TECHNISCHE UNIVERSITÄT DARMSTADT

Computation of Eisenstein series associated with discriminant forms

Vom Fachbereich Mathematik
der Technischen Universität Darmstadt
zur Erlangung des Grades eines
Doktors der Naturwissenschaften
(Dr. rer. nat.)
genehmigte Dissertation

von

M.Sc. Sebastian Opitz

aus Rüsselsheim

Referent:	Prof. Dr. Jan Hendrik Bruinier
Korreferent:	Prof. Dr. Nils Scheithauer
Tag der Einreichung:	18. Oktober 2018
Tag der mündlichen Prüfung:	27. November 2018

Darmstadt 2018
D 17

Opitz, Sebastian: Computation of Eisenstein series associated with discriminant forms
Darmstadt, Technische Universität Darmstadt
Jahr der Veröffentlichung der Dissertation auf TUprints: 2018
Tag der mündlichen Prüfung: 27. November 2018

Danksagung

Viele Menschen haben maßgeblich durch ihre fortwährende Unterstützung zum Entstehen dieser Dissertation beigetragen. Euch allen vielen Dank!

Meinem Doktorvater Professor Dr. Jan Hendrik Bruinier – Ich danke Dir für alle Anregungen, Gespräche und Motivation zu diesem spannenden Thema. Insbesondere danke ich Dir dafür, dass Du alle mein Fragen der Art „Das könnte man doch auch noch für die Diss implementieren!?“ stets mit „Ja“ beantwortet hast, sowie für alle Unterstützung im Rahmen interessanter Konferenzen und eines angenehmen Arbeitsplatzes.

Meinem Korreferenten Professor Dr. Nils R. Scheithauer – Vielen Dank für die interessanten Gespräche, Anregungen, Theorien und Vermutungen, die Du mit mir geteilt hast und auch für die Bereitschaft diese Arbeit zu begutachten.

Dr. Stephan Ehlen – Gerade in der Anfangszeit hast Du mich unterstützt und mir vieles erklärt. Auch den Einstieg in `sage` und `python` hast Du mir bedeutend erleichtert.

Dr. Markus Schwagenscheidt – Vielen Dank für sämtliche Erklärungen zum Einstieg in die Theorie der Borcherdsprodukte. Die gemeinsamen Diskussionen und Rechnungen am Whiteboard waren immer locker, aber erfolgreich. Du bist ein super Koautor, vielen Dank auch für das fleißige Korrekturlesen dieser Arbeit.

Ph.D. Brandon Williams – Vielen Dank für die vielen Erläuterungen zu Deiner eigenen Arbeit und die hilfreichen Diskussionen.

Einen riesigen Dank an die Arbeitsgruppe Algebra mit ihrer lockeren Arbeitsatmosphäre, viel zu seltenen Spiele- und Pokerabenden und tollen Konferenzen in Frankreich.

Weiter bedanke ich mich bei der Deutschen Forschungsgemeinschaft, aus deren Mitteln meine Stelle an der Technischen Universität Darmstadt zu Teilen im Rahmen der Forschergruppe „Symmetrie, Geometrie und Arithmetik“ finanziert wurde.

Ganz besonders bedanke ich mich bei meinem Eltern, Gabriele und Lothar Opitz, meiner Schwester Melanie und meiner Frau Kristina. Ihr habt mich immer unterstützt und ich weiß, dass ich mich immer auf euch verlassen kann. Dir Kristina danke ich auch für das Korrekturlesen dieser Arbeit, aber vor allem für alltäglichen Dinge, durch die Du mich tagtäglich unterstützt, dein Verständnis und das Zuhause, das Du mir bietest. Danke!

Zusammenfassung

In der vorliegenden Dissertation werden Methoden entwickelt, um die Fourierkoeffizienten spezieller Reihen, namentlich vektorwertige Eisensteinreihen zur Weildarstellung eines geraden Gitters, zu berechnen. Die bisher bekannten Formeln gehen immer von einem geraden Gitter aus und leiten von diesem die „lokalen“ Daten des Gitters ab. Zur Berechnung dieser Formeln wurde ein Programm in der Sprache `python` zur Benutzung mit `sage` geschrieben. Die Eisensteinreihe selbst hängt nur von der Diskriminantenform des Gitters ab. Vor diesem Hintergrund untersuchen wir die „globalen“ Formeln, um zu verstehen, wie sie aus den „lokalen“ Daten des Gitters, wie zum Beispiel dem Geschlechtssymbol oder der Zerlegung in Jordankomponenten, berechnet werden können. Aus dem Vergleich verschiedener Ansätze zur Berechnung der Fourierkoeffizienten der Eisensteinreihen können wir Formeln für die lokale Igusazetafunktion ableiten. Zuletzt benutzen wir die geschriebenen Programme, um alle Borcherdsprodukte, die von einer gewissen Klasse von Gittern kommen, zu klassifizieren.

Contents

Introduction

In this thesis, we implement various algorithms to work with even lattices and finite quadratic modules, see [Opi18] for the implementation. The program is meant for use with `sage`, (cf. [SageMath]) and written in `python`. In particular, we compute the Fourier coefficients of vector valued Eisenstein series with respect to the Weil representation for an even lattice. The main use case for our algorithms is the search and classification of Borcherds products of singular weight. A copy of the written program is attached to this thesis, but we explicitly refer to the online reference for future releases and bug fixes.

As it turns out, we do not need a global $\mathbb{Z}$-lattice to compute Eisenstein series, a consistent choice of local $\mathbb{Z}_p$-lattices suffices. This makes it possible to compute Eisenstein series purely from the genus symbol of the lattice or purely from its discriminant form. For the local algorithms, we have incorporated code from [PSAGE] and [Ehl16]. For the global algorithms, we have built upon [SageMath] and compared our results with [Wilb].

To gain further insights, we study and compare the approaches to the computation of the Fourier coefficients of Eisenstein series given in [KY10] and [BK01]. A key component in the second approach is the computation of representation numbers for the lattice and in particular the computation of the Poincaré series for these representation numbers. These Poincaré series can be computed by means of the Igusa local zeta function $\mathbb{Z}_f(t)$ where f is a polynomial of degree 2. We have learned that the Igusa local zeta function can be used to compute the Fourier coefficients of Eisenstein series from [Wil18a]. The Igusa local zeta functions can be computed by means of [CKW17].

Representation numbers and the Igusa local zeta function

Different kinds of representation numbers appear in this thesis. The representation numbers of positive definite lattices appear in the Siegel-Weil formula (cf. Theorem 3.7.2), which we use to test the algorithms written in course of this thesis. The representation numbers of the discriminant form of a lattice (or a finite quadratic module) appear in the dimension formula for the associated space of modular forms (see Section 4.4). The third kind of representation numbers are obtained by counting zeros of polynomials modulo prime powers. They can be studied through the corresponding Igusa local zeta function. For polynomials of degree 2, these representation numbers are related to the representation numbers of discriminant forms and they also appear in the local Euler factors in the formulas of the Fourier coefficients of Eisenstein series. The study of these local factors motivated the following Theorem 1, which plays a central role in this thesis.

The Igusa local Zeta function for $f \in \mathbb{Z}_p[x_1, \ldots, x_n]$ is defined by

$$Z_f(s) = \int_{\mathbb{Z}_p^n} |f(x_1, \ldots, x_n)|_p^s dx_1 \ldots dx_n$$

and it is related to the representation numbers

$$N_k(f) = \# \left\{ x \in \mathbb{Z}_p/p^k\mathbb{Z}_p \;\middle|\; f(x) \equiv 0 \pmod{p^k} \right\}$$

by the the equality

$$\frac{1 - tZ_f(t)}{1-t} = \sum_{k \in \mathbb{Z}_{\geq 0}} \frac{N_k(f)}{p^{nk}} p^{-ks}.$$

See Section 2.1 for more details. For an overview of the properties of Igusa local zeta functions, we refer to [Den91]. We can write $Z_f(s)$ as a rational function in $t = p^{-s}$. A proof of this can be found in [Den84].

Theorem 1 (cf. Theorem 2.1.1)**.** *Let p be a prime and $f \in \mathbb{Z}_p[x_1, \ldots, x_n]$. Let H denote the hyperbolic plane, given by the polynomial $H = x_{n+1}x_{n+2} \in \mathbb{Z}_p[x_{n+1}, x_{n+2}]$. The Igusa local zeta functions of f and $f + H$ satisfy the relation*

$$\frac{1 - tZ_f(t)}{1-t}\left(1 - \frac{t}{p}\right) = \frac{1 - tZ_{f+H}(t)}{1-t}\left(1 - \frac{t}{p}\right)\bigg|_{t=pt}$$

or equivalently

$$\left(1 - \frac{t}{p}\right) \sum_{i \in \mathbb{Z}_{\geq 0}} \frac{N_i(f)}{p^{ni}} t^i = \left(1 - \frac{t}{p}\right) \sum_{i \in \mathbb{Z}_{\geq 0}} \frac{N_i(f+H)}{p^{(n+2)i}} t^i \Bigg|_{t=pt}$$

in terms of (the meromorphic extension of) the Poincaré series for $t = p^{-s}$ and $s \in \mathbb{C}$ (assume $\Re(s)$ large for convergence).

We exploit this result in three different ways. The first is to generalize some aspects in the approach of [BK01]. The second is to see in which way the formulas in [BK01] can be computed purely from the genus symbol or Jordan decomposition of the lattice. The third is to derive explicit formulas for the representation numbers of even 2-adic Jordan components of finite quadratic modules. These Jordan components are finite abelian groups with a quadratic form Q which takes values in $\mathbb{Q}/\mathbb{Z}$. The even 2-adic Jordan components are denoted by $q_{II}^{\varepsilon 2n}$ and their representation numbers

$$N(q_{II}^{\varepsilon 2n}, j) := \left| \left\{ \gamma \in q_{II}^{\varepsilon 2n} : Q(\gamma) \equiv j \pmod{1} \right\} \right|$$

are given by the following theorem.

Theorem 2 (cf. Theorem 2.3.3). *Let $q = 2^l$ with $l \geq 1$, $\varepsilon = \pm 1$. Then*

$$N(q_{II}^{\varepsilon 2n}, \frac{j}{q}) = \begin{cases} \left(\frac{q}{2}\right)^n \left[(2^n - \varepsilon)\varepsilon^l \sum_{k'=0}^{l} (\varepsilon 2^{n-1})^{k'} + \varepsilon^{l+1}\right], & \text{if } j = 0 \pmod{2^l}, \\ \left(\frac{q}{2}\right)^n (2^n - \varepsilon)\varepsilon^l \sum_{k'=l-k}^{l} (\varepsilon 2^{n-1})^{k'}, & \text{if } 2^k \parallel j \neq 0 \pmod{2^l}. \end{cases}$$

These representation numbers are an important ingredient in the computation of the dimension of spaces of vector valued modular forms for the Weil representation with respect to a finite quadratic module or lattice (see Section 4.4). By rephrasing the theorem slightly we see that Siegel's formulas for certain representation numbers ([Sie35, Hilfssatz 16]) also hold for $p = 2$ when 2 does not divide the discriminant of the lattice and one uses the Kronecker symbol in place of the Legendre symbol.

The representation numbers for odd 2-adic Jordan components can be obtained by means of [Opi13]. The representation numbers for Jordan components for odd primes can be computed by means of [Sch13], with a slight simplification from [Opi13]. We recall these results in Section 4.2.1 and Section 4.2.2.

Fourier coefficients of Eisenstein series

The simplifications we can apply to the approach in [BK01] (see Section 3.2) amount to the following theorem. It can be thought of to be half way in between [BK01, Theorem 4.6] and [BK01, Theorem 4.8], where we have also removed the special treatment of the prime 2 when it does not divide the discriminant of the lattice. Note that the Eisenstein series we use here is half of that in [BK01], as all Eisenstein series treated in this thesis are normalized to have constant coefficient $\mathfrak{e}_0$. We achieve this by setting

$$E = E_{\rho_L^*,k,0} = \frac{1}{4} \sum_{(M,\phi) \in \langle T \rangle \backslash Mp_2(\mathbb{Z})} \mathfrak{e}_0 |_{k,L}^* (M, \phi),$$

where the slash operator is defined with respect to the half-integer $k > 2$ (the weight) and the dual Weil representation ρ_L^* (see Section 1.6). The 0 in $E_{\rho_L^*,k,0}$ states that this Eisenstein has constant term $\mathfrak{e}_0$, its Fourier expansion is of the form

$$E = \mathfrak{e}_0 + \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z} - Q(\gamma) \\ n > 0}} a_E(\gamma, n) q^n \mathfrak{e}_\gamma.$$

Eisenstein series with respect to other isotropic elements β of the discriminant form of L would have constant term $\frac{1}{2}(\mathfrak{e}_\beta + \mathfrak{e}_{-\beta})$.

Theorem 3 (cf. Theorem 3.2.4). *Let $\gamma \in L'$ and $n \in \mathbb{Z} - q(\gamma)$ with $n > 0$. The coefficient*

$a_E(\gamma, n)$ of the Eisenstein series E of weight $k > 2$ for ρ_L^* is equal to

$$\frac{2^k \pi^k n^{k-1} (-1)^{(2k-b^-+b^+)/4}}{\sqrt{|L'/L|}\Gamma(k)}$$

times

$$\begin{cases} \frac{\sigma_{1-k}(\tilde{n},\chi_D)}{L(k,\chi_D)} \prod\limits_{p|\det(S)} L_{\gamma,n}(k,p), & \text{if } 2 \mid m, \\ \frac{L(k-1/2,\chi_{\mathcal{D}})}{\zeta(2k-1)} \prod\limits_{\substack{p|\tilde{n} \\ p\nmid \det S}} \left(\sigma_{2-2k}(p^{\mathrm{ord}_p(f)}) - \chi_{\mathcal{D}}(p) p^{1/2-k} \sigma_{2-2k}(p^{\mathrm{ord}_p(f)-1})\right) \prod\limits_{p|\det(S)} \frac{L_{\gamma,n}(k,p)}{1-p^{1-2k}}, & \text{if } 2 \nmid m. \end{cases}$$

Here

$$\begin{aligned} L_{\gamma,n}(k,p) &= \sum_{\nu \in \mathbb{Z}_{\geq 0}} \frac{N_\nu(f)}{p^{n\nu}} p^{-\nu s} \left(1 - \frac{p^{-s}}{p}\right)\Bigg|_{s=-1-\frac{m}{2}+k} \\ &= \frac{1 - tZ_f(t)}{1-t} \left(1 - \frac{t}{p}\right)\Bigg|_{t=p^{1+\frac{m}{2}-k}} \end{aligned}$$

in terms of the Igusa local zeta function for the polynomial $f = Q(x) - (x, \gamma) + m + Q(\gamma)$ and the representation numbers

$$N_\nu(f) = \#\left\{x \in (\mathbb{Z}/p^\nu\mathbb{Z})^m \mid Q(x) - (x,\gamma) + m + Q(\gamma) \equiv 0 \pmod{p^\nu}\right\}.$$

The values of $D, \mathcal{D}, f, \tilde{n}$ are as in Theorem 3.2.3.

When the weight is small, $k = 3/2$ or $k = 2$, the Eisenstein series does not converge absolutely. However, it can be defined by means of the usual "Hecke trick". The resulting Eisenstein series is no longer holomorphic in general, but a harmonic Maass form. Its image under the ξ-operator (cf. [BF04, Section 3]) belongs to M_{2-k,ρ_L}. The coefficients with positive index ($n > 0$) are still given by the formula of Theorem 3.

With Theorem 1 it is now immediate that the local factors only depend on local (p-adic) choices of Gram matrices and on the parity of the rank which is determined by the 2-adic data of the lattice (genus symbol or Jordan decomposition). We discuss this in Section 3.4.

Comparing Theorem 3 to the formulas derived from [KY10], we see that the local factors are essentially given by the same rational functions and we can exploit this to give formulas for the Igusa local zeta function (see Section 3.3). We believe that this connection has not been made explicitly yet.

Estimates of Fourier coefficients

We define the hyperbolic plane as the lattice

$$U = (\mathbb{Z}^2, (x,y) \mapsto xy)$$

and denote the scaled hyperbolic planes by

$$U(N) = (\mathbb{Z}^2, (x, y) \mapsto Nxy)$$

for an integer N.

In the case that a lattice splits a scaled hyperbolic plane, we can estimate the representation numbers appearing in Theorem 3. Specializing to the case that the weight is half of the rank of the lattice, the local factors essentially reduce to a representation number. This allows us to estimate the Fourier coefficients of the lattice when they are not zero. The result was obtained together with Markus Schwagenscheidt (cf. [OS18]), we give the proof in Section 3.5.

Theorem 4 (cf. Theorem 3.5.2)**.** *Let L be a lattice of signature (b_+, b_-) (b_+ even) with rank $2k = m \geq 3$ such that $L = L_1 \oplus U(N)$ for some even lattice L_1 of rank $m-2 \geq 1$. Let d be the determinant of the Gram matrix of L. Let $\gamma \in L'$ and $n \in \mathbb{Z} - q(\gamma)$ with $n > 0$. The coefficient $a_E(\gamma, n)$ of the Eisenstein series with respect to the dual Weil representation and of weight $k = m/2$ is either 0 or*

$$(-1)^{b_+/2} a_E(\gamma, n) \geq C_{k,d,N} \cdot n^{k-1},$$

where $C_{k,d,N}$ is given by

$$\frac{2^{k+1}\pi^k}{\sqrt{|d|}\Gamma(k)}$$

times

$$\begin{cases} \frac{2-\zeta(k-1)}{\zeta(k)} \prod_{p|2d} p^{(3-2k)\operatorname{ord}_p(N)}(1-1/p), & 2 \mid m, \\ \frac{2-\zeta(k-1/2)}{\zeta(k-1/2)} \prod_{p|2d} \frac{p^{(3-2k)\operatorname{ord}_p(N)}(1-1/p)}{1-p^{1-2k}}, & 2 \nmid m. \end{cases}$$

This theorem is the main ingredient in our search for Borcherds products of singular weight.

Searching for Borcherds products

The search for a specific class of Borcherds products can be reduced to solving the linear equations

$$\frac{n}{2} - 1 = -\frac{1}{2} \sum_{\gamma \in L'/L} \sum_{n<0} a_f(\gamma, n) a_E(\gamma, -n)$$

and

$$\sum_{\gamma \in L'/L} \sum_{n<0} a_f(\gamma, n) a_g(\gamma, -n) = 0$$

for every cusp form g with respect to the dual Weil representation for an even lattice L $(2, n)$. We only allow solutions $a_f(\gamma, n) \leq 0$ to ensure that the resulting Borcherds product is holomorphic. For a simple lattice, there are no cusp forms, so the second condition is trivial. Going through the finite list of simple lattices of type $(2,n)$ from [BEF16a] and estimating the Fourier coefficients of E with Theorem 4 gives only finitely many solutions to the above condition. Together with Markus Schwagenscheidt (cf. [OS18]), we obtain the following classification result.

Theorem 5 (cf. Theorem 5.0.5)**.** *Holomorphic Borcherds products (coming from vector valued modular forms with non-negative principal part) of singular weight* $\frac{n}{2}-1$ *for simple lattices* L *of type* $(2,n), n \geq 3$*, only exist in the following cases.*

n	*genus*	*lattice*	*level*
3	$2_7^{+1}4^{+2}$	$A_1(-1) \oplus U \oplus U(4)$	4
	$2_7^{+3}4^{+2}$	$A_1(-1) \oplus U(2) \oplus U(4)$	4
	$2_7^{+1}4^{+4}$	$A_1(-1) \oplus U(4) \oplus U(4)$	4
	$2^{+4}4_7^{+1}$	$A_1(-2) \oplus U(2) \oplus U(2)$	8
	8_7^{+1}	$A_1(-4) \oplus U \oplus U$	16
4	3^{+5}	$A_2(-1) \oplus U(3) \oplus U(3)$	3
6	2^{-6}	$D_4(-1) \oplus U(2) \oplus U(2)$	2
10	2^{+2}	$E_8(-1) \oplus U \oplus U(2)$	2
26	1^{+1}	$E_8(-1) \oplus E_8(-1) \oplus E_8(-1) \oplus U \oplus U$	1

Here U *denotes the hyperbolic plane* $\mathbb{Z}^2$ *with* $Q(x,y) = xy$*, and* A_1, A_2, D_4, E_8 *denote the usual root lattices. Further, if* (L,Q) *is a lattice and* N *a positive integer, we write* $L(N)$ *for the scaled lattice* (L, NQ)*.*

The Siegel-Weil formula

The Siegel-Weil formula (cf. [Wei64] and Section 3.1.2) is an important test case for our implementation of the Fourier coefficients of Eisenstein series. In the special case of positive definite even lattices of rank greater equal 5, we can write it in the following form.

Siegel-Weil formula, Theorem 6 (cf. Theorem 3.1.3 and Theorem 3.7.2)**.** *For a positive definite even lattice* L *of rank* $n \geq 5$*, we have*

$$\Theta_L^{sym,gen} = E_{\rho_L, \frac{n}{2}, 0},$$

where $E_{\rho_L,\frac{n}{2},0}$ *is the vector valued Eisenstein series for the Weil representation* ρ_L *and of weight* $\frac{n}{2}$*. The series* $\Theta_L^{sym,gen}$ *is a vector valued theta series for* L *symmetrized once with respect to all the classes in the genus of* L *and once with respect to the orthogonal group of the discriminant form of* L*.*

The equality in the μ-th component is essentially given by applying [Kud03, Theorem 4.1 and Proposition 4.22] to the Schwartz function $\phi = \frac{1}{|O(L'/L)|}\sum_{\sigma \in O(L'/L)} \mathrm{char}(\sigma\mu + \hat{L})$, where $\mu \in L'/L$ and $\hat{L} = L \otimes \hat{\mathbb{Z}}$. We use the notation from [Ros15], where this formulation of the Siegel-Weil formula is proved when 2 does not divide the discriminant of L.

We list the lattices we considered as test cases in Appendix C.

Outlook

In this thesis we have developed all the necessary ingredients to compute the vector valued Eisenstein series $E_{\rho_L^*,k,0}$ purely from the genus symbol of a lattice or finite quadratic module, by making a consistent choice of local Gram matrices. The same choice gives $E_{\rho_L^*,k,\beta}$ for isotropic $\beta \neq 0$ when using the formulas from [Sch18]. Using Poincaré square series as defined in [Wil18a] for the same choice can give us a basis of cusp forms. The local factors appearing in all formulas can be stated in terms of the Igusa local zeta function, which we have already tested versus the formulas for generalized local densities. These in turn have been tested versus the Siegel-Weil formula and versus Jacobi-Eisenstein series. Implementing these modular forms is the next natural step. Then we will have the whole space at our (computational) disposal.

Once we have a basis of cusp forms, we can generalize our search for Borcherds products by dropping the assumption that the given lattices are simple. This is work in progress together with Stephan Ehlen and Markus Schwagenscheidt. The setup here is to first enumerate the genus symbols of lattices of type $(2, n)$ with small obstruction space (small dimension of the space of cusp forms). We then check if these lattices split a scaled hyperbolic plane (this can be done purely by looking at the local data). If they do, we can search for Borcherds products by solving a system of linear equations, like before.

1. Preliminaries

In this chapter we introduce the basic objects studied in this thesis. These include quadratic forms, lattices, discriminant forms, the Weil representation and (vector valued) automorphic forms. The expositions in this chapter cover well documented basics. Therefore we usually give several references and no proofs.

1.1. Quadratic forms

We recall some basic facts about quadratic forms, quadratic spaces and lattices. Some standard references are [Ser73], [Cas78], [Kit93], [CS99] and [Kne02]. For further reading and some historical notes, [Cox13] can also be recommended. For an introduction in a modular forms based setting, we refer to [Bru+08]. Finally for the technique of discriminant forms, we refer to [Nik79].

Let R be a ring with unity 1 and let M be a finitely generated R-module.

Definition 1.1.1. A *quadratic form* Q on M is a map $Q : M \to R$ such that

(i) $Q(rx) = r^2 Q(x)$ for all $r \in R$ and $x \in M$,

(ii) $(x, y) := Q(x + y) - Q(x) - Q(y)$ is a bilinear form.

We refer to (x, y) as the *bilinear form associated to* Q. We call the pair (M, Q) a *quadratic module*. In the case that R is a field, (M, Q) is a *quadratic space*. Note that $(x, x) = 2Q(x)$.

From now on let (M, Q) denote a quadratic module.

Definition 1.1.2. Let $x, y \in M$.

(i) If $(x, y) = 0$, we say that x and y are *orthogonal* to each other.

(ii) If $Q(x) = 0$, we say that x is *isotropic*. If $Q(x) \neq 0$, x is *anisotropic*.

(iii) For a subset $U \subset M$, we define its *orthogonal complement* by

$$U^\perp := \{x \in M \mid (x, y) = 0 \text{ for all } y \in U\}.$$

For an element x we analogously denote by $x^\perp$ the set of all elements orthogonal to x.

(iv) If $M^\perp = \{0\}$, the quadratic module is said to be *non-degenerate*.

Note that a non-degenerate quadratic module can have isotropic elements.

Definition 1.1.3. Let $(\tilde{M}, \tilde{Q})$ be another quadratic module. An injective linear map $\sigma : M \to \tilde{M}$ satisfying $Q = \tilde{Q} \circ \sigma$ is called an *isometry*. If σ is bijective, the quadratic spaces are called *isometric* and we write $(M, Q) \simeq (\tilde{M}, \tilde{Q})$ which we sometimes abbreviate by $M \simeq \tilde{M}$. The *orthogonal group* of M is the group of isometries from M onto itself and denoted by $O(M)$.

Example 1.1.4. Let (b^+, b^-) be non-negative integers. We denote by $\mathbb{R}^{b^+,b^-}$ the quadratic space over $\mathbb{R}$ given by $\mathbb{R}^{b^++b^-}$ with the quadratic form

$$Q(x_1, \ldots, x_{b^++b^-}) = x_1^2 + \cdots + x_{b^+}^2 - x_{b^++1}^2 - \cdots - x_{b^++b^-}^2. \tag{1.1.1}$$

Proposition 1.1.5. *Any non-degenerate quadratic space (V, Q) over $\mathbb{R}$ is isometric to $\mathbb{R}^{b^+,b^-}$. The values b^+, b^- are unique and satisfy $b^+ + b^- = \dim(V)$.*

Definition 1.1.6. The *type* of $V \simeq \mathbb{R}^{b^+,b^-}$ is defined by the pair (b^+, b^-). The *signature* of V is the value

$$\operatorname{sign}(V) = b^+ - b^-. \tag{1.1.2}$$

1.2. Quadratic forms on finite abelian groups

Introductions to this topic can be found in the section on discriminant forms in [Sch09] or in the section on finite quadratic modules in [Str13].

Finite abelian groups are $\mathbb{Z}$-modules. However they are not freely generated. Therefore we need to modify our previous definition of quadratic forms in the following manner. It should always be clear from the context which kind of quadratic form is used.

Definition 1.2.1. A *quadratic form* Q on a finite abelian group D is a map $Q : D \to \mathbb{Q}/\mathbb{Z}$ such that

(i) $Q(rx) = r^2Q(x)$ for all $r \in \mathbb{Z}$ and $x \in D$,

(ii) $(x, y) := Q(x + y) - Q(x) - Q(y)$ is a bilinear form.

Again, we refer to (x, y) as the *bilinear form associated to* Q. *Orthogonality*, *orthogonal complements* and *isometric/anisotropic* elements are defined analogously to the case of quadratic modules. We say that Q is *non-degenerate* if $D^\perp = \{0\}$. This is equivalent to the map $D \to \operatorname{Hom}(D, \mathbb{Q}/\mathbb{Z})$, $\gamma \mapsto (\gamma, \cdot)$ being an isomorphism. If Q is non-generate, we call the pair (D, Q) a *finite quadratic module*.

Remark 1.2.2. Finite quadratic modules arise naturally as $(L'/L, Q \pmod 1)$ where L is an even lattice with quadratic form Q and L' is the dual lattice. By identifying elements in $\gamma \in L'/L$ with a representative in $\gamma \in L'$, the conditions $Q(\gamma) = 0 \pmod 1$, $Q(\gamma) = \mathbb{Z}$ and $Q(\gamma) \in \mathbb{Z}$ all mean the same. If D arises from an even lattice in this manner, it is also called the *discriminant form* of the lattice L. Every finite quadratic module can be obtained as a discriminant form.

Definition 1.2.3. Let (D, Q) and $(\tilde{D}, \tilde{Q})$ be finite quadratic modules. An injective homomorphism $\sigma : D \to \tilde{D}$ satisfying $Q = \tilde{Q} \circ \sigma$ is called an *isometry*. If σ is bijective, the quadratic spaces are called *isometric* and we write $D \simeq \tilde{D}$. The *orthogonal group* of D is the group of isometries from M onto itself and denoted by $O(D)$.

Definition 1.2.4. We define the (orthogonal) direct sum of two finite quadratic modules (D_1, Q_1) and (D_2, Q_2) by

$$(D_1 \oplus D_2, Q_1 \oplus Q_2) := (D_1 \times D_2, (\gamma_1, \gamma_2) \mapsto Q_1(\gamma_1) + Q_2(\gamma_2)). \tag{1.2.1}$$

We say that a finite quadratic module D *decomposes* into D_1 and D_2, if it is isometric to $D_1 \oplus D_2$. If D cannot be decomposed, it is called *irreducible*.

Definition 1.2.5. The *level* of D is the smallest positive integer N such that $N \cdot Q(\gamma) \in \mathbb{Z}$ for all $\gamma \in D$.

Any finite quadratic module D decomposes (orthogonally) into its maximal p-subgroups. We write

$$D = \bigoplus_{p|N} D_p, \tag{1.2.2}$$

where N is the level of D and p runs through the prime divisors of N. These D_p decompose into Jordan components for which we introduce the symbols $q^{\pm n}$ (p-adic components) where q is the power of an odd prime p and the symbols $q_t^{\pm n}$ (odd 2-adic components) and $q_{II}^{\pm 2n}$ (even 2-adic components) where q is a power of 2, $t \in \mathbb{Z}/8\mathbb{Z}$ and $n \equiv t \pmod 2$. The irreducible Jordan components are the ones with $n = 1$. Jordan components for the same p-power compose and decompose as follows.

$$q^{\varepsilon_1 n_1} \oplus q^{\varepsilon_2 n_2} \simeq q^{(\varepsilon_1 \cdot \varepsilon_2)(n_1+n_2)}, \tag{1.2.3}$$

$$q_{t_1}^{\varepsilon_1 n_1} \oplus q_{t_2}^{\varepsilon_2 n_2} \simeq q_{t_1+t_2}^{(\varepsilon_1 \cdot \varepsilon_2)(n_1+n_2)}, \tag{1.2.4}$$

$$q_{t_1}^{\varepsilon_1 n_1} \oplus q_{II}^{\varepsilon_2 2n_2} \simeq q_{t_1}^{(\varepsilon_1 \cdot \varepsilon_2)(n_1+2n_2)}. \tag{1.2.5}$$

The ranks and subscripts add and the signs multiply.

Definition 1.2.6. Let q be the power of an odd prime p. The irreducible p-adic Jordan components of exponent q are given by

$$q^{\varepsilon 1} := \left(\mathbb{Z}/q\mathbb{Z}, \gamma \mapsto \frac{a}{q}\gamma^2\right), \text{ with } \left(\frac{2a}{p}\right) = \varepsilon. \tag{1.2.6}$$

If q is a power of 2, the irreducible even 2-adic Jordan components of exponent q are

$$q_{II}^{+2} := ((\mathbb{Z}/q\mathbb{Z})^2, (\gamma_1, \gamma_2) \mapsto \frac{1}{q}\gamma_1\gamma_2), \tag{1.2.7}$$

$$q_{II}^{-2} := ((\mathbb{Z}/q\mathbb{Z})^2, (\gamma_1, \gamma_2) \mapsto \frac{1}{q}(\gamma_1^2 + \gamma_1\gamma_2 + \gamma_2^2)) \tag{1.2.8}$$

and the irreducible odd 2-adic Jordan components of exponent q are

$$q_t^{\varepsilon 1} := \left(\mathbb{Z}/q\mathbb{Z}, \gamma \mapsto \frac{t}{2q}\gamma^2\right), \text{ with } \left(\frac{t}{2}\right) = \varepsilon. \tag{1.2.9}$$

We will further need the following invariants.

Definition 1.2.7. For an odd prime p and a p-power q we define the *p-excess*. For $p = 2$ and q a power of 2 we define the *oddity*. These invariants are given by

$$p\text{-excess}(q^{\varepsilon n}) := \begin{cases} n(q-1) \pmod 8, & \text{if } q = \square \text{ or } \varepsilon = +1, \\ n(q-1) + 4 \pmod 8, & \text{else,} \end{cases} \tag{1.2.10}$$

$$\text{oddity}(q_{II}^{\varepsilon 2n}) := \begin{cases} 0 \pmod 8, & \text{if } q = \square \text{ or } \varepsilon = +1, \\ 4 \pmod 8, & \text{else,} \end{cases} \tag{1.2.11}$$

$$\text{oddity}(q_t^{\varepsilon n}) := \begin{cases} t \pmod 8, & \text{if } q = \square \text{ or } \varepsilon = +1, \\ t + 4 \pmod 8, & \text{else,} \end{cases} \tag{1.2.12}$$

where $q = \square$ means that q is a square. We extend these invariants to any finite quadratic module by adding the invariants of the components in its Jordan decomposition. This is well defined and yields additive invariants with respect to orthogonal direct sums. Further we define the multiplicative *Weil invariants* $\gamma_p(D)$ by setting

$$\gamma_p(D) = \begin{cases} e\left(\text{oddity}(D)/8\right), & \text{if } p = 2, \\ e\left(-p\text{-excess}/8\right), & \text{else.} \end{cases} \tag{1.2.13}$$

Remark 1.2.8. The Weil invariants of a finite quadratic module correspond to the classical Weil invariants defined in [Wei64]. A proof can be found in [Zem15, Prop. 4.1].

Definition 1.2.9. We define the *signature* of a finite quadratic module D by

$$\text{sign}(D) := \text{oddity}(D) - \sum_{p \geq 3} p\text{-excess}(D) \pmod 8. \tag{1.2.14}$$

If D is the discriminant form of an even lattice L, this agrees with the definition

$$\text{sign}(D) := \text{sign}(L) \pmod 8. \tag{1.2.15}$$

In this case equation (1.2.14) is called oddity formula (or Weil reciprocity law [Wei64, Prop. 5]).

1.3. Lattices

Let R be the ring of integers $\mathbb{Z}$ or a ring of p-adic integers $\mathbb{Z}_p$. Let F be the corresponding field of fractions, that is $\mathbb{Q}$ or $\mathbb{Q}_p$.

Definition 1.3.1. An *R-lattice* L is a finitely generated R-module with a quadratic form $Q : L \otimes_R F \to F$ such that the quadratic space $(L \otimes_R F, Q)$ is non-degenerate. By *lattice*, we mean a $\mathbb{Z}$-lattice and by p-adic lattice, we mean a $\mathbb{Z}_p$-lattice.

Definition 1.3.2. For an R-lattice L with associated quadratic form Q and a factor $f \in R$, we denote by $L(f)$ the lattice L with scaled quadratic form $f \cdot Q$.

Definition 1.3.3. For an R-lattice L, we define its *dual lattice* L' as

$$L' := \{\lambda \in L \otimes_R F \mid (\lambda, \mu) \in R \text{ for all } \mu \in L\} \tag{1.3.1}$$

which is contained in the quadratic space associated to L.

Definition 1.3.4. If we $L' = L$, we call the lattice L *unimodular*. We define the *hyperbolic plane* as the unimodular lattice

$$U = \left(\mathbb{Z}^2, (x, y) \mapsto xy\right).$$

Definition 1.3.5. We define the *Gram matrix* of an R-lattice L with respect to a basis $b_1, \ldots, b_n$ of L to be the matrix S with coefficients (b_i, b_j). This allows us to define

$$\det(L) := \det(S). \tag{1.3.2}$$

Further we define the *discriminant* of this lattice by

$$\operatorname{disc}(L) := (-1)^{\frac{n(n-1)}{2}} \det(L), \tag{1.3.3}$$

where n is the rank of the lattice L.

Choosing a different R-basis for the lattice can change the determinant by a factor in $(R^\times)^2$. If L is a $\mathbb{Z}$-lattice, the determinant is well-defined. Note that the discriminant remains unchanged when adding hyperbolic planes.

Definition 1.3.6. An R-lattice L is called *integral* if its bilinear form takes values in R, that is $(\lambda, \mu) \in R$ for all $\lambda, \mu \in L$. If its quadratic form takes values in R, it is called *even*.

We see that even lattices are integral. Also a lattice is integral if and only if $L \subset L'$. In this case we have the equality

$$|L'/L| = |\det(L)| = |\operatorname{disc}(L)|. \tag{1.3.4}$$

If the rank of L is odd, then its discriminant form must have an odd 2-adic component which is equivalent to $2|\det(L)$ or $2|\operatorname{disc}(L)$ (cf. [Kne02, chapter 2]).

Definition 1.3.7. The *discriminant form* of an even $\mathbb{Z}$-lattice L with quadratic form Q is the finite quadratic module $(L'/L, Q \pmod 1)$.

Definition 1.3.8. The *level* of an even lattice is the level of its discriminant form. This is the smallest positive integer N such that $N \cdot Q(\gamma) \in \mathbb{Z}$ for all $\gamma \in L'$.

From now on let L be an even lattice. We define the associated p-adic lattices to be $L_p := L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ with quadratic form $Q_p(\lambda \otimes s) = s^2 Q(\lambda)$. In the same manner we can allow real scalars, to associate the real quadratic space $L \otimes_{\mathbb{Z}} \mathbb{R}$ to L.

Definition 1.3.9. The *type* and *signature* $\mathrm{sign}(L)$ of a lattice L are given by the corresponding qualities of the real quadratic space associated to L.

The associated p-adic lattices also appear in the decomposition of the discriminant form of L, that is

$$L'/L \simeq \bigoplus_p L'_p/L_p \tag{1.3.5}$$

where the concrete isometry is given by the Chinese Remainder Theorem. The quadratic form on the components of the right hand side can be obtained by applying the Chinese Remainder Theorem as a group isomorphism and then taking the quadratic form on L'/L. This is isomorphic to choosing

$$L'_p/L_p \xrightarrow{Q_p} \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\simeq} \mathbb{Z}[p^{-1}]/\mathbb{Z} \xrightarrow{\subseteq} \mathbb{Q}/\mathbb{Z}. \tag{1.3.6}$$

An isometry proving (1.3.5) is constructed explicitly in Section 4.1, where we give change of basis $T_p \in \mathrm{GL}_n(\mathbb{Z}_p)$ such that

$$L'_p/L_p \xrightarrow{T} L'_p/L_p \xrightarrow{Q_p} \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\simeq} \mathbb{Z}[p^{-1}]/\mathbb{Z} \hookrightarrow \mathbb{Q}/\mathbb{Z} \tag{1.3.7}$$

gives the quadratic form for the right hand side of (1.3.5).

