Realizing Hyperbolic and Elliptic Eisenstein Series as Regularized Theta Lifts

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Abstract

The classical parabolic Eisenstein series is a non-holomorphic modular form of weight 0, which is associated to a cusp of a given Fuchsian group of the first kind. Recently, hyperbolic and elliptic analogs of parabolic Eisenstein series were studied by Jorgenson, Kramer and von Pippich. These are non-holomorphic modular forms of weight 0, which are associated to a geodesic or a point of the complex upper half-plane, respectively. In particular, Kronecker limit type formulas were investigated for elliptic Eisenstein series.

In the present thesis we show that hyperbolic and elliptic Eisenstein series for Hecke congruence subgroups can be realized as regularized theta lifts of non-holomorphic Poincaré series of Selberg type. More precisely, we present three different lifting results. Firstly, averaged versions of hyperbolic, parabolic and elliptic Eisenstein series are obtained as the regularized Borcherds lift of Selberg's Poincaré series in signature $(2, 1)$. Here the type of the Eisenstein series is solely determined by the sign of the index of the Poincaré series. Secondly, we realize a certain hyperbolic kernel function as a regularized Borcherds lift of a modified version of Selberg's Poincaré series in signature $(2, 2)$. Using known relations between this hyperbolic kernel function, and hyperbolic and elliptic Eisenstein series, we obtain realizations of the latter functions in terms of the mentioned Borcherds lift. Thirdly, we show that using a new Maass-Selberg type of Poincaré series as an input for the regularized Borcherds lift in signature $(2, 2)$, we obtain individual elliptic Eisenstein series.

In the final two chapters of this work we present a detailed study of the meromorphic continuation of Selberg's Poincaré series in the case of signature $(2, 1)$. Evaluating this continuation at a special harmonic point, we can express the linear term in the Laurent expansion of averaged hyperbolic, parabolic and elliptic Eisenstein series at this point in terms of certain Borcherds products. This method enables us to generalize known Kronecker limit formulas in the elliptic case to higher levels, and to establish new Kronecker limit formulas for hyperbolic Eisenstein series associated to infinite geodesics.

Zusammenfassung

Die klassische parabolische Eisensteinreihe ist eine nicht-holomorphe Modulform vom Gewicht 0, welche zu einer Spitze einer gegebenen Fuchsschen Gruppe der ersten Art assoziiert ist. Vor einiger Zeit wurden hyperbolische und elliptische Analoga von parabolischen Eisensteinreihen von Jorgenson, Kramer und von Pippich untersucht. Dabei handelt es sich ebenfalls um nicht-holomorphe Modulformen vom Gewicht 0, welche jedoch zu Geodäten oder Punkten in der komplexen oberen Halbebene assoziiert sind. In diesem Zusammenhang wurden insbesondere Kroneckersche Grenzformeln für elliptische Eisensteinreihen studiert.

In der vorliegenden Arbeit zeigen wir, dass hyperbolische und elliptische Eisensteinreihen zu Hecke-Kongruenzuntergruppen als regularisierte Thetaliftungen von bestimmten nicht-holomorphen Poincaré-Reihen dargestellt werden können. Dazu geben wir drei unterschiedliche Liftungsresultate an. Als erstes realisieren wir gemittelte hyperbolische, parabolische und elliptische Eisensteinreihen als regularisierten Borcherdslift der Selbergschen Poincaré-Reihe in Signatur $(2,1)$. Der Typ der Eisensteinreihe wird dabei einzig vom Vorzeichen des Index der Poincaré-Reihe bestimmt. Anschließend stellen wir eine bestimmte hyperbolische Kernfunktion als regularisierten Borcherdslift einer modifizierten Selbergschen Poincaré-Reihe in Signatur $(2,2)$ dar. Unter Zuhilfenahme von bekannten Relationen zwischen der genannten Kernfunktion sowie hyperbolischen und elliptischen Eisensteinreihen erhalten wir dadurch Darstellungen dieser Eisensteinreihen in Termen des entsprechenden Borcherdslifts. Schlussendlich zeigen wir noch, dass sich eine einzelne elliptische Eisensteinreihe auch als regularisierter Borcherdslift einer neuen Maass-Selbergschen Poincaré-Reihe in Signatur $(2,2)$ realisieren lässt.

In den letzten beiden Kapiteln dieser Arbeit präsentieren wir schließlich eine detaillierte Untersuchung der meromorphen Fortsetzung der Selbergschen Poincaré-Reihe in Signatur $(2,1)$. Indem wir diese Fortsetzung an einem speziellen harmonischen Punkt auswerten, können wir den linearen Term in der Laurententwicklung von gemittelten hyperbolischen, parabolischen und elliptischen Eisensteinreihen an diesem Punkt durch gewisse Borcherdsprodukte ausdrücken. Mithilfe dieser Methode lassen sich bekannte Kroneckersche Grenzformeln elliptischer Eisensteinreihen für höhere Stufen verallgemeinern, sowie neue Kroneckersche Grenzformeln hyperbolischer Eisensteinreihen zu unendlichen Geodäten beweisen.

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Contents

1 Introduction

This thesis is concerned with the study of certain complex valued functions having lots of symmetries, called modular forms. In the famous popular science book [Sin97] it is written that:

"Modular forms are some of the weirdest and most wonderful objects in mathematics. They are one of the most esoteric entities in mathematics and yet the 20th century number theorist Martin Eichler rated them as one of the 5 fundamental operations: addition, subtraction, multiplication, division and modular forms."

Even though it is questionable whether modular forms can be regarded to be as fundamental as basic addition and multiplication, the quote nevertheless emphasizes the key role modular forms play in modern number theory.

1.1 Modular forms and Maass forms

One of the simplest examples of a modular form is the classical holomorphic Eisenstein series of weight k, which is given by the infinite sum

$$E_k(\tau) := \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} \frac{1}{(c\tau + d)^k} \tag{1.1.1}$$

for $\tau \in \mathbb{H}$, where $\mathbb{H}$ denotes the complex upper half-plane. For $k \geq 4$ an even integer the above sum defines a holomorphic function, which is 1-periodic and satisfies the identity $\tau^k E_k(\tau) = E_k(-1/\tau)$, i.e., E_k is modular of weight k with respect to the modular group $\mathrm{SL}_2(\mathbb{Z})$, which is generated by the two elements $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ corresponding to the mentioned symmetries of E_k. Moreover, one can show that $E_k(\tau)$ tends to 1 as τ goes to $i\infty$, i.e., E_k is holomorphic at ∞. Together, these properties show that E_k is a holomorphic modular form of weight k for the group $\mathrm{SL}_2(\mathbb{Z})$.

Holomorphic Eisenstein series are well understood and play a fundamental role in the theory of classical modular forms. For example, the (finite dimensional) vector space of holomorphic modular forms of weight k for $\mathrm{SL}_2(\mathbb{Z})$ is generated by products of Eisenstein series. Further, in the Fourier expansion of E_k we discover the Riemann zeta function $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$ and the divisor sums $\sigma_s(n) := \sum_{d \mid n} d^s$, namely

$$E_k(\tau) = 1 + \frac{(2\pi i)^k}{(k-1)!\,\zeta(k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n \tau}$$

for $\tau \in \mathbb{H}$. Using relations between Eisenstein series of different weight, which are clear in the context of modular forms, one can for example deduce highly non-trivial identities between divisor sums.

Instead of defining a function by summing over powers of linear polynomials as in (1.1.1), one can also sum over powers of quadratic polynomials. This idea was realized by D. Zagier in [Zag75b], where he introduced modular forms associated to positive discriminants. More generally, given a discriminant $\Delta \in \mathbb{Z}$ one defines

$$f_{k,\Delta}(\tau) := \sum_{\substack{Q \in \mathcal{Q}_\Delta \\ Q \neq 0}} \frac{1}{Q(\tau,1)^k} \tag{1.1.2}$$

for $k \in \mathbb{Z}$ with $k \geq 2$ and $\tau \in \mathbb{H}$ with $Q(\tau,1) \neq 0$ for all $Q \in \mathcal{Q}_\Delta$. Here $\mathcal{Q}_\Delta$ denotes the set of integral binary quadratic forms of discriminant Δ. Zagier shows that if $\Delta > 0$ then $f_{k,\Delta}$ is a holomorphic modular form of weight $2k$, which vanishes at ∞ in the sense that $f_{k,\Delta}(\tau) \to 0$ as $\tau \to i\infty$, i.e., $f_{k,\Delta}$ is a cusp form of weight $2k$. Moreover, if $\Delta = 0$ then $f_{k,\Delta}$ is essentially the holomorphic Eisenstein series of weight $2k$, and if $\Delta < 0$ then $f_{k,\Delta}$ is a cusp form, which is not holomorphic, but meromorphic on $\mathbb{H}$ with poles exactly at the so-called Heegner (or CM) points of discriminant Δ, i.e., at the zeros of the quadratic forms in $\mathcal{Q}_\Delta$. Here the latter functions $f_{k,\Delta}$ with $\Delta < 0$ have only recently been studied by Bengoechea, a PhD student of Zagier (see [Ben13] and [Ben15]).

For positive discriminants the cusp forms $f_{k,\Delta}$ appear as the Fourier coefficients of the holomorphic kernel function of the Shimura and Shintani lift between half-integral and integral weight cusp forms (see [KZ81] and [Koh85]). In other words, for $k \in \mathbb{Z}$ with $k \geq 2$ and $\Delta > 0$ the Shimura theta lift of the holomorphic Poincaré series of weight $k + 1/2$ and index Δ is essentially given by Zagier's cusp form $f_{k,\Delta}$ (see for example [Oda77] and again [KZ81]). In the language of vector valued modular forms for the Weil representation we can write this result as

$$\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \left\langle P^L_{k+1/2,m}(\tau), \Theta_{L,k}(\tau,z)\right\rangle \mathrm{Im}(\tau)^{k+1/2} d\mu(\tau) = \frac{2(k-1)!}{\pi^k} f_{k,4m}(z) \tag{1.1.3}$$

for $z \in \mathbb{H}$ and $m \in \mathbb{Z}$. Here L is a certain even lattice of signature $(2,1)$ and level 4 and $\Theta_{L,k}(\tau,z)$ is the associated vector valued Shintani theta function, which is modular of weight $k+1/2$ for the Weil representation ρ_L in the variable τ, and its complex conjugate is modular of weight $2k$ in z (see Section 4.3 for details on the lattice and the corresponding theta function). Moreover, $P^L_{k+1/2,m}$ denotes the vector valued holomorphic Poincaré series of weight $k+1/2$ and index m for ρ_L, and $f_{k,4m}$ denotes the associated Zagier cusp form of weight $2k$ and discriminant $4m$. We note that if $m \leq 0$ the integral on the right-hand side of (1.1.3) does in fact not converge, and thus needs to be regularized.

There are various generalizations of modular forms, such as the vector valued modular forms for the Weil representation mentioned above, which provide a framework to incorporate different types of modular forms using the language of lattices. More classically, H. Maass introduced so-called Maass forms in 1949 (see [Maa49]). In contrast to the above modular forms, Maass forms need not be holomorphic on $\mathbb{H}$ or at ∞. Instead, they are smooth eigenfunctions of the hyperbolic Laplace operator

$$\Delta_0 := -v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right),$$

which are modular of weight 0, and grow at most polynomially in v as $\tau \to i\infty$. Here and in the following we use the notation $\tau = u + iv$ for $\tau \in \mathbb{H}$. One of the best-known Maass forms is the non-holomorphic Eisenstein series

$$E_\infty(\tau, s) := \frac{1}{2} \sum_{\substack{c,d\in\mathbb{Z}\\(c,d)=1}} \frac{v^s}{|c\tau + d|^{2s}}, \tag{1.1.4}$$

which is a priori defined for $\tau \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. The function $E_\infty(\tau, s)$ is modular of weight 0 in the variable τ and satisfies the differential equation

$$\Delta_0 E_\infty(\tau, s) = s(1 - s)E_\infty(\tau, s), \tag{1.1.5}$$

i.e., the non-holomorphic Eisenstein series $E_\infty(\tau, s)$ is an eigenfunction of the hyperbolic Laplace operator with eigenvalue $s(1 - s)$. Moreover, the Eisenstein series $E_\infty(\tau, s)$ is holomorphic in the parameter s for $\mathrm{Re}(s) > 1$, and it has a meromorphic continuation in s to the whole complex plane with a simple pole at $s = 1$. The Laurent expansion of $E_\infty(\tau, s)$ at this pole is given by the famous Kronecker limit formula, which states that

$$E_\infty(\tau, s) = \frac{3/\pi}{s-1} - \frac{1}{2\pi} \log\left|\Delta(\tau)\,\mathrm{Im}(\tau)^6\right| + \frac{6 - 72\zeta'(-1) - 6\log(4\pi)}{\pi} + O(s - 1)$$

as $s \to 1$. Here we surprisingly spot a holomorphic modular form, namely the unique normalized cusp form $\Delta(\tau)$ of weight 12 for $\mathrm{SL}_2(\mathbb{Z})$. Using the functional equation of the non-holomorphic Eisenstein series, relating s and $1 - s$, one obtains the cleaner Laurent expansion

$$E_\infty(\tau, s) = 1 + \log\left|\Delta(\tau)^{1/6}\,\mathrm{Im}(\tau)\right| \cdot s + O(s^2) \tag{1.1.6}$$

at $s = 0$. We note that it is natural to study the Laurent expansion of $E_\infty(\tau, s)$ at the special points $s = 0$ and $s = 1$, as by (1.1.5) these are exactly the points where the non-holomorphic Eisenstein series is harmonic with respect to Δ_0. Eventually, we also remark that the non-holomorphic Eisenstein series $E_\infty(\tau, s)$ plays an important role in the spectral theory of the hyperbolic Laplace operator Δ_0, where it is used to represent the continuous spectrum of Δ_0.

1.2 Non-holomorphic Eisenstein series

Up to now we have only worked with the full modular group $\mathrm{SL}_2(\mathbb{Z})$, i.e., we have considered functions which are modular of weight k with respect to $\mathrm{SL}_2(\mathbb{Z})$. Here the corresponding Riemann surface $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ can be compactified by adding one point at ∞, which we call cusp. More generally, given a Fuchsian group of the first kind $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ the corresponding Riemann surface $\Gamma\backslash\mathbb{H}$ can possibly have finitely many cusps. If there is no cusp then the quotient $\Gamma\backslash\mathbb{H}$ is compact.

For every cusp p of the Riemann surface $\Gamma\backslash\mathbb{H}$ there is an associated non-holomorphic Eisenstein series, which we can write as

$$E_p^{\mathrm{par}}(\tau, s) := \sum_{M\in\Gamma_p\backslash\Gamma} \mathrm{Im}(\sigma_p^{-1}M\tau)^s, \tag{1.2.1}$$

for $\tau \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, where Γ_p is the stabilizer of the cusp p in Γ and σ_p is a so-called scaling matrix for the cusp p. Here we introduce the additional superscript "par", and we call $E_p^{\mathrm{par}}(\tau, s)$ the parabolic Eisenstein series associated to the cusp or parabolic fixed point p, in order to distinguish it from the hyperbolic and elliptic Eisenstein series we are going to define in the following. We further note that if $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ then $E_\infty^{\mathrm{par}}(\tau, s)$ is simply the non-holomorphic Eisenstein series from (1.1.4). As before, non-holomorphic Eisenstein series are elementary (non-trivial) examples of Maass forms for the group Γ, and they are still fundamental for the study of the spectral decomposition of the corresponding hyperbolic Laplace operator Δ_0. However, if the quotient $\Gamma\backslash\mathbb{H}$ is compact there are no non-holomorphic Eisenstein series in the above sense as there are no cusps.

Addressing this issue, Kudla and Millson in 1979 introduced so-called hyperbolic Eisenstein series in their work [KM79], which can be understood as non-holomorphic Eisenstein series associated to geodesics of the upper half-plane instead of cusps. These hyperbolic Eisenstein series defined by Kudla and Millson are 1-forms, which correspond to functions modular of weight 2. In fact, Kudla and Millson remark that scalar valued weight 2 analogs of their hyperbolic Eisenstein series have already been studied by Petersson in [Pet43]. Even though the non-compact case is dealt with in Section 5 of [KM79], Kudla and Millson focus on the case that the Riemann surface $\Gamma\backslash\mathbb{H}$ is compact. In particular, they only work with geodesics whose image in $\Gamma\backslash\mathbb{H}$ is closed.

More recently, in 2004 scalar valued weight 0 analogs of Kudla's and Millson's hyperbolic Eisenstein series were introduced in [Ris04] for the case that the Riemann surface $\Gamma\backslash\mathbb{H}$ is compact, and in 2005 von Pippich defined hyperbolic Eisenstein series for general Fuchsian groups of the first kind associated to closed geodesics in her Diploma thesis [Pip05] (see also [JKP10]). More precisely, given a geodesic c the hyperbolic Eisenstein series associated to c is given by

$$E_c^{\mathrm{hyp}}(\tau, s) = \sum_{M \in \Gamma_c \backslash \Gamma} \cosh(d_{\mathrm{hyp}}(M\tau, c))^{-s}$$

for $\tau \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. Here we assume that the geodesic c is either closed in $\Gamma\backslash\mathbb{H}$, or it connects two cusps of the Riemann surface $\Gamma\backslash\mathbb{H}$. In the latter case we call c an infinite geodesic. The stabilizer Γ_c of the (oriented) geodesic c in Γ is infinite cyclic if c is closed, and trivial otherwise. Further, the function $d_{\mathrm{hyp}}(\tau, c)$ denotes the hyperbolic distance between some point $\tau \in \mathbb{H}$ and the given geodesic c in the upper half-plane.

As in the classical parabolic case the hyperbolic Eisenstein series $E_c^{\mathrm{hyp}}(\tau, s)$ is smooth and modular of weight 0 in the variable τ, and holomorphic in s for $\mathrm{Re}(s) > 1$. However, $E_c^{\mathrm{hyp}}(\tau, s)$ is not a Maass form, since it is not an eigenfunction of the hyperbolic Laplace operator, but satisfies the shifted Laplace equation

$$\Delta_0 E_c^{\mathrm{hyp}}(\tau, s) = s(1-s)E_c^{\mathrm{hyp}}(\tau, s) + s^2 E_c^{\mathrm{hyp}}(\tau, s+2).$$

Establishing the meromorphic continuation of the hyperbolic Eisenstein series $E_c^{\mathrm{hyp}}(\tau, s)$ in s, one can study its behaviour at the special point $s = 0$, where the hyperbolic Eisenstein is supposed to be harmonic according to the above differential equation. In the case that the geodesic c is closed, this question has been addressed in [JKP10], where the meromorphic continuation in s to all of $\mathbb{C}$ is established, and where the authors show that

$$E_c^{\mathrm{hyp}}(\tau, s) = O(s^2) \tag{1.2.2}$$

as $s \to 0$. Yet, in the case that the geodesic c is infinite, not much is known. In particular, the techniques from [JKP10] cannot be directly applied in this case, since the corresponding hyperbolic Eisenstein series is not square-integrable anymore.

In addition to the parabolic and hyperbolic Eisenstein series defined above, Jorgenson and Kramer also introduced elliptic analogs of these non-holomorphic Eisenstein series in their unpublished work [JK04] (see also [JK11]). These so-called elliptic Eisenstein series are associated to elliptic and more general to arbitrary points in the upper half-plane, instead of being associated to geodesics or cusps. More precisely, given $\omega \in \mathbb{H}$ the elliptic Eisenstein series associated to ω is given by

$$E_\omega^{\mathrm{ell}}(\tau, s) = \sum_{M \in \Gamma_\omega \backslash \Gamma} \sinh(d_{\mathrm{hyp}}(M\tau, \omega))^{-s}$$

for $\tau \in \mathbb{H} \setminus \Gamma\omega$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. In contrast to the parabolic and the hyperbolic case, elliptic Eisenstein series are not defined for all $\tau \in \mathbb{H}$, but they have singularities at the Γ-translates of the given point ω.

Following the idea of Jorgenson and Kramer, elliptic Eisenstein series were studied in great detail for arbitrary Fuchsian groups of the first kind by Kramer's student von Pippich in her PhD thesis [Pip10]. In particular, it is shown that the elliptic Eisenstein series $E_\omega^{\mathrm{ell}}(\tau, s)$ is smooth in τ whenever it is defined, modular of weight 0 in τ, and holomorphic in s for $\mathrm{Re}(s) > 1$. Further, $E_\omega^{\mathrm{ell}}(\tau, s)$ satisfies the differential equation

$$\Delta_0 E_\omega^{\mathrm{ell}}(\tau, s) = s(1-s) E_\omega^{\mathrm{ell}}(\tau, s) - s^2 E_\omega^{\mathrm{ell}}(\tau, s+2),$$

which, as in the parabolic and the hyperbolic case, motivates to study its meromorphic continuation to the special harmonic point $s = 0$. This has also been worked out by von Pippich in her thesis [Pip10], where it is shown that the elliptic Eisenstein series $E_\omega^{\mathrm{ell}}(\tau, s)$ has a meromorphic continuation in s to all of $\mathbb{C}$, and where the Laurent expansion of this continuation at $s = 0$ is given in a very explicit form. Since it is slightly technical, we present the Laurent expansion for the special case $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ given in [Pip16], namely,

$$E_\omega^{\mathrm{ell}}(\tau, s) = -\log\left(|j(\tau) - j(\omega)|^{2/|\Gamma_\omega|}\right) \cdot s + O(s^2) \tag{1.2.3}$$

as $s \to 0$. As in the parabolic case the linear term in the Laurent expansion of the Eisenstein series at $s = 0$ turns out to be the logarithm of the absolute value of some classical modular object, namely, we discover the well-known modular j-function $j(\tau)$, which is the unique normalized weakly holomorphic modular form of weight 0 for $\mathrm{SL}_2(\mathbb{Z})$ with a simple pole at ∞.

1.3 Regularized theta lifts

In the following we quickly explain the concept of a theta lift as it is used in this thesis. Generally, a theta lift is an operator which maps or "lifts" modular objects of one type to modular objects of another type, by using some theta function as an integral kernel. Here, given an even lattice (L, q) of signature $(2, n)$ and a non-negative integer k, we use a variant of the theta lift given in [Bor98], which maps vector valued modular forms of weight $1+k-n/2$ for the Weil representation associated to the given lattice L, to modular forms of weight k for the orthogonal group of L. The corresponding theta kernel $\Theta_{L,k}(\tau, Z)$ with $\tau \in \mathbb{H}$ and Z being an element of the generalized upper half-plane $\mathbb{H}_n$ induced by the lattice L, can be regarded as a vector valued version of the non-holomorphic Shintani theta function. Further, for $k = 0$ the theta function $\Theta_L(\tau, Z) := \Theta_{L,0}(\tau, Z)$ is simply a vector valued version of the classical Siegel theta function.

For an introduction to the theory of vector valued modular forms for the Weil representation and to modular forms for orthogonal groups we refer to Section 3.4 and Section 4.1 of this work, respectively. Moreover, for the precise definition of the theta function $\Theta_{L,k}(\tau, Z)$ we refer to Section 4.2.

More formally, the theta lift used in this work is given as follows. Let $F\colon \mathbb{H} \to \mathbb{C}[L'/L]$ be modular of weight $1 + k - n/2$ for the Weil representation ρ_L, where $\mathbb{C}[L'/L]$ denotes the group algebra of the finite abelian group L'/L. Then the regularized theta lift of F is given by

$$\Phi_k(Z; F) := \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}}^{\mathrm{reg}} \langle F(\tau), \Theta_{L,k}(\tau, Z)\rangle \, \mathrm{Im}(\tau)^{1+k-n/2} d\mu(\tau) \tag{1.3.1}$$

for $Z \in \mathbb{H}_n$, whenever the integral on the right-hand side exists. Here the superscript "reg" means that the integral is regularized in the sense of Borcherds (see [Bor98] or equivalently equation (3.4.4) in this work). The lift $\Phi_k^L(Z; F)$ is sometimes called Borcherds' generalized Shimura lift, or if $k = 0$ it is simply called Borcherds regularized theta lift. In this case we write $\Phi(Z; F) := \Phi_0(Z; F)$.

In the past 50 years various types of theta lifts have been used to prove relations between different types of modular forms. For example, the classical Shimura theta lift gives a connection between half-integral weight modular forms and modular forms of even weight (see [Shi73, Niw75]). In the above setting this lift can be realized by taking the lattice of signature $(2, 1)$ from Section 4.3. Generally speaking, theta lifts can be used to transfer knowledge from one space of modular forms to another one. It is the aim of this thesis to use the rich theory of theta lifts to study hyperbolic and elliptic Eisenstein series.

For the sake of convenience we use explicit lattices of signature $(2, 1)$ and $(2, 2)$ in this work (see Sections 4.3 and 4.4) such that the corresponding theta lifts are modular with respect to the well-known Hecke congruence subgroup $\Gamma_0(N)$. Hence we restrict to the study of hyperbolic and elliptic Eisenstein series for $\Gamma_0(N)$. However, it is possible to treat more general Fuchsian groups of the first kind using appropriate lattices. For simplicity we also often assume that N is squarefree.

1.4 Averaged Eisenstein series as theta lifts

The starting point for this thesis was the work [Mat99] by R. Matthes, which shows that a theta lift of some non-holomorphic Poincaré series is essentially given as the so-called hyperbolic kernel function $K(\tau, \omega, s)$, averaged over some Heegner points or geodesics. Here the hyperbolic kernel function is defined by

$$K(\tau, \omega, s) = \sum_{M \in \Gamma} \cosh(d_{\mathrm{hyp}}(M\tau, \omega))^{-s} \tag{1.4.1}$$

for $\tau, \omega \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. Since this kernel function is itself closely connected to hyperbolic, parabolic and elliptic Eisenstein series (see for example [Pip10, Remark 3.3.9] and [JPS16, Proposition 11]), the question arises whether these non-holomorphic Eisenstein series can also be realized as theta lifts of a non-holomorphic Poincaré series, following the ideas of [Mat99]. This thought was further supported by the fact that in the level 1 case the linear terms appearing in the Laurent expansions of the parabolic and elliptic Eisenstein series at $s = 0$, namely

$$\log\left|\Delta(\tau)^{1/6}\,\mathrm{Im}(\tau)\right| \qquad \text{and} \qquad \log\left|j(\tau) - j(\omega)\right|,$$

do indeed also look like theta lifts, by the theory of Borcherds products developed by Borcherds in [Bor98].

In [Mat99] Matthes uses the classical non-holomorphic Selberg Poincaré series introduced in [Sel65] as an input for the theta lift. Motivated by his work, we define a vector valued version of Selberg's Poincaré series. Therefore, we let (L, q) be a certain lattice of signature $(2, 1)$ and level $4N$ with N squarefree, such that the corresponding generalized upper half-plane can be identified with the usual upper half-plane $\mathbb{H}$, and such that functions which are modular of weight k for the orthogonal group of L are indeed modular of weight $2k$ for the group $\Gamma_0(N)$ in the classical sense. For further details on this lattice we refer to Section 4.3.

Now, given $\beta \in \mathbb{Z}/2N\mathbb{Z}$ and $m \in \mathbb{Z} + \beta^2/4N$ we define Selberg's vector valued Poincaré series by

$$U_{k+1/2,\beta,m}(\tau, s) = \sum_{M \in \langle T \rangle \backslash \mathrm{SL}_2(\mathbb{Z})} v^s e(m\tau)\mathfrak{e}_\beta \Big|_{k+1/2,L} \tilde{M}$$

for $\tau = u + iv \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 3/4 - k/2$. Here k is a non-negative integer as before, and we refer to Chapter 3 for the notation. The function $U_{k+1/2,\beta,m}(\tau, s)$ is modular of weight $k + 1/2$ for the Weil representation ρ_L, and can thus be used as an input for Borcherds' generalized Shimura lift. The lift is computed in Theorem 5.2.1, stating that

$$\Phi_k(z; U_{k+1/2,\beta,m}(\,\cdot\,, s)) = \frac{2N^s\Gamma(s+k)}{\pi^{s+k}} f_{k,\beta,4Nm}(z, 2s) \tag{1.4.2}$$

for $z \in \mathbb{H} \setminus H_{\beta,4Nm}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1/2 - k/2$. Here $H_{\beta,4Nm}$ denotes the set of all Heegner points of class β and discriminant $4Nm$, and the functions $f_{k,\beta,4Nm}(z, s)$ are non-holomorphic analogs of Zagier's functions $f_{k,\Delta}$ given in (1.1.2), namely

$$f_{k,\beta,4Nm}(z, s) := \sum_{\substack{Q \in \mathcal{Q}_{\beta,4Nm} \\ Q \neq 0}} \frac{\mathrm{Im}(z)^s}{Q(z,1)^k |Q(z,1)|^s}$$

for $z \in \mathbb{H} \setminus H_{\beta,4Nm}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1-k$, where $\mathcal{Q}_{\beta,4Nm}$ denotes the set of integral binary quadratic forms $Q(x,y) = aNx^2 + bxy + cy^2$ of class β and discriminant $4Nm$, i.e., with $b \equiv \beta \bmod 2N$ and with $b^2 - 4aNc = 4Nm$.

Remarkably, for $k = 0$ the function $f_{0,\beta,4Nm}(z,s)$ can be written as a finite sum of hyperbolic, parabolic or elliptic Eisenstein series if $m > 0$, $m = 0$ or $m < 0$, respectively, yielding that

$$\Phi(z; U_{1/2,\beta,m}(\,\cdot\,,s)) = \begin{cases} \dfrac{2\,\Gamma(s)}{(4\pi m)^s} \displaystyle\sum_{Q \in \mathcal{Q}_{\beta,4Nm}/\Gamma_0(N)} E^{\mathrm{hyp}}_{c_Q}(z,2s), & \text{if } m > 0, \\ \dfrac{4N^s\Gamma(s)\zeta(2s)}{\pi^s} \displaystyle\sum_{p \in C(\Gamma_0(N))} E^{\mathrm{par}}_{p}(z,2s), & \text{if } m = 0, \\ \dfrac{2\,\Gamma(s)}{(4\pi|m|)^s} \displaystyle\sum_{Q \in \mathcal{Q}_{\beta,4Nm}/\Gamma_0(N)} E^{\mathrm{ell}}_{\tau_Q}(z,2s), & \text{if } m < 0, \end{cases} \tag{1.4.3}$$

for $z \in \mathbb{H} \setminus H_{\beta,4Nm}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1/2$ (see Corollary 5.2.2). Here the three sums on the right-hand side are all finite, running over the set of Heegner geodesics c_Q or Heegner points τ_Q corresponding to quadratic forms $Q \in \mathcal{Q}_{\beta,4Nm}$ modulo $\Gamma_0(N)$ if $m > 0$ or $m < 0$, respectively, or over the set of cusps of $\Gamma_0(N)$ if $m = 0$, denoted by $C(\Gamma_0(N))$.

We emphazise that averaged versions of all three types of non-holomorphic Eisenstein series indeed arise as the theta lift of a single type of Poincaré series, namely Selberg's Poincaré series, and that the type of Eisenstein series is determined by the sign of the index m of the Poincaré series. This strongly supports the claim that hyperbolic and elliptic Eisenstein series are indeed natural generalizations of the classical parabolic one.

Next, we quickly comment on the relation between the theta lift given in (1.4.3) and the theta lift given by Matthes in [Mat99]. Firstly, we note that Matthes works with a symmetric space of signature $(1,n')$, which fits into the present setting if we consider the lattice $(L,-q)$ instead of (L,q). The index m of the Poincaré series defined above thus corresponds to the index $\nu = -4Nm$ of the Poincaré series used in [Mat99].

Now, if $m > 0$ (and correspondingly $\nu < 0$) the non-holomorphic Poincaré series used here and in [Mat99] agree, and the relation between the theta lift given in (1.4.3) and the theta lift given in part (b) of Theorem 1.1 in [Mat99] is explained by the non-trivial relation between the hyperbolic kernel function and the hyperbolic Eisenstein series given in Proposition 11 of [JPS16]. However, so far this relation is only known if the corresponding geodesic is closed, i.e., if $4Nm$ is not a square.

If $m < 0$ (and correspondingly $\nu > 0$) the situation is slightly different, which is explained as follows: Originally, Selberg defined his non-holomorphic Poincaré series for positive index, i.e., by averaging the function $v^s e(m\tau)$ for $m > 0$. If $m < 0$ one can either average over the function $v^s e(m\tau)$ as before, or over the function $v^s e^{-2\pi|m|v} e(mu)$, where the first one looks more natural, but the latter one has the advantage of decaying exponentially as $v \to \infty$. In the present work we indeed use both functions, calling them Selberg Poincaré series of the first and second kind (see Definition 3.6.3). Using this notation, we find that part (a) of Theorem 1.1 in [Mat99] computes the lift of Selberg's Poincaré series of the second kind, whereas we compute the lift of Selberg's Poincaré series of the first kind in (1.4.3). Therefore, in the case $m < 0$ (and $\nu > 0$) our results are only loosely connected.

Eventually, we also mention that for $k \geq 2$ we can simply evaluate both sides of (1.4.2) at $s = 0$, which yields a level N version of the identity given in (1.1.3), namely

$$\Phi_k(z; P_{k+1/2,\beta,m}) = \frac{2(k-1)!}{\pi^k} f_{\beta,4Nm}(z)$$

for $z \in \mathbb{H} \setminus H_{\beta,4Nm}$. Here $P_{k+1/2,\beta,m}$ is the vector valued holomorphic Poincaré series of weight $k + 1/2 > 2$ for ρ_L, and $f_{\beta,4Nm}$ is the level N analog of the holomorphic or meromorphic modular form given in (1.1.2). We remark that even though the above result is well-known for positive discriminants $4Nm$, it has only recently (in fact parallel to the work of this thesis and only in the case of level 1) been proven for discriminants $4Nm < 0$ in [BKP18, Theorem 1.1]. In this regard, we also mention the work [Zem16], where the functions $f_{\beta,4Nm}(z)$ with $m < 0$ are obtained by applying a combination of the above theta lift and so-called weight-raising operators to Maass Poincaré series.

1.5 Individual Eisenstein series as theta lifts

Though we were able to realize averaged versions of hyperbolic and elliptic Eisenstein series as theta lifts in a unified way in (1.4.3), the question remains whether we can also realize individual hyperbolic or elliptic Eisenstein series as theta lifts. We answer this question by using again the hyperbolic kernel function from (1.4.1).

Let (L, q) be the lattice of signature $(2, 2)$ and level N introduced in Section 4.4. Then the corresponding generalized upper half-plane can be identified with two copies of the usual upper half-plane $\mathbb{H}$, such that Borcherds' generalized Shimura lift can be written in the form $\Phi_k(z, z'; F)$ with $z, z' \in \mathbb{H}$ and F modular of weight k for ρ_L. Here the lift is modular of weight k in the classical sense in both variables.

Next, we define the non-holomorphic Poincaré series

$$Q_{-1}(\tau, s) := \sum_{M \in \langle T \rangle \setminus \mathrm{SL}_2(\mathbb{Z})} v^s e(-u) \mathfrak{e}_0 \Big|_{0,L} \tilde{M}$$

for $\tau = u + iv \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, which is modular of weight 0 for the Weil representation ρ_L. Hence we can use $Q_{-1}(\tau, s)$ as an input for Borcherds regularized theta lift. Actually, we consider a slightly more general version of the Poincaré series $Q_{-1}(\tau, s)$ in Section 5.3, but for simplicity we restrict to the case of weight 0 and index $(0, -1)$ in this introduction. Now Corollary 5.3.6 in this work shows that the Borcherds lift of $Q_{-1}(\tau, s)$ is essentially given by the hyperbolic kernel function, namely

$$\Phi(z, z'; Q_{-1}(\,\cdot\,, s)) = \frac{2\,\Gamma(s)}{(2\pi)^s} K(z, z', s) \tag{1.5.1}$$

for $z, z' \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. Hence we can use the relations given in [Pip10, Remark 3.3.9] and [JPS16, Proposition 11] to express hyperbolic Eisenstein series associated to closed geodesics and arbitrary elliptic Eisenstein series in terms of the Borcherds lift $\Phi(z, z'; Q_{-1}(\,\cdot\,, s))$. More precisely, given a closed geodesic c or a point

$w \in \mathbb{H}$ we find that

$$E_c^{\mathrm{hyp}}(z,s) = \frac{\pi^s}{\Gamma(s/2)^2} \int_{[c]} \Phi(z,z'; Q_{-1}(\,\cdot\,, s)) ds(z'), \tag{1.5.2}$$

$$E_w^{\mathrm{ell}}(z,s) = \frac{(2\pi)^s}{2\operatorname{ord}(w)} \sum_{n=0}^{\infty} \frac{(2\pi)^{2n}\,(s/2)_n}{n!\;\Gamma(s+2n)}\,\Phi(z,w; Q_{-1}(\,\cdot\,, s+2n)) \tag{1.5.3}$$

for $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, where we further assume that $z \not\equiv w$ modulo $\Gamma_0(N)$ in (1.5.3), and we refer to Proposition 5.3.7 for the precise notation. Roughly speaking, the hyperbolic Eisenstein series associated to a closed geodesic c is obtained by integrating the Borcherds theta lift $\Phi(z,z'; Q_{-1}(\,\cdot\,, s))$ over the geodesic c in one of its two hyperbolic variables z and z'. Correspondingly, the elliptic Eisenstein series associated to some point w is obtained by summing shifted Borcherds lifts $\Phi(z,z'; Q_{-1}(\,\cdot\,, s+2n))$, evaluated at the point w in z or z'.

For the sake of completeness, Proposition 5.3.7 also contains a parabolic version, which expresses an individual parabolic Eisenstein series in terms of the Borcherds lift of the Poincaré series $Q_{-1}(\tau, s)$, using the well-known relation between the parabolic Eisenstein series and the hyperbolic Green's function (see part (b) of Proposition 2.6.6).

Though the identities (1.5.2) and (1.5.3) are interesting on their own, we can still ask, whether it is possible to realize hyperbolic and elliptic Eisenstein series as single theta lifts. In fact, we can formally interchange summation and integration (coming from the theta lift) in (1.5.3), such that the right-hand side becomes the Borcherds lift of an infinite sum of shifted Poincaré series $Q_{-1}(\tau, s+2n)$. This motivates the definition of the non-holomorphic Maass-Selberg Poincaré series

$$M_{-1}(\tau,s) := \sum_{M \in \langle T\rangle \backslash \mathrm{SL}_2(\mathbb{Z})} v^{s/2} \mathcal{M}_{0,s/2}(-4\pi v) e(-u) \mathfrak{e}_0 \Big|_{0,L} \tilde{M}$$

for $\tau = u + iv \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, which can be seen as a Selberg-analog for weight 0 and index $(\beta, m) = (0, -1)$ of the Maass Poincaré series $F_{\beta,m}(\tau, s)$ defined for example in [Bru02, Definition 1.8]. Computing the Borcherds lift of $M_{-1}(\tau, s)$ we indeed obtain the elliptic Eisenstein series, namely

$$\Phi(z,w; M_{-1}(\,\cdot\,, s)) = \frac{2\operatorname{ord}(w)\Gamma(s)}{\pi^{s/2}}\, E_w^{\mathrm{ell}}(z,s) \tag{1.5.4}$$

for $z, w \in \mathbb{H}$ with $z \not\equiv w$ modulo $\Gamma_0(N)$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. This result is given as Corollary 5.4.3 in the present work.

1.6 Kronecker limit formulas

One of the motivations for the realization of non-holomorphic Eisenstein series as theta lifts, was the appearance of Borcherds products in the linear terms of the Laurent expansions of the parabolic and elliptic Eisenstein series at the special harmonic point $s = 0$, see (1.1.6) and (1.2.3). In the collaboration [PSV17] with A. von Pippich and M. Schwagenscheidt we have used the unified realization of averaged hyperbolic, parabolic and elliptic Eisenstein series given in (1.4.3) to systematically derive Kronecker limit formulas for these functions. Here the idea is as follows:

(1) Firstly, we establish the meromorphic continuation of the non-holomorphic Poincaré series $U_{1/2,\beta,m}(\tau, s)$ in s to the special point $s = 0$.

(2) Secondly, we explicitly identify the functions $U_{1/2,\beta,m}(\tau) := U_{1/2,\beta,m}(\tau, 0)$, which all turn out to be harmonic Maass forms of weight $1/2$, though their type strongly depends on the sign of m. If $m > 0$ the function $U_{1/2,\beta,m}$ is a holomorphic cusp form, which is characterized by the Petersson inner product formula given in (6.3.28). Further, if $m = 0$ then $U_{1/2,\beta,m}$ is a holomorphic modular form, which is orthogonal to cusp forms, and if $m < 0$ then $U_{1/2,\beta,m}$ is a non-holomorphic harmonic Maass form, which is also orthogonal to cusp forms, and thus uniquely determined by its principal part. We refer to Theorem 6.3.5 for the precise statement.

(3) Next, we show that the order of evaluating at $s = 0$ and lifting can be reversed, i.e., we show that

$$\Phi(z; U_{1/2,\beta,m}(\,\cdot\,, s))\Big|_{s=0} = \Phi(z; U_{1/2,\beta,m})$$

for $m \neq 0$ (some care has to be taken if $m = 0$, see Proposition 7.1.1 for the details).

(4) Because of the Gamma-factor appearing on the right-hand side of (1.4.3) we thus obtain that the linear term in the Laurent expansion at $s = 0$ of the averaged hyperbolic, parabolic and elliptic Eisenstein series is essentially given by the regularized Borcherds lift of the harmonic Maass form $U_{1/2,\beta,m}$.

(5) Finally, we compute this lift by using the powerful machinery of Borcherds products, which then yields explicit Kronecker limit type formulas for all three types of averaged non-holomorphic Eisenstein series for the group $\Gamma_0(N)$.

In the following we present special cases of the Kronecker limit type formulas obtained through the process explained above. For the general, more technical statements we refer to Chapter 7. We recall that N is always squarefree.

In the parabolic case we reobtain the known averaged Kronecker limit formula, which is given by

$$\sum_{p \in C(\Gamma_0(N))} E_p^{\mathrm{par}}(z, s) = 1 + \frac{1}{\sigma_0(N)} \sum_{d\,|\,N} \log\Big(|\Delta(dz)|^{1/6}\,\mathrm{Im}(z)\Big) \cdot s + O(s^2) \tag{1.6.1}$$

as $s \to 0$. Here $\sigma_0(N) = \sum_{d\,|\,N} 1$ denotes the number of (positive) divisors of N. For $N = 1$ this is simply the classical Kronecker limit formula given in (1.1.6), and for general squarefree N the formula (1.6.1) can for example be found in Section 1.5 of [JST16].

In order to present the elliptic case, we here assume that the compactification of the Riemann surface $\Gamma_0(N)\backslash\mathbb{H}$ has genus 0 for simplicity (see Theorem 7.5.3 for the general statement). This is exactly the case if

$$N = 1, 2, 3, 5, 6, 7, 10, 13.$$

For these N we find for each tuple (p, w), where p is a cusp of the Riemann surface and $w \in \mathbb{H}$, a so-called Hauptmodul $j_{N,p,w}(z)$. This is the unique normalized generator of the function field of weakly holomorphic modular forms of weight 0 and level N, which is holomorphic and non-vanishing up to a a simple pole at the cusp p and a simple zero at the point w (modulo $\Gamma_0(N)$). Now given $\beta \in \mathbb{Z}/2N\mathbb{Z}$ and $m \in \mathbb{Z} + \beta^2/4N$ with $m < 0$, the corresponding averaged elliptic Eisenstein series has the Laurent expansion

$$\sum_{Q \in \mathcal{Q}_{\beta,4Nm}/\Gamma_0(N)} E_w^{\mathrm{ell}}(z,s)$$
$$= -\frac{1}{\sigma_0(N)}\left(\sum_{Q \in \mathcal{Q}_{\beta,4Nm}/\Gamma_0(N)} \sum_{p \in C(\Gamma_0(N))} \log\left(\left|j_{N,p,\tau_Q}(z)\right|^{2/\operatorname{ord}(\tau_Q)}\right)\right) \cdot s + O(s^2)$$

at $s = 0$, for $z \in \mathbb{H} \setminus H_{\beta,4Nm}$. In particular, if $j(z)$ denotes the usual modular j function for the group $\mathrm{SL}_2(\mathbb{Z})$ then $j_{1,\infty,w}(z) = j(z) - j(w)$. Thus, in the case $N = 1$ the above Kronecker limit formula yields an averaged version of the elliptic Kronecker limit formula given in (1.2.3).

Finally, we turn to the hyperbolic case. Here we firstly note that given $\beta \in \mathbb{Z}/2N\mathbb{Z}$ and $m \in \mathbb{Z} + \beta^2/4N$ with $m > 0$ the corresponding Heegner geodesics of class β and discriminant $4Nm$ are all infinite if $4Nm$ is a square, and closed otherwise. In the latter case we simply find that the averaged hyperbolic Eisenstein series has a double zero at $s = 0$, which agrees with (1.2.2). Furthermore, Corollary 7.4.6 shows that the cusp form $U_{1/2,\beta,m}$ vanishes for trivial reasons if $N = 1$ or $N = p$ is a prime, inducing again a double zero of the averaged hyperbolic Eisenstein series at $s = 0$. Hence, as our method does not give any information about the second order term of the Laurent expansion at $s = 0$, our results are trivial in these cases.

It is thus reasonable to assume that $4Nm$ is a square, and that N is the product of at least two different primes. For the sake of simplicity, we further use the following assumptions: We set

$$N := pq, \qquad \beta := n, \qquad m := n^2/4N.$$

with p, q being different primes and n being a positive integer coprime to p and q. Then

$$\sum_{Q \in \mathcal{Q}_{n,n^2}/\Gamma_0(N)} E_{c_Q}^{\mathrm{hyp}}(z,s) = \frac{n}{4\varphi(N)} \log\left|\frac{\Delta(z)\,\Delta(Nz)}{\Delta(pz)\,\Delta(qz)}\right| \cdot s + O(s^2)$$

as $s \to 0$, for $z \in \mathbb{H}$. Here $\varphi(N) = (p-1)(q-1)$ denotes Euler's totient function.

1.7 Outline of this work

In the following we quickly describe the structure of this thesis. In Chapters 2, 3 and 4 we essentially recall known results and fix the notation. Afterwards, we present our main results in Chapter 5, the realization of hyperbolic and elliptic Eisenstein series as regularized theta lifts. Finally, we use one of these theta lifts to obtain Kronecker limit type formulas in Chapters 6 and 7. We remark that the results of the last two chapters are also contained in the collaboration [PSV17].

Let us be more precise. In Chapter 2 we give a quick introduction to the theory of classical elliptic modular forms, focussing on the definition of hyperbolic, parabolic and elliptic Eisenstein series, and their counterparts, the (in general non-holomorphic) modular forms associated to integral binary quadratic forms. Here we point out that to the best knowledge of the author the non-holomorphic analogs of Zagier's well-known cusp forms associated to quadratic forms or discriminants, namely the functions $f_{k,Q}(\tau, s)$ and $f_{k,\beta,\Delta}(\tau, s)$ introduced in Section 2.5.2, have not been defined in this generality before, though special cases (such as hyperbolic and elliptic Eisenstein series) are known. Finally, we introduce the hyperbolic kernel function at the end of the chapter, recalling its relation to hyperbolic, parabolic and elliptic Eisenstein series.

In Chapter 3 we further introduce vector valued modular forms for the Weil representation. Here we concentrate on the definition of different types of vector valued non-holomorphic Poincaré series in Section 3.6. In particular, we introduce a new type of Maass-Selberg Poincaré series $M^L_{k,\beta,m}(\tau, s)$ (see Definition 3.6.10), which has not been studied before. Moreover, we present a detailed translation of the work [Roe66, Roe67] by Roelcke to the setting of vector valued forms for the Weil representation in Section 3.7, which though well-known to the experts we could not find in the literature.

Afterwards we introduce a special case of the general regularized theta lift defined by Borcherds (see [Bor98]) in Chapter 4. Here we firstly explain the notation for modular forms on orthogonal groups, before we can then use the notation of Borcherds to introduce a vector valued version of Shintani's theta function and the corresponding generalized Shimura lift. Finally, we recall two specific lattices of signature $(2, 1)$ and signature $(2, 2)$ in Sections 4.3 and 4.4, which are fundamental in the following chapters.

Chapter 5 is the core of this thesis. Here we present the different realizations of hyperbolic and elliptic Eisenstein series as theta lifts of signature $(2, 1)$ and $(2, 2)$. Our main statements are the following:

- In Corollary 5.2.2 we give the realization of averaged hyperbolic, parabolic and elliptic Eisenstein series as the Borcherds lift of Selberg's Poincaré series $U^L_{0,\beta,m}(\tau, s)$ in signature $(2, 1)$.
- In Corollary 5.3.6 we obtain the hyperbolic kernel function as the Borcherds lift of the non-holomorphic Poincaré series $Q^L_{0,0,-1}(\tau, s)$ in signature $(2, 2)$. As a consequence we can write individual hyperbolic, parabolic and elliptic Eisenstein series as modifications of this theta lift in Proposition 5.3.7.
- In Corollary 5.4.3 we realize an individual elliptic Eisenstein series as the Borcherds lift of the Maass-Selberg Poincaré series $M^L_{0,0,-1}(\tau, s)$ in signature $(2, 2)$.

We claim that all of these results are original work (up to the mentioned relation between our lift in Corollary 5.2.2 and the work [Mat99] explained above in Section 1.4 of this introduction). In addition, we give further lifting results for the different non-holomorphic Poincaré series.

In Chapters 6 and 7 we finally present results which are included in the upcoming publication [PSV17]. Here the meromorphic continuation of Selberg's Poincaré series, given as Section 3 in [PSV17], takes the whole Chapter 6. However, in [PSV17] some of the technical details are omitted, whereas we here give a detailed presentation of the process:

(i) Section 6.1 deals with the vector valued non-holomorphic Eisenstein series and the corresponding Kloosterman zeta functions. Though most of the results of this section are known to the experts, we still give proofs, as we could not find exact statements matching our situation in the literature.

(ii) In the following Section 6.2 we use a modified version of our usual Selberg Poincaré series (which we call Selberg's Poincaré series of the second kind) to establish the meromorphic continuation as well as polynomial bounds of Kloosterman zeta functions in Theorem 6.2.6. This section is based on the work of Selberg [Sel65] and Pribitkin [Pri00] (see also [GS83]). However, we cannot directly employ their results, since both assume that the corresponding hyperbolic Laplace operator has only discrete spectrum, which is not true in our case. Also, we need to translate their results to the present vector valued setting.

(iii) In the third and final Section of this Chapter we prove two theorems, namely, using the bounds from Section 6.2 we show that the Fourier expansion of Selberg's Poincaré series yields its meromorphic continuation to the whole complex plane in Theorem 6.3.1, and afterwards we use the Fourier expansion to evaluate this continuation at the special point $s = 0$ in Theorem 6.3.5. Here the first result is probably known to the experts, though not given in the literature for the present setting. On the other hand, the characterization of the evaluation of the Poincaré series at $s = 0$ is original work, given as Theorem 3.9 in [PSV17].

In Chapter 7 we finally use the characterization of Selberg's Poincaré series at the point $s = 0$ established before, to obtain averaged Kronecker limit type formulas. We start by recalling the basics on Borcherds products using [BO10], which we subsequently employ to determine the Borcherds theta lift of the harmonic Maass form characterized by Theorem 6.3.5. Since the hyperbolic, parabolic and elliptic case turn out to be of rather different nature, we treat them separately, obtaining averaged Kronecker limit type formulas in all three cases. Here the parabolic case is known, but the hyperbolic and elliptic formulas are only partially known. In particular, we obtain new Kronecker limit formulas for averaged hyperbolic Eisenstein series corresponding to infinite geodesics (see Theorem 7.4.4), and we can give an averaged version in certain higher levels of the known elliptic Kronecker limit formula for level 1 (see Corollary 7.5.4). These results are also given in Section 5 of [PSV17].

1.8 Open problems

In this final section we shortly comment on open problems this thesis does not address. These will be part of our future research.

(a) It would be interesting to have a geometric understanding of the averaged hyperbolic Kronecker limit formula given in Theorem 7.4.4. We think that the formula has an interpretation in terms of the cusps, connected by infinite geodesics of the corresponding hyperbolic Eisenstein series.

(b) It is natural to ask whether similar techniques as used in Chapters 6 and 7 can also be applied to the realization of the individual elliptic Eisenstein series as a Borcherds lift of the Maass-Selberg Poincaré series. We will address this question in a subsequent work [PSV18].

(c) The relation [JPS16, Proposition 2016] between the hyperbolic kernel function and hyperbolic Eisenstein series is so far only known for closed geodesics. Generalizing this result to infinite geodesics would also imply a generalization of part (a) of Proposition 5.3.7.

(d) Let L be the lattice of signature $(2,2)$ from Section 4.4, and let c be a closed geodesic. Interchanging integration in part (a) of Proposition 5.3.7 we formally obtain

$$E_c^{\mathrm{hyp}}(z,s) = \frac{\pi^s}{\Gamma(s/2)^2} \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}}^{\mathrm{reg}} \left\langle Q_{-1}(\tau,s), \int_{[c]} \Theta_L(\tau,(z,z'))ds(z') \right\rangle d\mu(\tau) \tag{1.8.1}$$

for $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, where $Q_{-1}(\tau,s)$ is the Poincaré series defined in Section 1.5 of this introduction. This motivates the definition of the Borcherds theta function associated to the geodesic c via

$$\Theta_{c,L}(\tau,z) := \int_{[c]} \Theta_L(\tau,(z,z'))ds(z')$$

for $\tau, z \in \mathbb{H}$, which is then modular of weight 0 for ρ_L in τ, and modular of weight 0 for the group $\Gamma(N)$ in z. Moreover, by (1.8.1) integrating the Poincaré series $Q_{-1}(\tau,s)$ against the theta function $\Theta_{c,L}(\tau,z)$ in the variable τ essentially yields the hyperbolic Eisenstein series. It might thus be interesting to study the theta lift induced by this new theta function.

(e) A classical result by Kohnen and Zagier states that the generating series build out of the holomorphic $f_{k,\Delta}$'s for $\Delta > 0$ is a holomorphic modular form in two variables, which is in fact the holomorphic kernel function for the Shimura and Shintani lift between modular forms of half-integral weight, and modular forms of even weight (see [KZ81] for the details). We ask whether a similar result holds for a generating series build out of the non-holomorphic functions $f_{0,\Delta}(\tau,s)$ for $\Delta > 0$, i.e., a generating series build out of averaged hyperbolic Eisenstein series. This idea is further supported by the fact that if $4Nm$ is not a square the relation [JPS16, Proposition 2016] shows that

$$E_{\beta,m}^{\mathrm{hyp}}(\tau,s) = \frac{\Gamma(s)}{2^{s-1}\Gamma(s/2)^2} \sum_{Q \in \mathcal{Q}_{\beta,4Nm}/\Gamma_0(N)} \int_{[c]} K(\tau,\omega,s)ds(\omega),$$

i.e., averaged hyperbolic Eisenstein series can be seen as modular traces of the hyperbolic kernel function.

(f) One could use part (a) of Theorem 5.1.3 with $k = 0$ to define averaged hyperbolic, parabolic and elliptic Eisenstein series for more general lattices of signature $(2,n)$. Given such a lattice L these Eisenstein series would be living on the corresponding generalized upper half-plane $\mathbb{H}_n$, and they would be modular of weight 0 with respect to the discrete subgroup $\Gamma(L)$ of the orthogonal group $O(L)$ (see equation (4.1.4) for the notation). By construction, these generalized Eisenstein series could again be realized as the regularized Borcherds lift of Selberg's Poincaré series for the given lattice L and some index (β,m).

2 Scalar valued modular forms

In this first chapter we recall some basic facts on scalar valued modular forms. As the theory is well-known we omit most of the proofs. For a more detailed introduction we refer the reader for example to the textbooks [Miy06] and [DS05].

The focus of this chapter is on the introduction of non-holomorphic analogous of Zagier's cusp forms associated to discriminants. In the case of weight 0 these can be seen as generalized non-holomorphic Eisenstein series.

2.1 The upper half-plane

We denote the complex upper half-plane in $\mathbb{C}$ by $\mathbb{H}$, i.e.,

$$\mathbb{H} := \{\tau \in \mathbb{C} \colon\ \mathrm{Im}(\tau) > 0\}.$$

The special linear group $\mathrm{SL}_2(\mathbb{C})$ consisting of 2×2-matrices with complex coefficients and determinant one acts on the Riemann sphere $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ by Möbius transformations, i.e., via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau := \frac{a\tau + b}{c\tau + d} \quad \text{for } \tau \in \mathbb{C} \qquad \text{and} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \infty := \frac{a}{c}$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{C})$. In both cases we interpret the right-hand side as ∞ if the corresponding denominator vanishes. The subgroup $\mathrm{SL}_2(\mathbb{R})$ of real matrices in $\mathrm{SL}_2(\mathbb{C})$ fixes the upper half-plane $\mathbb{H}$ and its boundary $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$ in the Riemann sphere $\mathbb{S}^1$, yielding an action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ and $\mathbb{P}^1(\mathbb{R})$.

Up to constant multiplication there is a unique metric and a unique measure on $\mathbb{H}$, which is invariant under the action of the group $\mathrm{SL}_2(\mathbb{R})$. These are realized by the *hyperbolic metric* ds and the *hyperbolic measure* $d\mu$ given by

$$ds(\tau) := \frac{\sqrt{du^2 + dv^2}}{v} \qquad \text{and} \qquad d\mu(\tau) := \frac{du\, dv}{v^2},$$

where we write $\tau \in \mathbb{H}$ as $\tau = u + iv$.

An (oriented) *geodesic* or hyperbolic line in $\mathbb{H}$ is a vertical half-line parallel to the imaginary axis, or a half-circle centered on the real line, together with an orientation. In other words, a geodesic defines a unique path connecting any two distinct points on the boundary $\mathbb{P}^1(\mathbb{R})$. We emphasize that our geodesics will always have an orientation.

The *hyperbolic distance* function d_{hyp} induced by the metric ds is given by

$$d_{\mathrm{hyp}}(\tau, \omega) := \inf_{\gamma} \int_{\gamma} ds(\xi)$$

for $\tau, \omega \in \mathbb{H}$, where the infimum is taken over all paths γ in $\mathbb{H}$ from τ to ω. The shortest such path joining τ and ω is always realized by a geodesic, i.e., geodesics are "straight" lines in $\mathbb{H}$. Since the hyperbolic metric is $\mathrm{SL}_2(\mathbb{R})$-invariant, the same is true for the hyperbolic distance function, i.e., we have

$$d_{\mathrm{hyp}}(\alpha\tau, \alpha\omega) = d_{\mathrm{hyp}}(\tau, \omega)$$

for $\alpha \in \mathrm{SL}_2(\mathbb{R})$ and $\tau, \omega \in \mathbb{H}$. Moreover, an elementary calculation shows the formula

$$\cosh(d_{\mathrm{hyp}}(\tau, \omega)) = 1 + \frac{|\tau - \omega|^2}{2\,\mathrm{Im}(\tau)\,\mathrm{Im}(\omega)} \tag{2.1.1}$$

for $\tau, \omega \in \mathbb{H}$. Given a geodesic c in $\mathbb{H}$ we use the notation $d_{\mathrm{hyp}}(\tau, c)$ to denote the hyperbolic distance of the point $\tau \in \mathbb{H}$ to the geodesic c, i.e., the minimum length of a path from τ to c which is always realized by a unique geodesic through τ orthogonal to c. If $c_0 := (0, \infty)$ is the geodesic given by a vertical line from 0 to ∞, we have the formula

$$\cosh(d_{\mathrm{hyp}}(\tau, c_0)) = \frac{|\tau|}{\mathrm{Im}(\tau)} \tag{2.1.2}$$

for $\tau \in \mathbb{H}$, which can be easily deduced from the formula (2.1.1).

Non-scalar elements in $\mathrm{SL}_2(\mathbb{R})$ can be characterized as follows: We call $\alpha \in \mathrm{SL}_2(\mathbb{R})$ with $\alpha \neq \pm 1$ *elliptic*, *parabolic* or *hyperbolic* if

$$|\mathrm{tr}(\alpha)| < 2, \qquad |\mathrm{tr}(\alpha)| = 2 \qquad \text{or} \qquad |\mathrm{tr}(\alpha)| > 2,$$

respectively. Here we denote the identity matrix in $\mathrm{SL}_2(\mathbb{R})$ simply by 1. It is easy to check that $\alpha \in \mathrm{SL}_2(\mathbb{R})$ is elliptic, parabolic or hyperbolic if and only if α has a unique fixed point in the upper half-plane $\mathbb{H}$, a unique fixed point on the boundary $\mathbb{P}^1(\mathbb{R})$ or two distinct unique fixed points on the boundary $\mathbb{P}^1(\mathbb{R})$, respectively.

2.2 The modular curve $X_0(N)$

Let N be a positive integer. We define the *congruence subgroup* of level N as

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \colon c \equiv 0 \bmod N \right\}.$$

These are subgroups of the so-called *modular group* $\mathrm{SL}_2(\mathbb{Z}) = \Gamma_0(1)$. Furthermore, we call $\tau \in \mathbb{H} \cup \mathbb{R} \cup \{\infty\}$ *elliptic*, *parabolic* or *hyperbolic* with respect to $\Gamma_0(N)$ if τ is fixed by some elliptic, parabolic or hyperbolic element in $\Gamma_0(N)$, respectively. In the following we characterize these points and their stabilizers:

- Elliptic points for $\Gamma_0(N)$ lie in the upper half-plane $\mathbb{H}$. More precisely, if $\tau \in \mathbb{H}$ is elliptic then $\tau = Mi$ or $\tau = Me^{\pi i/3}$ for some $M \in \mathrm{SL}_2(\mathbb{Z})$, and the corresponding stabilizer of τ in $\Gamma_0(N)$ is a finite cyclic group of order 4 or 6 if τ is equivalent to i or $e^{\pi i/3}$, respectively. In particular, the stabilizer of i in $\Gamma_0(N)$ is generated by the matrix $S := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

For an arbitrary, not necessarily elliptic point $\tau \in \mathbb{H}$ we define the *order* of τ with respect to $\Gamma_0(N)$ as

$$\operatorname{ord}(\tau) := |(\Gamma_0(N))_\tau|\,.$$

Clearly, we have $\operatorname{ord}(\tau) \in \{2,4,6\}$ and τ is elliptic if and only if $\operatorname{ord}(\tau) > 2$.

- The set of parabolic points for $\Gamma_0(N)$ is exactly given by $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$, independent of N, and given $p \in \mathbb{P}^1(\mathbb{Q})$ there is an associated parabolic scaling matrix $\sigma_p \in \mathrm{SL}_2(\mathbb{R})$ such that the stabilizer of p in $\Gamma_0(N)$ is of the form

$$(\Gamma_0(N))_p = \sigma_p \left\{ \pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Z} \right\} \sigma_p^{-1}. \tag{2.2.1}$$

For $p \in \mathbb{P}^1(\mathbb{Q})$ we further call its orbit $[p] = \Gamma_0(N)p$ a *cusp* for $\Gamma_0(N)$, and we denote the set of all cusps for $\Gamma_0(N)$ by $C(\Gamma_0(N))$, i.e.,

$$C(\Gamma_0(N)) := \Gamma_0(N)\backslash\mathbb{P}^1(\mathbb{Q}).$$

Since $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on $\mathbb{P}^1(\mathbb{Q})$ the set of cusps $C(\Gamma_0(N))$ is always finite.

In abuse of notation we often simply write p instead of $[p]$. Further, we will often say that $\sigma_p \in \mathrm{SL}_2(\mathbb{R})$ is a parabolic scaling matrix corresponding to the cusp p, by which we mean that σ_p actually corresponds to a fixed parabolic element $p \in \mathbb{P}^1(\mathbb{Q})$.

If N is squarefree a set of representatives for the cusps of $\Gamma_0(N)$ is given by the fractions $1/d$ with d running through the positive divisors of N. In this case the matrix

$$\sigma_{1/d} := \sqrt{\frac{N}{d}} \begin{pmatrix} 1 & 0 \\ d & d/N \end{pmatrix} \tag{2.2.2}$$

is a scaling matrix in the above sense for the cusp $1/d$. More generally, for an arbitrary parabolic element $p = \frac{s}{t} \in \mathbb{Q}$ with $s, t \in \mathbb{Z}$ being coprime, a scaling matrix of p is given by

$$\sigma_p := \sqrt{\frac{N}{d}} \begin{pmatrix} s & \beta d/N \\ t & \delta d/N \end{pmatrix},$$

where $d = (t, N)$ and $\beta, \delta \in \mathbb{Z}$ with $\delta s - \beta t = 1$. For the cusp $p = \infty$ we can always choose $\sigma_\infty = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

- Hyperbolic points for $\Gamma_0(N)$ are real quadratic irrationals. They always come in so-called *hyperbolic pairs* (x, x') with $x, x' \in \mathbb{R} \setminus \mathbb{Q}$ being fixed by the same hyperbolic element in $\Gamma_0(N)$. Given such a pair we can choose a hyperbolic scaling matrix $\sigma_{x,x'} \in \mathrm{SL}_2(\mathbb{R})$ such that the stabilizer in $\Gamma_0(N)$ fixing both x and x' is of the form

$$(\Gamma_0(N))_{x,x'} = \sigma_{x,x'} \left\{ \pm \begin{pmatrix} \mu^{1/2} & 0 \\ 0 & \mu^{-1/2} \end{pmatrix}^n : n \in \mathbb{Z} \right\} \sigma_{x,x'}^{-1}$$

with $\mu > 1$. In particular, the stabilizer of the pair (x, x') in $\Gamma_0(N)/\{\pm 1\}$ is an infinite cyclic group.

Moreover, given a hyperbolic pair $(x, x') \in \mathbb{R}^2$ there is an *associated geodesic* $c_{x,x'}$ in $\mathbb{H}$ which is the geodesic path from x to x', and the stabilizer of the (oriented) geodesic $c_{x,x'}$ in $\Gamma_0(N)$ agrees with the stabilizer of the pair (x, x'), i.e.,

$$(\Gamma_0(N))_{c_{x,x'}} = (\Gamma_0(N))_{x,x'}.$$

Note that the geodesics $c_{x,x'}$ and $c_{x',x}$ have the same image in $\mathbb{H}$, but their orientation is inverted. However, their stabilizers in $\Gamma_0(N)$ clearly agree. Elements in $\Gamma_0(N)$ that fix the image of $c_{x,x'}$ but change its orientation are exactly the elements mapping $x \mapsto x'$ and $x' \mapsto x$. One easily checks that such an element is either scalar or elliptic of order 4, i.e., conjugate to $\pm S$.

It is now natural to consider the quotient space $Y_0(N) := \Gamma_0(N)\backslash\mathbb{H}$. The complex structure of the upper half-plane induces a complex structure on the quotient $Y_0(N)$, turning it into a non-compact Riemann surface, which can be compactified by adding all its cusps, i.e., by adding the finite quotient $C(\Gamma_0(N)) = \Gamma_0(N)\backslash\mathbb{P}^1(\mathbb{Q})$. We then obtain the so-called *modular curve* of level N, which is given by

$$X_0(N) := \Gamma_0(N)\backslash(\mathbb{H} \cup \mathbb{Q} \cup \{\infty\}).$$

Next we introduce geodesics on modular curves. A *geodesic* in $X_0(N)$ is a quotient of the form

$$[c] = (\Gamma_0(N))_c\backslash c,$$

where c is an arbitrary (oriented) geodesic in $\mathbb{H}$ and $(\Gamma_0(N))_c$ is the stabilizer of this geodesic in $\Gamma_0(N)$. In abuse of notation, we will sometimes denote the geodesic $[c]$ in the modular curve $X_0(N)$ simply by c.

Further, we call $[c]$ a *closed geodesic* if its stabilizer is infinite cyclic in $\Gamma_0(N)/\{\pm 1\}$, i.e., if there is a hyperbolic pair (x, x') for $\Gamma_0(N)$ such that $c = c_{x,x'}$. In this case we may identify $[c]$ with the geodesic path from z_0 to γz_0 in $\mathbb{H}$, where z_0 is an arbitrary point on c and γ is a generator of the infinite cyclic stabilizer of c. Moreover, if the geodesic c in $\mathbb{H}$ joins two parabolic points $x, x' \in \mathbb{P}_1(\mathbb{Q})$, we call c an *infinite geodesic* connecting the (not necessarily distinct) cusps x and x' of $X_0(N)$. In this case the stabilizer of c in $\Gamma_0(N)/\{\pm 1\}$ is trivial. Hence, the image of the geodesic c in the quotient $\Gamma_0(N)\backslash\mathbb{H}$ under the canonical projection does not contain any loop, and can thus be identified with the geodesic c itself.

Finally, we note that since the hyperbolic metric ds and the hyperbolic measure $d\mu$ are by construction $\mathrm{SL}_2(\mathbb{R})$-invariant, they directly define a metric and a measure on $Y_0(N)$ which we again denote by ds and $d\mu$.

2.3 Integral binary quadratic forms

Before we define modular forms we recall some basic properties about quadratic forms. We will need these afterwards in order to define Zagier's cusp forms associated to discriminants. Moreover, we introduce Heegner geodesics and Heegner points, which will be crucial in the course of this thesis.

An *integral binary quadratic form* Q is a homogeneous quadratic polynomial of two variables with integer coefficients, i.e.,

$$Q(x,y)=ax^2+bxy+cy^2$$

with $a,b,c\in\mathbb{Z}$. The *discriminant* of Q is given by $\Delta(Q):=b^2-4ac$. Given Q as above not identically zero and of discriminant $\Delta\in\mathbb{Z}$ we can distinguish three cases:

- If $\Delta>0$ then one can find $(x,y),(x',y')\in\mathbb{Z}^2$ such that $Q(x,y)>0$ and $Q(x',y')<0$. In this case Q is called *indefinite*.
- If $\Delta=0$ then either $Q(x,y)\geq 0$ or $Q(x,y)\leq 0$ for all $(x,y)\in\mathbb{Z}^2$, and Q is called *positive* or *negative semi-definite*, respectively. One can further check that Q is positive or negative semi-definite if and only if $a,c\geq 0$ or $a,c\leq 0$, respectively.
- If $\Delta<0$ then either $Q(x,y)>0$ or $Q(x,y)<0$ for all $(x,y)\in\mathbb{Z}^2$ with $(x,y)\neq 0$, and Q is called *positive* or *negative definite*, respectively. Here Q is positive or negative definite if and only if $a,c>0$ or $a,c<0$, respectively.

From now on we will only consider quadratic forms $Q(x,y)=ax^2+bxy+cy^2$ where the first coefficient a is divisible by N. We denote the set of all such forms by $\mathcal{Q}$, i.e.,

$$\mathcal{Q}:=\Big\{Q(x,y)=ax^2+bxy+cy^2\colon a,b,c\in\mathbb{Z}, N\mid a\Big\}.$$

For $Q\in\mathcal{Q}$ as above we call $\beta\in\mathbb{Z}/2N\mathbb{Z}$ with $\beta\equiv b \bmod 2N$ the *class* of Q, and for Q of class β we clearly have $\Delta(Q)\equiv\beta^2 \bmod 4N$.

Let $\beta\in\mathbb{Z}/2N\mathbb{Z}$ and $\Delta\in\mathbb{Z}$ with $\Delta\equiv\beta^2 \bmod 4N$. We define $\mathcal{Q}_{\beta,\Delta}$ as the set of integral binary quadratic forms of class β and discriminant Δ, i.e.,

$$\mathcal{Q}_{\beta,\Delta}:=\Big\{Q(x,y)=ax^2+bxy+cy^2\in\mathcal{Q}\colon \Delta(Q)=\Delta, b\equiv\beta \mod 2N\Big\}.$$

Note that for $N=1$ the class $\beta\in\mathbb{Z}/2\mathbb{Z}$ is uniquely determined by the discriminant Δ and may thus be omitted. In this case one simply writes $\mathcal{Q}_\Delta$.

The group $\Gamma_0(N)$ acts on $\mathcal{Q}_{\beta,\Delta}$ from the right via

$$Q.M=M^tQM$$

for $Q\in\mathcal{Q}_{\beta,\Delta}$ and $M\in\Gamma_0(N)$, where we identify $Q(x,y)=ax^2+bxy+cy^2$ with its associated matrix

$$Q=\begin{pmatrix} a & b/2 \\ b/2 & c\end{pmatrix}.$$

We write $[Q]$ for the $\Gamma_0(N)$-orbit of Q, and $Q\sim Q'$ if Q and Q' are equivalent modulo $\Gamma_0(N)$, i.e., if $Q\in[Q']$. For $\Delta\neq 0$ the quotient $\mathcal{Q}_{\beta,\Delta}/\Gamma_0(N)$ is always finite as its order is essentially given as a sum of generalized class numbers. However, for $\Delta=0$ the group $\Gamma_0(N)$ acts on $\mathcal{Q}_{\beta,0}$ with infinitely many orbits.

Also, one easily checks that $Q\in\mathcal{Q}$ is indefinite, positive (semi-)definite or negative (semi-)definite if and only if $Q.M$ is for all $M\in\Gamma_0(N)$. Thus, if $\Delta\leq 0$ then the only quadratic form Q of discriminant Δ with $Q\sim -Q$ is the zero-form $Q\equiv 0$ which is positive and negative semi-definite at the same time.

Let now $Q \in \mathcal{Q}_{\beta,\Delta}$ with $Q(x,y) = ax^2 + bxy + cy^2$ and $Q \not\equiv 0$. Depending on the sign of Δ we associate to Q a geodesic, a parabolic point or a point in $\mathbb{H}$. These associated objects are essentially given by the roots of $Q(x,1)$, where we understand ∞ as a root of $Q(x,1)$ if $Q(1,0) = 0$.

- If $\Delta > 0$ we associate to Q the unique geodesic c_Q in $\mathbb{H}$ joining the two distinct roots of $Q(x,1)$ in $\mathbb{P}^1(\mathbb{R})$, namely
$$x = \frac{-b + \sqrt{\Delta}}{2a} \qquad \text{and} \qquad x' = \frac{-b - \sqrt{\Delta}}{2a}$$
if $a \neq 0$, and $x = -\frac{c}{b}$ and $x' = \infty$ if $a = 0$. The orientation of c_Q is defined to go from x to x', except when $a = 0$ and $b < 0$, in which case it runs from x' to x. This guarantees that the geodesics associated to Q and $-Q$ have the same image in $\mathbb{H}$ but their orientations are inverted. As a subset of $\mathbb{H}$, the geodesic c_Q can also be written as
$$c_Q = \left\{\tau \in \mathbb{H} \colon a|\tau|^2 + b\,\mathrm{Re}(\tau) + c = 0\right\}.$$
We call c_Q the *Heegner geodesic* associated to Q.

- If $\Delta = 0$ we associate to Q the parabolic point p_Q which is given as the unique root of $Q(x,1)$ in $\mathbb{P}^1(\mathbb{Q})$, namely $p_Q = -\frac{b}{2a}$ if $a \neq 0$, and $p_Q = \infty$ otherwise. In abuse of notation we also call p_Q the cusp associated to Q.

- If $\Delta < 0$ we associate to Q the point $\tau_Q \in \mathbb{H}$ which is the unique root of $Q(x,1)$ in the upper half-plane, i.e.,
$$\tau_Q := -\frac{b}{2a} + i\frac{\sqrt{|\Delta|}}{2|a|}.$$
The point τ_Q is called the *Heegner point* or *CM point* associated to Q. We further define the set of all Heegner points associated to the class $[Q]$ as
$$H_{[Q]} := \{\tau_{Q'} \colon Q' \in [Q]\}. \tag{2.3.1}$$
In abuse of notation we simply write H_Q for the set $H_{[Q]}$, and we set $H_Q = \emptyset$ if $\Delta(Q) \geq 0$. Moreover, we define the set of all Heegner points of class β and discriminant Δ by
$$H_{\beta,\Delta} := \{\tau_{Q'} \colon Q' \in \mathcal{Q}_{\beta,\Delta}\}. \tag{2.3.2}$$
For $\Delta \in \mathbb{Z}$ with $\Delta \equiv \beta^2 \bmod 4N$ and $\Delta \geq 0$ we again simply set $H_{\beta,\Delta} = \emptyset$.

Let $Q \in \mathcal{Q}_{\beta,\Delta}$ with $Q \not\equiv 0$ and $M \in \Gamma_0(N)$. It is easy to verify that $x \in \mathbb{H} \cup \mathbb{R} \cup \{\infty\}$ is a root of $(Q.M)(x,1)$ if and only if Mx is a root of $Q(x,1)$. Therefore, the action of $\Gamma_0(N)$ on $\mathcal{Q}_{\beta,\Delta} \setminus \{0\}$ is compatible with the above identifications in the sense that
$$c_{Q.M} = M^{-1}c_Q, \qquad p_{Q.M} = M^{-1}p_Q \qquad \text{and} \qquad \tau_{Q.M} = M^{-1}\tau_Q \tag{2.3.3}$$
if $\Delta > 0$, $\Delta = 0$ or $\Delta < 0$, respectively. This also shows that the set of Heegner points associated to some class $[Q]$ defined above can indeed be written as
$$H_Q = \{M\tau_Q \colon M \in \Gamma_0(N)\}. \tag{2.3.4}$$

Here we recall that $H_Q = \emptyset$ if $\Delta(Q) \geq 0$.

Moreover, the stabilizer of the quadratic form Q in $\Gamma_0(N)$ agrees with the stabilizer of its associated object, i.e.,depending on the sign of Δ the stabilizer $(\Gamma_0(N))_Q$ equals the stabilizer $(\Gamma_0(N))_{c_Q}$, $(\Gamma_0(N))_{p_Q}$ or $(\Gamma_0(N))_{\tau_Q}$, respectively. Here it is important that for $Q \not\equiv 0$ we can only have $Q.M = -Q$ for some $M \in \Gamma_0(N)$ if $\Delta(Q) > 0$ in which case M does not fix the oriented geodesic c_Q but swaps its endpoints.

Lemma 2.3.1. *Let $Q \in \mathcal{Q}$ with $\Delta(Q) > 0$. Then the stabilizer of Q in $\Gamma_0(N)/\{\pm 1\}$ is trivial if $\Delta(Q)$ is a square, and infinite cyclic otherwise.*

Proof. Let $\Delta = \Delta(Q)$, and recall that $(\Gamma_0(N))_Q = (\Gamma_0(N))_{c_Q}$ as noted above. Therefore, the stabilizer $(\Gamma_0(N))_Q/\{\pm 1\}$ is either trivial or infinite cyclic, and every non-trivial element in this stabilizer is hyperbolic.

If Δ is a square then the endpoints of the geodesic c_Q are rational, and thus there is no hyperbolic element in $\Gamma_0(N)$ fixing the oriented geodesic c_Q. So $(\Gamma_0(N))_Q = \{\pm 1\}$. Conversely, suppose that Δ is not a square. Then Pell's equation $t^2 - \Delta u^2 = 1$ has a non-trivial solution $(t, u) \in \mathbb{Z}^2$. Writing $Q(x, y) = ax^2 + bxy + cy^2$ we set

$$M := \begin{pmatrix} t + bu & 2cu \\ -2au & t - bu \end{pmatrix},$$

which defines an element of $\Gamma_0(N)$ as N divides a. Now an easy computation shows that $Q.M = Q$. Further, $M \neq \pm 1$ since if $a = c = 0$ the discriminant $\Delta = b^2 - 4ac$ would be a square. Hence we have shown that the stabilizer of Q in $\Gamma_0(N)/\{\pm 1\}$ is non-trivial. Thus it has to be infinite cyclic. $\square$

Corollary 2.3.2. *Let c_Q be a Heegner geodesic. Then c_Q is infinite if $\Delta(Q)$ is a square, and closed otherwise. Conversely, every closed or infinite geodesic c can be realized as a Heegner geodesic $c_Q = c$ with $Q \in \mathcal{Q}$ and $\Delta(Q) > 0$.*

Proof. Let $Q \in \mathcal{Q}$ with $\Delta(Q) > 0$. If $\Delta(Q)$ is a square, then the endpoints of the geodesic c_Q are rational, and thus the geodesic c_Q is infinite. Further, if $\Delta(Q)$ is not a square, then the stabilizer of Q in $\Gamma_0(N)/\{\pm 1\}$ is non-trivial by Lemma 2.3.1, and thus c_Q is closed.

