}x^{\hat{\imath}_2}x^{\hat{\imath}_3} \\
&- \frac{1}{40}\Big[R^{\kappa}{}_{(\hat{\imath}_1\hat{\imath}_2|\hat{a};|\hat{\imath}_3\hat{\imath}_4)} - \tfrac{10}{3}\big(\tfrac{1}{c}\big)\Omega^{\kappa}{}_{(\hat{\imath}_1|}R^{\hat{0}}{}_{|\hat{\imath}_2\hat{\imath}_3|\hat{a};|\hat{\imath}_4)} - \tfrac{7}{9}R^{\kappa}{}_{(\hat{\imath}_1\hat{\imath}_2|\hat{\delta}}R^{\hat{\delta}}{}_{|\hat{\imath}_3\hat{\imath}_4)\hat{a}} \\
&\qquad + \tfrac{40}{9}\big(\tfrac{1}{c}\big)^2\Omega^{\kappa}{}_{(\hat{\imath}_1|}\Omega^{\hat{0}}{}_{|\hat{\imath}_2|}R^{\hat{0}}{}_{|\hat{\imath}_3\hat{\imath}_4)\hat{a}}\Big]\,x^{\hat{\imath}_1}x^{\hat{\imath}_2}x^{\hat{\imath}_3}x^{\hat{\imath}_4} \\
&+ \mathscr{O}(x^5)\,.
\end{aligned} \tag{6.139b}$$

For the inverse tetrad we have

$$\begin{aligned}
e^{\hat{\alpha}}{}_{0}(x) = \delta^{\hat{\alpha}}{}_{0} &- \tfrac{1}{c}\Omega^{\hat{\alpha}}{}_{\hat{\imath}}x^{\hat{\imath}} + \tfrac{1}{2!}R^{\hat{\alpha}}{}_{\hat{\imath}_1\hat{\imath}_2 0}x^{\hat{\imath}_1}x^{\hat{\imath}_2} + \frac{1}{3!}\Big[R^{\hat{\alpha}}{}_{(\hat{\imath}_1\hat{\imath}_2|0;|\hat{\imath}_3)} - \tfrac{1}{c}R^{\hat{\alpha}}{}_{(\hat{\imath}_1\hat{\imath}_2|\hat{\delta}}\Omega^{\hat{\delta}}{}_{|\hat{\imath}_3)}\Big]x^{\hat{\imath}_1}x^{\hat{\imath}_2}x^{\hat{\imath}_3} \\
&+ \frac{1}{4!}\Big[R^{\hat{\alpha}}{}_{(\hat{\imath}_1\hat{\imath}_2|0;|\hat{\imath}_3\hat{\imath}_4)} + R^{\hat{\alpha}}{}_{(\hat{\imath}_1\hat{\imath}_2|\hat{\delta}}R^{\hat{\delta}}{}_{|\hat{\imath}_3\hat{\imath}_4)0} - 2\tfrac{1}{c}R^{\hat{\alpha}}{}_{(\hat{\imath}_1\hat{\imath}_2|\hat{\delta};|\hat{\imath}_3|}\Omega^{\hat{\delta}}{}_{|\hat{\imath}_4)}\Big]x^{\hat{\imath}_1}x^{\hat{\imath}_2}x^{\hat{\imath}_3}x^{\hat{\imath}_4} \\
&+ \mathscr{O}(x^5)\,,
\end{aligned} \tag{6.140a}$$

$$\begin{aligned}
e^{\hat{\alpha}}{}_{k}(x) = \delta^{\hat{\beta}}{}_{\kappa} &+ \frac{1}{6}R^{\hat{\alpha}}{}_{\hat{\imath}_1\hat{\imath}_2 k}x^{\hat{\imath}_1}x^{\hat{\imath}_2} + \frac{1}{12}R^{\hat{\alpha}}{}_{(\hat{\imath}_1\hat{\imath}_2|k;|\hat{\imath}_3)}x^{\hat{\imath}_1}x^{\hat{\imath}_2}x^{\hat{\imath}_3} \\
&+ \frac{1}{40}\Big[R^{\hat{\alpha}}{}_{(\hat{\imath}_1\hat{\imath}_2|k;|\hat{\imath}_3\hat{\imath}_4)} + \tfrac{1}{3}R^{\hat{\alpha}}{}_{(\hat{\imath}_1\hat{\imath}_2|\hat{\delta}}R^{\hat{\delta}}{}_{|\hat{\imath}_3\hat{\imath}_4)k}\Big]x^{\hat{\imath}_1}x^{\hat{\imath}_2}x^{\hat{\imath}_3}x^{\hat{\imath}_4} \\
&+ \mathscr{O}(x^5)\,,
\end{aligned} \tag{6.140b}$$

The expansion of the metric is then calculated from the orthonormality condition of the inverse tetrad, i.e.,

$$g_{00}(x) = e^{\hat{\alpha}}{}_{0}(x)e_{\hat{\alpha}0}(x)\,, \quad g_{0k}(x) = e^{\hat{\alpha}}{}_{0}(x)e_{\hat{\alpha}k}(x)\,, \quad g_{k_1k_2}(x) = e^{\hat{\alpha}}{}_{k_1}(x)e_{\hat{\alpha}k_2}(x)\,. \tag{6.141}$$

For the time–time part of the metric, we then obtain,

$$
\begin{aligned}
g_{00}(x) = \eta_{\hat{0}\hat{0}} &- 2(\tfrac{1}{c})\Omega_{\hat{0}\hat{\imath}}x^{\hat{\imath}} + \Big[R_{\hat{0}\hat{\imath}_1\hat{\imath}_2\hat{0}} + (\tfrac{1}{c})^2\Omega^{\hat{\delta}}{}_{(\hat{\imath}_1|}\Omega_{\hat{\delta}|\hat{\imath}_2)}\Big]x^{\hat{\imath}_1}x^{\hat{\imath}_2} \\
&+ \frac{1}{3}\Big[R_{\hat{0}(\hat{\imath}_1\hat{\imath}_2|\hat{0};|\hat{\imath}_3)} - 4(\tfrac{1}{c})\Omega_{\hat{0}(\hat{\imath}_1|}R^{\hat{0}}{}_{|\hat{\imath}_2\hat{\imath}_3)\hat{0}}\Big]x^{\hat{\imath}_1}x^{\hat{\imath}_2}x^{\hat{\imath}_3} \\
&+ \frac{1}{12}\Big[R_{\hat{0}(\hat{\imath}_1\hat{\imath}_2|\hat{0};|\hat{\imath}_3\hat{\imath}_4)} + 4R_{\hat{0}(\hat{\imath}_1\hat{\imath}_2|\hat{\delta}}R^{\hat{\delta}}{}_{|\hat{\imath}_3\hat{\imath}_4)\hat{0}} - 2(\tfrac{1}{c})R_{\hat{0}(\hat{\imath}_1\hat{\imath}_2|\hat{\delta};|\hat{\imath}_3|}\Omega^{\hat{\delta}}{}_{|\hat{\imath}_4)} \\
&\qquad - 4(\tfrac{1}{c})\Omega_{\hat{\delta}(\hat{\imath}_1|}R^{\hat{\delta}}{}_{|\hat{\imath}_2\hat{\imath}_3|\hat{0};|\hat{\imath}_4)} + 4(\tfrac{1}{c})^2\Omega_{\hat{\delta}_1(\hat{\imath}_1|}R^{\hat{\delta}_1}{}_{|\hat{\imath}_2\hat{\imath}_3|\hat{\delta}_2}\Omega^{\hat{\delta}_2}{}_{|\hat{\imath}_4)}\Big]x^{\hat{\imath}_1}x^{\hat{\imath}_2}x^{\hat{\imath}_3}x^{\hat{\imath}_4} \\
&+ \mathscr{O}(x^5)\,,
\end{aligned}
\tag{6.142a}
$$

where the outermost numerical coefficient at each order is $2\frac{1}{n!}$. For the expansion of the mixed part of the metric, we obtain

$$
\begin{aligned}
g_{0k}(x) = -\tfrac{1}{c}\Omega_{\hat{k}\hat{\imath}}x^{\hat{\imath}} &+ \tfrac{2}{3}R_{\hat{0}(\hat{\imath}_1\hat{\imath}_2)\hat{k}}x^{\hat{\imath}_1}x^{\hat{\imath}_2} \\
&+ \frac{1}{4}\Big[R_{\hat{0}(\hat{\imath}_1\hat{\imath}_2|\hat{k};|\hat{\imath}_3)} - \tfrac{4}{3}(\tfrac{1}{c})\Omega^{\hat{\delta}}{}_{(\hat{\imath}_1|}R_{\hat{\delta}|\hat{\imath}_2\hat{\imath}_3)\hat{k}}\Big]x^{\hat{\imath}_1}x^{\hat{\imath}_2}x^{\hat{\imath}_3} \\
&+ \frac{1}{15}\Big[R_{\hat{0}(\hat{\imath}_1\hat{\imath}_2|\hat{k};|\hat{\imath}_3\hat{\imath}_4)} + 7R_{\hat{0}(\hat{\imath}_1\hat{\imath}_2|\hat{\delta}}R^{\hat{\delta}}{}_{|\hat{\imath}_3\hat{\imath}_4)\hat{k}} - \tfrac{5}{2}\Omega^{\hat{\delta}}{}_{(\hat{\imath}_1|}R_{\hat{\delta}|\hat{\imath}_2\hat{\imath}_3|\hat{k};|\hat{\imath}_4)}\Big]x^{\hat{\imath}_1}x^{\hat{\imath}_2}x^{\hat{\imath}_3}x^{\hat{\imath}_4} \\
&+ \mathscr{O}(x^5)\,.
\end{aligned}
\tag{6.142b}
$$

Here, the outermost coefficient is $\frac{1}{n!}\frac{2n}{n+1}$. Finally, the expansion of the spacial part of the metric is again just that of Riemann normal coordinates, i.e.,

$$
\begin{aligned}
g_{k_1k_2}(x) = \eta_{\hat{a}\hat{b}} &+ \tfrac{1}{3}R_{\hat{k}_1(\hat{\imath}_1\hat{\imath}_2)\hat{k}_2}x^{\hat{\imath}_1}x^{\hat{\imath}_2} + \tfrac{1}{6}R_{\hat{k}_1(\hat{\imath}_1\hat{\imath}_2|\hat{k}_2|;|\hat{\imath}_3)}x^{\hat{\imath}_1}x^{\hat{\imath}_2}x^{\hat{\imath}_3} \\
&+ \frac{1}{20}\Big[R_{\hat{k}_1(\hat{\imath}_1\hat{\imath}_2|\hat{k}_2;|\hat{\imath}_3\hat{\imath}_4)} + \tfrac{8}{9}R_{\hat{k}_1(\hat{\imath}_1\hat{\imath}_2|\hat{\delta}}R^{\hat{\delta}}{}_{|\hat{\imath}_3\hat{\imath}_4)\hat{k}_2}\Big]x^{\hat{\imath}_1}x^{\hat{\imath}_2}x^{\hat{\imath}_3}x^{\hat{\imath}_4} \\
&+ \mathscr{O}(x^5)\,,
\end{aligned}
\tag{6.142c}
$$

where, correspondingly, the outermost numerical coefficient at each order is $2\frac{1}{n!}\frac{n-1}{n+1}$. As with the expansion of the spacial co-vector of the tetrad above, the spacial part of the Fermi metric can be checked against the metric in Riemann normal coordinates (6.63). Note that all terms in the above expansion of the metric are already symmetrical in $\hat{k}_1$ and $\hat{k}_1$, and we have used the corresponding freedom in order to arrange the Riemann and inertial factors in a suggestive manner. The expansion of the inverse metric is then calculated from the orthonormality condition of the inverse tetrad, i.e.,

$$
g^{00}(x) = e_{\hat{0}}{}^{0}(x)e^{\hat{0}0}(x) + e_{\hat{a}}{}^{0}(x)e^{\hat{a}0}(x)\,, \quad g^{0k}(x) = e_{\hat{0}}{}^{0}(x)e^{\hat{0}k}(x) + e_{\hat{a}}{}^{0}(x)e^{\hat{a}k}(x)\,, \tag{6.143}
$$

$$
g^{k_1k_2}(x) = e_{\hat{0}}{}^{k_1}(x)e^{\hat{0}k_2}(x) + e_{\hat{a}}{}^{k_1}(x)e^{\hat{a}k_2}(x)\,. \tag{6.144}
$$

At this point, we shall only provide the expression for the time–time part of the inverse metric, which reads,

$$\begin{aligned}
g^{00}(x) = \eta^{\hat{0}\hat{0}} &+ 2(\tfrac{1}{c})\Omega^{\hat{0}}{}_{\hat{\imath}}x^{\hat{\imath}} - \Big[R^{\hat{0}}{}_{(\hat{\imath}_1\hat{\imath}_2)}{}^{\hat{0}} - 3(\tfrac{1}{c})^2\Omega^{\hat{0}}{}_{(\hat{\imath}_1|}\Omega^{\hat{0}}{}_{|\hat{\imath}_2)}\Big]x^{\hat{\imath}_1}x^{\hat{\imath}_2} \\
&- \frac{1}{3}\Big[R^{\hat{0}}{}_{(\hat{\imath}_1\hat{\imath}_2|}{}^{\hat{0}}{}_{;|\hat{\imath}_3)} + 2(\tfrac{1}{c})R^{\hat{0}}{}_{(\hat{\imath}_1\hat{\imath}_2|\hat{\delta}}\Omega^{\hat{\delta}}{}_{|\hat{\imath}_4)} + \tfrac{9}{2}(\tfrac{1}{c})\Omega^{\hat{0}}{}_{(\hat{\imath}_1|}R^{\hat{0}}{}_{|\hat{\imath}_2\hat{\imath}_3)}{}^{\hat{0}} \\
&\qquad\qquad - 9(\tfrac{1}{c})^3\Omega^{\hat{0}}{}_{(\hat{\imath}_1|}\Omega^{\hat{0}}{}_{|\hat{\imath}_2|}\Omega^{\hat{0}}{}_{|\hat{\imath}_3)}\Big]x^{\hat{\imath}_1}x^{\hat{\imath}_2}x^{\hat{\imath}_3} \\
&+ \frac{1}{12}\Big[R^{\hat{0}}{}_{(\hat{\imath}_1\hat{\imath}_2|}{}^{\hat{0}}{}_{;|\hat{\imath}_3\hat{\imath}_4)} + 4R^{\hat{0}}{}_{(\hat{\imath}_1\hat{\imath}_2|\hat{\delta}}R^{\hat{\delta}}{}_{|\hat{\imath}_3\hat{\imath}_4)}{}^{\hat{0}} - 2(\tfrac{1}{c})R^{\hat{0}}{}_{(\hat{\imath}_1\hat{\imath}_2|\hat{\delta};|\hat{\imath}_3|}\Omega^{\hat{\delta}}{}_{|\hat{\imath}_4)} \\
&\qquad - 4(\tfrac{1}{c})\Omega_{\hat{\delta}(\hat{\imath}_1|}R^{\hat{\delta}}{}_{|\hat{\imath}_2\hat{\imath}_3|}{}^{\hat{0}}{}_{;|\hat{\imath}_4)} + 4(\tfrac{1}{c})^2\Omega_{\hat{\delta}_1(\hat{\imath}_1|}R^{\hat{\delta}_1}{}_{|\hat{\imath}_2\hat{\imath}_3|\hat{\delta}_2}\Omega^{\hat{\delta}_2}{}_{|\hat{\imath}_4)}\Big]x^{\hat{\imath}_1}x^{\hat{\imath}_2}x^{\hat{\imath}_3}x^{\hat{\imath}_4} \\
&+ \mathscr{O}(x^5)\,,
\end{aligned} \tag{6.145a}$$

Our results for the metric agree with the third-order calculation of Li and Ni [190] including the inertial terms, and the fourth-order calculation without inertial terms in their second paper [191].

6.3.6. Covariant Recursions and their Closed-Form Solution in Fermi Coordinates

To begin with, we shall take a closer look at the second-order partial derivatives of the inverse tetrad's time-like co-vector in (6.137). In terms of the generalised connections and Christoffel symbols this reads explicitly,

$$\begin{aligned}
\partial_{(i_1}\partial_{i_2)}e^{\hat{\alpha}}{}_0(x)\big|_{\mathscr{W}} &= e^{\hat{\alpha}}{}_\delta(x)\Big[{}_{\parallel}\Gamma^\delta{}_{0(i_1,i_2)} + \Gamma^\delta{}_{\delta_1(i_1|}\Gamma^{\delta_1}{}_{0|i_2)}\Big]\Big|_{\mathscr{W}} \\
&= e^{\hat{\alpha}}{}_\delta(x)\Big[\Big(R^\delta{}_{(i_1i_2)0} - \Gamma^\delta{}_{\delta_1(i_1|}\Gamma^{\delta_1}{}_{0|i_2)}\Big) + \Gamma^\delta{}_{\delta_1(i_1|}\Gamma^{\delta_1}{}_{0|i_2)}\Big]\Big|_{\mathscr{W}} \\
&= e^{\hat{\alpha}}{}_\delta(x)R^\delta{}_{(i_1i_2)0}\big|_{\mathscr{W}}\,,
\end{aligned} \tag{6.146}$$

so all first-order, i.e. Christoffel terms cancel. This means that at least for the coefficients of the time-part of the inverse tetrad, we have a relation with a manifestly covariant right-hand side, which yields a second-order recursion for all its higher-order expansion coefficients in terms of *partial* derivatives of the Riemann tensor,

$$\boxed{\partial_{(i_1}\cdots\partial_{i_n)}e^{\hat{\alpha}}{}_0(x)\big|_{\mathscr{W}} = \partial_{(i_1}\cdots\partial_{i_{n-2}|}\Big(e^{\hat{\alpha}}{}_\delta(x)R^\delta{}_{|i_{n-1}i_n)0}\Big)\Big|_{\mathscr{W}}\,.} \tag{6.147}$$

In contrast, for the corresponding second partial derivative of the time-like leg of the tetrad itself, the two Christoffel terms from the second-order generalised connection and from the first iteration add up instead, due to the minus sign that occurs for every odd iteration, i.e.

$$\partial_{(i_1}\partial_{i_2)}e_{\hat{0}}{}^\kappa(x)\big|_{\mathscr{W}} = -\Big[R^\kappa{}_{(i_1i_2)\delta} - 2\Gamma^\kappa{}_{\delta_1(i_1|}\Gamma^{\delta_1}{}_{\delta|i_2)}\Big]e_{\hat{0}}{}^\delta(x)\big|_{\mathscr{W}}\,. \tag{6.148}$$

Thus, as with Riemann coordinates, we do not have a *covariant* recursion for the tetrad itself, since the non-linear, non-tensorial term $\Gamma^\kappa{}_{\delta_1(i_1|}\Gamma^{\delta_1}{}_{\delta|i_2)}$ makes a closed-form solution difficult. At least for the second-order recursion (6.147) above, we can now proceed in the same fashion as with the corresponding recursion (6.64) for Riemann coordinates in subsection 6.2.6. As with Riemann coordinates, in order to display the following equations and solutions in a more concise manner, we define the k^{th} order *modified* Riemann matrix,

$$
\begin{aligned}
\tilde{\mathsf{R}}_k &:= R^{\hat{\delta}_1}{}_{(\hat{\imath}_1\hat{\imath}_2|\,\delta_2,\,|\hat{\imath}_3\cdots\hat{\imath}_k)} x^{\hat{\imath}_1}\cdots x^{\hat{\imath}_k}\,, && (\textit{partial}\text{ Riemann derivatives}), && (6.149\text{a})\\
\mathsf{R}_k &:= R^{\hat{\delta}_1}{}_{(\hat{\imath}_1\hat{\imath}_2|\,\delta_2;\,|\hat{\imath}_3\cdots\hat{\imath}_k)} x^{\hat{\imath}_1}\cdots x^{\hat{\imath}_k}\,, && (\textit{covariant}\text{ Riemann derivatives}), && (6.149\text{b})\\
\Omega &:= \tfrac{1}{c}\Omega^{\hat{\delta}}{}_{\hat{\imath}} x^{\hat{\imath}}\,, && && (6.149\text{c})
\end{aligned}
$$

in terms of the $(k-2)$nd *partial* derivative of the Riemann tensor contracted with the k Fermi coordinates. Writing (6.147) for the inverse tetrad in terms of the Leibnitz rule again,

$$
\partial_{(i_1}\cdots\partial_{i_n)} e^{\hat{\alpha}}{}_0(x)\big|_{\mathscr{W}} = \sum_{k=0}^{n-2}\binom{n-2}{k}\Big(\partial_{(i_1}\cdots\partial_{i_k|} e^{\hat{\alpha}}{}_{\delta}(x)\Big) R^{\delta}{}_{|i_{k+1}i_{k+2}|0\,,|i_{k+3}\cdots i_n)}\big|_{\mathscr{W}}\,, \quad (6.150)
$$

it can be iterated in the same fashion as before. As before for the Riemann-coordinate recursion, this recursion in terms of *partial* derivatives of Riemann tensors produces strings of Riemann factors which grow to the left upon iteration. As far as *inertial* Fermi coordinates are concerned, the closed-form solution of (6.147), (6.150) reads,

$$
\partial_{(i_1}\cdots\partial_{i_n)} e^{\hat{\alpha}}{}_0(x)\big|_{\mathscr{W}} = \sum_{l=1}^{n}\sum_{\substack{k_1,\ldots,k_l\geq 2\\ k_1+\cdots+k_l=n}} \mathcal{C}_{\parallel}(k_1,\ldots,k_l)\big(\tilde{\mathsf{R}}_{k_l}\tilde{\mathsf{R}}_{k_{l-1}}\cdots\tilde{\mathsf{R}}_{k_1}\big)^{\hat{\alpha}}{}_0\,, \quad (6.151)
$$

with the only difference to the Riemann-coordinate version being that the numerical factor now consists only of the product of binomial coefficients, i.e. it is given by the "parallel" coefficient $\mathcal{C}_{\parallel}(k_1,\ldots,k_l)$ in (6.72), so the Riemann-coordinate-specific part coming from the factor $\frac{n-1}{n+1}$ at each iteration is absent here.

