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# Cayley Structures and the Expressiveness of Common Knowledge Logic

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## *Abstract*

Van Benthem's theorem states that basic modal logic $\mathsf{ML}$ is expressively equivalent to the bisimulation-invariant fragment of first-order logic $\mathsf{FO}/{\sim}$; we write $\mathsf{ML} \equiv \mathsf{FO}/{\sim}$ for short. Hence, $\mathsf{ML}$ can express every bisimulation-invariant first-order property and, moreover, $\mathsf{ML}$ can be considered an effective syntax for the undecidable fragment $\mathsf{FO}/{\sim}$. Over the years, many variations of this theorem have been established. Rosen proved that $\mathsf{ML} \equiv \mathsf{FO}/{\sim}$ is still true when restricted to finite transition systems. Going beyond first-order logic, Janin and Walukiewicz showed that the bisimulation-invariant fragment of monadic second-order logic $\mathsf{MSO}$ is precisely as expressive as the modal $\mu$-calculus $\mathsf{L}_\mu$, and several important fragments of $\mathsf{L}_\mu$ have been characterised classically in a similar vein. However, whether $\mathsf{L}_\mu \equiv \mathsf{MSO}/{\sim}$ is true over finite transition systems, remains an open problem.

This thesis is concerned with modal common knowledge logic $\mathsf{ML}[\mathsf{CK}]$, another fragment of $\mathsf{L}_\mu$ that is more expressive than $\mathsf{ML}$. The main result is a characterisation of $\mathsf{ML}[\mathsf{CK}]$ over $\mathsf{S5}$ structures, both classically and also in the sense of finite model theory. We achieved this result by showing that $\mathsf{ML} \equiv \mathsf{FO}/{\sim}$ over the non-elementary classes of (finite or arbitrary) common knowledge Kripke structures ($\mathsf{CK}$ Structures).

The fixpoint character of the derived accessibility relations of $\mathsf{CK}$ structures poses a novel challenge for the analysis of model-theoretic games. The technical core of this thesis deals with the development of a specific structure theory for specially adapted Cayley graphs, which we call Cayley structures. We show that questions regarding $\mathsf{CK}$ structures can be reduced, up to bisimulation, to Cayley structures. Specific acyclicity properties of Cayley structures make it possible to adapt and expand known locality-based methods, which leads to new techniques for playing first-order Ehrenfeucht-Fraïssé games over these non-elementary structures.

## *Zusammenfassung*

Nach dem Satz von van Benthem ist Modallogik $\mathsf{ML}$ genauso ausdrucksstark wie das bisimulationsinvariante Fragment der Logik erster Stufe $\mathsf{FO}/{\sim}$; abgekürzt schreiben wir $\mathsf{ML} \equiv \mathsf{FO}/{\sim}$. Damit kann $\mathsf{ML}$ alle bisimulationsinvarianten Aussagen der Logik erster Stufe ausdrücken und $\mathsf{ML}$ kann als eine effektive Syntax für das unentscheidbare Fragment $\mathsf{FO}/{\sim}$ betrachtet werden. Eine Reihe von Varianten dieses Satzes wurde im Laufe der Jahre bewiesen. Rosen hat gezeigt, dass $\mathsf{ML} \equiv \mathsf{FO}/{\sim}$ auch unter Einschränkung auf endliche Transitionssysteme gilt. Über die Ausdrucksstärke der ersten Stufe hinausgehend haben Janin und Walukiewicz gezeigt, dass das bisimulationsinvariante Fragment der monadischen Logik zweiter Stufe $\mathsf{MSO}$ äquivalent zum modalen $\mu$-Kalkül $\mathsf{L}_\mu$ ist. Weitere wichtige Fragmente von $\mathsf{L}_\mu$ wurden klassisch auf ähnliche Weise charakterisiert. Ob $\mathsf{L}_\mu \equiv \mathsf{MSO}/{\sim}$ auch über endlichen Transitionssystemen gilt, bleibt jedoch weiterhin ein offenes Problem.

Diese Dissertation beschäftigt sich mit modaler Common Knowledge Logik $\mathsf{ML}[\mathsf{CK}]$, einem weiteren Fragment von $\mathsf{L}_\mu$, das ausdrucksstärker als $\mathsf{ML}$ ist. Das Hauptresultat ist eine Charakterisierung von $\mathsf{ML}[\mathsf{CK}]$ über $\mathsf{S5}$-Strukturen, sowohl klassisch als auch im Sinne der endlichen Modelltheorie. Dieses Resultat wurde bewiesen, indem wir gezeigt haben, dass $\mathsf{ML} \equiv \mathsf{FO}/{\sim}$ über den nicht-elementaren Klassen der endlichen und beliebigen Common-Knowledge-Kripke-Strukturen ($\mathsf{CK}$-Strukturen) gilt.

Der Fixpunkt-Charakter der abgeleiteten Kantenrelationen der $\mathsf{CK}$-Strukuren bereitet eine neuartige Herausforderung für die Analyse modelltheoretischer Spiele. Der technische Kern dieser Dissertation befasst sich mit der Entwicklung einer spezifischen Strukturtheorie für speziell adaptierte Cayleygraphen, die wir als Cayleystrukturen bezeichnen. Wir zeigen, dass sich Fragen über $\mathsf{CK}$-Strukuren bis auf Bisimulation auf Cayleystrukturen reduzieren lassen. Spezifische Azyklizitätseigenschaften von Cayleystrukturen ermöglichen es, bekannte lokalitätsbasierte Methoden anzupassen und zu erweitern, was zu neuen Techniken für Ehrenfeucht-Fraïssé-Spiele über diesen nicht-elementaren Strukturen führt.

# Contents

# 1 Introduction

As a notion of equivalence, bisimulation captures the relevant properties of transition systems or Kripke structures that do not depend on a specific encoding. This makes bisimulation invariance the essential semantic property of any logic that is meant to deal with the relevant phenomena of transition systems and their wide range of applications, which include hardware verification, program synthesis, databases and reasoning about knowledge. In contrast to classical logics like first-order logic $\mathsf{FO}$ or monadic second-order logic $\mathsf{MSO}$, modal logics are typically bisimulation-invariant and, moreover, can often be characterised as fragments of classical logics that precisely capture the bisimulation-invariant properties of transition systems. This turns bisimulation invariance into a criterion of expressive completeness, and gives important undecidable fragments of classical logics an effective syntax.

The first significant example of such an expressive completeness result is van Benthem's theorem [29]. It states that basic modal logic $\mathsf{ML}$ is the bisimulation-invariant fragment of first-order logic. In other words, an $\mathsf{FO}$-formula $\varphi$ is bisimulation-invariant if and only if $\varphi$ is logically equivalent to an $\mathsf{ML}$-formula. We write $\mathsf{ML} \equiv \mathsf{FO}/{\sim}$ for short, where $\mathsf{FO}/{\sim}$ is the set $\mathsf{FO}$-formulae that are invariant under bisimulation equivalence $\sim$. Many variations of this theorem have been proven over the last decades. In [28] Rosen showed the finite model theory version, i.e. an $\mathsf{FO}$-formula $\varphi$ is $\sim$-invariant over *finite* Kripke structures if and only if it is logically equivalent to an $\mathsf{ML}$-formula over *finite* Kripke structures. Adding finiteness to both sides of the equation changes the meaning of the statement completely, and its proof requires a whole new approach; in this case, we say that $\mathsf{ML} \equiv \mathsf{FO}/{\sim}$ *over finite structures.* In general, if a class of structures $\mathcal{C}$ is $\mathsf{FO}$-axiomatizable, then $\mathsf{ML} \equiv \mathsf{FO}/{\sim}$ over $\mathcal{C}$ follows from a straightforward adaptation of van Benthem's original proof. If the class is not $\mathsf{FO}$-axiomatizable, such as the class of all finite structures, then the picture looks quite different. Van Benthem's proof of the classical version is based on compactness arguments, tools that are not available in finite model theory. Moreover, a preservation of the statement when restricted to finite structures does not simply follow from the finite model property of basic modal logic. The two-variable fragment of first-order logic $\mathsf{FO}^2$ also possesses the finite model property. However, although $\mathsf{FO}^2$ is precisely equivalent to the set of $\mathsf{FO}$-formulae that are classically invariant under 2-pebble game equivalence, this characterisation fails over finite structures.

Otto gave another constructive proof of $\mathsf{ML} \equiv \mathsf{FO}/{\sim}$ that works both classically and in the sense of finite model theory [24]. It is based solely on elementary model-theoretic methods, which simplifies Rosen's proof, and it even gives additional insight into the classical case. Adaptations of his approach have been fruitful for a wide range of variations of van Benthem's theorem. For example, it was applied to characterising

$\mathsf{FO}/\sim$ over other non-elementary classes like (finite) rooted structures and (finite) S5 structures [9], as well as to characterise extensions of basic modal logic like global modal logic and the guarded fragment, again both classically and in the sense of finite model theory [26]. The main idea of Otto's proof strategy is to show that an FO-formula $\varphi$ that is $\sim$-invariant over a class of structures $\mathcal{C}$ is, in fact, $\sim^\ell$-invariant over $\mathcal{C}$, for some finite approximation $\sim^\ell$ of full bisimulation equivalence $\sim$. Then the modal Ehrenfeucht-Fraïssé theorem implies that $\varphi$ is equivalent over $\mathcal{C}$ to some ML-formula of modal nesting depth $\ell$. The core of the proof is an upgrading argument that links $\sim^\ell$-equivalence to finite levels $\equiv_q$ of first-order equivalence, for some $\ell$ that depends on $q$. To be more precise: given a formula $\varphi \in \mathsf{FO}$ of quantifier depth $q$ that is $\sim$-invariant over $\mathcal{C}$, one shows that there is some $\ell$ such that for all pointed Kripke structures $\mathfrak{M}, w$ and $\mathfrak{N}, v$ in $\mathcal{C}$ with $\mathfrak{M}, w \sim^\ell \mathfrak{N}, v$ there are bisimilar coverings $\hat{\mathfrak{M}}, \hat{w} \sim \mathfrak{M}, w$ and $\hat{\mathfrak{N}}, \hat{v} \sim \mathfrak{N}, v$ in $\mathcal{C}$ such that $\hat{\mathfrak{M}}, \hat{w} \equiv_q \hat{\mathfrak{N}}, \hat{v}$. This directly implies $\sim^\ell$-invariance of $\varphi$ over $\mathcal{C}$. The challenges of upgrading are twofold: constructing suitable bisimilar coverings and proving their $\equiv_q$-equivalence. We apply the very same approach to prove characterisation theorems for epistemic modal logic with common knowledge modalities in both the classical and finite case. Before we shift our attention to the results of this thesis, we highlight some other modal characterisation theorems of a different flavour.

The famous result by Janin and Walukiewicz shows that there is an analogous characterisation available for the modal $\mu$-calculus $\mathsf{L}_\mu$ in relation to monadic second-order logic MSO [17]: $\mathsf{L}_\mu \equiv \mathsf{MSO}/\sim$. The proof of this theorem is based on automata-theoretic methods, in contrast to the proofs of van Benthem, Rosen and Otto. Janin and Walukiewicz gave automata-theoretic characterisations of MSO over trees [30] and of $\mathsf{L}_\mu$ over general Kripke structures [16], and showed that these two types of automata are equivalent over trees. Since every Kripke structure is bisimilar to a tree, the result follows. However, their approach has one drawback: it does not work over the class of finite structures. The question of whether $\mathsf{L}_\mu \equiv \mathsf{MSO}/\sim$ over finite structures is true, remains a prominent open problem in finite model theory. There is a series of partial results on this topic: Hirsch showed that every MSO-formula that is $\sim$-invariant over tree-unravellings of finite structures is equivalent to an $\mathsf{L}_\mu$-formula [15], Dawar and Janin showed that over finite unary transition systems the $\sim$-invariant fragment of existential monadic second-order logic is equivalent to $\mathsf{L}_\mu$ [8], and Blumensath and Wolf showed that MSO coincides with $\mathsf{L}_\mu$ over several other subclasses of finite structures [5]. For some other fragments of MSO, classical characterisations are available. Moller and Rabinovich showed that $\mathsf{CTL}^*$ is the $\sim$-invariant fragment of monadic path logic [22], and Carreiro showed that PDL is the $\sim$-invariant fragment of weak chain logic [7]. The finite case remains open for these characterisations too. Modal logic with common knowledge modalities, for short ML[CK], is, as $\mathsf{CTL}^*$ and PDL, another fragment of $\mathsf{L}_\mu$ that goes beyond the expressive power of first-order logic. A characterisation of this logic in an epistemic context, both classically and also in the sense of finite model theory, is the topic of this thesis.

Epistemic modal logics deal with information and knowledge in multi-agent settings. Such settings are usually modelled by so-called S5 structures, special instances of Kripke structures in which accessibility relations for the individual agents are equivalence relations. These equivalence relations encode an agent's knowledge by relating possible worlds that are indistinguishable from the agent's point of view. A characterisation theorem for basic modal logic ML in this epistemic setting was obtained by Dawar and Otto in [9], both classically and in the sense of finite model theory. Like the characterisation proofs of van Benthem and Rosen, this result deals with plain first-order logic but over the elementary class of S5 structures and the non-elementary class of finite S5 structures. A key aspect of the proof are arguments that are based on Gaifman locality of first-order logic. There are several parallels between the proof of Dawar and Otto and the proof in this thesis, although the case of ML[CK] is considerably more difficult.

Common knowledge is a notion of group knowledge that expresses that everybody in a group $\alpha$ knows some information $\varphi$, and also that everybody in $\alpha$ knows that everybody in $\alpha$ knows $\varphi$, and that everybody in $\alpha$ knows that everybody in $\alpha$ knows that everybody in $\alpha$ knows $\varphi$, and so on, for arbitrary iteration depth. The notion of common knowledge goes beyond the scope of FO but can be captured as a fixpoint construct, which is definable in $\mathsf{L}_\mu$ and MSO. Moreover, it can be captured in terms of basic modal logic over expanded S5 structures, with derived accessibility relations obtained as the transitive closures of unions of the original individual agents. We call these structures *common knowledge structures*, or CK structures for short. *Both* the classes of CK structures and finite CK structures are non-elementary. This view on epistemic modal common knowledge logic, as basic modal logic over CK structures, forms the backbone of our proof. However, no matter which perspective one chooses, be it fixpoints, MSO or CK structures, ML[CK] seems inherently locality averse, and each of these variations rules out a straightforward use of locality based methods.

Following Otto's strategy, we need to construct suitable (finite) bisimilar coverings for two pointed Kripke structures $\mathfrak{M}, w \sim^\ell \mathfrak{N}, v$. By suitable, we mean that the coverings are not merely $\ell$-bisimilar but $\mathsf{FO}_q$-equivalent. Hence, through the construction they must avoid features that are not controlled by $\ell$-bisimulation but can be defined by FO-formulae of quantifier rank $q$. These features concern small multiplicities and short cycles. Dealing with small multiplicities is easy. Short cycles are the hard part. The solution comes in the form of special Cayley groups that have a highly intricate yet regular edge pattern. We can regard Cayley groups with suitable generator sets, or rather relational encodings of their Cayley graphs, as special instances of CK frames. In these frames, generator combinations model coalitions of agents, and certain cosets model equivalence classes. CK frames, with their non first-order definable derived accessibility relations, have a very intricate edge pattern. In addition Cayley graphs imbue this edge pattern with a high degree of regularity that makes them amenable to structural analysis. Adding a propositional assignment to a relational encoding of a Cayley graph turns it into a Cayley structure. The first major result of this thesis states that every (finite) CK structure admits (finite) bisimilar coverings by Cayley structures. Thus, questions about CK structures can be reduced, up to bisimulation, to Cayley structures.

We already mentioned that it is hard to construct coverings that do not have short cycles. Moreover, in CK structures, and also in Cayley structures, it is not even immediately clear what we mean by avoiding short cycles. Certain cycles like loops or cycles within equivalence classes are inherent to these structures and cannot by avoided by any kind of construction. However, we are not at all interested in such cycles, which are basically trivial. Rather, we view the relational structure of CK frames as overlap patterns of equivalence classes w.r.t. different coalitions of agents. A suitable notion of acyclicity for these overlap patterns and Cayley groups that have the associated acyclicity property are available from [26]. This thesis presents a new analysis of these acyclic structures in terms of first-order expressiveness and Ehrenfeucht-Fraïssé games.

In playing Ehrenfeucht-Fraïssé games on relational structures, the main challenge is usually to match the distances between the pebbled elements up to certain thresholds. Ehrenfeucht-Fraïssé games on Cayley structures are no exception. However, as with cycles, one has to find the suitable notion of distance first. In Cayley structures, which are always connected, every pair of worlds is connected by some edge. But if one ignores these directly connecting edges, or rather equivalence classes, a look at the overlap patterns on a finer level of granularity leads to a meaningful notion of distance. Essentially, the challenge is to deal with locality issues on multiple scales simultaneously. A detailed analysis of these multi-scale acyclic overlap patterns allows for the proof of $\equiv_q$-equivalence for suitable Cayley structures that are $\ell$-bisimilar. That implies the main result of this thesis - a characterisation theorem for epistemic modal logic with common knowledge modalities:

**Theorem.** *Over* CK *structures, both classically and in the sense of finite model theory:*

$$\mathsf{ML} \equiv \mathsf{FO}/{\sim}$$

## Outline

Chapter 2 introduces modal logic and all the relevant related concepts and known results. It presents bisimulation, the theorems of van Benthem and Rosen and several epistemic modal logics. Amonge those logics, this thesis focuses particularly on common knowledge logic ML[CK]. The chapter ends with the introduction of common knowledge structures (CK structures), and a detailed presentation of our proof strategy for showing the characterisation theorem for ML[CK] in an epistemic context.

Chapter 3 is divided into two parts. The first part sketches Dawar's and Otto's proof of the characterisation of global modal logic over (finite) S5 structures. Along the way, several concepts like acyclicity and locality, which are important for characterising ML[CK], are introduced. The second part is concerned with constructing bisimilar coverings for CK structures. It defines Cayley structures and contains our first main result: every (finite) CK structure admits (finite) bisimilar coverings by Cayley structures; in particular, they admit coverings that possess special acyclicity properties.

These acyclic Cayley structures are the topic of Chapter 4. We develop a novel structure theory for acyclic Cayley structures. This theory is both interesting in its own right and crucial for playing first-order Ehrenfeucht-Fraïssé games on Cayley structures. The last section of this chapter focuses on a special property of Cayley structures called *freeness*. Essentially, freeness governs the responses in Ehrenfeucht-Fraïssé games on Cayley structures.

Chapter 5 is concerned with applying the results from Chapter 4 to Ehrenfeucht-Fraïssé games. It presents a complex invariant for these games and shows how to maintain it by employing freeness. This thesis culminates in Section 5.3, where the main theorem is proven. Chapter 5 closes with another characterisation theorem. We show how to adapt the methods from this thesis, without developing any more theory, to characterise relativized common knowledge, an extension of $\mathsf{ML}[\mathsf{CK}]$.

The results of this thesis appeared in [6]. The work was supported by the Deutsche Forschungsgemeinschaft (Project OT 147/6-1 Konstruktionen und Modelltheorie für Hypergraphen kontrollierter Azyklizität).

# 2 Preliminaries

In this chapter, we introduce our notation and recall some definitions and results that form the basis of this thesis. We assume that the reader is familiar with first-order logic and the associated standard notions, concepts, and model-theoretic methods; for an introduction we refer to [11, 12]. The necessary background on modal logic and related concepts, such as bisimulation, will be given in this chapter.

We denote the set of natural numbers by $\mathbb{N} = \{0, 1, 2, \dots\}$ and the set of integers by $\mathbb{Z}$. The *power set* of a set $A$ is denoted by $\mathcal{P}(A)$. For a set $A$ and a natural number $n \geq 1$, $A^n$ denotes the set of $A$-tuples of length $n$. Set inclusion is denoted by $\subseteq$ and strict inclusion by $\subsetneq$. For an equivalence relation $\approx$ on $A$, we denote the equivalence class of an element $a \in A$ by $[a]_\approx$ and write $A/{\approx} = \{[a]_\approx : a \in A\}$ for the set of all equivalence classes. For a binary relation $R \subseteq A^2$, we denote its transitive closure by $\mathrm{TC}(R) \subseteq A^2$. The set of $R$-successors $\{b \in A : (a, b) \in R\}$ of an element $a$ is denoted by $R[a]$. An *undirected Graph* is an ordered pair $(V, E)$, where $V$ is a set (the set of vertices) and $E \subseteq \mathcal{P}(V)$ (the edge relation) is comprised of 1-element and 2-element subsets of $V$; if $E$ is only comprised of 2-element subsets we call the graph *loop-free*. If $G$ is a graph, we also write $V[G]$ for its vertex set and $E[G]$ for its edge relation. A *walk* in a graph $G$ is a sequence $v_1, \dots, v_n \in V[G]$ of vertices such that $\{v_i, v_{i+1}\} \in E[G]$, for $1 \leq i < n$. A *path* is a walk that does not visit any vertex twice, with the possible exception of $v_1 = v_n$, in which case it is called a *cycle*. If a graph has no cycle, it is called *acyclic*. The *length* of a path is the number of its edges. The vertices $v_1$ and $v_n$ are called the endpoints of the walk or path, and we speak of a walk or path from $v_1$ to $v_n$. A set $W \subseteq V[G]$ is *connected* if for every $w, v \in W$ there is a path from $v$ to $w$ all of whose vertices are in $W$. The graph $G$ is *connected* if $V[G]$ is connected. A *tree* is an undirected, loop-free graph that is connected and acyclic.

Section 2.1 introduces basic modal logic, defines its syntax and semantics formally, and presents modal depth, a measure of complexity for modal formulae. Section 2.2 introduces bisimulation and bisimulation invariance, two of the central concepts of this thesis. The section also presents the modal Ehrenfeucht-Fraïssé theorem that characterises the expressive power of basic modal logic w.r.t. the graded version of bisimulation. The theorems of van Benthem and Rosen are discussed in Section 2.3. These results characterise basic modal logic in relation to first-order logic, van Benthem's theorem in the classical sense and Rosen's theorem in the sense of finite model theory. The two theorems are the starting point for the new research presented in this thesis. Section 2.4 introduces epistemic modal logic. It shows how to use modal logic and suitable Kripke structures to reason about knowledge, and it presents several extension of basic modal logic that play a prominent role in this thesis. The last section of this chapter, Section 2.5, presents the main result of this work and the central proof strategy for achieving it.

## 2.1 Basic modal logic

Propositional logic allows for building statements $\varphi$ out of propositional variables $p_i$ and the boolean connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$. Given a certain situation, such a propositional statement can be either true or false. Take as an example "it's raining in Berlin". We can imagine two possible situations, it's either raining or it isn't, but the statement is not necessarily true at all times. It is different with "all circles are round". This statement is necessarily true in any possible situation. Propositional logic cannot capture this difference. Modal logic extends propositional logic by the unary modal operators $\Box$ and $\Diamond$ (read "box" and "diamond") that can express *necessity* and *possibility*. Thus, if $\varphi$ expresses "all circles are round" and $\psi$ expresses "it's raining in Berlin", then $\Box\varphi$ means "all circles are necessarily round" and $\Diamond\psi$ would be "it's possibly raining in Berlin".

Observe that $\Box$ and $\Diamond$ are duals: something is necessarily true if and only if it is not possibly not true. Essentially, modal logic allows to qualify existing statements to build new ones. Other than necessity, there is a number of different interpretations for the $\Box$-operator. Depending on the context, $\Box\varphi$ could also mean "$\varphi$ is obligatory" (deontic), "in the future, $\varphi$ is always true" (temporal), "$\varphi$ is provable" (proof-theoretic), "one believes $\varphi$ to be true" (doxastic), or "one knows $\varphi$ to be true" (epistemic). These different interpretations allow for a wide range of applications for modal logics in mathematics, computer science, philosophy and linguistics. In this work, we will focus on the epistemic interpretation of modal logic.

In propositional logic, a certain situation is modelled by an *interpretation* over a set of variables $\{p_i : i \in I\}$ that maps each variable to either 1 or 0 (true or false). This model needs to be extended since modal logic allows to quantify over a number of different situations that are related. Such a semantics for modal logic was developed by Saul Kripke and André Joyal in the late 1950s and early 1960s. Intuitively, an interpretation for modal logic is a set of possible worlds, with each world being a propositional interpretation, together with a set of binary relations that connect the possible worlds. Formally:

**Definition 2.1.1.** A *modal signature* is a set that contains binary and unary relation symbols. For sets $\Gamma$ and $I$ we denote the modal signature over $\Gamma$ and $I$ as $\{(R_a)_{a\in\Gamma}, (P_i)_{i\in I}\}$. The individual elements $a \in \Gamma$, which index the binary relations, are refered to as *agents*.

A *Kripke structure* or *transition system* is a structure $\mathfrak{M} = (W, (R_a^{\mathfrak{M}})_{a\in\Gamma}, (P_i^{\mathfrak{M}})_{i\in I})$ over a modal signature. We refer to the elements of $W$ as *possible worlds* or simply *worlds*. A *Kripke frame* $(W, (R_a^{\mathfrak{M}})_{a\in\Gamma})$ is a reduct of a Kripke structure over a signature that contains only binary accessibility relations. A pointed Kripke structure is a pair $\mathfrak{M}, w$ where $\mathfrak{M}$ is a Kripke structure and $w$ is a distinguished world from $W$.

Together with the family $(P_i^{\mathfrak{M}})_{i\in I}$ each world $w \in W$ can be seen as a propositional interpretation $\mathfrak{I}_w\colon \{p_i : i \in I\} \rightarrow \{0, 1\}$ with $\mathfrak{I}_w(p_i) = 1$ if $w \in P_i^{\mathfrak{M}}$, and $\mathfrak{I}_w(p_i) = 0$ if $w \notin P_i^{\mathfrak{M}}$. The accessibility relations $(R_a^{\mathfrak{M}})_{a\in\Gamma}$ describe how the different worlds are related to each other. For example, in a temporal context $(w, v) \in R_a^{\mathfrak{M}}$ might mean that $v$ comes after $w$ in some process $a$, and in an epistemic context $(w, v) \in R_a^{\mathfrak{M}}$ means that agent $a$ cannot distinguish between the worlds $w$ and $v$. In this more general context that allows for a set of accessibility relations, there is not just one $\Box$-operator but an operator $\Box_a$ for each $a \in \Gamma$. Essentially, these $\Box_a$-operators are a locally restricted form of quantification: if $w$ is the distinguished world of a Kripke structure $\mathfrak{M}$, $\Box_a$ quantifies over all $v$ with $(w, v) \in R_a^{\mathfrak{M}}$. We we will give formal definitions for all this below.

Seeing modal operators as a local form of quantification leads to a comparison with another classical logic, first-order logic $\mathsf{FO}$. First-order logic allows for unrestricted quantification over all elements of a given structure: if $\varphi$ is an $\mathsf{FO}$-formula, then $\forall x \varphi$ is an $\mathsf{FO}$-formula that is true in a structure $\mathfrak{M}$ if $\varphi$ is true for all elements of $\mathfrak{M}$. In fact, basic modal logic can be regarded as a syntactic fragment of first-order logic (cf. Section 2.3). Coming from the point of view of $\mathsf{FO}$, the restriction from global to local quantification is accompanied by certain benefits and drawbacks. For example, some drawbacks are less expressive power and a restriction to structures over a signature that contains only unary and binary relation symbols. The benefits include decidable satisfiability and validity problems, tractable model checking, and nice model-theoretic properties such as the tree-model property and the finite model property. Furthermore, and very importantly, modal logic cannot distinguish between bisimilar structures (cf. Section 2.2). This is both an example for its limitations w.r.t. expressive power, but also one of its benefits when it comes to certain applications. The precise relation between basic modal logic and first-order logic is the content of van Benthem's theorem (cf. Section 2.3). This thesis further investigates this relation in a specific epistemic context.

*Some more conventions and notation*: If a Kripke structure is called $\mathfrak{M}$, its set of possible worlds is called $W$; if it is called $\mathfrak{N}$, its set of possible worlds is $V$. If it is clear from the context we often drop the superscript $\mathfrak{M}$ and write $R_a$ and $P_i$ instead of $R_a^{\mathfrak{M}}$ and $P_i^{\mathfrak{M}}$, respectively. We denote the set of $a$-successors $\{w' : (w, w') \in R_a^{\mathfrak{M}}\}$ of a world $w$ as $R_a^{\mathfrak{M}}[w]$. For the remainder of this chapter, we fix the sets $\Gamma$ and $I$ and assume that they are finite.

**Definition 2.1.2.** The *syntax of basic modal logic* $\mathsf{ML}$ over the signature $\{(R_a)_{a \in \Gamma}, (P_i)_{i \in I}\}$ is given by the following grammar:

$$\varphi ::= \quad \bot \quad | \quad p_i \quad | \quad \neg\varphi \quad | \quad (\varphi \wedge \varphi) \quad | \quad \Box_a \varphi$$

for all $i \in I$ and all $a \in \Gamma$.

We will make use of the following abbreviations:

- $\top := \neg\bot$
- $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$
- $\varphi \to \psi := \neg\varphi \vee \psi$
- $\Diamond_a \varphi := \neg\Box_a \neg\varphi$

**Definition 2.1.3.** Let $\mathfrak{M}, w$ be a pointed Kripke structure. The *semantics of* ML is defined inductively:

- $\mathfrak{M}, w \models \bot$ for no $w \in W$;
- $\mathfrak{M}, w \models p_i \quad \Leftrightarrow \quad w \in P_i^{\mathfrak{M}}$;
- $\mathfrak{M}, w \models \neg\varphi \quad \Leftrightarrow \quad \mathfrak{M}, w \not\models \varphi$;
- $\mathfrak{M}, w \models \varphi \wedge \psi \quad \Leftrightarrow \quad \mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$;
- $\mathfrak{M}, w \models \Box_a\varphi \quad \Leftrightarrow \quad \mathfrak{M}, w' \models \varphi$ for all $w' \in R_a^{\mathfrak{M}}[w]$.

For a formula $\varphi$ and a Kripke structure $\mathfrak{M}$ the set of worlds where $\varphi$ is satisfied is denoted by

$$[\![\varphi]\!]^{\mathfrak{M}} := \{w \in W : \mathfrak{M}, w \models \varphi\}.$$

The nesting depth of the modal operators $\Box_a$, or *modal depth*, measures the complexity of a modal formula. As we will later see, this syntactic property corresponds to expressive power.

**Definition 2.1.4.** The *modal depth* of an ML-formula is a mapping $\mathrm{md}\colon \mathsf{ML} \to \mathbb{N}$ that is inductively defined as follows:

- $\mathrm{md}(\bot) := 0$;
- $\mathrm{md}(p_i) := 0$;
- $\mathrm{md}(\neg\varphi) := \mathrm{md}(\varphi)$;
- $\mathrm{md}(\varphi \wedge \psi) := \max\{\mathrm{md}(\varphi), \mathrm{md}(\psi)\}$;
- $\mathrm{md}(\Box_a\varphi) := 1 + \mathrm{md}(\varphi)$.

The set of ML-formulae up to modal depth $\ell$ is denoted by $\mathsf{ML}_\ell$.

Two ML-formulae $\varphi, \psi$ are *logically equivalent* if they have exactly the same models, i.e. $\mathfrak{M}, w \models \varphi$ if and only if $\mathfrak{M}, w \models \psi$, for all pointed Kripke structures $\mathfrak{M}, w$. Two pointed Kripke structures $\mathfrak{M}, w$, $\mathfrak{N}, v$ are ML-equivalent, denoted $\mathfrak{M}, w \equiv_{\mathsf{ML}} \mathfrak{N}, v$, if they satisfy exactly the same ML-formulae, i.e. if

$$\mathfrak{M}, w \models \varphi \quad \Leftrightarrow \quad \mathfrak{N}, v \models \varphi$$

for all $\varphi \in \mathsf{ML}$. Two pointed Kripke structures are $\mathsf{ML}_\ell$-equivalent if they satisfy exactly the same ML-formulae up to modal depth $\ell$, denoted $\mathfrak{M}, w \equiv_{\mathsf{ML}}^{\ell} \mathfrak{N}, v$.

## 2.2 Bisimulation

Bisimulation invariance may be regarded as the crucial semantic property of modal logics with their many and diverse applications that range from specification of process behaviours to reasoning about knowledge. As a notion of equivalence, bisimulation captures the relevant properties of transition systems or Kripke structures that do not depend on some specific encoding of a structure. That makes bisimulation invariance the essential semantic property of any logic that is meant to deal with the relevant phenomena of transition systems. Various modal logics share this preservation property and can, moreover, often be characterised in relation to classical logics as precisely capturing the bisimulation invariant properties of transition systems. This turns bisimulation invariance into a criterion of expressive completeness that is the core theme of this work.

**Definition 2.2.1.** A *bisimulation* is a non-empty binary relation $Z \subseteq W \times V$ between two Kripke structures $\mathfrak{M} = (W, (R_a^{\mathfrak{M}})_{a\in\Gamma}, (P_i^{\mathfrak{M}})_{i\in I})$ and $\mathfrak{N} = (V, (R_a^{\mathfrak{N}})_{a\in\Gamma}, (P_i^{\mathfrak{N}})_{i\in I})$ such that for all $(w, v) \in Z$:

1. $w \in P_i^{\mathfrak{M}} \Leftrightarrow v \in P_i^{\mathfrak{N}}$ for all $i \in I$;
2. for all $a \in \Gamma$ and all $w' \in R_a^{\mathfrak{M}}[w]$ there is a world $v' \in R_a^{\mathfrak{N}}[v]$ with $(w', v') \in Z$;
3. for all $a \in \Gamma$ and all $v' \in R_a^{\mathfrak{N}}[v]$ there is a world $w' \in R_a^{\mathfrak{M}}[w]$ with $(w', v') \in Z$.

The second and third property are the so-called *back-and-forth properties.* $\mathfrak{M} \sim \mathfrak{N}$ denotes that there exists a bisimulation between $\mathfrak{M}$ and $\mathfrak{N}$. Two pointed Kripke structures $\mathfrak{M}, w$ and $\mathfrak{N}, v$ are *bisimilar*, denoted $\mathfrak{M}, w \sim \mathfrak{N}, v$, if there is a bisimulation $Z$ between them with $(w, v) \in Z$.

Additionally, there is a graded notion of bisimulation that describes bisimilarity between two structures up to a certain finite degree $\ell$.

**Definition 2.2.2.** An *$\ell$-bisimulation* is a family $(Z_k)_{0\leq k\leq\ell}$ of non-empty binary relations $Z_k \subseteq W \times V$, $0 \leq k \leq \ell$, between two Kripke structures $\mathfrak{M} = (W, (R_a^{\mathfrak{M}})_{a\in\Gamma}, (P_i^{\mathfrak{M}})_{i\in I})$ and $\mathfrak{N} = (V, (R_a^{\mathfrak{N}})_{a\in\Gamma}, (P_i^{\mathfrak{N}})_{i\in I})$ such that for all $0 \leq k \leq \ell$ and all $(w, v) \in Z_k$:

1. $w \in P_i^{\mathfrak{M}} \Leftrightarrow v \in P_i^{\mathfrak{N}}$ for all $i \in I$;

and for all $1 \leq k \leq \ell$ and all $(w, v) \in Z_k$:

2. for all $a \in \Gamma$ and all $w' \in R_a^{\mathfrak{M}}[w]$ there is a world $v' \in R_a^{\mathfrak{N}}[v]$ with $(w', v') \in Z_{k-1}$;
3. for all $a \in \Gamma$ and all $v' \in R_a^{\mathfrak{N}}[v]$ there is a world $w' \in R_a^{\mathfrak{M}}[w]$ with $(w', v') \in Z_{k-1}$.

$\mathfrak{M} \sim^\ell \mathfrak{N}$ denotes that there exists an $\ell$-bisimulation between $\mathfrak{M}$ and $\mathfrak{N}$. Two pointed Kripke structures $\mathfrak{M}, w$ and $\mathfrak{N}, v$ are *$\ell$-bisimilar*, denoted $\mathfrak{M}, w \sim^\ell \mathfrak{N}, v$ if there is an $\ell$-bisimulation $(Z_k)_{0\leq k\leq\ell}$ between them with $(w, v) \in Z_\ell$.

A straightforward induction shows that basic modal formulae are *bisimulation-invariant*, i.e. they cannot distinguish between bisimilar structures:

$$\mathfrak{M}, w \sim \mathfrak{N}, v \quad \Rightarrow \quad (\mathfrak{M}, w \models \varphi \Leftrightarrow \mathfrak{N}, v \models \varphi),$$

for all $\varphi \in \mathsf{ML}$.

Furthermore, $\mathsf{ML}_\ell$ formulae cannot distinguish between $\ell$-bisimilar structures:

$$\mathfrak{M}, w \sim^\ell \mathfrak{N}, v \quad \Rightarrow \quad (\mathfrak{M}, w \models \varphi \Leftrightarrow \mathfrak{N}, v \models \varphi),$$

for all $\varphi \in \mathsf{ML}_\ell$. Hence, bisimilarity implies $\mathsf{ML}$-equivalence. The other direction does not hold; there are $\mathsf{ML}$-equivalent structures that are not bisimilar. However, this direction holds if we consider certain subclasses like the class of all finitely branching Kripke structures. On such classes, bisimilarity and $\mathsf{ML}$-equivalence are equivalent. In contrast, $\ell$-bisimilarity and $\mathsf{ML}_\ell$-equivalence are always equivalent if the modal signature of the structures is finite. This fact is stated by the *modal Ehrenfeucht-Fraïssé theorem.*

### 2.2.1 The modal Ehrenfeucht-Fraïssé theorem

The modal Ehrenfeucht-Fraïssé theorem states that Kripke structures over a finite signature are $\ell$-bisimilar if and only if they are $\mathsf{ML}_\ell$-equivalent. The bisimulation game and characteristic formulae of structures are two other ways of characterising $\ell$-bisimilarity. The former rephrases bisimulation as a highly intuitive game and the latter expresses the $\ell$-bisimulation type of a pointed Kripke structure as an $\mathsf{ML}_\ell$-formula.

*The bisimulation game.* An alternative way to define bisimulation that is both practical and highly intuitive is the bisimulation game. Given two pointed Kripke structures $\mathfrak{M}, w_0$ and $\mathfrak{N}, v_0$ the game is played by two players, player **I** and player **II**. Player **I** aims to find a difference between the two structures with respect to bisimulation while player **II** aims to show that the structures are bisimilar.

The positions of the game are pairs $(w, v)$ where $w$ is world of $\mathfrak{M}$ and $v$ is world of $\mathfrak{N}$, with an intuitive meaning that the board of the game consists of the two structures and one pebble is placed in $\mathfrak{M}$ on $w$ and another in $\mathfrak{N}$ on $v$; the initial position is $(w_0, v_0)$.

In every round, the players move the pebbles along the edges of the board. One round is played as follows: **I** chooses a structure and a label $a \in \Gamma$ and moves the pebble in the structure of his choice along an $R_a$-edge. **II** responds by moving the pebble in the other structure along an $R_a$-edge; the result of the round is a successor position $(w', v')$. Either player loses when stuck; additionally, **II** loses if the play reaches a position $(w, v)$ where the pebbled elements do not satisfy the same atomic propositions, i.e. there is an $i \in I$ such that $w \in P_i^{\mathfrak{M}}$ and $v \notin P_i^{\mathfrak{N}}$, or vice versa.

The unbounded game continues indefinitely, and all infinite plays are won by player **II**. The $\ell$-round game, for some $\ell \in \mathbb{N}$, is won by player **II** if she does not lose for $\ell$ rounds.

A *winning strategy* for either player is a strategy that guarantees a win no matter what the opposing player does. Two pointed Kripke structures are bisimilar if and only if **II** has a winning strategy in the associated bisimulation game. In other words, bisimulations describe winning strategies and vice versa.

*Characteristic formulae.* The proof that $\mathsf{ML}_\ell$-equivalent structures are also $\ell$-bisimilar relies on the *characteristic formulae* $\chi^\ell_{\mathfrak{M},w}$. The formula $\chi^\ell_{\mathfrak{M},w}$ describes the structure $\mathfrak{M}, w$ up to $\ell$-bisimulation. Essentially, $\mathfrak{N}, v \models \chi^\ell_{\mathfrak{M},w}$ states that player **II** has a winning strategy in the $\ell$-round bisimulation game on $\mathfrak{M}$ and $\mathfrak{N}$ from position $(w, v)$. We can define these formulae by an induction on $\ell$, simultaneously for all pointed Kripke structures:

$$\chi^0_{\mathfrak{M},w} := \bigwedge_{\{i \in I : w \in P_i\}} p_i \wedge \bigwedge_{\{i \in I : w \notin P_i\}} \neg p_i$$

$$\chi^{\ell+1}_{\mathfrak{M},w} := \chi^0_{\mathfrak{M},w} \wedge \bigwedge_{a \in \Gamma} \bigwedge_{u \in R_a[w]} \Diamond_a \chi^\ell_{\mathfrak{M},u} \wedge \bigwedge_{a \in \Gamma} \Box_a \bigvee_{u \in R_a[w]} \chi^\ell_{\mathfrak{M},u}$$

To show that these formulae are in fact $\mathsf{ML}_\ell$-formulae one has to show by induction that up to logical equivalence there are only finitely many formulae of the form $\chi^\ell_{\mathfrak{M},w}$, for all $\ell \in \mathbb{N}$: Since $I$ is finite, there are only finitely many worlds with respect to atomic equivalence. If the induction hypothesis states that there are up to logical equivalence only finitely many formulae of type $\chi^\ell_{\mathfrak{M},u}$, then $\Gamma$ being finite implies that the conjunction of the set $\{\chi^\ell_{\mathfrak{M},u} : u \in R_a[w]\}$ is essentially also finite, even if $w$ has infinitely many $a$-successors.

**Theorem 2.2.3** (modal Ehrenfeucht-Fraïssé theorem)**.** *Let $\mathfrak{M}, w$ and $\mathfrak{N}, v$ be pointed Kripke structures over a finite signature. Then the following are equivalent:*

1. $\mathfrak{M}, w \sim^\ell \mathfrak{N}, v$
2. ***II** has a winning strategy in the $\ell$-round game on $\mathfrak{M}$ and $\mathfrak{N}$ from $(w, v)$*
3. $\mathfrak{N}, v \models \chi^\ell_{\mathfrak{M},w}$
4. $\mathfrak{M}, w \equiv^\ell_{\mathsf{ML}} \mathfrak{N}, v$

**Corollary 2.2.4.** *Every formula $\varphi$ over the finite modal signature $\{(R_a)_{a \in \Gamma}, (P_i)_{i \in I}\}$ that is $\ell$-bisimulation-invariant is equivalent to the disjunction*

$$\bigvee_{\mathfrak{M},w \models \varphi} \chi^\ell_{\mathfrak{M},w},$$

*which is in fact finite, and hence equivalent to an $\mathsf{ML}_\ell$-formula.*

## 2.3 The theorems of van Benthem and Rosen

Modal characterisation theorems state that modal logics like $\mathsf{ML}$ or the modal $\mu$-calculus $\mathsf{L}_\mu$ are expressively equivalent to the bisimulation-invariant fragment of some classical logic like $\mathsf{FO}$ or $\mathsf{MSO}$. The standard translation shows that basic modal logic can be regarded as a fragment of first-order logic. We define the mapping $\mathrm{st}\colon \mathsf{ML} \to \mathsf{FO}$ inductively:

- $\mathrm{st}(p_i) := P_i x$
- $\mathrm{st}(\neg\varphi) := \neg\mathrm{st}(\varphi)$
- $\mathrm{st}((\varphi \wedge \psi)) := (\mathrm{st}(\varphi) \wedge \mathrm{st}(\psi))$
- $\mathrm{st}(\Box_a\varphi) := \exists y(R_a xy \wedge \mathrm{st}(\varphi)[y/x])$, where $y$ is a fresh variable

Additionally, since every ML-formula is bisimulation-invariant and the syntactic translation implies $\mathfrak{M}, w \models \varphi$ if and only if $\mathfrak{M} \models \mathrm{st}(\varphi)(w)$, for all $\varphi \in \mathsf{ML}$ and all pointed Kripke structures, every formula of st(ML) is bisimulation-invariant. Hence, st(ML) is not only a fragment of first-order logic but of its bisimulation-invariant fragment

$$\mathsf{FO}/{\sim} \; = \{\varphi(x) \in \mathsf{FO} : \varphi(x) \text{ is bisimulation-invariant}\}.$$

The question remains: can ML actually express all first-order definable properties that are bisimulation-invariant? Van Benthem's theorem answers this question positively.