Conversely, let c be a closed or infinite geodesic. If c is closed then we find $M \in \Gamma_0(N)$ with $M \neq \pm 1$ which stabilizes c. Then $c = c_Q$ for $Q(x, y) = \pm(cx^2 + (d - a)xy - by^2)$ where we write $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, and where we choose the sign ± 1 such that the orientations of c and c_Q match. If on the other hand c is infinite, then we find $p, q \in \mathbb{P}_1(\mathbb{Q})$ with $c = c_{p,q}$. Hence $c = c_Q$ for $Q(x, y) = n(x - py)(x - qy)$ where we choose $n \in N\mathbb{Z}$, $n \neq 0$, such that Q is integral, and such that the orientations of c and c_Q agree. $\square$

Sometimes it will be useful to extend the action of $\Gamma_0(N)$ on $\mathcal{Q}$ to an action of $\mathrm{SL}_2(\mathbb{R})$ on the set of all not necessarily integral binary quadratic forms $Q(x, y) = ax^2 + bxy + cy^2$ with $a, b, c \in \mathbb{R}$. In this context we also extend the definitions of geodesics c_Q, cusps p_Q and points in the upper half-plane τ_Q to non-integral quadratic forms Q. One easily checks that the identities from (2.3.3) still hold, i.e., that

$$c_{Q.\alpha} = \alpha^{-1} c_Q, \qquad p_{Q.\alpha} = \alpha^{-1} p_Q \qquad \text{and} \qquad \tau_{Q.\alpha} = \alpha^{-1} \tau_Q$$

for $Q(x, y) = ax^2 + bxy + cy^2$ with $a, b, c \in \mathbb{R}$ and $\alpha \in \mathrm{SL}_2(\mathbb{R})$.

The following lemma shows that using scaling matrices associated to the geodesic c_Q, the parabolic point p_Q or the point $\tau_Q \in \mathbb{H}$ we can transform an integral binary quadratic form $Q \not\equiv 0$ into an (in general non-integral) binary quadratic form of one of the following standard types:

Lemma 2.3.3. *Let $Q \in \mathcal{Q}$ with $Q(x,y) = ax^2 + bxy + cy^2$, $Q \not\equiv 0$ and $\mu = |\Delta(Q)|^{1/2}$.*

(a) If $\Delta(Q) > 0$ then

$$Q.\sigma = \begin{pmatrix} 0 & \mu/2 \\ \mu/2 & 0 \end{pmatrix} \tag{2.3.5}$$

for some $\sigma \in \mathrm{SL}_2(\mathbb{R})$ if and only if σ maps c_0 to c_Q preserving orientations. Here c_0 is the standard geodesic from 0 to ∞.

(b) If $\Delta(Q) = 0$ then

$$Q.\sigma = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_Q \end{pmatrix} \tag{2.3.6}$$

for $\sigma \in \mathrm{SL}_2(\mathbb{R})$ if and only if σ is a parabolic scaling matrix for the cusp p_Q. Here $\lambda_Q \in \mathbb{Q}^*$ is given by

$$\lambda_Q := \operatorname{sign}(a+c)\frac{(a,c)(a/N,b/2)}{(a,b/2)},$$

which simplifies to $\lambda_Q = \operatorname{sign}(a+c)(a/N,c)$ if N is squarefree.

(c) If $\Delta(Q) < 0$ then

$$Q.\sigma = \operatorname{sign}(a)\begin{pmatrix} \mu/2 & 0 \\ 0 & -\mu/2 \end{pmatrix} \tag{2.3.7}$$

for $\sigma \in \mathrm{SL}_2(\mathbb{R})$ if and only if $\sigma i = \tau_Q$.

Proof. Firstly, let $\Delta(Q) > 0$ and $\sigma \in \mathrm{SL}_2(\mathbb{R})$ mapping c_0 to c_Q. Then $Q.\sigma$ needs to be of the form $\left(\begin{smallmatrix} 0 & \lambda \\ \lambda & 0 \end{smallmatrix}\right)$ with $\lambda \in \mathbb{R}^*$. Further, λ is independent of the choice of σ as $\Delta(Q.\sigma) = \Delta(Q)$ yields $\lambda = \pm\mu/2$. We find that $\lambda = \mu/2$ by choosing

$$\sigma = (4|a|\mu)^{-1/2}\begin{pmatrix} -\operatorname{sign}(a)(b+\mu) & b-\mu \\ 2|a| & -2a \end{pmatrix}$$

if $a \neq 0$, $\sigma = \left(\begin{smallmatrix} 1 & -c/b \\ 0 & 1 \end{smallmatrix}\right)$ if $a = 0$ and $b > 0$, and $\sigma = \left(\begin{smallmatrix} -c/b & -1 \\ 1 & 0 \end{smallmatrix}\right)$ if $a = 0$ and $b < 0$. This proves part (a).

Next let $\Delta(Q) = 0$ and let $\sigma \in \mathrm{SL}_2(\mathbb{R})$ be a parabolic scaling matrix for the cusp p_Q. Then $\sigma\infty = p_Q$, which implies $Q.\sigma = \left(\begin{smallmatrix} 0 & 0 \\ 0 & \lambda \end{smallmatrix}\right)$ for some $\lambda \in \mathbb{R}^*$. Moreover, if $\sigma' \in \mathrm{SL}_2(\mathbb{R})$ is another scaling matrix for p_Q then $\sigma^{-1}\sigma' = \left(\begin{smallmatrix} \pm 1 & h \\ 0 & \pm 1 \end{smallmatrix}\right)$ for some $h \in \mathbb{R}$, showing that λ does not depend on the choice of σ. If $a = 0$ the statement becomes trivial by choosing $\sigma = 1$. If on the other hand $a \neq 0$, we set $g = (a, b/2)$, $d = (a/g, N)$ and

$$\sigma = \sqrt{\frac{N}{d}}\begin{pmatrix} b/2g & \beta d/N \\ -a/g & \delta d/N \end{pmatrix}.$$

Here β, δ are integers with $a\beta + b\delta/2 = g$. Now a direct computation of $Q.\sigma$ shows that $\lambda = \frac{d}{N}Q(\beta,\delta)$. Since $d/N = (a/N, g)/g = (a/N, b/2)/g$ and $Q(\beta,\delta) = \operatorname{sign}(a)(a,c)$ we obtain the formula given in part (b) of the lemma.

For (c) let $\Delta(Q) < 0$ and $\sigma \in \mathrm{SL}_2(\mathbb{R})$ with $\sigma i = \tau_Q$. Then $(Q.\sigma)(i,1) = 0$ and thus $Q.\sigma = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & -\lambda \end{smallmatrix}\right)$ for some $\lambda \in \mathbb{R}^*$. Again, λ needs to be independent of the choice of σ since $\Delta(Q.\sigma) = \Delta(Q)$ implies $\lambda = \pm\mu/2$. By choosing

$$\sigma = (2\mu|a|)^{-1/2}\begin{pmatrix} \operatorname{sign}(a)\mu & -b \\ 0 & 2a \end{pmatrix}$$

we find that the sign of λ equals the sign of a.

Finally, we note that the converse of the proven implications in (a), (b) and (c) are indeed trivial, which is why we did not mention them. $\square$

In particular, the previous Lemma helps us to understand the quadratic forms of discriminant zero, which turn out to correspond to cusps of the underlying group $\Gamma_0(N)$. More precisely, if N is squarefree there is only one class $\mathcal{Q}_{\beta,0}$ of forms of discriminant zero, namely $\mathcal{Q}_{0,0}$, which can be identified with the set of cusps $C(\Gamma_0(N))$ as follows:

Lemma 2.3.4. *Let N be squarefree. Then the map*

$$\mathcal{Q}_{0,0} \setminus \{0\} \to \mathbb{P}^1(\mathbb{Q}) \times (\mathbb{Z} \setminus \{0\}),\ Q \mapsto (p_Q, \lambda_Q)$$

is a bijection, where $\lambda_Q = \operatorname{sign}(a+c)(a,c)$ for $Q(x,y) = aNx^2 + bxy + cy^2$. Further, the above map is compatible with the corresponding actions of $\Gamma_0(N)$ where $\Gamma_0(N)$ acts trivially on $\mathbb{Z} \setminus \{0\}$, i.e., for $M \in \Gamma_0(N)$ the element $Q.M$ is mapped to $(M^{-1}p_Q, \lambda_Q)$. Thus the induced map

$$\left(\mathcal{Q}_{0,0} \setminus \{0\}\right)/\Gamma_0(N) \to C(\Gamma_0(N)) \times (\mathbb{Z} \setminus \{0\}),\ [Q] \mapsto ([p_Q], \lambda_Q)$$

is again a bijection.

Proof. We start by noting that given $Q, Q' \in \mathcal{Q}_{0,0} \setminus \{0\}$ we have

$$Q.\sigma_{p_Q} = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_Q \end{pmatrix} \qquad \text{and} \qquad Q'.\sigma_{p_{Q'}} = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_{Q'} \end{pmatrix}$$

where $\sigma_{p_Q}, \sigma_{p_{Q'}} \in \mathrm{SL}_2(\mathbb{R})$ are parabolic scaling matrices for the cusps p_Q, $p_{Q'}$, respectively. Thus, if $p_Q = p_{Q'}$ and $\lambda_Q = \lambda_{Q'}$ we find

$$Q = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_Q \end{pmatrix}.\sigma_{p_Q}^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_{Q'} \end{pmatrix}.\sigma_{p_{Q'}}^{-1} = Q'$$

as σ_{p_Q} and $\sigma_{p_{Q'}}$ are both scaling matrices for the same cusp $p_Q = p_{Q'}$. Hence, the given map $Q \mapsto (p_Q, \lambda_Q)$ is indeed injective.

Further, given $p \in \mathbb{P}^1(\mathbb{Q})$ and $\lambda \in \mathbb{Z}$ with $\lambda \neq 0$ we can define $Q = \left(\begin{smallmatrix} 0 & 0 \\ 0 & \lambda \end{smallmatrix}\right)\sigma_p^{-1}$ where σ_p is a scaling matrix for the cusp p. Then $Q \in \mathcal{Q}$ since $\sigma_p\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\sigma_p^{-1} \in \Gamma_0(N)$, and by construction Q is mapped to (p,λ). So the given map is also surjective.

In order to show that the map $Q \mapsto (p_Q, \lambda_Q)$ is compatible with the actions of $\Gamma_0(N)$, we only need to check that $\lambda_{Q.M} = \lambda_Q$ for $Q \in \mathcal{Q}_{0,0}$ with $Q \not\equiv 0$ and $M \in \Gamma_0(N)$, since we already know that $p_{Q.M} = M^{-1}p_Q$. Let $\sigma_{p_Q} \in \mathrm{SL}_2(\mathbb{R})$ be a scaling matrix for p_Q. Then $M^{-1}\sigma_{p_Q}$ is a scaling matrix for $M^{-1}p_Q$ and thus

$$\begin{pmatrix} 0 & 0 \\ 0 & \lambda_{Q.M} \end{pmatrix} = (Q.M).\sigma_{Q.M} = (Q.M).(M^{-1}\sigma_{p_Q}) = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_Q \end{pmatrix}$$

proving the claim. □

Given $Q \in \mathcal{Q}$ with $Q(x, y) = ax^2 + bxy + cy^2$ we also define the real-valued rational function $p_Q(\tau)$ by

$$p_Q(\tau) := \frac{a|\tau|^2 + bu + c}{v} \tag{2.3.8}$$

for $\tau = u + iv \in \mathbb{H}$. It is easy to check that

$$p_Q(\tau)^2 := \frac{|Q(\tau, 1)|^2}{v^2} - \Delta(Q). \tag{2.3.9}$$

In particular, we note that if $\Delta(Q) > 0$ the geodesic c_Q is given as the zero set of the rational function $p_Q(\tau)$ in $\mathbb{H}$ (without orientation).

2.4 Holomorphic modular forms

Throughout this section let N be a positive integer. The action of $\mathrm{SL}_2(\mathbb{R})$ on the upper half-plane gives rise to a linear action of $\mathrm{SL}_2(\mathbb{R})$ on the space of complex-valued functions $f\colon \mathbb{H} \to \mathbb{C}$ via

$$(f\,|_k\,\alpha)(\tau) := j(\alpha, \tau)^{-k} f(\alpha\tau)$$

for $\tau \in \mathbb{H}$ where k is an arbitrary integer and $j(\alpha, \tau) = c\tau + d$ for $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$. We call this the *weight k action* of $\mathrm{SL}_2(\mathbb{R})$. Furthermore, we say that f is *modular* of weight k and level N if

$$f\,\big|_k\, M = f$$

for all $M \in \Gamma_0(N)$.

Let $f\colon \mathbb{H} \to \mathbb{C}$ be meromorphic and modular of weight k and level N. Further, let p be a cusp of $\Gamma_0(N)$ and let $\sigma_p \in \mathrm{SL}_2(\mathbb{R})$ be a parabolic scaling matrix corresponding to p. Then $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \sigma_p^{-1}\Gamma_0(N)\sigma_p$, and thus $(f\,|_k\,\sigma_p)(\tau + 1) = (f\,|_k\,\sigma_p)(\tau)$. Therefore, the meromorphic function $f\,|_k\,\sigma_p$ has a Fourier expansion of the form

$$(f\,|_k\,\sigma_p)(\tau) = \sum_{n\in\mathbb{Z}} a_f(n; \sigma_p) e(n\tau) \tag{2.4.1}$$

with Fourier coefficients $a_f(n; \sigma_p) \in \mathbb{C}$. Here we use the notation

$$e(z) := e^{2\pi i z}$$

for $z \in \mathbb{C}$. We say that f is *meromorphic*, *holomorphic* or *vanishing* at the cusp p if there is $T > 0$ such that $f\,|_k\,\sigma_p$ is holomorphic for $\mathrm{Re}(\tau) > T$, and if $a_f(n; \sigma_p) = 0$ for all but finitely many $n < 0$, for all $n < 0$ or for all $n \leq 0$, respectively. Moreover, if f is meromorphic at p and $f \not\equiv 0$ we define the *order* of f at the cusp p as

$$\operatorname{ord}_p(f) := \min\{n \in \mathbb{Z}\colon a_f(n; \sigma_p) \neq 0). \tag{2.4.2}$$

One can check that these definitions are indeed independent of the choice of representative p and the corresponding scaling matrix σ_p. However, the Fourier coefficients $a_f(n; \sigma_p) \in \mathbb{C}$ do depend on p and σ_p. If $p = \infty$ we always choose $\sigma_p = 1$ and simply write

$$a_f(n) := a_f(n; 1)$$

for the corresponding Fourier coefficients.

Let k be an integer. We call a function $f\colon \mathbb{H} \to \mathbb{C}$ *weakly holomorphic modular form*, *modular form* or *cusp form* of weight k and level N if f is holomorphic on $\mathbb{H}$, modular of weight k and level N, and meromorphic, holomorphic or vanishing at all cusps of $\Gamma_0(N)$, respectively. Correspondingly, we call $f\colon \mathbb{H} \to \mathbb{C}$ a *weakly meromorphic modular form*, *meromorphic modular form* or *meromorphic cusp form* of weight k and level N if f is modular of weight k and level N, and meromorphic, holomorphic or vanishing at all cusps of $\Gamma_0(N)$, respectively, but only meromorphic on $\mathbb{H}$.

We denote the complex vector spaces of weakly holomorphic modular form, modular forms or cusp forms of weight k and level N by $M_k^!(N)$, $M_k(N)$ or $S_k(N)$, respectively. Here the spaces $M_k(N)$ and $S_k(N)$ are finite-dimensional for every k and N, and trivial if k is negative or odd. Further, the only modular forms of weight 0 are the constant functions.

We will often write a weakly meromorphic modular form f in terms of its Fourier expansion at the cusp ∞, i.e., as

$$f(\tau) = \sum_{n \gg -\infty} a_f(n) e(n\tau).$$

Further, we recall that for $k \geq 2$ even the space of cusp forms $S_k(N)$ can be equipped with an inner product, the so-called *Petersson inner product*, given by

$$(f, g) := \int_{Y_0(N)} f(\tau)\overline{g(\tau)} \operatorname{Im}(\tau)^k d\mu(\tau) \tag{2.4.3}$$

for $f, g \in S_k(N)$. Here the integral is well-defined since the integrand is modular of weight 0, and since f and g vanish at ∞ the integral converges.

2.4.1 Holomorphic Eisenstein series of weight k

The simplest example of a modular form is obtained by averaging the constant one-function over the group $\Gamma_0(N)$ modulo the corresponding stabilizer, namely $(\Gamma_0(N))_\infty$, containing all elements of the form $\left(\begin{smallmatrix} \pm 1 & b \\ 0 & \pm 1 \end{smallmatrix}\right)$ with $b \in \mathbb{Z}$. The following definition generalizes this idea to arbitrary cusps of $\Gamma_0(N)$.

Definition 2.4.1. Let $k \in \mathbb{Z}$ be even with $k \geq 4$ and let p be a cusp of $\Gamma_0(N)$. The *Eisenstein series* of weight k and level N associated to the cusp p is defined by

$$E_{k,p}(\tau) = \sum_{M \in (\Gamma_0(N))_p \backslash \Gamma_0(N)} 1 \Big|_k \sigma_p^{-1} M$$

for $\tau \in \mathbb{H}$. Here $\sigma_p \in \mathrm{SL}_2(R)$ is a parabolic scaling matrix associated to the cusp p.

The sum defining the series is absolutely and locally uniformly convergent for $k \geq 4$ even. Moreover, it is independent of the choice of scaling matrix since for $\sigma_p, \sigma_p' \in \mathrm{SL}_2(\mathbb{R})$ both satisfying (2.2.1) we have $\sigma_p^{-1}\sigma_p' = \left(\begin{smallmatrix} \pm 1 & b \\ 0 & \pm 1 \end{smallmatrix}\right)$ for some $b \in \mathbb{R}$ and thus

$$1 \Big|_k \sigma_p^{-1} = \left(1 \Big|_k \begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix}\right) \Big|_k \sigma_p'^{-1} = (\pm 1)^{-k} \Big|_k \sigma_p'^{-1} = 1 \Big|_k \sigma_p'^{-1},$$

as k is even. Hence, the Eisenstein series $E_{k,p}$ is a well-defined holomorphic function on $\mathbb{H}$, which is by construction modular of weight k and level N.

Proposition 2.4.2. *Let $k \in \mathbb{Z}$ be even with $k \geq 4$ and let p be a cusp of $\Gamma_0(N)$. The Eisenstein series $E_{k,p}$ is a modular form of weight k and level N, vanishing at all cusps but p.*

We omit the corresponding proof and instead refer to Section 2.6 of [Miy06], where the above Proposition is given (in a more general setting) as Theorem 2.6.9.

Note that since $M, M' \in \Gamma_0(N)$ have the same bottom row if and only if MM'^{-1} is of the form $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ with $b \in \mathbb{Z}$, the holomorphic Eisenstein series $E_{k,\infty}$ can also be written as

$$E_{k,\infty}(\tau) = 1 + \sum_{\substack{c>0 \\ N|c}} \sum_{\substack{d \in \mathbb{Z} \\ (c,d)=1}} (c\tau + d)^{-k} = \frac{1}{2\zeta(k)} \sum_{(c,d) \in \mathbb{Z}^2 \setminus \{0\}} (Nc\tau + d)^{-k}. \tag{2.4.4}$$

Here $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the usual Riemann zeta function. Thus, up to the constant factor $\frac{1}{2\zeta(k)}$ we can understand $E_{k,\infty}$ as the sum of all $-k$'th powers of linear polynomials of the form $Nc\tau+d$. Therefore, it is natural to try to replace the sum of linear polynomials by quadratic ones.

2.4.2 Zagier's cusp forms associated to discriminants

In his famous work [Zag75b] Zagier introduced modular forms defined as the sum of all $-k$-th powers of quadratic forms of a given non-negative discriminant Δ. For $\Delta = 0$ this is essentially the holomorphic Eisenstein series of weight $2k$ and level 1, and for positive discriminant theses forms appear as the Fourier coefficients of the holomorphic kernel function of the Shimura and Shintani lift between half-integral and integral weight cusp forms (see [KZ81] and [Koh85]).

Later on analogous functions have been defined and studied in the case of negative discriminants by Bengoechea (see [Ben13] and [Ben15]). These functions turn out to be meromorphic on the upper half-plane with poles at all Heegner points of the given (negative) discriminant.

Definition 2.4.3. Let $k \in \mathbb{Z}$ with $k \geq 2$. Given $Q \in \mathcal{Q}$ with $Q \not\equiv 0$ we define the associated modular form as

$$f_{k,Q}(\tau) = \sum_{Q' \in [Q]} Q'(\tau, 1)^{-k}$$

for $\tau \in \mathbb{H} \setminus H_Q$, where H_Q is given as in (2.3.4).

One can check that for $k \geq 2$ the sum defining $f_{k,Q}(\tau)$ is absolutely and locally uniformly convergent for $\tau \in \mathbb{H} \setminus H_Q$. Thus, $f_{k,Q}$ is a well-defined meromorphic function on $\mathbb{H}$, which is holomorphic if $\Delta(Q) \geq 0$, and which is holomorphic up to poles at the $\Gamma_0(N)$-translates of the Heegner point τ_Q if $\Delta(Q) < 0$. Moreover, since

$$(Q'.M)(\tau, 1) = j(M, \tau)^2 Q'(M\tau, 1) \tag{2.4.5}$$

for $Q' \in \mathcal{Q}$, $M \in \Gamma_0(N)$ and $\tau \in \mathbb{H}$ the function $f_{k,Q}$ is modular of weight $2k$ and level N. Furthermore, if $k \geq 2$ is odd and $Q \sim -Q$ then all terms in the sum defining $f_{k,Q}$ cancel, giving $f_{k,Q} \equiv 0$. We noted earlier that this can only happen if $\Delta(Q) > 0$.

Proposition 2.4.4. *Let $k \geq 2$ and let $Q \in \mathcal{Q}$ with $Q \not\equiv 0$. The function $f_{k,Q}$ is a cusp form, modular form or meromorphic cusp form of weight $2k$ and level N if $\Delta(Q) > 0$, $\Delta(Q) = 0$ or $\Delta(Q) < 0$, respectively.*

As for the holomorphic Eisenstein series we omit the proof of this proposition. Instead, we refer to the Appendix 2 in [Zag75b] for $Q \in \mathcal{Q}$ with $\Delta(Q) \geq 0$, and to Proposition 2.1 and 2.2 in [Ben15] for the case of a negative discriminant. Though Zagier and Bengoechea only deal with the case $N = 1$, the general case stated here follows analogously.

As mentioned earlier, for $Q \in \mathcal{Q}$ with $Q \not\equiv 0$ and $\Delta(Q) = 0$ the function $f_{k,Q}$ is essentially the Eisenstein series of weight $2k$ associated to the corresponding cusp p_Q. More precisely, we have the following proposition:

Proposition 2.4.5. *Let $k \geq 2$ and let $Q \in \mathcal{Q}$ with $\Delta(Q) = 0$ and $Q \not\equiv 0$. Then up to a constant factor $f_{k,Q}$ equals the Eisenstein series of weight $2k$ and level N associated to the cusp p_Q, i.e.,*

$$f_{k,Q}(\tau) = \lambda_Q^{-k}\, E_{2k,p_Q}(\tau)$$

for $\tau \in \mathbb{H}$. Here λ_Q is the factor given in part (b) of Lemma 2.3.3.

We omit the corresponding proof since this will be a special case of Proposition 2.5.6 evaluated at $s = 0$ for $k \geq 2$.

Instead of just summing over a single equivalence class of a given quadratic form we may also sum over all quadratic forms of a given class and discriminant, which is what Zagier originally did in [Zag75b]. These functions can be regarded as averaged versions of the meromorphic modular forms $f_{k,Q}$ defined above.

Definition 2.4.6. Let $k \in \mathbb{Z}$ with $k \geq 2$. Given $\beta \in \mathbb{Z}/2N\mathbb{Z}$ and $\Delta \in \mathbb{Z}$ with $\Delta \equiv \beta^2 \bmod 4N$ we define the associated modular forms as

$$f_{k,\beta,\Delta}(\tau) = \sum_{\substack{Q \in \mathcal{Q}_{\beta,\Delta} \\ Q \not\equiv 0}} Q(\tau, 1)^{-k}$$

for $\tau \in \mathbb{H} \setminus H_{\beta,\Delta}$, where $H_{\beta,\Delta}$ is given as in (2.3.2).

Again, the above sum converges absolutely and locally uniformly for $k \geq 2$, and thus defines a meromorphic function on $\mathbb{H}$, which is holomorphic if $\Delta \geq 0$, and which is holomorphic up to poles at all Heegner points of class β and discriminant Δ if $\Delta < 0$. Clearly, we can write

$$f_{k,\beta,\Delta}(\tau) = \sum_{\substack{Q \in \mathcal{Q}_{\beta,\Delta}/\Gamma_0(N) \\ Q \not\equiv 0}} f_{k,Q}(\tau), \tag{2.4.6}$$

where the sum on the right-hand side is finite if $\Delta \neq 0$. Thus, Proposition 2.4.4 implies:

Corollary 2.4.7. *Let $k \geq 2$, $\beta \in \mathbb{Z}/2N\mathbb{Z}$ and $\Delta \in \mathbb{Z}$ with $\Delta \equiv \beta^2 \bmod 4N$. The function $f_{k,\beta,\Delta}(\tau)$ is a cusp form, modular form or meromorphic cusp form of weight $2k$ and level N if $\Delta > 0$, $\Delta = 0$ or $\Delta < 0$, respectively.*

We further note that, if $k \geq 2$ is odd and $\beta \equiv -\beta$ in $\mathbb{Z}/2N\mathbb{Z}$ then $\mathcal{Q}_{\beta,\Delta} = -\mathcal{Q}_{\beta,\Delta}$, and thus the modular form $f_{k,\beta,\Delta}$ vanishes completely in this case. In contrast to the functions $f_{k,Q}$ defined before, whose defining sum could only cancel if $\Delta(Q) > 0$, the vanishing of the functions $f_{k,\beta,\Delta}$ does not depend on sign of the discriminant Δ.

Moreover, if $\Delta = 0$ we can rewrite the sum in (2.4.6) as an (infinite) sum of holomorphic Eisenstein series using Proposition 2.4.5. Further assuming that N is squarefree, there is only one modular form $f_{k,\beta,0}$ of weight $2k$ and discriminant 0, namely $f_{k,0,0}$, and this function takes the following nice form:

Corollary 2.4.8. *Let N be squarefree and let $k \geq 2$. If k is even then*

$$f_{k,0,0}(\tau) = 2\zeta(k) \sum_{p \in C(\Gamma_0(N))} E_{2k,p}(\tau)$$

for $\tau \in \mathbb{H}$, and if k is odd the function $f_{k,0,0}$ vanishes identically, i.e, $f_{k,0,0} \equiv 0$.

Proof. By Proposition 2.4.5 we have

$$f_{k,0,0}(\tau) = \sum_{\substack{Q \in \mathcal{Q}_{0,0}/\Gamma_0(N) \\ Q \not\equiv 0}} \lambda_Q^{-k} E_{2k,p_Q}(\tau).$$

Further, Lemma 2.3.4 states that the map $[Q] \mapsto ([p_Q], \lambda_Q)$ is a bijection between classes of quadratic forms in $(\mathcal{Q}_{0,0} \setminus \{0\})/\Gamma_0(N)$ and tuples in $C(\Gamma_0(N)) \times (\mathbb{Z} \setminus \{0\})$, giving

$$f_{k,0,0}(\tau) = \left(\sum_{n \in \mathbb{Z} \setminus \{0\}} n^{-k} \right) \left(\sum_{p \in C(\Gamma_0(N))} E_{2k,p}(\tau) \right).$$

Now if k is even the first sum is simply twice the Riemann zeta function, and if k is odd the sum cancels completely. This proves the claimed statement. □

2.5 Non-holomorphic modular forms

In the present section we introduce weight 0 analogs of the functions $f_{k,Q}$ and $f_{k,\beta,\Delta}$ defined in Secion 2.4.2, whose defining sums do not converge for $k < 2$. As in the previous section we start by considering Eisenstein series which we may then generalize.

2.5.1 Non-holomorphic Eisenstein series of weight k

Since there are no holomorphic modular forms of weight $k < 4$ and level N (up to the constant functions which are modular forms of weight 0) it is natural to loosen some of the corresponding conditions in order to find interesting modular objects of smaller weight. Classically, an Eisenstein series of weight $k < 4$ was defined by replacing the constant one-function with some other simple $(\Gamma_0(N))_\infty$-invariant function to average over, namely the function $\tau \mapsto \mathrm{Im}(\tau)^s$, where s is some complex parameter guaranteeing convergence. However, because of the term $\mathrm{Im}(\tau)$ this new Eisenstein series is not holomorphic in τ.

Definition 2.5.1. Let $k \in \mathbb{Z}$ be even and let p be a cusp of $\Gamma_0(N)$. The *non-holomorphic Eisenstein series* of weight k and level N associated to the cusp p is defined by

$$E_{k,p}(\tau, s) = \sum_{M \in (\Gamma_0(N))_p \backslash \Gamma_0(N)} \mathrm{Im}(\tau)^s \Big|_k \sigma_p^{-1} M$$

for $\tau \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1 - k/2$. Here $\sigma_p \in \mathrm{SL}_2(\mathbb{R})$ is a parabolic scaling matrix for the cusp p.

For fixed $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1 - k/2$ the sum defining $E_{k,p}(\tau, s)$ is absolutely and locally uniformly convergent in τ. Thus, in the variable τ the function $E_{k,p}(\tau, s)$ is real analytic and modular of weight k and level N. Moreover, one can check that it satisfies the differential equation

$$\Delta_k E_{k,p}(\tau, s) = s(1 - k - s) E_{k,p}(\tau, s), \tag{2.5.1}$$

where Δ_k is the hyperbolic Laplace operator of weight k defined by

$$\Delta_k := -v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + ikv \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right) \tag{2.5.2}$$

for $\tau = u + iv$. It is invariant under the weight k action of $\mathrm{SL}_2(\mathbb{R})$, i.e., we have

$$\Delta_k \left(f \big|_k \alpha \right) = (\Delta_k f) \big|_k \alpha \tag{2.5.3}$$

for any $f \colon \mathbb{H} \to \mathbb{C}$ two times continuously differentiable in u and v and any $\alpha \in \mathrm{SL}_2(\mathbb{R})$.

For $k \geq 4$ even we may evaluate the non-holomorphic Eisenstein series $E_{k,p}(\tau, s)$ at $s = 0$, which clearly yields the holomorphic Eisenstein series $E_{k,p}(\tau)$. Thus the above definition indeed generalises Definition 2.4.1.

Fixing $\tau \in \mathbb{H}$ the Eisenstein series $E_{k,p}(\tau, s)$ defines a holomorphic function in s on the half-plane $\mathrm{Re}(s) > 1 - k/2$, which has a meromorphic continuation to the whole complex plane. It is an interesting problem to evaluate this continuation at the points $s = 0$ and $s = 1 - k$ since the continued Eisenstein series needs to be harmonic at these points according to equation (2.5.1). In particular, for $N = 1$, $k = 0$ and $p = \infty$ the classical Kronecker limit formula states that

$$E_{0,\infty}(\tau, s) = \frac{3/\pi}{s - 1} - \frac{1}{2\pi} \log(|\Delta(\tau)| \, \mathrm{Im}(\tau)^6) + C + O(s - 1) \tag{2.5.4}$$

as $s \to 1$, where $\Delta(\tau)$ is the unique normalized cusp form of weight 12 and level 1, and $C = (6 - 72\zeta'(-1) - 6\log(4\pi))/\pi$. Using the functional equation of $E_{0,\infty}(\tau, s)$ relating s and $1 - s$ we obtain the cleaner Laurent expansion

$$E_{0,\infty}(\tau, s) = 1 + \log(|\Delta(\tau)|^{1/6} \, \mathrm{Im}(\tau)) \cdot s + O(s^2) \tag{2.5.5}$$

at $s = 0$.

2.5.2 Non-holomorphic analogs of Zagier's cusp forms

To motivate the definition of non-holomorphic modular forms generalizing the meromorphic modular forms introduced in Definition 2.4.3 and Definition 2.4.6, we note that as in the holomorphic case (compare equation (2.4.4)) we can write

$$E_{2k,\infty}(\tau,s) = \frac{1}{2\zeta(2k+2s)} \sum_{(c,d)\in\mathbb{Z}^2\setminus\{0\}} \frac{v^s}{(Nc\tau+d)^{2k}|Nc\tau+d|^{2s}}$$

for $\tau \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1-k$. This motivates the following definition:

Definition 2.5.2. Let $k \in \mathbb{Z}$.

(a) Given $Q \in \mathcal{Q}$ with $Q \not\equiv 0$ we define the associated non-holomorphic modular forms as

$$f_{k,Q}(\tau,s) = \sum_{Q'\in[Q]} \frac{v^s}{Q'(\tau,1)^k|Q'(\tau,1)|^s}$$

for $\tau \in \mathbb{H} \setminus H_Q$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1-k$.

(b) Given $\beta \in \mathbb{Z}/2N\mathbb{Z}$ and $\Delta \in \mathbb{Z}$ with $\Delta \equiv \beta^2 \bmod 4N$ we define the associated non-holomorphic modular form as

$$f_{k,\beta,\Delta}(\tau,s) = \sum_{\substack{Q'\in\mathcal{Q}_{\beta,\Delta}\\ Q'\not\equiv 0}} \frac{v^s}{Q'(\tau,1)^k|Q'(\tau,1)|^s}$$

for $\tau \in \mathbb{H} \setminus H_{\beta,\Delta}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1-k$.

As for the Eisenstein series $E_{k,p}(\tau,s)$ the complex parameter s guarantees that for $\mathrm{Re}(s) > 1-k$ the above series both converge absolutely and locally uniformly in τ. Hence they define real analytic functions which are clearly modular of weight $2k$ and level N. For $k \geq 2$ we can simply evaluate these non-holomorphic modular forms at $s = 0$, and this evaluation yields the corresponding meromorphic modular forms from Section 2.4.2, i.e., we have

$$f_{k,Q}(\tau,0) = f_{k,Q}(\tau) \qquad \text{and} \qquad f_{k,\beta,\Delta}(\tau,0) = f_{k,\beta,\Delta}(\tau) \tag{2.5.6}$$

for $k \geq 2$. Moreover, we note that as in (2.4.6) we clearly have

$$f_{k,\beta,\Delta}(\tau,s) = \sum_{\substack{Q\in\mathcal{Q}_{\beta,\Delta}/\Gamma_0(N)\\ Q\not\equiv 0}} f_{k,Q}(\tau,s), \tag{2.5.7}$$

for $\tau \in \mathbb{H} \setminus H_{\beta,\Delta}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1-k$. As before, the sum on the right-hand side of (2.5.7) is finite if $\Delta \neq 0$.

In order to prove a differential equation for the functions $f_{k,Q}(\tau,s)$ and $f_{k,\beta,\Delta}(\tau,s)$ generalising the one given in (2.5.1), we firstly recall the differential operator ξ_k, which is defined by

$$\xi_k f(\tau) := 2iv^k \frac{\partial}{\partial\tau}\overline{f(\tau)} = 2iv^k \overline{\frac{\partial}{\partial\bar\tau} f(\tau)} \tag{2.5.8}$$

for $k \in \mathbb{Z}$, some function $f\colon \mathbb{H} \to \mathbb{C}$ and $\tau = u + iv$. Here the derivatives $\frac{\partial}{\partial \tau}$ and $\frac{\partial}{\partial \bar{\tau}}$ are given by

$$\frac{\partial}{\partial \tau} := \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right) \qquad \text{and} \qquad \frac{\partial}{\partial \bar{\tau}} := \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

If $f\colon \mathbb{H} \to \mathbb{C}$ is holomorphic then $\frac{\partial}{\partial \tau} f = f'$ and $\frac{\partial}{\partial \bar{\tau}} f = 0$. There is an interesting, though elementary relation between the hyperbolic Laplacian of weight k and the differential operators ξ_k and ξ_{2-k} which is given by

$$\Delta_k = -\xi_{2-k}\xi_k. \tag{2.5.9}$$

We now use this relation to compute the action of Δ_{2k} on the non-holomorphic modular forms given in Definition 2.5.2.

Lemma 2.5.3. *Let $k \in \mathbb{Z}$ and $Q \in \mathcal{Q}$ with $Q \not\equiv 0$. Then*

$$\Delta_{2k} f_{k,Q}(\tau, s) = s(1 - 2k - s) f_{k,Q}(\tau, s) + s(s + 2k)\Delta(Q) f_{k,Q}(\tau, s + 2)$$

for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1 - k$, and the same differential equation also holds for $f_{k,\beta,\Delta}(\tau, s)$ where $\beta \in \mathbb{Z}/2N\mathbb{Z}$ and $\Delta \in \mathbb{Z}$ with $\Delta \equiv \beta^2 \bmod 4N$.

Proof. Since Δ_{2k} is invariant under the weight $2k$ action of $\mathrm{SL}_2(\mathbb{R})$ it suffices to show that the above differential equation holds for $g(\tau, s) := v^s Q'(\tau, 1)^{-k}|Q'(\tau, 1)|^{-s}$ where $Q' \not\equiv 0$ is an arbitrary fixed quadratic form with $\Delta(Q') = \Delta(Q)$. To simplify notation we further set $g(\tau) := |Q'(\tau, 1)|^2/v^2$, such that $g(\tau, s) = Q'(\tau, 1)^{-k} g(\tau)^{-s/2}$. Firstly, we compute

$$\frac{\partial}{\partial \tau} p_{Q'}(\tau) = i\,\frac{Q'(\bar{\tau}, 1)}{2v^2} \qquad \text{and} \qquad \frac{\partial}{\partial \tau} g(\tau) = i\, p_{Q'}(\tau)\,\frac{Q'(\bar{\tau}, 1)}{v^2},$$

where $p_{Q'}(\tau)$ denotes the real valued function given in (2.3.8), and the second equality follows from the first one and the identity in (2.3.9). Recall that $\Delta_{2k} = -\xi_{2-2k}\xi_{2k}$ as in equation (2.5.9). We compute

$$\xi_{2k} g(\tau, s) = 2iv^{2k}\; Q'(\bar{\tau}, 1)^{-k}\left(-\frac{\bar{s}}{2}\right) g(\tau)^{-\bar{s}/2-1}\; \overline{\frac{\partial}{\partial \bar{\tau}} g(\tau)} = \bar{s}\; Q'(\tau, 1)^{k-1} g(\tau)^{-\bar{s}/2-k} p_{Q'}(\tau),$$

and thus

$$\Delta_{2k} g(\tau, s) = -2siv^{2-2k} Q'(\bar{\tau}, 1)^{k-1}\; \overline{\frac{\partial}{\partial \bar{\tau}}\left(g(\tau)^{-\bar{s}/2-k} p_{Q'}(\tau)\right)}. \tag{2.5.10}$$

Here

$$\overline{\frac{\partial}{\partial \bar{\tau}}\left(g(\tau)^{-\bar{s}/2-k} p_{Q'}(\tau)\right)} = g(\tau)^{-s/2-k-1}\left(\left(-\frac{s}{2} - k\right) p_{Q'}(\tau)\; \overline{\frac{\partial}{\partial \bar{\tau}} g(\tau)} + g(\tau)\; \overline{\frac{\partial}{\partial \bar{\tau}} p_{Q'}(\tau)}\right).$$

Hence equation (2.5.10) becomes

$$\begin{aligned}
\Delta_{2k} g(\tau, s) &= s \cdot g(\tau, s)\left((-s - 2k)\,\frac{p_{Q'}(\tau)^2}{g(\tau)} - \frac{2i\,v^2}{Q'(\bar{\tau}, 1)}\; \overline{\frac{\partial}{\partial \bar{\tau}} p_{Q'}(\tau)}\right)\\
&= s \cdot g(\tau, s)\left((-s - 2k)\left(1 - \frac{\Delta(Q')}{g(\tau)}\right) + 1\right)\\
&= s(1 - 2k - s) g(\tau, s) + s(s + 2k)\Delta(Q') g(\tau, s + 2).
\end{aligned}$$

This proves the claimed statement. ∎

In the following we establish a different representation of the non-holomorphic functions $f_{k,Q}(\tau,s)$, which turns out to be of particular interest in the case $k=0$. Recall that given $Q \in \mathcal{Q}$ with $\Delta(Q) \neq 0$ there is an associated geodesic c_Q or CM point τ_Q depending on the sign of $\Delta(Q)$. The following lemma realizes the quantity $v^{-1}|Q(\tau,1)|$ in a somewhat geometric way. Even though this identity is well-known we also give a proof as the proof is often omitted in the literature.

Lemma 2.5.4. *Let $Q \in \mathcal{Q}$ with $\Delta(Q) \neq 0$. Then*

$$\frac{|Q(\tau,1)|}{v} = \begin{cases} \Delta(Q)^{1/2} \cosh(d_{\mathrm{hyp}}(\tau, c_Q)), & \text{if } \Delta(Q) > 0, \\ |\Delta(Q)|^{1/2} \sinh(d_{\mathrm{hyp}}(\tau, \tau_Q)), & \text{if } \Delta(Q) < 0, \end{cases}$$

for $\tau = u + iv \in \mathbb{H}$. Here c_Q is the Heegner geodesic associated to Q, and τ_Q is the Heegner point associated to Q.

Proof. Let $\Delta(Q) > 0$. Further, let $\sigma \in \mathrm{SL}_2(\mathbb{R})$ be such that σ maps the standard geodesic c_0 from 0 to ∞ to the geodesic c_Q preserving orientations. Then $Q.\sigma = \left(\begin{smallmatrix} 0 & \mu/2 \\ \mu/2 & 0 \end{smallmatrix}\right)$ with $\mu = |\Delta(Q)|^{1/2}$ by Lemma 2.3.3, and thus

$$\cosh(d_{\mathrm{hyp}}(\tau, c_Q)) = \cosh(d_{\mathrm{hyp}}(\sigma^{-1}\tau, c_0)) = \frac{|\sigma^{-1}\tau|}{\mathrm{Im}(\sigma^{-1}\tau)} = \frac{|(Q.\sigma)(\sigma^{-1}\tau,1)|}{\mu\ \mathrm{Im}(\sigma^{-1}\tau)} = \frac{|Q(\tau,1)|}{\mu\ \mathrm{Im}(\tau)}.$$

Here we used the identity (2.1.2) for the second equality. On the other hand, if $\Delta(Q) < 0$ a direct computation shows that

$$\sinh^2(d_{\mathrm{hyp}}(\tau,\tau_Q)) = \cosh^2(d_{\mathrm{hyp}}(\tau,\tau_Q)) - 1 = \frac{p_Q(\tau)^2}{|\Delta(Q)|} - 1 = \frac{|Q(\tau,1)|^2}{v^2|\Delta(Q)|}$$

as claimed. Here $p_Q(\tau)$ is the rational function defined in (2.3.8). $\square$

In the following lemma we treat the case $\Delta(Q) = 0$, which is of slightly different nature.

Lemma 2.5.5. *Let $Q \in \mathcal{Q}$ with $\Delta(Q) = 0$ and $Q \not\equiv 0$. Then*

$$Q(\tau,1) = \lambda_Q\, j(\sigma_{p_Q}^{-1}, \tau)^2 \qquad \text{and} \qquad \frac{|Q(\tau,1)|}{v} = |\lambda_Q|\, \mathrm{Im}(\sigma_{p_Q}^{-1}\tau)^{-1}$$

for $\tau = u + iv \in \mathbb{H}$. Here λ_Q is the factor given in part (b) of Lemma 2.3.3, and σ_{p_Q} is a parabolic scaling matrix for the cusp p_Q.

Proof. We have

$$Q(\tau,1) = ((Q.\sigma_{p_Q}).\sigma_{p_Q}^{-1})(\tau,1) = j(\sigma_{p_Q}^{-1},\tau)^2 (Q.\sigma_{p_Q})(\sigma_{p_Q}^{-1}\tau,1) = \lambda_Q\, j(\sigma_{p_Q}^{-1},\tau)^2$$

as $Q.\sigma_{p_Q} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & \lambda_Q \end{smallmatrix}\right)$ by Lemma 2.3.3. $\square$

We now use the previous two lemmas to write the non-holomorphic function $f_{k,Q}(\tau,s)$ in a different, more geometric way:

Proposition 2.5.6. *Let $k \in \mathbb{Z}$ and $Q \in \mathcal{Q}$ with $Q \not\equiv 0$.*

(a) If $\Delta(Q) > 0$ *then*

$$f_{k,Q}(\tau, s) = \Delta(Q)^{-s/2} \sum_{M \in (\Gamma_0(N))_{c_Q} \backslash \Gamma_0(N)} Q(\tau, 1)^{-k} \cosh(d_{\mathrm{hyp}}(\tau, c_Q))^{-s} \Big|_{2k} M$$

for $\tau \in \mathbb{H}$ *and* $s \in \mathbb{C}$ *with* $\mathrm{Re}(s) > 1 - k$. *Here* c_Q *is the Heegner geodesic associated to* Q.

(b) If $\Delta(Q) = 0$ *then*

$$f_{k,Q}(\tau, s) = \lambda_Q^{-k} |\lambda_Q|^{-s} E_{2k,p_Q}(\tau, s).$$

for $\tau \in \mathbb{H}$ *and* $s \in \mathbb{C}$ *with* $\mathrm{Re}(s) > 1 - k$. *Here* p_Q *is the cusp associated to* Q, *and* λ_Q *is the factor defined in part (b) of Lemma 2.3.3.*

(c) If $\Delta(Q) < 0$ *then*

$$f_{k,Q}(\tau, s) = |\Delta(Q)|^{-s/2} \sum_{M \in (\Gamma_0(N))_{\tau_Q} \backslash \Gamma_0(N)} Q(\tau, 1)^{-k} \sinh(d_{\mathrm{hyp}}(\tau, \tau_Q))^{-s} \Big|_{2k} M$$

for $\tau \in \mathbb{H} \setminus H_Q$ *and* $s \in \mathbb{C}$ *with* $\mathrm{Re}(s) > 1 - k$. *Here* τ_Q *the Heegner point associated to* Q.

Proof. Recalling that $(Q'.M)(\tau, 1) = j(M, \tau)^2 Q'(M\tau, 1)$ for $Q' \in \mathcal{Q}$ and $M \in \Gamma_0(N)$ as in equation (2.4.5) we find

$$f_{k,Q}(\tau, s) = \sum_{M \in (\Gamma_0(N))_Q \backslash \Gamma_0(N)} Q(\tau, 1)^{-k} \left(\frac{|Q(\tau, 1)|}{v} \right)^{-s} \Big|_{2k} M \tag{2.5.11}$$

for $\tau = u + iv \in \mathbb{H} \setminus H_Q$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1 - k$. Depending on whether $\Delta(Q) \neq 0$ or $\Delta(Q) = 0$ we can now use Lemma 2.5.4 or Lemma 2.5.5 to obtain the new representation of $f_{k,Q}(\tau, s)$ given in the proposition. □

So part (b) of the previous proposition tells us that if $\Delta(Q) = 0$ the function $f_{k,Q}(\tau, s)$ is indeed simply a multiple of the non-holomorphic Eisenstein series of weight $2k$ associated to the cusp p_Q. Moreover, if we further assume that N is squarefree Lemma 2.3.4 implies the following analog of Corollary 2.4.8. We omit the corresponding proof since using the identity (2.5.7) the proof is completely analogously to the one of Corollary 2.4.8.

Corollary 2.5.7. *Let* N *be squarefree and let* $k \in \mathbb{Z}$. *If* k *is even then*

$$f_{k,0,0}(\tau, s) = 2\zeta(s + k) \sum_{p \in C(\Gamma_0(N))} E_{2k,p}(\tau, s)$$

for $\tau \in \mathbb{H}$ *and* $s \in \mathbb{C}$ *with* $\mathrm{Re}(s) > 1 - k$, *and if* k *is odd the function* $f_{k,0,0}(\tau, s)$ *vanishes identically, i.e,* $f_{k,0,0} \equiv 0$.

Also, for $k = 0$ and $\Delta(Q) \neq 0$ we can identify the new representations for the functions $f_{0,Q}(\tau, s)$ established in Proposition 2.5.6 as generalized non-holomorphic Eisenstein series of weight 0, which are called hyperbolic and elliptic Eisenstein series if $\Delta(Q) > 0$ or $\Delta(Q) < 0$, respectively. In the following final section of this chapter we give a brief introduction to these two types of Eisenstein series.

2.6 Non-holomorphic Eisenstein series of weight 0

In order to distinguish the classical non-holomorphic Eisenstein series of weight 0 defined in Definition 2.5.1 from the hyperbolic and elliptic Eisenstein series we are going to define in the following two subsections, we will call the classical Eisenstein series $E_{0,p}(\tau, s)$ *parabolic Eisenstein series* from now on, and given a cusp p we will use the notation

$$E_p^{\mathrm{par}}(\tau, s) := E_{0,p}(\tau, s) \tag{2.6.1}$$

for $\tau \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. In order to compare the parabolic Eisenstein series to the hyperbolic and elliptic Eisenstein series in the following subsections, we quickly recall from (2.5.1) that the parabolic Eisenstein series satisfies the differential equation

$$\Delta_0 E_p^{\mathrm{par}}(\tau, s) = s(1-s)E_p^{\mathrm{par}}(\tau, s), \tag{2.6.2}$$

and that for $N = 1$ the meromorphic continuation of the Eisenstein series $E_p^{\mathrm{par}}(\tau, s)$ in s to $\mathbb{C}$ has the Laurent expansion

$$E_\infty^{\mathrm{par}}(\tau, s) = 1 + \log(|\Delta(\tau)|^{1/6}\,\mathrm{Im}(\tau)) \cdot s + O(s^2) \tag{2.6.3}$$

at $s = 0$.

2.6.1 Hyperbolic Eisenstein series

In [KM79] Kudla and Millson introduced form valued non-holomorphic Eisenstein series associated to hyperbolic elements of a given Fuchsian group Γ of the first kind, calling their functions hyperbolic Eisenstein series. These Eisenstein series are modular 1-forms, which can be understood as non-holomorphic modular forms of weight 2, and the hyperbolic elements they are associated to correspond to closed geodesics in the in general non-compact Riemann surface $\Gamma\backslash\mathbb{H}$. As is mentioned in [KM79] (see the remark following Theorem 5.1 in [KM79]) scalar valued weight 2 analogs of these hyperbolic Eisenstein series have already been studied by Petersson in [Pet43].

More recently, scalar valued weight 0 analogs of Kudla's and Millson's hyperbolic Eisenstein series were investigated in [Ris04] for $\Gamma\backslash\mathbb{H}$ compact, and in [JKP10] for general Fuchsian groups of the first kind. We use their definition, i.e., the one given in [JKP10], restricting it to congruence subgroups $\Gamma_0(N)$, but also allowing the corresponding geodesics to be infinite.

Definition 2.6.1. Given a closed or infinite geodesic c in $\Gamma_0(N)\backslash\mathbb{H}$, we define the *hyperbolic Eisenstein series* of level N associated to c by

$$E_c^{\mathrm{hyp}}(\tau, s) = \sum_{M \in (\Gamma_0(N))_c \backslash \Gamma_0(N)} \cosh(d_{\mathrm{hyp}}(M\tau, c))^{-s}$$

for $\tau \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$.

The sum defining the hyperbolic Eisenstein series converges absolutely and locally uniformly in τ and s for $\tau \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. This is known if the geodesic c is closed (see for example [Ris04], Lemma 3.2), and has been shown in Section 3.3 of

[Fal11] if c is infinite for the case of weight 2, which easily generalizes to the present case of weight 0. So $E_c^{\mathrm{hyp}}(\tau, s)$ defines a real analytic function in τ, and a holomorphic function in s for $\mathrm{Re}(s) > 1$. Moreover, the hyperbolic Eisenstein series is by construction modular of level N and weight 0 in τ, and it satisfies the differential equation

$$\Delta_0 E_c^{\mathrm{hyp}}(\tau, s) = s(1-s)E_c^{\mathrm{hyp}}(\tau, s) + s^2 E_c^{\mathrm{hyp}}(\tau, s+2). \tag{2.6.4}$$

In contrast to the differential equation of the parabolic Eisenstein series given in (2.6.2), we discover the additional shifted term $s^2 E_c^{\mathrm{hyp}}(\tau, s+2)$. In particular, the hyperbolic Eisenstein series is not an eigenfunction of the hyperbolic Laplace operator Δ_0.

Furthermore, if c is a closed geodesic it has been shown in [JKP10] using spectral theory that the hyperbolic Eisenstein series $E_c^{\mathrm{hyp}}(\tau, s)$ has a meromorphic continuation in s to the whole complex plane, and that this continuation has a double zero at the distinguished point $s = 0$. Therefore, if c is closed the associated hyperbolic Eisenstein series has the Laurent expansion

$$E_c^{\mathrm{hyp}}(\tau, s) = O(s^2) \tag{2.6.5}$$

at $s = 0$. However, if the given geodesic c is infinite, [Fal11] only proves the meromorphic continuation of the corresponding hyperbolic Eisenstein series to the half-plane given by $\mathrm{Re}(s) > 1/2$. Indeed, the hyperbolic Eisenstein series associated to some infinite geodesic is not square-integrable, which is why the techniques from [JKP10] cannot be directly applied in this case.

2.6.2 Elliptic Eisenstein series

In 2004 Jorgenson and Kramer investigated elliptic analogs of the above hyperbolic Eisenstein series in their unpublished work [JK04] (see also [JK11]). These are non-holomorphic Eisenstein series, which are associated to elliptic (and more general arbitrary) points in the upper half-plane, instead of being associated to geodesics or cusps. Elliptic Eisenstein series were later studied in detail for arbitrary Fuchsian groups of the first kind by Kramer's student von Pippich in [Pip10] and [Pip16].

Definition 2.6.2. Given $\omega \in \mathbb{H}$, we define the *elliptic Eisenstein series* of level N associated to ω by

$$E_\omega^{\mathrm{ell}}(\tau, s) = \sum_{M \in (\Gamma_0(N))_\omega \backslash \Gamma_0(N)} \sinh(d_{\mathrm{hyp}}(M\tau, \omega))^{-s}$$

for $\tau \in \mathbb{H} \setminus \Gamma_0(N)\omega$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$.

The sum defining the elliptic Eisenstein series converges absolutely and locally uniformly in τ and s, and thus defines a real analytic function in τ and a holomorphic function in s for $\mathrm{Re}(s) > 1$. We note that it is indeed not necessary to assume that the point $\omega \in \mathbb{H}$ is elliptic. Moreover, the elliptic Eisenstein series satisfies the differential equation

$$\Delta_0 E_\omega^{\mathrm{ell}}(\tau, s) = s(1-s)E_\omega^{\mathrm{ell}}(\tau, s) - s^2 E_\omega^{\mathrm{ell}}(\tau, s+2), \tag{2.6.6}$$

which agrees with the differential equation of the hyperbolic Eisenstein series given in (2.6.4) up to a sign, and the elliptic Eisenstein series also has a meromorphic continuation in s to the whole complex plane which was proven in [Pip10]. Furthermore, von Pippich shows in [Pip16] that the elliptic Eisenstein series admits a Kronecker limit type formula as in (2.6.3), which for $N = 1$ and $\omega = i$ or $\omega = \rho = e^{\pi i/3}$ is given by

$$E_\omega^{\mathrm{ell}}(\tau, s) = -\log\left(|j(\tau) - j(\omega)|^{2/\operatorname{ord}(\omega)}\right) \cdot s + O(s^2) \tag{2.6.7}$$

as $s \to 0$. Here $j(\tau) := E_{4,\infty}(\tau)^3/\Delta(\tau)$ is the well-known modular j-function, which is the unique weakly holomorphic modular form of weight 0 and level 1 whose Fourier expansion at ∞ is of the form $j(\tau) = e(-\tau) + 744 + O(e(\tau))$.

2.6.3 Non-holomorphic Zagier cusp forms of weight 0

Using the parabolic, hyperbolic and elliptic Eisenstein series defined above, we may now restate Proposition 2.5.6 in the case of weight 0 in a simpler way:

Corollary 2.6.3. *Let $Q \in \mathcal{Q}$ with $Q \not\equiv 0$. Then*

$$f_{0,Q}(\tau, s) = \begin{cases} \Delta(Q)^{-s/2} E_{c_Q}^{\mathrm{hyp}}(\tau, s), & \text{if } \Delta(Q) > 0, \\ |\lambda_Q|^{-s} E_{p_Q}^{\mathrm{par}}(\tau, s), & \text{if } \Delta(Q) = 0, \\ |\Delta(Q)|^{-s/2} E_{\tau_Q}^{\mathrm{ell}}(\tau, s), & \text{if } \Delta(Q) < 0, \end{cases}$$

for $\tau \in \mathbb{H} \setminus H_Q$ and $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$.

Thus we may understand $f_{0,Q}(\tau, s)$ as a general type of non-holomorphic Eisenstein series of weight 0. This is clear in the parabolic case. Moreover, in the hyperbolic case this identification can for example be found in [Fal07]: Comparing our equation (2.5.11) with the first line of the proof of Proposition 1.1 in [Fal07], we find that for $Q \in \mathcal{Q}$ with $\Delta(Q) > 0$ the functions $f_{1,Q}(\tau, s)$ of weight 2 defined in this work can indeed be seen as scalar valued analogs of the form valued hyperbolic Eisenstein series defined in [Fal07].

We further remark that by Corollary 2.3.2 every hyperbolic Eisensteins series (in the sense of Definition 2.6.1) can be written in the form $E_c^{\mathrm{hyp}}(\tau, s) = \Delta(Q)^{s/2} f_{0,Q}(\tau, s)$ for some quadratic form Q. In particular, the convergence of the sum defining the non-holomorphic functions $f_{0,Q}(\tau, s)$ (see Definition 2.5.2) also implies the convergence of the sum defining the hyperbolic Eisenstein series $E_{c_Q}^{\mathrm{hyp}}(\tau, s)$, independent of whether the geodesic c_Q is closed or infinite. On the other hand, not every elliptic Eisenstein series can be written in the form $E_{\tau_Q}^{\mathrm{ell}}(\tau, s) = \Delta(Q)^{s/2} f_{0,Q}(\tau, s)$, since not every point in the upper half-plane is a Heegner point associated to some integral binary quadratic form.

At this point we also want to mention the work [IS09], where hyperbolic, parabolic and elliptic Poincaré series are recalled and studied. These are meromorphic modular forms which for parameter $m = 0$ become Eisenstein series, and they essentially agree with the meromorphic modular forms $f_{k,Q}(\tau)$ associated to quadratic forms defined in Section 2.4.2 of the present work, where the sign of the discriminant $\Delta(Q)$ determines whether the form $f_{k,Q}(\tau)$ corresponds to a hyperbolic, parabolic or elliptic Poincaré series in the sense of [IS09] (see for example Proposition 9 and 10 in [IS09] for the hyperbolic case).

Therefore, the hyperbolic, parabolic and elliptic Eisenstein series of weight 0 defined in this section can be seen as non-holomorphic weight 0 analogs of the mentioned hyperbolic, parabolic and elliptic Poincaré series for parameter $m = 0$.

Finally, we want to write the functions $f_{0,\beta,\Delta}(\tau, s)$ introduced in Definition 2.5.2 as averaged versions of non-holomorphic Eisenstein series of weight 0:

Corollary 2.6.4. *Let $\beta \in \mathbb{Z}/2N\mathbb{Z}$ and $\Delta \in \mathbb{Z}$ with $\Delta \equiv \beta^2 \bmod 4N$. Then*

$$f_{0,\beta,\Delta}(\tau, s) = \begin{cases} \Delta^{-s/2} \sum\limits_{Q \in \mathcal{Q}_{\beta,\Delta}/\Gamma_0(N)} E^{\mathrm{hyp}}_{c_Q}(\tau, s), & \text{if } \Delta > 0, \\ \sum\limits_{p \in C(\Gamma_0(N))} \lambda_{\beta,p}(s) E^{\mathrm{par}}_p(\tau, s), & \text{if } \Delta = 0, \\ |\Delta|^{-s/2} \sum\limits_{Q \in \mathcal{Q}_{\beta,\Delta}/\Gamma_0(N)} E^{\mathrm{ell}}_{\tau_Q}(\tau, s), & \text{if } \Delta < 0, \end{cases} \tag{2.6.8}$$

for $\tau \in \mathbb{H} \setminus H_{\beta,\Delta}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. *Here*

$$\lambda_{\beta,p}(s) := \sum_{\substack{Q \in (\mathcal{Q}_{\beta,0} \setminus \{0\})/\Gamma_0(N) \\ p_Q = p}} |\lambda_Q|^{-s}$$

for $p \in C(\Gamma_0(N))$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$.

Proof. The cases $\Delta > 0$ and $\Delta < 0$ follow directly from the identity (2.5.7) and Corollary (2.6.3). For the case $\Delta = 0$ we additionally note that every $Q \in \mathcal{Q}_{\beta,0}$ with $Q \not\equiv 0$ corresponds to a cusp p_Q. □

We note that the three sums given in equation (2.6.8) are all finite, running over the finitely many cusps of the modular curve $X_0(N)$ if $\Delta = 0$, or over the finitely many Heegner geodesics or Heegner points of class β and discriminant $\Delta > 0$ or $\Delta < 0$ modulo $\Gamma_0(N)$, respectively.

If $\Delta = 0$ the sum defining the factor $\lambda_{\beta,p}(s)$ given in Corollary 2.6.4 is either trivial (in case there is no $Q \in \mathcal{Q}_{\beta,0}$ with $p_Q = p$) or infinite. Assuming that the level N is squarefree $\beta = 0$ is the only class allowing $\Delta = 0$, and by Lemma 2.3.4 we have $\lambda_{0,p}(s) = 2\zeta(s)$ independent of the cusp p, giving

$$f_{0,0,0}(\tau, s) = 2\zeta(s) \sum_{p \in C(\Gamma_0(N))} E^{\mathrm{par}}_p(\tau, s) \tag{2.6.9}$$

for $\tau \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. This is a special case of Corollary 2.5.7.

2.6.4 The hyperbolic kernel function

We have seen in the Corollaries 2.6.3 and 2.6.4 that hyperbolic, parabolic and elliptic Eisenstein series are somehow deeply connected, even though their ad hoc definitions look quite different. We further support this idea by introducing the so-called hyperbolic kernel function for the group $\Gamma_0(N)$.

Definition 2.6.5. The *hyperbolic kernel function* of level N is defined by

$$K(\tau,\omega,s) = \sum_{M\in\Gamma_0(N)} \cosh(d_{\mathrm{hyp}}(M\tau,\omega))^{-s}$$

for $\tau,\omega\in\mathbb{H}$ and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1$.

The series converges absolutely and locally uniformly for $\mathrm{Re}(s)>1$. Thus $K(\tau,\omega,s)$ is a real analytic function in the variables τ and ω, and holomorphic in s for $\mathrm{Re}(s)>1$. Further, it is by definition modular of level N and weight 0 in both variables τ and ω, and it satisfies the differential equation

$$\Delta_0 K(\tau,\omega,s) = s(1-s)K(\tau,\omega,s) + s(s+1)K(\tau,\omega,s+2). \tag{2.6.10}$$

Also, the hyperbolic kernel function has a meromorphic continuation in s to all of $\mathbb{C}$ (see for example Proposition 5.1.4 in [Pip10]).

The following proposition shows that the hyperbolic, parabolic and elliptic Eisenstein series can indeed all be expressed in terms of the hyperbolic kernel function $K(\tau,\omega,s)$. However, in the hyperbolic case this relation only holds if the corresponding geodesic is closed.

Proposition 2.6.6.

(a) Let c be a closed geodesic in $\mathbb{H}$. Then

$$E_c^{\mathrm{hyp}}(\tau,s) = \frac{\Gamma(s)}{2^{s-1}\,\Gamma(s/2)^2}\int_{[c]} K(\tau,\omega,s)ds(\omega)$$

for $\tau\in\mathbb{H}$ and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1$. Here $\Gamma(z)$ denotes the usual Gamma function.

(b) Let p be a cusp and let $y>1$. Then

$$E_p^{\mathrm{par}}(\tau,s) = \frac{2^s(2s-1)\Gamma(s)^2}{4\pi\,\Gamma(2s)}y^{s-1}\sum_{n=0}^{\infty}\frac{(s/2)_n(s/2+1/2)_n}{n!\,(s+1/2)_n}\int_0^1 K(\tau,\sigma_p(x+iy),s+2n)dx$$

for $\tau\in\mathbb{H}$ with $\mathrm{Im}(M\tau)<y$ for all $M\in\Gamma_0(N)$ and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1$. Here $\sigma_p\in\mathrm{SL}_2(\mathbb{R})$ is a scaling matrix for the cusp p, and the right-hand side of the equation is independent of y. Moreover, $\Gamma(z)$ denotes the Gamma function as in part (a), and

$$(z)_n := \frac{\Gamma(z+n)}{\Gamma(z)} = z(z+1)\cdots(z+n-1) \tag{2.6.11}$$

denotes the Pochhammer symbol defined for $z\in\mathbb{C}$ and $n\in\mathbb{N}_0$.

(c) Let $\omega\in\mathbb{H}$. Then

$$E_\omega^{\mathrm{ell}}(\tau,s) = \frac{1}{\mathrm{ord}(\omega)}\sum_{n=0}^{\infty}\frac{(s/2)_n}{n!}K(\tau,\omega,s+2n)$$

for $\tau\in\mathbb{H}\setminus\Gamma_0(N)\omega$ and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1$. Here $(z)_n$ denotes the Pochhammer symbol as in part (b).

Proof. Part (a) is given as Proposition 11 in [JPS16], and part (c) as Remark 3.3.9 in [Pip10]. We thus only comment on part (b). By the well-known relation between the parabolic Eisenstein series and the hyperbolic Green's function we have

$$E_p^{\mathrm{par}}(\tau, s) = (2s-1)y^{s-1}\int_0^1 G_s(\tau, \sigma_p(x+iy))dx, \tag{2.6.12}$$

for $\tau \in \mathbb{H}$, $y \in \mathbb{R}$ with $y > \mathrm{Im}(M\tau)$ for all $M \in \Gamma_0(N)$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. This identity can for example be found as Proposition 24 in [JPS16]. Here we also recall that the hyperbolic Green's function of level N is given by

$$G_s(\tau, \omega) = \frac{1}{2\pi}\sum_{M \in \Gamma_0(N)} Q_{s-1}(\cosh(d_{\mathrm{hyp}}(M\tau, \omega)))$$

for $\tau, \omega \in \mathbb{H}$ with $\tau \not\equiv \omega$ modulo $\Gamma_0(N)$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, where $Q_\nu(z) = Q_\nu^0(z)$ is the associated Legendre function of the second kind (see for example [GR07], Section 8.7). By Lemma 7.1 in [Pip16] there is also a relation between the hyperbolic Green's function and the hyperbolic kernel function $K(\tau, \omega, s)$, namely

$$G_s(\tau, \omega) = \frac{2^s\,\Gamma(s)^2}{4\pi\,\Gamma(2s)}\sum_{n=0}^{\infty}\frac{(s/2)_n(s/2+1/2)_n}{n!(s+1/2)_n}\,K(\tau, \omega, s+2n) \tag{2.6.13}$$

for $\tau, \omega \in \mathbb{H}$ with $\tau \not\equiv \omega$ modulo $\Gamma_0(N)$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. Combining (2.6.12) and (2.6.13), and interchanging summation and integration we obtain the identity claimed in part (b) of the proposition. □

3 Vector valued modular forms

In the present chapter we introduce holomorphic and non-holomorphic vector valued modular forms for the Weil representation. In particular, we define different types of non-holomorphic Poincaré series in Section 3.6, whose theta lifts we study in the following chapters. Moreover, we present a translation of the spectral theory of automorphic forms given in [Roe66, Roe67] to the setting of vector valued modular forms in Section 3.7.

Our main references for this chapter are [Bru02] and [BF04], as well as the mentioned work of Roelcke.

3.1 The metaplectic group

The set of pairs

$$\mathrm{Mp}_2(\mathbb{R}) := \left\{(M,\phi)\colon M \in \mathrm{SL}_2(\mathbb{R}), \phi\colon \mathbb{H} \to \mathbb{C} \text{ holomorphic with } \phi(\tau)^2 = j(M,\tau)\right\}$$

together with the operation $(M,\phi) \circ (M',\phi') = (MM', \phi(M'\tau)\phi'(\tau))$ is the so-called *metaplectic group*, which is a double cover of $\mathrm{SL}_2(\mathbb{R})$. For $M \in \mathrm{SL}_2(\mathbb{R})$ we write

$$\tilde{M} := (M, \phi_M)$$

for the corresponding element in $\mathrm{Mp}_2(\mathbb{R})$ where $\phi_M(\tau) := \sqrt{c\tau + d}$ for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Here $z \mapsto \sqrt{z}$ denotes the usual choice of the complex square root, i.e., $\arg(\sqrt{z}) \in (-\pi/2, \pi/2]$. We further denote the inverse image of the modular group $\mathrm{SL}_2(\mathbb{Z})$ under the covering map $(M,\phi) \mapsto M$ by $\mathrm{Mp}_2(\mathbb{Z})$. One can check that it is generated by the two elements

$$\tilde{T} := (T, 1) \qquad \text{and} \qquad \tilde{S} := (S, \sqrt{\tau}),$$

where $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ are the usual generators of $\mathrm{SL}_2(\mathbb{Z})$.

3.2 Quadratic spaces

Let V be an n-dimensional $\mathbb{Q}$-vector space. We call (V,q) a *quadratic space* if $q\colon V \to \mathbb{Q}$ is a non-degenerate quadratic form on V, i.e., if $q(\lambda x) = \lambda^2 q(x)$ for all $\lambda \in \mathbb{Q}$, $x \in V$ and if the pairing $(x,y) = q(x+y) - q(x) - q(y)$ for $x, y \in V$ is a symmetric, non-degenerate $\mathbb{Q}$-bilinear form. Here we say that $(\cdot,\cdot)$ is non-degenerate if $(x,y) = 0$ for all $y \in V$ implies $x = 0$. We call $(\cdot,\cdot)$ the bilinear form associated to q.

By Silvester's law of inertia, given a quadratic space (V,q) there is a unique tuple of non-negative integers (b^+, b^-) such that the $n \times n$-matrix $((b_i, b_j))_{i,j}$ with $(b_1, \ldots, b_n)$ a basis of V has b^+ positive eigenvalues and b^- negative eigenvalues. The tuple (b^+, b^-) is called the *signature* of the space (V,q) or simply of q. Since q is non-degenerate the matrix $((b_i, b_j))_{i,j}$ has full rank, i.e., we have $b^+ + b^- = n$.

A map $\sigma\colon V \to V$ is called an isometry of the quadratic space (V,q) if it is an automorphism of the $\mathbb{Q}$-vector space V satisfying $q(\sigma(x)) = q(x)$ for all $x \in V$. The set of all such isometries is called the *orthogonal group* of (V,q), which we denote by $O(V)$.

From now on let (V,q) be a quadratic space of dimension n and signature (b^+, b^-). A *lattice* L in V is a $\mathbb{Z}$-module of rank n lying in the rational space V, i.e.,

$$L = \mathbb{Z}b_1 \oplus \ldots \mathbb{Z}b_n$$

with $(b_1,\ldots,b_n)$ a basis of V. The *signature* of L is defined as the signature of its surrounding quadratic space, i.e., as the signature of V, and the *orthogonal group* of L is the set of all isometries of (V,q) fixing L, i.e.,

$$O(L) = \{\sigma \in O(V)\colon \sigma(L) = L\}.$$

The lattice L is called *integral* if $(x,y) \in \mathbb{Z}$ for all $x,y \in L$, and it is called *even* if $q(x) \in \mathbb{Z}$ for all $x \in L$. Clearly, every even lattice is integral. Further, we call a vector $x \in L$ *isotropic* if $q(x) = 0$, and we call $x \in L$ *primitive* if $\mathbb{Q}x \cap L = \mathbb{Z}x$.

The *dual lattice* of L is defined by

$$L' := \{x \in V\colon (x,y) \in \mathbb{Z} \text{ for all } y \in L\}.$$

One can check that L' is indeed a lattice by noting that if $(b_1,\ldots,b_n)$ is a basis for L then the so called dual-basis $(b_1',\ldots,b_n')$ of V characterised by $(b_i,b_j') = \delta_{i,j}$ for $1 \leq i,j \leq n$ is a basis of L'. Now the *level* of L is defined as the smallest positive integer N with $Nq(x) \in \mathbb{Z}$ for all $x \in L'$.

Let L be an even lattice. Then $L \subseteq L'$ and by the elementary divisor theorem the quotient L'/L is a finite abelian group of order $|\det(((b_i,b_j))_{i,j})|$ where $(b_1,\ldots,b_n)$ is an arbitrary $\mathbb{Z}$-basis of L. We call L'/L the *discriminant group* of L. The quadratic form q induces a well-defined map $L'/L \to \mathbb{Q}/\mathbb{Z}$ via $q(x+L) = q(x) \bmod \mathbb{Z}$ for $x \in L'$. By construction, we then have $q(\lambda\gamma) = \lambda^2 q(\gamma) \bmod \mathbb{Z}$ for all $\lambda \in \mathbb{Z}$, $\gamma \in L'/L$, and the map

$$(\gamma,\delta) = q(\gamma+\delta) - q(\gamma) - q(\delta) \mod \mathbb{Z}$$

for $\gamma,\delta \in L'/L$ defines a symmetric, non-degenerate $\mathbb{Z}$-bilinear form on L'/L. The group L'/L together with the quadratic form $q\colon L'/L \to \mathbb{Q}/\mathbb{Z}$ is called the *discriminant form* induced by L. We further note that the level N of L is the smallest positive integer N such that $Nq(\gamma) = 0 \bmod \mathbb{Z}$ for all $\gamma \in L'/L$.

3.3 The Weil representation

Let (V,q) be a quadratic space of signature (b^+,b^-) and let L be an even lattice in V. We write $\mathbb{C}[L'/L]$ for the so-called group algebra of the finite abelian group L'/L, which is the $\mathbb{C}$-vector space of formal linear combinations of basis elements $\mathfrak{e}_\gamma$ for $\gamma \in L'/L$, i.e.,

$$\mathbb{C}[L'/L] := \left\{ \sum_{\gamma \in L'/L} \lambda_\gamma \mathfrak{e}_\gamma \colon \lambda_\gamma \in \mathbb{C} \text{ for all } \gamma \in L'/L \right\}.$$

The multiplication on $\mathbb{C}[L'/L]$ is defined by bilinear continuation of $\mathfrak{e}_\gamma \cdot \mathfrak{e}_\delta = \mathfrak{e}_{\gamma+\delta}$ for $\gamma, \delta \in L'/L$. Further, one can equip the space $\mathbb{C}[L'/L]$ via

$$\left\langle \sum_{\gamma\in L'/L} \lambda_\gamma \mathfrak{e}_\gamma, \sum_{\delta\in L'/L} \mu_\delta \mathfrak{e}_\delta \right\rangle := \sum_{\gamma\in L'/L} \lambda_\gamma \overline{\mu_\gamma}$$

with a natural scalar product.

We now introduce the *Weil representation* associated to the lattice L which is a representation of the group $\mathrm{Mp}_2(\mathbb{Z})$ on the group algebra $\mathbb{C}[L'/L]$. Firstly, we set

$$\begin{aligned} \rho_L(\tilde{T})\mathfrak{e}_\gamma &:= e(q(\gamma))\mathfrak{e}_\gamma, \\ \rho_L(\tilde{S})\mathfrak{e}_\gamma &:= \frac{e((b^- - b^+)/8)}{\sqrt{|L'/L|}} \sum_{\delta\in L'/L} e(-(\gamma,\delta))\mathfrak{e}_\delta \end{aligned} \tag{3.3.1}$$

for the generators $\tilde{T}$ and $\tilde{S}$ of $\mathrm{Mp}_2(\mathbb{Z})$ and $\gamma \in L'/L$. Here $e(z) = e^{2\pi i z}$ for $z \in \mathbb{C}$ as before. The linear continuations of $\rho_L(\tilde{T})$ and $\rho_L(\tilde{S})$ to $\mathbb{C}[L'/L]$ are automorphisms of the $\mathbb{C}$-vector space $\mathbb{C}[L'/L]$. Moreover, one can check that the natural extension of ρ_L to the group $\mathrm{Mp}_2(\mathbb{Z})$ generated by $\tilde{T}$ and $\tilde{S}$ is well-defined, i.e., it is compatible with the relations

$$\tilde{S}^2 = \left(\tilde{S}\tilde{T}\right)^3 = \widetilde{-1} = \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, i\right),$$

which completely determine the group $\mathrm{Mp}_2(\mathbb{Z})$.

We remark that if N is the level of L, then ρ_L factors through $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ if $b^+ - b^-$ is even, and through a double cover of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ if $b^+ - b^-$ is odd. In any case, ρ_L can in fact be regarded as the representation of a finite group. Further, a direct computation shows that ρ_L is a unitary representation, i.e., we have

$$\langle \rho_L(M,\phi)x, y\rangle = \langle x, \rho_L(M,\phi)^{-1}y\rangle$$

for all $(M,\phi) \in \mathrm{Mp}_2(\mathbb{Z})$ and all $x, y \in \mathbb{C}[L'/L]$. We also compute

$$\rho_L(\widetilde{-1})\mathfrak{e}_\gamma = e\left(\frac{b^- - b^+}{4}\right) e_{-\gamma} \tag{3.3.2}$$

for $\gamma \in L'/L$. Finally, we note that the Weil representation ρ_L does not depend on the complete lattice L, but only on its induced discriminant form L'/L. This follows from the fact that for lattices (L,q) and (M,r) of signature (b^+,b^-) and (c^+,c^-) we have

$$b^+ - b^- \equiv c^+ - c^- \mod 8$$

if the discriminant forms induced by L and M are isomorphic as finite quadratic modules (see for example [Nik80], Theorem 1.3.3).

Next, we define the generalised Kloosterman sum $H^L_{c,k}(\gamma,m,\delta,n)$ associated to the Weil representation ρ_L as in [Bru02], equation (1.38), and its associated Kloosterman zeta function.

Definition 3.3.1. Let $k \in \frac{1}{2}\mathbb{Z}$. Given $c \in \mathbb{Z}$ with $c \neq 0$ the Kloosterman sum associated to the Weil representation ρ_L is defined by

$$H^L_{c,k}(\gamma, m, \delta, n) = \frac{e(-\operatorname{sign}(c)k/4)}{|c|} \sum_{\substack{d \in (\mathbb{Z}/c\mathbb{Z})^* \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})}} \rho_{\delta,\gamma}\left(\widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}^{-1}\right) e\left(\frac{ma + nd}{c}\right)$$

for $\gamma, \delta \in L'/L$ and $m \in \mathbb{Z} + q(\gamma)$, $n \in \mathbb{Z} + q(\delta)$. Here the sum on the right-hand side runs over all integers d modulo c which are coprime to c, and for each such d we choose a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Moreover, the factor $\rho_{\delta,\gamma}(M, \phi)$ denotes the (δ, γ)'s entry of the matrix representing $\rho_L(M, \phi)$ with respect to the standard basis, i.e., we set

$$\rho_{\delta,\gamma}(M, \phi) := \langle \rho_L(M, \phi)\mathfrak{e}_\gamma, \mathfrak{e}_\delta \rangle$$

for $\gamma, \delta \in L'/L$ and $(M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$.

Since ρ_L factors through a double cover of the finite group $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ as mentioned above, the coefficients $\rho_{\delta,\gamma}(M, \phi)$ are universally bounded. Therefore, the Kloosterman sum $H^L_{c,k}(\gamma, m, \delta, n)$ is in fact bounded by some constant only depending on the underlying lattice L.