We now turn to to the more general case of the solution for *non-inertial* Fermi coordinates, i.e. the solution of (6.150) in the presence of accelerations and rotations of the fiducial Fermi observer's tetrad, which means that the first derivative of the inverse tetrad is non-zero and given by the transport matrix $-\Omega$. To this end it is important to realise, that the recursions (6.64), (6.82), (6.150) are all "second-order" in the sense, that they essentially result from acting with partial derivatives on the geodesic deviation equation (5.108). This means that upon iteration, the solution is built up with only "second-order" and higher terms, i.e. with only Riemann factors R_k, at least for all but the last iteration. However, it can happen that in the last iteration, there is one "dangling" first derivative of the inverse tetrad left, which in the case of Riemann coordinates was not an issue, simply because there the Christoffel symbols and thus the first partial derivative of the inverse tetrad vanish, and because of this, all the sums could be taken to start at their respective index $k_i = 2$. In the present case, however, these single first derivatives become single first-order terms of $-\Omega$ at the very left (or right for the manifestly covariant recursion) of some of the strings of Riemann factors, depending on the specific integer composition that is responsible for the string. For example, for $n = 10$, we have

$$\begin{aligned} \mathsf{R}_3\,\mathsf{R}_3\,\mathsf{R}_4 \qquad & \text{composition } 4+3+3=10, \\ -\Omega\,\mathsf{R}_3\,\mathsf{R}_2\,\mathsf{R}_4 \qquad & \text{composition } 4+2+3+1=10. \end{aligned}$$

This generally occurs whenever $k_l = 1$ in the corresponding composition $k_1+k_2+\cdots+k_l = n$. In these cases we can trivially rewrite the l-part integer composition of n as the $(l-1)$-part integer composition $k_1 + k_2 + \cdots + k_{l-1} = n-1$ of $n-1$. We thus expect all the Riemann strings of the $(n-1)^{\text{st}}$ order to appear with a $-\Omega$ appended (prepended) also at n^{th} order.

In this vain, the first few terms in the expansion of the time-part of the inverse tetrad in terms of partial derivatives of the Riemann tensor is calculated to be,

$$\begin{aligned} e^{\hat{\alpha}}{}_0(x) = \big[1 - \Omega + \tfrac{1}{2!}\tilde{\mathsf{R}}_2 + \tfrac{1}{3!}(\tilde{\mathsf{R}}_3 - \Omega\tilde{\mathsf{R}}_2) + \tfrac{1}{4!}(\tilde{\mathsf{R}}_4 - 2\Omega\tilde{\mathsf{R}}_3 + \tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2) \\ + \tfrac{1}{5!}(\tilde{\mathsf{R}}_5 - 3\Omega\tilde{\mathsf{R}}_4 + \tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_2 + 3\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_3 - \Omega\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2) \\ + \tfrac{1}{6!}(\tilde{\mathsf{R}}_6 - 4\Omega\tilde{\mathsf{R}}_5 + \tilde{\mathsf{R}}_4\tilde{\mathsf{R}}_2 + 6\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_4 + 4\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_3 - 2\Omega\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_2 - 4\Omega\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_3 + \tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2) \\ + \tfrac{1}{7!}(\tilde{\mathsf{R}}_7 - 5\Omega\tilde{\mathsf{R}}_6 + \tilde{\mathsf{R}}_5\tilde{\mathsf{R}}_2 + 10\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_5 + 5\tilde{\mathsf{R}}_4\tilde{\mathsf{R}}_3 + 10\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_4 - 3\Omega\tilde{\mathsf{R}}_4\tilde{\mathsf{R}}_2 - 10\Omega\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_4 \\ - 10\Omega\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_3 + \tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2 + 3\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_2 + 5\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_3 - \Omega\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2) \\ + \tfrac{1}{8!}(\tilde{\mathsf{R}}_8 - 6\Omega\tilde{\mathsf{R}}_7 + \tilde{\mathsf{R}}_6\tilde{\mathsf{R}}_2 + 15\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_6 + 6\tilde{\mathsf{R}}_5\tilde{\mathsf{R}}_3 + 20\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_5 - 4\Omega\tilde{\mathsf{R}}_5\tilde{\mathsf{R}}_2 \\ - 20\Omega\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_5 + 15\tilde{\mathsf{R}}_4\tilde{\mathsf{R}}_4 - 18\Omega\tilde{\mathsf{R}}_4\tilde{\mathsf{R}}_3 - 30\Omega\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_4 + \tilde{\mathsf{R}}_4\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2 \\ + 6\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_4\tilde{\mathsf{R}}_2 + 15\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_4 + 4\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_2 + 6\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_3 + 18\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_3 \\ - 2\Omega\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2 - 4\Omega\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_2 - 6\Omega\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_3 + \tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2)\big]^{\hat{\alpha}}{}_0 + \mathscr{O}(x^9)\,, \end{aligned}$$

etc. For the recursion in terms of *covariant* derivatives of the Riemann tensor, we again use the method in (6.80), (6.81) from subsection 6.2.7, i.e. we recall the tetrads' role as parallel propagator along the space-like geodesic coordinate lines, so the n^{th} tensor coefficient in the expansion of the Riemann tensor is given by,

$$\partial_{(i_1} \cdots \partial_{i_{n-1}|} \Big[e^{\hat{\alpha}}{}_{\delta_1}(x) R^{\delta_1}{}_{|i_{n-1} i_n)\delta_2} e_{\hat{\beta}}{}^{\delta_2}(x) \Big] = e^{\hat{\alpha}}{}_{\delta_1}(x) R^{\delta_1}{}_{(i_1 i_2|\delta_2;\,|i_3 \cdots i_n)} e_{\hat{\beta}}{}^{\delta_2}(x)\,. \qquad (6.152)$$

Here, we stress that $\partial_{(i_1|} R^{\hat{\alpha}}{}_{|i_2 i_3)\hat{\beta}}$ are the partial derivatives of the *frame components* of the Riemann tensor. The fact that their partial derivatives are covariant is clear since $R^{\hat{\alpha}}{}_{i_1 i_2 \hat{\beta}}$ is effectively a coordinate-scalar due to the all-spacial part of the tetrad being trivial, i.e. we have $e_{\hat{i}}{}^{i}(x) = \delta_{\hat{i}}{}^{i}$ in Fermi coordinates, so i-indices do not matter. This fact has a profound impact. This means that, when we actually want to *evaluate* the Fermi expansion of a quantity in some *given* background metric, we do not need to calculate any covariant derivatives of the Riemann tensor. All we need to know are the Riemann tensor's *frame components* in some frame. Fortunately, this ties in well with what we have worked out in subsection 5.6.4 and subsection 5.6.5, where we saw that in all vacuum space-times, there exists one particular curvature-adapted frame, in which the electric and magnetic parts E and B of the Weyl tensor take on particularly simple (diagonal) forms. Thus, any concrete Fermi expansion can be written down in terms of (Lorentz transformations of) partial derivatives of E and B alone.

As with Riemann coordinates in (6.81), we thus obtain a manifestly covariant recursion for the inverse tetrad's time-part, i.e. one in terms of *covariant* Riemann derivatives,

$$\boxed{\partial_{(i_1}\cdots\partial_{i_n)}e^{\hat{\alpha}}{}_0(x)\big|_{\mathscr{W}} = \partial_{(i_1}\cdots\partial_{i_{n-2}|}\Big(R^{\hat{\alpha}}{}_{|i_{n-1}i_n)\hat{\delta}}\,e^{\hat{\delta}}{}_0(x)\Big)\Big|_{\mathscr{W}}\,.} \tag{6.153}$$

This recursion was also derived in a non-coordinate basis in a recent paper by Mukhopadhyay [193], who generalised a method developed for Riemann coordinates by Müller, Schubert and van de Ven in [170] to the case of Fermi-like coordinates around a (not necessarily one-dimensional) sub-manifold. While their method is somewhat more general (they derive a differential equation and its corresponding integral form, before Taylor-expanding these to obtain the recursion), our straightforward derivation of the above recursion for the time-like part of the inverse tetrad is performed in a coordinate basis and is perhaps simpler. We also display the expansion of the time-like part of the inverse tetrad in terms of *covariant* derivatives of Riemann tensors, written in terms of our covariant Riemann matrix R_n and the transport matrix Ω. To 8^{th} order in the Fermi coordinates, it reads,

$$\begin{aligned}
e^{\hat{\alpha}}{}_0(x) = \Big[&1 - \Omega + \tfrac{1}{2!}\mathsf{R}_2 + \tfrac{1}{3!}\big(\mathsf{R}_3 - \mathsf{R}_2\Omega\big) + \tfrac{1}{4!}\big(\mathsf{R}_4 - 2\mathsf{R}_3\Omega + \mathsf{R}_2\mathsf{R}_2\big)\\
&+ \tfrac{1}{5!}\big(\mathsf{R}_5 - 3\mathsf{R}_4\Omega + 3\mathsf{R}_3\mathsf{R}_2 + \mathsf{R}_2\mathsf{R}_3 - \mathsf{R}_2\mathsf{R}_2\Omega\big)\\
&+ \tfrac{1}{6!}\big(\mathsf{R}_6 - 4\mathsf{R}_5\Omega + 6\mathsf{R}_4\mathsf{R}_2 + \mathsf{R}_2\mathsf{R}_4 + 4\mathsf{R}_3\mathsf{R}_3 - 4\mathsf{R}_3\mathsf{R}_2\Omega - 2\mathsf{R}_2\mathsf{R}_3\Omega + \mathsf{R}_2\mathsf{R}_2\mathsf{R}_2\big)\\
&+ \tfrac{1}{7!}\big(\mathsf{R}_7 - 5\mathsf{R}_6\Omega + 10\mathsf{R}_5\mathsf{R}_2 + \mathsf{R}_2\mathsf{R}_5 + 10\mathsf{R}_4\mathsf{R}_3 + 5\mathsf{R}_3\mathsf{R}_4 - 10\mathsf{R}_4\mathsf{R}_2\Omega\\
&\qquad - 3\mathsf{R}_2\mathsf{R}_4\Omega - 10\mathsf{R}_3\mathsf{R}_3\Omega + 5\mathsf{R}_3\mathsf{R}_2\mathsf{R}_2 + 3\mathsf{R}_2\mathsf{R}_3\mathsf{R}_2 + \mathsf{R}_2\mathsf{R}_2\mathsf{R}_3 - \mathsf{R}_2\mathsf{R}_2\mathsf{R}_2\Omega\big)\\
&+ \tfrac{1}{8!}\big(\mathsf{R}_8 - 6\mathsf{R}_7\Omega + 15\mathsf{R}_6\mathsf{R}_2 + \mathsf{R}_2\mathsf{R}_6 + 20\mathsf{R}_5\mathsf{R}_3 + 6\mathsf{R}_3\mathsf{R}_5 - 20\mathsf{R}_5\mathsf{R}_2\Omega - 4\mathsf{R}_2\mathsf{R}_5\Omega\\
&\qquad + 15\mathsf{R}_4\mathsf{R}_4 - 30\mathsf{R}_4\mathsf{R}_3\Omega - 18\mathsf{R}_3\mathsf{R}_4\Omega + 15\mathsf{R}_4\mathsf{R}_2\mathsf{R}_2 + 6\mathsf{R}_2\mathsf{R}_4\mathsf{R}_2 + \mathsf{R}_2\mathsf{R}_2\mathsf{R}_4\\
&\qquad + 18\mathsf{R}_3\mathsf{R}_3\mathsf{R}_2 + 6\mathsf{R}_3\mathsf{R}_2\mathsf{R}_3 + 4\mathsf{R}_2\mathsf{R}_3\mathsf{R}_3 - 6\mathsf{R}_3\mathsf{R}_2\mathsf{R}_2\Omega - 4\mathsf{R}_2\mathsf{R}_3\mathsf{R}_2\Omega\\
&\qquad - 2\mathsf{R}_2\mathsf{R}_2\mathsf{R}_3\Omega + \mathsf{R}_2\mathsf{R}_2\mathsf{R}_2\mathsf{R}_2\big)\Big]^{\hat{\alpha}}{}_0 + \mathscr{O}(x^{11})\,.
\end{aligned} \tag{6.154}$$

The number of terms at each order n is again related to the Fibonacci sequence, here it is given directly by the Fibonacci number F_n. For the expansion of the time-like tetrad vector, we restrict ourselves to the $\Omega = 0$ case. Its expansion to 10^{th} order reads,

$$
\begin{aligned}
e_{\hat{0}}{}^{\kappa}(x) = \Big[1 &- \tfrac{1}{2!}\mathsf{R}_2 - \tfrac{1}{3!}\mathsf{R}_3 - \tfrac{1}{4!}(\mathsf{R}_4 - \mathsf{R}_2\mathsf{R}_2) - \tfrac{1}{5!}(\mathsf{R}_5 - 3\mathsf{R}_3\mathsf{R}_2 - \mathsf{R}_2\mathsf{R}_3) \\
&- \tfrac{1}{6!}(\mathsf{R}_6 - 6\mathsf{R}_4\mathsf{R}_2 - \mathsf{R}_2\mathsf{R}_4 - 4\mathsf{R}_3\mathsf{R}_3 + \mathsf{R}_2\mathsf{R}_2\mathsf{R}_2) \\
&- \tfrac{1}{7!}(\mathsf{R}_7 - 10\mathsf{R}_5\mathsf{R}_2 - \mathsf{R}_2\mathsf{R}_5 - 10\mathsf{R}_4\mathsf{R}_3 - 5\mathsf{R}_3\mathsf{R}_4 + 5\mathsf{R}_3\mathsf{R}_2\mathsf{R}_2 \\
&\qquad + 3\mathsf{R}_2\mathsf{R}_3\mathsf{R}_2 + \mathsf{R}_2\mathsf{R}_2\mathsf{R}_3) \\
&- \tfrac{1}{8!}(\mathsf{R}_8 - 15\mathsf{R}_6\mathsf{R}_2 - \mathsf{R}_2\mathsf{R}_6 - 20\mathsf{R}_5\mathsf{R}_3 - 6\mathsf{R}_3\mathsf{R}_5 - 15\mathsf{R}_4\mathsf{R}_4 \\
&\qquad + 15\mathsf{R}_4\mathsf{R}_2\mathsf{R}_2 + 6\mathsf{R}_2\mathsf{R}_4\mathsf{R}_2 + \mathsf{R}_2\mathsf{R}_2\mathsf{R}_4 + 18\mathsf{R}_3\mathsf{R}_3\mathsf{R}_2 \\
&\qquad + 6\mathsf{R}_3\mathsf{R}_2\mathsf{R}_3 + 4\mathsf{R}_2\mathsf{R}_3\mathsf{R}_3 - \mathsf{R}_2\mathsf{R}_2\mathsf{R}_2\mathsf{R}_2) \\
&- \tfrac{1}{9!}(\mathsf{R}_9 - 21\mathsf{R}_7\mathsf{R}_2 - \mathsf{R}_2\mathsf{R}_7 - 35\mathsf{R}_6\mathsf{R}_3 - 7\mathsf{R}_3\mathsf{R}_6 - 35\mathsf{R}_5\mathsf{R}_4 - 21\mathsf{R}_4\mathsf{R}_5 \\
&\qquad + 35\mathsf{R}_5\mathsf{R}_2\mathsf{R}_2 + 10\mathsf{R}_2\mathsf{R}_5\mathsf{R}_2 + \mathsf{R}_2\mathsf{R}_2\mathsf{R}_5 + 63\mathsf{R}_4\mathsf{R}_3\mathsf{R}_2 \\
&\qquad + 21\mathsf{R}_4\mathsf{R}_2\mathsf{R}_3 + 42\mathsf{R}_3\mathsf{R}_4\mathsf{R}_2 + 7\mathsf{R}_3\mathsf{R}_2\mathsf{R}_4 \\
&\qquad + 10\mathsf{R}_2\mathsf{R}_4\mathsf{R}_3 + 5\mathsf{R}_2\mathsf{R}_3\mathsf{R}_4 + 28\mathsf{R}_3\mathsf{R}_3\mathsf{R}_3) \\
&- \tfrac{1}{10!}(\mathsf{R}_{10} - 28\mathsf{R}_8\mathsf{R}_2 - \mathsf{R}_2\mathsf{R}_8 - 56\mathsf{R}_7\mathsf{R}_3 - 8\mathsf{R}_3\mathsf{R}_7 - 70\mathsf{R}_6\mathsf{R}_4 - 28\mathsf{R}_4\mathsf{R}_6 \\
&\qquad - 56\mathsf{R}_5\mathsf{R}_5 + 70\mathsf{R}_6\mathsf{R}_2\mathsf{R}_2 + 15\mathsf{R}_2\mathsf{R}_6\mathsf{R}_2 + \mathsf{R}_2\mathsf{R}_2\mathsf{R}_6 + 168\mathsf{R}_5\mathsf{R}_3\mathsf{R}_2 \\
&\qquad + 56\mathsf{R}_5\mathsf{R}_2\mathsf{R}_3 + 80\mathsf{R}_3\mathsf{R}_5\mathsf{R}_2 + 20\mathsf{R}_2\mathsf{R}_5\mathsf{R}_3 + 8\mathsf{R}_3\mathsf{R}_2\mathsf{R}_5 + 6\mathsf{R}_2\mathsf{R}_3\mathsf{R}_5 \\
&\qquad + 168\mathsf{R}_4\mathsf{R}_4\mathsf{R}_2 + 28\mathsf{R}_4\mathsf{R}_2\mathsf{R}_4 + 15\mathsf{R}_2\mathsf{R}_4\mathsf{R}_4 + 112\mathsf{R}_4\mathsf{R}_3\mathsf{R}_3 + 80\mathsf{R}_3\mathsf{R}_4\mathsf{R}_3 \\
&\qquad + 40\mathsf{R}_3\mathsf{R}_3\mathsf{R}_4 - 28\mathsf{R}_4\mathsf{R}_2\mathsf{R}_2\mathsf{R}_2 - 15\mathsf{R}_2\mathsf{R}_4\mathsf{R}_2\mathsf{R}_2 - 6\mathsf{R}_2\mathsf{R}_2\mathsf{R}_4\mathsf{R}_2 - \mathsf{R}_2\mathsf{R}_2\mathsf{R}_2\mathsf{R}_4 \\
&\qquad - 40\mathsf{R}_3\mathsf{R}_3\mathsf{R}_2\mathsf{R}_2 - 24\mathsf{R}_3\mathsf{R}_2\mathsf{R}_3\mathsf{R}_2 - 18\mathsf{R}_2\mathsf{R}_3\mathsf{R}_3\mathsf{R}_2 - 8\mathsf{R}_3\mathsf{R}_2\mathsf{R}_2\mathsf{R}_3 \\
&\qquad - 6\mathsf{R}_2\mathsf{R}_3\mathsf{R}_2\mathsf{R}_3 - 4\mathsf{R}_2\mathsf{R}_2\mathsf{R}_3\mathsf{R}_3 + \mathsf{R}_2\mathsf{R}_2\mathsf{R}_2\mathsf{R}_2\mathsf{R}_2)\Big]^{\kappa}{}_{\hat{0}} + \mathscr{O}(x^{11})\,.
\end{aligned}
\tag{6.155}
$$