**Theorem 2.3.1** (van Benthem, [29])**.** *Basic modal logic is expressively equivalent to the bisimulation-invariant fragment of first-order logic; in short* $\mathsf{ML} \equiv \mathsf{FO}/{\sim}$.

An alternative phrasing is that an FO-formula is bisimulation-invariant if and only if it is logically equivalent to an ML-formula. Over the years, many variations of this theorem have been shown. Some characterised more expressive modal logics like $\mathsf{L}_\mu$, others looked at the equivalence between ML and FO/∼ over special classes of Kripke structures. One important theorem of the latter variety is Rosen's theorem, the finite model version of van Benthem's theorem: $\mathsf{ML} \equiv \mathsf{FO}/{\sim}$ over the class of finite Kripke structures, i.e. an FO-formula is bisimulation-invariant over the class of *finite* Kripke structures if and only if it is logically equivalent to an ML-formula over the class of *finite* Kripke structures. In general, if $L_1$ and $L_2$ are logics and $\mathcal{C}$ is a class of pointed Kripke structures, we write $L_1 \equiv L_2$ over $\mathcal{C}$ if for every $\varphi \in L_1$ there is a $\varphi' \in L_2$ such that $\mathfrak{M}, w \models \varphi$ if and only if $\mathfrak{M}, w \models \varphi'$, for every $\mathfrak{M}, w \in \mathcal{C}$, and for every $\varphi' \in L_2$ there is a $\varphi \in L_1$ such that $\mathfrak{M}, w \models \varphi'$ if and only if $\mathfrak{M}, w \models \varphi$, for every $\mathfrak{M}, w \in \mathcal{C}$.

**Theorem 2.3.2** (Rosen, [28])**.** $\mathsf{ML} \equiv \mathsf{FO}/{\sim}$ *over the class of finite Kripke structures.*

For every class $\mathcal{C}$ that is FO-axiomatizable, the statement $\mathsf{ML} \equiv \mathsf{FO}/{\sim}$ over $\mathcal{C}$ follows from a straightforward adaptation of van Benthem's original proof. However, with Rosen's theorem this is not the case. The finite model version changes both sides of the equivalence, gives it a different meaning and requires very different proof techniques.

Another constructive proof that shows the classical as well as the finite model version at the same time was given by Otto in [24]. This proof only requires prior knowledge of the modal Ehrenfeucht-Fraïssé theorem and of Ehrenfeucht-Fraïssé games for first-order logic. For both versions, it further shows that every FO/∼-formula of quantifier rank $q$ is equivalent to an ML-formula of modal depth $\ell := 2^q - 1$. This is established by showing that every bisimulation-invariant FO-formula $\varphi$ of quantifier rank $q$ is in fact $\ell$-bisimulation-invariant; that $\varphi$ is equivalent to an $\mathsf{ML}_\ell$-formula follows then from

Corollary 2.2.4. The core idea is to show that $\varphi$ is $\ell$-local, i.e. whether or not $\varphi$ is satisfied in $\mathfrak{M}, w$ only depends on the $\ell$-neighbourhood $\mathcal{N}^\ell(w)$ of $w$:

$$\mathfrak{M}, w \models \varphi \quad \Leftrightarrow \quad \mathfrak{M} \upharpoonright \mathcal{N}^\ell(w), w \models \varphi$$

This can be established by elementary arguments that use Ehrenfeucht-Fraïssé games for FO and a basic model construction that preserves bisimilarity. This model construction unravels the $\ell$-neighbourhood of $w$ into a tree; in other words, it makes the $\ell$-neighbourhood acyclic.

Otto's proof shows that the satisfiability of a bisimulation-invariant FO-formula is an entirely local matter. Furthermore, the proof only makes use of model constructions that manipulate a small neighbourhood around the distinguished world of pointed Kripke structure. Stronger variants of bisimulation, such as global bisimulation, need technically more advanced locality arguments, and more intricate model constructions that make a structure locally acyclic for every small neighbourhood. We will explain these arguments more in-depth in Chapter 3 as a preparation for our multi-scale acyclicity arguments for the characterisation of common knowledge logic.

## 2.4 Epistemic modal logic

Epistemic modal logic is concerned with the reasoning about knowledge. We represent knowledge with special Kripke structures, so-called *epistemic* or S5 structures, and reason about knowledge with various *epistemic modal logics* that we will introduce in this section. In this work, we approach the field of epistemic modal logic from a model-theoretic point of view, compared with the more traditional proof-theoretic one. Instead of the power of proof calculi, this work is concerned with the *expressive* power of epistemic modal logics.

**Definition 2.4.1.** An *epistemic (or* S5*) frame* is a Kripke frame $(W, (R_a)_{a \in \Gamma})$ where every accessibility relation $R_a$ is an equivalence relation. An *epistemic (or* S5*) structure* is a Kripke structure that is based on an epistemic frame.

The logic S5 is defined as the set of all basic modal logic formulae that are valid in all pointed S5 structures:

$$\mathsf{S5} = \{\varphi \in \mathsf{ML} : \mathfrak{M}, w \models \varphi, \text{ for all S5 structures } \mathfrak{M} \text{ and all } w \in \mathfrak{M}\}$$

This is the model-theoretic way to define S5. The arguably more traditional, yet equivalent, way would be via some proof calculus (cf. Chapter 7 of [10]): define S5 as the set of ML-formulae that can be inferred via modus ponens (from $\varphi$ and $\varphi \to \psi$, infer $\psi$) and $\Box_a$-generalisation (from $\varphi$, infer $\Box_a\varphi$) from all instantiations of propositional tautologies, the axiom $K$ ($\Box_a(\varphi \to \psi) \to (\Box_a\varphi \to \Box_a\psi)$), and the three epistemic axioms. These three axioms are: the axiom of veridicality, the positive introspection axiom and the negative introspection axiom. All three of them are valid in S5 structures.

1. $\Box_a\varphi \to \varphi$ veridicality

2. $\Box_a\varphi \to \Box_a\Box_a\varphi$ positive introspection
3. $\neg\Box_a\varphi \to \Box_a\neg\Box_a\varphi$ negative introspection

In the epistemic context, we interpret the $\Box_a$-operator as "agent $a$ knows something" and the $\Diamond_a$-operator as "agent $a$ considers something possible". Hence, the axiom of veridicality expresses that "if agent $a$ knows something, then it is true", the positive introspection axiom expresses that "if agent $a$ knows something, then she knows that she knows it", and the negative introspection axiom expresses that "if agent $a$ does not know something, then she knows that she does not know it".

Every epistemic axiom corresponds to a frame property: veridicality corresponds to reflexivity ($(w, w) \in R_a$ for all $w$), positive introspection corresponds to transitivity ($(w, v) \in R_a$ and $(v, u) \in R_a$ imply $(w, u) \in R_a$), and negative introspection corresponds to a euclidean relation ($(w, v) \in R_a$ and $(w, u) \in R_a$ imply $(v, u) \in R_a$). This correspondence works in the follwing sense: first, $\Box_a\varphi \to \varphi$ is true, for every $a \in \Gamma$ and every $\varphi \in \mathsf{ML}$, at a pointed Kripke structure that is based on a reflexive frame, irrespective of the atomic propositions. Second, for a frame $(W, (R_a)_{a\in\Gamma})$ with a world $w$ such that $(w, w) \notin R_a$ we can construct a set $P \subseteq W$ such that $(W, (R_a)_{a\in\Gamma}, P), w \models \Box_a p \to p$ is not true. The same is true for the other two correspondences. Furthermore, if a relation is reflexive and euclidean, it is symmetric, and if it is transitive and symmetric it is euclidean.

With S5 structures we represent knowledge that the agents possess, and we use modal logic to reason about that knowledge. For a world $w$ in an epistemic frame we denote by $[w]_a$ the *a-equivalence* class or the *a-cluster* $\{v \in W : v \in R_a[w]\}$ of $w$. The interpretation is that the worlds of $[w]_a$ look locally indistinguishable to agent $a$. Hence, she knows something in world $w$ if it is true in all worlds that look indistinguishable to her:

$$\mathfrak{M}, w \models \Box_a\varphi \quad \Leftrightarrow \quad \mathfrak{M}, v \models \varphi \text{ for all } v \in [w]_a$$

Agent $a$ considers something possible in $w$ if it is true in at least one world that looks indistinguishable from $w$:

$$\mathfrak{M}, w \models \Diamond_a\varphi \quad \Leftrightarrow \quad \mathfrak{M}, v \models \varphi \text{ for some } v \in [w]_a$$

This representation of knowledge might seem counterintuitive at first. More possible worlds do not mean more knowledge, but more *uncertainty*. Hence, the fewer worlds a cluster contains the more knowledge an agent has. Moreover, basic modal logic allows for expressing an agent's knowledge about another agent's knowledge. For example, $\Box_a\Diamond_b\varphi$ expresses that agent $a$ knows that agent $b$ considers $\varphi$ possible. However, basic modal logic does not possess the expressive power to say that something is common knowledge among a certain coalition of agents.

### 2.4.1 Common knowledge logic

Common knowledge (CK) is a notion of group knowledge that we add to basic modal logic in order to express that some formula $\varphi$ is common knowledge among the agents of a set $\alpha \subseteq \Gamma$, i.e. every agent of $\alpha$ knows that every agent of $\alpha$ knows that every agent of $\alpha$ knows... that $\varphi$. More formally, to define the syntax of *common knowledge logic* $\mathsf{ML}[\mathsf{CK}]$ we add, for every set $\alpha \subseteq \Gamma$, the modal operator $\Box_\alpha$ to our modal language. The semantics of $\mathsf{ML}[\mathsf{CK}]$ is defined as follows:

$$\mathfrak{M}, w \models \Box_\alpha \varphi \quad :\Leftrightarrow \quad \mathfrak{M}, w \models \Box_{a_1}\Box_{a_2}\dots\Box_{a_\ell}\varphi$$

for all $\ell \in \mathbb{N}$ and all $a_1, a_2, \dots, a_\ell \in \alpha$. Essentially, common knowledge logic goes beyond the expressive power of $\mathsf{ML}$ because it speaks about knowledge that agents possess about other agents' knowledge of *arbitrary* depth. This is a concept that cannot be expressed at the level of first-order logic.

Common knowledge adds a new feature to our epistemic language that is essential if we want to speak about knowledge that is necessary for certain group interactions. Consider the following example from [13] Chapter 6.1. Two allied armies $A$ and $B$ are stationed on two hills with a valley between them that contains a third army $C$ that is hostile to $A$ and $B$. The commanders of $A$ and $B$ know respectively that their combined strength suffices to defeat $C$ if they coordinate their attack. However, they also know that both their armies are not strong enough to defeat $C$ on their own. In order to coordinate their attack, the commander of $A$ sends a carrier to $B$ with a message that he wants both armies to attack together at the next sunrise. However, sending the message is not enough because commander $A$ also needs to know that $B$ received his message. Otherwise, for example if the carrier got caught by army $C$, only $A$ would attack at sunrise and lose the battle. Imagine the commander of $B$ receives the message and sends a response to $A$ that confirms the plans. In this case, commander $B$ could not attack at sunrise because she does not know if commander $A$ received her response. Even if commander $A$ received commander $B$'s response he could not attack because $A$ knows that $B$ does not know that $A$ received the response. In fact, no finite number of sending a carrier with a message that confirms the plan would suffice because the respective commander would not know if the other commander received her last message. The key is to somehow establish common knowledge. Only if the battle plan is common knowledge among the two commanders can they proceed with their attack.

Common knowledge logic has another characterisation in terms of graph theory that is both highly intuitive and practical. In order to describe it, we need some terminology from graph theory. A *(labelled) path* in a Kripke structure is a finite sequence $p$ of the form $w_1, a_1, w_2, a_2, \dots, a_\ell, w_{\ell+1}$, where the $w_i$ are possible worlds and the $a_i$ are agents, with $w_{i+1} \in R_{a_i}[w_i]$, for all $1 \leq i \leq \ell$; $p$ is an *$\alpha$-path*, $\alpha \subseteq \Gamma$, if $a_i \in \alpha$, for all $1 \leq i \leq \ell$. A world $w'$ is $\alpha$-reachable from a world $w$ if there is an $\alpha$-path that starts at $w$ and ends at $w'$. In epistemic structures, we write $[w]_\alpha$, $\alpha \subseteq \Gamma$, for the set of worlds that are $\alpha$-reachable from $w$; since all relations $R_a$ are equivalence relations, every set of agents $\alpha$ induces another equivalence relation on the set of possible worlds of an epistemic structure by partitioning it into sets $[w]_\alpha$. It is straightforward to see the following equivalence (cf. Chapter 2 in [13]):

$$\mathfrak{M}, w \models \Box_\alpha \varphi \quad \Leftrightarrow \quad \mathfrak{M}, v \models \varphi \text{ for all } v \in [w]_\alpha$$

We like to point out that $[w]_a \subseteq [w]_\alpha$, for all $a \in \alpha$. This means that the common knowledge of a group at a world $w$ is always less or equal the knowledge of the individual agents at $w$, because agent $a$ might know $\varphi$ but agent $b$ might not know that $a$ knows $\varphi$.

This characterisation of common knowledge in terms of reachability in graphs is a well-known reason why common knowledge is not expressible in $\mathsf{ML}$ since graph-reachability goes beyond the expressive power of first-order logic. However, common knowledge can be captured as a fixpoint construct based on the formula $\Box_\alpha\varphi \leftrightarrow \bigwedge_{a\in\alpha}\Box_a(\varphi \wedge \Box_\alpha\varphi)$. Thus, $\mathsf{ML}[\mathsf{CK}]$ is a fragment of the modal $\mu$-calculus $\mathsf{L}_\mu$, in particular its alternation-free fragment, and the bisimulation-invariant fragment of monadic second-order logic $\mathsf{MSO}$.

Similar to basic modal logic, the expressive power of $\mathsf{ML}[\mathsf{CK}]$ can be characterised by a bisimulation game. The *common knowledge bisimulation game* is played like the bisimulation game with an additional *common knowledge move*: player **I** chooses a structure and a *set* $\alpha \subseteq \Gamma$ and moves the pebble along an *$\alpha$-path* in the structure of his choice. Player **II** responds by moving the pebble in the other structure along an $\alpha$-path. All other rules stay the same. Modal depth of $\mathsf{ML}[\mathsf{CK}]$-formulae is defined as follows.

**Definition 2.4.2.** The *modal depth* of an $\mathsf{ML}[\mathsf{CK}]$-formula is a map $\mathrm{md}\colon \mathsf{ML}[\mathsf{CK}] \to \mathbb{N}$ that is inductively defined as follows:

- $\mathrm{md}(\bot) := 0$;
- $\mathrm{md}(p_i) := 0$;
- $\mathrm{md}(\neg\varphi) := \mathrm{md}(\varphi)$;
- $\mathrm{md}(\varphi \wedge \psi) := \max\{\mathrm{md}(\varphi), \mathrm{md}(\psi)\}$;
- $\mathrm{md}(\Box_a\varphi) := 1 + \mathrm{md}(\varphi)$;
- $\mathrm{md}(\Box_\alpha\varphi) := 1 + \mathrm{md}(\varphi)$.

The set of $\mathsf{ML}[\mathsf{CK}]$-formulae up to modal depth $\ell$ is denoted by $\mathsf{ML}[\mathsf{CK}]_\ell$.

Analogously, we have an Ehrenfeucht-Fraïssé theorem for common knowledge logic.

**Theorem 2.4.3.** *[2] Let $\mathfrak{M}, w$ and $\mathfrak{N}, v$ be pointed Kripke structures over a finite signature. Then the following are equivalent:*

1. ***II** has a winning strategy in the $\ell$-round $\mathsf{CK}$ game on $\mathfrak{M}$ and $\mathfrak{N}$ from $(w, v)$*
2. *$\mathfrak{M}, w \equiv^\ell_{\mathsf{ML}[\mathsf{CK}]} \mathfrak{N}, v$*

We will use the common knowledge bisimulation game to compare the expressive power of $\mathsf{ML}[\mathsf{CK}]$ to even more expressive modal logics.

### 2.4.2 Public announcement

Making a public announcement is a possible action to create common knowledge. The idea is that after a public announcement is made every agent knows the content of the announcement and it becomes common knowledge. However, it is more complicated than that. On the formal side, after a fact $\varphi$ is announced only the possible worlds remain where $\varphi$ is true, all other worlds are eliminated. But the elimination of all worlds where $\varphi$ is not true might result in a world where $\varphi$ is not true *after* the announcement, although it had been true before the announcement. To illustrate, we return to the example of the armies that need to coordinate their attack. Imagine that commander $A$ knows that the attack will take place at sunrise but commander $B$ does not know this yet. This situation can be formalised by two worlds $w_1$ and $w_2$ and a proposition $P$ that is interpreted as "the attack takes place at sunrise"; $P$ is true at $w_1$ but not at $w_2$. Commander $A$ is able to distinguish these two worlds but $B$ is not. Imagine that, somehow, $A$ makes the public announcement "you don't know it yet, but we will attack at sunrise", formally $p \wedge \neg\Box_b p$. This is true at $w_1$ because $w_1 \models p$ and we also have $w_2 \models \neg p$ and $w_1$ and $w_2$ look indistinguishable to $B$. But it is not true in $w_2$, hence $B$ regards $w_2$ no longer possible and it can be eliminated. Now, $p$ is common knowledge, $w_1 \models \Box_{\{a,b\}} p$. However, the formula $p \wedge \neg\Box_b p$ is no longer true at $w_1$ because $B$ knows $p$ after the announcement. Thus, the announcement of $p \wedge \neg\Box_b p$ changed the truth of itself at $w_1$. This is what we mean when we speak about the *dynamic* nature of public announcement.

Formally, to add public announcement to a modal logic, we add the syntax rule

$$\varphi \quad ::= \quad [\varphi]\varphi.$$

Intuitively, a formula $[\varphi]\psi$ is true if $\psi$ is true after the public announcement of $\varphi$. In order to define proper semantics, we need to introduce the notion of a *relativized structure*: if $\mathfrak{M} = (W, (R_a)_{a\in\Gamma}, (P_i)_{i\in I})$ is a Kripke structure and $\varphi$ a modal formula, then $\mathfrak{M}$ *relativized to* $\varphi$ is the structure $\mathfrak{M}^\varphi = (W^\varphi, (R_a^\varphi)_{a\in\Gamma}, (P_i^\varphi)_{i\in I})$ with

- $W^\varphi = \{w \in W : \mathfrak{M}, w \models \varphi\}$,
- $R_a^\varphi = R_a \cap (W^\varphi \times W^\varphi)$ and
- $P_i^\varphi = P_i \cap W^\varphi$.

The formal semantics is given by

$$\mathfrak{M}, w \models [\varphi]\psi \quad :\Leftrightarrow \quad (\mathfrak{M}, w \models \varphi \;\Rightarrow\; \mathfrak{M}^\varphi, w \models \psi).$$

Note that $\mathfrak{M}, w \models [\varphi]\psi$, for all $\psi$, if $\mathfrak{M}, w \models \neg\varphi$. By $\mathsf{ML}[\mathsf{PA}]$ we denote the logic that adds public announcement to basic modal logic. Although syntactically richer, $\mathsf{ML}[\mathsf{PA}]$ is not more expressive than $\mathsf{ML}$ which might be surprising at first glance (first shown in [27]). Before we continue with the comparison of expressive power, we would like to introduce some helpful notation: for two logics $L_1$ and $L_2$, we write $L_1 \leq L_2$ if for every formula $\varphi_1 \in L_1$ there is a logically equivalent formula $\varphi_2 \in L_2$. We write $L_1 \equiv L_2$ if $L_1 \leq L_2$ and $L_2 \leq L_1$; we write $L_1 \lneq L_2$ if $L_1 \leq L_2$ but not $L_1 \equiv L_2$.

Consider the following equivalences; their proofs are straightforward.

**Proposition 2.4.4.** [27] The following equivalences hold:

$$\begin{aligned}
[\varphi]p &\leftrightarrow (\varphi \rightarrow p) \\
[\varphi]\neg\psi &\leftrightarrow (\varphi \rightarrow \neg[\varphi]\psi) \\
[\varphi](\psi \wedge \chi) &\leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi) \\
[\varphi]\Box_a\psi &\leftrightarrow (\varphi \rightarrow \Box_a[\varphi]\psi) \\
[\varphi][\psi]\chi &\leftrightarrow [\varphi \wedge [\varphi]\psi]\chi
\end{aligned}$$

These validities provide us with a rewrite system that allows us to remove public announcements by pushing them, one by one, to the inside where they can be eliminated. The result is an equivalent formula $\varphi^* \in \mathsf{ML}$, for every $\varphi \in \mathsf{ML[PA]}$. Yet, $\mathsf{ML[PA]}$ allows us to express certain statements in a more succinct way.

Of course, it is also possible to add public announcement to $\mathsf{ML[CK]}$ in order to obtain the logic $\mathsf{ML[CK, PA]}$. One might assume that a straightforward generalisation of the principle $[\varphi]\Box_a\psi \leftrightarrow (\varphi \rightarrow \Box_a[\varphi]\psi)$ to $[\varphi]\Box_\alpha\psi \leftrightarrow (\varphi \rightarrow \Box_\alpha[\varphi]\psi)$ is possible. However, the latter schema is not valid. $\mathsf{ML[CK, PA]}$ is in fact more expressive than $\mathsf{ML[CK]}$ [2]. As mentioned above, we can show this by using the bisimulation game for $\mathsf{ML[CK]}$.

Consider the $\mathsf{ML[CK, PA]}$-formula $\varphi := [\neg p \rightarrow \Box_a\neg p]\Box_{a,b}\neg p$. We will present two families of pointed Kripke structures that can be distinguished by $\varphi$ but not by any $\mathsf{ML[CK]}$-formula. For every $n \in \mathbb{N}$ we define the $\mathsf{S5}$ structure $\text{Hairpin}(n) := (W, R_a, R_b, P)$ with

- $W := \{s_m : m \leq (n + (n \bmod 2))\} \cup \{t_m : m \leq (n + (n \bmod 2))\} \cup \{u, v\}$;
- $(s_m, s_k) \in R_a$ if and only if $\min(m, k) \bmod 2 = 0$ and $|m - k| = 1$,
  $(t_m, t_k) \in R_a$ if and only if $\min(m, k) \bmod 2 = 0$ and $|m - k| = 1$,
  $(u, v) \in R_a$;
- $(s_m, s_k) \in R_b$ if and only if $\min(m, k) \bmod 2 = 1$ and $|m - k| = 1$,
  $(t_m, t_k) \in R_b$ if and only if $\min(m, k) \bmod 2 = 1$ and $|m - k| = 1$,
  $(u, s_{n+(n \bmod 2)}) \in R_b$, $(v, t_{n+(n \bmod 2)}) \in R_b$;
- $P = \{u\}$.

This hairpin model can be viewed as two finite $\{a, b\}$-paths in equivalence structures with alternating edge-labels that are joined at a unique $p$-state. The two families of pointed Kripke structures that we consider are $(\text{Hairpin}(n), s_0)_{n\in\mathbb{N}}$ and $(\text{Hairpin}(n), t_0)_{n\in\mathbb{N}}$; note that, for every $n \in \mathbb{N}$, the two structures only differ in their distinguished worlds.

The formula $\varphi$ distinguishes the structures $\text{Hairpin}(n), s_0$ and $\text{Hairpin}(n), t_0$, for every $n \in \mathbb{N}$: the public announcement of the subformula $\neg p \rightarrow \Box_a\neg p$ separates the two paths at $v$ because $\text{Hairpin}(n), w \models \neg p \rightarrow \Box_a\neg p$ if and only if $w \neg v$. After the announcement the world $u$, the only worlds where $p$ is true, is reachable from $s_0$ but not from $t_0$, hence $\text{Hairpin}(n), s_0 \models [\neg p \rightarrow \Box_a\neg p]\Box_{a,b}\neg p$ but not $\text{Hairpin}(n), t_0 \models [\neg p \rightarrow \Box_a\neg p]\Box_{a,b}\neg p$. In contrast, $\mathsf{ML[CK]}$ is not expressive enough to separate the two structures.

**Lemma 2.4.5.** *For all $n \in \mathbb{N}$,*

$$\text{Hairpin}(n), s_0 \equiv^n_{\mathsf{ML[CK]}} \text{Hairpin}(n), t_0.$$

*Proof.* We describe a winning strategy for player **II** in the $n$-round common knowledge game. Essentially, player **I** can choose a set $\alpha \in \{\{a\}, \{b\}, \{a, b\}\}$ and move the pebble of his choice. The set $\{a, b\}$ is of no use because the whole structure Hairpin($n$) is one single $\{a, b\}$-equivalence class and player **II** could just move the other pebble to the exact same world. The $\{a\}$- and $\{b\}$-moves boil down to regular moves. But with only those player **I** is not able to reach the $p$-world before the game is over. □

**Lemma 2.4.6.** *There is no* ML[CK]*-formula $\psi$ such that*

$$\text{Hairpin}(n), s_0 \models \psi \textit{ and } \text{Hairpin}(n), t_0 \not\models \psi,$$

*for all $n \in \mathbb{N}$.*

*Proof.* Suppose there is such a formula $\psi$. This formula has a finite modal depth $\ell$. Hence, Lemma 2.4.5 implies Hairpin($n$), $s_0 \models \psi$ if and only if Hairpin($n$), $t_0 \models \psi$. This contradicts the assumption. □

**Proposition 2.4.7.** $\mathsf{ML[CK]} \lneq \mathsf{ML[CK, PA]}$.

*Proof.* $\mathsf{ML[CK]} \leq \mathsf{ML[CK, PA]}$ because ML[CK] is a syntactic fragment of ML[CK, PA]. We argued above that $[\neg p \to \Box_a \neg p]\Box_{a,b}\neg p$ distinguishes Hairpin($n$), $s_0$ and Hairpin($n$), $t_0$, for all $n \in \mathbb{N}$. Additionally, Lemma 2.4.6 implies that there is no such formula in ML[CK]. Thus, $\mathsf{ML[CK]} \lneq \mathsf{ML[CK, PA]}$. □

A more expressive generalisation of common knowledge that allows us to eliminate public announcements is relativized common knowledge.

### 2.4.3 Relativized common knowledge

*Relativized common knowledge* (RC) is a logic that generalises common knowledge logic in order to relativize $\alpha$-paths over worlds that satisfy a certain formula. It can be seen as a syntactic generalisation of ML[CK] and it is more expressive than both ML[CK] and ML[CK, PA] over epistemic structures [21]. To obtain RC we add the following new rule to ML:

$$\varphi \quad ::= \quad \Box^{\varphi}_{\alpha}\varphi$$

To define the semantics, we need to generalise the notion of an $\alpha$-path. An *$\alpha$-$\varphi$-path*, for $\alpha \subseteq \Gamma$, $\varphi \in \mathsf{RC}$, is an $\alpha$-path $w_1, a_1, \ldots, a_\ell, w_{\ell+1}$ such that $\mathfrak{M}, w_i \models \varphi$, for all $1 \leq i \leq \ell + 1$. A world $v$ is $\alpha$-$\varphi$-reachable from $w$ if there is an $\alpha$-$\varphi$-path from $w$ to $v$. For a world $w$, the set $[w]^{\varphi}_{\alpha}$ is the set of worlds that are $\alpha$-$\varphi$-reachable or $\alpha$-$\neg\varphi$-reachable from $w$. By adding also the $\alpha$-$\neg\varphi$-reachable worlds to the set $[w]^{\varphi}_{\alpha}$, every pair $(\alpha, \varphi)$ induces an equivalence relation on the set of possible worlds. The semantics of RC is:

$$\mathfrak{M}, w \models \Box^{\psi}_{\alpha}\varphi \quad :\Leftrightarrow \quad \mathfrak{M}, v \models \varphi \text{ for all } v \in [w]^{\psi}_{\alpha}$$

Obviously, $[w]_\alpha = [w]_\alpha^\top$, for all $w$ and all $\alpha$. Thus, the ML[CK]-formula $\Box_\alpha\varphi$ and the RC-formula $\Box_\alpha^\top\varphi$ are logically equivalent and we can regard ML[CK] as a fragment of RC.

Our definition of relativized common knowledge differs slightly from other definitions in the literature, yet it defines a logic of the same expressive power. In [10] $\mathfrak{M}, w \models \Box_\alpha^\psi\varphi$ is true if $\mathfrak{M}, v \models \varphi$, for all $v$ such that $(w, v) \in (\bigcup_{a\in\alpha} R_a \cap (W \times [\![\psi]\!]^{\mathfrak{M}}))^+$, i.e. all $v$ that are reachable via an $\alpha$-path $w_1, a_1, w_2, a_2, \ldots, a_\ell, w_{\ell+1}$ with $w_i \in [\![\psi]\!]^{\mathfrak{M}}$, for all $1 < i \leq \ell + 1$; we denote this alternative RC-operator as $C_\alpha(\psi, \varphi)$. We can easily use one operator to express the other.

**Proposition 2.4.8.** *The following are equivalent:*

1. $\Box_\alpha^\psi\varphi \equiv (\psi \to C_\alpha(\psi, \varphi)) \wedge (\neg\psi \to C_\alpha(\neg\psi, \varphi))$
2. $C_\alpha(\psi, \varphi) \equiv \bigwedge_{a\in\alpha} \Box_a(\psi \to \Box_\alpha^\psi\varphi)$

Proposition 2.4.8 implies that both definitions of relativized common knowledge have the same expressive power. However, in general the set

$$\{v \in W : (w, v) \in (\bigcup_{a\in\alpha} R_a \cap (W \times [\![\psi]\!]^{\mathfrak{M}}))^+\}$$

is not an equivalence class in contrast to $[w]_\alpha^\psi$. The set $[w]_\alpha^\psi$ being an equivalence class is not merely algebraically nicer, it also gives us an important advantage when we construct coverings in the following chapter.

We saw that public announcements increase the expressive power of ML[CK]. The case is different with RC. Let RC[PA] denote the logic of relativized common knowledge with public announcement. With ML[PA]-formulae we could find equivalent ML-formula by pushing the public announcements further and further to the inside where we could eliminate them at atoms. This approach does not work with regular common knowledge because, in essence, there is no way to push a public announcement past a common knowledge modality in the case $[\psi]\Box_\alpha\varphi$. However, the relativization of the common knowledge modalities in RC gives us a way to do this.

**Proposition 2.4.9.** *The following equivalence holds:*

$$[\chi]\Box_\alpha^\psi\varphi \;\leftrightarrow\; (\chi \to \Box_\alpha^{\chi\wedge[\chi]\psi}[\chi]\varphi)$$

*Proof.* Let $\mathfrak{M}, w$ be a pointed Kripke structure. We prove $\mathfrak{M}, w \models [\chi]\Box_\alpha^\psi\varphi$ if and only if $\mathfrak{M}, w \models \chi \to \Box_\alpha^{\chi\wedge[\chi]\psi}[\chi]\varphi$. The statement is obviously true if $\mathfrak{M}, w \not\models \chi$.

Assume $\mathfrak{M}, w \models \chi$. In this case we need to compare $\alpha$-$\psi$-paths in $\mathfrak{M}^\chi$ to $\alpha$-$(\chi \wedge [\chi]\psi)$-paths in $\mathfrak{M}$. Let $v$ be $\alpha$-$\psi$-reachable from $w$ in $\mathfrak{M}^\chi$ and let $w_1, a_1, \ldots, a_{\ell-1}, w_\ell$ be an $\alpha$-$\psi$-path from $w$ to $v$ in $\mathfrak{M}^\chi$. In particular,

$$\mathfrak{M}^\chi, w_i \models \psi \qquad \Leftrightarrow \qquad \mathfrak{M}, w_i \models \chi \wedge [\chi]\psi,$$

for all $1 \leq i \leq \ell$, which means that $w_1, a_1, \dots, a_{\ell-1}, w_\ell$ is an $\alpha$-$(\chi \wedge [\chi]\psi)$-path in $\mathfrak{M}$. Analogously, every $\alpha$-$(\chi \wedge [\chi]\psi)$-path in $\mathfrak{M}$ is an $\alpha$-$\psi$-path in $\mathfrak{M}^\chi$. Hence, $v$ is $\alpha$-$\psi$-reachable from $w$ in $\mathfrak{M}^\chi$ if and only if it is $\alpha$-$(\chi \wedge [\chi]\psi)$-reachable from $w$ in $\mathfrak{M}$. Thus,

$$\begin{aligned}
\mathfrak{M}, w \models [\chi]\Box_\alpha^\psi \varphi &\Leftrightarrow \mathfrak{M}^\chi, w \models \Box_\alpha^\psi \varphi \\
&\Leftrightarrow \mathfrak{M}^\chi, v \models \varphi \text{ for all } v \in [w]_\alpha^\psi \text{ in } \mathfrak{M}^\chi \\
&\Leftrightarrow \mathfrak{M}, v \models [\chi]\varphi \text{ for all } v \in [w]_\alpha^{\chi\wedge[\chi]\psi} \text{ in } \mathfrak{M} \\
&\Leftrightarrow \mathfrak{M}, w \models \Box_\alpha^{\chi\wedge[\chi]\psi}[\chi]\varphi \\
&\Leftrightarrow \mathfrak{M}, w \models \chi \to \Box_\alpha^{\chi\wedge[\chi]\psi}[\chi]\varphi.
\end{aligned}$$

□

As with ML[PA]-formulae, we can eliminate every occurrence of a public announcement from a RC[PA]-formula and find an equivalent RC-formula. In particular, every ML[CK, PA]-formula is equivalent to a RC-formula [19]. To summarise, over S5 structures:

$$\mathsf{ML} \equiv \mathsf{ML[PA]} \lneq \mathsf{ML[CK]} \lneq \mathsf{ML[CK, PA]} \lneq \mathsf{RC} \equiv \mathsf{RC[PA]}$$

## 2.5 A new characterisation theorem

The main theorem of this work is a new modal characterisation theorem that characterises common knowledge logic over the class of epistemic structures as the bisimulation-invariant fragment of a suitable extension of first-order logic, both classically and in the sense of finite model theory. As mentioned above, the expressive power of CK goes beyond basic modal logic and hence beyond first-order logic. There are several other modal logics of this kind that have been characterised as the bisimulation-invariant fragment of some classical logic. The most famous example is arguably the Janin-Walukiewicz theorem [17]: the modal $\mu$-calculus $\mathsf{L}_\mu$ is expressively equivalent to the bisimulation-invariant fragment of monadic second-order logic MSO. Other examples include the characterisation of the computation tree logic CTL* as the bisimulation-invariant fragment of monadic path logic MPL [22] by Moller and Rabinovich, and Carreiro's characterisation of propositional dynamic logic PDL as the bisimulation-invariant fragment of weak chain logic [7]. However, all these older theorems have one thing in common: there is no finite model theory version.

The modal $\mu$-calculus $\mathsf{L}_\mu$ is the extension of ML by least and greatest fixpoint operators. It is a highly expressive fragment of MSO with a decidable satisfiability problem that encompasses many modal logics that are used in practical applications like LTL, CTL, CTL* and PDL. The Janin-Walukiewicz theorem states that it can express all MSO-properties that are bisimulation-invariant. The logic CTL* is a fragment of $\mathsf{L}_\mu$ that freely combines temporal operators like "next time" and "until" with path quantifiers. It encompasses both the linear time logic LTL and CTL. Propositional dynamic logic PDL is an extension of ML that allows for the construction of so-called *programs* out of formulae and regular expressions over a set of agents to specify new modalities.

The characterisation of $\mathsf{L}_\mu$ is based on techniques from automata theory. The core idea of the proof are characterisations of $\mathsf{L}_\mu$ and MSO by certain automata in the following sense. For every $\varphi \in \mathsf{L}_\mu$ there is an automaton $\mathfrak{A}_\varphi$ that accepts a Kripke structure $\mathfrak{M}$ if and only if $\mathfrak{M} \models \varphi$ [16], and for every $\psi \in$ MSO there is an automaton $\mathfrak{A}_\psi$ that accepts a *tree* $\mathcal{T}$ if and only if $\mathcal{T} \models \psi$ [30]. Using these characterisations, Janin and Walukiewicz show that for every MSO-sentence $\varphi$ there is an $\mathsf{L}_\mu$-sentence $\varphi^*$ such that $\mathfrak{M} \models \varphi^*$ if and only if $\mathcal{T}^{\mathfrak{M}} \models \varphi$, for every Kripke structure $\mathfrak{M}$ with its bisimilar unravelling into a tree $\mathcal{T}^{\mathfrak{M}}$. If we assume the MSO-sentence $\varphi$ to be bisimulation-invariant we obtain the following equivalences:

$$\mathfrak{M} \models \varphi \quad \Leftrightarrow \quad \mathcal{T}^{\mathfrak{M}} \models \varphi \quad \Leftrightarrow \quad \mathfrak{M} \models \varphi^*$$

The first equivalence fails in the case of finite model theory because for finite $\mathfrak{M}$ the unravelling $\mathcal{T}^{\mathfrak{M}}$ is infinite if and only if $\mathfrak{M}$ is cyclic. Hence, if $\varphi$ is only bisimulation-invariant over finite structures, $\mathfrak{M} \models \varphi \Leftrightarrow \mathcal{T}^{\mathfrak{M}} \models \varphi$ is no longer true in general. In contrast to this, one key aspect of our new characterisation are constructions of bisimilar coverings that preserve finiteness.

For his characterisation of PDL, as the bisimulation-invariant fragment of weak-chain logic WCL (a fragment of MSO), Carreiro employs fairly similar techniques using automata-theoretic characterisation of PDL and WCL on trees.

Moller and Rabinovich base the proof of their characterisation of CTL$^*$ on a composition theorem as an alternative to methods from automata theory. Their composition theorem states that for every property $\varphi$ over wide trees that is expressible in monadic path logic MPL (a fragment of MSO over trees where one can quantify over paths) there is an equivalent FO-property $\psi$ over $(\omega)$-words. A *wide tree* is a tree in which for every edge $(v, w)$ there are infinitely many edges $(v, w')$ such that the subtree that is rooted at $w$ is isomorphic to the one at $w'$. The FO-property $\psi$ gets translated to an LTL-property $\psi'$ via Kamp's theorem [18] that can be easily translated to an equivalent CTL$^*$-property over wide trees. Since every tree is bisimilar to a wide tree, they obtain their translation from bisimulation-invariant MPL-formulae to CTL$^*$-formulae over trees.

### 2.5.1 A characterisation of common knowledge logic

For our characterisation of common knowledge logic over S5 structures, we use an approach that differs strongly from the ones described above. The core ideas of our approach are constructions of bisimilar models, which preserve finiteness, and playing Ehrenfeucht-Fraïsse games over non FO-axiomatizable structures.

The first thing to do is to define a suitable logic $\mathcal{L}$ such that ML[CK] is expressively equivalent to $\mathcal{L}/{\sim}$ over S5 structures. Since ML[CK] is more expressive than basic modal logic, we need a logic that is more expressive than FO. Hence, we add to first-order logic over the signature $\{(R_a)_{a\in\Gamma}, (P_i)_{i_I}\}$ the binary relations $R_\alpha$, for all $\alpha \subseteq \Gamma$, with the semantics

$$\mathfrak{M} \models R_\alpha vw \quad :\Leftrightarrow \quad (v, w) \in \mathrm{TC}(\bigcup_{a\in\alpha} R_a),$$

where TC denotes the transitive closure operator. I.e. $\mathfrak{M} \models R_\alpha vw$ is true if and only if there is an $\alpha$-path in $\mathfrak{M}$ from $v$ to $w$. We call this logic *first-order common knowledge logic* and abbreviate it with $\mathsf{FO}[\mathsf{CK}]$. Note that the relations $R_\alpha$ are not first-order definable from the relations $R_a$, for $|\alpha| \geq 2$. Thus, the main result of this thesis can be stated as follows.

**Theorem 2.5.1** (Main theorem)**.** *Over* S5 *structures, classically and in the sense of finite model theory:*

$$\mathsf{ML}[\mathsf{CK}] \equiv \mathsf{FO}[\mathsf{CK}]/{\sim}$$

The first step to proving this theorem is a simple rephrasing. Instead of strengthening the expressive power of our logics, we can alternatively specialize the class of structures over which we consider those logics and arrive at an equivalent statement.

### 2.5.2 Common knowledge structures

As we did in this thesis, common knowledge logic is usually introduced as an expansion of basic modal logic with semantics for epistemic Kripke structures. In order to prove our characterisation theorem we choose a different approach. Instead, we view common knowledge logic as basic modal logic with semantics over *Common Knowledge structures.*

**Definition 2.5.2.** With every S5 Kripke frame (or structure) we associate the CK frame (or structure) obtained as the expansion by the family $(R_a)_{a\in\Gamma}$ to the family $(R_\alpha)_{\alpha\in\tau}$, for $\tau := \mathcal{P}(\Gamma)$, where $R_\alpha = \mathsf{TC}(\bigcup_{a\in\alpha} R_a)$.