1.4. Canonical forms

Every p-adic lattice is an orthogonal direct sum of canonical forms. For proofs on how to obtain such a direct sum (including the needed change of basis), see [Cas78]. We construct the missing change of basis in the $p = 2$ case in Section A in the appendix. The canonical forms correspond to Jordan components of discriminant forms and are as follows.

Lemma 1.4.1. *Let $p \neq 2$ and $L_p \simeq \mathbb{Z}_p^n$ with Gram matrix S_p. Then S_p is $\mathrm{GL}_n(\mathbb{Z}_p)$-equivalent to a matrix of the form*

$$\mathrm{diag}(2\varepsilon_1 p^{l_1}, \ldots, 2\varepsilon_n p^{l_n}), \tag{1.4.1}$$

with $\varepsilon_i \in \mathbb{Z}_p^\times$ and $l_i \in \mathbb{Z}$. If L_p is even, we have $l_i \geq 0$. A block of length r of the form $p^l \cdot \mathrm{diag}(4, \ldots, 4, 4a)$ corresponds to $(p^l)^{\varepsilon r}$ in the Jordan decomposition, where $\varepsilon = \left(\frac{a}{p}\right)$.

Let $p = 2$ and $L_2 \simeq \mathbb{Z}_2^n$ with Gram matrix S_2. Then S_2 is $\mathrm{GL}_n(\mathbb{Z}_2)$-equivalent to a matrix of the form

$$\mathrm{diag}(\varepsilon_1 p^{l_1}, \ldots, \varepsilon_n p^{l_H}) \oplus \left(\bigoplus_{i=1}^{M} 2^{m_i} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \oplus \left(\bigoplus_{j=1}^{N} 2^{n_j} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \right), \tag{1.4.2}$$

with $\varepsilon_h \in \mathbb{Z}_2^\times$ and $l_h, m_i, n_j \in \mathbb{Z}$. If L_2 is even, we have $l_h \geq 1$, $m_i \geq 0$ and $n_j \geq 0$. A block of the form $(t2^l)$ corresponds to $(p^l)_t^{\varepsilon 1}$ in the Jordan decomposition, where $\varepsilon = \left(\frac{t}{2}\right)$. Blocks of the form $2^m \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ correspond to components $(2^m)_{II}^{+2}$. Analogously the form $2^m \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ corresponds to $(2^m)_{II}^{-2}$.

Conversely, this lemma allows us to “guess” local Gram matrices for discriminant forms. We will use such a guess to compute Eisenstein series for even lattices purely from local data, that is, purely from the genus symbol of the lattice in Section 3.4.

1.5. Genus symbols

Definition 1.5.1. Two R-lattices L, M are *isometric* if there is an isometry σ of the associated quadratic spaces mapping one lattice to the other, that is $\sigma(L) = M$. In this case we write $L \simeq M$ and say that these lattices are in the same (isometry) class. We say that two $\mathbb{Z}$-lattices L and M are in the same *genus*, if we have $L_p \simeq M_p$ for all primes p and $L \otimes \mathbb{R} \simeq M \otimes \mathbb{R}$.

Every genus consists of finitely many isometry classes [Kne02, Satz 21.3]. If the lattices L and M are in the same genus, then $L + H \simeq M + H$ are isomorphic, i.e. they belong to the same class [Kne02, (27.6)]. From this we can deduce that L and M represent the same element in the Witt group and hence have the same discriminant.

Definition 1.5.2. We define the *genus symbol* of a discriminant form or a lattice to be the symbol of its Jordan decomposition. A genus is determined by its genus symbol together with the type of its lattices.

Remark 1.5.3. It is possible, to obtain the discriminant of a lattice L purely from its genus symbol. This is done as follows:

The genus symbol encodes the Jordan decomposition of the discriminant form of L. From the oddity formula, we know the signature $s = b^+ - b^- \pmod 8$ of L and in particular the parity of the rank $n = b^+ + b^-$ of L. We have

$$\mathrm{sign}(\mathrm{disc}(L)) = (-1)^{\frac{n(n-1)}{2}} \mathrm{sign}(\det(L)) = (-1)^{\frac{n(n-1)}{2}} (-1)^{-b^-} = \begin{cases} (-1)^{\frac{b^+ - b^-}{2}}, & n \text{ even}, \\ (-1)^{\frac{b^+ - b^- - 1}{2}}, & n \text{ odd}. \end{cases}$$

From this we deduce

$$\operatorname{disc}(L) = \left\{ \begin{array}{ll} (-1)^{\frac{b^+ - b^-}{2}} |L'/L|, & n \text{ even}, \\ (-1)^{\frac{b^+ - b^- - 1}{2}} |L'/L|, & n \text{ odd}, \end{array} \right\} = (-1)^{\frac{s(s-1)}{2}} |L'/L|. \tag{1.5.1}$$

The right hand side can be computed purely from the genus symbol without knowledge of the rank of L.

Remark 1.5.4. The parity of the rank of L is determined by the 2-adic components in the Jordan decomposition.

Example 1.5.5. (i) The genus symbol 3^{-1} stands for the finite quadratic module

$$\left(\mathbb{Z}/3\mathbb{Z},\ x \mapsto \frac{x^2}{3} \pmod 1 \right),$$

which we can also realize as

$$\left(\frac{1}{3}\mathbb{Z}/\mathbb{Z},\ x \mapsto 3x^2 \pmod 1 \right).$$

We calculate

$$\operatorname{sign}(3^{-1}) = 0 - 3\text{-excess}(3^{-1}) = 0 - 6 \equiv 2 \pmod 8$$

and

$$\operatorname{disc}(3^{-1}) = (-1)^{\frac{2}{2}} |\mathbb{Z}/3\mathbb{Z}| = -3.$$

If we want to realize 3^{-1} as the discriminant form of an even lattice L, then L must have even rank. It is easily seen, that $L = (\mathbb{Z}^2, x \mapsto x^t \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right) x)$ fulfills the above criteria. Over $\mathbb{Z}_3$, the Gram matrix of the lattice is equivalent to $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 6 \end{smallmatrix}\right)$, which is itself equivalent to $\left(\begin{smallmatrix} 4\cdot 2 & 0 \\ 0 & 4\cdot 2\cdot 3 \end{smallmatrix}\right)$ with $\left(\frac{2}{3}\right) = -1$ in accordance with Lemma 1.4.1.

(ii) The genus symbol $2_3^{-1}8^{-4}3^{-1}$ can be decomposed into irreducible components as $2_3^{-1}8^{+2}8^{-2}3^{-1}$. This means we can realize it as

$$\begin{aligned} &\mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/4\mathbb{Z})^2 \times \mathbb{Z}/3\mathbb{Z}, \\ &Q(x_1, \ldots, x_6) = \frac{3}{4}x_1^2 + \frac{1}{8}x_2x_3 + \frac{1}{8}\left(x_4^2 + x_4x_5 + x_5^2\right) + \frac{1}{3}x_6^2 \pmod 1. \end{aligned}$$

This finite quadratic module has level 12. Its signature is

$$\begin{aligned} s &\equiv \operatorname{sign}(2_3^{-1}8^{-4}3^{-1}) \\ &\equiv \operatorname{oddity}(2_3^{-1}) + \operatorname{oddity}(8^{-4}) - 3\text{-excess}(3^{-1}) \\ &\equiv (3 + 4) + 4 - 6 \equiv 5 \pmod 8 \end{aligned}$$

and the discriminant is given by

$$\mathrm{disc}(2_3^{-1}8^{-4}3^{-1}) = (-1)^{\frac{s(s-1)}{2}} \cdot 2 \cdot 8^4 \cdot 3 = 24576.$$

1.6. The Weil representation

For the definition of the Weil representation in our case, we follow [Bru02a]. Other aspects of the Weil representation are introduced where needed. We realize the metaplectic double cover $\mathrm{Mp}_2(\mathbb{R})$ of $\mathrm{SL}_2(\mathbb{R})$ as the group of pairs $(M, \phi(\tau))$, where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$, and ϕ is a holomorphic square root of $c\tau + d$ for $\tau \in \mathbb{H}$. Further we define $\mathrm{Mp}_2(\mathbb{Z})$ as the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ under the covering map $\mathrm{Mp}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$. It is well known that $\mathrm{Mp}_2(\mathbb{Z})$ is generated by

$$T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right) \quad \text{and} \quad S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right) \tag{1.6.1}$$

The Weil representation ρ_L with respect to an even lattice L is a unitary representation of $\mathrm{Mp}_2(\mathbb{Z})$ on the group ring $\mathbb{C}[L'/L]$ which we endow with the standard scalar product. If we denote the standard basis of $\mathbb{C}[L'/L]$ by $(\mathfrak{e}_\gamma)_{\gamma \in L'/L}$, the Weil representation is defined by

$$\rho_L(T)\mathfrak{e}_\gamma = e(Q(\gamma))\mathfrak{e}_\gamma, \qquad \rho_L(S)\mathfrak{e}_\gamma = \frac{e((b^- - b^+)/8)}{\sqrt{|L'/L|}} \sum_{\delta \in L'/L} e(-(\gamma, \delta))\mathfrak{e}_\delta, \tag{1.6.2}$$

where $b^+ - b^-$ is the signature of L. We see that the signature of L is only needed modulo 8 and that the Weil representation only depends on the discriminant form of L.

We denote the dual Weil representation by ρ_L^* which is essentially the same as the Weil representation for the lattice $L(-1)$.

Definition 1.6.1. For a half-integer $k \in \frac{1}{2}\mathbb{Z}$ and an even lattice L, we define actions of $Mp_2(\mathbb{Z})$ on $\mathbb{C}[L'/L]$-valued functions on $\mathbb{H}$ via the Petersson slash operator

$$f|_{k,L}(M, \phi)(\tau) = \phi(\tau)^{-2k} \rho_L(M, \phi)^{-1} f(M\tau), \tag{1.6.3}$$

$$f|_{k,L}^*(M, \phi)(\tau) = \phi(\tau)^{-2k} \rho_L^*(M, \phi)^{-1} f(M\tau). \tag{1.6.4}$$

1.7. Vector valued modular forms

Definition 1.7.1. A holomorphic function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ is called a *(weakly holomorphic) modular form* of *weight* $k \in \frac{1}{2}\mathbb{Z}$ with respect to ρ_L if

(i) $f|_k(M, \phi) = f$ for all $(M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$ and

(ii) f has at most a pole at ∞.

More precisely, this means that f has a Fourier expansion of the form

$$f(\tau) = \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n \gg -\infty}} a_f(\gamma, n) e(n\tau) \mathfrak{e}_\gamma, \tag{1.7.1}$$

that is, $a_f(\gamma, n) = 0$ for all but finitely many $n < 0$. The finite Fourier polynomial

$$P_f(\tau) = \sum_{\gamma \in L'/L} \sum_{n<0} a_f(\gamma, n) e(n\tau) \mathfrak{e}_\gamma \tag{1.7.2}$$

is called the principal part of f. If the principal part is 0, we call f a *(holomorphic) modular form*. Additionally if all Fourier coefficients $a_f(\gamma, n)$ vanish for $n \leq 0$, we call f a *cusp form*. We denote the space of cusp (resp. holomorphic modular, resp. weakly holomorphic modular) forms by S_{k,ρ_L}, (resp. M_{k,ρ_L}, resp. $M^!_{k,\rho_L}$) and we have the natural inclusions

$$S_{k,\rho_L} \subset M_{k,\rho_L} \subset M^!_{k,\rho_L}.$$

Note that the coefficients of f satisfy the symmetry $a_f(\gamma, n) = (-1)^{k-(b^+-b^-)/2} a_f(-\gamma, n)$ for all $n \in \mathbb{Q}, \gamma \in L'/L$. Every finite sum as in (1.7.2) which satisfies this symmetry will be called a *formal principal part*.

1.8. Vector valued Eisenstein series

We fix an even lattice L and a weight k such that $2 < k \in \frac{1}{2}\mathbb{Z}$. In analogy to the case of classical modular forms, we define vector valued *Eisenstein series of weight k* with respect to ρ_L by

$$E_{\rho_L,k,\beta} = \frac{1}{4} \sum_{(M,\phi) \in \langle T \rangle \backslash Mp_2(\mathbb{Z})} \mathfrak{e}_\beta|_{k,L}(M, \phi) \tag{1.8.1}$$

for any isotropic vector $\beta \in L'/L$. The E_β define holomorphic modular forms and as in the classical case, every holomorphic modular form for ρ_L can be written as a linear combination of Eisenstein series and cusp forms.

In this thesis, we use formulas from [KY10], to compute the Fourier coefficients of $E = E_{\rho_L,k,0}$ with `sage`. For a positive definite lattice of rank greater than 4, the Eisenstein series E appears in the Siegel-Weil formula as a weighted sum of vector valued theta series for representatives of classes in the same genus. We use the Siegel-Weil formula to test our program. A second test is provided using the theory of Jacobi forms.

The Fourier coefficients of $2E_{\rho_L^*,k,0}$ with respect to the dual Weil representation have also been previously computed in [BK01]. Note that these formulas can be evaluated using Igusa local zeta functions and that this approach is described in [Wil18c]. We use these formulas to estimate the Fourier coefficients in the case that the lattice L splits a scaled hyperbolic plane. As these Fourier coefficients are an obstruction to the existence of certain Borcherds products, their estimates play a crucial role in trying to find new

Borcherds products systematically.

1.9. Jacobi Forms

The Algorithm for the computation of vector valued modular forms for the Weil representation can be tested against the theory of Jacobi forms. Our standard reference is [EZ85].

Jacobi forms of even weight $k \in \mathbb{Z}_{\geq 4}$ and index $m \in \mathbb{Z}_{>0}$ for the group $\mathrm{SL}_2(\mathbb{Z})$ are holomorphic functions

$$f : \mathbb{H} \times \mathbb{C} \to \mathbb{C}.$$

that satisfy the following transformation laws

(i) $f(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}) = (c\tau+d)^k e^{\frac{2\pi i m c z^2}{c\tau+d}} f(\tau, z)$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$,

(ii) $f(\tau, z+\lambda\tau+\mu) = (c\tau+d)^k e^{-2\pi i m(\lambda^2\tau+2\lambda z)} f(\tau, z)$ for $(\lambda, \tau) \in \mathbb{Z}^2$,

and have a Fourier expansion of the form

(iii) $f(\tau, z) = \sum_{n=0}^{\infty} \sum_{\substack{r\in\mathbb{Z} \\ r^2 \leq 4nm}} c(n,r) e^{2\pi i(n\tau+rz)}$.

As in the case of (vector valued) modular forms, we obtain examples of Jacobi forms by defining Jacobi-Eisenstein series. This approach yields

$$E_{k,m} = \frac{1}{2} \sum_{\substack{c,d\in\mathbb{Z} \\ (c,d)=1}} \sum_{\lambda\in\mathbb{Z}} (c\tau+d)^{-k} e^m \left(\lambda^2 \frac{a\tau+b}{c\tau+d} + 2\lambda \frac{z}{c\tau+d} - \frac{c^2}{c\tau+d}\right), \tag{1.9.1}$$

the Jacobi-Eisenstein series of weight k and index m. These series correspond to vector valued Eisenstein series for a certain lattice (cf. Section 3.6). This allows us to test the Fourier coefficients of these series against each other. The Fourier expansion is given by the following theorem.

Theorem 1.9.1. *The Fourier coefficients $e_{k,m}(n,r)$ of the Jacobi-Eisenstein series of weight k and index m are given by 1 if $4nm = r^2$ and $r \equiv 0 \mod 2m$, and by 0 if $4nm = r^2$ and $r \not\equiv 0 \mod 2m$. If $4nm > r^2$, we have*

$$e_{k,1}(n,r) = \frac{L_{r^2-4n}(2-k)}{\zeta(3-2k)}, \tag{1.9.2}$$

$$e_{k,m}(n,r) = m^{-k+1} \prod_{p|m} \left(1+p^{-k+1}\right)^{-1} \sum_{\substack{d^2|m \\ d|r}} \mu(d) \sum_{a|(n,\frac{r}{d},\frac{m}{d^2})} a^{k-1} e_{k,1}(\frac{nm}{a^2d^2}, \frac{r}{ad}). \tag{1.9.3}$$

The Dirichlet L-series $L_D(s)$ is the usual one if D is a fundamental discriminant, i.e. $D = 1$ or D is the discriminant of the quadratic field $\mathbb{Q}(\sqrt{D})$. If D is not a fundamental *discriminant, we may write* $D = D_0 f^2$ *where* $f \in \mathbb{Z}_{\geq 0}$ *and* D_0 *is the discriminant of* $\mathbb{Q}(\sqrt{D})$*. We extend the definition of Dirichlet L-series to*

$$L_D(s) = \begin{cases} 0, & \text{if } D \not\equiv 0,1 \mod 4, \\ \zeta(2s-1), & \text{if } D = 0, \\ L_{D_0}(s) \sum_{d|f} \mu(d) \left(\frac{D_0}{d}\right) d^{-s} \sigma_{1-2s}(\frac{f}{d}), & \text{if } D \equiv 0,1 \mod 4,\ D \neq 0. \end{cases} \tag{1.9.4}$$

Proof. The formula for $e_{k,1}$ is given in [EZ85, Theorem 2.1], the general formula follows from equation (13) in [EZ85, in Chapter 4] together with the formulas for Fourier coefficients under Hecke operations in the same chapter. □

Remark 1.9.2. If m is square-free, the factors in front of the sum simplify to

$$m^{-k+1} \prod_{p|m} \left(1 + p^{-k+1}\right)^{-1} = \sigma_{1-k}(m)^{-1} \tag{1.9.5}$$

in accordance with equation (7) after Theorem 2.1 in [EZ85].

1.10. Idele class characters and Dirichlet characters

It is a well-known fact that idele class characters with finite image correspond one-to-one to primitive Dirichlet characters. We work out this correspondence explicitly in the case of quadratic characters. In this case, an idele class character given by a Hilbert symbol corresponds to a Dirichlet character given by a Kronecker symbol. The general correspondence can be found in [Dei10]. More information on Hilbert symbols can be found in [Ser73]. In the treatment of Dirichlet characters we follow [Zag81].

Let $\hat{\mathbb{Z}} = \prod_{p<\infty} \mathbb{Z}_p$. We denote the finite adeles by $\mathbb{A}_f = \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ and the adeles by $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$. For an element $x = (x_p)_{p\leq\infty} \in \mathbb{A}$, we denote by $x_f = (x_p)_{p<\infty} \in \mathbb{A}_f$ the finite part of x. We denote the ideles by $\mathbb{A}^\times$ and refer to [Dei10, Chapter 5] for the topology of $\mathbb{A}$ and $\mathbb{A}^\times$ which is given by the topology of restricted products. The rational numbers can be embedded diagonally into $\mathbb{A}$ and we have $\mathbb{Q}^\times \subset \mathbb{A}^1 = \{a \in \mathbb{A}^\times \mid |a| = 1\}$ where the norm is given by

$$|a| = \prod_{p\leq\infty} |a_p|_p.$$

The isomorphism

$$\begin{aligned} \mathbb{A}^\times/\mathbb{Q}^\times &\simeq \mathbb{A}^1/\mathbb{Q}^\times \times (0,\infty), \\ (x_f, x_\infty)\mathbb{Q}^\times &\mapsto ((x_f, \frac{x_\infty}{|x|})\mathbb{Q}^\times, |x|), \\ (x_f, r x_\infty)\mathbb{Q}^\times &\hookleftarrow ((x_f, x_\infty)\mathbb{Q}^\times, r) \end{aligned}$$

together with the fact that characters of $\mathbb{A}^1/\mathbb{Q}^\times$ have finite image shows that characters of

$\mathbb{A}^1/\mathbb{Q}^\times$ correspond to characters of $\mathbb{A}^\times/\mathbb{Q}^\times$ with finite image. The isomorphism

$$\begin{aligned}\mathbb{A}^1/\mathbb{Q}^\times &\simeq \hat{\mathbb{Z}}^\times = (\prod_{p<\infty} \mathbb{Z}_p)^\times = \prod_{p<\infty} \mathbb{Z}_p^\times,\\ (x_p)\mathbb{Q}^\times &\mapsto \frac{1}{x_\infty} x_f,\\ (x_f, 1)\mathbb{Q}^\times &\leftarrow\!\shortmid x_f\end{aligned}$$

allows us to use the Chinese Remainder Theorem. We have

$$\begin{aligned}&\prod_{p<\infty} \mathbb{Z}_p^\times \simeq \varprojlim_{N\in\mathbb{Z}_{\geq 0}} (\mathbb{Z}/N\mathbb{Z})^\times,\\ &((z_{p^j})_j)_p \mapsto (z_N)_N,\ z_N := \sum_{i=1}^n z_{p_i^{j_i}} \frac{N}{p_i^{j_i}} \left(\frac{N}{p_i^{j_i}} \pmod{p_i^{j_i}}\right)^{-1} \pmod N \ \text{ for } \ N = \prod_{i=1}^n p_i^{j_i},\\ &((z_{p^j})_j)_p \leftarrow\!\shortmid (z_N)_N.\end{aligned}$$

The correspondence of characters uses the following lemma.

Lemma 1.10.1 ([Dei10, Lemma 6.3.2]). *The isomorphism*

$$\mathbb{A}^1/\mathbb{Q}^\times \simeq \prod_{p<\infty} \mathbb{Z}_p^\times \simeq \varprojlim_{N\in\mathbb{Z}_{\geq 0}} (\mathbb{Z}/N\mathbb{Z})^\times$$

induces a bijection between the set of characters of the group $\mathbb{A}^1/\mathbb{Q}^\times$ and the set of all primitive Dirichlet characters as follows: For a primitive Dirichlet character χ with modulus N_0, the composition

$$\mathbb{A}^1/\mathbb{Q}^\times \simeq \varprojlim_{N\in\mathbb{Z}_{\geq 0}} (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\text{proj.}} (\mathbb{Z}/N_0\mathbb{Z})^\times \xrightarrow{\chi} \mathbb{T}$$

defines a character of $\mathbb{A}^1/\mathbb{Q}^\times$.

Let $\kappa \in \mathbb{Q}\backslash\{0\}$. The character $(\,\cdot\,, \kappa) : \mathbb{A}^\times/\mathbb{Q}^\times \to \mathbb{T}$ has finite image. Without loss of generality we may assume that κ is a square-free integer. Working through the various isomorphisms yields the following correspondence.

Correspondence 1.10.2.

$$
\begin{aligned}
(\,\cdot\,,\kappa) : \mathbb{A}^\times/\mathbb{Q}^\times &\to \mathbb{T}, \\
(x_p) &\mapsto \prod_{p\le\infty} (x_p,\kappa)_p
\end{aligned}
$$

$$\updownarrow$$

$$
\begin{aligned}
(\,\cdot\,,\kappa) : \mathbb{A}^1/\mathbb{Q}^\times &\to \mathbb{T}, \\
(x_p) &\mapsto \prod_{p\le\infty} (x_p,\kappa)_p
\end{aligned}
$$

$$\updownarrow$$

$$
\begin{aligned}
(\,\cdot\,,\kappa) : \hat{\mathbb{Z}}^\times = \prod_{p<\infty} \mathbb{Z}_p^\times &\to \mathbb{T}, \\
(x_p) &\mapsto \prod_{p<\infty} (x_p,\kappa)_p
\end{aligned}
$$

$$\updownarrow$$

$$
\begin{aligned}
(\,\cdot\,,\kappa) : \varprojlim_{N\in\mathbb{Z}_{\ge 0}} (\mathbb{Z}/N\mathbb{Z})^\times &\to \mathbb{T}, \\
(z_N) &\mapsto \prod_{p<\infty} (x_p,\kappa)_p = (z_8,\kappa)_2 \prod_{2<p<\infty} (z_p,\kappa)_p \\
&\quad \textit{where } x_p = (z_p, z_{p^2}, z_{p^3}, \ldots)
\end{aligned}
$$

$$\updownarrow$$

$$
\begin{aligned}
\chi_D = \left(\frac{D}{\cdot}\right) : \mathbb{Z}/D\mathbb{Z} &\to \mathbb{T}, \\
n &\mapsto \left(\frac{D}{n}\right),
\end{aligned}
$$

where D is the fundamental discriminant defined by

$$
D = \begin{cases} \kappa, & \textit{if } \kappa \equiv 1 \mod 4, \\ 4\kappa, & \textit{else.} \end{cases}
$$

The last step in this correspondence deserves some more attention. The Hilbert symbol on the fields $\mathbb{Q}_p$ for $p \le \infty$ can be evaluated using Legendre symbols as in the following theorem.

Theorem 1.10.3 ([Ser73, Chapter III, Theorem 1]). *For $a, b \in \mathbb{R}^\times$, the Hilbert symbol $(a,b)_\infty$ is 1 if $a > 0$ or $b > 0$ and -1 if $a < 0$ and $b < 0$. For $a, b \in \mathbb{Q}_p^\times$ we write $a = p^\alpha u$ and $b = p^\beta v$ with $u = (u_p, u_{p^2}, \ldots), v = (v_p, v_{p^2}, \ldots) \in \mathbb{Z}_p^\times$. We then have*

$$\begin{aligned} (a,b)_p &= (-1)^{\alpha\beta\varepsilon(p)} \left(\frac{u_p}{p}\right)^\beta \left(\frac{v_p}{p}\right)^\alpha && \text{if } 2 < p < \infty, \\ (a,b)_2 &= (-1)^{\varepsilon(u_4)\varepsilon(v_4)+\alpha\omega(v_8)+\beta\omega(u_8)} && \text{if } p = 2. \end{aligned}$$

Here $\varepsilon(t) := \frac{t-1}{2} \pmod 2$ and $\omega(t) := \frac{t^2-1}{8} \pmod 2$.

We see that 2-adic integers are only needed modulo 8 in the 2-adic Hilbert symbol, for all other primes p the p-adic integers are only needed modulo p. The last step in Correspondence 1.10.2 is now a matter of case distinction. Recall that κ is a square-free integer and $z_p \in \mathbb{Z}_p^\times$.

If p is odd and $p \nmid \kappa$, we have

$$(x_p, \kappa)_p = (-1)^{0 \cdot 0 \cdot \varepsilon(p)} \left(\frac{z_p}{p}\right)^0 \left(\frac{\kappa}{p}\right)^0 = 1.$$

If p is odd and $p \mid \kappa$, we get

$$(x_p, \kappa)_p = (-1)^{0 \cdot 1 \cdot \varepsilon(p)} \left(\frac{z_p}{p}\right)^1 \left(\frac{p^{-1}\kappa}{p}\right)^0 = \left(\frac{z_p}{p}\right) = \left(\frac{p'}{z_p}\right) = \chi_{p'}(z_p),$$

using quadratic reciprocity with $p' = (-1)^{\varepsilon(p)}p$.

If $p = 2$, we see that

$$\begin{aligned} (x_2, \kappa)_2 &= (-1)^{\varepsilon(z_4)\varepsilon(2^{-\nu_2(\kappa)}\kappa)+\nu_2(\kappa)\omega(z_8)} \\ &= \begin{cases} (-1)^{\varepsilon(z_4)\varepsilon(\kappa)}, & \text{if } 2 \nmid \kappa, \\ (-1)^{\varepsilon(z_4)\varepsilon(2^{-1}\kappa)+\omega(z_8)}, & \text{if } 2 \mid\mid \kappa, \end{cases} \\ &= \begin{cases} 1, & \text{if } \kappa \equiv 1 \mod 4, \\ \left(\frac{-4}{z_4}\right) = \chi_{-4}(z_4), & \text{if } \kappa \equiv 3 \mod 4, \\ \left(\frac{8}{z_8}\right) = \chi_8(z_8), & \text{if } \kappa \equiv 2 \mod 8, \\ \left(\frac{-8}{z_8}\right) = \chi_{-8}(z_8), & \text{if } \kappa \equiv 6 \mod 8, \end{cases} \end{aligned}$$

is given in terms of Kronecker symbols.

Putting all cases together using the fact that primitive Dirichlet characters with coprime modulus are multiplicative yields

$$
\begin{aligned}
(z_8,\kappa)_2 \prod_{2<p<\infty} (z_p,\kappa)_p &= (z_8,\kappa)_2 \prod_{\substack{p\mid\kappa \\ p\neq 2}} (z_p,\kappa)_p \\
&= \begin{cases}
\prod_{p\mid\kappa} \chi_{p'}(z_p), & \text{if } \kappa \equiv 1 \mod 4, \\
\chi_{-4}(z_4) \prod_{p\mid\kappa} \chi_{p'}(z_p), & \text{if } \kappa \equiv 3 \mod 4, \\
\chi_8(z_8) \prod_{p\mid\frac{\kappa}{2}} \chi_{p'}(z_p), & \text{if } \kappa \equiv 2 \mod 8, \\
\chi_{-8}(z_8) \prod_{p\mid\frac{\kappa}{2}} \chi_{p'}(z_p), & \text{if } \kappa \equiv 6 \mod 8,
\end{cases} \\
&= \begin{cases}
\chi_\kappa(z_\kappa), & \text{if } \kappa \equiv 1 \mod 4, \\
\chi_{4\kappa}(z_{4\kappa}), & \text{else.}
\end{cases}
\end{aligned}
$$

This proves the last step in Correspondence 1.10.2.

2. Representation numbers

In this chapter we study different types of representation numbers. For polynomials, these numbers count how often it takes the value 0 in a certain sense. Their Poincaré series is related to the Igusa local zeta function. The representation numbers of lattices can be viewed as the representation numbers for polynomials of degree 2.

Counting how often discriminant forms take certain values can be reduced to computing these representation numbers for the Jordan components. We derive explicit formulas for the 2-adic Jordan components. The Jordan components for odd primes can be treated by computing the lengths of orbits with respect to the action of the orthogonal group.

2.1. The Igusa local zeta function - representation numbers of polynomials

For our purposes it suffices to define the *Igusa local zeta function* as

$$Z_f(s) = \int_{\mathbb{Z}_p^n} |f(x_1, \dots, x_n)|_p^s dx_1 \dots dx_n, \tag{2.1.1}$$

where the measure is normalized such that $\mathrm{vol}(\mathbb{Z}_p, dx_i) = 1$, $f \in \mathbb{Z}_p[x_1, \dots, x_n]$ and $s \in \mathbb{C}$ with $\Re(s) > 0$. For an overview and generalizations, we refer to [Den91]. The Igusa local zeta function is a polynomial in $t = p^{-s}$ (cf. [Den84]). Whenever we state $s \in \mathbb{C}$ without a condition on its real part, we talk about meromorphic extensions of the regarding functions. Whenever convergence is an issue, we assume s to have a large real part and usually $\Re(s) > 0$ suffices.

If we define $N_k(f) = \#\left\{x \in \mathbb{Z}_p/p^k\mathbb{Z}_p \;\middle|\; f(x) \equiv 0 \pmod{p^k}\right\}$, we have the equality

$$\frac{1 - tZ_f(t)}{1 - t} = \sum_{k \in \mathbb{Z}_{\geq 0}} \frac{N_k(f)}{p^{nk}} p^{-ks}. \tag{2.1.2}$$

The term on the right hand side appears in the computations of [BK01], where it is evaluated without use of the Igusa local zeta function.

Note that the Igusa local zeta function can be computed for quadratic polynomials by means of [CKW17]. These formulas were implemented in [Wilb].

2.1.1. Adding hyperbolic planes

The following theorem which was inspired by Observation 3.0.7 and Remark 3.1.7 shows the behaviour of the Igusa local zeta function and the Poincaré series under addition of a hyperbolic plane.

Theorem 2.1.1. *Let p be a prime and $f \in \mathbb{Z}_p[x_1, \ldots, x_n]$. Let H denote the hyperbolic plane, given by the polynomial $H = x_{n+1}x_{n+2} \in \mathbb{Z}_p[x_{n+1}, x_{n+2}]$. The Igusa local zeta functions of f and $f+H$ satisfy the relation*

$$\frac{1 - tZ_f(t)}{1-t}\left(1 - \frac{t}{p}\right) = \frac{1 - tZ_{f+H}(t)}{1-t}\left(1 - \frac{t}{p}\right)\Bigg|_{t=pt} \tag{2.1.3}$$

or equivalently

$$\left(1 - \frac{t}{p}\right)\sum_{i \in \mathbb{Z}_{\geq 0}} \frac{N_i(f)}{p^{ni}} t^i = \left(1 - \frac{t}{p}\right)\sum_{i \in \mathbb{Z}_{\geq 0}} \frac{N_i(f+H)}{p^{(n+2)i}} t^i\Bigg|_{t=pt} \tag{2.1.4}$$

in terms of the Poincaré series for $t = p^{-s}$ and $s \in \mathbb{C}$.

The proof relies on the following identity of representation numbers.

Lemma 2.1.2. *Let p be a prime and $f \in \mathbb{Z}_p[x_1, \ldots, x_n]$. Let H denote the hyperbolic plane, given by the polynomial $H = x_{n+1}x_{n+2} \in \mathbb{Z}_p[x_{n+1}, x_{n+2}]$. The representation numbers of f and $f+H$ satisfy the relation*

$$p^i\left(N_i(f) - p^{n-1}N_{i-1}(f)\right) = N_i(f+H) - p^{(n+2)-1}N_{i-1}(f+H)$$

for $i > 0$.

Proof. We recall the well known formula (cf. Lemma 3.5.3)

$$N_i(H - \alpha) = \begin{cases} (\mathrm{ord}_p(\alpha)+1)(1-\frac{1}{p})p^i, & \mathrm{ord}_p(\alpha) < i, \\ i(1-\frac{1}{p})p^i + p^i, & \mathrm{ord}_p(\alpha) \geq i, \end{cases} \tag{2.1.5}$$

for any $n \in \mathbb{Z}_p$ and $i \in \mathbb{Z}_{\geq 0}$. It is important that the representation number depends only on i and $\mathrm{ord}_p(\alpha)$, i.e. $N_i(H-\alpha) = N_i(H - p^{\mathrm{ord}_p(\alpha)})$. For $i > 0$, we deduce

$$\begin{aligned}
& N_{i-1}(f+H) \\
&= N_{i-1}(f)N_{i-1}(H) + \sum_{\substack{x \in (\mathbb{Z}_p/p^{i-1}\mathbb{Z}_p)^n \\ f(x) \not\equiv 0 \pmod{p^{i-1}}}} N_{i-1}(H - f(x)) \\
&= N_{i-1}(f)N_{i-1}(H) + \sum_{\nu=0}^{i-2} \#\left\{x \in (\mathbb{Z}_p/p^{i-1}\mathbb{Z}_p)^n \;\middle|\; \mathrm{ord}_p(f(x)) = \nu\right\} N_{i-1}(H - p^\nu) \\
&= \left((i-1)(1-\frac{1}{p})p^{i-1} + p^{i-1}\right) N_{i-1}(f) \\
&\quad + \sum_{\nu=0}^{i-2} \#\left\{x \in (\mathbb{Z}_p/p^{i-1}\mathbb{Z}_p)^n \;\middle|\; \mathrm{ord}_p(f(x)) = \nu\right\} (\nu+1)(1-\frac{1}{p})p^{i-1}
\end{aligned}$$

and

$$\begin{aligned}
& N_i(f+H) \\
&= N_i(f)N_i(H) + \sum_{\substack{x \in (\mathbb{Z}_p/p^{i}\mathbb{Z}_p)^n \\ f(x) \not\equiv 0 \pmod{p^{i}}}} N_i(H - f(x)) \\
&= N_i(f)N_i(H) + \sum_{\nu=0}^{i-1} \#\left\{x \in (\mathbb{Z}_p/p^{i}\mathbb{Z}_p)^n \;\middle|\; \mathrm{ord}_p(f(x)) = \nu\right\} N_i(H - p^\nu) \\
&= \left(i(1-\frac{1}{p})p^{i} + p^{i}\right) N_i(f) + \sum_{\nu=0}^{i-1} \#\left\{x \in (\mathbb{Z}_p/p^{i}\mathbb{Z}_p)^n \;\middle|\; \mathrm{ord}_p(f(x)) = \nu\right\} (\nu+1)(1-\frac{1}{p})p^{i} \\
&= \left(i(1-\frac{1}{p})p^{i} + p^{i}\right) N_i(f) + \#\left\{x \in (\mathbb{Z}_p/p^{i}\mathbb{Z}_p)^n \;\middle|\; \mathrm{ord}_p(f(x)) = i-1\right\} i(1-\frac{1}{p})p^{i} \\
&\quad + \sum_{\nu=0}^{i-2} \#\left\{x \in (\mathbb{Z}_p/p^{i}\mathbb{Z}_p)^n \;\middle|\; \mathrm{ord}_p(f(x)) = \nu\right\} (\nu+1)(1-\frac{1}{p})p^{i} \\
&= \left(i(1-\frac{1}{p})p^{i} + p^{i}\right) N_i(f) + (p^n N_{i-1}(f) - N_i(f))\, i(1-\frac{1}{p})p^{i} \\
&\quad + \sum_{\nu=0}^{i-2} p^n \#\left\{x \in (\mathbb{Z}_p/p^{i-1}\mathbb{Z}_p)^n \;\middle|\; \mathrm{ord}_p(f(x)) = \nu\right\} (\nu+1)(1-\frac{1}{p})p^{i} \\
&= p^i N_i(f) + p^n N_{i-1}(f) i(1-\frac{1}{p})p^{i} \\
&\quad + \sum_{\nu=0}^{i-2} p^{n+1} \#\left\{x \in (\mathbb{Z}_p/p^{i-1}\mathbb{Z}_p)^n \;\middle|\; \mathrm{ord}_p(f(x)) = \nu\right\} (\nu+1)(1-\frac{1}{p})p^{i-1}.
\end{aligned}$$

Together, this yields

$$\begin{aligned}&N_i(f+H) - p^{n+1} N_{i-1}(f+H)\\ =&p^i N_i(f) + p^n N_{i-1}(f) i(1-\frac{1}{p})p^i - p^{n+1}\left((i-1)(1-\frac{1}{p})p^{i-1} + p^{i-1}\right) N_{i-1}(f)\\ =&p^i\left(N_i(f) - p^{n-1}N_{i-1}(f)\right),\end{aligned}$$

which finishes the proof. □

Example 2.1.3. Let $p=3$, $n=1$, $i=2$ and $f(x_1)=x_1^2$. By counting, we get

$$\begin{aligned}N_1(f) = 1, \quad N_1(f+H) = 9,\\ N_2(f) = 3, \quad N_2(f+H) = 99.\end{aligned}$$

We see that

$$3^2(3 - 3^0 \cdot 1) = 18 = 99 - 3^2 \cdot 9,$$

and no simpler relation between the representation numbers seems possible.