Definition 3.3.2. Let $k \in \frac{1}{2}\mathbb{Z}$. Given $\gamma, \delta \in L'/L$ and $m \in \mathbb{Z} + q(\gamma)$, $n \in \mathbb{Z} + q(\delta)$ the Kloosterman zeta function associated to the Weil representation ρ_L is defined by

$$Z^L_k(s; \gamma, m, \delta, n) = \sum_{c \in \mathbb{Z} \setminus \{0\}} |c|^{1-k-2s} H^L_{c,k}(\gamma, m, \delta, n)$$

for $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1 - k/2$.

Here the sum converges for $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1 - k/2$ since the Kloosterman sum $H^L_{c,k}(\gamma, m, \delta, n)$ is bounded by some constant independent of $c \in \mathbb{Z} \setminus \{0\}$, $\gamma, \delta \in L'/L$ and $m \in \mathbb{Z} + q(\gamma)$, $n \in \mathbb{Z} + q(\delta)$. More precisely, we have

$$\left|Z^L_k(s; \gamma, m, \delta, n)\right| \leq C_L\, \zeta(2\operatorname{Re}(s) + k - 1) \tag{3.3.3}$$

for $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1 - k/2$ where $C_L > 0$ only depends on the lattice L and $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the usual Riemann zeta function.

We also remark that given two lattices L and M whose induced discriminant forms are isomorphic the corresponding Kloosterman sums and Kloosterman zeta functions agree as they only depend on the Weil representation $\rho_L = \rho_M$.

3.4 Holomorphic modular forms

Let (V, q) be a quadratic space of signature (b^+, b^-) and let L be an even lattice in V. Further, let k be an integer or a half-integer. The group $\mathrm{Mp}_2(\mathbb{Z})$ acts on the space of vector valued functions $F\colon \mathbb{H} \to \mathbb{C}[L'/L]$ via

$$\big(F\,|_{k,L}\,(M, \phi)\big)(\tau) := \phi(\tau)^{-2k} \rho_L(M, \phi)^{-1} F(M\tau)$$

for $\tau \in \mathbb{H}$. We call this the *weight k action* of $\mathrm{Mp}_2(\mathbb{Z})$ with respect to the Weil representation ρ_L. Moreover, we say that $F\colon \mathbb{H} \to \mathbb{C}[L'/L]$ is modular of weight k with respect to the Weil representation ρ_L if

$$F\,|_{k,L}\,(M,\phi) = F$$

for all $(M,\phi) \in \mathrm{Mp}_2(\mathbb{Z})$. Note that $F\,|_{k,L}\,(\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), -1) = (-1)^{b^+-b^--2k}F$ by (3.3.2). Thus we may always assume that

$$2k \equiv b^+ - b^- \mod 2 \tag{3.4.1}$$

since otherwise the only function modular of weight k with respect to ρ_L is the zero-function.

Let $F = \sum_{\gamma\in L'/L} f_\gamma \mathfrak{e}_\gamma$ be a vector valued function on $\mathbb{H}$ with component functions $f_\gamma\colon \mathbb{H} \to \mathbb{C}$. Then F is holomorphic if every f_γ is. Suppose that F is holomorphic and $F\,|_{k,L}\,\tilde{T} = F$. Then the shifted component functions $g_\gamma(\tau) = e(-q(\gamma)\tau)f_\gamma(\tau)$ for $\gamma \in L'/L$ are 1-periodic, and thus F has a Fourier expansion of the form

$$F(\tau) = \sum_{\gamma\in L'/L}\sum_{n\in\mathbb{Z}+q(\gamma)} a_F(\gamma,n)e(n\tau)\mathfrak{e}_\gamma$$

for $\tau \in \mathbb{H}$ with Fourier coefficients $a_F(\gamma,n) \in \mathbb{C}$ given by

$$a_F(\gamma,n) = \int_0^1 \langle F(\tau), e(n\bar{\tau})\mathfrak{e}_\gamma\rangle\, du$$

where $\gamma \in L'/L$, $n \in \mathbb{Z} + q(\gamma)$ and $\tau = u + iv \in \mathbb{H}$. As in the scalar valued case we say that F is *meromorphic*, *holomorphic* or *vanishing* at ∞ if $a_F(\gamma,n) = 0$ for all $\gamma \in L'/L$ and all but finitely many $n < 0$, for all $n < 0$ or for all $n \leq 0$, respectively. If F is meromorphic at ∞ we call the Fourier polynomial

$$\sum_{\gamma\in L'/L}\sum_{\substack{n\in\mathbb{Z}+q(\gamma)\\ n\leq 0}} a_F(\gamma,n)e(n\tau)\mathfrak{e}_\gamma \tag{3.4.2}$$

the *principal part* of F.

Definition 3.4.1. A function $F\colon \mathbb{H} \to \mathbb{C}[L'/L]$ is called *weakly holomorphic modular form*, *modular form* or *cusp form* of weight k for the Weil representation ρ_L if F is holomorphic on $\mathbb{H}$, modular of weight k with respect to ρ_L, and meromorphic, holomorphic or vanishing at ∞, respectively.

The corresponding vector spaces of vector valued weakly holomorphic modular forms, modular forms and cusp forms are denoted by $M^!_{k,L}$, $M_{k,L}$ and $S_{k,L}$, respectively. As in the scalar valued case, the latter two spaces $M_{k,L}$ and $S_{k,L}$ are finite-dimensional for every k, and the space of vector valued cusp forms $S_{k,L}$ can be equipped with an inner product via

$$(F,G) := \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \langle F(\tau), G(\tau)\rangle \operatorname{Im}(\tau)^k d\mu(\tau) \tag{3.4.3}$$

for $F, G \in S_{k,L}$. If only one of F and G is a cusp form and the other one is a weakly holomorphic modular form the above integral does in general not converge. However, we can define a regularized version of this inner product following Section 6 of [Bor98] which converges for much more general functions:

Let $\mathcal{F}$ be the usual fundamental domain for the action of the modular group $\mathrm{SL}_2(\mathbb{Z})$ on the upper half-plane, i.e., $\mathcal{F} := \{\tau \in \mathbb{H}\colon |\tau| \geq 1, |\operatorname{Re}(\tau)| \leq 1/2\}$, and set

$$\mathcal{F}_T := \{\tau \in \mathcal{F}\colon \operatorname{Im}(\tau) \leq T\}$$

for $T > 1$. Moreover, we use the notation $\mathrm{CT}_{t=0}\, h(t)$ to denote the constant term of the Laurent expansion of the meromorphic continuation of some analytic function $h(t)$ at $t = 0$. Now, given two real analytic functions $F, G\colon \mathbb{H} \to \mathbb{C}[L'/L]$ both modular of weight k with respect to ρ_L we define the *regularized inner product* of F and G by

$$(F, G)^{\mathrm{reg}} := \mathrm{CT}_{t=0} \left[\lim_{T\to\infty} \int_{\mathcal{F}_T} \langle F(\tau), G(\tau)\rangle \operatorname{Im}(\tau)^{k-t} d\mu(\tau) \right], \tag{3.4.4}$$

whenever the inner limit exists for some $t \in \mathbb{C}$ and can be continued to a meromorphic function defined on a half-plane $\{t \in \mathbb{C}\colon \operatorname{Re}(t) > -\delta\}$ with $\delta > 0$. Note that the integrand in (3.4.4) is only modular in τ if $t = 0$. Thus it is crucial that we indeed use a fixed fundamental domain. However, we can work with a simpler regularization if the limit in equation (3.4.4) converges for $t = 0$ and $T \to \infty$, in which case we have

$$(F, G)^{\mathrm{reg}} = \lim_{T\to\infty} \int_{\mathcal{F}_T} \langle F(\tau), G(\tau)\rangle \operatorname{Im}(\tau)^{k} d\mu(\tau) \tag{3.4.5}$$

since $\mathrm{CT}_{t=0}\, h(t) = h(0)$ if $h(t)$ is holomorphic at $t = 0$. Eventually, we also remark that if the integral in (3.4.3) defining the inner product (F, G) exists, then it agrees with the regularized inner products given above. In particular, we have $(F, G) = (F, G)^{\mathrm{reg}}$ if F and G are both cusp forms.

As in the scalar valued case we now define (vector valued) holomorphic Eisenstein series and Poincaré series, which are basic examples of vector valued modular forms.

Definition 3.4.2. Let $k \in \frac{1}{2}\mathbb{Z}$ with $k > 2$ satisfying (3.4.1). Given $\beta \in L'/L$ with $q(\beta) = 0$ the *vector valued Eisenstein series* of weight k associated to the vector β is defined by

$$E^L_{k,\beta}(\tau) = \frac{1}{2} \sum_{(M,\phi)\in\langle\tilde{T}\rangle\backslash \mathrm{Mp}_2(\mathbb{Z})} \mathfrak{e}_\beta \Big|_{k,L} (M, \phi)$$

for $\tau \in \mathbb{H}$.

Here the condition $q(\beta) = 0$ implies $\mathfrak{e}_\beta\,|_{k,L}\,\tilde{T} = \mathfrak{e}_\beta$, i.e., it guarantees that the series is well-defined. Clearly, the series converges absolutely and locally uniformly as in the scalar valued case, defining a holomorphic function which is by construction modular of weight k with respect to ρ_L. Moreover, it is also holomorphic at ∞. Thus $E^L_{k,\beta}$ is indeed a vector valued modular form of weight k for ρ_L.

Note that since the element $\left(\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),-1\right)$ in $\mathrm{Mp}_2(\mathbb{Z})$ acts trivially on $\mathfrak{e}_\beta$ as we assume that the weight k satisfies (3.4.1), we can also write

$$E^L_{k,\beta}(\tau)=\sum_{M\in\langle T\rangle\setminus\mathrm{SL}_2(\mathbb{Z})}\mathfrak{e}_\beta\Big|_{k,L}\tilde{M},$$

which explains why we introduced the additional factor $\frac{1}{2}$ in Definition 3.4.2. On the other hand, the element $\left(\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right),i\right)$ in $\mathrm{Mp}_2(\mathbb{Z})$ does not act trivially on $\mathfrak{e}_\beta$ but as

$$\mathfrak{e}_\beta\Big|_{k,L}\widetilde{-1}=(-1)^{b^+/2-b^-/2-k}\mathfrak{e}_{-\beta}. \tag{3.4.6}$$

Since $\widetilde{-1}$ also commutes with any element of $\mathrm{Mp}_2(\mathbb{Z})$ we thus have

$$E^L_{k,\beta}=(-1)^{b^+/2-b^-/2-k}E^L_{k,-\beta}.$$

Therefore, the Eisenstein series $E^L_{k,\beta}$ vanishes identically if the sign $(-1)^{b^+/2-b^-/2-k}$ is negative and $\beta=-\beta$ in L'/L. Thus, for $\beta=0$ or β of order two we always assume that

$$2k\equiv b^+-b^-\mod 4, \tag{3.4.7}$$

which refines the condition on the weight k given in (3.4.1).

Instead of generalising Eisenstein series in the way we did in Section 2.4.2, we now use a different, more elementary approach, defining holomorphic Poincaré series.

Definition 3.4.3. Let $k\in\frac{1}{2}\mathbb{Z}$ with $k>2$ satisfying (3.4.1). Given $\beta\in L'/L$ and $m\in\mathbb{Z}+q(\beta)$ the *vector valued Poincaré series* of weight k and index (β,m) is defined by

$$P^L_{k,\beta,m}(\tau)=\frac{1}{2}\sum_{(M,\phi)\in\langle\tilde{T}\rangle\setminus\mathrm{Mp}_2(\mathbb{Z})}e(m\tau)\mathfrak{e}_\beta\Big|_{k,L}(M,\phi)$$

for $\tau\in\mathbb{H}$.

Again, the series converges absolutely and locally uniformly, and thus defines a holomorphic function which is clearly modular of weight k with respect to ρ_L. Therefore, computing its Fourier expansion (compare for example [Bru02], Theorem 1.4) one can check that the Poincaré series $P^L_{k,\beta,m}$ is a weakly holomorphic modular form, a modular form or a cusp form of weight k for ρ_L if $m<0$, $m=0$ or $m>0$, respectively. More precisely, depending on the sign of m the Poincaré series $P^L_{k,\beta,m}$ can be characterized as follows:

Proposition 3.4.4. *Let $k\in\frac{1}{2}\mathbb{Z}$ with $k>2$ satisfying* (3.4.1)*, $\beta\in L'/L$ and $m\in\mathbb{Z}+q(\beta)$.*

(a) If $m>0$ then $P^L_{k,\beta,m}(\tau)$ is the unique cusp form of weight k for ρ_L, which satisfies the inner product formula

$$(F,P^L_{k,\beta,m})=2\,\frac{\Gamma(k-1)}{(4\pi m)^{k-1}}\,a_F(\beta,m) \tag{3.4.8}$$

for all $F\in S_{k,L}$.

(b) If $m = 0$ then $P^L_{k,\beta,0}(\tau) = E^L_{k,\beta}(\tau)$ is the unique holomorphic Eisenstein series of weight k for ρ_L associated to the vector $\beta \in L'/L$.

(c) If $m < 0$ then $P^L_{k,\beta,m}(\tau)$ is the unique weakly holomorphic modular form of weight k for ρ_L, which has principal part $e(m\tau)(\mathfrak{e}_\beta + \mathfrak{e}_{-\beta})$, and which is orthogonal to cusp forms with respect to the regularized inner product defined in (3.4.5).

Proof. Part (a) is well-known (see for example [Bru02], Proposition 1.5), and part (b) is trivial. Moreover, part (c) easily follows from the fact that two weakly holomorphic modular forms, whose principal parts agree, can only differ by a cusp form. □

As for the vector valued Eisenstein series, the Poincaré series $P^L_{k,\beta,m}$ vanishes identically if $b^+ - b^- - 2k \equiv 2 \bmod 4$ and $2\beta = 0$ in L'/L. However, this does not contradict the above proposition since in this case $a_F(\beta, m) = 0$ for all $F \in S_{k,L}$ and $\mathfrak{e}_\beta + \mathfrak{e}_{-\beta} = 0$.

3.5 Harmonic Maass forms

We quickly recall from [BF04] the definition and basic properties of harmonic Maass forms of integral or half-integral weight. However, we also note that we will not need these until the end of Chapter 6. As before, let (V, q) be a quadratic space of signature (b^+, b^-), let L be an even lattice in V, and let k be an integer or a half-integer.

Definition 3.5.1. A smooth function $F\colon \mathbb{H} \to \mathbb{C}[L'/L]$ is called *harmonic Maass form* of weight k, if $\Delta_k F(\tau) = 0$, if F is modular of weight k with respect to ρ_L, and if for every $\gamma \in L'/L$ there is an $\varepsilon > 0$ such that

$$|\langle F(\tau), \mathfrak{e}_\gamma \rangle| = O(e^{\varepsilon v})$$

as $v \to \infty$, uniformly in u for $u \in \mathbb{R}$, where $\tau = u + iv \in \mathbb{H}$.

Here the hyperbolic Laplace operator Δ_k is defined as in the scalar valued case (see (2.5.2)), and it simply acts componentwise on vector valued functions. We denote the vector space of harmonic Maass forms by $H_{k,L}$. Clearly, the space $M^!_{k,L}$ of weakly holomorphic modular forms defined in the previous section is a subspace of $H_{k,L}$.

In [BF04] it is shown that every harmonic Maass form $F \in H_{k,L}$ has a unique decomposition of the form

$$F = F^+ + F^-$$

into a *holomorphic part* F^+ and a *non-holomorphic part* F^-, which are characterized by their Fourier expansions. These are of the form

$$F^+(\tau) = \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z} + q(\gamma) \\ n \gg -\infty}} a^+_F(\gamma, n) e(n\tau) \mathfrak{e}_\gamma$$

and

$$F^-(\tau) = \sum_{\gamma \in L'/L} \left(a^-_F(\gamma, 0) v^{1-k} + \sum_{\substack{n \in \mathbb{Z} + q(\gamma) \\ n \ll \infty \\ n \neq 0}} a^-_F(\gamma, n) H_k(2\pi n v) e(nu) \right) \mathfrak{e}_\gamma,$$

where the term $a_F^-(\gamma,0)v^{1-k}$ is replaced by $a_F^-(\gamma,0)\log(v)$ if $k=1$. Here the notation $n \gg -\infty$ means that all but finitely many of the coefficients $a_F^+(\gamma,n)$ with $n<0$ vanish, and correspondingly $n \ll \infty$ means that all but finitely many of the coefficients $a_F^-(\gamma,n)$ with $n>0$ vanish. Moreover, $H_k(w)$ is the integral function defined in Section 3 of [BF04]. In particular, $F \in H_{k,L}$ is holomorphic on $\mathbb{H}$ if and only if $F^- \equiv 0$. Thus, the space $M^!_{k,L}$ contains exactly those harmonic Maass forms, which are holomorphic on $\mathbb{H}$.

Next we consider the differential operator ξ_k defined in (2.5.8), which also acts componentwise on vector valued functions $F\colon \mathbb{H} \to \mathbb{C}[L'/L]$. In Lemma 3.1 of [BF04] it is shown that given a harmonic Maass form $F \in H_{k,L}$ the operator ξ_k acts on the Fourier expansion of F via

$$\xi_k F(\tau) = -2 \sum_{\gamma\in L'/L} \left((k-1)\overline{a_F^-(\gamma,0)} + \sum_{\substack{n\in\mathbb{Z}-q(\gamma)\\ n\gg-\infty\\ n\neq 0}} \overline{a_F^-(\gamma,-n)}(4\pi n)^{1-k} e(n\tau) \right)\mathfrak{e}_\gamma, \tag{3.5.1}$$

and by Proposition 3.2 and Theorem 3.7 in [BF04] the induced map

$$\xi_k\colon H_{k,L} \to M^!_{2-k,L^*},\ F \mapsto \xi_k F$$

is surjective with kernel given by $M^!_{k,L}$. Here $M^!_{2-k,L^*}$ denotes the space of weakly holomorphic modular forms of weight $2-k$ for the dual Weil representation ρ_L^*, which is simply given by the complex conjugate of the representation ρ_L, i.e., we have

$$\rho_L^*(M,\phi) = \Big(\rho_L(M,\phi)\Big)^* \tag{3.5.2}$$

for $(M,\phi)\in \mathrm{Mp}_2(\mathbb{Z})$, where we understand both sides as complex matrices and the star on the right-hand side denotes the complex conjugate of the matrix $\rho_L(M,\phi)$. As in [BF04] we denote the subspace of functions in $H_{k,L}$ which map to cusp forms under ξ_k by $H^+_{k,L}$, i.e., we set

$$H^+_{k,L} := \xi_k^{-1}\Big(S_{2-k,L^*}\Big). \tag{3.5.3}$$

Now if $F \in H^+_{k,L}$ then $a_F^-(\gamma,n)=0$ for all $\gamma\in L'/L$ and $n \in \mathbb{Z}+q(\gamma)$ with $n \geq 0$. In particular, if $k<1$ the Fourier expansion of a harmonic Maass form $F \in H^+_{k,L}$ can be written in the form

(3.5.4)

$$F(\tau) = \sum_{\gamma\in L'/L} \left(\sum_{\substack{n\in\mathbb{Z}+q(\gamma)\\ n\gg-\infty}} a_F^+(\gamma,n)e(n\tau) + \sum_{\substack{n\in\mathbb{Z}+q(\gamma)\\ n<0}} a_F^-(\gamma,n)\Gamma(1-k,4\pi|n|v)e(n\tau) \right)\mathfrak{e}_\gamma$$

for $\tau = u+iv \in \mathbb{H}$. Here $\Gamma(s,x)=\int_x^\infty t^{s-1}e^{-t}dt$ denotes the usual incomplete Gamma function. As in the holomorphic case (compare (3.4.2)) we call the finite Fourier polynomial

$$\sum_{\gamma\in L'/L} \sum_{\substack{n\in\mathbb{Z}+q(\gamma)\\ n\leq 0}} a_F^+(\gamma,n)e(n\tau)\mathfrak{e}_\gamma$$

the *principal part* of F.

We also remark that by [BF04, Proposition 3.5] we have that

$$(F, \xi_k(G)) = \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z}+q(\gamma) \\ n \leq 0}} a_F(\gamma, -n)\, a_G^+(\gamma, n) \tag{3.5.5}$$

for $F \in M_{2-k,L^*}$ and $G \in H_{k,L}^+$, where the inner product on the right-hand side is the inner product given in (3.4.3) of weight $2-k$ for the dual Weil representation ρ_L^*. Here the integral defining the inner product converges without regularization since $\xi_k(G)$ is a cusp form, and the sum on the right-hand side of (3.5.5) is clearly finite.

Lemma 3.5.2. *Let $F \in H_{k,L}^+$ with $a_F^+(\gamma, n) = 0$ for all $\gamma \in L'/L$ and $n \in \mathbb{Z} + q(\gamma)$ with $n < 0$. Then F is a holomorphic modular form of weight k for ρ_L. If additionally $a_F^+(\gamma, 0) = 0$ for all $\gamma \in L'/L$ then F is a cusp form.*

Proof. By (3.5.5) we have

$$(\xi_k(F), \xi_k(F)) = \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z}+q(\gamma) \\ n \leq 0}} a_{\xi_k(F)}(\gamma, -n)\, a_F^+(\gamma, n) = 0,$$

since $a_{\xi_k(F)}(\gamma, 0) = 0$ as $\xi_k(F)$ is a cusp form, and since $a_F^+(\gamma, n) = 0$ for all $n < 0$ by assumption. So $\xi_k(F) = 0$ as the inner product on S_{2-k,L^*} is non-degenerate, and thus we find $F \in \ker(\xi_k) = M_{k,L}^!$. Using again that $a_F^+(\gamma, n) = 0$ for all $n < 0$ or $n \leq 0$, we obtain that F is in fact a holomorphic modular form or cusp form of weight k for ρ_L, respectively. □

3.6 Non-holomorphic Poincaré series

In the present section we introduce (vector valued) non-holomorphic analogs of the (vector valued) holomorphic Poincaré series defined at the end of Section 3.4. We note that these Poincaré series will in general not be harmonic, i.e., they do not fit into the framework of harmonic Maass forms introduced in the previous section.

As before, we let (V, q) be a quadratic space of signature (b^+, b^-), and we let L be an even lattice in V. Further, we assume that k is an integer or a half-integer, which satisfies the condition given in (3.4.1), and if $\beta \in L'/L$ has order one or two, we further assume that k satisfies (3.4.7). We start by defining vector valued non-holomorphic Eisenstein series.

Definition 3.6.1. Given $\beta \in L'/L$ with $q(\beta) = 0$ the *vector valued non-holomorphic Eisenstein series* of weight k associated to the vector β is defined by

$$E_{k,\beta}^L(\tau, s) = \frac{1}{2} \sum_{(M,\phi) \in \langle \tilde{T} \rangle \backslash \operatorname{Mp}_2(\mathbb{Z})} \operatorname{Im}(\tau)^s \mathfrak{e}_\beta \Big|_{k,L} (M, \phi)$$

for $\tau \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1 - k/2$.

As in the scalar valued case the series converges absolutely and locally uniformly in both variables, and thus defines a real analytic function in τ and a holomorphic function in s for $\mathrm{Re}(s) > 1-k/2$, which is modular of weight k with respect to ρ_L. The hyperbolic Laplace operator defined in (2.5.2) acts component wise on vector valued functions, and $\Delta_k(F\,|_{k,L}\,(M,\phi)) = (\Delta_k F)\,|_{k,L}\,(M,\phi)$ for $F\colon \mathbb{H} \to \mathbb{C}[L'/L]$ real analytic and $(M,\phi) \in \mathrm{Mp}_2(\mathbb{Z})$. Thus we easily see that the Eisenstein series $E^L_{k,\beta}(\tau,s)$ is an eigenfunction of the hyperbolic Laplacian, i.e., we have

$$\Delta_k E^L_{k,\beta}(\tau,s) = s(1-k-s)E^L_{k,\beta}(\tau,s). \tag{3.6.1}$$

Moreover, it is well-known that the Eisenstein series $E^L_{k,\beta}(\tau,s)$ has a meromorphic continuation in s to all of $\mathbb{C}$, which is holomorphic for $\mathrm{Re}(s) \geq 1/2 - k/2$ up to finitely many simple poles in the real interval $(1/2-k/2, 1-k/2]$ (see for example [Sel56] and [Roe66, Roe67]).

For convenience, we quickly define the modified W-Whittaker function $\mathcal{W}_{k,s}(x)$ for $x \neq 0$ via

$$\mathcal{W}_{k,s}(x) = |x|^{-k/2} W_{\mathrm{sign}(x)\,k/2,s-1/2}(|x|)$$

as in [Bru02, equation (1.28)]. Here $W_{\kappa,\mu}(z)$ is the usual W-Whittaker function defined for $\kappa,\mu,z \in \mathbb{C}$ with $z \neq 0$ (see for example [GR07], Section 9.22), whose asymptotic behaviour (see for example [Bru02], equation (1.26)) directly implies that

$$\mathcal{W}_{k,s}(x) = \begin{cases} O(e^{-|x|/2}), & \text{for } x > 0, \\ O(e^{-|x|/2}|x|^{-k}), & \text{for } x < 0, \end{cases} \tag{3.6.2}$$

as $x \to \pm\infty$. Now, the Fourier expansion of the vector valued non-holomorphic Eisenstein series $E^L_{k,\beta}(\tau,s)$, which is for example computed in [BK03, Proposition 3.1], can be written in the following form:

Proposition 3.6.2. *Let $\beta \in L'/L$ with $q(\beta) = 0$. The Eisenstein series $E^L_{k,\beta}(\tau,s)$ has a Fourier expansion of the form*

$$E^L_{k,\beta}(\tau,s) = v^s(\mathfrak{e}_\beta + (-1)^{(b^+-b^--2k)/2}\mathfrak{e}_{-\beta}) + \sum_{\gamma\in L'/L}\ \sum_{n\in\mathbb{Z}+q(\gamma)} b(\gamma,n,v,s)e(nu)\mathfrak{e}_\gamma,$$

for $\tau = u+iv \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1-k/2$, where the Fourier coefficients are given by

$$b(\gamma,n,v,s) = \begin{cases} \dfrac{2^k\pi^{s+k}|n|^{s+k-1}}{\Gamma(s+k)}\ \mathcal{W}_{k,s+k/2}(4\pi n v)\ Z^L_k(s;\beta,0,\gamma,n), & \text{if } n > 0, \\ 4^{1-k/2-s}\pi v^{1-k-s}\ \dfrac{\Gamma(2s+k-1)}{\Gamma(s)\,\Gamma(s+k)}\ Z^L_k(s;\beta,0,\gamma,0), & \text{if } n = 0, \\ \dfrac{2^k\pi^{s+k}|n|^{s+k-1}}{\Gamma(s)}\ \mathcal{W}_{k,s+k/2}(4\pi n v)\ Z^L_k(s;\beta,0,\gamma,n), & \text{if } n < 0. \end{cases}$$

Here $Z^L_k(s;\beta,m,\gamma,n)$ denotes the Kloosterman zeta function given in Definition 3.3.2.

In Section 6.1 we study the non-holomorphic Eisenstein series $E^L_{1/2,0}(\tau, s)$ of weight $1/2$ associated to the zero-vector and a certain lattice L of signature $(2, 1)$ in detail, using results from Section 3 of [BK03].

Next, we define vector valued non-holomorphic Poincaré series. However, in contrast to the holomorphic case there is no natural choice of a Poincaré series here. Thus, we introduce different types of non-holomorphic Poincaré series.

Definition 3.6.3. Let $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$.

(a) The Poincaré series $Q^L_{k,\beta,m}(\tau, s)$ is defined by

$$Q^L_{k,\beta,m}(\tau, s) = \frac{1}{2} \sum_{(M,\phi)\in\langle\tilde{T}\rangle\setminus \mathrm{Mp}_2(\mathbb{Z})} v^s e(mu)\mathfrak{e}_\beta \Big|_{k,L} (M, \phi)$$

for $\tau = u + iv \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1 - k/2$.

(b) The Poincaré series $U^L_{k,\beta,m}(\tau, s)$ is defined by

$$U^L_{k,\beta,m}(\tau, s) = \frac{1}{2} \sum_{(M,\phi)\in\langle\tilde{T}\rangle\setminus \mathrm{Mp}_2(\mathbb{Z})} v^s e(m\tau)\mathfrak{e}_\beta \Big|_{k,L} (M, \phi)$$

for $\tau = u+iv \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1-k/2$. We call $U^L_{k,\beta,m}$ *Selberg's Poincaré series of the first kind.*

(c) The Poincaré series $V^L_{k,\beta,m}(\tau, s)$ is defined by

$$V^L_{k,\beta,m}(\tau, s) = \frac{1}{2} \sum_{(M,\phi)\in\langle\tilde{T}\rangle\setminus \mathrm{Mp}_2(\mathbb{Z})} v^s e^{-2\pi|m|v} e(mu)\mathfrak{e}_\beta \Big|_{k,L} (M, \phi)$$

for $\tau = u+iv \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1-k/2$. We call $V^L_{k,\beta,m}$ *Selberg's Poincaré series of the second kind.*

Since the series defining $Q^L_{k,\beta,m}$, $U^L_{k,\beta,m}$ and $V^L_{k,\beta,m}$ can be majorized by the vector valued non-holomorphic Eisenstein series $E^L_{k,\beta}(\tau, \mathrm{Re}(s))$, the corresponding Poincaré series are real analytic in τ and holomorphic in s for $\mathrm{Re}(s) > 1 - k/2$ as is the Eisenstein series. Also, all three Poincaré series are by definition modular of weight k with respect to ρ_L.

Remark 3.6.4.

(1) For $m = 0$ the Poincaré series $Q^L_{k,\beta,0}$, $U^L_{k,\beta,0}$ and $V^L_{k,\beta,0}$ all agree with the non-holomorphic Eisenstein series of weight k associated to β, i.e., we have

$$Q^L_{k,\beta,0}(\tau, s) = U^L_{k,\beta,0}(\tau, s) = V^L_{k,\beta,0}(\tau, s) = E^L_{k,\beta}(\tau, s)$$

for $\tau \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1 - k/2$.

(2) The Poincaré series $U^L_{k,\beta,m}$ and $V^L_{k,\beta,m}$, which trivially agree for m non-negative, can both be understood as vector valued versions of Selberg's non-holomorphic Poincaré series introduced in his famous work [Sel65]. We sometimes use the simplified notation *Selberg's Poincaré series* to refer to Selberg's Poincaré series of the first kind $U^L_{k,\beta,m}(\tau,s)$, omitting the appendix "first kind". In fact, Selberg's Poincaré series of the second kind will solely be used in Chapter 6.

(3) For $\beta \in L'/L$ and $m \in \mathbb{Z}+q(\beta)$ with $m<0$ we have the identity

$$v^k\,\overline{V^L_{k,\beta,m}(\tau,\bar{s}-k)} = \frac{1}{2}\sum_{(M,\phi)\in\langle\tilde{T}\rangle\setminus \mathrm{Mp}_2(\mathbb{Z})} v^s e(-m\tau)\mathfrak{e}_\beta\Big|^*_{-k,L}(M,\phi) \tag{3.6.3}$$

for $\tau = u+iv\in\mathbb{H}$ and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1+k/2$, where $|^*_{-k,L}$ denotes the weight $-k$ action of $\mathrm{Mp}_2(\mathbb{Z})$ with respect to the dual Weil representation ρ^*_L, i.e.,

$$\big(F\,|^*_{k,L}\,(M,\phi)\big)(\tau) = \phi(\tau)^{-2k}\rho^*_L(M,\phi)^{-1}F(M\tau)$$

for $F\colon\mathbb{H}\to\mathbb{C}[L'/L]$, $(M,\phi)\in\mathrm{Mp}_2(\mathbb{Z})$ and $\tau\in\mathbb{H}$. Here the dual Weil representation ρ^*_L is given as in (3.5.2).

Therefore, the right-hand side of (3.6.3) can be understood as the Selberg's Poincaré series of the first kind of weight $-k$ for the dual Weil representation ρ^*_L and index $(\beta,-m)$. In other words, Selberg's Poincaré series of the second kind for negative index m is indeed a normalized version of the corresponding dual Selberg Poincaré series of the first kind of positive index $-m$.

Next we determine the action of the hyperbolic Laplace operator Δ_k on these non-holomorphic Poincaré series introduced above.

Lemma 3.6.5. *Let $\beta\in L'/L$ and $m\in\mathbb{Z}+q(\beta)$. Then*

$$(\Delta_k - s(1-k-s))Q^L_{k,\beta,m}(\tau,s) = (2\pi m)^2 Q^L_{k,\beta,m}(\tau,s+2) - 2\pi mk Q^L_{k,\beta,m}(\tau,s+1),$$

$$(\Delta_k - s(1-k-s))U^L_{k,\beta,m}(\tau,s) = 4\pi ms\,U^L_{k,\beta,m}(\tau,s+1),$$

$$(\Delta_k - s(1-k-s))V^L_{k,\beta,m}(\tau,s) = \begin{cases} 4\pi ms\,V^L_{k,\beta,m}(\tau,s+1), & \text{if } m\geq 0,\\ 4\pi|m|(s+k)\,V^L_{k,\beta,m}(\tau,s+1), & \text{if } m<0,\end{cases}$$

for $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1-k/2$.

Proof. As the operator Δ_k is invariant under the weight k action of $\mathrm{Mp}_2(\mathbb{Z})$ with respect to ρ_L it suffices to prove the above identities for $g_j(\tau,s) := v^s\varphi^{(j)}_m(v)e(mu)$ where

$$\varphi^{(1)}_m(v) := 1,\qquad \varphi^{(2)}_m(v) := e^{-2\pi mv},\qquad \varphi^{(3)}_m(v) := e^{-2\pi|m|v}.$$

Now a tedious, but straightforward computation shows that

(3.6.4)

$$\begin{aligned}&(\Delta_k - s(1-k-s))g_j(\tau,s)\\ &\quad= v^s e(mu)\left(\left((2\pi mv)^2 - 2\pi mkv\right)\varphi^{(j)}_m(v) - (2s+k)v\frac{\partial}{\partial v}\varphi^{(j)}_m(v) - v^2\frac{\partial^2}{\partial v^2}\varphi^{(j)}_m(v)\right)\end{aligned}$$

for $j = 1, 2, 3$. Computing the derivatives $\frac{\partial}{\partial v}\varphi_m^{(j)}(v)$ and $\frac{\partial^2}{\partial v^2}\varphi_m^{(j)}(v)$ we obtain the claimed differential equations. □

Next we compute a unified Fourier expansion valid for all three types of Poincaré series. Even though the proof is essentially given by a standard unfolding argument we present it for the sake of completeness.

Proposition 3.6.6. *Let $\beta \in L'/L$, $m \in \mathbb{Z} + q(\beta)$ and*

$$\varphi_m^{(1)}(v) := 1, \qquad \varphi_m^{(2)}(v) := e^{-2\pi m v}, \qquad \varphi_m^{(3)}(v) := e^{-2\pi|m|v}.$$

For $j = 1, 2, 3$ the Poincaré series

$$P_{k,\beta,m}^{L,(j)}(\tau, s) := \frac{1}{2} \sum_{(M,\phi)\in\langle\tilde{T}\rangle\backslash \operatorname{Mp}_2(\mathbb{Z})} v^s \varphi_m^{(j)}(v) e(mu)\mathfrak{e}_\beta \Big|_{k,L} (M, \phi)$$

has a Fourier expansion of the form

$$v^s \varphi_m^{(j)}(v) e(mu)(\mathfrak{e}_\beta + (-1)^{(b^+ - b^- - 2k)/2}\mathfrak{e}_{-\beta}) + \sum_{\gamma\in L'/L} \sum_{n\in\mathbb{Z}+q(\gamma)} b^{(j)}(\gamma, n, v, s) e(nu)\mathfrak{e}_\gamma,$$

where the Fourier coefficients are given by

$$b^{(j)}(\gamma, n, v, s) = v^s \sum_{c\in\mathbb{Z}\backslash\{0\}} |c|^{1-k-2s} H_{c,k}^L(\beta, m, \gamma, n) J_{k,m}^{(j)}(n, v, s, c).$$

Here $H_{c,k}^L(\beta, m, \gamma, n)$ is the generalised Kloosterman sum from Definition 3.3.1, and the integral function

$$J_{k,m}^{(j)}(n, v, s, c) := i^k \int_{-\infty}^{\infty} \tau^{-k} |\tau|^{-2s} \varphi_m^{(j)}\left(\frac{v}{c^2|\tau|^2}\right) e\left(-\frac{mu}{c^2|\tau|^2} - nu\right) du \tag{3.6.5}$$

is analytic in v for $v > 0$ and holomorphic in s for $\operatorname{Re}(s) > 1/2 - k/2$.

Proof. Splitting the sum over matrices $(M, \phi) \in \langle\tilde{T}\rangle\backslash \operatorname{Mp}_2(\mathbb{Z})$ defining the given Poincaré series into matrices M with lower left entry $c = 0$ and $c \neq 0$, and using that the element $(\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), -1)$ acts trivially on $\mathfrak{e}_\beta$, we can write

$$P_{k,\beta,m}^{L,(j)}(\tau, s; \varphi_m) = v^s \varphi_m^{(j)}(v) e(mu)(\mathfrak{e}_\beta + (-1)^{(b^+ - b^- - 2k)/2}\mathfrak{e}_{-\beta}) + H(\tau, s) \tag{3.6.6}$$

with

$$H(\tau, s) := \sum_{\substack{M\in\langle T\rangle\backslash \operatorname{SL}_2(\mathbb{Z})\\ M\neq\pm1}} v^s \varphi_m^{(j)}(v) e(mu)\mathfrak{e}_\beta \Big|_{k,L} \tilde{M}.$$

We now want to compute $b^{(j)}(\gamma, n, v, s)$, which is the (γ, n)'th Fourier coefficient of $H(\tau, s)$, i.e.,

$$b^{(j)}(\gamma, n, v, s) = \int_0^1 \langle H(\tau, s), e(nu)\mathfrak{e}_\gamma\rangle\, du.$$

Fix $v > 0$, $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1 - k/2$, $\gamma \in L'/L$ and $n \in \mathbb{Z} + q(\gamma)$. Then

$$\begin{aligned} b^{(j)}(\gamma, n, v, s) &= \sum_{\substack{M \in \langle T \rangle \backslash \operatorname{SL}_2(\mathbb{Z}) \\ M \neq \pm 1}} \int_0^1 \left\langle v^s \varphi_m^{(j)}(v) e(mu) \mathfrak{e}_\beta \Big|_{k,L} \tilde{M}, e(nu) \mathfrak{e}_\gamma \right\rangle du \\ &= \sum_{\substack{M \in \langle T \rangle \backslash \operatorname{SL}_2(\mathbb{Z}) / \langle T \rangle \\ M \neq \pm 1}} \sum_{\ell \in \mathbb{Z}} \int_0^1 \left\langle v^s \varphi_m^{(j)}(v) e(mu) \mathfrak{e}_\beta \Big|_{k,L} \widetilde{MT^\ell}, e(nu) \mathfrak{e}_\gamma \right\rangle du. \end{aligned}$$

We denote the inner sum over $\ell \in \mathbb{Z}$ by $\tilde{b}(\gamma, n, v, s; M)$ for $M \in \operatorname{SL}_2(\mathbb{Z})$. Since $\widetilde{MT^\ell}$ acts as $\tilde{M}\tilde{T}^\ell$ with $\tilde{T}^\ell = (T^\ell, 1)$, and since

$$e(nu) \rho_L(\tilde{T}^\ell) \mathfrak{e}_\gamma = e(nu + \ell q(\gamma)) \mathfrak{e}_\gamma = e(n(u + \ell)) \mathfrak{e}_\gamma$$

for $\ell \in \mathbb{Z}$ as we chose $n \in \mathbb{Z} + q(\gamma)$, we obtain

$$\begin{aligned} \tilde{b}(\gamma, n, v, s; M) &= \sum_{\ell \in \mathbb{Z}} \int_0^1 \left\langle \left(v^s \varphi_m^{(j)}(v) e(mu) \mathfrak{e}_\beta \Big|_{k,L} \tilde{M} \right) (\tau + \ell), e(nu) \rho_L(\tilde{T}^\ell) \mathfrak{e}_\gamma \right\rangle du \\ &= \int_{-\infty}^{\infty} \left\langle v^s \varphi_m^{(j)}(v) e(mu) \mathfrak{e}_\beta \Big|_{k,L} \tilde{M}, e(nu) \mathfrak{e}_\gamma \right\rangle du. \end{aligned}$$

Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $c \neq 0$. Then we can write the above integral as

$$\int_{-\infty}^{\infty} \phi_M(\tau)^{-2k} \operatorname{Im}(M\tau)^s \varphi_m^{(j)}(\operatorname{Im}(M\tau)) e(m \operatorname{Re}(M\tau) - nu) du \ \cdot \ \rho_{\gamma,\beta}(\tilde{M}^{-1})$$

where $\rho_{\gamma,\beta}(\tilde{M}^{-1}) = \langle \rho_L(\tilde{M}^{-1}) \mathfrak{e}_\beta, \mathfrak{e}_\gamma \rangle$. Recalling that $z \mapsto \sqrt{z}$ denotes the usual complex square root with $\arg(\sqrt{z}) \in (-\pi/2, \pi/2]$ we find that

$$\phi_M(\tau)^{-2k} = \sqrt{c\tau + d}^{-2k} = \operatorname{sign}(c)^k |c|^{-k} (\tau + d/c)^{-k}.$$

Moreover, we use the simple identity $M\tau = \frac{a}{c} - \frac{1}{c^2(\tau + d/c)}$ to get

$$\operatorname{Re}(M\tau) = \frac{a}{c} - \frac{u + d/c}{c^2 |\tau + d/c|^2} \qquad \text{and} \qquad \operatorname{Im}(M\tau) = \frac{v}{c^2 |\tau + d/c|^2}$$

for $\tau = u + iv$. Thus the substitution $u + d/c \mapsto u$ yields

$$\begin{aligned} \tilde{b}(\gamma, n, v, s; M) = \operatorname{sign}(c)^k |c|^{-2s-k} v^s e\left(\frac{ma + nd}{c} \right) \rho_{\gamma,\beta}(\tilde{M}^{-1}) \\ \times \int_{-\infty}^{\infty} \tau^{-k} |\tau|^{-2s} \varphi_m^{(j)} \left(\frac{v}{c^2 |\tau|^2} \right) e\left(-\frac{mu}{c^2 |\tau|^2} - nu \right) du. \end{aligned}$$

Recalling the definition of the generalized Kloosterman sum $H_{c,k}^L(\beta, m, \gamma, n)$ and noting that the sum over matrices $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in the double coset $\langle T \rangle \backslash \operatorname{SL}_2(\mathbb{Z}) / \langle T \rangle$ with $c \neq 0$ can be separated into an infinite sum over all non-zero integers c and a finite sum over all $d \in (\mathbb{Z}/c\mathbb{Z})^*$ we obtain the claimed expression for the Fourier coefficients of $P_{k,\beta,m}^{L,(j)}(\tau, s)$.

It remains to note that for fixed $v > 0$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1/2 - k/2$ the remaining integral can be estimated by

$$\sup_{u\in\mathbb{R}} \left| \varphi_m^{(j)}\left(\frac{v}{c^2|\tau|^2}\right)\right| \cdot \int_{-\infty}^{\infty} (u^2+v^2)^{-\mathrm{Re}(s)-k/2} du. \tag{3.6.7}$$

Here the latter integral is for example computed in [GR07, formula 3.251.2], giving

$$\int_{-\infty}^{\infty} (u^2+v^2)^{-\mu} du = v^{1-2\mu}\sqrt{\pi}\,\frac{\Gamma(\mu-1/2)}{\Gamma(\mu)} \tag{3.6.8}$$

for $v > 0$ and $\mu \in \mathbb{C}$ with $\mathrm{Re}(\mu) > 1/2$. This proves the claimed statement. □

In order to obtain more specific Fourier expansions for the present non-holomorphic Poincaré series one needs to evaluate the integral function $J_{k,m}^{(j)}(n,v,s,c)$ given in (3.6.5). For example, if $j = 2,3$ we can use equation (8) in [Pri99] to compute the corresponding integrals, which yields simplified Fourier expansions for Selberg's Poincaré series of the first and second kind.

For the sake of clarity, we state these expansions in separate propositions even though they are very similar. Nevertheless, we only prove the first one, and since $U_{k,\beta,m}^L(\tau,s) = V_{k,\beta,m}^L(\tau,s)$ for $m \geq 0$, we give the Fourier expansion of $V_{k,\beta,m}^L(\tau,s)$ only for $m < 0$. Moreover, we note that a scalar valued version of the following Fourier expansion is for example given in Lemma 5 of [Pri99].

Proposition 3.6.7. *Let $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$. The Poincaré series $U_{k,\beta,m}^L(\tau,s)$ has a Fourier expansion of the form*

$$U_{k,\beta,m}^L(\tau,s) = v^s e(m\tau)(\mathfrak{e}_\beta + (-1)^{(b^+-b^--2k)/2}\mathfrak{e}_{-\beta}) + \sum_{\gamma\in L'/L}\sum_{n\in\mathbb{Z}+q(\gamma)} b(\gamma,n,v,s)e(nu)\mathfrak{e}_\gamma$$

for $\tau = u+iv \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1-k/2$, where the Fourier coefficients are given by

$$b(\gamma,0,v,s) = \frac{4^{1-k/2-s}\pi v^{1-k-s}}{\Gamma(s)}\sum_{j=0}^{\infty}\frac{1}{j!}\left(-\frac{\pi m}{v}\right)^j Z_k^L(s+j;\beta,m,\gamma,0)\frac{\Gamma(2s+k+j-1)}{\Gamma(s+k+j)}$$

for $n = 0$, and by

$$b(\gamma,n,v,s)$$
$$= \begin{cases} 2^k\pi^{s+k}|n|^{s+k-1}\displaystyle\sum_{j=0}^{\infty}\frac{(-4\pi^2 m|n|)^j}{j!}Z_k^L(s+j;\beta,m,\gamma,n)\frac{\mathcal{W}_{k+j,s+k/2+j/2}(4\pi nv)}{\Gamma(s+k+j)}, & \text{if } n > 0,\\ 2^k\pi^{s+k}|n|^{s+k-1}\displaystyle\sum_{j=0}^{\infty}\frac{(-4\pi^2 m|n|)^j}{j!}Z_k^L(s+j;\beta,m,\gamma,n)\frac{\mathcal{W}_{k+j,s+k/2+j/2}(4\pi nv)}{\Gamma(s)}, & \text{if } n < 0,\end{cases}$$

for $n \neq 0$.

Proof. By Proposition 3.6.6 we only need to show that

$$i^k v^s \sum_{c\in\mathbb{Z}\setminus\{0\}} |c|^{1-k-2s} H^L_{c,k}(\beta,m,\gamma,n) \int_{-\infty}^{\infty} \frac{e(-nu)}{\tau^{s+k}\overline{\tau}^s}\, e\left(-\frac{m}{c^2\tau}\right) du \tag{3.6.9}$$

equals the Fourier coefficient $b(\gamma,n,v,s)$ given in the proposition. Firstly, we expand the exponential function $e(-\frac{m}{c^2\tau}) = \sum_{j=0}^{\infty}\frac{1}{j!}(-\frac{2\pi i m}{c^2\tau})^j$. Next we want to interchange summation and integration in (3.6.9). Therefore we note that

$$\begin{aligned}&\sum_{c\in\mathbb{Z}\setminus\{0\}} \left||c|^{1-k-2s} H^L_{c,k}(\beta,m,\gamma,n)\right| \int_{-\infty}^{\infty} \left|\frac{e(-nu)}{\tau^{s+k}\overline{\tau}^s}\right| \sum_{j=0}^{\infty}\frac{1}{j!}\left|-\frac{2\pi i m}{c^2\tau}\right|^j du \\ &\leq \sum_{c\in\mathbb{Z}\setminus\{0\}} |c|^{1-k-2\operatorname{Re}(s)} \sum_{j=0}^{\infty}\frac{1}{j!}\left|\frac{2\pi m}{c^2}\right|^j \int_{-\infty}^{\infty} (u^2+v^2)^{-\operatorname{Re}(s)-k/2-j/2} du.\end{aligned} \tag{3.6.10}$$

Computing the latter integral as in (3.6.8) and using that the fraction $\frac{\Gamma(\mu-1/2)}{\Gamma(\mu)}$ is bounded by some constant C for all $\mu \geq 1/2+\varepsilon$ with $\varepsilon > 0$, we find that (3.6.10) can further be estimated by

$$2C\sqrt{\pi}\, v^{1-2\operatorname{Re}(s)-k} e^{2\pi|m|/v}\zeta(2\operatorname{Re}(s)+k-1)$$

for fixed $v>0$ and $s\in\mathbb{C}$ with $\operatorname{Re}(s) > 1-k/2$. Hence, we may write (3.6.9) as

$$i^k v^s \sum_{j=0}^{\infty}\frac{(-2\pi i m)^j}{j!} \sum_{c\in\mathbb{Z}\setminus\{0\}} |c|^{1-k-2(s+j)} H^L_{c,k}(\beta,m,\gamma,n) \int_{-\infty}^{\infty} \frac{e(-nu)}{\tau^{s+k+j}\overline{\tau}^s} du. \tag{3.6.11}$$

Now the remaining integral is computed in [Pri99], equation (8), which states that

(3.6.12)

$$\int_{-\infty}^{\infty} \frac{e(-nu)}{(u+iv)^\alpha(u-iv)^\beta} du = \begin{cases} (-2i)^{\alpha-\beta}\pi^\alpha |n|^{\alpha-1} v^{-\beta}\dfrac{1}{\Gamma(\alpha)}\mathcal{W}_{\alpha-\beta,(\alpha+\beta)/2}(4\pi n v), & \text{if } n>0,\\ 2\pi(-i)^{\alpha-\beta}(2v)^{1-\alpha-\beta}\dfrac{\Gamma(\alpha+\beta-1)}{\Gamma(\alpha)\Gamma(\beta)}, & \text{if } n=0,\\ (-2i)^{\alpha-\beta}\pi^\alpha |n|^{\alpha-1} v^{-\beta}\dfrac{1}{\Gamma(\beta)}\mathcal{W}_{\alpha-\beta,(\alpha+\beta)/2}(4\pi n v), & \text{if } n<0,\end{cases}$$

for $n\in\mathbb{R}$, $\alpha,\beta\in\mathbb{C}$ with $\operatorname{Re}(\alpha+\beta)>1$ and $v>0$. Setting $\alpha = s+k+j$ and $\beta = s$ we obtain the claimed statement. □

Proposition 3.6.8. *Let $\beta\in L'/L$ and $m\in\mathbb{Z}+q(\beta)$ with $m<0$. The Poincaré series $V^L_{k,\beta,m}(\tau,s)$ has a Fourier expansion of the form*

$$V^L_{k,\beta,m}(\tau,s) = v^s e(m\overline{\tau})(\mathfrak{e}_\beta + (-1)^{(b^+-b^--2k)/2}\mathfrak{e}_{-\beta}) + \sum_{\gamma\in L'/L}\sum_{n\in\mathbb{Z}+q(\gamma)} b(\gamma,n,v,s)e(nu)\mathfrak{e}_\gamma$$

for $\tau = u+iv\in\mathbb{H}$ and $s\in\mathbb{C}$ with $\operatorname{Re}(s)>1-k/2$, where the Fourier coefficients are given by

$$b(\gamma,0,v,s) = \frac{4^{1-k/2-s}\pi v^{1-k-s}}{\Gamma(s+k)}\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{\pi m}{v}\right)^j Z^L_k(s+j;\beta,m,\gamma,0)\frac{\Gamma(2s+k+j-1)}{\Gamma(s+j)}$$

for $n = 0$, and by

$$b(\gamma, n, v, s)$$

$$= \begin{cases} 2^k \pi^{s+k} |n|^{s+k-1} \displaystyle\sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\pi m}{v}\right)^j Z_k^L(s+j; \beta, m, \gamma, n) \frac{\mathcal{W}_{k-j, s+k/2+j/2}(4\pi n v)}{\Gamma(s+k)}, & \text{if } n > 0, \\ 2^k \pi^{s+k} |n|^{s+k-1} \displaystyle\sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\pi m}{v}\right)^j Z_k^L(s+j; \beta, m, \gamma, n) \frac{\mathcal{W}_{k-j, s+k/2+j/2}(4\pi n v)}{\Gamma(s+j)}, & \text{if } n < 0, \end{cases}$$

for $n \neq 0$.

Eventually, we remark that for $m > 0$ Selberg's Poincaré series $U_{k,\beta,m}^L = V_{k,\beta,m}^L$, and for $m < 0$ Selberg's Poincaré series of the second kind $V_{k,\beta,m}^L$ are square-integrable in the variable τ with respect to the vector valued inner product defined in (3.4.3). Though this is well-known for $m > 0$ (compare for example [Sel65], page 10) we present a proof as this result will be crucial in Section 6.2.

Lemma 3.6.9. *Let $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m \neq 0$. For fixed $s \in \mathbb{C}$ with* $\mathrm{Re}(s) > 1 - k/2$ *the Poincaré series $V_{k,\beta,m}^L(\tau, s)$ is square-integrable in τ with respect to the inner product defined in* (3.4.3). *Further, given $\Omega \subseteq \{s \in \mathbb{C}\colon \mathrm{Re}(s) > 1 - k/2\}$ compact we have*

$$|(V_{\beta,m}(\tau, s), V_{\beta,m}(\tau, s))| = O(1)$$

as $|m| \to \infty$, uniformly in s for $s \in \Omega$. Here the implied constant depends on τ and Ω.

Proof. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1 - k/2$. Splitting the sum over matrices $M \in \langle T \rangle \backslash \mathrm{SL}_2(\mathbb{Z})$ defining the Poincaré series $V_{k,\beta,m}^L(\tau, s)$ into matrices M with lower left entry $c = 0$ and $c \neq 0$ we can write $V_{k,\beta,m}^L(\tau, s)$ as

(3.6.13)
$$v^s e^{-2\pi |m| v} e(mu)(\mathfrak{e}_\beta + (-1)^{(b^+ - b^- - 2k)/2} \mathfrak{e}_{-\beta}) + \sum_{\substack{M \in \langle T \rangle \backslash \mathrm{SL}_2(\mathbb{Z}) \\ M \neq \pm 1}} v^s e^{-2\pi |m| v} e(mu) \mathfrak{e}_\beta \Big|_{k,L} \tilde{M}$$

for $\tau \in \mathbb{H}$. Here the first summand of (3.6.13) behaves as $v^s e^{-2\pi |m| v}$ as $v \to \infty$, uniformly in u. In order to estimate the second summand of (3.6.13) we note that

$$\text{(3.6.14)} \qquad \Big| \Big\langle \sum_{\substack{M \in \langle T \rangle \backslash \mathrm{SL}_2(\mathbb{Z}) \\ M \neq \pm 1}} v^s e^{-2\pi |m| v} e(mu) \mathfrak{e}_\beta \Big|_{k,L} \tilde{M}, \mathfrak{e}_\gamma \Big\rangle \Big| \leq \tilde{C} \sum_{\substack{M \in \langle T \rangle \backslash \mathrm{SL}_2(\mathbb{Z}) \\ M \neq \pm 1}} \Big| \Big(v^s \Big|_k M \Big) \Big|$$

for $\gamma \in L'/L$. Here we have used the simple estimate $e^{-2\pi |m| v} < 1$ for $v > 0$, and the fact that the coefficients $\langle \rho_L(\tilde{M})^{-1} \mathfrak{e}_\beta, \mathfrak{e}_\gamma \rangle$ are universally bounded by some constant $\tilde{C} > 0$ (see Section 3.3). Now the right-hand side of (3.6.14) can be estimated as the usual non-holomorphic Eisenstein series:

Let $\mathcal{F} = \{\tau \in \mathbb{H}\colon |\tau| \geq 1, \mathrm{Re}(\tau) \leq 1/2\}$ be the usual fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$. We recall that instead of summing over the quotient $\langle T \rangle \backslash \mathrm{SL}_2(\mathbb{Z})$

we can sum over tuples of coprime integers (c,d) which represent the matrix $\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$. Here the condition $M \neq \pm 1$ corresponds to $c \neq 0$. We thus find

$$\sum_{\substack{M \in \langle T \rangle \backslash \mathrm{SL}_2(\mathbb{Z}) \\ M \neq \pm 1}} \left| \left(v^s \Big|_k M \right) \right| \leq 2 v^{\sigma} \sum_{c=1}^{\infty} \sum_{\substack{d \in \mathbb{Z} \\ (c,d)=1}} |c\tau + d|^{-2\sigma - k} \tag{3.6.15}$$

for $\tau \in \mathcal{F}$ where $\sigma := \mathrm{Re}(s)$. Let $c \in \mathbb{N}$. We estimate the quantity $|c\tau + d|$ in two ways: On the one hand we clearly have $|c\tau + d| \geq cv$ for any $d \in \mathbb{Z}$ and $\tau \in \mathbb{H}$, and on the other hand one can also check that $|c\tau + d| \geq |d|/2$ for $d \in \mathbb{Z}$ with $|d| \geq c$ and $\tau \in \mathcal{F}$. Splitting the sum over d in (3.6.15) for fixed $\tau \in \mathcal{F}$ into two sums over integers d with $|d| \leq 2cv$ and with $|d| > 2cv$ we thus obtain

(3.6.16)

$$\sum_{\substack{d \in \mathbb{Z} \\ (c,d)=1}} |c\tau + d|^{-2\sigma - k} \leq \sum_{\substack{d \in \mathbb{Z} \\ (c,d)=1 \\ |d| \leq 2cv}} (cv)^{-2\sigma - k} + \sum_{\substack{d \in \mathbb{Z} \\ (c,d)=1 \\ |d| > 2cv}} \left| \frac{d}{2} \right|^{-2\sigma - k} \leq C_1 (cv)^{1 - 2\sigma - k} + C_2 \sum_{d = \lceil 2cv \rceil}^{\infty} d^{-2\sigma - k},$$

where the constants $C_1, C_2 > 0$ do not depend c or v. Since

$$\sum_{d=N}^{\infty} d^{-\alpha} \leq \int_{N-1}^{\infty} x^{-\alpha} dx = \frac{(N-1)^{1-\alpha}}{\alpha - 1}$$

for $N \in \mathbb{N}$ with $N \geq 2$ and $\alpha \in (1, \infty)$, we can estimate the remaining sum in (3.6.16) for $2cv > 1$ by $C_3 (cv)^{1-2\sigma-k}$ with $C_3 > 0$ independent of c and v. Combining this with (3.6.15) we get

$$\sum_{\substack{M \in \langle T \rangle \backslash \mathrm{SL}_2(\mathbb{Z}) \\ M \neq \pm 1}} \left| \left(v^s \Big|_k M \right) \right| \leq 2 (C_1 + C_2 C_3) \zeta(2\sigma + k - 1) v^{1 - \sigma - k}$$

where the Riemann zeta function converges as $\sigma = \mathrm{Re}(s) > 1 - k/2$. Recalling (3.6.13) and (3.6.14) we therefore find $C > 0$ such that

$$\left| \left\langle V^L_{k,\beta,m}(\tau, s), \mathfrak{e}_\gamma \right\rangle \right| \leq C v^{1 - \mathrm{Re}(s) - k}$$

for all $\gamma \in L'/L$ and $\tau \in \mathcal{F}$. Here the constant C does not depend on γ or τ. Hence, we can finally estimate

$$\begin{aligned} \left(V^L_{k,\beta,m}(\tau, s), V^L_{k,\beta,m}(\tau, s) \right) &= \int_{\mathcal{F}} \sum_{\gamma \in L'/L} \left| \left\langle V^L_{k,\beta,m}(\tau, s), \mathfrak{e}_\gamma \right\rangle \right|^2 v^k d\mu(\tau) \\ &\leq C^2 |L'/L| \int_{1/2}^{\infty} v^{-2\,\mathrm{Re}(s) - k} dv, \end{aligned}$$

where the latter integral converges as $\mathrm{Re}(s) > 1/2 - k/2$.

Eventually, we note that the claimed estimate for the growth of the norm

$$|(V^L_{k,\beta,m}(\tau, s), V^L_{k,\beta,m}(\tau, s))|$$

can easily be deduced from the given proof. More precisely, it suffices to recall that the constant $\tilde{C}$ from equation (3.6.14) does not depend on m. □

At the end of this section we introduce another non-holomorphic Poincaré series, which differs significantly from the previous three, which is why we treat it separately. It can be regarded as a Selberg-analog of the vector valued Maass Poincaré series, which is for example studied in [Bru02]. Though the Maass Poincaré series is an eigenfunction of the hyperbolic Laplace operator Δ_k, the present Selberg analog will again only satisfy a shifted Laplace equation (see Lemma 3.6.12 for the case $k=0$).

For the sake of convenience, we start by defining the modified M-Whittaker function $\mathcal{M}_{k,s}(x)$ for $x<0$ and $s \neq 0,-1,-2,\dots$ via

$$\mathcal{M}_{k,s}(x) = |x|^{-k/2} M_{-k/2,s-1/2}(|x|)$$

as in [Bru02, equation (1.27)]. Here $M_{\kappa,\mu}(z)$ denotes the well-known M-Whittaker function defined for $\kappa,\mu,z \in \mathbb{C}$ with $\mu \neq -\frac{1}{2},-\frac{3}{2},\dots$ and $z \neq 0$ (see for example [GR07], Section 9.22). Recalling the asymptotic behaviour of the M-Whittaker function (see for example [Bru02], equations (1.23) and (1.25)) we find that

$$\mathcal{M}_{k,s}(x) = O(e^{|x|/2}) \tag{3.6.17}$$

as $x \to -\infty$, and

$$\mathcal{M}_{k,s}(x) = O(|x|^{\mathrm{Re}(s)-k/2}) \tag{3.6.18}$$

as $x \to 0$ for $x<0$.

Definition 3.6.10. Let $\beta \in L'/L$ and $m \in \mathbb{Z}+q(\beta)$ with $m<0$. The Poincaré series $M^L_{k,\beta,m}(\tau,s)$ is defined by

$$M^L_{k,\beta,m}(\tau,s) = \frac{1}{2} \sum_{(M,\phi)\in\langle\tilde{T}\rangle\backslash \mathrm{Mp}_2(\mathbb{Z})} v^{s/2}\mathcal{M}_{k,s/2}(4\pi m v)e(mu)\mathfrak{e}_\beta \Big|_{k,L} (M,\phi)$$

for $\tau = u+iv \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s)>1$. We call $M^L_{k,\beta,m}(\tau,s)$ the *Maass-Selberg Poincaré series.*

The Poincaré series $M^L_{k,\beta,m}$ can be majorized by the Eisenstein series $E^L_{k,\beta}(\tau,\mathrm{Re}(s)-k/2)$ since by (3.6.18) the Whittaker function $\mathcal{M}_{k,s/2}(4\pi m v)$ behaves as $v^{s/2-k/2}$ as $v \to 0$. Hence the function $M^L_{k,\beta,m}(\tau,s)$ is real analytic in τ, holomorphic in s for $\mathrm{Re}(s)>1$ and modular of weight k with respect to ρ_L.

Remark 3.6.11. For $\beta \in L'/L$, $m \in \mathbb{Z}+q(\beta)$ with $m<0$, and $k=0$ we can write the Poincaré series $M^L_{0,\beta,m}(\tau,s)$ as a sum of shifted Poincaré series $Q^L_{0,\beta,m}(\tau,s)$, namely

$$M^L_{0,\beta,m}(\tau,s) = (4\pi|m|)^{s/2} \sum_{n=0}^{\infty} \frac{(\pi|m|)^{2n}}{n!\,(s/2+1/2)_n} Q^L_{0,\beta,m}(\tau,s+2n) \tag{3.6.19}$$

for $\tau \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s)>1$. Here $(z)_n$ denotes the Pochhammer symbol given in (2.6.11), and the identity (3.6.19) follows directly from [GR07, formula 9.226]. We note that we are not aware of a generalization of this identity to non-zero weight.

Next we determine the action of the hyperbolic Laplace operator Δ_k on the present Maass-Selberg Poincaré series. For simplicity, we restrict to the case of weight 0.

Lemma 3.6.12. *Let $\beta \in L'/L$ and $m \in \mathbb{Z}+q(\beta)$ with $m<0$. Then*

$$\Big(\Delta_0 - s(1-s)\Big) M^L_{0,\beta,m}(\tau,s) = -\frac{\pi|m|s}{1+s} M^L_{0,\beta,m}(\tau,s+2)$$

for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$.

Proof. We follow the proof of Lemma 3.6.5, setting

$$g(\tau,s) := v^{s/2} M_{0,s/2-1/2}(4\pi|m|v) e(mu).$$

By (3.6.4) we then find that

$$\left(\Delta_0 - \frac{s}{2}\left(1-\frac{s}{2}\right)\right) g(\tau,s) = v^{s/2}\psi(v,s)e(mu)$$

with

$$\begin{aligned} &\psi(v,s) \\ &:= (2\pi mv)^2 M_{0,s/2-1/2}(4\pi|m|v) - sv\frac{\partial}{\partial v} M_{0,s/2-1/2}(4\pi|m|v) - v^2\frac{\partial^2}{\partial v^2} M_{0,s/2-1/2}(4\pi|m|v). \end{aligned}$$

It thus remains to compute the above derivatives of the given M-Whittaker function. Recalling the connection between the M-Whittaker function and the confluent hypergeometric function of the first kind $\Phi(a,c;z)$ (see for example [GR07], formula 9.220.2), and combining the recursion formulas 9.212.2 and 9.212.3 from [GR07] to get

$$\Phi(a+1,c+1;z) = \Phi(a,c;z) + \frac{c-a}{c(1+c)}\, z\, \Phi(a+1,c+2;z),$$

we can eventually apply [GR07, formula 9.213] to obtain

$$\frac{\partial}{\partial z} M_{\kappa,\mu-1/2}(z) = \left(\frac{\mu}{z} - \frac{\kappa}{2\mu}\right) M_{\kappa,\mu-1/2}(z) + \frac{\mu^2-\kappa^2}{4\mu^2(1+2\mu)} M_{\kappa,\mu+1/2}(z).$$

Therefore, we find

$$sv\,\frac{\partial}{\partial v} M_{0,s/2-1/2}(4\pi|m|v) = \frac{s^2}{2} M_{0,s/2-1/2}(4\pi|m|v) + \frac{\pi|m|sv}{1+s} M_{0,s/2+1/2}(4\pi|m|v).$$

Moreover, the M-Whittaker function $M_{\kappa,\mu}(z)$ is by construction a solution of the differential equation

$$\frac{\partial^2}{\partial z^2} f + \left(\frac{1/4-\mu^2}{z^2} + \frac{\kappa}{z} - \frac{1}{4}\right) f = 0,$$

showing that

$$v^2\,\frac{\partial^2}{\partial v^2} M_{0,s/2-1/2}(4\pi|m|v) = -\left(\frac{s}{2}\left(1-\frac{s}{2}\right) - (2\pi|m|v)^2\right) M_{0,s/2-1/2}(4\pi|m|v).$$

Putting everything back together, we thus get

$$\psi(v,s) = \frac{s}{2}\left(1-\frac{3s}{2}\right) M_{-k/2,s/2-1/2}(4\pi|m|v) - \frac{\pi|m|sv}{1+s} M_{-k/2,s/2+1/2}(4\pi|m|v),$$

which proves the claimed differential equation. $\square$

Finally, we present an analog of Proposition 3.6.6 for the Maass-Selberg Poincaré series.

Proposition 3.6.13. *Let $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m < 0$. Then the Poincaré series $M^L_{k,\beta,m}(\tau, s)$ has a Fourier expansion of the form*

$$v^{s/2}\mathcal{M}_{k,s/2}(4\pi mv)e(mu)(\mathfrak{e}_\beta + (-1)^{(b^+-b^--2k)/2}\mathfrak{e}_{-\beta}) + \sum_{\gamma\in L'/L}\sum_{n\in\mathbb{Z}+q(\gamma)} b^{(j)}(\gamma, n, v, s)e(nu)\mathfrak{e}_\gamma,$$

where the Fourier coefficients are given by

$$b(\gamma, n, v, s) = v^{s/2}\sum_{c\in\mathbb{Z}\setminus\{0\}} |c|^{1-k-s}H^L_{c,k}(\beta, m, \gamma, n)J_{k,m}(n, v, s, c).$$

Here $H^L_{c,k}(\beta, m, \gamma, n)$ is the generalised Kloosterman sum from Definition 3.3.1, and the integral function

$$J_{k,m}(n, v, s, c) := i^k\int_{-\infty}^{\infty}\tau^{-k}|\tau|^{-s}\mathcal{M}_{k,s/2}\left(\frac{4\pi mv}{c^2|\tau|^2}\right)e\left(-\frac{mu}{c^2|\tau|^2} - nu\right)du$$

is analytic in v for $v > 0$ and holomorphic in s for $\mathrm{Re}(s) > 1/2$.

Proof. The proof runs completely analogous to the the one of Proposition 3.6.6, where we only have to consider the expression

$$\int_{-\infty}^{\infty}|\tau|^{-k-s}\left|\mathcal{M}_{k,s/2}\left(\frac{4\pi mv}{c^2|\tau|^2}\right)\right|du \tag{3.6.20}$$

instead of (3.6.7) to prove the existence of the present integral function $J_{k,m}(n, v, s, c)$. We estimate (3.6.20) by

$$\sup_{u\in\mathbb{R}}\left|\tau^{s-k}\mathcal{M}_{k,s/2}\left(\frac{4\pi mv}{c^2|\tau|^2}\right)\right|\cdot\int_{-\infty}^{\infty}(u^2+v^2)^{-\mathrm{Re}(s)}du. \tag{3.6.21}$$

Using (3.6.18) we find $C > 0$ such that

$$\left|\mathcal{M}_{k,s/2}\left(\frac{4\pi mv}{c^2|\tau|^2}\right)\right| \leq C|\tau|^{-\mathrm{Re}(s)+k}$$

for all $u \in \mathbb{R}$. Thus, by (3.6.8) the expression in (3.6.21) is bounded for $\mathrm{Re}(s) > 1/2$. □

3.7 Vector valued spectral theory

In the following we translate Roelckes results from [Roe66] and [Roe67] to the modern language of (non-holomorphic) vector valued modular forms for the Weil representation. As before let (V, q) be a quadratic space of signature (b^+, b^-), let L be an even lattice in V and let k be an integer or a half-integer with $2k \equiv b^+ - b^- \bmod 2$.

Recall that $\phi_M(\tau) = \sqrt{c\tau + d}$ for $M = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$. We set

$$\sigma(M, M') := \frac{\phi_M(M'\tau)\phi_{M'}(\tau)}{\phi_{MM'}(\tau)} \tag{3.7.1}$$

for $M, M' \in \mathrm{SL}_2(\mathbb{Z})$ and any $\tau \in \mathbb{H}$. Since $(M, \phi_M) \circ (M', \phi_{M'}) = (MM', \pm\phi_{MM'}(\tau))$ the quantity $\sigma(M, M')$ is simply a sign, i.e., $\sigma(M, M') = \pm 1$, and this sign does not depend on the choice of $\tau \in \mathbb{H}$ as the right-hand side of (3.7.1) is continuous in τ. We further set

$$\sigma_k(M, M') = \sigma(M, M')^{2k} \tag{3.7.2}$$

for $M, M' \in \mathrm{SL}_2(\mathbb{Z})$. Then $\rho_L(E_2, \sigma(M, M')) = \sigma_k(M, M') \, \mathrm{id}_{\mathbb{C}[L'/L]}$ as we assume that $2k \equiv b^+ - b^- \bmod 2$ and thus

$$\rho_L(\widetilde{MM'}) = \sigma_k(M, M')\rho_L(\widetilde{M})\rho_L(\widetilde{M'}) \tag{3.7.3}$$

for $M, M' \in \mathrm{SL}_2(\mathbb{Z})$, i.e., the Weil representation ρ_L satisfies equation (1.14) in [Roe66]. However, ρ_L does not satisfy equation (1.15) in [Roe66] as the negative identity matrix does not act as a scalar multiplication on $\mathbb{C}[L'/L]$ but as

$$\mathfrak{e}_\gamma \big|_{k,L} \widetilde{-1} = (-1)^{b^+/2 - b^-/2 - k} \mathfrak{e}_{-\gamma} \tag{3.7.4}$$

for $\gamma \in L'/L$, as was already noted in (3.4.6). Therefore we need to work with a subspace of $\mathbb{C}[L'/L]$ on which $\rho_L(\widetilde{-1})$ does act as a simple scalar multiplication.

To motivate the definition of this subspace we first note that if $F \colon \mathbb{H} \to \mathbb{C}[L'/L]$ is modular of weight k with respect to ρ_L then $F|_{k,L} \widetilde{-1} = F$ and thus

$$f_\gamma = \xi \, f_{-\gamma}$$

for all $\gamma \in L'/L$ where $F = \sum_{\gamma \in L'/L} f_\gamma \mathfrak{e}_\gamma$ and

$$\xi := (-1)^{b^+/2 - b^-/2 - k}$$

is the fixed sign, determined by the weight k action of $\widetilde{-1}$ as in (3.7.4). So the image of F does always lie in the subspace

$$\mathcal{U} := \left\{ \sum_{\gamma \in L'/L} \lambda_\gamma \mathfrak{e}_\gamma \in \mathbb{C}[L'/L] \colon \lambda_\gamma = \xi \, \lambda_{-\gamma} \text{ for all } \gamma \in L'/L \right\}$$

of $\mathbb{C}[L'/L]$. We fix the following notation: Let

$$L'/L = \{\gamma_1, -\gamma_1, \ldots, \gamma_q, -\gamma_q, \gamma_{q+1}, \ldots, \gamma_r\}$$

with $\gamma_1, -\gamma_1, \ldots, \gamma_q, -\gamma_q, \gamma_{q+1}, \ldots, \gamma_r$ pairwise distinct and $2\gamma_j = 0$ for $j = q+1, \ldots, r$, and set

$$\hat{\mathfrak{e}}_j := \begin{cases} \frac{1}{2}(\mathfrak{e}_{\gamma_j} + \xi \, \mathfrak{e}_{-\gamma_j}), & \text{if } 1 \le j \le q, \\ \mathfrak{e}_{\gamma_j}, & \text{if } q+1 \le j \le r. \end{cases}$$

Then $(\hat{\mathfrak{e}}_1, \ldots, \hat{\mathfrak{e}}_m)$ with $m = r$ if $\xi = 1$ and $m = q$ otherwise defines an orthonormal basis of the subspace $\mathcal{U}$ where we equip $\mathcal{U}$ with the scalar product coming from $C[L'/L]$. Moreover, we find

$$\rho_L(\tilde{T})\hat{\mathfrak{e}}_j = e(q(\gamma_j))\hat{\mathfrak{e}}_j$$

and

$$\rho_L(\tilde{S})\hat{\mathfrak{e}}_j = \begin{cases} \dfrac{e((b^- - b^+)/8)}{\sqrt{|L'/L|}} \displaystyle\sum_{\ell=1}^{r} \left(e(-(\gamma_j,\gamma_\ell)) + \xi\, e((\gamma_j,\gamma_\ell))\right)\hat{\mathfrak{e}}_\ell, & \text{if } 1 \le j \le q,\\ \dfrac{e((b^- - b^+)/8)}{\sqrt{|L'/L|}} \displaystyle\sum_{\ell=1}^{r} e(-(\gamma_j,\gamma_\ell))\hat{\mathfrak{e}}_\ell, & \text{if } q+1 \le j \le m, \end{cases}$$

for $j = 1,\dots,m$. Thus, the Weil representation ρ_L fixes the subspace $\mathcal{U}$, and we can therefore define the map

$$\nu\colon\ \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Aut}(\mathcal{U}),\ M \mapsto \rho_L(\tilde{M}), \tag{3.7.5}$$

which now satisfies

$$\nu(-1)\hat{\mathfrak{e}}_j = \rho_L(\widetilde{-1})\hat{\mathfrak{e}}_j = e(-k/2)\hat{\mathfrak{e}}_j \tag{3.7.6}$$

for $j = 1,\dots,m$. So by (3.7.3) and (3.7.6) the mapping ν is indeed a unitary multiplier system of weight k for the group $\mathrm{SL}_2(\mathbb{Z})$ and for the space $\mathcal{U}$ in the sense of Roelcke (compare [Roe66], Section 1.6).

As in equation (1.16) of [Roe66] we call a vector valued function $G\colon \mathbb{H} \to \mathcal{U}$ modular of weight k in the sense of Roelcke if $G\,|^{\mathrm{R}}_{k,L}\, M = G$ for all $M \in \mathrm{SL}_2(\mathbb{Z})$ where

$$\left(G\,|^{\mathrm{R}}_{k,L}\, M\right)(\tau) := \left(\frac{\phi_M(\tau)}{|\phi_M(\tau)|}\right)^{-2k} \nu(M)^{-1} G(M\tau) \tag{3.7.7}$$

for $M \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$. We then have the following identification:

Proposition 3.7.1. *A function $F\colon \mathbb{H} \to \mathbb{C}[L'/L]$ is modular of weight k with respect to the Weil representation ρ_L if and only if the function $G(\tau) := \mathrm{Im}(\tau)^{k/2}F(\tau)$ is modular of weight k in the sense of Roelcke.*

Proof. Since the elements of $\mathrm{Mp}_2(\mathbb{Z})$ can be written as $(M, \pm\phi_M)$ with $M \in \mathrm{SL}_2(\mathbb{Z})$, and since $(1,-1) \in \mathrm{Mp}_2(\mathbb{Z})$ acts trivially on any vector valued function $\mathbb{H} \to \mathbb{C}[L'/L]$ as we assume that $2k \equiv b^+ - b^- \bmod 2$, a function $F\colon \mathbb{H} \to \mathbb{C}[L'/L]$ is modular of weight k with respect to ρ_L if and only if $F\,|_{k,L}\,(M,\phi_M) = F$ for all $M \in \mathrm{SL}_2(\mathbb{Z})$. Moreover, it is easy to check that $F\,|_{k,L}\,(M,\phi_M) = F$ if and only if

$$\mathrm{Im}(\tau)^{k/2}F(\tau) = \left(\frac{\phi_M(\tau)}{|\phi_M(\tau)|}\right)^{-2k} \rho_L(M)^{-1}\left(\mathrm{Im}(M\tau)^{k/2}F(M\tau)\right)$$

for $M \in \mathrm{SL}_2(\mathbb{Z})$. This proves the claimed statement. □

We denote the corresponding identification map by Π_k, i.e., for $F\colon \mathbb{H} \to \mathbb{C}[L'/L]$ we define

$$\Pi_k(F)(\tau) := \mathrm{Im}(\tau)^{k/2}F(\tau) \tag{3.7.8}$$

for $\tau \in \mathbb{H}$. Clearly, $\Pi_k^{-1}(G) = \operatorname{Im}(\tau)^{-k/2} G(\tau)$ for $G\colon \mathbb{H} \to \mathcal{U}$, and by the above considerations $\Pi_k(F)$ is modular of weight k in the sense of Roelcke if and only if F is modular of weight k with respect to ρ_L.

In order to state the spectral theorem given as a combination of Satz 7.2 and Satz 12.3 in [Roe67], which we will then transfer to the setting of (non-holomorphic) vector valued modular forms for the Weil representation, we need to introduce some more of Roelckes notation. For $G, G'\colon \mathbb{H} \to \mathcal{U}$ modular of weight k in the sense of Roelcke and measurable with respect to μ we define the scalar product

$$(G, G')^R := \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \langle G(\tau), G'(\tau)\rangle \, d\mu(\tau)$$

whenever the integral on the right-hand side exists. We then denote the Hilbert space of μ-measurable functions $G\colon \mathbb{H} \to \mathcal{U}$ that are modular of weight k in the sense of Roelcke and square-integrable with respect to the above scalar product, i.e., $(G, G)^R < \infty$, by $\mathcal{H}_k(\mathrm{SL}_2(\mathbb{Z}), \mathcal{U}, \nu)$. Further, we write $\mathcal{C}_k^2(\mathrm{SL}_2(\mathbb{Z}), \mathcal{U}, \nu)$ for the space of functions $G\colon \mathbb{H} \to \mathcal{U}$ modular of weight k in the sense of Roelcke which are two times continuously partially differentiable.