The relations $R_a$, for $a \in \Gamma$, are called the *basic-agent* relations, the S5 structure $\mathfrak{M} = (W, (R_a)_{a\in\Gamma}, (P_i)_{i\in I})$ is called the *basic-agent structure*, and the notation $\mathfrak{M}^{\mathsf{CK}}$ is used to indicate the passage from the basic-agent structure to its associated CK structure

$$\mathfrak{M}^{\mathsf{CK}} = (W, (R_\alpha)_{\alpha\in\tau}, (P_i)_{i\in I}),$$

which is again an S5 structure; we also call $\mathfrak{M}$ the *basic-agent reduct* of $\mathfrak{M}^{\mathsf{CK}}$. The resulting class of CK structures is non-elementary since transitive closures are not first-order definable. A formula $\Box_\alpha\varphi$ can be considered as an $\mathsf{ML}[\mathsf{CK}]$-formula with semantics for regular S5 structures with the set of agents $\Gamma$, or as an ML-formula with semantics over the associated CK structures; they are just two different points of view for the same thing. Hence, we can rephrase the main theorem the following way:

**Theorem 2.5.3** (Main theorem)**.** *Over* CK *structures, classically and in the sense of finite model theory:*

$$\mathsf{ML} \equiv \mathsf{FO}/{\sim}$$

This rephrasing allows us to employ methods like the modal Ehrenfeucht-Fraïssé theorem and first-order Ehrenfeucht-Fraïssé games for the proof of Theorem 2.5.3. A further rephrasing states that an FO-formula $\varphi$ is logically equivalent to an ML-formula over (finite) CK structures if and only if it is $\sim$-invariant over (finite) CK structures. If $\varphi$

$$\begin{array}{ccc}
\mathfrak{M}, w & \text{——} \sim^{\ell} \text{——} & \mathfrak{N}, v \\
| & & | \\
\wr & & \wr \\
| & & | \\
\hat{\mathfrak{M}}, \hat{w} & \text{——} \equiv_q \text{——} & \hat{\mathfrak{N}}, \hat{v}
\end{array}$$

Figure 2.1: Upgrading $\sim^{\ell}$ to $\equiv_q$.

is equivalent to an ML-formula over a class of structures $\mathcal{C}$, than it is also $\sim$-invariant over $\mathcal{C}$. However, the real challenge lies in showing that $\varphi$ is equivalent to an ML-formula over (finite) CK structures if it is $\sim$-invariant over (finite) CK structures.

The modal Ehrenfeucht-Fraïssé theorem reduces this task to showing that $\varphi$ is $\sim^{\ell}$-invariant over (finite) CK structures, for some $\ell \in \mathbb{N}$. Thus, our proof boils down to showing the following compactness statement for $\varphi \in \mathsf{FO}$ over (finite) CK structures:

$$\varphi \text{ is } \sim\text{-invariant} \quad \Leftrightarrow \quad \varphi \text{ is } \sim^{\ell}\text{-invariant for some } \ell \in \mathbb{N}$$

As mentioned above, the direction from right to left is trivial because a formula that cannot distinguish between $\ell$-bisimilar structures cannot distinguish between bisimilar structures. Thus, it remains to show that

$$\varphi \text{ is } \sim\text{-invariant} \quad \Rightarrow \quad \varphi \text{ is } \sim^{\ell}\text{-invariant for some } \ell \in \mathbb{N}.$$

For this, let $\varphi \in \mathsf{FO}/\sim$ with quantifier rank $q$, and let $\mathfrak{M}, w$, $\mathfrak{N}, v$ be arbitrary (finite) pointed CK structures that are $\ell$-bisimilar, for some $\ell$ that depends on $q$. If we can show

$$\mathfrak{M}, w \models \varphi \quad \Leftrightarrow \quad \mathfrak{N}, v \models \varphi,$$

we are done. To do this, we construct a detour. The strategy is to construct two bisimilar CK structures $\hat{\mathfrak{M}}, \hat{w} \sim \mathfrak{M}, w$ and $\hat{\mathfrak{N}}, \hat{v} \sim \mathfrak{M}, w$ that are first-order equivalent up to quantifier rank $q$. Then $\varphi$ being a bisimulation-invariant first-order formula of quantifier rank $q$ implies

$$\begin{aligned}
\mathfrak{M}, w \models \varphi \;&\Leftrightarrow\; \hat{\mathfrak{M}}, \hat{w} \models \varphi \\
&\Leftrightarrow\; \hat{\mathfrak{N}}, \hat{v} \models \varphi \\
&\Leftrightarrow\; \mathfrak{N}, v \models \varphi.
\end{aligned}$$

We *upgrade* $\ell$-bisimilarity to $\mathsf{FO}_q$-equivalence (see Figure 2.1). This strategy poses two challenges: constructing the suitable bisimilar companions $\hat{\mathfrak{M}}, \hat{w}$ and $\hat{\mathfrak{N}}, \hat{v}$, and showing that they are $\mathsf{FO}_q$-equivalent. The construction needs to avoid properties of CK structures that can be defined in $\mathsf{FO}_q$ but are not controlled by $\ell$-bisimilarity. As usual, with upgradings of this kind, these properties are different multiplicities and short cycles. Taking care of different multiplicities is easy. Taking care of short cycles is rather difficult in the case of CK structures, and the main topic of Chapter 3.

Essentially, we prove $\mathsf{FO}_q$-equivalence of $\hat{\mathfrak{M}}, \hat{w}$ and $\hat{\mathfrak{N}}, \hat{v}$ by describing a winning strategy for player **II** in the $q$-round Ehrenfeucht-Fraïssé game on $\hat{\mathfrak{M}}, \hat{w}$ and $\hat{\mathfrak{N}}, \hat{v}$. In order to do this, we need to develop a structure theory for CK structures without short cycles that is of interest in its own right; this is the content of Chapter 4. The $\mathsf{FO}_q$-equivalence, which is the final part of the proof, is shown in Chapter 5.

# 3 Coverings

At the end of Chapter 2, we describe our proof strategy for showing the characterisation of common knowledge logic: an upgrading of $\ell$-bisimulation to $\mathsf{FO}_q$-equivalence over the non-elementary class of (finite) common knowledge structures, for all $q \in \mathbb{N}$ and some $\ell$ that depends on $q$. The first part of the upgrading argument is the construction of suitable bisimilar *coverings*. The construction and analysis of bisimilar coverings lies at the heart of the upgrading method.

**Definition 3.0.4.** A homomorphism $\pi\colon \hat{\mathfrak{M}} \to \mathfrak{M}$ is a *bisimilar covering* of $\mathfrak{M}$ by $\hat{\mathfrak{M}}$ if it is surjective and its graph is a bisimulation w.r.t. both $R_a$ and its inverse relation $R_a^{-1}$, for all $a \in \Gamma$.

A bisimilar covering $\pi\colon \hat{\mathfrak{M}} \to \mathfrak{M}$ is *faithful* in a world $\hat{w} \in \hat{\mathfrak{M}}$ if the incidence degrees of $\hat{w}$ with both $R_a$ and $R_a^{-1}$ in $\hat{\mathfrak{M}}$ are the same as those at $\pi(\hat{w})$ in $\mathfrak{M}$, for all $a \in \Gamma$; $\pi$ is *faithful* if it is faithful in all $\hat{w} \in \hat{\mathfrak{M}}$.

If $\pi\colon \hat{\mathfrak{M}} \to \mathfrak{M}$ is a bisimilar covering, we also often refer to the structure $\hat{\mathfrak{M}}$ as a bisimilar covering of $\mathfrak{M}$. The main result of the current chapter is that suitable coverings for $\mathsf{CK}$ structures can be constructed based on Cayley graphs of specific Cayley groups, both for the classical and finite model theory case, and that Cayley groups themselves can be regarded as universal representatives of common knowledge structures up to bisimulation.

The method of upgrading, together with the construction of bisimilar coverings, has been fruitful for proving many different variations of van Benthem's theorem. Characterisations of $\mathsf{ML}$ over several special classes of frames and characterisations of extensions of $\mathsf{ML}$, such as global modal logic or the guarded fragment, were shown in [23], [25], [9] and [26]. Before we dive into the case of $\mathsf{CK}$ structures, we want to introduce and develop the main ideas and methods like locality, acyclicity, etc. for the simpler, yet related case of *global modal logic* $\mathsf{ML}^\forall$ over (finite) $\mathsf{S5}$ structures proven by Dawar and Otto in [9].

**Theorem 3.0.5.** $\mathsf{ML}^\forall \equiv \mathsf{FO}/{\sim_\forall}$ *over (finite)* $\mathsf{S5}$ *structures.*

In the course of this chapter we will present several different coverings, for general Kripke structures, for $\mathsf{S5}$ structures and $\mathsf{CK}$ structures. We would like to begin with the arguably quintessential model construction that preserves bisimilarity: the *tree unravelling* of a pointed Kripke structure. Remember that a walk $p$ in a frame is a finite sequence $w_1, a_1, w_2, \dots, w_{\ell-1}, a_{\ell-1}, w_\ell$, with worlds $w_i$ and agents $a_i$ such that $w_{i+1} \in R_{a_i}[w_i]$. The *terminal state map* $t$ is the map that maps every walk to its terminal state, i.e. if $p = w_1, a_1, \dots, a_{\ell-1}, w_\ell$, then $t(p) = w_\ell$.

**Definition 3.0.6.** The *tree unravelling* of a pointed Kripke structure $\mathfrak{M}, w$ from $w$, denoted $\mathfrak{M}^*_w$, is defined as follows: the universe of $\mathfrak{M}^*_w$ consists of all directed walks

$w_1, a_2, w_2, a_2, \ldots, a_{\ell-1}, w_\ell$, starting at $w_1 = w$. For the interpretations of the relations we put

- an $a$-edge from $p = w_1, a_1, \ldots, w_\ell$ to exactly all its extensions by one $a$-edge from $w_\ell$ in $\mathfrak{M}$, $p' = w_1, a_1, \ldots, w_\ell, a, w_{\ell+1}$, and
- $p = w_1, a_1, \ldots, a_{\ell-1}, w_\ell$ in $P_i$ in $\mathfrak{M}^*_w$ if and only if $w_\ell \in P_i^{\mathfrak{M}}$.

Then the terminal state map $t \colon \mathfrak{M}^*_w \to \mathfrak{M}$ induces a bisimulation between $\mathfrak{M}^*_w$ and $\mathfrak{M}$.

The unravelling $\mathfrak{M}^*_w$ is a rooted tree that is bisimilar to $\mathfrak{M}$. It is finite if $\mathfrak{M}$ is finite and acyclic; in general, it is infinite. Hence, every Kripke structure is bisimilar to a tree, which implies that every satisfiable, bisimulation-invariant formula has a tree model. Unravelling a structure gives it the nice and simple shape of a tree, which is useful for many application, e.g. proving van Benthem's theorem. Unfortunately, the finite model theory variations of van Benthem's theorem restrict us to constructions that preserve finiteness, which rules out the straightforward standard unravelling. However, most, if not all, coverings that we present in this chapter can be considered a variation of tree unravelling because they give certain restricted neighbourhoods a tree-like shape. Section 3.1 sketches how to use such coverings to prove Theorem 3.0.5, and Section 3.2 presents suitable coverings for CK structures.

As in the previous chapter, we usually consider all structures to be Kripke structures over the finite, fixed modal signature $\{(R_a)_{a\in\Gamma}, (P_i)_{i\in I}\}$, except if stated otherwise. We denote Kripke structures by $\mathfrak{M}$ and $\mathfrak{N}$, and their sets of worlds by $W$ and $V$, respectively.

## 3.1 Finite S5 structures

*Global modal logic* $\mathsf{ML}^{\forall}$ is the extension of ML by the global modality $\forall$ with semantics

$$\mathfrak{M}, w \models \forall\varphi \quad \Leftrightarrow \quad \mathfrak{M}, v \models \varphi \text{ for all } v \in W.$$

Its bisimulation counterpart is global bisimulation. A *global bisimulation* $Z \subseteq W \times V$ between two Kripke structures $\mathfrak{M}$ and $\mathfrak{N}$ is a bisimulation such that for every world $w \in W$ there is a world $v \in V$ with $(w, v) \in Z$ and for every world $v \in V$ there is a world $w \in W$ with $(w, v) \in Z$. We write $\mathfrak{M} \sim_\forall \mathfrak{N}$ if there is a global bisimulation between $\mathfrak{M}$ and $\mathfrak{N}$, and $\mathfrak{M}, w \sim_\forall \mathfrak{N}, v$ if there is a global bisimulation that contains the pair $(w, v)$. In the associated global bisimulation game the players have an additional type of move, the *global move*. They can use the global move to go from the position $(w, v)$ to any other position $(w', v')$.

*Two-way modal logic* $\mathsf{ML}^{-}$ is the extension of ML with backward modalities $\Box_{-a}$ for the inverses of the given accessibility relations, i.e.

$$\mathfrak{M}, w \models \Box_{-a}\varphi \quad \Leftrightarrow \quad \mathfrak{M}, v \models \varphi \text{ for all } v \text{ with } w \in R_a[v];$$

we also write $R_a^{-1}$ for the backward relation of $R_a$, $\{(v, w) \in W \times W : (w, v) \in R_a\}$. A bisimulation is a *two-way bisimulation* if it also satisfies the back-and-forth conditions with respect to all $R_a^{-1}$, for $a \in \Gamma$. We write $\mathfrak{M} \sim_{\rightleftarrows} \mathfrak{N}$ if there is a two-way bisimulation between $\mathfrak{M}$ and $\mathfrak{N}$, and $\mathfrak{M}, w \sim_{\rightleftarrows} \mathfrak{N}, v$ if there is a two-way bisimulation that contains the pair $(w, v)$. In the associated two-way bisimulation game the players have an additional type of move, the *inverse move*, in which both players use an inverse accessibility relation $R_a^{-1}$.

The logic $\mathsf{ML}^{-\forall}$ is the combined extension of $\mathsf{ML}$ by both global and inverse modalities. Its associated *global two-way bisimulation* is denoted $\mathfrak{M} \approx \mathfrak{N}$, and the global two-way bisimulation game allows both global and inverse moves.

*Remark* 3.1.1. The standard translation, modal depth, graded bisimulation and the Ehrenfeucht-Fraïssé theorem generalize for $\mathsf{ML}^{-}$, $\mathsf{ML}^{\forall}$ and $\mathsf{ML}^{-\forall}$ in the obvious way.

Since S5 structures have, in particular, symmetric accessibility relations, Theorem 3.0.5 implies that over finite S5 structures

$$\mathsf{ML}^{\forall} \equiv \mathsf{FO}/{\sim_{\forall}} \equiv \mathsf{FO}/{\approx}.$$

Again, the standard translation implies that $\mathsf{ML}^{\forall}$ can be regarded as a fragment of $\mathsf{FO}/{\sim_{\forall}}$. The direction from $\mathsf{FO}/{\sim_{\forall}}$ to $\mathsf{ML}^{\forall}$ can be shown by an upgrading argument. In this case, we upgrade graded global bisimulation $\sim_{\forall}^{\ell}$ to an approximation of FO-equivalence that is based on the Gaifman local form of an FO-formula.

### 3.1.1 Locality

Global modal logic $\mathsf{ML}^{\forall}$ and basic modal logic ML over CK structures share a kind of global nature: in the global bisimulation game the players are allowed to make global moves that can transport the current position to any part of the structure, and in the standard bisimulation game on CK structures the players can move to any world within the current component via the relation $R_{\Gamma}$, induced by the set of all agents $\Gamma$. This leads to new challenges if one wants to characterise $\mathsf{ML}^{\forall}$ or ML over CK structures. For the remainder of Section 3.1, we will focus on $\mathsf{ML}^{\forall}$. We will continue to deal with CK structures in Section 3.2.

Proving $\mathsf{ML}^{\forall} \equiv \mathsf{FO}/{\sim_{\forall}}$ requires much more intricate model constructions than proving the theorems of van Benthem and Rosen. To show $\mathsf{ML} \equiv \mathsf{FO}/{\sim}$ over general or finite structures one, essentially, has to transform only the $\ell$-neighbourhood, for some $\ell \in \mathbb{N}$, of the distinguished world $w$ of a pointed structure $\mathfrak{M}, w$ and unravel it into a tree. This local transformation does not suffice to upgrade $\sim_{\forall}^{\ell}$ to an approximation of FO-equivalence. In this case, we need a kind of global transformation that makes *every* $\ell$-neighbourhood tree-like simultaneously. We use the Gaifman graph of a relational structure to define notions as distance and neighbourhoods formally.

**Definition 3.1.2.** The *Gaifman graph* of a relational structure $\mathfrak{M}$ is the undirected graph with vertex set $W$ and an edge between two vertices $w$ and $v$ if $w \neq v$ and both vertices are among the components of some tuple in one of the relations of $\mathfrak{M}$.

The *Gaifman distance* in $\mathfrak{M}$ is the usual graph theoretic distance in the Gaifman graph of $\mathfrak{M}$, which we denote by $d(\cdot,\cdot)$.

The $\ell$-neighbourhood $N^\ell(w)$ of a world $w$ in $\mathfrak{M}$ is the set of all worlds that are at Gaifman distance up to $\ell$ from $w$, i.e. $N^\ell(w) = \{w' \in W : d(w, w') \leq \ell\}$.

**Definition 3.1.3.** A subset of $\mathfrak{M}$ is *$\ell$-scattered* if the $\ell$-neighbourhoods of any two distinct members of this set are disjoint. An *$\ell$-scattered subset for $\psi(x)$* is an $\ell$-scattered subset whose members each satisfy $\psi$ in their $\ell$-neighbourhoods: $w_1, \ldots, w_m$ such that $d(w_i, w_j) > 2\ell$, for $i \neq j$, and $\mathfrak{M} \restriction N^\ell(w_i), w_i \models \psi$, for all $1 \leq i \leq m$.

**Definition 3.1.4.** $\mathfrak{M}, w \equiv_{q,n}^{(\ell)} \mathfrak{N}, v$ if

1. $\mathfrak{M} \restriction N^\ell(w), w \equiv_q \mathfrak{N} \restriction N^\ell(v), v$, i.e. $\mathsf{FO}$-formulae of quantifier rank $q$ cannot distinguish $w$ and $v$ in their respective $\ell$-neighbourhoods, and
2. $\mathfrak{M}$ and $\mathfrak{N}$ realise exactly the same quantifier rank $q$ formulae in $k$-scattered sets of size $m$, for $k \leq \ell$ and $m \leq n$, i.e. for every $\psi(x) \in \mathsf{FO}_q$ and every $k \leq \ell$, $m \leq n$: $\mathfrak{M}$ has a $k$-scattered subset of size $m$ for $\psi$ if and only if $\mathfrak{N}$ has one.

Gaifman's theorem [12, 14] implies that every first-order formula $\psi(x)$ (with one free variable) is invariant under $\equiv_{q,n}^{(\ell)}$, for some $\ell, q, n \in \mathbb{N}$. The relation $\equiv_{q,n}^{(\ell)}$ was specifically designed to capture the expressiveness of formulae in Gaifman local form. For given $\ell, q, n$, Dawar and Otto showed in [9] that $\sim_\forall^{\ell+1}$-bisimilarity can be upgraded to $\equiv_{q,n}^{(\ell)}$-equivalence over (finite) $\mathsf{S5}$ structures. Together with the modal Ehrenfeucht-Fraïssé theorem, this immediately implies their characterisation theorem.

But how does Gaifman locality help? Essentially, with the help of Gaifman locality proving $\equiv_{q,n}^{(\ell)}$-equivalence boils down to proving $\equiv_q$-equivalence of small induced substructures. I.e., while playing first-order Ehrenfeucht-Fraïssé games we do not have to take whole structures into account but only small neighbourhoods around certain elements. This can be useful in the classical case, but it is especially important in the case of finite model theory. Given a finite structure $\mathfrak{M}$, its bisimilar covering $\hat{\mathfrak{M}}$ must, in particular, also be finite for the upgrading argument to go through. This is a heavy restriction on the methods of construction we can employ. The coverings have to avoid features that are $\mathsf{FO}_q$-definable but cannot be controlled by $\ell$-bisimulation. Among these are different multiplicities and short cycles. Different multiplicities can easily be taken care of by boosting the multiplicities of every world above a certain finite threshold. The finite case does not complicate this any further. The following covering boosts the multiplicity of every world up to $q$.

**Definition 3.1.5.** For a Kripke structure $\mathfrak{M}$ and a positive integer $q$, we define the structure $\mathfrak{M} \otimes q$ to be the structure with universe $W \times \{0, \ldots, q-1\}$ and

- an $R_a$-edge from $(w, j)$ to $(v, k)$ in $\mathfrak{M} \otimes q$ if and only if $(w, v) \in R_a^{\mathfrak{M}}$, and
- $(w, j) \in P_i^{\mathfrak{M} \otimes q}$ if and only if $w \in P_i^{\mathfrak{M}}$.

It is easy to see that $\mathfrak{M} \otimes q \to \mathfrak{M}, (w, i) \mapsto w$ is a bisimilar covering of $\mathfrak{M}$; in particular $\mathfrak{M}, w \sim \mathfrak{M} \otimes q, (w, i)$, for all $w \in W$ and $0 \leq i < q$. We often identify a world $w \in W$ with its copy $(w, 0)$ in $\mathfrak{M} \otimes q$ to simplify notation.

*Remark* 3.1.6. The class of (finite) S5 Kripke structures is closed under the operation $\mathfrak{M} \mapsto \mathfrak{M} \otimes q$.

Recall that we can unravel every structure $\mathfrak{M}, w$ into a bisimilar directed tree $\mathfrak{M}^*_w, w$. This is how one usually takes care of cycles. However, the complete unravelling of a cycle results in an infinite path. Hence, we need a different approach in the finite case. It is not possible to avoid cycles completely in finite coverings, but it is possible to avoid *short* cycles. Gaifman locality implies that this is enough since we only need to play Ehrenfeucht-Fraïssé games in small neighbourhoods, and if a structure does not have short cycles, its small neighbourhoods are acyclic. The following lemma from [9] is a key component in the upgrading.

**Lemma 3.1.7.** *If* $\mathfrak{M}, w \sim^\ell \mathfrak{N}, v$, *then* $(\mathfrak{M} \otimes q)^*_w \restriction N^\ell(w), w \equiv_q (\mathfrak{N} \otimes q)^*_v \restriction N^\ell(v), v$.

Lemma 3.1.7 shows that the $\mathsf{FO}_q$-type of the $\ell$-neighbourhood $N^\ell(w)$ in an unravelling of a boosted structure $(\mathfrak{M} \otimes q)^*_w$ is determined by the $\ell$-bisimulation-type of $\mathfrak{M}, w$. If one takes different multiplicities into account, the $\ell$-bisimulation between $\mathfrak{M}$ and $\mathfrak{N}$ can easily be translated into a winning strategy for duplicator in the first-order Ehrenfeucht-Fraïssé game because the $\ell$-neighbourhoods $N^\ell(w)$ and $N^\ell(v)$ in $(\mathfrak{M} \otimes q)^*_w$ and $(\mathfrak{N} \otimes q)^*_v$, respectively, are acyclic. Dawar and Otto showed in [9], using Lemma 3.1.7 and Gaifman locality, that it suffices to make $\ell$-neighbourhoods acyclic, or rather to avoid short cycles.

### 3.1.2 Acyclicity

We argued in the last section that it essentially suffices for the upgrading to construct bisimilar coverings without short cycles. Our main tools in these kinds of constructions are Cayley groups and their associated Cayley graphs. Before we describe how to construct acyclic Cayley graphs we want to make precise what we mean by acyclic and $k$-acyclic Kripke structures.

**Definition 3.1.8.** Let $\mathfrak{M}$ be a Kripke structure and $G(\mathfrak{M})$ its Gaifman graph.

1. A *cycle of length* $k$ in $\mathfrak{M}$ is a $k$-cycle in $G(\mathfrak{M})$ in the graph theoretic sense: a sequence of worlds $w_0, \ldots, w_{k-1}$, where for each consecutive pair of indices $(i, i+1)$ from $\mathbb{Z}_k$ we have $(w_i, w_{i+1}) \in R_a$ or $(w_i, w_{i+1}) \in R_a^{-1}$, for some $a \in \Gamma$. A cycle of length 1 is called a *loop*.

2. A cycle is *non-degenerate* if $w_{i-1} \neq w_{i+1}$, for all $i \in \mathbb{Z}_k$.

3. $\mathfrak{M}$ is *acyclic* if it is loop-free has no non-degenerate cycles.

4. $\mathfrak{M}$ is *$k$-acyclic* if it is loop-free has no non-degenerate cycles of length $\leq k$.

*Remark* 3.1.9. Every edge gives rise to a degenerate cycle of length 2. Hence, degenerate cycles cannot be avoided completely. A structure is 2-acyclic if it is loop-free.

The following proposition is one of the key results from [25].

**Proposition 3.1.10.** For all $k \geq 2$: for every finite $\mathfrak{M}$ there is a faithful bisimilar covering $\pi\colon \hat{\mathfrak{M}} \to \mathfrak{M}$ of $\mathfrak{M}$ by a finite $k$-acyclic $\hat{\mathfrak{M}}$.

In general, the covering $\hat{\mathfrak{M}}$ of an S5 structure $\mathfrak{M}$ obtained by Proposition 3.1.10 is not an S5 structure. Constructing suitable $k$-acyclic coverings for S5 structures needs some additional work. The construction of $k$-acyclic coverings for general Kripke structures is based on a product of a given finite Kripke structure with a $k$-acyclic Cayley graph.

**Definition 3.1.11.** A *Cayley group* is a group $\mathbb{G} = (G, \circ, 1)$ with an associated generator set $E$ that consists of non-trivial involutions, i.e. $e \neq 1$ and $e \circ e = 1$, for all $e \in E$. That $\mathbb{G}$ is generated by the set $E$ means that every group element can be represented as a product of generators. In other words, every $g \in G$ can be represented as a word in $E^*$; w.l.o.g. such a representation is reduced in the sense that is does not have any factors $e^2$.

With every Cayley group $\mathbb{G} = (G, \circ, 1)$ one associates its *Cayley graph* $(G, (R_e)_{e\in E})$: its vertex set is the set of group elements $G$, and its edge relations are

$$R_e = \{(g, ge) \in G \times G : g \in G\}.$$

In our case, all edge relations are symmetric and complete matchings on $G$. Since $E$ generates $\mathbb{G}$, the edge coloured graph $(G, (R_e)_{e\in E})$ is connected. Furthermore, it is homogeneous in the sense that every two vertices $g$ and $h$ are related by a graph homomorphism that is induced by multiplication from the left with $hg^{-1}$.

We would like to sketch the proof of Proposition 3.1.10 as it is similar in spirit to our construction of coverings for CK structures (cf. Theorem 3.2.19). The first step is to show that for every finite set $E$ and every positive integer $k$ there is a Cayley group generated by $E$ with a $k$-acyclic Cayley graph (see [1]): let $T$ be a regular $\ell$-edge-coloured undirected, infinite tree, in which every vertex has exactly one neighbour for every edge-colour $e_1, \ldots, e_\ell$. If we distinguish a vertex $\lambda$ as the root of the tree and truncate it at distance $k \geq 1$ from $\lambda$, we obtain a finite tree of height $k$ with the set of nodes $V = \{v \in T : d(\lambda, v) \leq k\}$. Every edge-colour $e_i$ induces a permutation $\pi_i$ on $V$ by swapping every pair of nodes that are connected by an $e_i$-edge. These operations are well defined since every node is incident with at most one edge of every colour. Furthermore, each of the permutations $\pi_i$ is a non-trivial involution, and every leaf is fixed by all except one of the $\pi_i$. Hence, the set $\{\pi_i \in \mathrm{Sym}(V) : 1 \leq i \leq \ell\}$ generates a Cayley group $\mathbb{G}$ that is a subgroup of $\mathrm{Sym}(V)$, the symmetric group on $V$. The Cayley graph of $\mathbb{G}$ is $k$-acyclic since no reduced sequence of generators of length up to $k$ can represent its neutral element: every operation $\pi_i$ moves $\lambda$ exactly one step closer to a leaf. Hence, a sequence of up to $k$ generators can never operate as the identity mapping since it does not fix the root (in fact, $\mathbb{G}$ is even $4k + 2$-acyclic). The coverings from Proposition 3.1.10 are based on products of Kripke structures with these $k$-acyclic Cayley graphs (for technical details see [25]).

However, this construction does not suffice for S5 structures. First, S5 structures can never be acyclic or $k$-acyclic in the sense of Definition 3.1.8 since, by definition, they contain cliques. Second, if $\mathfrak{M}$ is the S5 structure to be covered, the resulting covering $\hat{\mathfrak{M}}$ is, in general, not an S5 structure itself. This leads us to the following notion of acyclicity that is specifically tailored to S5 structures.

**Definition 3.1.12.** Let $k \geq 2$, and $\mathfrak{M}$ be an S5 structure. A labelled cycle

$$w_0, a_0, w_1, a_1, \ldots, a_{n-1}, w_n$$

in $\mathfrak{M}$ is *non-trivial* if $w_{i+1} \neq w_i$ and $a_{i+1} \neq a_i$, for all $i \in \mathbb{Z}_n$. $\mathfrak{M}$ is $k$*-acyclic* if every non-trivial cycle is of length $> k$.

Intuitively, a non-trivial cycle in an S5 structure changes the vertex and the edge-colour with every step. In other words, we are only interested in how equivalence classes of *different* colours are connected to each other. For example, in a 2-acyclic S5 structure two different equivalence classes always intersect in at most one world. Regarding the upgrading argument, short non-trivial cycles are not controlled by $\ell$-bisimulation, for any $\ell$, but can be defined in $\mathsf{FO}_q$, for $q \geq k$. Hence, they have to be avoided in the coverings. Single cliques $[w]_a$, inherent to S5 structures, considered individually do not pose a problem for the upgrading. If we have two different equivalence classes of the same colour $[w]_a$ and $[v]_a$ in two different structures, $\mathsf{FO}_q$ can only differentiate between them if there is a world of one atomic type in $[w]_a$ that does not occur in $[v]_a$, or if the multiplicities do not match. The former gets taken care of by $\ell$-bisimilarity and the latter by boosting multiplicities as before.

In order to take care of short cycles, we need to define an auxiliary structure. We write $W/R_a$ for the set of $a$-equivalence classes $\{[w]_a : w \in W\}$.

**Definition 3.1.13.** With an S5 structure $\mathfrak{M} = (W, (R_a)_{a\in\Gamma}, (P_i)_{i\in I})$ we associate the Kripke structure $\mathfrak{M}^+ = (W^+, R, U, (Q_a)_{a\in\Gamma}, (P_i)_{i\in I})$ with a new binary relation $R$, a new unary relation $U$, and one new unary relation $Q_a$, for each $a \in \Gamma$ as follows:

$$\begin{aligned} W^+ &= W \,\dot\cup\, \dot{\bigcup}_{a\in\Gamma} W/R_a, \\ U^{\mathfrak{M}^+} &= W, \\ P_i^{\mathfrak{M}^+} &= P_i^{\mathfrak{M}}, \\ Q_a^{\mathfrak{M}^+} &= W/R_a, \\ R^{\mathfrak{M}^+} &= \bigcup_{a\in\Gamma} \{(\mu, w) : w \in \mu \in W/R_a\}. \end{aligned}$$

$\mathfrak{M}^+$ is a two-sorted structure. Its universe is the disjoint union of the worlds of $\mathfrak{M}$ and the sets of $a$-equivalence classes in $\mathfrak{M}$, for all $a \in \Gamma$. There is an $R$-edge in $\mathfrak{M}^+$ from an equivalence class $\mu$ to a world $w$ if $w$ is an element of $\mu$, i.e. $[w]_a = \mu$, for some $a \in \Gamma$. Furthermore, all elements of $\mathfrak{M}^+$ are coloured by unary predicates such that one can reconstruct $\mathfrak{M}$ from $\mathfrak{M}^+$. The following observation is crucial but easy to see.

*Observation* 3.1.14. Let $\mathfrak{M}$ be an S5 structure and $k \geq 2$: $\mathfrak{M}$ is $k$-acyclic in the sense of Definition 3.1.12 if and only if $\mathfrak{M}^+$ is $2k$-acyclic in the sense of Definition 3.1.8.

Proposition 3.1.10 gives us a finite $2k$-acyclic covering $\hat{\mathfrak{M}}^+$ of the Kripke structure $\mathfrak{M}^+$. Since this covering is in particular faithful, one can reconstruct its associated S5 structure $\hat{\mathfrak{M}}$ which is in turn a $k$-acyclic covering of $\mathfrak{M}$.

**Proposition 3.1.15.** [9] Let $k \geq 2$. Every finite $\mathsf{S5}$ structure $\mathfrak{M}$ possesses a faithful bisimilar covering by a finite $k$-acyclic $\mathsf{S5}$ structure $\hat{\mathfrak{M}}$.

The coverings from Proposition 3.1.15 and a variation of Lemma 3.1.7 provide the key arguments to upgrade $\sim_{\forall}^{\ell+1}$-bisimilarity to $\equiv_{q,n}^{(\ell)}$-equivalence over (finite) $\mathsf{S5}$ structures. As argued above, this upgrading immediately implies the characterisation theorem 3.0.5. The following section deals with $\mathsf{CK}$ structures, a special subclass of $\mathsf{S5}$ structures that was introduced at the end of Chapter 2. We will explain why this subclass seems inherently locality averse and how a generalisation of $k$-acyclicity for $\mathsf{S5}$ structures helps to mend this problem.

## 3.2 Cayley structures

At first glance, locality techniques seem useless for working with $\mathsf{CK}$ structures. Since every pointed $\mathsf{CK}$ structure $\mathfrak{M}, w$ is bisimilar to the component of $w$, we can w.l.o.g. assume that every pointed $\mathsf{CK}$ structure is connected and one clique with respect to the relation $R_\Gamma$, which is induced by the set of all agents. Hence, every small neighbourhood is already the whole structure and the notion of Gaifman locality is trivialised. However, suitable Cayley structures, which can be seen as special $\mathsf{CK}$ structures, provide a promising approach to proving an upgrading theorem for $\mathsf{CK}$ structures because their edge pattern is not only very dense but also highly regular and therefore amenable to structured analysis.

Again, in order to prove the modal characterisation theorem for $\mathsf{ML}[\mathsf{CK}]$ we need to prove an upgrading for (finite) $\mathsf{CK}$ structures. In this case, we upgrade $\sim^{\ell}$-bisimilarity to $\equiv_q$-equivalence, for some $\ell$ that depends on $q$. The core ideas in the proof are the same as in the case of $\mathsf{ML}^{\forall}$ over (finite) $\mathsf{S5}$ structures but the technical details are considerably more difficult. Also with $\mathsf{CK}$ structures, we face the challenges of avoiding small multiplicities and short non-trivial cycles. Small multiplicities are taken care of by copying every edge multiple times. The challenge of avoiding small cycles can be viewed as a generalisation of the case of $\mathsf{S5}$ structures. With $\mathsf{S5}$ structures, we were not interested in the usual graph theoretic notion of acyclicity but in the overlap patterns of different equivalence classes (cf. Definition 3.1.12). The same is true for $\mathsf{CK}$ structures but the possible overlap patterns that we encounter are much more intricate because we need to deal with different levels of granularity: within every $\alpha$-equivalence class $[w]_\alpha$ we must be able to control the overlap patterns of classes $[w']_\beta \subseteq [w]_\alpha$, for $\beta \subsetneq \alpha$, and for every class $[w']_\beta \subseteq [w]_\alpha$, $\beta \subsetneq \alpha$, we must be able to control the overlap patterns of the classes that are subsets of $[w']_\beta$, and so forth. Cayley groups and coset acyclicity provide the right tools to deal with this kind of acyclicity.

### 3.2.1 From groups to structures

We introduced Cayley groups and Cayley graphs in Section 3.1.2 to sketch how they can be used to make Kripke structures $k$-acyclic via some product construction. In this section and throughout the rest of this work, we see Cayley structures as special instances of CK structures. The bisimilar coverings for CK structures will be Cayley structures that feature the kind of acyclicity we need to deal with common knowledge logic. Furthermore, acyclic Cayley structures are of independent interest and will be closely investigated in Chapter 4.

If we partition the generator set $E$ of a Cayley group $\mathbb{G} = (G, \circ, 1)$ into subsets $E_a$, associated with the agents $a \in \Gamma$, we may regard cosets with respect to the subgroups $\mathbb{G}_a = \langle e \colon e \in E_a \rangle$ as $a$-equivalence classes over $G$, and turn $G$ into the set of possible worlds of an S5 frame. The associated equivalence relation

$$R_a := \{(g, gh) \colon h \in \mathbb{G}_a\} = \mathsf{TC}\big(\bigcup\{R_e \colon e \in E_a\}\big)$$

is the (reflexive, symmetric) transitive closure of the edge relation induced by the generators $E_a$ in the Cayley graph. For sets of agents $\alpha \subseteq \Gamma$, this pattern naturally extends to subgroups $\mathbb{G}_\alpha = \langle e \colon e \in E_a, a \in \alpha \rangle$. The equivalence relations

$$R_\alpha := \{(g, gh) \colon h \in \mathbb{G}_\alpha\} = \mathsf{TC}\Big(\bigcup\{R_a \colon a \in \alpha\}\Big),$$

are the accessibility relations in the CK-expansion. Their equivalence classes are the cosets w.r.t. the subgroups generated by corresponding unions of sets of generators from the $\Gamma$-partition of $E$.

**Definition 3.2.1.** With any Cayley group $\mathbb{G} = (G, \circ, 1)$ with generator set $E$ that is $\Gamma$-partitioned, i.e. $E = \dot{\bigcup}_{a \in \Gamma} E_a$, we associate the *Cayley* CK*-frame* (Cayley frame, for short) $\mathbb{G}^{\mathsf{CK}}$ over the set $G$ of possible worlds with accessibility relations $R_\alpha$ for $\alpha \in \tau = \mathcal{P}(\Gamma)$. We also say that the Cayley frame $\mathbb{G}^{\mathsf{CK}}$ is *based on* the Cayley group $\mathbb{G}$. A *Cayley structure* consists of a Cayley frame together with a propositional assignment.

*Remark* 3.2.2. In Cayley frames, which are special instances of S5 frames, the $\mathbb{G}_\alpha$-coset of an element $g$ is the same as the $\alpha$-equivalence class of $g$ and the set of $\alpha$-successors of $g$, i.e.

$$g\mathbb{G}_\alpha = [g]_\alpha = R_\alpha[g].$$

*Notation* 3.2.3. From now on we write $\tau := \mathcal{P}(\Gamma)$ for the set of all sets of agents.

### 3.2.2 Coset acyclicity

As with S5 structures we aim to avoid certain cyclic overlap patterns between different equivalence classes. In the case of CK structures, the possible overlap patterns are more intricate yet highly regular. Consider a connected CK structure on the level of the accessibility relation $R_\Gamma$. Two different possible worlds are always at distance 1 from each other with respect to $R_\Gamma$. Non-trivial distances only arise when we zoom in and consider $\alpha$-steps, for $\alpha \subsetneq \Gamma$. Assume we have two Cayley structures $\mathfrak{M}$ and $\mathfrak{N}$ with possible world $w \in W$ and $v \in V$ such that $\mathfrak{M}, w \sim^\ell \mathfrak{N}, v$, and imagine there is a cycle $w, \alpha_1, w_2, \ldots, w_n, \alpha_n, w$ in $\mathfrak{M}$ with $n \leq \ell$. $\mathfrak{M}, w \sim^\ell \mathfrak{N}, v$ implies a path $v, \alpha_1, v_2, \ldots, v_n, \alpha_n, v_{n+1}$ in $\mathfrak{N}$ with $\mathfrak{M}, w_i \sim \mathfrak{N}, v_i$, for $2 \leq i \leq n$, and $\mathfrak{M}, w \sim \mathfrak{N}, v_{n+1}$. However, this path in $\mathfrak{N}$ is not necessarily a cycle. This difference might be expressible in $\mathsf{FO}_q$ and must therefore be avoided in coverings of $\mathfrak{M}$ and $\mathfrak{N}$. To be more precise, the coverings of $\mathfrak{M}$ and $\mathfrak{N}$ must be locally acyclic w.r.t. non-trivial overlaps of $\alpha$-cosets, for various $\alpha$. Simultaneously, every coset $[w]_\alpha$ of the coverings must be locally acyclic in the same sense w.r.t. $\beta$-cosets, for $\beta \subsetneq \alpha$.

The suitable notion of acyclicity to capture all these overlap patterns for all levels of granularity is called *coset acyclicity* and was first defined in [26] for Cayley groups.

**Definition 3.2.4.** Let $\mathbb{G}$ be a Cayley group with generator set $E$. A *coset cycle of length $m$ in* $\mathbb{G}$ is a cyclic tuple $((g_i, \alpha_i))_{i \in \mathbb{Z}_m}$ with $g_i \in \mathbb{G}$ and $\alpha_i \subseteq E$, for all $i \in \mathbb{Z}_m$, where $g_i g_{i+1}^{-1} \in \mathbb{G}_{\alpha_i}$ and

$$g_i \mathbb{G}_{\alpha_{i-1} \cap \alpha_i} \cap g_{i+1} \mathbb{G}_{\alpha_i \cap \alpha_{i+1}} = \emptyset.$$

**Definition 3.2.5.** A Cayley group is *acyclic* if it does not contain a coset cycle, and *$n$-acyclic* if it does not contain a coset cycle of length up to $n$.

For every generator set $E$ one can construct a finite, $n$-acyclic Cayley group generated by $E$ in an inductive manner. One starts with some finite Cayley group $\mathbb{G}$ that is generated by $E$ and makes it, in some sense, more and more acyclic in every step. The resulting Cayley group $\hat{\mathbb{G}}$ is also generated by $E$, finite, $n$-acyclic and compatible with $\mathbb{G}$ in the sense that there is a surjective group homomorphism $\pi \colon \hat{\mathbb{G}} \to \mathbb{G}$. This means that every Cayley structure that is based on $\mathbb{G}$ has a covering that is based on $\hat{\mathbb{G}}$. This has been shown by Otto in [26]:

**Lemma 3.2.6.** *For every finite Cayley group $\mathbb{G}$ with finite generator set $E$ and every $n \in \mathbb{N}$, there is a finite, $n$-acyclic Cayley group $\hat{\mathbb{G}}$ with generator set $E$ such that there is a surjective homomorphism $\pi \colon \hat{\mathbb{G}} \to \mathbb{G}$.*

Based on Lemma 3.2.6 we will construct suitable finite coverings for finite $\mathsf{CK}$ structures in Section 3.2.4. The definition of a coset cycle in 3.2.4 is very general. The sets $\alpha_i$ in a cyclic tuple can be any subset of the generator set, as long as the tuple as a whole fulfils the coset acyclic property. This is far more general than we need for our purposes. A more restrictive version tailored for Cayley frames suffices, where the arbitrary sets of generators become sets of agents.

**Definition 3.2.7.** Let $\mathfrak{M}$ be a Cayley frame. A *coset cycle of length $m$ in* $\mathfrak{M}$ is a cyclic tuple $((w_i, \alpha_i))_{i \in \mathbb{Z}_m}$ with $w_i \in W$ and $\alpha_i \in \tau$, for all $i \in \mathbb{Z}_m$, where $(w_i, w_{i+1}) \in R_{\alpha_i}$ and

$$[w_i]_{\alpha_{i-1} \cap \alpha_i} \cap [w_{i+1}]_{\alpha_i \cap \alpha_{i+1}} = \emptyset.$$

**Definition 3.2.8.** A Cayley frame is *acyclic* if it does not contain a coset cycle, and *$n$-acyclic* if it does not contain a coset cycle of length up to $n$.

If a Kripke structure is acyclic in the sense of Definition 3.1.8, any two connected worlds give rise to a unique connecting path. If a Kripke structure is $2k+1$-acyclic, any two worlds that share some $k$-neighbourhood give rise to a unique shortest path. These notions of unique path and unique shortest path within $k$-neighbourhoods, respectively, can be generalized to acyclic and $n$-acyclic Cayley structures. This is one of the main results of Chapter 4, which is essential for proving $\mathsf{FO}_q$-equivalence between suitable $\ell$-bisimilar coverings.