We also display the 7^{th}-order expansion of the time–time part of the metric in Fermi normal coordinates in terms of the partial-derivative Riemann matrix $\tilde{\mathsf{R}}_k$ and the transport matrix Ω, which can be calculated from that of the inverse tetrad in (6.154). It reads,

$$
\begin{aligned}
g_{00}(x) = [1 &- 2\Omega + (\tilde{\mathsf{R}}_2 + \Omega\Omega) + \tfrac{2}{3!}(\tilde{\mathsf{R}}_3 - 2\tilde{\mathsf{R}}_2\Omega) \\
&+ \tfrac{2}{4!}(\tilde{\mathsf{R}}_4 - 4\tilde{\mathsf{R}}_3\Omega + 2\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2 + 2\tilde{\mathsf{R}}_2\Omega\Omega) \\
&+ \tfrac{2}{5!}(\tilde{\mathsf{R}}_5 - 6\tilde{\mathsf{R}}_4\Omega + 6\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_2 - 2\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_3 + 6\tilde{\mathsf{R}}_3\Omega\Omega - 4\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2\Omega) \\
&+ \tfrac{2}{6!}(\tilde{\mathsf{R}}_6 - 8\tilde{\mathsf{R}}_5\Omega + 12\tilde{\mathsf{R}}_4\tilde{\mathsf{R}}_2 + 2\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_4 + 12\tilde{\mathsf{R}}_4\Omega\Omega + 8\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_3 - 16\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_2\Omega \\
&\qquad - 8\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_3\Omega + 4\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2 + 4\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2\Omega\Omega) \\
&+ \tfrac{2}{7!}(\tilde{\mathsf{R}}_7 - 10\tilde{\mathsf{R}}_6\Omega + 20\tilde{\mathsf{R}}_5\tilde{\mathsf{R}}_2 + 2\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_5 + 20\tilde{\mathsf{R}}_5\Omega\Omega + 20\tilde{\mathsf{R}}_4\tilde{\mathsf{R}}_3 + 10\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_4 \\
&\qquad - 40\tilde{\mathsf{R}}_4\tilde{\mathsf{R}}_2\Omega - 12\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_4\Omega - 40\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_3\Omega + 20\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2 + 12\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_2 \\
&\qquad + 4\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_3 + 20\tilde{\mathsf{R}}_3\tilde{\mathsf{R}}_2\Omega\Omega + 12\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_3\Omega\Omega - 8\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2\tilde{\mathsf{R}}_2\Omega)]_{00} \\
&+ \mathscr{O}(x^8)\,.
\end{aligned}
\tag{6.156}
$$

Here, we recall that in the single 2^{nd}-order inertial-only term $\Omega\Omega$, the contraction is over the full tetrad index, i.e., (up to inverse factors of c) it stands for $\Omega^{\hat{\delta}}{}_{(\hat{\imath}_1|}\Omega_{\hat{\delta}|\hat{\imath}_2)}$, as initially displayed in the component expressions (6.142a). The same holds true for an Ω-matrix which multiplies a Riemann factor from the right, i.e., $\tilde{\mathsf{R}}_k\Omega$ with $k \geq 2$ always stands for $R_{\hat{0}(\hat{\imath}_1\hat{\imath}_2|\hat{\delta},\,|\hat{\imath}_3\cdots\hat{\imath}_k|}\Omega^{\hat{\delta}}{}_{|\hat{\imath}_{k+1})}$.

Linearised Background Metric: Rederiving Marzlin's Formula

Now that we know the structure of the inertial-force–curvature coupling terms in the inverse tetrad and the metric, we recover a result by Marzlin [208]. He considers a background metric which is a first-order perturbation of the form (5.164), but around flat space-time, i.e.,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \tag{6.157}$$

so that the Riemann tensor becomes linear and all its covariant derivatives reduce to partial ones. We can deduce the corresponding expression for its Fermi expansion to arbitrary order in the above linearised setting simply by noting the following points: (1) All terms of quadratic and higher order in the Riemann tensor vanish (recalling that these non-linear terms and their numerical prefactors are essentially what makes Fermi-coordinate expansions complicated in the first place), and, (2) since the Fermi expansion of the metric is at most quadratic in Ω, we just have to consider the terms linear in the Riemann tnesor of the above expansion up to $n = 4$, in order to capture the complete inertial–curvature interaction present at linear order.

If we now split all contraction indices between Riemann factors and Ω-matrices into space and time parts, we can write the metric in these *linearised Fermi coordinates* compactly as,

$$\begin{aligned} g_{00}(x) &= 1 - 2\big(\tfrac{1}{c}\big)\Omega_{\hat{0}\hat{\imath}}x^{\hat{\imath}} + \big(\tfrac{1}{c}\big)^2\Omega^{\hat{\delta}}{}_{\hat{\imath}_1}\Omega_{\hat{\delta}\hat{\imath}_2}x^{\hat{\imath}_1}x^{\hat{\imath}_2} \\ &\quad + \sum_{n=2}^{\infty}\frac{2}{(n+1)!}R_{\hat{0}(\hat{\imath}_1\hat{\imath}_2|\hat{0},\,|\hat{\imath}_3\cdots\hat{\imath}_n|}\,x^{\hat{\imath}_1}\cdots x^{\hat{\imath}_n} \\ &\qquad \times \Big[(n+1) - 2n\big(\tfrac{1}{c}\Omega^{\hat{0}}{}_{|\hat{\imath})}\,x^{\hat{\imath}}\big) + (n-1)\big(\tfrac{1}{c}\big)^2\Omega^{\hat{0}}{}_{|\hat{\imath}|}\Omega_{\hat{0}|\hat{\jmath})}x^{\hat{\imath}}x^{\hat{\jmath}}\Big]\,, \end{aligned} \tag{6.158a}$$

$$\begin{aligned} g_{0k}(x) &= -\tfrac{1}{c}\Omega_{\hat{k}\hat{\imath}}x^{\hat{\imath}} + \sum_{n=2}^{\infty}\frac{2}{(n+1)!}\big[n\,\delta^{\hat{\delta}}{}_{\hat{0}} - (n-1)\big(\tfrac{1}{c}\Omega^{\hat{\delta}}{}_{(\hat{\imath}|}\,x^{\hat{\imath}}\big)\big] \\ &\qquad \times R_{\hat{\delta}|\hat{\imath}_1\hat{\imath}_2|\hat{k},\,|\hat{\imath}_3\cdots\hat{\imath}_n)}\,x^{\hat{\imath}_1}\cdots x^{\hat{\imath}_n}\,, \end{aligned} \tag{6.158b}$$

$$g_{k_1k_2}(x) = \eta_{\hat{k}_1\hat{k}_2} + \sum_{n=2}^{\infty}2\frac{(n-1)}{(n+1)!}R_{\hat{k}_1(\hat{\imath}_1\hat{\imath}_2|\hat{k}_2,\,|\hat{\imath}_3\cdots\hat{\imath}_n)}\,x^{\hat{\imath}_1}\cdots x^{\hat{\imath}_n}\,, \tag{6.158c}$$

Marzlin's form [208, eqs. (16), (17)] is then recovered by splitting the contraction index $\hat{\delta}$ into $\hat{0}$ and $\hat{d}$ and using $\Omega^{\hat{0}}{}_{\hat{\imath}}\,x^{\hat{\imath}} = \frac{1}{c}a_{\hat{\imath}}x^{\hat{\imath}} = -\frac{1}{c}\boldsymbol{a}\cdot\boldsymbol{x}$ and $\Omega^{\hat{d}}{}_{\hat{\imath}}\,x^{\hat{\imath}} = (\boldsymbol{\omega}\times\boldsymbol{x})^{\hat{d}}$, as well as shifting the summation indices to $r = n - 2$.

6.3.7. Range of Validity of the Riemann and Fermi Normal Expansions

The Riemann and Fermi normal coordinate expansion are well-defined i.e., they define valid local coordinate charts, where the coordinate transformation is non-singular, cf. (6.33). This is generally the case as long as the spacial geodesics through the origin that are taken as coordinate lines do not intersect (and don't run into singularities of the space-time, so we have to be sufficiently far away from these, which we shall assume). This, in turn, will hold as long as the scale of the expansion parameter is small compared to the typical length scales associated with curvature of the background space-time on the one side, and inertial forces on the other. The typical length scales associated with gravity are the curvature radius $\mathscr{R}$, and the corresponding n^{th}-order (inverse) rates-of-change $\mathscr{D}_n$ of curvature of space-time [209] given by,

$$\mathscr{R} := \frac{1}{\sqrt{\max|R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}|}}\,, \qquad \mathscr{D}_n := \frac{\max|R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta};\,\hat{\kappa}_1\cdots\hat{\kappa}_{n-1}}|}{\max|R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta};\,\hat{\kappa}_1\cdots\hat{\kappa}_n}|}\,, \tag{6.159}$$

where $\max|R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}|$ refers to the largest frame component of the Riemann tensor. $\mathscr{R}$ is then the scale of the second-order coefficient, $\mathscr{D}_1$ the relative scale of the third-order coefficient (we can set $\mathscr{D}_0 := \mathscr{R}$ for convenience), etc., in the Riemann and Fermi normal coordinate expansions of the tetrads and the metric. Heuristically, $\mathscr{R}$ corresponds to the length scale where geodesics start intersecting. This can be seen if, for the sake of this argument, we approximately think of the space-time as being de Sitter , i.e. a hypersphere, on which the geodesics will intersect at $|x^{\hat{\alpha}}| = 2\pi\mathscr{R}$.

As far as Riemann and *inertial* Fermi normal coordinates are concerned, the general requirement for the expansions to converge then is that the largest coordinate value be (much) smaller than the smallest of these length scales, i.e. we require that

$$\max|x^{\hat{\alpha}}| \ll \min\{\mathscr{R}, \mathscr{D}_1, \ldots, \mathscr{D}_n\}\,. \tag{6.160}$$

To give an example, for the Schwarzschild space-time that we will consider below, the radius of curvature is smallest at the surface of the Earth $R = R_{♁} = 6378\,\mathrm{km}$ (equatorial Earth radius), and the local rate-of-change of curvature grows linearly with the Schwarzschild radial coordinate R, yielding

$$\mathscr{R}_{♁} = \sqrt{R_{♁}^3/R_{\mathrm{S}}^{♁}} = \sqrt{\frac{c^2 R_{♁}^3}{2GM_{♁}}} \approx 1.7\times 10^8\,\mathrm{km}\,, \qquad \mathscr{D}_{1,♁} = \frac{R_{♁}}{3} \approx 2126\,\mathrm{km}\,.$$

This value for $\mathscr{R}_{♁}$ corresponds roughly to 450 times the Earth–Moon distance. More generally for Petrov type D space-times, i.e. those that are of the Coulomb-like central-field type (cf. Table 5.2) where the Riemann tensor falls off proportional to $1/R^3$, one easily infers that $\mathscr{D}_n \sim R$, and – at first sight – this seems to be the most restrictive of the length scales. Fortunately, Nesterov [210] showed by using integral formulae for the expansions, that the actual radius of convergence of the Riemann and Fermi normal expansions is given by $\mathscr{R}$ alone.

In Fermi coordinates with non-vanishing acceleration $\boldsymbol{a}$ and possibly also rotations $\boldsymbol{\omega}$, we additionally have to consider in (6.161) the characteristic length scales associated with these inertial forces, that is, the acceleration and rotation lengths (4.98). For an expansion to n^{th} order, we thus have the following restrictions on the admissible coordinate radius,

$$\boxed{\max|x^{\hat{a}}| \equiv s_{\max} \ll \min\{\mathscr{R}, \mathscr{R}_a, \mathscr{R}_\omega\}\,,} \tag{6.161}$$

where $r \equiv \sqrt{x^{\hat{a}} x_{\hat{a}}}$, for the maximum value of proper distance $s_{\max}$ (being the expansion parameter), and the Fermi normal coordinate $x^{\hat{a}}$ derived from it.

6.4. Equation of Motion for General World-Lines in Fermi Coordinates

In section 4.4 we showed how the equation of motion for an accelerating and rotating observer in special relativity can be obtained from considering the geodesic equation in non-affine parametrisation in terms of coordinate time x^0. This resulted in (4.149). Here we are going to use the same approach in Fermi coordinates, which includes the respective curvature terms, and yields the equation of motion for an arbitrary world-line that is close to the central world-line $\mathscr{W}$ of the fiducial Fermi observer in some sense, but does not intersect it in general.

6.4.1. Expansion of Christoffel Symbols in Fermi Coordinates

In order to write down the geodesic equation for arbitrary geodesics in Fermi coordinates in section 6.4, we shall also need the expansion of the Christoffel symbols,

$$\Gamma^{\kappa}{}_{\alpha_1\alpha_2}(x) = \Gamma^{\hat{\kappa}}{}_{\hat{\alpha}_1\hat{\alpha}_2,\,\hat{\imath}}\big|_{\mathscr{W}} x^{\hat{\imath}} + \tfrac{1}{2!}\Gamma^{\hat{\kappa}}{}_{\hat{\alpha}_1\hat{\alpha}_2,\,(\hat{\imath}_1\hat{\imath}_2)}\big|_{\mathscr{W}} x^{\hat{\imath}_1}x^{\hat{\imath}_2} + \tfrac{1}{3!}\Gamma^{\hat{\kappa}}{}_{\hat{\alpha}_1\hat{\alpha}_2,\,(\hat{\imath}_1\hat{\imath}_2\hat{\imath}_3)}\big|_{\mathscr{W}} x^{\hat{\imath}_1}x^{\hat{\imath}_2}x^{\hat{\imath}_3} + \mathscr{O}(x^3)\,, \tag{6.162}$$

where we have suppressed the dependence on proper time as usual. The coefficients $\Gamma^{\hat{\kappa}}{}_{\hat{\alpha}_1\hat{\alpha}_2,\,(\hat{\imath}_1\cdots\hat{\imath}_n)}$ are more general, but also much harder to calculate (at least for $n \geq 2$) than the generalised connection coefficients (6.125), in which the symmetrisation extends also over one of the Christoffel's lower indices. However, they too are tensors and given in terms of certain non-trivial index permutations of Riemann tensors and their covariant derivatives, which reduce to the $\Gamma^{\hat{\kappa}}{}_{\alpha(i_1,i_2\cdots i_n)}$ when symmetrised also over $\hat{\alpha}_2$.

At zeroth order, the Christoffel symbols on the central world-line – when non-vanishing – are given in terms of the transport matrix (4.49) and can be written compactly as

$$\Gamma^{\hat{\kappa}}{}_{\hat{\alpha}_1\hat{\alpha}_2}\big|_{\mathscr{W}} = -\tfrac{1}{c}\Big[\eta_{\hat{\alpha}_1\hat{0}}\Omega^{\hat{\kappa}}{}_{\hat{\alpha}_2} + \eta_{\hat{0}\hat{\alpha}_2}\Omega^{\hat{\kappa}}{}_{\hat{\alpha}_1} - \eta_{\hat{\alpha}_1\hat{0}}\eta_{\hat{0}\hat{\alpha}_2}\Omega^{\hat{\kappa}}{}_{\hat{0}}\Big]\,, \tag{6.163}$$

where only those with one lower zero index are non-zero, as usual. Explicitly we have,

$$\Gamma^{\hat{0}}{}_{\hat{0}\hat{0}} = 0\,, \tag{6.164a}$$
$$\Gamma^{\hat{k}}{}_{\hat{0}\hat{0}} = -\tfrac{1}{c}\Omega^{\hat{k}}{}_{\hat{0}}\,, \tag{6.164b}$$
$$\Gamma^{\hat{\kappa}}{}_{\hat{a}\hat{0}} = -\tfrac{1}{c}\Omega^{\hat{\kappa}}{}_{\hat{a}}\,, \tag{6.164c}$$
$$\Gamma^{\hat{\kappa}}{}_{\hat{a}_1\hat{a}_2} = 0\,, \tag{6.164d}$$

In order to calculate the first partial derivatives of the Christoffel symbols, one employs the definition of the Riemann tensor (3.80) again, rearranging it as

$$\Gamma^{\kappa}{}_{(\gamma\beta),\,\alpha} = R^{\kappa}{}_{(\gamma|\alpha|\beta)} + \Gamma^{\kappa}{}_{(\gamma|\alpha,\,|\beta)} - \Gamma^{\kappa}{}_{\delta\alpha}\Gamma^{\delta}{}_{(\gamma\beta)} + \Gamma^{\kappa}{}_{\delta(\beta|}\Gamma^{\delta}{}_{|\gamma)\alpha} \tag{6.165}$$

and uses the relation between the Christoffel symbols with one lower index zero and the transport matrix (6.163) above. The spacial derivatives, i.e. the coefficients of the expansion (6.162), are then found to be,

$$
\begin{aligned}
\Gamma^{\hat{0}}{}_{\hat{0}\hat{0},\hat{\imath}} &= -\tfrac{1}{c}\Omega^{\hat{0}}{}_{\hat{\imath},\hat{0}} - \tfrac{1}{c^2}\Omega^{\hat{0}}{}_{\hat{d}}\Omega^{\hat{d}}{}_{\hat{\imath}}\,, && (6.166a)\\
\Gamma^{\hat{k}}{}_{\hat{0}\hat{0},\hat{\imath}} &= R^{\hat{k}}{}_{\hat{0}\hat{\imath}\hat{0}} - \tfrac{1}{c}\Omega^{\hat{k}}{}_{\hat{\imath},\hat{0}} + \tfrac{1}{c^2}\Omega^{\hat{k}}{}_{\hat{\delta}}\Omega^{\hat{\delta}}{}_{\hat{\imath}}\,, && (6.166b)\\
\Gamma^{\hat{\kappa}}{}_{\hat{a}\hat{0},\hat{\imath}} &= R^{\hat{\kappa}}{}_{\hat{a}\hat{\imath}\hat{0}} - \tfrac{1}{c^2}\Omega^{\hat{\kappa}}{}_{\hat{\imath}}\Omega^{\hat{0}}{}_{\hat{a}}\,, && (6.166c)\\
\Gamma^{\hat{\kappa}}{}_{\hat{a}_1\hat{a}_2,\hat{\imath}} &= -\tfrac{2}{3}R^{\hat{\kappa}}{}_{(\hat{a}_1\hat{a}_2)\hat{\imath}}\,. && (6.166d)
\end{aligned}
$$