Proof of Theorem 2.1.1. The left hand side of (2.1.4) is

$$\begin{aligned}\left(1-\frac{t}{p}\right)\sum_{i\in\mathbb{Z}_{\geq 0}} \frac{N_i(f)}{p^{ni}} t^i &= \sum_{i\in\mathbb{Z}_{\geq 0}} \frac{N_i(f)}{p^{ni}} t^i - \frac{N_i(f)}{p^{ni+1}} t^{i+1}\\ &= 1 + \sum_{i>0}\left(\frac{N_i(f)}{p^{ni}} - \frac{N_{i-1}(f)}{p^{n(i-1)+1}}\right) t^i\\ &= 1 + \sum_{i>0}\left(N_i(f) - p^{n-1}N_{i-1}(f)\right)\frac{t^i}{p^{ni}}.\end{aligned}$$

The right hand side is

$$\begin{aligned}\left(1-\frac{t}{p}\right)\sum_{i\in\mathbb{Z}_{\geq 0}} \frac{N_i(f+H)}{p^{(n+2)i}} t^i\bigg|_{t=pt} &= \sum_{i\in\mathbb{Z}_{\geq 0}} \frac{N_i(f+H)}{p^{(n+2)i}}(pt)^i - \frac{N_i(f+H)}{p^{(n+2)i+1}}(pt)^{i+1}\\ &= 1 + \sum_{i>0}\left(\frac{N_i(f+H)}{p^{(n+1)i}} - \frac{N_{i-1}(f+H)}{p^{(n+2)(i-1)+1-i}}\right) t^i\\ &= 1 + \sum_{i>0}\left(N_i(f+H) - p^{(n+2)-1}N_{i-1}(f+H)\right)\frac{t^i}{p^{(n+1)i}}.\end{aligned}$$

By Lemma 2.1.2, both sides agree. □

Applying Theorem 2.1.1 repeatedly, yields the following result.

Corollary 2.1.4. *Let p be a prime and $f \in \mathbb{Z}_p[x_1, \dots, x_n]$. Let $H^{(h)}$ denote $h \in \mathbb{Z}_{\geq 0}$ copies of the hyperbolic plane, given by the polynomial*

$$H^{(h)} = x_{n+1}x_{n+2} + \dots + x_{n+2h-1}x_{n+2h} \in \mathbb{Z}_p[x_{n+1}, \dots, x_{n+2h}].$$

The Igusa local zeta functions of f and $f + H^{(h)}$ satisfy the relation

$$\frac{1 - tZ_f(t)}{1-t}\left(1 - \frac{t}{p}\right) = \frac{1 - tZ_{f+H^{(h)}}(t)}{1-t}\left(1 - \frac{t}{p}\right)\Bigg|_{t=p^h t}, \tag{2.1.6}$$

or equivalently

$$\left(1 - \frac{t}{p}\right)\sum_{i \in \mathbb{Z}_{\geq 0}} \frac{N_i(f)}{p^{ni}} t^i = \left(1 - \frac{t}{p}\right)\sum_{i \in \mathbb{Z}_{\geq 0}} \frac{N_i(f + H^{(h)})}{p^{(n+2h)i}} t^i \Bigg|_{t=p^h t} \tag{2.1.7}$$

in terms of the Poincaré series for $t = p^{-s}$ and $s \in \mathbb{C}$.

Choosing $f = 0 \in \mathbb{Z}_p$ (as a polynomial in 0 variables), we can compute the Igusa local zeta function of h copies of the hyperbolic plane. Note that

$$Z_0(t) = 0. \tag{2.1.8}$$

Corollary 2.1.5. *For $h \in \mathbb{Z}_{\geq 0}$ we have*

$$Z_{H^{(h)}}(t) = \frac{\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{p^h}\right)}{\left(1 - \frac{t}{p}\right)\left(1 - \frac{t}{p^h}\right)}.$$

This formula agrees with [CKW17, Lemma 5.7], where hyperbolic planes are represented by double the polynomials we use here.

Choosing any $f \in \mathbb{Z}_p$, we could also recover Lemma 3.5.3 by computing $Z_{-f+H}(t)$. This is a nice exercise. As a first step one would compute

$$Z_f(t) = t^{\mathrm{ord}_p(f)}, \tag{2.1.9}$$

which is 0 if $f = 0$. We now combine this with Corollary 2.1.4.

Proposition 2.1.6. *Let p be a prime and $f \in \mathbb{Z}_p$. Let $H^{(h)}$ denote $h \in \mathbb{Z}_{\geq 0}$ copies of the hyperbolic plane, given by the polynomial*

$$H^{(h)} = x_1x_2 + \dots + x_{2h-1}x_{2h} \in \mathbb{Z}_p[x_1, \dots, x_{2h}].$$

The representation numbers of $-f + H^{(h)}$ *are given by*

$$
\begin{aligned}
&N_i(-f + H^{(h)}) \\
=&\begin{cases} p^{h(i-1)}\left[(p^h-1)\sum_{k'=0}^{i} p^{(h-1)k'} + 1\right], & \text{if } \mathrm{ord}_p(f) \geq i, \\ p^{h(i-1)}(p^h-1)\sum_{k'=i-\mathrm{ord}_p(f)}^{i} p^{(h-1)k'}, & \text{if } \mathrm{ord}_p(f) < i, \end{cases} \\
=&\begin{cases} p^i\left(i\left(1-\frac{1}{p}\right)+1\right), & \text{if } \mathrm{ord}_p(f) \geq i \text{ and } h=1, \\ p^i\left(\mathrm{ord}_p(f)+1\right)\left(1-\frac{1}{p}\right), & \text{if } \mathrm{ord}_p(f) < i \text{ and } h=1, \\ p^{h(i-1)}\left[\frac{p^h-1}{p^{h-1}-1}\left(p^{(h-1)(i+1)}-1\right)+1\right], & \text{if } \mathrm{ord}_p(f) \geq i \text{ and } h>1, \\ p^{h(i-1)}\frac{p^h-1}{p^{h-1}-1}\left(p^{(h-1)(i+1)}-p^{(h-1)(i-\mathrm{ord}_p(f))}\right), & \text{if } \mathrm{ord}_p(f) < i \text{ and } h>1. \end{cases}
\end{aligned}
$$

Proof. Let $j = \mathrm{ord}_p(f)$. The case $f = 0$ follows from $f \neq 0$ for large j, so we assume $f \neq 0$, i.e. $j < \infty$. We have $Z_{-f}(t) = Z_f(t) = t^{\mathrm{ord}_p(f)}$, so by Corollary 2.1.4, the equality

$$
\frac{1-t^{j+1}}{1-t}\left(1-\frac{t}{p}\right) = \frac{1-tZ_{-f+H^{(h)}}(t)}{1-t}\left(1-\frac{t}{p}\right)\Big|_{t=p^h t}
$$

holds. Replacing by the Poincaré series on the right hand side gives

$$
\frac{(1-t^{j+1})\left(1-\frac{t}{p}\right)}{(1-t)\,(1-p^{h-1}t)} = \sum_{i\in\mathbb{Z}_{\geq 0}} \frac{N_i(-f+H^{(h)})}{p^{2hi}}(p^h t)^i.
$$

Carrying out the multiplication in the numerator and replacing the denominator on the left hand side by geometric series, we get

$$
\left(1-\frac{t}{p}-t^{j+1}+\frac{t^{j+2}}{p}\right)\sum_{k\in\mathbb{Z}_{\geq 0}}\sum_{l=0}^{k} p^{(h-1)l}t^k = \sum_{i\in\mathbb{Z}_{\geq 0}} \frac{N_i(-f+H^{(h)})}{p^{hi}}t^i.
$$

Comparing the coefficients of both sides finishes the proof. □

Note that for odd primes this result is a special case of [Sie35, Hilfssatz 16] (compare [BK01, Theorem 4.5]). We could also have obtained it analogously to [BM17, Lemma 2.4].

2.1.2. A special case for $p = 2$

The aim of this section is to prove version of [Sie35, Hilfssatz 16] for $p = 2$ (cf. Theorem 2.3.3).

Proposition 2.1.7. *Let* $\alpha \in \mathbb{Z}_2$ *and* $f = -\alpha + x_1^2 + x_1x_2 + x_2^2 \in \mathbb{Z}_2[x_1, x_2]$. *Let* $H^{(h-1)}$

denote $h-1 \in \mathbb{Z}_{\geq 0}$ copies of the hyperbolic plane, given by the polynomial

$$H^{(h-1)} = x_3x_4 + \cdots + x_{2h-1}x_{2h} \in \mathbb{Z}_p[x_3, \ldots, x_{2h}].$$

The representation numbers of $f + H^{(h-1)}$ are given by

$$\begin{aligned} & N_i(f + H^{(h-1)}) \\ &= \begin{cases} 2^{h(i-1)} \left[(2^h+1)(-1)^i \sum_{k'=0}^{l} (-2^{h-1})^{k'} + (-1)^{i+1} \right], & \text{if } \operatorname{ord}_2(\alpha) \geq i, \\ 2^{h(i-1)} (2^h+1)(-1)^i \sum_{k'=i-\operatorname{ord}_2(\alpha)}^{i} (-2^{h-1})^{k'}, & \text{if } \operatorname{ord}_2(\alpha) < i, \end{cases} \\ &= \begin{cases} 2^{h(i-1)} \left[\frac{2^h+1}{2^{h-1}+1} \left(2^{(h-1)(i+1)} - (-1)^{i+1} \right) + (-1)^{i+1} \right], & \text{if } \operatorname{ord}_2(\alpha) \geq i, \\ 2^{h(i-1)} \frac{2^h+1}{2^{h-1}+1} \left(2^{(h-1)(i+1)} - (-1)^{\operatorname{ord}_2(\alpha)+1} 2^{(h-1)(i-\operatorname{ord}_2(\alpha))} \right), & \text{if } \operatorname{ord}_2(\alpha) < i. \end{cases} \end{aligned}$$

Proof. Let $j = \operatorname{ord}_2(\alpha)$. The case $\alpha = 0$ follows from $f \neq 0$ for arbitrarily large j, so we assume $f \neq 0$, i.e. $j < \infty$. For $h = 1$, we have Proposition 2.3.2, i.e.

$$N_i(f) = \begin{cases} 2^{i-1} \ \frac{3+(-1)^i}{2}, & \text{if } \operatorname{ord}_2(\alpha) \geq i, \\ 2^{i-1} \ 3 \ \frac{1+(-1)^{\operatorname{ord}_2(\alpha)}}{2}, & \text{if } \operatorname{ord}_2(\alpha) < i. \end{cases}$$

This gives the Poincaré series

$$\sum_{i \in \mathbb{Z}_{\geq 0}} \frac{N_i(f)}{2^{2i}} t^i = \frac{\left(1 + \frac{(-1)^j}{2^{j+1}} t^{j+1}\right)\left(1 + \frac{t}{4}\right)}{\left(1 - \frac{t}{2}\right)\left(1 + \frac{t}{2}\right)}.$$

Applying Corollary 2.1.4, we get

$$\frac{\left(1 + \frac{(-1)^j}{2^{j+1}} t^{j+1}\right)\left(1 + \frac{t}{4}\right)}{\left(1 - \frac{t}{2}\right)\left(1 + \frac{t}{2}\right)} \left(1 - \frac{t}{2}\right) = \sum_{i \in \mathbb{Z}_{\geq 0}} \frac{N_i(f + H^{(h-1)})}{2^{2hi}} t^i \left(1 - \frac{t}{2}\right)\Bigg|_{t = 2^{h-1}t},$$

which simplifies to

$$\frac{\left(1 + \frac{(-1)^j}{2^{j+1}} t^{j+1}\right)\left(1 + \frac{t}{4}\right)}{(1 - 2^{h-2}t)\left(1 + \frac{t}{2}\right)} = \sum_{i \in \mathbb{Z}_{\geq 0}} \frac{N_i(f + H^{(h-1)})}{2^{2hi}} (2^{h-1}t)^i,$$

Carrying out the multiplication in the numerator and replacing the denominator on the left hand side by geometric series, we get

$$\left(1 + \frac{t}{4} + \frac{(-1)^j}{2^{j+1}} t^{j+1} + \frac{(-1)^j}{2^{j+3}} t^{j+2}\right) \sum_{k \in \mathbb{Z}_{\geq 0}} \frac{1 - (-2^{1-h})^{k+1}}{1 + 2^{1-h}} t^k = \sum_{i \in \mathbb{Z}_{\geq 0}} \frac{N_i(f + H^{(h-1)})}{2^{-(h+1)i}} t^i.$$

Comparing the coefficients of both sides finishes the proof. □

2.2. Representation numbers associated to lattices

Let L be an even lattice and $\gamma \in L'$. Following [BK01], we define the representation numbers

$$N_{\gamma,n}(a) = N^L_{\gamma,n}(a) = \#\left\{r \in L/aL \mid Q(r-\gamma) + n \equiv 0 \pmod{a}\right\} \tag{2.2.1}$$

for an integer a. Identifying L with $(\mathbb{Z}^n, x \mapsto \frac{1}{2}x^t Sx)$, we can write

$$N_{\gamma,n}(a) = N^L_{\gamma,n}(a) = \#\left\{r \in (\mathbb{Z}/a\mathbb{Z})^n \;\middle|\; (r-\gamma)^t S(r-\gamma) + n \equiv 0 \pmod{a}\right\}. \tag{2.2.2}$$

These representation numbers are multiplicative, i.e.

$$N_{\gamma,n}(a_1 a_2) = N_{\gamma,n}(a_1) N_{\gamma,n}(a_2)$$

for coprime a_1 and a_2. For prime powers, these are special cases of the representation numbers of the last section. We will use this relation in Section 3.2 to compute vector valued Eisenstein series for the dual Weil representation.

2.3. Representation numbers associated to finite quadratic modules

In this section, we recall how to compute representation numbers of finite quadratic modules from [Opi13] and derive a closed formula for the representation numbers of even 2-adic Jordan components. Let $D = \bigoplus_p D_p$ be a finite quadratic module decomposed into its maximal p-subgroups.

We define *representation numbers* of D by

$$N(D, j) := |\{\gamma \in D : Q(\gamma) \equiv j \pmod{1}\}| \tag{2.3.1}$$

for any $j \in \mathbb{Q}/\mathbb{Z}$. They count how often the quadratic form Q takes the value $j \pmod{1}$ on D.

One need for computing representation numbers arises from their appearance in the dimension formula for spaces of vector valued modular form with respect to the Weil representation associated to D (cf. Section 4.4).

The representation numbers of orthogonal direct sums can be obtained by repeatedly applying

$$N(D \oplus D', j) = \sum_{\substack{j_1 \pmod{1} \\ j_2 \pmod{1} \\ j_1 + j_2 = j \pmod{1}}} N(D, j_1) \cdot N(D', j_2). \tag{2.3.2}$$

Hence it suffices to compute the representation numbers of the D_p.

If p is odd and D_p is a finite quadratic module of order p^l, then the representation numbers can be computed with the theory from Section 4.2.1 and Section 4.2.2 as follows. The orbits $\mathcal{O}$ denote orbits of D_p with respect to the action of the orthogonal group $O(D_p)$.

$$\begin{aligned} N(D_p, j) &= \sum_{\substack{v_1 | v_p | \cdots | v_{p^k} | p^l \\ \frac{v_{p^j}}{v_{p^i}} t_{p^j} \equiv p^{j-i} t_{p^i} \pmod 1 \\ v_1 t_1 \equiv j \pmod 1}} |\mathcal{O}(D, p^{k+1}, v_1, \ldots, v_{p^k}, t_1, \ldots, t_{p^k})| \\ &= \sum_{\mathcal{O} \text{ an orbit of } D \text{ with } v_1 t_1 \equiv j \pmod 1} |\mathcal{O}| \end{aligned} \tag{2.3.3}$$

where elements $\gamma \in \mathcal{O}(D, p^{k+1}, v_1, \ldots, v_{p^k}, t_1, \ldots, t_{p^k})$ have norm $Q(\gamma) = v_1 t_1 \pmod 1$.

2.3.1. 2-adic Jordan components

We know that we can decompose odd 2-adic Jordan components as follows.

$$q_t^{\pm n} \simeq \begin{cases} q_t^{\pm n}, & \text{if } n = 1, \\ q_{t_1}^{\pm 1} \oplus q_{t_2}^{+1}, & \text{if } n = 2, \\ q_{t_1}^{\pm 1} \oplus q_{t_2}^{+1} \oplus q_{t_3}^{+1} \oplus q_{II}^{+2\frac{n-3}{2}}, & \text{if } n \geq 3 \text{ is odd}, \\ q_{t_1}^{\pm 1} \oplus q_{t_2}^{+1} \oplus q_{t_3}^{+1} \oplus q_{t_4}^{+1} \oplus q_{II}^{+2\frac{n-4}{2}}, & \text{if } n \geq 4 \text{ is even}, \end{cases} \tag{2.3.4}$$

where $\sum_i t_i = t \pmod 8$, $\left(\frac{t_1}{2}\right) = \pm 1$ and $\left(\frac{t_i}{2}\right) = +1$ if $i \geq 2$. If such t_i do not exist then $q_t^{\pm n}$ is not a valid Jordan component. This can only happen if $n = 1$ or $n = 2$ as long as $t \equiv n \pmod 2$. Recall that for even 2-adic Jordan components, we have

$$q_{II}^{\pm 2n} \simeq \begin{cases} q_{II}^{+2n}, & \text{if } \pm = +, \\ q_{II}^{-2} \oplus q_{II}^{+2(n-1)}, & \text{if } \pm = -. \end{cases} \tag{2.3.5}$$

It suffices to know the representation numbers of all the components occurring in (2.3.4) and (2.3.5) since we can combine them using equation (2.3.2).

Proposition 2.3.1 ([Opi13, Theorem 3.3]). *Let $q = 2^l$ with $l \geq 1$. Then*

$$
\begin{aligned}
&N(q_{II}^{+2n}, \frac{j}{q}) \\
&= \begin{cases} \left(\frac{q}{2}\right)^n \left[(2^n - 1)\sum_{k'=0}^{l} 2^{(n-1)k'} + 1\right], & \text{if } j = 0 \pmod{2^l}, \\ \left(\frac{q}{2}\right)^n (2^n - 1)\sum_{k'=l-k}^{l} 2^{(n-1)k'}, & \text{if } 2^k \parallel j \neq 0 \pmod{2^l}, \end{cases} \\
&= \begin{cases} \frac{q}{2}(l+2), & \text{if } j = 0 \pmod{2^l} \text{ and } n = 1, \\ \frac{q}{2}(k+1), & \text{if } 2^k \parallel j \neq 0 \pmod{2^l} \text{ and } n = 1, \\ \left(\frac{q}{2}\right)^n \left[\frac{2^n-1}{2^{n-1}-1}\left(2^{(n-1)(l+1)} - 1\right) + 1\right], & \text{if } j = 0 \pmod{2^l} \text{ and } n > 1, \\ \left(\frac{q}{2}\right)^n \frac{2^n-1}{2^{n-1}-1}\left(2^{(n-1)(l+1)} - 2^{(n-1)(l-k)}\right), & \text{if } 2^k \parallel j \neq 0 \pmod{2^l} \text{ and } n > 1. \end{cases}
\end{aligned}
$$

Proof. This is Proposition 2.1.6 for $p = 2$. The original proof used induction on n. The case $n = 1$ can be obtained by induction on l and is well known (cf. Lemma 3.5.3). The induction step was done by a lengthy computation using equation (2.3.2). □

Proposition 2.3.2 ([Opi13, Theorem 3.4]). *Let $q = 2^l$ with $l \geq 1$. Then*

$$
\begin{aligned}
&N(q_{II}^{-2}, \frac{j}{q}) \\
&= \begin{cases} \frac{q}{2}\ \frac{3+(-1)^l}{2}, & \text{if } j = 0 \pmod{2^l}, \\ \frac{q}{2}\ 3^{\frac{1+(-1)^k}{2}}, & \text{if } 2^k \parallel j \neq 0 \pmod{2^l}, \end{cases} \\
&= \begin{cases} 2^l, & \text{if } j = 0 \pmod{2^l} \text{ and } l \text{ is even}, \\ 2^{l-1}, & \text{if } j = 0 \pmod{2^l} \text{ and } l \text{ is odd}, \\ 3 \cdot 2^{l-1}, & \text{if } 2^k \parallel j \neq 0 \pmod{2^l} \text{ and } k \text{ is even}, \\ 0, & \text{if } 2^k \parallel j \neq 0 \pmod{2^l} \text{ and } k \text{ is odd}. \end{cases}
\end{aligned}
$$

Proof. We sketch the proof form [Opi13] for the convenience of the reader. For small l we obtain the representation numbers by counting appropriate elements. This yields

$$
\begin{aligned}
N(2_{II}^{-2}, 0) &= 1, \\
N(2_{II}^{-2}, \frac{1}{2}) &= 3
\end{aligned} \tag{2.3.6}
$$

and

$$\begin{aligned} N(4_{II}^{-2}, 0) &= 4, \\ N(4_{II}^{-2}, \frac{1}{4}) &= 6, \\ N(4_{II}^{-2}, \frac{1}{2}) &= 0, \\ N(4_{II}^{-2}, \frac{3}{4}) &= 6. \end{aligned} \tag{2.3.7}$$

For $l > 2$, we can prove the recursion formula

$$N(q_{II}^{-2}, j) = \begin{cases} 2 \cdot N(\left(\frac{q}{2}\right)_{II}^{-2}, j), & \text{if } j \text{ is odd}, \\ 0, & \text{if } 2 \parallel j \neq 0 \pmod{q}. \end{cases} \tag{2.3.8}$$

The complete formula follows by induction. □

Combining these propositions using equation (2.3.2), we get a closed formula for all even 2-adic components. Note that this is a generalization of [Sch06, Proposition 3.1].

Theorem 2.3.3. *Let $q = 2^l$ with $l \geq 1$, $\varepsilon = \pm 1$. Then*

$$\begin{aligned} &N(q_{II}^{\varepsilon 2n}, \frac{j}{q}) \\ &= \begin{cases} \left(\frac{q}{2}\right)^n \left[(2^n - \varepsilon)\varepsilon^l \sum_{k'=0}^{l} (\varepsilon 2^{n-1})^{k'} + \varepsilon^{l+1}\right], & \text{if } j = 0 \pmod{2^l}, \\ \left(\frac{q}{2}\right)^n (2^n - \varepsilon)\varepsilon^l \sum_{k'=l-k}^{l} (\varepsilon 2^{n-1})^{k'}, & \text{if } 2^k \parallel j \neq 0 \pmod{2^l}, \end{cases} \\ &= \begin{cases} \frac{q}{2}(l+2), & \text{if } j = 0 \pmod{2^l},\ \varepsilon = +1 \text{ and } n = 1, \\ \frac{q}{2}(k+1), & \text{if } 2^k \parallel j \neq 0 \pmod{2^l},\ \varepsilon = +1 \text{ and } n = 1, \\ \left(\frac{q}{2}\right)^n \left[\frac{2^n - \varepsilon}{2^{n-1} - \varepsilon}\left(2^{(n-1)(l+1)} - \varepsilon^{l+1}\right) + \varepsilon^{l+1}\right], & \text{if } j = 0 \pmod{2^l},\ \varepsilon = -1 \text{ or } n > 1, \\ \left(\frac{q}{2}\right)^n \frac{2^n - \varepsilon}{2^{n-1} - \varepsilon}\left(2^{(n-1)(l+1)} - \varepsilon^{k+1} 2^{(n-1)(l-k)}\right), & \text{if } 2^k \parallel j \neq 0 \pmod{2^l},\ \varepsilon = -1 \text{ or } n > 1. \end{cases} \end{aligned}$$

Proof. For $\varepsilon = +1$, this is Proposition 2.3.1. For $\varepsilon = -1$, this is Proposition 2.1.7. □

The representation numbers of irreducible odd 2-adic Jordan components are all closely related. They can be reduced to

$$N_{sq}(n, j) := |\{a \in \{0, \ldots, n-1\} : a^2 = j \pmod{n}\}| \tag{2.3.9}$$

for $n \in \mathbb{Z}_{\geq 1}$ and $j \in \mathbb{Z}$, which we call *square representation numbers.*

Lemma 2.3.4 ([Opi13, Lemma 3.6])**.** *Let $q = 2^l$ with $l \geq 1$. We have*

$$N(q_t^{\pm 1}, \frac{j}{2q}) = \frac{1}{2} N_{sq}(2q, jt^{-1}), \tag{2.3.10}$$

where t^{-1} *refers to the inverse of* $t \pmod{2q}$*.*

The square representation numbers can be given as follows.

Proposition 2.3.5 ([Opi13, Proposition 3.7])**.** *Let* $q = 2^l$ *with* $l \geq 1$*. We have*

$$N_{sq}(q,j) = \begin{cases} 2^{\frac{l}{2}}, & \text{if } l \text{ is even and } j = 0 \text{ or } j = 2^{l-2} \pmod{2^l}, \\ 2^{\frac{l-1}{2}}, & \text{if } l \text{ is odd and } j = 0 \text{ or } j = 2^{l-1} \pmod{2^l}, \\ 2^{\frac{k}{2}+2}, & \text{if } j = 2^k a \pmod{2^l} \text{ with } a = 1 \mod 8 \text{ and } k \leq l-3 \text{ is even}, \\ 0, & \text{else.} \end{cases}$$

We define 2-*torsion representation numbers* of D by

$$N_{\text{2-torsion}}(D,j) := |\{\gamma \in D : 2\gamma = 0 \text{ and } Q(\gamma) = j \pmod 1\}| \tag{2.3.11}$$

for any $j \in \mathbb{Q}/\mathbb{Z}$. It tells us how often the quadratic form Q takes the value $j \mod 1$ on D_2. The non trivial 2-torsions of D are elements of the 2-adic part of D. It suffices to calculate them for the Jordan components occurring in (2.3.4) and (2.3.5).

Lemma 2.3.6 ([Opi13, Bemerkung 5.1])**.** *Let* $q = 2^l$ *with* $l \geq 1$*. We have*

$$N_{\text{2-torsion}}\left(\left(q_{II}^{\varepsilon 2n}\right)_2, \frac{j}{q}\right) = \begin{cases} \frac{4^n+\varepsilon 2^n}{2}, & \text{if } l = 1 \text{ and } j = 0 \pmod 2, \\ \frac{4^n-\varepsilon 2^n}{2}, & \text{if } l = 1 \text{ and } j = 1 \pmod 2, \\ 4^n, & \text{if } l \geq 2 \text{ and } j = 0 \pmod{2^l}, \\ 0, & \text{if } l \geq 2 \text{ and } j \neq 0 \pmod{2^l}, \end{cases} \tag{2.3.12}$$

and

$$N_{\text{2-torsion}}\left(\left(q_t^{\pm 1}\right)_2, \frac{j}{2q}\right) = \begin{cases} 1, & \text{if } l = 1 \text{ and } j = 0 \text{ or } t \pmod 4, \\ 1, & \text{if } l = 2 \text{ and } j = 0 \text{ or } q \pmod{2q}, \\ 2, & \text{if } l \geq 3 \text{ and } j = 0 \pmod{2q}, \\ 0, & \text{else.} \end{cases} \tag{2.3.13}$$

3. Fourier coefficients of Eisenstein series

One way to obtain formulas for the vector valued Eisenstein series for the Weil representation is described in [KY10], where the computation of certain p-adic integrals (generalized local densities) is key. This setting gives a straight forward interpretation of the Siegel-Weil formula in the case of positive definite even lattices of rank greater or equal to 5.

A second way is obtained using the formulas from [BK01, Theorem 4.6] and to state them in terms of the Igusa local zeta functions which can be evaluated by means of [CKW17]. This approach has been used in [Wil18a; Wil18b; Wil18c] from which we can also get cusp forms. Using this approach with the formulas from [Sch18], we can also compute the Fourier coefficients of $E_{\rho_L^*,k,\beta}$ for any isotropic $\beta \in L'/L$ and a suitable weight.

By comparison of both ways and the respective formulas we can obtain a proof for the following observation. This leads to formulas for the Igusa local zeta function in terms of the generalized local densities, a connection we believe has not been made explicitly before.

Observation 3.0.7. Comparing the generalized local densities and the local factors in [Wilb](some documentation in [Wila], using the Igusa local zeta function) we observe the following equality of rational functions. For an even lattice $L = \mathbb{Z}^n$ with Gram matrix S and $\mu \in L' = S^{-1}\mathbb{Z}^n$, we have

$$W_{p,m,\mu}(X) = \frac{1 - tZ_{-\frac{1}{2}x^tSx + x^tS\mu + m - Q(\mu)}(t)}{1-t} \left(1 - \frac{t}{p}\right)\Bigg|_{t=pX}. \tag{3.0.1}$$

This observation in itself leads to much of the simplifications which can be made in the approach of [BK01]. This is achieved by translating properties which are immediate for the local densities $W_{p,m,\mu}(X)$ on the left hand side to properties regarding the Igusa local zeta function. Most notable is the case of “adding hyperbolic planes”, which we described in Section 2.1.1.

The algorithms to compute the Fourier coefficients of the vector valued Eisenstein series were first tested against [Bun98], where three bases for specific spaces of holomorphic modular forms are computed in terms of classical theta functions. As a second test case, the Fourier coefficients were tested against the Fourier coefficients of Jacobi forms. The most general test cases are obtained be the Siegel-Weil formula. We list the considered lattices in appendix C.

3.1. The approach of Kudla and Yang

In this section, we adapt the formulas in [KY10] to our setting. We want to compute the Fourier coefficients of the Eisenstein series

$$E_L(\tau, s; \ell) = \frac{1}{4} \sum_{\gamma \in \langle T \rangle \backslash Mp_2(\mathbb{Z})} \left[\Im(\tau)^{(s+1-\ell)/2} \mathfrak{e}_0 \right] \Big|_{\ell, L} \gamma. \tag{3.1.1}$$

When $s = \ell - 1 > 1$, we obtain the holomorphic vector valued Eisenstein $E_{\rho_L, \ell, 0}$.We do not restrict to this case too early in order to keep the implementations derived form these formulas open to future use cases.

We replace the group ring $\mathbb{C}[L'/L]$ with a ring of Schwartz functions. This amounts to replacing the standard generators $\mathfrak{e}_\mu$ by characteristic functions ϕ_μ in our notation.

3.1.1. Setup

The basic setup taken from [Kud03; BY09; KY10] is as follows.

Let $G = \mathrm{SL}_2$ viewed as an algebraic group over $\mathbb{Q}$, and let $P = NM$ be its parabolic subgroup of upper triangular matrices with notation

$$\begin{aligned} M &= \{m(a) \mid a \text{ invertible}\}, & m(a) &= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \\ N &= \{n(b) \mid b\}, & n(b) &= \begin{pmatrix} 1 & b \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

We have the Iwasawa decomposition $G(\mathbb{A}) = P(\mathbb{A})K$, where $K = K_\infty K_f$ is given by $K_\infty = \mathrm{SO}_2(\mathbb{R})$ at the Archimedean place and $K_f = \mathrm{SL}_2(\hat{\mathbb{Z}}) = \prod_{p<\infty} \mathrm{SL}_2(\mathbb{Z}_p)$ at the finite places. For $\theta \in \mathbb{R}$ and $\tau = u + iv \in \mathbb{H}$, we define

$$k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SO}_2(\mathbb{R}), \quad g_\tau = n(u)m(v^{\frac{1}{2}}) \in P(\mathbb{R}). \tag{3.1.2}$$

The Möbius transformation associated to k_θ fixes i, whereas g_τ sends i to τ.

We let

$$1 \to \mathbb{C}^1 \to G'_{\mathbb{A}} \to G(\mathbb{A}) \to 1 \tag{3.1.3}$$

be the metaplectic extension of $G(\mathbb{A})$ by $\mathbb{C}^1$, the unit circle in $\mathbb{C}$. For details on this extension, we refer to [KRY06, Chapter 8.5]. Elements of $G'_{\mathbb{A}}$ are written as $(g, z) \in G(\mathbb{A}) \times \mathbb{C}^1$ and the multiplication can be given in terms of the Leray cocycle.

We fix an even lattice L with quadratic form Q of signature $b^+ - b^-$ and rank n. We say that we are in the "even case", if the rank of L is even, otherwise we are in the "odd case". Define the set of "bad primes" to be $S = \{p \mid p \text{ prime and } p \mid \det(L)\}$. The prime 2 is always bad in the odd case. To L we associate the quadratic space $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$ and the quadratic characters

$$\chi_V(x) = (x, \mathrm{disc}(L))_{\mathbb{A}^\times}, \quad \chi_{V,p}(x) = (x, \mathrm{disc}(L))_p \tag{3.1.4}$$

given in terms of adelic and p-adic Hilbert symbols. We represent these characters by a square-free integer κ satisfying

$$\chi_V(x) = \begin{cases} (x, \kappa)_{\mathbb{A}}, & \text{in the even case,} \\ (x, 2\kappa)_{\mathbb{A}}, & \text{in the odd case.} \end{cases} \tag{3.1.5}$$

We assume

$$\begin{cases} (-1)^\ell = \mathrm{sign}(\kappa), & \text{in the even case,} \\ \ell \equiv \frac{1}{2}\mathrm{sign}(\kappa) \pmod 2, & \text{in the odd case,} \end{cases} \tag{3.1.6}$$

which is equivalent to $2\ell \equiv b^+ - b^- \pmod 4$.

Remark 3.1.1. If we were using the dual Weil representation instead, we would get the above condition for $L(-1)$, where the roles of b^+ and b^- are interchanged. This is equivalent to the condition $2\ell + b^+ - b^- \equiv 0 \pmod 4$ encountered in [BK01] for ρ_L^*.

For $s \in \mathbb{C}$, we have the principal series representation $I(s, \chi_V)$ of $G'_{\mathbb{A}}$. The sections $\Phi(s) \in I(s, \chi_V)$ are smooth functions on $G'_{\mathbb{A}}$ satisfying

$$\Phi((n(b)m(a), z)g', s) = \chi_V(a)|a|^{s+1}\Phi(g', s) \begin{cases} 1, & \text{in the even case,} \\ z, & \text{in the odd case,} \end{cases} \tag{3.1.7}$$

for all $b \in \mathbb{A}$ and $a \in \mathbb{A}^\times$. Such a section is called *standard*, if its restriction to K' (the inverse image of K in $G'_{\mathbb{A}}$) is independent of s. To a standard section $\Phi(s) \in I(s, \chi_V)$, we associate the Eisenstein series

$$E(g', s, \Phi) = \sum_{\gamma \in P(\mathbb{Q}) \backslash G(\mathbb{Q})} \Phi(\gamma g', s). \tag{3.1.8}$$

It is known that this series is convergent for $\Re(s) > 1$ and that it has a meromorphic continuation to the whole s-plane.

The components of the vector valued Eisenstein series can be given in terms of Eisenstein series for standard sections. We follow [BY09] for this identification. We fix the standard additive character $\psi : \mathbb{A}/\mathbb{Q} \to \mathbb{C}^\times$ given by $\psi_\infty(x_\infty) = e(x_\infty) = e^{2\pi i x_\infty}$ and $\psi_p(x_p) = e(-x_p)$ (cf. [Dei10, Kapitel 5.4]). We define the characteristic functions $\phi_\mu = \mathrm{char}(\mu + L \otimes \hat{\mathbb{Z}})$ for $\mu \in L'/L$. They span the space $S_L \subset S(V(\mathbb{A}_f))$ of Schwartz functions on $V(\mathbb{A}_f)$. There is a Weil representation $\omega = \otimes \omega_{V,\psi_p}$ acting on Schwartz functions and a $G(\mathbb{A})$-intertwining map

$$\lambda : S(V(\mathbb{A})) \to I(\frac{n}{2} - 1, \chi_V), \quad \lambda(\phi)(g') = (\omega(g')\phi)(0). \tag{3.1.9}$$

At the Archimedean place, we use the unique standard section $\Phi_\infty^\ell \in I(s, \chi_V)$ satisfying

$$\Phi_\infty^\ell(k_\theta, s) = e^{i\ell\theta}. \tag{3.1.10}$$

Using the Iwasawa decomposition, this amounts to

$$\Phi_\infty^\ell(n(b)m(a)k_\theta, s) = \chi_V(a)|a|^{s+1}e^{i\ell\theta}. \tag{3.1.11}$$

The vector valued Eisenstein series can now be written in the form (cf. [BY09, equations (2.16) and (2.17)])

$$\begin{aligned} E_L(\tau, s; \ell) &= \frac{1}{4} \sum_{\gamma \in \langle T \rangle \backslash Mp_2(\mathbb{Z})} \left[\Im(\tau)^{(s+1-\ell)/2} \phi_0 \right] \Big|_\ell \gamma \\ &= v^{-\ell/2} \sum_{\mu \in L'/L} E(g_\tau, s; \Phi_\infty^\ell \otimes \lambda_f(\phi_\mu))\phi_\mu. \end{aligned} \tag{3.1.12}$$

We denote the components by

$$E(\tau, s, \Phi_\infty^\ell \otimes \lambda_f(\phi_\mu)) := v^{-\ell/2} E(g'_\tau, s, \Phi_\infty^\ell \otimes \lambda_f(\phi_\mu)). \tag{3.1.13}$$

3.1.2. The Siegel-Weil formula

The Siegel-Weil formula identifies an integral of a theta function with an Eisenstein series at $s = s_0 := n/2 - 1$ (i.e. weight $n/2$). We will state it in a vector valued form in the case of positive definite even lattices of rank $n \geq 5$. We state the Siegel-Weil formula as given in [Kud03]. The mass of the genus of L appears naturally and we use methods and notation from [Kne02, Kapitel X] to discuss this.

Let L be a positive definite even lattice of rank $n \geq 5$. As before we set $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$. We define the adelic orthogonal groups by

$$\begin{aligned} O_{\mathbb{A}}(V) &= \left\{ u = (u_p)_p \in \prod_{p \leq \infty} O(V_p) \;\middle|\; u_p L_p = L_p \text{ almost all } p \right\}, \\ O_{\mathbb{A}_f}(V) &= \left\{ u = (u_p)_p \in \prod_{p < \infty} O(V_p) \;\middle|\; u_p L_p = L_p \text{ almost all } p \right\}. \end{aligned} \tag{3.1.14}$$

For a lattice $M \supset L$, we define the stabilizer

$$O_{\mathbb{A}_f}(V, M) = \left\{ u \in O_{\mathbb{A}_f}(V) \;\middle|\; uM = M \right\}. \tag{3.1.15}$$

For $g' \in G'_{\mathbb{A}}$, $h \in O_{\mathbb{A}}(V)$ and a Schwartz function $\phi \in S(V(\mathbb{A}))$, the theta series

$$\theta(g', h, \phi) = \sum_{x \in V(\mathbb{Q})} \omega(g')\phi(h^{-1}x) \tag{3.1.16}$$

defines a smooth function on $G'_{\mathbb{A}} \times O_{\mathbb{A}}(V)$. The theta integral

$$I(g', \phi) = \int_{O(V)\backslash O_{\mathbb{A}}(V)} \theta(g', h, \phi) \mathrm{d}h \tag{3.1.17}$$

converges absolutely by Weil's convergence criterion. The measure is normalized such that $\mathrm{vol}(O(V)\backslash O_{\mathbb{A}}(V), \mathrm{d}h) = 1$. In our particular case, the Siegel-Weil formula takes the following form.