### 3.2.3 The dual hypergraph and $\alpha$-acyclicity

In Section 3.1.2, we introduced an auxiliary structure $\mathfrak{M}^+$ associated with an S5 structure $\mathfrak{M}$. $\mathfrak{M}^+$ is $2k$-acyclic in the usual graph theoretic sense if and only if $\mathfrak{M}$ is a $k$-acyclic S5 structure. We used $\mathfrak{M}^+$ to construct $k$-acyclic coverings for S5 structures by constructing sufficiently acyclic coverings for $\mathfrak{M}^+$. In this section, we define for every Cayley structure $\mathfrak{M}$ an auxiliary structure $d(\mathfrak{M})$, the *dual hypergraph of* $\mathfrak{M}$. This hypergraph is $n$-acyclic with respect to a suitable notion of acyclicity for hypergraphs if $\mathfrak{M}$ is $n$-acyclic with respect to coset acyclicity. In contrast to the standard S5 case, we will not use $d(\mathfrak{M})$ to construct coverings for $\mathfrak{M}$. Instead, the dual hypergraph plays a key role in describing a winning strategy for player **II** in the Ehrenfeucht-Fraïssé game on Cayley structures.

In an undirected, loop-free graph every edge can be seen as a set that contains exactly two vertices. A hypergraph is a generalisation of a graph in which an edge can contain any number of vertices.

**Definition 3.2.9.** A *hypergraph* is a structure $\mathcal{A} = (A, S)$ with a set of vertices $A$ and a set of hyperedges $S \subseteq \mathcal{P}(A)$.

With a hypergraph $\mathcal{A} = (A, S)$ we associate its Gaifman graph $G(\mathcal{A}) = (A, G(S))$ with an undirected edge relation $G(S)$ that links two vertices $a \neq a'$ if $a, a' \in s$, for some $s \in S$. An $n$-cycle in a hypergraph is a cycle of length $n$ in its Gaifman graph, and an $n$-path in a hypergraph is a path of length $n$ in its Gaifman graph. The distance $d(X, Y)$ in a hypergraph between two subsets of vertices $X$ and $Y$ is the usual graph theoretic distance between $X$ and $Y$ in its Gaifman graph. A *chord* of an $n$-cycle or $n$-path is an edge between vertices that are not next neighbours along the cycle or path. The following definition of hypergraph acyclicity is the classical one from [4], also known as $\alpha$-acyclicity in [3]; $n$-acyclicity was first introduced in [26].

**Definition 3.2.10.** A hypergraph $\mathcal{A} = (A, S)$ is *acyclic* if it is *conformal* and *chordal*:

1. conformality requires that every clique in the Gaifman graph $G(\mathcal{A})$ is contained in some hyperedge $s \in S$;

2. chordality requires that every cycle in the Gaifman graph $G(\mathcal{A})$ of length greater than 3 has a chord.

For $n \geq 3$, $\mathcal{A} = (A, S)$ is *$n$-acyclic* if it is *$n$-conformal* and *$n$-chordal*:

3. $n$-conformality requires that every clique in $G(\mathcal{A})$ up to size $n$ is contained in some hyperedge $s \in S$;

4. $n$-chordality requires that every cycle in $G(\mathcal{A})$ of length greater than 3 and up to $n$ has a chord.

*Remark* 3.2.11. If a hypergraph is $n$-acyclic, then every induced substructure of size up to $n$ is acyclic [26].

A connected and acyclic graph is called a tree. If a hypergraph is acyclic, it is in some sense tree-like or *tree decomposable*.

**Definition 3.2.12.** A hypergraph $(A, S)$ is *tree decomposable* if it admits a tree decomposition $\mathcal{T} = (T, \delta)$: $T$ is a tree and $\delta \colon T \to S$ is a map such that image$(\delta) = S$ and, for every node $a \in A$, the set $\{v \in T : a \in \delta(v)\}$ is connected in $T$.

A well-known result from classical hypergraph theory states that a hypergraph is tree decomposable if and only if it is acyclic (see [4], [3]).

In the characterisation of $\mathsf{ML}^{\forall}$, Dawar and Otto used that small neighbourhoods of sufficiently acyclic structures are trees to show Gaifman equivalence. To prove the characterisation of $\mathsf{ML[CK]}$, we will use that small substructures of sufficiently acyclic hypergraphs are tree decomposable in a similar vein (cf. [26]). The auxiliary structure we need for this approach is the dual hypergraph of a Cayley frame.

**Definition 3.2.13.** In a Cayley frame $\mathfrak{M}$, define the *dual hyperedge* induced by a world $w$ to be the set of cosets

$$[\![w]\!] := \{[w]_\alpha \colon \alpha \in \tau\}.$$

**Definition 3.2.14.** Let $\mathfrak{M} = (W, (R_\alpha)_{\alpha\in\tau})$ be a Cayley frame. Its *dual hypergraph* is the vertex-coloured hypergraph

$$\begin{aligned} d(\mathfrak{M}) &:= (d(W), S, (Q_\alpha)_{\alpha\in\tau}) \text{ where} \\ d(W) &:= \dot{\bigcup}_{\alpha\in\tau} Q_\alpha \quad \text{for } Q_\alpha := W/R_\alpha, \\ S &:= \{[\![w]\!] \subseteq d(W) : w \in W\}. \end{aligned}$$

As the name suggests, everything in the dual hypergraph is flipped. The worlds of $\mathfrak{M}$ are the hyperedges of $d(\mathfrak{M})$, the equivalence classes of $\mathfrak{M}$ are the vertices of $d(\mathfrak{M})$, and if two worlds in $\mathfrak{M}$ are connected by an edge, their respective dual hyperedges in $d(\mathfrak{M})$ share a vertex.

*Remark* 3.2.15. In a Cayley frame $\mathfrak{M}$ for all $w, v \in W$ and all $\alpha \in \tau$:

$$[w]_\alpha = [v]_\alpha \;\Leftrightarrow\; w \in [v]_\alpha \;\Leftrightarrow\; [v]_\alpha \in [\![w]\!] \;\Leftrightarrow\; (w, v) \in R_\alpha \;\Leftrightarrow\; v \in R_\alpha[w]$$

The notions of acyclicity for Cayley frames and hypergraph acyclicity are directly connected by the following lemma.

**Lemma 3.2.16.** *[26] For $n \geq 3$, if $\mathfrak{M}$ is an $n$-acyclic Cayley frame, then $d(\mathfrak{M})$ is an $n$-acyclic hypergraph.*

*Proof.* We need to show that $d(\mathfrak{M})$ is $n$-chordal and $n$-conformal.

*$n$-chordality*: Let $([w_i]_{\alpha_i})_{i\in\mathbb{Z}_m}$ be a chordless cycle in $d(\mathfrak{M})$ of length $m > 3$ with hyperedges $([\![w_i]\!])_{i\in\mathbb{Z}_m}$ linking $[w_{i-1}]_{\alpha_{i-1}}$ and $[w_i]_{\alpha_i}$; in particular $[w_{i-1}]_{\alpha_{i-1}}, [w_i]_{\alpha_i} \in [\![w_i]\!]$, i.e. $w_i \in [w_{i-1}]_{\alpha_{i-1}} \cap [w_i]_{\alpha_i}$. We prove $m > n$ by showing that every chordless cycle of length $m > 3$ induces a coset cycle of the same length in $\mathfrak{M}$, which must be longer than $n$ because $\mathfrak{M}$ is $n$-acyclic.

That $[\![w_{i+1}]\!]$ links $[w_i]_{\alpha_i}$ and $[w_{i+1}]_{\alpha_{i+1}}$ implies

$$[w_i]_{\alpha_i} \in [\![w_{i+1}]\!] \;\Rightarrow\; w_{i+1} \in [w_i]_{\alpha_i} \;\Rightarrow\; (w_i, w_{i+1}) \in R_{\alpha_i}.$$

Hence, the sequence $((w_i, \alpha_i))_{i \in \mathbb{Z}_m}$ forms a cycle in $\mathfrak{M}$. Additionally, this cycle must also be a coset cycle. If it was not, there would be an $i \in \mathbb{Z}_m$ and some $w \in \mathfrak{M}$ with

$$w \in [w_i]_{\alpha_{i-1} \cap \alpha_i} \cap [w_{i+1}]_{\alpha_i \cap \alpha_{i+1}}.$$

By definition we have $[v]_\beta \subseteq [v]_\alpha$, for all worlds $v$ and all $\alpha, \beta \in \tau$ with $\beta \subseteq \alpha$. Hence,

$$w \in [w_{i+1}]_{\alpha_i \cap \alpha_{i+1}} \;\Rightarrow\; w \in [w_{i+1}]_{\alpha_{i+1}}$$

and

$$w \in [w_i]_{\alpha_{i-1} \cap \alpha_i} \;\Rightarrow\; w \in [w_i]_{\alpha_{i-1}} = [w_{i-1}]_{\alpha_{i-1}}$$

since $[w_{i-1}]_{\alpha_{i-1}} \in [\![w_i]\!]$. This means that $[\![w]\!]$ links $[w_{i-1}]_{\alpha_{i-1}}$ and $[w_{i+1}]_{\alpha_{i+1}}$ which contradicts the choice of $([w_i]_{\alpha_i})_{i \in \mathbb{Z}_m}$ as a chordless cycle. Thus, $((w_i, \alpha_i))_{i \in \mathbb{Z}_m}$ is a coset cycle of length $m > n$ since $\mathfrak{M}$ is $n$-acyclic.

*$n$-conformality*: We say that a clique $U \subseteq d(W)$ in $d(\mathfrak{M})$ is *guarded* if there is a hyperedge $[\![w]\!]$ with $U \subseteq [\![w]\!]$. Let $\{a_i : i \in \mathbb{Z}_m\} \subseteq d(W)$ be an unguarded clique in $d(\mathfrak{M})$ of size $m$ that is assumed to be minimal in the sense that all its sub-cliques of size $m-1$ are guarded by some hyperedge. Again, we will show $m > n$ by constructing some coset cycle of length $m$ in $\mathfrak{M}$.

For $i \in \mathbb{Z}_m$ let $[\![w_i]\!]$, $w_i \in W$, be a hyperedge such that $\{a_j : j \in \mathbb{Z}_m \setminus \{i\}\} \subseteq [\![w_i]\!]$. If we assume that $a_j$ is an $\alpha_j$-coset, $\alpha_j \in \tau$, then $a_j$ contains all $w_i$ with $i \neq j$, i.e. $a_j = [w_i]_{\alpha_j}$, for $i \neq j$. It follows that $w_i$ and $w_{i+1}$ are both elements of the same $\alpha_j$-cosets $a_j$, for all $j \neq i, i+1$. Hence, if we define $\beta_i := \bigcap_{j \neq i, i+1} \alpha_j$, for all $i \in \mathbb{Z}_m$, then $(w_i, w_{i+1}) \in R_{\beta_i}$. For the sequence $((w_i, \beta_i))_{i \in \mathbb{Z}_m}$ to be a coset cycle it remains to show that

$$[w_i]_{\beta_{i-1} \cap \beta_i} \cap [w_{i+1}]_{\beta_i \cap \beta_{i+1}} = \emptyset,$$

for all $i \in \mathbb{Z}_m$. Assume that there is an $i \in \mathbb{Z}_m$ and a $w \in W$ with

$$w \in [w_i]_{\beta_{i-1} \cap \beta_i} \cap [w_{i+1}]_{\beta_i \cap \beta_{i+1}}.$$

We show, contrary to our assumption, that $[\![w]\!]$ would guard the clique $\{a_j : j \in \mathbb{Z}_m\}$, i.e. $a_j \in [\![w]\!]$, for all $j \in \mathbb{Z}_m$. First, note that $\beta_{i-1} \cap \beta_i = \bigcap_{j \neq i} \alpha_j$ and $\beta_i \cap \beta_{i+1} = \bigcap_{j \neq i+1} \alpha_j$. Now if $j \neq i$, then

$$w \in [w_i]_{\beta_{i-1} \cap \beta_i} \subseteq [w_i]_{\alpha_j} = a_j \;\Rightarrow\; a_j \in [\![w]\!],$$

and if $j \neq i+1$, then

$$w \in [w_{i+1}]_{\beta_i \cap \beta_{i+1}} \subseteq [w_{i+1}]_{\alpha_j} = a_j \;\Rightarrow\; a_j \in [\![w]\!].$$

Thus, $((w_i, \beta_i))_{i \in \mathbb{Z}_m}$ is a coset cycle and $n$-acyclicity of $\mathfrak{M}$ implies $m > n$. □

Essentially, if a hypergraph is $n$-acyclic, its small subhypergraphs are acyclic and therefore tree decomposable. In the Ehrenfeucht-Fraïssé game on $n$-acyclic Cayley structures we will keep track of small substructures in the dual hypergraphs that are determined by the pebbled elements, and use tree decompositions of these substructures to describe a winning strategy for player **II**.

### 3.2.4 Cayley structures as coverings

As introduced in Section 3.2.1, Cayley structures are Kripke structures with frames that are based on Cayley groups. If the Cayley group has $E$ as its generator set and $E$ is $\Gamma$-partitioned, we can consider the Cayley structure as a special $\mathsf{CK}$ structure. Compared to general $\mathsf{CK}$ structures, Cayley structures have a highly regular and homogeneous edge pattern. We aim to exploit this regularity.

In the present section, we will show that for every $\mathsf{CK}$ structure there is a bisimilar covering by a Cayley structure. Thus, every question about $\mathsf{CK}$ structures reduces, up to bisimulation, to Cayley structures. Furthermore, we will show that every $\mathsf{CK}$ structure has an infinite covering that is acyclic, and every finite $\mathsf{CK}$ structure has a finite covering that is $n$-acyclic, for any $n \in \mathbb{N}$ (cf. Lemmas 3.2.22 and 3.2.24). These lemmas are among the main results of this thesis. Acyclic and $n$-acyclic Cayley structures as bisimilar coverings play a key role in the characterisation of $\mathsf{ML}[\mathsf{CK}]$. They are needed to upgrade $\ell$-bisimilarity to $\mathsf{FO}_q$-equivalence. This will be proven in Chapter 5.

The first step to our coverings is the following helpful lemma. It states that we only have to consider the basic-agent reduct of $\mathsf{CK}$ structures when we construct coverings for them. It will be used in the following way: in order to construct a covering for a $\mathsf{CK}$ structure $\mathfrak{M}^{\mathsf{CK}}$, one just needs to construct a covering $\mathfrak{N}$ for $\mathfrak{M}$. Then $\mathfrak{N}^{\mathsf{CK}}$ is a covering for $\mathfrak{M}^{\mathsf{CK}}$.

**Lemma 3.2.17.** *Let* $\mathfrak{M} = (W, (R_a^{\mathfrak{M}})_{a\in\Gamma}, (P_i^{\mathfrak{M}})_{i\in I}), w$ *and* $\mathfrak{N} = (V, (R_a^{\mathfrak{N}})_{a\in\Gamma}, (P_i^{\mathfrak{N}})_{i\in I}), v$ *be two pointed* $\mathsf{S5}$ *structures and* $\mathfrak{M}^{\mathsf{CK}}, w$ *and* $\mathfrak{N}^{\mathsf{CK}}, v$ *their* $\mathsf{CK}$*-expansions. Then*

$$\mathfrak{M}, w \sim \mathfrak{N}, v \quad \Leftrightarrow \quad \mathfrak{M}^{\mathsf{CK}}, w \sim \mathfrak{N}^{\mathsf{CK}}, v.$$

*Proof.* For the direction from left to right let $Z \subseteq W \times V$ be a bisimulation relation with $(w, v) \in Z$. We claim that $Z$ is also a bisimulation relation with respect to $\mathfrak{M}^{\mathsf{CK}}$ and $\mathfrak{N}^{\mathsf{CK}}$. If $\alpha \subseteq \Gamma$, $w' \in R_\alpha^{\mathfrak{M}}[w]$, there is an $\alpha$-path

$$w = w_0, a_1, w_1, a_2, w_2, \ldots, w_\ell = w',$$

$a_i \in \alpha$ for all $1 \leq i \leq \ell$, from $w$ to $w'$. Since $(w, v) \in Z$, there is an $\alpha$-path

$$v = v_0, a_1, v_1, a_2, v_2, \ldots, v_\ell$$

in $\mathfrak{N}$ with $(w_i, v_i) \in Z$, for all $0 \leq i \leq \ell$. By definition of a $\mathsf{CK}$-expansion we have $v_\ell \in R_\alpha^{\mathfrak{N}}[v]$. The direction from right to left is trivial. □

For technical reasons, the construction that is used to prove Theorem 3.2.19 and Lemma 3.2.24 involves hypercubes.

**Definition 3.2.18.** For a set $E$, the *hypercube* $\mathcal{Q}_E$ is the undirected, loop-free, $E$-edge-coloured graph $(Q_E, (F_e)_{e\in E})$ with vertex set $Q_E := \{f \colon E \to \{0,1\}\}$ and with an $e$-edge, $e \in E$, between any two vertices $f, g \in Q_E$ if $f(e) \neq g(e)$ and $f(e') = g(e')$, for all $e' \in E \setminus \{e\}$.

The construction of Cayley structures as coverings for $\mathsf{CK}$ structures bears similarities to the construction of acyclic Cayley groups described in Section 3.1.2. There we constructed an edge-coloured tree with vertex set $V$ and defined the Cayley group of interest as the subgroup of the symmetric group on $V$ that is generated by the swaps that are induced by the edge-colours. In the proof of Theorem 3.2.19, we define the Cayley group that is supposed to cover the given $\mathsf{CK}$ structure also as the subgroup of a symmetric group, but instead of some tree we use the given $\mathsf{CK}$ structure to get the suitable set of generators.

**Theorem 3.2.19.** *Every (finite) connected* $\mathsf{CK}$ *structure admits a bisimilar covering by a (finite) Cayley structure.*

*Proof.* Let $\mathfrak{M}^{\mathsf{CK}}$ be a connected $\mathsf{CK}$ structure. If we construct a bisimilar covering that is based on a Cayley group $\mathbb{G}$ for the reduct $\mathfrak{M} = (W, (R_a^{\mathfrak{M}})_{a\in\Gamma}, (P_i^{\mathfrak{M}})_{i\in I})$, we are done by Lemma 3.2.17. The generator set $E$ of $\mathbb{G}$ will be the disjoint union of all accessibility relations $R_a^{\mathfrak{M}}$, $a \in \Gamma$, i.e.

$$E := \bigcup_{a\in\Gamma} \big(R_a^{\mathfrak{M}} \times \{a\}\big),$$

which is partitioned into subsets $E_a := R_a^{\mathfrak{M}} \times \{a\}$ corresponding to the individual $R_a^{\mathfrak{M}}$. Essentially, $\mathbb{G}$ is a subgroup of the symmetric group on $W$ generated by the swaps that are induced by the edges of $\mathfrak{M}$. However, for our construction to be well defined we need to make sure that every element in $E$ induces a permutation that is distinct from the identity and from all other permutations induced by elements from $E$. This is where the hypercube $\mathcal{Q}_E = (Q_E, (F_e)_{e\in E})$ comes in. Let

$$\mathfrak{M} \oplus \mathcal{Q}_E := (V, (R_e)_{e\in E})$$

be the undirected $E$-edge-labelled graph formed by the disjoint union of $\mathfrak{M}$ with the $|E|$-dimensional hypercube $\mathcal{Q}_E$; in contrast to directed edges, we denote the undirected edges of $\mathfrak{M} \oplus \mathcal{Q}_E$ as sets of size 1 or 2:

- $V := W \,\dot\cup\, Q_E$
- $R_e := \{\{w, v\} \subseteq W : e = ((w, v), a) \in E, \text{ for some } a \in \Gamma\} \cup F_e$

As a side note, the graph $\mathfrak{M} \oplus \mathcal{Q}_E$ is not loop-free since there are loops in $R_a$ that induce loops in $R_e$, for $e = ((w, w), a)$. However, and more importantly, for all $e, e' \in E$,

1. if $e \neq e'$, then $R_e \neq R_{e'}$ because $F_e \cap F_{e'} = \emptyset$,
2. $F_e \neq \emptyset$, and $F_e$ does not contain loops, and
3. every edge in $R_e$ is induced either by an edge from $\mathfrak{M}$ or $\mathcal{Q}_E$.

With $\mathrm{Sym}(V)$ we denote the symmetric group on $V$, and with every $e \in E$ we associate the involutive permutation $\pi_e \in \mathrm{Sym}(V)$ that precisely swaps all pairs of vertices in $e$-labelled edges, i.e. $\pi_e(w) = v$ if and only if $\{w, v\} \in R_e$. From the properties of $R_e$ listed above, we immediately obtain for all $e, e' \in E$

1. $\pi_e \neq \pi_{e'}$ if $e \neq e'$,
2. $\pi_e$ is not the identity on $V$, and
3. $\pi_e(v) \in W$ if $v \in W$, and $\pi_e(v) \in Q_E$ if $v \in Q_E$.

We define $\mathbb{G} = (G, \circ, 1)$ as the subgroup of $\mathrm{Sym}(V)$ that is generated by the set of involutions $\{\pi_e \in \mathrm{Sym}(V) : e \in E\}$. If $\mathfrak{M}$ is finite and with it the sets $E$ and $V$, then $\mathbb{G}$ is also finite. The Cayley structure that is a covering of $\mathfrak{M}$ will be based on $\mathbb{G}$. In order to unclutter notation, we identify the permutation $\pi_e$ with the edge $e \in E$. If $(G, (R_a)^{\mathbb{G}}_{a\in\Gamma})$ is the Cayley frame that is based on $\mathbb{G}$, it remains to define a propositional assignment $(P_i^{\mathbb{G}})_{i\in I}$ for the elements of $G$ and prove that the structure $(G, (R_a^{\mathbb{G}})_{a\in\Gamma}, (P_i^{\mathbb{G}})_{i\in I})$ is a bisimilar covering of $\mathfrak{M}$.

In order to define the propositional assignment, we let $\mathbb{G}$ act on $V$ from the right: for a group element $g = e_1 \cdots e_n$ let

$$g \colon v \mapsto vg = ve_1 \cdots e_n := (\pi_{e_n} \circ \cdots \circ \pi_{e_1})(v).$$

This operation is well-defined ($vg$ does not depend on the decomposition of $g$ into generators) as a group action, since by definition of $\mathbb{G}$ we have $e_1 \cdots e_n = 1$ if and only if $\pi_{e_n} \circ \cdots \circ \pi_{e_1}$ fixes every $v \in V$. Since $\pi_e(W) \subseteq W$, for all $e \in E$, the group action can be restricted to $W$, and the map

$$\begin{aligned} \hat{\pi} \colon W \times G &\longrightarrow W, \\ (w, g) &\longmapsto wg \end{aligned}$$

is well-defined. We fix an arbitrary world $w_0$ to define the propositional assignment: the group $\mathbb{G}$ acts transitively on $W$, since $\mathfrak{M}$ is connected, which implies that the map

$$\pi \colon G \to W,\ g \mapsto w_0 g$$

is surjective. Set, for all $i \in I$,

$$P_i^{\mathbb{G}} := \pi^{-1}(P_i^{\mathfrak{M}}) \subseteq G.$$

We claim that $\pi \colon (G, (R_a^{\mathbb{G}})_{a\in\Gamma}, (P_i^{\mathbb{G}})_{i\in I}) \to \mathfrak{M}$ is a bisimilar covering. It remains to be proven that the two structures are bisimilar via the relation

$$Z := \{(g, w) \in G \times W : w = \pi(g) = w_0 g\}.$$

First, the definition of $P_i^{\mathbb{G}}$ implies

$$g \in P_i^{\mathbb{G}} \quad \Leftrightarrow \quad \pi(g) \in P_i^{\mathfrak{M}}.$$

Second, $Z$ has the back-and-forth properties:

*Forth*: Let $(g, w_0g) \in Z$, $a \in \Gamma$ and $g' \in R_a^{\mathbb{G}}[g]$. As $g' \in R_a^{\mathbb{G}}[g]$ there is an element $h \in \mathbb{G}_a$ with $g' = gh$ and a decomposition $h = e_1 \cdots e_n$ such that $e_i \in E_a$, for all $1 \leq i \leq n$. We need to show $w_0gh \in R_a^{\mathfrak{M}}[w_0g]$.

The decomposition of $h$ induces in $\mathfrak{M}$ the sequence of worlds $w_0g = v_0, v_1, \ldots, v_n$ with $v_i = \pi_{e_i}(v_{i-1})$, for all $1 \leq i \leq n$, and in particular $v_n = w_0gh$. If $v_i = v_{i-1}$, there is an $a$-edge between $v_{i-1}$ and $v_i$ because $R_a^{\mathfrak{M}}$ is reflexive. If $v_i \neq v_{i-1}$, then $\{v_{i-1}, v_i\} \in R_{e_i}$, which means that either $e_i = ((v_{i-1}, v_i), a)$ or $e_i = ((v_i, v_{i-1}), a)$. In any case, there is an $a$-edge between $v_{i-1}$ and $v_i$ because $R_a^{\mathfrak{M}}$ is symmetric.

Thus, the sequence $v_0, v_1, \ldots, v_n$ from $w_0g$ to $w_0gh$ is in fact an $a$-path in $\mathfrak{M}$, which implies $w_0gh \in R_a^{\mathfrak{M}}[w_0g]$ because $R_a^{\mathfrak{M}}$ is transitive.

*Back*: Let $(g, w_0g) \in Z$, $a \in \Gamma$ and $w' \in R_a^{\mathfrak{M}}[w_0g]$; set $w = w_0g$. That $w'$ is an $a$-successor of $w$ implies that $(w, w')$ is an element of $R_a^{\mathfrak{M}}$, which means $\{w, w'\} \in R_e$ and $\pi_e(w) = w'$, for $e = ((w, w'), a) \in E_a \subseteq \mathbb{G}_a$. Thus, $ge \in R_a^{\mathbb{G}}[g]$, $w_0ge = we = \pi_e(w) = w'$ and $(ge, w') \in Z$. □

Theorem 3.2.19 gives us the crucial insight that we can reduce every model-theoretic question about (finite) CK structures up to bisimulation to (finite) Cayley structures. With Cayley structures as the universal representatives of CK structures, we can take advantage of their highly regular, homogeneous edge patterns and of coset acyclicity. As we will show, coset acyclicity is the suitable notion of acyclicity for Cayley structures and for characterising common knowledge logic. For this purpose, we need to be able to construct suitable coverings with specific acyclicity properties: for the classical case, fully acyclic Cayley structures, which are easier to construct, suffice. For the finite model theory case, we need finite $n$-acyclic coverings.

*Acyclic coverings*: For the classical case of our characterisation theorem, we want to construct coverings that are fully acyclic and have infinitely many copies of every realised bisimulation type.

**Definition 3.2.20** ($\omega$-rich)**.** A Cayley structure $\mathfrak{M}$ is *$\omega$-rich* if, for all $\alpha \in \tau$, every $\alpha$-neighbour $u$ of any world $w$ has countably infinitely many $\alpha$-neighbours $u' \in [w]_\alpha$ that are bisimilar to $u$.

Acyclic and $\omega$-rich Cayley structures can be constructed by using a free group over some suitable generator set that boosts the original multiplicities. To define free groups, we need the notion of a reduced word: if $E$ is a set, then an $E$-word $w = e_1 \ldots e_n$ is *reduced* if $e_{i+1} \neq e_i$, for all $1 \leq i < n$.

**Definition 3.2.21** (Free group)**.** Let $E$ be a generator set. The *free group* $F(E)$ over $E$ is the group that consists of all reduced words over the alphabet $E$ without any non-trivial equalities, together with the (reduced) concatenation of words as its operation and the empty word as its neutral element.

That no non-trivial equalities exist means that two reduced words $e_1 \ldots e_n$ and $a_1 \ldots a_\ell$ are equivalent in $F(G)$ if and only if $n = \ell$ and $e_i = a_i$, for all $1 \leq i \leq n$. Hence, the Cayley graph of a free group is an infinite tree, i.e. it is acyclic in the usual graph-theoretic sense.

**Lemma 3.2.22.** *Every connected* CK *structure admits a bisimilar covering by an acyclic and $\omega$-rich Cayley structure.*

*Proof.* Let $\mathfrak{M}^{\mathsf{CK}}$ be a connected CK structure and $\mathfrak{M} = (W, (R_a^{\mathfrak{M}})_{a\in\Gamma}, (P_i^{\mathfrak{M}})_{i\in I})$ its basic-agent reduct. The generator set $E$ of the Cayley group $\mathbb{G}$ that the covering will be based on is the disjoint union of the relations $R_a^{\mathfrak{M}}$, $a \in \Gamma$, with multiplicities boosted by $\omega$, i.e.

$$E := \bigcup_{a\in\Gamma} (R_a^{\mathfrak{M}} \times \{a\} \times \omega),$$

which is partitioned into subsets $E_a := R_a^{\mathfrak{M}} \times \{a\} \times \omega$, for every agent $a \in \Gamma$. Let $\mathbb{G} = (G, \circ, 1)$ be the free group generated by $E$. Then $(G, (R_a^{\mathbb{G}})_{a\in\Gamma})$ is the suitable Cayley frame of the desired covering. The propositional assignment is defined as in Theorem 3.2.19: choose an arbitrary element $w_0 \in W$ and consider the group action from the right, $G \times W, (g, w) \mapsto wg$, to define the surjective map

$$\pi\colon G \to W,\ g \mapsto w_0 g.$$

If $P_i^{\mathbb{G}} := \pi^{-1}(P_i^{\mathfrak{M}})$, then

$$\pi\colon (G, (R_a^{\mathbb{G}})_{a\in\Gamma}, (P_i^{\mathbb{G}})_{i\in I}) \to (W, (R_a^{\mathfrak{M}})_{a\in\Gamma}, (P_i^{\mathfrak{M}})_{i\in I})$$

is a bisimilar covering of $\mathfrak{M}$. This can be shown analogously to the case of Theorem 3.2.19.

The full Cayley structure $(G, (R_\alpha^{\mathbb{G}})_{\alpha\in\tau}, (P_i^{\mathbb{G}})_{i\in I})$ is $\omega$-rich because we boosted the multiplicity of every generator to $\omega$. It is coset acyclic because the Cayley graph of $\mathbb{G}$ is acyclic: every coset cycle $w_1, \alpha_1, w_2, \ldots, w_\ell, \alpha_\ell, w_1$ in the Cayley structure can be decomposed into a cycle

$$w_1, a_{11}, \ldots, a_{1k_1}, w_2, \ldots, w_\ell, a_{\ell 1}, \ldots, a_{\ell k_\ell}, w_1$$

with $a_{ij} \in \alpha_i$, for all $1 \leq i \leq \ell$, $1 \leq j \leq k_i$, which is a cycle in the usual graph theoretic sense in the Cayley graph of $\mathbb{G}$. Hence, the Cayley structure is coset acyclic if the Cayley graph of $\mathbb{G}$ is acyclic. □

*Finite coverings*: For the finite model theory case of the characterisation theorem for common knowledge logic, we need to construct suitable *finite* coverings for the upgrading argument to go through. We cannot hope to obtain finite coverings that are $\omega$-rich or fully acyclic for obvious reasons. However, since we only need to play $q$-round Ehrenfeucht-Fraïssé games for some finite number $q$, the coverings only need to be *sufficiently* acyclic and *sufficiently* rich.

**Definition 3.2.23** ($k$-rich)**.** For $k \in \mathbb{N}$, a Cayley structure $\mathfrak{M}$ is *$k$-rich* if every $\alpha$-neighbour $u$ of any world $w$ has at least $k$ $\alpha$-neighbours $u' \in [w]_\alpha$ that are bisimilar to $u$.

Since finite, $n$-acyclic groups for arbitrary generator sets $E$ exist by Lemma 3.2.6, we can easily obtain finite coverings for $\mathsf{CK}$ structures that are $k$-rich and $n$-acyclic. Let $\mathfrak{M} = (W, (R_a^{\mathfrak{M}})_{a\in\Gamma}, (P_i^{\mathfrak{M}})_{i\in I})$ be a finite $\mathsf{S5}$ structure and set

$$E := \bigcup_{a\in\Gamma} (R_a^{\mathfrak{M}} \times \{a\} \times \{1, \dots, k\}),$$

with partitions $E_a := R_a^{\mathfrak{M}} \times \{a\} \times \{1, \dots, k\}$. First, we use the construction from Theorem 3.2.19 to obtain a finite Cayley group $\mathbb{G}$ with an associated Cayley frame that can be extended to a covering of $\mathfrak{M}$. Second, we apply Lemma 3.2.6 to $\mathbb{G}$ to obtain a finite, $n$-acyclic Cayley group $\hat{\mathbb{G}} = (\hat{G}, \circ, 1)$ with generator set $E$ and an associated Cayley frame $(\hat{G}, (R_a^{\hat{\mathbb{G}}})_{a\in\Gamma})$. Since there is, in particular, a surjective homomorphism $\pi \colon \hat{\mathbb{G}} \to \mathbb{G}$, we can extend $(\hat{G}, (R_a^{\hat{\mathbb{G}}})_{a\in\Gamma})$ for every covering of $\mathfrak{M}$ that is based on $\mathbb{G}$ to a covering $(\hat{G}, (R_a^{\hat{\mathbb{G}}})_{a\in\Gamma}, (P_i^{\hat{\mathbb{G}}})_{i\in I})$ of $\mathfrak{M}$ that is based on $\hat{\mathbb{G}}$. In particular, the Cayley structure $(\hat{G}, (R_\alpha^{\hat{\mathbb{G}}})_{\alpha\in\tau}, (P_i^{\hat{\mathbb{G}}})_{i\in I})$ is $k$-rich because we boosted the multiplicity of every edge of $\mathfrak{M}$ by $k$. Thus, we obtain the following lemma that is crucial for the upgrading in the case of finite model theory.

**Lemma 3.2.24.** *For all $k, n \in \mathbb{N}$, every finite, connected* $\mathsf{CK}$ *structure admits a finite bisimilar covering by an $n$-acyclic and $k$-rich Cayley structure.*

# 4 Structure theory for acyclic Cayley structures

In this chapter, we develop a structure theory for acyclic and, first and foremost, finite $n$-acyclic Cayley structures. The main goal is to provide the necessary tools to play first-order Ehrenfeucht-Fraïssé games on the non-elementary class of (finite) Cayley structures. Apart from its application in Ehrenfeucht-Fraïssé games, the closer investigation of coset acyclicity and Cayley structures is interesting from a purley theoretical perspective. Except for a reliance on the definitions of Cayley structures, dual hypergraphs, coset acyclicity and richness from the previous chapter, the current chapter can be read independently of the others.

Section 4.1 highlights the special role that 2-acyclicity plays for our structure theory. We give an alternative, very intuitive characterisation of 2-acyclicity and show some basic properties of 2-acyclic structures that provide the backbone for most of the following notions.

Section 4.2 introduces coset paths, which generalise the usual graph-theoretic notion of a path in the same way that coset cycles generalise usual cycles. The main result of this section is the zipper lemma. It states that in sufficiently acyclic Cayley structures two short coset paths that start and end at the same vertices overlap like a zipper from both ends. This is a generalisation of the fact that in graphs of large girth two close vertices are connected by a unique short path. Furthermore, we prove several corollaries of the zipper lemma for short coset paths in $n$-acyclic structures that are crucial for the third and last section of this chapter.

Section 4.3 is concerned with *freeness*, a property of special Cayley structures. Essentially, freeness governs the placement of a pebble in the Ehrenfeucht-Fraïssé game on Cayley structures. Freeness ensures that player **II** can match long distances in one structure with long distances in the other. The main result about freeness, and also the main result of the chapter, is that sufficiently acyclic and sufficiently rich Cayley structures are sufficiently free. We show in this section that all these notions like $n$-acyclicity, richness and freeness, which we investigate here, make it possible to use locality techniques in a scenario that seems inherently locality averse. The $\alpha$-relations, for a set of agents $\alpha$, render classical Gaifman locality trivial and useless in $\mathsf{CK}$ structures. However, $n$-acyclicity implies that in order to control the distance between two worlds we only need to look at a certain reduced substructure. Within this substructure we can enforce long distances, which suffices to win the Ehrenfeucht-Fraïssé game.

As in the previous chapters, we fix a finite set of agents $\Gamma$, which labels the accessibility relations $(R_a)_{a\in\Gamma}$, and some finite index set $I$, which labels the atomic propositions $(P_i)_{i\in I}$. The set of all sets of agents with respect to $\Gamma$, i.e. $\mathcal{P}(\Gamma)$, is denoted by $\tau$. We regard S5 structures without accessibility relations that respond to coalitions of multiple agents as Kripke structures over the modal signature $\{(R_a)_{a\in\Gamma}, (P_i)_{i\in I}\}$, and all CK structure are Kripke structures over the modal signatur $\{(R_\alpha)_{\alpha\in\tau}, (P_i)_{i\in I}\}$. We denote Kripke structures by $\mathfrak{M}$ or $\mathfrak{N}$ and their sets of possible worlds by $W$ and $V$, respectively.

For a world $w$ and a set of agents $\alpha$, the set of all $\alpha$-successors of $w$ is denoted by $[w]_\alpha$. Because the relation $R_\alpha$ is always an equivalence relation in this chapter, the set $[w]_\alpha$ is the $\alpha$-equivalence class or $\alpha$-cluster of $w$. The set $\{[w]_\alpha : \alpha \subseteq \Gamma\}$ of all equivalence classes of $w$ is denoted by $[\![w]\!]$. Finite sets of worlds are denoted in bold, e.g. $\mathbf{w}$ or $\mathbf{z}$.

*Remark* 4.0.25. In a Cayley frame $\mathfrak{M}$ with worlds $w, w_1, \ldots, w_k$ and a set of agents $\alpha$:

$$[w]_\alpha \in \bigcap_{1\le i\le k} [\![w_i]\!] \quad \Rightarrow \quad [w]_{\alpha'} \in \bigcap_{1\le i\le k} [\![w_i]\!],$$

for any $\alpha' \supseteq \alpha$. Because if $\alpha \subseteq \alpha'$, then by definition of CK frames

$$\begin{aligned} [w]_\alpha \in [\![w_i]\!] \quad &\Leftrightarrow \quad (w, w_i) \in R_\alpha \\ &\Rightarrow \quad (w, w_i) \in R_{\alpha'} \\ &\Leftrightarrow \quad [w]_{\alpha'} \in [\![w_i]\!]. \end{aligned}$$

## 4.1 2-acyclicity

A Cayley frame is 2-acyclic if there are no coset cycles of length 2, i.e. if for all worlds $w, v$ and all sets of agents $\alpha, \beta$ with $(w, v) \in R_\alpha$ and $(v, w) \in R_\beta$: $[w]_{\alpha\cap\beta} \cap [v]_{\alpha\cap\beta} \neq \emptyset$. Compared to the possibly completely arbitrary overlap patterns of clusters in general CK structures, 2-acyclicity imposes a high degree of order in Cayley structures. It immediately implies that two clusters that belong to different agents $a$ and $b$ intersect in at most one world. Otherwise, there would be a 2-cycle $w, a, v, b, w$. This notion is generalised in the following lemma. Essentially, it states that an $\alpha$-cluster and a $\beta$-clusters with a non-empty intersection intersect in exactly one $(\alpha \cap \beta)$-cluster. In fact, this property characterises 2-acyclicity.

**Lemma 4.1.1.** *A Cayley frame $\mathfrak{M}$ is 2-acyclic if and only if for all $w \in W, \alpha, \beta \in \tau$*

$$[w]_\alpha \cap [w]_\beta = [w]_{\alpha\cap\beta}.$$

*Proof.* "$\Leftarrow$": If there is a 2-cycle $w, \alpha, v, \beta, w$, then $v \in [w]_\alpha \cap [w]_\beta$ and $[w]_{\alpha\cap\beta} \cap [v]_{\alpha\cap\beta} = \emptyset$. In particular, this means $v \notin [w]_{\alpha\cap\beta}$, which implies $[w]_\alpha \cap [w]_\beta \neq [w]_{\alpha\cap\beta}$.

"$\Rightarrow$": Assume there are $w \in W$, $\alpha, \beta \in \tau$ such that $[w]_\alpha \cap [w]_\beta \neq [w]_{\alpha\cap\beta}$. Since by definition always $[w]_{\alpha\cap\beta} \subseteq [w]_\alpha \cap [w]_\beta$, there must be some $v \in [w]_\alpha \cap [w]_\beta \setminus [w]_{\alpha\cap\beta}$. In particular, $v \notin [w]_{\alpha\cap\beta}$ implies $[w]_{\alpha\cap\beta} \cap [v]_{\alpha\cap\beta} = \emptyset$. Hence $w, \alpha, v, \beta, w$ forms a 2-cycle. ☐

The characterisation of 2-acyclicity in Lemma 4.1.1 implies that the overlap patterns of 2-acyclic Cayley structures are already far form arbitrary. As mentioned above, 2-acyclicity provides the backbone of our locality techniques. Lemma 4.1.2 shows that in

2-acyclic structures two worlds $w, v$ are always connected by some unique minimal set of agents $\alpha$, i.e. $[w]_\beta = [v]_\beta$ if and only if $\beta \supseteq \alpha$.

**Lemma 4.1.2.** *In a 2-acyclic Cayley frame $\mathfrak{M}$ with worlds $w, w_1, \ldots, w_k$ and sets of agents $\alpha_1, \ldots, \alpha_k \in \tau$:*

1. *For $\beta := \bigcap_{1 \leq i \leq k} \alpha_i$:*

$$w \in \bigcap_{1\leq i\leq k} [w_i]_{\alpha_i} \qquad \Rightarrow \qquad \bigcap_{1\leq i\leq k} [w_i]_{\alpha_i} = [w]_\beta$$

2. *The set $\bigcap_{1\leq i\leq k} \llbracket w_i \rrbracket$ has a least element in the sense that there is an $\alpha_0 \in \tau$ such that $[w_1]_{\alpha_0} \in \bigcap_{1\leq i\leq k} \llbracket w_i \rrbracket$ and, for any $\alpha \in \tau$:*

$$[w_i]_\alpha \in \bigcap_{1\leq i\leq k} \llbracket w_i \rrbracket \qquad \Leftrightarrow \qquad \alpha_0 \subseteq \alpha'$$

*Proof.* 1. Lemma 4.1.1 implies $\bigcap_{1\leq i\leq k} [w_i]_{\alpha_i} = \bigcap_{1\leq i\leq k} [w]_{\alpha_i} = [w]_\beta$.

2. 2-acyclicity implies that the collection

$$\{\alpha \in \tau \colon [w_1]_\alpha \in \bigcap_{1\leq i\leq k} \llbracket w_i \rrbracket\}$$

is closed under intersections: otherwise there would be $\alpha, \beta \in \tau$ with

$$[w_1]_\alpha, [w_1]_\beta \in \bigcap_{1\leq i\leq k} \llbracket w_i \rrbracket \qquad \text{and} \qquad [w_1]_{\alpha\cap\beta} \notin \bigcap_{1\leq i\leq k} \llbracket w_i \rrbracket.$$

This implies $[w_1]_{\alpha\cap\beta} \notin \llbracket w_j \rrbracket$, but $[w_1]_\alpha, [w_1]_\beta \in \llbracket w_j \rrbracket$, for some $1 \leq j \leq k$. Hence, there would be a 2-cycle $w_1, \alpha, w_j, \beta, w_1$.

□

Lemma 4.1.2 justifies the following definition.

**Definition 4.1.3.** In a 2-acyclic Cayley frame we denote the unique *minimal set of agents that connects the worlds in* $\mathbf{w}$ by $\mathrm{agt}(\mathbf{w}) \in \tau$.