On $\mathcal{C}_k^2(\mathrm{SL}_2(\mathbb{Z}), \mathcal{U}, \nu)$ we consider Roelcke's hyperbolic Laplace operator

$$\Delta_k^R := -v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) - ikv \frac{\partial}{\partial u}, \tag{3.7.9}$$

which is invariant under the corresponding weight k action of $\mathrm{SL}_2(\mathbb{Z})$, i.e., we have

$$\Delta_k^R \Big(G \big|_{k,L}^{\mathrm{R}} M \Big) = \left(\Delta_k^R G \right) \big|_{k,L}^{\mathrm{R}} M$$

for $G\colon \mathbb{H} \to \mathcal{U}$ two times continuously partially differentiable and $M \in \mathrm{SL}_2(\mathbb{Z})$. An easy calculation shows that the relation between Roelcke's Laplace operator and the hyperbolic Laplace operator from (2.5.2) is given by

$$\Delta_k^R \left(\Pi_k(F)(\tau) \right) = \Pi_k \left(\left(\Delta_k - \frac{k^2 - 2k}{4} \right) F(\tau) \right) \tag{3.7.10}$$

for $F\colon \mathbb{H} \to \mathbb{C}[L'/L]$ two times continuously partially differentiable. As in [Roe66], equation (2.21), we now define the space

$$\mathcal{D}_k^2(\mathrm{SL}_2(\mathbb{Z}), \mathcal{U}, \nu) := \left\{ G \in \mathcal{C}_k^2(\mathrm{SL}_2(\mathbb{Z}), \mathcal{U}, \nu)\colon G, \Delta_k^R G \in \mathcal{H}_k(\mathrm{SL}_2(\mathbb{Z}), \mathcal{U}, \nu) \right\}.$$

In [Roe66], Satz 3.2, the author proves that the restriction of the operator Δ_k^R to the space $\mathcal{D}_k^2(\mathrm{SL}_2(\mathbb{Z}), \mathcal{U}, \nu)$ has a self-adjoint continuation $\tilde{\Delta}_k^R$, and this continuation is then used to show that Δ_k^R itself has a countable orthonormal system of eigenfunctions $(\psi_j)_{j \in \mathbb{N}_0}$ and a countable system of pairwise orthogonal eigenpackages $(\nu_{j,\lambda})_{j \in N_0}$ in the sense of [Roe66], Definition 5.1, such that the system of all these eigenfunctions ψ_j and eigenpackages $\nu_{j,\lambda}$ is complete in the Hilbert space $\mathcal{H}_k(\mathrm{SL}_2(\mathbb{Z}), \mathcal{U}, \nu)$ (see [Roe66], Satz 5.7). In particular, the eigenfunctions ψ_j are real analytic elements of the space $\mathcal{D}_k^2(\mathrm{SL}_2(\mathbb{Z}), \mathcal{U}, \nu)$.

In [Roe67] Roelcke uses Eisenstein series to calculate a nicer expression for the terms coming from the eigenpackages $\nu_{j,\lambda}$ in the spectral expansion of a square-integrable function (compare Satz 7.2 and 12.3 in [Roe67]). Let $\mathcal{L}$ be the eigenraum for the eigenvalue 1 of the automorphism $\nu(T)\colon \mathcal{U} \to \mathcal{U}$ with $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, i.e.,

$$\mathcal{L} := \{x \in \mathcal{U} \colon \nu(T)x = x\} = \left\{ \sum_{\substack{\gamma \in L'/L \\ q(\gamma)=0}} \lambda_\gamma \mathfrak{e}_\gamma \in \mathcal{U} \right\}.$$

Evidently, an orthonormal basis of $\mathcal{L}$ is given by the set

$$\{\hat{\mathfrak{e}}_j \colon 1 \le j \le m \text{ and } q(\gamma_j) = 0\}.$$

Note that $\mathcal{L}$ might indeed be the zero-space if $\xi = -1$ and the group L'/L does not contain any norm-zero element of order larger than two. In this case there are no Eisenstein series to consider.

Now, given an element $\mathfrak{h} \in \mathcal{L}$ Roelcke defines the non-holomorphic Eisenstein series associated to $\mathfrak{h}$ via

$$E^{L,R}_{k,\mathfrak{h}}(\tau, s) := \frac{1}{2} \sum_{M \in \langle T \rangle \backslash \operatorname{SL}_2(\mathbb{Z})} \operatorname{Im}(\tau)^s \mathfrak{h} \Big|^{\mathrm{R}}_{k,L} M \tag{3.7.11}$$

for $\tau \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$ (see [Roe67], eq. (10.1)). Note that we can omit some of the notation in [Roe67] as we only work with the full modular group $\operatorname{SL}_2(\mathbb{Z})$ which has exactly one cusp, namely ∞. One can check that there are exactly $\dim(\mathcal{L})$ linearly independent Eisenstein series of the above type. Thus, it suffices to consider the Eisenstein series $E^{L,R}_{k,\hat{\mathfrak{e}}_j}(\tau, s)$ for $j = 1, \dots, m$ with $q(\gamma_j) = 0$.

We can now finally state [Roe67], Satz 7.2, where we replace the second part of the spectral expansion by the expression given in [Roe67], Satz 12.3:

Theorem 3.7.2 ([Roe67], Satz 7.2 and 12.3)**.** *Let $G \in \mathcal{H}_k(\operatorname{SL}_2(\mathbb{Z}), \mathcal{U}, \nu)$. Then*

$$G(\tau) = \sum_{j=0}^{\infty} (G, \psi_j)^R \psi_j(\tau) + \frac{1}{4\pi} \sum_{\substack{j=1,\dots,m \\ q(\gamma_j)=0}} \int_{-\infty}^{\infty} \left(G, E^{L,R}_{k,\hat{\mathfrak{e}}_j}\left(\cdot, \frac{1}{2} + ir\right)\right)^R E^{L,R}_{k,\hat{\mathfrak{e}}_j}\left(\tau, \frac{1}{2} + ir\right) dr.$$

Here $(\psi_j)_{j \in \mathbb{N}_0} \subseteq \mathcal{D}^2_k(\operatorname{SL}_2(\mathbb{Z}), \mathcal{U}, \nu)$ is an orthonormal system of real analytic eigenfunctions of Roelcke's hyperbolic Laplace operator Δ^R_k as introduced above, and the elements $\hat{\mathfrak{e}}_j$ for $j = 1, \dots, m$ with $q(\gamma_j) = 0$ form an orthonormal basis of the 1-eigenraum $\mathcal{L}$ of $\nu(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right))$.

Using the identification given in (3.7.8) we can reformulate the above result in the setting of this work, i.e., for vector valued functions modular with respect to the Weil representation. In oder to do so we quickly comment on the necessary translations:

(1) For $F, F' \colon \mathbb{H} \to \mathbb{C}[L'/L]$ modular of weight k with respect to ρ_L and measurable with respect to μ we have

$$(\Pi_k(F), \Pi_k(F'))^R = (F, F').$$

In particular, F is square-integrable with respect to the scalar product $(\cdot,\cdot)$ if and only if $\Pi_k(F)$ is square-integrable with respect to $(\cdot,\cdot)^R$. Defining

$$\mathcal{H}_{k,L} := \{F \in \mathcal{A}_{k,L} \colon F \text{ is } \mu\text{-measurable and } (F,F) < \infty\}, \tag{3.7.12}$$

where $\mathcal{A}_{k,L}$ is the space of functions $F\colon \mathbb{H} \to \mathbb{C}[L'/L]$ modular of weight k with respect to ρ_L, we thus find that $\Pi_k(\mathcal{H}_{k,L}) = \mathcal{H}_k(\mathrm{SL}_2(\mathbb{Z}),\mathcal{U},\nu)$.

(2) As in [Roe66], Satz 3.2, the restriction of the hyperbolic Laplace operator Δ_k to the space

$$\mathcal{D}_{k,L} := \Pi_k^{-1}(\mathcal{D}_k^2(\mathrm{SL}_2(\mathbb{Z}),\mathcal{U},\nu)) \tag{3.7.13}$$

has a self-adjoint continuation which we denote by $\tilde{\Delta}_k$. Let $(\psi_j)_{j\in\mathbb{N}_0}$ be an orthonormal system of eigenfunctions of Roelcke's Laplace operator Δ_k^R as in the above theorem. Then $(\Pi_k^{-1}(\psi_j))_{j\in\mathbb{N}_0}$ is an orthonormal system of eigenfunctions of the hyperbolic Laplace operator Δ_k which is complete in the Hilbert space $\mathcal{H}_{k,L}$ up to some eigenpackages corresponding to Eisenstein series.

(3) A direct computation shows that

$$\Pi_k^{-1}\big(E^{L,R}_{k,\hat{\mathfrak{e}}_j}(\tau,s)\big) = \frac{1}{2}\, E^L_{k,\gamma_j}(\tau, s-k/2)$$

for $j=1,\dots,m$ with $q(\gamma_j)=0$.

Therefore, Theorem 3.7.2 translates to the following statement:

Theorem 3.7.3. *Let $F \in \mathcal{H}_{k,L}$. Then*

$$\begin{aligned} F(\tau) &= \sum_{j=0}^{\infty} (F,\psi_j)\psi_j(\tau) \\ &\quad + \frac{1}{16\pi} \sum_{\substack{j=1,\dots,m\\ q(\gamma_j)=0}} \int_{-\infty}^{\infty} \left(F, E^L_{k,\gamma_j}\left(\cdot, \frac{1-k}{2}+ir\right)\right) E^L_{k,\gamma_j}\left(\tau, \frac{1-k}{2}+ir\right) dr. \end{aligned}$$

Here $(\psi_j)_{j\in\mathbb{N}_0} \subseteq \mathcal{D}_{k,L}$ is an orthonormal system of real analytic eigenfunctions of the hyperbolic Laplace operator Δ_k, and

$$L'/L = \{\gamma_1, -\gamma_1, \dots, \gamma_q, -\gamma_q, \gamma_{q+1}, \dots, \gamma_r\}$$

with $\gamma_1, -\gamma_1, \dots, \gamma_q, -\gamma_q, \gamma_{q+1}, \dots, \gamma_r$ pairwise distinct and $2\gamma_j = 0$ for $j = q+1,\dots,r$. Further, we have $m = r$ if $(-1)^{b^+/2-b^-/2-k} = 1$ and $m = q$ otherwise.

To the end of this section we further study eigenvalues of the operator Δ_k and their corresponding eigenfunctions, following [Roe66]. Recall that $\tilde{\Delta}_k^R$ denotes the self-adjoint continuation of the restriction of the operator Δ_k^R to the space $\mathcal{D}_k^2(\mathrm{SL}_2(\mathbb{Z}),\mathcal{U},\nu)$. Combining Satz 5.4, Satz 5.5 and the considerations on page 335 in [Roe66] we find

$$\operatorname{spec}\big(\tilde{\Delta}_k^R\big) \subseteq \big\{\lambda_0^R, \lambda_1^R, \dots, \lambda_{m_k}^R\big\} \cup [\mu_k^R, \infty), \tag{3.7.14}$$

where the left-hand side denotes the spectrum of the operator $\tilde{\Delta}_k^R$, and λ_ℓ^R and μ_k^R are given by

$$\lambda_\ell^R := \frac{1}{4} - \left(\frac{|k|-1}{2} - \ell\right)^2, \qquad \mu_k^R := \frac{1}{4} - \left(\frac{1-k_0}{2}\right)^2 \tag{3.7.15}$$

for $\ell = 0, 1, \ldots, m_k$. Here $m_k := \lfloor \frac{|k|-1}{2} \rfloor$, and $k_0 \in [0, 2)$ such that $k_0 \equiv k \bmod 2$. In particular, the finite set $\{\lambda_0^R, \ldots, \lambda_{m_k}^R\}$ in (3.7.14) can be omitted if $|k| \leq 2$ as for $|k| < 1$ we have $m_k < 0$, and for $1 \leq |k| \leq 2$ we have $m_k = 0$ and $\lambda_0^R = \mu_k^R$.

We can reformulate the above result as follows: If $G \in \mathcal{D}_k^2(\mathrm{SL}_2(\mathbb{Z}), \mathcal{U}, \nu)$ is an eigenfunction of Δ_k^R we can write its eigenvalue λ^R as $\lambda^R = 1/4 + r^2$ with

$$r \in [0, \infty), \qquad r \in \left[i\,\frac{1-k_0}{2}, 0\right) \quad \text{or} \quad r = i\left(\frac{|k|-1}{2} - \ell\right) \tag{3.7.16}$$

for some $\ell = 0, 1, \ldots, m_k$.

Next we again translate these results to our vector valued setting. Recall that we write $\tilde{\Delta}_k$ for the self-adjoint continuation of the restriction of the Laplace operator Δ_k to the space $\mathcal{D}_{k,L}$. Using the relation (3.7.10) it is easy to see that $G\colon \mathbb{H} \to \mathcal{U}$ is an eigenfunction of Roelckes hyperbolic Laplace operator Δ_k^R with eigenvalue λ^R if and only if $\Pi_k^{-1}(G)$ is an eigenfunction of the operator Δ_k with eigenvalue $\lambda = \lambda^R + \frac{k^2-2k}{4}$. Therefore, the spectrum of $\tilde{\Delta}_k$ is simply shifted by $\frac{k^2-2k}{4}$, i.e., (3.7.14) translates to

$$\operatorname{spec}\left(\tilde{\Delta}_k\right) \subseteq \{\lambda_0, \lambda_1, \ldots, \lambda_{m_k}\} \cup [\mu_k, \infty). \tag{3.7.17}$$

with

$$\lambda_\ell := \left(\frac{1-k}{2}\right)^2 - \left(\frac{|k|-1}{2} - \ell\right)^2, \qquad \mu_k := \left(\frac{1-k}{2}\right)^2 - \left(\frac{1-k_0}{2}\right)^2, \tag{3.7.18}$$

and where m_k and k_0 are given as above. Moreover, given an eigenfunction $F \in \mathcal{D}_{k,L}$ of Δ_k we can write its eigenvalue λ as

$$\lambda = \left(\frac{1-k}{2}\right)^2 + r^2 \tag{3.7.19}$$

with r as in (3.7.16).

In chapter 2 of [Roe66] the author also studies Fourier expansions of eigenfunctions. More precisely, it is shown that if $G\colon \mathbb{H} \to \mathcal{U}$ is modular of weight k in the sense of Roelcke and an eigenfunction of Δ_k^R with eigenvalue $\lambda^R = 1/4 + r^2$ for some $r \in \mathbb{C}$, then G has a Fourier expansion of the form

$$\begin{aligned} G(\tau) = u(v) + \sum_{j=1}^{m} \sum_{\substack{n \in \mathbb{Z}+q(\gamma_j) \\ n \neq 0}} \big(& b_G(\gamma_j, n) W_{-\operatorname{sign}(n)\,k/2, ir}(-4\pi|n|v) \\ &+ c_G(\gamma_j, n) W_{\operatorname{sign}(n)\,k/2, ir}(4\pi|n|v)\big)\, e(nu)\hat{\mathfrak{e}}_j \end{aligned} \tag{3.7.20}$$

for $\tau = u + iv \in \mathbb{H}$ and $b_G(\delta_j, n), c_G(\delta_j, n) \in \mathbb{C}$ (see [Roe66], equations (2.13) and (2.18)). Here the function $u(v)$ is given by

$$u(v) := v^{1/2-ir} \sum_{\substack{j=1,\dots,m \\ q(\gamma_j)=0}} b_G(\gamma_j, 0)\hat{\mathfrak{e}}_j + v^{1/2+ir} \sum_{\substack{j=1,\dots,m \\ q(\gamma_j)=0}} c_G(\gamma_j, 0)\hat{\mathfrak{e}}_j, \tag{3.7.21}$$

where we need to replace the term $v^{1/2+ir}$ by $v^{1/2}\ln(v)$ if $r = 0$ (compare [Roe66], equation (2.17)). If we further assume that $G(\tau) = O(v^{\alpha})$ as $v \to \infty$ uniformly in u for some $\alpha \in \mathbb{R}$ as in [Roe66], Definition 1.1 (b), then we find $b_G(\gamma_j, n) = 0$ for all $j = 1, \dots, m$ and $n \in \mathbb{Z} + q(\gamma_j)$, $n \neq 0$ (compare [Roe66], Lemma 2.1) , i.e.,

$$G(\tau) = u(v) + \sum_{j=1}^{m} \sum_{\substack{n \in \mathbb{Z}+q(\gamma_j) \\ n \neq 0}} c_G(\gamma_j, n) W_{\operatorname{sign}(n)\, k/2, ir}(4\pi|n|v) e(nu)\hat{\mathfrak{e}}_j, \tag{3.7.22}$$

where $u(v)$ is still given as in (3.7.21).

We now further assume that $G \in \mathcal{D}_k^2(\mathrm{SL}_2(\mathbb{Z}), \mathcal{U}, \nu)$. Then $\lambda^R = 1/4 + r^2$ with r as in (3.7.16), i.e., we either have $r \geq 0$ or $r \in i\mathbb{R}$. Moreover, G is square-integrable and thus needs to behave as $v^{1/2-\varepsilon}$ as $v \to \infty$ with $\varepsilon > 0$. Therefore G has a Fourier expansion as in (3.7.22) where if r is purely imaginary either the coefficients $b_G(\gamma_j, 0)$ or $c_G(\gamma_j, 0)$ of $u(v)$ all need to vanish, depending on whether $\mathrm{Im}(r)$ is positive or negative, and if r is real $u(v)$ needs to vanish identically. In other words, we have

(3.7.23)

$$G(\tau) = v^{1/2-|\operatorname{Im}(r)|} \sum_{\substack{j=1,\dots,m \\ q(\gamma_j)=0}} c_G(\gamma_j, 0)\hat{\mathfrak{e}}_j + \sum_{j=1}^{m} \sum_{\substack{n \in \mathbb{Z}+q(\gamma_j) \\ n \neq 0}} c_G(\gamma_j, n) W_{\operatorname{sign}(n)\, k/2, ir}(4\pi|n|v) e(nu)\hat{\mathfrak{e}}_j,$$

with $c_G(\gamma_j, 0) = 0$ for $j = 1, \dots, m$ if r is real.

All of these considerations translate directly to the setting of vector valued eigenfunctions for the Weil representation. We quickly summarise them in the following lemma:

Lemma 3.7.4. *Let $F\colon \mathbb{H} \to \mathbb{C}[L'/L]$ be modular of weight k with respect to the Weil representation and an eigenfunction of the hyperbolic Laplace operator Δ_k with eigenvalue*

$$\lambda = \left(\frac{1-k}{2}\right)^2 + r^2$$

with $r \in \mathbb{C}$.

(a) Then F has a Fourier expansion of the form

$$\begin{aligned} F(\tau) = u(v) + \sum_{j=1}^{m} \sum_{\substack{n \in \mathbb{Z}+q(\gamma_j) \\ n \neq 0}} \Big(& b_F(\gamma_j, n)(-4\pi|n|v)^{-k/2} W_{-\operatorname{sign}(n)\, k/2, ir}(-4\pi|n|v) \\ & + c_F(\gamma_j, n)(4\pi|n|v)^{-k/2} W_{\operatorname{sign}(n)\, k/2, ir}(4\pi|n|v) \Big) e(nu)\hat{\mathfrak{e}}_j \end{aligned}$$

with

$$u(v) = v^{1/2-k/2-ir} \sum_{\substack{j=1,\dots,m \\ q(\gamma_j)=0}} b_F(\gamma_j, 0)\hat{\mathfrak{e}}_j + v^{1/2-k/2+ir} \sum_{\substack{j=1,\dots,m \\ q(\gamma_j)=0}} c_F(\gamma_j, 0)\hat{\mathfrak{e}}_j.$$

(b) If $F(\tau) = O(v^\alpha)$ as $v \to \infty$ uniformly in u for some $\alpha \in \mathbb{R}$ then F has a Fourier expansion of the form

$$F(\tau) = u(v) + \sum_{j=1}^{m} \sum_{\substack{n \in \mathbb{Z}+q(\gamma_j) \\ n \neq 0}} c_F(\gamma_j, n) \mathcal{W}_{k,1/2+ir}(4\pi n v) e(nu) \hat{\mathfrak{e}}_j,$$

where $u(v)$ is given as in part (a).

(c) If $F \in \mathcal{D}_{k,L}$ then $\lambda = (\frac{1-k}{2})^2 + r^2$ with

$$r \in [0, \infty), \qquad r \in \left[i\,\frac{1-k_0}{2}, 0\right) \quad \text{or} \quad r = i\left(\frac{|k|-1}{2} - \ell\right),$$

where $k_0 \in [0,2)$ with $k_0 \equiv k$ mod 2 and $\ell \in \{0, 1, \ldots, m_k\}$ with $m_k = \lfloor \frac{|k|-1}{2} \rfloor$. In this case F has a Fourier expansion of the form

$$F(\tau) = v^{1/2-k/2-|\operatorname{Im}(r)|} \sum_{\substack{j=1,\ldots,m \\ q(\gamma_j)=0}} c_F(\gamma_j, 0)\hat{\mathfrak{e}}_j + \sum_{j=1}^{m} \sum_{\substack{n \in \mathbb{Z}+q(\gamma_j) \\ n \neq 0}} c_F(\gamma_j, n) \mathcal{W}_{k,1/2+ir}(4\pi n v) e(nu) \hat{\mathfrak{e}}_j,$$

where $c_F(\gamma_j, 0) = 0$ for $j = 1, \ldots, m$ if r is real.

4 Borchderds' generalized Shimura lift

In the following we give a quick introduction to the theory of regularized theta lifts in the sense of Borcherds, focussing on a generalized version of the classical Shimura lift from half-integral weight modular forms to forms of even weight. In the second half of this chapter, we specialize to certain lattices of signature $(2,1)$ and $(2,2)$.

Our main reference for this chapter is the original work of Borcherds, namely [Bor98]. Furthermore, we again use the notation from [Bru02].

4.1 Modular forms on orthogonal groups

Let (V,q) be a quadratic space of signature $(2,n)$ with $n \geq 1$. The *Grassmannian* of V is given by the set of 2-dimensional positive definite subspaces of $V(\mathbb{R}) := V \otimes \mathbb{R}$, i.e., by

$$\mathrm{Gr}(V) := \{v \subseteq V(\mathbb{R})\colon \dim(v) = 2 \text{ and } q|_v > 0\}.$$

For $v \in \mathrm{Gr}(V)$ we write $v^\perp$ for the orthogonal complement of v in $V(\mathbb{R})$ such that $V(\mathbb{R}) = v \oplus v^\perp$. Clearly, $v^\perp$ is a negative definite subspace of dimension n. Further, we denote the orthogonal projection of some vector $X \in V(\mathbb{R})$ onto v or $v^\perp$ by X_v or $X_{v^\perp}$, respectively. In particular, we have $q(X) = q(X_v) + q(X_{v^\perp})$.

Next we equip the Grassmannian $\mathrm{Gr}(V)$ with a complex structure: Firstly, we extend the bilinear form on V to a $\mathbb{C}$-bilinear form on the complexification $V(\mathbb{C}) = V \otimes \mathbb{C}$ of V, which we again denote by $(\cdot,\cdot)$, and we let $\mathbb{P}(V(\mathbb{C}))$ be the projective space over $V(\mathbb{C})$. We write $Z \mapsto [Z]$ for the canonical projection of $V(\mathbb{C}) \setminus \{0\}$ onto $\mathbb{P}(V(\mathbb{C}))$. Now, the complex manifold

$$\mathcal{K} := \left\{[Z] \in \mathbb{P}(V(\mathbb{C}))\colon (Z,Z) = 0 \text{ and } (Z,\overline{Z}) > 0\right\} \tag{4.1.1}$$

has two connected components which we denote by $\mathcal{K}^\pm$, and it is easy to check that the two maps

$$\mathcal{K}^\pm \to \mathrm{Gr}(V),\ [X+iY] \mapsto \mathbb{R}X \oplus \mathbb{R}Y$$

are both bijections, inducing a complex structure on the Grassmannian $\mathrm{Gr}(V)$.

Next we realize $\mathcal{K}^+$ as a so-called tube domain: Let L be an even lattice in V and choose vectors $e_1 \in L$ and $e_2 \in L'$ such that e_1 is primitive with $q(e_1) = 0$ and $(e_1,e_2) = 1$. We then define the sublattice

$$K := L \cap e_1^\perp \cap e_2^\perp$$

and let $U := K \otimes \mathbb{Q}$. Then K is an even lattice lying in the quadratic space (U,q) of signature $(1,n-1)$, and we have $V = U \oplus \mathbb{Q}e_1 \oplus \mathbb{Q}e_2$. Further, we define the complex plane

$$\mathcal{H} := \{X + iY \in U(\mathbb{C})\colon q(Y) > 0\}$$

of vectors in $U(\mathbb{C})$ with positive imaginary part, and for $Z = X + iY \in U(\mathbb{C})$ we set

$$Z_L := X_L + iY_L = Z - (q(Z) + q(e_2))e_1 + e_2.$$

Here $X_L = X - (q(X) - q(Y) + q(e_2))e_1 + e_2$ and $Y_L = Y - (X, Y)e_1$. Then $(Z_L, Z_L) = 0$ and $(Z_L, \overline{Z_L}) = 4q(Y) > 0$ for $Z = X + iY \in \mathcal{H}$, and thus the mapping

$$\mathcal{H} \to \mathcal{K},\ Z \mapsto [Z_L]$$

is well-defined. Moreover, one can check that this map is indeed biholomorphic, giving us an identification between the complex plane $\mathcal{H}$ and the complex manifold $\mathcal{K}$. We let $\mathbb{H}_n$ be the component of $\mathcal{H}$ which is mapped to $\mathcal{K}^+$, and we call $\mathbb{H}_n$ a *generalized upper half-plane*. Note that the concrete realization of $\mathbb{H}_n$ does depend on the choice of vectors e_1 and e_2.

We further introduce some notation. For $Z \in \mathbb{H}_n$ we write

$$Z = \mathrm{Re}(Z) + i\,\mathrm{Im}(Z)$$

with $\mathrm{Re}(Z), \mathrm{Im}(Z) \in U(\mathbb{R})$ and where $q(\mathrm{Im}(Z)) > 0$. Moreover, for $Z \in \mathbb{H}_n$ with $Z_L = X_L + iY_L$ and $\lambda \in V$ we denote the orthogonal projection of λ onto the positive definite subspace $\mathbb{R}X_L \oplus \mathbb{R}Y_L$ by λ_Z. Correspondingly, we write $\lambda_{Z^\perp}$ for the orthogonal projection of λ onto the negative definite subspace $(\mathbb{R}X_L \oplus \mathbb{R}Y_L)^\perp$. Then an easy computation shows that

$$q(\lambda_Z) = \frac{|(\lambda, Z_L)|^2}{4q(\mathrm{Im}(Z))} \tag{4.1.2}$$

for $\lambda \in V$ and $Z \in \mathbb{H}_n$. Finally, we introduce the notation

$$q_Z(\lambda) := q(\lambda_Z) - q(\lambda_{Z^\perp}) = 2q(\lambda_Z) - q(\lambda) \tag{4.1.3}$$

for $\lambda \in V$ and $Z \in \mathbb{H}_n$. Then q_Z is a majorant of the quadratic form q associated to Z. In particular, the quadratic form q_Z is positive definite.

In the following, we introduce an automorphy factor for the action of an orthogonal group on the generalized upper half-plane defined above. Recall that $O(V)$ is the orthogonal group of the quadratic space (V, q). We write $\mathrm{SO}(V)$ for the special orthogonal group of q, i.e., for the subgroup of $O(V)$ of elements of determinant 1. It is well-known that for $n \geq 1$ the group $\mathrm{SO}(V)$ is not connected. As usual we denote the connected component of the identity in $\mathrm{SO}(V)$ by $\mathrm{SO}^+(V)$.

The action of $O(V)$ on $V(\mathbb{R})$ naturally induces actions on the Grassmannian $\mathrm{Gr}(V)$ and on the complex manifold $\mathcal{K}$ which are by construction compatible with the identification $[X+iY] \mapsto \mathbb{R}X \oplus \mathbb{R}Y$, and the subgroup $\mathrm{SO}^+(V)$ preserves the two connected components $\mathcal{K}^\pm$ of $\mathcal{K}$, whereas $\mathrm{SO}(V) \setminus \mathrm{SO}^+(V)$ interchanges them. Since the mapping $Z \mapsto [Z_L]$ defines a bijection between the generalized upper half-plane $\mathbb{H}_n$ and the manifold $\mathcal{K}^+$, the action of the group $\mathrm{SO}^+(V)$ on $\mathcal{K}^+$ further induces an action of $\mathrm{SO}^+(V)$ on $\mathbb{H}_n$ which we denote by $\sigma.Z$ for $\sigma \in \mathrm{SO}^+(V)$ and $Z \in \mathbb{H}_n$ in order to avoid confusion with the operation of $\mathrm{SO}^+(V)$ on $V(\mathbb{C})$ which we simply denote by $\sigma(Z)$. By construction we then have

$$[(\sigma.Z)_L] = [\sigma(Z_L)]$$

for $\sigma \in \mathrm{SO}^+(V)$ and $Z \in \mathbb{H}_n$, i.e., there is $\mu \in \mathbb{C}^\times$ such that $\mu \cdot (\sigma.Z)_L = \sigma(Z_L)$ where we understand $(\sigma.Z)_L$ and $\sigma(Z_L)$ as elements of the cone over $\mathcal{K}^+$ given by

$$C(\mathcal{K}^+) := \left\{W \in V(\mathbb{C}) \setminus \{0\} \colon [W] \in \mathcal{K}^+\right\}.$$

Using that $\mathbb{H}_n \subseteq U(\mathbb{C})$ it is easy to check that the factor μ from above is given by the scalar product $(\sigma(Z_L), e_1)$, i.e., we have

$$(\sigma(Z_L), e_1) \cdot (\sigma.Z)_L = \sigma(Z_L)$$

for all $\sigma \in \mathrm{SO}^+(V)$ and $Z \in \mathbb{H}_n$. We give this factor a name, i.e., we define the complex valued function

$$j(\sigma, Z) := (\sigma(Z_L), e_1)$$

for $\sigma \in \mathrm{SO}^+(V)$ and $Z \in \mathbb{H}_n$. By construction $j(\sigma, Z)$ is non-vanishing and satisfies the so-called cocycle relation

$$j(\sigma_1\sigma_2, Z) = j(\sigma_1, \sigma_2.Z)j(\sigma_2, Z)$$

for $\sigma_1, \sigma_2 \in \mathrm{SO}^+(V)$ and $Z \in \mathbb{H}_n$. Thus we can call the function j an automorphy factor for the group $\mathrm{SO}^+(V)$. We will later see that it indeed generalizes the usual automorphy factor $j(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \tau) = c\tau + d$ defined on $\mathrm{SL}_2(\mathbb{R}) \times \mathbb{H}$ at the beginning of Section 2.4.

Lemma 4.1.1. *Let $\sigma \in \mathrm{SO}^+(V)$. Then*

$$q(\mathrm{Im}(\sigma.Z)) = \frac{q(\mathrm{Im}(Z))}{|j(\sigma, Z)|^2} \qquad \text{and} \qquad q(\sigma(\lambda)_Z) = q(\lambda_{\sigma^{-1}.Z}).$$

for $Z \in \mathbb{H}_n$ and $\lambda \in V$.

Proof. We find that

$$q(\mathrm{Im}(\sigma.Z)) = \frac{((\sigma.Z)_L, \overline{(\sigma.Z)_L})}{4} = \frac{(\sigma(Z_L), \overline{\sigma(Z_L)})}{4|j(\sigma, Z)|^2} = \frac{(Z_L, \overline{Z_L})}{4|j(\sigma, Z)|^2} = \frac{q(\mathrm{Im}(Z))}{|j(\sigma, Z)|^2},$$

which proves the first equality. Further, we obtain

$$q(\sigma(\lambda)_Z) = \frac{|(\sigma(\lambda), Z_L)|^2}{4q(\mathrm{Im}(Z))} = \frac{|(\lambda, \sigma^{-1}(Z_L))|^2}{4|j(\sigma^{-1}, Z)|^2 q(\mathrm{Im}(\sigma^{-1}.Z))} = q(\lambda_{\sigma^{-1}.Z})$$

using equation (4.1.2). □

Finally, we recall that $O(L) = \{\sigma \in O(V) \colon \sigma(L) = L\}$, and we let $O_d(L)$ be the finite index subgroup of $O(L)$ consisting of all elements that act trivially on the discriminant group L'/L. We then define the group

$$\Gamma(L) := \mathrm{SO}^+(V) \cap O_d(L). \tag{4.1.4}$$

Moreover, we set $\mathrm{SO}^+(L) := \mathrm{SO}^+(V) \cap O(L)$. One can show that the two groups $\Gamma(L)$ and $\mathrm{SO}^+(L)$ are discrete subgroups of $\mathrm{SO}^+(V)$.

Definition 4.1.2. Let $k \in \mathbb{Z}$ and let $\Gamma \leq \Gamma(L)$ be a finite index subgroup. We say that a function $F\colon \mathbb{H}_n \to \mathbb{C}$ is *modular of weight k with respect to* Γ if

$$F(\gamma.Z) = j(\gamma, Z)^k F(Z)$$

for all $\gamma \in \Gamma$ and $Z \in \mathbb{H}_n$.

Lemma 4.1.3. *Let $G\colon C(\mathcal{K}^+) \to \mathbb{C}$ be homogeneous of degree $-k \in \mathbb{Z}$ and let Γ be a finite index subgroup of $\Gamma(L)$. Then G is invariant under the action of Γ, i.e., $G(\gamma(W)) = G(W)$ for all $\gamma \in \Gamma$ and $W \in C(\mathcal{K}^+)$, if and only if the corresponding function $F\colon \mathbb{H}_n \to \mathbb{C}$ defined by $F(Z) = G(Z_L)$ is modular of weight k with respect to Γ.*

Proof. Suppose that G is invariant under the action of Γ. Then

$$F(\gamma.Z) = G((\gamma.Z)_L) = G\big(j(\gamma, Z)^{-1}\gamma(Z_L)\big) = j(\gamma, Z)^k G(\gamma(Z_L)) = j(\gamma, Z)^k F(Z)$$

for $\gamma \in \Gamma$ and $Z \in \mathbb{H}_n$. Conversely, given $W \in C(\mathcal{K}^+)$ there is $Z \in \mathbb{H}_n$ such that $[Z_L] = [W]$ in $\mathcal{K}^+$, i.e., we find $\mu \in \mathbb{C}^\times$ with $W = \mu \cdot Z_L$. Hence

$$G(\gamma(W)) = \mu^{-k} G(\gamma(Z_L)) = \mu^{-k} j(\gamma, Z)^{-k} G((\gamma.Z)_L) = \mu^{-k} j(\gamma, Z)^{-k} F(\gamma.Z)$$

for $\gamma \in \Gamma$. Assuming that F is modular of weight k with respect to Γ we therefore obtain

$$G(\gamma(W)) = \mu^{-k} F(Z) = \mu^{-k} G(Z_L) = G(\mu \cdot Z_L) = G(W)$$

for all $\Gamma \in \Gamma$. This proves the claimed equivalence. $\square$

4.2 Borcherds regularized theta lift

In the present section we want to use Borcherds' language to define a certain regularized theta lift, using a vector valued version of Shintani's theta function. For the moment, let (V, q) be a quadratic space of dimension n and signature (b^+, b^-), and let L be an even lattice in V. In Section 4 of his celebrated work [Bor98] Borcherds defines the theta function

$$\Theta_L(\tau, v; p) := \sum_{\beta \in L'/L} \sum_{\lambda \in L+\beta} \exp\left(-\frac{\Delta}{8\pi \operatorname{Im}(\tau)}\right)(p)(v(\lambda))\ e(\tau q(\lambda_{v^+}) + \overline{\tau} q(\lambda_{v^-}))\mathfrak{e}_\beta \tag{4.2.1}$$

for $\tau \in \mathbb{H}$, an isometry $v\colon V(\mathbb{R}) \to \mathbb{R}^{b^+,b^-}$ and a polynomial $p\colon \mathbb{R}^{b^+,b^-} \to \mathbb{R}$ homogeneous of degree (m^+, m^-). We quickly explain the notation: We write $\mathbb{R}^{b^+,b^-}$ for the standard real vector space of signature (b^+, b^-), i.e., the space $\mathbb{R}^n$ equipped with the quadratic form $x \mapsto x_1^2 + \ldots + x_{b^+}^2 - x_{b^++1}^2 - \ldots - x_n^2$, and we call p homogeneous of degree (m^+, m^-) if p is homogeneous of degree m^+ in the first b^+ variables and of degree m^- in the last b^- variables where m^+ and m^- are non-negative integers. Moreover, Δ denotes the usual Laplace operator on $\mathbb{R}^n$, i.e., $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$, and

$$\exp\left(-\frac{\Delta}{8\pi \operatorname{Im}(\tau)}\right)(p)(v(\lambda)) = \sum_{j=0}^\infty \frac{(\Delta^j p)(v(\lambda))}{j!\,(-8\pi \operatorname{Im}(\tau))^j}.$$

Finally, we write X_{v^+} and X_{v^-} for the orthogonal projection of $X \in V(\mathbb{R})$ onto the subspaces v^+ and v^-, respectively, and by v^+ and v^- we mean the inverse images of $\mathbb{R}^{b^+,0}$ and $\mathbb{R}^{0,b^-}$ in $V(\mathbb{R})$ under the isometry v with $\mathbb{R}^{b^+,0}$ and $\mathbb{R}^{0,b^-}$ being the obvious choices of subspaces of $\mathbb{R}^{b^+,b^-}$.

In [Bor98], Theorem 4.1, Borcherds proves that $\Theta_L(\tau, v; p)$ transforms as

$$\Theta_L(M\tau, v; p) = \phi_M(\tau)^{b^+/2+m^+}\overline{\phi_M(\tau)}^{b^-/2+m^-}\rho_L(M, \phi_M)\Theta_L(\tau, v; p)$$

for $(M, \phi_M) \in \mathrm{Mp}_2(\mathbb{Z})$. Equivalently, we can say that the function

$$\tau \mapsto \mathrm{Im}(\tau)^{b^-/2+m^-}\Theta_L(\tau, v; p)$$

is modular of weight $k = b^+/2 - b^-/2 + m^+ - m^-$ with respect to the Weil representation ρ_L. Moreover, the group $\mathrm{SO}^+(V)$ acts on the set of isometries $V(\mathbb{R}) \to \mathbb{R}^{b^+,b^-}$ via $\sigma.v = v \circ \sigma$ for $\sigma \in \mathrm{SO}^+(V)$. Since $(\sigma.v)(X) = v(\sigma(X))$ and $q(X_{(\sigma.v)^\pm}) = q((\sigma(X))_{v^\pm})$ for $X \in V(\mathbb{R})$ the theta function $\Theta_L(\tau, v; p)$ is invariant in v under the action of the group $\Gamma(L)$ defined in (4.1.4), which simply changes the order of summation.

We now turn back to the case of (V, q) being a quadratic space of dimension $n + 2$ and signature $(2, n)$ as in the previous section. In addition, we fix a polynomial $p\colon \mathbb{R}^{2,n} \to \mathbb{R}$, and we assume that p is harmonic and homogeneous of degree $(m, 0)$ for some non-negative integer m. Then $\Delta^j p = 0$ for $j \geq 1$ giving $\exp(-\frac{\Delta}{8\pi y})(p)(v(\lambda)) = p(v(\lambda))$, and $p(x_1, \ldots, x_{n+2})$ does only depend on x_1 and x_2, i.e., p is actually a polynomial in 2 variables homogeneous of degree m. In this situation we can write Borcherds theta function defined in (4.2.1) in the form

$$\Theta_L(\tau, v; p) = \sum_{\beta \in L'/L} \sum_{\lambda \in L+\beta} p(v_1(\lambda), v_2(\lambda))e(\tau q(\lambda_{v^+}) + \overline{\tau} q(\lambda_{v^-}))\mathfrak{e}_\beta$$

for $\tau \in \mathbb{H}$ and an isometry $v\colon V(\mathbb{R}) \to \mathbb{R}^{2,n}$. However, since $v^- = (v^+)^\perp$ this theta function does not depend on the complete isometry v, but only on its first and second component. More precisely, given an isometry $v\colon V \to \mathbb{R}^{2,n}$ let $X = v^{-1}(e_1)$ and $Y = v^{-1}(e_2)$ be the preimages of the usual basis vectors e_1 and e_2 of the vector space $\mathbb{R}^{2,n}$. Then $v_1(\lambda) = (\lambda, X)$ and $v_2(\lambda) = (\lambda, Y)$ for $\lambda \in V(\mathbb{R})$, and $v^+ = \mathbb{R}X \oplus \mathbb{R}Y$. Moreover, we have $q(X) = q(Y) = 1$ and $(X, Y) = 0$, and thus the element $W = X + iY$ lies in the cone

$$C(\mathcal{K}) := \{W \in V(\mathbb{C}) \setminus \{0\}\colon [W] \in \mathcal{K}\}.$$

Hence, we may understand $\Theta_L(\tau, \cdot\,; p)$ as a function on the cone $C(\mathcal{K})$, i.e.,

$$\Theta_L(\tau, W; p) = \sum_{\beta \in L'/L} \sum_{\lambda \in L+\beta} p((\lambda, X), (\lambda, Y))e(\tau q(\lambda_W) + \overline{\tau} q(\lambda_{W^\perp}))\mathfrak{e}_\beta$$

for $\tau \in \mathbb{H}$ and $W \in C(\mathcal{K})$, where in slight abuse of notation we write λ_W and $\lambda_{W^\perp}$ instead of $\lambda_{\mathbb{R}X\oplus\mathbb{R}Y}$ and $\lambda_{(\mathbb{R}X\oplus\mathbb{R}Y)^\perp}$ for $W = X + iY \in C(\mathcal{K})$.

Since the action of $\Gamma(L)$ on isometries $V(\mathbb{R}) \to \mathbb{R}^{2,n}$ and elements in the cone $C(\mathcal{K})$ is clearly compatible, the theta function $\Theta_L(\tau, W; p)$ is still invariant under the action of $\Gamma(L)$ in the variable W. Further, as p is homogeneous of degree m we have

$$\Theta_L(\tau, \mu W; p) = \mu^m \Theta_L(\tau, W; p)$$

for $\mu \in \mathbb{C}^\times$. Therefore, by Lemma 4.1.3 the induced theta function

$$\Theta_L(\tau, Z; p) = \sum_{\beta \in L'/L} \sum_{\lambda \in L+\beta} p((\lambda, X_L), (\lambda, Y_L)) e(\tau q(\lambda_Z) + \overline{\tau} q(\lambda_{Z^\perp})) \mathfrak{e}_\beta$$

for $\tau \in \mathbb{H}$ and $Z \in \mathbb{H}_n$ with $Z_L = X_L + iY_L$ is modular of weight $-m$ with respect to $\Gamma(L)$, i.e., we have

$$\Theta_L(\tau, \gamma.Z; p) = j(\gamma, Z)^{-m} \Theta_L(\tau, Z; p)$$

for $\tau \in \mathbb{H}$, $\gamma \in \Gamma(L)$ and $Z \in \mathbb{H}_n$. We summarise our considerations in the following proposition which is due to Borcherds:

Proposition 4.2.1. *Let $m \in \mathbb{Z}$ with $m \geq 0$ and let $p \in \mathbb{R}[x_1, x_2]$ be harmonic and homogeneous of degree m. Then the modified theta function*

$$\tilde{\Theta}_L(\tau, Z; p) = \mathrm{Im}(\tau)^{n/2} \sum_{\beta \in L'/L} \sum_{\lambda \in L+\beta} p((\lambda, X_L), (\lambda, Y_L)) e(\tau q(\lambda_Z) + \overline{\tau} q(\lambda_{Z^\perp})) \mathfrak{e}_\beta$$

defined for $\tau \in \mathbb{H}$ and $Z \in \mathbb{H}_n$ is modular of weight $1 + m - n/2$ with respect to the Weil representation ρ_L in the variable τ, and modular of weight $-m$ with respect to $\Gamma(L)$ in the variable Z.

In the following we work with a specific choice of polynomial p which yields a vector valued, non-holomorphic version of Shintani's theta function. Let k be a non-negative integer. We choose

$$p(x, y) := p_k(x, y) := (x + iy)^k$$

for $x, y \in \mathbb{R}$. This polynomial is clearly homogeneous of degree k and an easy computation shows that p_k is also harmonic. Moreover, we note that $p_k((\lambda, X_L), (\lambda, Y_L)) = (\lambda, Z_L)^k$ for $\lambda \in V(\mathbb{R})$ and $Z \in \mathbb{H}_n$ with $Z_L = X_L + iY_L$.

Definition 4.2.2. Let $k \in \mathbb{Z}$ with $k \geq 0$. We define *Shintani's theta function* as

$$\Theta_{L,k}(\tau, Z) = \mathrm{Im}(\tau)^{n/2} q(\mathrm{Im}(Z))^{-k} \sum_{\beta \in L'/L} \sum_{\lambda \in L+\beta} (\lambda, Z_L)^k e(\tau q(\lambda_Z) + \overline{\tau} q(\lambda_{Z^\perp})) \mathfrak{e}_\beta$$

for $\tau \in \mathbb{H}$ and $Z \in \mathbb{H}_n$.

By Proposition 4.2.1 and Lemma 4.1.1 our vector valued, non-holomorphic version of Shintani's theta function is modular in the following sense:

Corollary 4.2.3. *Let $k \in \mathbb{Z}$ with $k \geq 0$. Shintani's theta function $\Theta_{L,k}(\tau, Z)$ is modular of weight $1 + k - n/2$ with respect to ρ_L in the variable τ, and in the variable Z it transforms as*

$$\Theta_{L,k}(\tau, \gamma.Z) = \overline{j(\gamma, Z)}^k \Theta_{L,k}(\tau, Z)$$

for $\tau \in \mathbb{H}$, $\gamma \in \Gamma(L)$ and $Z \in \mathbb{H}_n$, i.e., the conjugated function $\overline{\Theta_{L,k}(\tau, Z)}$ is modular of weight k with respect to $\Gamma(L)$ in the variable Z.

We quickly study the asymptotic behaviour of Shintani's theta function $\Theta_{L,k}(\tau, Z)$ in the variable τ. Note that

$$e(\tau q(\lambda_Z) + \overline{\tau} q(\lambda_{Z^\perp})) = e(uq(\lambda))e^{-2\pi v q_Z(\lambda)}$$

for $\tau = u + iv \in \mathbb{H}$, $Z \in \mathbb{H}_n$ and $\lambda \in L'$, where q_Z is the majorant of q associated to the element Z which was introduced in (4.1.3). In particular, $q_Z(\lambda) \geq 0$ with equality if and only if $\lambda = 0$. Therefore, the only term in the sum defining the theta function $\Theta_{L,k}(\tau, Z)$ that contributes to its asymptotic behaviour in τ for $\mathrm{Im}(\tau) \to \infty$ is the one corresponding to $\beta = \lambda = 0$. So we obtain the following lemma:

Lemma 4.2.4. *Let $k \in \mathbb{Z}$ with $k \geq 0$. For $\beta \in L'/L$ and $Z \in \mathbb{H}_n$ Shintani's theta function $\Theta_{L,k}(\tau, Z)$ satisfies*

$$\langle \Theta_{L,k}(u + iv, Z), \mathfrak{e}_\beta \rangle = O(v^{n/2})$$

as $v \to \infty$, uniformly in u.

Using standard arguments we conclude:

Corollary 4.2.5. *Let $k \in \mathbb{Z}$ with $k \geq 0$. For $\beta \in L'/L$ and $Z \in \mathbb{H}_n$ Shintani's theta function $\Theta_{L,k}(\tau, Z)$ satisfies*

$$\langle \Theta_{L,k}(u + iv, Z), \mathfrak{e}_\beta \rangle = O(v^{-k-1})$$

as $v \to 0$, uniformly in u.

Proof. Set $\kappa = 1 + k - n/2$ and $f(\tau) = \mathrm{Im}(\tau)^{\kappa/2} |\Theta_{L,k}(\tau, Z)|$. Then f is by construction invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$, and by Lemma 4.2.4 there is a constant $C > 0$ such that $f(\tau) \leq C v^{\kappa/2 + n/2}$ for all $\tau \in \mathcal{F}$. Here $\mathcal{F} = \{\tau' \in \mathbb{H} \colon |\tau'| \geq 1, |\mathrm{Re}(\tau')| \leq 1/2\}$ is the usual fundamental domain of the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$.

Next we fix $\tau = u + iv \in \mathbb{H}$ with $v < 1/2$. Then we find $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ such that $M\tau \in \mathcal{F}$. Since $v < 1/2$ we need to have $c \neq 0$ and thus $|c\tau + d| \geq v$. Therefore we obtain

$$|\Theta_{L,k}(\tau, Z)| = v^{-\kappa/2} f(\tau) = v^{-\kappa/2} f(M\tau) \leq \frac{C v^{n/2}}{|c\tau + d|^{\kappa + n}} \leq C v^{-\kappa - n/2}$$

as $\kappa + n = 1 + k + n/2 \geq 0$. This proves the claimed statement. □

Finally, we define the regularized theta lift corresponding to Shintani's theta function $\Theta_{L,k}(\tau, Z)$ as in Section 6 of [Bor98]. Recall that for $F, G \colon \mathbb{H} \to \mathbb{C}[L'/L]$ real analytic and modular of weight k with respect to ρ_L we defined in (3.4.4) the regularized inner product of F and G as in [Bor98] by

$$(F, G)^{\mathrm{reg}} = \mathrm{CT}_{t=0} \left[\lim_{T \to \infty} \int_{\mathcal{F}_T} \langle F(\tau), G(\tau) \rangle \, \mathrm{Im}(\tau)^{k-t} d\mu(\tau) \right].$$

Here $\mathrm{CT}_{t=0}\, h(t)$ denotes the constant term of the Laurent expansion of the analytic continuation of $h(t)$ at $t = 0$.

Definition 4.2.6. Let $k \in \mathbb{Z}$ with $k \geq 0$. Given a real analytic function $F\colon \mathbb{H} \to \mathbb{C}[L'/L]$ modular of weight $1+k-n/2$ with respect to ρ_L we define the *regularized theta lift* of F as

$$\Phi_k^L(Z;F) = (F, \Theta_{L,k}(\cdot, Z))^{\mathrm{reg}}$$

for $Z \in \mathbb{H}_n$, whenever the regularized inner product exists.

Assuming that the theta lift $\Phi_k^L(Z;F)$ exists up to some singularities on $\mathbb{H}_n$ for a given function F modular of weight $1+k-n/2$ with respect to ρ_L, Corollary 4.2.3 implies that $\Phi_k^L(Z;F)$ is modular of weight k with respect to $\Gamma(L)$. We further note that Φ_k^L is essentially the theta lift considered in Section 14 of [Bor98]. It is sometimes called *Borcherds' generalized Shimura lift*. However, for F being a holomorphic cusp form the lift $\Phi_k^L(Z;F)$ has already been studied in [Oda77]. Further, for $k=0$ the corresponding theta lift $\Phi_0^L(Z;F)$ is usually simply called *Borcherds regularized theta lift* of F. It is invariant under the action of $\Gamma(L)$ in the variable Z if F is modular of weight $1-n/2$ with respect to ρ_L.

4.3 The symmetric space of signature $(2,1)$

In the following we introduce a particularly interesting quadratic space (V,q) of signature $(2,1)$. Working out the theory developed in the previous two sections we will see that the corresponding generalized upper half-plane $\mathbb{H}_1$ can be identified with the usual upper half-plane $\mathbb{H}$, and that the corresponding automorphy factors essentially agree. Moreover, the induced theta lift is essentially the classical Shimura lift.

Let V be the vector space of 2×2-matrices with rational entries and trace 0, i.e.,

$$V := \{X \in \mathbb{Q}^{2\times 2}\colon\ \mathrm{tr}(X) = 0\}.$$

Further, let N be a positive integer. We equip V with the quadratic form

$$q(X) := -N\det(X)$$

for $X \in V$. Then (V,q) is a quadratic space of signature $(2,1)$ with associated bilinear form given by $(X,Y) = N\,\mathrm{tr}(XY)$ for $X, Y \in V$. In V we consider the lattice L given by

$$L := \left\{ \begin{pmatrix} b & c/N \\ -a & -b \end{pmatrix} : a,b,c \in \mathbb{Z} \right\}.$$

Then L is an even lattice of level $4N$ with dual lattice given by

$$L' = \left\{ \begin{pmatrix} b/2N & c/N \\ -a & -b/2N \end{pmatrix} : a,b,c \in \mathbb{Z} \right\}.$$

The discriminant group L'/L is isomorphic to the quotient $\mathbb{Z}/2N\mathbb{Z}$ equipped with the quadratic form $x \mapsto x^2/4N \bmod \mathbb{Z}$. We will often identify the two groups L'/L and $\mathbb{Z}/2N\mathbb{Z}$, viewing elements of L'/L as elements of $\mathbb{Z}/2N\mathbb{Z}$ and vice versa.

It is well-known that the action of the identity component $\mathrm{SO}^+(V)$ of the orthogonal group of V can be realized by the action of the group $\mathrm{SL}_2(\mathbb{R})$ on $V(\mathbb{R})$ via

$$\sigma(X) := \sigma X \sigma^{-1}$$

for $\sigma \in \mathrm{SL}_2(\mathbb{R})$ and $X \in V(\mathbb{R})$. Here σ and $-\sigma$ define the same element in $\mathrm{SO}^+(V)$. Recall that $\mathrm{SO}^+(L) = \mathrm{SO}^+(V) \cap O(L)$, and that the group $\Gamma(L)$ is the subgroup of $\mathrm{SO}^+(L)$ of elements acting trivially on the discriminant group L'/L. One can show that under the identification of $\mathrm{SO}^+(V)$ with $\mathrm{SL}_2(\mathbb{R})/\{\pm 1\}$ we have

$$\Gamma(L) \simeq \Gamma_0(N)/\{\pm 1\} \qquad \text{and} \qquad \mathrm{SO}^+(L) \simeq \Gamma_0^*(N)/\{\pm 1\}, \tag{4.3.1}$$

where $\Gamma_0^*(N)$ is the so-called Fricke group of level N, i.e., the extension of $\Gamma_0(N)$ by all Atkin-Lehner involutions given by

$$\Gamma_0^*(N) := \left\{ \frac{1}{\sqrt{D}} \begin{pmatrix} aD & b \\ cN & dD \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})\colon a,b,c,d \in \mathbb{Z} \text{ and } D||N \right\}. \tag{4.3.2}$$

Here the notation $D||N$ means that D is a (positive) divisor of N such that D and N/D are coprime. For the identifications in (4.3.1) we refer to [BO10], Proposition 2.2.

We now choose a basis (e_1, e_2, e_3) for the lattice L, namely

$$e_1 := \begin{pmatrix} 0 & 1/N \\ 0 & 0 \end{pmatrix}, \qquad e_2 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad e_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then e_1 and e_2 span a hyperbolic plane, i.e., we have $q(e_1) = q(e_2) = 0$ and $(e_1, e_2) = 1$, and this plane is orthogonal to the positive line spanned by e_3, i.e., $(e_1, e_3) = (e_2, e_3) = 0$. Moreover, e_1 is primitive and $q(e_3) = N$. As in Section 4.1 we set $K = L \cap e_1^\perp \cap e_2^\perp$ to be the sublattice of L induced by e_1 and e_2, $U = K \otimes \mathbb{Q}$ to be the corresponding quadratic space of signature $(1,0)$ and $\mathcal{H} = \{X + iY \in U(\mathbb{C})\colon q(Y) > 0\}$ to be the induced complex plane. Then $K = \mathbb{Z}e_3$, $U = \mathbb{Q}e_3$ and thus

$$\mathcal{H} = \{ze_3\colon z \in \mathbb{C} \text{ with } \mathrm{Im}(z) \neq 0\}.$$

Therefore we can choose $\mathbb{H}_1$ to be the component $\mathbb{H}e_3$ of $\mathcal{H}$, i.e., we can naturally identify the generalized upper half-plane $\mathbb{H}_1$ with our usual upper half-plane $\mathbb{H}$. Let $\mathcal{K}$ be the complex manifold of elements $[W] \in \mathbb{P}(V(\mathbb{C}))$ with $(W, W) = 0$ and $(W, \overline{W}) > 0$. Then $\mathcal{K}$ has two connected components $\mathcal{K}^\pm$ and we choose $\mathcal{K}^+$ to be the image of $\mathbb{H}_1$ in $\mathcal{K}$ under the biholomorphic map $Z \mapsto [Z_L]$ where

$$Z_L = Z - (q(Z) + q(e_2))e_1 + e_2 = \begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix}$$

for $Z = ze_3 \in \mathbb{H}_1$. Here Z_L is an element in the cone $C(\mathcal{K}^+)$ of elements $W \in V(\mathbb{C})$ with $W \neq 0$ and $[W] \in \mathcal{K}^+$. Given $z = x + iy \in \mathbb{H}$ we set

$$X_L(z) := \begin{pmatrix} x & y^2 - x^2 \\ 1 & -x \end{pmatrix}, \quad Y_L(z) := \begin{pmatrix} y & -2xy \\ 0 & -y \end{pmatrix} \quad \text{and} \quad X(z) := \begin{pmatrix} x & -x^2 - y^2 \\ 1 & -x \end{pmatrix}.$$

Then $X_L(z) = \mathrm{Re}(Z_L)$ and $Y_L(z) = \mathrm{Im}(Z_L)$ for $Z = ze_3 \in \mathbb{H}_1$, and $(X_L(z), Y_L(z), X(z))$ is an orthogonal basis of the space $V(\mathbb{R})$ with

$$q(X_L(z)) = q(Y_L(z)) = -q(X(z)) = N\,\mathrm{Im}(z)^2.$$

Therefore, the identification of the upper half-plane $\mathbb{H}$ with the complex manifold $\mathcal{K}^+$ and with the Grassmannian $\mathrm{Gr}(V)$ is given by the maps

$$z \mapsto [X_L(z) + iY_L(z)] \qquad \text{and} \qquad z \mapsto \mathbb{R}X_L(z) \oplus \mathbb{R}Y_L(z),$$

respectively. We also note that $\mathbb{R}X_L(z) \oplus \mathbb{R}Y_L(z) = X(z)^\perp$.

Next we want to understand the action of the orthogonal group $\mathrm{SO}^+(V)$ on $\mathbb{H}_1$, where we identify $\mathrm{SO}^+(V)$ with the group $\mathrm{SL}_2(\mathbb{R})$ as before. Given $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and $Z = ze_3 \in \mathbb{H}_1$ the automorphy factor $j(\sigma, Z)$ can be computed as

$$j(\sigma, Z) = (\sigma(Z_L), e_1) = N \operatorname{tr}\left(\sigma \begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix} \sigma^{-1} \begin{pmatrix} 0 & 1/N \\ 0 & 0 \end{pmatrix}\right) = (cz+d)^2. \tag{4.3.3}$$

Thus we obtain

$$(\sigma.Z)_L = j(\sigma, Z)^{-1}\sigma(Z_L) = \begin{pmatrix} \sigma z & -(\sigma z)^2 \\ 1 & -\sigma z \end{pmatrix} = ((\sigma z)e_3)_L$$

for $\sigma \in \mathrm{SL}_2(\mathbb{R})$ and $Z = ze_3 \in \mathbb{H}_1$, i.e., $\sigma.Z = (\sigma z)e_3$. Therefore, the identification of $\mathbb{H}_1$ with the upper half-plane $\mathbb{H}$ is $\mathrm{SL}_2(\mathbb{R})$-equivariant.

However, we remark that for $k \in \mathbb{Z}$ a function $F\colon \mathbb{H}_1 \to \mathbb{C}$ is modular of weight k with respect to $\Gamma_0(N)$ in the sense of Definition 4.1.2 if and only if the function $f(z) = F(ze_3)$ for $z \in \mathbb{H}$ is modular of weight $2k$ and level N in the classical sense. Here the factor 2 comes in because of the square in equation (4.3.3).

For $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ we let

$$L_{\beta,m} := \{\lambda \in L + \beta\colon q(\lambda) = m\}.$$

According to the identification in (4.3.1) the group $\Gamma_0(N)$ acts on $L_{\beta,m}$. We can identify lattice elements in $L_{\beta,m}$ with integral binary quadratic forms via the bijection

$$L_{\beta,m} \to \mathcal{Q}_{\beta,4Nm}, \; \lambda \mapsto Q_\lambda = \begin{pmatrix} 0 & N \\ -N & 0 \end{pmatrix} \lambda, \tag{4.3.4}$$

i.e., $\lambda = \left(\begin{smallmatrix} b/2N & c/N \\ -a & -b/2N \end{smallmatrix}\right)$ corresponds to the quadratic form $Q_\lambda(x,y) = aNx^2 + bxy + cy^2$. Here we understand $\beta \in L'/L$ as an element of $\mathbb{Z}/2N\mathbb{Z}$. Recall that $\Gamma_0(N)$ acts on $\mathcal{Q}_{\beta,4Nm}$ from the right via $Q.M = M^tQM$. It is easy to check that the above identification of $L_{\beta,m}$ and $\mathcal{Q}_{\beta,4Nm}$ is compatible with the corresponding actions of $\Gamma_0(N)$ in the sense that

$$Q_\lambda.M = Q_{M^{-1}(\lambda)}$$

for $\lambda \in L_{\beta,m}$ and $M \in \Gamma_0(N)$. In particular, the map $\lambda \mapsto Q_\lambda$ gives rise to a bijection between $\Gamma_0(N)\backslash L_{\beta,m}$ and $\mathcal{Q}_{\beta,4Nm}/\Gamma_0(N)$. As in Section 2.3 we now further associate to $\lambda \in L_{\beta,m}$ with $\lambda \neq 0$ a geodesic, a parabolic element or a point in $\mathbb{H}$ as follows:

- If $m > 0$ we let c_λ be the set of 2-dimensional positive definite subspaces of $V(\mathbb{R})$ containing λ, i.e.,
$$c_\lambda := \{v \in \mathrm{Gr}(V)\colon \lambda \in v\} \subseteq \mathrm{Gr}(V).$$
Under the identification $z \mapsto X(z)^\perp$ of the upper half-plane $\mathbb{H}$ with the Grassmannian $\mathrm{Gr}(V)$ the set c_λ becomes a geodesic in $\mathbb{H}$. More precisely, one can check that it is exactly the Heegner geodesic c_{Q_λ} associated to the quadratic form Q_λ.

- If $m = 0$ we let p_λ be the isotropic line $\mathbb{R}\lambda$. We can identify the set of rational isotropic lines $\mathrm{Iso}(V) = \{\mathbb{R}Y : Y \in V \setminus \{0\}, q(Y) = 0\}$ with the set of parabolic points for $\Gamma_0(N)$ via the map

$$P^1(\mathbb{Q}) \to \mathrm{Iso}(V),\ (\alpha : \beta) \mapsto \mathbb{R}\begin{pmatrix} \alpha\beta & \alpha^2 \\ -\beta^2 & -\alpha\beta \end{pmatrix},$$

and this identification respects the corresponding actions of $\Gamma_0(N)$. In particular, it induces a bijection between $\Gamma_0(N)\backslash \mathrm{Iso}(V)$ and the set of cusps of $\Gamma_0(N)$.

Using this identification, we can understand p_λ as the parabolic point (or cusp) corresponding to the line $\mathbb{R}\lambda$. It turns out that p_λ is exactly the parabolic element (or cusp) p_{Q_λ} associated to the quadratic form Q_λ.

- If $m < 0$ we let τ_λ be the orthogonal complement of λ in $V(\mathbb{R})$, i.e.,

$$\tau_\lambda := \lambda^\perp \in \mathrm{Gr}(V).$$

Identifying $\mathrm{Gr}(V)$ with $\mathbb{H}$ we can view τ_λ as a point in the upper half-plane, namely as the unique point $\tau_\lambda \in \mathbb{H}$ satisfying $\lambda^\perp = \mathbb{R}X_L(\tau_\lambda) \oplus \mathbb{R}Y_L(\tau_\lambda)$. Indeed, one can check that it is exactly the Heegner point τ_{Q_λ} associated to the quadratic form Q_λ.

In abuse of notation we treat c_λ, p_λ and τ_λ as a geodesic in $\mathbb{H}$, a parabolic point (or cusp) of $\Gamma_0(N)$ and a point in the upper half-plane, respectively. Furthermore, we note that the identities in (2.3.3) translate to

$$c_{M(\lambda)} = Mc_\lambda, \qquad p_{M(\lambda)} = Mp_\lambda \qquad \text{and} \qquad \tau_{M(\lambda)} = M\tau_\lambda$$

for $M \in \Gamma_0(N)$ and $\lambda \in L_{\beta,m} \setminus \{0\}$ if $m > 0$, $m = 0$ or $m < 0$, respectively, and the corresponding stabilizers also agree.

Eventually, we remark that given $\lambda \in L'$ with $\lambda \neq 0$ an easy computation shows that

$$(\lambda, Z_L) = Q_\lambda(z, 1) \tag{4.3.5}$$

for $Z = ze_3 \in \mathbb{H}_1$. We use this to prove the following lemma:

Lemma 4.3.1. *Let $\lambda \in L'$ with $\lambda = \left(\begin{smallmatrix} b/2N & c/N \\ -a & -b/2N \end{smallmatrix}\right)$ and $\lambda \neq 0$. Then*

$$q(\lambda_z) = \frac{|Q_\lambda(z,1)|^2}{4Ny^2} \qquad \text{and} \qquad q(\lambda_{z^\perp}) = -\frac{1}{4N}\, p_{Q_\lambda}(z)^2$$

for $z \in \mathbb{H}$, $z = x + iy$. Here λ_z and $\lambda_{z^\perp}$ denote the orthogonal projections of λ onto the subspaces $\mathbb{R}X_L(z) \oplus \mathbb{R}Y_L(z)$ and $\mathbb{R}X(z)$ of $V(\mathbb{R})$, and $p_{Q_\lambda}(z)$ denotes the quantity defined in (2.3.8).

Proof. By equation (4.1.2) and the above identity (4.3.5) we have

$$q(\lambda_z) = \frac{|(\lambda, Z_L)|^2}{4q(\mathrm{Im}(Z))} = \frac{|Q_\lambda(z,1)|^2}{4q(\mathrm{Im}(Z))}$$

for $Z = ze_3 \in \mathbb{H}_1$. This proves the first equality as $q(\mathrm{Im}(Z)) = Ny^2$. Further, we thus find

$$q(\lambda_{z^\perp}) = q(\lambda) - q(\lambda_z) = -\frac{1}{4N}\left(\frac{|Q_\lambda(z,1)|^2}{y^2} - \Delta(Q_\lambda)\right).$$

Now the second equality follows from (2.3.9). $\square$

Finally, we state the theta lift from Definition 4.2.6 in the situation of the current lattice L. Let k be a non-negative integer. Identifying $\mathbb{H}$ with $\mathbb{H}_1$ via the map $z \mapsto ze_3$, and using the identity from (4.3.5) and the fact that $q(\operatorname{Im}(ze_3)) = N \operatorname{Im}(z)^2$ for $z \in \mathbb{H}$ we can write Shintani's theta function for the lattice L as

$$\Theta_{L,k}(\tau, z) = N^{-k} \operatorname{Im}(\tau)^{1/2} \operatorname{Im}(z)^{-2k} \sum_{\beta \in L'/L} \sum_{\lambda \in L+\beta} Q_\lambda(z, 1)^k e(\tau q(\lambda_z) + \overline{\tau} q(\lambda_{z^\perp}))\mathfrak{e}_\beta \tag{4.3.6}$$

for $\tau, z \in \mathbb{H}$. Here the quantities $q(\lambda_z)$ and $q(\lambda_{z^\perp})$ are given as in Lemma 4.3.1. By Corollary 4.2.3 the theta function $\Theta_{L,k}(\tau, z)$ is modular of weight $k + 1/2$ with respect to ρ_L in τ, and in the variable z it transforms as

$$\Theta_{L,k}(\tau, \gamma z) = \overline{j(\gamma, z)}^{2k} \Theta_{L,k}(\tau, z)$$

for $\tau, z \in \mathbb{H}$ and $\gamma \in \Gamma_0(N)$. Accordingly, the (regularized) theta lift of a real analytic function $F \colon \mathbb{H} \to \mathbb{C}[L'/L]$ modular of weight $k + 1/2$ with respect to ρ_L is given by

$$\Phi_k^L(z; F) = (F, \Theta_{L,k}(\cdot, z))^{\mathrm{reg}}$$

for $z \in \mathbb{H}$, and assuming that the (regularized) inner product exists the lift $\Phi_k^L(z; F)$ of F is modular of weight $2k$ and level N.

For an integer $k \geq 2$ the theta lift $\Phi_k^L(z; \cdot)$ is essentially the classical Shimura theta lift, mapping (vector valued) holomorphic cusp forms of weight $k + 1/2$ to holomorphic cusp forms of weight $2k$. It was introduced by Shimura in [Shi73] and later expressed as a theta lift by Niwa in [Niw75]. In particular, the Shimura lift of a holomorphic Poincaré series is up to some constant the corresponding Zagier cusp form introduced in Section 2.4.2 (see for example [Oda77] or [KZ81]). More precisely, let $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m > 0$. Then an easy unfolding argument shows that

$$\Phi_k^L(z; P_{k+1/2,\beta,m}) = \frac{2\,(k-1)!}{\pi^k} f_{k,\beta,4Nm}(z) \tag{4.3.7}$$

for $k \in \mathbb{Z}$ with $k \geq 2$ and $z \in \mathbb{H}$. Here $f_{k,\beta,4Nm}(z)$ is the meromorphic modular form introduced in Definition 2.4.6. Note that we can ignore the regularization of the inner product defining the theta lift in this case since $P_{k+1/2,\beta,m}(\tau)$ is a cusp form for $m > 0$.

4.4 The orthogonal space of signature $(2, 2)$

Finally, we present a quadratic space of signature $(2, 2)$ whose associated generalized upper half-plane is simply given as a product of two normal upper half-planes. In this situation Borcherds' generalized Shimura lift yields functions which are modular of parallel weight in two complex variables.

Let $V = \mathbb{Q}^{2\times 2}$ be the vector space of 2×2-matrices with rational entries equipped with the quadratic form

$$q(X) := -\det(X)$$

for $X \in V$. Then (V, q) is a quadratic space of signature $(2,2)$ with associated bilinear form given by $(X, Y) = -\operatorname{tr}(XY^{\#})$ for $X, Y \in V$ where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{\#} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. Given a positive integer N we consider the lattice

$$L := \left\{ \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$$

in V. Then L is an even lattice of level N with dual lattice given by

$$L' = \left\{ \begin{pmatrix} a & b/N \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\},$$

and the discriminant group L'/L is isomorphic to the quotient $(\mathbb{Z}/N\mathbb{Z})^2$. It is well-known that the action of the identity component $\mathrm{SO}^+(V)$ of the orthogonal group of V can be realized by the action of the group $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ on $V(\mathbb{R})$ via

$$(\sigma, \sigma').X := \sigma X \sigma'^{-1}$$

for $\sigma, \sigma' \in \mathrm{SL}_2(\mathbb{R})$ and $X \in V(\mathbb{R})$. Here the four elements $(\pm\sigma, \pm\sigma')$ define the same element in $\mathrm{SO}^+(V)$. Further, one easily checks that the discrete subgroup $\Gamma_0(N) \times \Gamma_0(N)$ of $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ acts on L', and that its subgroup $\Gamma(N) \times \Gamma(N)$ fixes the classes of the discriminant group L'/L, i.e., we have

$$\Gamma(N) \times \Gamma(N) \subseteq \Gamma(L) \qquad \text{and} \qquad \Gamma_0(N) \times \Gamma_0(N) \subseteq \mathrm{SO}^+(L), \tag{4.4.1}$$

under the above identification of $\mathrm{SO}^+(V)$ with the product $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$. Here $\Gamma(N)$ is the usual *principal congruence subgroup* given by

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})\colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N \right\},$$

which clearly is a subgroup of $\Gamma_0(N)$.

We now choose vectors

$$e_1 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad e_2 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad e_3 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad e_4 := \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $e_1 \in L$ is primitive and isotropic and $e_2 \in L'$ is isotropic with $(e_1, e_2) = 1$. Moreover, the hyperbolic plane spanned by e_1 and e_2 is orthogonal to the plane spanned by the isotropic vectors e_3 and e_4 where $(e_3, e_4) = 1$. We define the sublattice $K = L \cap e_1^{\perp} \cap e_2^{\perp}$, the corresponding quadratic space $U = K \otimes \mathbb{Q}$ of signature $(1,1)$ and the induced complex plane $\mathcal{H} = \{X + iY \in U(\mathbb{C})\colon q(Y) > 0\}$. Then $K = \{\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)\colon a, d \in \mathbb{Z}\}$, $U = \{\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)\colon a, d \in \mathbb{Q}\}$ and thus

$$\mathcal{H} = \left\{ \begin{pmatrix} z & 0 \\ 0 & z' \end{pmatrix} : z, z' \in \mathbb{C} \text{ with } \operatorname{Im}(z)\operatorname{Im}(z') < 0 \right\}.$$

Hence by choosing $\mathbb{H}_2$ as the connected component $\mathbb{H}e_3 \oplus \mathbb{H}e_4$ of $\mathcal{H}$ we can identify the generalized upper half-plane $\mathbb{H}_2$ with the product $\mathbb{H} \times \mathbb{H}$. Let $\mathcal{K}$ be the complex manifold defined in (4.1.1) associated to the space $V(\mathbb{C})$. We choose $\mathcal{K}^+$ to be the image of $\mathbb{H}_2$ in $\mathcal{K}$ under the map $Z \mapsto [Z_L]$ where

$$Z_L = Z - (q(Z) + q(e_2))e_1 + e_2 = \begin{pmatrix} z & -zz' \\ 1 & -z' \end{pmatrix}$$

for $Z = ze_3 + z'e_4 \in \mathbb{H}_2$. Conversly, given $z, z' \in \mathbb{H}$ with $z = x + iy$, $z' = x' + iy'$ we set

$$X_L(z,z') := \begin{pmatrix} x & yy' - xx' \\ 1 & -x' \end{pmatrix} \qquad \text{and} \qquad Y_L(z,z') := \begin{pmatrix} y & -xy' - x'y \\ 0 & -y' \end{pmatrix}$$

such that $X_L(z,z') = \mathrm{Re}(Z_L)$, $Y_L(z,z') = \mathrm{Im}(Z_L)$ for $Z = ze_3+z'e_4$. Then $q(X_L(z,z')) = q(Y_L(z,z')) = yy'$, and the identification of $\mathbb{H}_2$ with the Grassmannian $\mathrm{Gr}(V)$ is given by the map $ze_3 + z'e_4 \mapsto \mathbb{R}X_L(z,z') \oplus \mathbb{R}Y_L(z,z')$.

In order to understand the action of the group $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}_2$ we compute

$$j((\sigma,\sigma'),Z) = ((\sigma,\sigma').Z_L, e_1) = j(\sigma,z)j(\sigma',z') \tag{4.4.2}$$

for $\sigma, \sigma' \in \mathrm{SL}_2(\mathbb{R})$ and $Z = ze_3 + z'e_4 \in \mathbb{H}_2$. Here the automorphy factors on the right are the classical ones defined at the beginning of Section 2.4. Hence we find

$$((\sigma,\sigma').Z)_L = j((\sigma,\sigma'),Z)^{-1}(\sigma,\sigma').Z_L = \begin{pmatrix} \sigma z & -\sigma z \cdot \sigma' z' \\ 1 & -\sigma' z' \end{pmatrix} = (\sigma z\, e_3 + \sigma' z'\, e_4)_L \tag{4.4.3}$$

for $\sigma, \sigma' \in \mathrm{SL}_2(\mathbb{R})$ and $Z = ze_3 + z'e_4 \in \mathbb{H}_2$, i.e., $(\sigma,\sigma').Z = \sigma z\, e_3 + \sigma' z'\, e_4$. Therefore, the identification of $\mathbb{H}_2$ with the product $\mathbb{H} \times \mathbb{H}$ is $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$-equivariant where the latter group naturally acts on $\mathbb{H} \times \mathbb{H}$ via $(\sigma,\sigma').(z,z') = (\sigma z, \sigma' z')$.

In particular, for $k \in \mathbb{Z}$ a function $F\colon \mathbb{H}_2 \to \mathbb{C}$ is modular of weight k with respect to the group $\Gamma(N) \times \Gamma(N)$ in the sense of Definition 4.1.2 if and only if the function $f(z,z') = F(ze_3 + z'e_4)$ for $z, z' \in \mathbb{H}$ is modular of weight k for the group $\Gamma(N)$ in the classical sense in both variables, i.e., if

$$f(\gamma z, \gamma' z') = j(\gamma,z)^k j(\gamma',z')^k f(z,z')$$

for all $\gamma, \gamma' \in \Gamma(N)$ and $z, z' \in \mathbb{H}$.

For $\lambda \in V$ and $z, z' \in \mathbb{H}$ we denote the orthogonal projection of λ onto the positive or negative definite subspace $\mathbb{R}X_L(z,z') \oplus \mathbb{R}Y_L(z,z')$ or $X_L(z,z')^\perp \cap Y_L(z,z')^\perp$ in $V(\mathbb{R})$ by $\lambda_{z,z'}$ or $\lambda_{(z,z')^\perp}$, respectively. As $q(\mathrm{Im}(Z)) = \mathrm{Im}(z)\,\mathrm{Im}(z')$ we can restate equation (4.1.2) in the present setting as

$$q(\lambda_{z,z'}) = \frac{|(\lambda, Z_L)|^2}{4\,\mathrm{Im}(z)\,\mathrm{Im}(z')} \tag{4.4.4}$$

for $\lambda \in V$ and $z, z' \in \mathbb{H}$ with $Z = ze_3 + z'e_4$. Further, given $z, z' \in \mathbb{H}$ we denote the majorant of q associated to the pair (z,z') as defined in (4.1.3) by

$$q_{z,z'}(\lambda) = q(\lambda_{z,z'}) - q(\lambda_{(z,z')^\perp}),$$

and using (4.4.4) we thus find

$$q_{z,z'}(\lambda) = \frac{|(\lambda, Z_L)|^2}{2\,\mathrm{Im}(z)\,\mathrm{Im}(z')} - q(\lambda) \tag{4.4.5}$$

for $\lambda \in V$ and $z, z' \in \mathbb{H}$ with $Z = ze_3 + z'e_4$.

Eventually, we present the theta lift $\Phi_k^L(Z;F)$ from Section 4.2 in the case of L being our current lattice of signature $(2,2)$. Recall that $q(\mathrm{Im}(Z)) = \mathrm{Im}(z)\,\mathrm{Im}(z')$ for $Z = ze_3 + z'e_4$ with $z, z' \in \mathbb{H}$. Let now k be a non-negative integer. Identifying $\mathbb{H}_2$ with the product $\mathbb{H} \times \mathbb{H}$ we can write Shintani's theta function for the lattice L as

$$\Theta_{L,k}(\tau, z, z') = v\ \mathrm{Im}(z)^{-k}\,\mathrm{Im}(z')^{-k} \sum_{\beta \in L'/L} \sum_{\lambda \in L+\beta} (\lambda, Z_L)^k e\Big(uq(\lambda) + ivq_{z,z'}(\lambda)\Big)\mathfrak{e}_\beta \tag{4.4.6}$$

for $\tau, z, z' \in \mathbb{H}$ with $\tau = u + iv$. The theta function $\Theta_{L,k}(\tau, z)$ is modular of weight k with respect to ρ_L in τ, and in the variables z and z' it transforms as

$$\Theta_{L,k}(\tau, \gamma z, \gamma' z') = \overline{j(\gamma, z)}^k\,\overline{j(\gamma', z')}^k\,\Theta_{L,k}(\tau, z, z')$$

for $\tau, z, z' \in \mathbb{H}$ and $\gamma, \gamma' \in \Gamma(N)$. Therefore, the (regularized) theta lift of a real analytic function $F\colon \mathbb{H} \to \mathbb{C}[L'/L]$ modular of weight k with respect to ρ_L is given by

$$\Phi_k^L(z, z'; F) = (F, \Theta_{L,k}(\cdot, z, z'))^{\mathrm{reg}}$$

for $z, z' \in \mathbb{H}$, and the lift $\Phi_k^L(z, z'; F)$ of F is modular of weight k for the group $\Gamma(N)$ in both variables z and z' whenever the (regularized) inner product exists. In this setting, the lift $\Phi_k^L(z, z'; F)$ is sometimes called the *Doi-Naganuma lift* of F, which classically is a lift from scalar valued modular forms to Hilbert modular forms (see [DN67, DN69]).