Collecting the above results into (6.162), the spacial expansion of the non-vanishing Christoffel symbols away from the central world-line then reads,

$$
\begin{aligned}
\Gamma^{k}{}_{00}(x) &= -\tfrac{1}{c}\Omega^{\hat{k}}{}_{\hat{0}} + [R^{\hat{k}}{}_{\hat{0}\hat{\imath}\hat{0}} - \tfrac{1}{c}\Omega^{\hat{k}}{}_{\hat{\imath},\hat{0}} + \tfrac{1}{c^2}\Omega^{\hat{k}}{}_{\hat{\delta}}\Omega^{\hat{\delta}}{}_{\hat{\imath}}]x^{\hat{\imath}} + \mathscr{O}(x^2)\,, && (6.167a)\\
\Gamma^{\kappa}{}_{a0}(x) &= -\tfrac{1}{c}\Omega^{\hat{\kappa}}{}_{\hat{a}} + [R^{\hat{\kappa}}{}_{\hat{a}\hat{\imath}\hat{0}} - \tfrac{1}{c^2}\Omega^{\hat{\kappa}}{}_{\hat{\imath}}\Omega^{\hat{0}}{}_{\hat{a}}]x^{\hat{\imath}} + \mathscr{O}(x^2)\,, && (6.167b)\\
\Gamma^{k}{}_{a_1a_2}(x) &= -\tfrac{2}{3}R^{\hat{k}}{}_{(\hat{a}_1\hat{a}_2)\hat{\imath}}x^{\hat{\imath}} + \mathscr{O}(x^2)\,, && (6.167c)\\
\Gamma^{0}{}_{a_1a_2}(x) &= -\tfrac{2}{3}R^{\hat{0}}{}_{(\hat{a}_1\hat{a}_2)\hat{\imath}}x^{\hat{\imath}} + \mathscr{O}(x^2)\,. && (6.167d)
\end{aligned}
$$

6.4.2. The Generalised Jacobi Equation

In contrast to the geodesic equation (4.140), we shall initially allow for the particle to be accelerated with four-acceleration $\tilde{a}^\kappa$. We thus start with,

$$\frac{dv^\kappa}{d\lambda} + \Gamma^{\kappa}{}_{\alpha_1\alpha_2}\,v^{\alpha_1}v^{\alpha_2} = \tilde{a}^\kappa\,, \tag{6.168}$$

where we stress that $\tilde{a}^\kappa$ is the particle's proper four-acceleration that it experiences in its own frame, to be contrasted with the untilded acceleration a^κ of the present Fermi-coordinate frame from which $\tilde{a}^\kappa$ is observed. The coordinate velocity v^k defined in (4.142), (4.147a) for a geodesic particle, is then augmented to,

$$
\begin{aligned}
\frac{d\gamma(x)}{dx^0} &= -\gamma(x)\Gamma^{0}{}_{\alpha_1\alpha_2}(x)\,\mathrm{v}^{\alpha_1}\mathrm{v}^{\alpha_2} + \gamma^{-1}\tilde{a}^0\,, && (6.169a)\\
\frac{d(\gamma(x)\mathrm{v}^k(x))}{dx^0} &= \gamma\frac{d\mathrm{v}^k}{dx^0} + \left(\frac{d\gamma}{dx^0}\right)\mathrm{v}^k = -\gamma(x)\Gamma^{k}{}_{\alpha_1\alpha_2}\,\mathrm{v}^{\alpha_1}\mathrm{v}^{\alpha_2} + \gamma^{-1}\tilde{a}^k\,, && (6.169b)
\end{aligned}
$$

and the geodesic equation in non-affine parametrisation (4.148) becomes,

$$\frac{d^2x^k_{\mathscr{G}}}{(dx^0)^2} \equiv \frac{d\mathrm{v}^k}{dx^0} = -\Big(\Gamma^{k}{}_{\alpha_1\alpha_2} - \Gamma^{0}{}_{\alpha_1\alpha_2}\mathrm{v}^k\Big)\mathrm{v}^{\alpha_1}\mathrm{v}^{\alpha_2} + \gamma^{-2}(\tilde{a}^k - \tilde{a}^0\mathrm{v}^k)\,. \tag{6.170}$$

Here, the prefactor of $\gamma^{-2} = (d\lambda/dx^0)^2$ in the last term can be seen to account for the fact that $\tilde{a}^\kappa$ is still affinely parametrised in terms of λ, since in coming from the equation of motion (6.168), we have merely performed a $1+3$ split of its index up to this point.

If we want to describe arbitrary time-like world-lines in the neighbourhood of our fiducial central world-line, we have to Taylor-expand the Christoffel symbols on the right-hand side away from the central world-line. Setting now $\tilde{a}^\kappa = 0$ for simplicity, this yields,

$$\frac{dv^k}{dx^0} = -\Big[\Gamma^k{}_{00}(x) + 2\Gamma^k{}_{a_1 0}(x)v^{a_1}(x) + \Gamma^k{}_{a_1 a_2}(x)v^{a_1}(x)v^{a_2}(x) \\ - \Big(\Gamma^0{}_{00}(x) + 2\Gamma^0{}_{a_1 0}(x)v^{a_1}(x) + \Gamma^0{}_{a_1 a_2}(x)v^{a_1}(x)v^{a_2}(x)\Big)v^k(x)\Big], \quad (6.171)$$

where we have set the particle's proper acceleration to zero, $\tilde{a}^\kappa = 0$, for simplicity. Expanding the Christoffel symbols to first order in the Fermi coordinate $x^{\hat{\imath}}$ according to,

$$\frac{dv^k}{dx^0} = -\Big[(\Gamma^{\hat{k}}{}_{\hat{0}\hat{0}} + \Gamma^{\hat{k}}{}_{\hat{0}\hat{0},\hat{\imath}}x^{\hat{\imath}}) + 2(\Gamma^{\hat{k}}{}_{\hat{a}_1\hat{0}} + \Gamma^{\hat{k}}{}_{\hat{a}_1\hat{0},\hat{\imath}}x^{\hat{\imath}})v^{\hat{a}_1} + (\Gamma^{\hat{k}}{}_{\hat{a}_1\hat{a}_2} + \Gamma^{\hat{k}}{}_{\hat{a}_1\hat{a}_2,\hat{\imath}}x^{\hat{\imath}})v^{\hat{a}_1}v^{\hat{a}_2}\Big] \\ + \Big[(\Gamma^{\hat{0}}{}_{\hat{0}\hat{0}} + \Gamma^{\hat{0}}{}_{\hat{0}\hat{0},\hat{\imath}}x^{\hat{\imath}}) + 2(\Gamma^{\hat{0}}{}_{\hat{a}_1\hat{0}} + \Gamma^{\hat{0}}{}_{\hat{a}_1\hat{0},\hat{\imath}}x^{\hat{\imath}})v^{\hat{a}_1} \\ + (\Gamma^{\hat{0}}{}_{\hat{a}_1\hat{a}_2} + \Gamma^{\hat{0}}{}_{\hat{a}_1\hat{a}_2,\hat{\imath}}x^{\hat{\imath}})v^{\hat{a}_1}v^{\hat{a}_2}\Big]v^{\hat{k}}, \quad (6.172)$$

yields, upon inserting the relations (6.167), the result

$$\frac{dv^k}{dx^0} = \tfrac{1}{c}\Omega^{\hat{k}}{}_{\hat{0}} + \Big(R^{\hat{k}}{}_{\hat{0}\hat{\imath}\hat{0}} + \tfrac{1}{c}\Omega^{\hat{k}}{}_{\hat{\imath},\hat{0}} + \tfrac{1}{c^2}\Omega^{\hat{k}}{}_{\hat{\delta}}\Omega^{\hat{\delta}}{}_{\hat{\imath}}\Big)x^{\hat{\imath}} + 2\Big[\tfrac{1}{c}\Omega^{\hat{k}}{}_{\hat{a}_1} - \Big(R^{\hat{k}}{}_{\hat{a}_1\hat{\imath}\hat{0}} - \tfrac{1}{c^2}\Omega^{\hat{k}}{}_{\hat{\imath}}\Omega^{\hat{0}}{}_{\hat{a}_1}\Big)x^{\hat{\imath}}\Big]v^{\hat{a}_1} \\ + \tfrac{2}{3}R^{\hat{k}}{}_{(\hat{a}_1\hat{a}_2)\hat{\imath}}x^{\hat{\imath}}v^{\hat{a}_1}v^{\hat{a}_2} - \Big\{\Big(\tfrac{1}{c}\Omega^{\hat{0}}{}_{\hat{\imath},\hat{0}} + \tfrac{1}{c^2}\Omega^{\hat{0}}{}_{\hat{d}}\Omega^{\hat{d}}{}_{\hat{\imath}}\Big)x^{\hat{\imath}} \\ + 2\Big[\tfrac{1}{c}\Omega^{\hat{0}}{}_{\hat{a}_1} - \Big(R^{\hat{0}}{}_{\hat{a}_1\hat{\imath}\hat{0}} - \tfrac{1}{c^2}\Omega^{\hat{0}}{}_{\hat{a}_1}\Omega^{\hat{0}}{}_{\hat{\imath}}\Big)x^{\hat{\imath}}\Big]v^{\hat{a}_1} + \tfrac{2}{3}R^{\hat{0}}{}_{(\hat{a}_1\hat{a}_2)\hat{\imath}}x^{\hat{\imath}}v^{\hat{a}_1}v^{\hat{a}_2}\Big\}v^{\hat{k}}. \quad (6.173)$$

This is the equation of motion for test particles in Fermi coordinates [189–191] and generally known as the *generalized Jacobi equation* [211–214], where the Jacobi equation is the mathematicians' name for the geodesic deviation equation as noted in subsection 5.6.1.

7 Free-Falling Quantum Gases in Local Frames

In the present chapter, we apply what we have studied in the first part of this thesis to the description of Bose-Einstein condensates in free fall along geodesics of curved space-time near the Earth, which we approximately model in terms of a Schwarzschild metric. As discussed in subsection 5.7.1, the Earth's true (or "physical") space-time metric is – albeit a weak deviation from flat Minkowski space – more complicated and includes contributions from: (1) the Earth's mass multipole moments; (2) its spin multipole moments, the lowest-order of which being the Kerr-like term sourced by its rotation and leading to frame-dragging; (3) contributions from other solar system bodies such as the Moon, the Sun, and other planets, together with their multipole moments. However, the main and leading-order contribution to space-time curvature is the zeroth-order mass moment or "mass monopole" which the Schwarzschild metric captures. Although the Earth's mass multipole moments would appear in the non-relativistic limit, together with tidal contributions from the Sun and the Moon, this would require us to use a full post-Minkowskian or post-Newtonian background metric, which is fairly complicated as noted in subsection 5.7.3, and thus beyond the scope of this thesis.

In order to exhibit the leading-order general-relativistic corrections for Bose-Einstein condensates in the mean-field description, in the following section 7.1, we first generalise the Gross-Pitaevskii equation to curved space-time in terms of the covariant non-linear Klein-Gordon equation. We then calculate the metric in Fermi normal coordinates to quadratic order for inertial frames that fall along radial and circular Schwarzschild geodesics in section 7.2, before in section 7.3, we expand the non-linear Klein-Gordon equation in a perturbation-like approach, which allows for compact and general representation. Subsequently, we perform the non-relativistic limit and exhibit the arising corrections.

7.1. Gross-Pitaevskii Equation in Curved Space-Time

In the present section, we show how the Gross-Pitaevskii equation (2.9) that describes Bose-Einstein condensates in the mean-field approximation is generalised to curved space-time. Here, we shall follow the usual strategy for turning a Newtonian equation into a generally covariant, i.e. general relativistic, tensor equation: in a first step, one generalises the equation to special relativity and in the second step one applies the substitution rules of the correspondence principle (4.35). While this first step was displayed already in (2.10), what remains is to apply the correspondence principle (4.35). Because of the scalar nature of equation (2.10), we only have to deal with the kinetic terms, i.e. the d'Alembertian. Substituting $\eta_{\alpha\beta} \to g_{\mu\nu}$ and $\partial_\alpha \to \nabla_\sigma$ finally yields the generally covariant, non-linear

Klein-Gordon equation,

$$\boxed{\nabla_\sigma \nabla^\sigma \phi + \Big(\frac{mc}{\hbar}\Big)^2 \phi + \xi |\phi|^2 \phi = 0\,.} \tag{7.1}$$

The covariant generalisation of the Klein-Gordon equation and its non-linear counterpart (7.1) above have been known for a long time in the context of quantum field theory in curved space-time, see e.g. the textbook [215]. In a BEC-related context (but long before BEC was experimentally achieved in 1995), equation (7.1) appears as a generalisation of the Gross-Pitaevskii equation in two papers by Anandan [216, 217], where it is used to investigate the interaction of superfluid helium with space-time curvature. We note that because the kinetic operator in (7.1) is applied to a scalar field, the inner covariant derivative reduces to a partial one and we can write,

$$\nabla_\sigma \nabla^\sigma \phi = g^{\mu\nu} \nabla_\mu \partial_\nu \phi = g^{\mu\nu} (\partial_\mu \partial_\nu + \Gamma^\lambda{}_{\lambda\mu} \partial_\nu) \phi\,. \tag{7.2}$$

Further, the contracted Christoffel symbol therein can be expressed in terms of a logarithmic derivative of the square root of the metric determinant g as,

$$\Gamma_\sigma \coloneqq \Gamma^\nu{}_{\nu\sigma} = \frac{1}{\sqrt{-g}} (\partial_\sigma \sqrt{-g}\,) = \partial_\sigma \ln\big(\sqrt{-g}\,\big)\,, \tag{7.3}$$

so that equation (7.1) is seen to expand to (see e. g. [215]),

$$\boxed{g^{\mu\nu} (\partial_\nu \partial_\mu + \Gamma_\mu \partial_\nu) \phi + \Big(\frac{mc}{\hbar}\Big)^2 \phi + \xi |\phi|^2 \phi = 0\,.} \tag{7.4}$$

The non-linear Klein-Gordon equation (7.1) above follows from the corresponding principle of least action of $|\phi|^4$ theory in special relativity with one minor modification to the Lagrangean, which consists in the correspondence-principle replacement of the integral measure $d^4x \longrightarrow \sqrt{-g}\, d^4X$ of (4.35). In addition, we note that there is generically a non-minimal coupling the Ricci scalar R_{Ric}, with an (essentially arbitrary) coupling parameter α, which is needed in order to make the curved-space–time path integral consistent, and which we have suppressed in equations (7.1), (7.4). The full Lagrangean density of a massive, complex scalar field with quartic interaction in curved space-time then reads,

$$\mathcal{L}[\phi, \phi^*, \nabla_\mu \phi, \nabla_\mu \phi^*] = \tfrac{1}{2} \sqrt{-g}\, \Big\{ g^{\mu\nu} (\nabla_\mu \phi^*)(\nabla_\nu \phi) - \Big[\Big(\frac{mc}{\hbar}\Big)^2 + \alpha R_{\mathrm{Ric}}\Big] \phi^* \phi - \tfrac{1}{2} \xi (\phi^* \phi)^2 \Big\}\,. \tag{7.5}$$

Varying this action and setting $R_{\mathrm{Ric}} = 0$ for vacuum space-times, then yields (7.1) as the equation of motion.

7.2. Fermi Metric for Radial and Circular Geodesics

In this section we calculate the Fermi metric in Schwarzschild space-time, for frames adapted to purely radial and circular equatorial geodesics, respectively.

7.2.1. Fermi Metric for Radial Geodesics

At first, we are going to calculate the metric in Fermi normal coordinates for frames that are in free fall along purely radial geodesics in Schwarzschild space-time, noting that this can be used to approximately model the near-radial free fall in a drop tower.

The Weyl tensor in standard Schwarzschild coordinates can be read off our general coordinate-independent expression for spherically symmetric space-times in 2+2 split (A.16). Its non-vanishing frame components in the natural static metric-adapted frame (A.30), are found to be simply,

$$R_{\hat{1}\hat{0}\hat{1}\hat{0}} = \frac{R_\mathrm{S}}{R^3}\,, \quad R_{\hat{2}\hat{0}\hat{2}\hat{0}} = R_{\hat{3}\hat{0}\hat{3}\hat{0}} = -\frac{R_\mathrm{S}}{2R^3}\,, \quad R_{\hat{2}\hat{3}\hat{2}\hat{3}} = -\frac{R_\mathrm{S}}{R^3}\,, \quad R_{\hat{1}\hat{2}\hat{1}\hat{2}} = R_{\hat{1}\hat{3}\hat{1}\hat{3}} = \frac{R_\mathrm{S}}{2R^3}\,. \tag{7.6}$$

When adapting the Weyl tensor of Schwarzschild space-time to the radial free-fall frame by contracting with the corresponding tetrad (5.87), one finds that its frame components in the radial free-fall frame actually coincide with those for the metric-adapted frame (7.6). This is explained by the fact that the Weyl tensor in a spherically symmetric space-time is totally invariant under Lorentz boosts in the radial direction, as is obvious from inspecting its closed-form expression (A.16) that we derive in Appendix A

Using the expansion of the Fermi metric (6.142a), (6.142b) and (6.142c) to $\mathscr{O}(x^3)$ in the Fermi coordinates along a radial geodesic given by, $X^\mu_\mathrm{rad}(\tau) = (cT, R(\tau), \pi/2, \Phi_0)$, in Schwarzschild space-time, we then easily obtain,

$$\begin{aligned} &\qquad\qquad g^\mathrm{rad}_{00} = 1 - \frac{1}{2}\frac{R_\mathrm{S}}{R^3(\tau)}(2x^2 - y^2 - z^2)\,, \\ g^\mathrm{rad}_{12} &= \frac{1}{6}\frac{R_\mathrm{S}}{R^3(\tau)}\,yx\,, & g^\mathrm{rad}_{11} &= -1 - \frac{1}{6}\frac{R_\mathrm{S}}{R^3(\tau)}(y^2 + z^2)\,, \\ g^\mathrm{rad}_{23} &= -\frac{1}{3}\frac{R_\mathrm{S}}{R^3(\tau)}\,yz\,, & g^\mathrm{rad}_{22} &= -1 - \frac{1}{6}\frac{R_\mathrm{S}}{R^3(\tau)}(x^2 - 2z^2)\,, \\ g^\mathrm{rad}_{31} &= \frac{1}{6}\frac{R_\mathrm{S}}{R^3(\tau)}\,xz\,, & g^\mathrm{rad}_{33} &= -1 - \frac{1}{6}\frac{R_\mathrm{S}}{R^3(\tau)}(x^2 - 2y^2)\,, \end{aligned} \tag{7.7}$$

where (x, y, z) are the Fermi normal coordinates, which – by construction – are initially aligned with the Cartesian coordinate axes (X, Y, Z) of the background Schwarzschild metric and the tetrad vector in x-direction pointing radially outwards from the Earth for $\Phi = 0$. This metric (7.7) for free-falling observers along radial Schwarzschild geodesics is probably the simplest non-trivial case of a Fermi metric and was already exhibited in the well-known paper by Manasse and Misner [188], together with its more compact spherical-coordinate form (7.10). In order to obtain this form, we perform a suitable orthogonal transformation $O^{a'}{}_a$ on the Fermi metric (7.7) in order to diagonalise the spacial part. This transformation happens to be identical with the transformation to polar spherical coordinates $x^{k'} = (r, \theta, \varphi)$ with the x axis as the polar axis, i. e.,

$$x = r\cos\theta\,, \qquad y = r\sin(\theta)\cos(\varphi)\,, \qquad z = r\sin(\theta)\sin(\varphi)\,, \tag{7.8}$$

$O^{a'}{}_a$ is given by the Jacobian matrix of this transformation to spherical coordinates. The spacial part of the metric (7.7) then transforms as,

$$g^{\text{rad}}_{\hat{\imath}'\hat{\jmath}'}(r,\theta,\varphi) = \frac{\partial x^{\hat{k}}}{\partial x^{\hat{\imath}'}}\frac{\partial x^{\hat{l}}}{\partial x^{\hat{\jmath}'}}\, g^{\text{rad}}_{\hat{k}\hat{l}}(x,y,z)\,, \tag{7.9}$$

and we thus obtain the Fermi metric in spherical Fermi normal coordinates [188], also written up to $\mathscr{O}(x^3)$,

$$\begin{aligned}
g^{\text{rad}}_{00} &= 1 - \frac{1}{2}\frac{R_\text{S}}{R}\Big(\frac{r}{R}\Big)^2\big[3\cos^2\theta - 1\big]\\
g^{\text{rad}}_{11} &= -1\\
g^{\text{rad}}_{22} &= -r^2\Big[1 + \frac{1}{6}\frac{R_\text{S}}{R}\Big(\frac{r}{R}\Big)^2\Big]\\
g^{\text{rad}}_{33} &= -(r\sin^2\theta)^2\Big[1 + \frac{1}{6}\frac{R_\text{S}}{R}\Big(\frac{r}{R}\Big)^2(3\cos^2\theta - 2)\Big].
\end{aligned} \tag{7.10}$$

with inverse

$$\begin{aligned}
g_{\text{rad}}^{00} &= 1 + \frac{1}{2}\frac{R_\text{S}}{R}\Big(\frac{r}{R}\Big)^2\big[3\cos^2\theta - 1\big]\\
g_{\text{rad}}^{11} &= -1\\
g_{\text{rad}}^{22} &= -\frac{1}{r^2}\Big[1 - \frac{1}{6}\frac{R_\text{S}}{R}\Big(\frac{r}{R}\Big)^2\Big]\\
g_{\text{rad}}^{33} &= -\frac{1}{(r\sin\theta)^2}\Big[1 - \frac{1}{6}\frac{R_\text{S}}{R}\Big(\frac{r}{R}\Big)^2(3\cos^2\theta - 2)\Big].
\end{aligned} \tag{7.11}$$

At second order in the Fermi coordinates, the coefficients of the Fermi metric in Schwarzschild space-time feature a product of two scales. The first scale is the *curvature scale* R_S/R of the background Schwarzschild metric, and given by the ratio of Schwarzschild radius R_S to Schwarzschild-coordinate radius $R(\tau)$ at the central world-line in terms of proper time τ, as discussed in subsection 5.3.2. In the vicinity of the Earth, its value is as given in (5.41), i.e., $R_\text{S}/R_{\oplus} \approx 10^{-9}$.