Intuitively, $\mathrm{agt}(\mathbf{w})$ sets the scale for zooming-in on the minimal substructure that connects the worlds $\mathbf{w}$. Regarding distances between the worlds $\mathbf{w}$, we only need to control cycles with $\beta$-steps, for $\beta \subsetneq \mathrm{agt}(\mathbf{w})$, within the cluster $[w]_{\mathrm{agt}(\mathbf{w})}$, $w \in \mathbf{w}$. For dual hypergraphs of Cayley structures and intersections between hyperedges, Lemma 4.1.2 implies that every intersection can be described by the unique set of agents $\mathrm{agt}(\mathbf{w})$. This means, for every $w \in \mathbf{w}$:

$$[w]_\alpha \in \bigcap_{w\in\mathbf{w}} \llbracket w \rrbracket \qquad \Leftrightarrow \qquad \alpha \supseteq \mathrm{agt}(\mathbf{w})$$

Furthermore, in 2-acyclic frames, the set $\mathrm{agt}(\mathbf{w})$ can be controlled in a regular manner.

**Lemma 4.1.4.** *In a 2-acyclic Cayley frame for worlds $w, v$:*

1. *For every agent $a \notin \mathrm{agt}(w, v)$ and every $v' \in [v]_a \setminus \{v\}$:*

$$\mathrm{agt}(w, v') = \mathrm{agt}(w, v) \cup \{a\}$$

2. *For every agent $a \in \mathrm{agt}(w, v)$ there is at most one $v' \in [v]_a$ such that*

$$\mathrm{agt}(w, v') = \mathrm{agt}(w, v) \setminus \{a\}.$$

*Proof.* Set $\alpha := \mathrm{agt}(w, v)$.

1. Let $a \in \Gamma \setminus \mathrm{agt}(w, v)$, $v' \in [v]_a \setminus \{v\}$, and set $\beta := \mathrm{agt}(w, v')$. The choice of $v'$ implies an $(\alpha \cup \{a\})$-path from $w$ to $v'$. Hence, $\beta \subseteq (\alpha \cup \{a\})$ because of 2-acyclicity.

   Assume $(\alpha \cup \{a\}) \not\subseteq \beta$. First, if $a \notin \beta$, then $\beta \subseteq \alpha$, which means there is an $\alpha$-path from $w$ to $v'$ that can be combined with the $\alpha$-path from $w$ to $v$ to an $\alpha$-path from $v$ to $v'$. Furthermore, $[v]_{\alpha \cap \{a\}} = [v]_\emptyset = \{v\}$ and $[v']_{\{a\} \cap \alpha} = [v']_\emptyset = \{v'\}$ since $a \notin \alpha$. Together with $v \neq v'$ this implies that $v, \alpha, v', a, v$ forms a 2-cycle. Thus, $a \in \beta$ since $\mathfrak{M}$ is 2-acyclic.
   Second, assume there is some agent $b \in \alpha$ with $b \notin \beta$; in particular $\beta \subsetneq \alpha \cup \{a\}$. Furthermore, $a \in \beta$ implies

$$[v]_a = [v']_a \quad \Rightarrow \quad [v]_\beta = [v']_\beta = [w]_\beta.$$

   However, if $\beta \subsetneq \alpha \cup \{a\}$, then a $\beta$-path from $w$ to $v$ contradicts the minimality property of $\alpha$.

2. Let $a \in \alpha$ and assume there are two different worlds $v', v'' \in [v]_a \setminus \{v\}$ such that $\beta := \mathrm{agt}(w, v') = \mathrm{agt}(w, v'') = \alpha \setminus \{a\}$. This implies a $\beta$-path from $v'$ to $v''$ and an $a$-path from $v'$ to $v''$. Additionally,

$$[v']_{\beta \cap \{a\}} \cap [v'']_{\{a\} \cap \beta} = [v']_\emptyset \cap [v'']_\emptyset = \{v'\} \cap \{v''\} = \emptyset$$

   since $a \notin \beta$ and $v' \neq v''$. Thus, $v', \beta, v'', a, v'$ forms a 2-cycle.

□

In order to prove the freeness theorem in Section 4.3, we need to be able to manipulate the set $\mathrm{agt}(w, v)$. Assuming sufficient richness, Lemma 4.1.4 makes it possible to find for all $w, v$ and all $\alpha \supseteq \mathrm{agt}(w, v)$ a world $v' \sim v$ with $\mathrm{agt}(w, v') = \alpha$ in 2-acyclic structures. Lemma 4.1.5 gives us some additional useful insight into the structure of 2-acyclic and 2-rich Cayley structures.

**Lemma 4.1.5.** *Let $\mathfrak{M}$ be a 2-acyclic and 2-rich Cayley structure. Then, for all worlds $w$, for all $\alpha, \beta \in \tau$,*

$$\beta \subseteq \alpha \quad \Leftrightarrow \quad [w]_\beta \subseteq [w]_\alpha.$$

*Proof.* The direction from left to right is true in general Cayley structures because every $\beta$-path is an $\alpha$-path if $\beta \subseteq \alpha$.

For the converse direction, let $a \in \beta$, and assume $a \notin \alpha$. Since $\mathfrak{M}$ is 2-rich, there exists a world $w' \in [w]_a$ that is different from $w$. Additionally, $w' \in [w]_a \subseteq [w]_\beta \subseteq [w]_\alpha$ implies an $\alpha$-path from $w$ to $w'$. However, this means that $w, \{a\}, w', \alpha, w$ is a coset path of length 2 since

$$[w]_{\{a\}\cap\alpha} \cap [w']_{\alpha\cap\{a\}} = [w]_\emptyset \cap [w']_\emptyset = \{w\} \cap \{w'\} = \emptyset,$$

which contradicts the assumption of 2-acyclicity. □

We finish this section with Lemma 4.1.6. It provides a helpful tool in dealing with coset cycles.

**Lemma 4.1.6.** *If $\mathfrak{M}$ is a 2-acyclic Cayley frame and $(w_i, \alpha_i)_{i\in\mathbb{Z}_m}$ a cycle, then for all $i \in \mathbb{Z}_m$*

$$[w_i]_{\alpha_{i-1}\cap\alpha_i} \cap [w_{i+1}]_{\alpha_i\cap\alpha_{i+1}} = [w_{i-1}]_{\alpha_{i-1}} \cap [w_i]_{\alpha_i} \cap [w_{i+1}]_{\alpha_{i+1}}.$$

*Proof.* 2-acyclicity and $[w_i]_{\alpha_i} = [w_{i+1}]_{\alpha_i}$, for all $i \in \mathbb{Z}_m$, imply

$$\begin{aligned} &[w_i]_{\alpha_{i-1}\cap\alpha_i} \cap [w_{i+1}]_{\alpha_i\cap\alpha_{i+1}} \\ =&[w_i]_{\alpha_{i-1}} \cap [w_i]_{\alpha_i} \cap [w_{i+1}]_{\alpha_i} \cap [w_{i+1}]_{\alpha_{i+1}} \\ =&[w_{i-1}]_{\alpha_{i-1}} \cap [w_i]_{\alpha_i} \cap [w_i]_{\alpha_i} \cap [w_{i+1}]_{\alpha_{i+1}} \\ =&[w_{i-1}]_{\alpha_{i-1}} \cap [w_i]_{\alpha_i} \cap [w_{i+1}]_{\alpha_{i+1}}. \end{aligned}$$

□

## 4.2 Coset paths and the zipper lemma

Coset cycles generalise the usual graph-theoretic notion of a cycle for Cayley structures. Instead of taking one edge, or one generator, at a time, in coset cycles a sequence of edges, or an arbitrary group element, counts as one step in the cycle, as long as the associated cosets overlap only in a certain way (cf. Definition 3.2.7). *Coset paths* generalise the usual notion of a path in the same way as coset cycles generalise the usual cycles. These coset paths and their behaviour in $n$-acyclic Cayley structures are the subject of this section.

Many of the various definitions and notions that we will introduce from now on only make sense in 2-acyclic Cayley frames. This is the case because they are based on the notion of the unique minimal connecting set of agents $\text{agt}(\mathbf{w})$ defined in the previous section. Therefore, and because every Cayley structure has a 2-acyclic covering, we assume that every Cayley frame is 2-acyclic, for the remainder of this chapter.

**Definition 4.2.1** (Coset path)**.** Let $\mathfrak{M}$ be a Cayley frame. A *coset path of length* $\ell \geq 1$ is a labelled path $w_1, \alpha_1, w_2, \alpha_2, \ldots, \alpha_\ell, w_{\ell+1}$ such that, for $1 \leq i \leq \ell$,

$$[w_i]_{\alpha_{i-1} \cap \alpha_i} \cap [w_{i+1}]_{\alpha_i \cap \alpha_{i+1}} = \emptyset,$$

with $\alpha_0 = \alpha_{\ell+1} = \emptyset$. A coset path $w_1, \alpha_1, \ldots, \alpha_\ell, w_{\ell+1}$ of length $\ell \geq 2$ is *non-trivial* if, for $\alpha = \mathrm{agt}(w_1, w_{\ell+1})$, for all $1 \leq i \leq \ell$,

$$[w_1]_\alpha \not\subseteq [w_i]_{\alpha_i}.$$

A coset path $w_1, \alpha_1, \ldots, \alpha_\ell, w_{\ell+1}$ of length $\ell \geq 2$ is an *inner* path if, for $\alpha = \mathrm{agt}(w_1, w_{\ell+1})$, for all $1 \leq i \leq \ell$,

$$[w_i]_{\alpha_i} \subsetneq [w_1]_\alpha.$$

A non-trivial coset path from $w$ to $v \neq w$ is *minimal* if there is no shorter non-trivial coset path from $w$ to $v$.

*Remark* 4.2.2. Non-trivial and inner coset paths are only well-defined in 2-acyclic frames.

*Observation* 4.2.3. Inner coset paths are non-trivial.

Every Cayley frame is one clique with respect to $R_\Gamma$, the accessibility relation induced by the set of all agents $\Gamma$. Therefore, a world $w$ has at most distance 1 from all other worlds. This makes the *usual* notion of distance trivial and renders locality techniques seemingly useless.

However, we find a remedy in 2-acyclicity and its implications. Using 2-acyclicity and the set $\mathrm{agt}(w, v)$, for worlds $w, v$, we defined non-trivial coset paths. Intuitively, these are the coset paths from $w$ to $v$ that remain if one forbids to use all edges that connect $w$ and $v$ in one step, i.e. the trivial connections between the two worlds. In particular, this forbids $\Gamma$-edges. Thus, non-trivial coset paths lead us to a non-trivial notion of distance in 2-acyclic Cayley frames that is of crucial importance.

**Definition 4.2.4** (Distance in Cayley frames)**.** Let $\mathfrak{M}$ be a Cayley frame. The *distance* $d(w, v)$ between two worlds $w \neq v$ is defined as the length of a minimal non-trivial coset path from $w$ to $v$.

*Remark* 4.2.5. Definition 4.2.4 does not allow for $d(w, v) = 1$. This might seem peculiar compared to other distance measures. However, in structures where worlds are always connected by some edge, the measure $d(w, v)$ is precisely designed to capture the length of the *non-trivial* connections between two worlds, and their length is always at least 2.

**Definition 4.2.6.** Let $\mathfrak{M}$ be a Cayley frame that is $2n$-acyclic. We call a coset path *short* if its length is $\leq n$.

Often, we do not make it explicit to what degree a Cayley frame is acyclic. Instead, we write that a Cayley frame $\mathfrak{M}$ is *sufficiently* acyclic, i.e. there is some $n \in \mathbb{N}$ such that $\mathfrak{M}$ is $n$-acyclic and all the arguments go through.

Above, we mentioned briefly that the main obstacle in playing Ehrenfeucht-Fraïssé games on Cayley structures is to match long distances in one structure with long distances in the other. We will show that this task actually boils down to avoiding inner coset paths of a certain length. To be more precise: Assume we play on two $n$-acyclic Cayley structures $\mathfrak{M}$ and $\mathfrak{N}$ with worlds $w, w' \in \mathfrak{M}$ and $v, v' \in \mathfrak{N}$ such that

- $\mathfrak{M}, w \sim \mathfrak{N}, v$,
- $\mathfrak{M}, w' \sim \mathfrak{N}, v'$, and
- $\mathrm{agt}(w, w') = \mathrm{agt}(v, v')$.

If $d(v, v') > m$, i.e. no non-trivial coset paths of length $\leq m$ from $v$ to $v'$ in $\mathfrak{N}$, for some threshold $m \in \mathbb{N}$, but $d(w, w') \leq m$, we need to find some world $w^* \in \mathfrak{M}$ with $\mathfrak{M}, w^* \sim \mathfrak{M}, w'$ and $\mathrm{agt}(w, w^*) = \mathrm{agt}(w, w')$ such that $d(w, w^*) > m$. We refer to this as matching the long distance between $v$ and $v'$ with a long distance between $w$ and $w^*$.

In the remainder of this section, we argue that for sufficiently acyclic frames it suffices to look only at inner coset paths to avoid short distances (cf. Lemma 4.2.14), and we introduce the crucial tool to find a suitable copy $w^*$ of $w'$ that has a sufficient distance from $w$. We begin with an analogue of Lemma 4.1.6 for coset paths.

**Lemma 4.2.7.** *If $\mathfrak{M}$ is a Cayley frame and $w_1, \alpha_1, w_2, \ldots, w_\ell, \alpha_\ell, w_{\ell+1}$ a path, then, for all $2 \leq i \leq \ell$,*

$$[w_i]_{\alpha_{i-1} \cap \alpha_i} \cap [w_{i+1}]_{\alpha_i \cap \alpha_{i+1}} = [w_{i-1}]_{\alpha_{i-1}} \cap [w_i]_{\alpha_i} \cap [w_{i+1}]_{\alpha_{i+1}},$$

*with $\alpha_{\ell+1} = \emptyset$.*

*Proof.* Exactly as Lemma 4.1.6. □

In order to understand the importance of short inner coset paths, we need to understand how short coset paths (the ones we would like to avoid) behave in sufficiently acyclic Cayley structures. As a jumping-off point, we look at short paths in the usual graph-theoretic sense.

If a graph is acyclic, then two connected vertices are always connected by a unique path. If a graph is $2k+1$-acyclic, then every $k$-neighbourhood is acyclic and every two vertices at distance $\leq k$ are connected by a unique minimal path of length up to $k$. These notions can be generalised to Cayley frames and coset acyclicity. If $\mathfrak{M}$ is an acyclic Cayley frame, then the coset path $w_1, \{a_1\}, w_2, \ldots, w_\ell, \{a_\ell\}, w_{\ell+1}$ with agents $a_i \in \Gamma$, for $1 \leq i \leq \ell$, is unique and every coset path from $w_1$ to $w_{\ell+1}$ is a contraction of this path. If $\mathfrak{M}$ is $2n$-acyclic, short coset paths between two vertices are not necessarily unique but they always overlap, in some sense. This is the content of the zipper lemma (Lemma 4.2.9).

Essentially, the zipper lemma states that in a sufficiently acyclic Cayley frame two short coset paths that both start at the same world $w$ and end at the same world $v$ overlap at *both* ends. Thus, multiple applications of the zipper lemma imply that two short coset paths of this kind behave like a zipper that can be closed from both ends. Furthermore, and maybe most importantly, the zipper lemma implies that, for all pairs of worlds $(w, v)$, there is a unique minimal set of agents $\alpha_0$ such that $\alpha_0 \subseteq \alpha_1$, for all *short* coset paths $w, \alpha_1, \dots, \alpha_\ell, v$. Since $w, \mathrm{agt}(w, v), v$ is always a short coset path, $\alpha_0 \subseteq \mathrm{agt}(w, v)$, and it might even be the case that $\alpha_0 = \mathrm{agt}(w, v)$. However, if there is a short inner coset path from $w$ to $v$, then $\alpha_0$ is a proper subset of $\mathrm{agt}(w, v)$. This set $\alpha_0$ can be interpreted as the direction one *has to* take if one wants to move from $w$ to $v$ on a short inner coset path. We will make all these statements more precise down below.

In order to prove the zipper lemma, we begin with considering short coset paths

$$w_1, \alpha_1, w_2, \dots, w_\ell, \alpha_\ell, w_1$$

that start and end at the same vertex $w_1$. Such a path differs from a coset cycle regarding the overlaps at the ends. If $w_1, \alpha_1, w_2, \dots, w_\ell, \alpha_\ell, w_1$ is just a *path*, we can by definition only assume

$$[w_1]_{\emptyset \cap \alpha_1} \cap [w_2]_{\alpha_1 \cap \alpha_2} = \emptyset \quad \text{and} \quad [w_\ell]_{\alpha_{\ell-1} \cap \alpha_\ell} \cap [w_1]_{\alpha_\ell \cap \emptyset} = \emptyset,$$

i.e. $w_1 \notin [w_2]_{\alpha_1 \cap \alpha_2}$ and $w_1 \notin [w_\ell]_{\alpha_{\ell-1} \cap \alpha_\ell}$, but not that it is a complete coset *cycle*, i.e. that also

$$[w_1]_{\alpha_\ell \cap \alpha_1} \cap [w_2]_{\alpha_1 \cap \alpha_2} = \emptyset \quad \text{and} \quad [w_\ell]_{\alpha_{\ell-1} \cap \alpha_\ell} \cap [w_1]_{\alpha_\ell \cap \alpha_1} = \emptyset.$$

Hence, these cyclic coset paths are not directly ruled out by acyclicity but by the following lemma.

**Lemma 4.2.8.** *Let $w$ be a world in a Cayley frame $\mathfrak{M}$. If $\mathfrak{M}$ is $n$-acyclic, then there is no coset path of length up to $n$ that starts at $w$ and ends at $w$.*

*Proof.* The claim is shown by induction on the length $\ell$ of the coset path, for $1 \leq \ell \leq n$.

For $\ell = 1$, Definition 4.2.1 rules out coset loops $w, \alpha, w$ because it implies

$$\emptyset = [w]_{\emptyset \cap \alpha} \cap [w]_{\alpha \cap \emptyset} = \{w\}.$$

For $\ell = 2$, coset paths $w_1, \alpha_1, w_2, \alpha_2, w_1$ with $w_1 \notin [w_2]_{\alpha_1 \cap \alpha_2}$ are ruled out by 2-acyclicity because it implies

$$[w_1]_{\alpha_1 \cap \alpha_2} = [w_2]_{\alpha_1 \cap \alpha_2},$$

leading to the contradiction $w_1 \notin [w_1]_{\alpha_1 \cap \alpha_2}$.

For $2 < \ell \leq n$, assume there are no coset paths of length up to $\ell - 1$ from any world back to itself. Consider a coset path

$$w_1, \alpha_1, w_2, \dots, w_\ell, \alpha_\ell, w_{\ell+1}$$

of length $\ell$ with $w_1 = w_{\ell+1}$. $n$-acyclicity of $\mathfrak{M}$ implies

$$[w_1]_{\alpha_\ell \cap \alpha_1} \cap [w_2]_{\alpha_1 \cap \alpha_2} \neq \emptyset \quad \text{or} \quad [w_\ell]_{\alpha_{\ell-1} \cap \alpha_\ell} \cap [w_1]_{\alpha \ell \cap \alpha_1} \neq \emptyset.$$

W.l.o.g. we assume there is some $v \in [w_1]_{\alpha_\ell \cap \alpha_1} \cap [w_2]_{\alpha_1 \cap \alpha_2}$. If $v \notin [w_\ell]_{\alpha_{\ell-1} \cap \alpha_\ell}$, then

$$v, \alpha_2, w_3, \alpha_3, w_4, \dots, w_\ell, \alpha_\ell, v$$

is a coset path of length $\ell - 1$ from $v$ to itself. Otherwise,

$$v, \alpha_2, w_3, \alpha_3, w_4, \ldots, w_{\ell-1}, \alpha_{\ell-1}, v$$

is a coset path of length $\ell - 2$ from $v$ to itself. In both cases, such a coset path cannot exist according to the induction hypothesis. □

The proof of Lemma 4.2.8 shows that a short cyclic path cannot exist in a sufficiently acyclic frame because it would collapse onto itself. With this lemma at our disposal, it is easy to show that two short coset paths that both start at some world $w$ and both end at a world $v$ would collapse in a similar fashion.

**Lemma 4.2.9** (Zipper lemma)**.** *Let $\mathfrak{M}$ be a $2n$-acyclic Cayley frame, $w, v \in W$, and*

$$w, \alpha_1, u_2, \alpha_2, u_3, \ldots, u_\ell, \alpha_\ell, v \quad \textit{and} \quad w, \beta_1, r_2, \beta_2, r_3, \ldots, r_k, \beta_k, v$$

*be two coset paths from $w$ to $v$ of length up to $n$. Then*

1. $[w]_{\beta_1 \cap \alpha_1} \cap [u_2]_{\alpha_1 \cap \alpha_2} \neq \emptyset$ *or* $[w]_{\alpha_1 \cap \beta_1} \cap [r_2]_{\beta_1 \cap \beta_2} \neq \emptyset$*;*
2. $[v]_{\beta_k \cap \alpha_\ell} \cap [u_\ell]_{\alpha_\ell \cap \alpha_{\ell-1}} \neq \emptyset$ *or* $[v]_{\alpha_\ell \cap \beta_k} \cap [r_k]_{\beta_k \cap \beta_{k-1}} \neq \emptyset$*.*

*Proof.* Since $\mathfrak{M}$ is $2n$-acyclic we know that

- $[w]_{\beta_1 \cap \alpha_1} \cap [u_2]_{\alpha_1 \cap \alpha_2} \neq \emptyset$, or
- $[w]_{\alpha_1 \cap \beta_1} \cap [r_2]_{\beta_1 \cap \beta_2} \neq \emptyset$, or
- $[v]_{\beta_k \cap \alpha_\ell} \cap [u_\ell]_{\alpha_\ell \cap \alpha_{\ell-1}} \neq \emptyset$, or
- $[v]_{\alpha_\ell \cap \beta_k} \cap [r_k]_{\beta_k \cap \beta_{k-1}} \neq \emptyset$

occurs because otherwise the two coset paths would form a coset cycle of length up to $2n$. W.l.o.g. assume $[w]_{\beta_1 \cap \alpha_1} \cap [u_2]_{\alpha_1 \cap \alpha_2} \neq \emptyset$, and

$$[v]_{\beta_k \cap \alpha_\ell} \cap [u_\ell]_{\alpha_\ell \cap \alpha_{\ell-1}} = \emptyset \quad \text{and} \quad [v]_{\alpha_\ell \cap \beta_k} \cap [r_k]_{\beta_k \cap \beta_{k-1}} = \emptyset.$$

This implies a cyclic coset path of length up to $2n$, contradicting Lemma 4.2.8. □

The zipper lemma states that the two coset paths overlap both at the start and at the end, i.e they behave like a zipper that is closed from both ends simultaneously. In some sense, they can be considered as two recombinations of the constituents of a common core path. The zipper lemma has several interesting and crucially important consequences.

**Corollary 4.2.10.** *Let $\mathfrak{M}$ be a $2n$-acyclic Cayley frame, $w, v \in W$. If there are two short coset paths*

$$w, \alpha_1, u_2, \alpha_2, u_3, \ldots, u_\ell, \alpha_\ell, v \quad \textit{and} \quad w, \beta_1, r_2, \beta_2, r_3, \ldots, r_k, \beta_k, v$$

*from $w$ to $v$ with $\ell, k \leq n$, then there is a short coset paths from $w$ to $v$ that starts with an $(\alpha_1 \cap \beta_1)$-edge.*

*Proof.* W.l.o.g. we can assume that there is some $w_2 \in [w]_{\beta_1 \cap \alpha_1} \cap [u_2]_{\alpha_1 \cap \alpha_2}$ by Lemma 4.2.9. First, the choice of $w_2$ and the coset property of the original path imply

$$[w_2]_{\alpha_1 \cap \alpha_2} \cap [u_3]_{\alpha_2 \cap \alpha_3} = [u_2]_{\alpha_1 \cap \alpha_2} \cap [u_3]_{\alpha_2 \cap \alpha_3} = \emptyset.$$

Second,

$$w \notin [u_2]_{\alpha_1 \cap \alpha_2} = [w_2]_{\alpha_1 \cap \alpha_2} \supseteq [w_2]_{\alpha_1 \cap \alpha_2 \cap \beta_1}$$

implies

$$[w]_{\emptyset \cap (\alpha_1 \cap \beta_1)} \cap [w_2]_{(\alpha_1 \cap \beta_1) \cap \alpha_2} = \emptyset.$$

Thus, $w, (\alpha_1 \cap \beta_1), w_2, \alpha_2, u_3, \ldots, u_\ell, \alpha_\ell, v$ is a short coset path. □

Let $\mathfrak{M}$ be a 2-acyclic Cayley frame and $w, v \in W$. Based on Corollary 4.2.10 we define the unique minimal set of agents $\mathrm{short}(w, v) \in \tau$ such that every short coset path from $w$ to $v$ starts with an $\alpha$-edge, for $\alpha \supseteq \mathrm{short}(w, v)$. Formally:

**Definition 4.2.11.** A set of agents $\alpha$ is a *first edge set for* $(w, v)$ if there is a short coset path from $w$ to $v$ that starts with an $\alpha$-edge. The *minimal first edge set for* $(w, v)$ $\mathrm{short}(w, v)$ is the intersection of all first edge sets:

$$\mathrm{short}(w, v) := \bigcap \{\alpha \in \tau : \ \alpha \text{ is a first edge set for } (w, v)\}$$

The unique set $\mathrm{short}(w, v)$ is well-defined because the intersection of two first edge sets is again a first edge set by Corollary 4.2.10. In general, $\mathrm{short}(w, v) \neq \mathrm{short}(v, w)$ but

$$\mathrm{short}(w, v), \mathrm{short}(v, w) \subseteq \mathrm{agt}(w, v) = \mathrm{agt}(v, w)$$

because $\mathrm{agt}(w, v)$ is a first edge set for $(w, v)$ and $(v, w)$. One of the main results of Section 4.3 states that for all $m \in \mathbb{N}$ and for all worlds $w, v$ in sufficiently acyclic and sufficiently rich Cayley structures there is some world $v^* \sim v$ with $\mathrm{agt}(w, v) = \mathrm{agt}(w, v^*)$ such that $d(w, v^*) > m$ (cf. Lemma 4.3.18). The set $\mathrm{short}(w, v)$ is one of the crucial ingredients in finding such a world $v^*$.

Furthermore, the zipper lemma also implies that all short coset paths of length $\geq 2$ can be assumed to be inner paths. In particular, this applies to short non-trivial paths.

**Corollary 4.2.12.** *Let $\mathfrak{M}$ be a $2n$-acyclic Cayley structure, $2 \leq \ell \leq n$,*

$$w_1, \alpha_1, w_2, \alpha_2, w_3, \ldots, w_\ell, \alpha_\ell, w_{\ell+1}$$

*be a coset path and $\alpha = \mathrm{agt}(w_1, w_{\ell+1})$. Then $\alpha_i \not\supseteq \alpha$, for $1 \leq i \leq \ell$, and there are $w_i' \in [w_i]_{\alpha_{i-1} \cap \alpha_i}$, for $1 < i \leq \ell$, such that*

$$w_1, (\alpha_1 \cap \alpha), w_2', (\alpha_2 \cap \alpha), w_3', \ldots, w_\ell', (\alpha_\ell \cap \alpha), w_{\ell+1}$$

*is an inner coset path.*

*Proof.* First, $\alpha_1 \supseteq \alpha$ cannot be the case: if $\ell = 2$, then $w_1, \alpha_1, w_2, \alpha_2, w_3$ would not be a coset path since $w_3 \in [w_2]_{\alpha_1 \cap \alpha_2}$, and $\ell > 2$ would imply a short cyclic coset path from $w_{\ell+1}$ to itself, contradicting Lemma 4.2.8. Hence, in both cases

$$\alpha_1 \not\supseteq \alpha \quad \Rightarrow \quad \alpha_1 \cap \alpha \subsetneq \alpha \quad \Rightarrow \quad [w_1]_{\alpha_1 \cap \alpha} \subsetneq [w_1]_\alpha.$$

Second, analogously to the proof of Corollary 4.2.10 one can show that there is some $w_2' \in [w_2]_{\alpha_1 \cap \alpha_2}$ such that

$$w_1, (\alpha_1 \cap \alpha), w_2', \alpha_2, w_3, \ldots, w_\ell, \alpha_\ell, w_{\ell+1}$$

is a coset path because $w_1, \alpha, w_{\ell+1}$ is also a short coset path from $w_1$ to $w_{\ell+1}$. Applying the same argument iteratively to the paths $w_i', \alpha_i, w_{i+1}, \ldots, w_\ell, \alpha_\ell, w_{\ell+1}$ and $w_i', \alpha, w_{\ell+1}$, for $2 \leq i \leq \ell$, shows $\alpha_i \not\supseteq \alpha$ and results in the desired worlds. □

Corollary 4.2.12 illustrates why it suffices to regard only the substructure induced by $[w]_{\text{agt}(w,v)}$ if one wants to control the short non-trivial coset paths between two worlds. Its converse direction states: if a coset path has a link that is disjoint from $[w]_{\text{agt}(w,v)}$, then it cannot be short.

**Corollary 4.2.13.** *Let $\mathfrak{M}$ be a $2n$-acyclic Cayley frame. If $w_1, \alpha_1, \ldots, \alpha_\ell, w_{\ell+1}$ is a coset path with*

$$[w_1]_{\text{agt}(w_1, w_{\ell+1})} \cap [w_i]_{\alpha_{i-1} \cap \alpha_i} = \emptyset,$$

*for some $2 \leq i \leq \ell$, then $\ell > n$.*

In other words, in sufficiently acyclic Cayley frames the distance between two worlds must be long if there are no short inner coset paths.

**Lemma 4.2.14.** *Let $m \in \mathbb{N}$, $\mathfrak{M}$ be a sufficiently acyclic Cayley frame and $w, v$ two worlds. If there are no inner coset paths from $w$ to $v$ of length $\leq m$, then*

$$d(w, v) > m.$$

*Proof.* Let $\ell \leq m$, and assume there is a non-trivial coset path $w_1, \alpha_1, \ldots, \alpha_\ell, w_{\ell+1}$ of length $\ell$ from $w = w_1$ to $v = w_{\ell+1}$. First, any non-trivial coset path has at least length 2. Second, we can assume that the path is an inner coset path by Lemma 4.2.12 since $\mathfrak{M}$ is sufficiently acyclic. This contradicts our assumption. Thus, $d(w, v) > m$. □

To conclude this section: in order to win the Ehrenfeucht-Fraïssé game on Cayley structures, we need to be able to avoid short non-trivial coset paths, i.e. for some threshold $m \in \mathbb{N}$, and for worlds $w, v$, we need to be able to find a world $v^* \sim v$ with $\text{agt}(w, v^*) = \text{agt}(w, v)$ such that $d(w, v^*) > m$. First, Lemma 4.2.14 states that in sufficiently acyclic structures the only short non-trivial paths that need to be avoided are in fact *inner* paths. Second, the unique set of agents $\text{short}(v, w)$, which is implied by the zipper lemma, is the crucial tool for finding a bisimilar world $v^*$, as desired. How to apply this set is one of the main topics of the following section.

## 4.3 Freeness

Assume we play the $i$-th round of an Ehrenfeucht-Fraïssé game on Cayley structures $\mathfrak{M}$ and $\mathfrak{N}$ with worlds $w_1, \ldots, w_{i-1} \in W$ and $v_1, \ldots, v_{i-1} \in V$ already pebbled. The main question is this: if player **I** chooses the world $w_i \in W$ in his next move, how does player **II** respond to this with a world $v_i \in v$ such that she not only survives the current round but wins the whole play in the end? As usual, **II** has to maintain a partial bijection between the pebbled worlds and match short distances between worlds exactly, and match long distances with long distances. Since we play on Cayley structures, she has to maintain these distances on multiple scales.

We call the special property of Cayley structures that allows for making a suitable choice: *freeness*, or to be more precise $(m, k)$-freeness, for $m, k \in \mathbb{N}$. To give some first intuition, freeness roughly means that for a world $v$ and a set of $k$ worlds $\mathbf{z}$ there is some world $v^* \sim v$ such that $d(v^*, z) > m$, for all $z \in \mathbf{z}$. In the scenario of the Ehrenfeucht-Fraïssé game, the worlds in $\mathbf{z}$ are not only the worlds pebbled so far, but a certain small substructure of $\mathfrak{N}$ spanned by the pebbled worlds. In Cayley structures that are sufficiently free, this substructure can be extended properly from round to round, resulting in a win for player **II**. The main result of this section (Theorem 4.3.19, also referred to as the freeness theorem) states that a Cayley structure is $(m, k)$-free if it is sufficiently acyclic and sufficiently rich.

In the context of freeness, distance is actually defined by means of the dual hypergraph of a Cayley structure. We use the dual hypergraph as an auxiliary structure to describe a winning strategy for player **II** in Chapter 5. However, we will further investigate the connections between Cayley structures and their dual hypergraphs in this section and show that the notion of distance for hypergraphs is closely connected to the notion of distance for Cayley frames defined in 4.2.4.

Some notation before we present the formal definition of freeness: for $t, X, Y \subseteq A$ in a hypergraph $\mathcal{A} = (A, S)$, we denote as $d_t(X, Y)$ the distance between $X \setminus t$ and $Y \setminus t$ in the induced sub-hypergraph $\mathcal{A} \setminus t := \mathcal{A} \restriction (A \setminus t)$ (distance in hypergraphs was defined in Section 3.2.3). For a set of worlds $\mathbf{z} \subseteq W$, we write $[\![\mathbf{z}]\!]$ for the set $\{[\![z]\!] : z \in \mathbf{z}\}$ of associated hyperedges. A *pointed set (of worlds)* is a pair $(\mathbf{z}, z_0)$, where $\mathbf{z}$ is a set of worlds and $z_0 \in \mathbf{z}$.

**Definition 4.3.1** (Freeness)**.** Let $\mathfrak{M}$ be a 2-acyclic Cayley structure, and $m, k \in \mathbb{N}$.

- Let $v \in W$ and $(\mathbf{z}, z_0)$ be a pointed set of worlds. We say that $(\mathbf{z}, z_0)$ and $v$ are *m-free* if $d_t(\bigcup[\![\mathbf{z}]\!], [\![v]\!]) > m$, where $t = [\![v]\!] \cap [\![z_0]\!]$. In short: $(\mathbf{z}, z_0) \perp_m v$.
- We say that $\mathfrak{M}$ is $(m, k)$*-free* if for all $v \in W$, all pointed sets $(\mathbf{z}, z_0)$ with $|\mathbf{z}| \leq k$, and all sets of agents $\gamma \supseteq \mathrm{agt}(v, z_0)$, there is some $v^* \sim v$ such that
  - $\mathrm{agt}(v^*, z_0) = \gamma$, and
  - $(\mathbf{z}, z_0) \perp_m v^*$.

As mentioned above, the finite set of worlds $\mathbf{z}$ represent a set in $\mathfrak{M}$ that is spanned by the worlds that have already been pebbled in the play. The world $v$ is a possible next move for **II** that is not entirely suitable because it is too close to $\mathbf{z}$, in the sense that $v$ and $(\mathbf{z}, z_0)$ are not $m$-free; $z_0$ is the world in $\mathbf{z}$ that is, in some sense, closest to $v$.

Freeness, in the sense presented here, originated in [26] and was used to define a winning strategy for an Ehrenfeucht-Fraïssé game played on $n$-acyclic hypergraphs, in order to show a characterisation theorem for guarded logic in the sense of finite model theory. We adapted the idea for our purposes to use it for Cayley structures. Essentially, freeness is applied in the same way as in [26], but the proof that sufficiently acyclic and sufficiently rich Cayley structures are $(m, k)$-free is entirely different. Definition 4.3.1 speaks both about worlds in the Cayley structure and distances in the Gaifman graph of the dual hypergraph. Our proof of the freeness theorem finds the desired world $v^*$, which is far enough away from $\mathbf{z}$, through constructions on the original Cayley structure. Therefore, investigating the connections between coset paths in a Cayley structure and chordless paths in its dual hypergraph is crucial to proving the freeness theorem.

A world $v$ and a pointed set $(\mathbf{z}, z_0)$ are $m$-free if the distance between $[\![v]\!] \setminus t$ and $(\bigcup[\![\mathbf{z}]\!]) \setminus t$ in $d(\mathfrak{M}) \setminus t$ is strictly larger than $m$, for $t = [\![v]\!] \cap [\![z_0]\!]$. In other words, a minimal, and in particular chordless, path between $[\![v]\!] \setminus t$ and $(\bigcup[\![\mathbf{z}]\!]) \setminus t$ in $d(\mathfrak{M}) \setminus t$ must be strictly longer than $m$.

We would like to emphasize the role of the set $t$: we are only interested in those paths between $[\![v]\!]$ and $\bigcup[\![\mathbf{z}]\!]$ that *avoid* $t$. Essentially, the paths that go through $t$ are all the trivial paths between $[\![v]\!]$ and $\bigcup[\![\mathbf{z}]\!]$. The goal is to find some $v^* \sim v$ such that all the non-trivial paths are long. The set $t$ is a set of equivalence classes in $\mathfrak{M}$. Because $\mathfrak{M}$ is 2-acyclic, $t$ contains exactly those classes that contain both $v$ and $z_0$: some class $[v]_\beta$ is an element of $t$ if and only if $(v, z_0) \in R_\beta^{\mathfrak{M}}$, i.e.

$$t = \{[v]_\beta : \beta \supseteq \mathrm{agt}(z_0, v)\} = \{[v]_\beta : [v]_\beta \supseteq [v]_{\mathrm{agt}(z_0,v)}\}.$$

The classes in $t$ represent the coset paths of length 1 from $z_0$ to $v$. These are the trivial paths, the ones we cannot and do not need to avoid. But in order to win the Ehrenfeucht-Fraïssé game we need to be able to ensure that all other paths are long. Hence, we cut out $t$ and take a look at the distance between $[\![v]\!] \setminus t$ and $(\bigcup[\![\mathbf{z}]\!]) \setminus t$. If it is too short, we need a different $v^* \sim v$ that is farther away.

In order to find a suitable $v^*$, we will deal with every world $z \in \mathbf{z}$ one after the other. First, we find a copy $v_0$ of $v$ such that $d_t([\![v_0]\!], [\![z_0]\!]) > m$, then we find a copy $v_1$ such that $d_t([\![v_1]\!], [\![z]\!]) > m$, for another world $z \in \mathbf{z}$, while maintaining $d_t([\![v_1]\!], [\![z_0]\!]) > m$, and so forth. The last of these copies will be $v^*$. Take note of the fact that we always need to avoid the same set $t = [\![v]\!] \cap [\![z_0]\!]$, and not $[\![v]\!] \cap [\![z]\!]$ if we want to increase the distance between $[\![v]\!]$ and $[\![z]\!]$. This complicates things on a technical level. However, the requirement $d_t([\![v]\!], [\![z]\!]) > 1$ implies that we want $[\![v]\!] \cap [\![z]\!] \subseteq t$, which means that all the classes that directly connect $v$ and $z$ will be cut out too.

As mentioned above, we will work with the Cayley structure $\mathfrak{M}$ to find some suitable world $v^*$. Hence, we need to represent the paths in $d(\mathfrak{M}) \setminus t$ that we want to avoid as paths in $\mathfrak{M}$. As a first step, we want an alternative way to describe the set of equivalence classes $t$ (mainly for technical reasons). Coming from the definition of freeness, $t$ was defined in terms of $v$ and the distinguished world $z_0$ ($t = [\![v]\!] \cap [\![z_0]\!]$). Since we assume $\mathfrak{M}$

to be 2-acyclic, $t$ can also be described in terms of $v$ and the set of agents $\mathrm{agt}(v, z_0) = \gamma$, i.e. $t = \{[v]_\beta : \beta \supseteq \gamma\}$. The following definition gives us a mapping that defines such a set of classes based on a world and a set of agents in a general way. This mapping will be useful to state the upcoming lemmas.

**Definition 4.3.2.** For a 2-acyclic Cayley frame $\mathfrak{M}$ with a dual hypergraph $d(\mathfrak{M})$, we define the following mapping:

$$\rho^{\mathfrak{M}} : W \times \tau \to \mathcal{P}(d(W)), (v, \gamma) \mapsto \{[v]_\beta : \beta \supseteq \gamma\}$$

If it is clear from the context, we drop the superscript $\mathfrak{M}$ and just write $\rho$ instead.

The following lemma characterises the relationship of the sets $[\![v]\!] \cap [\![z_0]\!]$ and $[\![v]\!] \cap [\![z]\!]$ in $d(\mathfrak{M})$ in terms of $\mathrm{agt}(v, z_0)$ and $\mathrm{agt}(v, z)$. We can observe the usual duality in the transition from Cayley structures to their dual hypergraphs.

**Lemma 4.3.3.** *Let $\mathfrak{M}$ be a 2-acyclic Cayley frame, $v, z$ two worlds and $\gamma$ a set of agents, then*

$$[\![z]\!] \cap [\![v]\!] \subseteq \rho(v, \gamma) \quad \Leftrightarrow \quad \gamma \subseteq \mathrm{agt}(z, v).$$

*Proof.* Put $\alpha := \mathrm{agt}(z, v)$. From right to left: assume $\gamma \subseteq \alpha$. Together with 2-acyclicity this implies

$$[\![z]\!] \cap [\![v]\!] = \{[v]_\beta : \beta \supseteq \alpha\} \subseteq \{[v]_\beta : \beta \supseteq \gamma\} = \rho(v, \gamma).$$

From left to right: assume $[\![z]\!] \cap [\![v]\!] \subseteq \rho(v, \gamma)$. As before, $[\![z]\!] \cap [\![v]\!] = \{[v]_\beta : \beta \supseteq \alpha\}$ because of 2-acyclicity. Hence, for all $\beta \in \tau$

$$\begin{aligned} \beta \supseteq \alpha \quad &\Leftrightarrow \quad [v]_\beta \in [\![w]\!] \cap [\![v]\!] \\ &\Rightarrow \quad [v]_\beta \in \rho(v, \gamma) \\ &\Leftrightarrow \quad \beta \supseteq \gamma, \end{aligned}$$

which implies, in particular, $\gamma \subseteq \alpha$. □

Recall that for $v$ and $(\mathbf{z}, z_0)$ to be $m$-free, every minimal path $[w_1]_{\alpha_1}, \ldots, [w_\ell]_{\alpha_\ell}$ from $[\![v]\!]$ to $[\![z]\!]$, for all $z \in \mathbf{z}$, with $[w_i]_{\alpha_i} \notin t$, for all $1 \leq i \leq \ell$, needs to be strictly longer than $m$. These paths in $d(\mathfrak{M}) \setminus t$ correspond to the following coset paths in $\mathfrak{M}$.