5 Realizing non-holomorphic Eisenstein series as theta lifts

In this chapter we realize the non-holomorphic Eisenstein series introduced in Section 2.6 as regularized theta lifts in two different ways. On the one hand we obtain averaged versions of these Eisenstein series as the theta lift of Selberg's Poincaré series of the first kind using the lattice of signature $(2,1)$ from Section 4.3. Remarkably, we indeed obtain averaged versions of hyperbolic, parabolic an elliptic Eisenstein series as the theta lift of a single type of Poincaré series.

On the other hand we realize the hyperbolic kernel function defined in Section 2.6.4 as the theta lift of a more elementary Poincaré series introduced in Section 3.6, using the lattice of signature $(2,2)$ from Section 4.4. We can then use this representation of the hyperbolic kernel function to obtain individual hyperbolic, parabolic and elliptic Eisenstein series as variations of this theta lift.

5.1 Regularized theta lifts of non-holomorphic Poincaré series

Let (V,q) be a quadratic space of signature $(2,n)$ and let L be an even lattice in V. We further fix vectors $e_1 \in L$ and $e_2 \in L'$ such that e_1 is primitive with $q(e_1)=0$ and $(e_1,e_2)=1$ as in Section 4.1, and we let $\mathbb{H}_n$ be the induced generalized upper half-plane. For $\beta \in L'/L$ and $m \in \mathbb{Z}+q(\beta)$ we set

$$L_{\beta,m} := \{\lambda \in L+\beta \colon q(\lambda)=m\},$$

and we define the subset $H^L_{\beta,m}$ of $\mathbb{H}_n$ by

$$H^L_{\beta,m} := \bigcup_{\substack{\lambda \in L_{\beta,m} \\ \lambda \neq 0}} \{Z \in \mathbb{H}_n \colon (\lambda, Z_L)=0\}. \tag{5.1.1}$$

Note that $H^L_{\beta,m}=\emptyset$ if $m \geq 0$ since $\lambda \in L_{\beta,m}$ can only be orthogonal to some $Z_L = X_L+iY_L$ with $Z \in \mathbb{H}_n$ if $\lambda = 0$ or $q(\lambda)<0$ as $\mathbb{R}X_L \oplus \mathbb{R}Y_L$ defines a 2-dimensional positive definite subspace of $V(\mathbb{R})$.

Let k be a non-negative integer and set $\kappa := 1+k-n/2$. Then κ satisfies the condition given in (3.4.1). In the present section we study the regularized theta lift of two of the non-holomorphic Poincaré series given in Definition 3.4.3, namely Selberg's Poincaré series of the first kind $U^L_{\kappa,\beta,m}(\tau,s)$ and the simpler, non-standard Poincaré series $Q^L_{\kappa,\beta,m}(\tau,s)$.

Definition 5.1.1. Let $k \in \mathbb{Z}$ with $k \geq 0$, $\kappa = 1+k-n/2$, $\beta \in L'/L$ and $m \in \mathbb{Z}+q(\beta)$.

(a) We define the regularized theta lift of Selberg's Poincaré series of the first kind $U^L_{\kappa,\beta,m}(\tau, s)$ by

$$\Phi^{\mathrm{Sel},L}_{k,\beta,m}(Z, s) = \Phi^L_k(Z; U^L_{\kappa,\beta,m}(\,\cdot\,, s))$$

for $Z \in \mathbb{H}_n \setminus H^L_{\beta,m}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1 + n/2$.

(b) We define the regularized theta lift of the Poincaré series $Q^L_{\kappa,\beta,m}(\tau, s)$ by

$$\Phi^{\mathrm{Q},L}_{k,\beta,m}(Z, s) = \Phi^L_k(Z; Q^L_{\kappa,\beta,m}(\,\cdot\,, s))$$

for $Z \in \mathbb{H}_n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1 + n/2$.

Here $\Phi^L_k(Z;\,\cdot\,)$ denotes the regularized theta lift from Definition 4.2.6. We will see in Theorem 5.1.3 that the above definition is indeed well-defined, i.e., that the corresponding (regularized) integrals exist in all the given cases. Thus, $\Phi^{\mathrm{Sel},L}_{k,\beta,m}(Z, s)$ and $\Phi^{\mathrm{Q},L}_{k,\beta,m}(Z, s)$ are modular of weight k in the variable Z with respect to the group $\Gamma(L)$ defined in (4.1.4).

Before we start studying the above lifts, we quickly recall some technicalities. For $Z \in \mathbb{H}_n$ the majorant of the present quadratic form q associated to Z is given by $q_Z(\lambda) = q(\lambda_Z) - q(\lambda_{Z^\perp})$ for $\lambda \in V$ (see equation (4.1.3)). Here λ_Z and $\lambda_{Z^\perp}$ denote the orthogonal projections of λ onto the positive or negative definite subspaces of V corresponding to Z and $Z^\perp$, respectively. The majorant q_Z defines a positive definite quadratic form on V, and thus the Epstein zeta function associated to q_Z and the lattice L' defined by

$$Z(s; q_Z, L') := \sum_{\lambda \in L' \setminus \{0\}} q_Z(\lambda)^{-s} \tag{5.1.2}$$

converges absolutely and locally uniformly for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1 + n/2$ (see for example [Eps03]).

Lemma 5.1.2. *Let $\beta \in L'/L$, $m \in \mathbb{Z} + q(\beta)$ and $Z \in \mathbb{H}_n \setminus H^L_{\beta,m}$. Then the sum*

$$\sum_{\lambda \in L_{\beta,m} \setminus \{0\}} q(\lambda_Z)^{-s}$$

converges absolutely and locally uniformly for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1 + n/2$.

Proof. Firstly, we note that for $\lambda \in L_{\beta,m}$ we have $q(\lambda_Z) = 0$ if and only if $(\lambda, Z_L) = 0$ by (4.1.2), which is the case if and only if $\lambda = 0$ or $Z \in H^L_{\beta,m}$. Thus the given sum is well-defined. Next, we can choose a finite subset $K \subset L'$ such that $q_Z(\lambda) \geq 2|m|$ for all $\lambda \in L' \setminus K$ since q_Z is positive definite. Hence we obtain

$$q(\lambda_Z) = \frac{q_Z(\lambda) + m}{2} = \frac{q_Z(\lambda)}{4} + \frac{q_Z(\lambda) + 2m}{4} \geq \frac{q_Z(\lambda)}{4}$$

for $\lambda \in L_{\beta,m} \setminus K$, giving

$$\sum_{\substack{\lambda \in L_{\beta,m} \\ \lambda \neq 0}} \left| q(\lambda_Z)^{-s} \right| \leq \sum_{\substack{\lambda \in L_{\beta,m} \cap K \\ \lambda \neq 0}} q(\lambda_Z)^{-\mathrm{Re}(s)} + 4^{\mathrm{Re}(s)} \sum_{\substack{\lambda \in L_{\beta,m} \setminus K \\ \lambda \neq 0}} q_Z(\lambda)^{-\mathrm{Re}(s)}.$$

Here the first sum is finite and the second sum is dominated by the Epstein zeta function associated to q_Z and L' given in (5.1.2), which converges absolutely and locally uniformly for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1 + n/2$. $\square$

We can now prove that the theta lifts given in Definition 5.1.1 are indeed well-defined. Moreover, if $k > 0$ we already obtain a meromorphic continuation of the two lifts to the half-plane defined by $\mathrm{Re}(s) > 1 + n/2 - k/2$

Theorem 5.1.3. *Let $k \in \mathbb{Z}$ with $k \geq 0$, $\kappa = 1 + k - n/2$, $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$.*

(a) The regularized theta lift of Selberg's Poincaré series $U^L_{\kappa,\beta,m}(\tau, s)$ defines a real analytic function in $Z \in \mathbb{H}_n \setminus H^L_{\beta,m}$ and a holomorphic function in s for $\mathrm{Re}(s) > 1 + n/2$, which is given by

$$\Phi^{\mathrm{Sel},L}_{k,\beta,m}(Z, s) = \frac{2\,\Gamma(s + k)}{4^s\,\pi^{s+k}} \sum_{\substack{\lambda \in L_{\beta,m} \\ \lambda \neq 0}} q(\lambda_Z)^{-s}(\lambda, Z_L)^{-k}.$$

For $k > 0$ and $Z \in \mathbb{H}_n \setminus H^L_{\beta,m}$ the sum on the right-hand side yields a holomorphic continuation of $\Phi^{\mathrm{Sel},L}_{k,\beta,m}(Z, s)$ in s to the half-plane $\mathrm{Re}(s) > 1 + n/2 - k/2$.

(b) The regularized theta lift of the Poincaré series $Q^L_{\kappa,\beta,m}(\tau, s)$ defines a real analytic function in $Z \in \mathbb{H}_n$ and a holomorphic function in s for $\mathrm{Re}(s) > 1 + n/2$, which is given by

$$\Phi^{\mathrm{Q},L}_{k,\beta,m}(Z, s) = \frac{2\,\Gamma(s + k)}{(2\pi)^{s+k}}\; q(\mathrm{Im}(Z))^{-k} \sum_{\substack{\lambda \in L_{\beta,m} \\ \lambda \neq 0}} q_Z(\lambda)^{-s-k}\, \overline{(\lambda, Z_L)}^k.$$

For $k > 0$ and $Z \in \mathbb{H}_n$ the sum on the right-hand side yields a holomorphic continuation of $\Phi^{\mathrm{Q},L}_{k,\beta,m}(Z, s)$ in s to the half-plane $\mathrm{Re}(s) > 1 + n/2 - k/2$.

Proof. Since (a) and (b) are poven analogously, we only give a detailed proof for part (a), and afterwards comment on the necessary adaptions of the given proof in the case of (b).

For (a) fix $Z \in \mathbb{H}_n \setminus H^L_{\beta,m}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1 + n/2$. Evaluating the inner product $\left\langle U^L_{\kappa,\beta,m}(\tau, s), \Theta_{L,k}(\tau, Z)\right\rangle$ and splitting the sum over matrices $M \in \langle T\rangle\backslash \mathrm{SL}_2(\mathbb{Z})$ coming from the Poincaré series into matrices M with lower left entry $c \neq 0$ and $c = 0$ we can write the theta lift $\Phi^{\mathrm{Sel},L}_{k,\beta,m}(Z, s)$ as

$$\begin{aligned} \text{(5.1.3)}\qquad \mathrm{CT}_{t=0}&\left[\lim_{T\to\infty} \int_{\mathcal{F}_T} \sum_{\substack{M \in \langle T\rangle\backslash \mathrm{SL}_2(\mathbb{Z}) \\ M \neq \pm 1}} \mathrm{Im}(M\tau)^{s+\kappa} e(mM\tau)\overline{\Theta_{L,k,\beta}(M\tau, Z)}v^{-t} d\mu(\tau)\right] \\ &+ 2\,\mathrm{CT}_{t=0}\left[\lim_{T\to\infty} \int_{\mathcal{F}_T} v^{s+\kappa-t} e(m\tau)\overline{\Theta_{L,k,\beta}(\tau, Z)} d\mu(\tau)\right]. \end{aligned}$$

Here we use the notation $\tau = u + iv$, $\kappa = 1 + k - n/2$ and $\Theta_{L,k,\beta}(\tau, Z) = \langle\Theta_{L,k}(\tau, Z), \mathfrak{e}_\beta\rangle$. Let $\mathcal{R} := \{\tau \in \mathbb{H} \colon |\,\mathrm{Re}(\tau)| \leq 1/2\}$. Then $\mathcal{R}$ is a rectangular fundamental domain for the action of $\langle T\rangle$ on $\mathbb{H}$, and by Corollary 4.2.5 we have

$$\text{(5.1.4)}\qquad \int_{\mathcal{R}\setminus\mathcal{F}} \left|v^{s+\kappa} e(m\tau)\overline{\Theta_{L,k,\beta}(\tau, Z)}\right| d\mu(\tau) \leq C \int_0^1 v^{\mathrm{Re}(s)-n/2-2} dv < \infty$$

for some constant $C > 0$ as $\mathrm{Re}(s) > 1 + n/2$. Hence we can plug in $t = 0$ in the first term in (5.1.3), take the limit $T \to \infty$, interchange summation and integration and apply the usual unfolding trick to find

$$\mathrm{CT}_{t=0}\left[\lim_{T\to\infty}\int_{\mathcal{F}_T}\sum_{\substack{M\in\langle T\rangle\backslash\mathrm{SL}_2(\mathbb{Z})\\ M\neq\pm1}}\mathrm{Im}(M\tau)^{s+\kappa}e(mM\tau)\overline{\Theta_{L,k,\beta}(M\tau,Z)}v^{-t}d\mu(\tau)\right]$$
$$= 2\int_{\mathcal{R}\backslash\mathcal{F}}v^{s+\kappa}e(m\tau)\overline{\Theta_{L,k,\beta}(\tau,Z)}d\mu(\tau).$$

In the second term in (5.1.3) we split the integral at the horizontal line $v = 1$, i.e.,

$$\mathrm{CT}_{t=0}\left[\lim_{T\to\infty}\int_{\mathcal{F}_T}v^{s+\kappa-t}e(m\tau)\overline{\Theta_{L,k,\beta}(\tau,Z)}d\mu(\tau)\right]$$
$$=\int_{\mathcal{F}_1}v^{s+\kappa}e(m\tau)\overline{\Theta_{L,k,\beta}(\tau,Z)}d\mu(\tau)+\mathrm{CT}_{t=0}\left[\lim_{T\to\infty}\int_{\mathcal{F}_T\backslash\mathcal{F}_1}v^{s+\kappa-t}e(m\tau)\overline{\Theta_{L,k,\beta}(\tau,Z)}d\mu(\tau)\right].$$

Here we can drop the regularization in the first term on the right-hand side as $\mathcal{F}_1$ is compact. Putting everything back together we obtain

$$\Phi^{\mathrm{Sel},L}_{k,\beta,m}(Z,s) = 2\int_0^1\int_{-1/2}^{1/2}v^{s+\kappa}e(m\tau)\overline{\Theta_{L,k,\beta}(\tau,Z)}\frac{dudv}{v^2}$$
$$+\,2\ \mathrm{CT}_{t=0}\left[\int_1^\infty\int_{-1/2}^{1/2}v^{s+\kappa-t}e(m\tau)\overline{\Theta_{L,k,\beta}(\tau,Z)}\frac{dudv}{v^2}\right]. \tag{5.1.5}$$

In both terms the integral with respect to u is simply given by

$$\int_{-1/2}^{1/2}e(mu)\overline{\Theta_{L,k,\beta}(\tau,Z)}du.$$

Plugging in the definition of the β-th component of the theta function $\Theta_{L,k}(\tau,Z)$ and using that

$$\sum_{\lambda\in L+\beta}\int_{-1/2}^{1/2}\left|e(mu)\overline{(\lambda,Z_L)}^k e(\tau q(\lambda_Z)+\overline{\tau}q(\lambda_{Z^\perp}))\right|du = \sum_{\lambda\in L+\beta}\left|(\lambda,Z_L)^k e^{-2\pi v q_Z(\lambda)}\right| < \infty$$

as the sum defining $\Theta_{L,k,\beta}(\tau,Z)$ is absolutely convergent, we obtain

$$\int_{-1/2}^{1/2}e(mu)\overline{\Theta_{L,k,\beta}(\tau,Z)}du = \frac{v^{n/2}}{q(\mathrm{Im}(Z))^k}\sum_{\lambda\in L+\beta}\overline{(\lambda,Z_L)}^k e^{-2\pi v q_Z(\lambda)}\int_{-1/2}^{1/2}e(u(m-q(\lambda))du.$$

Here $q_Z(\lambda) = q(\lambda_Z) - q(\lambda_{Z^\perp})$ as in (4.1.3), and the integral on the right-hand side is 1 if $q(\lambda) = m$ and vanishes otherwise. Therefore, we can write (5.1.5) as

$$\Phi^{\mathrm{Sel},L}_{k,\beta,m}(Z,s) = \frac{2}{q(\mathrm{Im}(Z))^k}\Bigg(\int_0^1 v^{s+k-1}\sum_{\lambda\in L_{\beta,m}}\overline{(\lambda,Z_L)}^k e^{-4\pi v q(\lambda_Z)}dv$$
$$+\,\mathrm{CT}_{t=0}\left[\int_1^\infty v^{s+k-t-1}\sum_{\lambda\in L_{\beta,m}}\overline{(\lambda,Z_L)}^k e^{-4\pi v q(\lambda_Z)}dv\right]\Bigg).$$

Here the contribution of the element $\lambda = 0$ appearing only if $m = \beta = 0$ is zero since

$$\int_0^1 v^{s+k-1} dv + \mathrm{CT}_{t=0} \int_1^\infty v^{s+k-t-1} dv = \frac{1}{s+k} - \mathrm{CT}_{t=0}\left(\frac{1}{s+k-t}\right) = 0.$$

Hence we can exclude the element $\lambda = 0$ in each of the two sums above. Moreover, we reunite the remaining terms, giving

$$\Phi^{\mathrm{Sel},L}_{k,\beta,m}(Z,s) = \frac{2}{q(\mathrm{Im}(Z))^k} \mathrm{CT}_{t=0}\left[\int_0^\infty v^{s+k-t-1} \sum_{\substack{\lambda \in L_{\beta,m} \\ \lambda \neq 0}} \overline{(\lambda, Z_L)}^k e^{-4\pi v q(\lambda_Z)} dv\right]. \tag{5.1.6}$$

Next we note that $q(\lambda_Z) > 0$ and $(\lambda, Z_L) \neq 0$ for $\lambda \in L_{\beta,m}$ with $\lambda \neq 0$ as $Z \in \mathbb{H}_n \setminus H^L_{\beta,m}$. Thus, using equation (4.1.2) we find that

(5.1.7)
$$\sum_{\substack{\lambda \in L_{\beta,m} \\ \lambda \neq 0}} \int_0^\infty \left| v^{s+k-1} \overline{(\lambda, Z_L)}^k e^{-4\pi v q(\lambda_Z)} \right| dv = \frac{\Gamma(\mathrm{Re}(s)+k) q(\mathrm{Im}(Z))^{k/2}}{4^{\mathrm{Re}(s)+k/2} \pi^{\mathrm{Re}(s)+k}} \sum_{\substack{\lambda \in L_{\beta,m} \\ \lambda \neq 0}} q(\lambda_Z)^{-\mathrm{Re}(s)-k/2}$$

for $\mathrm{Re}(s)+k > 0$, and by Lemma 5.1.2 the sum on the right-hand side converges absolutely and locally uniformly for $\mathrm{Re}(s) > 1 + n/2 - k/2$. Therefore we can simply plug in $t = 0$ on the right-hand side of (5.1.6), interchange summation and integration and use again (4.1.2) to proof the statement given in part (a) of the theorem.

In order to also proof part (b) we need to modify the above proof of (a) as follows: As in equation (5.1.6) we obtain

$$\Phi^{\mathrm{Q},L}_{k,\beta,m}(Z,s) = \frac{2}{q(\mathrm{Im}(Z))^k} \mathrm{CT}_{t=0}\left[\int_0^\infty v^{s+k-t-1} \sum_{\substack{\lambda \in L_{\beta,m} \\ \lambda \neq 0}} \overline{(\lambda, Z_L)}^k e^{-2\pi v q_Z(\lambda)} dv\right] \tag{5.1.8}$$

for $Z \in \mathbb{H}_n$ (without the exclusion of $H^L_{\beta,m}$) and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1 + n/2$, where in (5.1.6) we find the term $e^{-4\pi v q(\lambda_Z)} = e^{-2\pi v q_Z(\lambda)} e^{-2\pi v m}$ instead of $e^{-2\pi v q_Z(\lambda)}$ as in (5.1.8). Here the missing factor, namely $e^{-2\pi v m}$, is exactly the term by which the two building blocks $v^s e(m\tau)\mathfrak{e}_\beta$ and $v^s e(mu)\mathfrak{e}_\beta$ defining the Poincaré series $U^L_{\kappa,\beta,m}(\tau,s)$ and $Q^L_{\kappa,\beta,m}(\tau,s)$ differ. Now as in equation (5.1.7) we see that

$$\begin{aligned} \sum_{\substack{\lambda \in L_{\beta,m} \\ \lambda \neq 0}} \int_0^\infty \left| v^{s+k-1} \overline{(\lambda, Z_L)}^k e^{-2\pi v q_Z(\lambda)} \right| dv \\ = \frac{\Gamma(\mathrm{Re}(s)+k) q(\mathrm{Im}(Z))^{k/2}}{2^{\mathrm{Re}(s)} \pi^{\mathrm{Re}(s)+k}} \sum_{\substack{\lambda \in L_{\beta,m} \\ \lambda \neq 0}} q(\lambda_Z)^{k/2} q_Z(\lambda)^{-\mathrm{Re}(s)-k}, \end{aligned} \tag{5.1.9}$$

where we have used again the identity (4.1.2). As $q(\lambda_Z) \leq q_Z(\lambda) = q(\lambda_Z) + |q(\lambda_{Z^\perp})|$, and since the sum $\sum_{\lambda \in L' \setminus \{0\}} q_Z(\lambda)^{-s}$ converges for $\mathrm{Re}(s) > 1 + n/2$ as remarked in (5.1.2), we find that the right-hand side of (5.1.9) converges for $\mathrm{Re}(s) > 1 + n/2 - k/2$. Therefore we can plug in $t = 0$ on the right-hand side of (5.1.8), interchange summation and integration, and compute the remaining integral. This finishes the proof of part (b). □

Remark 5.1.4.

(1) If $\beta = -\beta$ in L'/L and $2-n-2\kappa \equiv 2 \bmod 4$ then the Poincaré series $U^L_{\kappa,\beta,m}$ and $Q^L_{\kappa,\beta,m}$ vanish identically. This matches the two formulas given in the previous theorem since if $\beta = -\beta$ then $L_{\beta,m} = -L_{\beta,m}$ and thus the sums in (a) and (b) of Theorem 5.1.3 cancel completely if k is odd as

$$q((-\lambda)_Z) = q(\lambda_Z), \qquad q_Z(-\lambda) = q_Z(\lambda), \qquad (-\lambda, Z_L) = -(\lambda, Z_L)$$

for $\lambda \in V$ and $Z \in \mathbb{H}_n$. On the other hand, the congruence $2-n-2\kappa \equiv 2 \bmod 4$ is satisfied if and only if the corresponding non-negative integer k (with $\kappa = 1+k-n/2$) is odd.

(2) If $m = 0$ then $U^L_{\kappa,\beta,0}(\tau, s) = Q^L_{\kappa,\beta,0}(\tau, s)$, and thus also the corresponding lifts need to agree, i.e., we have

$$\Phi^{\mathrm{Sel},L}_{k,\beta,0}(Z, s) = \Phi^{\mathrm{Q},L}_{k,\beta,0}(Z, s)$$

for $Z \in \mathbb{H}_n$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1 + n/2 - k/2$. This agrees with the formulas given in Theorem 5.1.3 since

$$q(\lambda_Z) = \frac{q_Z(\lambda)}{2} \qquad \text{and} \qquad (\lambda, Z_L) = \frac{2q_Z(\lambda)q(\mathrm{Im}(Z))}{\overline{(\lambda, Z_L)}}$$

for $\lambda \in L_{\beta,0}$ with $\lambda \neq 0$ and $Z \in \mathbb{H}_n$. Here the first equality is clear as $q(\lambda) = 0$, and the second equality follows from the identity in (4.1.2).

In the following two sections we now evaluate the above theta lifts for the special lattices of signature $(2, 1)$ and $(2, 2)$ from Section 4.3 and Section 4.4, respectively. Though we are mainly interested in the lift of Selberg's Poincaré series in the case of signature $(2, 1)$, and in the lift of the Poincaré series $Q^L_{\kappa,\beta,m}(\tau, s)$ in the case of signature $(2, 2)$, we also present the opposite cases for the sake of completeness.

5.2 Averaged non-holomorphic Eisenstein series as theta lifts of signature $(2, 1)$

In the present section we restrict to the symmetric space (V, q) of signature $(2, 1)$ introduced in Section 4.3, i.e., we let V be the vector space of 2×2-matrices with rational entries and trace 0, equipped with the quadratic form $q(X) = -N \det(X)$ for some positive integer N. In V we fix the lattice L given by matrices $\left(\begin{smallmatrix} b & c/N \\ -a & -b \end{smallmatrix}\right)$ for $a, b, c \in \mathbb{Z}$ and we choose lattice vectors $e_1 = \left(\begin{smallmatrix} 0 & 1/N \\ 0 & 0 \end{smallmatrix}\right)$ and $e_2 = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ such that the induced generalized upper half-plane $\mathbb{H}_1$ can naturally be identified with the usual upper half-plane $\mathbb{H}$.

Recall that given $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ there is a natural identification of lattice elements in $L_{\beta,m}$ with quadratic forms in $\mathcal{Q}_{\beta,4Nm}$ via $\lambda \mapsto Q_\lambda$ as in (4.3.4), where we understand $\beta \in L'/L$ as an element of $\mathbb{Z}/2N\mathbb{Z}$. Then

$$(\lambda, Z_L) = Q_\lambda(z, 1)$$

as in (4.3.5) for $\lambda \in L_{\beta,m}$ and $Z \in \mathbb{H}_1$ corresponding to $z \in \mathbb{H}$. Thus, the subset $H^L_{\beta,m}$ of $\mathbb{H}_1$ from (5.1.1) corresponds to the set of Heegner points associated to quadratic forms Q_λ with $\lambda \in L_{\beta,m}$, i.e.,

$$H^L_{\beta,m} = H_{\beta,4Nm} \tag{5.2.1}$$

with $H_{\beta,4Nm}$ being defined as in (2.3.2). Furthermore, we recall that

$$q(\operatorname{Im}(Z)) = N\operatorname{Im}(z)^2 \qquad \text{and} \qquad q(\lambda_Z) = \frac{|Q_\lambda(z,1)|^2}{4N\operatorname{Im}(z)^2}$$

for $\lambda \in L_{\beta,m}$ and $Z \in \mathbb{H}_1$ corresponding to $z \in \mathbb{H}$, by Lemma 4.3.1. Here the second identity also implies

$$q_Z(\lambda) = 2q(\lambda_Z) - q(\lambda) = \frac{|Q_\lambda(z,1)|^2}{2N\operatorname{Im}(z)^2} - m. \tag{5.2.2}$$

We now restate part (a) of Theorem 5.1.3 in the present setting. Even though this is essentially a corollary we state it as a theorem to highlight its importance for the present work.

Theorem 5.2.1. *Let $k \in \mathbb{Z}$ with $k \geq 0$, $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$. The regularized theta lift of Selberg's Poincaré series $U^L_{k+1/2,\beta,m}(\tau,s)$ for the present lattice L of signature $(2,1)$ is given by*

$$\Phi^{\mathrm{Sel},L}_{k,\beta,m}(z,s) = \frac{2N^s\,\Gamma(s+k)}{\pi^{s+k}}\; f_{k,\beta,4Nm}(z,2s)$$

for $z \in \mathbb{H} \setminus H_{\beta,4Nm}$ and $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 3/2 - k/2$. Here the right-hand side yields a holomorphic continuation of $\Phi^{\mathrm{Sel},L}_{k,\beta,m}(z,s)$ in s to the half-plane $\operatorname{Re}(s) > 1/2 - k/2$.

Proof. Using the expressions for $q(\lambda_Z)$ and (λ, Z_L) for $Z \in \mathbb{H}_1$ corresponding to $z \in \mathbb{H}$ recalled above, the given identity is a direct consequence of part (a) of Theorem 5.1.3. Further, the holomorphic continuation of $\Phi^{\mathrm{Sel},L}_{k,\beta,m}(z,s)$ follows as the functions $f_{k,\beta,4Nm}(z,2s)$ are defined for $\operatorname{Re}(2s) > 1-k$. □

For $k = 0$ we may rewrite Theorem 5.2.1 in terms of Corollary 2.6.4, showing that the regularized theta lift of Selberg's Poincaré series of the first kind of weight $\kappa = 1/2$ is indeed an averaged version of hyperbolic, parabolic or elliptic Eisenstein series of weight 0, where the type of Eisenstein series depends on the sign of the parameter m of the corresponding Poincaré series. In fact, the following special case of Theorem 5.2.1 was the reason to consider the theta lift of Selberg's Poincaré series in the first place.

Corollary 5.2.2. *Let $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$.*

(a) Then

$$\Phi^{\mathrm{Sel},L}_{0,\beta,m}(z,s) = \begin{cases} \dfrac{2\,\Gamma(s)}{(4\pi m)^s} \displaystyle\sum_{Q \in \mathcal{Q}_{\beta,4Nm}/\Gamma_0(N)} E^{\mathrm{hyp}}_{c_Q}(z,2s), & \text{if } m > 0, \\ \dfrac{2N^s\,\Gamma(s)}{\pi^s} \displaystyle\sum_{p \in C(\Gamma_0(N))} \lambda_{\beta,p}(2s) E^{\mathrm{par}}_p(z,2s), & \text{if } m = 0, \\ \dfrac{2\,\Gamma(s)}{(4\pi|m|)^s} \displaystyle\sum_{Q \in \mathcal{Q}_{\beta,4Nm}/\Gamma_0(N)} E^{\mathrm{ell}}_{\tau_Q}(z,2s), & \text{if } m < 0, \end{cases}$$

for $z \in \mathbb{H} \setminus H_{\beta,4Nm}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1/2$. Here the coefficients $\lambda_{\beta,p}(2s)$ are defined as in Corollary 2.6.4.

(b) *Assume that N is squarefree and let $m = 0$. Then $\beta = 0$ and*

$$\Phi^{\mathrm{Sel},L}_{0,0,0}(z,s) = 4N^s \zeta^*(2s) \sum_{p \in C(\Gamma_0(N))} E^{\mathrm{par}}_p(z,2s)$$

for $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1/2$. Here $\zeta^*(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is the completed Riemann zeta function.

Proof. Part (a) is a simple application of Corollary 2.6.4 to Theorem 5.2.1. Analogously, if N is squarefree part (b) follows from the identity given in (2.6.9). □

Another special case of Theorem 5.2.1 is given if $k \geq 2$. In this case the non-holomorphic function $f_{k,\beta,4Nm}(z,2s)$ can simply be evaluated at $s = 0$, and by (2.5.6) we have

$$f_{k,\beta,4Nm}(z,0) = f_{k,\beta,4Nm}(z)$$

for $z \in \mathbb{H} \setminus H_{\beta,4Nm}$. Here the function on the right-hand side is the holomorphic (or meromorphic if $m < 0$) modular form associated to the class β and the discriminant $4Nm$ given in Definition 2.4.6.

Corollary 5.2.3. *Let $k \in \mathbb{Z}$ with $k \geq 2$, $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$. Then*

$$\Phi^{\mathrm{Sel},L}_{k,\beta,m}(z,0) = \frac{2\,\Gamma(k)}{\pi^k}\; f_{k,\beta,4Nm}(z).$$

for $z \in \mathbb{H} \setminus H_{\beta,4Nm}$.

This is essentially the Shimura theta lift of the vector valued (holomorphic) Poincaré series $P^L_{\kappa,\beta,m}(\tau)$ of weight $\kappa = k + 1/2$ (compare Proposition 3.4.4 and equation (4.3.7)). More precisely, we can understand the above Corollary in the following way: Let $k \in \mathbb{Z}$ with $k \geq 2$, $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$. Then

$$\Phi^L_k(z; U^L_{k+1/2,\beta,m}(\,\cdot\,,s))\bigg|_{s=0} = \Phi^L_k(z; U^L_{k+1/2,\beta,m}(\,\cdot\,,0)).$$

In other words, the process of lifting and evaluating at $s = 0$ can be reversed if $k \geq 2$. We will later see that this is in general not true if $k = 0$ (compare Proposition 7.1.1).

For the sake of completeness, we also present the theta lift of the non-standard Poincaré series $Q^L_{\kappa,\beta,m}(\tau,s)$ given in part (b) of Theorem 5.1.3 in the situation of the current lattice.

Proposition 5.2.4. *Let $k \in \mathbb{Z}$ with $k \geq 0$, $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$. The regularized theta lift of the Poincaré series $Q^L_{k+1/2,\beta,m}(\tau,s)$ for the present lattice L of signature $(2,1)$ is given by*

$$\Phi^{\mathrm{Q},L}_{k,\beta,m}(z,s) = \frac{2N^s\,\Gamma(s+k)}{\pi^{s+k}} \sum_{\substack{Q \in \mathcal{Q}_{\beta,4Nm} \\ Q \neq 0}} \left(\frac{|Q(z,1)|^2}{\mathrm{Im}(z)^2} - 2Nm\right)^{-s-k} \left(\frac{Q(\bar{z},1)}{\mathrm{Im}(z)^2}\right)^k$$

for $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 3/2 - k/2$.

Proof. This is a direct consequences of part (b) of Theorem 5.1.3 where we only have to apply the expressions for the quantities $q(\mathrm{Im}(Z))$, $q_Z(\lambda)$ and (λ, Z_L) for $Z \in \mathbb{H}_1$ corresponding to $z \in \mathbb{H}$ recalled at the beginning of the present section. $\square$

As in Corollary 5.2.2 we also present Proposition 5.2.4 for the special case $k = 0$. However, we ignore the case $m = 0$, which is simply given by

$$\Phi^{\mathrm{Q},L}_{k,\beta,0}(z,s) = \Phi^{\mathrm{Sel},L}_{k,\beta,0}(z,s) = \frac{2N^s\,\Gamma(s+k)}{\pi^{s+k}}\; f_{k,\beta,0}(z,2s)$$

for $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1/2 - k/2$ (compare part (2) of Remark 5.1.4).

Corollary 5.2.5. *Let $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m \neq 0$. Then*

$$\Phi^{\mathrm{Q},L}_{0,\beta,m}(z,s) = \frac{2\,\Gamma(s)}{(2\pi|m|)^s} \sum_{Q \in \mathcal{Q}_{\beta,4Nm}/\Gamma_0(N)} g_Q(z,s) \tag{5.2.3}$$

for $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 3/2$. Here given $Q \in \mathcal{Q}_{\beta,4Nm}$ the associated function $g_Q(z,s)$ is defined by

$$g_Q(z,s) := \begin{cases} \displaystyle\sum_{M \in (\Gamma_0(N))_{c_Q}\backslash\Gamma_0(N)} \Big(\sinh(d_{\mathrm{hyp}}(Mz,c_Q))^2 + \cosh(d_{\mathrm{hyp}}(Mz,c_Q))^2\Big)^{-s}, & \text{if } m > 0,\\ \displaystyle\sum_{M \in (\Gamma_0(N))_{\tau_Q}\backslash\Gamma_0(N)} \Big(\sinh(d_{\mathrm{hyp}}(Mz,\tau_Q))^2 + \cosh(d_{\mathrm{hyp}}(Mz,\tau_Q))^2\Big)^{-s}, & \text{if } m < 0, \end{cases}$$

for $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1/2$. In particular, the right-hand side of equation (5.2.3) yields a holomorphic continuation of $\Phi^{\mathrm{Q},L}_{0,\beta,m}(z,s)$ is s to the half-plane $\mathrm{Re}(s) > 1/2$, and the functions $g_Q(z,s)$ are modular of weight 0 and level N.

Proof. For $k = 0$ and $m \neq 0$ Proposition 5.2.4 simplifies to

$$\Phi^{\mathrm{Q},L}_{0,\beta,m}(z,s) = \frac{2N^s\,\Gamma(s)}{\pi^s} \sum_{Q \in \mathcal{Q}_{\beta,4Nm}} \left(\frac{|Q(z,1)|^2}{\mathrm{Im}(z)^2} - 2Nm\right)^{-s}.$$

for $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 3/2$. Now, an application of Lemma 2.5.4 to the term $|Q(z,1)|^2/\mathrm{Im}(z)^2$ together with the identity $\cosh(z)^2 - \sinh(z)^2 = 1$ yields the claimed formula. Moreover, since $\sinh(x)^2 \geq 0$ for $x \in \mathbb{R}$, the sum defining the function $g_Q(z,s)$ with $Q \in \mathcal{Q}_{\beta,4Nm}$ is either dominated by the hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{c_Q}(z, 2\,\mathrm{Re}(s))$ if $m > 0$, or by the hyperbolic kernel function $K(z, \tau_Q, 2\,\mathrm{Re}(s))$ if $m < 0$. Therefore, the function $g_Q(z,s)$ is indeed well-defined for $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1/2$. $\square$

Remark 5.2.6. There are different, trivially equivalent ways of writing the functions $g_Q(z,s)$ appearing in the previous corollary, since

$$\sinh(z)^2 + \cosh(z)^2 = 2\sinh(z)^2 + 1 = 2\cosh(z)^2 - 1 = \cosh(2z)$$

for $z \in \mathbb{H}$. In particular, if c is a geodesic in $\mathbb{H}$ which is either closed or infinite then

$$g_Q(z,s) = \sum_{M \in (\Gamma_0(N))_c\backslash\Gamma_0(N)} \cosh(2\,d_{\mathrm{hyp}}(Mz,c))^{-s}$$

for $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1/2$, where we choose $Q \in \mathcal{Q}$ such that $c_Q = c$ (see Corollary 2.3.2). So for $Q \in \mathcal{Q}$ with discriminant $\Delta(Q) > 0$ the function $g_Q(z,s)$ is similar to the corresponding hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{c_Q}(z,2s)$.

5.3 The hyperbolic kernel function as a theta lift of signature $(2,2)$

Let now (V,q) be the orthogonal space of signature $(2,2)$ from Section 4.4, i.e., let V be the vector space of 2×2-matrices with rational entries equipped with the quadratic form $q(X) = -\det(X)$, and given some positive integer N let L be the lattice in V given by matrices of the form $\left(\begin{smallmatrix} a & b \\ Nc & d \end{smallmatrix}\right)$ with $a,b,c,d \in \mathbb{Z}$. Fixing $e_1 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in L$ and $e_2 = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \in L'$ we can identify the induced generalized upper half-plane $\mathbb{H}_2$ with the product of two copies of the upper half-plane, i.e., with $\mathbb{H}\times\mathbb{H}$.

We further recall from Section 4.4 that the discriminant form L'/L can be identified with the quotient $(\mathbb{Z}/N\mathbb{Z})^2$, and that the dual lattice L' of L consists of matrices of the form $\left(\begin{smallmatrix} a & b/N \\ c & d \end{smallmatrix}\right)$ with $a,b,c,d\in\mathbb{Z}$. Thus

$$L_{\beta,m} = \left\{ \begin{pmatrix} a & b/N \\ c & d \end{pmatrix} : a,b,c,d\in\mathbb{Z}, b\equiv\beta_1 \mod N, c\equiv\beta_2 \mod N, \frac{bc}{N} - ad = m \right\}$$

for $\beta = (\beta_1,\beta_2)\in(\mathbb{Z}/N\mathbb{Z})^2$. In particular, we find $L_{0,-1} = \Gamma_0(N)$ and

$$L_{0,1} = \left\{ \begin{pmatrix} a & b \\ cN & d \end{pmatrix} : a,b,c,d\in\mathbb{Z}, ad-bcN = -1 \right\} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Gamma_0(N).$$

We concentrate on these two special cases from now on, namely $\beta = 0$ and $m=\pm 1$. One can check that the general case $\beta\in L'/L$ and $m\in\mathbb{Z}+q(\beta)$ with $m\neq 0$ can indeed be reduced to these special cases, even though we do in general not end up with the group $\Gamma_0(N)$, but with some scaled version of it.

Since we assume that $\beta = 0$ we trivially have $\beta = -\beta$ in L'/L. Hence the theta lifts from Theorem 5.1.3 vanish completely if k is odd (compare Remark 5.1.4). So we additionally assume that k is even from now on.

In the following we quickly recall some more notation from Section 4.4, concentrating on the special case $\beta = 0$ and $m = \pm 1$. Given $z,z'\in\mathbb{H}$ the tuple (z,z') in $\mathbb{H}\times\mathbb{H}$ corresponds to the element $Z = \left(\begin{smallmatrix} z & 0 \\ 0 & -z' \end{smallmatrix}\right)$ in the generalized upper half-plane $\mathbb{H}_2$. As in Section 4.4 we write $q_{z,z'}$ for the majorant q_Z associated to Z, and for $\lambda\in V$ we denote the orthogonal projection of λ onto the 2-dimensional positive definite subspace corresponding to Z by $\lambda_{z,z'}$. Further, we use the notation

$$Z_L(z,z') = Z_L = \begin{pmatrix} z & -zz' \\ 1 & -z' \end{pmatrix},$$

and we recall that $q(\operatorname{Im}(Z)) = \operatorname{Im}(z)\operatorname{Im}(z')$ for $z,z'\in\mathbb{H}$ with $Z = \left(\begin{smallmatrix} z & 0 \\ 0 & -z' \end{smallmatrix}\right)$.

Lemma 5.3.1. *Let $\lambda\in L_{0,\pm1}$ and $z,z'\in\mathbb{H}$. Then*

$$(\lambda, Z_L(z,z')) = j(M_1(\lambda),z')(M_1(\lambda)z' \pm z) = j(M_2(\lambda),z)(z' \pm M_2(\lambda)z)$$

where $M_1(\lambda) := \left(\begin{smallmatrix} 1 & 0 \\ 0 & \mp 1 \end{smallmatrix}\right)\lambda$ and $M_2(\lambda) := \left(\begin{smallmatrix} 1 & 0 \\ 0 & \mp 1 \end{smallmatrix}\right)\lambda^{-1}$ are both elements of $\Gamma_0(N)$.

Proof. By definition we have $(\lambda, Z_L(z,z')) = az' + b - czz' - dz$ for $\lambda = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Now the lemma follows by a straightforward computation. $\square$

In particular, the previous lemma implies that we have $(\lambda, Z_L(z,z')) = 0$ for $\lambda \in L_{0,\pm 1}$ if and only if $z = \mp M_1(\lambda)z'$. So $H^L_{0,1} = \emptyset$ as $-M_1(\lambda)z' \notin \mathbb{H}$ for $\lambda \in L_{0,1}$, and

$$H^L_{0,-1} = \{(z,z') \in \mathbb{H} \times \mathbb{H} \colon z \equiv z' \mod \Gamma_0(N)\}.$$

Here the set $H^L_{0,\pm 1}$ is defined as in (5.1.1). Since $H^L_{0,-1}$ can be understood as the diagonal in the product of modular curves $Y_0(N) \times Y_0(N)$, we use the notation

$$D_{-1} := H^L_{0,-1} = \{(z,z') \in \mathbb{H} \times \mathbb{H} \colon z \equiv z' \mod \Gamma_0(N)\}. \tag{5.3.1}$$

To simplify notation we also set $D_1 := \emptyset$, such that $D_{\pm 1} = H^L_{0,\pm 1}$.

Next we consider the quantities $q(\lambda_{z,z'})$ and $q_{z,z'}(\lambda)$. Applying the previous lemma to the identity given in (4.4.4) we directly find that

$$q(\lambda_{z,z'}) = \frac{|z \pm M_1(\lambda)z'|^2}{4\operatorname{Im}(z)\operatorname{Im}(M_1(\lambda)z')} = \frac{|M_2(\lambda)z' \pm z)|^2}{4\operatorname{Im}(M_2(\lambda)z)\operatorname{Im}(z')} \tag{5.3.2}$$

for $z, z' \in \mathbb{H}$ and $\lambda \in L_{0,\pm 1}$, where the matrices $M_1(\lambda), M_2(\lambda) \in \Gamma_0(N)$ are given as in Lemma 5.3.1. Moreover, $q_{z,z'}(\lambda)$ is given as follows:

Lemma 5.3.2. *Let $z, z' \in \mathbb{H}$ and $\lambda \in L_{0,m}$ with $m = \pm 1$.*

(a) If $m = 1$ then

$$q_{z,z'}(\lambda) = \cosh\left(d_{\mathrm{hyp}}\left(z, -\overline{M_1(\lambda)z'}\right)\right) = \cosh\left(d_{\mathrm{hyp}}\left(M_2(\lambda)z, -\overline{z'}\right)\right).$$

(b) If $m = -1$ then

$$q_{z,z'}(\lambda) = \cosh(d_{\mathrm{hyp}}(z, M_1(\lambda)z')) = \cosh(d_{\mathrm{hyp}}(M_2(\lambda)z, z')).$$

Here the matrices $M_1(\lambda), M_2(\lambda) \in \Gamma_0(N)$ are defined as in Lemma 5.3.1.

Proof. Since $q_{z,z'}(\lambda) = 2q(\lambda_{z,z'}) - q(\lambda)$ equation (5.3.2) implies that

$$q_{z,z'}(\lambda) = \frac{|z \pm M_1(\lambda)z'|^2}{2\operatorname{Im}(z)\operatorname{Im}(M_1(\lambda)z')} \mp 1 = \frac{|M_2(\lambda)z' \pm z)|^2}{2\operatorname{Im}(M_2(\lambda)z)\operatorname{Im}(z')} \mp 1$$

for $\lambda \in L_{0,\pm 1}$. Now part (b) follows directly from the identity given in (2.1.1). Moreover, an easy computation using again (2.1.1) shows that

$$\frac{|z + z'|^2}{2\operatorname{Im}(z)\operatorname{Im}(z')} - 1 = \cosh\left(d_{\mathrm{hyp}}\left(z, -\overline{z'}\right)\right)$$

for $z, z' \in \mathbb{H}$, which proves part (a). $\square$

We now restate part (a) of Theorem 5.1.3 for the present lattice L of signature $(2,2)$, and for $\beta = 0$ and $m = \pm 1$:

Proposition 5.3.3. *Let $k \in \mathbb{Z}$ with $k \geq 0$ even. The regularized theta lift of Selberg's Poincaré series $U^L_{k,0,\pm1}(\tau, s)$ for the present lattice L of signature $(2,2)$ is given by*

$$\Phi^{\mathrm{Sel},L}_{k,0,\pm1}(z,z',s) = \frac{2\,\Gamma(s+k)}{\pi^{s+k}} \sum_{M \in \Gamma_0(N)} \left(\frac{\mathrm{Im}(z)\,\mathrm{Im}(z')}{|z \pm z'|^2} \right)^s (z \pm z')^{-k} \Big|_k M$$

for $(z,z') \in (\mathbb{H} \times \mathbb{H}) \setminus D_{\pm1}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 2 - k/2$. Here the weight k action on the right-hand side of the above identity can either be seen as an action in the variable z, or in the variable z'.

Proof. By part (a) of Theorem 5.1.3, equation (5.3.2) and Lemma 5.3.1 we find

$$\begin{aligned}\Phi^{\mathrm{Sel},L}_{k,0,\pm1}(z,z',s) &= \frac{2\,\Gamma(s+k)}{\pi^{s+k}} \sum_{\lambda \in L_{0,\pm1}} \left(\frac{\mathrm{Im}(z)\,\mathrm{Im}(M_1(\lambda)z')}{|z \pm M_1(\lambda)z'|^2} \right)^s j(M_1(\lambda), z')^{-k} (z \pm M_1(\lambda)z')^{-k} \\ &= \frac{2\,\Gamma(s+k)}{\pi^{s+k}} \sum_{M \in \Gamma_0(N)} \left(\frac{\mathrm{Im}(z)\,\mathrm{Im}(z')}{|z \pm z')|^2} \right)^s (z \pm z')^{-k} \Big|_k M,\end{aligned}$$

since the map $\lambda \mapsto M_1(\lambda)$ with $M_1(\lambda)$ as in Lemma 5.3.1 gives a bijection $L_{0,\pm1} \to \Gamma_0(N)$. Here we understand the given weight k action as an action in the variable z'.

Analogously, using the bijection $\lambda \mapsto M_2(\lambda)$ we obtain the same expression for the lift $\Phi^{\mathrm{Sel},L}_{k,0,\pm1}(z,z',s)$ with the given weight k action being an action in the variable z. □

We remark that given a function $f \colon \mathbb{H} \to \mathbb{C}[L'/L]$ modular of weight k with respect to ρ_L, the (regularized) theta lift of f for the present lattice L is in general only modular of weight k with respect to the group $\Gamma(N)$ (see Section 4.4). However, the theta lift of the Poincaré series $U^L_{k,0,\pm1}(\tau,s)$ is indeed modular of weight k with respect to the larger group $\Gamma_0(N)$.

For the sake of completeness we quickly treat the special cases $k = 0$, and $s = 0$ if $k \geq 3$, of the previous Proposition.

Corollary 5.3.4. *Let $m = \pm1$.*

(a) If $k = 0$ then

$$\Phi^{\mathrm{Sel},L}_{0,0,m}(z,z',s) = \begin{cases} \dfrac{2\,\Gamma(s)}{(2\pi)^s} \displaystyle\sum_{M \in \Gamma_0(N)} \Big(\cosh\left(d_{\mathrm{hyp}}\left(z, -\overline{z'}\right)\right) + 1 \Big)^{-s} \Big|_0 M, & \text{if } m = 1, \\ \dfrac{2\,\Gamma(s)}{(2\pi)^s} \displaystyle\sum_{M \in \Gamma_0(N)} \Big(\cosh\left(d_{\mathrm{hyp}}\left(z, z'\right)\right) - 1 \Big)^{-s} \Big|_0 M, & \text{if } m = -1, \end{cases}$$

for $(z,z') \in (\mathbb{H} \times \mathbb{H}) \setminus D_{\pm1}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 2$. Here the weight 0 action on the right-hand side can either be seen as an action in the variable z, or in the variable z'. Further, the expression on the right-hand side yields a holomorphic continuation of the lift $\Phi^{\mathrm{Sel},L}_{0,0,m}(z,z',s)$ in s to the half-plane $\mathrm{Re}(s) > 1$.

(b) If $k \geq 3$ then

$$\Phi^{\mathrm{Sel},L}_{k,0,m}(z,z',0) = \frac{2\,\Gamma(k)}{\pi^k} \sum_{M \in \Gamma_0(N)} (z \pm z')^{-k} \Big|_k M$$

for $(z,z') \in (\mathbb{H} \times \mathbb{H}) \setminus D_{\pm1}$. Here the weight k action on the right-hand side can either be seen as an action in the variable z, or in the variable z'.

Proof. As in the proof of Lemma 5.3.2 we find that

$$\frac{|z \pm z'|^2}{2\operatorname{Im}(z)\operatorname{Im}(z')} = \begin{cases} \cosh(d_{\mathrm{hyp}}(z, -\overline{z'})) + 1, & \text{if } \pm 1 = 1, \\ \cosh(d_{\mathrm{hyp}}(z, z')) - 1, & \text{if } \pm 1 = -1, \end{cases}$$

for $z, z' \in \mathbb{H}$. Thus, the formula in part (a) follows directly from Proposition 5.3.3. Moreover, if $m = 1$ the given sum is clearly dominated by the hyperbolic kernel function $K(z, -\overline{z'}, \operatorname{Re}(s))$, which is defined for $\operatorname{Re}(s) > 1$. This proves the holomorphic continuation of the given theta lift for $k = 0$ and $m = 1$ as claimed in (a). If $m = -1$ we need to be more carefully:

Let $m = -1$ and fix $z, z' \in \mathbb{H}$ with $z \not\equiv z'$ modulo $\Gamma_0(N)$. Then we find $\varepsilon > 0$ such that the set

$$\{M \in \Gamma_0(N) \colon d_{\mathrm{hyp}}(Mz, z') < \varepsilon\}$$

is finite, and $C > 0$ such that $\tanh(x/2) \geq C$ for all $x \geq \varepsilon$. Hence

$$\cosh(d_{\mathrm{hyp}}(Mz, z')) - 1 = \sinh(d_{\mathrm{hyp}}(Mz, z')) \tanh\left(\frac{d_{\mathrm{hyp}}(Mz, z')}{2}\right) \geq C \sinh(d_{\mathrm{hyp}}(Mz, z'))$$

for all $M \in \Gamma_0(N)$ with $d_{\mathrm{hyp}}(Mz, z') \geq \varepsilon$, and thus

$$\begin{aligned} &\left| \sum_{M \in \Gamma_0(N)} \Big(\cosh(d_{\mathrm{hyp}}(Mz, z')) - 1\Big)^{-s} \right| \\ &\leq \sum_{\substack{M \in \Gamma_0(N) \\ d_{\mathrm{hyp}}(Mz, z') < \varepsilon}} \Big(\cosh(d_{\mathrm{hyp}}(Mz, z')) - 1\Big)^{-\operatorname{Re}(s)} + C^{-\operatorname{Re}(s)} \sum_{\substack{M \in \Gamma_0(N) \\ d_{\mathrm{hyp}}(Mz, z') \geq \varepsilon}} \sinh(d_{\mathrm{hyp}}(Mz, z'))^{-\operatorname{Re}(s)} \end{aligned}$$

for all $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 2$. Here the first sum is finite, and the second sum is dominated by the elliptic Eisenstein series $E^{\mathrm{ell}}_{z'}(z, \operatorname{Re}(s))$, which converges for $\operatorname{Re}(s) > 1$. This also proves the holomorphic continuation claimed in (a) in the case $m = -1$.

Finally, we remark that part (b) is a trivial consequence of Proposition 5.3.3. □

We now turn our attention to the theta lift of the Poincaré series $Q^L_{k,0,\pm 1}(\tau, s)$ in the situation of the present lattice L. It turns out that for $m = -1$ we exactly obtain the hyperbolic kernel function $K(z, z', s)$, which we defined in Section 2.6.4. Using the relation between the hyperbolic kernel function and non-holomorphic Eisenstein series of weight 0 as given in Proposition 2.6.6 we are able to realize individual hyperbolic, parabolic and elliptic Eisenstein series.

Theorem 5.3.5. *Let $k \in \mathbb{Z}$ with $k \geq 0$ even.*

(a) The regularized theta lift of the Poincaré series $Q^L_{k,0,1}(\tau, s)$ for the present lattice L of signature $(2, 2)$ is given by

$$\Phi^{\mathrm{Q},L}_{k,0,1}(z, z', s) = \frac{2\,\Gamma(s+k)}{(2\pi)^{s+k}} \sum_{M \in \Gamma_0(N)} \cosh\left(d_{\mathrm{hyp}}\left(z, -\overline{z'}\right)\right)^{-s-k} \left(\frac{\overline{z + z'}}{\operatorname{Im}(z)\operatorname{Im}(z')}\right)^k \Bigg|_k M$$

for $z, z' \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 2 - k/2$. Here the weight k action on the right-hand side can either be seen as an action in the variable z, or in the variable z'.

(b) *The regularized theta lift of the Poincaré series* $Q^L_{k,0,-1}(\tau,s)$ *for the present lattice* L *of signature* $(2,2)$ *is given by*

$$\Phi^{\mathrm{Q},L}_{k,0,-1}(z,z',s) = \frac{2\,\Gamma(s+k)}{(2\pi)^{s+k}} \sum_{M\in\Gamma_0(N)} \cosh(d_{\mathrm{hyp}}(z,z'))^{-s-k} \left(\frac{\overline{z-z'}}{\mathrm{Im}(z)\,\mathrm{Im}(z')}\right)^k \Bigg|_k M$$

for $z,z'\in\mathbb{H}$ *and* $s\in\mathbb{C}$ *with* $\mathrm{Re}(s)>2-k/2$. *Here the weight* k *action on the right-hand side can either be seen as an action in the variable* z, *or in the variable* z'.

Proof. By (4.4.4) we have

$$q(\mathrm{Im}(Z))^{-k}\overline{(\lambda,Z_L)}^k = 4^k q(\lambda_{z,z'})^k (\lambda, Z_L(z,z'))^{-k}$$

for $\lambda\in V$ and $Z=\left(\begin{smallmatrix} z & 0\\ 0 & -z'\end{smallmatrix}\right)$ with $z,z'\in\mathbb{H}$. Thus, part (b) of Theorem 5.1.3 states that

$$\Phi^{\mathrm{Q},L}_{k,0,\pm1}(Z,s) = \frac{2\,\Gamma(s+k)}{(2\pi)^{s+k}} \sum_{\lambda\in L_{0,\pm1}} q_{z,z'}(\lambda)^{-s-k}\left(\frac{4q(\lambda_{z,z'})}{(\lambda, Z_L(z,z'))}\right)^k$$

for $z,z'\in\mathbb{H}$ and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>2-k/2$. Applying Lemma 5.3.1, equation (5.3.2) and Lemma 5.3.2 we obtain the claimed statement. □

We are mainly interested in the following special case of part (b) of the previous theorem, which gives a realization of the hyperbolic kernel function of level N defined in Section 2.6.4 as the (regularized) theta lift of the non-holomorphic Poincaré series $Q^L_{0,0,-1}(\tau,s)$ of weight 0.

Corollary 5.3.6. *The regularized theta lift of the Poincaré series* $Q^L_{0,0,-1}(\tau,s)$ *for the present lattice* L *of signature* $(2,2)$ *is given by*

$$\Phi^{\mathrm{Q},L}_{0,0,-1}(z,z',s) = \frac{2\,\Gamma(s)}{(2\pi)^s} K(z,z',s)$$

for $z,z'\in\mathbb{H}$ *and* $s\in\mathbb{C}$ *with* $\mathrm{Re}(s)>2$. *Here right-hand side yields a holomorphic continuation of the lift* $\Phi^{\mathrm{Q},L}_{0,0,-1}(z,z',s)$ *in* s *to the half-plane* $\mathrm{Re}(s)>1$.

Proof. Part (b) of Theorem 5.3.5 states that

$$\Phi^{\mathrm{Q},L}_{0,0,-1}(z,z',s) = \frac{2\,\Gamma(s)}{(2\pi)^s} \sum_{M\in\Gamma_0(N)} \cosh(d_{\mathrm{hyp}}(Mz,z'))^{-s}$$

for $z,z'\in\mathbb{H}$ and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>2$, which proves the given identity. □

Applying the previous corollary to Proposition 2.6.6 we obtain the following technical realization of individual hyperbolic, parabolic and elliptic Eisenstein series:

Proposition 5.3.7.

(a) *Let c be a closed geodesic in $\mathbb{H}$. Then*

$$E_c^{\mathrm{hyp}}(z,s) = \frac{\pi^s}{\Gamma(s/2)^2} \int_{[c]} \Phi_{0,0,-1}^{\mathrm{Q},L}(z,w,s)ds(w)$$

for $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$.

(b) *Let p be a cusp and let $v > 1$. Then*

$$E_p^{\mathrm{par}}(z,s) = \frac{2s-1}{4}\, v^{s-1} \sum_{n=0}^{\infty} \frac{\pi^{s+2n-1/2}}{n!\,\Gamma(s+n+1/2)} \int_0^1 \Phi_{0,0,-1}^{\mathrm{Q},L}(z,\sigma_p(u+iv),s+2n)du$$

for $z \in \mathbb{H}$ with $\mathrm{Im}(Mz) < v$ for all $M \in \Gamma_0(N)$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. Here $\sigma_p \in \mathrm{SL}_2(\mathbb{R})$ is a scaling matrix for the cusp p, and the right-hand side of the equation is independent of v.

(c) *Let $w \in \mathbb{H}$. Then*

$$E_w^{\mathrm{ell}}(z,s) = \frac{(2\pi)^s}{2\,\mathrm{ord}(w)} \sum_{n=0}^{\infty} \frac{(2\pi)^{2n}\,(s/2)_n}{n!\;\Gamma(s+2n)}\, \Phi_{0,0,-1}^{\mathrm{Q},L}(z,w,s+2n)$$

for $z \in \mathbb{H} \setminus \Gamma_0(N)w$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$.

Proof. By Corollary 5.3.6 we have

$$K(z,z',s) = \frac{(2\pi)^s}{2\,\Gamma(s)}\, \Phi_{0,0,-1}^{\mathrm{Q},L}(z,z',s)$$

for $z, z' \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. Thus, part (a) and (c) follow directly from the corresponding parts (a) and (c) of Proposition 2.6.6. Furthermore, several applications of the well-known duplication formula of the Gamma function yield

$$\frac{(2\pi)^{s+2n}}{2\,\Gamma(s+2n)} \cdot \frac{2^s\,\Gamma(s)^2}{4\pi\,\Gamma(2s)} \cdot \frac{(s/2)_n(s/2+1/2)_n}{(s+1/2)_n} = \frac{\pi^{s+2n-1/2}}{4\,\Gamma(s+n+1/2)}.$$

Together with part (b) of Proposition 2.6.6 we hence also obtain part (b) of the present proposition. □

Remark 5.3.8.

(1) If we could establish the meromorphic continuation of the regularized theta lift $\Phi_{0,0,-1}^{\mathrm{Q},L}(z,w,s)$ in s to $s=0$, part (a) of Proposition 5.3.7 would yield another proof of the Kronecker limit type formula

$$E_c^{\mathrm{hyp}}(z,s) = O(s^2)$$

as $s \to 0$, for c a closed geodesic (compare equation (2.6.5)).

(2) Ignoring convergence, part (c) of Proposition 5.3.7 yields the formal identity

$$E_w^{\mathrm{ell}}(z,s) = \frac{1}{2\operatorname{ord}(w)} \lim_{T\to\infty} \int_{\mathcal{F}_T} \left\langle \tilde{Q}(\tau,s), \Theta_{L,0}(\tau,(z,w)) \right\rangle d\mu(\tau),$$

where we set

$$\tilde{Q}(\tau,s) := (2\pi)^s \sum_{n=0}^{\infty} \frac{(2\pi)^{2n}\,(s/2)_n}{n!\;\Gamma(s+2n)}\, Q^L_{0,0,-1}(\tau,s+2n).$$

Applying two times the duplication formula of the Gamma function we find that

$$\frac{(2\pi)^{2n}(s/2)_n}{\Gamma(s+2n)} = \frac{\pi^{2n}}{(s/2+1/2)_n\Gamma(s)}.$$

Hence, we can use Remark 3.6.11 to see that the function $\tilde{Q}(\tau,s)$ is essentially the Maass-Selberg Poincaré series $M^L_{0,0,-1}(\tau,s)$ given in Definition 3.6.10, namely

$$\tilde{Q}(\tau,s) = \frac{(2\pi)^s}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{\pi^{2n}}{n!\,(s/2+1/2)_n}\, Q^L_{0,0,-1}(\tau,s+2n) = \frac{\pi^{s/2}}{\Gamma(s)}\, M^L_{0,0,-1}(\tau,s).$$

Therefore, up to the factor $\pi^{s/2}/(2\operatorname{ord}(w)\Gamma(s))$ the elliptic Eisenstein series $E_w^{\mathrm{ell}}(z,s)$ is the formal Borcherds lift of the Poincaré series $M^L_{0,0,-1}(\tau,s)$.

5.4 The elliptic Eisenstein series as a theta lift of signature $(2,2)$

Motivated by the considerations of part (2) of Remark 5.3.8 we also investigate the regularized theta lift of the Maass-Selberg Poincaré series $M^L_{\kappa,\beta,m}(\tau,s)$, which was introduced in Definition 3.6.10 for $m<0$.

We start with a general quadratic space (V,q) of signature $(2,n)$, some even lattice L in V, and the corresponding generalized upper half-plane $\mathbb{H}_n$ as in Section 5.1. After computing the regularized theta lift of the Poincaré series $M^L_{\kappa,\beta,m}(\tau,s)$ in this general setting, we will return to the orthogonal space of signature $(2,2)$ from the previous section. As before, we let k be a non-negative integer, and we set $\kappa := 1+k-n/2$ such that κ satisfies the congruence (3.4.1).

Definition 5.4.1. Let $k\in\mathbb{Z}$ with $k\geq 0$, $\kappa = 1+k-n/2$, $\beta\in L'/L$ and $m\in\mathbb{Z}+q(\beta)$ with $m<0$. We define the regularized theta lift of the Maass-Selberg Poincaré series $M^L_{\kappa,\beta,m}(\tau,s)$ by

$$\Phi^{\mathrm{M},L}_{k,\beta,m}(Z,s) = \Phi^L_k(Z; M^L_{\kappa,\beta,m}(\,\cdot\,,s))$$

for $Z\in\mathbb{H}_n\setminus H^L_{\beta,m}$ and $s\in\mathbb{C}$ with $\operatorname{Re}(s) > \max(3/2+k/2+n/4, 2+n-k)$.

We quickly recall that the classical hypergeometric function ${}_2F_1(a,b,c;z)$ is defined by

$$(5.4.1) \qquad {}_2F_1(a,b,c;z) := \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{n!\,(c)_n} z^n$$

for $a,b,c\in\mathbb{C}$ with $c\neq 0,-1,-2,\ldots$ and $z\in\mathbb{C}$ with $|z|<1$. Here $(z)_n$ denotes the Pochhammer symbol from (2.6.11).

Theorem 5.4.2. *Let $k \in \mathbb{Z}$ with $k \geq 0$, $\kappa = 1 + k - n/2$, $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m < 0$. The regularized theta lift of the Maass-Selberg Poincaré series $M^L_{\kappa,\beta,m}(\tau, s)$ defines a real analytic function in $Z \in \mathbb{H}_n \setminus H^L_{\beta,m}$ and a holomorphic function in s for* $\mathrm{Re}(s) > \max(3/2 + k/2 + n/4, 2 + n - k)$, *which is given by*

$$\Phi^{\mathrm{M},L}_{k,\beta,m}(Z, s) = \frac{2\,\Gamma(s + k/2)}{(4\pi|m|)^{s/2+k}} \sum_{\lambda \in L_{\beta,m}} \left(\frac{\overline{(\lambda, Z_L)}}{q(\mathrm{Im}(Z))} \right)^k \left(\frac{|m|}{q(\lambda_Z)} \right)^{s+k/2} \times {}_2F_1\left(s/2 + k/2, s + k/2, s; \frac{m}{q(\lambda_Z)} \right).$$

Here the sum on the right-hand side yields a holomorphic continuation of $\Phi^{\mathrm{M},L}_{k,\beta,m}(Z, s)$ in s to the half-plane $\mathrm{Re}(s) > 1 + n/2$.

Proof. The proof resembles the proof of Theorem 5.1.3. Thus, we only comment on the necessary adaptions. Firstly, we note that we can omit the regularization coming from the additional complex parameter t and simply set $t = 0$, since we do not consider the case $m = 0$. Moreover, we always need to replace the term $e^{-2\pi mv}$ by $v^{-s/2}\mathcal{M}_{\kappa,s/2}(4\pi mv)$, since these are the terms by which the two Poincaré series $U^L_{\kappa,\beta,m}(\tau, s)$ and $M^L_{\kappa,\beta,m}(\tau, s)$ differ.

We start by checking that the integral corresponding to (5.1.4) converges. Indeed, using Corollary 4.2.5 and the fact that $\mathcal{M}_{\kappa,s/2}(4\pi mv) = O(v^{\mathrm{Re}(s)/2-\kappa/2})$ as $v \to 0$ as in (3.6.18) we find

(5.4.2)
$$\int_{\mathcal{R}\setminus\mathcal{F}} \left| v^{s/2+\kappa}\mathcal{M}_{\kappa,s/2}(4\pi mv)e(mu)\overline{\Theta_{L,k,\beta}(\tau, Z)} \right| d\mu(\tau) \leq C \int_0^1 v^{\sigma-k/2-n/4-5/2} dv < \infty$$

for $s = \sigma + it \in \mathbb{C}$ with $\sigma > 3/2 + k/2 + n/4$ and some constant $C > 0$. Now, everything up to equation (5.1.6) works just the same, giving

$$\Phi^{\mathrm{M},L}_{k,\beta,m}(Z, s) = \frac{2}{q(\mathrm{Im}(Z))^k} \int_0^\infty v^{s/2+k-1}\mathcal{M}_{\kappa,s/2}(4\pi mv) \sum_{\lambda \in L_{\beta,m}} \overline{(\lambda, Z_L)}^k e^{-2\pi v q_Z(\lambda)} dv. \tag{5.4.3}$$

Using the estimates given in (3.6.17) and (3.6.18) for the modified M-Whittaker function $\mathcal{M}_{\kappa,s/2}(4\pi mv)$, and the fact that $q(\lambda_Z) \leq q_Z(\lambda)$, we obtain

(5.4.4)
$$\begin{aligned}
&\sum_{\lambda \in L_{\beta,m}} \int_0^\infty \left| v^{s/2+k-1}\mathcal{M}_{\kappa,s/2}(4\pi mv)\overline{(\lambda, Z_L)}^k e^{-2\pi v q_Z(\lambda)} \right| dv \\
&\quad \leq C'(s) \sum_{\lambda \in L_{\beta,m}} q(\lambda_Z)^{k/2} \left(\int_0^1 v^{\sigma+k/2+n/4-3/2} e^{-2\pi v q_Z(\lambda)} dv + \int_1^\infty v^{\sigma/2+k-1} e^{-4\pi v q(\lambda_Z)} dv \right) \\
&\quad \leq C''(s) \Bigg(\gamma(\sigma + k/2 + n/4 - 1/2, 1) \sum_{\lambda \in L_{\beta,m}} q_Z(\lambda)^{1/2-\sigma-n/4} \\
&\qquad\qquad + \Gamma(\sigma/2 + k, 1) \sum_{\lambda \in L_{\beta,m}} q(\lambda_Z)^{-\sigma/2-k/2} \Bigg)
\end{aligned}$$

for $\sigma > 1/2 - k/2 - n/4$. Here $\gamma(s,x)$ and $\Gamma(s,x)$ denote the usual lower and upper incomplete Gamma functions, respectively, and $C'(s)$, $C''(s)$ are positive, real valued functions, which are continuous in s. Now the first remaining sum is dominated by the Epstein zeta function associated to q_Z and the dual lattice L', and thus converges for $\sigma > 3/2 + n/4$. Further, by Lemma 5.1.2 the second remaining sum converges for $\sigma > 2 + n - k$. Altogether this shows that the sum in (5.4.4) converges for

$$\sigma > \max(3/2 + n/4, 2 + n - k),$$

Taking into account the estimate from (5.4.2), we find that the regularized theta lift $\Phi^{\mathrm{M},L}_{k,\beta,m}(Z,s)$ is well-defined for

$$\mathrm{Re}(s) > \max(3/2 + k/2 + n/4, 2 + n - k).$$

Furthermore, the estimate (5.4.4) shows that we may interchange summation and integration in (5.4.3), which yields

$$(5.4.5)\quad \Phi^{\mathrm{M},L}_{k,\beta,m}(Z,s) = \frac{2}{q(\mathrm{Im}(Z))^k} \sum_{\lambda \in L_{\beta,m}} \overline{(\lambda, Z_L)}^k \int_0^\infty v^{s/2+k-1} \mathcal{M}_{\kappa,s/2}(4\pi m v) e^{-2\pi v q_Z(\lambda)} dv.$$

Writing $\mathcal{M}_{\kappa,s/2}(4\pi m v)$ in terms of the confluent hypergeometric function of the first kind $\Phi(a,c;z)$ (see for example [GR07], formula 9.220.2), the remaining integral takes the form

$$\begin{aligned}\int_0^\infty & v^{s/2+k-1}\mathcal{M}_{\kappa,s/2}(4\pi m v) e^{-2\pi v q_Z(\lambda)} dv \\ &= (4\pi|m|)^{s/2-k/2} \int_0^\infty v^{s+k/2-1}\Phi(s/2-k/2, s; 4\pi|m|v) e^{-2\pi v(q_Z(\lambda)+|m|)} dv\end{aligned}$$

Up to the factor $(4\pi|m|)^{s/2-k/2}$ the right-hand side is the Laplace transform of the function

$$v \mapsto v^{s+k/2-1}\Phi(s/2-k/2, s; 4\pi|m|v),$$

evaluated at $2\pi(q_Z(\lambda)+|m|)$. This Laplace transform is for example computed in [PBM86, formula 3.35.1.2], giving

$$\begin{aligned}\int_0^\infty & v^{s+k/2-1}{}_1F_1(s/2-k/2, s; 4\pi|m|v) e^{-2\pi v(q_Z(\lambda)+|m|)} dv \\ &= \frac{\Gamma(s+k/2)}{(2\pi(q_Z(\lambda)-|m|))^{s+k/2}}\, {}_2F_1\left(s/2+k/2, s+k/2, s; \frac{4\pi|m|}{2\pi(|m|-q_Z(\lambda))}\right)\end{aligned}$$

for $\mathrm{Re}(s) > 0$ and $\lambda \in L_{\beta,m}$ with $q(\lambda_Z) > 0$ and $q(\lambda_{Z^\perp}) < 0$. The latter two conditions are always fulfilled since $Z \notin H_{\beta,m}$ implies $q(\lambda_Z) \neq 0$, and $m < 0$ implies $q(\lambda_{Z^\perp}) \neq 0$. Moreover, we note that $q_Z(\lambda) - |m| = 2q(\lambda_Z)$ as $m < 0$. Therefore, we can write (5.4.5) as

$$\begin{aligned}\Phi^{\mathrm{M},L}_{k,\beta,m}(Z,s) = \frac{2\,\Gamma(s+k/2)}{(4\pi|m|)^{s/2+k}} \sum_{\lambda \in L_{\beta,m}} & \left(\frac{\overline{(\lambda, Z_L)}}{q(\mathrm{Im}(Z))}\right)^k \left(\frac{|m|}{q(\lambda_Z)}\right)^{s+k/2} \\ & \times {}_2F_1\left(s/2+k/2, s+k/2, s; \frac{m}{q(\lambda_Z)}\right),\end{aligned}$$

and since

$$\left|\left(\frac{\overline{(\lambda, Z_L)}}{q(\mathrm{Im}(Z))}\right)^k \left(\frac{|m|}{q(\lambda_Z)}\right)^{s+k/2} {}_2F_1\left(s/2+k/2, s+k/2, s; \frac{m}{q(\lambda_Z)}\right)\right| = O\Big(q(\lambda_Z)^{-\operatorname{Re}(s)}\Big)$$

as $q(\lambda_Z) \to 0$ for $\lambda \in L_{\beta,m}$, the remaining sum converges for $\operatorname{Re}(s) > 1 + n/2$ by Lemma 5.1.2. Now it remains to note that if $\operatorname{Re}(s) > \max(3/2 + k/2 + n/4, 2 + n - k)$ then

$$\operatorname{Re}(s) = \frac{2\operatorname{Re}(s)}{3} + \frac{\operatorname{Re}(s)}{3} > \frac{2}{3}\left(\frac{3}{2} + \frac{k}{2} + \frac{n}{4}\right) + \frac{2+n-k}{3} = \frac{5}{3} + \frac{n}{2} > 1 + \frac{n}{2}.$$

This finishes the proof of the theorem. $\square$

For $k = 0$ the hypergeometric function appearing in the previous theorem simplifies considerably, yielding the following special case of Theorem 5.4.2:

Corollary 5.4.3. *Let $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m < 0$. The regularized theta lift of the Maass-Selberg Poincaré series $M^L_{1-n/2,\beta,m}(\tau, s)$ is given by*

$$\Phi^{\mathrm{M},L}_{0,\beta,m}(Z, s) = \frac{2\,|m|^{s/2}\,\Gamma(s)}{\pi^{s/2}} \sum_{\lambda \in L_{\beta,m}} (q_Z(\lambda)^2 - m^2)^{-s/2}$$

for $Z \in \mathbb{H}_n \setminus H^L_{\beta,m}$ and $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1 + n/2$.

Proof. Specializing Theorem 5.4.2 for $k = 0$ we find

$$\Phi^{\mathrm{M},L}_{0,\beta,m}(Z, s) = \frac{2\,\Gamma(s)}{(4\pi|m|)^{s/2}} \sum_{\lambda \in L_{\beta,m}} \left(\frac{|m|}{q(\lambda_Z)}\right)^s {}_2F_1\left(s/2, s, s; \frac{m}{q(\lambda_Z)}\right)$$

for $Z \in \mathbb{H}_n \setminus H^L_{\beta,m}$ and $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1 + n/2$. Here the remaining hypergeometric function simplifies as

$${}_2F_1\left(s/2, s, s; \frac{m}{q(\lambda_Z)}\right) = \left(1 - \frac{m}{q(\lambda_Z)}\right)^{-s/2},$$

see for example [GR07, formula 9.121.1]. Hence, we have shown that

$$\Phi^{\mathrm{M},L}_{0,\beta,m}(Z, s) = \frac{2\,|m|^{s/2}\,\Gamma(s)}{(4\pi)^{s/2}} \sum_{\lambda \in L_{\beta,m}} \Big(q(\lambda_Z)(q(\lambda_Z) - m)\Big)^{-s/2}.$$

Recalling that $2q(\lambda_Z) = q_Z(\lambda) + m$ for $\lambda \in L_{\beta,m}$ we now find

$$q(\lambda_Z)(q(\lambda_Z) - m) = \frac{q_Z(\lambda) + m}{2} \cdot \frac{q_Z(\lambda) - m}{2} = \frac{q_Z(\lambda)^2 - m^2}{4},$$

which proves the claimed statement. $\square$

Let now (V, q) be again the orthogonal space of signature $(2, 2)$ from Section 4.4, and let L be the corresponding even lattice of level N. For $\beta = 0$ and $m = -1$ the previous corollary now takes the following form:

Proposition 5.4.4. *The regularized theta lift of the Poincaré series $M^L_{0,0,-1}(\tau, s)$ for the lattice L of signature $(2,2)$ from Section 4.4 is given by*

$$\Phi^{\mathrm{M},L}_{0,0,-1}(z, z', s) = \frac{2\,\Gamma(s)}{\pi^{s/2}} \sum_{M \in \Gamma_0(N)} \sinh(d_{\mathrm{hyp}}(Mz, z'))^{-s}$$

for $z, z' \in \mathbb{H}$ with $z \neq z'$ modulo $\Gamma_0(N)$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 2$. *Here the right-hand side yields a holomorphic continuation of the lift $\Phi^{\mathrm{M},L}_{0,0,-1}(z, z', s)$ to the half-plane* $\mathrm{Re}(s) > 1$.

Proof. By Corollary 5.4.3 we have

$$\Phi^{\mathrm{M},L}_{0,0,-1}(z, z', s) = \frac{2\,\Gamma(s)}{\pi^{s/2}} \sum_{\lambda \in L_{0,-1}} (q_{z,z'}(\lambda)^2 - 1)^{-s/2}$$

for $z, z' \in \mathbb{H}$ with $z \neq z'$ modulo $\Gamma_0(N)$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 2$. Now, the given statement follows directly from part (b) of Lemma 5.3.2, and since $\cosh(z)^2 - \sinh(z)^2 = 1$ for $z \in \mathbb{C}$. □

Eventually, we present a rigorous version of the idea explained in part (2) of Remark 5.3.8: Given an arbitrary point $w \in \mathbb{H}$, we can realize the elliptic Eisenstein series associated to w as the evaluation of the (regularized) theta lift $\Phi^{\mathrm{M},L}_{0,0,-1}(z, z', s)$ in one of the two variables z and z' at the point w.

Corollary 5.4.5. *Let $w \in \mathbb{H}$. Then*

$$E^{\mathrm{ell}}_w(z, s) = \frac{\pi^{s/2}}{2\,\mathrm{ord}(w)\,\Gamma(s)}\ \Phi^{\mathrm{M},L}_{0,0,-1}(z, w, s)$$

for $z \in \mathbb{H} \setminus \Gamma_0(N)w$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$.

Proof. Since the stabilizer $(\Gamma_0(N))_w$ is always finite with $\mathrm{ord}(w) = |(\Gamma_0(N))_w|$ the statement is a direct consequence of Proposition 5.4.4. □

Remark 5.4.6. Assuming that the theta lift $\Phi^{\mathrm{M},L}_{0,0,-1}(z, w, s)$ has a meromorphic continuation in s to $s = 0$, such that this continuation is holomorphic at $s = 0$, Corollary 5.4.5 would imply that

$$E^{\mathrm{ell}}_w(z, s) = \frac{1}{2\,\mathrm{ord}(w)}\ \Phi^{\mathrm{M},L}_{0,0,-1}(z, w, 0) \cdot s + O(s^2)$$

as $s \to 0$. This could be used to reprove known Kronecker limit formulas for elliptic Eisenstein series in the case of level 1 (compare equation (2.6.7)), and also to obtain new Kronecker limit type formulas for higher levels.