The second scale is what we call the *expansion scale*, $r/R(\tau)$, the ratio between radial proper distance from the centre of the world-line to the point in question, and the Schwarzschild coordinate radius.

7.2.2. Fermi Metric for Circular World-Lines and Geodesics

While the frame components of the Riemann tensor (7.6) for the radial free fall frame were almost trivial to obtain since no actual transformation had to be carried out in this special case, transforming the Riemann (i.e. Weyl) tensor to a specific given frame can be quite a tedious task. It is for this reason, that we went to some length in discussing the Petrov classification and the resulting normal forms of the Weyl tensor in certain unique curvature-adapted frames in sections 5.6.3 and 5.6.4, since these provide us with a simple and general starting point.

In subsection 5.6.5 we then showed how to express general Lorentz boosts as linear transformations in bi-vector space adapted to the above normal forms in terms of two Cartesian matrices K and H, which resulted in the general transformation formula (5.161) for the Weyl tensor's electric and magnetic parts E and B, where in Schwarzschild space-time B vanishes and E is given in (5.148).

Circular World-Lines

Using now these transformation relations in (5.161), it is actually straightforward to calculate the Weyl tensor's frame components in the frame (5.98) adapted to circular equatorial world-lines, and then making it parallel-transported by undoing the rotation via (5.105). With our conventions, the resulting components of the Weyl tensor in this *non-rotating* frame read,

$$\mathsf{E}^{\text{circ}}_{11} = \tilde{R}^{\text{circ}}_{\hat{1}\hat{0}\hat{1}\hat{0}} = \frac{R_{\text{S}}\mathit{\Gamma}^2}{2R_0^3}\Big[\tfrac{1}{2}B(R_0)\big(1+3\cos 2\mathit{\Phi}(\tau)\big) + \Big(\frac{\Omega}{c}\Big)^2 (R_0\sin\mathit{\Theta})^2\Big] \tag{7.12a}$$

$$\mathsf{E}^{\text{circ}}_{22} = \tilde{R}^{\text{circ}}_{\hat{2}\hat{0}\hat{2}\hat{0}} = -\frac{R_{\text{S}}\mathit{\Gamma}^2}{2R_0^3}\Big[B(R_0) + 2\Big(\frac{\Omega}{c}\Big)^2 (R_0\sin\mathit{\Theta})^2\Big] \tag{7.12b}$$

$$\mathsf{E}^{\text{circ}}_{33} = \tilde{R}^{\text{circ}}_{\hat{3}\hat{0}\hat{3}\hat{0}} = \frac{R_{\text{S}}\mathit{\Gamma}^2}{2R_0^3}\Big[\tfrac{1}{2}B(R_0)\big(1-3\cos 2\mathit{\Phi}(\tau)\big) + \Big(\frac{\Omega}{c}\Big)^2 (R_0\sin\mathit{\Theta})^2\Big] \tag{7.12c}$$

$$\mathsf{E}^{\text{circ}}_{13} = \tilde{R}^{\text{circ}}_{\hat{1}\hat{0}\hat{3}\hat{0}} = \frac{3R_{\text{S}}}{4R_0^3}\mathit{\Gamma}^2 B(R_0)\sin 2\mathit{\Phi}(\tau) \tag{7.12d}$$

$$\mathsf{B}^{\text{circ}}_{12} = \tilde{R}^{\text{circ}}_{\hat{1}\hat{0}\hat{1}\hat{3}} = \frac{3R_{\text{S}}}{2R_0^2}\mathit{\Gamma}^2\sqrt{B(R_0)}\Big(\frac{\Omega}{c}\Big)\sin(\mathit{\Theta})\cos\mathit{\Phi}(\tau) \tag{7.12e}$$

$$\mathsf{B}^{\text{circ}}_{23} = \tilde{R}^{\text{circ}}_{\hat{2}\hat{0}\hat{1}\hat{2}} = -\frac{3R_{\text{S}}}{2R_0^2}\mathit{\Gamma}^2\sqrt{B(R_0)}\Big(\frac{\Omega}{c}\Big)\sin(\mathit{\Theta})\sin\mathit{\Phi}(\tau)\,, \tag{7.12f}$$

in terms of the Schwarzschild function (5.36), the general orbital coordinate-frequency Ω, and the coordinate-red–shift factor (5.95). These frame components can be seen to reduce to those in the natural static frame of the Schwarzschild metric (7.6), i.e. to (5.148) for $\Omega \to 0$ and $\mathit{\Phi} = 0$. Using (7.12a) with (6.142b) then leads initially to the following expression for the expansion of the metric in Fermi normal coordinates along general circular world-lines,

$$\begin{aligned}
\tilde{g}^{\text{circ}}_{00} = 1 - \frac{R_{\text{S}}}{2R^3}\mathit{\Gamma}^2\Big\{&\Big[\tfrac{1}{2}B(R)\big(1+3\cos 2\mathit{\Phi}(\tau)\big) + \Big(\frac{\Omega}{c}\Big)^2 R^2\sin^2\mathit{\Theta}\Big]x^2 \\
&- \Big[B(R) + 2\Big(\frac{\Omega}{c}\Big)^2 R^2\sin^2\mathit{\Theta}\Big]y^2 \\
&+ \Big[\tfrac{1}{2}B(R)\big(1-3\cos 2\mathit{\Phi}(\tau)\big) + \Big(\frac{\Omega}{c}\Big)^2 R^2\sin^2\mathit{\Theta}\Big]z^2 \\
&+ 3B(R)\sin[2\mathit{\Phi}(\tau)]\,xz\Big\},
\end{aligned}$$

$$\begin{aligned}
\tilde{g}^{\text{circ}}_{11} = -1 + \frac{R_{\text{S}}}{6R^3}\mathit{\Gamma}^2\Big\{&\Big[\tfrac{1}{2}B(R)\big(1-3\cos 2\mathit{\Phi}(\tau)\big) + \Big(\frac{\Omega}{c}\Big)^2 R^2\sin^2\mathit{\Theta}\Big]y^2 \\
&- \Big[B(R) + 2\Big(\frac{\Omega}{c}\Big)^2 R^2\sin^2\mathit{\Theta}\Big]z^2\Big\},
\end{aligned}$$

$$\begin{aligned}
\tilde{g}^{\text{circ}}_{22} = -1 + \frac{R_{\text{S}}}{6R^3}\mathit{\Gamma}^2\Big\{&\Big[\tfrac{1}{2}B(R)\big(1-3\cos 2\mathit{\Phi}(\tau)\big) + \Big(\frac{\Omega}{c}\Big)^2 R^2\sin^2\mathit{\Theta}\Big]x^2 \\
&- \tfrac{3}{2}B(R)\sin[2\mathit{\Phi}(\tau)]\,xz + \Big[\tfrac{1}{2}B(R)\big(1+3\cos 2\mathit{\Phi}(\tau)\big) + \Big(\frac{\Omega}{c}\Big)^2 R^2\sin^2\mathit{\Theta}\Big]z^2\Big\},
\end{aligned}$$

$$
\begin{aligned}
\tilde{g}_{33}^{\mathrm{circ}} &= -1 + \frac{R_{\mathrm{S}}}{12R^3}\varGamma^2\Big\{\Big[\tfrac{1}{2}B(R)(1+3\cos 2\varPhi(\tau)) + \Big(\frac{\Omega}{c}\Big)^2 R^2 \sin^2\varTheta\Big]y^2 \\
&\qquad\qquad - \Big[B(R) + 2\Big(\frac{\Omega}{c}\Big)^2 R^2\sin^2\varTheta\Big]x^2\Big\}, \\
\tilde{g}_{01}^{\mathrm{circ}} &= \frac{R_{\mathrm{S}}}{R^2}\varGamma^2\sqrt{B(R)}\Big(\frac{\Omega}{c}\Big)\sin(\varTheta)\{\cos[\varPhi(\tau)]\,xz - \sin[\varPhi(\tau)]\,y^2\}, \\
\tilde{g}_{02}^{\mathrm{circ}} &= \frac{R_{\mathrm{S}}}{R^2}\varGamma^2\sqrt{B(R)}\Big(\frac{\Omega}{c}\Big)\sin(\varTheta)\{\sin[\varPhi(\tau)]\,yx - \cos[\varPhi(\tau)]\,yz\}, \\
\tilde{g}_{03}^{\mathrm{circ}} &= \frac{R_{\mathrm{S}}}{R^2}\varGamma^2\sqrt{B(R)}\Big(\frac{\Omega}{c}\Big)\sin(\varTheta)\{(y^2-x^2)\cos\varPhi(\tau)\}, \\
\tilde{g}_{23}^{\mathrm{circ}} &= \frac{R_{\mathrm{S}}}{6R^3}\varGamma^2\Big\{\tfrac{3}{2}B(R)\sin[2\varPhi(\tau)]\,xy \\
&\qquad - \Big[\tfrac{1}{2}B(R)(1+3\cos 2\varPhi(\tau)) + \Big(\frac{\Omega}{c}\Big)^2 R^2\sin^2\varTheta\Big]yz\Big\}, \\
\tilde{g}_{31}^{\mathrm{circ}} &= \frac{R_{\mathrm{S}}}{6R^3}\varGamma^2\Big\{\Big[B(R) + 2\Big(\frac{\Omega}{c}\Big)^2 R^2\sin^2\varTheta\Big]xz - \tfrac{3}{2}B(R)\sin[2\varPhi(\tau)]\,y^2\Big\}, \\
\tilde{g}_{12}^{\mathrm{circ}} &= \frac{R_{\mathrm{S}}}{6R^3}\varGamma^2\Big\{\tfrac{3}{2}B(R)\sin[2\varPhi(\tau)]\,yz - \Big[\tfrac{1}{2}B(R)(1-3\cos 2\varPhi(\tau)) + \Big(\frac{\Omega}{c}\Big)^2 R^2\sin^2\varTheta\Big]yx\Big\}.
\end{aligned} \tag{7.13}
$$

The general expression (7.13) for the metric expansion is obviously somewhat more complicated than the corresponding expression for the Fermi metric for radial geodesics (7.7). Much of the complication in the form of the trigonometric functions comes from the additional retrograde rotation in (5.105) with respect to the static Schwarzschild coordinates, which makes the tetrad parallel transported, i.e. *non-rotating* and thus inertially pointing towards, e.g., a distant fixed star, as discussed in subsection 5.5.5 and displayed in our Figure 5.1.

Circular Geodesics

We now restrict ourselves to circular *geodesics*, noting that these approximately model the free fall of satellites in near-circular orbits, such as for the ISS. For these circular geodesics, the orbital *coordinate*-frequency Ω was shown in (5.101) to actually coincide with the Keplerian one, i.e.,

$$
\Omega = \Omega_{\mathrm{Kepler}} = \pm\sqrt{GM/R_0^3}\,.
$$

With this, the Weyl tensor's frame components in the general circular-world–line frame (7.12a) simplify to,

$$
\begin{aligned}
\tilde{R}_{\hat{1}\hat{0}\hat{1}\hat{0}} &= \frac{R_{\mathrm{S}}}{4R^3}\varGamma^2[1+3B(R)\cos 2\varPhi(\tau)]\,, &\quad \tilde{R}_{\hat{2}\hat{0}\hat{2}\hat{0}} &= -\frac{R_{\mathrm{S}}}{2R^3}\varGamma^2\,, \\
\tilde{R}_{\hat{3}\hat{0}\hat{3}\hat{0}} &= \frac{R_{\mathrm{S}}}{4R^3}\varGamma^2[1-3B(R)\cos 2\varPhi(\tau)]\,, &\quad \tilde{R}_{\hat{1}\hat{0}\hat{3}\hat{0}} &= \frac{3R_{\mathrm{S}}}{4R^3}\varGamma^2 B(R)\sin 2\varPhi(\tau)\,, \\
\tilde{R}_{\hat{1}\hat{0}\hat{1}\hat{3}} &= \frac{3R_{\mathrm{S}}}{2R^3}\varGamma^2\sqrt{B(R)}\sqrt{\frac{R_{\mathrm{S}}}{2R}}\cos\varPhi(\tau)\,, &\quad \tilde{R}_{\hat{2}\hat{0}\hat{1}\hat{2}} &= -\frac{3R_{\mathrm{S}}}{2R^3}\varGamma^2\sqrt{B(R)}\sqrt{\frac{R_{\mathrm{S}}}{2R}}\sin\varPhi(\tau)\,,
\end{aligned} \tag{7.14}
$$

which agrees with the results of [218], (who use the other metric sign convention). From its general circular-world–line expression (7.13), we find that the Fermi metric for circular geodesics reduces to,

$$\tilde{g}_{00}^{\mathrm{circ}} = 1 - \frac{R_{\mathrm{S}}\varGamma^2}{4R_0^3}\big\{x^2 - 2y^2 + z^2 + 3B(R_0)\big[(x^2 - z^2)\cos 2\varPhi(\tau) + 2\,xz\sin 2\varPhi(\tau)\big]\big\}, \tag{7.15a}$$

$$\tilde{g}_{11}^{\mathrm{circ}} = -1 + \frac{R_{\mathrm{S}}\varGamma^2}{12R_0^3}\{y^2 - 2z^2 + 3B(R_0)y^2\cos 2\varPhi(\tau)\}, \tag{7.15b}$$

$$\tilde{g}_{22}^{\mathrm{circ}} = -1 + \frac{R_{\mathrm{S}}\varGamma^2}{12R_0^3}\big\{x^2 + z^2 + 3B(R_0)\big[(z^2 - x^2)\cos 2\varPhi(\tau) - 2\,xz\sin 2\varPhi(\tau)\big]\big\}, \tag{7.15c}$$

$$\tilde{g}_{33}^{\mathrm{circ}} = -1 + \frac{R_{\mathrm{S}}\varGamma^2}{12R_0^3}\{y^2 - 2x^2 + 3B(R_0)y^2\cos 2\varPhi(\tau)\}, \tag{7.15d}$$

$$\tilde{g}_{01}^{\mathrm{circ}} = \frac{R_{\mathrm{S}}\varGamma^2}{R_0^3}\sqrt{B(R_0)}\sqrt{\frac{R_{\mathrm{S}}}{2R_0}}\big[xz\cos\varPhi(\tau) + (z^2 - y^2)\sin\varPhi(\tau)\big], \tag{7.15e}$$

$$\tilde{g}_{02}^{\mathrm{circ}} = \frac{R_{\mathrm{S}}\varGamma^2}{R_0^3}\sqrt{B(R_0)}\sqrt{\frac{R_{\mathrm{S}}}{2R_0}}y\,\big[x\sin\varPhi(\tau) - z\cos\varPhi(\tau)\big], \tag{7.15f}$$

$$\tilde{g}_{03}^{\mathrm{circ}} = -\frac{R_{\mathrm{S}}\varGamma^2}{R_0^3}\sqrt{B(R_0)}\sqrt{\frac{R_{\mathrm{S}}}{2R_0}}\big[(x^2 - y^2)\cos\varPhi(\tau) + xz\sin\varPhi(\tau)\big], \tag{7.15g}$$

$$\tilde{g}_{23}^{\mathrm{circ}} = -\frac{R_{\mathrm{S}}\varGamma^2}{12R_0^3}\big\{yz + 3B(R_0)\big[yz\cos 2\varPhi(\tau) - xy\sin 2\varPhi(\tau)\big]\big\}, \tag{7.15h}$$

$$\tilde{g}_{31}^{\mathrm{circ}} = \frac{R_{\mathrm{S}}\varGamma^2}{12R_0^3}\{2xz - 3B(R_0)y^2\sin 2\varPhi(\tau)\}, \tag{7.15i}$$

$$\tilde{g}_{12}^{\mathrm{circ}} = -\frac{R_{\mathrm{S}}\varGamma^2}{12R_0^3}\big\{xy - B(R_0)\big[3xy\cos 2\varPhi(\tau) + yz\sin 2\varPhi(\tau)\big]\big\}, \tag{7.15j}$$

where the geodesic coordinate-red–shift factor $\varGamma$ is given by (5.102).

7.3. Non-Linear Klein-Gordon Equation in Fermi Coordinates

In order to evaluate the non-linear Klein-Gordon equation (7.4) in Fermi normal coordinates, it is extremely convenient to represent the Fermi expansion of the metric and its inverse in terms of the general perturbation expansions in subsection 5.7.2, with the Minkowski metric playing the role of background metric $\bar{g}_{\alpha\beta}$. In what follows, we shall initially keep $\bar{g}_{\alpha\beta}$ and its determinant $\bar{g}$ general, since this allows one to easily describe also non-Cartesian Fermi coordinates, in which the Minkowski metric is no longer constant, before we specialise to $\bar{g}_{\alpha\beta} = \eta_{\alpha\beta}$ in the end.

Thus in the present context, the tensor ${h^{(n)}}_{\alpha\beta}$ would then simply be an abbreviation for the n^{th}-order term in the Fermi metric, and we note that before actually specifying ${h^{(n)}}_{\alpha\beta}$ to be a term in the Fermi expansion of the metric, our result will equally be valid also for the case that it represents a true perturbation of space-time around the flat Minkowski metric, independently of any Fermi coordinate expansion. This approach has two benefits, the first being that it keeps the notation concise and avoids cluttering our general calculation of the non-linear Klein-Gordon equation with Riemann tensors and numerical factors. The second benefit is, that the expansion of most derived quantities like the inverse metric, the Christoffels, the metric determinant, etc. can be calculated straightforwardly in terms of ${h^{(n)}}_{\alpha\beta}$ with this approach, i.e., all that one really needs to know explicitly is the expansion of the metric itself.