**Definition 4.3.4.** Let $\mathfrak{M}$ be a Cayley frame, $w_1, w_{\ell+1}$ two worlds, $\gamma$ a set of agents and $t = \rho(w_{\ell+1}, \gamma)$. A coset path

$$w_1, \alpha, w_2, \alpha_2, \ldots, \alpha_\ell, w_{\ell+1}$$

*avoids* $t$ if, for all $1 \leq i \leq \ell$,

$$[w_{\ell+1}]_\gamma \not\subseteq [w_i]_{\alpha_i}.$$

Coset paths that avoid $t$ are a generalisation of non-trivial coset paths (cf. Definition 4.2.1). Every non-trivial coset path from $w$ to $v$ is a coset path that avoids $t$, for $t = \rho(v, \mathrm{agt}(w, v))$. Lemmas 4.3.5 and 4.3.6 prove a correspondence between these paths and chordless paths in $d(\mathfrak{M}) \setminus t$. The former states that the minimal paths in question in $d(\mathfrak{M}) \setminus t$ induce coset paths that avoid $t$. Its contraposition plays a key role in our argument: if there are no short coset paths from $w$ to $v$ that avoid $t$, then there cannot be short minimal paths from $[\![w]\!] \setminus t$ to $[\![v]\!] \setminus t$ in $d(\mathfrak{M}) \setminus t$.

**Lemma 4.3.5.** *Let $\mathfrak{M}$ be a 2-acyclic Cayley frame, $w_1 \neq w_{\ell+1}$ two worlds, $\gamma$ a set of agents and $t = \rho(w_{\ell+1}, \gamma)$. Then a minimal (in particular chordless) path of length $\ell \geq 0$*

$$[w_1]_{\alpha_1}, [\![w_2]\!], [w_2]_{\alpha_2}, [\![w_3]\!], \ldots, [\![w_\ell]\!], [w_\ell]_{\alpha_\ell}$$

*in $d(\mathfrak{M}) \setminus t$ from $[\![w_1]\!] \setminus t$ to $[\![w_{\ell+1}]\!] \setminus t$ induces a coset path*

$$w_1, \alpha_1, w_2, \ldots, w_\ell, \alpha_\ell, w_{\ell+1}$$

*of length $\ell + 1$ in $\mathfrak{M}$ that avoids $t$.*

*Proof.* We assumed that the path ends in $[\![w_{\ell+1}]\!] \setminus t$, which means $[w_\ell]_{\alpha_\ell} \in [\![w_{\ell+1}]\!]$. Since $[w_i]_{\alpha_i} \in [\![w_{i+1}]\!]$ implies $w_{i+1} \in [w_i]_{\alpha_i}$, for all $1 \leq i \leq \ell$,

$$w_1, \alpha_1, w_2, \ldots, w_\ell, \alpha_\ell, w_{\ell+1}$$

is a path in $\mathfrak{M}$. First, we need to prove that it is also a coset path. If there is a world

$$v \in [w_1]_{\emptyset \cap \alpha_1} \cap [w_2]_{\alpha_1 \cap \alpha_2} = \{w_1\} \cap [w_1]_{\alpha_1} \cap [w_2]_{\alpha_2},$$

then $v = w_1$ and $[w_2]_{\alpha_2} \in [\![w_1]\!]$, and together with $[w_2]_{\alpha_2} \in d(W) \setminus t$ this implies $[w_2]_{\alpha_2} \in [\![w_1]\!] \setminus t$, which contradicts the minimality of the path in $d(\mathfrak{M})$. Analogously, one proves $[w_\ell]_{\alpha_{\ell-1} \cap \alpha_\ell} \cap [w_{\ell+1}]_{\alpha_\ell \cap \emptyset} = \emptyset$. If there is an $1 < i \leq \ell$ and some world

$$v \in [w_i]_{\alpha_{i-1} \cap \alpha_i} \cap [w_{i+1}]_{\alpha_i \cap \alpha_{i+1}} = [w_{i-1}]_{\alpha_{i-1}} \cap [w_i]_{\alpha_i} \cap [w_{i+1}]_{\alpha_{i+1}},$$

then $[w_{i-1}]_{\alpha_{i-1}}, [w_{i+1}]_{\alpha_{i+1}} \in [\![v]\!]$, which makes $[\![v]\!]$ a chord for the path in $d(\mathfrak{M})$, contradicting its minimality. Second, the coset path also avoids $t$ because, for all $1 \leq i \leq \ell$,

$$[w_i]_{\alpha_i} \notin t \quad \Leftrightarrow \quad [w_{\ell+1}]_\gamma \nsubseteq [w_i]_{\alpha_i}.$$

□

Lemma 4.3.6 states the converse direction: a minimal coset path that avoids $t$ in a Cayley structure $\mathfrak{M}$ induces a chordless path in $d(\mathfrak{M}) \setminus t$.

**Lemma 4.3.6.** *Let $\mathfrak{M}$ be a 2-acyclic Cayley frame, $w_1, w_{\ell+1}$ two worlds, $\gamma \subseteq \mathrm{agt}(w_1, w_{\ell+1})$ a set of agents and $t = \rho(v, \gamma)$. A minimal coset path of length $\ell \geq 1$*

$$w_1, \alpha_1, w_2, \ldots, w_\ell, \alpha_\ell, w_{\ell+1}$$

*that avoids $t$ induces a chordless path of length $\ell - 1$*

$$[w_1]_{\alpha_1}, [\![w_2]\!], [w_2]_{\alpha_2}, [\![w_3]\!], \ldots, [\![w_\ell]\!], [w_\ell]_{\alpha_\ell}$$

*in $d(\mathfrak{M}) \setminus t$.*

*Proof.* For all $1 \leq i \leq \ell$, $w_{i+1} \in R_{\alpha_i}[w_i]$ implies $[w_i]_{\alpha_i} \in [\![w_{i+1}]\!]$, hence

$$[w_1]_{\alpha_1}, [\![w_2]\!], \ldots, [\![w_\ell]\!], [w_\ell]_{\alpha_\ell}$$

is indeed a path in $d(\mathfrak{M})$. Furthermore, the coset path avoids $t$ because, for all $1 \leq i \leq \ell$,

$$[w_i]_{\alpha_i} \notin t \quad \Leftrightarrow \quad [w_{\ell+1}]_\gamma \nsubseteq [w_i]_{\alpha_i}.$$

It remains to show that the path is chordless.

Assume there is a chord, i.e. a hyperedge $[\![w]\!] \subseteq d(\mathfrak{M})$ that contains the vertices $[w_i]_{\alpha_i}, [w_j]_{\alpha_j}$, for some $1 \leq i, j \leq \ell$ with $j > i+1$. Then

- $[w]_{\alpha_i} = [w_i]_{\alpha_i} = [w_{i+1}]_{\alpha_i}$, and
- $[w]_{\alpha_j} = [w_j]_{\alpha_j} = [w_{j+1}]_{\alpha_j}$.

Hence,

$$w_1, \alpha_1, w_2, \ldots, w_i, \alpha_i, w, \alpha_j, w_{j+1}, \ldots, w_\ell, \alpha_\ell, w_{\ell+1}$$

is a shorter, non-trivial path from $w_1$ to $w_{\ell+1}$. If we can show that it is also a coset path, we are done because this would be a contradiction to the minimality of the original path.

We choose the chord $[\![w]\!]$ and $i$ such that $i$ is the minimal index such that $[w_i]_{\alpha_i}$ is incident with any chord of the path, and $j$ is the maximal index such that there is a chord that connects $[w_i]_{\alpha_i}$ with $[w_j]_{\alpha_j}$. Formally: $i$ is the minimal index such that there is no other chord $[\![v]\!]$ and no $1 \leq k, k' \leq \ell$ with $k < i$, $k' > k+1$ and $[w_k]_{\alpha_k}, [w_{k'}]_{\alpha_{k'}} \in [\![v]\!]$; depending on $i$, we choose $j$ such that there is no other chord $[\![v]\!]$ and no $i+1 < k < j$ with $[w_i]_{\alpha_i}, [w_k]_{\alpha_k} \in [\![v]\!]$.

We need to show

1. $[w_i]_{\alpha_{i-1} \cap \alpha_i} \cap [w]_{\alpha_i \cap \alpha_j} = \emptyset$, and
2. $[w]_{\alpha_i \cap \alpha_j} \cap [w_{j+1}]_{\alpha_j \cap \alpha_{j+1}} = \emptyset$.

1. Assume there is some world $v \in [w_i]_{\alpha_{i-1} \cap \alpha_i} \cap [w]_{\alpha_i \cap \alpha_j}$. If $i \geq 2$, then by Lemma 4.2.7

$$\begin{aligned} [w_i]_{\alpha_{i-1} \cap \alpha_i} \cap [w]_{\alpha_i \cap \alpha_j} &= [w_{i-1}]_{\alpha_{i-1}} \cap [w_i]_{\alpha_i} \cap [w]_{\alpha_j} \\ &= [w_{i-1}]_{\alpha_{i-1}} \cap [w_i]_{\alpha_i} \cap [w_j]_{\alpha_j} \end{aligned}$$

implies $[w_{i-1}]_{\alpha_{i-1}}, [w_j]_{\alpha_j} \in [\![v]\!]$, which contradicts the choice of $i$. If $i = 1$, then

$$\begin{aligned} [w_i]_{\alpha_{i-1} \cap \alpha_i} \cap [w]_{\alpha_i \cap \alpha_j} &= [w_1]_{\emptyset \cap \alpha_1} \cap [w]_{\alpha_i} \cap [w]_{\alpha_j} \\ &= \{w_1\} \cap [w_1]_{\alpha_1} \cap [w_j]_{\alpha_j} \end{aligned}$$

implies $v = w_1$ and $[w_{j+1}]_{\alpha_j} = [w_j]_{\alpha_j} \in [\![w_1]\!]$. Hence, there is a shorter path

$$w_1, \alpha_j, w_{j+1}, \alpha_{j+1}, \ldots, \alpha_\ell, w_{\ell+1}$$

from $w_1$ to $w_{\ell+1}$. If there is some $v \in [w_1]_{\alpha_0 \cap \alpha_j} \cap [w_{j+1}]_{\alpha_j \cap \alpha_{j+1}}$, then

$$[w_1]_{\alpha_0 \cap \alpha_j} \cap [w_{j+1}]_{\alpha_j \cap \alpha_{j+1}} = [w_1]_\emptyset \cap [w_1]_{\alpha_j} \cap [w_{j+1}]_{\alpha_{j+1}}$$

implies $[w_1]_{\alpha_1}, [w_{j+1}]_{\alpha_{j+1}} \in [\![v]\!]$, which contradicts the maximality of $j$.
2. Assume there is some world $v \in [w]_{\alpha_i \cap \alpha_j} \cap [w_{j+1}]_{\alpha_j \cap \alpha_{j+1}}$. If $j < \ell$, then by Lemma 4.2.7

$$\begin{aligned} [w]_{\alpha_i \cap \alpha_j} \cap [w_{j+1}]_{\alpha_j \cap \alpha_{j+1}} &= [w_i]_{\alpha_i} \cap [w]_{\alpha_j} \cap [w_{j+1}]_{\alpha_{j+1}} \\ &= [w_i]_{\alpha_i} \cap [w_j]_{\alpha_j} \cap [w_{j+1}]_{\alpha_{j+1}} \end{aligned}$$

implies $[w_i]_{\alpha_i}, [w_{j+1}]_{\alpha_{j+1}} \in [\![v]\!]$, which contradicts the maximality of $j$. If $j = \ell$, then $[w_{j+1}]_{\alpha_{j+1}} = [w_{\ell+1}]_{\alpha_{\ell+1}} = [w_{\ell+1}]_{\emptyset} = \{w_{\ell+1}\}$, which implies together with the equalities above that $v = w_{\ell+1}$ and $[w_i]_{\alpha_i} \in [\![w_{\ell+1}]\!]$. Hence, there is a shorter path

$$w_1, \alpha_1, w_2, \ldots, w_i, \alpha_i, w_{\ell+1}$$

from $w_1$ to $w_{\ell+1}$. If there is some $v \in [w_i]_{\alpha_{i-1} \cap \alpha_i} \cap [w_{\ell+1}]_{\alpha_i \cap \alpha_{\ell+1}}$, then

$$\begin{aligned} [w_i]_{\alpha_{i-1} \cap \alpha_i} \cap [w_{\ell+1}]_{\alpha_i \cap \alpha_{\ell+1}} &= [w_{i-1}]_{\alpha_{i-1}} \cap [w_i]_{\alpha_i} \cap [w_{\ell+1}]_{\alpha_{\ell+1}} \\ &= [w_{i-1}]_{\alpha_{i-1}} \cap [w_i]_{\alpha_i} \cap [w_{\ell+1}]_{\emptyset} \\ &= [w_{i-1}]_{\alpha_{i-1}} \cap [w_i]_{\alpha_i} \cap \{w_{\ell+1}\} \end{aligned}$$

implies $v = w_{\ell+1}$ and $[w_{i-1}]_{\alpha_{i-1}} \in [\![w_{\ell+1}]\!]$, which together with $[w_\ell]_{\alpha_\ell} \in [\![w_{\ell+1}]\!]$ contradicts the minimality of $i$. □

So far, in this section, we illustrated how coset paths in a Cayley structure and chordless paths in its dual hypergraph are linked. We would like to use these results to connect different notions of distance. Based on the generalisation of non-trivial coset paths to coset paths that avoid $t$, we can generalise the notion of distance from the previous chapter to a notion that depends on $t$ in a straightforward manner.

**Definition 4.3.7.** Let $\mathfrak{M}$ be a 2-acyclic Cayley frame, $w \neq v$ two worlds, $\gamma \subseteq \Gamma$ and $t = \rho(v, \gamma)$. The *t-distance* $d_t(w, v)$ between $w$ and $v$ is defined as the length of a minimal coset path from $w$ to $v$ that avoids $t$. For a set of worlds $\mathbf{z}$, the *t-distance* $d_t(\mathbf{z}, v)$ between $\mathbf{z}$ and $v$ is defined as

$$d_t(\mathbf{z}, v) := \min_{z \in \mathbf{z}} d_t(z, v).$$

*Remark* 4.3.8. $t$-distance generalises the notion of distance from Definition 4.2.4 in the sense that

$$d_t(w, v) = d(w, v),$$

for $t = \rho(v, \mathrm{agt}(w, v)) = \rho(w, \mathrm{agt}(w, v))$.

*Remark* 4.3.9. Depending on $t$, $t$-distance allows for distance 1: $d_t(w, v) = 1$ if and only if $[v]_{\mathrm{agt}(w,v)} \notin t$. However, we are usually interested in cases where $\gamma \subseteq \mathrm{agt}(w, v)$, which implies $[v]_{\mathrm{agt}(w,v)} \in t$, for $t = \rho(v, \gamma)$.

As the notation suggests, there is a close connection between $d_t(w, v)$ and $d_t([\![w]\!], [\![v]\!])$. Lemma 4.3.5 shows that chordless paths of length $\ell + 1$ from $[\![w]\!] \setminus t$ to $[\![v]\!] \setminus t$ in $d(\mathfrak{M}) \setminus t$ induce coset paths of length $\ell$ from $w$ to $v$ that avoid $t$, and Lemma 4.3.6 shows that coset paths from $w$ to $v$ of length $\ell + 1$ that avoid $t$ induce chordless paths of length $\ell$ from $[\![w]\!] \setminus t$ to $[\![v]\!] \setminus t$ in $d(\mathfrak{M}) \setminus t$. Thus, the two different notions of distance are in fact equivalent.

**Corollary 4.3.10.** *Let $\mathfrak{M}$ be Cayley frame, $w \neq v$ two worlds, $\gamma \subseteq \Gamma$ and $t = \rho(v, \gamma)$. For $\ell \geq 1$,*

$$d_t(w, v) = \ell \quad \Leftrightarrow \quad d_t([\![w]\!], [\![v]\!]) = \ell - 1.$$

Hence, given $w, v$ and $t$, finding some $v^* \sim v$ such that $d_t([\![w]\!], [\![v^*]\!]) > \ell - 1$ reduces to finding some $v^* \sim v$ such that $d_t(w, v^*) > \ell$. To do that we generalise a result about inner paths from the previous section: for a threshold $\ell \in \mathbb{N}$, $d_t(w, v)$ depends only on the length of minimal *inner* coset paths that avoid $t$ if the frame is sufficiently acyclic. The following lemma subsumes everything we did so far, and is the first central result of this section.

**Lemma 4.3.11.** *Let $\ell \geq 1$, $\mathfrak{M}$ be a sufficiently acyclic Cayley frame, $w, v$ two worlds, $\gamma \subseteq \Gamma$ and $t = \rho(v, \gamma)$. If there is no inner coset path of length $\ell$ that avoids $t$, then*

$$d_t(w, v) > \ell \quad \textit{and} \quad d_t([\![w]\!], [\![v]\!]) > \ell - 1.$$

*Proof.* Assume $d_t(w, v) = k \leq \ell$, and let $w_1, \alpha_1, w_2, \ldots, w_k, \alpha_k, w_{k+1}$, with $w_1 = w$ and $w_{k+1} = v$, be a coset path that avoids $t$. Since $\mathfrak{M}$ is sufficiently acyclic, this path is short. Hence, Corollary 4.2.12 implies there are $w'_i \in [w_i]_{\alpha_i \cap \alpha}$, for $\alpha = \mathrm{agt}(w, v)$ and $1 < i \leq k$, such that

$$w_1, \alpha'_1, w'_2, \alpha'_2, w'_3, \ldots, w'_k, \alpha'_k, w_{k+1},$$

for $\alpha'_i = \alpha_i \cap \alpha$, $1 \leq i \leq k$, is a short inner coset path. This inner coset path also avoids $t$ because $[w_{\ell+1}]_\gamma \nsubseteq [w_i]_{\alpha_i}$ and $[w_i]_{\alpha_i \cap \alpha} = [w'_i]_{\alpha_i \cap \alpha} \subseteq [w_i]_{\alpha_i \cap \alpha}$ imply $[w_{\ell+1}]_\gamma \nsubseteq [w'_i]_{\alpha_i \cap \alpha}$. However, we assumed that such inner paths do not exist. Thus, $d_t(w, v) > \ell$ and by Corollary 4.3.10 also $d_t([\![w]\!], [\![v]\!]) > \ell - 1$. □

Proving the freeness theorem involves several steps. Let $\mathfrak{M}$ be a Cayley structure, $v$ a world, $(\mathbf{z}, z_0)$ a pointed set with $v \notin \mathbf{z}$ and $\gamma = \mathrm{agt}(z_0, v)$. The challenge is to find a world $v^* \sim v$ with $\gamma = \mathrm{agt}(z_0, v^*)$ such that $v^*$ and $(\mathbf{z}, z_0)$ are $m$-free, assuming $\mathfrak{M}$ is sufficiently acyclic and sufficiently free depending on $m$ and $|\mathbf{z}|$. Hence, we need a suitable $v^*$ such that $d_t(\bigcup [\![\mathbf{z}]\!], [\![v^*]\!]) > m$, for $t = [\![z_0]\!] \cap [\![v^*]\!]$, and by Lemma 4.3.11 it suffices to have a $v^*$ such that $d_t(\mathbf{z}, v^*) > m+1$. Since we need such a $v^*$ for arbitrary $m$, we will show how to obtain one such that $d_t(\mathbf{z}, v^*) > m$ in order to make the following more readable.

The first step is to find some $v_1 \sim v$ with $\mathrm{agt}(v_1, z_0) = \gamma$ such that $d_t(\mathbf{z}, v_1) > 1$, for $t = [\![z_0]\!] \cap [\![v]\!] = \rho(v, \gamma) = \rho(v_1, \gamma)$. The choice of $t$ immediately implies $d_t(z_0, v) > 1$, but we need to look for the right bisimilar copy of $v$ in $[v]_\gamma$ to increase the $t$-distance to the other worlds of $\mathbf{z}$. The condition $d_t(\mathbf{z}, v_1) > 1$ can be equivalently rephrased as $\mathrm{agt}(v_1, z_0) \subseteq \mathrm{agt}(v_1, z)$, for all $z \in \mathbf{z}$. 2-acyclicity of $\mathfrak{M}$ implies a triangle inequality with respect to the smallest connecting sets of agents, namely

$$\mathrm{agt}(v, z) \subseteq \mathrm{agt}(v, z_0) \cup \mathrm{agt}(z_0, v).$$

Hence, if we find a bisimilar copy $v_1$ of $v$ with $\mathrm{agt}(z_0, v_1) = \gamma$ such that

$$\mathrm{agt}(v_1, z) = \mathrm{agt}(v_1, z_0) \cup \mathrm{agt}(z_0, z),$$

then $\mathrm{agt}(v_1, z_0) \subseteq \mathrm{agt}(v_1, z)$. In other words, we need to increase the distance, with regards to connecting agents, between $v$ and $z$ without changing the distance between $v$ and $z_1$. Lemma 4.3.12 shows that this can be done in 2-acyclic structures for multiple $z \in \mathbf{z}$ simultaneously. In the Lemma, the worlds of $\mathbf{z}$ are the ones that have already been taken care of and $u$ is the world the will be processed next.

**Lemma 4.3.12.** *Let $\mathfrak{M}$ be a Cayley structure, $v, u$ worlds and $(\mathbf{z}, z_0)$ a finite pointed set with $\mathrm{agt}(v, z) = \mathrm{agt}(v, z_0) \cup \mathrm{agt}(z_0, z)$, for all $z \in \mathbf{z}$. If $\mathfrak{M}$ is 2-acyclic and sufficiently rich, then there is a world $v^*$ with*

- $\mathfrak{M}, v^* \sim \mathfrak{M}, v$,
- $\mathrm{agt}(v^*, z) = \mathrm{agt}(v, z)$, *for all* $z \in \mathbf{z}$,

*such that*

$$\mathrm{agt}(v^*, u) = \mathrm{agt}(v^*, z_0) \cup \mathrm{agt}(z_0, u).$$

*Proof.* Put $\alpha_1 := \mathrm{agt}(v, z_0)$, $\alpha_2 := \mathrm{agt}(z_0, u)$ and $\alpha_3 := \mathrm{agt}(u, v)$. By Lemma 4.1.2 2-acyclicity implies

- $\alpha_1 \subseteq \alpha_2 \cup \alpha_3$,
- $\alpha_2 \subseteq \alpha_1 \cup \alpha_3$ and
- $\alpha_3 \subseteq \alpha_1 \cup \alpha_2$.

We show that if $\alpha_3 \subsetneq \alpha_1 \cup \alpha_2$, then for every agent $a \in (\alpha_1 \cup \alpha_2) \setminus \alpha_3$ there is a world $v' \in [v]_a$ with $v' \sim v$ such that

- $\mathrm{agt}(u, v') = \alpha_3 \cup \{a\}$,
- $\mathrm{agt}(v', z_0) = \alpha_1$,
- $\mathrm{agt}(v', z) = \mathrm{agt}(v, z)$, for all $z \in \mathbf{z}$,
- $\alpha_1 \subseteq \alpha_2 \cup \mathrm{agt}(u, v')$,
- $\alpha_2 \subseteq \alpha_1 \cup \mathrm{agt}(u, v')$,
- $\mathrm{agt}(u, v') \subseteq \alpha_1 \cup \alpha_2$, and
- $|(\alpha_1 \cup \alpha_2) \setminus \mathrm{agt}(u, v')| < |(\alpha_1 \cup \alpha_2) \setminus \alpha_3|$.

In other words, $v'$ increases the distance from $u$ by $a$, with respect to the minimal connecting set of agents, and keeps all the other relevant properties of $v$ fixed. Since $(\alpha_1 \cup \alpha_2) \setminus \alpha_3$ is finite, applying this argument a finite number of times leads to a suitable world $v^*$ with, in particular, $(\alpha_1 \cup \alpha_2) \setminus \mathrm{agt}(u, v^*) = \emptyset$.

Let $a \in (\alpha_1 \cup \alpha_2) \setminus \alpha_3$. We only need to consider the case $a \in \alpha_1$ because if $a \notin \alpha_1$, then

$$a \notin \alpha_1 \xrightarrow{a \in \alpha_1 \cup \alpha_2} a \notin \alpha_1, i \in \alpha_2 \xrightarrow{\alpha_2 \subseteq \alpha_1 \cup \alpha_3} a \in \alpha_3.$$

Since $\mathfrak{M}$ is sufficiently rich and 2-acyclic, there is some $v' \in [v]_a \setminus \{v\}$ by Lemma 4.1.4 such that still $\mathrm{agt}(v', z) = \mathrm{agt}(v, z)$, for all $z \in \mathbf{z}$, because $a \in \alpha_1 \subseteq \mathrm{agt}(v, z)$. Lemma 4.1.4 also implies $\mathrm{agt}(u, v') = \alpha_3 \cup \{a\}$ because $a \notin \alpha_3$ and $\mathfrak{M}$ is 2-acyclic, which in turn implies $|(\alpha_1 \cup \alpha_2) \setminus \mathrm{agt}(u, v')| < |(\alpha_1 \cup \alpha_2) \setminus \alpha_3|$. Furthermore, $a \in \alpha_1$ implies $\mathrm{agt}(v', z_0) = \alpha_1$, and 2-acyclicity gives us again

- $\alpha_1 \subseteq \alpha_2 \cup \mathrm{agt}(u, v')$,
- $\alpha_2 \subseteq \alpha_1 \cup \mathrm{agt}(u, v')$ and
- $\mathrm{agt}(u, v') \subseteq \alpha_1 \cup \alpha_2$.

□

The second step, and the more difficult one by far, is to establish $d_t(\mathbf{z}, v^*) > m$, while maintaining $\mathrm{agt}(v^*, z_0) = \gamma$. By Lemma 4.3.11 this means that we need to eliminate short inner coset paths that avoid $t$ between $v$ and the worlds in $\mathbf{z}$ by moving only to bisimilar copies of $v$ within the class $[v]_\gamma$.

The question arises: how do we find a suitable $v^*$ in $[v]_\gamma$? For all $a \in \gamma$, there are many possible bisimilar copies of $v$ to choose from. But which choice is the right one, or brings us at least closer to our goal? Furthermore, before we decide on some world in $[v]_a$, which $a \in \gamma$ do we choose for that? And is it possible to make a wrong choice? Is there an $a \in \gamma$ such that there is some $v' \in [v]_a$ that is actually closer to $\mathbf{z}$ instead of further away, as needed? We would like to be able to describe the direction one has to take if one wants to move on a short path from $v$ to $\mathbf{z}$. If we can do that, we just move in the other direction.

The set of agents $\mathrm{short}(v, z)$, defined in the previous section, gives us an answer to these questions. Implied by the zipper lemma, $\mathrm{short}(v, z)$ is the unique intersection of all first edge sets of short coset paths from $v$ to $z$. In other words, if $v, \alpha, \ldots, z$ is a short path, then $\mathrm{short}(v, z) \subseteq \alpha$. However, with short inner coset paths that avoid $t$ we handle a special kind of short coset path. Hence, we need a specialized version of $\mathrm{short}(v, z)$. As a reminder: $\alpha \subseteq \Gamma$ is a first edge set for the pair worlds $(v, z)$ if there is a short coset path from $v$ to $z$ that starts with an $\alpha$-edge.

**Definition 4.3.13.** Let $\mathfrak{M}$ be a 2-acyclic Cayley structure, $v, z \in \mathfrak{M}$ and $\gamma \subseteq \mathrm{agt}(v, z)$ a set of agents. For $t = \rho(v, \gamma)$, we define the set of agents

$$\mathrm{short}_t(v, z)$$

as the intersection of all the first edge sets of short coset paths from $v$ to $z$ that avoid $t$.

As argued in the previous section, $\mathrm{short}(v, z)$ is always a well-defined subset of $\mathrm{agt}(v, z)$ since $v, \mathrm{agt}(v, z), z$ is a short coset path. To define $\mathrm{short}_t(v, z)$, for some $t = \rho(v, \gamma)$, we consider a certain subset of all short coset paths from $v$ to $z$, namely the ones that avoid $t$. If there are no such paths, then this subset is empty and $\mathrm{short}_t(v, z)$ is not defined. This does not pose a problem because we only need $\mathrm{short}_t(v, z)$ if there are short coset paths that avoid $t$. However, assume there are short coset paths $v, \alpha, \ldots, z$ and $v, \beta, \ldots, z$ that avoid $t$. Then there is a short coset path $v, \alpha \cap \beta, \dots, z$ by Corollary 4.2.10, which also avoids $t$ because $[v]_\gamma \nsubseteq [v]_\alpha$ and $[v]_\gamma \nsubseteq [v]_\beta$ imply $[v]_\gamma \nsubseteq [v]_{\alpha\cap\beta}$. Thus, $\mathrm{short}_t(v,z)$ is well-defined if short coset paths from $v$ to $z$ that avoid $t$ exist.

Intuitively, $\mathrm{short}_t(v,z)$ describes the direction one has to take if one wants to move on a short coset path that avoids $t$ from $v$ to $z$. More formally, if $v, \alpha, \dots, z$ is a short coset path that avoids $t$, then $\mathrm{short}_t(v,z) \subseteq \alpha$. However, we would like to *increase* the $t$-distance between $v$ and $z$. This means that we must take a different direction, i.e. some $a \notin \mathrm{short}_t(v,z)$, and move to a bisimilar copy of $v$ in $[v]_a$. The idea is to repeat this procedure again and again, with different suitable agents, until we reach a copy of $v$ that has a sufficiently long $t$-distance to $z$.

Furthermore, the agent $a \notin \mathrm{short}_t(v,z)$ can be chosen to be in $\gamma$: if $v, \mathrm{short}_t(v,z), \dots, z$ is a short coset path that avoids $t$ (remember $t = \rho(v,\gamma)$), then $[v]_\gamma \nsubseteq [v]_{\mathrm{short}_t(v,z)}$ implying $\gamma \nsubseteq \mathrm{short}_t(v,z)$. The case $\gamma \subseteq \mathrm{agt}(v,z)$ is of particular interest in the proof of the freeness theorem.

*Remark* 4.3.14. Let $\mathfrak{M}$ be a 2-acyclic Cayley structure, $v, z \in \mathfrak{M}$ and $\gamma \subseteq \mathrm{agt}(v,z)$ a set of agents. Then, for $t = \rho(v,\gamma)$,

$$\gamma \nsubseteq \mathrm{short}_t(v,z) \subseteq \mathrm{agt}(v,z).$$

We continue with investigating the properties of $\mathrm{short}_t(v,z)$. Similar to the set $\mathrm{agt}(v,z)$ in 2-acyclic structures, $\mathrm{short}_t(v,z)$ behaves in a controlled manner in sufficiently acyclic structures.

**Lemma 4.3.15.** *Let $m \in \mathbb{N}$, $\mathfrak{M}$ be a Cayley frame, $z, v$ two worlds, $\gamma \subseteq \mathrm{agt}(v,z)$ and $t = \rho(v,\gamma)$. Assume $\mathfrak{M}$ is $2m+1$-acyclic, $d_t(z,v) \leq m$, and that there are*

- *$a \notin \mathrm{short}_t(v,z)$ and*
- *$v' \in [v]_a \setminus \{v\}$*

*such that $d_t(v',z) \leq m$, then*

$$a \in \mathrm{short}_t(v',z).$$

*Proof.* Let $\ell, k \leq m$, and $w_1, \alpha_1, \dots, \alpha_\ell, w_{\ell+1}$ and $u_1, \beta_1, \dots, \beta_k, u_{k+1}$ be two coset paths that avoid $t$ with

- $w_1 = u_1 = z$, $w_{\ell+1} = v$, $u_{k+1} = v'$, and
- $\alpha_\ell = \mathrm{short}_t(v,z)$, $\beta_k = \mathrm{short}_t(v',z)$.

By choice of $z, v$ and $v'$ and Definition 4.3.13 such paths must exist. If we assume $a \notin \mathrm{short}_t(v',z)$, then $a \notin \alpha_\ell \cup \beta_k$. Together with $w_{\ell+1} \notin [w_\ell]_{\alpha_{\ell-1}\cap\alpha_\ell}$, $u_{k+1} \notin [u_k]_{\beta_{k-1}\cap\beta_k}$ and $w_{\ell+1} \neq u_{k+1}$ this implies

- $[w_\ell]_{\alpha_{\ell-1}\cap\alpha_\ell} \cap [w_{\ell+1}]_{\alpha_\ell\cap\{a\}} = \emptyset$,
- $[w_{\ell+1}]_{\alpha_\ell\cap\{a\}} \cap [u_{k+1}]_{\{a\}\cap\beta_k} = \emptyset$, and

- $[u_{k+1}]_{\{a\}\cap\beta_k} \cap [u_k]_{\beta_k\cap\beta_{k-1}} = \emptyset$.

Hence,

$$w_1, \alpha_1, w_2, \ldots, w_\ell, \alpha_\ell, w_{\ell+1}, a, u_{k+1}, \beta_k, u_k, \ldots, u_2, \beta_1, u_1$$

is a coset path of length $\ell+k+1 \leq 2m+1$ from $z$ to $z$, which cannot exist by Lemma 4.2.8 in a $2m+1$-acyclic Cayley frame. □

If we choose $\gamma = \text{agt}(v, w)$ in the lemma above, we obtain this special case:

**Corollary 4.3.16.** *Let $m \in \mathbb{N}$, $\mathfrak{M}$ be a Cayley frame and $z, v$ two worlds. Assume $\mathfrak{M}$ is $2m+1$-acyclic, $d(z, v) \leq m$, and that there are*

- *$a \notin \text{short}(v, z)$ and*
- *$v' \in [v]_a \setminus \{v\}$*

*such that $d(z, v') \leq m$, then $a \in \text{short}(v', w)$.*

We explained that the agents in $\text{short}_t(v, z)$ are the ones that represent the direction one needs to take if one wants to move from $v$ to $z$ on a short coset path that avoids $t$. Lemma 4.3.15 makes this notion precise and tells us how to use $\text{short}_t(v, z)$. We choose an agent $a \notin \text{short}_t(v, z)$ and move to a world $v' \in [v]_a \setminus \{v\}$. If the structure is sufficiently acyclic, every short path from $v'$ to $z$ that avoids $t$ must start with a set that includes agent $a$.

Lemma 4.3.15 is the cornerstone for proving the second step of the freeness theorem: establishing $d_t(\mathbf{z}, v^*) > m$. We will utilise it as follows. Let $w_1, \alpha_1, \ldots, \alpha_\ell, w_{\ell+1}$ be a short inner coset path from $z$ to $v$ that avoids $t$, for $t = \rho(v, \gamma)$. First, every set $\alpha_i$, $1 \leq i \leq \ell$, is a proper subset of $\text{agt}(v, z)$. Second, no class $[w_i]_{\alpha_i}$ contains $[v]_\gamma$; in other words, if $[w_i]_{\alpha_i} \cap [v]_\gamma \neq \emptyset$, then $\gamma \not\subseteq \alpha_i$. Hence, the size of any set of agents $\alpha_i$ is bounded in terms of $\text{agt}(v, z)$ and $\gamma$. Assume that we move along an $a_1$-edge from $v$ to $v_1$, then along an $a_2$-edge from $v_1$ to $v_2$ and so forth, for suitable agents $a_i \in \gamma$. Then the set $\text{short}_t(v_1, z)$ must contain $a_1$, if the $t$-distance between $z$ and $v_2$ is still $d_t(z, v)$, then $\text{short}_t(v_2, z)$ must additionally contain $a_2$, etc. Intuitively, we force the sets $\text{short}_t(v_i, z)$ to grow by $a_i$ in every step. Yet, the $t$-distance between $z$ and $v_i$, might still be $d_t(z, v)$ because other short coset paths that avoid $t$ remain. However, the sets $\text{short}_t(v_i, z)$ cannot grow indefinitely by the agents $a_i$ because anyone of these agents will, in particular, be chosen as an element of $\gamma$ and no $\text{short}_t(v_i, z)$ can contain all of $\gamma$ by definition. Essentially, we force the set $\text{short}_t(v_i, z)$ to grow until it becomes too large. At this stage, we can show that $d_t(z, v_i)$ is actually greater than $d_t(z, v)$. This is the content of Lemma 4.3.18.

However, finding some $v^* \sim v$ such that $v^*$ and $(\mathbf{z}, z_0)$ are $m$-free involves increasing the $t$-distance, $t = \rho(v, \text{agt}(v, z_0))$, to a whole set $\mathbf{z}$. We start with increasing $d_t(z_0, v)$ and continue with increasing $d_t(z, v)$, for the other worlds in $z \in \mathbf{z}$, one after another. Hence, before we prove Lemma 4.3.18, we need to show that we are able to move away from several worlds simultaneously, provided the Cayley structure is sufficiently rich. In Lemma 4.3.17 below, $\mathbf{z}$ plays the role of the set of worlds that have are already been processed ($d_t(\mathbf{z}, v) > m$) and $w$ is the world to be processed next (still $d_t(w, v) \leq m$). The point is that we can move further away from $w$ without decreasing $d_t(\mathbf{z}, v)$.

**Lemma 4.3.17.** *Let $m \geq 2$, $\mathfrak{M}$ be a Cayley structure, $w, v$ two worlds, $(\mathbf{z}, z_0)$ a finite pointed set and $t = \rho(v, \mathrm{agt}(z_0, v))$. Assume that $\mathfrak{M}$ is sufficiently acyclic and sufficiently rich, and*

- $\mathrm{agt}(z_0, v) \subseteq \mathrm{agt}(w, v)$,
- $\mathrm{agt}(z_0, v) \subseteq \mathrm{agt}(z, v)$, *for all* $z \in \mathbf{z}$,
- $d_t(\mathbf{z}, v) > m$, *and*
- $d_t(w, v) \leq m$.

*Then* $\mathrm{agt}(z_0, v) \setminus \mathrm{short}_t(v, w) \neq \emptyset$, *and for every* $a \in \mathrm{agt}(z_0, v) \setminus \mathrm{short}_t(v, w)$ *there is some* $v' \in [v]_a \setminus \{v\}$ *such that*

- $\mathfrak{M}, v' \sim \mathfrak{M}, v$, *and*
- $d_t(\mathbf{z}, v') > m$.

*Proof.* Put $\gamma := \mathrm{agt}(z_0, v)$, let $\ell \leq m$ and $w, \alpha_1, \dots, \alpha_\ell, v$ be an inner coset path that avoids $t$. In particular, this means $\gamma \nsubseteq \alpha_\ell$ and also $\gamma \nsubseteq \mathrm{short}_t(v, w)$ since $\emptyset \neq \mathrm{short}_t(v, w) \subseteq \alpha_\ell$. Thus, we obtain the first statement: $\gamma \setminus \mathrm{short}_t(v, w) \neq \emptyset$.

Let $a \in \gamma \setminus \mathrm{short}_t(v, w)$, $z \in \mathbf{z}$ and assume there is some $u \in [v]_a$ and $k \leq m$ such that there is a coset path $z, \beta_1, \dots, \beta_k, u$ that avoids $t$. First, 2-acyclicity and the choice of $a$ and $u$ imply

$$\mathrm{agt}(z, u) = \mathrm{agt}(z, v) \setminus \{a\} \quad \text{or} \quad \mathrm{agt}(z, u) = \mathrm{agt}(z, v).$$

If $\mathrm{agt}(z, u) = \mathrm{agt}(z, v) \setminus \{a\}$, then $z, \mathrm{agt}(z, u), u, a, v$ is a coset path of length 2 that avoids $t$ implying $d_t(z, v) \leq 2$. This cannot be the case since $d_t(z, v) > m$. Hence, $\mathrm{agt}(z, u) = \mathrm{agt}(z, v)$.

Second, we claim $d_t(z, u') > m$, for all $u' \in [v]_a \setminus \{u\}$: $a \in \mathrm{short}_t(u, z)$ cannot be the case because $z, \beta_1, \dots, \beta_k, v$ or $z, \beta_1, \dots, \beta_{k-1}, v$ would be a short coset path that avoids $t$, which implies $d_t(z, v) \leq m$. Hence, if $d_t(z, u') \leq m$, for some $u' \in [v]_a \setminus \{u\}$, then $a \in \mathrm{short}_t(u', z)$ follows from Lemma 4.3.15. However, this implies, again, $d_t(z, v) \leq m$, contrary to assumption.

Thus, for any $z \in \mathbf{z}$ there is at most one world $u_z \in [v]_a$ such that $d_t(z, u_z) \leq m$. Since $\mathfrak{M}$ is sufficiently rich, there remains a world $v' \in [v]_a \setminus \{v\}$ such that $v' \sim v$ and $d_t(\mathbf{z}, v') > m$. □

In Lemma 4.3.18 $\mathbf{z}$ plays once again the role of the worlds with an already increased distance to $v$, and $w$ is the world to be taken care of next.

**Lemma 4.3.18.** *Let $m \in \mathbb{N}$, $\mathfrak{M}$ be a Cayley structure, $w, v$ two worlds, $(\mathbf{z}, z_0)$ a finite pointed set, and $t = \rho(v, \mathrm{agt}(z_0, v))$. Assume that $\mathfrak{M}$ is sufficiently acyclic and sufficiently rich, and*

- $\mathrm{agt}(z_0, v) \subseteq \mathrm{agt}(w, v)$,

- $\operatorname{agt}(z_0, v) \subseteq \operatorname{agt}(z, v)$, *for all* $z \in \mathbf{z}$,
- $d_t(\mathbf{z}, v) > m$,
- $d_t(w, v) \leq m$.

*Then there is a world* $v^* \in [v]_{\operatorname{agt}(v,z_0)}$ *with*

- $\mathfrak{M}, v^* \sim \mathfrak{M}, v$,
- $\operatorname{agt}(z, v^*) = \operatorname{agt}(z, v)$, *for all* $z \in \mathbf{z}$, *and*
- $\operatorname{agt}(w, v^*) = \operatorname{agt}(w, v)$ *such that*
- $d_t(\mathbf{z}, v^*) > m$, *and*
- $d_t(w, v^*) > m$.

*Proof.* The assumptions $\operatorname{agt}(z_0, v) \subseteq \operatorname{agt}(w, v)$ and $d_t(w, v) \leq m$ imply $d_t(w, v) = \ell$, for some $1 < \ell \leq m$. Hence, since $\mathfrak{M}$ is sufficiently acyclic there is an inner coset path from $w$ to $v$ that avoids $t$ of length $\ell$ but no such path of length $< \ell$. We need to show that there is a suitable world $v^*$ such that there is no inner coset path from $w$ to $v^*$ that avoids $t$ of length up to $\ell$. If we can do that, the statement follows from multiple applications of the same argument.

First, we inductively construct a sequence of worlds $(v_n)_{n\geq 1}$ in $[v]_{\operatorname{agt}(z_0,v)}$ that are bisimilar to $v$, along with three auxiliary sequences: two sequences of sets of agents $(\beta_n)_{n\geq 1}, (\gamma_n)_{n\geq 1}$ and a sequence of agents $(a_n)_{n\geq 1}$ in $\operatorname{agt}(z_1, v)$.

Second, we show that these sequences must be finite and that the last of the $v_n$ is the desired world $v^*$. Intuitively, every $v_n$ will be, in some sense, farther away from $v$, $\beta_n$ describes the direction back to $w$ on short paths that avoid $t$, $\gamma_n$ the steps that still have to be taken to get far enough away from $w$ and $a_n$ is the direction we take to get from $v_{n-1}$ to $v_n$.