For the sake of completeness we also present Corollary 5.4.3 in the situation of the lattice L of signature $(2,1)$ from Section 4.3. The result is similar to part (a) of Corollary 5.2.2 in the case $m < 0$. In fact, we obtain averaged versions of certain almost elliptic Eisenstein series.

Proposition 5.4.7. *Let $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m < 0$. The regularized theta lift of the Maass-Selberg Poincaré series $M^{L}_{1/2,\beta,m}(\tau, s)$ for the lattice L of signature $(2,1)$ from Section 4.3 is given by*

$$\Phi^{\mathrm{M},L}_{0,\beta,m}(z,s) = \frac{2\,\Gamma(s)}{(4\pi|m|)^{s/2}} \sum_{Q\in\mathcal{Q}_{\beta,4Nm}/\Gamma_0(N)} h_{\tau_Q}(z,s) \tag{5.4.6}$$

for $z \in \mathbb{H} \setminus H_{\beta,4Nm}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 3/2$. Here given $w \in \mathbb{H}$ the associated function $h_w(z,s)$ is defined by

$$h_w(z,s) := \sum_{M\in(\Gamma_0(N))_w\backslash\Gamma_0(N)} \sinh(d_{\mathrm{hyp}}(Mz,w))^{-s}\cosh(d_{\mathrm{hyp}}(Mz,w))^{-s}$$

for $z \in \mathbb{H} \setminus \Gamma_0(N)w$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1/2$. In particular, the right-hand side of equation (5.4.6) yields a holomorphic continuation of the lift $\Phi^{\mathrm{M},L}_{0,\beta,m}(z,s)$ in s to the half-plane $\mathrm{Re}(s) > 1/2$, and the functions $h_w(z,s)$ are modular of weight 0 and level N.

Proof. By Corollary 5.4.3 we have

$$\Phi^{\mathrm{M},L}_{0,\beta,m}(Z,s) = \frac{2\,|m|^{s/2}\,\Gamma(s)}{\pi^{s/2}} \sum_{\lambda\in L_{\beta,m}} (q_Z(\lambda)^2 - m^2)^{-s/2} \tag{5.4.7}$$

for $Z \in \mathbb{H}_1 \setminus H^L_{\beta,m}$ corresponding to $z \in \mathbb{H}$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 3/2$. Firstly, we recall that $H^L_{\beta,m} = H_{\beta,4Nm}$ as in (5.2.1). Moreover, we can identify lattice elements $\lambda \in L_{\beta,m}$ with quadratic forms $Q_\lambda \in \mathcal{Q}_{\beta,4Nm}$ as in (4.3.4). By (5.2.2) we then have

$$q_Z(\lambda)^2 - m^2 = (q_Z(\lambda)+m)(q_Z(\lambda)-m) = 4m^2\,\frac{|Q_\lambda(z,1)|^2}{4Nm\,\mathrm{Im}(z)^2}\left(\frac{|Q_\lambda(z,1)|^2}{4Nm\,\mathrm{Im}(z)^2} - 1\right).$$

Here

$$\frac{|Q_\lambda(z,1)|^2}{4N|m|\,\mathrm{Im}(z)^2} = \sinh(d_{\mathrm{hyp}}(z,\tau_{Q_\lambda}))^2$$

by Lemma 2.5.4, and since $\cosh(z)^2 - \sinh(z)^2 = 1$ for $z \in \mathbb{C}$ we thus find

$$q_Z(\lambda)^2 - m^2 = 4m^2\sinh(d_{\mathrm{hyp}}(z,\tau_{Q_\lambda}))^2\cosh(d_{\mathrm{hyp}}(z,\tau_{Q_\lambda}))^2.$$

Replacing the sum over lattice elements in $L_{\beta,m}$ in (5.4.7) by the corresponding sum over quadratic forms in $\mathcal{Q}_{\beta,4Nm}$, we obtain

$$\Phi^{\mathrm{M},L}_{0,\beta,m}(Z,s) = \frac{2\,\Gamma(s)}{(4\pi|m|)^{s/2}} \sum_{Q\in\mathcal{Q}_{\beta,4Nm}} \sinh(d_{\mathrm{hyp}}(z,\tau_Q))^{-s}\cosh(d_{\mathrm{hyp}}(z,\tau_Q))^{-s}.$$

Recalling that $d_{\mathrm{hyp}}(z,\tau_{Q.M}) = d_{\mathrm{hyp}}(z,M^{-1}\tau_Q) = d_{\mathrm{hyp}}(Mz,\tau_Q)$ for $Q \in \mathcal{Q}_{\beta,4Nm}$ and $M \in \Gamma_0(N)$ we thus obtain the claimed formula for the present theta lift.

It remains to note that given $w \in \mathbb{H}$ the sum defining the function $h_w(z,s)$ converges absolutely and locally uniformly for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1/2$ since it is dominated by the elliptic Eisenstein series $E^{\mathrm{ell}}_w(z, 2\,\mathrm{Re}(s))$, as $\cosh(x) \geq \sinh(x)$ for all $x \in \mathbb{R}$. $\square$

Remark 5.4.8. We note that given $w \in \mathbb{H}$ we could also write the functions $h_w(z,s)$ defined in Proposition 5.4.7 in the form

$$h_w(z,s) = 2^s \sum_{M \in (\Gamma_0(N))_w \backslash \Gamma_0(N)} \sinh(2\, d_{\mathrm{hyp}}(Mz, w))^{-s}$$

for $z \in \mathbb{H} \setminus \Gamma_0(N)w$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1/2$, since $2\sinh(z)\cosh(z) = \sinh(2z)$ for $z \in \mathbb{C}$. This representation emphasises the similarity between the two non-holomorphic functions $h_w(z,s)$ and $E_w^{\mathrm{ell}}(z,2s)$.

6 Meromorphic continuation of Selberg's Poincaré series

In the final two chapters we want to use Corollary 5.2.2 from the previous chapter to obtain Kronecker limit type formulas for averaged hyperbolic, parabolic and elliptic Eisenstein series. Both chapters are based on the work [PSV17]. In particular, the present chapter deals with the content of Section 3 of [PSV17], giving a detailed presentation of the meromorphic continuation of Selberg's Poincaré series of the first kind of weight $1/2$ to the special point $s = 0$. In the next chapter this continuation will be used to evaluate the meromorphic continuation of the regularized theta lift of Selberg's Poincaré series at $s = 0$, which in combination with Corollary 5.2.2 yields the claimed Kronecker limit type formulas.

Since we aim to apply Corollary 5.2.2 we do not work with a general quadratic space (V, q) of signature $(2, n)$, but we restrict to the symmetric space of signature $(2, 1)$ introduced in Section 4.3. More precisely, we let N be a positive integer, and we let V be the vector space of 2×2-matrices with rational entries and trace 0, equipped with the quadratic form $q(X) = -N \det(X)$. In V we fix the even lattice L of level $4N$ from Section 4.3, and we identify the induced generalized upper half-plane $\mathbb{H}_1$ with the usual upper half-plane $\mathbb{H}$.

As in Corollary 5.2.2 we further restrict to the case of weight $\kappa = 1/2$. In order to simplify notation, we thus from now on drop the index κ, and in case no confusion is possible, we also drop the superscript L referring to the currently used lattice. Moreover, we always assume that the fixed integer N is squarefree. Then $\beta = 0$ is the only element of norm 0 in the discriminant form L'/L, which greatly simplifies the presentation. In particular, there is only one non-holomorphic Eisenstein series of weight $1/2$ in the sense of Definition 3.6.1, namely the Eisenstein series $E_0(\tau, s)$.

6.1 Vector valued non-holomorphic Eisenstein series revisited

In the present section we establish the meromorphic continuation of the vector valued non-holomorphic Eisenstein series $E_0(\tau, s)$ of weight $1/2$ for the lattice L via its Fourier expansion, which is given in Proposition 3.6.2. Therefore, we first need to study the Kloosterman zeta functions appearing in the Fourier expansion of $E_0(\tau, s)$, namely the functions $Z(s; 0, 0, \gamma, n)$ for $\gamma \in L'/L$ and $n \in \mathbb{Z} + q(\gamma)$.

We start by translating the results from Section 4 of [BK01] and Section 3 of [BK03] to the present situation. Note that in [BK01] and [BK03] the authors work with the dual representation ρ_L^* of the Weil representation ρ_L, and they assume that their weight κ^* is of the form $2\kappa^* \equiv 3 \bmod 4$ and $\kappa^* > 2$. We can deduce the non-dual case from the dual one, and by doing so the condition $2\kappa^* \equiv 3 \bmod 4$ becomes $2\kappa \equiv 1 \bmod 4$ which $\kappa = 1/2$ clearly satisfies. Further, for the results we use from their work the assumption $\kappa^* > 2$ can indeed be dropped.

Recall that we remarked in Section 3.3 that the Weil representation ρ_L and its associated Kloosterman sum and Kloosterman zeta function do not depend on the complete lattice L but only on its induced discriminant form. In fact, also the Eisenstein series $E_0(\tau, s) = E^L_{1/2,0}(\tau, s)$ does only depend on the discriminant form induced by L. Therefore we can temporarily replace the current lattice (L, q) by another more appropiate lattice (M, r) as long as the corresponding discriminant forms are isomorphic. Recall that we have seen in Section 4.3 that L'/L can be identified with the quotient $\mathbb{Z}/2N\mathbb{Z}$ together with the quadratic form $q(\gamma) = \gamma^2/4N \bmod \mathbb{Z}$. We set

$$M := \mathbb{Z} \qquad \text{and} \qquad r(x) := Nx^2 \tag{6.1.1}$$

for $x \in M$. Then $(M \otimes \mathbb{Q}, r)$ is a quadratic space of signature $(1, 0)$, and M is a lattice in this space with dual lattice given by $M' = \frac{1}{2N}\mathbb{Z}$. Hence the discriminant form induced by M can also be identified with the group $\mathbb{Z}/2N\mathbb{Z}$ where $r(\gamma) = \gamma^2/4N$ for $\gamma \in \mathbb{Z}/2N\mathbb{Z}$. In particular, the discriminant forms M'/M and L'/L are indeed isomorphic, and we thus have

$$H^L_{c,1/2}(\beta, m, \gamma, n) = H^M_{c,1/2}(\beta, m, \gamma, n) \tag{6.1.2}$$

and

$$Z^L_{1/2}(s; \beta, m, \gamma, n) = Z^M_{1/2}(s; \beta, m, \gamma, n) \tag{6.1.3}$$

for $\beta, \gamma \in \mathbb{Z}/2N\mathbb{Z}$, $m \in \mathbb{Z}+\beta^2/4N$, $n \in \mathbb{Z}+\gamma^2/4N$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 3/4$. Moreover, the corresponding Eisenstein series agree, i.e., we have $E^L_{1/2,0}(\tau, s) = E^M_{1/2,0}(\tau, s)$.

Next, we import some notation and results from Section 3 of [BK03] using the lattice (M, r) introduced above. For the sake of clarity we will again use indices to denote the current weight and lattice for the notation of Kloosterman sums and Kloosterman zeta functions, until we change back to our usual lattice (L, q).

Firstly, we define the Kloosterman sum $H^{M,*}_{c,3/2}(\beta, m, \gamma, n)$ associated to the dual Weil representation ρ^*_M as in equation (3.6) of [BK03] by

$$H^{M,*}_{c,3/2}(\beta, m, \gamma, n) := \frac{e(-3\,\mathrm{sign}(c)/8)}{|c|} \sum_{\substack{d \in (\mathbb{Z}/c\mathbb{Z})^* \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})}} \rho_{\beta,\gamma}\left(\widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}\right) e\left(\frac{ma + nd}{c}\right) \tag{6.1.4}$$

for $c \neq 0$, $\beta, \gamma \in \mathbb{Z}/2N\mathbb{Z}$ and $m \in \mathbb{Z} - \beta^2/4N$, $n \in \mathbb{Z} - \gamma^2/4N$. Correspondingly, we define the Kloosterman zeta function $Z^{M,*}_{3/2}(s; \beta, m, \gamma, n)$ associated to ρ^*_M by

$$Z^{M,*}_{3/2}(s; \beta, m, \gamma, n) := \sum_{c \in \mathbb{Z}\setminus\{0\}} |c|^{-1/2-2s} H^{M,*}_{c,1/2}(\beta, m, \gamma, n) \tag{6.1.5}$$

for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1/4$, $\beta, \gamma \in \mathbb{Z}/2N\mathbb{Z}$ and $m \in \mathbb{Z} - \beta^2/4N$, $n \in \mathbb{Z} - \gamma^2/4N$. These are the dual versions of the Kloosterman sum and the Kloosterman zeta function defined in Section 3.3. In particular, the two zeta functions are related by the following identity:

Lemma 6.1.1. *Let* $\beta, \gamma \in \mathbb{Z}/2N\mathbb{Z}$ *and* $m \in \mathbb{Z} + \beta^2/4N$, $n \in \mathbb{Z} + \gamma^2/4N$. *Then*

$$Z_{1/2}^{M}(s; \beta, m, \gamma, n) = -Z_{3/2}^{M,*}(s - 1/2; \beta, -m, \gamma, -n)$$

for $s \in \mathbb{C}$ *with* $\mathrm{Re}(s) > 3/4$.

Proof. Let $c \neq 0$. By Lemma 1.15 in [Bru02] the dual and the non-dual Kloosterman sum are related by the identity

$$H_{c,1/2}^{M}(\beta, m, \gamma, n) = -H_{-c,3/2}^{M,*}(\gamma, -n, \beta, -m). \tag{6.1.6}$$

Moreover, we have the relations

$$\overline{H_{c,1/2}^{M}(\beta, m, \gamma, n)} = H_{-c,1/2}^{M}(\beta, m, \gamma, n), \quad \overline{H_{c,1/2}^{M,*}(\beta, -m, \gamma, -n)} = H_{-c,1/2}^{M,*}(\gamma, -n, \beta, -m),$$

where the first one is given as Lemma 1.13 in [Bru02] and the second one can easily be proven directly. Combining all of these we find that

$$H_{c,1/2}^{M}(\beta, m, \gamma, n) = -H_{-c,3/2}^{M,*}(\beta, -m, \gamma, -n). \tag{6.1.7}$$

Plugging this into the definition of the dual and the non-dual Kloosterman zeta function we obtain the claimed statement. □

Given $\gamma \in \mathbb{Z}/2N\mathbb{Z}$ and $n \in \mathbb{Z}+\gamma^2/4N$ we now introduce the L-series $L_{\gamma,-n}(s)$ associated to the representation number $N_{\gamma,-n}(a)$ defined in equation (3.14) of [BK03], i.e., we set

$$L_{\gamma,-n}(s) := \sum_{a=1}^{\infty} \frac{N_{\gamma,-n}(a)}{a^s} \tag{6.1.8}$$

for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 2$, where

$$N_{\gamma,-n}(a) := \#\left\{x \in \mathbb{Z}/a\mathbb{Z} \colon N\left(x - \frac{\gamma}{2N}\right)^2 - n \equiv 0 \mod a\right\}$$

for $a \in \mathbb{N}$. According to Proposition 4 in [BK01] (see also equation (3.15) in [BK03]) this L-series is closely connected to the Kloosterman zeta function we are interested in. More precisely, we can use our Lemma 6.1.1 to translate their result, namely Proposition 4 in [BK01], to the present non-dual setting of weight $\kappa = 1/2$, giving

$$Z_{1/2}^{M}(s; 0, 0, \gamma, n) = \frac{2}{\sqrt{2N}\ \zeta(2s)}\ L_{\gamma,-n}(2s) \tag{6.1.9}$$

for $\gamma \in \mathbb{Z}/2N\mathbb{Z}$, $n \in \mathbb{Z} + \gamma^2/4N$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. Since the representation number $N_{\gamma,-n}(a)$ is multiplicative in a its associated L-series admits an Euler product expansion, i.e., we have

$$L_{\gamma,-n}(s) = \prod_{p \text{ prime}} \sum_{\nu=0}^{\infty} \frac{N_{\gamma,-n}(p^\nu)}{p^{\nu s}}$$

for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 2$. Here the product on the right runs over all primes p. Given such a prime p we define the corresponding local Euler factor

$$L^{(p)}_{\gamma,-n}(X) := 1 + \sum_{\nu=1}^{\infty} \left(N_{\gamma,-n}(p^\nu) - N_{\gamma,-n}(p^{\nu-1})\right) X^\nu. \tag{6.1.10}$$

Then $\sum_{\nu=0}^{\infty} N_{\gamma,-n}(p^\nu)X^\nu = (1-X)^{-1}L^{(p)}_{\gamma,-n}(X)$ and thus

$$L_{\gamma,-n}(s) = \zeta(s) \prod_{p \text{ prime}} L^{(p)}_{\gamma,-n}(p^{-s}) \tag{6.1.11}$$

for $\gamma \in \mathbb{Z}/2N\mathbb{Z}$, $n \in \mathbb{Z} + \gamma^2/4N$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 2$ (see equation (21) in [BK01], or equation (3.19) in [BK03]). In order to evaluate the local Euler factor $L^{(p)}_{\gamma,-n}(p^{-s})$ we first need to study the corresponding representation numbers for prime powers.

Lemma 6.1.2. *Let $\gamma \in \mathbb{Z}/2N\mathbb{Z}$, $n \in \mathbb{Z} + \gamma^2/4N$ and let p be a prime.*

(a) If $n = \gamma = 0$ then

$$N_{0,0}(p^\nu) = \begin{cases} p^{\lfloor \nu/2 \rfloor}, & \text{if } (p,N) = 1, \\ p^{\lceil \nu/2 \rceil}, & \text{if } p \text{ divides } N. \end{cases}$$

for all $\nu \in \mathbb{N}$.

(b) If $n \neq 0$ set $w_p := 1 + 2\nu_p(2n\,\mathrm{ord}(\gamma))$. Then

$$N_{\gamma,-n}(p^\nu) = N_{\gamma,-n}(p^{\nu-1}).$$

for all $\nu \in \mathbb{N}$ with $\nu > w_p$.

(c) If $n \neq 0$ and $(p, 2N) = 1$ then

$$N_{\gamma,-n}(p^\nu) = \begin{cases} p^{\lfloor \nu/2 \rfloor}, & \text{if } \nu \leq \nu_p(4Nn), \\ (\chi_{D_0}(p) + \chi_{D_0}(p)^2)p^{\nu_p(f)}, & \text{if } \nu > \nu_p(4Nn). \end{cases}$$

Here D_0 is the unique fundamental discriminant determined by $D_0 f^2 = 4Nn\,\mathrm{ord}(\gamma)^2$ for some $f \in \mathbb{N}$, and χ_{D_0} is the Kronecker symbol associated to D_0, i.e., $\chi_{D_0} = (\frac{D_0}{\cdot})$.

Proof. We first recall that $n = 0$ implies $\gamma = 0$ since we assume that N is squarefree. In this case

$$N_{0,0}(p^\nu) = \#\{x \in \mathbb{Z}/p^\nu\mathbb{Z} \colon Nx^2 \equiv 0 \mod p^\nu\}$$

for $\nu \in \mathbb{N}$. Distinguishing the cases $(p, N) = 1$ and $(p, N) = p$ we obtain part (a). Furthermore, part (b) is given as Lemma 5 in [BK01].

We now assume that $n \neq 0$ and $(p, 2N) = 1$. Let $\mathcal{D} = 4Nn_0\,\mathrm{ord}(\gamma)^2$ with $n_0 \in \mathbb{Q}$ such that $n = n_0 g^2$ for some $g \in \mathbb{N}$ with $(g, 4N) = 1$ and $\nu_\ell(n_0) \in \{0, 1\}$ for all primes ℓ coprime to $2N$. Then Theorem 6 in [BK01] (which is essentially a reformulation of the Hilfssatz 16 in [Sie35]) yields that

$$N_{\gamma,-n}(p^\nu) = (\chi_{\mathcal{D}}(p) + \chi_{\mathcal{D}}(p)^2)p^{\nu_p(g)}$$

for $\nu > \nu_p(4Nn)$. Let D_0 and f be defined as in the statement, i.e., let D_0 be the unique fundamental discriminant determined by $D_0 f^2 = 4Nn \operatorname{ord}(\gamma)^2$ for some $f \in \mathbb{N}$. Then g divides f and $\mathcal{D} = D_0(f/g)^2$. More precisely, g is the part of f which is coprime to $2N$, i.e.,

$$g = \prod_{\substack{\ell \text{ prime} \\ \ell \mid f, (\ell, 2N)=1}} \ell^{\nu_\ell(f)}.$$

In particular, we have $\nu_p(g) = \nu_p(f)$ as $(p, 2N) = 1$, and

$$\chi_{\mathcal{D}}(p) = \left(\frac{\mathcal{D}}{p}\right) = \left(\frac{D_0}{p}\right)\left(\frac{f/g}{p}\right)^2 = \chi_{D_0}(p)$$

since $(p, f/g) = 1$. This proves the case $\nu > \nu_p(4Nn)$. In order to also compute $N_{\gamma,-n}(p^\nu)$ for $\nu \le \nu_p(4Nn)$ recall that

$$N_{\gamma,-n}(p^\nu) = \#\{x \in \mathbb{Z}/p^\nu\mathbb{Z} \colon N(x - \gamma/2N)^2 \equiv n \mod p^\nu\}.$$

As we assume that p and $2N$ are coprime we have $N(x-\gamma/2N)^2 \equiv n \bmod p^\nu$ if and only if $(2Nx - \gamma)^2 \equiv 4Nn \bmod p^\nu$. Using the substitution $y = 2Nx - \gamma$ we find

$$N_{\gamma,-n}(p^\nu) = \#\{y \in \mathbb{Z}/p^\nu\mathbb{Z} \colon y^2 \equiv 4Nn \mod p^\nu\}.$$

Now if $\nu \le \nu_p(4Nn)$ then $4Nn \equiv 0 \bmod p^\nu$ and thus the above congruence is satisfied if and only if y is of the form $p^{\lceil \nu/2 \rceil}\tilde{y}$ with $\tilde{y} \in \mathbb{Z}/p^{\lfloor \nu/2 \rfloor}\mathbb{Z}$. So $N_{\gamma,-n}(p^\nu) = p^{\lfloor \nu/2 \rfloor}$ in this case. This finishes the proof of part (c). $\square$

Corollary 6.1.3. *Let $\gamma \in \mathbb{Z}/2N\mathbb{Z}$, $n \in \mathbb{Z} + \gamma^2/4N$ and let p be a prime.*

(a) If $n = \gamma = 0$ then

$$L_{0,0}^{(p)}(X) = \frac{(1 + (p,N)X)(1 - X)}{1 - pX^2}.$$

(b) If $n \neq 0$ then

$$L_{\gamma,-n}^{(p)}(X) = 1 + \sum_{\nu=1}^{w_p} \left(N_{\gamma,-n}(p^\nu) - N_{\gamma,-n}(p^{\nu-1})\right) X^\nu.$$

with $w_p = 1 + 2\nu_p(2n \operatorname{ord}(\gamma))$ as in Lemma 6.1.2.

(c) If $n \neq 0$ and $(p, 2N) = 1$ then

$$L_{\gamma,-n}^{(p)}(X) = \frac{1 - X^2}{1 - \chi_{D_0}(p)X}\left(\sum_{j=0}^{\nu_p(f)} (pX^2)^j - \chi_{D_0}(p)X \sum_{j=0}^{\nu_p(f)-1} (pX^2)^j\right).$$

Here χ_{D_0} and $f \in \mathbb{N}$ are defined as in part (c) of Lemma 6.1.2. If $(p, 4Nn) = 1$ then $(p, D_0) = 1$ and the above formula simplifies to

$$L_{\gamma,-n}^{(p)}(X) = \frac{1 - X^2}{1 - \chi_{D_0}(p)X}.$$

Proof. Let $n = \gamma = 0$ and $(p, N) = 1$. Then

$$L_{0,0}^{(p)}(X) = 1 + \sum_{\nu \in 2\mathbb{N}} \left(p^{\nu/2} - p^{\nu/2-1}\right) X^{\nu} = 1 + \left(1 - \frac{1}{p}\right) \sum_{j=1}^{\infty} (pX^2)^j$$

by part (a) of Lemma 6.1.2. Since $\sum_{j=1}^{\infty}(pX^2)^j = (1-pX^2)^{-1} - 1$ we obtain

$$L_{0,0}^{(p)}(X) = 1 + \left(1 - \frac{1}{p}\right) \frac{pX^2}{1-pX^2} = \frac{1-X^2}{1-pX^2}$$

as claimed. The case $(p, N) = p$ follows analogously. This proves part (a). Further, part (b) is a direct consequence of Lemma 6.1.2, (b).

We now assume that $n \neq 0$ and $(p, 2N) = 1$. Let D_0 be the fundamental discriminant determined by $D_0 f^2 = 4Nn \operatorname{ord}(\gamma)^2$ for some $f \in \mathbb{N}$. Then

$$\nu_p(4Nn) = \nu_p(4Nn \operatorname{ord}(\gamma)^2) = \nu_p(D_0 f^2) = \nu_p(D_0) + 2\nu_p(f) \tag{6.1.12}$$

as $\operatorname{ord}(\gamma) \mid 2N$ and $(p, 2N) = 1$. In other words, $\nu_p(4Nn)$ is even if and only if p divides D_0, and $\lfloor \nu_p(4Nn)/2 \rfloor = \nu_p(f)$. Thus, using part (c) of Lemma 6.1.2 we see that in the sum defining the local Euler factor $L_{\gamma,-n}^{(p)}(X)$ the summands vanish if ν is odd or if $\nu > \nu_p(4Nn) + 1$, i.e., we find

$$\begin{aligned} L_{\gamma,-n}^{(p)}(X) &= 1 + \left(1 - \frac{1}{p}\right) \sum_{j=1}^{\nu_p(f)} (pX^2)^j + \left((\chi_{D_0}(p) + \chi_{D_0}(p)^2) p^{\nu_p(f)} - p^{\nu_p(f)}\right) X^{\nu_p(4Nn)+1} \\ &= \sum_{j=0}^{\nu_p(f)} (pX^2)^j - X^2 \sum_{j=0}^{\nu_p(f)-1} (pX^2)^j + \left(\chi_{D_0}(p) + \chi_{D_0}(p)^2 - 1\right) X^{1+\nu_p(D_0)} (pX^2)^{\nu_p(f)}. \end{aligned}$$

It is easy to check that

$$1 + \left(\chi_{D_0}(p)^2 + \chi_{D_0}(p) - 1\right) X^{1+\nu_p(D_0)} = \frac{1-X^2}{1-\chi_{D_0}(p)X},$$

which yields

$$L_{\gamma,-n}^{(p)}(X) = (1-X^2) \sum_{j=0}^{\nu_p(f)-1} (pX^2)^j + \frac{1-X^2}{1-\chi_{D_0}(p)X} (pX^2)^{\nu_p(f)}.$$

It remains to note that if $(p, 4Nn) = 1$ then $\nu_p(D_0) = \nu_p(f) = 0$ by (6.1.12) since $4Nn$, D_0 and f are all integers. $\square$

We can now use the previous Corollary to evaluate the Euler product in (6.1.11). Using equation (6.1.9) we thus obtain formulas for Kloosterman zeta functions. Moreover, we now turn back to our lattice L of signature $(2, 1)$ introduced in Section 4.3. Therefore, we again drop the indices referring to the weight and the lattice. However, we always understand $\gamma \in L'/L$ as an element of $\mathbb{Z}/2N\mathbb{Z}$ with $q(\gamma) = \gamma^2/4N \bmod \mathbb{Z}$.

Further, we quickly recall that given a primitive Dirichlet character χ of modulus m, the Dirichlet L-function associated to the character χ is given by

$$L(\chi, s) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \tag{6.1.13}$$

for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. It is well-known that $L(\chi, s)$ has a meromorphic continuation in s to all of $\mathbb{C}$, which is holomorphic everywhere if χ is not the principal character, i.e., if $m > 1$. Otherwise, i.e, if $m = 1$ then $L(\chi, s) = \zeta(s)$ is simply the Riemann zeta function, whose meromorphic continuation to $\mathbb{C}$ is holomorphic up to a simple pole at $s = 1$.

Moreover, given a fundamental discriminant $D \in \mathbb{Z}$ we will always denote the Kronecker symbol associated to D by $\chi_D := (\frac{D}{\cdot})$, as before. Clearly χ_D is a primitive Dirichlet character of modulus $|D|$.

Proposition 6.1.4. *Let $\gamma \in L'/L$ and $n \in \mathbb{Z} + q(\gamma)$.*

(a) If $n = \gamma = 0$ then

$$Z(s; 0, 0, 0, 0) = \frac{2}{\sqrt{2N}} \frac{\zeta(4s-1)}{\zeta(4s)} \prod_{p \mid N} \frac{1 + p^{1-2s}}{1 + p^{-2s}}$$

for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. The product on the right-hand side runs over all primes p dividing N.

(b) If $n \neq 0$ then

$$Z(s; 0, 0, \gamma, n) = \frac{2}{\sqrt{2N}\,\zeta(4s)} L(\chi_{D_0}, 2s)\, \sigma^{(0)}_{\gamma,-n}(2s)\, \sigma^{(1)}_{\gamma,-n}(2s)$$

for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. Here D_0 is the unique fundamental discriminant determined by $D_0 f^2 = 4Nn\,\mathrm{ord}(\gamma)^2$ for some $f \in \mathbb{N}$, and $\sigma^{(0)}_{\gamma,-n}(s)$ and $\sigma^{(1)}_{\gamma,-n}(s)$ are the generalized divisor sums defined by

$$\sigma^{(0)}_{\gamma,-n}(s) := \prod_{\substack{p \mid 4Nn \\ (p,2N)=1}} \left(\sigma_{1-2s}\left(p^{\nu_p(f)}\right) - \chi_{D_0}(p) p^{-s} \sigma_{1-2s}\left(p^{\nu_p(f)-1}\right) \right),$$

$$\sigma^{(1)}_{\gamma,-n}(s) := \prod_{p \mid 2N} \frac{1 - \chi_{D_0}(p) p^{-s}}{1 - p^{-2s}} L^{(p)}_{\gamma,-n}(p^{-s}).$$

The two products on the right-hand side run over all primes p dividing $4Nn$ with $(p, 2N) = 1$ or $(p, 2N) = p$, respectively. Moreover, $\sigma_s(m)$ denotes the usual divisor sum $\sigma_s(m) = \sum_{d \mid m} d^s$ defined for $s \in \mathbb{C}$ and $m \in \mathbb{N}$.

Proof. Let $n = \gamma = 0$. Then

$$L_{0,0}(s) = \zeta(s) \left(\prod_{(p,N)=1} \frac{1 - p^{-2s}}{1 - p^{1-2s}} \right) \left(\prod_{p \mid N} \frac{(1 + p^{1-s})(1 - p^{-s})}{1 - p^{1-2s}} \right)$$

by part (a) of Corollary 6.1.3. Since $\zeta(s) = \prod_p (1-p^{-s})^{-1}$ we obtain

$$L_{0,0}(s) = \frac{\zeta(s)\zeta(2s-1)}{\zeta(2s)} \prod_{p\,|\,N} \frac{1+p^{1-s}}{1+p^{-s}}.$$

Analogously we find

$$\begin{aligned} L_{\gamma,-n}(s) &= \zeta(s) \left(\prod_{(p,4Nn)=1} \frac{1-p^{-2s}}{1-\chi_{D_0}(p)p^{-s}} \right) \left(\prod_{p\,|\,2N} L^{(p)}_{\gamma,-n}(p^{-s}) \right) \\ &\quad \times \left(\prod_{\substack{p\,|\,4Nn \\ (p,2N)=1}} \frac{1-p^{-2s}}{1-\chi_{D_0}(p)p^{-s}} \left(\sigma_{1-2s}\left(p^{\nu_p(f)}\right) - \chi_{D_0}(p)p^{-s}\sigma_{1-2s}\left(p^{\nu_p(f)-1}\right)\right) \right) \\ &= \frac{\zeta(s)}{\zeta(2s)} L(\chi_{D_0}, s)\, \sigma^{(0)}_{\gamma,-n}(s)\, \sigma^{(1)}_{\gamma,-n}(s) \end{aligned}$$

if $n \neq 0$ using part (c) of Corollary 6.1.3 and the well-known Euler product expansion of the Dirichlet L-function associated to χ_{D_0}, namely $L(\chi_{D_0}, s) = \prod_p (1-\chi_{D_0}(p)p^{-s})^{-1}$. Applying these formulas to the identity given in equation (6.1.9), and recalling that

$$Z^L_{1/2}(s;0,0,\gamma,n) = Z^M_{1/2}(s;0,0,\gamma,n)$$

by (6.1.3), we obtain the claimed statement. □

We note that part (a) of the previous proposition is essentially given as equation (3.15) in [Ara90]. Moreover, part (b) is a reformulation of Theorem 7 in [BK01]. However, we have used the ideas from Example 10 in [BK01] to further split the remaining product over primes p dividing the discriminant $4Nn$ into an in general large product $\sigma^{(0)}_{\gamma,-n}(s)$ which is holomorphic on all of $\mathbb{C}$, and into an in general small product $\sigma^{(1)}_{\gamma,-n}(s)$ which might still have poles on the imaginary axis $\mathrm{Re}(s) = 0$. In other words, up to normalization in s we have

$$\sigma_{\gamma,-n}(s) = \sigma^{(0)}_{\gamma,-n}(s) \cdot \sigma^{(1)}_{\gamma,-n}(s),$$

where $\sigma_{\gamma,-n}(s)$ is the generalized divisor sum defined in [BK03], equation (3.28).

Theorem 6.1.5. *Let $\gamma \in L'/L$ and $n \in \mathbb{Z} + q(\gamma)$.*

(a) The function

$$\left(\prod_{p\,|\,2N} (1-p^{-4s}) \right) \zeta(4s)\, Z(s;0,0,\gamma,n)$$

admits a meromorphic continuation in s to $\mathbb{C}$ which is holomorphic up to a possible simple pole at $s = 1/2$ occurring if and only if $4Nn$ is a non-negative square.

(b) In particular, the Kloosterman zeta function $Z(s;0,0,\gamma,n)$ admits a meromorphic continuation in s to $\mathbb{C}$ which vanishes at $s = 1/4$, and has at most a simple pole at $s = 1/2$ occurring if and only if $4Nn$ is a non-negative square.

Proof. If $n = \gamma = 0$ then part (a) follows directly from part (a) of Proposition 6.1.4 and the trivial fact that $1 - p^{-4s} = (1 + p^{-2s})(1 - p^{-2s})$.

For $n \neq 0$ let D_0 be the fundamental discriminant with $D_0 f^2 = 4Nn \operatorname{ord}(\gamma)^2$ for some $f \in \mathbb{N}$. As recalled before the Dirichlet L-function $L(\chi_{D_0}, s)$ has a meromorphic continuation in s to all of $\mathbb{C}$, which is holomorphic up to a possible simple pole at $s = 1$, occurring if and only if $D_0 = 1$. Now part (a) follows from part (b) of Proposition 6.1.4 since $D_0 = 1$ if and only if $4Nn$ is a positive square. Here we also have to note that if $D_0 = 1$ the pole coming from the Dirichlet L-function cannot be compensated for by the remaining product

$$\left(\prod_{p \mid 2N} (1 - p^{-4s})\right) \frac{2}{\sqrt{2N}}\, \sigma^{(0)}_{\gamma,-n}(2s)\, \sigma^{(1)}_{\gamma,-n}(2s) \tag{6.1.14}$$

since one easily checks that (6.1.14) evaluates to a positive real number at $s = 1/2$.

Finally, part (a) directly implies part (b) of the present corollary. In particular, the possible pole at $s = 1/2$ from part (a) carries over to part (b), and the pole of the Riemann zeta function $\zeta(4s)$ at $s = 1/4$ forces the corresponding Kloosterman zeta function $Z(s; 0, 0, \gamma, n)$ to vanish at this point. □

We also need to study the asymptotic behaviour of the meromorphic continuation of the Kloosterman zeta function in n. In order to do so we first study the behaviour of the special functions appearing in part (b) of Proposition 6.1.4, namely the behaviour of the generalized divisor sums $\sigma^{(0)}_{\gamma,-n}(s)$ and $\sigma^{(1)}_{\gamma,-n}(s)$ in n, and the behaviour of the Dirichlet L-function $L(\chi_{D_0}, s)$ in D_0.

Lemma 6.1.6.

(a) *Let $\gamma \in L'/L$ and let $\Omega \subseteq \mathbb{C}$ be compact. Then we find $C > 0$ such that*

$$\left|\sigma^{(0)}_{\gamma,-n}(s)\right| \leq C\,|n|^{5/2 + 2\ln(2) - \sigma_0}$$

for all $n \in \mathbb{Z} + q(\gamma)$ with $n \neq 0$ and all $s \in \Omega$. Here $\sigma_0 = \min(0, \min_{s\in\Omega} \operatorname{Re}(s))$, and the constant C depends on γ and Ω, but not on n or s.

(b) *Let $\gamma \in L'/L$ and let $\Omega \subseteq \{s \in \mathbb{C} \colon \prod_{p \mid 2N}(1 - p^{-2s}) \neq 0\}$ be compact. Then we find $C > 0$ such that*

$$\left|\sigma^{(1)}_{\gamma,-n}(s)\right| \leq C\,|n|^{\omega(2N)(3 - 2\sigma_1)}$$

for all $n \in \mathbb{Z} + q(\gamma)$ with $n \neq 0$ and all $s \in \Omega$. Here $\sigma_1 = \min(1, \min_{s\in\Omega} \operatorname{Re}(s))$, and the constant C depends on γ and Ω, but not on n or s. Further, $\omega(2N)$ denotes the number of primes dividing $2N$.

Proof. Let $\gamma \in L'/L$, $n \in \mathbb{Z} + q(\gamma)$ with $n \neq 0$ and let D_0 and f be defined as in part (c) of Lemma 6.1.2, i.e., let D_0 be the unique fundamental discriminant determined by $D_0 f^2 = 4Nn \operatorname{ord}(\gamma)^2$ for some $f \in \mathbb{N}$.

We start with part (a). Let $\Omega \subseteq \mathbb{C}$ be compact, and set $\sigma := \min_{s\in\Omega} \operatorname{Re}(s)$ and $\sigma_0 := \min(0, \sigma)$. Further, let p be a prime dividing $4Nn$ with $(p, 2N) = 1$. Then

$$\begin{aligned}\left|\sigma_{1-2s}\left(p^{\nu_p(f)}\right) - \chi_{D_0}(p)p^{-s}\sigma_{1-2s}\left(p^{\nu_p(f)-1}\right)\right| &\le (1+p^{-\sigma})\sum_{j=0}^{\nu_p(f)-1} p^{(1-2\sigma)j} + p^{(1-2\sigma)\nu_p(F)}\\ &\le (1+2\nu_p(f))p^{(1-2\sigma_0)\nu_p(f)}\end{aligned}$$

for all $s \in \Omega$, and thus

$$\left|\sigma^{(0)}_{\gamma,-n}(s)\right| \le \Bigg(\prod_{\substack{p\,|\,4Nn\\(p,2N)=1}} (1+2\nu_p(f))\Bigg)\Bigg(\prod_{\substack{p\,|\,4Nn\\(p,2N)=1}} p^{\nu_p(f)}\Bigg)^{1-2\sigma_0}.$$

Recall from equation (6.1.12) that $2\nu_p(f) \le \nu_p(4Nn)$ as $(p, 2N) = 1$. Hence we find

$$1 + 2\nu_p(f) \le 2\nu_p(4Nn) \le 2\,\frac{\ln|4Nn|}{\ln(p)} \le 2\,\ln|4Nn|$$

for p dividing $4Nn$ but not $2N$, yielding

$$\left|\sigma^{(0)}_{\gamma,-n}(s)\right| \le \Bigg(\prod_{\substack{p\,|\,4Nn\\(p,2N)=1}} 2\ln|4Nn|\Bigg)\Bigg(\prod_{\substack{p\,|\,4Nn\\(p,2N)=1}} p^{\nu_p(4Nn)}\Bigg)^{1/2-\sigma_0}$$

for $s \in \Omega$. We can now drop the condition $(p, 2N) = 1$ in both products to obtain

$$\left|\sigma^{(0)}_{\gamma,-n}(s)\right| \le (2\ln|4Nn|)^{\omega(4Nn)}\ |4Nn|^{1/2-\sigma_0}. \tag{6.1.15}$$

Here $\omega(m)$ denotes the number of primes dividing m, i.e.,

$$\omega(m) := \#\{p\,|\,m\colon p \text{ prime}\} \tag{6.1.16}$$

for $m \in \mathbb{Z}$ with $m \ne 0$, and by [Rob83, Theorem 11] we have

$$\omega(m) \le 2\,\frac{\ln|m|}{\ln(\ln|m|)} \tag{6.1.17}$$

for all $m \in \mathbb{Z}$ with $|m| \ge 3$. Now part (a) of the lemma follows by applying the estimate in (6.1.17) to the function $\omega(4Nn)$ in (6.1.15).

Next we prove part (b). Let $\Omega \subseteq \{s \in \mathbb{C}\colon \prod_{p\,|\,2N}(1-p^{-2s}) \ne 0\}$ be compact, and set $\sigma := \min_{s\in\Omega} \operatorname{Re}(s)$ and $\sigma_1 := \min(1, \sigma)$. Further, let p be a prime dividing $2N$. By Corollary 6.1.3, part (b), we have

$$\left|L^{(p)}_{\gamma,-n}(p^{-s})\right| \le \sum_{\nu=0}^{w_p} p^{(1-\sigma)\nu} \le (1+w_p)p^{(1-\sigma_1)w_p}$$

for all $s \in \Omega$ as $|N_{\gamma,-n}(a)| \le a$ for $a \in \mathbb{N}$. Here $w_p = 1 + 2\nu_p(2n\operatorname{ord}(\gamma))$. Since $2n\operatorname{ord}(\gamma)$ is an integer we have

$$w_p = 1 + 2\nu_p(2n\operatorname{ord}(\gamma)) \le 1 + 2\,\frac{\ln|2n\operatorname{ord}(\gamma)|}{\ln(p)} = \frac{\ln(4n^2\operatorname{ord}(\gamma)^2p)}{\ln(p)},$$

and thus

$$\left|L^{(p)}_{\gamma,-n}(p^{-s})\right| \le \frac{2\ln|2n\,\mathrm{ord}(\gamma)p|}{\ln(p)}\,\left(4n^2\,\mathrm{ord}(\gamma)^2p\right)^{1-\sigma_1} \le 2p^{2-\sigma_1}|2n\,\mathrm{ord}(\gamma)|^{3-2\sigma_1}$$

for all $s\in\Omega$. Therefore we obtain

$$\left|\sigma^{(1)}_{\gamma,-n}(s)\right| \le 2^{\omega(2N)}(2N)^{2-\sigma_1}|2n\,\mathrm{ord}(\gamma)|^{\omega(2N)(3-2\sigma_1)}\prod_{p\,|\,2N}\left(\sup_{s\in\Omega}\frac{1+p^{-\sigma}}{|1-p^{-2s}|}\right).$$

Here $\omega(2N)$ denotes the number of prime divisors of $2N$ as in (6.1.16). Further, the supremum on the right exists since the fraction is continuous in s, as Ω is compact and as $p^{-2s}\neq 1$ for all $s\in\Omega$. This finishes the proof of (b). $\square$

Lemma 6.1.7. *Let $\Omega\subseteq\mathbb{C}$ be compact and let χ be a primitive Dirichlet character of modulus m with $m>1$. Then we find $C>0$ such that*

$$|L(\chi,s)|\le C\,m^{3/2}(1+m^{1/2-\sigma})$$

for all $s\in\Omega$. Here $\sigma:=\min_{s\in\Omega}\mathrm{Re}(s)$, and the constant C depends on Ω, but not on χ or s.

Proof. Let $q=0$ if $\chi(-1)=1$, and $q=1$ otherwise. By the considerations in [Neu99, Section 7.2] the meromorphic continuation of the Dirichlet L-function $L(\chi,s)$ can be written as

(6.1.18)
$$L(\chi,s)=\left(\frac{\pi}{m}\right)^{s/2+q/2}\frac{1}{\Gamma(s/2+q/2)}\int_1^\infty\left(\theta(\chi,iy)y^{s/2+q/2}+W(\chi)\theta(\overline{\chi},iy)y^{q/2+1/2-s/2}\right)\frac{dy}{y}$$

for all $s\in\mathbb{C}$. Here the theta function $\theta(\chi,z)$ associated to the character χ is given by

$$\theta(\chi,z):=\sum_{n=1}^{\infty}\chi(n)n^qe^{\pi in^2z/m}$$

for $z\in\mathbb{H}$, and $W(\chi)=i^{-q}m^{-1/2}\tau(\chi)$ with $\tau(\chi)=\sum_{j=1}^{m-1}\chi(j)e(j/m)$ being the usual Gauss sum associated to the character χ. In particular, $|W(\chi)|=1$.

Now, let $\Omega\subseteq\mathbb{C}$ be compact. By (6.1.18) we find $C>0$ such that

$$|L(\chi,s)|\le Cm^{-\mathrm{Re}(s)/2-q/2}\int_1^\infty\left(|\theta(\chi,iy)|\,y^{\mathrm{Re}(s)/2+q/2}+|\theta(\chi,iy)|\,y^{q/2+1/2-\mathrm{Re}(s)/2}\right)\frac{dy}{y}.$$

for all $s\in\Omega$ with C depending on Ω, but not on χ or s. Next, we estimate the theta function $\theta(\chi,iy)$ associated to χ: We obtain

$$|\theta(\chi,iy)|\le\sum_{n=1}^{\infty}ne^{-\pi n^2y/m}=e^{-\pi y/m}\left(1+\sum_{n=1}^{\infty}(n+1)e^{-\pi n(n+2)y/m}\right)\le e^{-\pi y/m}\Big(1+\theta_0(iy/m)\Big)$$

with $\theta_0(z):=2\sum_{n=1}^\infty ne^{\pi in^2z}$ for $z\in\mathbb{H}$. Here the modified theta function $\theta_0(z)$ is a holomorphic modular form of weight $3/2$, giving that $\theta_0(iy)=O(1)$ as $y\to\infty$ and $\theta_0(iy)=O(y^{-3/2})$ as $y\to 0$. Hence, we find $C'>0$ not depending on χ or m such that

$$|\theta(\chi,iy)|\le C'm^{3/2}e^{-\pi y/m}$$

for all $y \geq 1$. Therefore, we obtain

$$\begin{aligned}|L(\chi,s)| &\leq Cm^{-\operatorname{Re}(s)/2-q/2}\, C'm^{3/2}\int_1^\infty \Big(y^{\operatorname{Re}(s)/2+q/2}+y^{q/2+1/2-\operatorname{Re}(s)/2}\Big)e^{-\pi y/m}\frac{dy}{y}\\ &\leq C''m^{3/2}\Big(1+m^{1/2-\operatorname{Re}(s)}\Big)\end{aligned}$$

for all $s\in\Omega$ with $C''>0$ depending on Ω, but not on χ or s. This proves the claimed estimate. $\square$

Theorem 6.1.8. *Let $\gamma\in L'/L$ and let $\Omega\subseteq\mathbb{C}$ be compact with*

$$\Omega\subseteq\left\{s\in\mathbb{C}\colon\left(\prod_{p\,|\,2N}(1-p^{-4s})\right)\zeta(4s)\ (s-1/2)\neq 0\right\}.$$

Then we find $\delta>0$ such that

$$|Z(s;0,0,\gamma,n)|=O(|n|^\delta)$$

as $|n|\to\infty$ with $n\in\mathbb{Z}+q(\gamma)$, uniformly in s for $s\in\Omega$. Here the implied constant depends on γ and Ω.

Proof. Let $n\in\mathbb{Z}+q(\gamma)$ with $n\neq 0$, and let D_0 be the fundamental discriminant determined by $D_0f^2=4Nn\operatorname{ord}(\gamma)^2$ with $f\in\mathbb{N}$. By Theorem 6.1.5 the Kloosterman zeta function $Z(s;0,0,\gamma,n)$ is holomorphic on Ω, and by Proposition 6.1.4 we have

$$|Z(s;0,0,\gamma,n)|=\frac{2}{\sqrt{2N}\,|\zeta(4s)|}\left|L(\chi_{D_0},2s)\right|\left|\sigma^{(0)}_{\gamma,-n}(2s)\right|\left|\sigma^{(1)}_{\gamma,-n}(2s)\right|$$

for $s\in\Omega$. Using the estimates from Lemma 6.1.6 and the fact that $\zeta(4s)\neq 0$ for all $s\in\Omega$ we find $C>0$ and $\delta>0$, both depending on Ω but neither on s nor on n, such that

$$|Z(s;0,0,\gamma,n)|\leq C|n|^\delta\left|L(\chi_{D_0},2s)\right|$$

for $s\in\Omega$. Moreover, if $D_0\neq 1$ then χ_{D_0} is a primitive Dirichlet character mod $|D_0|$, and thus Lemma 6.1.7 yields constants $C'>0$ and $\delta'>0$, depending again on Ω but neither on s nor on n with

$$|L(\chi_{D_0},2s)|\leq C'\,|D_0|^{\delta'}\leq C'(4N\operatorname{ord}(\gamma)^2)^{\delta'}\,|n|^{\delta'}$$

for all $s\in\Omega$. Conversely, if $D_0=1$ then $L(\chi_{D_0},2s)=\zeta(2s)$ is bounded on Ω, since Ω is compact and $1/2\notin\Omega$. This proves the claimed statement. $\square$

Eventually, we turn back to the study of the Eisenstein series $E_0(\tau,s)$. However, instead of considering $E_0(\tau,s)$ itself, we define the corresponding completed non-holomorphic Eisenstein series $E_0^*(\tau,s)$ of weight $1/2$ for ρ_L by

$$E_0^*(\tau,s):=\frac{1}{2}\,\varphi(s)\,E_0(\tau,s) \tag{6.1.19}$$

for $\tau \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 3/4$, where the function $\varphi(s)$ is given by

$$\varphi(s) := N^s \left(\prod_{p \mid N} (1 + p^{-2s}) \right) \zeta^*(4s). \tag{6.1.20}$$

Here the product runs over all primes p dividing N, and $\zeta^*(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is the completed Riemann zeta function.

Given a primitive Dirichlet character χ of modulus m, we further recall that the *completed Dirichlet L-function* $L^*(\chi, s)$ associated to χ is given by

$$L^*(\chi, s) = \left(\frac{m}{\pi}\right)^{s/2} \Gamma\left(\frac{s+q}{2}\right) L(\chi, s) \tag{6.1.21}$$

for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ (see for example [Neu99], Section 7.2). Here $q = 0$ if $\chi(-1) = 1$, and $q = 1$ otherwise. If $m = 1$ then $L^*(\chi, s) = \zeta^*(s)$ is simply the completed Riemann zeta function, whose meromorphic continuation in s to $\mathbb{C}$ is holomorphic up to simple poles at $s = 0, 1$. Otherwise, i.e., if $m > 1$ then $L^*(\chi, s)$ has a holomorphic continuation in s to all of $\mathbb{C}$.

Proposition 6.1.9. *The completed non-holomorphic Eisenstein series $E_0^*(\tau, s)$ has a Fourier expansion of the form*

$$E_0^*(\tau, s) = \sum_{\gamma \in L'/L} \sum_{n \in \mathbb{Z}+q(\gamma)} b^*(\gamma, n, v, s) e(nu) \mathfrak{e}_\gamma,$$

for $\tau = u+iv \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 3/4$, where the Fourier coefficients $b^*(\gamma, n, v, s)$ are given by

$$b^*(0, 0, v, s) = \varphi(s)v^s + \varphi(1/2 - s)v^{1/2-s}$$

if $n = \gamma = 0$, where the function $\varphi(s)$ is defined as in (6.1.20), and by

$$\begin{aligned} &b^*(\gamma, n, v, s) \\ &\quad = \frac{(4N|n|)^{s-1/2}}{|D_0|^s} \left(\prod_{p \mid N} (1 + p^{-2s}) \right) L^*(\chi_{D_0}, 2s)\, \sigma^{(0)}_{\gamma,-n}(2s)\, \sigma^{(1)}_{\gamma,-n}(2s)\, \mathcal{W}_{1/2,s+1/4}(4\pi nv) \end{aligned}$$

if $n \neq 0$. Here the given product runs over all primes p dividing N, D_0 is the unique fundamental discriminant determined by $D_0 f^2 = 4Nn\,\mathrm{ord}(\gamma)^2$ with $f \in \mathbb{N}$, $\sigma^{(0)}_{\gamma,-n}(s)$ and $\sigma^{(1)}_{\gamma,-n}(s)$ are the generalized divisor sums defined in Proposition 6.1.4, and $L^*(\chi_{D_0}, s)$ is the completed Dirichlet L-function associated to the Kronecker symbol $\chi_{D_0} := (\frac{D_0}{\cdot})$.

Proof. Recall that by Proposition 3.6.2 the Fourier coefficients $b(\gamma, n, v, s)$ of the Eisenstein series $E_0(\tau, s)$ are given by

$$b(\gamma, n, v, s) = \begin{cases} \dfrac{2^{1/2}\pi^{s+1/2}|n|^{s-1/2}}{\Gamma(s+1/2)} \mathcal{W}_{1/2,s+1/4}(4\pi nv)\ Z(s; 0, 0, \gamma, n), & \text{if } n > 0, \\ 2v^s + 4^{3/4-s}\pi v^{1/2-s} \dfrac{\Gamma(2s-1/2)}{\Gamma(s)\,\Gamma(s+1/2)}\ Z(s; 0, 0, 0, 0), & \text{if } n = 0, \\ \dfrac{2^{1/2}\pi^{s+1/2}|n|^{s-1/2}}{\Gamma(s)} \mathcal{W}_{1/2,s+1/4}(4\pi nv)\ Z(s; 0, 0, \gamma, n), & \text{if } n < 0. \end{cases}$$

Further, the Fourier coefficients of the completed Eisenstein series $E_0^*(\tau,s)$ are by construction given by $b^*(\gamma,n,v,s)=\varphi(s)/2\cdot b(\gamma,n,v,s)$. Using part (a) of Proposition 6.1.4 we thus find that

$$b^*(0,0,v,s)=\varphi(s)v^s+\frac{2^{1-2s}\pi^{1/2}\Gamma(2s)}{\Gamma(s)\Gamma(s+1/2)}N^{s-1/2}\left(\prod_{p\,|\,N}(1+p^{1-2s})\right)\zeta^*(4s-1)v^{1/2-s}$$

for $n=\gamma=0$. Here the given fraction vanishes, since $\Gamma(s)\Gamma(s+1/2)=2^{1-2s}\pi^{1/2}\Gamma(2s)$ by the duplication formula of the Gamma function, and

$$N^{s-1/2}\left(\prod_{p\,|\,N}(1+p^{1-2s})\right)\zeta^*(4s-1)=N^{1/2-s}\left(\prod_{p\,|\,N}(1+p^{2s-1})\right)\zeta^*(2-4s)=\varphi(1/2-s),$$

as N is squarefree, and as the completed Riemann zeta function satisfies the functional equation $\zeta^*(s)=\zeta^*(1-s)$. So the coefficient $b^*(0,0,v,s)$ is of the claimed form.

For $n\neq 0$ we use part (b) of Proposition 6.1.4 to find that

$$b^*(\gamma,n,v,s)=\frac{\pi^{1/2}|n|^{s-1/2}N^{s-1/2}}{|D_0|^s}\;\frac{\Gamma(2s)}{\Gamma(s)\Gamma(s+1/2)}\left(\prod_{p\,|\,N}(1+p^{-2s})\right)$$
$$\times\, L^*(\chi_{D_0},2s)\sigma^{(0)}_{\gamma,-n}(2s)\sigma^{(1)}_{\gamma,-n}(2s)\mathcal{W}_{1/2,s+1/4}(4\pi nv).$$

Here the completed Dirichlet L-function $L^*(\chi_{D_0},s)$ is defined as in (6.1.21). Applying again the duplication formula of the Gamma function to the given expression we obtain the claimed Fourier expansion. □

Theorem 6.1.10. *The Fourier expansion given in Proposition 6.1.9 yields a meromorphic continuation of the completed Eisenstein series $E_0^*(\tau,s)$ in s to all of $\mathbb{C}$. This continuation is holomorphic up to simple poles at the two points $s=0$ and $s=1/2$, and it satisfies the functional equation*

$$E_0^*(\tau,s)=E_0^*(\tau,1/2-s)$$

for $\tau\in\mathbb{H}$ and $s\in\mathbb{C}\setminus\{0,1/2\}$.

Proof. Firstly, we note that the function $\varphi(s)$ defined in (6.1.20) is clearly holomorphic on all of $\mathbb{C}$ up to simple poles at $s=0$ and $s=1/4$ coming from the completed Riemann zeta function. Thus the constant term $b^*(0,0,v,s)$ of the Fourier expansion of the completed Eisenstein series given in the previous proposition is holomorphic on all of $\mathbb{C}$ up to simple poles at $s=0$ and $s=1/2$, and a possible simple pole at $s=1/4$. Indeed, the pole at $s=1/4$ vanishes since a direct computation shows that

$$\operatorname{Res}_{s=1/4}\left(\varphi(s)v^s+\varphi(1/2-s)v^{1/2-s}\right)=0$$

for $v>0$.

Next we consider the Fourier coefficients $b^*(\gamma,n,v,s)$ for $n\neq 0$. Again, these clearly define meromorphic functions in s on all of $\mathbb{C}$, with possible poles coming from the generalized divisor sum $\sigma^{(1)}_{\gamma,-n}(2s)$ and the completed Dirichlet L-function $L^*(\chi_{D_0}, 2s)$. More precisely, the divisor sum $\sigma^{(1)}_{\gamma,-n}(2s)$ is holomorphic on $\mathbb{C}$ up to possible simple poles at points $s \in i\mathbb{R}$ with $p^{-4s} = 1$ for some prime p dividing $2N$, and the L-function $L^*(\chi_{D_0}, 2s)$ is holomorphic on $\mathbb{C}$ up to possible simple poles at the points $s = 0$ and $s = 1/2$ which occur if and only if $D_0 = 1$.

Summarising we have seen that all Fourier coefficients $b^*(\gamma, n, v, s)$ define meromorphic functions on the domain

$$\Omega := \mathbb{C} \setminus \left(\left\{s \in i\mathbb{R}\colon p^{-4s} = 1 \text{ for some prime } p \text{ with } p \,|\, 2N\right\} \cup \{1/2\}\right),$$

and this domain Ω does not depend on n. In particular, the Fourier coefficients are holomorphic on the half-plane defined by $\mathrm{Re}(s) > 0$ up to a possible simple pole at $s = 1/2$ which occurs if and only if $n = 0$ or $D_0 = 1$, i.e., if $4Nn$ is a non-negative square.

We further claim that the coefficients $b^*(\gamma, n, v, s)$ decay exponentially in n as $|n| \to \infty$, locally uniformly in s: Let $K \subseteq \Omega$ be compact. By (3.6.2) we find $C > 0$ such that

$$\left|\mathcal{W}_{1/2,s+1/4}(4\pi n v)\right| \leq C e^{-2\pi |n| v}$$

for all $s \in K$ and $n \neq 0$ with C depending on K but not on s. Moreover, the estimates in Lemma 6.1.6 and Lemma 6.1.7 imply that the remaining terms defining the coefficients $b^*(\gamma, n, v, s)$ grow at most polynomially in n as $|n| \to \infty$, locally uniformly in s. Therefore, the Fourier expansion given in Proposition 6.1.9 converges absolutely and locally uniformly in s whenever the coefficients are holomorphic, and thus defines a meromorphic continuation of the completed Eisenstein series $E_0^*(\tau, s)$ to all of $\mathbb{C}$, which is holomorphic on the half-plane defined by $\mathrm{Re}(s) > 0$ up to a simple pole at $s = 1/2$.

In order to obtain the functional equation we now consider the difference

$$f(\tau, s) := E_0^*(\tau, 1/2 - s) - E_0^*(\tau, s)$$

for $\tau \in \mathbb{H}$ and $s \in \Omega \cap (1/2 - \Omega)$. By Proposition 6.1.9 f has a Fourier expansion of the form

$$f(\tau, s) = \sum_{\gamma \in L'/L} \sum_{n \in \mathbb{Z} + q(\gamma)} c(\gamma, n, v, s) e(nu) \mathfrak{e}_\gamma$$

with $c(\gamma, n, v, s) = b^*(\gamma, n, v, 1/2 - s) - b^*(\gamma, n, v, s)$. In particular, we have

$$c(0, 0, v, s) = b^*(0, 0, v, s) - b^*(0, 0, v, 1/2 - s) = 0,$$

i.e., the constant Fourier coefficient of f vanishes. As the remaining Fourier coefficients $c(\gamma, n, v, s)$ with $n \neq 0$ decay exponentially in n as $|n| \to \infty$, the function f vanishes at ∞, i.e., for fixed s we have $f(\tau, s) \to 0$ as $v \to \infty$, uniformly in u. Therefore, the function $f(\tau, s)$ is square-integrable in τ with respect to the scalar product defined in (3.4.3), i.e., we have $f(\,\cdot\,, s) \in \mathcal{H}_{1/2,L}$ for fixed s where we refer to equation (3.7.12) for the definition of the latter space of functions.

Moreover, because of (2.5.1) the completed Eisenstein series $E_0^*(\tau, s)$ is an eigenfunction of the hyperbolic Laplace operator $\Delta_{1/2}$ with eigenvalue $s(1/2 - s)$, and thus the same is true for the difference defining $f(\tau, s)$. In particular, also $\Delta_{1/2} f$ lies in the space $\mathcal{H}_{1/2,L}$, and we thus have $f \in \mathcal{D}_{1/2,L}$ with $\mathcal{D}_{1/2,L}$ being defined in (3.7.13). However, we have seen in (3.7.17) that the spectrum of the continuation of $\Delta_{1/2}$ to the space $\mathcal{D}_{1/2,L}$ lies in the interval $[0,\infty)$. Therefore, the function $f(\tau,s)$ needs to vanish identically, as for $s \in \Omega \cap (1/2-\Omega)$ with $\mathrm{Re}(s)$ large enough the eigenvalue $s(1/2-s)$ of $f(\tau,s)$ has negative real part which cannot lie in the interval $[0,\infty)$. This proves the claimed functional equation.

Recall that we have seen that the completed Eisenstein series $E_0^*(\tau,s)$ is holomorphic on the half-plane defined by $\mathrm{Re}(s)>0$ up to a simple pole at $s=1/2$. Now the functional equation deduced above implies, that the meromorphic continuation of $E_0^*(\tau,s)$ is indeed holomorphic on all of $\mathbb{C}$ up to simple poles at the two points $s=0$ and $s=1/2$. □

We quickly summarize what we know about the meromorphic continuation of the non-completed Eisenstein series $E_0(\tau,s)$ and the corresponding Kloosterman zeta functions.

Corollary 6.1.11. *The non-holomorphic Eisenstein series $E_0(\tau,s)$ has a meromorphic continuation in s to all of $\mathbb{C}$ which is given by its Fourier expansion*

$$E_0(\tau,s)=\sum_{\gamma\in L'/L}\sum_{n\in\mathbb{Z}+q(\gamma)} b(\gamma,n,v,s)e(nu)\mathfrak{e}_\gamma$$

for $\tau=u+iv\in\mathbb{H}$. Here the Fourier coefficients are given by

$$b(\gamma,n,v,s)$$
$$=\begin{cases}\dfrac{2}{\sqrt{N}}\dfrac{(4|n|)^{s-1/2}}{|D_0|^s}\dfrac{L^*(\chi_{D_0},2s)}{\zeta^*(4s)}\,\sigma^{(0)}_{\gamma,-n}(2s)\,\sigma^{(1)}_{\gamma,-n}(2s)\,\mathcal{W}_{1/2,s+1/4}(4\pi nv), & \text{if } n\neq 0,\\ 2v^s+\dfrac{2}{\sqrt{N}}v^{1/2-s}\dfrac{\zeta^*(4s-1)}{\zeta^*(4s)}\displaystyle\prod_{p\,|\,N}\frac{1+p^{1-2s}}{1+p^{-2s}}, & \text{if } n=0,\end{cases}$$

and the notation is the same as in Theorem 6.1.10. This continuation is holomorphic up to a simple pole at $s=1/2$, and possible further simple poles at

- *$s\in i\mathbb{R}$ with $p^{-2s}=-1$ for some prime p dividing N, and*
- *$s\in\mathbb{C}$ with $\zeta(4s)=0$ and $0<\mathrm{Re}(s)<1/4$.*

Moreover, the Eisenstein series $E_0(\tau,s)$ is holomorphic at $s=0$ and vanishes at $s=1/4$.

Proof. Recall that

$$E_0(\tau,s)=\frac{2}{N^s}\left(\prod_{p\,|\,N}\frac{1}{1+p^{-2s}}\right)\frac{1}{\zeta^*(4s)}\,E_0^*(\tau,s) \tag{6.1.22}$$

for $\tau\in\mathbb{H}$ and $\mathrm{Re}(s)>3/4$. By Theorem 6.1.10 the completed Eisenstein series $E_0^*(\tau,s)$ has a meromorphic continuation in s to all of $\mathbb{C}$, which is holomorphic up to simple poles at $s=0$ and $s=1/2$, and which is given by its Fourier expansion. Therefore, also the left-hand side of (6.1.22), namely the Eisenstein series $E_0(\tau,s)$, admits a meromorphic continuation in s to all of $\mathbb{C}$, and this continuation is again given by its Fourier expansion, where the corresponding Fourier coefficients are simply given by

$$b(\gamma,n,v,s)=\frac{2}{N^s}\left(\prod_{p\,|\,N}\frac{1}{1+p^{-2s}}\right)\frac{1}{\zeta^*(4s)}\,b^*(\gamma,n,v,s)$$

with $b^*(\gamma, n, v, s)$ as in Proposition 6.1.9. This directly implies the claimed form of the Fourier coefficients in the case $n \neq 0$, and for $n = \gamma = 0$ we only need to recall that $\zeta^*(2-4s) = \zeta^*(4s-1)$ by the functional equation of the completed Riemann zeta function.

Moreover, by (6.1.22) the meromorphic continuation of $E_0(\tau, s)$ is holomorphic up to possible simple poles at $s \in \mathbb{C}$ with $1 + p^{-2s} = 0$ for some prime p dividing N, at $s \in \mathbb{C}$ with $\zeta^*(4s) = 0$, or at the poles of $E_0^*(\tau, s)$, namely at $s = 0$ and $s = 1/2$. We note that $\zeta^*(4s) = 0$ for $s \in \mathbb{C}$ if and only if s is a non-trivial zero of the Riemann zeta function $\zeta(4s)$ lying in the strip $0 < \mathrm{Re}(s) < 1/4$, since the trivial zeros of the Riemann zeta function $\zeta(4s)$ at $s = -1/2, -1, -3/2, \ldots$ are canceled by the poles of the corresponding Gamma function $\Gamma(2s)$ defining $\zeta^*(4s)$. Furthermore, the possible pole at $s = 0$ is compensated for by the corresponding pole of the completed Riemann zeta function $\zeta^*(4s)$, and the possible pole at $s = 1/2$ cannot be compensated for. □

The following Corollary is a refinement of part (b) of Theorem 6.1.5.

Corollary 6.1.12. *Let $\gamma \in L'/L$ and $n \in \mathbb{Z} + q(\gamma)$. The Kloosterman zeta function $Z(s; 0, 0, \gamma, n)$ has a meromorphic continuations in s to all of $\mathbb{C}$ which is holomorphic up to a possible simple pole at $s = 1/2$, which appears if and only if $4Nn$ is a non-negative square, and possible further simple poles at*

- *$s \in i\mathbb{R}$ with $p^{-2s} = -1$ for some prime p dividing N, and*
- *$s \in \mathbb{C}$ with $\zeta(4s) = 0$, i.e., $s = -1/2, -1, -3/2, \ldots$ or s some non-trivial zero of the Riemann zeta function $\zeta(4s)$ lying in the interval $0 < \mathrm{Re}(s) < 1/4$.*

Here the function $Z(s; 0, 0, \gamma, n)$ is holomorphic at the points $s = -1, -2, \ldots$ if $n > 0$, and at the points $s = -1/2, -3/2, \ldots$ if $n < 0$. Moreover, the Kloosterman zeta function $Z(s; 0, 0, \gamma, n)$ is holomorphic at $s = 0$ and vanishes at $s = 1/4$.

Proof. Firstly, we recall that

$$Z(s; 0, 0, 0, 0) = \frac{2}{\sqrt{2N}} \frac{\zeta(4s-1)}{\zeta(4s)} \prod_{p \mid N} \frac{1 + p^{1-2s}}{1 + p^{-2s}}$$

by part (a) of Proposition 6.1.4. Here the right-hand side gives a meromorphic continuation of $Z(s; 0, 0, 0, 0)$ in s to all of $\mathbb{C}$, which is holomorphic up to simple poles at $s = 1/2$ and at $s \in \mathbb{C}$ with $\zeta(4s) = 0$ or $p^{-2s} = -1$ for some prime p dividing N. Further, non of these poles can be compensated for by the Riemann zeta function $\zeta(4s-1)$ or by some term $1 + p^{1-2s}$ with p prime dividing N. We also note that $Z(s; 0, 0, 0, 0)$ is clearly holomorphic at $s = 0$ and vanishes at $s = 1/4$.

Let now $n \neq 0$. By Theorem 6.1.5 the function

$$g(s; \gamma, n) := \zeta(4s)\, Z(s; 0, 0, \gamma, n) \prod_{p \mid 2N} (1 - p^{-4s})$$

has a meromorphic continuation in s to $\mathbb{C}$ which is holomorphic up to a possible simple pole at $s = 1/2$ occurring if and only if $4Nn$ is a non-negative square. Hence the Kloosterman zeta function $Z(s; 0, 0, \gamma, n)$ is holomorphic up to possible simple poles at

$s = 1/2$, and at $s \in \mathbb{C}$ with $\zeta(4s) = 0$ or $p^{-4s} = 1$ for some prime p dividing $2N$. Here the pole at $s = 1/2$ occurs exactly if $4Nn$ is a non-negative square. Moreover, $Z(s; 0, 0, \gamma, n)$ needs to vanish at $s = 1/4$ in order to compensate for the pole of the Riemann zeta function $\zeta(4s)$ at $s = 1/4$.

In order to further investigate the possible poles at $s \in \mathbb{C}$ with $\zeta(4s) = 0$ or $p^{-4s} = 1$ for some prime p dividing $2N$, we recall that by Corollary 6.1.11 the Fourier expansion of the Eisenstein series $E_0(\tau, s)$ yields a meromorphic continuation of $E_0(\tau, s)$ in s to all of $\mathbb{C}$, which is holomorphic up to a simple pole at $s = 1/2$, and possible simple poles at $s \in i\mathbb{R}$ with $p^{-2s} = -1$ for some prime p dividing N, and at $s \in \mathbb{C}$ with $\zeta(4s) = 0$ and $0 < \mathrm{Re}(s) < 1/4$. Moreover, by Proposition 3.6.2 the Fourier coefficients $b(\gamma, n, v, s)$ from Corollary 6.1.11 can also be written as

$$b(\gamma, n, v, s) = \begin{cases} \dfrac{2^{1/2}\pi^{s+1/2}|n|^{s-1/2}}{\Gamma(s+1/2)} \, \mathcal{W}_{1/2, s+1/4}(4\pi n v) \; Z(s; 0, 0, \gamma, n), & \text{if } n > 0, \\[2ex] \dfrac{2^{1/2}\pi^{s+1/2}|n|^{s-1/2}}{\Gamma(s)} \, \mathcal{W}_{1/2, s+1/4}(4\pi n v) \; Z(s; 0, 0, \gamma, n), & \text{if } n < 0, \end{cases}$$

for $n \neq 0$. Here the coefficients $b(\gamma, n, v, s)$ need to be holomorphic in s whenever the Eisenstein series $E_0(\tau, s)$ is. Thus, the function $Z(s; 0, 0, \gamma, n)$ cannot have a pole at $s \in \mathbb{C}$ with $p^{-2s} = 1$ for some prime p dividing $2N$, or at $s \in \mathbb{C}$ with $2^{-2s} = -1$ if N is odd, since $E_0(\tau, s)$ is holomorphic at these points. In particular, $Z(s; 0, 0, \gamma, n)$ is holomorphic at $s = 0$.

Suppose that $n > 0$. Then the Kloosterman zeta function $Z(s; 0, 0, \gamma, n)$ can indeed have poles at the points $s = -1/2, -3/2, \ldots$, since these poles would be compensated for by the Gamma function $\Gamma(s + 1/2)$ appearing in the corresponding Fourier coefficient $b(\gamma, n, v, s)$. On the other hand, $Z(s; 0, 0, \gamma, n)$ cannot have poles at $s = -1, -2, \ldots$ as these could not be compensated for, but $E_0(\tau, s)$ is holomorphic at these points. Analogously, we find that if $n < 0$ the function $Z(s; 0, 0, \gamma, n)$ can have poles at the points $s = -1, -2, \ldots$, but not at the points $s = -1/2, -3/2, \ldots$. □

To the end of this section we evaluate the meromorphic continuation of $E_0(\tau, s)$ at the special point $s = 0$, which is possible, since $E_0(\tau, s)$ is holomorphic at $s = 0$ by Corollary 6.1.11.

Corollary 6.1.13. *The evaluation of $E_0(\tau, s)$ at $s = 0$ has a Fourier expansion of the form*

$$E_0(\tau, 0) = 2\mathfrak{e}_0 + \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z} + q(\gamma) \\ n > 0}} \tilde{b}(\gamma, n) e(n\tau) \mathfrak{e}_\gamma$$

for $\tau \in \mathbb{H}$, where the Fourier coefficients $\tilde{b}(\gamma, n)$ are given by

$$\tilde{b}(\gamma, n) = 2^{1/2} |n|^{-1/2} Z(0; 0, 0, \gamma, n)$$

In particular, the coefficients $\tilde{b}(\gamma, n)$ are all real.

Proof. By Corollary 6.1.11 the meromorphic continuation of the Eisenstein series $E_0(\tau, s)$ is indeed given by its Fourier expansion. Thus, we can also use the Fourier expansion given in Proposition 3.6.2, namely

$$E_0(\tau,s) = 2v^s\mathfrak{e}_0 + \sum_{\gamma\in L'/L}\sum_{n\in\mathbb{Z}+q(\gamma)} b(\gamma,n,v,s)e(nu)\mathfrak{e}_\gamma,$$

with

$$b(\gamma,n,v,s) = \begin{cases} \dfrac{2^{1/2}\pi^{s+1/2}|n|^{s-1/2}}{\Gamma(s+1/2)}\ \mathcal{W}_{1/2,s+1/4}(4\pi nv)\ Z(s;0,0,\gamma,n), & \text{if } n>0,\\ 4^{3/4-s}\pi v^{1/2-s}\ \dfrac{\Gamma(2s-1/2)}{\Gamma(s)\,\Gamma(s+1/2)}\ Z(s;0,0,0,0), & \text{if } n=0,\\ \dfrac{2^{1/2}\pi^{s+1/2}|n|^{s-1/2}}{\Gamma(s)}\ \mathcal{W}_{1/2,s+1/4}(4\pi nv)\ Z(s;0,0,\gamma,n), & \text{if } n<0.\end{cases}$$

By Corollary 6.1.12 the Kloosterman zeta functions $Z(s;0,0,\gamma,n)$ are all holomorphic at $s=0$. Hence we find $b(\gamma,n,v,0)=0$ for all $n\le 0$ because of the Gamma factor $1/\Gamma(s)$ appearing if $n=0$ or if $n<0$. Moreover, if $n>0$ we have $\Gamma(1/2)=\pi^{1/2}$ and

$$\mathcal{W}_{1/2,1/4}(4\pi nv) = (4\pi nv)^{-1/4}W_{1/4,-1/4}(4\pi nv) = e^{-2\pi nv},$$

where the second equality is for example given in [GR07, formula 9.237.3] with $L_0^\alpha(x)=1$ being the trivial Laguerre polynomial. This proves the claimed Fourier expansion. It remains to note that the Fourier coefficients $\tilde{b}(\gamma,n)$ are indeed real, since

$$\overline{Z(0;0,0,\gamma,n)} = Z(\overline{0};0,0,\gamma,n) = Z(0;0,0,\gamma,n)$$

for all $\gamma\in L'/L$ and $n\in\mathbb{Z}+q(\gamma)$. □

6.2 Continuation of Selberg's Poincaré series of the second kind via its spectral expansion

In order to establish the meromorphic continuation of the Kloosterman zeta function $Z(s;\beta,m,\gamma,n)$ for $m\ne 0$, it turns out to be useful to work with Selberg's Poincaré series of the second kind $V_{\beta,m}(\tau,s)$ of weight $1/2$ for the Weil representation ρ_L, which is square-integrable for all $m\ne 0$, and can thus be meromorphically continued via its spectral expansion. We will use this continuation in the following section to treat the corresponding Kloosterman zeta functions.

Let $\beta\in L'/L$ and $m\in\mathbb{Z}+q(\beta)$. We recall that Selberg's Poincaré series of the second kind $V_{\beta,m}(\tau,s)$ is given by

$$V_{\beta,m}(\tau,s) = \frac{1}{2}\sum_{(M,\phi)\in\langle\tilde{T}\rangle\backslash\operatorname{Mp}_2(\mathbb{Z})} v^s e^{-2\pi|m|v}e(mu)\mathfrak{e}_\beta\Big|_{1/2,L}(M,\phi)$$

for $\tau=u+iv\in\mathbb{H}$ and $s\in\mathbb{C}$ with $\operatorname{Re}(s)>3/4$. If $m=0$ then $\beta=0$ as we assume N to be squarefree, and $V_{0,0}(\tau,s)=E_0(\tau,s)$ is the non-holomorphic Eisenstein series from the previous section. Thus, we assume that $m\ne 0$ throughout the present section.

By Lemma 3.6.9 the Poincaré series $V_{\beta,m}(\tau,s)$ is square-integrable with respect to the inner product given in (3.4.3), i.e., we have $V_{\beta,m}(\,\cdot\,,s)\in\mathcal{H}_{1/2,L}$ where $\mathcal{H}_{1/2,L}$ is defined as in (3.7.12). So by Theorem 3.7.3 Selberg's Poincaré series of the second kind admits a spectral expansion. In the following we use this spectral expansion to establish the meromorphic continuation of the Poincaré series $V_{\beta,m}(\tau,s)$ in s to all of $\mathbb{C}$.

Let $\mathcal{D}_{1/2,L}$ be defined as in (3.7.13). As in Theorem 3.7.3 we fix an orthonormal system $(\psi_j)_{j\in\mathbb{N}_0}\subseteq\mathcal{D}_{1/2,L}$ of real analytic eigenfunctions of the hyperbolic Laplace operator $\Delta_{1/2}$, and we denote the eigenvalue of the eigenfunction ψ_j by λ_j. According to (3.7.19) we can write λ_j in the form

$$\lambda_j=\frac{1}{16}+r_j^2$$

with $r_j\geq 0$ if $\lambda_j\in[1/4,\infty)$ and $r_j\in[i/4,0)$ if $\lambda_j\in[0,1/4)$. Note that the distinguished eigenvalues defined in (3.7.18) do not appear for weight 1/2. Further, using the notation of Lemma 3.7.4 we write $c_{\psi_j}(\gamma,n)\mathcal{W}_{1/2,1/2+ir_j}(4\pi nv)$ for the (γ,n)'th Fourier coefficient of ψ_j if $n\neq 0$.