Siegel-Weil formula, Theorem 3.1.2 (cf. [Kud03, Theorem 4.1])**.**

$$E(g', s_0, \lambda(\phi)) = I(g', \phi)$$

We follow the proof of [Kud03, Proposition 4.22] to further evaluate the right hand side of the Siegel Weil formula in the case that $\phi = \phi_\mu$ for $\mu \in L'/L$. As a special case of [Kne02, Satz 31.13], we have the decomposition

$$O_{\mathbb{A}_f}(V) = \bigcup_{j=1}^{h(L)} O(V) h_j O_{\mathbb{A}_f}(V, L) \tag{3.1.18}$$

where $h(L)$ is the class number of L. The h_j can be chosen such that the lattices $h_j L$ represent all classes in the genus of L. Their orthogonal groups are given by

$$O(h_j L) \simeq \Gamma_j := O(V) \cap \left(O_{\mathbb{R}}(V) h_j O_{\mathbb{A}_f}(V, L) h_j^{-1} \right). \tag{3.1.19}$$

These are finite groups because the $h_j L$ are positive definite. We let $\phi_0(x) = e^{-2\pi Q(x)}$ denote the Gaussian and calculate

$$\begin{aligned} I(g', \lambda(\phi_0 \otimes \phi_\mu)) \mathrm{d}h &= \int_{O(V)\backslash O_{\mathbb{A}}(V)} \theta(g', h, \lambda(\phi_0 \otimes \phi_\mu)) \mathrm{d}h \\ &= \sum_j \int_{\Gamma_j \backslash O_{\mathbb{R}} h_j O_{\mathbb{A}_f}(V,L)} \theta(g', h, \lambda(\phi_0 \otimes \phi_\mu)) \mathrm{d}h \end{aligned} \tag{3.1.20}$$

By [Nik79, Corollary 1.9.6], the canonical homomorphism $O_{\mathbb{A}_f}(V, L) \to O(L'/L)$ is surjective. Hence we can symmetrize ϕ_μ with respect to $O(L'/L)$ without changing the integral.

$$\begin{aligned} &\sum_j \int_{\Gamma_j \backslash O_{\mathbb{R}} h_j O_{\mathbb{A}_f}(V,L)} \theta(g', h, \lambda(\phi_0 \otimes \phi_\mu)) \mathrm{d}h \\ &= \sum_j \int_{\Gamma_j \backslash O_{\mathbb{R}} h_j O_{\mathbb{A}_f}(V,L)} \theta \left(g', h, \lambda \left(\phi_0 \otimes \frac{1}{|O(L'/L)|} \sum_{\gamma \in O(L'/L)} \phi_{\gamma\mu} \right) \right) \mathrm{d}h \\ &= \mathrm{vol}(O_{\mathbb{R}}(V) O_{\mathbb{A}_f}(V, L)) \sum_j \frac{1}{|O(h_j L)|} \theta \left(g', h_j, \lambda \left(\phi_0 \otimes \frac{1}{|O(L'/L)|} \sum_{\gamma \in O(L'/L)} \phi_{\gamma\mu} \right) \right) \end{aligned} \tag{3.1.21}$$

For $g' = g'_\tau$, we have

$$(\omega_\infty(g'_\tau)\phi_0)(x) = v^{\ell/2} e^{2\pi i Q(x)\tau}. \tag{3.1.22}$$

We recall the normalization of the measure dh, and infer

$$\begin{aligned} 1 &= \mathrm{vol}(O(V)\backslash O_{\mathbb{A}}(V), \mathrm{d}h) \\ &= \mathrm{vol}(O_{\mathbb{R}}(V)O_{\mathbb{A}_f}(V, L)), \end{aligned} \tag{3.1.23}$$

which implies

$$\mathrm{vol}(O_{\mathbb{R}}(V)O_{\mathbb{A}_f}(V, L)) = \left(\sum_j \frac{1}{|O(h_j L)|}\right)^{-1} \tag{3.1.24}$$

which is the reciprocal of the mass of the genus of L in the sense of [Kne02, (35.1)]. Summarizing, we have the following expression for the Eisenstein series of weight $\ell = n/2$,

$$\begin{aligned} &E(\tau, s_0, \Phi_\infty^\ell \otimes \lambda_f(\phi_\mu)) \\ &= v^{-\ell/2} E(g'_\tau, s_0, \lambda(\phi_0 \otimes \phi_\mu)) \\ &= v^{-\ell/2} I(g'_\tau, \phi_0 \otimes \phi_\mu) \\ &= \frac{1}{|O(L'/L)|}\left(\sum_j \frac{1}{|O(h_j L)|}\right)^{-1} \sum_{\gamma \in O(L'/L)} \sum_j \frac{1}{|O(h_j L)|} \sum_{x \in V(\mathbb{Q})} \mathrm{char}(h_j\gamma(\mu + L))(x) q^{Q(x)} \\ &= \frac{1}{|O(L'/L)|}\left(\sum_j \frac{1}{|O(h_j L)|}\right)^{-1} \sum_{\gamma \in O(L'/L)} \sum_j \frac{1}{|O(h_j L)|} \sum_{x \in h_j\gamma(\mu+L)} q^{Q(x)}. \end{aligned} \tag{3.1.25}$$

We can think of this as the sum $\sum_{x\in\mu+L} q^{Q(x)}$ symmetrized once with respect to $O(L'/L)$ and once with respect to the classes in the genus of L. In analogy to [Kud03, Proposition 4.22] we obtain the following theorem.

Siegel-Weil formula, Theorem 3.1.3. For a positive definite even lattice of rank $n \geq 5$, the Eisenstein series of weight $\ell = n/2$ has the Fourier expansion

$$E(\tau, \ell - 1, \Phi_\infty^\ell \otimes \lambda_f(\phi_\mu)) = \sum_{m \in \mathbb{Q}_{\geq 0}} r_\mu(m) q^m,$$

where

$$r_\mu(m) = \frac{1}{|O(L'/L)|}\left(\sum_j \frac{1}{|O(h_j L)|}\right)^{-1} \sum_{\gamma \in O(L'/L)} \sum_j \frac{1}{|O(L_j)|} \sum_{\substack{x \in h_j\gamma(\mu+L) \\ Q(x) = m}} 1.$$

Equivalently, if $L_1, \ldots, L_{h(L)}$ are representatives for the classes in the genus of L and we identify their discriminant forms such that $\mu_j \in L_j'/L_j$ corresponds to $\mu \in L'/L$, we have

$$r_\mu(m) = \frac{1}{|O(L'/L)|}\left(\sum_j \frac{1}{|O(L_j)|}\right)^{-1} \sum_j \frac{1}{|O(L_j)|} \sum_{\gamma \in O(L_j'/L_j)} \sum_{\substack{x \in \gamma(\mu_j + L_j) \\ Q(x)=m}} 1.$$

In Section 3.7, we give a vector valued version of this Siegel-Weil formula and some examples.

3.1.3. Fourier coefficients

For $m \in \mathbb{Q}$ we write

$$4\kappa m = dc^2, \tag{3.1.26}$$

where d is the fundamental discriminant of $\mathbb{Q}(\sqrt{\kappa m})$. For a prime p we let $k = \mathrm{ord}_p(c)$, $v_p = \left(\frac{d}{p}\right)$ and $X = p^{-s}$ and define $b_p(\kappa m, s; D)$ as

$$\begin{cases} \frac{1 - v_pX + v_pp^kX^{2k+1} - p^{k+1}X^{2k+2}}{1-pX^2}, & p \nmid D \text{ and } k \geq 0, \\ \frac{(1-v_pX)(1-p^2X^2) - v_pp^{k+1}X^{2k+1} + p^{k+2}X^{2k+2} + v_pp^{k+1}X^{2k+3} - p^{2k+2}X^{2k+4}}{1-pX^2}, & p \mid D \text{ and } k \geq 0, \\ 1, & k < 0. \end{cases} \tag{3.1.27}$$

We set $b_p(\kappa m, s) = b_p(\kappa m, s; 1)$ and $b(\kappa m, s) = \prod_p b_p(\kappa m, s)$. The following functions are given by products of primes which are not in S. Any appearing χ is a Dirichlet character. In our setup, it will always be χ_V or $\chi_{\kappa m}$, the Dirichlet character associated to $\mathbb{Q}(\sqrt{\kappa m})$.

$$\zeta^S(s) = \prod_{p \notin S}(1-p^{-s})^{-1} = \zeta(s) \cdot \prod_{p \in S}(1-p^{-s}), \tag{3.1.28}$$

$$L^S(s,\chi) = \prod_{p \notin S}(1-\chi(p)p^{-s})^{-1} = L(s,\chi) \cdot \prod_{p \in S}(1-\chi(p)p^{-s}), \tag{3.1.29}$$

$$\sigma_s^S(m,\chi) = \prod_{\substack{\mathrm{ord}_p(m) \geq 0 \\ p \notin S}} \sum_{r=0}^{\mathrm{ord}_p(m)} (\chi(p)p^s)^r, \tag{3.1.30}$$

$$b^S(\kappa m, s) = \prod_{p \notin S} b_p(\kappa m, s) = \prod_{\substack{p \mid c \\ p \notin S}} b_p(\kappa m, s) = \prod_{\substack{\mathrm{ord}_p(m) \geq 0 \\ p \notin S}} b_p(\kappa m, s). \tag{3.1.31}$$

We also need the following product over generalized local densities $W_p(s, m, \mu)$ for the even lattice L given by

$$W_{m,S}(s) = \prod_{p \in S} \gamma_p(L)|\det(L)|_p^{\frac{1}{2}} W_p(s,m,\mu). \tag{3.1.32}$$

The values of these local densities have been computed in [KY10]. We give the formulas in the next section. The $\gamma_p(L)$ are the Weil invariants of the discriminant form L'/L.

Fixing the quadratic Dirichlet characters χ associated to the Hilbert symbol $(\,\cdot\,,\kappa)$ by Correspondence 1.10.2 and $\chi_{\kappa m}$ associated to $\mathbb{Q}(\sqrt{\kappa m})$, we can give the following formulas for the Eisenstein series.

Theorem 3.1.4 ([KY10, Proposition 2.6])**.** *Let L be an even lattice L and $\mu \in L'/L$. When $\ell > 2$, the the special value*

$$E(\tau, \ell-1, \Phi_\infty^\ell \otimes \lambda_f(\phi_\mu)) = \mathfrak{e}_0 + \sum_{\mu \in L'/L} \sum_{m>0} a_E(\mu, m) q^n \mathfrak{e}_\gamma \tag{3.1.33}$$

is holomorphic. In the even case its Fourier coefficients are given by

$$a_E(\mu, m) = \frac{(-2\pi i)^\ell m^{\ell-1}}{\Gamma(\ell) L^S(\ell, \chi)} \sigma_{1-\ell}^S(m, \chi) W_{m,S}(\ell - 1). \tag{3.1.34}$$

In the odd case, they are given by

$$a_E(\mu, m) = \frac{(-2\pi i)^\ell m^{\ell-1}}{\Gamma(\ell)\zeta^S(2\ell-1)} L^S(\ell - \frac{1}{2}, \chi_{\kappa m}) b^S(\kappa m, \ell - \frac{1}{2}) W_{m,S}(\ell-1). \tag{3.1.35}$$

3.1.4. Generalized local densities

In this section, we explain how to compute the local densities (also called local Whittaker functions) $W_p(s, m, \mu)$ which are needed for the above formulas for the Fourier coefficients of vector valued Eisenstein series for the Weil representation. Recall that L is an even lattice and $\mu \in L'/L$. We identify L_p with $\mathbb{Z}_p^n$ such that the Gram matrix S_p is in canonical form, as given in Lemma 1.4.1. By this identification we have $\mu \in S^{-1}\mathbb{Z}_p^n/\mathbb{Z}_p^n$. An appropriate change of basis can be computed by following the steps in [Cas78, Chapter 8, Section 4]. We have computed the missing change of basis for the even 2-adic case in Appendix A.

The case $p \neq 2$

We assume $L = \mathbb{Z}_p^n$ with Gram matrix

$$S = \operatorname{diag}(2\varepsilon_1 p^{l_1}, \ldots, 2\varepsilon_n p^{l_n}), \tag{3.1.36}$$

where $\varepsilon_i \in \mathbb{Z}_p^\times$ and $l_i \in \mathbb{Z}_{\geq 0}$. Write $\mu = (\mu_1, \ldots, \mu_n) \in L' = S^{-1}\mathbb{Z}_p^n = \oplus p^{-l_i}\mathbb{Z}_p$, where we identify $\mu \in L'/L$ with a representative in L'. We fix the values

$$s_0 = \frac{n}{2} - 1, \tag{3.1.37}$$

$$H_\mu = \{i \mid \mu_i \in \mathbb{Z}_p\}, \tag{3.1.38}$$

$$K_0 = K_0(\mu) = \min(\{l_i + \operatorname{ord}_p(\mu_i) \mid i \notin H_\mu\} \cup \{\infty\}), \tag{3.1.39}$$

$$L_\mu(k) = \{i \in H_\mu \mid l_i - k < 0 \text{ is odd }\}, \tag{3.1.40}$$

$$l_\mu(k) = \# L_\mu(k), \tag{3.1.41}$$

$$d_\mu(k) = k + \frac{1}{2} \sum_{i \in H_\mu} \min(l_i - k, 0), \tag{3.1.42}$$

$$\varepsilon_\mu(k) = \left(\frac{-1}{p}\right)^{[\frac{l_\mu(k)}{2}]} \prod_{i \in L_\mu(k)} \left(\frac{\varepsilon_i}{p}\right), \tag{3.1.43}$$

$$t_\mu(m) = m - \sum_{i \notin H_\mu} \varepsilon_i p^{l_i} \mu_i^2, \tag{3.1.44}$$

$$a_\mu(m) = \operatorname{ord}_p(t_\mu) \tag{3.1.45}$$

for $k \in \mathbb{Z}_{\geq 0}$ and $m \in \mathbb{Q}$. Furthermore, we need the function

$$f_1(\alpha p^a) = \begin{cases} -\frac{1}{p}, & \text{if } l_\mu(a+1) \text{ is even,} \\ (\alpha, p)_p \frac{1}{\sqrt{p}}, & \text{if } l_\mu(a+1) \text{ is odd,} \end{cases} \tag{3.1.46}$$

where $\alpha \in \mathbb{Z}_p^\times$. With this notation the p-adic local Whittaker function can be computed as follows.

Theorem 3.1.5 ([KY10, Theorem 4.3])**.** *If $m \notin Q(\mu) + \mathbb{Z}_p$, then $W_p(s + s_0, m, \mu) = 0$. If $m \in Q(\mu) + \mathbb{Z}_p$, we set $X = p^{-s}$ and distinguish two cases:*

(i) If $0 \leq a < K_0$, then

$$\begin{aligned} W_p(s + s_0, m, \mu) = 1 + \left(1 - \frac{1}{p}\right) &\sum_{\substack{0 < k \leq a \\ l_\mu(k) \text{ even}}} \varepsilon_\mu(k) p^{d_\mu(k)} X^k \\ &+ \varepsilon_\mu(a+1) f_1(t_\mu(m)) p^{d_\mu(a+1)} X^{a+1}. \end{aligned} \tag{3.1.47}$$

(ii) If $a \geq K_0$, then

$$W_p(s + s_0, m, \mu) = 1 + \left(1 - \frac{1}{p}\right) \sum_{\substack{0 < k \leq K_0 \\ l_\mu(k) \text{ even}}} \varepsilon_\mu(k) p^{d_\mu(k)} X^k. \tag{3.1.48}$$

The case $p = 2$

We assume $L = \mathbb{Z}_2^n$ with Gram matrix

$$S = \operatorname{diag}(\varepsilon_1 p^{l_1}, \dots, \varepsilon_n p^{l_H}) \oplus \left(\bigoplus_{i=1}^{M} 2^{m_i} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \oplus \left(\bigoplus_{j=1}^{N} 2^{n_j} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \right), \tag{3.1.49}$$

where $\varepsilon_h \in \mathbb{Z}_2^\times$, $1 \le l_h \in \mathbb{Z}_{\ge 0}$ and $m_i, n_j \in \mathbb{Z}_{\ge 0}$. Write

$$\begin{aligned} \mu &= (\mu_1, \dots, \mu_H, \mu'_1, \dots, \mu'_M, \mu''_1, \dots, \mu''_N) \\ &\in \oplus_{h=1}^{H} p^{-l_i} \mathbb{Z}_2 \oplus \left(\oplus_{i=1}^{M} 2^{-m_i} \mathbb{Z}_2^2 \right) \oplus \left(\oplus_{j=1}^{N} 2^{-n_j} \mathbb{Z}_2^2 \right) \\ &= S^{-1} \mathbb{Z}_2^n = L', \end{aligned} \tag{3.1.50}$$

where we identify $\mu \in L'/L$ with a representative in L'. Note that the coefficients of μ with dashes are 2-dimensional. For these coefficients, we define

$$\operatorname{ord}_2(t_1, t_2) = \min(\operatorname{ord}_2(t_1), \operatorname{ord}_2(t_2)).$$

We fix the values

$$s_0 = \frac{n}{2} - 1, \tag{3.1.51}$$

$$H_\mu = \{h \mid \mu_h \in \mathbb{Z}_2\}, \tag{3.1.52}$$

$$M_\mu = \left\{ i \;\middle|\; \mu'_i \in \mathbb{Z}_2^2 \right\}, \tag{3.1.53}$$

$$N_\mu = \left\{ i \;\middle|\; \mu''_j \in \mathbb{Z}_2^2 \right\}, \tag{3.1.54}$$

$$\begin{aligned} K_0 = K_0(\mu) = \min(&\{l_h + \operatorname{ord}_2(\mu_h) \mid \operatorname{ord}_2(\mu_h) < -1\} \cup \{l_h \mid \operatorname{ord}_2(\mu_h) = -1\} \quad (3.1.55) \\ &\cup \{m_i + \operatorname{ord}_2(\mu'_i) \mid i \notin M_\mu\} \cup \left\{ n_j + \operatorname{ord}_2(\mu''_j) \;\middle|\; j \notin N_\mu \right\} \cup \{\infty\}), \end{aligned} \tag{3.1.56}$$

$$L_\mu(k) = \{i \in H_\mu \mid l_i - k < 0 \text{ is odd }\}, \tag{3.1.57}$$

$$l_\mu(k) = \# L_\mu(k), \tag{3.1.58}$$

$$d_\mu(k) = k + \frac{1}{2} \sum_{h \in H_\mu} \min(l_h - k, 0) + \sum_{i \in M_\mu} \min(m_i - k, 0) + \sum_{j \in N_\mu} \min(n_j - k, 0), \tag{3.1.59}$$

$$p_\mu(k) = (-1)^{\sum_{j \in N_\mu} \min(n_j - k, 0)}, \tag{3.1.60}$$

$$\varepsilon_\mu(k) = \prod_{h \in L_\mu(k)} \varepsilon_h, \tag{3.1.61}$$

$$\delta_\mu(k) = \begin{cases} 0, & \text{if } l_h = k \text{ for some } h \in H_\mu, \\ 1, & \text{else}, \end{cases} \tag{3.1.62}$$

$$t_\mu(m) = m - \sum_{h \notin H_\mu} \varepsilon_h 2^{l_h - 1} \mu_h^2 - \sum_{i \notin M_\mu} 2^{m_i} \mu'_{i1} \mu'_{i2} - \sum_{j \notin N_\mu} 2^{n_j} ((\mu''_{i1})^2 + \mu'_{i1}\mu'_{i2} + (\mu''_{i2})^2), \tag{3.1.63}$$

$$a_\mu(m) = \operatorname{ord}_2(t_\mu), \tag{3.1.64}$$

$$\nu = \nu(m,k) = t_\mu 2^{3-k} - \sum_{\substack{h \in H_\mu \\ l_h < k}} \varepsilon_h, \tag{3.1.65}$$

$$\alpha(\nu) = \begin{cases} 0, & \nu \notin 4\mathbb{Z}_2, \\ -1, & \nu \in 4 + 8\mathbb{Z}_2, \\ 1, & \nu \in 8\mathbb{Z}_2, \end{cases} \tag{3.1.66}$$

for $k \in \mathbb{Z}_{\geq 0}$ and $m \in \mathbb{Q}$. With this notation the 2-adic local Whittaker function is as follows.

Theorem 3.1.6 ([KY10, Theorem 4.3])**.** *If $m \notin Q(\mu) + \mathbb{Z}_2$, then $W_2(s + s_0, m, \mu) = 0$. If $m \in Q(\mu) + \mathbb{Z}_2$, we set $X = 2^{-s}$ and have*

$$\begin{aligned} &W_2(s+s_0,m,\mu) \\ &= 1 + \sum_{\substack{0<k\leq \min(K_0,a+3) \\ l_\mu(k) \text{ odd}}} \delta_\mu(k) p_\mu(k) 2^{d_\mu(k) - \frac{3}{2}} \left(\frac{2}{\varepsilon_\mu(k)\nu}\right) X^k \\ &\quad + \sum_{\substack{0<k\leq \min(K_0,a+3) \\ l_\mu(k) \text{ even}}} \delta_\mu(k) p_\mu(k) 2^{d_\mu(k)-1} \left(\frac{2}{\varepsilon_\mu(k)}\right) \alpha(\nu) X^k. \end{aligned} \tag{3.1.67}$$

3.1.5. Properties of local densities

Let $\mu \in L'$ and p a prime. We have seen in the previous theorems, that the local densities are given in terms of polynomials, which we call Whittaker polynomials and denote them by $W_{p,m,\mu}(X)$ such that

$$W_p(s+s_0,m,\mu) = W_{p,m,\mu}(p^{-s}). \tag{3.1.68}$$

For now, these polynomials depend on an even lattice. The Eisenstein series only depends on the discriminant form, this should also hold for the generalized local densities. To prove this, we collect some of their properties. This will allow us to compute the generalized local densities purely from the discriminant form.

Remark 3.1.7. Tracking changes when adding a hyperbolic plane, we see that

$$W_{p,L}(s+s_0,m,\mu) = W_{p,L\oplus H}((s-1)+(s_0+1),m,(\mu,0,0)) \tag{3.1.69}$$

stays invariant. Note that this fact is used in the proofs for the above formulas of local densities in [KY10].

Proof. Evaluation of the right hand side (using Theorem 3.1.5 and Theorem 3.1.6) gives the formula of the left hand side, where $d_\mu(k)$ has been replaced by $d_\mu(k)-k$ and $X=p^{-s}$ has been replaced by $p^{-(s-1)}=pX$. These changes cancel. □

Lemma 3.1.8. *For fixed μ, the local densities $W_p(s+s_0,m,\mu)$ and their corresponding polynomials only depend on $m \pmod{p^{K_0}}$, that is*

$$W_p(s+s_0,m+p^{K_0},\mu) = W_p(s+s_0,m,\mu) \tag{3.1.70}$$

for any $s\in\mathbb{C}$, $m\in\mathbb{Q}$, $\mu\in L'$ and $K_0=K_0(\mu)\neq\infty$.

Proof. If p is odd, we look at the changes in the formulas of Theorem 3.1.5, when replacing m by $m+p^{K_0}$. The value of $t_\mu(m)$ changes to $t_\mu(m)+p^{K_0}$.

If $a=\mathrm{ord}_p(t_\mu(m))<K_0$, then $\mathrm{ord}_p(t_\mu(m))=\mathrm{ord}_p(t_\mu(m)+p^{K_0})$ and a remains unchanged. In the computation of $f_1(t_\mu(m)+p^{K_0})$, the value α changes to $\alpha+p^{K_0-a}$ which has no effect on the Hilbert symbol.

On the other hand, if $a\geq K_0$, it might be replaced by a bigger value which has no effect on the case distinction and on (3.1.48).

If $p=2$, , we look at the changes in the formulas of Theorem 3.1.6, when replacing m by $m+2^{K_0}$. The value of $t_\mu(m)$ changes to $t_\mu(m)+2^{K_0}$.

If $a=\mathrm{ord}_2(t_\mu(m))<K_0$, then $\mathrm{ord}_2(t_\mu(m))=\mathrm{ord}_2(t_\mu(m)+2^{K_0})$ and a remains unchanged. If $a\geq K_0$, then a it might be replaced by a bigger value. In both these cases, the value $\min(a+3,K_0)$ remains unchanged.

By verifying that ν changes by a multiple of 8, we see that formula (3.1.67) is invariant. □

3.2. The approach of Bruinier and Kuss

In this section, we adapt the formulas in [BK01] to our setting. Let L be an even lattice of type (b^+,b^-), rank m and with Gram matrix S. Let $2k-b^-+b^+\equiv 0 \pmod 4$. As in equation (1.8.1), we define the Eisenstein series of weight $k>2$ by

$$E = E_{\rho_L^*,k,0} = \frac{1}{4}\sum_{(M,\phi)\in\langle T\rangle\backslash Mp_2(\mathbb{Z})} \mathfrak{e}_0|^*_{k,L}(M,\phi), \tag{3.2.1}$$

which is normalized to have constant coefficient 1 and is thus half of the Eisenstein series in [BK01]. It has a Fourier expansion of the form

$$E = \mathfrak{e}_0 + \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z} - Q(\gamma) \\ n > 0}} a_E(\gamma, n) q^n \mathfrak{e}_\gamma. \tag{3.2.2}$$

The Fourier coefficients can be obtained as follows.

Proposition 3.2.1 (cf. [BK01, Proposition 4.3])**.** *Let $\gamma \in L'$ and $n \in \mathbb{Z} - Q(\gamma)$ with $n > 0$. The coefficient $a_E(\gamma, n)$ equals the value at $s = k$ of the analytic continuation in s of*

$$\frac{2^k \pi^k n^{k-1} (-1)^{(2k - b^- + b^+)/4}}{\sqrt{|L'/L|}\Gamma(k)\zeta(s - m/2)} L_{\gamma,n}(s). \tag{3.2.3}$$

Here $\zeta(s)$ denotes the Riemann zeta function and $L_{\gamma,n}(s)$ the L-series

$$L_{\gamma,n}(s) = \sum_{a \geq 1} N_{\gamma,n}(a) a^{1-m/2-s} = \prod_p \left(\sum_{\nu \geq 0} N_{\gamma,n}(p^\nu) p^{\nu(1-m/2-s)} \right) \tag{3.2.4}$$

for the representation numbers

$$N_{\gamma,n}(a) = \# \left\{ r \in L/aL \mid Q(r - \gamma) + n \equiv 0 \pmod{a} \right\}. \tag{3.2.5}$$

We may write the Euler product as

$$\begin{aligned} L_{\gamma,n}(s) &= \zeta(s - m/2) \prod_p \left(\left(1 - p^{m/2-s}\right) \sum_{\nu \geq 0} N_{\gamma,n}(p^\nu) p^{\nu(1-m/2-s)} \right) \\ &= \zeta(s - m/2) \prod_p \left(\left(1 - p^{m/2-s}\right) \sum_{\nu \geq 0} \frac{N_{\gamma,n}(p^\nu)}{p^{m\nu}} p^{\nu(1+m/2-s)} \right) \\ &= \zeta(s - m/2) \prod_p \left(\left(1 - \frac{t}{p}\right) \sum_{\nu \geq 0} \frac{N_{\gamma,n}(p^\nu)}{p^{m\nu}} t^\nu \bigg|_{t = p^{1+m/2-s}} \right) \\ &= \zeta(s - m/2) \prod_p \left(\left(1 - \frac{t}{p}\right) \frac{1 - tZ_f(t)}{1 - t} \bigg|_{t = p^{1+m/2-s}} \right) \end{aligned} \tag{3.2.6}$$

using an appropriate polynomial f and the Igusa local zeta function $Z_f(t)$. The coefficient $a_{E_0}(\gamma, n)$ can be obtained by the analytic continuation of

$$\frac{2^k \pi^k n^{k-1} (-1)^{(2k - b^- + b^+)/4}}{\sqrt{|L'/L|}\Gamma(k)} \prod_p L_{\gamma,n}(s, p) \tag{3.2.7}$$

with

$$L_{\gamma,n}(s,p) = \left(\left(1 - \frac{t}{p}\right) \sum_{\nu \geq 0} \frac{N_{\gamma,n}(p^\nu)}{p^{m\nu}} t^\nu \right)\Bigg|_{t=p^{1+m/2-s}}. \tag{3.2.8}$$

By Theorem 2.1.1, the $L_{\gamma,n}(s,p)$ are invariant under addition of hyperbolic planes to L (where we add two zeros to γ). This insight allows us to obtain the simplifications stated in [BK01] for the case $k = m/2$ for all weights.

Lemma 3.2.2 ([BK01, Lemma 4.4], cf. [Sie35, Hilfssatz 13]). *Let p be a prime and let d_γ denote the order of γ in L'/L. Put*

$$w_p = 1 + 2\operatorname{ord}_p(2nd_\gamma).$$

Then the equality

$$N_{\gamma,n}(p^{\alpha+1}) = p^{m-1} N_{\gamma,n}(p^\alpha)$$

holds for any $\alpha \geq w_p$.

Using geometric series, the Euler factors are hence given by

$$L_{\gamma,n}(s,p) = \left(1 - p^{m/2-s}\right) \sum_{\nu=0}^{w_p-1} N_{\gamma,n}(p^\nu) p^{\nu(1-m/2-s)} + N_{\gamma,n}(p^{w_p}) p^{w_p(1-m/2-s)}. \tag{3.2.9}$$

In the case $s = m/2$, the factor in front of the sum vanishes and we can compute what is left with the following theorem. It is a generalization of [Sie35, Hilfssatz 16] in the version of [BK01, Theorem 4.5] where $p = 2$ was excluded. We include the case $p = 2$.

Theorem 3.2.3. *Let p be a prime not dividing $\det(S)$ and $\alpha \in \mathbb{Z}$ with $\alpha > \operatorname{ord}_p(n)$.*

(i) Suppose that m is even. Put $D = (-1)^{m/2}\det(S)$. Then

$$p^{\alpha(1-m)} N_{\gamma,n}(p^\alpha) = \left(1 - \chi_D(p)p^{-m/2}\right)\left(1 + \chi_D(p)p^{1-m/2} + \cdots + \left(\chi_D(p)p^{1-m/2}\right)^{\operatorname{ord}_p(n)}\right).$$

(ii) Suppose that m is odd. Write $n = n_0 f^2$ (where $n_0 \in \mathbb{Q}$ and $f \in \mathbb{Z}_{\geq 0}$) such that $(f, \det(S)) = 1$ and $\operatorname{ord}_\ell(n_0) \in \{0,1\}$ for all primes ℓ with $(\ell, \det(S)) = 1$. Let $\tilde{n}_0 = n_0 d_\gamma^2$ and $\mathcal{D} = 2(-1)^{(m+1)/2}\tilde{n}_0 \det(S)$. If $m \geq 3$, then

$$p^{\alpha(1-m)} N_{\gamma,n}(p^\alpha) = \frac{1 - p^{1-m}}{1 - \chi_{\mathcal{D}}(p)p^{(1-m)/2}} \left(\sigma_{2-m}(p^{\operatorname{ord}_p(f)}) - \chi_{\mathcal{D}}(p)p^{(1-m)/2}\sigma_{2-m}(p^{\operatorname{ord}_p(f)-1})\right).$$

If $m = 1$, we have

$$N_{\gamma,n}(p^\alpha) = \left(\chi_{\mathcal{D}}(p) + \chi_{\mathcal{D}}(p)^2\right) p^{\operatorname{ord}_p(f)}.$$

Proof. If 2 is factor of $\det(S)$, this is [BK01, Theorem 4.5]. This is always the case if the rank of L is odd.

If 2 does not divide $\det(S)$, then the Gram matrix S of L is $\mathbb{Z}_2$-equivalent to one of

$$\mathrm{diag}\,(H,\dots,H) \text{ or } \mathrm{diag}\left(\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}, H,\dots,H\right).$$

The first case occurs when $\mathrm{disc}(L) \equiv 1 \pmod 8$, which implies $\chi_D(2) = 1$. The second case occurs when $\mathrm{disc}(L) \equiv 5 \pmod 8$, which implies $\chi_D(2) = -1$. Since 2 does not divide $\det(S)$, we have $\gamma \in \mathbb{Z}_2^m$ and we can assume $\gamma = 0$. Setting $\varepsilon = \chi_D(2)$ and using Theorem 2.3.3, we get

$$\begin{aligned}
2^{\alpha(1-m)}N_{\gamma,n}(2^\alpha) &= 2^{\alpha(1-m)}N((2^\alpha)_{II}^{\varepsilon 2(m/2)}, -n)\\
&= 2^{\alpha(1-m)}2^{(\alpha-1)m/2}(2^{m/2}-\varepsilon)\varepsilon^\alpha \sum_{k'=\alpha-\mathrm{ord}_2(n)}^{\alpha} (\varepsilon 2^{m/2-1})^{k'}\\
&= 2^{\alpha(1-m)}2^{(\alpha-1)m/2}2^{m/2}(1-\varepsilon 2^{-m/2})\varepsilon^\alpha(\varepsilon 2^{m/2-1})^\alpha \sum_{k'=\alpha-\mathrm{ord}_2(n)}^{\alpha} (\varepsilon 2^{1-m/2})^{k'}\\
&= (1-\varepsilon 2^{-m/2}) \sum_{k'=0}^{\mathrm{ord}_2(n)} (\varepsilon 2^{1-m/2})^{k'}\\
&= (1-\chi_D(2)2^{-m/2}) \sum_{k'=0}^{\mathrm{ord}_2(n)} (\chi_D(2)2^{1-m/2})^{k'}
\end{aligned}$$

which finishes the proof. □

We know from the theory of Igusa local zeta functions (and equation (3.2.9)) that the local factors $L_{\gamma,n}(s,p)$ are rational functions in p^{-s}. Let p be a prime not dividing $\det(S)$ and let $2 < s = \frac{m}{2} + h \in \frac{m}{2} + \mathbb{Z}_{\geq 0}$. We add h hyperbolic planes to L and argue

$$\begin{aligned}
L_{\gamma,n}(s,p) &= \left(\left(1-\frac{t}{p}\right)\sum_{\nu\geq 0}\frac{N_{\gamma,n}^L(p^\nu)}{p^{m\nu}}t^\nu\right)\Bigg|_{t=p^{1+m/2-s}}\\
&= \left(\left(1-\frac{t}{p}\right)\sum_{\nu\geq 0}\frac{N_{\gamma,n}^{L+H^{(h)}}(p^\nu)}{p^{(m+2h)\nu}}t^\nu\right)\Bigg|_{t=p^{1+(m+2h)/2-s}}\\
&= N_{\gamma,n}^{L+H^{(h)}}(p^{w_p})p^{w_p(1-m/2-h-s)}\\
&= N_{\gamma,n}^{L+H^{(h)}}(p^{w_p})p^{w_p(1-m-2h)}
\end{aligned} \tag{3.2.10}$$

by Corollary 2.1.4 and equation (3.2.9). Applying Theorem 3.2.3, this yields

$$L_{\gamma,n}(s,p) = \left(1-\chi_D(p)p^{-s}\right)\left(1+\chi_D(p)p^{1-s}+\dots+\left(\chi_D(p)p^{1-s}\right)^{\mathrm{ord}_p(n)}\right) \tag{3.2.11}$$

in the even case and

$$L_{\gamma,n}(s,p) = \frac{1-p^{1-2s}}{1-\chi_{\mathcal{D}}(p)p^{1/2-s}} \left(\sigma_{2-2s}(p^{\mathrm{ord}_p(f)}) - \chi_{\mathcal{D}}(p)p^{1/2-s}\sigma_{2-2s}(p^{\mathrm{ord}_p(f)-1})\right) \tag{3.2.12}$$

in the odd case. Since these are rational functions in p^{-s}, these equations hold for all $s \in \mathbb{C}$. Note that this is equivalent to [KY10, Proposition 2.1]. In analogy to [BK01, Theorem 4.8], the Fourier coefficients of the Eisenstein series are given as follows.

Theorem 3.2.4. *Let $\gamma \in L'$ and $n \in \mathbb{Z} - q(\gamma)$ with $n > 0$. The coefficient $a_{E_0}(\gamma, n)$ of the Eisenstein series E_0 of weight $k > 2$ for ρ_L^* is equal to*

$$\frac{2^k \pi^k n^{k-1}(-1)^{(2k-b^-+b^+)/4}}{\sqrt{|L'/L|}\Gamma(k)} \tag{3.2.13}$$

times

$$\begin{cases} \frac{\sigma_{1-k}(\tilde{n},\chi_D)}{L(k,\chi_D)} \prod\limits_{p|\det(S)} L_{\gamma,n}(k,p), & \text{if } 2 \mid m, \\ \frac{L(k-1/2,\chi_{\mathcal{D}})}{\zeta(2k-1)} \prod\limits_{\substack{p|\tilde{n} \\ p\nmid \det S}} \left(\sigma_{2-2k}(p^{\mathrm{ord}_p(f)}) - \chi_{\mathcal{D}}(p)p^{1/2-k}\sigma_{2-2k}(p^{\mathrm{ord}_p(f)-1})\right) \prod\limits_{p|\det(S)} \frac{L_{\gamma,n}(k,p)}{1-p^{1-2k}}, & \text{if } 2 \nmid m. \end{cases} \tag{3.2.14}$$

Here

$$\begin{aligned} L_{\gamma,n}(k,p) &= \sum_{\nu \in \mathbb{Z}_{\geq 0}} \frac{N_\nu(f_{\gamma,n})}{p^{m\nu}} p^{-\nu s} \left(1 - \frac{p^{-s}}{p}\right)\Bigg|_{s=-1-\frac{m}{2}+k} \\ &= \frac{1-tZ_{f_{\gamma,n}}(t)}{1-t}\left(1-\frac{t}{p}\right)\Bigg|_{t=p^{1+\frac{m}{2}-k}} \end{aligned} \tag{3.2.15}$$

in terms of the Igusa local zeta function for the polynomial $f_{\gamma,n} = Q(x) - (x,\gamma) + n + Q(\gamma)$ and the representation numbers

$$N_\nu(f_{\gamma,n}) = \#\left\{x \in (\mathbb{Z}/p^\nu\mathbb{Z})^m \mid Q(x) - (x,\gamma) + n + Q(\gamma) \equiv 0 \pmod{p^\nu}\right\}. \tag{3.2.16}$$

The values of $D, \mathcal{D}, f, \tilde{n}$ are as in Theorem 3.2.3.