*Construction*: Set $\gamma = \operatorname{agt}(z_0, v)$. For $n = 1$, Lemma 4.3.17 implies $\gamma \setminus \operatorname{short}_t(v, w) \neq \emptyset$; let $a_1 \in \gamma \setminus \operatorname{short}_t(v, w)$. As $\mathfrak{M}$ is sufficiently rich, there is a world $v_1 \in [v]_{a_1} \setminus \{v\}$ that is bisimilar to $v$ such that $\operatorname{agt}(v_1, w) = \operatorname{agt}(v, w)$ and $\operatorname{agt}(v_1, z) = \operatorname{agt}(v, z)$, for all $z \in \mathbf{z}$, and $d_t(\mathbf{z}, v_1) > m$ (cf. Lemmas 4.1.4 and 4.3.17). If $d_t(w, v_1) \leq \ell$, set $\beta_1 := \operatorname{short}_t(v_1, w)$ and $\gamma_1 := (\gamma \setminus \operatorname{short}_t(v, w)) \setminus \beta_1$. If $d_t(w, v_1) > \ell$ or $\gamma_1 = \emptyset$, then $v_1$ is the only world in our sequence.

For $n > 1$, assume the worlds $v_1, \ldots, v_{n-1}$ and the sets $\beta_1, \ldots, \beta_{n-1}$, $\gamma_1, \ldots, \gamma_{n-1}$ have been defined and are non-empty. Let $a_n \in \gamma_{n-1}$. Since $\mathfrak{M}$ is sufficiently rich, by Lemmas 4.1.4 and 4.3.17 there is again a $v_n \in [v_{n-1}]_{a_n} \setminus \{v_{n-1}\}$, bisimilar to $v_{n-1}$ such that $\operatorname{agt}(v_n, w) = \operatorname{agt}(v_{n-1}, w)$ and $\operatorname{agt}(v_n, z) = \operatorname{agt}(v_{n-1}, z)$, for all $z \in \mathbf{z}$, and $d_t(\mathbf{z}, v_n) > m$. If $d_t(w, v_n) \leq \ell$, set $\beta_n := \operatorname{short}_t(v_n, w)$, and $\gamma_n := \gamma_{n-1} \setminus \beta_n$. If $d_t(w, v_n) > \ell$ or $\gamma_n = \emptyset$, then $v_n$ is the last world in our sequence.

Thus, we constructed the four sequences:

- $(v_n)_{n\geq 1}$ in $[v]_{\mathrm{agt}(w,v)}$, all bisimilar to $v$;
- $(a_n)_{n\geq 1}$ in $\Gamma$;
- $(\beta_n)_{n\geq 1}$ in $\tau$;
- $(\gamma_n)_{n\geq 1}$ in $\tau$.

*Correctness*: Set $v_0 := v$, $\beta_0 := \mathrm{short}_t(v,w)$ and $\gamma_0 := \gamma \setminus \beta_0$. We show the following properties of the sequences by induction on $n \geq 1$.

(1) $\beta_n = \{a_j, a_{j+1}, \ldots, a_n\}$, for some $1 \leq j \leq n$, or $\beta_0 \cup \{a_1, \ldots, a_n\} \subseteq \beta_n$.

(2) The worlds $v_0, \ldots, v_n$ occur on every short inner coset path that avoids $t$ from $w$ to $v_n$ in the order of their indices: let $w_1, \alpha_1, w_2, \ldots, w_m, \alpha_m, w_{m+1}$ be such a path, and $0 \leq i < j \leq n$. If $1 \leq k_i, k_j \leq m$ are minimal such that $v_i \in [w_{k_i}]_{\alpha_{k_i}}$ and $v_j \in [w_{k_j}]_{\alpha_{k_j}}$, then $k_i \leq k_j$.

(3) $\gamma_n \subsetneq \gamma_{n-1}$.

For $n = 1$: *(1) and (2)*: Together with $a_1 \in \gamma \setminus \mathrm{short}_t(v,w)$ and $v_1 \neq v_0$, Lemma 4.3.15 implies $a_1 \in \beta_1 = \mathrm{short}_t(v_1, w)$. For every inner short coset path

$$w_1, \alpha_1, w_2, \ldots, w_k, \alpha_k, w_{k+1}$$

that avoids $t$ with $w = w_1$ and $v_1 = w_{k+1}$ we have $v_0 \in [v_1]_{\alpha_k}$ because $\alpha_k \supseteq \beta_1 \ni a_1$. Furthermore, since $k$ is the minimal index such that $v_1 \in [w_k]_{\alpha_k}$, the minimal index for $v_0$ can only be smaller or equal. If there is one such path with $v_0 \in [w_k]_{\alpha_{k-1} \cap \alpha_k}$, we have $\beta_1 = \{a_1\}$, because

$$w_1, \alpha_1, w_2, \ldots, v_0, \{a_1\}, w_{k+1}$$

would be a short inner coset path from $w$ to $v_1$. If $v_0 \in [w_k]_{\alpha_k} \setminus [w_{k-1}]_{\alpha_{k-1}}$, for all short inner coset paths from $w$ to $v_1$, then $\beta_0 = \mathrm{short}_t(v_0, w) \subseteq \alpha_k$ since every such path is a short inner coset path from $w$ to $v_0$ that avoids $t$. Thus, $\beta_1 = \{a_1\}$ or $\beta_0 \cup \{a_1\} \subseteq \beta_1$.

*(3)*: Note that the $\gamma_1 = (\gamma_0 \setminus \beta_0) \setminus \beta_1 = \gamma_0 \setminus (\beta_0 \cup \beta_1)$ implies $\gamma_1 \subseteq \gamma_0$, which together with $a_1 \in \gamma_0 \cap \beta_1$ implies $\gamma_1 \subsetneq \gamma_0$.

For $n > 1$: assume all the properties hold for $1, \ldots, n-1$.

*(1)*: we chose

$$a_n \in \gamma_{n-1} = \gamma_{n-2} \setminus \beta_{n-1} = \gamma_{n-2} \setminus \mathrm{short}(v_{n-1}, w) \quad \text{and} \quad v_n \in [v_{n-1}]_{a_n} \setminus \{v_{n-1}\}.$$

Hence, Lemma 4.3.15 implies $a_n \in \beta_n = \mathrm{short}_t(v_n, w)$. Let $1 \leq j < n$ be the largest index such that $a_j \notin \beta_n$. Thus, there is a short inner coset path

$$w_1, \alpha_1, w_2, \ldots, v_j, \{a_{j+1}, \ldots, a_n\}, v_n$$

from $w$ to $v_n$ that avoids $t$ which implies $\beta_n = \{a_{j+1}, \ldots, a_n\}$. If $\{a_1, \ldots, a_n\} \subseteq \beta_n$, then $\beta_n = \{a_1, \ldots, a_n\}$ or $\beta_0 \cup \{a_1, \ldots, a_n\} \subseteq \beta_n$, similar to the base case.

*(2)*: let $w_1, \alpha_1, w_2, \ldots, w_k, \alpha_k, w_{k+1}$ be a short inner coset path from $w$ to $v_n$ that avoids $t$. We showed $a_n \in \beta_n \subseteq \alpha_k$ which implies $v_{n-1} \in [w_{k+1}]_{\alpha_k}$. Thus,

$$w_1, \alpha_1, w_2, \ldots, w_k, \alpha_k, v_{n-1} \quad \text{or} \quad w_1, \alpha_1, w_2, \ldots, w_{k-1}, \alpha_{k-1}, v_{n-1}$$

is a short inner coset path from $w$ to $v_{n-1}$ that avoids $t$. By induction hypothesis the worlds $v_0, \ldots, v_{n-1}$ must occur on such a path in order of their indices. The smallest index $i$ such that $v_n \in [w_i]_{\alpha_i}$ is $k$. Thus, all worlds $v_0, \ldots, v_{n-1}$ occur in equivalence classes $[w_i]_{\alpha_i}$ with $i \leq k$.

*(3)*: follows from $\gamma_n = \gamma_{n-1} \setminus \beta_n$ and the fact that $a_n \in \gamma_{n-1} \cap \beta_n$.

First of all, property (3) implies that the four constructed sequences have a finite length $k$ since there are only finitely many agents. We claim $d_t(w, v_k) > \ell$:

There cannot be an inner coset path that avoids $t$ of length $< \ell$ from $w$ to $v_k$ because that would imply an inner coset path from $w$ to $v$ that avoids $t$ of length $< \ell$ by property (2), which cannot exist by assumption. Hence, for the sake of contradiction, we assume that there is an inner coset path of length $\ell$

$$w_1, \alpha_1, w_2, \ldots, w_\ell, \alpha_\ell, w_{\ell+1}$$

from $w = w_1$ to $v_k = w_{\ell+1}$ that avoids $t$. Again, property (2) implies that $v$ occurs somewhere on this path. Furthermore, the smallest index $i$ such that $v \in [w_i]_{\alpha_i}$ must be $\ell$, otherwise there would be an inner coset path from $w$ to $v$ that avoids $t$ of length $< \ell$. In particular, $v = v_0 \in [w_\ell]_{\alpha_\ell} \setminus [w_{\ell-1}]_{\alpha_{\ell-1}}$. Property (2) states that all worlds $v_1, \ldots, v_k$ must occur after $v_0$ on all short inner coset paths from $w$ to $v_k$ that avoid $t$, hence $v_i \in [w_\ell]_{\alpha_\ell} \setminus [w_{\ell-1}]_{\alpha_{\ell-1}}$, for all $1 \leq i \leq k$. This implies $\bigcup_{i=0}^{k} \beta_i \subseteq \alpha_\ell$ because $\beta_i = \mathrm{short}_t(v_i, w)$, for all $0 \leq i \leq k$. Furthermore,

$$\emptyset = \gamma_k = \gamma_0 \setminus \bigcup_{i=1}^{k} \beta_i \qquad \Rightarrow \qquad \gamma_0 \subseteq \bigcup_{i=1}^{k} \beta_i.$$

Thus, together we obtain

$$\gamma = \beta_0 \cup \gamma_0 \subseteq \beta_0 \cup \bigcup_{i=1}^{k} \beta_i = \bigcup_{i=0}^{k} \beta_i \subseteq \alpha_\ell.$$

However, we also have $\gamma \nsubseteq \alpha_\ell$ because we assumed that the coset path $w_1, \alpha_1, \ldots, \alpha_\ell, w_{\ell+1}$ with $w_{\ell+1} = v_k$ avoids $t = \rho(v, \gamma) = \rho(v_k, \gamma)$, contradicting the assumption $d_t(w, v_k) \leq \ell$.

Furthermore, since each agent $a_i$, $1 \leq i \leq k$, is an element of $\gamma = \mathrm{agt}(z_1, v)$, and each $v_i$, $1 \leq i \leq k$, was chosen such that $\mathrm{agt}(w, v_i) = \mathrm{agt}(w, v)$ and $\mathrm{agt}(z, v_i) = \mathrm{agt}(z, v)$, for all $z \in \mathbf{z}$, $d_t(\mathbf{z}, v_i) > m$ and $\mathfrak{M}, v \sim \mathfrak{M}, v_i$, the world $v_k$ is the desired world $v^*$. □

This section concludes with the proof of the freeness theorem, the crucial tool for choosing suitable answers in the Ehrenfeucht-Fraïssé game on Cayley structures. The proof makes heavy use of the structure theory for $n$-acyclic Cayley structures we developed so far in this section. The main ingredients are Lemma 4.3.12 for the first step and Lemma 4.3.18 for the second step.

**Theorem 4.3.19** (Freeness theorem)**.** *Let $m, k \in \mathbb{N}$. If a Cayley structure $\mathfrak{M}$ is sufficiently acyclic and sufficiently rich, then $\mathfrak{M}$ is $(m,k)$-free.*

*Proof.* Let $v$ be a world, $(\mathbf{z}, z_0)$ a pointed set with $|\mathbf{z}| = k$ and an enumeration $(z_i)_{0 \leq i < k}$, and $\gamma \supseteq \mathrm{agt}(v, z_0)$. We show that there is a world $v^* \sim v$ with $\mathrm{agt}(v^*, z_0) = \gamma$ such that $v^*$ and $(\mathbf{z}, z_0)$ are $m$-free.

2-acyclicity and Lemma 4.1.4, together with sufficient richness imply the existence of some $v' \sim v$ with $\mathrm{agt}(v', z_0) = \gamma$. Replace $v$ by this world $v'$.

The next step is to make $v$ and $(\mathbf{z}, z_0)$ $m$-free: for $t := \rho(v, \gamma) = [\![v]\!] \cap [\![z_0]\!]$, we must find some world $v^* \sim v$ such that $d_t(\bigcup [\![\mathbf{z}]\!], [\![v^*]\!]) > m$. By Lemma 4.3.11 it suffices to show $d_t(\mathbf{z}, v^*) > m + 1$. We do this in two steps. Step 1 ensures $\mathrm{agt}(z_0, v^*) \subseteq \mathrm{agt}(z, v^*)$, for all $z \in \mathbf{z}$, or equivalently $d_t(\mathbf{z}, v^*) > 1$. Step 2 ensures $d_t(\mathbf{z}, v^*) > m + 1$.

*Step 1*: Through induction on $0 \leq j < k$ we find worlds $v_j \sim v$ such that $d_t(v_j, z_i) > 1$, for all $0 \leq i \leq j$. For $j = 0$, $v_0 := v$ works trivially. Let $j \geq 1$, and assume there is a world $v_{j-1} \sim v$ with $\mathrm{agt}(v_{j-1}, z_0) = \gamma$ such that $\mathrm{agt}(z_0, v_{j-1}) \subseteq \mathrm{agt}(z_i, v_{j-1})$ for all $0 \leq i < j$. With Lemma 4.3.12 at our disposal, the desired world can be obtained easily. We need to find a world $v_j$ with

- $v_j \sim v_{j-1}$,
- $\mathrm{agt}(v_j, z_0) = \mathrm{agt}(v_{j-1}, z_0)$, and
- $\mathrm{agt}(v_j, z_i) = \mathrm{agt}(v_{j-1}, z_i)$, for all $0 \leq i < j$

such that $\mathrm{agt}(v_j, z_j) \supseteq \mathrm{agt}(v_j, z_0)$. The induction hypothesis implies

- $\mathrm{agt}(v_{j-1}, z_i) \supseteq \mathrm{agt}(v_{j-1}, z_0)$, for all $0 \leq i < j$, and

2-acyclicity implies

- $\mathrm{agt}(z_0, z_i) \subseteq \mathrm{agt}(v_{j-1}, z_i) \cup \mathrm{agt}(v_{j-1}, z_0)$.

Hence, $\mathrm{agt}(v_{j-1}, z_i) = \mathrm{agt}(v_{j-1}, z_0) \cup \mathrm{agt}(z_0, z_i)$, for all $0 \leq i < j$. This means that all premises of Lemma 4.3.12 are satisfied and it yields a suitable world $v_j$ with

$$\mathrm{agt}(v_j, z_j) = \mathrm{agt}(v_j, z_0) \cup \mathrm{agt}(z_0, z_j) \quad \Rightarrow \quad \mathrm{agt}(v_j, z_j) \supseteq \mathrm{agt}(v_j, z_0).$$

Thus, by induction we obtain a world $v_{k-1} \sim v$ such that

- $\mathrm{agt}(v_{k-1}, z_0) = \gamma$ and
- $\mathrm{agt}(z_0, v_{k-1}) \subseteq \mathrm{agt}(z_i, v_{k-1})$, for all $0 \leq i \leq k - 1$.

Set the new $v$ to be $v_{k-1}$.

*Step 2:* Step 1 established $d_t(\mathbf{z}, v) > 1$. The next step is to find worlds $v_i \in [v]_\gamma$, $0 \leq i \leq k$, inductively such that

- $v_i \sim v$,

- $\mathrm{agt}(v_i, z_j) = \mathrm{agt}(v, z_j)$, for all $0 \leq j < i$, and
- $d_t(z_j, v_i) > m + 1$, for all $0 \leq j < i$.

Then the world $v_k$ will be bisimilar to the original $v$, $\mathrm{agt}(v_k, z_0) = \gamma$ and $v_k$ and $(\mathbf{z}, z_0)$ will be $m$-free.

Establishing $d_t(z_j, v_i) > m+1$ can be done by multiple applications of Lemma 4.3.18. Let $0 \leq i < k$ and assume there is some world $v_i \in [v]_\gamma$ with $v_i \sim v$ such that, for all $j < i$,

- $\mathrm{agt}(z_j, v_i) = \mathrm{agt}(z_j, v)$ and
- $d_t(z_j, v_i) > m + 1$.

Since $\gamma \subseteq \mathrm{agt}(z_j, v)$, for all $1 \leq j \leq i$, Lemma 4.3.18 implies a world $v_{i+1} \in [v_i]_\gamma = [v]_\gamma$ with

- $v_{i+1} \sim v_i$,
- $\mathrm{agt}(z_j, v_{i+1}) = \mathrm{agt}(z_j, v_i)$, for all $0 \leq j \leq i$, such that
- $d_t(z_j, v_{i+1}) > m + 1$, for all $0 \leq j \leq i$.

Since these assumptions are fulfilled by $v_0 = v$, applying Lemma 4.3.18 $k$ times yields the desired world $v^* = v_k$. □

# 5 The characterisation theorem

The main result of this work is a modal characterisation theorem for common knowledge logic ML[CK] over (finite) S5 structures. This chapter contains the final step of its proof. We described the strategy for the proof at the end of Chapter 2: if we can show that an FO-formula $\varphi$ that is $\sim$-invariant over (finite) CK structures is $\sim^\ell$-invariant over (finite) CK structures, for some $\ell \in \mathbb{N}$, then $\varphi$ must be equivalent to an ML-formula over (finite) CK structures by the modal Ehrenfeucht-Fraïssé theorem. This is done by upgrading $\ell$-bisimilarity to $\mathsf{FO}_q$-equivalence over (finite) Cayley structures, i.e. we show for suitable pointed Cayley structures $\mathfrak{M}, w$ and $\mathfrak{N}, v$ that

$$\mathfrak{M}, w \sim^\ell \mathfrak{N}, v \quad \Rightarrow \quad \mathfrak{M}, w \equiv_q \mathfrak{N}, v,$$

where $q$ is the quantifier rank of $\varphi$ and $\ell$ depends on $q$. Upgrading over Cayley structures suffices because we showed that Cayley structures are, up to bisimulation, the universal representatives of CK structures (cf. Theorem 3.2.19). For the upgrading, we regard a Cayley structure as suitable if it is $n$-acyclic and $k$-rich, for sufficiently large $n, k \in \mathbb{N}$ that depend on $q$. Constructing sufficiently acyclic and sufficiently rich (finite) coverings for (finite) CK structures is the first part of the upgrading argument. This was done in Chapter 3. The second part is showing that sufficiently acyclic and sufficiently rich $\ell$-bisimilar Cayley structures are also $\mathsf{FO}_q$-equivalent. The necessary structure theory for playing first-order Ehrenfeucht-Fraïssé games on the non-elementary class of Cayley structures was developed in Chapter 4. The central notion of that chapter is freeness, a special property of Cayley structures that plays a crucial role in the Ehrenfeucht-Fraïssé game on Cayley structures. The current chapter deals with the final step of the upgrading: applying the structure theory from Chapter 4 and proving $\mathsf{FO}_q$-equivalence.

As usual, in order to win the $q$-round Ehrenfeucht-Fraïssé game on $\mathfrak{M}, w$ and $\mathfrak{N}, v$ player **II** needs to maintain a certain invariant for the entire game. In our case, the invariant has to keep track of several things. First, there need to be isomorphic substructures $\mathfrak{M}_i \subseteq \mathfrak{M}$ and $\mathfrak{N}_i \subseteq \mathfrak{N}$, via some isomorphism $\sigma_i$, that include the worlds pebbled in the first $i$ rounds. But that alone does not suffice because the invariant needs to be set up in a way such that it can be maintained for $q$ rounds. For that reason, $\mathfrak{M}_i$ and $\mathfrak{N}_i$ must also include worlds that lie on short paths between two pebbled worlds $w_k$ and $w_j$, and $\sigma_i$ must map them to worlds that lie on short paths between the pebbled worlds $\sigma_i(w_k)$ and $\sigma_i(w_j)$. Additionally, for every world $w' \in \mathfrak{M}_i$, $\mathfrak{M}, w'$ and $\mathfrak{N}, \sigma_i(w')$ need to be $\ell_i$-bisimilar, for some $\ell_i \in \mathbb{N}$, to find suitable responses round after round. In particular, $\mathfrak{M}, w$ and $\mathfrak{N}, v$ need to be $\ell_0$-bisimilar initially, and we must show that all the $\ell_i$, $0 \leq i \leq q$, can be bounded in terms of $q$. Furthermore, in order to determine **II**'s next response, substructures of the dual hypergraphs $d(\mathfrak{M})$ and $d(\mathfrak{N})$ (cf. Section 3.2.3) that are essentially the dual images of $\mathfrak{M}_i$ and $\mathfrak{N}_i$ are also part of the invariant.

In Section 5.1, we show that the necessary degrees of bisimulation $\ell_i$ can be bounded in terms of $q$. We introduce the notion of the convex closure in a hypergraph, and use results from [26] to show how the convex closure can be applied to bound the $\ell_i$. In Section 5.2, we present the invariant formally and show how to maintain it from round to round. Section 5.3 contains the proof of the upgrading and the characterisation theorem. The final section shows how the methods developed up to that point can be applied in a more general setting to characterise relativized common knowledge.

As in the previous chapters, we fix a finite set of agents $\Gamma$, which labels the accessibility relations $(R_a)_{a\in\Gamma}$, and some finite index set $I$, which labels the atomic propositions $(P_i)_{i\in I}$. The set of all sets of agents with respect to $\Gamma$, i.e. $\mathcal{P}(\Gamma)$, is denoted by $\tau$. We regard S5 structures without accessibility relations that respond to coalitions of multiple agents as Kripke structures over the modal signature $\{(R_a)_{a\in\Gamma}, (P_i)_{i\in I}\}$, and we regard Cayley structures as Kripke structures over the modal signature $\{(R_\alpha)_{\alpha\in\tau}, (P_i)_{i\in I}\}$. We denote Kripke structures by $\mathfrak{M}$ or $\mathfrak{N}$ and their sets of possible worlds by $W$ and $V$, respectively.

## 5.1 Convex closure

Matching short distances is not the real problem in Ehrenfeucht-Fraïssé games on Cayley structures. However, we needed the structure theory from Chapter 4 to make explicit what we mean by short distances in these structures. This is not obvious because two different worlds have, in some sense, always distance 1 since they are connected by a $\Gamma$-edge. To obtain a meaningful notion of distance between worlds $w$ and $v$ we used the minimal connecting set of agents $\mathrm{agt}(w, v)$ that is unique in 2-acyclic structures. We say that the distance $d(w, v)$ between $w$ and $v$ is $m$ if $m$ is the length of a minimal non-trivial coset path from $w$ to $v$. Roughly speaking, we want to close the substructures $\mathfrak{M}_i$ and $\mathfrak{N}_i$ of our invariant under such paths.

In order to define the closure condition on $\mathfrak{M}_i$ and $\mathfrak{N}_i$ formally, we consider the correspondence between coset paths in Cayley structures and chordless paths in their dual hypergraphs, and use a tool called the *convex closure* from [26].

**Definition 5.1.1.** (Convex closure) Let $\mathcal{A} = (A, S)$ be a hypergraph.

1. A subset $Q \subseteq A$ is *$m$-closed* if every chordless path up to length $m$ between vertices $a, a' \in Q$ is contained in $Q$.

2. For $m \in \mathbb{N}$, the *convex $m$-closure* of a subset $P \subseteq A$ is the minimal $m$-closed subset that contains $P$:

$$\mathrm{cl}_m(P) := \bigcap\{Q \supseteq P \colon Q \subseteq A \ m\text{-closed } \}.$$

The convex $m$-closure of a set $P$ of vertices in a hypergraph is the minimal set that contains $P$ and is closed under chordless paths up to length $m$. Part of the invariant are auxiliary sets $Q_i \subseteq d(W)$, $0 \leq i \leq q$, that are $m_i$-closed, for some distance $m_i$, that is considered short in the $i$-th round. $Q_i$ can be roughly viewed as the dual image of $\mathfrak{M}_i$ in $d(\mathfrak{M})$.

If $w_1, \ldots, w_q$ are the worlds pebbled in $\mathfrak{M}$, then the invariant will be defined such that $Q_0$ is the singleton set that contains $[w]_\emptyset$, $Q_1$ is the $m_1$-closure of $\{[w]_\emptyset, [w_1]_\emptyset\}$, $Q_2$ is the $m_2$-closure $Q_1 \cup \{[w_2]_\emptyset\}$, and so forth. If **I** chooses to play in $\mathfrak{M}$ in the $i$-th round, we need to analyse the resulting set $Q_i = \mathrm{cl}_{m_i}(Q_{i-1} \cup \{[w_i]_\emptyset\})$, find out how $Q_{i-1}$ changes into $Q_i$ in order to find a suitable response $v_i \in V$ for **II**. This analysis will focus on two questions: what is the structure of the sub-hypergraph induced by $Q_i$ compared to the one induced by $Q_{i-1}$? And can we bound the size of $Q_i$ in terms of $q$? Answers to these questions go back to earlier work by Otto in [26]. The bound on the size of the sets $Q_i$ (under certain assumptions) is crucial in bounding the bisimulation degree $\ell$ such that $\mathfrak{M}, w \sim^\ell \mathfrak{N}, v$ implies $\mathfrak{M}, w \equiv_q \mathfrak{N}, v$.

In the Ehrenfeucht-Fraïssé game on $\mathfrak{M}$ and $\mathfrak{N}$, the auxiliary sets $Q_{i-1} \subseteq d(\mathfrak{M})$ will be chosen to be $2m_i + 1$-closed. Lemmas 5.1.2 and 5.1.3 show that in sufficiently acyclic hypergraphs, the addition of $[w_i]_\emptyset$ to $Q_{i-1}$ and closure under chordless paths of length $m_i$ updates $Q_{i-1}$ to $Q_i$ in a well-behaved manner. The following three lemmas are from [26]. Here $N^1(P) = \bigcup\{N^1(p) \colon p \in P\}$ refers to the 1-neighbourhood of the set $P$ in the Gaifman graph; distance $d(P, q) = \min\{d(p, q) \colon p \in P\}$ between a set and a vertex similarly refers to distance in the Gaifman graph.

**Lemma 5.1.2.** *Let $m > 1$, $\mathcal{A} = (A, S)$ be a hypergraph that is sufficiently acyclic, $Q \subseteq A$ $m$-closed and $a \in A$ some vertex with $1 \leq d(Q, a) \leq m$. Let $\hat{Q} := \mathrm{cl}_m(Q \cup \{a\})$ and consider the region in which this extended closure attaches to $Q$:*

$$D := Q \cap N^1(\hat{Q} \setminus Q).$$

*Then*

- *$\hat{Q} \setminus Q$ is connected, and*
- *$D$ separates $\hat{Q} \setminus Q$ from $Q \setminus D$, hence $\hat{Q} = Q \cup \mathrm{cl}_m(D \cup \{a\})$.*

Since we additionally assume that $Q_{i-1}$ is $2m_i + 1$-closed, we can employ the following lemma.

**Lemma 5.1.3.** *Let $\mathcal{A} = (A, S)$ be a hypergraph that is sufficiently acyclic, $Q \subseteq A$ $m$-closed, $a \in A$ some vertex with $1 \leq d(Q, a) \leq m$ and $\hat{Q} := \mathrm{cl}_m(Q \cup \{a\})$. If $Q$ is even $(2m + 1)$-closed, then*

$$D = Q \cap N^1(\hat{Q} \setminus Q)$$

*is a clique.*

If the set $Q_{i-1}$ is sufficiently small, then the sub-hypergraph $d(\mathfrak{M}) \restriction Q_{i-1}$ is fully acyclic and therefore tree decomposable. That the set $D$, from the lemmas above, is a clique means that it is contained in a single bag of the tree decomposition. This bag will be our starting point for finding **II**'s response in $\mathfrak{N}$ to **I**'s move to $w_i$ in $\mathfrak{M}$.

Now that we know $Q_i = Q_{i-1} \cup \mathrm{cl}_{m_i}(D \cup \{[w_i]_\emptyset\})$ ($[w_i]_\emptyset$ plays the role of $a$ from the lemmas above), for some clique $D \subseteq Q_{i-1}$, we want to obtain a bound on the size of the extension $\mathrm{cl}_{m_i}(D \cup \{[w']_\emptyset\})$ that can occur in a single round; such a bound is critical in bounding the required level of $\ell$-bisimulation. Since the dual hypergraphs have a uniform width of $|\tau|$, which we regard as constant, we seek functions $f_{m_i}(k)$ that bound the size of $m_i$-closures of sets or tuples of size $k$ in those hypergraphs, provided they are sufficiently acyclic.

**Lemma 5.1.4.** *For fixed width, there are functions $f_m(k)$ such that in hypergraphs $\mathcal{A}$ of that width that are sufficiently acyclic, $|\mathrm{cl}_m(P)| \leq f_m(k)$, for all $P \subseteq A$ of size $|P| \leq k$.*

The closures that we encounter in a single round of the game will be generated by at most $|\tau|+1$ vertices, a hyperedge (all hyperedges of dual hypergraphs are of size $|\tau|$) and an additional vertex. Hence, we can bound the size of these closures and therefore the size of the relevant sub-hypergraphs induced by the closures. In turn, this allows us to bound the bisimulation degree $\ell$ because the bound on the size of the sub-hypergraphs translates to a bound on the depth of their tree decompositions. Availability of suitable tree-like extensions of bounded depth, which can be used to cover the newly pebbled worlds, is controlled by bisimulation types of corresponding depth.

Additionally, Lemma 5.1.4 gives us important information about the degree of acyclicity $n$ needed to prove the upgrading theorem. First, $n$ must be large enough such that Lemma 5.1.4 is applicable. Keep in mind that a dual hypergraph is $n$-acyclic if its associated Cayley structure is (cf. Lemma 3.2.16). Second, $n$ must be large enough such that all sub-hypergraphs induced by the closed sets that we could encounter in the $q$-round game are fully acyclic, and hence tree decomposable. Most importantly, Lemma 5.1.4 implies that for all $q \in \mathbb{N}$ there is some finite $n$ that suffices to make all relevant sub-hypergraphs in the $q$-round game fully acyclic.

## 5.2 The invariant

### Definition

Player **II** wins a play in the $q$-round Ehrenfeucht-Fraïssé game on the pointed Kripke structures $\mathfrak{M}, w_0$ and $\mathfrak{N}, v_0$ if $w_i \mapsto v_i$, $0 \leq i \leq q$, induces a partial isomorphism $\sigma_q$, for the pebbled worlds $w_0, w_1, \ldots, w_q \in W$ and $v_0, v_1, \ldots, v_q \in V$. Starting with the mapping $\sigma_0 = \{w_0 \mapsto v_0\}$ before the first round is played, **II** extends $\sigma_i$, the partial isomorphism after the $i$-th round, from round to round. In order to be able to do that for the entire $q$ rounds, properly and in a foresighted manner, she needs to keep track of more information than just $\sigma_i$.

Two finite, decreasing sequences of natural numbers play an important role: the sequence $(m_i)_{0 \leq i \leq q}$ indicates that distances up to $m_i$ are considered short in the $i$-th round, and $(\ell_i)_{0 \leq i \leq q}$ is the degree of bisimilarity that two worlds $w' \in W$ and $v' \in V$ are supposed to have if $\sigma_i(w') = v'$ in round $i$, i.e. $\mathfrak{M}, w' \sim^{\ell_i} \mathfrak{N}, v'$. As usual in Ehrenfeucht-Fraïssé games, $m_i$ decreases by about half in every round:

- $m_q := 2$

- $m_{i-1} := 2m_i + 1$, for $q \geq i > 0$

The sequence $(\ell_i)_{0\leq i\leq q}$ depends on the function $f_m$ from Lemma 5.1.4 that bounds the size of $m$-closed sets:

- $\ell_q := 1$
- $\ell_{i-1} := \ell_i + f_{m_i}(|\tau| + 1)$, for $q \geq i > 0$

With these sequences at hand we can describe the rest of the invariant. Assume that after the $i$-th round the worlds $w_0, w_1, \ldots, w_i \in W$ and $v_0, v_1, \ldots, v_i \in V$ are pebbled. Player **II** has to uphold the following invariant:

- two induced substructures $\mathfrak{M}_i \subseteq \mathfrak{M}$ and $\mathfrak{N}_i \subseteq \mathfrak{N}$;
- an isomorphism $\sigma_i \colon \mathfrak{M}_i \to \mathfrak{N}_i$;
- $m_i$-closed subsets $Q_i \subseteq d(W)$ and $Q'_i \subseteq d(V)$;
- isomorphic tree decompositions $\mathcal{T}_i$ and $\mathcal{T}'_i$ of $d(\mathfrak{M}) \restriction Q_i$ and $d(\mathfrak{N}) \restriction Q'_i$, respectively.

The induced substructures $\mathfrak{M}_i$ and $\mathfrak{N}_i$ contain the worlds that are pebbled in the $i$-th round, together with worlds that are added if one, essentially, closes these substructures under short coset paths. The isomorphism $\sigma_i$ maps the world $w_k$, which was pebbled in $\mathfrak{M}$ in the $k$-th round, to $v_k$, which was pebbled in the same round in $\mathfrak{N}$, for all $1 \leq k \leq i$; $\sigma_i$ also maps $w_0$ to $v_0$. Additionally, and important for maintaining the invariant, $\sigma_i$ maps worlds from $\mathfrak{M}_i$ to worlds from $\mathfrak{N}_i$ that are $\ell_i$-bisimilar in $\mathfrak{M}$ and $\mathfrak{N}$, i.e.

$$\mathfrak{M}, w' \sim^{\ell_i} \mathfrak{N}, \sigma_i(w'),$$

for all $w' \in \mathfrak{M}_i$. If player **II** is able to maintain these isomorphic substructures throughout the $q$-round game, she wins since $\mathfrak{M}_q$ and $\mathfrak{N}_q$ are induced substructures.

We choose the worlds that define $\mathfrak{M}_i$ and $\mathfrak{N}_i$, and determine how to extend $\sigma_i$ through the auxiliary sets $Q_i$ and $Q'_i$ in the dual hypergraphs and their associated, isomorphic tree decompositions $\mathcal{T}_i = (T_i, \delta_i)$ and $\mathcal{T}'_i = (T'_i, \delta'_i)$. No matter what moves player **I** makes, player **II** chooses her responses to maintain the following equivalences for $Q_i$ in $d(\mathfrak{M})$, $0 \leq i \leq q$, and analogously for the $Q'_i$ in $d(\mathfrak{N})$:

- $Q_0 = \{[w_0]_\emptyset\}$
- $Q_i = \mathrm{cl}_{m_i}(Q_{i-1} \cup \{[w_i]_\emptyset\})$, for $i \geq 1$

Since $\mathfrak{M}$ is sufficiently acyclic, so is the hypergraph $d(\mathfrak{M})$, which means that the substructures $d(\mathfrak{M}) \restriction Q_i$ are fully acyclic (cf. Lemma 5.1.4) and decomposable into a tree $\mathcal{T}_i = (T_i, \delta_i)$. Every bag $\delta_i(u)$, for $u \in V[T_i]$, is a hyperedge of $d(\mathfrak{M}) \restriction Q_i$ and with that a subset of some hyperedge $[\![w_u]\!]$ of $d(\mathfrak{M})$. Hence, for every vertex of the tree $u \in V[T_i]$, we can choose some world $w_u \in W$ such that $\delta_i(u) \subseteq [\![w_u]\!]$ and define a mapping

$$\hat{\delta}_i \colon V[T_i] \to W,\ u \mapsto w_u.$$

In general, the choice for $\hat{\delta}_i$ is not unique, but that does not create any problems. However, if a vertex $[w']_\emptyset$ is in $\delta_i(u)$, then $[\![w']\!]$ is the only hyperedge of $d(\mathfrak{M})$ that is a superset of $\delta_i(u)$ because it is the only one that contains $[w']_\emptyset$. Since for every pebbled world $w_i$ the vertex $[w_i]_\emptyset$ is an element of $Q_i$, every $w_i$ must be in the image of $\hat{\delta}_i$. Therefore, we define $\mathfrak{M}_i$ as the substructure of $\mathfrak{M}$ that is induced by the set of worlds $\mathrm{im}(\hat{\delta}_i)$. We can regard $\mathfrak{M}_i$ as a representation of $\mathcal{T}_i$ in $\mathfrak{M}$. The isomorphism $\sigma_i \colon \mathfrak{M}_i \to \mathfrak{N}_i$ is defined by

$$\sigma_i(\hat{\delta}(u)) := \hat{\delta}'(u), \text{ for all } u \in V[T_i].$$

The challenges for **II** lie in choosing her responses to **I**'s moves in such a way that all the conditions described above are fulfilled for both structures, with the addition that $d(\mathfrak{M}) \upharpoonright Q_i$ and $d(\mathfrak{N}) \upharpoonright Q_i'$ must be isomorphic. That means, if **I** chooses $w_i$ in $\mathfrak{M}$ in the $i$-th round, we define $Q_i := \mathrm{cl}_{m_i}(Q_{i-1} \cup \{[w_i]_\emptyset\})$. But how does **II** find a suitable $v_i$ in $\mathfrak{N}$ such that $Q_i' = \mathrm{cl}_{m_i}(Q_{i-1}' \cup \{[v_i]_\emptyset\})$, $d(\mathfrak{M}) \upharpoonright Q_i \simeq d(\mathfrak{N}) \upharpoonright Q_i'$ and $\sigma_i$ fulfils the bisimilarity condition? This can be accomplished by analysing $Q_i$ and copying the tree decomposition $\mathcal{T}_i$ vertex by vertex to $d(\mathfrak{N})$. The following section presents a detailed discussion.

### Preserving the invariant

Assume we play the $i$-th round of an Ehrenfeucht-Fraïssé game on sufficiently acyclic and sufficiently rich pointed Cayley structures $\mathfrak{M}, w_0$ and $\mathfrak{N}, v_0$. So far, the worlds $w_1, \dots, w_{i-1} \in W$ and $v_1, \dots, v_{i-1} \in V$ have been pebbled and player **II** was able to maintain the invariant defined above, i.e. there are

- induced, isomorphic substructures $\sigma_{i-1} \colon \mathfrak{M}_{i-1} \simeq \mathfrak{N}_{i-1}$,
- $m_{i-1}$-closed subsets $Q_{i-1} \subseteq d(W)$ and $Q_{i-1}' \subseteq d(V)$,
- isomorphic tree decompositions $\mathcal{T}_{i-1} = (T_{i-1}, \delta_{i-1})$ and $\mathcal{T}_{i-1}' = (T_{i-1}', \delta_{i-1}')$ of the isomorphic substructures $d(\mathfrak{M}) \upharpoonright Q_{i-1}$ and $d(\mathfrak{N}) \upharpoonright Q_{i-1}'$, respectively,

such that

- $w_0, w_1, \dots, w_{i-1} \in \mathfrak{M}_{i-1}$ and $v_0, v_1, \dots, v_{i-1} \in \mathfrak{N}_{i-1}$;
- $\mathfrak{M}_{i-1}$ is induced by $\mathrm{im}(\hat{\delta}_{i-1})$ and $\mathfrak{N}_{i-1}$ is induced by $\mathrm{im}(\hat{\delta}_{i-1}')$, for some arbitrary, but fixed $\hat{\delta}_{i-1} \colon V[T_{i-1}] \to W$ and $\hat{\delta}_{i-1}' \colon V[T_{i-1}] \to V$ such that
  - $\delta_{i-1}(u) \subseteq [\![\hat{\delta}_{i-1}(u)]\!]$, for all $u \in V[T_{i-1}]$, and
  - $\delta_{i-1}'(u) \subseteq [\![\hat{\delta}_{i-1}'(u)]\!]$, for all $u \in V[T_{i-1}]$;
- $\sigma_{i-1}(\hat{\delta}(u)) = \hat{\delta}'(u)$, for all $u \in V[T_{i-1}]$;
- $\mathfrak{M}, w' \sim^{\ell_{i-1}} \mathfrak{N}, \sigma_{i-1}(w')$, for all $w' \in \mathfrak{M}_{i-1}$.

W.l.o.g. player **I** chooses for his move in the $i$-th round a world $w_i \in W$, which is not contained in $\mathfrak{M}_{i-1}$. It is player **II**'s objective to find some $v_i \in V$ such that the invariant still holds after the $i$-th round. On the side of $\mathfrak{M}$, the changes from $Q_{i-1}$ to $Q_i$ and $\mathfrak{M}_{i-1}$ to $\mathfrak{M}_i$ etc. are determined by the move of player **I**. On the side of $\mathfrak{N}$, player **II** needs to copy these changes in the same way to maintain the isomorphisms and closure properties. The key to this are the lemmas on convex closures from Section 5.1.

We start with analysing $Q_i := \mathrm{cl}_{m_i}(Q_{i-1} \cup \{[w_i]_\emptyset\})$. First, since $w_i$ is not an element of $\mathfrak{M}_{i-1}$, the vertex $[w_i]_\emptyset$ cannot be in $Q_{i-1}$. Second, the distance between two vertices in the dual hypergraph is always at most 2, hence $1 \leq d(Q_{i-1}, [w_i]_\emptyset) \leq m_i$. Third, $d(\mathfrak{M})$ is sufficiently acyclic because $\mathfrak{M}$ is. Hence, Lemma 5.1.2 can be applied. Thus, if $D := Q_{i-1} \cap N^1(Q_i \setminus Q_{i-1})$ is the region in which the extended closure attaches to $Q_{i-1}$, then $Q_i \setminus Q_{i-1}$ is connected, and $D$ separates $Q_i \setminus Q_{i-1}$ from $Q_{i-1} \setminus D$, respectively $Q_i = Q_{i-1} \cup \mathrm{cl}_{m_i}(D \cup \{[w_i]_\emptyset\})$. Furthermore, $D$ is a clique since $Q_{i-1}$ is $(2m_i+1)$-closed ($m_{i-1} = 2m_i + 1$) by Lemma 5.1.3, and the size of $\mathrm{cl}_{m_i}(D \cup \{[w_i]_\emptyset\})$ is bounded by $f_{m_i}(|\tau|+1)$ (Lemma 5.1.4), which means that $d(\mathfrak{M}) \upharpoonright Q_i$ is also tree decomposable. Let $\mathcal{T}_i = (T_i, \delta_i)$ be a tree decomposition of $d(\mathfrak{M}) \upharpoonright Q_i$ that extends $\mathcal{T}_{i-1} = (T_{i-1}, \delta_{i-1})$, and $\hat{\delta}_i \colon V[T_i] \to \mathfrak{M}$ be a mapping with $\delta_i(u) \subseteq [\![\hat{\delta}_i(u)]\!]$, for all $u \in V[T_i]$, that extends $\hat{\delta}_{i-1}$. Such extensions exist because $Q_i = Q_{i-1} \cup \mathrm{cl}_{m_i}(D \cup \{[w_i]_\emptyset\})$ and $D$ is a clique.

Let $Q := (Q_i \setminus Q_{i-1}) \cup D$ be the new part of $Q_i$ including the region of attachment, and let $\mathcal{T} = (T, \delta)$ be a tree decomposition of $d(\mathfrak{M}) \upharpoonright Q$ that is a restriction of $\mathcal{T}_i$, with an associated mapping $\hat{\delta} \colon V[T] \to \mathfrak{M}$. The set $D$ has an isomorphic image in $d(\mathfrak{N})$ by assumption. Starting from this image, we need to find a suitable isomorphic image for the rest of $Q$. We do this through an induction on the structure of the tree $T$. Starting with a child of the root, we add vertex after vertex in a breadth-first manner. In other words, we find a suitable extension of $\mathcal{T}'_{i-1}$.