In describing a free-falling BEC, we are of course implicitly employing a co-moving inertial frame, so that – by the equivalence principle – any first-order contribution ${h^{(1)}}_{\alpha\beta}$ in the Fermi expansion of the metric is absent. Thus, we can further restrict ourselves to the first-order formulae of subsection 5.7.2, by taking the second-order terms of the metric expansion as our actual only first-order "perturbation", $h_{\alpha\beta} \equiv {h^{(2)}}_{\alpha\beta}$. Thus, we write the Fermi metric expansion to second order as,

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + h_{\alpha\beta} + \mathscr{O}(h^2)\,, \qquad \text{and} \quad g^{\alpha\beta} = \bar{g}^{\alpha\beta} - \bar{g}^{\alpha\gamma}\bar{g}^{\beta\delta}h_{\gamma\delta} + \mathscr{O}(h^2)\,, \tag{7.16}$$

in terms of the spacial Fermi coordinates (6.12), in which we then have,

$$h_{\alpha\beta} = h_{\alpha\beta}(x) = \mathscr{O}(x^2)\,. \tag{7.17}$$

At the end, we will then split the indices of $h_{\alpha\beta}$ into time and space parts and insert its different Fermi-coordinate components, i.e.,

$$h_{00}(x) = R_{\hat{0}\hat{\imath}_1\hat{\imath}_2\hat{0}}x^{\hat{\imath}_1}x^{\hat{\imath}_2}\,, \qquad h_{0k}(x) = \tfrac{2}{3}R_{\hat{0}\hat{\imath}_1\hat{\imath}_2\hat{k}}x^{\hat{\imath}_1}x^{\hat{\imath}_2}\,, \qquad h_{k_1k_2}(x) = \tfrac{1}{3}R_{\hat{k}_1\hat{\imath}_1\hat{\imath}_2\hat{k}_2}x^{\hat{\imath}_1}x^{\hat{\imath}_2}\,. \tag{7.18}$$

Proceeding now with the expansion of the non-linear Klein-Gordon equation (7.4) in Fermi coordinates, we first turn to the term $\Gamma_\alpha = \partial_\alpha \ln(\sqrt{-g}\,)$ coming from the contracted Christoffel symbol in (7.3). In the "perturbation" approach, the expansion of the metric determinant g is easily calculated. We have,

$$g = \bar{g}\big[1 + h + \mathscr{O}(h^2)\big]\,, \qquad \text{with } h \coloneqq \bar{g}^{\gamma\delta}h_{\gamma\delta}\,, \tag{7.19}$$

where $\bar{g} \coloneqq \det(\bar{g}_{\alpha\beta})$ is the determinant of the Minkowski metric in general coordinates and non-inertial frames, and h the trace of the metric "perturbation" introduced in subsection 5.7.2. From equation (7.19) we obtain the expansion of the contracted connection coefficient (7.3),

$$\Gamma_\sigma = \partial_\sigma\big[\ln(\sqrt{-\bar{g}}\,) + \tfrac{1}{2}h + \mathscr{O}(h^2)\big]. \tag{7.20}$$

The non-linear Klein-Gordon equation (7.4), more precisely its kinetic term, then splits into a Minkowski part and a "perturbation" part,

$$\begin{aligned} &\bar{g}^{\alpha\beta}\{\partial_\alpha\partial_\beta + [\partial_\alpha \ln(\sqrt{-\bar{g}})]\partial_\beta\}\phi + \tfrac{1}{2}\bar{g}^{\alpha\beta}(\partial_\alpha h)\partial_\beta\phi \\ &\qquad - h^{\alpha\beta}\{\partial_\alpha\partial_\beta + [\partial_\alpha \ln(\sqrt{-\bar{g}}\,)]\partial_\beta\}\phi + \Big(\frac{mc}{\hbar}\Big)^2\phi + \xi|\phi|^2\phi = 0\,. \end{aligned} \tag{7.21}$$

where we have kept only terms of first order in $h^{\alpha\beta}$ and h. In preparation for carrying out the non-relativistic limit, the next step is to perform the 1+3 split into time and space by expanding out $\bar{g}^{\alpha\beta}$ and the metric "perturbation" $h^{\alpha\beta}$ in terms of their time–time,

time–space and space–space components $\bar{g}^{00}$, $\bar{g}^{0j}$, $\bar{g}^{ij}$ and h^{00}, h^{0j}, h^{ij}, respectively. In order not to clutter our equations too much, we shall temporarily focus only on the terms in (7.21) that come from the differential operator $\nabla^\alpha \nabla_\alpha \phi$, i.e. on the kinetic term (7.2). Carrying out the split initially triples the number of terms, as can be expected, i.e. we have,

$$\begin{aligned}\nabla^\alpha \nabla_\alpha \phi = {} & \bar{g}^{00}\big(\tfrac{1}{c}\big)^2\Big[\ddot{\phi} + (\partial_t \ln\sqrt{-\bar{g}}\,)\dot{\phi}\Big] + \bar{g}^{ij}\,[\partial_i\partial_j + (\partial_i \ln\sqrt{-\bar{g}}\,)\partial_j]\phi \\ & + \bar{g}^{0i}\tfrac{1}{c}\Big[2\partial_i\dot{\phi} + (\partial_i \ln\sqrt{-\bar{g}}\,)\dot{\phi} + (\partial_t \ln\sqrt{-\bar{g}}\,)\partial_i\phi\Big] + \tfrac{1}{2}\bar{g}^{00}\big(\tfrac{1}{c}\big)^2\dot{h}\dot{\phi} \\ & + \tfrac{1}{2}\bar{g}^{0i}\tfrac{1}{c}\,[\dot{h}\partial_i\phi + (\partial_i h)\dot{\phi}] - h^{00}\big(\tfrac{1}{c}\big)^2\Big[\ddot{\phi} + (\partial_t \ln\sqrt{-\bar{g}}\,)\dot{\phi}\Big] \\ & - h^{0i}\tfrac{1}{c}\Big[2\partial_i\dot{\phi} + (\partial_i \ln\sqrt{-\bar{g}}\,)\dot{\phi} + (\partial_t \ln\sqrt{-\bar{g}}\,)\partial_i\phi\Big] \\ & - h^{ij}\Big[\partial_i\partial_j + (\partial_i \ln\sqrt{-\bar{g}}\,)\partial_j\Big]\phi\,. \end{aligned} \tag{7.22}$$

We now assume that the Minkowski metric in general coordinates $\bar{g}^{\alpha\beta}$ is time-independent, which means that we are in an inertial frame, and, in particular that – in orthogonal coordinates, to which we restrict ourselves – it will also be diagonal. In this context, the inverse metric's time–time component also becomes unity, $\bar{g}^{00} = 1$. With these simplifications, the kinetic term (7.22) becomes, much more manageable and we have,

$$\begin{aligned}\nabla^\alpha \nabla_\alpha \phi = {} & (1 - h^{00})\big(\tfrac{1}{c}\big)^2\ddot{\phi} + \tfrac{1}{2}\big(\tfrac{1}{c}\big)^2\dot{h}\dot{\phi} - h^{0i}\tfrac{1}{c}\,[2\partial_i\dot{\phi} + (\partial_i \ln\sqrt{-\bar{g}}\,)\dot{\phi}\,] \\ & + (\bar{g}^{ij} - h^{ij})\,[\partial_i\partial_j + (\partial_i \ln\sqrt{-\bar{g}}\,)\partial_j]\phi\,. \end{aligned} \tag{7.23}$$

With this, the non-linear Klein-Gordon equation (7.21) in 1+3 split is now much more compact and given by,

$$\begin{aligned} & (1 - h^{00})\big(\tfrac{1}{c}\big)^2\ddot{\phi} + \tfrac{1}{2}\big(\tfrac{1}{c}\big)^2\dot{h}\dot{\phi} - h^{0i}\tfrac{1}{c}\,[2\partial_i\dot{\phi} + (\partial_i \ln\sqrt{-\bar{g}}\,)\dot{\phi}\,] \\ & \quad + (\bar{g}^{ij} - h^{ij})\,[\partial_i\partial_j + (\partial_i \ln\sqrt{-\bar{g}}\,)\partial_j]\phi + \Big(\frac{mc}{\hbar}\Big)^2\phi + \xi|\phi|^2\phi = 0\,. \end{aligned} \tag{7.24}$$

We now proceed to performing the non-relativistic limit in the usual way, by separating off the phase associated with the rest energy $E_0 = mc^2$ in terms of the ansatz (2.12), i.e.,

$$\phi = \psi\,\exp\Big(-i\frac{mc^2}{\hbar}\tau\Big)\,. \tag{7.25}$$

Using the first and second time derivatives of the relativistic field ϕ in terms of the non-relativistic wave function ψ in equations (2.14), we initially obtain,

$$\begin{aligned} & (1 - h^{00})\Big[\big(\tfrac{1}{c}\big)^2\ddot{\psi} - 2i\Big(\frac{m}{\hbar}\Big)\dot{\psi} - \Big(\frac{mc}{\hbar}\Big)^2\psi\Big] + \tfrac{1}{2}\dot{h}\Big[\big(\tfrac{1}{c}\big)^2\dot{\psi} - i\Big(\frac{m}{\hbar}\Big)\psi\Big] \\ & \quad - h^{0i}\Big\{2\Big[\tfrac{1}{c}\partial_i\dot{\psi} - i\Big(\frac{mc}{\hbar}\Big)\partial_i\psi\Big] + (\partial_i \ln\sqrt{-\bar{g}}\,)\Big[\tfrac{1}{c}\dot{\psi} - i\Big(\frac{mc}{\hbar}\Big)\psi\Big]\Big\} \\ & \quad + (\bar{g}^{ij} - h^{ij})\,[\partial_i\partial_j + (\partial_i \ln\sqrt{-\bar{g}}\,)\partial_j]\psi + \Big(\frac{mc}{\hbar}\Big)^2\psi + \xi|\psi|^2\psi = 0\,, \end{aligned} \tag{7.26}$$

which we now rearrange and where we multiply through with $-\frac{\hbar^2}{2m}$. This reduces the non-linear Klein-Gordon equation to,

$$\begin{aligned}(1-h^{00})i\hbar\dot{\psi} = \frac{\hbar^2}{2m}\Big\{&(\bar{g}^{ij}-h^{ij})\big[\partial_i\partial_j + (\partial_i \ln\sqrt{-\bar{g}}\,)\partial_j\big]\psi + \tfrac{1}{2}\dot{h}\big[(\tfrac{1}{c})^2\dot{\psi} - i\Big(\frac{m}{\hbar}\Big)\psi\big] \\ &- h^{0i}\{2\big[\tfrac{1}{c}\partial_i\dot{\psi} - i\Big(\frac{mc}{\hbar}\Big)\partial_i\psi\big] + (\partial_i \ln\sqrt{-\bar{g}}\,)\big[\tfrac{1}{c}\dot{\psi} - i\Big(\frac{mc}{\hbar}\Big)\psi\big]\} \\ &+ (1-h^{00})\ddot{\psi}\Big\} + V_{\mathrm{tidal}}\psi - \frac{\hbar^2\xi}{2m}|\psi|^2\psi\,, \end{aligned} \tag{7.27}$$

where the Klein-Gordon mass term has cancelled, and where a tidal potential,

$$V_{\mathrm{tidal}} = \frac{mc^2}{2}h^{00} = \frac{mc^2}{2}h_{00}\,, \tag{7.28}$$

appears. The fact that in the non-relativistic limit one generally obtains the full relativistically correct tidal potential,

$$V_{\mathrm{tidal}} = \frac{mc^2}{2}(g_{00}-1)\,, \tag{7.29}$$

from the g_{00} component of the Fermi metric, which of course includes the Newtonian part, is a major benefit of employing Fermi normal coordinates.

At this point we are also going to get rid of the term with $\frac{1}{c}\dot{h}$ in (7.27). Since we have that $h \propto R_{\hat{\alpha}\hat{\imath}_1\hat{\imath}_2\hat{\alpha}}x^{\hat{\imath}_1}x^{\hat{\imath}_2}$, this means that this term should essentially be regarded as a partial time derivative of the Riemann tensor,

$$\frac{1}{c}\dot{h} \propto R_{\hat{\alpha}\hat{\imath}_1\hat{\imath}_2\hat{\alpha}\,;\hat{0}}x^{\hat{\imath}_1}x^{\hat{\imath}_2}\,. \tag{7.30}$$

If we agree to treat spacial Riemann tensor derivatives on the same footing as time ones for consistency reasons, this term would thus correspond to a third-order term in the Fermi normal coordinate expansion of the metric, an order that we have neglected from the start. Another term that we have also routinely neglected already in the spacial-relativistic case in section 2.3, is the second time derivative $(\frac{1}{c})^2\ddot{\psi}$, since we assume that we are in the deeply non-relativistic limit with our ultracold quantum gas. Thus, dropping these terms, we can further divide through with the red-shift factor $(1-h^{00})$ on the left-hand side of equation (7.27) and re-linearise the right-hand side, in order to bring the result into a more Gross–Pitaevskii-like form. In terms of the non-Relativistic Laplacean, defined by

$$\boldsymbol{\nabla}^2\psi = -\bar{g}^{ij}\big[\partial_i\partial_j + (\partial_i \ln\sqrt{-\bar{g}}\,)\partial_j\big]\psi\,, \tag{7.31}$$

this yields the general expression for the non-relativistic limit of our non-linear Klein-Gordon, to second order in the Fermi normal coordinates,

$$\begin{aligned} i\hbar\dot{\psi} = &-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\psi + (1+h^{00})\frac{\hbar^2\xi}{2m}|\psi|^2\psi + V_{\mathrm{tidal}}\psi \\ &+ \frac{\hbar^2}{2m}\Big\{(h^{00}\bar{g}^{ij}-h^{ij})\big[\partial_i\partial_j + (\partial_i \ln\sqrt{-\bar{g}}\,)\partial_j\big]\psi \end{aligned}$$

$$- h^{0i}\{2[\tfrac{1}{c}\partial_i\dot{\psi} - i\Big(\frac{mc}{\hbar}\Big)\partial_i\psi] + (\partial_i \ln\sqrt{-\bar{g}}\,)[\tfrac{1}{c}\dot{\psi} - i\Big(\frac{mc}{\hbar}\Big)\psi]\}\Big\}\,. \tag{7.32}$$

We see that the above re-linearisation only leads to two changes on the right-hand side of (7.32) as compared to equation (7.27): The h^{00}-correction from the red-shift factor combines with the Minkowski part of the kinetic operator, which we could combine to the correction term with $(h^{00}\bar{g}^{ij} - h^{ij})$ in the second line of (7.32), and additionally, the non-linearity now also picks up a correction. All other terms are of higher order in the "perturbation" $h^{\alpha\beta}$ and were thus neglected.

In a final step, we now specialise the above general expressions to the usual Cartesian Fermi coordinates $x^{\hat{\alpha}} = (c\tau, x, y, z)$, which means that we take $\bar{g}^{ij} = \eta^{ij}$. Since all logarithmic derivatives of the square root of $\bar{g} = \eta$ vanish, equation (7.32) can be further simplified. This yields,

$$\begin{aligned} i\hbar\dot{\psi} &= -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\psi + [1 + h^{00}(x)]\frac{\hbar^2\xi}{2m}|\psi|^2\psi + V_{\text{tidal}}\psi \\ &\quad + \frac{\hbar^2}{2m}\Big\{[h^{00}(x)\eta^{ij} - h^{ij}(x)]\partial_i\partial_j\psi - 2h^{0i}(x)\Big[\tfrac{1}{c}\partial_i\dot{\psi} - i\Big(\frac{mc}{\hbar}\Big)\partial_i\psi\Big]\Big\}\,. \end{aligned} \tag{7.33}$$

Finally, we may write the Hamiltonian of (7.33) as a sum of the Gross-Pitaevskii Hamiltonian $\hat{H}_0$ and a perturbation $\hat{H}_1$ as,

$$i\hbar\,\dot{\psi} = [\hat{H}_0 + \hat{H}_1]\psi \tag{7.34}$$

where $\hat{H}_0$ and $\hat{H}_1$ are given, respectively, by

$$\begin{aligned} \hat{H}_0 &= -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + \frac{4\pi\hbar^2 a_{\text{s}}}{m}|\psi|^2 + \bar{V}_{\text{tidal}}\,, \\ \hat{H}_1 &= \frac{\hbar^2}{2m}\{[h^{00}(x)\eta^{ij} - h^{ij}(x)]\partial_i\partial_j\psi - 2h^{0i}(x)[\tfrac{1}{c}\partial_i\dot{\psi} - i\Big(\frac{mc}{\hbar}\Big)\partial_i\psi]\} \\ &\qquad + \delta V_{\text{tidal}}(x) + h^{00}(x)\frac{4\pi\hbar^2 a_{\text{s}}}{m}|\psi|^2\,. \end{aligned} \tag{7.35}$$

and where we have also identified $\xi = 8\pi a_{\text{s}}$. Above, we have split the tidal potential into its Newtonian part and a part coming from the re-linearisation in the non-linear Klein-Gordon equation as,

$$\bar{V}_{\text{tidal}} = \bar{V}_{\text{tidal}} + \delta V_{\text{tidal}}\,. \tag{7.36}$$

7.3.1. Evaluation for Radial Free-Fall Observers

The Fermi metric in the Radial free-fall frame (7.7) is simple enough that we can write down the actual correction terms that appear in the Gross-Pitaevskii equation, in particular the time–space part of the Fermi metric happens to be diagonal in this case, so that we have $h^{i0}_{\text{rad}} = 0$. The only surviving correction terms are then,

$$h^{00}_{\text{rad}}(\tau, x) = R_{\hat{0}\hat{\imath}_1\hat{\imath}_2\hat{0}}(\tau)x^{\hat{\imath}_1}x^{\hat{\imath}_2} = -\frac{R_{\text{S}}}{R^3(\tau)}(2x^2 - y^2 - z^2)\,, \tag{7.37a}$$

$$(h^{00}_{\mathrm{rad}}\eta^{ij} - h^{ij}_{\mathrm{rad}}) = \frac{1}{6}\frac{R_{\mathrm{S}}}{R^3}\begin{pmatrix} 6x^2 - 2y^2 - 2z^2 & -yx & -xz \\ -yx & 7x^2 - 3y^2 - 5z^2 & 2yz \\ -xz & 2yz & 7x^2 - 5y^2 - 3z^2 \end{pmatrix}. \tag{7.37b}$$

7.3.2. Type and Magnitude of Corrections

Because it is a special case as discussed above, the Fermi metric for purely radial geodesics is in fact not very interesting. For this reason, we shall now briefly turn to the metric for circular geodesics instead Since the Fermi metric in the circular geodesic frame (7.15a) is – in contrast to the radial one – already fairly complicated, with many non-diagonal components, and exhibits in particular now also cross-terms between time and space, we refrain from inserting it into our result (7.33) for the non-relativistic limit of the non-linear Klein-Gordon equation. Instead we shall focus on the more compact and interesting tidal potential that arises from this metric.