Remark 3.2.5. In the odd case we have the equality

$$\begin{aligned} &\prod_{\substack{p|\tilde{n} \\ p\nmid \det S}} \left(\sigma_{2-2k}(p^{\mathrm{ord}_p(f)}) - \chi_{\mathcal{D}}(p)p^{1/2-k}\sigma_{2-2k}(p^{\mathrm{ord}_p(f)-1})\right) \\ =&\sum_{d|f} \mu(d)\chi_{\mathcal{D}}(d)d^{1/2-k}\sigma_{2-2k}(f/d). \end{aligned} \tag{3.2.17}$$

If $k = m/2$, Theorem 3.2.4 reduces to [BK01, Theorem 4.8].

3.3. Comparing the approaches

In this section, we compare the approach of Kudla and Yang with the approach, we derived from Bruinier and Kuss. Let L be an even lattice of type (b^+, b^-) and rank n. Further, let $\mu \in L'/L$ and $0 < m \in Q(\mu) + \mathbb{Z}$. Let S denote the set of prime divisors of $\det(L)$ and suppose $2\ell \equiv b^+ - b^- \pmod 4$. The Eisenstein series $E = E_{\rho_L,\ell,0}$ and $E = E_{\rho^*_{L(-1)},\ell,0}$ are equal by definition, so their Fourier coefficients match. By Theorem 3.1.4 and Theorem 3.2.4, these Fourier coefficients are given by

$$\begin{aligned} a_E(\mu, m) &= \frac{(-2\pi i)^\ell m^{\ell-1}}{\Gamma(\ell) L^S(\ell, \chi)} \sigma^S_{1-\ell}(m, \chi) W_{m,S}(\ell - 1), \\ a_E(\mu, m) &= \frac{2^\ell \pi^\ell m^{\ell-1} (-1)^{(2\ell - b^+ + b^-)/4}}{\sqrt{|L'/L|}\Gamma(\ell)} \cdot \frac{\sigma_{1-\ell}(\tilde{m}, \chi_D)}{L(\ell, \chi_D)} \prod_{p \in S} L_{\mu,m}(\ell, p) \end{aligned} \tag{3.3.1}$$

in the even case. In the odd case, they are given by

$$\begin{aligned} a_E(\mu, m) &= \frac{(-2\pi i)^\ell m^{\ell-1}}{\Gamma(\ell) \zeta^S(2\ell - 1)} L^S(\ell - \frac{1}{2}, \chi_{\kappa m}) b^S(\kappa m, \ell - \frac{1}{2}) W_{m,S}(\ell - 1), \\ a_E(\mu, m) &= \frac{2^\ell \pi^\ell m^{\ell-1} (-1)^{(2\ell - b^+ + b^-)/4}}{\sqrt{|L'/L|}\Gamma(\ell)} \cdot \frac{L(\ell - 1/2, \chi_{\mathcal{D}})}{\zeta(2\ell - 1)} \\ &\quad \cdot \prod_{\substack{p \mid \tilde{m} \\ p \notin S}} \left(\sigma_{2-2\ell}(p^{\mathrm{ord}_p(f)}) - \chi_{\mathcal{D}}(p) p^{1/2-\ell} \sigma_{2-2\ell}(p^{\mathrm{ord}_p(f)-1}) \right) \prod_{p \in S} \frac{L_{\mu,m}(\ell, p)}{1 - p^{1-2\ell}}. \end{aligned} \tag{3.3.2}$$

We recall equation (3.1.32)

$$W_{m,S}(s) = \prod_{p \in S} \gamma_p(L) |\det(L)|_p^{\frac{1}{2}} W_p(s, m, \mu).$$

The local factors for $L(-1)$ are given by (3.2.15),

$$\begin{aligned} L_{\mu,m}(\ell, p) &= \sum_{\nu \in \mathbb{Z}_{\geq 0}} \frac{N_\nu(f_{\mu,m})}{p^{n\nu}} p^{-\nu s} \left(1 - \frac{p^{-s}}{p}\right) \Bigg|_{s = -1 - \frac{n}{2} + \ell} \\ &= \frac{1 - t Z_{f_{\mu,m}}(t)}{1 - t} \left(1 - \frac{t}{p}\right) \Bigg|_{t = p^{1 + \frac{n}{2} - \ell}}, \end{aligned}$$

in terms of the Igusa local zeta function for the polynomial $f_{\mu,m} = -Q(x) + (x, \mu) + m - Q(\mu)$. Using the oddity formula (1.2.14), we can identify the occurring terms by checking

$$(-i)^\ell \prod_{p \in S} \gamma_p(L) = (-1)^{(2\ell - b^+ + b^-)/4}, \qquad \prod_{p \in S} |\det(L)|_p^{\frac{1}{2}} = \frac{1}{\sqrt{|L'/L|}},$$

$$L^S(\ell,\chi) = L(\ell,\chi_D), \qquad \sigma^S_{1-\ell}(m,\chi) = \sigma_{1-\ell}(\tilde{m},\chi_D),$$
$$\zeta^S(2\ell-1) = \zeta(2\ell-1)\prod_{p\in S} 1-p^{1-2\ell}, \qquad L^S(\ell-\frac{1}{2},\chi_{\kappa m}) = L(\ell-1/2,\chi_{\mathcal{D}}),$$

and finally

$$b^S(\kappa m,\ell-\frac{1}{2}) = \prod_{\substack{p\mid\tilde{m}\\ p\notin S}} \left(\sigma_{2-2\ell}(p^{\mathrm{ord}_p(f)}) - \chi_{\mathcal{D}}(p)p^{1/2-\ell}\sigma_{2-2\ell}(p^{\mathrm{ord}_p(f)-1})\right).$$

From this we infer the equality

$$\prod_{p\in S} W_p(\ell-1,m,\mu) = \prod_{p\in S} L_{\mu,m}(\ell,p) \tag{3.3.3}$$

for infinitely many possible values of ℓ. Using the fact that all factors are rational functions in $p^{-\ell}$ we can deduce

$$W_p(\ell-1,m,\mu) = L_{\mu,m}(\ell,p) \tag{3.3.4}$$

for all $p \in S$, we leave the details to the reader. The equality for all primes p can be obtained by realizing that the formulas for the Fourier still hold, when enlarging the set S. An alternative is the direct comparison of equations (3.2.11) and (3.2.12) with [KY10, Proposition 2.1]. We have proved Observation 3.0.7.

Theorem 3.3.1. *For an even lattice $L = \mathbb{Z}^n$ with Gram matrix S and $\mu \in L' = S^{-1}\mathbb{Z}^n$, the equality*

$$W_{p,m,\mu}(X) = \frac{1-tZ_{-\frac{1}{2}x^tSx+x^tS\mu+m-Q(\mu)}(t)}{1-t}\left(1-\frac{t}{p}\right)\Bigg|_{t=pX}$$

holds for all primes p.

3.4. Exact formulas depending on discriminant forms

In this section, we discuss in which sense the above formulas only depend on the discriminant form L'/L (or the genus symbol) of an even lattice L. The ingredients which depend on L are the rank $n = \mathrm{rank}(L)$, the discriminant $\mathrm{disc}(L)$ and the Gram matrices S_p in canonical form together with the corresponding changes of basis for every $p \mid \mathrm{disc}(L)$. The change of basis is only needed, to associate local coordinates $\mu_p \in S_p^{-1}\mathbb{Z}_p^n$ to an element $\mu \in L'$.

By Remark 1.5.3 and Remark 1.5.4, we can recover the parity of the rank, $n \pmod 2$, and the discriminant of L from the genus symbol via the oddity formula. We know from Lemma 1.4.1, how Jordan components relate to canonical Gram matrices. For the sake of computing vector valued Eisenstein with respect to a discriminant form we need a consistent choice of Gram matrices S_p.

A precise condition on the minimal rank of any lattice having a fixed discriminant form is given by [Nik79, Theorem 1.10.1]. For the sake of computational simplicity, we choose a possibly higher rank.

Lemma 3.4.1. *Let D be a finite quadratic module generated by r elements. Then D is the discriminant form of an even lattice L of rank $n \geq 2 + r$ with $n \equiv \operatorname{sign}(D) \pmod 2$.*

Proof. For such an n, we can always find $t_{(+)}, t_{(-)}$ with $n = t_{(+)} + t_{(-)}$ satisfying [Nik79, Corollary 1.10.2]. Hence a lattice L with signature $t_{(+)} - t_{(-)}$ and discriminant form D exists. □

With this in mind, we can choose local Gram matrices for a finite quadratic module in the following way.

Lemma 3.4.2. *Let D be a finite quadratic module. Let n be the minimal rank satisfying Lemma 3.4.1 and L be a lattice of rank n such that $L'/L \simeq D$. Further, we let C_p be the $r_p \times r_p$ block diagonal matrix consisting of the canonical choices for Gram matrices of the Jordan decomposition of D_p as given by Lemma 1.4.1, where r_p is the minimal number of generators of D_p. There is a $\mathbb{Z}_p$-basis such that the Gram matrix of L can be presented in the form*

$$S_p = \begin{pmatrix} H & & & & \\ & \ddots & & & \\ & & H & & \\ & & & U_p & \\ & & & & C_p \end{pmatrix}, \quad U_p = \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, & p = 2, \\ \begin{pmatrix} 2 & & & \\ & \ddots & & \\ & & 2 & \\ & & & 2\varepsilon \end{pmatrix}, & p \neq 2, \end{cases} \tag{3.4.1}$$

where $\varepsilon \in \mathbb{Z}_p^\times$ is chosen such that $\operatorname{disc}(S_p) \equiv \operatorname{disc}(D) \pmod{(\mathbb{Z}_p^\times)^2}$ is satisfied.

Proof. All given Gram matrices give the right discriminant form by Lemma 1.4.1 and they have the same dimension. The rest follows from the classification of p-adic quadratic forms. □

Given a finite quadratic module D, we can now compute the corresponding local densities by fixing local Gram matrices S_p as in Lemma 3.4.2 and representing $\mu \in D$ by the local coordinates $\mu_p \in S_p^{-1}\mathbb{Z}_p^n/\mathbb{Z}_p^n$ for every prime p.

When computing or implementing this, we do not need to keep track of the hyperbolic planes by Remark 3.1.7 when we use the approach of Kudla and Yang. When using the approach of Bruinier and Kuss, this remains true by Theorem 2.1.1.

3.5. Estimates

In this section we estimate the Fourier coefficients of Eisenstein series with respect to the dual Weil representation for an even lattice of type (b^+, b^-) and of weight $k = \frac{b^+ + b^-}{2}$. We need the assumptions that b^+ is even and that the lattice L splits a scaled hyperbolic plane. The case that L splits a hyperbolic plane is treated in [BM17]. In particular, we are interested in the case $b^+ = 2$ to search for Borcherds products in a systematic manner. The results of this section were obtained together with Markus Schwagenscheidt (cf. [OS18]).

Theorem 3.5.1 ([BK01, Thm. 4.8]). *Let $\gamma \in L'$ and $n \in \mathbb{Z} - Q(\gamma)$ with $n > 0$. The coefficient $a_E(\gamma, n)$ of the Eisenstein series E of weight $k = m/2$ for ρ_L^* is equal to*

$$\frac{2^{k+1}\pi^k n^{k-1}(-1)^{b_+/2}}{\sqrt{|L'/L|}\Gamma(k)} \tag{3.5.1}$$

times

$$\begin{cases} \frac{\sigma_{1-k}(\tilde{n},\chi_{4D})}{L(k,\chi_{4D})} \prod\limits_{p|2\det(S)} p^{w_p(1-2k)} N^L_{\gamma,n}(p^{w_p}), & \text{if } 2 \mid m, \\ \frac{L(k-1/2,\chi_{\mathcal{D}})}{\zeta(2k-1)} \sum\limits_{d|f} \mu(d)\chi_{\mathcal{D}}(d) d^{1/2-k}\sigma_{2-2k}(f/d) \prod\limits_{p|2\det(S)} \frac{p^{w_p(1-2k)}N^L_{\gamma,n}(p^{w_p})}{1-p^{1-2k}}, & \text{if } 2 \nmid m. \end{cases} \tag{3.5.2}$$

Here S is the Gram matrix of L and

$$N^L_{\gamma,n}(a) = \#\{r \in L/aL : Q(r-\gamma) + n \equiv 0 \pmod{a}\} \tag{3.5.3}$$

is a representation number. The definition of w_p, D, $\mathcal{D}$, f and $\tilde{n}$ can be found in Section 3.2, Theorem 3.2.3.

If the lattice in question splits a rescaled hyperbolic plane $U(N) = (\mathbb{Z}^2, (x,y) \mapsto Nxy)$, we may estimate the coefficients as follows.

Theorem 3.5.2. *Let L be a lattice of signature (b_+, b_-) (b_+ even) with rank $2k = m \geq 3$ such that $L = L_1 \oplus U(N)$ for some even lattice L_1 of rank $m - 2 \geq 1$. Let d be the determinant of the Gram matrix of L. Let $\gamma \in L'$ and $n \in \mathbb{Z} - q(\gamma)$ with $n > 0$. The coefficient $a_{E_0}(\gamma, n)$ of the Eisenstein series with respect to the dual Weil representation and of weight $k = m/2$ is either 0 or*

$$(-1)^{b_+/2} a_{E_0}(\gamma, n) \geq C_{k,d,N} \cdot n^{k-1}, \tag{3.5.4}$$

where $C_{k,d,N}$ is given by

$$\frac{2^{k+1}\pi^k}{\sqrt{|d|}\Gamma(k)} \tag{3.5.5}$$

times

$$\begin{cases} \frac{2-\zeta(k-1)}{\zeta(k)} \prod_{p\mid 2d} p^{(3-2k)\operatorname{ord}_p(N)}(1-1/p), & 2\mid m,\\ \frac{2-\zeta(k-1/2)}{\zeta(k-1/2)} \prod_{p\mid 2d} \frac{p^{(3-2k)\operatorname{ord}_p(N)}(1-1/p)}{1-p^{1-2k}}, & 2\nmid m. \end{cases} \tag{3.5.6}$$

The proof is accomplished using the following lemmas. The methods follow and generalize the estimates in [BM17]. The following lemma is well known.

Lemma 3.5.3. *Let $n \in \mathbb{Z}$ and $\nu \in \mathbb{Z}_{\geq 0}$. Then*

$$N_{0,n}^{U(1)}(p^\nu) = \begin{cases} (\operatorname{ord}_p(n)+1)(1-\frac{1}{p})p^\nu, & \operatorname{ord}_p(n) < \nu,\\ \nu(1-\frac{1}{p})p^\nu + p^\nu, & \operatorname{ord}_p(n) \geq \nu. \end{cases} \tag{3.5.7}$$

Corollary 3.5.4. *We have*

$$(1-\frac{1}{p}) \leq p^{-\nu} N_{0,n}^{U(1)}(p^\nu) \leq \nu + 1. \tag{3.5.8}$$

Lemma 3.5.5. *Let p be a prime, $\nu \in \mathbb{Z}_{\geq 0}$, $N \in \mathbb{Z}$ and $\gamma = \left(\frac{\gamma_1}{N}, \frac{\gamma_2}{N}\right) \in U(N)' = \frac{1}{N}\mathbb{Z}^2$. We write $p^{\nu_N} \| N$, $p^{\nu_\gamma} \| (\gamma_1, \gamma_2)$ ($\nu_\gamma = \infty$ for $\gamma = (0,0)$) and $n = \ell - \frac{\gamma_1\gamma_2}{N}$ with $\ell \in \mathbb{Z}$. Furthermore, we define $\nu_{\min} = \min(\nu, \nu_\gamma, \nu_N)$. The representation numbers for a rescaled hyperbolic plane are given by*

$$N_{\gamma,n}^{U(N)}(p^\nu) = \begin{cases} 0, & p^{\nu_{\min}} \nmid \ell,\\ p^{2\nu_N} N_{0,\tilde{n}}^{U(1)}(p^{\nu-\nu_N}), & \nu_N \leq \min(\nu,\nu_\gamma) \text{ and } p^{\nu_{\min}} \mid \ell,\\ p^{\nu+\min(\nu,\nu_\gamma)}, & \nu_N > \min(\nu,\nu_\gamma) \text{ and } p^{\nu_{\min}} \mid \ell, \end{cases} \tag{3.5.9}$$

where $\tilde{n} = Nnp^{-2\nu_N}$.

Proof. We may write

$$N_{\gamma,n}^{U(N)}(p^\nu) = \#\{(a,b) \in (\mathbb{Z}/p^\nu\mathbb{Z})^2 : Nab - (a\gamma_2 + b\gamma_1) \equiv -\ell \pmod{p^\nu}\}. \tag{3.5.10}$$

If $p^{\nu_{\min}} \nmid \ell$, then the condition for (a,b) implies $0 \equiv -l \not\equiv 0 \pmod{p^{\nu_{\min}}}$. This condition cannot be fulfilled and the representation number is 0 in this case.

If $\nu_N \leq \min(\nu, \nu_\gamma)$, we write $N = N' \cdot p^{\nu_N}$ and find an integer $\overline{N}'$ such that

$$\overline{N}' \equiv (N')^{-1} \pmod{p^\nu}. \tag{3.5.11}$$

The bijection

$$(a_1, b_1) \mapsto (\overline{N}'(a_1 + \frac{\gamma_1}{p^{\nu_N}}), \overline{N}'(b_1 + \frac{\gamma_2}{p^{\nu_N}})) =: (a,b) \tag{3.5.12}$$

shows that

$$\begin{aligned} N_{\gamma,n}^{U(N)}(p^\nu) &= \#\{(a_1,b_1) \in (\mathbb{Z}/p^\nu\mathbb{Z})^2 : a_1b_1 \equiv -Np^{-2\nu_N}n \pmod{p^{\nu-\nu_N}}\} \\ &= p^{2\nu_N} N_{0,\tilde{n}}^{U(1)}(p^{\nu-\nu_N}). \end{aligned} \tag{3.5.13}$$

This proves the second case.

If $\nu_N > \min(\nu,\nu_\gamma)$, we distinguish two cases. If $\nu_\gamma \geq \nu$, the condition for (a,b) is trivial and we have $p^{2\nu} = p^{\nu+\min(\nu,\nu_\gamma)}$ solutions. If $\nu_\gamma < \nu$, we may assume that $p^{\nu_\gamma} \| \gamma_2$. We see

$$\begin{aligned} N_{\gamma,n}^{U(N)}(p^\nu) &= \#\{(a,b) \in (\mathbb{Z}/p^\nu\mathbb{Z})^2 : a \equiv \left(\frac{N}{p^{\nu_\gamma}}b - \frac{\gamma_2}{p^{\nu_\gamma}}\right)^{-1} \frac{b\gamma_1 - \ell}{p^{\nu_\gamma}} \pmod{p^{\nu-\nu_\gamma}}\} \\ &= p^{\nu+\nu_\gamma}, \end{aligned} \tag{3.5.14}$$

so again, we have $p^{\nu+\min(\nu,\nu_\gamma)}$ solutions. □

Lemma 3.5.6. *Let L be a lattice of rank $2k = m \geq 3$ such that $L = L_1 \oplus U(N)$ for some even lattice L_1 of rank $m-2 \geq 1$. Then either $N_{\gamma,n}^L(p^\nu) = 0$, or*

$$p^{\nu(1-2k)} N_{\gamma,n}^L(p^\nu) \geq p^{(3-2k)\nu_N}(1-\frac{1}{p}). \tag{3.5.15}$$

An upper bound is given by

$$p^{\nu(1-2k)} N_{\gamma,n}^L(p^\nu) \leq p^{\nu_{\min}}(\nu - \nu_{\min} + 1). \tag{3.5.16}$$

Proof. Write $\gamma = \gamma_1 + \gamma_2$ with $\gamma_1 \in L_1'$ and $\gamma_2 \in U(N)'$. We may write

$$N_{\gamma,n}^L(p^\nu) = \sum_{\lambda_1 \in L_1/p^\nu L_1} N_{\gamma_2,n+Q(\lambda_1-\gamma_1)}^{U(N)}(p^\nu). \tag{3.5.17}$$

To estimate the summands we define $\nu_\gamma := \nu_{\gamma_2}$, ν_N and $\nu_{\min}$ as in Lemma 3.5.5. If all summands are 0, there is nothing to prove. Therefore, we may assume that there is a λ_1 such that the corresponding summand $N_{\gamma_2,n+Q(\lambda_1-\gamma_1)}^{U(N)}(p^\nu)$ is nonzero. This implies

$$p^{\nu_{\min}} \mid \ell = n + Q(\lambda_1 - \gamma_1) + Q(\gamma_2). \tag{3.5.18}$$

If we change λ_1 modulo $p^{\nu_{\min}}L_1$, this remains true. This gives at least $p^{(\nu-\nu_{\min})(m-2)}$ nonzero summands, which we can estimate using Lemma 3.5.5.

We distinguish the cases $\nu_N \leq \min(\nu,\nu_\gamma)$ and $\nu_N > \min(\nu,\nu_\gamma)$.

In the first case, the nonzero summands are of the form

$$p^{2\nu_N} N_{0,\tilde{n}}^{U(1)}(p^{\nu-\nu_N}) \geq p^{2\nu_N} p^{\nu-\nu_N}(1-\frac{1}{p}) \tag{3.5.19}$$

where $\tilde{n}$ might depend on λ_1 and we use Corollary 3.5.4 for the estimate. This yields

$$N^L_{\gamma,n}(p^\nu) \geq p^{(\nu-\nu_{\min})(m-2)} p^{2\nu_N} p^{\nu-\nu_N}(1-\frac{1}{p}) = p^{(m-1)\nu} p^{(3-m)\nu_N}(1-\frac{1}{p}) \tag{3.5.20}$$

for the sum.

In the second case, the nonzero summands are of the form

$$p^{\nu+\min(\nu,\nu_\gamma)} = p^{\nu+\nu_{\min}}. \tag{3.5.21}$$

This yields

$$N^L_{\gamma,n}(p^\nu) \geq p^{(\nu-\nu_{\min})(m-2)} p^{\nu+\nu_{\min}} = p^{(m-1)\nu} p^{(3-m)\nu_{\min}} \geq p^{(m-1)\nu} p^{(3-m)\nu_N}(1-\frac{1}{p}) \tag{3.5.22}$$

where we have used $3-m \leq 0$ and $\nu_{\min} \leq \nu_N$.

The upper bound can be found by similar estimates using Lemma 3.5.5 and the upper bound of Corollary 3.5.4. □

Note that the characters χ_{4D} and $\chi_{\mathcal{D}}$ appearing in the Fourier expansion of $E(z)$ given in Theorem 3.5.1 are quadratic Dirichlet characters. For even rank, we need the following estimate.

Lemma 3.5.7. *Let χ be a real Dirichlet character and $n \in \mathbb{Z}_{\geq 0}, s \geq 2$. Then*

$$\zeta(s) \geq \sigma_{-s}(n,\chi) \geq 2-\zeta(s). \tag{3.5.23}$$

Proof. We have

$$\sigma_{-s}(n,\chi) = \sum_{d|n} \chi(d) d^{-s} \geq 2 - \sum_{d|n} d^{-s} \geq 2-\zeta(s) \tag{3.5.24}$$

and

$$\sigma_{-s}(n,\chi) = \sum_{d|n} \chi(d) d^{-s} \leq \sum_{d\geq 1} d^{-s} = \zeta(s). \tag{3.5.25}$$

This finishes the proof. □

For odd signature, the following two estimates are useful.

Lemma 3.5.8. *Let χ be a real Dirichlet character, let $f \in \mathbb{Z}_{\geq 0}$, and let $k \geq 5/2$ be a half-integer. Then we have*

$$\sum_{d|f} \mu(d)\chi(d) d^{1/2-k} \sigma_{2-2k}(f/d) > 2-\zeta(k-1/2). \tag{3.5.26}$$

Proof. We split off the term for $d = 1$ on the left hand side and estimate

$$\begin{aligned}\sum_{d|f} \mu(d)\chi(d)d^{1/2-k}\sigma_{2-2k}(f/d) &= \sigma_{2-2k}(f) + \sum_{\substack{d|f\\ d\neq 1}} \mu(d)\chi(d)d^{1/2-k}\sigma_{2-2k}(f/d) \\ &\geq 2\sigma_{2-2k}(f) - \sum_{d|f} d^{1/2-k}\sigma_{2-2k}(f/d).\end{aligned} \tag{3.5.27}$$

Now $\sigma_{2-2k}(f/d) \leq \sigma_{2-2k}(f)$ for $d \mid f$, so the last expression is greater or equal than

$$\sigma_{2-2k}(f)\left(2 - \sum_{d|f} d^{1/2-k}\right) > \sigma_{2-2k}(f)\left(2 - \zeta(k-1/2)\right) \geq 2 - \zeta(k-1/2). \tag{3.5.28}$$

This finishes the proof. □

Lemma 3.5.9. *Let χ be a real Dirichlet character and let $s \in \mathbb{R}, s > 1$. Then*

$$\zeta(s) \geq L(s,\chi) \geq \frac{\zeta(2s)}{\zeta(s)}. \tag{3.5.29}$$

Proof. For $s > 1$ we have

$$L(s,\chi) = \prod_p \frac{1}{1-\chi(p)p^{-s}} \geq \prod_p \frac{1}{1+p^{-s}} = \frac{\zeta(2s)}{\zeta(s)} \tag{3.5.30}$$

and

$$L(s,\chi) = \prod_p \frac{1}{1-\chi(p)p^{-s}} \leq \prod_p \frac{1}{1-p^{-s}} = \zeta(s). \tag{3.5.31}$$

This completes the proof. □

3.6. Vector valued Eisenstein series vs. Jacobi-Eisenstein series

By [EZ85, theorem 5.1] Jacobi forms of weight k and index $m \geq 1$ correspond to vector valued modular forms for the Weil representation of weight $k - \frac{1}{2}$ with respect to the even lattice $(\mathbb{Z}, Q(x) = -mx^2)$. The concrete correspondence of the Fourier coefficients is given by

$$a_{\text{lattice}}(\frac{r}{2m}, n) \leftrightarrow a_{\text{Jacobi}}(\frac{[n]+r^2}{4m}, r), \tag{3.6.1}$$

where $\gamma = \frac{r}{2m} \in L'/L$ and $n \in Q(\gamma) + \mathbb{Z}$. In particular, this correspondence sends the vector valued Eisenstein series to the Jacobi-Eisenstein series, allowing us to test our implementation of the Fourier coefficients against Theorem 1.9.1. The Fourier coefficients we used as test are the first 1000 coefficients in every coordinate for all combinations of index $1, 2 \dots, 15$ and weights $4, 6, \dots, 50$. All these test cases only test the formulas for lattices of odd rank (rank 1 to be specific). Nonetheless, with Jacobi forms, the formulas for $p = 2$ are tested quite thoroughly. The Siegel-Weil formula provides more general test cases.

We give an example of the above Correspondence 3.6.1. The given coefficients also agree with [EZ85, Table 1].

Example 3.6.1. The Jacobi-Eisenstein series of weight $k = 4$ and index $m = 1$ is given by

$$\begin{aligned} E_{4,1}(\tau, z) &= \sum_{n=0}^{\infty} \sum_{\substack{r \in \mathbb{Z} \\ r^2 \leq 4n}} e_{4,1}(n, r) q^n \zeta^r \\ &= 1 + \left(1\zeta^{-2} + 56\zeta^{-1} + 126\zeta^0 + 56\zeta^1 + 1\zeta^2\right) q^1 \\ &\quad + \left(126\zeta^{-2} + 576\zeta^{-1} + 756\zeta^0 + 576\zeta^1 + 126\zeta^2\right) q^2 \\ &\quad + \left(56\zeta^{-3} + 756\zeta^{-2} + 1512\zeta^{-1} + 2072\zeta^0 + 1512\zeta^1 + 756\zeta^2 + 56\zeta^3\right) q^3 \\ &\quad + O(q^4), \end{aligned} \tag{3.6.2}$$

whereas the vector valued Eisenstein series for the lattice $L = (\mathbb{Z}, Q(x) = -x^2)$ of weight $\frac{7}{2}$ is given by

$$\begin{aligned} E_L &= (1 + 126q^1 + 756q^2 + 2072q^3 + O(q^4))\mathfrak{e}_{0+L} \\ &\quad + (56q^{3/4} + 576q^{7/4} + 1512q^{11/4} + 4032q^{15/4} + O(q^{19/4}))\mathfrak{e}_{\frac{1}{2}+L} \end{aligned} \tag{3.6.3}$$

and we see that the coefficients agree in the sense of (3.6.1).

3.7. The Siegel-Weil formula

In this section, we introduce vector valued theta series and state the Siegel Weil formula 3.1.3 in its vector valued version.

Definition 3.7.1. Let L be a positive definite even lattice of rank $n \geq 5$. We define

$$\Theta_{\gamma+L} = \sum_{\mu \in \gamma + L} q^{Q(\mu)} \text{ for } \gamma \in L'/L, \tag{3.7.1}$$

$$\Theta_L = \sum_{\gamma \in L'/L} \Theta_{\gamma+L} \mathfrak{e}_\gamma, \tag{3.7.2}$$

$$\varphi^*(\Theta_L) = \sum_{\gamma \in L'/L} \Theta_{\gamma+L} \mathfrak{e}_{\varphi(\gamma)} \text{ for } \varphi \in O(L'/L), \tag{3.7.3}$$

$$\Theta_L^{sym} = \frac{1}{|O(L'/L)|} \sum_{\varphi \in O(L'/L)} \varphi^*(\Theta_L), \tag{3.7.4}$$

$$\Theta_L^{sym,gen} = \left(\sum_{M \in \mathrm{gen}(L)/\sim} \frac{1}{|O(M)|} \right)^{-1} \sum_{M \in \mathrm{gen}(L)/\sim} \frac{1}{|O(M)|} \Theta_M^{sym}. \tag{3.7.5}$$

Note that in the above formula, we need to identify the discriminant forms M'/M for all M representing classes in the genus of L. If there is only one class in the genus, this becomes unnecessary and we also do not need the size of the orthogonal groups of the lattices.

Siegel-Weil formula, Theorem 3.7.2. For a positive definite even lattice L of rank $n \geq 5$, we have

$$\Theta_L^{sym,gen} = E_L, \tag{3.7.6}$$

where E_L is the vector valued Eisenstein series for the Weil representation ρ_L associated to L of weight $\frac{n}{2}$.

Proof. This is a reformulation of the Siegel-Weil formula as stated in Theorem 3.1.3. An alternative proof using vector valued Hecke operators in the case of odd level (and hence even rank) can be found in [Ros15, Theorem 4.18]. This second approach generalizes the arguments for unimodular lattices in [KK07, chapter 5]. □

The Siegel-Weil formula provides test cases for our implementation of the Fourier coefficients of the vector valued Eisenstein series. In most considered tests, there is only one class in the genus of the lattice. We also give an example with class number 2.

Example 3.7.3. Let $L = E_8$ denote the even unimodular E_8 root lattice, which can be realized as $(\mathbb{Z}^8, x \mapsto \frac{1}{2}x^t S x)$ with Gram matrix

$$S = \begin{pmatrix} 4 & -2 & 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \tag{3.7.7}$$

as can be deduced from [CS99, chapter 4, equation (99)]. As for all unimodular lattices, the vector valued modular forms for the Weil representation associated to this lattice are scalar valued modular forms. All lattices in the genus of E_8 are isomorphic, i.e. there is only one class in this genus. Since $L' = L$ there is no need to symmetrize and the Siegel-Weil formula states that the theta series associated to E_8 is the Eisenstein series of weight 4 associated to this lattice. This is the classical holomorphic Eisenstein series of weight 4. By enumerating elements of $\gamma \in L' = L$ with $Q(\gamma) < 6$, we see that

$$E_L = \Theta_L = \left(1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + 30240q^5 + O(q^6)\right) \mathfrak{e}_{0+L} \tag{3.7.8}$$

which we recognize as the classical holomorphic Eisenstein series of weight 4.

In the last example there was no need to sort elements of $\mu \in L'$ with respect to their cosets $\mu \in \gamma + L$. To show how this can be done, we look at the E_7 root lattice.

Example 3.7.4. Let $L = E_7$ denote the even E_7 root lattice, which can be realized as $(\mathbb{Z}^7, x \mapsto \frac{1}{2}x^t Sx)$ with Gram matrix

$$S = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}, \tag{3.7.9}$$

as can be deduced from [CS99, chapter 4, equation (110)]. This lattice is also known to have class number 1, i.e. there is only one class in its genus. The discriminant form L'/L has only 2 elements which we can represent by $0 = (0,0,0,0,0,0,0)$ and $\gamma = (1/2, 0, 1/2, 0, 0, 0, 1/2)$. The genus symbol of E_7 is 2_3^{-1} and it has level 4.

In order to enumerate short vectors, we identify $L' = S^{-1}\mathbb{Z}^7$ with $(\mathbb{Z}^n, x \mapsto \frac{1}{2}x^t S^{-1}x)$ by sending $x \mapsto Sx$. Since the algorithm (Q.short_vector_list_up_to_length for a quadratic form Q in `sage`) we use to enumerate short vectors needs an even lattice, we scale the above lattice by the level $N = 4$ and look at $(\mathbb{Z}^n, Q(x) = x^t NS^{-1}x)$. This yields

$$\Theta_L = \sum_{\nu \in \mathbb{Z}^n} q^{\frac{Q(\nu)}{N}} \mathfrak{e}_{S^{-1}\nu + \mathbb{Z}^n} \tag{3.7.10}$$

and in order to compute this theta series to precision ω, we need to sort the elements of $\mathbb{Z}^n$ with $Q(\nu) < N\omega$ with respect to their cosets $S^{-1}\nu + \mathbb{Z}^n$. Choosing precision $\omega = 5$, we get

$$\begin{aligned} \Theta_L = &\Big(1 + 126q + 756q^2 + 2072q^3 + 4158q^4 + O(q^5)\Big)\,\mathfrak{e}_0 \\ &+ \Big(56q^{3/4} + 576q^{7/4} + 1512q^{11/4} + 4032q^{15/4} + 5544q^{19/4} + O(q^5)\Big)\,\mathfrak{e}_\gamma, \end{aligned} \tag{3.7.11}$$

where

$$\begin{aligned} 0 &= (0,0,0,0,0,0,0) + \mathbb{Z}^7, \\ \gamma &= (1/2, 0, 1/2, 0, 0, 0, 1/2) + \mathbb{Z}^7. \end{aligned}$$

Due to the simple structure of the discriminant form, there are no nontrivial isometries, so Θ_L is already symmetrized and we have

$$E_L = \Theta_L, \tag{3.7.12}$$

where E_L is the vector valued Eisenstein series of weight $\frac{7}{2}$ for the Weil representation ρ_L.

We give one final and more general example for the Siegel-Weil formula.

Example 3.7.5. Let $L = A_8$ denote the even A_8 root lattice, which can be realized as $(\mathbb{Z}^8, x \mapsto \frac{1}{2}x^t S_0 x)$ with Gram matrix

$$S_0 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}, \tag{3.7.13}$$

as given in [CS99, chapter 4, equation (53)]. This lattice is known to have class number 2 and the second class in its genus is represented by lattice L_1 with Gram matrix

$$S_1 = \begin{pmatrix} 2 & 1 & -1 & 0 & 0 & -1 & 1 & 1 \\ 1 & 2 & -1 & 1 & 1 & -1 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 2 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 2 & -1 & 1 & 0 \\ -1 & -1 & 0 & -1 & -1 & 2 & -1 & 0 \\ 1 & 1 & -1 & 1 & 1 & -1 & 2 & 1 \\ 1 & 0 & -1 & 0 & 0 & 0 & 1 & 6 \end{pmatrix}, \tag{3.7.14}$$

which can be found on [LMFDB]. The discriminant form L'/L has 9 elements. The genus symbol is 9^{-1} and it has level 9. Sending $x \mapsto -x$ is the only non trivial isometry of the discriminant form. There are 5 orbits, each containing 2 elements except for the orbit of 0. The 5 orbits are

$0 \in L'/L$ represented by $(0,0,0,0,0,0,0,0)$,

$0 \neq \gamma_0 \in L'/L$ with $Q(\gamma_0) = 0 + \mathbb{Z}$ repr. by $\pm\,(1/3, 2/3, 0, 1/3, 2/3, 0, 1/3, 2/3) + \mathbb{Z}^8$,

$\gamma_1 \in L'/L$ with $Q(\gamma_1) = 1/9 + \mathbb{Z}$ repr. by $\pm\,(4/9, 8/9, 1/3, 7/9, 2/9, 2/3, 1/9, 5/9) + \mathbb{Z}^8$,

$\gamma_2 \in L'/L$ with $Q(\gamma_2) = 4/9 + \mathbb{Z}$ repr. by $\pm\,(1/9, 2/9, 1/3, 4/9, 5/9, 2/3, 7/9, 8/9) + \mathbb{Z}^8$,

$\gamma_3 \in L'/L$ with $Q(\gamma_3) = 7/9 + \mathbb{Z}$ repr. by $\pm\,(2/9, 4/9, 2/3, 8/9, 1/9, 1/3, 5/9, 7/9) + \mathbb{Z}^8$.

For both Gram matrices, we can determine the corresponding theta series as in the last example. We get

$$\begin{aligned}\Theta_L = &\Big(1 + 72q + 756q^2 + O(q^3)\Big)\,\mathfrak{e}_0\\ &+ \Big(84q + 402q^2 + O(q^3)\Big)\,(\mathfrak{e}_{\gamma_0} + \mathfrak{e}_{-\gamma_0})\\ &+ \Big(0q^{1/9} + 126q^{10/9} + 756q^{19/9} + O(q^3)\Big)\,(\mathfrak{e}_{\gamma_1} + \mathfrak{e}_{-\gamma_1})\\ &+ \Big(9q^{4/9} + 252q^{13/9} + 1332q^{22/9} + O(q^3)\Big)\,(\mathfrak{e}_{\gamma_2} + \mathfrak{e}_{-\gamma_2})\\ &+ \Big(36q^{7/9} + 513q^{16/9} + 1764q^{25/9} + O(q^3)\Big)\,(\mathfrak{e}_{\gamma_3} + \mathfrak{e}_{-\gamma_3})\end{aligned} \tag{3.7.15}$$

and

$$\begin{aligned}\Theta_{L_1} = &\Big(1 + 126q + 756q^2 + O(q^3)\Big)\,\mathfrak{e}_0\\ &+ \Big(57q + 702q^2 + O(q^3)\Big)\,(\mathfrak{e}_{\gamma_0} + \mathfrak{e}_{-\gamma_0})\\ &+ \Big(1q^{1/9} + 126q^{10/9} + 812q^{19/9} + O(q^3)\Big)\,(\mathfrak{e}_{\gamma_1} + \mathfrak{e}_{-\gamma_1})\\ &+ \Big(1q^{4/9} + 182q^{13/9} + 1332q^{22/9} + O(q^3)\Big)\,(\mathfrak{e}_{\gamma_2} + \mathfrak{e}_{-\gamma_2})\\ &+ \Big(56q^{7/9} + 577q^{16/9} + 1639q^{25/9} + O(q^3)\Big)\,(\mathfrak{e}_{\gamma_3} + \mathfrak{e}_{-\gamma_3}).\end{aligned} \tag{3.7.16}$$

The vector valued Eisenstein series of weight 4 for the Weil representation ρ_L is

$$\begin{aligned}E_L = &\Big(1 + 78q + 756q^2 + O(q^3)\Big)\,\mathfrak{e}_0\\ &+ \Big(81q + 702q^2 + O(q^3)\Big)\,(\mathfrak{e}_{\gamma_0} + \mathfrak{e}_{-\gamma_0})\\ &+ \Big(\frac{1}{9}q^{1/9} + 126q^{10/9} + \frac{6860}{9}q^{19/9} + O(q^3)\Big)\,(\mathfrak{e}_{\gamma_1} + \mathfrak{e}_{-\gamma_1})\\ &+ \Big(\frac{73}{9}q^{4/9} + \frac{2198}{9}q^{13/9} + 1332q^{22/9} + O(q^3)\Big)\,(\mathfrak{e}_{\gamma_2} + \mathfrak{e}_{-\gamma_2})\\ &+ \Big(\frac{344}{9}q^{7/9} + \frac{4681}{9}q^{16/9} + \frac{15751}{9}q^{25/9} + O(q^3)\Big)\,(\mathfrak{e}_{\gamma_3} + \mathfrak{e}_{-\gamma_3}).\end{aligned} \tag{3.7.17}$$

Both theta series are already symmetrized and we have

$$E_L = \frac{8}{9}\Theta_L + \frac{1}{9}\Theta_{L_1}. \tag{3.7.18}$$

4. Finite quadratic modules revisited

In this chapter, we take a second look at finite quadratic modules. In particular we are interested in their isometries and orbits under the action of the orthogonal group. The formulas for orbit lengths are from [Sch13]. They were tested in [Opi13], where some representation numbers for the 2-adic case have also been computed. We generalize the formula for representation numbers of even 2-adic Jordan components.