In order to make the extension suitable, we describe $\mathcal{T}$ by an $\mathsf{ML}$-formula. We make use of the fact that in dual hypergraphs of 2-acyclic Cayley structures the set of equivalence classes $[\![u]\!] \cap [\![u']\!]$ is fully determined by the set of agents $\mathrm{agt}(u, u')$ (cf. Lemma 4.1.2), i.e.

$$[\![u]\!] \cap [\![u']\!] = \{[u]_\beta : \beta \supseteq \mathrm{agt}(u, u')\} = \{[u']_\beta : \beta \supseteq \mathrm{agt}(u, u')\}.$$

This allows for describing the overlap between two bags of the tree decomposition by a single set of agents. We will follow this description to find a suitable bisimilar image of $\mathrm{im}(\hat{\delta})$ in $\mathfrak{N}$ in order to extend $\mathfrak{N}_{i-1}$ to $\mathfrak{N}_i$.

Let $w_u := \hat{\delta}(u)$, for $u \in V[T]$, and $\lambda \in V[T]$ be the vertex with $D \subseteq \delta(\lambda)$. We regard $\lambda$ as the root of $T$. We describe the finite substructure $\mathfrak{M}{\upharpoonright}\mathrm{im}(\hat{\delta}), w_\lambda$ by a formula $\varphi_{\mathcal{T}} := \varphi_{\mathcal{T},\lambda} \in \mathsf{ML}$ of modal depth $\ell_{i-1}$. For every vertex $u \in V[T]$ we define a formula $\varphi_{\mathcal{T},u}$ by induction on the depth of $u$ in $T$:

- For a leaf $u$, let $\varphi_{\mathcal{T},u}$ be the formula of modal depth $\ell_i$ that describes the $\ell_i$-bisimulation type of $w_u$.
- For a non-leaf $u$ with children $u_1, \dots, u_k$ and their associated formulae $\varphi_{\mathcal{T},u_j}$, let $\alpha_j = \mathrm{agt}(w_u, w_{u_j})$. Let $\chi \in \mathsf{ML}_{\ell_i}$ be the formula that describes the $\ell_i$-bisimulation type of $w_u$, then
$$\varphi_{\mathcal{T},u} := \chi \wedge \textstyle\bigwedge_{1 \leq j \leq k} \Diamond_{\alpha_j} \varphi_{\mathcal{T},u_j}.$$

Note that the modal nesting depth of $\varphi_{\mathcal{T}}$ is uniformly bounded by $\ell_i$ plus the depth of $T$, which in turn is bounded by $f_{m_i}(|\tau|+1)$, the size of the relevant $m_i$-closure. For this reason the value $\ell_{i-1}$, the degree of bisimulation that needs to be respected by $\sigma_{i-1}\colon \mathfrak{M}_{i-1} \simeq \mathfrak{N}_{i-1}$, can be chosen such that in round $i$ of the game, $\varphi_{\mathcal{T}}$ is preserved by $\sigma_{i-1}$. Furthermore, by adding a subformula $\chi$ that describes the $\ell_i$-bisimulation type of every world $w_u$, we make sure that, after adding all the necessary worlds to $\mathfrak{N}_{i-1}$ to define $\mathfrak{N}_i$, starting at $\sigma_{i-1}(w_\lambda)$, the resulting isomorphism $\sigma_i\colon \mathfrak{M}_i \to \mathfrak{N}_i$ maps worlds from $\mathfrak{M}$ to $\ell_i$-bisimilar worlds from $\mathfrak{N}$, i.e. $\mathfrak{M}, w' \sim^{\ell_i} \mathfrak{N}, \sigma_i(w')$, for all $w' \in \mathfrak{M}_i$. This works with a formula $\varphi_T$ of modal depth $\ell_i + f_{m_i}(|\tau|+1)$ because the depth of the tree is bounded by $f_{m_i}(|\tau|+1)$, which means that we only need to take at most $f_{m_i}(|\tau|+1)$ steps away from $\sigma_{i-1}(w_\lambda)$ in $\mathfrak{N}$ to construct the extension.

How does this extension work exactly? $\mathfrak{M}, w_\lambda \models \varphi_{\mathcal{T}}$ is true by construction of $\varphi_{\mathcal{T}}$, which implies $\mathfrak{N}, v_\lambda \models \varphi_{\mathcal{T}}$, for $v_\lambda := \sigma_{i-1}(w_\lambda)$, because $\mathrm{md}(\varphi_{\mathcal{T}}) \leq \ell_{i-1}$. Eventually, this will allow us to expand $\mathfrak{N}_{i-1}$ to $\mathfrak{N}_i$ to keep the invariant alive. Starting from the root $\lambda$, we add for every child $u$ of $\lambda$ a suitable world $v_u \in V$ with $\mathfrak{N}, v_u \sim^{\ell_{i-1}-1} \mathfrak{M}, w_u$ to $\mathfrak{N}_{i-1}$, and extend $Q'_{i-1}$, $\mathcal{T}'$ and $\hat{\delta}'$ accordingly:

- for every equivalence class $[w_u]_\beta \in Q_i$, we add the class $[v_u]_\beta$ to $Q'_{i-1}$,
- add $u$ as a child of $\lambda$ in $T'_{i-1}$ (remember $T_{i-1} = T'_{i-1}$),
- extend $\delta'_{i-1}$ by a bag for $u$ that contains the newly added vertices, and
- extend $\hat{\delta}'_{i-1}$ by $u \mapsto [\![v_u]\!]$.

However, we cannot choose any $v_u$ that is $\ell_i$-bisimilar to $w_u$ because that might violate the $m_i$-closure condition on $Q'_i$ and lead to a substructure $\mathfrak{N}_i$ that is not isomorphic to $\mathfrak{M}_i$. Consider a subformula $\Diamond_\alpha \varphi_{\mathcal{T},u}$ of $\varphi_{\mathcal{T}}$ with $\alpha = \mathrm{agt}(w_\lambda, w_u)$. Since $\mathfrak{N}, v_\lambda \models \Diamond_\alpha \varphi_{\mathcal{T},u}$ and $\mathfrak{M}, w_\lambda \sim^{\ell_{i-1}} \mathfrak{N}, v_\lambda$, there must be some $v_u \in [v_\lambda]_\alpha$ with $\mathfrak{N}, v_u \sim^{\ell_i} \mathfrak{M}, w_u$ such that $\mathfrak{N}, v_u \models \varphi_{\mathcal{T},u}$. This $v_u$ might not be a suitable choice for several reasons.

First, $\mathrm{agt}(v_\lambda, v_u) \subsetneq \mathrm{agt}(w_\lambda, w_u)$ could be the case, and if not, there could still be some other $s \in V[T_{i-1}]$ such that $\mathrm{agt}(v_s, v_u) \subsetneq \mathrm{agt}(w_s, w_u)$. This would violate the isomorphism condition on $\sigma_i$ immediately, because it implies an $\mathrm{agt}(v_s, v_u)$-edge between $\sigma_i(w_s) = v_s$ and $\sigma_i(w_u) = v_u$ in $\mathfrak{N}$ that does not exist between $w_s$ and $w_u$ in $\mathfrak{M}$. Therefore, we would like to have some world $v_u$ such that

$$\mathrm{agt}(v_s, v_u) = \mathrm{agt}(w_s, w_u),$$

for all $s \in V[T_{i-1}]$, and the isomorphism condition on $\sigma_i$ is respected for at least another round. It remains to be proven that we can make such a choice. And there might be a second problem with $v_u$.

Ideally, we would not only like to respect the isomorphism condition for another round but all the way to end of the game. This is were the closure condition on the sets $Q_i$ and $Q'_i$ chimes in. $Q_i$ was defined to be $m_i$-closed in $d(\mathfrak{M})$, where we consider distances up to $m_i$ as short in the $i$-th round. We need to extend $Q'_{i-1}$ to $Q'_i$ such that $Q'_i$ will be $m_i$-closed too, and such that the induced sub-hypergraphs $d(\mathfrak{M}) \restriction Q_i$ and $d(\mathfrak{N}) \restriction Q'_i$

are isomorphic. Essentially, $Q_i$ and $Q'_i$ are $m_i$-closed so that player **I** cannot exploit short distances that exist between two pebbled elements in one structure but not in the other. Closing both sets under chordless paths of length $m_i$ entails that short distances are matched exactly and long distances are matched with long distances.

To make this problem more explicit, we return to $v_u$: assume we found a world $v_u$ such that $\operatorname{agt}(v_s, v_u) = \operatorname{agt}(w_s, w_u)$, for all $s \in V[T_{i-1}]$. On the side of the dual hypergraph this translates to $[\![v_s]\!] \cap [\![v_u]\!] = [\![w_s]\!] \cap [\![w_u]\!]$, for all $s \in V[T_{i-1}]$. Furthermore, since $\mathcal{T}_i$ is a tree decomposition and $u$ is a child of $\lambda$, the bag $\delta_i(u)$ intersects all bags of $\mathcal{T}_{i-1}$ only within $\delta_i(\lambda)$, i.e. $\delta_i(s) \cap \delta_i(u) \subseteq \delta_i(\lambda) \cap \delta_i(u)$, for all $s \in V[T_{i-1}]$. Together with $Q_i$ being 2-closed this implies $[\![w_s]\!] \cap [\![w_u]\!] \subseteq [\![w_\lambda]\!] \cap [\![w_u]\!]$ which in turn implies $[\![v_s]\!] \cap [\![v_u]\!] \subseteq [\![v_\lambda]\!] \cap [\![v_u]\!]$, for all $s \in V[T_{i-1}]$. So far so good. The next thing to do would be to add a vertex $[v_u]_\beta$, $\beta \in \tau$, to $Q'_i$ if and only if $[w_u]_\beta \in Q_i$. However, this might result in a set that is *not* $m_i$-closed. Since $Q_i$ is $m_i$-closed, there are no short paths of length up to $m_i$ from $Q_{i-1}$ to $\delta_i(u) \setminus \delta_i(\lambda)$ that leave $Q_i$. In other words, if $x_1, x_2 \ldots, x_\ell$ is a short path in $d(\mathfrak{M})$ from $Q_{i-1}$ to $\delta_i(u)$, then $x_i \in Q_i$, for all $1 \le i \le \ell$. In particular, such a short path must necessarily go through $\delta_i(u) \cap \delta_i(\lambda)$ since $\mathcal{T}_i$ is a tree decomposition, i.e. there is some $1 \le i \le \ell$ such that $x_i \in \delta_i(u) \cap \delta_i(\lambda)$. Hence, for $t = \delta_i(u) \cap \delta_i(\lambda)$, the distance between $\delta_i(w_u) \setminus t$ and $Q_{i-1} \setminus t$ in $d(\mathfrak{M}) \setminus t$ is greater than $m_i$, in short $d_t(\delta_i(w_u), Q_{i-1}) > m_i$. We need to transfer this situation to $\mathfrak{N}$.

The key to overcoming both obstacles is, of course, freeness (cf. Definition 4.3.1). Since we assumed $\mathfrak{M}$ and $\mathfrak{N}$ to be sufficiently acyclic and sufficiently rich, the freeness theorem (Theorem 4.3.19) states that both structures are sufficiently free. Let $v'$ be some world in $[v_\lambda]_\alpha$ that is $\ell_i$-bisimilar to $w_u$, and define $\mathbf{z} := \operatorname{im}(\hat{\delta}_{i-1}) = \{v_s : s \in V[T_{i-1}]\}$. Then freeness of $\mathfrak{N}$ implies that there is some $v_u \sim v'$ such that

- $\operatorname{agt}(v_\lambda, v_u) = \alpha = \operatorname{agt}(w_\lambda, w_u)$, and
- $(\mathbf{z}, v_\lambda) \perp_{m_i} v_u$, i.e. $(\mathbf{z}, v_\lambda)$ and $v_u$ are $m_i$-free.

This world $v_u$ is a suitable choice for extending $\mathfrak{N}_{i-1}$: for $t = [\![v_\lambda]\!] \cap [\![v_u]\!]$,

$$(\mathbf{z}, v_\lambda) \perp_{m_i} v_u \quad \Rightarrow \quad d_t([\![v_u]\!], \bigcup [\![\mathbf{z}]\!]) > m_i.$$

If we set $\delta'_i(u) := \{[v_u]_\beta : [w_u]_\beta \in \delta_i(u)\}$, then

$$d_t([\![v_u]\!], \bigcup [\![\mathbf{z}]\!]) > m_i \quad \Rightarrow \quad d_t(\delta'_i(u), \bigcup \operatorname{im}(\delta'_{i-1})) > m_i$$

because $\delta'_i(u) \subseteq [\![v_u]\!]$ and $\bigcup \operatorname{im}(\delta'_{i-1}) \subseteq \bigcup [\![\mathbf{z}]\!]$, which implies that the set $\bigcup \operatorname{im}(\delta'_{i-1}) \cup \delta'_i(u)$ is $m_i$-closed. Furthermore,

$$\begin{aligned} & d_t([\![v_u]\!], \bigcup [\![\mathbf{z}]\!]) > 1 \\ \Rightarrow \quad & [\![z]\!] \cap [\![v_u]\!] \subseteq [\![v_\lambda]\!] \cap [\![v_u]\!], \text{ for all } z \in \mathbf{z}, \\ \Rightarrow \quad & \operatorname{agt}(z, v_u) \supseteq \operatorname{agt}(v_\lambda, v_u), \text{ for all } z \in \mathbf{z}. \end{aligned}$$

Together with $\operatorname{agt}(v_\lambda, v_s) = \operatorname{agt}(w_\lambda, w_s)$, for all $s \in V[T_{i-1}]$, and $\operatorname{agt}(v_\lambda, v_u) = \operatorname{agt}(w_\lambda, w_u)$ we obtain, for all $s \in V[T_{i-1}]$,

$$\operatorname{agt}(v_u, v_s) = \operatorname{agt}(w_u, w_s),$$

which means that defining $\sigma_i(w_u) = v_u$ preserves the isomorphism condition on $\sigma_i$. Additionally, adding $\{[v_u]_\beta : [w_u]_\beta \in \delta_i(u)\}$ to $Q'_{i-1}$ as a first step in defining $Q'_i$ preserves the isomorphism condition on $d(\mathfrak{M}) \restriction Q_i$ and $d(\mathfrak{N}) \restriction Q'_i$, at least for the part of $Q'_i$ that has been defined so far. Thus, we have shown that we can find a counterpart for $w_u$ in $\mathfrak{N}$ such that invariant is preserved in every way.

The remainder of the tree $T$ is treated in the same way in a breadth-first manner. All the vertices processed so far are added to $\mathbf{z}$ and the distinguished world of the pointed set $(\mathbf{z}, z_0)$, which is the set to be extended in the current step, is the world that is associated with the father of the vertex that is currently processed. The freeness argument works for the whole tree $T$, vertex by vertex, because we could bound the size of $Q$ in terms of $q$ and, essentially, assumed $\mathfrak{M}$ and $\mathfrak{N}$ to be sufficiently free. Adding worlds to $\mathfrak{N}_{i-1}$ for every vertex of $T$ that is not the root $\lambda$ to define $\mathfrak{N}_i$ gives us a suitable response for $w_i$, **I**'s move in the $i$-th round, in $\mathfrak{N}$. Since $T$ has at most depth $f_{m_i}(|\tau|+1)$ and $\mathfrak{M}, w_\lambda \sim^{\ell_{i-1}} \mathfrak{N}, v_\lambda$, for the associated isomorphism $\sigma_i$ holds

$$\mathfrak{M}, w' \sim^{\ell_i} \mathfrak{N}, \sigma_i(w'),$$

for all $w' \in \mathfrak{M}_i$ because $\ell_{i-1} = \ell_i + f_{m_i}(|\tau|+1)$. Thus, player **II** is able to preserve the invariant in the $i$-th round. The following lemma summarises this entire section.

**Lemma 5.2.1.** *Let $q \in \mathbb{N}$, and $\mathfrak{M}, w_0$ and $\mathfrak{N}, v_0$ be pointed Cayley structures that are sufficiently acyclic and sufficiently rich. If the invariant described in Section 5.2 is true in the $(i-1)$-th round of the $q$-round Ehrenfeucht-Fraïssé game on $\mathfrak{M}, w_0$ and $\mathfrak{N}, v_0$, then player **II** has a strategy to preserve this invariant in the $i$-th round.*

## 5.3 Upgrading and characterisation

This section can be regarded as the culmination of all the work done so far: the upgrading theorem and the characterisation of basic modal logic over (finite) Cayley structures. Chapters 3, 4 and the previous sections of the current chapter all contain different building blocks for proving those two theorems. In Chapter 3, we showed that every (finite) CK structure can be covered by a (finite) Cayley structure that is arbitrarily acyclic and arbitrarily rich. In particular, Cayley structures are up to bisimulation the universal representatives of CK structures. Cayley structures were further investigated in Chapter 4. Its main result is the freeness theorem, which states that sufficient acyclicity and sufficient richness imply $(m, k)$-freeness, a special property of Cayley structures that is essential for the upgrading. Finally, the previous sections of the current chapter describe an invariant for player **II** in the Ehrenfeucht-Fraïssé game on suitable pointed Cayley structures and how to preserve it using freeness. The upgrading theorem follows easily from that.

**Theorem 5.3.1** (Upgrading theorem)**.** *Let $q \in \mathbb{N}$. For some suitable choice of $\ell = \ell(q)$, any sufficiently acyclic and sufficiently rich Cayley structures $\mathfrak{M}$ and $\mathfrak{N}$ satisfy:*

$$\mathfrak{M}, w \sim^{\ell} \mathfrak{N}, v \quad \Rightarrow \quad \mathfrak{M}, w \equiv_q \mathfrak{N}, v$$

*Proof.* Let $(\ell_k)_{0\le k\le q}$ be the sequence of the same name from Section 5.2. Set $\ell := \ell_0$ and let $\mathfrak{M}, w$, $\mathfrak{N}, v$ be two sufficiently acyclic and sufficiently rich pointed Cayley structures such that $\mathfrak{M}, w \sim^\ell \mathfrak{N}, v$. In order to prove that these structures are $\mathsf{FO}_q$-equivalent we provide a winning strategy for player **II** in the $q$-round Ehrenfeucht-Fraïssé game on $\mathfrak{M}, w$ and $\mathfrak{N}, v$. Her strategy is to preserve the invariant from Section 5.2.

Before any pebbles have been placed, the invariant holds: $\mathfrak{M}, w \sim^\ell \mathfrak{N}, v$ implies that the substructures induced by $\{w\}$ and $\{v\}$, respectively, are isomorphic via $\sigma_0 = \{w \mapsto v\}$ because $w$ and $v$ are atomically equivalent and all accessibility relations are reflexive. Additionally, $\sigma_0$ respects the condition $\mathfrak{M}, w \sim^{\ell_0} \mathfrak{N}, \sigma_0(w)$. The sets $Q_0 = \{[w]_\emptyset\}$ and $Q_0' = \{[v]_\emptyset\}$ in the dual hypergraphs are $m_0$-closed because the Cayley structures, and with them their dual hypergraphs (cf. Lemma 3.2.16), are sufficiently acyclic. The induced sub-hypergraphs $d(\mathfrak{M}) \restriction Q_0$ and $d(\mathfrak{N}) \restriction Q_0'$ are, of course, isomorphic and tree decomposable.

Lemma 5.2.1 states that **II** can preserve the invariant round after round for the entire game. Thus, in the end, no matter what choices player **I** makes, there are induced substructures $\mathfrak{M}_q$ and $\mathfrak{N}_q$ and an isomorphism $\sigma_q \colon \mathfrak{M}_q \to \mathfrak{N}_q$ such that $\sigma_q(w_i) = v_i$, for the worlds $w_i$ and $v_i$ pebbled in the $i$-th round, which means player **II** wins. □

The upgrading theorem, together with the existence of suitable bisimilar coverings, implies the main theorem of this thesis: the characterisation of $\mathsf{ML}$ over (finite) Cayley structures.

**Theorem 5.3.2.** *Over the class of (finite) Cayley structures, and hence over the class of (finite) CK-structures:*

$$\mathsf{ML}[\mathsf{CK}] \equiv \mathsf{ML} \equiv \mathsf{FO}/{\sim}$$

*Proof.* The standard translation (cf. Section 2.3) implies $\mathsf{ML} \subseteq \mathsf{FO}/{\sim}$. For the other direction, let $\varphi$ be an $\mathsf{FO}$-formula with $\mathrm{qr}(\varphi) = q$ that is bisimulation-invariant over (finite) Cayley structures. If we can show that $\varphi$ is $\sim^\ell$-invariant, for some $\ell \in \mathbb{N}$, over (finite) Cayley structures, there is an $\mathsf{ML}$ formula of modal depth $\ell$ that is logically equivalent to $\varphi$ over (finite) Cayley structures (cf. Theorem 2.2.3).

We choose the $\ell = \ell(q)$ from the upgrading theorem and let $\mathfrak{M}, w$ and $\mathfrak{N}, v$ be pointed Cayley structures that are $\ell$-bisimilar (compare Figure 5.1). By Lemmas 3.2.22 and 3.2.24 there are bisimilar coverings $\hat{\mathfrak{M}}, \hat{w} \sim \mathfrak{M}, w$ and $\hat{\mathfrak{N}}, \hat{v} \sim \mathfrak{N}, v$ that are sufficiently acyclic and sufficiently rich such that theorem 5.3.1 applies. In particular, Lemma 3.2.24 gives us finite coverings if $\mathfrak{M}$ and $\mathfrak{N}$ are finite.

Since in particular $\hat{\mathfrak{M}}, \hat{w} \sim^\ell \hat{\mathfrak{N}}, \hat{v}$, Theorem 5.3.1 implies $\hat{\mathfrak{M}}, \hat{w} \equiv_q \hat{\mathfrak{N}}, \hat{v}$, hence

$$\begin{aligned} \mathfrak{M}, w \models \varphi \;&\Leftrightarrow\; \hat{\mathfrak{M}}, \hat{w} \models \varphi \\ &\Leftrightarrow\; \hat{\mathfrak{N}}, \hat{v} \models \varphi \\ &\Leftrightarrow\; \mathfrak{N}, v \models \varphi, \end{aligned}$$

which implies that $\varphi$ is $\sim^\ell$-invariant over (finite) Cayley structures, as desired. □

$$\begin{array}{ccc} \mathfrak{M}, w & \text{——} \sim^{\ell} \text{——} & \mathfrak{N}, v \\ | & & | \\ \wr & & \wr \\ | & & | \\ \hat{\mathfrak{M}}, \hat{w} & \text{——} \equiv_q \text{——} & \hat{\mathfrak{N}}, \hat{v} \end{array}$$

Figure 5.1: Upgrading $\sim^{\ell}$ to $\equiv_q$.

## 5.4 A characterisation of relativized common knowledge

In Chapter 2, we introduce in addition to ML and ML[CK] two other epistemic logics: common knowledge logic with public announcement ML[CK, PA] and relativized common knowledge RC. As shown in Section 2.4.2, adding public announcement to ML results in an expressively equivalent logic, but adding public announcement to ML[CK] actually increases the expressive power. The modal operators of RC are a generalisation of the usual common knowledge operators of ML[CK] and extend the expressive power of ML[CK]. Furthermore, a syntactic translation shows that ML[CK, PA] can be considered a fragment of RC; it is, in fact, a proper fragment. To summarize, the situation over S5 structures is as follows (cf. [10], [20], [21]):

$$\mathsf{ML} \equiv \mathsf{ML[PA]} \lneq \mathsf{ML[CK]} \lneq \mathsf{ML[CK, PA]} \lneq \mathsf{RC}$$

After characterising ML[CK], one might ask: can we also characterise ML[CK, PA] and RC as the bisimulation-invariant fragments of suitable extensions of first-order logic? We do not know how to precisely characterise ML[CK, PA], but we can give a characterisation for RC.

We rephrased characterising ML[CK] over (finite) S5 structure as characterising ML over (finite) Cayley structures. This approach was successful because we figured out how to play first-order Ehrenfeucht-Fraïssé games on these non-elementary graph classes. For ML[CK, PA] we would need an extension of FO that is capable of making public announcements. In the associated Ehrenfeucht-Fraïssé game, these announcements would correspond to moves that exclude certain parts of both structures for the remainder of a play (for an example see Chapter 8 of [10]). So far, we do not know how to handle such games.

In contrast, the characterisation of RC can be considered a rather straightforward generalisation of our characterisation of ML[CK]. Its proof is based on the same strategy and methods and even uses exactly the same bisimilar coverings. We do not need to develop any new theory. However, there are technical difficulties that we have to address.

First, we need a suitable extension of FO, basically a logic that plays the role of FO[CK]. In the case of FO[CK], we added the relations $(R_\alpha)_{\alpha \subseteq \Gamma}$ to the basic accessibility relations $(R_a)_{a \in \Gamma}$. This is equivalent to expanding basic-agent S5 structures to CK structures. The approach works because this expansion is compatible with bisimulation, in the sense that $\mathfrak{M}, w \sim \hat{\mathfrak{M}}, \hat{w}$ if and only if $\mathfrak{M}^{\mathsf{CK}}, w \sim \hat{\mathfrak{M}}^{\mathsf{CK}}, \hat{w}$ (cf. Lemma 3.2.17). For RC, we need to add relativized accessibility $R_\alpha^\psi$, which translates to expanding basic-agent S5 structures by such relations. However, if the relativizing formula $\psi$ is not bisimulation-invariant, this expansion is not compatible with bisimulation in the sense of Lemma 3.2.17, i.e. we need to restrict the relativization somehow. Thus, we will define the logic $\mathsf{FO}[\mathsf{RC}^\sim]$ that only allows to relativize over bisimulation-invariant formulae, and show that over (finite) $\mathsf{S5}$ structures $\mathsf{RC} \equiv \mathsf{FO}[\mathsf{RC}^\sim]/\sim$.

Compared to $\mathsf{ML}[\mathsf{CK}]$, the logic $\mathsf{RC}$ allows for relativizing the common knowledge modalities over arbitrary $\mathsf{RC}$-formulae. Hence, we need to add an analogous feature to the relations of $\mathsf{FO}[\mathsf{CK}]$ to obtain $\mathsf{FO}[\mathsf{RC}^\sim]$. We define the logic $\mathsf{FO}[\mathsf{RC}^\sim]$ formally as an extension of first-order logic: to its syntax over a modal signature $\{(R_a)_{a\in\Gamma}, (P_i)_{i\in I}\}$ we add, for all $\alpha \subseteq \Gamma$ and all $\psi \in \mathsf{RC}$, the binary relation symbols $R_\alpha^\psi$. To define the semantics we use the set $[w]_\alpha^\psi$ defined in Section 2.4.3: if $w$ is a world in an $\mathsf{S5}$ structure $\mathfrak{M}$, then $[w]_\alpha^\psi$ is the set of worlds that are reachable on an $\alpha$-$\psi$-path or $\alpha$-$(\neg\psi)$-path from $w$. Then

$$\mathfrak{M} \models R_\alpha^\psi wv \quad :\Leftrightarrow \quad v \in [w]_\alpha^\psi.$$

It is important to note that the relation $R_\alpha^\psi$ defines an equivalence relation on an $\mathsf{S5}$ structure $\mathfrak{M}$. With $\mathsf{FO}[\mathsf{RC}^\sim]$ defined, we can state the characterisation theorem for $\mathsf{RC}$.

**Theorem 5.4.1.** *Over the class of (finite)* $\mathsf{S5}$ *structures:*

$$\mathsf{RC} \equiv \mathsf{FO}[\mathsf{RC}^\sim]/\sim$$

The proof of the theorem follows the same strategy as the proof of Theorem 5.3.2: rephrase the statement to $\mathsf{ML} \equiv \mathsf{FO}/\sim$ over a suitable non-elementary subclass of $\mathsf{S5}$ structures, and upgrade $\sim^\ell$-equivalence to $\equiv_q$-equivalence over this class. Essentially, these structures are Cayley structures adapted for the case of $\mathsf{RC}$. Although $\mathsf{RC}$ is more expressive than $\mathsf{ML}[\mathsf{CK}]$ over $\mathsf{S5}$ structures, we do not need to develop any new theory. A closer look at the original acyclic Cayley groups constructed by Otto in [26] reveals the underlying reason for this. His Cayley groups are coset acyclic in the sense of Definition 3.2.5. Our definition of coset acyclicity for Cayley frames is a restricted version of this (cf. Definition 3.2.8). Hence, roughly speaking, there is some more acyclicity left in these Cayley groups that we did not use. This additional acyclicity is enough to make the adapted Cayley structures sufficiently acyclic too. However, there are some more technical difficulties to deal with.

Coming back to the rephrasing of Theorem 5.4.1: as with $\mathsf{FO}[\mathsf{CK}]$, we can regard any $\mathsf{FO}[\mathsf{RC}^\sim]$-formula as an $\mathsf{FO}$-formula over an extended signature and define a suitable non-elementary subclass of $\mathsf{S5}$ structures such that the semantics is preserved. But with $\mathsf{FO}[\mathsf{RC}^\sim]$ we run into a problem: since there are infinitely many $\mathsf{RC}$-formulae, $\mathsf{FO}[\mathsf{RC}^\sim]$ allows for infinitely many relation symbols. This interferes with the application of the modal Ehrenfeucht-Fraïssé theorem since it is only true over finite signatures. But since every single formula $\varphi$ contains only finitely many relation symbols, we can define suitable classes of structures depending on $\varphi$.

Let $\sigma = \{(R_a)_{a\in\Gamma}, (P_i)_{i\in I}\}$ be a finite, fixed modal signature and $\varphi$ be an $\mathsf{FO}[\mathsf{RC}^\sim]$-formula over $\sigma$. Based on $\sigma$ and $\varphi$ we will define a finite modal signature $\sigma(\varphi)$ for our new class of structures. This signature will contain every relation symbol that occurs in $\varphi$, but for technical reasons we need to add further symbols. The signatures of $\mathsf{CK}$ structures are closed under intersection in the following sense: if $R_\alpha, R_\beta$ are relation symbols, then so is $R_{\alpha\cap\beta}$. The signature $\sigma(\varphi)$ needs to have the same property. However, the relation symbols in $\varphi$ are not just associated with a set of agents, but with a set of agents *and* an $\mathsf{RC}$-formula: if $\mathfrak{M} \models R_\alpha^\psi wv$, then there is a path $w_1, a_1, \ldots, a_\ell, w_{\ell+1}$ from $w$ to $v$ in $\mathfrak{M}$ such that,

- $a_i \in \alpha$, for all $1 \leq i \leq \ell$, and
- $\mathfrak{M}, w_i \models \psi \;\Leftrightarrow\; \mathfrak{M}, w \models \psi$, for all $1 \leq i \leq \ell + 1$.

The intersection of $R_{\alpha_1}^{\psi_1}$ and $R_{\alpha_2}^{\psi_2}$ is supposed to mean that a path $w_1, a_1, \ldots, a_\ell, w_{\ell+1}$ is *simultaneously* an $R_{\alpha_1}^{\psi_1}$-path and an $R_{\alpha_2}^{\psi_2}$-path, i.e.

- $a_i \in \alpha_1 \cap \alpha_2$, for all $1 \leq i \leq \ell$, and
- $\mathfrak{M}, w_i \models \psi \;\Leftrightarrow\; \mathfrak{M}, w_1 \models \psi$, for all $\psi \in \{\psi_1, \psi_2\}$ and all $1 \leq i \leq \ell + 1$.

Hence, we generalise the notions of $\alpha$-$\varphi$-path and $\alpha$-$(\neg\varphi)$-path to $\alpha$-$\Phi$-path, for a set of formulae $\Phi$.

**Definition 5.4.2.** Let $\mathfrak{M}$ be a $\sigma$-structure, $\alpha \subseteq \Gamma$, $\Phi \subseteq \mathsf{RC}$. A path $w_1, a_1, \ldots, a_\ell, w_{\ell+1}$ in $\mathfrak{M}$ is an *$\alpha$-$\Phi$-path* if

- $a_i \in \alpha$, for all $1 \leq i \leq \ell$, and
- $\mathfrak{M}, w_i \models \psi \;\Leftrightarrow\; \mathfrak{M}, w_1 \models \psi$, for all $\psi \in \Phi$ and all $1 \leq i \leq \ell + 1$.

We write $v \in [w]_\alpha^\Phi$ if there is an $\alpha$-$\Phi$-path from $w$ to $v$.

Definition 5.4.2 suggests a symbol for the intersection of $R_{\alpha_1}^{\psi_1}$ and $R_{\alpha_2}^{\psi_2}$, namely $R_{\alpha_1\cap\alpha_2}^{\{\psi_1,\psi_2\}}$. Observe that the union of sets of formulae increases the restriction on paths, in contrast to the union of sets of agents.

We continue with $\sigma(\varphi)$: let $\Psi := \{\psi \in \mathsf{RC} : \text{for some } \alpha, R_\alpha^\psi \text{ occurs in } \varphi\} \cup \{\top\}$. Then $\sigma(\varphi)$ is defined as the modal signature that contains the families $(P_i)_{i\in I}$ and $(R_a)_{a\in\Gamma}$ and a symbol $R_\alpha^\Phi$ for every pair $(\alpha, \Phi)$ with $\alpha \subseteq \Gamma$, $\Phi \subseteq \Psi$. An $\mathsf{RC}(\varphi)$ *structure (or frame)* $\mathfrak{M}$ is an $\mathsf{S5}$ structure (or frame) over $\sigma(\varphi)$ that is an expansion of an $\mathsf{S5}$ structure (or frame) over $\sigma$ by binary relations $(R_\alpha^\Phi)^\mathfrak{M} \subseteq W \times W$, for every $R_\alpha^\Phi \in \sigma(\varphi)$, such that

$$(R_\alpha^\Phi)^\mathfrak{M} := \{(w, v) \in W \times W \;:\; v \in [w]_\alpha^\Phi\}.$$

The relations of an $\mathsf{RC}(\varphi)$ structure are no longer just associated with a set of agents, but with a pair $(\alpha, \Phi) \in \mathcal{P}(\Gamma) \times \mathcal{P}(\Psi)$; $\tau(\varphi) := \mathcal{P}(\Gamma) \times \mathcal{P}(\Psi)$ denotes the set of all these pairs. The intersection of two such pairs $(\alpha_1, \Phi_1), (\alpha_2, \Phi_2) \in \tau(\varphi)$ is defined as $(\alpha_1, \Phi_1) \cap (\alpha_2, \Phi_2) := (\alpha_1 \cap \alpha_2, \Phi_1 \cup \Phi_2)$. We regard $R_\alpha^\Phi$ as a refinement of $R_\alpha^\top$, and the original relations $R_a$ are considered to be associated with $(\{a\}, \top)$.

Finally, we arrive at the adapted rephrasing: $\mathsf{FO}[\mathsf{RC}^\sim]/\!\sim\; \subseteq \mathsf{RC}$ over $\mathsf{S5}$ structures if and only if for every $\varphi \in \mathsf{FO}[\mathsf{RC}^\sim]/\!\sim$, the formula $\varphi$ regarded as an $\mathsf{FO}$-formula over $\sigma(\varphi)$ is logically equivalent to an $\mathsf{ML}$-formula over $\mathsf{RC}(\varphi)$ structures.

The remainder of the proof follows the known pattern: for any $\varphi$ with quantifier rank $q$, we need to show that there is some $\ell \in \mathbb{N}$ such that $\sim^\ell$-invariance can be upgraded to $\mathsf{FO}_q$-equivalence over $\mathsf{RC}(\varphi)$ structures. Given two $\ell$-bisimilar structures $\mathfrak{M}, w$ and $\mathfrak{N}, v$, the upgrading involves two steps. First, constructing bisimilar coverings $\hat{\mathfrak{M}}, \hat{w} \sim \mathfrak{M}, w$ and $\hat{\mathfrak{N}}, \hat{v} \sim \mathfrak{N}, v$. Second, showing $\hat{\mathfrak{M}}, \hat{w} \equiv_q \hat{\mathfrak{N}}, \hat{v}$. Luckily, we do not have to put any new actual work into both steps. The coverings constructed for the standard $\mathsf{CK}$ structures already suffice and the equivalence proof goes through exactly as before. As mentioned above, we only have to take a closer look at coset acyclicity for Cayley groups as defined by Otto in [26] to convince ourselves that our coverings already have the acyclic properties that we need for upgrading over $\mathsf{RC}(\varphi)$ structures.

Let $\mathbb{G}$ be a Cayley group with generator set $E$. A *coset cycle* in $\mathbb{G}$ was defined as a cyclic tuple $((g_i, \alpha_i))_{i \in \mathbb{Z}_m}$ with $g_i \in \mathbb{G}$ and $\alpha_i \subseteq E$, for all $i \in \mathbb{Z}_m$, where $g_i g_{i+1}^{-1} \in \mathbb{G}_{\alpha_i}$ and

$$g_i \mathbb{G}_{\alpha_{i-1} \cap \alpha_i} \cap g_{i+1} \mathbb{G}_{\alpha_i \cap \alpha_{i+1}} = \emptyset.$$

For the generalisation of our method to $\mathsf{RC}(\varphi)$ structures, it is important to observe that the $\alpha_i$ can be arbitrary subsets of $E$, as long as the cyclic tuple fulfills the coset property. Our definition of coset acyclicity for Cayley frames (Definition 3.2.7) merely uses a restricted version of this. Only sets are admitted that contain all the generators that are associated with some set of agents. We soften this restriction for $\mathsf{RC}(\varphi)$ frames. Admittable sets of generators are no longer induced by sets of agents, but by pairs $(\alpha, \Phi) \in \tau(\varphi)$, i.e. we allow for the refinement that is suitable for $\mathsf{RC}(\varphi)$ frames.

**Definition 5.4.3.** Let $\varphi \in \mathsf{RC}$, and $\mathfrak{M}$ be an $\mathsf{RC}(\varphi)$ frame. A *coset cycle of length $m$ in $\mathfrak{M}$* is a cyclic tuple $((w_i, (\alpha_i, \Phi_i)))_{i \in \mathbb{Z}_m}$ with $w_i \in W$ and $(\alpha_i, \Phi_i) \in \tau(\varphi)$, for all $i \in \mathbb{Z}_m$, where $(w_i, w_{i+1}) \in R_{\alpha_i}^{\Phi_i}$ and

$$[w_i]_{\alpha_{i-1} \cap \alpha_i}^{\Phi_{i-1} \cup \Phi_i} \cap [w_{i+1}]_{\alpha_i \cap \alpha_{i+1}}^{\Phi_i \cup \Phi_{i+1}} = \emptyset.$$

An $\mathsf{RC}(\varphi)$ frame is *acyclic* if it does not contain a coset cycle, and *$n$-acyclic* if it does not contain a coset cycle of length up to $n$.

The coverings that we constructed for $\mathsf{CK}$ structures are based on Cayley groups that are acyclic in the sense of Definition 3.2.5. If we expand the very same coverings to $\mathsf{RC}(\varphi)$ structures over $\sigma(\varphi)$, they work just as well w.r.t. to coset acyclicity for $\mathsf{RC}(\varphi)$ structures.

**Lemma 5.4.4.** *Let $\varphi \in \mathsf{RC}$. Every connected $\mathsf{RC}(\varphi)$ structure admits a bisimilar covering by an acyclic and $\omega$-rich $\mathsf{RC}(\varphi)$ structure.*

**Lemma 5.4.5.** *Let $\varphi \in \mathsf{RC}$. For all $k, n \in \mathbb{N}$, every finite, connected $\mathsf{RC}(\varphi)$ structure admits a finite bisimilar covering by an $n$-acyclic and $k$-rich $\mathsf{RC}(\varphi)$ structure.*

The structure theory from Chapter 4 can be applied to these structures too because their binary relations are closed under intersection and they are sufficiently acyclic. Essentially, these structures fulfil the same requirements as $\mathsf{CK}$ structures. Closure under intersection is necessary to define coset cycles and prove analogons of the lemmas from Section 4.1 regarding 2-acyclicity, unique connecting pairs $(\alpha, \Phi)$, etc. Results like the zipper lemma and its corollaries, and the freeness theorem go through just the same. As before, these results lead to the desired upgrading over $\mathsf{RC}(\varphi)$ structures, and the upgrading implies the characterisation theorem.

**Theorem 5.4.6.** *Let* $\varphi \in \mathsf{RC}$. *Over the class of (finite)* $\mathsf{RC}(\varphi)$ *structures:*

$$\mathsf{ML} \equiv \mathsf{FO}/\sim$$

This is equivalent to Theorem 5.4.1: $\mathsf{RC} \equiv \mathsf{FO}[\mathsf{RC}^{\sim}]/\sim$ over the class of (finite) $\mathsf{S5}$ structures.

# 6 Conclusion

The main result of this thesis is a characterisation theorem for epistemic modal logic with common knowledge modalities. As $\mathsf{CTL}^*$ and $\mathsf{PDL}$, $\mathsf{ML}[\mathsf{CK}]$ is a fragment of the modal $\mu$-calculus $\mathsf{L}_\mu$ that is more expressive than $\mathsf{ML}$. Unlike $\mathsf{CTL}^*$, $\mathsf{PDL}$ or $\mathsf{L}_\mu$, which only have a classical characterisation, $\mathsf{ML}[\mathsf{CK}]$ is characterised both classically and in the sense of finite model theory.

We achieved this characterisation in terms of showing $\mathsf{ML} \equiv \mathsf{FO}/\!\sim$ over (finite and arbitrary) $\mathsf{CK}$ structures. First, we showed that we can reduce questions about $\mathsf{CK}$ structures, up to bisimulation, to Cayley structures (cf. Section 3.2). These structures are special instances of $\mathsf{CK}$ structures that imbue their very intricate edge pattern with a high degree of regularity. Second, this regularity allowed us to develop a structure theory for Cayley structures (cf. Chapter 4) that made it possible to play first-order Ehrenfeucht-Fraïssé games successfully over these (cf. Chapter 5). Thus, we could upgrade $\sim^\ell$-equivalence to $\equiv_q$-equivalence over (finite and arbitrary) Cayley structures, which implies our characterisation theorem.

The question arises: how can we further adapt Cayley structures and develop the associated theory? Section 5.4 gives a first example. In this section, we prove a characterisation for relativized common knowledge $\mathsf{RC}$. Although $\mathsf{RC}$ is more expressive than $\mathsf{ML}[\mathsf{CK}]$ over $\mathsf{S5}$ structures, we could achieve the result without the development of any new theory. It simply took a closer look at the bisimilar coverings we already constructed. It revealed that the very same coverings that we used for $\mathsf{CK}$ structures were also suitable for characterising $\mathsf{RC}$. The additional arguments were mere technicalities.

In Section 5.4, we also mention that the current methods do not suffice to characterise $\mathsf{ML}[\mathsf{CK}, \mathsf{PA}]$ or to give a more classical characterisation of $\mathsf{RC}$ that does not restrict relativization to bisimulation-invariant formulae. An extension of these methods, in order to play suitable extended Ehrenfeucht-Fraïssé games over Cayley structures, might make the aforementioned characterisations possible.

Furthermore, Cayley structures could be adapted to characterise $\mathsf{ML}[\mathsf{CK}]$ over other frame classes than $\mathsf{S5}$. Conceivable options might the class of reflexive and transitive frames, which corresponds to the axiom system $\mathsf{S4}$, or the class of transitive, serial and euclidean frames, which corresponds to the axiom system $\mathsf{KD45}$, which describes doxastic modal logic, the logic of belief.

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