"Length Contracted" Tidal Potential in the Circular-Geodesic Frame

As discussed above, the relativistic tidal potential (7.29) follows from the time–time component of the Fermi metric. For the observer on circular geodesics in Schwarzschild space-time, $\tilde{g}^{\mathrm{circ}}_{00}$ from equations (7.15a) yields,

$$V_{\mathrm{tidal}} = \frac{GMm}{4R_0^3}\Gamma^2\{2x^2 - z^2 - y^2 - 3B(R_0)[(z^2-y^2)\cos 2\Phi(\tau) + yz\sin 2\Phi(\tau)]\}\,. \tag{7.38}$$

We now simplify (7.38) by considering short times, $\tau \ll 1/(\Gamma\Omega)$, so that we can linearise the cos and sin functions, and also expand the coordinate red-shift factor Γ of equation (5.102), as well as the Schwarzschild function $B(R_0)$ to lowest order in the dimensionless curvature scale R_{S}/R_0 of Schwarzschild space-time. This yields,

$$V_{\mathrm{tidal}} \approx -\frac{GMm}{2R_0^3}\left\{2z^2 - x^2 - y^2 + \frac{3}{2}\frac{R_{\mathrm{S}}}{R_0}(z^2 - x^2) + \mathscr{O}\left[(R_{\mathrm{S}}/R_0)^2\right]\right\}, \tag{7.39}$$

so at zeroth order, we have the usual Newtonian tidal potential,

$$\bar{V}_{\mathrm{tidal}} = -\frac{GMm}{4R_0^3}(2z^2 - x^2 - y^2)\,. \tag{7.40}$$

To second order in Fermi normal coordinates, the classical $\bar{V}_{\mathrm{tidal}}$ is clearly a normal, i.e. attractive, harmonic oscillator potential in the x and y directions, whereas it is an inverted, repulsive one in the z direction. Picking out the the x-direction, its angular frequency ω_x and corresponding oscillator length a^{osc}_x are of the order of,

$$\omega_x = \sqrt{\frac{GM}{R_0^3}} \approx 2\pi\cdot 1\,\mathrm{mHz}\,, \qquad a^{\mathrm{osc}}_x = \sqrt{\frac{\hbar}{m\omega_x}} \approx 800\,\mu\mathrm{m}\,,$$

the measurement of which seems to be at the very edge of current experimental possibilities. The lowest-order *relativistic* correction to the Newtonian part in V_{tidal} for circular geodesics is given by,

$$\delta V_{\text{tidal}} = -\frac{3}{4}\frac{GMm}{R_0^3}\frac{R_\text{S}}{R_0}(z^2 - x^2) + \mathscr{O}\big[(R_\text{S}/R_0)^2\big]\,. \tag{7.41}$$

This terms can actually be seen to be a special-relativistic velocity correction due to the frame's Keplerian orbital velocity Ω_{Kepler} in (7.2.2), by tracing it back e.g. to (7.13). One can easily convince oneself that these types of special-relativistic velocity corrections generically appear in the fully relativistic tidal potential as obtained from the Fermi metric according to (7.29), for general, i.e. non-radial orbital motion in Petrov type D metrics, such as Schwarzschild and Kerr. One can also argue that these tidal potentials should also be gauge invariant, i.e. not subject to the possibility of performing a linearised coordinate transformation, since the time–time component g_{00} of the Fermi metric should have this property (in contrast to g_{0i} and g_{ij}); compare to the related discussion in [219].

To conclude this brief discussion, one interestingly finds that in expansions "beyond the equivalence principle", such as in terms of Fermi normal coordinates to second order and higher, the tidal potentials in the non-relativistic limit also get "length contracted", due to the Lorentz boost that was originally applied to the Riemann tensor's frame components in order to adapt it from the initial metric-adapted static Schwarzschild frame to the target frame that moves with a certain orbital velocity.

Order of Magnitude of Residual Curvature Corrections in $\hat{H}_1$

The general problem of the arbitrariness of coordinates in General Relativity, that we briefly discussed in our subsection 5.3.2 on the Schwarzschild metric, clearly also appears in the context of expansions, such as with Fermi normal coordinates, and equally in metric perturbation theory, where it is referred to as the gauge problem. As noted above, the residual curvature corrections coming from the mixed and purely spacial components of the Fermi metric cannot be expected to be gauge invariant, i.e. to have any physical meaning, and thus they cannot ultimately be trusted.

That said, we can nevertheless provide the order-of-magnitude of these curvature corrections appearing in $\hat{H}_1$. For an extremely large spherical BEC ($r = 1\,\text{cm}$) at the surface of Earth ($R = R_{\oplus} \approx 6 \times 10^6\,\text{m}$) these residual time and spacial coordinate corrections are of order

$$\frac{R_\text{S}}{2R_{\oplus}}\Big(\frac{r}{R_{\oplus}}\Big)^2 \approx 4 \times 10^{-27}\,.$$

8 Conclusions and Outlook

In this thesis we have pursued the question of a full general-relativistic description for local free-falling experiments in the weak gravitational field of the Earth, and in particular of a general relativistic extension of the usual mean-field description of Bose-Einstein condensates in terms of the Gross-Pitaevskii equation. This question was stipulated by the experimental quantum project QUANTUS, which has put quantum gases in drop towers and into extended free fall in a recent sounding rocket mission, during which the first BECs in space were created.

During our long work on this subject, we have used the chance to gain an extensive and in-depth knowledge of all the different aspects of the theories of Special and General Relativity, that play a role for the general question above. Recognising the fundamental importance and experimental relevance of local inertial and non-inertial frames in which essentially all physics takes place, we have chosen to base our theoretical description on these frames and their attached Fermi normal coordinates, which capture the tidal gravitational physics in terms of a curvature expansion around flat space-time.

In particular, we have spent a considerable amount of time in order to thoroughly understand the Riemann and Fermi normal expansions, which resulted in our extensive chapter 6 on this subject. There, we were able to find an interesting new interpretation for the expansion coefficients of the inverse tetrad and the metric, which could aid in unifying the whole subject of these expansions under a common combinatorial framework.

We have also developed an extremely convenient approach to the generally very tedious calculation of the Weyl tensor's frame components in frames adapted to arbitrarily moving observers in exact vacuum space-times such as Schwarzschild and Kerr. Note that these calculations always arise in the explicit evaluation of Fermi metrics, where the Weyl tensor's components in the chosen frame appear generically at second order. This approach, which we introduced in section 5.6, and which seems to have been largely unnoticed previously, starts from the unique curvature-adapted frame, that is connected to the Petrov classification of vacuum space-times, and in which the Weyl tensor can be represented in terms of two Cartesian matrices which are trace-less and diagonal for the relevant space-times

In order to give a (preliminary) answer to the initial questions asked above, in chapter 7, we have considered a relativistic generalisation of the usual Gross-Pitaevskii equation in terms of the non-linear Klein-Gordon equation in curved space-time, expanding it in terms of a convenient perturbation approach. This enabled us to compactly perform the non-relativistic limit of this relativistic wave equation, while at the same time keeping our calculations as general and systematic as possible. Calculating the Fermi metrics for BECs in free fall along purely radial and circular equatorial geodesics in Schwarzschild space-time, we used these metrics with our expansion approach to exhibited the different types of local tidal corrections to flat-space-time physics that appear in the Gross-Pitaevskii equation.

Outlook

During our work on this thesis, we have naturally encountered a number of subjects and directions that suggest themselves for further research, and which have either not made it into the present thesis, and/or would have exceeded its scope too much. In this short last section, we would like to provide an outlook to these and list some literature references.

First would be the experimentally relevant continuation of this thesis' subject, i.e. extending our Fermi-coordinate description, for which we have restricted ourselves mostly to exact background metrics, also to approximate metrics in terms of first-order metric perturbation theory around flat space-time, expressed with the tools of the post-Minkowskian and (parametrised) post-Newtonian expansions. As discussed and motivated in section 5.7, only these systematic expansions can ultimately provide a totally realistic description of the weak, but complicated and multi-polar tidal-type space-time curvature around the Earth and within the solar system, which includes multiple sources (Earth, Moon, Sun, planets, etc.) that are each described in terms of their own sets of relativistic multipole moments.

A second project, on which we have already spent quite some time but which we could not finish, is the calculation of the classical centre-of-mass phase shift in a matter-wave interferometer, described within Fermi normal coordinates, by explicitly expanding the point-particle action along the atomic clouds' centre-of-mass geodesics, and treating the light forces acting on the atoms during the laser beam-splitter and mirror pulses to lowest order as active Lorentz boosts.

Lastly, we would like to mention two more ambitious potential projects. The first of these is in extension of our discussion of multi-polar equations of motion in subsection 5.4.1 and subsection 5.4.2, namely to use not the scalar non-linear Klein-Gordon equation for the description of Bose-Einstein condensate in curved space-time, but instead a suitable covariant higher-spin wave equation. While in chapter 7 we only treat Bose-Einstein condensates in terms of a relativistic *scalar field*, in reality the atomic species that BECs are created with experimentally have a non-zero spin, for example, the "working horse" ^{87}Rb has nuclear spin $3/2$ and electron spin $1/2$, coupling to a total spin of $F = 1$ or $F = 2$, of which only the high-field-seeking $F = 2$ species can be reliably trapped. A convenient, suitably general, and systematic starting point for this seems to be the so-called Joos-Weinberg equation (cf. Joos [220] and Weinberg [221–223], as well as Jeffery [224]).

As a final direction for future research – and probably the most ambitious of all – it would be interesting to try and express the quantum mechanical propagator in curved space-time in terms of Fermi coordinates. In order to start from a firm footing, one would use its path-integral representation and attempt to express this in terms of arbitrary geodesics within Fermi coordinates. In this context, see the works by Bekenstein and Parker [225, 226], and also Singh and Mobed [227–229] for an interesting approach to the scalar-particle path integral in terms of Lie transport and Riemann and Fermi normal coordinates.

A 2 + 2 Decomposition of a Spherically Symmetric Space-Time and Closed-Form of Weyl Tensor

In the following we show how in spherical symmetry, i.e. in Schwarzschild space-time, one can make use of an elegant formalism in terms of a $2+2$ split of the manifold adapted to the symmetry, to calculate simple closed forms of the Weyl tensor and its first covariant derivative in a coordinate-invariant fashion. This type of decomposition is well known and the resulting manifold is generally called a *warped-product*, since it is a product manifold up to a scalar function, which "warps" it. A spherically symmetric space-time $\mathcal{M}$ can be decomposed as the product $\mathcal{M} = \mathrm{M}^2 \times \mathbb{S}^2$, where M^2 is the two-dimensional Lorentzian (T–R) manifold and $\mathbb{S}^2$ is the two-sphere. Note that in this appendix, we use upper-case Latin letters, $A, B, C, \ldots \in \{0, 1\}$ for indices in M^2, and lower case ones, i.e., $a, b, c, \ldots \in \{2, 3\}$ for indices on $\mathbb{S}^2$. Using a coordinate system $(X^\mu) = \big(X^A, X^a\big)$ that is adapted to the $\mathbb{S}^2$ orbits of spherical symmetry, the two coordinates $X^A = X^1, X^2$ of the $1+1$ Lorentzian sub-manifold M^2 are *scalars* in terms of coordinate transformations confined to $\mathbb{S}^2$, and vice versa, $X^a = X^2, X^3$, which can be thought of as angular coordinates, are scalars with respect to M^2. The metric splits as,

$$\begin{aligned} g_{\mu\nu}(X^\sigma)\, dX^\mu dX^\nu &= g_{AB}\, dX^A dX^B + g_{ab}\, dX^a dX^b \\ &= g_{AB}\, dX^A dX^B - f(X^C)^2 \gamma_{ab}\, dX^a dX^b\,, \end{aligned} \tag{A.1}$$

where g_{AB} is the metric on M^2, and

$$g_{ab} = -f(X^C)^2\, \gamma_{ab}\,, \tag{A.2}$$

is the metric on the warped manifold, which is written in terms of the metric on the two-sphere $\mathbb{S}^2$,

$$\gamma_{ab} \equiv \gamma_{ab}(X^c) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \Theta \end{pmatrix}\,. \tag{A.3}$$

Here, $f(X^C)$ is the above-mentioned *warp function*, which multiplies the spherical sector but depends on the coordinates in M^2.

The Christoffel symbols are then straightforward to calculate with the usual definition (3.75). We leave $\Gamma^C{}_{AB}$ general, since we do not specify the metric of the submanifold M^2 at this point. The same applies to $\Gamma^c{}_{ab}$ on $\mathbb{S}^2$, although it is not difficult to calculate, as the metric (A.3) is diagonal and contains only one nontrivial component. Thus we have,

$$\Gamma^C{}_{AB} = \tfrac{1}{2} g^{CD} \left[\partial_A g_{BD} + \partial_B g_{AD} - \partial_D g_{AB}\right], \tag{A.4a}$$

$$\Gamma^c{}_{ab} = \tfrac{1}{2}\gamma^{cd}\left[\partial_a\gamma_{bd} + \partial_b\gamma_{ad} - \partial_d\gamma_{ab}\right] . \tag{A.4b}$$

The non-zero mixed Christoffels contain the gradient of the warp function $f_A \coloneqq \widetilde{\nabla}_A f$,

$$\Gamma^c{}_{Ab} = f^{-1}\delta^c{}_b\, f_A \,, \tag{A.4c}$$
$$\Gamma^C{}_{ab} = f\gamma_{ab} f^C = -f^{-1} g_{ab} f^C \,, \tag{A.4d}$$
$$\Gamma^C{}_{Ab} = 0 = \Gamma^c{}_{AB} \,. \tag{A.4e}$$

Since the Riemann tensor of a two dimensional manifold is determined only by the latter's Ricci scalar (or equivalently by its Gaussian curvature), we use the closed form (5.119) for $R^C{}_{DAB}$ and $R^c{}_{dab}$,

$$R^C{}_{DAB} = \tfrac{1}{2}R_{\mathrm{M}^2}\left(\delta^C{}_A\, g_{DB} - \delta^C{}_B\, g_{DA}\right) , \tag{A.5a}$$
$$R^c{}_{dab} = \left[\tfrac{1}{2}R_{\mathbb{S}^2} + f^L f_L\right]\left(\delta^c{}_a\, \gamma_{db} - \delta^c{}_b\, \gamma_{da}\right) , \tag{A.5b}$$

where the Ricci scalars on M^2 and $\mathbb{S}^2$ are denoted by R_{M^2} and $R_{\mathbb{S}^2}$, respectively, with $R_{\mathbb{S}^2} = 2$. The mixed Riemann tensor terms incorporate the curvature of the warp function in the form of the covariant Hessian of f,

$$R^c{}_{DaB} = -f^{-1}\delta^c{}_a \widetilde{\nabla}_B f_D \tag{A.5c}$$
$$R^C{}_{dAb} = f\gamma_{db}\widetilde{\nabla}_A f^C \,, \tag{A.5d}$$
$$R^C{}_{daB} = -R^C{}_{dBa} \,, \qquad R^c{}_{DBa} = -R^c{}_{DaB} \,. \tag{A.5e}$$

Note that, due to the symmetries of the Riemann tensor, the mixed terms must have an even number of indices of each type, so $R^C{}_{dab} = 0 = R^c{}_{DAB}$, etc. Additionally, the cross terms $R^C{}_{Dab}$ and $R^c{}_{dAB}$ vanish. The Riemann tensor with all indices lowered is then given by

$$R_{CDAB} = \tfrac{1}{2}R_{\mathrm{M}^2}\left(g_{CA}\, g_{DB} - g_{CB}\, g_{DA}\right) , \tag{A.6a}$$
$$R_{cdab} = -f^2\left[1 + f^L f_L\right]\left(\gamma_{ca}\, \gamma_{db} - \gamma_{cb}\, \gamma_{da}\right) , \tag{A.6b}$$
$$R_{cDaB} = f\gamma_{ca}\widetilde{\nabla}_B f_D \,, \tag{A.6c}$$
$$R_{CdaB} = -f\gamma_{da}\widetilde{\nabla}_B f_C \,. \tag{A.6d}$$

The covariant derivatives ∇_C and ∇_c denote the restriction of the covariant derivative ∇_σ on the product space $\mathcal{M}$ to M^2 and $\mathbb{S}^2$, respectively, whereas we use the symbol $\widetilde{\nabla}_C$ for the covariant derivative on M^2, and D_c for the covariant derivative on $\mathbb{S}^2$. Consider an arbitrary rank-2 tensor $h_{\mu\nu}$ on $\mathcal{M}$. Its covariant derivative $\nabla_\sigma h_{\mu\nu}$ in the $2+2$ split can be displayed in terms of the different parts as

$$\nabla_C h_{ab} = \widetilde{\nabla}_C h_{ab} - 2f^{-1}\,(\widetilde{\nabla}_C f) h_{ab} \tag{A.7a}$$
$$\nabla_c h_{ab} = D_c h_{ab} - 2f\,(\widetilde{\nabla}^L f)\gamma_{c(a|} h_{L|b)} \tag{A.7b}$$
$$\nabla_C h_{Ab} = \widetilde{\nabla}_C h_{Ab} - f^{-1}\,(\widetilde{\nabla}_C f) h_{Ab} \tag{A.7c}$$
$$\nabla_c h_{Ab} = D_c h_{Ab} - f^{-1}\,(\widetilde{\nabla}_A f) h_{cb} - f\gamma_{cb}(\widetilde{\nabla}^L f) h_{AL} \tag{A.7d}$$

$$\nabla_c h_{AB} = D_c h_{AB} - 2f^{-1}\,(\widetilde{\nabla}_{(A|} f) h_{c|B)} \tag{A.7e}$$

$$\nabla_C h_{AB} = \widetilde{\nabla}_C h_{AB}\,, \tag{A.7f}$$

where we have explicitly written out the covariant derivatives as in

$$\begin{aligned} \nabla_C h_{ab} &= \widetilde{\nabla}_C h_{ab} - \Gamma^\lambda{}_{Ca} h_{\lambda b} - \Gamma^\lambda{}_{Cb} h_{a\lambda} \\ &= \widetilde{\nabla}_C h_{ab} - \Gamma^L{}_{Ca} h_{Lb} - \Gamma^L{}_{Cb} h_{aL} - \Gamma^l{}_{Ca} h_{lb} - \Gamma^l{}_{Cb} h_{al}\,, \end{aligned} \tag{A.8}$$

and used equations (A.4). (Note that $\widetilde{\nabla}_C h_{ab} = \partial_C h_{ab}$, since h_{ab} is a scalar on M^2.) Metric compatibility now leads to $\widetilde{\nabla}_C g_{AB} = 0$ on M^2 and $D_c \gamma_{ab} = 0$ on $\mathbb{S}^2$. We also have $\nabla_c g_{ab} = -f^2 D_c \gamma_{ab} = 0$, since the warp function f does not depend on the coordinates X^c of $\mathbb{S}^2$. On the other hand, we get a contribution from the warp function in the cross term $\nabla_C g_{ab} = -2f\, f_C \gamma_{ab}$.