4.1. Enumerating representatives of L'/L

Let L be an even lattice of rank n with Gram matrix S represented by $(\mathbb{Z}^n, x \mapsto \frac{1}{2}x^t Sx)$. The dual lattice L' is given by $S^{-1}\mathbb{Z}^n$ and the discriminant form is $L'/L = S^{-1}\mathbb{Z}^n/\mathbb{Z}^n$. To find a set of representatives, we write S in Smith normal form, that is

$$D = USV, \tag{4.1.1}$$

where $U, V \in \mathrm{GL}_n(\mathbb{Z})$ and

$$D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix} \tag{4.1.2}$$

is the diagonal matrix consisting of the elementary divisors of S. We see that

$$S^{-1}\mathbb{Z}^n = VD^{-1}U\mathbb{Z}^n = VD^{-1}\mathbb{Z}^n. \tag{4.1.3}$$

Writing the columns of V as $V = (v_1, \ldots, v_n)$, we can represent $S^{-1}\mathbb{Z}^n/\mathbb{Z}^n$ by the elements

$$\sum_{i=1}^{n} \frac{j_i}{d_i} v_i + \mathbb{Z}^n, \quad j_i \in \mathbb{Z}. \tag{4.1.4}$$

Two such elements represent the same coset if and only if $j_i \equiv j_i' \pmod{d_i}$ for all $i = 1, \ldots, n$. This implies that a set of representatives is given by

$$\sum_{i=1}^{n} \frac{j_i}{d_i} v_i, \quad j_i \in \{0, \ldots, d_i - 1\}. \tag{4.1.5}$$

To enumerate representatives for L_p'/L_p, the maximal p-subgroup of L'/L, we can restrict the above to elements whose denominators are a power of p.

$$\sum_{i=1}^{n} \frac{j_i}{p^{\mathrm{ord}_p(d_i)}} v_i + \mathbb{Z}^n, \quad j_i \in \mathbb{Z}. \tag{4.1.6}$$

Analogously to the above, two such elements represent the same coset if and only if $j_i \equiv j_i' \pmod{p^{\mathrm{ord}_p(d_i)}}$ for all $i = 1, \ldots, n$. This implies that a set of representatives is given by

$$\sum_{i=1}^{n} \frac{j_i}{p^{\mathrm{ord}_p(d_i)}} v_i, \quad j_i \in \{0, \ldots, p^{\mathrm{ord}_p(d_i)} - 1\}. \tag{4.1.7}$$

The above ways to write elements give rise to the group isomorphism

$$\begin{aligned} L'/L &\simeq \bigoplus_p L_p'/L_p, \\ \sum_{i=1}^{n} \frac{j_i}{d_i} v_i + \mathbb{Z}^n &\mapsto \left(\sum_{i=1}^{n} \frac{j_i}{p^{\mathrm{ord}_p(d_i)}} v_i + \mathbb{Z}_p^n \right)_p, \\ \sum_{i=1}^{n} \frac{\sum_p j_{p,i} \frac{d_i}{p^{\mathrm{ord}_p(d_i)}} \left(\left(\frac{d_i}{p^{\mathrm{ord}_p(d_i)}} \right)^{-1} \pmod{p^{\mathrm{ord}_p(d_i)}} \right)}{d_i} v_i + \mathbb{Z}^n &\leftarrow\!\shortmid \left(\sum_{i=1}^{n} \frac{j_{p,i}}{p^{\mathrm{ord}_p(d_i)}} v_i + \mathbb{Z}_p^n \right)_p, \end{aligned} \tag{4.1.8}$$

where we applied n instances of the Chinese Remainder Theorem. Evaluating the quadratic form on L'/L after applying the group isomorphism to an element of the form

$$(0, \ldots, 0, x_p, 0 \ldots, 0) \in \bigoplus_p L_p'/L_p, \tag{4.1.9}$$

where

$$x_p = \sum_{i=1}^{n} \frac{j_{p,i}}{p^{\mathrm{ord}_p(d_i)}} v_i + \mathbb{Z}_p^n = V D_p^{-1} J_p + \mathbb{Z}_p^n \in L_p'/L_p \tag{4.1.10}$$

and

$$D_p = \begin{pmatrix} p^{\mathrm{ord}_p(d_1)} & & 0 \\ & \ddots & \\ 0 & & p^{\mathrm{ord}_p(d_n)} \end{pmatrix}, \tag{4.1.11}$$

gives the same result as the composition

$$\begin{aligned} L_p'/L_p &\xrightarrow{\simeq} L_p'/L_p \xrightarrow{Q_p} \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\simeq} \mathbb{Z}[p^{-1}]/\mathbb{Z} \hookrightarrow \mathbb{Q}/\mathbb{Z}, \\ x_p &\mapsto T_p x_p \end{aligned} \tag{4.1.12}$$

with

$$T_p = V \begin{pmatrix} \left(\left(\frac{d_1}{p^{\mathrm{ord}_p(d_1)}}\right)^{-1} \pmod{p^{\mathrm{ord}_p(d_1)}}\right) & & 0 \\ & \ddots & \\ 0 & & \left(\left(\frac{d_n}{p^{\mathrm{ord}_p(d_n)}}\right)^{-1} \pmod{p^{\mathrm{ord}_p(d_n)}}\right) \end{pmatrix} V^{-1}$$
$$\in \mathrm{GL}_n(\mathbb{Z}_p). \tag{4.1.13}$$

Remark 4.1.1. We see that if we start from a lattice, there is no way (known to us) around using the Chinese Remainder Theorem. However, if we were to ignore this, we would still get a finite quadratic module which is isometric to the discriminant form we started with.

To see that the isomorphism given by T_p is necessary, we look at the the discriminant form of the even lattice of rank 1 with Gram matrix (18).

Example 4.1.2. Let $L = (\mathbb{Z}, x \mapsto 9x^2)$. We have $S = (18)$, $D = (18)$, $U = V = (1)$, $L' = \frac{1}{18}\mathbb{Z}$ and

$$\begin{aligned} \frac{1}{18}\mathbb{Z}/\mathbb{Z} &\simeq \frac{1}{2}\mathbb{Z}_2/\mathbb{Z}_2 \oplus \frac{1}{9}\mathbb{Z}_3/\mathbb{Z}_3, \\ \frac{x}{18} + \mathbb{Z} &\mapsto \left(\frac{x}{2} + \mathbb{Z}_2, \frac{x}{9} + \mathbb{Z}_3\right), \\ \frac{9x_2 + 10x_3}{18} + \mathbb{Z} &\leftarrow\!\shortmid \left(\frac{x_2}{2} + \mathbb{Z}_2, \frac{x_3}{9} + \mathbb{Z}_3\right). \end{aligned} \tag{4.1.14}$$

To get an orthogonal decomposition, we need to choose the quadratic forms for the p-adic lattices as

$$\left(\frac{1}{2}\mathbb{Z}_2/\mathbb{Z}_2, \frac{x_2}{2} + \mathbb{Z}_2 \mapsto 9\left(\frac{9x_2}{18}\right)^2 = \frac{1}{4}x_2^2 \pmod{\mathbb{Z}_2}\right) \tag{4.1.15}$$

and

$$\left(\frac{1}{9}\mathbb{Z}_3/\mathbb{Z}_3, \frac{x_3}{9} + \mathbb{Z}_3 \mapsto 9\left(\frac{10x_3}{18}\right)^2 = \frac{7}{9}x_3^2 \pmod{\mathbb{Z}_3}\right). \tag{4.1.16}$$

This truly gives an orthogonal decomposition, since

$$Q(\frac{x}{18}) = \frac{1}{36}x^2 \equiv \frac{1}{4}x^2 + \frac{7}{9}x^2 \pmod{1}. \tag{4.1.17}$$

If we made the "naive" choice

$$\left(\frac{1}{9}\mathbb{Z}_3/\mathbb{Z}_3, \frac{x_3}{9} + \mathbb{Z}_3 \mapsto 9\left(\frac{2x_3}{18}\right)^2 = \frac{1}{9}x_3^2 \pmod{\mathbb{Z}_3}\right), \tag{4.1.18}$$

we would not have the "right" orthogonal decomposition, since

$$Q(\frac{x}{18}) = \frac{1}{36}x^2 \not\equiv \frac{13}{36}x^2 = \frac{1}{4}x^2 + \frac{1}{9}x^2 \pmod{1}. \tag{4.1.19}$$

However, if we construct a finite quadratic module orthogonally from the "false" choices, we would get the finite quadratic module

$$D = (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}, (x_2, x_3) \mapsto \frac{1}{4}x_2^2 + \frac{1}{9}x_3^2 \pmod 1) \tag{4.1.20}$$

which is isometric to L'/L under the isometry

$$\begin{aligned} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z} &\to \frac{1}{18}\mathbb{Z}/\mathbb{Z}, \\ (x_2, x_3) &\mapsto \frac{9x_2 + 2x_3}{18} + \mathbb{Z}. \end{aligned} \tag{4.1.21}$$

4.2. Isometries

In this section, we collect information about the isometries of a finite quadratic module and some invariants of the elements under the action of the orthogonal group. The formulas for orbit lengths and representation numbers have been implemented using `magma` in [Opi13]. In the course of this thesis, a `python` version for use within `sage` has been implemented. This code has been used in [BEF16a; BEF16b] for which it was incorporated into the class for genus symbols in [Ehl16].

Let D be a finite quadratic module. We have already mentioned that D decomposes orthogonally into its maximal p-subgroups

$$D = \bigoplus_{p|\mathrm{ord}(D)} D_p. \tag{4.2.1}$$

If D is the discriminant form of an even lattice L, the previous section shows how to get the D_p. The orthogonal group $O(D)$ decomposes into the Cartesian product

$$O(D) = \prod_{p|\mathrm{ord}(D)} O(D_p). \tag{4.2.2}$$

We can enumerate the isometries of a discriminant form or finite quadratic module by choosing appropriate images of generators successively. This can be done as follows.

Algorithm 4.2.1.

(i) Choose generators $\mu_1, \ldots, \mu_n$ of D. Set $i = 1$.

(ii) If $i = 0$: Stop.

(iii) For all γ_i with $\mathrm{ord}(\gamma_i) = \mathrm{ord}(\mu_i)$ and $Q(\gamma_i) = Q(\mu_i) \pmod 1$ such that $\gamma_1, \ldots, \gamma_i$ have not been previously checked do:

   (a) If $B(\gamma_j, \gamma_k) = B(\mu_j, \mu_k)$ for all $1 \leq j \leq k \leq i$:

      i. If $i = n$, we have found an isometry by mapping $\mu_1 \mapsto \gamma_1, \ldots, \mu_n \mapsto \gamma_n$.

ii. If $i < n$, set $i = i + 1$ and go to (iii).

(b) Else we choose the next γ_i. If we have already checked all of them, set $i = i - 1$ and go to (ii).

Note that it is much faster, to apply Algorithm 4.2.1 to the maximal p-subgroups D_p and to combine the results using the Chinese Remainder Theorem. For odd primes, one can also use reflections as generators of $O(D_p)$.

Remark 4.2.2. By a theorem in [Sch13], the orthogonal group of a finite quadratic module of odd level is generated by reflections. If the level is even, this does not hold in general.

4.2.1. Orbits

Following [Sch13], we want to describe the orbits of a finite quadratic module D under the action of its orthogonal group $O(D)$. By equation (4.2.2), we can assume that D is a discriminant form of level p^l for a prime p. Additionally we need to assume that p is odd.

Any integer m acts by multiplication on D and we denote the kernel of this action by D_m and the image by D^m. This gives rise to the exact sequence

$$0 \to D_m \to D \to D^m \to 0. \tag{4.2.3}$$

The quotient D/D_m is a finite quadratic module with quadratic form given by

$$\begin{aligned} Q : D/D_m &\to \mathbb{Q}/\mathbb{Z}, \\ x + D_m &\mapsto mQ(x). \end{aligned} \tag{4.2.4}$$

We denote the canonical projection by $\pi_m : D \to D/D_m$. The finite quadratic module D/D_m is isomorphic to D^m, which we endow with the quadratic form

$$\begin{aligned} Q : D^m &\to \mathbb{Q}/\mathbb{Z}, \\ mx &\mapsto mQ(x). \end{aligned} \tag{4.2.5}$$

For an element $\gamma \in D$, we define the *multiplicity*

$$v(\gamma) = \max \left(\left\{ p^k \;\middle|\; k \in \mathbb{Z}_{\geq 0} \text{ and there is } \mu \in D \text{ with } p^k\mu = \gamma \right\} \right) \tag{4.2.6}$$

and the *reduced norm*

$$Q^{\text{red}}(\gamma) = v(\gamma)Q(\mu) \text{ for some } \mu \in D \text{ with } v(\gamma)\mu = \gamma. \tag{4.2.7}$$

The reduced norm is well defined (for odd p) and these values are invariant under isometries of D. We can generalize these invariants to obtain

$$\begin{aligned} t_{p^j} &= v_{p^j}(\gamma) = p^{-j}v(p^j\gamma), \\ v_{p^j} &= Q^{\text{red}}_{p^j}(\gamma) = Q^{\text{red}}(p^j\gamma), \end{aligned} \tag{4.2.8}$$

for $j = 0, \dots, k$ where $\operatorname{ord}(\gamma) = p^{k+1}$. Note that these are the multiplicities and reduced norms of $p^j\gamma \in D^{p^j}$ and also of $\pi_{p^j}(\gamma)$. By [Sch13], these invariants identify the orbit of γ (with respect to the action of the orthogonal group $O(D)$) and we write

$$O(D).\gamma = \mathcal{O}(D, p^{k+1}, v_1, \dots, v_{p^k}, t_1, \dots, t_{p^k}) \tag{4.2.9}$$

Note that the orbit invariants fulfill the relations

$$\begin{aligned} &v_1 \mid v_p \mid \cdots \mid v_{p^k} \mid p^l, \\ &\frac{v_{p^j}}{v_{p^i}} t_{p^j} \equiv p^{j-i} t_{p^i} \pmod 1, \end{aligned} \tag{4.2.10}$$

for $0 \le i \le j < k$. This allows us to predict the orbits which we can expect from a discriminant form. We make this precise in the following two technical lemmas. The first predicts the multiplicities which can occur, the second predicts fitting reduced norms.

Lemma 4.2.3 (cf. [Opi13, Lemma 4.4])**.** *Let p be a prime and let*

$$0 \neq \gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{Z}/p^{i_1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^{i_n}\mathbb{Z}$$

be an element of order p^{k+1}. We have

$$v(\pi_{p^k}(\gamma)) \in \{p^{i_r-k-1} \;:\; r \in \{1, \dots, n\} \text{ with } i_r > k\}$$

and if $j \in \{0, \dots, k-1\}$, we have

$$v(\pi_{p^j}(\gamma)) \in \{p^{i_r-j-1} \;:\; r \in \{1, \dots, n\} \text{ with } i_r > j \text{ and } p^{i_r-j-1} \le v(\pi_{p^{j+1}}(\gamma))\}.$$

Lemma 4.2.4 (cf. [Opi13, Lemma 4.5])**.** *Let $\gamma \in \mathcal{O}(D, m, v_1, \dots, v_{p^k}, t_1, \dots, t_{p^k})$. We have*

$$t_{p^k} \in \frac{1}{p}\mathbb{Z}$$

and for $j \in \{0, \dots, k-1\}$. For $j \in \{0, \dots, k-1\}$ let d denote the denominator of $t_{p^{j+1}}$. We then have

$$t_{p^j} \in \left(\min\left(\frac{v_{p^{j+1}}}{dpv_{p^j}}, \frac{1}{p}\right)\right)\mathbb{Z}.$$

4.2.2. Orbit lengths

We summarise some results from [Sch13] and [Opi13] on how to compute the lengths of the orbits from the last section. These computations depend purely on the Jordan decomposition.

Proposition 4.2.5 ([Sch13])**.** *Let D be a finite quadratic module of order p^l where p is an* *odd prime. We have*

$$|\mathcal{O}(D, p^{k+1}, v_1, \ldots, v_{p^k}, t_1, \ldots, t_{p^k})| = |\mathcal{O}(D^{v_1}, p^{k+1}, \frac{v_1}{v_1}, \ldots, \frac{v_{p^k}}{v_1}, t_1, \ldots, t_{p^k})|.$$

Proposition 4.2.6 ([Sch13], cf. [Sch06, Proposition 3.2])**.** *Let D be a finite quadratic module with Jordan decomposition $p^{\varepsilon n}$ where p is an odd prime. Then*

$$|\mathcal{O}(D, p, 1, t)| = \begin{cases} p^{n-1} - \varepsilon \left(\frac{-1}{p}\right)^{n/2} p^{(n-2)/2}, & \text{if } n \text{ is even and } t \neq 0, \\ p^{n-1} + \varepsilon \left(\frac{-1}{p}\right)^{n/2} \left(p^{n/2} - p^{(n-2)/2}\right) - 1, & \text{if } n \text{ is even and } t = 0, \\ p^{n-1} + \varepsilon \left(\frac{-1}{p}\right)^{(n-1)/2} \left(\frac{2}{p}\right) \left(\frac{pt}{p}\right) p^{(n-1)/2}, & \text{if } n \text{ is odd and } t \neq 0, \\ p^{n-1} - 1, & \text{if } n \text{ is odd and } t = 0. \end{cases} \tag{4.2.11}$$

Proposition 4.2.7 ([Sch13])**.** *Let D be a finite quadratic module of order p^l where p is an odd prime. Choose a Jordan decomposition of D. Define A as the sum over the Jordan components of order p and B as the sum over the remaining components. Then $D = A \oplus B$ and*

$$|\mathcal{O}(D, p, 1, t)| = \begin{cases} |\mathcal{O}(A, p, 1, t)||B_p|, & \text{if } A \neq 0, \\ 0, & \text{if } A = 0. \end{cases}$$

Theorem 4.2.8 ([Sch13])**.** *Let D be a finite quadratic module of order p^l where p is an odd prime. The orbit length*

$$|\mathcal{O}(D, m, 1, \ldots, 1, t, pt, \ldots, p^k t)|$$

depends only on $\left(\frac{mt}{p}\right)$.

Proposition 4.2.9 ([Sch13])**.** *Let D be a finite quadratic module of order p^l where p is an odd prime. We have*

$$|\mathcal{O}(D, m, 1, \ldots, 1, t, pt, \ldots, p^k t)| = \frac{|D_{p^k}|}{p^k} |\mathcal{O}(D/D_{p^k}, p, 1, p^k t)|.$$

Theorem 4.2.10 ([Sch13])**.** *Let D be a finite quadratic module of order p^l where p is an odd prime. Suppose $v_1 = \cdots = v_{p^j} = 1$ and $v_{p^{j+1}} > 1$ for some $j \in \{0, \ldots, k-1\}$. Further suppose that $t_{p^i} = p^i t_1 \pmod 1$ for all $i = 1, \ldots, j$. Choose a Jordan decomposition of D. Let A denote the sum over the Jordan components whose level divides $v_{p^{j+1}} p^{j+1}$ and B the* *sum of the remaining components. Then*

$$|\mathcal{O}(D, p^{k+1}, v_1, \ldots, v_{p^k}, t_1, \ldots, t_{p^k})| = \sum_{\substack{p^{j+1}r = t_{p^{j+1}} \pmod 1 \\ s + v_{p^{j+1}} r = t_1 \pmod 1}} |\mathcal{O}(A, p^{j+1}, 1, \ldots, 1, s, ps, \ldots, p^j s)| \; |\mathcal{O}(B^{v_{p^{j+1}}}, p^{k+1}, 1, \ldots, 1, \frac{v_{p^{j+1}}}{v_{p^{j+1}}}, \ldots, \frac{v_{p^k}}{v_{p^{j+1}}}, r, pr, \ldots, p^j r, t_{p^{j+1}}, \ldots, t_{p^k})|.$$

In the case that we have a valid orbit description in the sense of (4.2.10). These formulas can be further simplified.

Proposition 4.2.11 ([Sch13])**.** *Suppose the assumptions of Theorem 4.2.10 hold.*
If $v_{p^{j+1}} t_{p^{j+1}} = p^{j+1} t_1 \pmod 1$ *then*

$$|\mathcal{O}(D, m, v_1, \ldots, v_{p^k}, t_1, \ldots, t_{p^k})| = |\mathcal{O}(A, p^{j+1}, 1, \ldots, 1, a, pa, \ldots, p^j a)| \sum_{p^{j+1}r = t_{p^{j+1}} \pmod 1} |\mathcal{O}(B^{v_{p^{j+1}}}, m, 1, \ldots, 1, \frac{v_{p^{j+1}}}{v_{p^{j+1}}}, \ldots, \frac{v_{p^k}}{v_{p^{j+1}}}, r, pr, \ldots, p^j r, t_{p^{j+1}}, \ldots, t_{p^k})|$$

where a is any element in $\mathbb{Q}/\mathbb{Z}$ such that $p^j a = p^j t_1 - \frac{v_{p^{j+1}}}{p} t_{p^{j+1}}$.
If $v_{p^{j+1}} t_{p^{j+1}} \neq p^{j+1} t_1 \pmod 1$*, then* $|\mathcal{O}(D, m, v_1, \ldots, v_{p^k}, t_1, \ldots, t_{p^k})| = 0$.

Theorem 4.2.12 ([Opi13, Theorem 2.9, Korollar 2.10])**.** *Suppose the assumptions of Theorem 4.2.10 hold.*
If $v_{p^{j+1}} t_{p^{j+1}} = p^{j+1} t_1 \pmod 1$ *then*

$$|\mathcal{O}(D, m, v_1, \ldots, v_{p^k}, t_1, \ldots, t_{p^k})| = p^{j+1} \cdot |\mathcal{O}(A, p^{j+1}, 1, \ldots, 1, a, pa, \ldots, p^j a)| \cdot |\mathcal{O}(B^{v_{p^{j+1}}}, m, 1, \ldots, 1, \frac{v_{p^{j+1}}}{v_{p^{j+1}}}, \ldots, \frac{v_{p^k}}{v_{p^{j+1}}}, b, pb, \ldots, p^j b, t_{p^{j+1}}, \ldots, t_{p^k})|$$

where a is any element in $\mathbb{Q}/\mathbb{Z}$ such that $p^j a = p^j t_1 - \frac{v_{p^{j+1}}}{p} t_{p^{j+1}} \pmod 1$ *and b is any element in $\mathbb{Q}/\mathbb{Z}$ such that* $p^{j+1} b = t_{p^{j+1}} \pmod 1$.
If $v_{p^{j+1}} t_{p^{j+1}} \neq p^{j+1} t_1 \pmod 1$*, then* $|\mathcal{O}(D, m, v_1, \ldots, v_{p^k}, t_1, \ldots, t_{p^k})| = 0$.

Proof. This is obtained by applying Proposition 4.2.11 to its own summands. One shows that the p^{j+1} summands are all the same. ☐

Note that when computing the lengths of orbits which have the same multiplicities, the discriminant forms appearing in the above recursion are always the same. Hence they only have to be computed once.

Also note that the above formulas depend only on the Jordan decomposition D. The orbits can be computed purely from the genus symbol.

4.3. Gauss sums

We define the *Gauss sums* of a finite quadratic module D by

$$G(c, D) = \sum_{\mu \in D} e(cQ(\mu)) \tag{4.3.1}$$

for any integer c. One way to calculate them is to use the representation numbers of D. This approach gives

$$G(c, D) = \sum_{j \pmod 1} e(cj)N(D, j), \tag{4.3.2}$$

which is a finite sum. There is also a closed formula, for which we define

$$x_c = \left(2^{k-1}, \dots, 2^{k-1}\right) \in (2^k)_t^{\pm n} \tag{4.3.3}$$

if $2^k || c$ and D has a maximal odd component of this form. If there is no such odd component, we set $x_c = 0$. We define the coset $D^{c*} = x_c + D^c$ and for any integer q, we denote the greatest common divisor of q and c by $q_c = (q, c)$. Furthermore, we define the value $Q'(\alpha_c) = cQ(\gamma) + (x_c, \gamma)$ for any $\alpha = x_c + c\gamma \in D^{c*}$. This is well defined by [Sch09, Proposition 2.2].

Theorem 4.3.1 ([Sch09, Theorem 3.8 and 3.9]). *Let D be a finite quadratic module, $c \in \mathbb{Z}$ and $\alpha \in D$. We have*

$$\sum_{\mu \in D} e(cQ(\mu) + (\alpha, \mu)) = \begin{cases} 0, & \text{if } \alpha \notin D^{c*}, \\ \varepsilon_c e(-Q'(\alpha_c))\sqrt{|D_c||D|}, & \text{if } \alpha \in D^{c*}, \end{cases} \tag{4.3.4}$$

where

$$\begin{aligned} \varepsilon_c = \prod_{2 | q \nmid c} \gamma_2 \left((q/q_c)_{t/II}^{\varepsilon_q n_q}\right) e\left((c/q_c - 1) \operatorname{oddity}\left((q/q_c)_{t/II}^{\varepsilon_q n_q}\right)/8\right) \left(\frac{c/q_c}{(q/q_c)^{n_q}}\right) \\ \prod_{\substack{p | q \nmid c \\ p \text{ odd}}} \gamma_p\left((q/q_c)^{\varepsilon_q n_q}\right) \left(\frac{c/q_c}{(q/q_c)^{n_q}}\right). \end{aligned} \tag{4.3.5}$$

and the products run over the Jordan components $q^{\varepsilon_q n_q}$ of D.

Since $0 \in D^{c*}$ if and only $x_c = 0$ we can calculate the Gauss sums as follows.

Corollary 4.3.2. *Let D be a finite quadratic module, $c \in \mathbb{Z}$ and $\alpha \in D$. We have*

$$\begin{aligned} G(D,c) &= \sum_{\mu \in D} e(cQ(\mu)) \\ &= \begin{cases} 0, & \text{if } D \text{ has an odd 2-adic Jordan} \\ & \text{component of level } 2^{k+1} \text{ where } 2^k || c, \\ \varepsilon_c \sqrt{|D_c||D|}, & \text{else.} \end{cases} \end{aligned} \tag{4.3.6}$$

4.4. The dimension of $M_{\rho_L^*,k}$

The dimension of the vector space of vector valued modular forms of weight k for the dual Weil representation $M_{\rho_L^*,k}$ is computed in [Bru02b] for the case $2k + \operatorname{sign}(L) \equiv 0 \pmod 4$. Note that the formula only depends on the discriminant form $D = L'/L$. Let N denote the level of D. The dimension of the modular forms (resp. cusp forms) are

$$\begin{aligned} \dim_{\mathbb{C}}(M_{\rho_L^*,k}) &= d + \frac{dk}{12} - \alpha_1 - \alpha_2 - \alpha_3, \\ \dim_{\mathbb{C}}(S_{\rho_L^*,k}) &= d + \frac{dk}{12} - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4, \end{aligned} \tag{4.4.1}$$

where

$$\begin{aligned} d &= |D/\{\pm 1\}| = \frac{1}{2}(|D| + |\{\gamma \in D \mid 2\gamma = 0\}|), \\ \alpha_1 &= \frac{d}{4} - \frac{1}{4\sqrt{|D|}} \, e((2k + \operatorname{sign}(D))/8) \, \Re(G(2,D)), \\ \alpha_2 &= \frac{d}{3} + \frac{1}{3\sqrt{3|D|}} \, \Re(e((4k + 3\operatorname{sign}(D) - 10)/24) \, (G(1,D) + G(-3,D))), \\ \alpha_3 &= \sum_{\gamma \in D/\{\pm 1\}} -Q(\gamma) - [-Q(\gamma)] \\ &= \frac{1}{2} \sum_{j=1}^{N} \frac{N-j}{N} \left(N\left(D, \frac{j}{N}\right) + N_{\text{2-torsion}}\left(D, \frac{j}{N}\right)\right), \\ \alpha_4 &= \left| \left\{ \gamma \in D/\{\pm 1\} \;\middle|\; \gamma^2/2 = 0 \mod 1 \right\} \right| \\ &= \frac{1}{2}(N(D,0) + N_{\text{2-torsion}}(D,0)). \end{aligned} \tag{4.4.2}$$

4.5. Local densities as orbit invariants

In this section, we show that generalized local densities can be used as orbit invariants. Let L be an even lattice with discriminant form $D = L'/L$. By Remark 3.1.7, the Whittaker polynomials for a prime p depend mainly on the parity of the rank of L and on the maximal p-subgroup D_p of D.

Lemma 4.5.1. *Let p be a prime and L an even lattice with discriminant form $D = L'/L$ of order p^l. If there is an isometry of D mapping $\mu \in D$ to $\mu' \in D$, then*

$$W_{p,m,\mu}(X) = W_{p,m,\mu'}(X) \tag{4.5.1}$$

for all $m \in \mathbb{Q}$.

Proof. The vector valued Eisenstein series associated to L of weight k is invariant under $O(D)$, which means that the Fourier coefficients at $\mathfrak{e}_\mu$ and $\mathfrak{e}_{\mu'}$ coincide. Since all other factors in the formulas of Theorem 3.1.5 and Theorem 3.1.6 are non zero, the local densities must coincide for an infinite number of possible weights. This means that the Whittaker polynomials satisfy

$$W_{p,m,\mu}(p^{-s}) = W_{p,m,\mu'}(p^{-s}) \tag{4.5.2}$$

for infinite values of s, so the polynomials must be the same. □

Since the last lemma is only useful if the rank of L is even or if D has order 2^l, we need a different approach for odd rank and odd primes. We look at the even lattice $L_2 = (\mathbb{Z}, x \mapsto x^2)$. We have $L_2' = \frac{1}{2}\mathbb{Z}$ and using Lemma 3.1.8 we compute

$$W_{2,m,\frac{1}{2}}(X) = \begin{cases} X+1, & \text{if } m \equiv \frac{1}{4} \pmod 2, \\ -X+1, & \text{if } m \equiv \frac{5}{4} \pmod 2. \end{cases} \tag{4.5.3}$$

Lemma 4.5.2. *Let p be an odd prime and L_p an even lattice with discriminant form $D = L_p'/L_p$ of order p^l. Let $L = L_2 \oplus L_p$. If there is an isometry of L_p'/L_p mapping $\mu_p \in D$ to $\mu_p' \in D$, then the local Whittaker polynomials of L satisfy*

$$W_{p,m,(\frac{1}{2},\mu)}(X) = W_{p,m,(\frac{1}{2},\mu')}(X) \tag{4.5.4}$$

for all $m \in \mathbb{Q}$.

Proof. The vector valued Eisenstein series associated to L of weight k is invariant under $O(D)$, which means that the Fourier coefficients at $\mathfrak{e}_{(\frac{1}{2},\mu)}$ and $\mathfrak{e}_{(\frac{1}{2},\mu')}$ coincide. Since all other factors in the formulas of Theorem 3.1.5 and Theorem 3.1.6 are non zero, the local densities must coincide for an infinite number of possible weights. This means that the Whittaker polynomials satisfy

$$W_{2,m,(\frac{1}{2},\mu)}(2^{-s})W_{p,m,(\frac{1}{2},\mu)}(p^{-s}) = W_{2,m,(\frac{1}{2},\mu')}(2^{-s})W_{p,m,(\frac{1}{2},\mu')}(p^{-s}) \tag{4.5.5}$$

for infinitely many values of s. Since $W_{2,m,(\frac{1}{2},\mu)}(2^{-s}) = W_{2,m,(\frac{1}{2},\mu')}(2^{-s})$ is non zero for infinite values of s by equation (4.5.3), the Whittaker polynomials for p must be equal. □

Let L be an even lattice with discriminant form $D = L'/L$.

Proposition 4.5.3. *If there is an isometry of D mapping $\mu \in D$ to $\mu' \in D$, then*

$$W_{p,m,\mu}(X) = W_{p,m,\mu'}(X) \tag{4.5.6}$$

for all primes p and all $m \in \mathbb{Q}$.

Proof. For a fixed prime p, the above equation is only concerned with the p-adic properties of L and μ. We can therefore assume that it is an isometry of D_p which sends μ to μ'.

If the rank of L is even or $p = 2$, then there is an n such that L plus n hyperbolic planes is p-adically isometric to a lattice where Lemma 4.5.1 applies and Remark 3.1.7 implies (4.5.6).

If the rank of L is odd and $p \neq 2$, then there is an n such that L plus n hyperbolic planes is p-adically isometric to a lattice where Lemma 4.5.2 applies and again Remark 3.1.7 implies (4.5.6). □

For odd primes p, the Whittaker polynomials can be viewed as a generalization of the orbit invariants described in Section 4.2.1. To see this, recall that for odd primes p, an orbit with respect to the orthogonal group of a finite quadratic module of order p^l is uniquely determined by the order of its elements together with the multiplicities and reduced norms of multiples of its elements.

Let $\mu = (\mu_p)_p \in \bigoplus_p D_p$. For odd p we have

$$v(\mu_p) = p^{K_0}, \tag{4.5.7}$$

so by Lemma 3.1.8, the Whittaker polynomials $W_{p,m,\mu_p}(X)$ have modulus $v(\mu_p)$ in m.

Taking a look at Theorem 3.1.5, we see that out of the Whittaker polynomials

$$W_{p,Q(\mu)+m,\mu}(X), \quad m = 0, \ldots, v(\mu) - 1 \tag{4.5.8}$$

only $W_{p,Q(\mu),\mu}(X)$ has a coefficient with even numerator in front of X^{K_0}. This implies that the Whittaker polynomials determine $Q(\mu) \pmod{v(\mu)\mathbb{Z}_p}$, from which we can deduce the reduced norm of $Q^{\text{red}}(\mu_p) = \frac{Q(\mu_p)}{v(\mu_p)} \pmod 1$.

The same argument shows that the multiplicity of μ_p (for odd p) is the smallest modulus satisfied by the Whittaker polynomials $W_{p,m,\mu}(X)$ in the sense of Lemma 3.1.8, so we can also recover the multiplicity $v(\mu_p)$ from them.

We have proved the following theorem.

Theorem 4.5.4. *Let D be a finite quadratic module of order p^l for an odd prime p. Let $\mu, \mu' \in D$ have the same order p^{k+1} and the same multiplicities $v_{p^j}(\mu) = v_{p^j}(\mu')$ for $j = 0, \ldots, k$. There is an isometry mapping $\mu \in D$ to $\mu' \in D$ if and only if*

$$W_{p,m,p^j\mu}(X) = W_{p,m,p^j\mu'}(X) \tag{4.5.9}$$

for all $m \in \mathbb{Q}$ and $j = 0, \ldots, k$. It suffices to check $m = Q(p^j\mu), Q(p^j\mu) + 1, \ldots, Q(p^j\mu) + v_{p^j}(\mu) - 1$.

For odd primes p both the Whittaker polynomials and the reduced norms are well defined. For $p = 2$, the reduced norms are not generally well defined, so the Whittaker polynomials are new orbit invariants. We do not yet know however, if they determine a unique orbit.

5. Borcherds products

In this chapter, we use our algorithm for the computation of vector valued Eisenstein series to classify Borcherds products of singular weight arising from simple lattices. This is joint work with Markus Schwagenscheidt (cf. [OS18]).

In his celebrated work [Bor98], Borcherds defined a multiplicative lifting map from vector valued modular forms for the Weil representation associated to an even lattice L to modular forms on the hermitian symmetric domain corresponding to L. The resulting modular forms have infinite product expansions at the cusps and are therefore called automorphic (or Borcherds) products. The smallest possible weight of a non-constant holomorphic modular form for the orthogonal group of an even lattice L of type $(2, n)$ with $n \geq 3$, called the singular weight, is given by $\frac{n}{2} - 1$, compare [Bun01]. Borcherds products of singular weight have interesting Fourier and product expansions which often yield denominator identities of generalized Kac-Moody algebras [Sch06]. Furthermore, there are not many known holomorphic Borcherds products of singular weight, and it is a folklore conjecture that there are only finitely many of them, which makes it an interesting problem to classify them all.

Scheithauer [Sch06] obtained a complete list of the symmetric and reflective Borcherds products of singular weight for lattices of square free level. In [Sch17], he classified all reflective (not necessarily symmetric) holomorphic automorphic products of singular weight for lattices of prime level, and he gave an effective bound for the possible types of lattices of prime level (with prescribed discriminant group) allowing holomorphic Borcherds products of singular weight. His student M. Dittmann informed us that he was able to remove the requirement of being symmetric for all lattices of square free level [Dit18].

Following a somewhat different direction, Dittmann, Hagemeier and Schwagenscheidt in [DHS15] classified the simple lattices of square free level (hence even signature) and the corresponding holomorphic Borcherds products of singular weight. Here, an even lattice L of type $(2, n)$ is called simple if the space of cusp forms of weight $\frac{n}{2} + 1$ for the dual Weil representation of L vanishes. For a simple lattice, every formal principal part is the principal part of a vector valued modular form, which implies that a simple lattice allows many Borcherds products. One of the main result of [DHS15] is a list of 15 (isomorphy classes) of simple lattices of square free level. It was further proven that only four of them admit holomorphic automorphic products of singular weight, which were then constructed explicitly. Bruinier, Ehlen and Freitag [BEF16b] determined all simple lattices of arbitrary level and type $(2, n)$. The main result of this chapter is the classification of the holomorphic Borcherds products of singular weight for all simple lattices. To ensure that the Borcherds product is holomorphic, we assume that the corresponding vector valued modular form has only non-negative coefficients in its principal part.