Proposition 6.2.1. *Let $\beta\in L'/L$ and $m\in\mathbb{Z}+q(\beta)$ with $m\neq 0$. Then $V_{\beta,m}(\tau,s)$ has a spectral expansion of the form*

$$V_{\beta,m}(\tau,s)=\sum_{j=0}^{\infty}a_j(s)\psi_j(\tau)+\frac{1}{16\pi}\int_{-\infty}^{\infty}a_\infty(s,r)E_0(\tau,1/4+ir)dr$$

for $\tau\in\mathbb{H}$ and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>3/4$. The spectral coefficients are given by

$$a_j(s)=2(4\pi|m|)^{1/2-s}\,\overline{c_{\psi_j}(\beta,m)}\begin{cases}\dfrac{\Gamma(s-1/4+ir_j)\Gamma(s-1/4-ir_j)}{\Gamma(s)}, & \text{if } m>0,\\[2ex] \dfrac{\Gamma(s-1/4+ir_j)\Gamma(s-1/4-ir_j)}{\Gamma(s+1/2)}, & \text{if } m<0,\end{cases}$$

and

$$\begin{aligned}&a_\infty(s,r)\\ &=\frac{2\pi(4\pi|m|)^{3/4-s}}{(\pi|m|)^{1/2+ir}}Z\left(1/4-ir;0,0,\beta,m\right)\begin{cases}\dfrac{\Gamma(s-1/4+ir)\;\Gamma(s-1/4-ir)}{\Gamma(s)\;\Gamma(3/4-ir)}, & \text{if } m>0,\\[2ex] \dfrac{\Gamma(s-1/4+ir)\;\Gamma(s-1/4-ir)}{\Gamma(s+1/2)\;\Gamma(1/4-ir)}, & \text{if } m<0.\end{cases}\end{aligned}$$

Proof. Let $s\in\mathbb{C}$ with $\mathrm{Re}(s)>3/4$, and let $j\in\mathbb{N}_0$. Then $a_j(s)=(V_{\beta,m}(\,\cdot\,,s),\psi_j)$. As the inner product exists we can simply unfold the corresponding integral against the Poincaré series, which yields

(6.2.1)
$$\begin{aligned}a_j(s)&=\sum_{M\in\langle T\rangle\backslash\mathrm{SL}_2(\mathbb{Z})}\int_{\mathcal{F}}\mathrm{Im}(M\tau)^{s+1/2}e^{-2\pi|m|\,\mathrm{Im}(M\tau)}e(m\,\mathrm{Re}(M\tau))\left\langle\mathfrak{e}_\beta,\psi_j(M\tau)\right\rangle d\mu(\tau)\\ &=2\int_0^\infty v^{s-3/2}e^{-2\pi|m|v}\int_0^1\left\langle\mathfrak{e}_\beta,\psi_j(\tau)\right\rangle e(mu)dudv.\end{aligned}$$

Here the remaining inner integral is simply the complex conjugate of the (β, m)-th Fourier coefficient of ψ_j, which is of the form $c_{\psi_j}(\beta, m)\mathcal{W}_{1/2,1/2+ir_j}(4\pi mv)$. Hence we obtain

$$a_j(s) = 2\,\overline{c_{\psi_j}(\beta, m)}\,(4\pi|m|)^{-1/4}\int_0^\infty v^{s-7/4}e^{-2\pi|m|v}\overline{W_{\mathrm{sign}(m)/4,ir_j}(4\pi|m|v)}dv.$$

Substituting $v' = 4\pi|m|v$ in the remaining integral we can apply formula 7.621.11 in [GR07], which gives

$$a_j(s) = 2\,\overline{c_{\psi_j}(\beta, m)}\,(4\pi|m|)^{1/2-s}\begin{cases}\dfrac{\Gamma(s-1/4+ir_j)\Gamma(s-1/4-ir_j)}{\Gamma(s)}, & \text{if } m > 0,\\ \dfrac{\Gamma(s-1/4+ir_j)\Gamma(s-1/4-ir_j)}{\Gamma(s+1/2)}, & \text{if } m < 0,\end{cases}$$

for $\mathrm{Re}(s) > 1/4 + |\operatorname{Im}(r_j)|$, proving the claimed formula as $|\operatorname{Im}(r_j)| \leq 1/4$. It only remains to note that $\overline{r_j} = \pm r_j$, where the sign on the right-hand side depends on whether r_j is real, or purely imaginary.

As in (6.2.1), we also obtain

$$\langle V_{\beta,m}(\cdot, s), E_0(\cdot, \xi)\rangle = 2\int_0^\infty v^{s-3/2}e^{-2\pi|m|v}\int_0^1 \langle \mathfrak{e}_\beta, E_0(\tau, \xi)\rangle\, e(mu)dudv. \tag{6.2.2}$$

Again, the inner integral is simply the complex conjugate of the (β, m)-th Fourier coefficient of $E_0(\tau, \xi)$, which is given by

$$\begin{cases}\dfrac{2^{1/2}\pi^{\xi+1/2}|m|^{\xi-1/2}}{\Gamma(\xi+1/2)}\mathcal{W}_{1/2,\xi+1/4}(4\pi mv)Z(\xi; 0, 0, \beta, m), & \text{if } m > 0,\\ \dfrac{2^{1/2}\pi^{\xi+1/2}|m|^{\xi-1/2}}{\Gamma(\xi)}\mathcal{W}_{1/2,\xi+1/4}(4\pi mv)Z(\xi; 0, 0, \beta, m), & \text{if } m < 0.\end{cases}$$

Plugging this into (6.2.2) and using formula 7.621.11 in [GR07] as above we see that the inner product $\langle V_{\beta,m}(\cdot, s), E_0(\cdot, \xi)\rangle$ equals

$$\begin{cases}2^{5/2-2s}\pi^{\bar{\xi}-s+1}|m|^{\bar{\xi}-s}\,\dfrac{\Gamma(s-\bar{\xi})\Gamma(s-1/2+\bar{\xi})}{\Gamma(s)\Gamma(\bar{\xi}+1/2)}\,\overline{Z(\xi; 0, 0, \beta, m)}, & \text{if } m > 0,\\ 2^{5/2-2s}\pi^{\bar{\xi}-s+1}|m|^{\bar{\xi}-s}\,\dfrac{\Gamma(s-\bar{\xi})\Gamma(s-1/2+\bar{\xi})}{\Gamma(s+1/2)\Gamma(\bar{\xi})}\,\overline{Z(\xi; 0, 0, \beta, m)}, & \text{if } m < 0.\end{cases} \tag{6.2.3}$$

Now $a_\infty(s, r) = (V_{\beta,m}(\,\cdot\,, s), E_0(\,\cdot\,, 1/4 + ir))$ with $r \in \mathbb{R}$, and it remains to note that

$$\overline{Z(\xi; 0, 0, \beta, m)} = Z(\bar{\xi}; 0, 0, \beta, m)$$

as in [Bru02], Lemma 1.13. □

We now use the above spectral expansion to establish the meromorphic continuation of $V_{\beta,m}(\tau, s)$ in s to the whole complex plane. The idea of the proof is taken from [Pip10], Proposition 5.1.4.

Proposition 6.2.2. *Let $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m \neq 0$. Then $V_{\beta,m}(\tau, s)$ has a meromorphic continuation in s to $\mathbb{C}$, which is given by its spectral expansion for* $\mathrm{Re}(s) > 1/4$. *Moreover, on the vertical strip defined by $-3/4 < \mathrm{Re}(s) < 1/4$ this continuation is given by*

$$\sum_{j=0}^{\infty} a_j(s)\psi_j(\tau) + \frac{1}{16\pi}\int_{-\infty}^{\infty} a_\infty(s,r)E_0(\tau, 1/4+ir)dr$$

$$+ \frac{(2\pi|m|)^{1/2-2s}\pi}{2} Z(1/2-s;0,0,\beta,m)E_0(\tau,s)\begin{cases} \dfrac{\Gamma(2s-1/2)}{\Gamma(s)\Gamma(1-s)}, & \textit{if } m>0, \\ \dfrac{\Gamma(2s-1/2)}{\Gamma(s+1/2)\Gamma(1/2-s)}, & \textit{if } m<0, \end{cases}$$

$$+ \frac{\sqrt{2\pi}}{4}\ \frac{\Gamma(2s-1/2)}{\Gamma(2s)}\ Z(s;0,0,\beta,m)E_0(\tau,1/2-s).$$

Proof. We use the spectral expansion of $V_{\beta,m}(\tau,s)$ to derive its meromorphic continuation, starting with the part coming from the discrete spectrum, i.e., with the sum

$$\sum_{j=0}^{\infty} a_j(s)\psi_j(\tau), \tag{6.2.4}$$

where

$$a_j(s) = 2(4\pi|m|)^{3/4-s}\,\overline{c_{\psi_j}(\beta,m)}\begin{cases} \dfrac{\Gamma(s-1/4+ir_j)\Gamma(s-1/4-ir_j)}{\Gamma(s)}, & \text{if } m>0, \\ \dfrac{\Gamma(s-1/4+ir_j)\Gamma(s-1/4-ir_j)}{\Gamma(s+1/2)}, & \text{if } m<0. \end{cases}$$

Here $c_{\psi_j}(\beta,m)\mathcal{W}_{1/2,1/2+ir_j}(4\pi m v)$ is the (β,m)'th Fourier coefficient of the eigenfunction ψ_j, and $\lambda_j = 1/16 + r_j^2$ is the eigenvalue of ψ_j. By Stirling's formula we have

$$|\Gamma\left(s-1/4 \pm ir_j\right)| = O\Big(|r_j|^{\mathrm{Re}(s)-3/4}e^{-\pi|r_j|/2}\Big)$$

as $j \to \infty$ for fixed $s \in \mathbb{C}$. Moreover, the Fourier coefficients and the sup-norms of the eigenfunctions ψ_j behave as

$$|c_{\psi_j}(\beta,m)| = O(1) \qquad \text{and} \qquad \sup_{\tau\in\mathbb{H}}|\psi_j(\tau)| = O\big(|r_j|^{\delta}\big)$$

for some $\delta > 0$ as $j \to \infty$, for fixed β and m, respectively. Hence we find

$$\sup_{\tau\in\mathbb{H}}|a_j(s)\psi_j(\tau)| = O\Big(|r_j|^{2\,\mathrm{Re}(s)-3/2+\delta}e^{-\pi|r_j|}\Big)$$

as $j \to \infty$ away from the poles of $a_j(s)$, and therefore the sum in (6.2.4) has a meromorphic continuation in s to $\mathbb{C}$.

Fix $\tau \in \mathbb{H}$. We need to consider the part of the spectral expansion coming from the continuous spectrum, which is given by

$$\mathcal{C}(s) := \frac{1}{16\pi}\int_{-\infty}^{\infty} a_\infty(s,r)E_0(\tau,1/4+ir)dr = \int\limits_{1/4+i\mathbb{R}} g(s,w)dw. \tag{6.2.5}$$

Here the path integral on the right-hand side runs from $1/4 - i\infty$ to $1/4 + i\infty$, and the integrand $g(s, w)$ is given by

$$(6.2.6) \qquad g(s,w) := \frac{(4\pi|m|)^{3/4-s}}{8i(\pi|m|)^{1/4+w}} \, Z(1/2-w;0,0,\beta,m) \; E_0(\tau,w) \times \begin{cases} \dfrac{\Gamma(s+w-1/2)\;\Gamma(s-w)}{\Gamma(s)\;\Gamma(1-w)}, & \text{if } m>0, \\ \dfrac{\Gamma(s+w-1/2)\;\Gamma(s-w)}{\Gamma(s+1/2)\;\Gamma(1/2-w)}, & \text{if } m<0. \end{cases}$$

For fixed $w \in \mathbb{C}$ the poles of $g(s, w)$ in s are of the form $s = w - n$ and $s = 1/2 - w - n$ for $n \in \mathbb{N}_0$. Thus the line integral $\mathcal{C}(s)$ from (6.2.5) is holomorphic in s away from the vertical lines $1/4-n+i\mathbb{R}$ with $n \in \mathbb{N}_0$. In particular, $\mathcal{C}(s)$ is holomorphic for $\mathrm{Re}(s) > 1/4$, i.e., the spectral expansion from Proposition 6.2.1 yields a holomorphic continuation of the Poincaré series $V_{\beta,m}(\tau, s)$ in s to the half-plane $\mathrm{Re}(s) > 1/4$.

We now want to locally shift the line of integration in (6.2.5) to the right using the residue theorem. Firstly, we recall that by Corollary 6.1.11 and Corollary 6.1.12 the two functions $E_0(\tau, w)$ and $Z(w; 0, 0, \beta, m)$ are both holomorphic in w on the vertical strip $0 < \mathrm{Re}(w) < 1/2$ up to possible simple poles coming from non-trivial zeros of the classical Riemann zeta function $\zeta(4w)$ in the strip $0 < \mathrm{Re}(w) < 1/4$. However, given $T > 0$ we find $\varepsilon > 0$ such that $\zeta(4w)$ is non-vanishing on the open rectangle $R_{2T}(1/4 - 2\varepsilon, 1/4)$ where

$$R_\delta(a,b) := \{w \in \mathbb{C} \colon a < \mathrm{Re}(w) < b, |\,\mathrm{Im}(w)| < \delta\}$$

for $\delta > 0$ and $a, b \in \mathbb{R}$ with $a < b$. So the functions $E_0(\tau, w)$ and $Z(w; 0, 0, \beta, m)$ are holomorphic on the open rectangle $R_{2T}(1/4 - 2\varepsilon, 1/2)$. In particular, the product

$$E_0(\tau,w)\, Z(1/2-w;0,0,\beta,m)$$

appearing in (6.2.6) is holomorphic in w on the open rectangle $R_{2T}(1/4 - 2\varepsilon, 1/4 + 2\varepsilon)$. Here we use the factor 2 such that we can integrate along the boundary of the smaller rectangle $R_T(1/4 - \varepsilon, 1/4 + \varepsilon)$ in the following.

Next we fix $s \in R_T(1/4, 1/4+\varepsilon)$. Then the function $g(s, w)$ from (6.2.6) is holomorphic on the open rectangle $R_T(1/4 - \varepsilon, 1/4 + \varepsilon)$ up to possible simple poles at the two points $w = s$ and $w = 1/2 - s$ coming from the Gamma factors $\Gamma(s - w)$ and $\Gamma(s + w - 1/2)$, respectively. Let now $\gamma(T, \varepsilon)$ be the piecewise linear path from $1/4 - i\infty$ to $1/4 + i\infty$ passing through the corners of the rectangle $R_T(1/4, 1/4+\varepsilon)$ in the following order: From $1/4 - i\infty$ via $1/4 - iT$, $1/4 + \varepsilon - iT$, $1/4 + \varepsilon + iT$ and $1/4 + iT$ to $1/4 + i\infty$. Then the residue theorem yields

(6.2.7)

$$\int_{\gamma(T,\varepsilon)} g(s,w)dw - \int_{1/4+i\mathbb{R}} g(s,w)dw$$

$$= 2\pi i \ \mathrm{Res}_{w=s}\, g(s,w)$$

$$= -\frac{(2\pi|m|)^{1/2-2s}\ \pi}{2}\ Z(1/2-s;0,0,\beta,m)E_0(\tau,s)\begin{cases}\dfrac{\Gamma(2s-1/2)}{\Gamma(s)\ \Gamma(1-s)}, & \text{if } m>0,\\ \dfrac{\Gamma(2s-1/2)}{\Gamma(s+1/2)\ \Gamma(1/2-s)}, & \text{if } m<0.\end{cases}$$

Thus, rearranging terms we find

(6.2.8)

$$\mathcal{C}(s) = \int_{\gamma(T,\varepsilon)} g(s,w)dw + \frac{(2\pi|m|)^{1/2-2s}\ \pi}{2}\ Z(1/2-s;0,0,\beta,m)E_0(\tau,s)\frac{\Gamma(2s-1/2)}{\Gamma(s)\ \Gamma(1-s)},$$

where we need to replace the term $\Gamma(s)\Gamma(1-s)$ in the denominator by $\Gamma(s+1/2)\Gamma(1/2-s)$ if $m<0$.

It is now easy to check that the path integral over $\gamma(T,\varepsilon)$ is indeed holomorphic in s on the rectangle $R_T(1/4-\varepsilon,1/4+\varepsilon)$, and that the remaining expression on the right-hand side of (6.2.8) is clearly meromorphic in s on all of $\mathbb{C}$. Therefore, equation (6.2.8) yields a meromorphic continuation of the path integral $\mathcal{C}(s)$ to the rectangle $R_T(1/4-\varepsilon,1/4+\varepsilon)$, since $\mathcal{C}(s)$ and the path integral over $\gamma(T,\varepsilon)$ are both defined on the smaller rectangle $R_T(1/4,1/4+\varepsilon)$. We note that this continuation clearly depends on the choice of the parameters T and ε. Further, we remark that given any point s on the vertical line $\mathrm{Re}(s)=1/4$ we can use the above construction to establish the meromorphic continuation of $\mathcal{C}(s)$ to some small neighbourhood of this point by choosing $T>0$ large enough.

Next we move the path of integration back to its original path. As before we start by fixing s, but this time we choose $s\in R_T(1/4-\varepsilon,1/4)$. Using again the residue theorem we then find

(6.2.9)

$$\int_{\gamma(T,\varepsilon)} g(s,w)dw = \int_{1/4+i\mathbb{R}} g(s,w)dw + 2\pi i\ \mathrm{Res}_{w=1/2-s}\, g(s,w)$$

$$= \mathcal{C}(s) + \frac{2^{1/2-2s}\ \pi}{2}\ \frac{\Gamma(2s-1/2)}{\Gamma(s)\ \Gamma(s+1/2)}\ Z(s;0,0,\beta,m)E_0(\tau,1/2-s).$$

Here, the path integral over $\gamma(T,\varepsilon)$ on the left-hand side is holomorphic on the rectangle $R_T(1/4-\varepsilon,1/4+\varepsilon)$, the path integral $\mathcal{C}(s)$ is holomorphic on the strip $-3/4<\mathrm{Re}(s)<1/4$ as mentioned above, and the second summand on the right-hand side is clearly meromorphic on all of $\mathbb{C}$. Therefore, equation (6.2.9) yields a meromorphic continuation of the path integral over $\gamma(T,\varepsilon)$ to the strip $-3/4<\mathrm{Re}(s)<1/4$.

Combining (6.2.8) and (6.2.9) we can thus define

$$\mathcal{C}^*(s) := \mathcal{C}(s) + \frac{\sqrt{2\pi}}{4}\,\frac{\Gamma(2s-1/2)}{\Gamma(2s)}\,Z(s;0,0,\beta,m)E_0(\tau,1/2-s)$$
$$+\frac{(2\pi|m|)^{1/2-2s}\pi}{2}Z(1/2-s;0,0,\beta,m)E_0(\tau,s)\begin{cases}\dfrac{\Gamma(2s-1/2)}{\Gamma(s)\Gamma(1-s)}, & \text{if } m>0,\\ \dfrac{\Gamma(2s-1/2)}{\Gamma(s+1/2)\Gamma(1/2-s)}, & \text{if } m<0,\end{cases}$$

for $s \in \mathbb{C}$ with $-3/4 < \mathrm{Re}(s) < 1/4$, away from the poles of the summands on the right-hand side of the equation. Here we have also used the duplication formula of the Gamma function to simplify the right-hand side of (6.2.9). By construction $\mathcal{C}^*(s)$ is the meromorphic continuation of the integral $\mathcal{C}(s)$ to the strip $-3/4 < \mathrm{Re}(s) < 1/4$, which is now independent of the choice of parameters T and ε.

Finally, we can continue this two-step process to obtain the meromorphic continuation of the path integral $\mathcal{C}(s)$ in s to all of $\mathbb{C}$, which then gives the claimed meromorphic continuation of the Poincaré series $V_{\beta,m}(\tau,s)$ to the whole complex plane. □

Corollary 6.2.3. *Let $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m \neq 0$. Then the meromorphic continuation of $V_{\beta,m}(\tau,s)$ is holomorphic at $s = 0$.*

Proof. We need to investigate the different terms appearing in the meromorphic continuation of $V_{\beta,m}(\tau,s)$ given in Proposition 6.2.2. Clearly, the sum corresponding to the discrete spectrum, namely

$$\sum_{j=0}^{\infty} a_j(s)\psi_j(\tau), \tag{6.2.10}$$

is holomorphic at $s = 0$ if and only if the spectral coefficients $a_j(s)$ are. By Proposition 6.2.1 we have

$$a_j(s) = 2(4\pi|m|)^{1/2-s}\,\overline{c_{\psi_j}(\beta,m)}\begin{cases}\dfrac{\Gamma(s-1/4+ir_j)\Gamma(s-1/4-ir_j)}{\Gamma(s)}, & \text{if } m>0,\\ \dfrac{\Gamma(s-1/4+ir_j)\Gamma(s-1/4-ir_j)}{\Gamma(s+1/2)}, & \text{if } m<0.\end{cases}$$

Thus, the coefficients $a_j(s)$ are holomorphic up to possible poles coming from the Gamma factors $\Gamma(s-1/4 \pm ir_j)$. As $r_j \geq 0$ or $r_j \in [i/4, 0)$, exactly one of these two factors can have a pole at $s = 0$, and this pole occurs if and only if $r_j = i/4$, which is exactly the case if $\lambda_j = 0$.

If $m > 0$ this possible pole at $s = 0$ is always compensated for by the corresponding factor $\Gamma(s)$ in the denominator. In fact, if $m > 0$ we thus have

$$a_j(0) = \begin{cases}-8\pi\sqrt{m}\;\overline{c_{\psi_j}(\beta,m)}, & \text{if } \lambda_j = 0,\\ 0, & \text{if } \lambda_j > 0.\end{cases}$$

as $\Gamma(-1/2) = -2\sqrt{\pi}$. In order to also deal with the case $m < 0$, we now suppose that $\lambda_j = 0$. Then the corresponding eigenfunction ψ_j is harmonic. Since ψ_j is also square-integrable, it grows at most polynomially at ∞, i.e., ψ_j is a square-integrable harmonic

Maass form of weight 1/2 for ρ_L, whose holomorphic part is of the form

$$\psi_j^+(\tau) = \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z}+q(\gamma) \\ n \geq 0}} a_{\psi_j}^+(\gamma, n) e(n\tau) \mathfrak{e}_\gamma,$$

as the terms $e(n\tau)$ grow exponentially if $n < 0$, and whose non-holomorphic part is of the form

$$\psi_j^-(\tau) = \sum_{\gamma \in L'/L} \left(a_{\psi_j}^-(\gamma, 0) v^{1/2} + \sum_{\substack{n \in \mathbb{Z}+q(\gamma) \\ n < 0}} a_{\psi_j}^-(\gamma, n) \Gamma(1/2, 4\pi|n|v) e(n\tau) \right) \mathfrak{e}_\gamma,$$

as the integral function $H_{1/2}(w)$ from [BF04, Section 3] grows exponentially for $w \to \infty$ with $w > 0$, and if $w < 0$ then $H_{1/2}(w) = e^{-w}\Gamma(1/2, -2w)$. Further, one can check that for weight 1/2 the non-holomorphic constant Fourier coefficients $a_{\psi_j}^-(\gamma, 0)$ also need to vanish, since $v^{1/2}$ is not square-integrable with respect to the Petersson inner product of weight 1/2. So by (3.5.1) the eigenfunction ψ_j is mapped to a cusp form under the differential operator $\xi_{1/2}$, i.e., we have $\psi_j \in H_{1/2,L}^+$. As we have also seen that the negative holomorphic Fourier coefficients $a_{\psi_j}^+(\gamma, n)$ with $n < 0$ vanish, we can apply Lemma 3.5.2, which yields $\psi_j \in M_{1/2,L}$. Therefore, the eigenfunction ψ_j does not have negative Fourier coefficients, i.e., if $\lambda_j = 0$ and $m < 0$ we have $c_{\psi_j}(\beta, m) = 0$. So the spectral coefficient $a_j(s)$ vanishes identically in this case, and is thus trivially holomorphic at $s = 0$.

Hence we have shown that the sum in (6.2.10) is holomorphic at $s = 0$. Furthermore, the integral

$$\frac{1}{16\pi} \int_{-\infty}^{\infty} a_\infty(s, r) E_0(\tau, 1/4 + ir)\, dr$$

is clearly holomorphic at the point $s = 0$, since the spectral coefficient $a_\infty(s, r)$ is holomorphic at $s = 0$ for all $r \in \mathbb{R}$. Therefore it remains to consider the additional terms

(6.2.11)

$$\frac{(2\pi|m|)^{1/2-2s}\pi}{2} Z(1/2 - s; 0, 0, \beta, m) E_0(\tau, s) \begin{cases} \dfrac{\Gamma(2s - 1/2)}{\Gamma(s)\ \Gamma(1 - s)}, & \text{if } m > 0, \\ \dfrac{\Gamma(2s - 1/2)}{\Gamma(s + 1/2)\Gamma(1/2 - s)}, & \text{if } m < 0, \end{cases}$$

and

(6.2.12)
$$\frac{\sqrt{2\pi}}{4} \frac{\Gamma(2s - 1/2)}{\Gamma(2s)} Z(s; 0, 0, \beta, m)\ E_0(\tau, 1/2 - s).$$

By Corollary 6.1.11 the Eisenstein series $E_0(\tau, s)$ is holomorphic at $s = 0$ and has a pole at $s = 1/2$, whereas by Corollary 6.1.12 the Kloosterman zeta function $Z(s; 0, 0, \beta, m)$ is also holomorphic at $s = 0$ and has a pole at $s = 1/2$ if and only if $4Nm$ is a non-negative square. Hence the expression in (6.2.12) is holomorphic at $s = 0$ since the pole coming from the Eisenstein series at $s = 1/2$ is compensated for by the factor $1/\Gamma(2s)$. Correspondingly, the expression in (6.2.11) is holomorphic at $s = 0$, since the Kloosterman zeta function $Z(1/2 - s; 0, 0, \beta, m)$ can only have a pole at $s = 0$ if m is positive, in which case the pole is compensated for by the factor $1/\Gamma(s)$. This proves that the meromorphic continuation of $V_{\beta,m}(\tau, s)$ is indeed holomorphic at $s = 0$. $\square$

Later on we further need the following technical lemma.

Lemma 6.2.4. *Let $\beta, \gamma \in L'/L$, $m \in \mathbb{Z} + q(\beta)$ with $m \neq 0$, and ℓ a positive integer.*

(a) For $n \in \mathbb{Z} + q(\gamma)$ with $n \neq 0$ the inner product

$$(V_{\beta,m}(\tau, s), V_{\gamma,n}(\tau, \bar{s} + \ell))$$

defines a meromorphic function in s for $\mathrm{Re}(s) > 3/4 - \ell$, which is holomorphic at $s = 0$.

(b) Let $\Omega \subseteq \{s \in \mathbb{C} \colon \mathrm{Re}(s) > 3/4 - \ell\}$ be compact sucht that the inner product given in part (a) is holomorphic on Ω for all $n \in \mathbb{Z} + q(\gamma)$ with $n \neq 0$. Then we find $\delta > 0$ with

$$|(V_{\beta,m}(\tau, s), V_{\gamma,n}(\tau, \bar{s} + \ell))| = O\big(|n|^{\delta}\big)$$

as $|n| \to \infty$ with $n \in \mathbb{Z} + q(\gamma)$, uniformly in s for $s \in \Omega$. Here the implied constant depends on β, γ, m, ℓ, τ and Ω.

Proof. Fix $\tau \in \mathbb{H}$ and let $s_0 \in \mathbb{C}$ with $\mathrm{Re}(s_0) > 3/4 - \ell$ and such that $V_{\beta,m}(\tau, s)$ is holomorphic at s_0. We further let U be a small neighbourhood of s_0, which satisfies the following conditions:

- The function $V_{\beta,m}(\tau, s)$ is holomorphic on U, and $\mathrm{Re}(s) > 3/4 - \ell$ for all $s \in U$.
- If $\mathrm{Re}(s_0) \neq 1/4, -3/4, \dots$ then either U lies in the half plane $\mathrm{Re}(s) > 1/4$, or U lies in a vertical strip of the form $-3/4 - n < \mathrm{Re}(s) > 1/4 - n$ for some $n \in \mathbb{N}_0$.
- If $\mathrm{Re}(s_0) = 1/4 - n$ with $n \in \mathbb{N}_0$ then $U \subseteq R - n$ with

$$R := \{s \in \mathbb{C} \colon 1/4 - 2\varepsilon < \mathrm{Re}(s) < 1/4, |\operatorname{Im}(s)| < 2T\}.$$

  Here we set $T := 1 + |\operatorname{Im}(s_0)|$, and we choose $\varepsilon > 0$ such that $\zeta(4s)$ is non-vanishing on the open rectangle R, as in the proof of Proposition 6.2.2.

Clearly, it now suffices to show that there is $\delta > 0$ such that

$$|(V_{\beta,m}(\tau, s), V_{\gamma,n}(\tau, \bar{s} + \ell))| = O\big(|n|^{\delta}\big)$$

as $|n| \to \infty$, uniformly in s for $s \in U$. In particular, this proves that the inner product on the left-hand side is holomorphic in s on U. Moreover, the inner product is then also holomorphic at $s = 0$, since $V_{\beta,m}(\tau, s)$ is holomorphic at $s = 0$ by Corollary 6.2.3, and $V_{\gamma,n}(\tau, \bar{s} + \ell)$ is clearly holomorphic at $s = \ell$ since $\ell > 3/4$, i.e., we can choose $s_0 = 0$.

In order to prove the above estimate, we define two functions $V_{\beta,m}^{(1)}(s)$ and $V_{\beta,m}^{(2)}(s)$ on U: If $\mathrm{Re}(s_0) \neq 1/4, -3/4, \dots$ we set

$$V_{\beta,m}^{(1)}(s) := \sum_{j=0}^{\infty} a_j(s)\psi_j(\tau) + \frac{1}{16\pi} \int_{-\infty}^{\infty} a_{\infty}(s, r) E_0(\tau, 1/4 + ir) dr,$$

for $s \in U$, and if $\operatorname{Re}(s_0) = 1/4, -3/4, \ldots$ we set

$$V^{(1)}_{\beta,m}(s) := \sum_{j=0}^{\infty} a_j(s)\psi_j(\tau) + \int_{\gamma(T,\varepsilon)} g(s,w)dw$$

for $s \in U$. Here the path $\gamma(T,\varepsilon)$ and the integrand $g(s,w)$ are defined as in the proof of Proposition 6.2.2. Further, we set

$$V^{(2)}_{\beta,m}(s) := V_{\beta,m}(\tau,s) - V^{(1)}_{\beta,m}(s) \tag{6.2.13}$$

for $s \in U$. In the sense of Proposition 6.2.2, $V^{(1)}_{\beta,m}(s)$ is the spectral part of the meromorphic continuation of the Poincaré series $V_{\beta,m}(\tau,s)$, and $V^{(2)}_{\beta,m}(s)$ is the additional Eisenstein part appearing while continuing onto and over the vertical lines $\operatorname{Re}(s) = 1/4 - n$ for $n \in \mathbb{N}_0$.

By the Cauchy-Schwarz inequality we now find

(6.2.14)
$$|(V_{\beta,m}(\tau,s), V_{\gamma,n}(\tau,\bar{s}+\ell))| \leq \|V^{(1)}_{\beta,m}(s)\| \cdot \|V_{\gamma,n}(\tau,\bar{s}+\ell)\| + |(V^{(2)}_{\beta,m}(s), V_{\gamma,n}(\tau,\bar{s}+\ell))|$$

for $s \in U$, assuming that all the terms appearing on the right-hand side exist. Here $\|\cdot\|$ denotes the usual norm induced by the inner product $(\cdot,\cdot)$.

Firstly, we note that $\|V_{\gamma,n}(\tau,\bar{s}+\ell)\| = O(1)$ by Lemma 3.6.9 since $\operatorname{Re}(\bar{s}+\ell) > 3/4$ for all $s \in U$. Secondly, one can check that the function $V^{(1)}_{\beta,m}(s)$ defined above is indeed square-integrable, giving $\|V^{(1)}_{\beta,m}(s)\| = O(1)$ as $V^{(1)}_{\beta,m}(s)$ does not depend on n. Thus it remains to show that there is $\delta' > 0$ such that

$$|(V^{(2)}_{\beta,m}(s), V_{\gamma,n}(\tau,\bar{s}+\ell))| = O(|n|^{\delta'}) \tag{6.2.15}$$

as $|n| \to \infty$, uniformly in s for $s \in U$. We show in the following that this is indeed true, even though the function $V^{(2)}_{\beta,m}(s)$ itself is in general not square-integrable.

Suppose that $\operatorname{Re}(s_0) > 1/4$. Then $V^{(2)}_{\beta,m} \equiv 0$ by Proposition 6.2.2 and thus (6.2.15) is trivially true. Next, we suppose that $\operatorname{Re}(s_0) = 1/4$. Then

$$V^{(2)}_{\beta,m} = \frac{(2\pi|m|)^{1/2-2s}\pi}{2} Z(1/2-s;0,0,\beta,m)E_0(\tau,s) \begin{cases} \dfrac{\Gamma(2s-1/2)}{\Gamma(s)\Gamma(1-s)}, & \text{if } m > 0, \\ \dfrac{\Gamma(2s-1/2)}{\Gamma(s+1/2)\Gamma(1/2-s)}, & \text{if } m < 0, \end{cases}$$

as was shown in the proof of Proposition 6.2.2, and hence we find

$$\begin{aligned}(V^{(2)}_{\beta,m}(s), &V_{\gamma,n}(\tau,\bar{s}+\ell)) \\ &= \frac{(2\pi|m|)^{1/2-2s}\pi}{2} Z(1/2-s;0,0,\beta,m)\frac{\Gamma(2s-1/2)}{\Gamma(s)\Gamma(1-s)}(E_0(\tau,s), V_{\gamma,n}(\tau,\bar{s}+\ell)),\end{aligned}$$

where we have to replace the denominator $\Gamma(s)\Gamma(1-s)$ by $\Gamma(s+1/2)\Gamma(1/2-s)$ if $m<0$. Here the remaining inner product on the right-hand side was already computed in the proof of Proposition 6.2.1, namely in equation (6.2.3), giving

$$\begin{aligned}&(E_0(\tau,s), V_{\gamma,n}(\tau,\bar{s}+\ell))\\ &\qquad = 2^{5/2-2s-2\ell}\pi^{1-\ell}|n|^{-\ell}Z(s;0,0,\gamma,n)\begin{cases}\dfrac{\Gamma(\ell)\Gamma(2s+\ell-1/2)}{\Gamma(s+\ell)\Gamma(s+1/2)}, & \text{if } m>0,\\ \dfrac{\Gamma(\ell)\Gamma(2s+\ell-1/2)}{\Gamma(s+\ell+1/2)\Gamma(s)}, & \text{if } m<0.\end{cases}\end{aligned}$$

So the inner product $(V^{(2)}_{\beta,m}(s), V_{\gamma,n}(\tau,\bar{s}+\ell))$ exists, and behaves as $|n|^{-\ell}Z(s;0,0,\gamma,n)$ in the variable n. Thus, the estimate in (6.2.15) follows from Theorem 6.1.8.

Now suppose that $-3/4<\mathrm{Re}(s_0)<1/4$. Then

$$\begin{aligned}V^{(2)}_{\beta,m}(s) = {}&\frac{(2\pi|m|)^{1/2-2s}\pi}{2}Z(1/2-s;0,0,\beta,m)E_0(\tau,s)\frac{\Gamma(2s-1/2)}{\Gamma(s)\Gamma(1-s)}\\ &+\frac{\sqrt{2\pi}}{4}\frac{\Gamma(2s-1/2)}{\Gamma(2s)}Z(s;0,0,\beta,m)E_0(\tau,1/2-s).\end{aligned}$$

by Proposition 6.2.2, with $\Gamma(s)\Gamma(1-s)$ being replaced by $\Gamma(s+1/2)\Gamma(1/2-s)$ if $m<0$. Hence, the inner product $(V^{(2)}_{\beta,m}(s), V_{\gamma,n}(\tau,\bar{s}+\ell))$ is now given by

$$\begin{aligned}&\frac{(2\pi|m|)^{1/2-2s}\pi}{2}Z(1/2-s;0,0,\beta,m)\frac{\Gamma(2s-1/2)}{\Gamma(s)\Gamma(1-s)}(E_0(\tau,s),V_{\gamma,n}(\tau,\bar{s}+\ell))\\ &\quad+\frac{\sqrt{2\pi}}{4}\frac{\Gamma(2s-1/2)}{\Gamma(2s)}Z(s;0,0,\beta,m)(E_0(\tau,1/2-s),V_{\gamma,n}(\tau,\bar{s}+\ell)),\end{aligned}$$

with $\Gamma(s)\Gamma(1-s)$ being replaced by $\Gamma(s+1/2)\Gamma(1/2-s)$ if $m<0$. Here the first summand has already been computed above. Analogously, we find that the inner product of the second summand exists, and behaves as $|n|^{1/2-2s-\ell}Z(1/2-s;0,0,\gamma,n)$ in the variable n. So the estimate (6.2.15) follows again from Theorem 6.1.8.

Eventually, we note that if $\mathrm{Re}(s_0)\le -3/4$ we can argue analogously, as the additional terms defining the function $V^{(2)}_{\beta,m}(s)$ are essentially given by Eisenstein series $E_0(\tau,s+j)$ and $E_0(\tau,1/2-s-j)$ for $j\in\mathbb{N}_0$, which can again be integrated against $V_{\gamma,n}(\tau,\bar{s}+\ell)$ using equation (6.2.3). The estimate (6.2.15) then follows again from Theorem 6.1.8.

Finally, using the estimate (6.2.15), the statement of the present lemma follows from (6.2.14). □

We can now use the previous lemma to prove the meromorphic continuation of Kloosterman zeta functions $Z(s;\beta,m,\gamma,n)$ of weight $1/2$ associated to the Weil representation ρ_L for $m\neq 0$. This generalizes Theorem 6.1.5. However, we quickly present another lemma, which shows that the Kloosterman zeta function $Z(s;\beta,m,\gamma,n)$ is symmetric in the pairs (β,m) and (γ,n).

Lemma 6.2.5. *Let $\beta,\gamma\in\mathbb{Z}/2N\mathbb{Z}$ and $m\in\mathbb{Z}+\beta^2/4N$, $n\in\mathbb{Z}+\gamma^2/4N$. Then*

$$Z(s;\beta,m,\gamma,n)=Z(s;\gamma,n,\beta,m)$$

for $s\in\mathbb{C}$ with $\mathrm{Re}(s)>3/4$.

Proof. Let $c \neq 0$, and recall that (M, r) is the lattice $M = \mathbb{Z}$ with $r(x) = Nx^2$ from (6.1.1). Using the identities (6.1.6) and (6.1.7) we find that

$$H^M_{c,1/2}(\beta, m, \gamma, n) = -H^{M,*}_{-c,3/2}(\beta, -m, \gamma, -n) = H^M_{c,1/2}(\gamma, n, \beta, m),$$

where $H^{M,*}_{c,3/2}(\beta, -m, \gamma, -n)$ is the dual Kloosterman sum defined in equation (6.1.4). Now it remains to recall that the discriminant forms M'/M and L'/L are isomorphic, giving $H^L_{c,1/2}(\beta, m, \gamma, n) = H^M_{c,1/2}(\beta, m, \gamma, n)$ as in (6.1.2). Hence, also the corresponding Kloosterman zeta functions agree for $\mathrm{Re}(s) > 3/4$. □

The idea of the proof of the following theorem is taken from [Pri00]. We remark that in [Pri00] the author assumes that the given multiplier system does not allow the existence of Eisenstein series, i.e., he assumes that the hyperbolic Laplace operator $\Delta_{1/2}$ has only discrete spectrum. This is not true in our case. However, in [Pri00] this assumption is mainly used in part (c) of Lemma 3, in order to derive good estimates for the Petersson norms $\|V_{\beta,m}(\tau, s)\|$ as $m \to \infty$. Since m is fixed in our setting, this bound is indeed trivial here. Moreover, in the setting of [Pri00] the proof of Lemma 6.2.4 simplifies considerably, since the function $V^{(2)}_{\beta,m}$ defined in (6.2.13) in the course of our proof of the lemma is always trivial in the setting of [Pri00].

Theorem 6.2.6. *Let $\beta, \gamma \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m \neq 0$.*

(a) For $n \in \mathbb{Z} + q(\gamma)$ the Kloosterman zeta function $Z(s; \beta, m, \gamma, n)$ has a meromorphic continuation in s to all of $\mathbb{C}$, which is holomorphic at $s = 0$ if $m > 0$ or $n \geq 0$. Otherwise, i.e., if $m, n < 0$, this continuation has at most a simple pole at $s = 0$.

(b) Let $\Omega \subseteq \mathbb{C}$ be compact such that $Z(s; \beta, m, \gamma, n)$ is holomorphic on Ω for all $n \in \mathbb{Z} + q(\gamma)$. Then we find $\delta > 0$ such that

$$|Z(s; \beta, m, \gamma, n)| = O\big(|n|^\delta\big)$$

as $|n| \to \infty$ with $n \in \mathbb{Z} + q(\gamma)$, uniformly in s for $s \in \Omega$. Here the implied constant depends on β, γ, m and Ω.

(c) Let $m < 0$, and let $\Omega \subseteq \mathbb{C}$ be compact with $0 \in \Omega$ such that $Z(s; \beta, m, \gamma, n)/\Gamma(s)$ is holomorphic on Ω for all $n \in \mathbb{Z} + q(\gamma)$ with $n < 0$. Then we find $\delta > 0$ such that

$$\left| \frac{Z(s; \beta, m, \gamma, n)}{\Gamma(s)} \right| = O\big(|n|^\delta\big)$$

as $n \to -\infty$ with $n \in \mathbb{Z} + q(\gamma)$ and $n < 0$, uniformly in s for $s \in \Omega$. Here the implied constant depends on β, γ, m and Ω.

Proof. In addition to β, γ and m we fix $n \in \mathbb{Z} + q(\gamma)$. If $n = \gamma = 0$ we have

$$Z(s; \beta, m, 0, 0) = Z(s; 0, 0, \beta, m).$$

for $\mathrm{Re}(s) > 3/4$ by Lemma 6.2.5. Thus, for $n = 0$ part (a) follows directly from the given identity and Corollary 6.1.12. Moreover, we can ignore the case $n = 0$ in part (b) and (c), since we are only interested in the asymptotic behaviour of $Z(s; \beta, m, \gamma, n)$ as $n \to \pm\infty$.

So let $n \neq 0$, and let ℓ be a positive integer. As in Lemma 6.2.4 we consider the inner product

$$(V_{\beta,m}(\tau, s), V_{\gamma,n}(\tau, \bar{s} + \ell)).$$

For $\mathrm{Re}(s) > 3/4$ we may unfold the integral against the Poincaré series $V_{\gamma,n}(\tau, \bar{s} + \ell)$, finding

(6.2.16)

$$(V_{\beta,m}(\tau, s), V_{\gamma,n}(\tau, \bar{s} + \ell)) = 2 \int_0^\infty v^{s+\ell-3/2} e^{-2\pi|n|v} \int_0^1 \langle V_{\beta,m}(\tau, s), \mathfrak{e}_\gamma \rangle \, e(-nu) du dv.$$

as in equation (6.2.1). The inner integral is the (γ, n)'th Fourier coefficient of $V_{\beta,m}(\tau, s)$, which by Proposition 3.6.6 (for $j = 3$) is given by

$$b(\gamma, n, v, s) = v^s e^{-2\pi|m|v}(\delta_{\beta,\gamma} + \delta_{\beta,-\gamma})\delta_{m,n} + v^s \sum_{c \in \mathbb{Z}\setminus\{0\}} |c|^{1/2-2s} H_c(\beta, m, \gamma, n) J_m(n, v, s, c).$$

Here $J_m(n, v, s, c)$ is the integral function defined in (3.6.5), namely

$$J_m(n, v, s, c) = i^{1/2} \int_{-\infty}^{\infty} \tau^{-1/2}|\tau|^{-2s} e\left(-\frac{m}{c^2\tau^*}\right) e\left(-nu\right) du,$$

with $\tau^* := \tau$ if $m > 0$ and $\tau^* := \bar{\tau}$ if $m < 0$. In Proposition 3.6.6 it is further shown that $J_m(n, v, s, c)$ is real analytic in v for $v > 0$ and holomorphic in s for $\mathrm{Re}(s) > 1/4$.

As in the proof of Proposition 3.6.7 and Proposition 3.6.8 we expand the exponential function $e(-\frac{m}{c^2\tau^*})$ as

$$e\left(-\frac{m}{c^2\tau^*}\right) = \sum_{j=0}^{\infty} \frac{1}{j!} \left(-\frac{2\pi i m}{c^2\tau^*}\right)^j.$$

Now, instead of interchanging the complete sum over j with the integral over u and the sum over c as in the proof of Proposition 3.6.7 and Proposition 3.6.8, we only pull out the terms $j = 0, \dots, \ell - 1$, giving

$$\begin{aligned} b(\gamma, n, v, s) &= v^s e^{-2\pi|m|v}(\delta_{\beta,\gamma} + \delta_{\beta,-\gamma})\delta_{m,n} \\ &\quad + v^s \sum_{j=0}^{\ell-1} \frac{(-2\pi i m)^j}{j!} Z(s+j; \beta, m, \gamma, n) \cdot i^{1/2} \int_{-\infty}^{\infty} \frac{e(-nu)}{\tau^{1/2}(\tau^*)^j|\tau|^{2s}} du \\ &\quad + v^s \sum_{c \in \mathbb{Z}\setminus\{0\}} |c|^{1/2-2s} H_c(\beta, m, \gamma, n) \cdot i^{1/2} \int_{-\infty}^{\infty} \frac{e(-nu)}{\tau^{1/2}|\tau|^{2s}} \sum_{j=\ell}^{\infty} \frac{1}{j!} \left(-\frac{2\pi i m}{c^2\tau^*}\right)^j du. \end{aligned}$$

Now the first integral can be computed for $j = 0, \dots, \ell - 1$ using the integral formula given in (3.6.8), which yields

$$
\begin{aligned}
&i^{1/2}\int_{-\infty}^{\infty}\frac{e(-nu)}{\tau^{1/2}(\tau^*)^j|\tau|^{2s}}du\\
&\quad= 2^{1/2}\pi^{s+1/2}|n|^{s-1/2}v^{-s}(\operatorname{sign}(m)2\pi i|n|)^j\\
&\qquad\times\begin{cases}
\dfrac{(4\pi|n|v)^{-1/4-j/2}}{\Gamma(s+j+1/2)}W_{1/4+j/2,s-1/4+j/2}(4\pi|n|v), & \text{if } m,n>0,\\
\dfrac{(4\pi|n|v)^{-1/4-j/2}}{\Gamma(s)}W_{-1/4-j/2,s-1/4+j/2}(4\pi|n|v), & \text{if } m>0,\ n<0,\\
\dfrac{(4\pi|n|v)^{-1/4-j/2}}{\Gamma(s+1/2)}W_{1/4-j/2,s-1/4-j/2}(4\pi|n|v), & \text{if } m<0,\ n>0,\\
\dfrac{(4\pi|n|v)^{-1/4-j/2}}{\Gamma(s+j)}W_{-1/4+j/2,s-1/4-j/2}(4\pi|n|v), & \text{if } m,n<0.
\end{cases}
\end{aligned}
$$

Plugging all of this into equation (6.2.16) we obtain

(6.2.17)

$$
\begin{aligned}
&(V_{\beta,m}(\tau,s),V_{\gamma,n}(\tau,\bar{s}+\ell))\\
&\quad=\frac{2\,\Gamma(2s+\ell-1/2)}{(4\pi|m|)^{2s+\ell-1/2}}(\delta_{\beta,\gamma}+\delta_{\beta,-\gamma})\delta_{m,n}+\sum_{j=0}^{\ell-1}A_\ell^{(j)}(s;m,n)Z(s+j;\beta,m,\gamma,n)\\
&\qquad\qquad+R_\ell(s;\beta,m,\gamma,n),
\end{aligned}
$$

where the functions $A_\ell^{(j)}(s;m,n)$ and $R_\ell(s;\beta,m,\gamma,n)$ are given by

$$
A_\ell^{(j)}(s;m,n):=\frac{4^{5/4-s}\pi}{(4\pi|n|)^\ell}\ \frac{(4\pi^2|mn|)^j}{j!}\begin{cases}
\dfrac{\Gamma(2s+\ell-1/2)\,\Gamma(\ell-j)}{\Gamma(s+j+1/2)\,\Gamma(s+\ell-j)}, & \text{if } m,n>0,\\
\dfrac{\Gamma(2s+\ell-1/2)\,\Gamma(\ell-j)}{\Gamma(s)\,\Gamma(s+\ell+1/2)}, & \text{if } m>0,\ n<0,\\
\dfrac{\Gamma(2s+\ell-1/2)\,\Gamma(\ell-j)}{\Gamma(s+1/2)\,\Gamma(s+\ell)}, & \text{if } m<0,\ n>0,\\
\dfrac{\Gamma(2s+\ell-1/2)\,\Gamma(\ell-j)}{\Gamma(s+j)\,\Gamma(s+\ell+1/2-j)}, & \text{if } m,n<0,
\end{cases}
$$

and

$$
\begin{aligned}
R_\ell(s;\beta,m,\gamma,n):=2i^{1/2}\sum_{c\in\mathbb{Z}\setminus\{0\}}|c|^{1/2-2s}H_c(\beta,m,\gamma,n)\int_0^\infty v^{2s+\ell-3/2}e^{-2\pi|n|v}&\\
\times\int_{-\infty}^{\infty}\frac{e(-nu)}{\tau^{1/2}|\tau|^{2s}}\sum_{j=\ell}^{\infty}\frac{1}{j!}\left(-\frac{2\pi im}{c^2\tau^*}\right)^j dudv.&
\end{aligned}
$$

Here we have used the formula 7.621.11 in [GR07] to compute the remaining integral over v for the functions $A_\ell^{(j)}(s;m,n)$, assuming that $\operatorname{Re}(s)>1/4-j/2$ and $j<\ell$. We now rewrite equation (6.2.17) as

(6.2.18)

$$
\begin{aligned}
&Z(s;\beta,m,\gamma,n)\\
&\quad= \frac{1}{A_\ell^{(0)}(s;m,n)}\Bigg((V_{\beta,m}(\tau,s),V_{\gamma,n}(\tau,\bar{s}+\ell)) - \frac{2\,\Gamma(2s+\ell-1/2)}{(4\pi|m|)^{2s+\ell-1/2}}(\delta_{\beta,\gamma}+\delta_{\beta,-\gamma})\delta_{m,n}\\
&\qquad\qquad - \sum_{j=1}^{\ell-1} A_\ell^{(j)}(s;m,n) Z(s+j;\beta,m,\gamma,n) - R_\ell(s;\beta,m,\gamma,n)\Bigg).
\end{aligned}
$$

Next, we collect the meromorphic properties of terms on the right-hand side of (6.2.18):

- By part (a) of Lemma 6.2.4 the inner product $(V_{\beta,m}(\tau,s),V_{\gamma,n}(\tau,\bar{s}+\ell))$ defines a meromorphic function in s for $\mathrm{Re}(s) > 3/4-\ell$.
- The possible extra term $2(4\pi|m|)^{1/2-2s-\ell}\Gamma(2s+\ell-1/2)$, which only appears if $\beta = \pm\gamma$ and $m = n$, is clearly meromorphic on all of $\mathbb{C}$.
- Also the functions $A_\ell^{(j)}(s;m,n)$ with $j = 0,\dots,\ell-1$ are by definition meromorphic in s on the whole complex plane.
- Assuming that $Z(s;\beta,m,\gamma,n)$ is meromorphic for $\mathrm{Re}(s) > \alpha$ for some $\alpha \in \mathbb{R}$, the shifted functions $Z(s+j;\beta,m,\gamma,n)$ with $j = 1,\dots,\ell-1$ are meromorphic for $\mathrm{Re}(s) > \alpha - 1$.

Hence, it remains to consider the function $R_\ell(s;\beta,m,\gamma,n)$. By definition, we have

$$
\begin{aligned}
&|R_\ell(s;\beta,m,\gamma,n)|\\
&\leq 2\sum_{c\in\mathbb{Z}\setminus\{0\}} \frac{|H_c(\beta,m,\gamma,n)|}{|c|^{2\sigma-1/2}} \int_0^\infty v^{2\sigma+\ell-3/2}e^{-2\pi|n|v}\int_{-\infty}^\infty |\tau|^{-2\sigma-1/2}\left|\sum_{j=\ell}^\infty \frac{1}{j!}\left(-\frac{2\pi i m}{c^2\tau^*}\right)^j\right| du dv
\end{aligned}
$$

with $\sigma := \mathrm{Re}(s)$. Now, one can check that given $r \in [\ell-1,\ell)$ we have

$$
\left|\sum_{j=\ell}^\infty \frac{z^j}{j!}\right| = \left|e^z - \sum_{j=0}^{\ell-1}\frac{z^j}{j!}\right| < (e+1)|z|^r < 4|z|^r, \tag{6.2.19}
$$

for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) < 0$ (see for example [Pri00], page 454). Recalling that $\tau^* = \tau$ if $m > 0$ and $\tau^* = \bar{\tau}$ if $m < 0$, we indeed find

$$
\mathrm{Re}\left(-\frac{2\pi i m}{c^2\tau^*}\right) = -\frac{2\pi|m|v}{c^2|\tau|^2} < 0.
$$

Hence, given $r \in [\ell-1,\ell)$ we can apply the estimate given in (6.2.19), giving

(6.2.20)

$$\begin{aligned}
&|R_\ell(s;\beta,m,\gamma,n)| \\
&\quad \le 8(2\pi|m|)^r \sum_{c\in\mathbb{Z}\setminus\{0\}} \frac{|H_c(\beta,m,\gamma,n)|}{|c|^{2\sigma+2r-1/2}} \int_0^\infty v^{2\sigma+\ell-3/2} e^{-2\pi|n|v} \int_{-\infty}^\infty |\tau|^{-2\sigma-r-1/2} du dv \\
&\quad = 8\sqrt{\pi}\,(2\pi|m|)^r \frac{\Gamma(\sigma+r/2-1/4)}{\Gamma(\sigma+r/2+1/4)} \int_0^\infty v^{\ell-r-1} e^{-2\pi|n|v} dv \sum_{c\in\mathbb{Z}\setminus\{0\}} \frac{|H_c(\beta,m,\gamma,n)|}{|c|^{2\sigma+2r-1/2}} \\
&\quad \le 16\sqrt{\pi}\, C_L \frac{(4\pi^2|mn|)^r}{(2\pi|n|)^\ell} \frac{\Gamma(\sigma+r/2-1/4)\,\Gamma(\ell-r)}{\Gamma(\sigma+r/2+1/4)} \zeta(2\sigma+2r-1/2)
\end{aligned}$$

for $\sigma > 1/4 - r/2$, $r < \ell$ and $\sigma > 3/4 - r$. Here we have used the integral formula (3.6.8) to compute the integral over u in the second step, and the fact that the Kloosterman sums $H_c(\beta,m,\gamma,n)$ are unviersally bounded by some constant $C_L > 0$ only depending on the underlying lattice L in the third step. The condition $r < \ell$ is clearly satisfied since $r \in [\ell-1,\ell)$. We write $r = \ell - \varepsilon$ with $\varepsilon \in (0,1)$. Now the other two conditions $\sigma > 1/4 - r/2$ and $\sigma > 3/4 - r$ can be summarized as

$$\sigma > \max(1/4 - r/2, 3/4 - r) = \begin{cases} -1/4+\varepsilon, & \text{if } \ell = 1, \\ 1/4 - \ell/2 + \varepsilon/2, & \text{if } \ell > 1. \end{cases}$$

Since we can choose $\varepsilon > 0$ arbitrary small, the function $R_\ell(s;\beta,m,\gamma,n)$ is indeed well-defined for every $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1/4 - \ell/2$, and thus defines a holomorphic function on the half-plane $\mathrm{Re}(s) > 1/4 - \ell/2$.

We have now shown, that under the assumption that $Z(s;\beta,m,\gamma,n)$ is meromorphic for $\mathrm{Re}(s) > 5/4 - \ell/2$, the right-hand side of (6.2.18) defines a meromorphic function for $\mathrm{Re}(s) > 1/4 - \ell/2$, which yields the meromorphic continuation of the Kloosterman zeta function $Z(s;\beta,m,\gamma,n)$ to the corresponding half-plane $\mathrm{Re}(s) > 1/4 - \ell/2$.

Recall that $Z(s;\beta,m,\gamma,n)$ is by definition holomorphic for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 3/4$. Thus, choosing $\ell = 1$ equation (6.2.18) yields a meromorphic continuation of the Kloosterman zeta function to all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > -1/4$. Now, choosing $\ell = 3$, equation (6.2.18) yields a meromorphic continuation to the half-plane $\mathrm{Re}(s) > -5/4$, and so forth. By induction over the positive odd integers $\ell = 1, 3, \ldots$ we hence obtain the meromorphic continuation of the function $Z(s;\beta,m,\gamma,n)$ to all of $\mathbb{C}$.

In order to finish the proof of part (a) we further need to study the behaviour of the meromorphic continuation of $Z(s;\beta,m,\gamma,n)$ at $s = 0$. For $\ell = 1$ equation (6.2.18) becomes

(6.2.21)

$$\begin{aligned}
Z(s;\beta,m,\gamma,n) = \frac{1}{A_1^{(0)}(s;m,n)} \Bigg(&(V_{\beta,m}(\tau,s), V_{\gamma,n}(\tau,\bar{s}+1)) - R_1(s;\beta,m,\gamma,n) \\
&- \frac{2\,\Gamma(2s+1/2)}{(4\pi|m|)^{2s+1/2}} (\delta_{\beta,\gamma} + \delta_{\beta,-\gamma})\delta_{m,n} \Bigg).
\end{aligned}$$

By part (a) of Lemma 6.2.4 the inner product on the right-hand side is holomorphic at $s = 0$, and by the above considerations the function $R_1(s;\beta,m,\gamma,n)$ is holomorphic for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > -1/4$, hence also at the point $s = 0$. Further, the possible extra term $2(4\pi|m|)^{-2s-1/2}\Gamma(2s+1/2)$ is clearly holomorphic at $s = 0$, too. So it remains to consider the factor in front, namely

$$\frac{1}{A_1^{(0)}(s;m,n)} = 4^{s-1/4}|n| \begin{cases} \dfrac{\Gamma(s+1/2)\,\Gamma(s+1)}{\Gamma(2s+1/2)}, & \text{if } n > 0, \\ \dfrac{\Gamma(s)\,\Gamma(s+3/2)}{\Gamma(2s+1/2)}, & \text{if } n < 0. \end{cases} \tag{6.2.22}$$

Thus, if $n > 0$ the given term is holomorphic at $s = 0$, but if $n < 0$ it has a simple pole at $s = 0$ coming from the Gamma factor $\Gamma(s)$. Therefore, equation (6.2.21) shows that the meromorphic continuation of $Z(s;\beta,m,\gamma,n)$ is holomorphic at $s = 0$ if $n > 0$, but might have a simple pole at $s = 0$ if $n < 0$. We claim that this pole vanishes if $m > 0$. Instead of showing that the difference

$$(V_{\beta,m}(\tau,s), V_{\gamma,n}(\tau,\bar{s}+1)) - R_1(s;\beta,m,\gamma,n)$$

has a zero at $s = 0$ if $m > 0$ and $n < 0$, we use again Lemma 6.2.5, which states that

$$Z(s;\beta,m,\gamma,n) = Z(s;\gamma,n,\beta,m)$$

for $\mathrm{Re}(s) > 3/4$. Hence, also the analytic continuations of the functions $Z(s;\beta,m,\gamma,n)$ and $Z(s;\gamma,n,\beta,m)$ need to agree whenever they are defined. In particular, the Kloosterman zeta function $Z(s;\beta,m,\gamma,n)$ needs to be holomorphic at $s = 0$ if $m > 0$ and $n < 0$, since $Z(s;\gamma,n,\beta,m)$ is holomorphic at $s = 0$ if $m > 0$ by the above considerations. This finishes the proof of part (a).

For the proof of part (b) we let $\Omega \subseteq \mathbb{C}$ be compact such that $Z(s;\beta,m,\gamma,n)$ is holomorphic on Ω for all $n \in \mathbb{Z} + q(\gamma)$. Further, we let ℓ be a positive integer as before. By equation (6.2.18) we have

$$\begin{aligned} &|Z(s;\beta,m,\gamma,n)| \\ &\leq \left|\frac{1}{A_\ell^{(0)}(s;m,n)}\right| \Bigg(|(V_{\beta,m}(\tau,s), V_{\gamma,n}(\tau,\bar{s}+\ell))| + \left|\frac{2\,\Gamma(2s+\ell-1/2)}{(4\pi|m|)^{2s+\ell-1/2}}\right| (\delta_{\beta,\gamma}+\delta_{\beta,-\gamma})\delta_{m,n} \\ &\qquad + \sum_{j=1}^{\ell-1} \left|A_\ell^{(j)}(s;m,n)\right| \cdot |Z(s+j;\beta,m,\gamma,n)| + |R_\ell(s;\beta,m,\gamma,n)| \Bigg) \end{aligned}$$

for $\mathrm{Re}(s) > 1/4 - \ell/2$. We quickly collect the asymptotic properties of the terms on the right-hand side as $|n| \to \infty$ with $n \in \mathbb{Z} + q(\gamma)$. Here we can assume that $n \neq 0$ and $n \neq m$.

- Part (b) of Lemma 6.2.4 states that $|(V_{\beta,m}(\tau,s), V_{\gamma,n}(\tau,\bar{s}+\ell))| = O(|n|^\delta)$ for some $\delta > 0$ as $|n| \to \infty$, uniformly in s for $s \in \Omega$.
- The possible extra term vanishes since we can assume that $n \neq m$ as $|n| \to \infty$.

- By definition we have $|A_\ell^{(0)}(s;m,n)|^{-1} = O(|n|^\ell)$, and $|A_\ell^{(j)}(s;m,n)| = O(|n|^{j-\ell})$ for $j = 1, \ldots, \ell-1$, as $|n| \to \infty$, uniformly in s for $s \in \Omega$.
- The estimate given in (6.2.20) proves that $|R_\ell(s;\beta,m,\gamma,n)| = O(1)$ as $|n| \to \infty$, uniformly in s for $s \in \Omega$.

Therefore, we can again use induction over the positive odd integers $\ell = 1, 3, \ldots$ to show that $|Z(s;\beta,m,\gamma,n)| = O(|n|^\delta)$ for some $\delta > 0$ as $|n| \to \infty$, uniformly in s for $s \in \Omega$. This proves part (b) of the theorem.

Finally, let $m < 0$, and let $\Omega \subseteq \mathbb{C}$ be compact with $0 \in \Omega$ such that $Z(s;\beta,m,\gamma,n)/\Gamma(s)$ is holomorphic on Ω for all $n \in \mathbb{Z} + q(\gamma)$ with $n < 0$. Without loss of generality we may assume that Ω lies in the half-plane defined by $\mathrm{Re}(s) > -1/4$. Then

$$\left|\frac{Z(s;\beta,m,\gamma,n)}{\Gamma(s)}\right| \leq \left|\frac{1}{\Gamma(s)A_1^{(0)}(s;m,n)}\right| \Big(|(V_{\beta,m}(\tau,s), V_{\gamma,n}(\tau,\bar{s}+1))| + |R_1(s;\beta,m,\gamma,n)| \Big)$$

for all $s \in \Omega$ by equation (6.2.21). Here we can ignore the additional term containing the factor $\delta_{m,n}$, since m is fixed and we are only interested in the asymptotic behaviour in n as $n \to -\infty$. As before, we find that the remaining inner product behaves as $|n|^\delta$ for some $\delta > 0$ as $n \to -\infty$, uniformly in s for $s \in \Omega$, and that $|R_1(s;\beta,m,\gamma,n)| = O(1)$ as $n \to -\infty$, uniformly in s for $s \in \Omega$. Moreover, by (6.2.22) we see that

$$\left|\frac{1}{\Gamma(s)A_1^{(0)}(s;m,n)}\right| = \left|4^{s-1/4}|n|\frac{\Gamma(s+3/2)}{\Gamma(2s+1/2)}\right| = O(|n|)$$

as $n \to -\infty$ with $n \in \mathbb{Z} + q(\gamma)$ and $n < 0$, uniformly in s for $s \in \Omega$. This also proves part (c) of the theorem. □

6.3 Continuation of Selberg's Poincaré series of the first kind via its Fourier expansion

Recall that we have shown in Proposition 6.2.2 that for $m \neq 0$ Selberg's Poincaré series of the second kind $V_{\beta,m}(\tau,s)$ has a meromorphic continuation in s to all of $\mathbb{C}$, which is essentially given by its spectral expansion, and that $U_{\beta,m}(\tau,s) = V_{\beta,m}(\tau,s)$ if $m > 0$. In the present section we show that for $m < 0$ Selberg's Poincaré series of the first kind $U_{\beta,m}(\tau,s)$ also has a meromorphic continuation in s to the whole complex plane. However, we cannot use spectral theory here since $U_{\beta,m}(\tau,s)$ is not square-integrable if $m < 0$. Instead, we establish the continuation of the Poincaré series $U_{\beta,m}(\tau,s)$ via its Fourier expansion. In particular, we also obtain that for $m > 0$ the known meromorphic continuation of $U_{\beta,m}(\tau,s)$ is also given by its Fourier expansion.

Since the Fourier expansion of $U_{\beta,m}(\tau,s)$ is crucial in the present section, we recall it: By Proposition 3.6.7 the Fourier expansion of Selberg's Poincaré series of the first kind is given by

$$U_{\beta,m}(\tau,s) = v^s e(m\tau)(\mathfrak{e}_\beta + \mathfrak{e}_{-\beta}) + \sum_{\gamma \in L'/L} \sum_{n \in \mathbb{Z}+q(\gamma)} b(\gamma,n,v,s) e(nu) \mathfrak{e}_\gamma \tag{6.3.1}$$

for $\tau = u + iv \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 3/4$. Here the Fourier coefficients $b(\gamma, n, v, s)$ are of the form

$$b(\gamma, n, v, s) = \sum_{j=0}^{\infty} c_j(n, v, s) Z(s + j; \beta, m, \gamma, n) \tag{6.3.2}$$

with

(6.3.3)

$$c_j(n, v, s) := \begin{cases} \dfrac{2^{1/2}\pi^{s+1/2}|n|^{s-1/2}}{j!} \dfrac{(-4\pi^2 m|n|)^j}{\Gamma(s + 1/2 + j)} \mathcal{W}_{1/2+j,s+1/4+j/2}(4\pi n v), & \text{if } n > 0, \\ \dfrac{2^{3/2-2s}\pi v^{1/2-s}}{j!} \left(-\dfrac{\pi m}{v}\right)^j \dfrac{\Gamma(2s - 1/2 + j)}{\Gamma(s)\,\Gamma(s + 1/2 + j)}, & \text{if } n = 0, \\ \dfrac{2^{1/2}\pi^{s+1/2}|n|^{s-1/2}}{j!} \dfrac{(-4\pi^2 m|n|)^j}{\Gamma(s)} \mathcal{W}_{1/2+j,s+1/4+j/2}(4\pi n v), & \text{if } n < 0. \end{cases}$$

Theorem 6.3.1. *Let $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m \neq 0$. Then the Fourier expansion given in* (6.3.1) *yields a meromorphic continuation of the Poincaré series $U_{\beta,m}(\tau, s)$ in s to all of $\mathbb{C}$.*

Proof. Let $\gamma \in L'/L$ and $v > 0$ be fixed throughout this proof. Further, we let $\Omega \subseteq \mathbb{C}$ be compact such that the functions $c_j(n, v, s)$ from (6.3.3) and the Kloosterman zeta functions $Z(s + j; \beta, m, \gamma, n)$ are both holomorphic on Ω for all $j \in \mathbb{N}_0$ and $n \in \mathbb{Z} + q(\gamma)$, and we choose a positive integer ℓ such that $\mathrm{Re}(s) > 3/4 - \ell$ for all $s \in \Omega$. By the estimate in (3.3.3) and since $\mathrm{Re}(s + j) > 3/4$ for all $j \geq \ell$, we then find $C_Z > 0$ such that

$$|Z(s + j; \beta, m, \gamma, n)| \leq C_Z \tag{6.3.4}$$

for all $s \in \Omega$, $j \geq \ell$ and $n \in \mathbb{Z} + q(\gamma)$. Here the constant C_Z depends on Ω, but not on s, j or n.

Let $n = \gamma = 0$. Clearly, the finitely many Kloosterman zeta functions $Z(s+j; \beta, m, 0, 0)$ with $j = 0, \ldots, \ell - 1$ can be bounded by some universal constant $C_0 > 0$, depending on Ω, but not on j. Together with the estimate in (6.3.4) we thus obtain

$$|b(0, 0, v, s)| \leq C_0' \sum_{j=0}^{\infty} \frac{1}{j!} \left| \frac{(2s - 1/2)_j}{(s + 1/2)_j} \right| \left(\frac{\pi|m|}{v} \right)^j$$

for all $s \in \Omega$, with $C_0' > 0$ depending on Ω, but not on s. Here the remaining sum converges absolutely and uniformly for $s \in \Omega$, since the series defining the corresponding confluent hypergeometric function of the first kind $\Phi(2s - 1/2, s + 1/2; \pi|m|/v)$ does (see for example [GR07], Section 9.2).

Let now $n \in \mathbb{Z} + q(\gamma)$ with $n \neq 0$. In this case we split the sum in (6.3.2) defining the Fourier coefficients at $j = \ell$, namely we set $b(\gamma, n, v, s) = b_1(n, s) + b_2(n, s)$ with

$$b_1(n, s) := \sum_{j=0}^{\ell-1} c_j(n, v, s) Z(s + j; \beta, m, \gamma, n),$$

$$b_2(n, s) := \sum_{j=\ell}^{\infty} c_j(n, v, s) Z(s + j; \beta, m, \gamma, n),$$

for $n \in \mathbb{Z}+q(\gamma)$ with $n \neq 0$ and $s \in \Omega$. In the following we consider the two sums $b_1(n,s)$ and $b_2(n,s)$ separately.

By Theorem 6.2.6 we find $\delta > 0$ such that $|Z(s+j;\beta,m,\gamma,n)| = O(|n|^\delta)$ as $|n| \to \infty$, uniformly in s for $s \in \Omega$ and for all $j = 0,\dots,\ell-1$. Moreover, using the estimate given in (3.6.2) we find $C_1 > 0$ such that

$$\left|\mathcal{W}_{1/2+j,s+1/4+j/2}(4\pi nv)\right| \leq C_1 e^{-2\pi|n|v}$$

for all $j = 0,\dots,\ell-1$, $s \in \Omega$ and $n \in \mathbb{Z}+q(\gamma)$ with $n \neq 0$, where the constant C_1 depends on Ω, but not on j, s or n. Hence we obtain

$$|b_1(n,s)| \leq C_1'|n|^{\mathrm{Re}(s)+\ell+\delta-3/2}e^{-2\pi|n|v}$$

for $n \in \mathbb{Z}+q(\gamma)$ with $n \neq 0$ and $s \in \Omega$, where the constant $C_1' > 0$ again depends on Ω, but not on n or s. This implies the following asymptotic behaviour:

(i) For $\varepsilon > 0$ we have
$$|b_1(n,s)| = O\big(e^{-2\pi|n|v+\varepsilon|n|}\big)$$
as $|n| \to \infty$ with $n \in \mathbb{Z}+q(\gamma)$, uniformly in s for $s \in \Omega$.

Next we study the asymptotic behaviour of the infinite sum $b_2(n,s)$ as $|n| \to \infty$. As before, we always assume that $n \in \mathbb{Z}+q(\gamma)$ with $n \neq 0$. By (6.3.4) the Kloosterman zeta functions $Z(s+j;\beta,m,\gamma,n)$ are bounded for all $s \in \Omega$ and all $j \geq \ell$. Hence we find $C_2 > 0$ such that

(6.3.5)
$$|b_2(n,s)| \leq \begin{cases} C_2|n|^{\mathrm{Re}(s)-1/2}\displaystyle\sum_{j=\ell}^{\infty}\frac{(4\pi^2|mn|)^j}{j!\,|\Gamma(s+1/2+j)|}\left|\mathcal{W}_{1/2+j,s+1/4+j/2}(4\pi nv)\right|, & \text{if } n > 0,\\ C_2|n|^{\mathrm{Re}(s)-1/2}\displaystyle\sum_{j=\ell}^{\infty}\frac{(4\pi^2|mn|)^j}{j!}\left|\mathcal{W}_{1/2+j,s+1/4+j/2}(4\pi nv)\right|, & \text{if } n < 0,\end{cases}$$

for all $n \in \mathbb{Z}+q(\gamma)$ with $n \neq 0$ and $s \in \Omega$, where the constant C_2 depends on Ω, but not on n or s. Recalling the definition of the modified W-Whittaker function $\mathcal{W}_{\kappa,\mu}(z)$ and the relation between the W-Whittaker function $W_{\kappa,\mu}(z)$ and the confluent hypergeometric function of the second kind $\Psi(a,b;z)$ (see for example [EMOT55], Section 6.9), we see that

(6.3.6)
$$\mathcal{W}_{1/2+j,s+1/4+j/2}(4\pi nv) = (4\pi|n|v)^s\,e^{-2\pi|n|v}\begin{cases}\Psi(s,2s+1/2+j;4\pi|n|v), & \text{if } n > 0,\\ \Psi(s+1/2+j,2s+1/2+j;4\pi|n|v), & \text{if } n < 0.\end{cases}$$

We need to find estimates for these confluent hypergeometric functions for $s \in \Omega$, $j \geq \ell$, and $n \in \mathbb{Z}+q(\gamma)$ with $n \neq 0$.

Let $a,b \in \mathbb{C}$ with $\mathrm{Re}(a) > 0$ and $x > 0$. Then the confluent hypergeometric function $\Psi(a,b;x)$ has the integral representation

$$\Psi(a,b;x) = \frac{1}{\Gamma(a)}\int_0^\infty e^{-xt}t^{a-1}(1+t)^{b-a-1}dt, \tag{6.3.7}$$

see for example [GR07, formula 9.211.4]. If $\operatorname{Re}(a) \geq \operatorname{Re}(b-1)$ then $|(1+t)^{b-a-1}| \leq 1$ for all $t > 0$, which yields the estimate

$$|\Psi(a, b; x)| \leq x^{-\operatorname{Re}(a)} \frac{\Gamma(\operatorname{Re}(a))}{|\Gamma(a)|}. \tag{6.3.8}$$

Conversely, if $\operatorname{Re}(a) < \operatorname{Re}(b-1)$ we split the integral in (6.3.7) at $t = 1$, and use the simple estimates $1 + t \leq 2$ if $t \in (0, 1)$, and $1 + t \leq 2t$ if $t \in (1, \infty)$, to obtain

$$\begin{aligned} |\Psi(a, b; x)| &\leq \frac{2^{\operatorname{Re}(b-a)-1}}{|\Gamma(a)|} \left(\int_0^1 e^{-xt} t^{\operatorname{Re}(a)-1} dt + \int_1^\infty e^{-xt} t^{\operatorname{Re}(b)-2} dt \right) \\ &\leq \frac{2^{\operatorname{Re}(b-a)-1}}{|\Gamma(a)|} \left(\frac{1}{\operatorname{Re}(a)} + \frac{\Gamma(\operatorname{Re}(b)-1)}{x^{\operatorname{Re}(b)-1}} \right). \end{aligned} \tag{6.3.9}$$

We want to apply these estimates to the hypergeometric functions given on the right-hand side of equation (6.3.6).

Let $n \in \mathbb{Z} + q(\gamma)$ with $n < 0$. Given $s \in \Omega$ and $j \geq \ell$ we set $a := s + 1/2 + j$ and $b := 2s + 1/2 + j$. Then $\operatorname{Re}(a) > 5/4$, and $\operatorname{Re}(a) \geq \operatorname{Re}(b-1)$ if and only if $\operatorname{Re}(s) \leq 1$. Thus, applying the estimates from (6.3.8) and (6.3.9) to the hypergeometric function given in (6.3.6) we obtain

$$\begin{aligned} &\left|\mathcal{W}_{1/2+j, s+1/4+j/2}(4\pi n v)\right| \\ &\leq \begin{cases} \dfrac{2^{\operatorname{Re}(s)-1}\, e^{-2\pi|n|v}}{|\Gamma(s+1/2+j)|} \left(\dfrac{(4\pi|n|v)^{\operatorname{Re}(s)}}{\operatorname{Re}(s)+1/2+j} + \dfrac{\Gamma(2\operatorname{Re}(s)-1/2+j)}{(4\pi|n|v)^{\operatorname{Re}(s)-1/2+j}} \right), & \text{if } \operatorname{Re}(s) > 1/2, \\ (4\pi|n|v)^{-1/2-j}\, e^{-2\pi|n|v}\, \dfrac{\Gamma(\operatorname{Re}(s)+1/2+j)}{|\Gamma(s+1/2+j)|}, & \text{if } \operatorname{Re}(s) \leq 1/2, \end{cases} \end{aligned}$$

for $s \in \Omega$ and $j \geq \ell$. If $\operatorname{Re}(s) > 1/2$ and $j \geq \ell$ then $|\Gamma(s+1/2+j)| \geq |\Gamma(s+1/2+\ell)|$ and

$$\frac{\Gamma(2\sigma-1/2+j)}{|\Gamma(s+1/2+j)|} = \frac{\Gamma(2\sigma-1/2+\ell)}{|\Gamma(s+1/2+\ell)|} \frac{(2\sigma-1/2+\ell)_{j-\ell}}{|(s+1/2+\ell)_{j-\ell}|} \leq 2^{j-\ell} \frac{\Gamma(2\sigma-1/2+\ell)}{|\Gamma(s+1/2+\ell)|}, \tag{6.3.10}$$

where we use the notation $\sigma := \operatorname{Re}(s)$. Therefore, we find $C_2' > 0$ such that

$$\left|\mathcal{W}_{1/2+j, s+1/4+j/2}(4\pi n v)\right| \leq C_2'\, e^{-2\pi|n|v} \left(\frac{|n|^{\operatorname{Re}(s)}}{|\Gamma(s+1/2+j)|} + (4\pi|n|v)^{-j} \right) \tag{6.3.11}$$

for all $j \geq \ell$, $s \in \Omega$ with $\operatorname{Re}(s) > 1/2$ and $n \in \mathbb{Z} + q(\gamma)$ with $n < 0$. Here the constant C_2' depends on Ω, but not on j, s or n.