Closed-Form Expression for the Weyl Tensor

The full index-lowered Riemann tensor on $\mathcal{M}$ can then be written as the sum of the Riemann tensors on $\mathbb{S}^2$, M^2 and the respective non-vanishing mixed terms,

$$R_{\sigma\rho\mu\nu} = R_{CDAB} + R_{cdab} + R_{CdAb} + R_{cDaB} + R_{CdaB} + R_{cDAb} \tag{A.9a}$$

$$\begin{aligned} &= \frac{\tilde{R}_{\mathrm{M}^2}}{2}(\tilde{g}_{CA}\tilde{g}_{DB} - \tilde{g}_{CB}\tilde{g}_{DA}) - f^2[1 + f^L f_L](\gamma_{ca}\gamma_{db} - \gamma_{cb}\gamma_{da}) \\ &\quad + f\,[\gamma_{db}\widetilde{\nabla}_A\widetilde{\nabla}_C f + \gamma_{ca}\widetilde{\nabla}_B\widetilde{\nabla}_D f - \gamma_{da}\widetilde{\nabla}_B\widetilde{\nabla}_C f - \gamma_{cb}\widetilde{\nabla}_A\widetilde{\nabla}_D f]\,, \end{aligned} \tag{A.9b}$$

where we have split the indices σ, ρ, μ, ν on $\mathcal{M}$ according to the naming scheme $\sigma = (C, c), \rho = (D, d), \mu = (A, a), \nu = (B, b)$ into indices (C, D, A, B) on M^2 and (c, d, a, b) on $\mathbb{S}^2$. On the other hand, contracting (A.9b) once, we obtain the Ricci tensor on $\mathcal{M}$,

$$R_{\mu\nu} \equiv R^\lambda{}_{\mu\lambda\nu} = R^L{}_{ALB} + R^l{}_{AlB} + R^L{}_{aLb} + R^l{}_{alb} \tag{A.10a}$$

$$= \tfrac{1}{2}\tilde{R}_{\mathrm{M}^2}\tilde{g}_{AB} - 2f^{-1}(\widetilde{\nabla}_B\widetilde{\nabla}_A f) + [1 + f^L f_L + f\widetilde{\nabla}^L f_L]\gamma_{ab}\,. \tag{A.10b}$$

Contracting once more yields the Ricci scalar,

$$R_{\mathcal{M}} \equiv R^\lambda{}_\lambda = g^{AB}(R^L{}_{ALB} + R^l{}_{AlB}) + g^{ab}(R^L{}_{aLb} + R^l{}_{alb}) \tag{A.11a}$$

$$= \tilde{R}_{\mathrm{M}^2} - 4(f^{-1})\widetilde{\nabla}^L f_L - 2f^{-2}[1 + f^L f_L]\,. \tag{A.11b}$$

For vacuum space-times, we must have $R_{\mathcal{M}} = 0 = R_{\mu\nu}$. On the one hand, this condition together with equation (A.11b) yields an expression for the unknown Ricci scalar $\tilde{R}_{\mathrm{M}^2}$ of submanifold M^2 in (A.9b),

$$\tilde{R}_{\mathrm{M}^2} = 4(f^{-1})\widetilde{\nabla}^L f_L + 2(f^{-2})[1 + f^L f_L]\,. \tag{A.12}$$

On the other hand, we obtain from (A.10b) an expression for the covariant Hessian of f,

$$f\,\widetilde{\nabla}_B f_A = \tfrac{1}{4}f^2\tilde{R}_{\mathrm{M}^2}\tilde{g}_{AB} + \tfrac{1}{2}f^2[1 + f^L f_L + f\widetilde{\nabla}^L f_L]\gamma_{ab}\,, \tag{A.13}$$

with which we may replace these terms in the last line of (A.9b), by terms with the metrics $\tilde{g}_{CA}$ and γ_{ca} of the submanifolds, thereby considerably simplifying the prefactors. As a result we obtain a covariant closed-form expression for the Riemann, i.e., Weyl tensor of a spherically symmetric vacuum space-time in $2+2$ split,

$$\begin{aligned} R_{\sigma\rho\mu\nu} = -f^{-2}\left[1+f^L f_L\right]\big\{&\tilde{g}_{CA}\tilde{g}_{DB} - \tilde{g}_{CB}\tilde{g}_{DA} + g_{ca}g_{db} - g_{cb}g_{da} \\ &- \tfrac{1}{2}\big(g_{ca}\tilde{g}_{DB} - g_{cb}\tilde{g}_{DA} + g_{db}\tilde{g}_{CA} - g_{da}\tilde{g}_{CB}\big)\big\}\,, \end{aligned} \tag{A.14a}$$

or, in terms of the metric on $\mathbb{S}^2$, using (A.2), this reads

$$\begin{aligned} R_{\sigma\rho\mu\nu} = -f^{-2}\left[1+f^L f_L\right]\big\{&\big(\tilde{g}_{CA}\tilde{g}_{DB} - \tilde{g}_{CB}\tilde{g}_{DA}\big) + f^4\left(\gamma_{ca}\gamma_{db} - \gamma_{cb}\gamma_{da}\right) \\ &+ \tfrac{1}{2}f^2\left(\gamma_{ca}\tilde{g}_{DB} - \gamma_{cb}\tilde{g}_{DA} + \gamma_{db}\tilde{g}_{CA} - \gamma_{da}\tilde{g}_{CB}\right)\big\}\,. \end{aligned} \tag{A.14b}$$

Note, that we may also write (A.14) in a mixed form, using

$$g_{\sigma\mu}g_{\rho\nu} = g_{CA}g_{DB} + g_{ca}g_{db} + g_{ca}g_{DB} + g_{db}g_{CA}\,, \tag{A.15}$$

which yields the shorter expression

$$\boxed{\begin{aligned} R_{\sigma\rho\mu\nu} = -f^{-2}\left[1+f^L f_L\right]\big\{&g_{\sigma\mu}g_{\rho\nu} - g_{\sigma\nu}g_{\rho\mu} \\ &- \tfrac{3}{2}\big(g_{ca}\tilde{g}_{DB} - g_{cb}\tilde{g}_{DA} + g_{db}\tilde{g}_{CA} - g_{da}\tilde{g}_{CB}\big)\big\}\,. \end{aligned}} \tag{A.16}$$

Here, we observe that the term $g_{\sigma\mu}g_{\rho\nu} - g_{\sigma\nu}g_{\rho\mu}$ in (A.16) is locally isotropic and the second term in the round brackets encodes the deviations from isotropy.

In view of (A.10a) and its contraction (A.11a), the scalar factor determining the Weyl tensor in equations (A.14) and (A.16) can be written in terms of the Ricci scalars on the submanifolds $\mathbb{S}^2$, M^2, and the two cross terms as

$$\begin{aligned} -f^{-2}\left[1+f^L f_L\right] &= -\tfrac{1}{2}\left[R^{AB}{}_{AB} + R^{aB}{}_{aB} + R^{Ab}{}_{Ab}\right] && \text{(A.17a)} \\ &= \tfrac{1}{2}R^{ab}{}_{ab}\,. && \text{(A.17b)} \end{aligned}$$

Closed Form for First Covariant Derivative of the Weyl Tensor

Starting from (A.16), we pull out the warp function from the terms g_{ab} using its definition (A.2) in order to make all occurances of f explicit. With $\nabla_\kappa = \widetilde{\nabla}_K + D_k$, and metric compatibility of the respective covariant derivatives, we see that we only have to act on the scalar factors in front of the two terms with $\widetilde{\nabla}_K$, since they do not depend on the "angular" coordinates X^a of the sphere $\mathbb{S}^2$, thus it remains to calculate

$$\begin{aligned} \nabla_\kappa R_{\sigma\rho\mu\nu} = &-\big[\widetilde{\nabla}_K f^{-2}\left(1+f^L f_L\right)\big]\left(g_{\sigma\mu}g_{\rho\nu} - g_{\sigma\nu}g_{\rho\mu}\right) \\ &+ \tfrac{3}{2}\big[\widetilde{\nabla}_K\big(1+f^L f_L\big)\big]\big(\gamma_{ca}\tilde{g}_{DB} - \gamma_{cb}\tilde{g}_{DA} + \gamma_{db}\tilde{g}_{CA} - \gamma_{da}\tilde{g}_{CB}\big)\,. \end{aligned} \tag{A.18}$$

Carrying out the covariant derivatives of the two prefactors in (A.18) then yields

$$\begin{aligned} \widetilde{\nabla}_K\big(1+f^L f_L\big) &= 2f^L\widetilde{\nabla}_K f_L\,, && \text{(A.19a)} \\ -\widetilde{\nabla}_K(f^{-2})\left(1+f^L f_L\right) &= f^{-3}f_K\left(1+f^L f_L\right) - 2(f^{-2})f^L\widetilde{\nabla}_K f_L\,, && \text{(A.19b)} \end{aligned}$$

in terms of the contraction $f^L f_{LK} = f^L \widetilde{\nabla}_K f_L$ in which the second covariant gradient of f appears. From equation (A.13) for the covariant Hessian of f, with the expression (A.12) for the Ricci scalar on M^2 inserted, we obtain the second covariant gradient of f,

$$f_{K_1K_2} = \widetilde{\nabla}_{K_2} f_{K_1} = [f_L{}^L + \tfrac{1}{2}(f^{-1})(1 + f^L f_L)]\tilde{g}_{K_1K_2} \\ + f^2[\tfrac{1}{2}f_L{}^L + \tfrac{1}{2}(f^{-1})(1 + f^L f_L)]\gamma_{k_1k_2} \, , \quad \text{(A.20)}$$

in terms of its contraction, the metrics of the two submanifolds and its lower order covariant gradients. Setting $K_1 = L_1$ and contracting with f^{L_1} yields the contraction

$$f^L f_{LK_2} = [f_L{}^L + \tfrac{1}{2}(f^{-1})(1 + f^L f_L)] f_{K_2} \, , \quad \text{(A.21)}$$

which appears in the covariant derivative of the square of the gradient $f^L f_L$ above. Putting everything together, we thus obtain from (A.18) and (A.21) the result

$$\nabla_\kappa R_{\sigma\rho\mu\nu} = -f^{-2}\,[2f_L{}^L - (f^{-1})(1 + f^L f_L)] f_{K_1}\,(g_{\sigma\mu}g_{\rho\nu} - g_{\sigma\nu}g_{\rho\mu}) \\ + 3[f_L{}^L + \tfrac{1}{2}(f^{-1})(1 + f^L f_L)] f_{K_1}\,(\gamma_{ca}\tilde{g}_{DB} - \gamma_{cb}\tilde{g}_{DA} + \gamma_{db}\tilde{g}_{CA} - \gamma_{da}\tilde{g}_{CB}) \, . \quad \text{(A.22)}$$

for the first covariant derivative of the Riemann tensor. The third-order covariant gradient of f then follows straightforwardly. We obtain

$$f_{K_1K_2K_3} = [f_L{}^L{}_{K_3} + (f^{-1})f_L{}^L f_{K_3}]\tilde{g}_{K_1K_2} + f^2[\tfrac{1}{2}f_L{}^L{}_{K_3} + (f^{-1})f_L{}^L f_{K_3}]\gamma_{k_1k_2} \\ + f f_{K_3}\,[f_L{}^L + (f^{-1})(1 + f^L f_L)]\gamma_{k_1k_2} \, , \quad \text{(A.23)}$$

where we find that some terms cancel after using (A.21) for $f^L f_{LK_3}$.

Closed Forms for Frame Components of the Weyl Tensor and its Covariant Derivatives

The tetrad adapted to the metric in $2+2$ split (A.1) then reads

$$e_{\hat{\alpha}}{}^{\mu} = e_{\hat{A}}{}^{A} + e_{\hat{a}}{}^{a} \, , \qquad e_{\hat{a}}{}^{a} = f^{-1}\sigma_{\hat{a}}{}^{a} \, , \qquad \sigma_{\hat{a}}{}^{a} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin\Theta} \end{pmatrix} \, , \quad \text{(A.24)}$$

where the tetrads on M^2 and $\mathbb{S}^2$ are defined in the usual way through

$$\eta_{\hat{A}\hat{B}} = g_{AB}\, e_{\hat{A}}{}^{A} e_{\hat{B}}{}^{B} \, , \qquad \text{and} \qquad \eta_{\hat{a}\hat{b}} = \gamma_{ab}\, \sigma_{\hat{a}}{}^{a} \sigma_{\hat{b}}{}^{b} \, , \quad \text{(A.25)}$$

$\sigma_{\hat{a}}{}^{a}$ being the tetrad on $\mathbb{S}^2$. Projecting indices with the tetrad (A.24) in the expression for the Riemann tensor (A.14) according to

$$R_{\hat{\gamma}\hat{\delta}\hat{\alpha}\hat{\beta}} = R_{\sigma\rho\mu\nu} e_{\hat{\gamma}}{}^{\sigma} e_{\hat{\delta}}{}^{\rho} e_{\hat{\alpha}}{}^{\mu} e_{\hat{\beta}}{}^{\nu} \quad \text{(A.26)}$$

leads to the corresponding closed-form expression for the frame components of the Riemann (i.e. Weyl) tensor in the metric-adapted frame just introduced,

$$R_{\hat{\gamma}\hat{\delta}\hat{\alpha}\hat{\beta}} = -f^{-2}\,[1 + f^L f_L]\{\eta_{\hat{C}\hat{A}}\eta_{\hat{D}\hat{B}} - \eta_{\hat{C}\hat{B}}\eta_{\hat{D}\hat{A}} + \eta_{\hat{c}\hat{a}}\eta_{\hat{d}\hat{b}} - \eta_{\hat{c}\hat{b}}\eta_{\hat{d}\hat{a}}$$

$$- \tfrac{1}{2}\big(\eta_{\hat{c}\hat{a}}\eta_{\hat{D}\hat{B}} - \eta_{\hat{c}\hat{b}}\eta_{\hat{D}\hat{A}} + \eta_{\hat{d}\hat{b}}\eta_{\hat{C}\hat{A}} - \eta_{\hat{d}\hat{a}}\eta_{\hat{C}\hat{B}}\big)\big\}\,, \qquad (A.27)$$

where we have employed the obvious index naming scheme $\hat{\gamma} = (\hat{C}, \hat{c}), \hat{\delta} = (\hat{D}, \hat{d})$, and $\hat{\alpha} = (\hat{A}, \hat{a}), \hat{\beta} = (\hat{B}, \hat{b})$. The equivalent mixed form of (A.27) then follows as

$$\boxed{\begin{aligned} R_{\hat{\gamma}\hat{\delta}\hat{\alpha}\hat{\beta}} = -f^{-2}\left[1 + f^L f_L\right]\big\{&\eta_{\hat{\gamma}\hat{\alpha}}\eta_{\hat{\delta}\hat{\beta}} - \eta_{\hat{\gamma}\hat{\beta}}\eta_{\hat{\delta}\hat{\alpha}} \\ &- \tfrac{3}{2}\big(\eta_{\hat{c}\hat{a}}\eta_{\hat{D}\hat{B}} - \eta_{\hat{c}\hat{b}}\eta_{\hat{D}\hat{A}} + \eta_{\hat{d}\hat{b}}\eta_{\hat{C}\hat{A}} - \eta_{\hat{d}\hat{a}}\eta_{\hat{C}\hat{B}}\big)\big\}\,. \end{aligned}} \qquad (A.28)$$

Evaluation for Standard and Isotropic Schwarzschild Coordinates

We now evaluate the factor in front (A.28) for standard and isotropic Schwarzschild coordinates. For standard Schwarzschild coordinate $(cT, R, \mathit{\Theta}, \mathit{\Phi})$, the warp function is just $f(R) = R$ and the metric on M^2 reads

$$g^{AB} = \begin{pmatrix} B(R)^{-1} & 0 \\ 0 & -B(R) \end{pmatrix}\,. \qquad (A.29)$$

We then have $\widetilde{\nabla}_A f = \delta^1{}_A$, so that we easily obtain $-(f^{-2})\left[1 + g^{AB}(\widetilde{\nabla}_A f)(\widetilde{\nabla}_B f)\right] = -\frac{R_{\mathrm{S}}}{R^3}$. The Riemann tensor's frame components in standard Schwarzschild coordinates then read,

$$\boxed{R_{\hat{\gamma}\hat{\delta}\hat{\alpha}\hat{\beta}} = -\frac{R_{\mathrm{S}}}{R^3}\big\{\eta_{\hat{\gamma}\hat{\alpha}}\eta_{\hat{\delta}\hat{\beta}} - \eta_{\hat{\gamma}\hat{\beta}}\eta_{\hat{\delta}\hat{\alpha}} - \tfrac{3}{2}\big(\eta_{\hat{c}\hat{a}}\eta_{\hat{D}\hat{B}} - \eta_{\hat{c}\hat{b}}\eta_{\hat{D}\hat{A}} + \eta_{\hat{d}\hat{b}}\eta_{\hat{C}\hat{A}} - \eta_{\hat{d}\hat{a}}\eta_{\hat{C}\hat{B}}\big)\big\}\,.} \qquad (A.30)$$

In the second case, i.e. for isotropic coordinates $(cT, R_{\mathrm{iso}}, \mathit{\Theta}, \mathit{\Phi})$, we have the warp function $f = \sqrt{B_2(R_{\mathrm{iso}})}$, and its covariant gradient is easily found to be,

$$\widetilde{\nabla}_A f = -\frac{R_{\mathrm{S}}}{2R_{\mathrm{iso}}^2}\Big(1 + \frac{R_{\mathrm{S}}}{4R_{\mathrm{iso}}}\Big)\,, \qquad (A.31)$$

where the metric on the Lorentzian submanifold M^2 reads,

$$g^{AB} = \begin{pmatrix} B_1(R_{\mathrm{iso}})^{-1} & 0 \\ 0 & -B_2(R_{\mathrm{iso}})^{-1} \end{pmatrix}\,. \qquad (A.32)$$

For the scalar product of the gradient of f we first calculate the expression,

$$g^{AB}(\widetilde{\nabla}_A f)(\widetilde{\nabla}_B f) = -\left[\frac{R_{\mathrm{iso}}^2}{4}\Big(\frac{B_2'}{B_2}\Big)^2 + R_{\mathrm{iso}}\Big(\frac{B_2'}{B_2}\Big) + 1\right]\,, \qquad (A.33)$$

which then yields for the curvature factor the expression,

$$-(f^{-2})\left[1 + g^{AB}(\widetilde{\nabla}_A f)(\widetilde{\nabla}_B f)\right] = -\frac{R_{\mathrm{S}}}{R_{\mathrm{iso}}^3 B_2(R_{\mathrm{iso}})^{3/2}}\,, \qquad (A.34)$$

so that the frame components of the Weyl tensor in the metric-adapted frame (A.24) are written as,

$$\boxed{\begin{aligned} R_{\hat{\gamma}\hat{\delta}\hat{\alpha}\hat{\beta}} = -\frac{R_{\mathrm{S}}}{R_{\mathrm{iso}}^3 B_2(R_{\mathrm{iso}})^{3/2}} \{ &\eta_{\hat{\gamma}\hat{\alpha}}\eta_{\hat{\delta}\hat{\beta}} - \eta_{\hat{\gamma}\hat{\beta}}\eta_{\hat{\delta}\hat{\alpha}} \\ &- \tfrac{3}{2}\left(\eta_{\hat{c}\hat{a}}\eta_{\hat{D}\hat{B}} - \eta_{\hat{c}\hat{b}}\eta_{\hat{D}\hat{A}} + \eta_{\hat{d}\hat{b}}\eta_{\hat{C}\hat{A}} - \eta_{\hat{d}\hat{a}}\eta_{\hat{C}\hat{B}}\right)\} \,. \end{aligned}} \tag{A.35}$$

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Posters, Talks, and Publications

- O. Gabel, R. Walser:
  General Relativistic Description of Bose-Einstein Condensates
  Poster at DPG spring meeting 2011
- O. Gabel, R. Walser:
  Relativistic Corrections for BECs in μ-Gravity
  Poster at DPG spring meeting 2012
- O. Gabel, R. Walser:
  Tidal Corrections for Free Falling Relativistic Bose-Einstein Condensates
  Poster at DPG spring meeting 2013
- O. Gabel, R. Walser:
  Free Falling Bose-Einstein Condensates in General Relativity
  Poster at DPG spring meeting 2014
- O. Gabel, R. Walser:
  General Relativistic Corrections for Bose-Einstein Condensates in Local Frames
  Poster at DPG spring meeting 2015
- O. Gabel, R. Walser:
  General Relativistic Corrections for Free Falling Bose-Einstein Condensates in Fermi Coordinates
  Poster at DPG spring meeting 2016
- O. Gabel
  Tidal and Relativistic Corrections for Free Falling Bose-Einstein Condensates
  Talk at "Zentrum für angewandte Raumfahrttechnologie und Mikrogravitation (ZARM)", University of Bremen, 2014
- O. Gabel:
  General Relativistic Corrections for BECs in Fermi Coordinates
  Talk at conference "Frontiers in Matter Wave Optics (FOMO) 2016"
- O. Gabel, R. Walser:
  Tidal Corrections for Free Falling Bose Einstein Condensates in General Relativity
  Paper in Preparation

Erklärung gemäß §9 Promotionsordnung

Hiermit versichere ich, dass ich die vorliegende Dissertation selbstständig angefertigt und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet habe. Alle wörtlichen und paraphrasierten Zitate wurden angemessen kenntlich gemacht. Die Arbeit hat bisher noch nicht zu Prüfungszwecken gedient.

Darmstadt, den 2019,

Oliver Gabel