Theorem 5.0.5. *Holomorphic Borcherds products (coming from vector valued modular forms with non-negative principal part) of singular weight $\frac{n}{2} - 1$ for simple lattices L of type $(2, n), n \geq 3$, only exist in the following cases.*

n	*genus*	*lattice*	*level*
3	$2_7^{+1}4^{+2}$	$A_1(-1) \oplus U \oplus U(4)$	4
	$2_7^{+3}4^{+2}$	$A_1(-1) \oplus U(2) \oplus U(4)$	4
	$2_7^{+1}4^{+4}$	$A_1(-1) \oplus U(4) \oplus U(4)$	4
	$2^{+4}4_7^{+1}$	$A_1(-2) \oplus U(2) \oplus U(2)$	8
	8_7^{+1}	$A_1(-4) \oplus U \oplus U$	16
4	3^{+5}	$A_2(-1) \oplus U(3) \oplus U(3)$	3
6	2^{-6}	$D_4(-1) \oplus U(2) \oplus U(2)$	2
10	2^{+2}	$E_8(-1) \oplus U \oplus U(2)$	2
26	1^{+1}	$E_8(-1) \oplus E_8(-1) \oplus E_8(-1) \oplus U \oplus U$	1

Here U denotes the hyperbolic plane $\mathbb{Z}^2$ with $Q(x, y) = xy$, and A_1, A_2, D_4, E_8 denote the usual root lattices. Further, if (L, Q) is a lattice and N a positive integer, we write $L(N)$ for the scaled lattice (L, NQ).

We remark that the automorphic products for the lattices with $n \geq 4$ in the above table were already found in [DHS15].

We briefly explain the idea of the proof. Let L be an even lattice of type $(2, n)$, let L' denote its dual lattice and let L'/L be its discriminant form. We let $\mathbb{C}[L'/L]$ be the group ring of L'/L, which is generated by the basis elements $\mathfrak{e}_\gamma$ for $\gamma \in L'/L$. Let

$$f(z) = \sum_{\gamma \in L'/L} \sum_{n \gg -\infty} a_f(\gamma, n) e^{2\pi i n z} \mathfrak{e}_\gamma,$$

be a weakly holomorphic modular form of weight $1 - \frac{n}{2}$ for the Weil representation of L with real coefficients $a_f(\gamma, n)$. To make sure that the associated Borcherds product Ψ_f is holomorphic we assume that the coefficients $a_f(\gamma, n)$ with $n < 0$, i.e., the coefficients of the principal part of f, are non-negative integers. The weight of Ψ_f is given by the linear combination

$$-\frac{1}{2} \sum_{\gamma \in L'/L} a_f(\gamma, n) a_E(\gamma, -n), \tag{5.0.1}$$

where $a_E(\gamma, -n)$ are the Fourier coefficients of an Eisenstein series of weight $\frac{n}{2} + 1$ for the dual Weil representation (see Section 3.2). In Section 3.5, we have given an explicit lower bound for the absolute value of the coefficients of this Eisenstein series of the following form.

Theorem 5.0.6 (cf. Theorem 3.5.2)**.** *Let L be an even lattice of type (b^+, b^-) (b^+ even) and rank $m \geq 3$, and let $d = |L'/L|$. Suppose that L splits a scaled hyperbolic plane $U(N)$. Let $d = |L'/L|$. Let $\gamma \in L'/L$ and $n \in \mathbb{Z} - Q(\gamma)$ with $n > 0$. The coefficient $a_E(\gamma, n)$ of the Eisenstein series of weight $k = \frac{m}{2}$ for the dual Weil representation is either 0 or satisfies the estimate*

$$(-1)^{b^+/2} a_E(\gamma, n) \geq C_{k,d,N} \cdot n^{k-1}$$

for some explicit constant $C_{k,d,N} > 0$ depending on k, d, and N, but not on γ and n.

The theorem implies that the weight of Ψ_f is bigger than the singular weight for all but finitely many choices of the principal part of f. For the few remaining choices of the principal part of f, we explicitly compute the coefficients of the Eisenstein series to check whether Ψ_f has singular weight.

5.1. Orthogonal modular forms

Let L be an even lattice of type $(2, n)$ with $n \geq 3$ and let $V = L \otimes \mathbb{R}$. We let $\mathrm{Gr}(V)$ be the Grassmannian of positive definite planes in V. Choose some primitive isotropic vector $z \in L$ and some vector $z' \in L'$ with $(z, z') = 1$, and let $K = L \cap z^\perp \cap z'^\perp$. The complex manifold $\{Z = X + iY \in K \otimes \mathbb{C} : (Y, Y) > 0\}$ has two connected components. We pick one of them and denote it by $\mathcal{H}_n$. It can be viewed as a generalized upper half-plane. There is a bijection $\mathrm{Gr}(V) \cong \mathcal{H}_n$ which endows $\mathrm{Gr}(V)$ with a complex structure, compare [Bru02a, Section 3.2].

We let $O(L)^+ = O(L) \cap O(V)^+$ be the intersection of the orthogonal group $O(L)$ of L with the identity component of $O(V)$, and we let Γ_L be the kernel of the natural map $O(L)^+ \to O(L'/L)$. It has finite index in $O(L)^+$. The action of $O(L)^+$ on $\mathrm{Gr}(V)$ induces an action on $\mathcal{H}_n$. Further, there is a natural factor of automorphy $j(\sigma, Z)$ for $\sigma \in O(L)^+$ and $Z \in \mathcal{H}_n$, see [Bru02a, Section 3.3]. A meromorphic function $\Psi : \mathcal{H}_n \to \mathbb{C}$ is called a modular form of weight $k \in \frac{1}{2}\mathbb{Z}$ for Γ_L and multiplier system χ if $\Psi(\sigma Z) = \chi(\sigma) j(\sigma, Z)^k \Psi(Z)$ for all $\sigma \in \Gamma_L$ and $Z \in \mathcal{H}_n$. The smallest possible positive weight of a non-trivial holomorphic modular form for Γ_L is called the singular weight. It is given by $\frac{n}{2} - 1$, compare [Bun01].

For $\gamma \in L'/L$ and $n < 0$ we define the Heegner divisor of index (γ, n) by

$$H_L(\gamma, n) = \sum_{\substack{X \in L+\gamma \\ Q(X)=n}} X^\perp \subset \mathrm{Gr}(V).$$

Here $X^\perp \subset \mathrm{Gr}(V)$ denotes the set of all positive definite planes orthogonal to X. The corresponding divisor in $\mathcal{H}_n$ will be denoted by the same symbol $H_L(\gamma, n)$.

Theorem 5.1.1 ([Bor98, Theorem 13.3])**.** *Let $f = \sum_{\gamma,n} a_f(\gamma, n) e(nz) \mathfrak{e}_\gamma$ be a weakly holomorphic modular form of weight $1 - \frac{n}{2}$ for ρ_L with $a_f(\gamma, n) \in \mathbb{Z}$ for all $n \leq 0, \gamma \in L'/L$. Then there exists a meromorphic function $\Psi_f : \mathcal{H}_n \to \mathbb{C}$ with the following properties:*

(i) Ψ_f is a meromorphic modular form of weight $a_f(0, 0)/2$ for Γ_L with some multiplier system of finite order.

(ii) The divisor of Ψ_f *is given by*

$$\frac{1}{2}\sum_{\gamma\in L'/L}\sum_{n<0} a_f(\gamma,n)H_L(\gamma,n).$$

Here $H_L(\gamma,n)$ *has multiplicity* 2 *if* $2\gamma=0$ *in* L'/L*, and multiplicity* 1 *otherwise.*

(iii) Ψ_f *has an infinite product expansion.*

The product expansion of Ψ_f can be written down explicitly, but we did not include it here since we will not use it. The modular form Ψ_f is called the Borcherds product or automorphic product associated to f. We remark that it depends on the choice of the primitive isotropic vector z, and different choices of z correspond to expansions of Ψ_f at other cusps.

We are particularly interested in Borcherds products of singular weight $\frac{n}{2}-1$. Therefore, we need to control the constant coefficient $a_f(0,0)$ of f. Let $\kappa=\frac{n}{2}+1$, and recall the definition of the vector valued Eisenstein series for the dual Weil representation ρ_L^*,

$$E(z)=\frac{1}{4}\sum_{(M,\phi)\in\langle T\rangle\backslash\operatorname{Mp}_2(\mathbb{Z})}\mathfrak{e}_0|_k^*(M,\phi).$$

It is a modular form of weight κ for ρ_L^* and it has a Fourier expansion of the form

$$E(z)=\mathfrak{e}_0+\sum_{\gamma\in L'/L}\sum_{n>0}a_E(\gamma,n)e(nz)\mathfrak{e}_\gamma.$$

If f is a weakly holomorphic modular form of weight $k=1-\frac{n}{2}=2-\kappa$, then the function $\sum_{\gamma\in L'/L}f_\gamma(z)E_\gamma(z)dz$ is a meromorphic 1-form on $\operatorname{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. By the residue theorem its residue vanishes, which yields the formula

$$a_f(0,0)=-\sum_{\gamma\in L'/L}\sum_{n<0}a_f(\gamma,n)a_E(\gamma,-n).$$

Therefore, the constant coefficient of f, and hence the weight of the associated Borcherds product, is determined by the principal part of f and certain coefficients of an Eisenstein series.

5.2. Simple lattices

An even lattice L of type $(2,n)$ is called simple if the space of cusp forms of weight $\frac{n}{2}+1$ for ρ_L^* is trivial. This space of cusp forms is also called the obstruction space for L. The significance of this notion is the fact that a formal principal part as in (1.7.2) is the principal part of a weakly holomorphic modular form if and only if

$$\sum_{\gamma \in L'/L} \sum_{n<0} a_f(\gamma, n) a_g(\gamma, -n) = 0$$

for every cusp form g in the obstruction space. Hence, for a simple lattice every formal principal part is the principal part of a weakly holomorphic modular form, and hence every $\mathbb{Z}$-linear combination of Heegner divisors is the divisor of a Borcherds product.

The simple lattices of square free level and the corresponding holomorphic Borcherds products of singular weight were determined in [DHS15]. Later, all simple lattices of arbitrary level were computed in [BEF16b], but the corresponding Borcherds products of singular weight were not studied. For convenience of the reader, we give a list of the simple lattices of type $(2, n)$ with $n \geq 3$ at the end of this chapter in Section 5.5.

5.3. An example

For a given lattice of type $(2, n)$ we are interested in solutions to the equation

$$\frac{n}{2} - 1 = \frac{1}{2} a_f(0, 0) = -\frac{1}{2} \sum_{\gamma \in L'/L} \sum_{n<0} a_f(\gamma, n) a_E(\gamma, -n)$$

where f has a non-negative principal part, i.e. $a_f(\gamma, n) \in \mathbb{Z}_{\geq 0}$. We know from Theorem 3.5.1 that the Eisenstein coefficients on the right hand side are non-positive.

For any $\gamma \in L'/L$ and any n, we have $a_f(\gamma, n) = a_f(-\gamma, n)$ and $a_E(\gamma, -n) = a_E(-\gamma, -n)$. Hence any nonzero summand for a γ of order greater than 2 occurs twice, once for γ and again for $-\gamma \neq \gamma$.

We can only find a solution to the above equation, if there is an Eisenstein coefficient satisfying $2 - n \leq a_E(\gamma, -n) < 0$ and $2\gamma = 0$ or an Eisenstein coefficient satisfying $1 - \frac{n}{2} \leq a_E(\gamma, -n) < 0$ and $2\gamma \neq 0$.

The lattice $L = A_1(-1) \oplus 2U(4)$ has genus symbol $2_7^{+1} 4^{+4}$ and type $(2, 3)$. We need to check for Eisenstein coefficients satisfying $-1 \leq a_E(\gamma, -n) < 0$ and $2\gamma = 0$ or Eisenstein coefficients satisfying $-\frac{1}{2} \leq a_E(\gamma, -n) < 0$ and $2\gamma \neq 0$. The lattice splits a hyperbolic plane scaled by 4, which leads to the estimate

$$-a_E(\gamma, n) \geq C_{4,512,\frac{5}{2}} \cdot n^{\frac{3}{2}}$$

for the non zero coefficients of the Eisenstein series with respect to the dual Weil representation. We have

$$C_{4,512,\frac{5}{2}} = -\frac{1}{90} \pi^2 + \frac{2}{15} \approx 0.023671.$$

For

$$n \geq \left[C_{4,512,\frac{5}{2}}^{-\frac{2}{3}} \right] + 1 = 13$$

this implies $a_E(\gamma, n) < -1$.

Let f be a modular form of weight $-\frac{1}{2}$ for ρ_L with coefficients $a_f(\gamma, n)$. In view of the discussion above and formula (5.0.1) for the weight of the Borcherds products Ψ_f, we see that the weight of Ψ_f will be bigger than the singular weight $\frac{1}{2}$ if $a_f(\gamma, n) > 0$ for some $n \leq -13$. Hence it suffices to compute the Eisenstein cofficients $a_E(\gamma, n)$ for $n < 13$. The discriminant form of L is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/4\mathbb{Z})^4$ and it has 8 orbits with respect to the action of its orthogonal group. Since the Eisenstein series is invariant under the orthogonal group it suffices to list the coefficients of the Eisenstein series once for each orbit. The following table gives a representative for each orbit, the size of the orbit and the coefficients of the Eisenstein series for an element in this orbit.

orbit repr.	#orbit	q-expansion
$(0,0,0,0,0)$	1	$1 - 10\,q^1 - 70\,q^4 - 48\,q^5 - 120\,q^8 - 250\,q^9$ $-240\,q^{12} + O(q^{13})$
$(0,1,0,0,0)$	120	$-4\,q^1 - 8\,q^2 - 16\,q^3 - 32\,q^4 - 32\,q^5 - 48\,q^6$ $-64\,q^7 - 64\,q^8 - 100\,q^9 - 112\,q^{10} - 112\,q^{11}$ $-128\,q^{12} + O(q^{13})$
$(0,2,0,0,0)$	15	$-4\,q^1 - 8\,q^2 - 16\,q^3 - 32\,q^4 - 32\,q^5 - 48\,q^6$ $-64\,q^7 - 64\,q^8 - 100\,q^9 - 112\,q^{10} - 112\,q^{11}$ $-128\,q^{12} + O(q^{13})$
$(0,1,1,0,0)$	120	$-2\,q^{\frac{3}{4}} - 8\,q^{\frac{7}{4}} - 14\,q^{\frac{11}{4}} - 24\,q^{\frac{15}{4}} - 38\,q^{\frac{19}{4}}$ $-40\,q^{\frac{23}{4}} - 56\,q^{\frac{27}{4}} - 80\,q^{\frac{31}{4}} - 76\,q^{\frac{35}{4}} - 104\,q^{\frac{39}{4}}$ $-126\,q^{\frac{43}{4}} - 112\,q^{\frac{47}{4}} - 156\,q^{\frac{51}{4}} + O(q^{\frac{55}{4}})$
$(0,1,2,0,0)$	120	$-1\,q^{\frac{1}{2}} - 6\,q^{\frac{3}{2}} - 14\,q^{\frac{5}{2}} - 20\,q^{\frac{7}{2}} - 31\,q^{\frac{9}{2}}$ $-46\,q^{\frac{11}{2}} - 50\,q^{\frac{13}{2}} - 68\,q^{\frac{15}{2}} - 92\,q^{\frac{17}{2}} - 82\,q^{\frac{19}{2}}$ $-108\,q^{\frac{21}{2}} - 148\,q^{\frac{23}{2}} - 131\,q^{\frac{25}{2}} + O(q^{\frac{27}{2}})$
$(1,0,0,1,0)$	120	$-\frac{1}{2}\,q^{\frac{1}{4}} - 4\,q^{\frac{5}{4}} - \frac{25}{2}\,q^{\frac{9}{4}} - 20\,q^{\frac{13}{4}} - 24\,q^{\frac{17}{4}}$ $-40\,q^{\frac{21}{4}} - \frac{121}{2}\,q^{\frac{25}{4}} - 60\,q^{\frac{29}{4}} - 72\,q^{\frac{33}{4}} - 100\,q^{\frac{37}{4}}$ $-96\,q^{\frac{41}{4}} - 124\,q^{\frac{45}{4}} - \frac{337}{2}\,q^{\frac{49}{4}} + O(q^{\frac{53}{4}})$
$(1,0,0,0,0)$	10	$-1\,q^{\frac{1}{4}} - 25\,q^{\frac{9}{4}} - 48\,q^{\frac{17}{4}} - 121\,q^{\frac{25}{4}} - 144\,q^{\frac{33}{4}}$ $-192\,q^{\frac{41}{4}} - 337\,q^{\frac{49}{4}} + O(q^{\frac{53}{4}})$
$(1,2,2,0,0)$	6	$-8\,q^{\frac{5}{4}} - 40\,q^{\frac{13}{4}} - 80\,q^{\frac{21}{4}} - 120\,q^{\frac{29}{4}}$ $-200\,q^{\frac{37}{4}} - 248\,q^{\frac{45}{4}} + O(q^{\frac{49}{4}})$

We see that there are exactly two possibilities to obtain holomorphic Borcherds products of singular weight, namely by setting

$$a_f(\gamma, \tfrac{1}{4}) = a_f(-\gamma, \tfrac{1}{4}) = 1$$

for any γ in the 6th orbit or by setting

$$a_f(\gamma, \tfrac{1}{4}) = 1$$

for any γ in the 7th orbit. The coefficients $a_f(\gamma, n)$ for all other $\gamma \in L'/L, n < 0$ have to be set to 0. We call these elements γ (which lead to Borcherds products of singular weight) *good elements*. The other simple lattices can be treated analogously. We obtain the list given in Theorem 5.0.5. The automorphic products for the simple lattices of type $(2, 3)$ will be described more explicitly in the next section.

5.4. Automorphic products as Siegel modular forms

In order to identify the automorphic products in our list, we follow the setup of [Lip08]. We consider the real quadratic space

$$V = \left\{ \begin{pmatrix} x_5 & -x_3 & 0 & -x_1 \\ x_4 & -x_5 & x_1 & 0 \\ 0 & -x_2 & x_5 & x_4 \\ x_2 & 0 & -x_3 & -x_5 \end{pmatrix} : x_i \in \mathbb{R} \right\}$$

with the quadratic form

$$Q(X) = -\frac{1}{4}\operatorname{tr}(X^2) = x_1x_2 + x_3x_4 - x_5^2.$$

It has type $(2, 3)$. Occasionally, we identify V with $\mathbb{R}^5$ and write $X = (x_1, x_2, x_3, x_4, x_5) \in V$ to ease the notation. The group $\mathrm{Sp}_4(\mathbb{R})$ acts as isometries on V by conjugation. In fact, the identity component $O(V)^+$ of the orthogonal group of V is isomorphic to $\mathrm{Sp}_4(\mathbb{R})/\{\pm 1\}$.

Let $\mathbb{H}_2$ be the Siegel upper half-space of genus 2. For $Z = \left(\begin{smallmatrix} z_1 & z_2 \\ z_2 & z_3 \end{smallmatrix}\right) \in \mathbb{H}_2$ we let

$$X(Z) = \frac{1}{\sqrt{\det(Y)}} \begin{pmatrix} z_2 & -z_1 & 0 & \det(Z) \\ z_3 & -z_2 & -\det(Z) & 0 \\ 0 & -1 & z_2 & z_3 \\ 1 & 0 & -z_1 & -z_2 \end{pmatrix} \in V(\mathbb{C}).$$

Note that $X(Z)$ has norm 0, and that the real and the imaginary part have norm 1 and are orthogonal. The map

$$Z \mapsto \operatorname{span}\Big(\Re X(Z), \Im X(Z)\Big)$$

gives a bijection between $\mathbb{H}_2$ and the Grassmannian $\mathrm{Gr}(V)$ of positive definite planes in V, which is compatible with the corresponding actions of $\mathrm{Sp}_4(\mathbb{R})$. Note that the Siegel upper half-plane $\mathbb{H}_2$ can be naturally identified with the orthogonal half-plane $\mathcal{H}_3$ corresponding to the primitive isotropic vector $z = (1, 0, 0, 0, 0)$ and $z' = (0, 1, 0, 0, 0)$. Thus, orthogonal modular forms on $\mathcal{H}_3$ can be viewed as Siegel modular forms of genus 2.

Let L be an even lattice in V, and let $\gamma \in L'/L$ and $n \in \mathbb{Z} + Q(\beta)$ with $m < 0$. In the Siegel upper-plane model of $\mathrm{Gr}(V)$, the Heegner divisor $H_L(\gamma, n)$ corresponds to the set

$$\sum_{\substack{X \in \gamma + L \\ Q(X) = n}} \left\{ \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \in \mathbb{H}_2 : x_2(z_2^2 - z_1 z_3) + x_4 z_1 - 2x_5 z_2 + x_3 z_3 + x_1 = 0 \right\}.$$

The ten even theta constants

$$\vartheta_{a,b}(Z) = \sum_{g \in \mathbb{Z}^2} \exp\left(\pi i \left(Z[g + a/2] + b^t(g + a/2)\right)\right),$$

with $a, b \in \{0, 1\}^2$, $a_1 b_1 + a_2 b_2 \equiv 0 \pmod 2$, are Siegel modular forms of weight $1/2$ for the principal congruence subgroup $\Gamma(2)$, see [Fre83, Satz 3.2]. The divisor of $\vartheta_{1,1,1,1}(Z)$ on $\mathbb{H}_2$ is given by $\Gamma_\vartheta\{Z \in \mathbb{H}_2 : z_2 = 0\}$, where Γ_ϑ is the theta group, see [Fre83, Bemerkung A 2.3]. Since $\mathrm{Sp}_4(\mathbb{Z})$ acts transitively on the even theta constants, we can easily determine the divisors of the other theta functions from this. The following result is well known.

Lemma 5.4.1. *The divisor of $\vartheta_{a,b}(Z)$ on $\mathbb{H}_2$ is given by the set of all $Z \in \mathbb{H}_2$ satisfying an equation*

$$x_2(z_2^2 - z_1 z_3) + x_4 z_1 - 2x_5 z_2 + x_3 z_3 = 0$$

for some $(x_1, \dots, x_5) \in \mathbb{Z}^5$ satisfying $x_1 x_2 + x_3 x_4 - x_5^2 = -1$ and the following congruences mod 4:

$\vartheta_{a_1,a_2,b_1,b_2}$	x_1	x_2	x_3	x_4	x_5
$\vartheta_{0,0,0,0}$	2	2	2	2	± 1
$\vartheta_{0,0,0,1}$	0	2	2	0	± 1
$\vartheta_{0,0,1,0}$	0	2	0	2	± 1
$\vartheta_{0,0,1,1}$	0	2	0	0	± 1
$\vartheta_{0,1,0,0}$	2	0	0	2	± 1
$\vartheta_{0,1,1,0}$	0	0	0	2	± 1
$\vartheta_{1,0,0,0}$	2	0	2	0	± 1
$\vartheta_{1,0,0,1}$	0	0	2	0	± 1
$\vartheta_{1,1,0,0}$	2	0	0	0	± 1
$\vartheta_{1,1,1,1}$	0	0	0	0	± 1

We remark that the divisors of the ten even theta constants can be written as Heegner divisors with respect to the lattice $\sqrt{2}\mathbb{Z}^5 \subset V$, but we chose the above formulation to make everything as explicit as possible.

We now describe the Borcherds products of singular weight $\frac{1}{2}$ found in Theorem 5.0.5 in terms of theta constants. In each case, we first realize the simple lattice under consideration as a sublattice of V, which amounts to choosing a cusp at which we expand the Borcherds products for this lattice. We will frequently use the fact that, by the Koecher principle, a holomorphic Siegel modular form of weight 0 for some finite index subgroup of $\mathrm{Sp}_4(\mathbb{Z})$ and some multiplier system of finite order is constant. Hence, in order to show that our Borcherds products of weight $\frac{1}{2}$ are given by theta constants, it suffices to compare their divisors.

5.4.1. The lattice $A_1(-4) \oplus U \oplus U$

We realize L as the subset of V consisting of those $X = (x_1, \dots, x_5) \in V$ with $x_1, \dots, x_4 \in \mathbb{Z}$ and $x_5 \in 2\mathbb{Z}$. Then the dual lattice L' is then given by those $X \in V$ with $x_1, \dots, x_4 \in \mathbb{Z}$ and $x_5 \in \frac{1}{4}\mathbb{Z}$. There are two good elements in L'/L, which are inverses of each other, namely $\pm\gamma = \pm(0, 0, 0, 0, \frac{1}{4}) + L$. The corresponding Heegner divisor $H_L(\gamma, -\frac{1}{16})$ translates into the set

$$\{Z \in \mathbb{H}_2 : x_2(z_2^2 - z_1 z_3) + x_4 z_1 - 2x_5 z_2 + x_3 z_3 + x_1 = 0\,,\ x_i \in \mathbb{Z},$$
$$x_1 x_2 + x_3 x_4 - x_5^2 = -1,\ x_1 \equiv \ldots \equiv x_4 \equiv 0(4),\ x_5 \equiv \pm 1(8)\}.$$

This is the divisor of the theta constant $\vartheta_{1,1,1,1}(Z)$, which implies that the Borcherds product of weight $\frac{1}{2}$ with Heegner divisor $H_L(\gamma, -\frac{1}{16})$ equals $\vartheta_{1,1,1,1}(Z)$ up to multiplication by a constant.

5.4.2. The lattice $A_1(-1) \oplus U(4) \oplus U$

We realize L as the subset of V with $x_1, x_3, x_4, x_5 \in \mathbb{Z}$ and $x_2 \in 4\mathbb{Z}$. There is one good element of order 2 in L'/L, namely $\gamma = (\frac{1}{2}, 2, 0, 0, \frac{1}{2}) + L$. By comparing the Heegner divisor $H_L(\gamma, -\frac{1}{4})$ in $\mathbb{H}_2$ to the divisors of the theta constants as above, we see that the corresponding Borcherds product is given by $\vartheta_{0,0,0,0}(2Z)$.

5.4.3. The lattice $A_1(-1) \oplus U(4) \oplus U(2)$

We realize L as the subset of V with $x_1, x_3, x_5 \in \mathbb{Z}$ and $x_2 \in 4\mathbb{Z}, x_4 \in 2\mathbb{Z}$. There are eight good elements γ with order 2 in L'/L. The corresponding Heegner divisors $H(\gamma, -\frac{1}{4})$ in $\mathbb{H}_2$ can be worked out and compared to the divisors of the theta constants as before. The resulting Borcherds products are given by

$$\vartheta_{0,0,0,0}(2Z), \quad \vartheta_{0,0,0,0}\begin{pmatrix} 4z_1 & 2z_2 \\ 2z_2 & z_3 \end{pmatrix}, \quad \vartheta_{0,0,1,0}(2Z), \quad \vartheta_{0,0,0,1}\begin{pmatrix} 4z_1 & 2z_2 \\ 2z_2 & z_3 \end{pmatrix},$$
$$\vartheta_{0,1,0,0}(2Z), \quad \vartheta_{1,0,0,0}\begin{pmatrix} 4z_1 & 2z_2 \\ 2z_2 & z_3 \end{pmatrix}, \quad \vartheta_{0,1,1,0}(2Z), \quad \vartheta_{1,0,0,1}\begin{pmatrix} 4z_1 & 2z_2 \\ 2z_2 & z_3 \end{pmatrix}.$$

5.4.4. The lattice $A_1(-2) \oplus U(2) \oplus U(2)$

We realize L as the subset of V with $x_1, \dots, x_5 \in \sqrt{2}\mathbb{Z}$. There are 20 good elements in L'/L which come in pairs $\pm\gamma$. The corresponding Borcherds products are exactly the ten even theta constants. This case has been treated in detail in the Diploma thesis of Lippolt [Lip08], written under the supervision of Freitag.

5.4.5. The lattice $A_1(-1) \oplus U(4) \oplus U(4)$

We realize L as the subset of V with $x_1, x_2, x_3, x_4 \in 2\mathbb{Z}$ and $x_5 \in \mathbb{Z}$. There are 10 good elements of order 2 in L'/L, which lead to the ten even theta constants, and 120 good elements which do not have order 2, and which form a single orbit under the action of $O(L'/L)$. For example, one pair of good elements is given by $\pm\gamma = \pm(1, 0, 1, 0, \frac{1}{2}) + L$, and the Borcherds product corresponding to the Heegner divisor $H_L(\gamma, -\frac{1}{4})$ is given by $\vartheta_{0,0,0,0}\left(\begin{smallmatrix} 2z_1 & z_2 \\ z_2 & \frac{z_3}{2} \end{smallmatrix}\right)$. The remaining Borcherds products can be determined analogously.

5.5. A list of simple lattices

We list the simple even lattices of type $(2, n)$, $n \geq 3$. They have been determined by Bruinier, Ehlen and Freitag and can be found in the appendix of the extended online version [BEF16a] of their journal article [BEF16b].

Every genus in the following list contains exactly one class. We describe the corresponding lattices in terms of the hyperbolic plane $U = (\mathbb{Z}^2, Q(x, y) = xy)$, the standard positive definite root lattices A_n, D_n, E_6, E_7, E_8 and the lattice

$$S_8 = \begin{pmatrix} -8 & -4 & 0 \\ -4 & -2 & -1 \\ 0 & -1 & -2 \end{pmatrix}$$

with genus symbol 8_3^{-1}. For a lattice (L, Q) and an integer N we let $L(N) = (L, NQ)$

denote the scaled lattice.

n	genus	lattice
3	2_7^{+1}	$A_1(-1) \oplus U \oplus U$
	2_7^{+3}	$A_1(-1) \oplus U(2) \oplus U$
	$2_7^{+1}4^{+2}$	$A_1(-1) \oplus U(4) \oplus U$
	2_7^{+5}	$A_1(-1) \oplus U(2) \oplus U(2)$
	$2_7^{+3}4^{+2}$	$A_1(-1) \oplus U(2) \oplus U(4)$
	$2_7^{+1}4^{+4}$	$A_1(-1) \oplus U(4) \oplus U(4)$
	4_7^{+1}	$A_1(-2) \oplus U \oplus U$
	$2^{+2}4_7^{+1}$	$A_1(-2) \oplus U(2) \oplus U$
	$2^{+4}4_7^{+1}$	$A_1(-1) \oplus U(2) \oplus U(2)$
	$2_1^{+1}3^{+1}$	$A_1(-3) \oplus U \oplus U$
	$2_7^{+1}3^{-2}$	$A_1(-1) \oplus U(3) \oplus U$
	$2_7^{+1}3^{+4}$	$A_1(-1) \oplus U(3) \oplus U(3)$
	8_7^{+1}	$A_1(-4) \oplus U \oplus U$
	8_3^{-1}	$S_8 \oplus U$
	$2^{+2}8_3^{-1}$	$S_8 \oplus U(2)$
4	3^{+1}	$A_2(-1) \oplus U \oplus U$
	3^{-3}	$A_2(-1) \oplus U(3) \oplus U$
	3^{+5}	$A_2(-1) \oplus U(3) \oplus U(3)$
	$2^{+2}3^{+1}$	$A_2(-1) \oplus U(2) \oplus U$
	$2^{+4}3^{+1}$	$A_2(-1) \oplus U(2) \oplus U(2)$

n	genus	lattice
5	4_5^{-1}	$A_3(-1) \oplus U \oplus U$
	$2^{+2}4_5^{-1}$	$A_3(-1) \oplus U(2) \oplus U$
	$2^{+4}4_5^{-1}$	$A_3(-1) \oplus U(2) \oplus U(2)$
6	2^{-2}	$D_4(-1) \oplus U \oplus U$
	2^{-4}	$D_4(-1) \oplus U(2) \oplus U$
	2^{-6}	$D_4(-1) \oplus U(2) \oplus U(2)$
	5^{+1}	$A_4(-1) \oplus U \oplus U$
7	4_3^{-1}	$D_5(-1) \oplus U \oplus U$
	$2_1^{+1}3^{-1}$	$A_5(-1) \oplus U \oplus U$
8	3^{-1}	$E_6(-1) \oplus U \oplus U$
	2_2^{+2}	$D_6(-1) \oplus U \oplus U$
	7^{+1}	$A_6(-1) \oplus U \oplus U$
9	2_1^{+1}	$E_7(-1) \oplus U \oplus U$
	4_1^{+1}	$D_7(-1) \oplus U \oplus U$
	8_1^{+1}	$A_7(-1) \oplus U \oplus U$
10	1^{+1}	$E_8(-1) \oplus U \oplus U$
	2^{+2}	$E_8(-1) \oplus U(2) \oplus U$
18	1^{+1}	$2E_8(-1) \oplus U \oplus U$
26	1^{+1}	$3E_8(-1) \oplus U \oplus U$

A. An integral 2-adic change of basis

In [Cas78, Chapter 8, Section 4, Lemma 4.1], Cassels states that the canonical for $\mathbb{Q}_2$-valued forms under $\mathbb{Z}_2$-equivalence are

$$2^e x^2, \quad 2^e(3x^2), \quad 2^e(5x^2), \quad 2^e(7x^2),$$
$$2^e(2x_1x_2),$$
$$2^e(x_1^2 + 2x_1x_2 + x_2^2)$$

where $e \in \mathbb{Z}$. In his proof, he also deals with bilinear forms

$$h(y_1, y_2) = h_{11}y_1^2 + 2h_{12}y_1y_2 + h_{22}y_2^2$$

with $h_{11}, h_{12}, h_{22} \in \mathbb{Z}_2$ and

$$|h_{12}|_2 = 1, \quad |h_{11}|_2 < 1, \quad |h_{22}|_2 < 1.$$

He leaves it to the reader to verify that $h(y_1, y_2)$ is $\mathbb{Z}_2$-equivalent to

$$\begin{cases} 2y_1y_2, & \text{if } h_{12}^2 - h_{11}h_{22} \equiv 1 \pmod 8, \\ y_1^2 + 2y_1y_2 + y_2^2, & \text{if } h_{12}^2 - h_{11}h_{22} \equiv 5 \pmod 8. \end{cases}$$

In both cases we will compute a basis to prove these $\mathbb{Z}_2$-equivalences.

Let $h, h_{11}, h_{12}, h_{22}$ be as above. Then h_{11} is a 2-adic unit, $2 \mid h_{11}$ and $2 \mid h_{22}$. We will frequently use the following well known fact which can for example be found [Ser73].

Remark A.0.1. An element $u \in \mathbb{Z}_2^*$ is a square, if and only if $u \equiv 1 \pmod 8$.

As a first consequence, we get $h_{12}^2 - h_{11}h_{22} \equiv 1 - 4\frac{h_{11}}{2}\frac{h_{22}}{2} \equiv 1 \text{ or } 5 \pmod 8$.

A.0.1. Case 1

In the case $h_{12}^2 - h_{11}h_{22} \equiv 1 \pmod 8$, we need to find a $\mathbb{Z}_2$-basis (b_1, b_2) of $\mathbb{Z}_2^2$ such that

$$h(b_1, b_1) = 0, \quad h(b_1, b_2) = 1 \quad \text{and} \quad h(b_2, b_2) = 0.$$

If $h_{11} = 0$, the elements

$$\tilde{b}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \tilde{b}_2 = \begin{pmatrix} \frac{-h_{22}}{2} \\ h_{12} \end{pmatrix}$$

are isotropic,

$$\det\begin{pmatrix}1 & \frac{-h_{22}}{2}\\ 0 & h_{12}\end{pmatrix} = h_{12} \in \mathbb{Z}_2^\times$$

and

$$h(\tilde{b}_1, \tilde{b}_2) = 0 \cdot 1 \cdot \frac{-h_{22}}{2} + h_{12}(1 \cdot h_{12} + \frac{-h_{22}}{2} \cdot 0) + h_{22} \cdot 0 \cdot \frac{-h_{22}}{2} = h_{12}^2 \in \mathbb{Z}_2^\times.$$

In this case

$$b_1 = \frac{1}{h_{12}}\tilde{b}_1 = \begin{pmatrix}\frac{1}{h_{12}}\\ 0\end{pmatrix}, \quad b_2 = \frac{1}{h_{12}}\tilde{b}_2 = \begin{pmatrix}\frac{-h_{22}}{2h_{12}}\\ 1\end{pmatrix}$$

gives a basis with the desired properties.

If $h_{11} \neq 0$, the elements

$$\tilde{b}_1 = \begin{pmatrix}-h_{12} + \sqrt{h_{12}^2 - h_{11}h_{22}}\\ h_{11}\end{pmatrix}, \quad \tilde{b}_2 = \begin{pmatrix}-h_{12} - \sqrt{h_{12}^2 - h_{11}h_{22}}\\ h_{11}\end{pmatrix}$$

are isotropic,

$$\det\begin{pmatrix}-h_{12} + \sqrt{h_{12}^2 - h_{11}h_{22}} & -h_{12} - \sqrt{h_{12}^2 - h_{11}h_{22}}\\ h_{11} & h_{11}\end{pmatrix} = 2h_{11}\sqrt{h_{12}^2 - h_{11}h_{22}} \neq 0$$

and

$$\begin{aligned} h(\tilde{b}_1, \tilde{b}_2) = {} & h_{11}(-h_{12} + \sqrt{h_{12}^2 - h_{11}h_{22}})(-h_{12} - \sqrt{h_{12}^2 - h_{11}h_{22}}) \\ & + h_{12}h_{11}(-h_{12} + \sqrt{h_{12}^2 - h_{11}h_{22}} + -h_{12} - \sqrt{h_{12}^2 - h_{11}h_{22}}) + h_{22}h_{11}^2 \\ = {} & h_{11}(h_{12}^2 - (h_{12}^2 - h_{11}h_{22})) - 2h_{12}^2h_{11} + h_{22}h_{11}^2 \\ = {} & -2h_{11}(h_{12}^2 - h_{22}h_{11}) \neq 0. \end{aligned}$$

If $h_{11} \equiv 2 \pmod 4$,

$$b_1 = \frac{1}{-2(h_{12}^2 - h_{22}h_{11})}\tilde{b}_1, \quad b_2 = \frac{1}{h_{11}}\tilde{b}_2$$

gives a basis with the desired properties.

If $h_{11} \equiv 0 \pmod 4$ and $h_{22} \equiv 0 \pmod 4$, the transformation

$$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}h_{11} & h_{12}\\ h_{12} & h_{22}\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} = \begin{pmatrix}h_{11} + 2h_{12} + h_{22} & h_{12} + h_{22}\\ h_{12} + h_{22} & h_{22}\end{pmatrix} =: \begin{pmatrix}\tilde{h}_{11} & \tilde{h}_{12}\\ \tilde{h}_{12} & \tilde{h}_{22}\end{pmatrix}$$

yields a quadratic form $\tilde{h}$ with $\tilde{h}_{11} \equiv 2 \pmod 4$. We already know how to deal with this case.

If $h_{11} \equiv 0 \pmod 4$ and $h_{22} \equiv 2 \pmod 4$, we use the transformation

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} h_{22} & -h_{12} \\ -h_{12} & h_{11} \end{pmatrix} =: \begin{pmatrix} \tilde{h}_{11} & \tilde{h}_{12} \\ \tilde{h}_{12} & \tilde{h}_{22} \end{pmatrix}$$

to get $\tilde{h}$ with $\tilde{h}_{11} \equiv 2 \pmod 4$.

A.0.2. Case 2

In the case $h_{12}^2 - h_{11}h_{22} \equiv 5 \pmod 8$, we need to find a $\mathbb{Z}_2$-basis (b_1, b_2) of $\mathbb{Z}_2^2$ such that

$$h(b_1, b_1) = 2, \quad h(b_1, b_2) = 1 \quad \text{and} \quad h(b_2, b_2) = 2.$$

We have $h_{11} \equiv h_{22} \equiv 2 \pmod 4$. We know that a root $\sqrt{c^2(h_{12}^2 - h_{11}h_{22}) + 2h_{11}} \in \mathbb{Z}_2^\times$ exists for all $c \in \mathbb{Z}_2^\times$. Define

$$a_i(c) = \frac{-ch_{12} + (-1)^i\sqrt{c^2(h_{12}^2 - h_{11}h_{22}) + 2h_{11}}}{h_{11}}$$

for $c \in \mathbb{Z}_2^\times$, $i \in \{0, 1\}$. We have $h(a_i(c), c) = 2$ for all $c \in \mathbb{Z}_2^\times$ and $i \in \{0, 1\}$.

If $h_{11} \equiv 2 \pmod{16}$, a square root $c = \sqrt{\frac{-3h_{11}}{2(h_{12}^2 - h_{11}h_{22})}}$ exists and

$$b_1 = \begin{pmatrix} a_0(c) \\ c \end{pmatrix}, \quad b_1 = \begin{pmatrix} a_1(c) \\ c \end{pmatrix}$$

gives a basis with the desired properties.

If $h_{11} \equiv 6 \pmod{16}$, a square root $c = \sqrt{\frac{-h_{11}}{2(h_{12}^2 - h_{11}h_{22})}}$ exists and

$$b_1 = \begin{pmatrix} a_0(c) \\ c \end{pmatrix}, \quad b_1 = \begin{pmatrix} a_0(-c) \\ -c \end{pmatrix}$$

gives a basis with the desired properties.

If $h_{11} \equiv 10$ or $14 \pmod{16}$ and $h_{22} \equiv 2 \pmod 4$, we use the transformation

$$\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} h_{11} + 8h_{12} + 16h_{22} & h_{12} + 4h_{22} \\ h_{12} + 4h_{22} & h_{22} \end{pmatrix} =: \begin{pmatrix} \tilde{h}_{11} & \tilde{h}_{12} \\ \tilde{h}_{12} & \tilde{h}_{22} \end{pmatrix}$$

to get $\tilde{h}$ with $\tilde{h}_{11} \equiv 2$ or $6 \pmod{16}$. We already know how to deal with this case.

B. Computing Eisenstein series for the Weil representation

To use the algorithms provided for integer lattices, start sage from within the directory containing the source files. From then on the first steps could look as follows:

```
sage: from eisenstein import EisensteinSeries
```

The classical Eisenstein series of weight 4:

```
sage: EisensteinSeries('', 4)

A vector valued q-series with coefficients
(): 1 q^(0) + 240 q^(1) + 2160 q^(2) + O(q^(3))
```

These Fourier coefficients were tested with the Siegel-Weil formula:

```
sage: EisensteinSeries(matrix(2,2,[2,1,1,2]), weight = 5)

A vector valued q-series with coefficients
(0, 0): 1 q^(0) + 246 q^(1) + 3600 q^(2) + O(q^(3))
(1/3, 1/3): 3 q^(1/3) + 723 q^(4/3) + 7206 q^(7/3) + O(q
   ^(10/3))
(2/3, 2/3): 3 q^(1/3) + 723 q^(4/3) + 7206 q^(7/3) + O(q
   ^(10/3))
```

Using the negative Gram matrix and the dual Weil representation gives the same result:

```
sage: EisensteinSeries(- matrix(2,2,[2,1,1,2]), weight = 5,
   dual = True)

A vector valued q-series with coefficients
(0, 0): 1 q^(0) + 246 q^(1) + 3600 q^(2) + O(q^(3))
(1/3, 1/3): 3 q^(1/3) + 723 q^(4/3) + 7206 q^(7/3) + O(q
   ^(10/3))
(2/3, 2/3): 3 q^(1/3) + 723 q^(4/3) + 7206 q^(7/3) + O(q
   ^(10/3))
```

We can also compute the Eisenstein series from a genus symbol:

```
sage: EisensteinSeries('3^-1', 5)

A vector valued q-series with coefficients
((3, (0, 0, 0, 0)),): 1 q^(0) + 246 q^(1) + 3600 q^(2) + O(q
   ^(3))
((3, (0, 0, 0, 1/3)),): 3 q^(1/3) + 723 q^(4/3) + 7206 q^(7/3)
    + O(q^(10/3))
((3, (0, 0, 0, 2/3)),): 3 q^(1/3) + 723 q^(4/3) + 7206 q^(7/3)
    + O(q^(10/3))
```

Using the genus symbol for the finite quadratic module twisted with -1 and the dual Weil representation gives the same result:

```
sage: EisensteinSeries('3^1', 5, dual = True)

A vector valued q-series with coefficients
((3, (0, 0, 0, 0)),): 1 q^(0) + 246 q^(1) + 3600 q^(2) + O(q
   ^(3))
((3, (0, 0, 0, 1/3)),): 3 q^(1/3) + 723 q^(4/3) + 7206 q^(7/3)
    + O(q^(10/3))
((3, (0, 0, 0, 2/3)),): 3 q^(1/3) + 723 q^(4/3) + 7206 q^(7/3)
    + O(q^(10/3))
```

For lattices, the Eisenstein series is computed by the class “Lattice”:

```
sage: from integer_lattice import Lattice
sage: L = Lattice(matrix(2,2,[0,1,1,0]))
sage: L.eisenstein_series(4, prec = 5)
{(0, 0): {0: 1, 1: 240, 2: 2160, 3: 6720, 4: 17520}}
sage: L.eisenstein_series(6, prec = 5)
{(0, 0): {0: 1, 1: -504, 2: -16632, 3: -122976, 4: -532728}}
sage: L.eisenstein_series(8, prec = 5)
{(0, 0): {0: 1, 1: 480, 2: 61920, 3: 1050240, 4: 7926240}}
sage: eis = _
sage: eis[(0,0)][1]
480
```

Lattices are always assumed to be $\mathbb{Z}^n$ and even. Different lattices are given by different Gram matrices. In our example, we have chosen a hyperbolic plane an recovered the usual Eisenstein series of weight 4, 6 and 8 respectively. Note that the program computes the Eisenstein series for the Weil representation using the formulas from [KY10]. The formulas in [BK01] will give you an Eisenstein series for the dual Weil representation. If we want to compute an Eisenstein series for the dual Weil representation, we can use the negative of our Gram matrix. The computed Eisenstein series

$$\sum_{\gamma \in L'/L} \sum_{\substack{n \in Q(\gamma)+\mathbb{Z} \\ n \geq 0}} a(\gamma, n) q^n e_\gamma$$

is given by the dictionary of the form

$$
\begin{array}{l}
\texttt{\{} \\
\quad \gamma_1\texttt{:} \quad \texttt{\{}n_{\gamma_1,1}\texttt{:} \quad a(\gamma_1, n_{\gamma_1,0})\texttt{,} \quad \ldots\texttt{,} \quad n_{\gamma_1,\texttt{prec}-1)}\texttt{:} \quad a(\gamma_1, n_{\gamma_1,\texttt{prec}-1})\texttt{\}} \\
\quad \texttt{,} \quad \ldots \texttt{,} \\
\quad \gamma_m\texttt{:} \quad \texttt{\{}n_{\gamma_m,1}\texttt{:} \quad a(\gamma_1, n_{\gamma_1,0})\texttt{,} \quad \ldots\texttt{,} \quad n_{\gamma_m,\texttt{prec}-1)}\texttt{:} \quad a(\gamma_m, n_{\gamma_m,\texttt{prec}-1})\texttt{\}} \\
\texttt{\}},
\end{array}
$$

where `prec` is the precision to which this Eisenstein series was computed. If `d` is this dictionary, then `d[`γ`][n]` will yield $a(\gamma, n)$. Using the class "EisensteinSeries", we can print the dictionary as a vector of q-series:

```
sage: L = Lattice(matrix(2,2,[0,1,1,0]))
sage: L.eisenstein_series(4, prec = 10)
{(0, 0): {0: 1,
  1: 240,
  2: 2160,
  3: 6720,
  4: 17520,
  5: 30240,
  6: 60480,
  7: 82560,
  8: 140400,
  9: 181680}}
sage: EisensteinSeries(_)

A vector valued q-series with coefficients
(0, 0): 1 q^(0) + 240 q^(1) + 2160 q^(2) + 6720 q^(3) + 17520
   q^(4) + 30240 q^(5) + 60480 q^(6) + 82560 q^(7) + 140400 q
   ^(8) + 181680 q^(9) + O(q^(10))

sage: L.eisenstein_series(6, prec = 10)
{(0, 0): {0: 1,
  1: -504,
  2: -16632,
  3: -122976,
  4: -532728,
  5: -1575504,
```

```
  6: -4058208,
  7: -8471232,
  8: -17047800,
  9: -29883672}}
sage: EisensteinSeries(_)

A vector valued q-series with coefficients
(0, 0): 1 q^(0) + -504 q^(1) + -16632 q^(2) + -122976 q^(3) +
   -532728 q^(4) + -1575504 q^(5) + -4058208 q^(6) + -8471232
   q^(7) + -17047800 q^(8) + -29883672 q^(9) + O(q^(10))
```

Note that we had to use different weights to get a meaningful result. This is due to a congruence condition on the weight and the signature of the lattice which is equally present in [KY10] and [BK01]. Since some of the computations done by the Lattice class are quite expensive, the class will save some data. This data can then be stored in a file from which the Lattice object can be restored. If no file name is given, the save method will return a StringIO from which the Lattice can also be recovered.

```
sage: m = matrix(2,2,[0,1,1,0])
sage: L = Lattice(m)
sage: s = L.save().getvalue()
sage: print s[s.find('lattice_data'):]
lattice_data=\
{\
    'gram_matrix' : \
        matrix([\
            [0, 1],\
            [1, 0],\
        ]),\
}
sage: LL = Lattice(s)
sage: LL
Lattice given by "Ambient free module of rank 2 over the
   principal ideal domain Integer Ring" endowed with the
   quadratic form "Quadratic form in 2 variables over Integer
   Ring with coefficients:
[ 0 1 ]
[ * 0 ]"
sage: LL == L
True
```

When loading from a file, the file is expected to have a .sage extension.

```
sage: f = open(tmp_filename(ext = '.sage'))
```

```
sage: L.save(f.name)
sage: LL = Lattice(f)
sage: L == LL
True
```

From within the Lattice class, we can compute the genus symbol of the lattice. This uses the FiniteQuadraticModule class from psage. The lattice can also access the L-functions and modular forms database [LMFDB] to find the lmfdb label of the lattice.

```
sage: L = Lattice(matrix(1,1,[2]))
sage: L.genus_symbol()
'2_1'
sage: L.lmfdb_label()
'1.2.4.1.1'
```

Some more functions are as follows.

```
sage: L.gram_matrix()
[2]
sage: L.dim()
1
sage: L.det()
2
sage: L.is_positive_definite()
True
sage: L.level()
4
sage: L.smith_form()
([2], [1], [1])
sage: for el in L.discriminant_form_iterator():
....:     print el
....:
(0)
(1/2)
sage: L.local_normal_form(2)
([2], [1], [1])
sage: L.p_excess(3)
0
sage: L.oddity()
1
sage: L.weil_index(2)
e^(1/4*I*pi)
sage: L.isometry_orbits()
[[(0)], [(1/2)]]
```

```
sage: L.eisenstein_series_by_orbits(5/2, prec = 5)
{0: {0: 1, 1: -70, 2: -120, 3: -240, 4: -550},
 1: {1/4: -10, 5/4: -48, 9/4: -250, 13/4: -240, 17/4: -480}}
```

The last two functions are particularly useful if the discriminant form of the lattice has large orbits. Since the q-expansion of the Eisenstein series is the same at any two elements belonging to the same orbit (with respect to isometries of the discriminant form), we only list this q-expansion once. The index of the q-expansion is the same as the index for the orbit in the list of orbits.

C. Test cases Siegel-Weil

The positive definite lattices with class number one which we used to test the computation of the Fourier coefficients of Eisenstein series for the Weil representation were obtained from [LMFDB]. They are given in the following tables, where we give the lmfdb label (rank, determinant, level, class number, some number), the genus symbol and the precision to which the theta and Eisenstein series were computed.

Rank 7 to 9

lmfdb label	genus symbol	precision	lmfdb label	genus symbol	precision
7.2.4.1.2	2_3^{-1}	7	7.64.32.1.1	$2^2.16_7$	4
7.4.8.1.2	4_7	7	7.128.16.1.3	$2_0^4.8_3^{-1}$	4
7.6.12.1.1	$2_1.3$	6	7.128.16.1.4	$4^2.8_3^{-1}$	4
7.8.4.1.1	2_7^3	6	7.128.16.1.5	$4^2.8_7$	4
7.8.16.1.1	8_7	6	8.4.2.1.1	2^2	5
7.8.16.1.2	8_3^{-1}	6	8.4.4.1.1	2_4^{-2}	5
7.8.16.1.3	8_7	6	8.5.5.1.1	5^{-1}	5
7.10.20.1.1	$2_3^{-1}.5^{-1}$	6	8.9.3.1.1	3^{-2}	5
7.12.24.1.1	$4_5^{-1}.3^{-1}$	5	8.9.9.1.1	9	4
7.16.8.1.2	$2^2.4_7$	5	8.16.2.1.1	2^4	4
7.16.32.1.1	16_7	5	8.16.4.1.1	2_0^4	4
7.18.36.1.1	$2_3^{-1}.9$	5	8.16.4.1.2	4^2	4
7.24.48.1.1	$8_5^{-1}.3$	5	8.16.4.1.3	4^{-2}	4
7.32.4.1.1	2_3^{-5}	5	8.64.4.1.2	$2^2.4^2$	3
7.32.4.1.2	$2_3^{-1}.4^2$	5	8.64.4.1.3	$2^2.4^{-2}$	3
7.32.16.1.1	$2_4^{-2}.8_7$	5	8.64.4.1.4	$2_4^{-2}.4^2$	3
7.32.16.1.2	$2^2.8_7$	5	9.2.4.1.1	2_1	4
7.32.16.1.3	$2^2.8_3^{-1}$	5	9.8.16.1.1	8_5^{-1}	4

Rank 6

lmfdb label	genus symbol	precision	lmfdb label	genus symbol	precision
6.3.3.1.1	3	11	6.48.6.1.1	$2^4.3$	7
6.4.4.1.1	2_6^2	10	6.48.6.1.2	$2^{-4}.3^{-1}$	7
6.7.7.1.1	7^{-1}	9	6.48.12.1.1	$2_0^4.3$	7
6.8.8.1.1	$2_3^{-1}.4_7$	9	6.48.12.1.2	$4^{-2}.3$	6
6.11.11.1.1	11	8	6.48.12.1.3	$4^2.3$	7
6.12.6.1.1	$2^2.3$	8	6.64.4.1.3	$2_6^2.4^2$	6
6.12.6.1.2	$2^{-2}.3^{-1}$	8	6.64.8.1.2	$2^{-2}.4_2^2$	6
6.12.12.1.1	$2_4^{-2}.3$	8	6.64.8.1.4	$2^{-2}.4_2^{-2}$	6
6.15.15.1.1	$3^{-1}.5$	8	6.64.8.1.7	$2_2^2.4_4^{-2}$	6
6.15.15.1.2	3.5^{-1}	8	6.64.16.1.1	$2_7^3.8_3^{-1}$	6
6.16.4.1.1	2_2^{-4}	8	6.64.16.1.2	8_6^2	6
6.16.8.1.3	4_6^2	8	6.64.16.1.3	8_2^{-2}	6
6.16.8.1.5	4_6^{-2}	8	6.108.6.1.1	$2^2.3^{-3}$	6
6.16.16.1.1	$2_3^{-1}.8_7$	8	6.112.14.1.1	$2^4.7^{-1}$	6
6.20.20.1.1	$2_6^2.5^{-1}$	8	6.128.32.1.1	$8_5^{-1}.16_5^{-1}$	5
6.23.23.1.1	23^{-1}	7	6.192.12.1.1	$2^2.4^2.3$	5
6.27.3.1.1	3^{-3}	7	6.192.12.1.2	$2^2.4^{-2}.3$	5
6.27.9.1.1	3.9	7	6.192.12.1.3	$2_4^{-2}.4^2.3$	5
6.27.9.1.2	3.9^{-1}	7	6.243.9.1.1	3.9^2	5
6.28.14.1.1	$2^2.7^{-1}$	7	6.243.9.1.2	3.9^{-2}	5
6.32.16.1.1	$4_7.8_7$	7	6.256.8.1.4	$2_6^2.8^2$	5
6.32.32.1.1	$2_1.16_5^{-1}$	7	6.256.16.1.3	$2_4^{-2}.8_6^2$	5
6.32.32.1.2	$2_3^{-1}.16_7$	7	6.432.6.1.1	$2^{-4}.3^3$	4

Rank 5

lmfdb label	genus symbol	precision	lmfdb label	genus symbol	precision
5.4.8.1.3	4_5^{-1}	19	5.40.80.1.1	$8_5^{-1}.5$	12
5.6.12.1.1	$2_3^{-1}.3$	17	5.48.24.1.1	$2^{-2}.4_7.3^{-1}$	11
5.8.4.1.1	2_1^{-3}	16	5.48.96.1.1	$16_7.3$	11
5.10.20.1.1	$2_1.5$	16	5.54.12.1.1	$2_3^{-1}.3^{-3}$	11
5.12.24.1.1	$4_7.3$	15	5.54.36.1.1	$2_3^{-1}.3.9$	11
5.12.24.1.3	$4_3^{-1}.3^{-1}$	15	5.54.36.1.2	$2_3^{-1}.3.9^{-1}$	11
5.14.28.1.1	$2_3^{-1}.7^{-1}$	15	5.56.112.1.1	$8_7.7^{-1}$	11
5.16.8.1.2	$2_2^2.4_3^{-1}$	14	5.64.8.1.3	4_5^3	11
5.16.8.1.7	$2^{-2}.4_1$	14	5.64.8.1.6	4_5^{-3}	11
5.16.32.1.1	16_5^{-1}	14	5.64.32.1.1	$2_2^2.16_3^{-1}$	11
5.18.12.1.1	$2_1.3^2$	14	5.64.32.1.2	$2_6^2.16_7$	11
5.20.40.1.1	$4_5^{-1}.5^{-1}$	14	5.64.32.1.3	$2_4^{-2}.16_5^{-1}$	11
5.22.44.1.1	$2_3^{-1}.11$	13	5.64.32.1.5	$2^2.16_5^{-1}$	11
5.24.12.1.1	$2_7^3.3$	13	5.72.48.1.1	$8_5^{-1}.3^2$	10
5.24.48.1.1	$8_3^{-1}.3$	13	5.96.12.1.1	$2_3^{-1}.4^2.3$	10
5.24.48.1.2	$8_7.3$	13	5.96.48.1.1	$2_4^{-2}.8_3^{-1}.3$	10
5.28.56.1.1	$4_7.7^{-1}$	13	5.96.48.1.2	$2^2.8_7.3$	10
5.30.60.1.1	$2_3^{-1}.3.5^{-1}$	13	5.96.48.1.3	$2^2.8_3^{-1}.3$	10
5.32.8.1.2	$2_1.4_4^{-2}$	12	5.108.24.1.1	$4_7.3^{-3}$	10
5.32.16.1.5	$2^{-2}.8_5^{-1}$	12	5.108.72.1.1	$4_7.3.9^{-1}$	10
5.32.16.1.6	$2^{-2}.8_1$	12	5.128.8.1.1	$2_1.8^{-2}$	9
5.32.16.1.7	$2_6^2.8_7$	12	5.128.16.1.4	$4_4^{-2}.8_1$	9
5.36.24.1.1	$4_5^{-1}.3^{-2}$	12	5.128.16.1.5	$4_4^{-2}.8_5^{-1}$	9
5.40.80.1.1	$8_5^{-1}.5$	12	5.128.16.1.7	$2_3^{-1}.8_6^2$	9
5.48.24.1.1	$2^{-2}.4_7.3^{-1}$	11	5.128.64.1.2	$2_6^2.32_7$	9
5.48.96.1.1	$16_7.3$	11	5.162.12.1.1	$2_1.3^{-4}$	8
5.54.12.1.1	$2_3^{-1}.3^{-3}$	11	5.216.48.1.1	$8_7.3^{-3}$	8
5.54.36.1.1	$2_3^{-1}.3.9$	11	5.216.144.1.1	$8_7.3.9$	8
5.54.36.1.2	$2_3^{-1}.3.9^{-1}$	11	5.256.8.1.5	$4_1.8^{-2}$	8
5.56.112.1.1	$8_7.7^{-1}$	11	5.256.32.1.1	$4^2.16_5^{-1}$	8
5.64.8.1.3	4_5^3	11	5.256.32.1.2	$4^{-2}.16_5^{-1}$	8
5.64.8.1.6	4_5^{-3}	11	5.256.32.1.4	$4_2^{-2}.16_3^{-1}$	8
5.64.32.1.1	$2_2^2.16_3^{-1}$	11	5.256.32.1.6	$4_2^2.16_3^{-1}$	8

List of Symbols

(D, Q)	A finite quadratic module, 10
(a, b)	$= \gcd(a, b)$, the greatest common divisor of a and b
$(a, b)_p$	the p-adic Hilbert symbol of a and b
(x, y)	A bilinear form, 9
$\mathbb{A}$	The adeles of $\mathbb{Q}$, 21
$a_f(\gamma, n)$	Fourier coefficients of f, 18
A_n	The A_n root lattice, 82
(b^+, b^-)	The type of a lattice of space, 10
$b^S(\kappa m, s)$	A function used to compute local densities, 43
$\mathbb{C}$	The field of complex numbers
χ	A Dirichlet character
$C_{k,d,N}$	A constant used to estimate Fourier coefficients of Eisenstein series, 57
D	A diagonal matrix of elementary divisors, 67
D/D_m	A finite quadratic module derived from D, 71
$\det(L)$	The determinant of the lattice L, 13
$\operatorname{disc}(L)$	The discriminant of the lattice L, 13
D^m	A finite quadratic module derived from D, 71
D_n	The D_n root lattice, 82
$\left(\frac{D}{p}\right)$	The Kronecker symbol
$e(x)$	$= e^{2\pi i x}$
$E_{\rho_L,k,\beta}$	The vector valued Eisenstein series for an isotropic β, 18

$E_{k,m}$	The Jacobi-Eisenstein series of weight k and index m, 19
$E_L(\tau, s; \ell)$	An Eisenstein series associated to L, 38
E_n	The E_n root lattice, 82
$G(c, D)$	A Gauss sum, 75
$\mathbb{H}$	The complex upper half-plane, $\mathbb{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$
H_L	A Heegner divisor, 83
$\Im(z)$	The imaginary part of z
k	Usually a half-integer (a weight)
κ	A square-free integer associated to the discriminant of L, 39
L	A lattice (usually even), 13
L'	The dual lattice of L, 13
L'/L	$= (L'/L, Q \pmod 1)$, the discriminant form of the lattice L, 14
$L(f)$	A lattice with quadratic form scaled by f, 13
L_p	The p-adic lattice associated to L, 14
$L^S(m\chi)$	A modified L-function, 43
N	The level of a lattice or finite quadratic module, 11
$N(D, j)$	A representation number, 32
$N_{\text{2-torsion}}(D, j)$	A representation number with regard to 2-torsions, 36
$N_{\gamma,n}(a)$	A representation number, 56
$N_{sq}(n, j)$	A square representation number $\pmod n$, 35
$O(D)$	The orhtogonal group of D, 70
$\mathcal{O}(D, p^{k+1}, v_1, \ldots, v_{p^k}, t_1, \ldots, t_{p^k})$	The description of an orbit of D, 72
oddity(D)	The oddity of D, 12
p	Usually a prime
p-excess(D)	The p-excess of D, 12
P_f	The principal part of f, 18

Q	A quadratic form, 9
q	Usually $q = e^{2\pi i\tau}$ or $q = p^l$
$q^{\pm n}$	A Jordan component for an odd prime, 11
$q_{II}^{\pm 2n}$	An even 2-adic Jordan component, 12
$q_t^{\pm n}$	An odd 2-adic Jordan component, 12
$\mathbb{Q}_p$	The field of p-adic numbers
$Q^{\mathrm{red}}(\gamma)$	The reduced norm of γ, 71
$\Re(z)$	The real part of z
ρ_L	The Weil representation associated to the lattice L, 17
ρ_L^*	The dual Weil representation associated to the lattice L, 17
S	Usually a set of “bad” primes or a Gram matrix, 43
$\sigma^S(m, \chi)$	A modified divisor sum, 43
$\operatorname{sign}(D)$	The signature of the finite quadratic module D, 12
$\operatorname{sign}(L)$	The signature of the lattice L, 14
$\operatorname{sign}(V)$	The signature of the real space V, 10
$\square$	A square
τ	$\tau = u + iv \in \mathbb{H}$
$\theta_{a,b}(Z)$	An even theta constant, 88
$\Theta_{\gamma+L}$	A theta series for the coset $\gamma + L$, 62
Θ_L	A vector valued theta series, 62
Θ_L^{sym}	A symmetrized vector valued theta series, 62
$\Theta_L^{sym,gen}$	A special weighted sum of symmetrized vector valued theta series, 62
$U(N)$	A scaled hyperbolic plane, 56
$U^\perp$	The orthogonal complement of U, 9
$v(\gamma)$	The multiplicity of γ, 71
$W_{m,S}(s)$	A product of local densities, 44

$W_{p,m,\mu,X}$	A Whittaker polynomial, 47
$W_p(s+s_0)$	A generalized local density or Whittaker function, 45
$\mathbb{Z}$	The ring of integers
$\mathbb{Z}_{>0}$	The set of positive integers
$\zeta^S(s)$	A modified zeta function, 43
$Z_f(s)$	The Igusa local zeta function, 25
$\mathbb{Z}_p$	The p-adic integers

Bibliography

[BEF16a] Jan H. Bruinier, Stephan Ehlen, and Eberhard Freitag. *Lattices with many Borcherds products, extended version*. [Online; accessed March 2018]. 2016. URL: https://github.com/sehlen/sfqm/blob/master/bruinier_ehlen_freitag_extended.pdf.

[BEF16b] Jan Hendrik Bruinier, Stephan Ehlen, and Eberhard Freitag. Lattices with many Borcherds products. In: *Math. Comp.* 85.300 (2016), pp. 1953–1981. DOI: 10.1090/mcom/3059.

[BF04] Jan Hendrik Bruinier and Jens Funke. On two geometric theta lifts. In: *Duke Math. J.* 125.1 (2004), pp. 45–90. DOI: 10.1215/S0012-7094-04-12513-8.

[BK01] Jan Hendrik Bruinier and Michael Kuss. Eisenstein series attached to lattices and modular forms on orthogonal groups. In: *Manuscripta Math.* 106.4 (2001), pp. 443–459. DOI: 10.1007/s229-001-8027-1.

[BM17] Jan Hendrik Bruinier and Martin Möller. Cones of Heegner divisors. In: *ArXiv e-prints* (May 2017). arXiv: 1705.05534 [math.AG].

[Bor98] Richard E. Borcherds. Automorphic forms with singularities on Grassmannians. In: *Invent. Math.* 132.3 (1998), pp. 491–562. DOI: 10.1007/s002220050232.

[Bru+08] Jan Hendrik Bruinier et al. *The 1-2-3 of modular forms*. Universitext. Lectures from the Summer School on Modular Forms and their Applications held in Nordfjordeid, June 2004, Edited by Kristian Ranestad. Springer-Verlag, Berlin, 2008, pp. x+266. DOI: 10.1007/978-3-540-74119-0.

[Bru02a] Jan H. Bruinier. *Borcherds products on O(2, l) and Chern classes of Heegner divisors*. Vol. 1780. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2002, pp. viii+152. DOI: 10.1007/b83278. URL: https://doi.org/10.1007/b83278.

[Bru02b] Jan Hendrik Bruinier. On the rank of Picard groups of modular varieties attached to orthogonal groups. In: *Compositio Math.* 133.1 (2002), pp. 49–63. DOI: 10.1023/A:1016357029843. URL: https://doi.org/10.1023/A:1016357029843.

[Bun01] Michael Bundschuh. Über die Endlichkeit der Klassenzahl gerader Gitter der Signatur (2,n) mit einfachem Kontrollraum. Inaugural-Dissertation. Heidelberg: Ruprecht-Karls-Universität Heidelberg, 2001.

[Bun98] Michael Bundschuh. Über die Existenz von Modulformen zu orthogonalen Gruppen mit vorgegebnenen Null- und Polstellen. Diplomarbeit. Ruprecht-Karls-Universität Heidelberg, 1998.

[BY09] Jan Hendrik Bruinier and Tonghai Yang. Faltings heights of CM cycles and derivatives of L-functions. In: *Invent. Math.* 177.3 (2009), pp. 631–681. DOI: 10.1007/s00222-009-0192-8.

[Cas78] J. W. S. Cassels. *Rational quadratic forms.* Vol. 13. London Mathematical Society Monographs. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York, 1978, pp. xvi+413.

[CKW17] Raemeon A. Cowan, Daniel J. Katz, and Lauren M. White. A new generating function for calculating the Igusa local zeta function. In: *Advances in Mathematics* 304 (2017), pp. 355–420. DOI: https://doi.org/10.1016/j.aim.2016.09.003.

[Cox13] David A. Cox. *Primes of the form* $x^2 + ny^2$. Second. Pure and Applied Mathematics (Hoboken). Fermat, class field theory, and complex multiplication. John Wiley & Sons, Inc., Hoboken, NJ, 2013, pp. xviii+356. DOI: 10.1002/9781118400722.

[CS99] J. H. Conway and N. J. A. Sloane. *Sphere packings, lattices and groups.* Third. Vol. 290. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. With additional contributions by E. Bannai, R. E. Borcherds, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen and B. B. Venkov. Springer-Verlag, New York, 1999, pp. lxxiv+703. DOI: 10.1007/978-1-4757-6568-7.

[Dei10] Anton Deitmar. *Automorphe Formen.* Springer, Heidelberg, 2010, pp. viii+250. DOI: 10.1007/978-3-642-12390-0.

[Den84] J. Denef. The rationality of the Poincaré series associated to the p-adic points on a variety. In: *Invent. Math.* 77.1 (1984), pp. 1–23. DOI: 10.1007/BF01389133.

[Den91] Jan Denef. Report on Igusa's local zeta function. In: *Astérisque* 201-203 (1991). Séminaire Bourbaki, Vol. 1990/91, Exp. No. 741, 359–386 (1992).

[DHS15] Moritz Dittmann, Heike Hagemeier, and Markus Schwagenscheidt. Automorphic products of singular weight for simple lattices. In: *Math. Z.* 279.1-2 (2015), pp. 585–603. DOI: 10.1007/s00209-014-1383-6.

[Dit18] Moritz Dittmann. Reflective automorphic products of squarefree level. In: *Trans. Amer. Math. Soc.* (to appear 2018).

[Ehl16] Stephan Ehlen. *Finite Quadratic Modules and Simple Lattices.* [Online; accessed 2016-2018]. 2016. URL: https://github.com/sehlen/sfqm.

[EZ85] Martin Eichler and Don Zagier. *The theory of Jacobi forms*. Vol. 55. Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 1985, pp. v+148. DOI: 10.1007/978-1-4684-9162-3.

[Fre83] E. Freitag. *Siegelsche Modulfunktionen*. Vol. 254. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1983, pp. x+341. DOI: 10.1007/978-3-642-68649-8.

[Kit93] Yoshiyuki Kitaoka. *Arithmetic of quadratic forms*. Vol. 106. Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 1993, pp. x+268. DOI: 10.1017/CBO9780511666155.

[KK07] Max Koecher and Aloys Krieg. *Elliptische Funktionen und Modulformen*. revised. Springer-Verlag, Berlin, 2007, pp. viii+331.

[Kne02] Martin Kneser. *Quadratische Formen*. Revised and edited in collaboration with Rudolf Scharlau. Springer-Verlag, Berlin, 2002, pp. viii+164. DOI: 10.1007/978-3-642-56380-5.

[KRY06] Stephen S. Kudla, Michael Rapoport, and Tonghai Yang. *Modular forms and special cycles on Shimura curves*. Vol. 161. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2006, pp. x+373. DOI: 10.1515/9781400837168.

[Kud03] Stephen S. Kudla. Integrals of Borcherds forms. In: *Compositio Math.* 137.3 (2003), pp. 293–349. DOI: 10.1023/A:1024127100993.

[KY10] Stephen S. Kudla and TongHai Yang. Eisenstein series for SL(2). In: *Sci. China Math.* 53.9 (2010), pp. 2275–2316. DOI: 10.1007/s11425-010-4097-1.

[Lip08] Denis Lippolt. Thetanullwerte 2. Grades als Borcherds-Produkte. Diplomarbeit. Universität Heidelberg, 2008.

[LMFDB] The LMFDB Collaboration. *The L-functions and Modular Forms Database*. [Online; accessed 2014-2018]. 2018. URL: http://www.lmfdb.org.

[Nik79] V. V. Nikulin. Integer symmetric bilinear forms and some of their geometric applications. In: *Izv. Akad. Nauk SSSR Ser. Mat.* 43.1 (1979). English translation: Math USSR-Izv. 14(1):103–167, 1980., pp. 111–177, 238.

[Opi13] Sebastian Opitz. Diskriminantenformen und vektorwertige Modulformen. Masterarbeit. Technische Universität Darmstadt, 2013.

[Opi18] Sebastian Opitz. *Even Lattices and Vector Valued Eisenstein Series*. 2018. DOI: 10.5281/zenodo.1464927. URL: https://github.com/s-opitz/eisenstein_series.

[OS18] S. Opitz and M. Schwagenscheidt. Holomorphic Borcherds products of singular weight for simple lattices of arbitrary level. In: *ArXiv e-prints* (Oct. 2018). arXiv: 1810.06290 [math.NT].

[PSAGE] The PSAGE Developers. *The PSAGE library.* [Online; accessed 2013-2018]. 2018. URL: `https://github.com/fredstro/psage`.

[Ros15] Maximilian Rössler. Hecke Operators and Vector Valued Modular Forms. Masters Thesis. Technische Universität Darmstadt, 2015.

[SageMath] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 8.3).* 2018. URL: `http://www.sagemath.org`.

[Sch06] Nils R. Scheithauer. On the classification of automorphic products and generalized Kac-Moody algebras. In: *Invent. Math.* 164.3 (2006), pp. 641–678. DOI: `10.1007/s00222-006-0500-5`.

[Sch09] Nils R. Scheithauer. The Weil representation of $SL_2(\mathbb{Z})$ and some applications. In: *Int. Math. Res. Not. IMRN* 8 (2009), pp. 1488–1545. DOI: `10.1093/imrn/rnn166`.

[Sch13] Nils R. Scheithauer. *Discriminant forms and their automorphisms.* In preparation, 2013.

[Sch17] Nils R. Scheithauer. Automorphic products of singular weight. In: *Compos. Math.* 153.9 (2017), pp. 1855–1892. DOI: `10.1112/S0010437X17007266`.

[Sch18] Markus Schwagenscheidt. Eisenstein series for the Weil representation. In: *J. Number Theory* 193 (2018), pp. 74–90. DOI: `10.1016/j.jnt.2018.05.014`.

[Ser73] J.-P. Serre. *A course in arithmetic.* Translated from the French, Graduate Texts in Mathematics, No. 7. Springer-Verlag, New York-Heidelberg, 1973, pp. viii+115.

[Sie35] Carl Ludwig Siegel. Über die analytische Theorie der quadratischen Formen. In: *Ann. of Math. (2)* 36.3 (1935), pp. 527–606. DOI: `10.2307/1968644`.

[Str13] Fredrik Strömberg. Weil representations associated with finite quadratic modules. In: *Math. Z.* 275.1-2 (2013), pp. 509–527. DOI: `10.1007/s00209-013-1145-x`.

[Wei64] André Weil. Sur certains groupes d'opérateurs unitaires. In: *Acta Math.* 111 (1964), pp. 143–211. DOI: `10.1007/BF02391012`.

[Wila] Brandon Williams. *PSS examples.* [Online; accessed August 2018]. URL: `https://math.berkeley.edu/~btw/PSS-documentation.pdf`.

[Wilb] Brandon Williams. *PSS.sagews.* [Online; accessed June 2018]. URL: `https://math.berkeley.edu/~btw/PSS.sagews`.

[Wil18a] Brandon Williams. Poincaré square series for the Weil representation. In: *The Ramanujan Journal* (Mar. 20, 2018). DOI: `10.1007/s11139-017-9986-2`.

[Wil18b] Brandon Williams. Poincaré square series of small weight. In: *The Ramanujan Journal* (May 4, 2018). DOI: `10.1007/s11139-018-0002-2`.

[Wil18c] Brandon Williams. Vector-valued Eisenstein series of small weight. In: *Int. J. Number Theory* (Sept. 2018). DOI: `10.1142/S1793042119500118`.

[Zag81] D. B. Zagier. *Zetafunktionen und quadratische Körper*. Eine Einführung in die höhere Zahlentheorie. [An introduction to higher number theory], Hochschultext. [University Text]. Springer-Verlag, Berlin-New York, 1981, pp. viii+144.

[Zem15] Shaul Zemel. A p-adic approach to the Weil representation of discriminant forms arising from even lattices. In: *Ann. Math. Qué.* 39.1 (2015), pp. 61–89. DOI: 10.1007/s40316-015-0034-6.

Wissenschaftlicher Werdegang

Juni 2007	Allgemeine Hochschulreife
Oktober 2007 - Mai 2013	Studium der Mathematik (Nebenfach Informatik) an der Technischen Universität Darmstadt
Mai 2013	M.Sc. Mathematik Titel der Masterarbeit: „Diskriminantenformen und vektorwertige Modulformen“
seit Oktober 2013	Wissenschaftlicher Mitarbeiter am Fachbereich Mathematik der Technischen Universität Darmstadt, zeitweise Mitglied der DFG-Forschergruppe „Symmetrie, Geometrie und Arithmetik“
18. Oktober 2018	Einreichung der Dissertation an der Technischen Universität Darmstadt
27. November 2018	Mündliche Doktorprüfung Gesamturteil: „sehr gut bestanden“