Next, suppose that $s \in \Omega$ with $\operatorname{Re}(s) \leq 1/2$. Then

$$\frac{\Gamma(\sigma+1/2+j)}{|\Gamma(s+1/2+j)|} = \frac{\Gamma(\sigma+1/2+\ell)}{|\Gamma(s+1/2+\ell)|} \frac{(\sigma+1/2+\ell)_{j-\ell}}{|(s+1/2+\ell)_{j-\ell}|} \leq \frac{\Gamma(\sigma+1/2+\ell)}{|\Gamma(s+1/2+\ell)|}.$$

for $j \geq \ell$, where we again write $\sigma := \operatorname{Re}(s)$. Hence we find $C_2'' > 0$ such that

$$\left|\mathcal{W}_{1/2+j, s+1/4+j/2}(4\pi n v)\right| \leq C_2''\, (4\pi|n|v)^{-j}\, e^{-2\pi|n|v} \tag{6.3.12}$$

for all $j \geq \ell$, $s \in \Omega$ with $\mathrm{Re}(s) \leq 1/2$ and $n \in \mathbb{Z} + q(\gamma)$ with $n < 0$. As before the constant C_2'' depends on Ω, but not on j, s or n. Combining the above estimates (6.3.11) and (6.3.12) we obtain

$$\left|\mathcal{W}_{1/2+j,s+1/4+j/2}(4\pi nv)\right| \leq \max(C_2', C_2'')\, e^{-2\pi|n|v} \left(\frac{|n|^{\mathrm{Re}(s)}}{|\Gamma(s+1/2+j)|} + (4\pi|n|v)^{-j} \right)$$

for all $j \geq \ell$, $s \in \Omega$ and $n \in \mathbb{Z} + q(\gamma)$ with $n < 0$. Therefore, we can estimate the sum given in (6.3.5) by

(6.3.13)

$$\begin{aligned} &|b_2(n,s)| \\ &\quad \leq C_2 \max(C_2', C_2'')|n|^{\mathrm{Re}(s)-1/2} e^{-2\pi|n|v} \left(|n|^{\mathrm{Re}(s)} \sum_{j=\ell}^{\infty} \frac{(4\pi^2|mn|)^j}{j!\,|\Gamma(s+1/2+j)|} + \sum_{j=\ell}^{\infty} \frac{(\pi|m|/v)^j}{j!} \right) \end{aligned}$$

for $n \in \mathbb{Z} + q(\gamma)$ with $n < 0$ and $s \in \Omega$. Here the sum $\sum_{j=\ell}^{\infty} \frac{(\pi|m|/v)^j}{j!}$ is clearly bounded by $e^{\pi|m|/v}$. Moreover, we find that

$$\begin{aligned} \sum_{j=\ell}^{\infty} \frac{(4\pi^2|mn|)^j}{j!\,|\Gamma(s+1/2+j)|} &\leq \frac{(4\pi^2|mn|)^\ell}{|\Gamma(s+1/2+\ell)|} \sum_{k=0}^{\infty} \frac{(4\pi^2|mn|)^k}{(\ell+k)!\,k!} \\ &= \frac{(4\pi^2|mn|)^{\ell/2-1/4}}{4^{\ell+1/2}\,\ell!\,|\Gamma(s+1/2+\ell)|} M_{0,\ell}\left(8\pi\sqrt{|mn|}\right), \end{aligned}$$

where the first inequality follows since $|(s+1/2+\ell)_k| \geq k!$ for $k \in \mathbb{N}_0$, and the second identity is given as formula 9.226 in [GR07]. The remaining M-Whittaker function grows as $M_{\kappa,\mu}(z) = O(e^{|z|/2}|z|^{-\kappa})$ as $|z| \to \infty$ (see for example [Bru02], equation (1.25)). Hence, we find $C_2''' > 0$ such that

$$\sum_{j=\ell}^{\infty} \frac{(4\pi^2|mn|)^j}{j!\,|\Gamma(s+1/2+j)|} \leq C_2'''|n|^{\ell/2-1/4}\, e^{4\pi\sqrt{|mn|}} \tag{6.3.14}$$

for $n \in \mathbb{Z} + q(\gamma)$ with $n < 0$ and $s \in \Omega$, with C_2''' depending on Ω, but not on n or s. Therefore, we can write (6.3.13) as

$$|b_2(n,s)| \quad \leq C_2''''|n|^{\mathrm{Re}(s)-1/2} e^{-2\pi|n|v} \left(|n|^{\mathrm{Re}(s)+\ell/2-1/4} e^{4\pi\sqrt{|mn|}} + 1 \right)$$

for $n \in \mathbb{Z} + q(\gamma)$ with $n < 0$ and $s \in \Omega$,where the constant $C_2'''' > 0$ depends on Ω, but not on n or s. This implies the following asymptotic behaviour:

(ii) For $\varepsilon > 0$ we have
$$|b_2(n,s)| = O\big(e^{-2\pi|n|v+\varepsilon|n|}\big)$$
as $n \to -\infty$ with $n \in \mathbb{Z} + q(\gamma)$ and $n < 0$, uniformly in s for $s \in \Omega$.

It remains to study the behaviour of the infinite sum $b_2(n,s)$ as $n \to \infty$. Using the relation between the W-Whittaker function and the confluent hypergeometric function of the second kind given in (6.3.6) together with the estimate shown in (6.3.5), we see that

(6.3.15)
$$|b_2(n,s)| \le C_3 |n|^{2\operatorname{Re}(s)-1/2} e^{-2\pi|n|v} \sum_{j=\ell}^{\infty} \frac{(4\pi^2|mn|)^j}{j!\,|\Gamma(s+1/2+j)|}\ |\Psi(s, 2s+1/2+j; 4\pi|n|v)|$$

for $n \in \mathbb{Z} + q(\gamma)$ with $n > 0$ and $s \in \Omega$. Here $C_3 > 0$ is a constant depending on Ω, but not on n or s. However, in contrast to the case $n < 0$ we cannot directly apply the estimates from (6.3.8) and (6.3.9) to the hypergeometric function given in (6.3.15), since we can have $\operatorname{Re}(s) < 0$, in which case the integral representation from (6.3.7) is not valid.

Instead, we recall that the hypergeometric function $\Psi(a,b;x)$ satisfies the recurrence relation

$$\Psi(a,b;x) = x\,\Psi(a+1,b+1;x) - (b-a-1)\Psi(a+1,b;x), \tag{6.3.16}$$

which is for example given in [AS84, formula 13.4.18]. Using induction we deduce from (6.3.16) that

$$\Psi(a,b;x) = \sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} \frac{\Gamma(b-a)}{\Gamma(b-a-k)}\, x^{\ell-k}\, \Psi(a+\ell, b+\ell-k; x).$$

We can now apply this higher order recurrence relation to the estimate in (6.3.15), which yields

$$\begin{aligned}
|b_2(n,s)| &\le C_3' |n|^{2\operatorname{Re}(s)-1/2} e^{-2\pi|n|v} \sum_{k=0}^{\ell} \binom{\ell}{k} \sum_{j=\ell}^{\infty} \frac{(4\pi^2|m|)^j |n|^{j+\ell-k}}{j!\,|\Gamma(s+1/2+j-k)|} \\
&\qquad \times |\Psi(s+\ell, 2s+1/2+j+\ell-k; 4\pi|n|v)| \\
&= C_3' |n|^{2\operatorname{Re}(s)-1/2+\ell} e^{-2\pi|n|v} \sum_{k=0}^{\ell} \binom{\ell}{k} (4\pi^2|m|)^k \sum_{j=\ell-k}^{\infty} \frac{(4\pi^2|mn|)^j}{(j+k)!\,|\Gamma(s+1/2+j)|} \\
&\qquad \times |\Psi(s+\ell, 2s+1/2+j+\ell; 4\pi|n|v)|
\end{aligned}$$

for $n \in \mathbb{Z} + q(\gamma)$ with $n > 0$ and $s \in \Omega$. Here $C_3' > 0$ depends on Ω, but not on n or s. Hence, using the trivial estimate $(j+k)! \ge j!$, and adding extra terms such that the sum over j runs over all non-negative integers, we find $C_3'' > 0$ such that

$$|b_2(n,s)| \le C_3'' |n|^{2\operatorname{Re}(s)-1/2+\ell} e^{-2\pi|n|v} S(n,s) \tag{6.3.17}$$

for all $n \in \mathbb{Z} + q(\gamma)$ with $n > 0$ and $s \in \Omega$, where the constant C_3'' depends on Ω, but not on n or s, and the function $S(n,s)$ is given as the sum

$$S(n,s) := \sum_{j=0}^{\infty} \frac{(4\pi^2|mn|)^j}{j!\,|\Gamma(s+1/2+j)|}\ |\Psi(s+\ell, 2s+1/2+j+\ell; 4\pi|n|v)|\,.$$

We again split $S(n,s)$ into a finite and an infinite part, namely $S(n,s) = S_1(n,s)+S_2(n,s)$ with

$$S_1(n,s) := \sum_{j=0}^{\ell-1} \frac{(4\pi^2|mn|)^j}{j!\,|\Gamma(s+1/2+j)|}\ |\Psi(s+\ell, 2s+1/2+j+\ell; 4\pi|n|v)|\,, \tag{6.3.18}$$

$$S_2(n,s) := \sum_{j=\ell}^{\infty} \frac{(4\pi^2|mn|)^j}{j!\,|\Gamma(s+1/2+j)|}\ |\Psi(s+\ell, 2s+1/2+j+\ell; 4\pi|n|v)|\,. \tag{6.3.19}$$

Eventually, we reuse the estimates from (6.3.8) and (6.3.9). Note that this is indeed possible since $\mathrm{Re}(s+\ell) > 0$ for all $s \in \Omega$.

Let $n \in \mathbb{Z} + q(\gamma)$ with $n > 0$ and $j \in \mathbb{N}_0$. Given $s \in \Omega$ we set $a := s + \ell$ and $b := 2s + 1/2 + j + \ell$. Then $\mathrm{Re}(a) \geq \mathrm{Re}(b-1)$ if and only if $\mathrm{Re}(s) \leq 1/2 - j$, and the estimates (6.3.8) and (6.3.9) translate to

(6.3.20)

$$\begin{aligned} &|\Psi(s+\ell, 2s+1/2+j+\ell; 4\pi|n|v)| \\ &\quad\leq \begin{cases} \dfrac{2^{\mathrm{Re}(s)-1/2+j+\ell}}{|\Gamma(s+\ell)|}\left(\dfrac{1}{\mathrm{Re}(s)+\ell} + \dfrac{\Gamma(2\,\mathrm{Re}(s)-1/2+j+\ell)}{(4\pi|n|v)^{2\,\mathrm{Re}(s)-1/2+j+\ell}}\right), & \text{if } \mathrm{Re}(s) > 1/2-j, \\ (4\pi|n|v)^{-\mathrm{Re}(s)-\ell}\,\dfrac{\Gamma(\mathrm{Re}(s)+\ell)}{|\Gamma(s+\ell)|}, & \text{if } \mathrm{Re}(s) \leq 1/2-j. \end{cases} \end{aligned}$$

As $\mathrm{Re}(s) + \ell > 3/4$ the expression in the second case is clearly bounded for all n, j and s, i.e., we find $C_4 > 0$ such that

$$(4\pi|n|v)^{-\mathrm{Re}(s)-\ell}\,\frac{\Gamma(\mathrm{Re}(s)+\ell)}{|\Gamma(s+\ell)|} < C_4 \tag{6.3.21}$$

for all $n \in \mathbb{Z} + q(\gamma)$ with $n > 0$, $j \in \mathbb{N}_0$ and $s \in \Omega$, where C_4 does depend on Ω, but not on n, j or s. It remains to consider the first case of the above estimate for the hypergeometric function, i.e., the case when $\mathrm{Re}(s) > 1/2 - j$.

Let $s \in \Omega$ with $\mathrm{Re}(s) > 1/2 - j$, and let us further assume that $j < \ell$. Then

$$1/2 < 2\,\mathrm{Re}(s) - 1/2 + j + \ell < 2\,\mathrm{Re}(s) - 1/2 + 2\ell,$$

and thus we also find $C_4' > 0$ such that

$$\frac{2^{\mathrm{Re}(s)-1/2+j+\ell}}{|\Gamma(s+\ell)|}\left(\frac{1}{\mathrm{Re}(s)+\ell} + \frac{\Gamma(2\,\mathrm{Re}(s)-1/2+j+\ell)}{(4\pi|n|v)^{2\,\mathrm{Re}(s)-1/2+j+\ell}}\right) < C_4'$$

for all $n \in \mathbb{Z} + q(\gamma)$ with $n > 0$, $j = 0, 1, \ldots, \ell - 1$ and $s \in \Omega$ with $\mathrm{Re}(s) > 1/2 - j$. Again, the constant C_4' does depend on Ω, but not on n, j or s.

Hence we can estimate the sum $S_1(n,s)$ given in (6.3.18), since by the above considerations the hypergeometric function $\Psi(s+\ell, 2s+1/2+j+\ell; 4\pi|n|v)$ is universally bounded for all $j < \ell$, giving

$$|S_1(n,s)| < \max(C_4, C_4') \sum_{j=0}^{\ell-1} \frac{(4\pi^2|mn|)^j}{j!\,|\Gamma(s+1/2+j)|} < C_4''|n|^{\ell-1} \tag{6.3.22}$$

for all $n \in \mathbb{Z} + q(\gamma)$ with $n > 0$ and $s \in \Omega$. Here the new constant $C_4'' > 0$ depends on Ω, but not on n or s.

Next we estimate the sum $S_2(n, s)$ defined in (6.3.19) using the estimate given in (6.3.20). However, since $\mathrm{Re}(s) > 1/2 - j$ for all $j \geq \ell$ we can indeed always apply the first case of the estimate (6.3.20), which gives

(6.3.23)

$$|S_2(n,s)| \leq \frac{2^{\mathrm{Re}(s)-1/2+\ell}}{|\Gamma(s+\ell)|} \sum_{j=\ell}^{\infty} \frac{(8\pi^2|mn|)^j}{j!\,|\Gamma(s+1/2+j)|} \left(\frac{1}{\mathrm{Re}(s)+\ell} + \frac{\Gamma(2\,\mathrm{Re}(s)-1/2+j+\ell)}{(4\pi|n|v)^{2\,\mathrm{Re}(s)-1/2+j+\ell}} \right)$$
$$\leq C_5 \left(\sum_{j=\ell}^{\infty} \frac{(8\pi^2|mn|)^j}{j!|\Gamma(s+1/2+j)|} + |n|^{1/2-2\,\mathrm{Re}(s)-\ell} \sum_{j=\ell}^{\infty} \frac{(2\pi|m|/v)^j}{j!} \frac{\Gamma(2\,\mathrm{Re}(s)-1/2+j+\ell)}{|\Gamma(s+1/2+j)|} \right)$$

for all $n \in \mathbb{Z} + q(\gamma)$ with $n > 0$ and $s \in \Omega$. Here the constant $C_5 > 0$ depends on Ω, but not on n or s. By (6.3.14) we find that

$$\sum_{j=\ell}^{\infty} \frac{(8\pi^2|mn|)^j}{j!|\Gamma(s+1/2+j)|} \leq C_2'''|2n|^{\ell/2-1/4} e^{4\pi\sqrt{2|mn|}}.$$

for all $n \in \mathbb{Z} + q(\gamma)$ with $n > 0$ and $s \in \Omega$, where the estimate does indeed hold for all $n \neq 0$. Moreover, as in (6.3.10) we obtain

$$\frac{\Gamma(2\,\mathrm{Re}(s)-1/2+j+\ell)}{|\Gamma(s+1/2+j)|} \leq 2^{j-\ell} \frac{\Gamma(2\,\mathrm{Re}(s)-1/2+2\ell)}{|\Gamma(s+1/2+\ell)|}$$

for $s \in \Omega$ and $j \geq \ell$. Applying those estimates to (6.3.23), we get

$$|S_2(n,s)| \leq C_5' \left(|n|^{\ell/2-1/4} e^{4\pi\sqrt{2|mn|}} + |n|^{1/2-2\,\mathrm{Re}(s)-\ell} e^{4\pi|m|/v} \right) \tag{6.3.24}$$

or all $n \in \mathbb{Z} + q(\gamma)$ with $n > 0$ and $s \in \Omega$, where $C_5' > 0$ is a constant depending on Ω, but not on n or s.

Finally, recalling that $S(n,s) = S_1(n,s) + S_2(n,s)$ we can apply the estimates (6.3.22) and (6.3.24) to the estimate of the sum $b_2(n,s)$ for $n > 0$ given in (6.3.17). Thereby we obtain the following asymptotic behaviour:

(iii) For $\varepsilon > 0$ we have
$$|b_2(n,s)| = O\big(e^{-2\pi|n|v+\varepsilon|n|}\big)$$
as $n \to \infty$ with $n \in \mathbb{Z} + q(\gamma)$ and $n > 0$, uniformly in s for $s \in \Omega$.

We recall that $b(\gamma, n, v, s) = b_1(n,s) + b_2(n,s)$ for $n \in \mathbb{Z} + q(\gamma)$ with $n \neq 0$ and $s \in \Omega$. Thus, combining the asymptotic estimates given in (i), (ii) and (iii) we have

$$|b(\gamma, n, v, s)| = O\big(e^{-2\pi|n|v+\varepsilon|n|}\big)$$

as $|n| \to \infty$ with $n \in \mathbb{Z} + q(\gamma)$, uniformly in s for $s \in \Omega$, for any $\varepsilon > 0$. In particular, we have also shown that the Fourier coefficients $b(\gamma, n, v, s)$ with $n \in \mathbb{Z} + q(\gamma)$ (including the case $n = 0$) define holomorphic functions on Ω. Therefore we have shown that the Fourier series given in (6.3.1) defines a meromorphic function in s on $\mathbb{C}$, and this function is by construction the meromorphic continuation of $U_{\beta,m}(\tau, s)$ in s. $\square$

Remark 6.3.2. Let $\beta, \gamma \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m \neq 0$. Further, let $b(\gamma, n, v, s)$ be the Fourier coefficients of the Poincaré series $U_{\beta,m}(\tau, s)$ as in (6.3.2). Going carefully through the proof of Theorem 6.3.1 with $v > 1$ not being fixed we can deduce that

$$|b(\gamma, n, v, s)| = O\big(e^{-2\pi|n|v+\varepsilon|n|}\big) \tag{6.3.25}$$

as $|n| \to \infty$ with $n \in \mathbb{Z} + q(\gamma)$, locally uniformly in s and uniformly in v for $v > 1$, with $\varepsilon > 0$ not depending on v. In particular, the Fourier coefficients $b(\gamma, n, v, s)$ decay exponentially as $v \to \infty$, uniformly in n for $n \in \mathbb{Z} + q(\gamma)$ with $n \neq 0$. Moreover, if $n = \gamma = 0$ we can infer from the beginning of the proof of Theorem 6.3.1 that

$$|b(0, 0, v, s)| = O(v^{1/2-s}) \tag{6.3.26}$$

as $v \to \infty$, locally uniformly in s.

In addition, we note that the above estimates (6.3.25) and (6.3.26) also hold for $m = \beta = 0$, in which case they can be directly deduced from the Fourier expansion of the non-holomorphic Eisenstein series $E_0(\tau, s)$ (see Proposition 3.6.2) and the asymptotic behaviour of the corresponding Kloosterman zeta functions (see Theorem 6.1.8).

Corollary 6.3.3. *Let $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m \neq 0$. The meromorphic continuation of $U_{\beta,m}(\tau, s)$ is holomorphic at $s = 0$, and the evaluation of $U_{\beta,m}(\tau, s)$ at $s = 0$ has a Fourier expansion of the form*

$$\begin{aligned} U_{\beta,m}(\tau, 0) = e(m\tau)(\mathfrak{e}_\beta + \mathfrak{e}_{-\beta}) &+ \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z}+q(\gamma) \\ n>0}} \tilde{b}(\gamma, n) e(n\tau) \mathfrak{e}_\gamma \\ &+ \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z}+q(\gamma) \\ n<0}} \tilde{b}(\gamma, n) \Gamma(1/2, 4\pi|n|v) e(n\tau) \mathfrak{e}_\gamma \end{aligned}$$

for $\tau \in \mathbb{H}$. Here $\Gamma(s, x)$ denotes the usual incomplete Gamma function, and the Fourier coefficients $\tilde{b}(\gamma, n)$ are given by

$$\tilde{b}(\gamma, n) = \begin{cases} 2^{1/2}|n|^{-1/2} \left(\displaystyle\sum_{j=0}^{\infty} \frac{(-16\pi^2 m|n|)^j}{(2j)!} Z(j; \beta, m, \gamma, n) \right), & \text{if } n > 0, \\ (2\pi)^{1/2}|n|^{-1/2} \displaystyle\lim_{s \to 0} \left(\frac{Z(s; \beta, m, \gamma, n)}{\Gamma(s)} \right), & \text{if } n < 0, \end{cases}$$

for $\gamma \in L'/L$ and $n \in \mathbb{Z} + q(\gamma)$ with $n \neq 0$. In particular, the coefficients $\tilde{b}(\gamma, n)$ are all real, and if $m > 0$ then $\tilde{b}(\gamma, n) = 0$ for all $\gamma \in L'/L$ and $n \in \mathbb{Z} + q(\gamma)$ with $n < 0$.

Proof. We write the Fourier expansion of the Poincaré series $U_{\beta,m}(\tau, s)$ given in (6.3.1) in the form

$$\begin{aligned} U_{\beta,m}(\tau, s) = v^s e(m\tau)(\mathfrak{e}_\beta + \mathfrak{e}_{-\beta}) &+ \sum_{\gamma \in L'/L} \sum_{n \in \mathbb{Z}+q(\gamma)} \tilde{b}(\gamma, n, v, s) e(nu) \mathfrak{e}_\gamma \\ &+ \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z}+q(\gamma) \\ n<0}} \tilde{b}_0(\gamma, n, v, s) e(nu) \mathfrak{e}_\gamma, \end{aligned}$$

with modified Fourier coefficients $\tilde{b}(\gamma, n, v, s)$ and $\tilde{b}_0(\gamma, n, v, s)$ given by

$$\tilde{b}(\gamma, n, v, s) := \begin{cases} b(\gamma, n, v, s), & \text{if } m > 0 \text{ or if } n \geq 0, \\ \displaystyle\sum_{j=1}^{\infty} c_j(n, v, s) Z(s + j; \beta, m, \gamma, n), & \text{if } m, n < 0, \end{cases}$$

for $\gamma \in L'/L$ and $n \in \mathbb{Z} + q(\gamma)$, and

$$\tilde{b}_0(\gamma, n, v, s) := \begin{cases} 0, & \text{if } m > 0, \\ c_0(n, v, s) Z(s; \beta, m, \gamma, n), & \text{if } m < 0, \end{cases} \tag{6.3.27}$$

for $\gamma \in L'/L$ and $n \in \mathbb{Z} + q(\gamma)$ with $n < 0$. Here the Fourier coefficients $b(\gamma, n, v, s)$ and the functions $c_j(n, v, s)$ are given as in (6.3.2) and (6.3.3), respectively.

By part (a) of Theorem 6.2.6 the Kloosterman zeta function $Z(s; \beta, m, \gamma, n)$ is holomorphic at $s = 0$ if $m > 0$ or if $n \geq 0$, and has at most a simple pole at $s = 0$ if $m, n < 0$. Thus, we find $r > 0$ such that the functions $c_j(n, v, s)$ and the Kloosterman zeta functions $Z(s + j; \beta, m, \gamma, n)$ with $j \geq 1$ if $m, n < 0$ are all holomorphic on $\Omega := \{s \in \mathbb{C} \colon |s| \leq r\}$. Without loss of generality, we can further assume that $r < 1/4$ such that $\mathrm{Re}(s+1) > 3/4$ for all $s \in \Omega$. Hence, choosing $\ell = 1$ and Ω as above, we can apply the methods from the proof of Theorem 6.3.1 to see that the sum defining the (modified) Fourier coefficients $\tilde{b}(\gamma, n, v, s)$ is absolutely and locally uniformly convergent for $s \in \Omega$ with

$$\left| \tilde{b}(\gamma, n, v, s) \right| = O\big(e^{-\varepsilon |n|}\big)$$

as $|n| \to \infty$, uniformly in s for $s \in \Omega$ (see also (6.3.25)). Therefore, the Fourier expansion

$$\sum_{\gamma \in L'/L} \sum_{n \in \mathbb{Z} + q(\gamma)} \tilde{b}(\gamma, n, v, s) e(nu) \mathfrak{e}_\gamma$$

defines a holomorphic function in s on Ω, which we can simply evaluate at $s = 0$.

Firstly, let $n = \gamma = 0$. Then

$$\tilde{b}(0, 0, v, 0) = \lim_{s \to 0} \left(\frac{2^{3/2 - 2s} \pi v^{1/2 - s}}{\Gamma(s)} \sum_{j=0}^{\infty} \frac{1}{j!} \left(-\frac{\pi m}{v} \right)^j \frac{\Gamma(2s - 1/2 + j)}{\Gamma(s + 1/2 + j)} Z(s + j; \beta, m, 0, 0) \right),$$

and since the sum over j is holomorphic at $s = 0$ by the above considerations, the Fourier coefficient vanishes because of the factor $1/\Gamma(s)$ in front, i.e., we find $\tilde{b}(0, 0, v, 0) = 0$. Next, let $\gamma \in L'/L$ and $n \in \mathbb{Z} + q(\gamma)$ with $n < 0$. Then

$$\tilde{b}(\gamma, n, v, 0) = \lim_{s \to 0} \Bigg(\frac{2^{1/2} \pi^{s + 1/2} |n|^{s - 1/2}}{\Gamma(s)} \sum_{j=a}^{\infty} \frac{(-4\pi^2 m |n|)^j}{j!} \mathcal{W}_{1/2 + j, s + 1/4 + j/2}(4\pi n v) \\ \times Z(s + j; \beta, m, \gamma, n) \Bigg),$$

where the sum over j starts at $a = 0$ if $m > 0$, or at $a = 1$ if $m < 0$. In either case the sum over j is holomorphic at $s = 0$, and thus the corresponding Fourier coefficient vanishes again, giving $\tilde{b}(\gamma, n, v, 0) = 0$ for all $n \in \mathbb{Z} + q(\gamma)$ with $n < 0$. Finally, let $n \in \mathbb{Z} + q(\gamma)$ with $n > 0$. Then

$$\begin{aligned}\tilde{b}(\gamma, n, v, 0) &= \lim_{s \to 0} \Bigg(2^{1/2} \pi^{s+1/2} |n|^{s-1/2} \sum_{j=0}^{\infty} \frac{(-4\pi^2 m|n|)^j}{j!\,\Gamma(s + 1/2 + j)} \mathcal{W}_{1/2+j, s+1/4+j/2}(4\pi n v) \\ &\qquad\qquad \times Z(s + j; \beta, m, \gamma, n) \Bigg), \\ &= 2^{1/2} \pi^{1/2} |n|^{-1/2} \sum_{j=0}^{\infty} \frac{(-4\pi^2 m|n|)^j}{j!\,\Gamma(1/2 + j)} \mathcal{W}_{1/2+j, 1/4+j/2}(4\pi n v)\, Z(j; \beta, m, \gamma, n).\end{aligned}$$

Here

$$\mathcal{W}_{1/2+j, 1/4+j/2}(4\pi n v) = (4\pi n v)^{-1/4-j/2} W_{1/4+j/2, -1/4+j/2}(4\pi n v) = e^{-2\pi n v}$$

for $n > 0$, where the second identity is for example given in [GR07, formula 9.237.3] with $L_0^{\alpha}(x) = 1$ being the trivial Laguerre polynomial. Moreover, it is well-known that $\Gamma(1/2 + j) = 4^{-j} \pi^{1/2} (2j)!/j!$. Hence we find

$$\tilde{b}(\gamma, n, v, 0) = 2^{1/2} |n|^{-1/2} \left(\sum_{j=0}^{\infty} \frac{(-16\pi^2 m|n|)^j}{(2j)!} Z(j; \beta, m, \gamma, n) \right) e^{-2\pi n v}$$

for $n \in \mathbb{Z} + q(\gamma)$ with $n > 0$.

It remains to consider the Fourier coefficients $\tilde{b}_0(\gamma, n, v, s)$ for $n \in \mathbb{Z} + q(\gamma)$ with $n < 0$ defined in (6.3.27). We may assume that $m < 0$ since $\tilde{b}_0(\gamma, n, v, s) = 0$ otherwise. In this case the Kloosterman zeta function $Z(s; \beta, m, \gamma, n)$ has a possible simple pole at $s = 0$. We recall that

$$\tilde{b}_0(\gamma, n, v, s) = 2^{1/2} \pi^{s+1/2} |n|^{s-1/2} \mathcal{W}_{1/2, s+1/4}(4\pi n v) \frac{Z(s; \beta, m, \gamma, n)}{\Gamma(s)}.$$

Here the possible pole of the Kloosterman zeta function at $s = 0$ is compensated for by the Gamma factor $1/\Gamma(s)$. Moreover, by (3.6.2) the Whittaker function $\mathcal{W}_{1/2, s+1/4}(4\pi n v)$ behaves as $|n|^{-1/2} e^{-2\pi |n| v}$ as $n \to -\infty$, and by part (c) of Theorem 6.2.6 the quotient $Z(s; \beta, m, \gamma, n)/\Gamma(s)$ grows at most polynomially in n as $n \to -\infty$. Hence, there is $\varepsilon' > 0$ such that

$$\left| \tilde{b}_0(\gamma, n, v, s) \right| = O\big(e^{-\varepsilon' |n|}\big)$$

as $n \to -\infty$, uniformly in s for s in some small neighbourhood of 0 (see also (6.3.25)). So the Fourier expansion

$$\sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z} + q(\gamma) \\ n < 0}} \tilde{b}_0(\gamma, n, v, s) e(nu) \mathfrak{e}_\gamma$$

defines a holomorphic function in s on this neighbourhood, which can be evaluated at $s = 0$. In particular, we have

$$\tilde{b}_0(\gamma, n, v, 0) = 2^{1/2} \pi^{1/2} |n|^{-1/2} \mathcal{W}_{1/2, 1/4}(4\pi n v) \lim_{s \to 0} \left(\frac{Z(s; \beta, m, \gamma, n)}{\Gamma(s)} \right).$$

Here

$$\mathcal{W}_{1/2,1/4}(4\pi nv) = e^{-2\pi|n|v}\Psi(1/2, 1/2; 4\pi|n|v) = e^{2\pi|n|v}\Gamma(1/2, 4\pi|n|v)$$

for $n < 0$, where the second identity is for example given in [GR07, formula 9.236.3]. Here $\Psi(a, b; x)$ denotes the confluent hypergeometric function of the second kind, and $\Gamma(s, x)$ denotes usual incomplete Gamma function. Thus, we get

$$\tilde{b}_0(\gamma, n, v, 0) = 2^{1/2}\pi^{1/2}|n|^{-1/2} \lim_{s\to 0}\left(\frac{Z(s; \beta, m, \gamma, n)}{\Gamma(s)}\right)\Gamma(1/2, 4\pi|n|v)\, e^{-2\pi nv}$$

for $n \in \mathbb{Z} + q(\gamma)$ with $n < 0$. In particular, we have $\tilde{b}_0(\gamma, n, v, 0) = 0$ if $Z(s; \beta, m, \gamma, n)$ is holomorphic at $s = 0$.

Putting everything back together, we find that the meromorphic continuation of the Poincaré series $U_{\beta,m}(\tau, s)$, which by Theorem 6.3.1 is given by its Fourier expansion, is indeed holomorphic at $s = 0$, and the evaluation of $U_{\beta,m}(\tau, s)$ at $s = 0$ has the Fourier expansion described in the corollary.

It remains to note that the Fourier coefficients $\tilde{b}(\gamma, n)$ given in the corollary are indeed real. Therefore, we recall that for $x \in \mathbb{R}$ we also have $Z(x; \beta, m, \gamma, n) \in \mathbb{R}$, since

$$\overline{Z(x; \beta, m, \gamma, n)} = Z(\overline{x}; \beta, m, \gamma, n) = Z(x; \beta, m, \gamma, n)$$

for all $\gamma \in L'/L$ and $n \in \mathbb{Z} + q(\gamma)$. Hence, the coefficients $\tilde{b}(\gamma, n)$ with $n > 0$ are real. Moreover, as the quotient $Z(s; \beta, m, \gamma, n)/\Gamma(s)$ is holomorphic on some neighbourhood of $s = 0$, we find that

$$\lim_{s\to 0}\left(\frac{Z(s; \beta, m, \gamma, n)}{\Gamma(s)}\right) = \lim_{k\to\infty}\left(\frac{Z(1/k; \beta, m, \gamma, n)}{\Gamma(1/k)}\right) \in \mathbb{R}.$$

Thus, also the coefficients $\tilde{b}(\gamma, n)$ with $n < 0$ are real. □

We note that the previous Corollary is indeed also true for $\beta = m = 0$, in which case $U_{0,0}(\tau, 0) = E_0(\tau, 0)$ is the evaluation of the meromorphic continuation of the non-holomorphic Eisenstein $E_0(\tau, s)$ at $s = 0$, which was given in Corollary 6.1.13. In fact, for $\beta = m = 0$ the expansions given in Corollary 6.1.13 and Corollary 6.3.3 agree.

Definition 6.3.4. Given $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ we define

$$U_{\beta,m}(\tau) = U_{\beta,m}(\tau, 0)$$

for $\tau \in \mathbb{H}$.

To the end of this section we present a characterization of the functions $U_{\beta,m}(\tau)$, which turn out to be special harmonic Maass forms. The following theorem can be seen as a weight 1/2 analog of Proposition 3.4.4.

Theorem 6.3.5. *Let $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$. Then $U_{\beta,m}(\tau)$ is a harmonic Maass form, which can be characterized as follows:*

(a) If $m > 0$ then $U_{\beta,m}(\tau)$ is the unique cusp form of weight $1/2$ for ρ_L, which satisfies the inner product formula given in (3.4.8), i.e., with

$$(F, U_{\beta,m}) = -8\pi\sqrt{m}\, a_F(\beta, m) \tag{6.3.28}$$

for all $F \in S_{1/2,L}$.

(b) *If* $m = \beta = 0$ *then* $U_{0,0}(\tau)$ *is the unique modular form of weight* $1/2$ *for* ρ_L*, which has principal part* $2\mathfrak{e}_0$*, and which is orthogonal to cusp forms with respect to the regularized inner product defined in* (3.4.4)*.*

(c) *If* $m < 0$ *then* $U_{\beta,m}(\tau)$ *is the unique harmonic Maass form of weight* $1/2$ *for* ρ_L*, which maps to a cusp form under the differential operator* $\xi_{1/2}$*, which has principal part* $e(m\tau)(\mathfrak{e}_\beta + \mathfrak{e}_{-\beta})$*, and which is orthogonal to cusp forms with respect to the regularized inner product defined in* (3.4.5)*.*

Proof. Firstly, we note that the meromorphic continuation of $U_{\beta,m}(\tau, s)$ in s is clearly modular of weight $1/2$ for ρ_L whenever it is defined since $U_{\beta,m}(\tau, s)$ is for $\mathrm{Re}(s) > 3/4$. In particular, the function $U_{\beta,m}(\tau)$ is modular of weight $1/2$ for ρ_L.

Next, let $\tilde{b}(\gamma, n)$ be the Fourier coefficients given in Corollary 6.3.3. We have seen in the proof of the corollary that there is $\varepsilon > 0$ such that $|\tilde{b}(\gamma, n)| = O(e^{-\varepsilon n})$ as $|n| \to \infty$. Thus, the Fourier expansion

$$U^+_{\beta,m}(\tau) := e(m\tau)(\mathfrak{e}_\beta + \mathfrak{e}_{-\beta}) + \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z}+q(\gamma) \\ n>0}} \tilde{b}(\gamma, n) e(n\tau)\mathfrak{e}_\gamma,$$

defines a holomorphic function on $\mathbb{H}$, which we call the holomorphic part of $U_{\beta,m}(\tau)$. However, if $m \geq 0$ then $U^+_{\beta,m} = U_{\beta,m}$ as $\tilde{b}(\gamma, n) = 0$ for all $n < 0$ in this case. Hence, as the function $U_{\beta,m}(\tau)$ is also modular we directly obtain that $U_{\beta,m}(\tau)$ is a cusp form or a modular form of weight $1/2$ for ρ_L if $m > 0$ or $m = 0$, respectively.

If on the other hand $m < 0$, we further note that

$$|\Gamma(1/2, 4\pi|n|v)e(n\tau)| = O\big(e^{-2\pi|n|v}\big)$$

as $n \to -\infty$. Hence, the Fourier expansion given in Corollary 6.3.3 still defines a smooth function if $m < 0$, and this function behaves as $O(e^{2\pi|m|v})$ as $v \to \infty$, uniformly in u. Furthermore, we can apply the differential operator $\Delta_{1/2}$ directly to the Fourier expansion of $U_{\beta,m}(\tau)$, which yields

$$\Delta_{1/2} U_{\beta,m}(\tau) = \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z}+q(\gamma) \\ n<0}} \tilde{b}(\gamma, n) \Delta_{1/2}\Big(\Gamma(1/2, 4\pi|n|v)e(n\tau)\Big)\mathfrak{e}_\gamma,$$

since the holomorphic part $U^+_{\beta,m}(\tau)$ clearly vanishes under the action of $\Delta_{1/2}$. Moreover, it is well-known that

$$\Delta_{1/2}(\Gamma(1/2, 4\pi|n|v)e(n\tau)) = 0$$

for all $n \in \mathbb{Z} + q(\gamma)$ with $n < 0$ (see for example the discussion at the beginning of Section 3 in [BF04]). Hence $\Delta_{1/2}U_{\beta,m}(\tau) = 0$. Therefore, if $m < 0$ the function $U_{\beta,m}(\tau)$ is a harmonic Maass form in the sense of Definition 3.5.1. Moreover, by (3.5.1) the function $U_{\beta,m}(\tau)$ is mapped to a cusp form under the differential operator $\xi_{1/2}$ as the non-holomorphic part of the Fourier expansion given in Corollary 6.3.3 has only negative Fourier coefficients. Thus, if $m < 0$ we have $U_{\beta,m} \in H^+_{1/2,L}$.

Let now $F \in S_{1/2,L}$. Carefully unfolding against the Poincaré series $U_{\beta,m}(\tau,s)$ for $\mathrm{Re}(s) \gg 0$ we find that

$$(F, U_{\beta,m}(\,\cdot\,,s))^{\mathrm{reg}} = \begin{cases} 2\,\dfrac{\Gamma(s-1/2)}{(4\pi m)^{s-1/2}}\,a_F(\beta,m), & \text{if } m>0,\\ 0, & \text{if } m \le 0. \end{cases} \tag{6.3.29}$$

Here the integral defining the inner product on the left-hand side needs to be regularized in the sense of (3.4.4) if $m=0$, and in the sense of (3.4.5) if $m \neq 0$. One can check that the left-hand side of (6.3.29) has a meromorphic continuation to $s=0$, and that

$$(F, U_{\beta,m}(\,\cdot\,,s))^{\mathrm{reg}}\Big|_{s=0} = (F, U_{\beta,m})^{\mathrm{reg}}.$$

Hence, if $m>0$ the function $U_{\beta,m}(\tau)$ satisfies the inner product formula claimed in the theorem, i.e.,

$$(F, U_{\beta,m})^{\mathrm{reg}} = (F, U_{\beta,m}(\,\cdot\,,0))^{\mathrm{reg}} = 2\,\frac{\Gamma(-1/2)}{(4\pi m)^{-1/2}}\,a_F(\beta,m) = -8\pi\sqrt{m}\,a_F(\beta,m),$$

and if $m \le 0$ then $(F,U_{\beta,m})^{\mathrm{reg}} = 0$, i.e., $U_{\beta,m}(\tau)$ is orthogonal to cusp forms.

As the inner product on $S_{1/2,L}$ is non-degenerate, the given inner product formula does indeed uniquely determine the cusp form $U_{\beta,m}(\tau)$ if $m>0$. Moreover, since the difference of two modular forms having the same principal part is a cusp form, the modular form $U_{0,0}(\tau)$ is uniquely determined by the fact that its principal part is $2\mathfrak{e}_0$, and that it is orthogonal to cusp forms.

It remains to show that $U_{\beta,m}(\tau)$ is uniquely determined by the given conditions if $m<0$. However, this is a direct consequence of Lemma 3.5.2, by which two harmonic Maass forms, which map to cusp forms under $\xi_{1/2}$, and which have the same principal part, can only differ by a cusp form. As $U_{\beta,m}(\tau)$ is orthogonal to cusp forms by (6.3.29) if $m<0$, it is indeed uniquely determined by the conditions given in the theorem. □

Remark 6.3.6. As in the classical case one can deduce from the inner product formula given in (6.3.28) that the cusp forms $U_{\beta,m}(\tau)$ for $\beta \in L'/L$ and $m \in \mathbb{Z}+q(\beta)$ with $m>0$ generate the (finite dimensional) space of cusp forms $S_{1/2,L}$ of weight 1/2 for ρ_L. Moreover, since two modular forms in $M_{1/2,L}$, which have the same constant Fourier coefficient, can only differ by a cusp form, the set of modular forms $U_{\beta,m}(\tau)$ for $\beta \in L'/L$ and $m \in \mathbb{Z}+q(\beta)$ with $m \ge 0$ generates the space of modular forms $M_{1/2,L}$. Finally, we also know that two harmonic Maass forms of weight 1/2 for ρ_L, which map to cusp forms under $\xi_{1/2}$, and which have the same principal part, can only differ by a cusp form. Therefore, the set of all harmonic Maass forms $U_{\beta,m}(\tau)$ with $\beta \in L'/L$ and $m \in \mathbb{Z}+q(\beta)$ does in fact generate the complete space $H^+_{1/2,L}$ of harmonic Maass forms which map to cusp forms under $\xi_{1/2}$.

7 Kronecker limit formulas for averaged Eisenstein series

In the present chapter we use the realization of averaged non-holomorphic Eisenstein series as the theta lift of Selberg's Poincaré series given in Section 5.2, and the meromorphic continuation of Selberg's Poincaré series developed in the previous chapter, to obtain Kronecker limit type formulas for these averaged Eisenstein series.

The content of this chapter is already given in Sections 4 and 5 of the unpublished work [PSV17], which is a collaboration of the present author with A. von Pippich and M. Schwagenscheidt. Instead of referencing every given statement and the corresponding proofs, we state once and for all that the current chapter is a collaboration with the other two authors.

As in Chapter 6 we always let (V, q) be the symmetric space of signature $(2, 1)$ introduced in Section 4.3, and we let L the corresponding even lattice of level $4N$ in V. Further, we assume that N is squarefree, and we only consider the case of weight $\kappa = 1/2$. As before, we also drop the corresponding index κ and the superscript L to simplify notation.

7.1 Continuation of the theta lift of Selberg's Poincaré series

We use the following simplified notation: For $k = 0$ Shintani's theta function $\Theta_{L,0}(\tau, z)$ given in (4.3.6) is simply the vector valued Siegel theta function for the lattice L, namely

(7.1.1) $$\Theta(\tau, z) := \Theta_{L,0}(\tau, z) = \mathrm{Im}(\tau)^{1/2} \sum_{\gamma \in L'/L} \sum_{\lambda \in L+\gamma} e(\tau q(\lambda_z) + \bar{\tau} q(\lambda_{z^\perp}))\mathfrak{e}_\gamma$$

for $\tau, z \in \mathbb{H}$, which is modular of weight $1/2$ for ρ_L in τ, and modular of weight 0 and level N in the variable z (see Corollary 4.2.3). Further, given a real analytic function $F\colon \mathbb{H} \to \mathbb{C}[L'/L]$ modular of weight $1/2$ for ρ_L we denote the regularized theta lift of F by

$$\Phi(z; F) := \Phi_0^L(z; F) = \mathrm{CT}_{t=0} \left[\lim_{T\to\infty} \int_{\mathcal{F}_T} \langle F(\tau), \Theta(\tau, z)\rangle \,\mathrm{Im}(\tau)^{1/2-t} d\mu(\tau) \right]$$

for $z \in \mathbb{H}$, whenever the regularized inner product exists, and given $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ we denote the regularized theta lift of Selberg's Poincaré series $U_{\beta,m}(\tau, s)$ of weight $1/2$ for ρ_L by

(7.1.2) $$\Phi^{\mathrm{Sel}}_{\beta,m}(z, s) := \Phi(z; U_{\beta,m}(\,\cdot\,, s))$$

for $z \in \mathbb{H} \setminus H_{\beta,4Nm}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 3/2$ (compare Theorem 5.2.1).

In order to use Corollary 5.2.2 to study the behaviour of averaged hyperbolic, parabolic and elliptic Eisenstein series at $s = 0$ we first need to establish the meromorphic continuation of the corresponding regularized theta lift $\Phi^{\mathrm{Sel}}_{\beta,m}(z,s)$. The idea of the proof of the following proposition is based on [Bru02], Proposition 2.8 and Proposition 2.11.

Proposition 7.1.1. *Let $\beta \in L'/L$ and $m \in \mathbb{Z}+q(\beta)$. For $z \in \mathbb{H} \setminus H_{\beta,4Nm}$ the regularized theta lift $\Phi^{\mathrm{Sel}}_{\beta,m}(z,s)$ has a meromorphic continuation in s to all of $\mathbb{C}$, whose Laurent expansion at $s = 0$ is given by*

$$\Phi^{\mathrm{Sel}}_{\beta,m}(z,s) = -\frac{2}{s}\,\delta_{0,m} + \Phi(z;U_{\beta,m}) + O(s).$$

In particular, the meromorphic continuation of $\Phi^{\mathrm{Sel}}_{\beta,m}(z,s)$ is holomorphic at $s = 0$ if $m \neq 0$, and has a simple pole at $s = 0$ if $m = 0$.

Proof. By Theorem 6.3.1 the Poincaré series $U_{\beta,m}(\tau,s)$ has a meromorphic continuation in s to all of $\mathbb{C}$. Hence, the integral

$$\int_{\mathcal{F}_1} \langle U_{\beta,m}(\tau,s), \Theta(\tau,z)\rangle\, v^{1/2} d\mu(\tau)$$

over the compact set $\mathcal{F}_1$ is clearly holomorphic in s whenever $U_{\beta,m}(\tau,s)$ is, and it suffices to consider the function

$$\varphi(z,s,t) := \int_1^\infty \int_{-1/2}^{1/2} \langle U_{\beta,m}(\tau,s), \Theta(\tau,z)\rangle\, v^{1/2-t} d\mu(\tau).$$

We insert the Fourier expansion of $U_{\beta,m}(\tau,s)$ from (6.3.1) and the defining series for $\Theta(\tau,z)$ from (7.1.1), and carry out the integral over u, yielding

$$\begin{aligned}\varphi(z,s,t) = 2\int_1^\infty \sum_{\lambda \in L_{\beta,m}} v^{s-1-t} e^{-4\pi v q(\lambda_z)} dv + \int_1^\infty b(0,0,v,s) v^{-1-t} dv \\ + \int_1^\infty \sum_{\gamma \in L'/L} \sum_{\substack{\lambda \in L+\gamma \\ \lambda \neq 0}} b(\gamma, q(\lambda), v, s) v^{-1-t} e^{-2\pi v q_z(\lambda)} dv.\end{aligned} \tag{7.1.3}$$

Here $b(\gamma,n,v,s)$ denotes the (γ,n)'th Fourier coefficient of $U_{\beta,m}(\tau,s)$ as in (6.3.2), and q_z is the positive definite majorant of q associated to z. Moreover, we have used in the first summand of (7.1.3) that $L_{-\beta,m} = -L_{\beta,m}$ and $q((-\lambda)_z) = q(\lambda_z)$.

For $\mathrm{Re}(t)$ large enough we can split of the summand for $\lambda = 0$ in the first integral in (7.1.3), which only appears if $m = \beta = 0$, interchange summation and integration in the remaining sum, and compute the resulting integrals, to find that

$$2\int_1^\infty \sum_{\lambda \in L_{\beta,m}} v^{s-1-t} e^{-4\pi v q(\lambda_z)} dv = -\frac{2}{s-t}\,\delta_{0,m} + 2 \sum_{\lambda \in L_{\beta,m}\setminus\{0\}} \frac{\Gamma(s-t, 4\pi q(\lambda_z))}{(4\pi q(\lambda_z))^{s-t}}. \tag{7.1.4}$$

Here $\Gamma(s,x)$ denotes the incomplete Gamma function, which for $x > 0$ defines an entire function in s. Further, we note that $q(\lambda_z) \neq 0$ as $\lambda \neq 0$ and $z \notin H_{\beta,4Nm}$. Hence, the right-hand side of (7.1.4) has a meromorphic continuation in t to $t=0$, and taking the constant term at $t=0$ we find that

$$2\,\mathrm{CT}_{t=0}\int_1^\infty \sum_{\lambda\in L_{\beta,m}} v^{s-1-t}e^{-4\pi vq(\lambda_z)}dv = -\frac{2}{s}\,\delta_{0,m}+2\sum_{\lambda\in L_{\beta,m}\setminus\{0\}}\frac{\Gamma(s,4\pi q(\lambda_z))}{(4\pi q(\lambda_z))^s}. \tag{7.1.5}$$

Here the sum on the right-hand side defines a meromorphic function in s on $\mathbb{C}$, which is holomorphic at $s=0$ since the incomplete Gamma function $\Gamma(s,x)$ behaves as $x^{s-1}e^{-x}$ as $x\to\infty$, and using the arguments from the proof of Lemma 5.1.2 it can easily be shown that the sum

$$\sum_{\lambda\in L_{\beta,m}\setminus\{0\}} e^{-4\pi q(\lambda_z)}$$

converges. In particular, if $m=0$ equation (7.1.5) yields the claimed pole of the meromorphic continuation of the regularized theta lift $\Phi_{0,0}^{\mathrm{Sel}}(z,s)$ at $s=0$.

Next we consider the second integral on the right-hand side of (7.1.3). As remarked in (6.3.26) the constant Fourier coefficient $b(0,0,v,s)$ behaves as $v^{1/2-s}$ as $v\to\infty$. Thus, for $\mathrm{Re}(s)>1/2$ the integral

$$\int_1^\infty b(0,0,v,s)v^{-1-t}dv$$

has a meromorphic continuation in t, which is holomorphic at $t=0$. So we can plug in $t=0$, insert the explicit formula for the Fourier coefficient $b(0,0,v,s)$ given in (6.3.2) and (6.3.3), and evaluate the integral over v, yielding

$$\frac{2^{3/2-2s}\pi}{\Gamma(s)}\sum_{j=0}^{\infty}\frac{(-\pi m)^j}{j!}\frac{\Gamma(2s-1/2+j)}{\Gamma(s+1/2+j)}\,Z(s+j;\beta,m,0,0)\,\frac{1}{s+j-1/2}. \tag{7.1.6}$$

As the Kloosterman zeta functions $Z(s+j;\beta,m,0,0)$ define meromorphic functions in s on $\mathbb{C}$, which are universally bounded for j large enough, the given sum is essentially a confluent hypergeometric function of the first kind $\Phi(a,c;z)$ (see for example [GR07], Section 9.2). In particular, the expression in (7.1.6) has a meromorphic continuation in s to $\mathbb{C}$. Moreover, as the function $Z(s;\beta,m,0,0)$ is holomorphic at $s=0$ (compare Corollary 6.1.12 and Theorem 6.2.6), this continuation is holomorphic and indeed vanishing at $s=0$.

Finally, we consider the third integral on the right-hand side of (7.1.3), namely

$$\varphi_3(z,s,t) := \int_1^\infty \sum_{\gamma\in L'/L}\sum_{\substack{\lambda\in L+\gamma\\ \lambda\neq 0}} b(\gamma,q(\lambda),v,s)v^{-1-t}e^{-2\pi vq_z(\lambda)}dv.$$

As in [Bru02], equation (2.20), we split the inner sum into lattice elements λ with $q(\lambda)=0$ and with $q(\lambda)\neq 0$, i.e.,

$$\begin{aligned}\varphi_3(z,s,t) = &\int_1^\infty \sum_{\lambda\in L_{0,0}\setminus\{0\}} b(0,0,v,s)v^{-1-t}e^{-2\pi vq_z(\lambda)}dv\\ &+\int_1^\infty \sum_{\gamma\in L'/L}\sum_{\substack{\lambda\in L+\gamma\\ q(\lambda)\neq 0}} b(\gamma,q(\lambda),v,s)v^{-1-t}e^{-2\pi vq_z(\lambda)}dv.\end{aligned}$$

Following the arguments of the proof of Proposition 2.8 in [Bru02], and using the asymptotic estimate for the Fourier coefficients given in (6.3.25), we find that the evaluation of $\varphi_3(z,s,t)$ at $t = 0$ has a meromorphic continuation in s to all of $\mathbb{C}$, which is holomorphic whenever the Fourier coefficients $b(\gamma,n,v,s)$ are. In particular, the continuation of $\varphi_3(z,s,0)$ is holomorphic at $s = 0$.

It remains to prove that the meromorphic continuation of $\Phi^{\mathrm{Sel}}_{\beta,m}(z,s)$ has the claimed Laurent expansion at $s = 0$. Therefore, we have to go through the same proof again, replacing $U_{\beta,m}(\tau,s)$ by $U_{\beta,m}(\tau)$, and the Fourier coefficients $b(\gamma,n,v,s)$ by the Fourier coefficients of $U_{\beta,m}(\tau)$ given in Corollary 6.3.3, namely by

$$\tilde{b}(\gamma,n,v) := \begin{cases} \tilde{b}(\gamma,n)e^{-2\pi nv}, & \text{if } n > 0, \\ 0, & \text{if } n = 0, \\ \tilde{b}(\gamma,n)\Gamma(1/2,4\pi|n|v)e^{-2\pi nv}, & \text{if } n < 0. \end{cases}$$

Hence, instead of $\varphi(z,s,t)$ we now have to consider the function $\tilde{\varphi}(z,t)$ given by

(7.1.7)
$$\tilde{\varphi}(z,t) := 2\int_1^\infty \sum_{\lambda\in L_{\beta,m}} v^{-1-t}e^{-4\pi vq(\lambda_z)}dv + \int_1^\infty \sum_{\gamma\in L'/L}\sum_{\substack{\lambda\in L+\gamma\\ \lambda\neq 0}} \tilde{b}(\gamma,q(\lambda),v)v^{-1-t}e^{-2\pi vq_z(\lambda)}dv.$$

Firstly, we recall that the meromorphic continuation of the second integral from (7.1.3) vanishes at $s = 0$, as remarked above. Moreover, using similar arguments as before, one can show that the second integral from (7.1.7) has a meromorphic continuation in t to $t = 0$, which agrees with the evaluation of $\varphi_3(z,s,0)$ at $s = 0$. Hence it remains to consider the first integral from (7.1.7). As in (7.1.4) we find that

$$2\int_1^\infty \sum_{\lambda\in L_{\beta,m}} v^{-1-t}e^{-4\pi vq(\lambda_z)}dv = \frac{2}{t}\,\delta_{0,m} + 2\sum_{\lambda\in L_{\beta,m}\setminus\{0\}} \frac{\Gamma(-t,4\pi q(\lambda_z))}{(4\pi q(\lambda_z))^{-t}},$$

and thus

$$2\,\mathrm{CT}_{t=0}\int_1^\infty \sum_{\lambda\in L_{\beta,m}} v^{-1-t}e^{-4\pi vq(\lambda_z)}dv = 2\sum_{\lambda\in L_{\beta,m}\setminus\{0\}} \Gamma(0,4\pi q(\lambda_z)),$$

which agrees with the meromorphic continuation of the right-hand side of (7.1.5) to $s = 0$ up to the pole coming from the term $-2/s$ appearing only if $m = \beta = 0$. So

$$\left(\Phi^{\mathrm{Sel}}_{\beta,m}(z,s) + \frac{2}{s}\delta_{0,m}\right)\bigg|_{s=0} = \Phi(z;U_{\beta,m}),$$

proving the claimed Laurent expansion. □

Applying Proposition 7.1.1 to Corollary 5.2.2 we obtain Kronecker limit type formulas for averaged hyperbolic, parabolic and elliptic Eisenstein series. In order to state these in a compact form, we introduce the following notation.

Definition 7.1.2.

(a) Given $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m > 0$ we define the *averaged hyperbolic Eisenstein series* of index (β, m) as

$$E^{\mathrm{hyp}}_{\beta,m}(z,s) := \sum_{Q \in \mathcal{Q}_{\beta,4Nm}/\Gamma_0(N)} E^{\mathrm{hyp}}_{c_Q}(z,s)$$

for $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$.

(b) We define the *averaged parabolic Eisenstein series* as

$$E^{\mathrm{par}}(z,s) := \sum_{p \in C(\Gamma_0(N))} E^{\mathrm{par}}_p(z,s)$$

for $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$.

(c) Given $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m < 0$ we define the *averaged elliptic Eisenstein series* of index (β, m) as

$$E^{\mathrm{ell}}_{\beta,m}(z,s) := \sum_{Q \in \mathcal{Q}_{\beta,4Nm}/\Gamma_0(N)} E^{\mathrm{ell}}_{\tau_Q}(z,s)$$

for $z \in \mathbb{H} \setminus H_{\beta,4Nm}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$.

According to Corollary 2.6.4 the averaged hyperbolic, parabolic and elliptic Eisenstein series defined above essentially agree with the non-holomorphic modular forms of weight 0 associated to discriminants defined in part (b) of Definition 2.5.2. More precisely, given $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ we have

$$f_{0,\beta,4Nm}(z,s) = \begin{cases} (4Nm)^{-s/2} E^{\mathrm{hyp}}_{\beta,m}(z,s), & \text{if } m > 0, \\ 2\zeta(s) E^{\mathrm{par}}(z,s), & \text{if } m = 0, \\ (4N|m|)^{-s/2} E^{\mathrm{ell}}_{\beta,m}(z,s), & \text{if } m < 0, \end{cases}$$

for $z \in \mathbb{H} \setminus H_{\beta,4Nm}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. Here the case $m = 0$ is given in (2.6.9).

Corollary 7.1.3.

(a) For $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m > 0$ the averaged hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{\beta,m}(z,s)$ has a meromorphic continuation in s to all of $\mathbb{C}$, whose Laurent expansion at $s = 0$ is given by

$$E^{\mathrm{hyp}}_{\beta,m}(z,s) = \frac{1}{4}\,\Phi(z; U_{\beta,m}) \cdot s + O(s^2)$$

for $z \in \mathbb{H}$.

(b) The averaged parabolic Eisenstein series $E^{\mathrm{par}}(z,s)$ has a meromorphic continuation in s to all of $\mathbb{C}$, whose Laurent expansion at $s = 0$ is given by

$$E^{\mathrm{par}}(z,s) = 1 - \frac{1}{4}\Big(\Phi(z; U_{0,0}) + 2\log(4\pi N) - 2\gamma\Big) \cdot s + O(s^2)$$

for $z \in \mathbb{H}$. Here γ denotes the Euler-Mascheroni constant.

(c) *For $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m < 0$ the averaged elliptic Eisenstein series $E^{\mathrm{ell}}_{\beta,m}(z,s)$ has a meromorphic continuation in s to all of $\mathbb{C}$, whose Laurent expansion at $s = 0$ is given by*

$$E^{\mathrm{ell}}_{\beta,m}(z,s) = \frac{1}{4}\,\Phi(z;U_{\beta,m}) \cdot s + O(s^2)$$

for $z \in \mathbb{H} \setminus H_{\beta,4Nm}$.

Proof. Let $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$. By Corollary 5.2.2 we have that

$$\Phi^{\mathrm{Sel}}_{\beta,m}(z,s/2) = \begin{cases} \dfrac{2\,\Gamma(s/2)}{(4N|m|)^{s/2}}\,E^{\mathrm{hyp}}_{\beta,m}(z,s), & \text{if } m > 0, \\ 4N^{s/2}\zeta^*(s)\,E^{\mathrm{par}}(z,s), & \text{if } m = 0, \\ \dfrac{2\,\Gamma(s/2)}{(4N|m|)^{s/2}}\,E^{\mathrm{ell}}_{\beta,m}(z,s), & \text{if } m < 0, \end{cases} \tag{7.1.8}$$

for $z \in \mathbb{H} \setminus H_{\beta,4Nm}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. Hence, the meromorphic continuation of averaged hyperbolic, parabolic and elliptic Eisenstein series follows directly from the meromorphic continuation of the corresponding theta lift $\Phi^{\mathrm{Sel}}_{\beta,m}(z,s)$, which was proven in Proposition 7.1.1. Moreover, it was also shown in Proposition 7.1.1 that

$$\Phi^{\mathrm{Sel}}_{\beta,m}(z,s/2) = -\frac{4}{s}\,\delta_{0,m} + \Phi(z;U_{\beta,m}) + O(s) \tag{7.1.9}$$

as $s \to 0$. Comparing (7.1.9) with the Laurent expansion of the right-hand side of (7.1.8), and noting that

$$\frac{2\,\Gamma(s/2)}{(4N|m|)^{s/2}} = \frac{4}{s} + O(1), \qquad 4N^{s/2}\zeta^*(s) = -\frac{4}{s} - 2\log(4\pi N) + 2\gamma + O(s)$$

as $s \to 0$, we obtain the Laurent expansions claimed in the corollary. □

7.2 Borcherds products

The Laurent expansions given in Corollary 7.1.3 can be understood as abstract Kronecker limit formulas for averaged hyperbolic, parabolic and elliptic Eisenstein series. In order to obtain more concrete formulas in the form of (2.6.3) and (2.6.7), we now recall the theory of Borcherds products of harmonic Maass forms, which enables us to realize the regularized theta lift $\Phi(z;U_{\beta,m})$ as the logarithm of a modular form.

Let (M,r) be the lattice defined in (6.1.1), namely $M = \mathbb{Z}$ and $r(x) = Nx^2$. Then the discriminant forms induced by M and L are isomorphic, and thus, the so-called *unary theta function* associated to the lattice M given by

$$\theta(\tau) := \sum_{\gamma \in M'/M} \sum_{\lambda \in M+\gamma} e(r(\lambda)\tau)\mathfrak{e}_\gamma = \sum_{\gamma \in \mathbb{Z}/2N\mathbb{Z}} \sum_{\substack{n \in \mathbb{Z} \\ n \equiv \gamma\,(2N)}} e(n^2\tau/4N)\mathfrak{e}_\gamma, \tag{7.2.1}$$

is a vector valued modular form of weight $1/2$ for ρ_L (see for example [Bor98], Theorem 4.1). Now, given a harmonic Maass form $F \in H^+_{1/2,L}$, the so-called *Weil vector* associated to F and the cusp ∞ is given by

$$\rho_{F,\infty} := \frac{\sqrt{N}}{8\pi}(F,\theta)^{\mathrm{reg}}, \tag{7.2.2}$$

see Section 4.1 of [BO10]. In order to also define Weil vectors associated to cusps other than ∞, we use ideas from [BS17].

The orthogonal group $O(L'/L)$ acts on the vector space $\mathbb{C}[L'/L]$ by linear continuation of $w(\mathfrak{e}_\gamma) := \mathfrak{e}_{w(\gamma)}$ for $w \in O(L'/L)$ and $\gamma \in L'/L$, and it is easy to check that this action is compatible with the usual vector valued weight k action of the group $\mathrm{Mp}_2(\mathbb{Z})$ on the space of functions $F\colon \mathbb{H} \to \mathbb{C}[L'/L]$. Hence, we obtain an action of the group $O(L'/L)$ on the space of vector valued functions modular of weight $1/2$ for ρ_L by

$$F^w := \sum_{\gamma \in L'/L} f_\gamma \mathfrak{e}_{w(\gamma)}$$

for $F = \sum_{\gamma \in L'/L} f_\gamma \mathfrak{e}_\gamma$ and $w \in O(L'/L)$. It can be shown that the elements of $O(L'/L)$ are all involutions, the so-called *Atkin-Lehner involutions*, and since N is squarefree they correspond to the positive divisors d of N. More precisely, the involution w_d corresponding to the divisor $d \mid N$ is determined by the congruences

$$w_d(\gamma) \equiv -\gamma \mod 2d \qquad \text{and} \qquad w_d(\gamma) \equiv \gamma \mod 2N/d \tag{7.2.3}$$

for $\gamma \in L'/L$ (compare [EZ85], Theorem 5.2). In particular, w_1 acts as the identity. We also note that given a divisor $d \mid N$, we have $F^{w_d} = F^{w_{N/d}}$ and

$$(F^{w_d}, G)^{\mathrm{reg}} = (F, G^{w_d})^{\mathrm{reg}} \tag{7.2.4}$$

for $F, G\colon \mathbb{H} \to \mathbb{C}[L'/L]$ modular of weight $1/2$ for ρ_L, whenever the above regularized inner products exist.

Recall that the cusps of $\Gamma_0(N)$ can be represented by the fractions $1/d$ with $d \mid N$, and let $F \in H^+_{1/2,L}$. By Corollary 5.4 in [BS17] the Weil vector associated to F and the cusp $1/d$ for $d \mid N$ is given by

$$\rho_{F,1/d} := \frac{\sqrt{N}}{8\pi}(F, \theta^{w_d})^{\mathrm{reg}}. \tag{7.2.5}$$

We further quote Lemma 2.1 from [BS17]:

Lemma 7.2.1. *The set of unary theta functions θ^{w_d} with d running through the positive divisors of N modulo the relation $d \sim N/d$ forms a basis of the space $M_{1/2,L}$.*

Eventually, we state Theorem 6.1 of [BO10] on Borcherds products of harmonic Maass forms, restricting to the case of $\Delta = 1$ (in the notation of [BO10]). Here, given a unitary character $\chi\colon \Gamma_0(N) \to \{z \in \mathbb{C}\colon |z| = 1\}$ of possibly infinite order, we say that a function $f\colon \mathbb{H} \to \mathbb{C}$ is modular of weight k, level N and character χ if

$$f \big|_k M = \chi(M)\, f$$

for all $M \in \Gamma_0(N)$. Correspondingly, we say that f is a *weakly meromorphic or holomorphic modular form of weight k, level N and character χ* if f is meromorphic or holomorphic on $\mathbb{H}$, meromorphic at all cusps of $\Gamma_0(N)$ and modular of weight k, level N and character χ, respectively. Here we note that even though such f is in general not 1-periodic, we still have $f(z+1) = \chi(T)f(z)$, showing that f does have a Fourier expansion at ∞, and similarly at all other cusps. Hence, given a cusp p the terms *meromorphic at p* and *order at p* (compare Section 2.4) are still well-defined for f transforming with a character. However, the order of f at p will in general not be an integer.

Theorem 7.2.2 ([BO10], Theorem 6.1)**.** *Let $F \in H^+_{1/2,L}$ be a harmonic Maass form with real Fourier coefficients $a_F^+(\gamma, n)$ for all $\gamma \in L'/L$ and $n \in \mathbb{Z} + q(\gamma)$. Further, we assume that the coefficients of the principal part of F are integral, i.e., that $a_F^+(\gamma, n) \in \mathbb{Z}$ for all $\gamma \in L'/L$ and $n \in \mathbb{Z}+q(\gamma)$ with $n \leq 0$. Then the Borcherds product associated to F given by*

$$\Psi(z; F) := e(\rho_{F,\infty} z) \prod_{n=1}^{\infty} (1 - e(nz))^{a_F^+(n,n^2/4N)}$$

converges for $\operatorname{Im}(z)$ sufficiently large, and has a meromorphic continuation to all of $\mathbb{H}$, which satisfies the following properties:

(i) The product $\Psi(z; F)$ is a weakly meromorphic modular form of weight $a_F^+(0, 0)$, level N and some unitary character of possibly infinite order.

(ii) The orders of vanishing of $\Psi(z; F)$ at points in $\mathbb{H}$ are determined by the divisor

$$\frac{1}{2} \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z}+q(\gamma) \\ n<0}} a_F^+(\gamma, n) \sum_{Q \in \mathcal{Q}_{\gamma,4Nn}} (\tau_Q).$$

(iii) The order of $\Psi(z; F)$ at the cusp $1/d$ for $d \,|\, N$ is given by the corresponding Weil vector, i.e.,

$$\operatorname{ord}_{1/d}(\Psi(z; F)) = \rho_{F,1/d}.$$

(iv) The regularized theta lift of F is given by

$$\Phi(z; F) = -a_F^+(0, 0)(\log(4\pi N) - \gamma) - 4 \log \left| \Psi(z; F) \operatorname{Im}(z)^{a_F^+(0,0)/2} \right|.$$

We quickly comment on the above theorem, namely, we quote a characterization from [BO10] for the finiteness of the order of the character appearing in part (i) of the previous theorem, and we show that the Borcherds product is in fact uniquely determined by its modularity and its divisor.

Proposition 7.2.3. *Let $F \in H^+_{1/2,L}$ be a harmonic Maass form such that $a_F^+(\gamma, n) \in \mathbb{R}$ for all $\gamma \in L'/L$ and $n \in \mathbb{Z}+q(\gamma)$ with $n > 0$, and such that $a_F^+(\gamma, n) \in \mathbb{Z}$ for all $\gamma \in L'/L$ and $n \in \mathbb{Z} + q(\gamma)$ with $n \leq 0$.*

(a) Suppose that F is orthogonal to cusp forms in $S_{1/2,L}$, and let χ be the character the Borcherds product $\Psi(z; F)$ transforms with. Then

$$\chi \text{ is of finite order} \qquad \Longleftrightarrow \qquad a_F^+(n, n^2/4N) \in \mathbb{Q} \text{ for all } n \in \mathbb{N}.$$

(b) *Let g be a weakly meromorphic modular form of weight $a_F^+(0,0)$, level N and some unitary character. If g satisfies the properties (ii) and (iii) of Theorem 7.2.2 then*

$$g(z) = a_g(n_0) \cdot \Psi(z; F),$$

where n_0 is the order of g at ∞.

Proof. Part (a) of the proposition follows from Theorem 6.2 and the subsequent Remark 16 in [BO10] (recall that we assume $\Delta = 1$ in the notation of [BO10]). For (b) we copy the argument from the proof of [BO10, Theorem 6.2]: As noted in Remark 16 we can restrict to the case $a_F^+(0,0) = 0$ by adding a suitable linear combination of unary theta functions θ^{w_d} with $d \,|\, N$ to the harmonic Maass form F. Let now g be a weakly meromorphic modular form of weight 0, level N and some unitary character, satisfying the properties (ii) and (iii) given in Theorem 7.2.2. Here $\Psi(z;F)$ and g can be modular with respect to different unitary characters. It is now easy to check that

$$H(z) := \log \left| \frac{\Psi(z;F)}{g(z)} \right|$$

defines a harmonic function on the modular curve $X_0(N)$ without singularities. Thus, the maximum principle for harmonic functions on $X_0(N)$ implies that $H(z)$ is constant. So $\Psi(z;F)$ and $g(z)$ agree up to a constant multiple. Now the claimed identity follows from the fact that the Fourier expansion of $\Psi(z;F)$ at ∞ is normalized. This can easily be deduced from its product expansion. □

Using Theorem 7.2.2 we can now restate Corollary 7.1.3 in the following way:

Corollary 7.2.4.

(a) *For $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m > 0$ the averaged hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{\beta,m}(z,s)$ has a meromorphic continuation in s to all of $\mathbb{C}$, whose Laurent expansion at $s = 0$ is given by*

$$E^{\mathrm{hyp}}_{\beta,m}(z,s) = -\log \left| \Psi(z; U_{\beta,m}) \right| \cdot s + O(s^2)$$

for $z \in \mathbb{H}$.

(b) *The averaged parabolic Eisenstein series $E^{\mathrm{par}}(z,s)$ has a meromorphic continuation in s to all of $\mathbb{C}$, whose Laurent expansion at $s = 0$ is given by*

$$E^{\mathrm{par}}(z,s) = 1 + \log \left| \Psi(z; U_{0,0}) \operatorname{Im}(z) \right| \cdot s + O(s^2)$$

for $z \in \mathbb{H}$. Here γ denotes the Euler-Mascheroni constant.

(c) *For $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m < 0$ the averaged elliptic Eisenstein series $E^{\mathrm{ell}}_{\beta,m}(z,s)$ has a meromorphic continuation in s to all of $\mathbb{C}$, whose Laurent expansion at $s = 0$ is given by*

$$E^{\mathrm{ell}}_{\beta,m}(z,s) = -\log \left| \Psi(z; U_{\beta,m}) \right| \cdot s + O(s^2)$$

for $z \in \mathbb{H} \setminus H_{\beta,4Nm}$.

Proof. Let $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$. By Theorem 6.3.5 we know that $U_{\beta,m}(\tau)$ is a harmonic Maass form, which maps to a cusp form under $\xi_{1/2}$, and whose principal part is given by 0, $2\mathfrak{e}_0$ or $e(m\tau)(\mathfrak{e}_\beta + \mathfrak{e}_{-\beta})$ if $m > 0$, $m = 0$ or $m < 0$, respectively. Moreover, by Corollary Corollary 6.1.13 and 6.3.3 the holomorphic Fourier coefficients $a^+_{U_{\beta,m}}(\gamma, n)$ are real for all $\gamma \in L'/L$ and $n \in \mathbb{Z} + q(\gamma)$. Therefore, using part (iv) of Theorem 7.2.2 we find that

$$\Phi(z; U_{\beta,m}) = \begin{cases} -4\log|\Psi(z; U_{\beta,m})|, & \text{if } m \neq 0, \\ -2(\log(4\pi N) - \gamma) - 4\log|\Psi(z; U_{0,0})\operatorname{Im}(z)|, & \text{if } m = 0. \end{cases}$$

Here γ denotes the Euler-Mascheroni constant. Applying this to the Laurent expansions given in Corollary 7.1.3 we obtain the claimed formulas. □

In the following we compute the Borcherds products of $U_{\beta,m}(\tau)$ appearing in the previous corollary for $m = 0$, $m > 0$ and $m < 0$, yielding Kronecker limit type formulas in all three cases.

7.3 A parabolic Kronecker limit formula

We start with the parabolic case, i.e., we want to determine the Borcherds product of the harmonic Maass form $U_{\beta,m}(\tau)$ for $m = \beta = 0$. By part (b) of Theorem 6.3.5 we already know that $U_{0,0}(\tau)$ is a modular form of weight 1/2 for ρ_L, which has principal part $2\mathfrak{e}_0$, and which is orthogonal to cusp forms.

Lemma 7.3.1. *We have*

$$U_{0,0}(\tau) = \frac{2}{\sigma_0(N)} \sum_{d\,|\,N} \theta^{w_d}(\tau),$$

for $\tau \in \mathbb{H}$, where $\sigma_0(N) := \sum_{d|N} 1$.

Proof. By Lemma 7.2.1 the space $M_{1/2,L}$ is generated by the unary theta functions θ^{w_d} with $d\,|\,N$. As $U_{0,0}(\tau)$ is a modular form of weight 1/2 for ρ_L (compare Theorem 6.3.5), we can thus write $U_{0,0}(\tau)$ as a linear combination of unary theta functions, i.e., we find coefficients $\lambda_d \in \mathbb{C}$ such that

$$U_{0,0} = \sum_{d\,|\,N} \lambda_d \theta^{w_d}. \tag{7.3.1}$$

On the other hand, for $\operatorname{Re}(s) > 3/4$ the Eisenstein series $E_0(\tau, s) = U_{0,0}(\tau, s)$ is clearly invariant under all Atkin-Lehner involutions as $(v^s\mathfrak{e}_0)^w = v^s\mathfrak{e}_0$ for all $w \in O(L'/L)$. Hence, by meromorphic continuation also the function $U_{0,0}(\tau) = U_{0,0}(\tau, 0)$ is invariant under the action of $O(L'/L)$, and thus the coefficients λ_d with $d\,|\,N$ all agree. Comparing the constant Fourier coefficient of both sides in (7.3.1) we obtain $\lambda_d = 2/\sigma_0(N)$, since the constant coefficient of $U_{0,0}(\tau)$ is given by $2\mathfrak{e}_0$ (see again Theorem 6.3.5), and the constant coefficient of the unary theta functions θ^{w_d} is simply $\mathfrak{e}_0$. □

Lemma 7.3.2. *For $d\,|\,N$ the Borcherds product associated to $\theta^{w_d}(\tau)$ is given by*

$$\Psi(z; \theta^{w_d}) = \eta(dz)\,\eta(Nz/d)$$

for $z \in \mathbb{H}$. Here $\eta(z) := e(z/24)\prod_{n=1}^{\infty}(1 - e(nz))$ is the usual Dedekind eta function.

Proof. As the unary theta functions θ^{w_d} are modular forms of weight $1/2$ for ρ_L with integral Fourier coefficients, we can apply Theorem 7.2.2. Let $d \mid N$. Using the characterization of the involution w_d given in (7.2.3), we can deduce from (7.2.1) that the Fourier coefficients of θ^{w_d} relevant for its Borcherds product are given by

$$a_{\theta^{w_d}}(n, n^2/4N) = \begin{cases} 1, & \text{if } n \equiv 0 \bmod d \text{ or } n \equiv 0 \bmod N/d, \text{ and } n \not\equiv 0 \bmod N, \\ 2, & \text{if } n \equiv 0 \bmod N, \\ 0, & \text{else,} \end{cases} \tag{7.3.2}$$

for $n \in \mathbb{N}$. Further, by (7.2.2) the Weyl vector associated to θ^{w_d} and the cusp ∞ is given by

$$\rho_{\theta^{w_d},\infty} = \frac{\sqrt{N}}{8\pi} (\theta^{w_d}, \theta)^{\mathrm{reg}} = \frac{d + N/d}{24},$$

where the inner product $(\theta^{w_d}, \theta)^{\mathrm{reg}}$ is for example computed in [BS17, Example 5.6]. Therefore, applying Theorem 7.2.2 we obtain the product expansion

$$\Psi(z; \theta^{w_d}) = e((d + N/d)z/24) \prod_{n=1}^{\infty} (1 - e(ndz)) \prod_{n=1}^{\infty} (1 - e(nNz/d)).$$

This proves the claimed formula. □

Proposition 7.3.3. *The Borcherds product associated to $U_{0,0}(\tau)$ is given by*

$$\Psi(z; U_{0,0}) = \left(\prod_{d \mid N} \eta(dz) \right)^{4/\sigma_0(N)}$$

for $z \in \mathbb{H}$.

Proof. Since the regularized theta lift $\Phi(z; F)$ is clearly linear in F, the Borcherds product $\Psi(z; F)$ is multiplicative in F by part (iv) of Theorem 7.2.2. Thus, using Lemma 7.3.1 and Lemma 7.3.2 we find that

$$\Psi(z; U_{0,0}) = \left(\prod_{d \mid N} \Psi(z; \theta^{w_d}) \right)^{2/\sigma_0(N)} = \left(\prod_{d \mid N} \eta(dz) \cdot \prod_{d \mid N} \eta(Nz/d) \right)^{2/\sigma_0(N)},$$

which proves the claimed formula. □

We can now use Corollary 7.2.4 to obtain an averaged Kronecker limit formula for parabolic Eisenstein series of level N.

Theorem 7.3.4. *The averaged parabolic Eisenstein series $E^{\mathrm{par}}(z, s)$ has the Laurent expansion*

$$E^{\mathrm{par}}(z, s) = 1 + \frac{1}{\sigma_0(N)} \sum_{d \mid N} \log \left(|\Delta(dz)|^{1/6} \operatorname{Im}(z) \right) \cdot s + O(s^2)$$

at $s = 0$, for $z \in \mathbb{H}$. Here $\Delta(z) := \eta(z)^{24}$ is the unique normalized cusp form of weight 12 and level 1.

Proof. This is a direct application of Proposition 7.3.3 to part (b) of Corollary 7.2.4. □

Remark 7.3.5.

(1) For $N = 1$ Theorem 7.3.4 becomes the classical Kronecker limit formula given in (2.5.5), namely

$$E_\infty^{\mathrm{par}}(z,s) = 1 + \log\Big(|\Delta(z)|^{1/6}\,\mathrm{Im}(z)\Big)\cdot s + O(s^2)$$

as $s \to 0$.

(2) Let $\Gamma_0^*(N)$ be the Fricke group of level N given in (4.3.2), i.e., the extension of $\Gamma_0(N)$ by all Atkin-Lehner involutions. The Fricke group has only one cusp, which is represented by ∞, and the corresponding non-holomorphic Eisenstein series of weight 0 is given by

$$E_\infty^{\mathrm{par},\Gamma_0^*(N)}(z,s) := \sum_{M\in(\Gamma_0^*(N))_\infty\backslash\Gamma_0^*(N)} \mathrm{Im}(Mz)^s.$$

Using the scaling matrices for the cusps of $\Gamma_0(N)$ given in (2.2.2) one easily checks that the unique parabolic Eisenstein series for the Fricke group splits into parabolic Eisenstein series for the group $\Gamma_0(N)$, namely

$$E_\infty^{\mathrm{par},\Gamma_0^*(N)}(z,s) = \sum_{p\in C(\Gamma_0(N))} E_p^{\mathrm{par}}(z,s) = E^{\mathrm{par}}(z,s).$$

Therefore, Theorem 7.3.4 actually states the Kronecker limit formula for the Fricke group $\Gamma_0^*(N)$, given for example in Section 1.5 of [JST16].

7.4 Hyperbolic Kronecker limit formulas

Next we aim to compute the Borcherds product of the harmonic Maass form $U_{\beta,m}(\tau)$ for $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m > 0$. Recall from Theorem 6.3.5 that in this case $U_{\beta,m}(\tau)$ is the unique cusp form of weight 1/2 for ρ_L, which satisfies the inner product formula (6.3.28).

In order to compute its Borcherds product, we again write $U_{\beta,m}(\tau)$ as a linear combination of unary theta functions. More precisely, we define

$$\theta_d(\tau) := \sum_{c\,|\,N} \mu((c,d))\theta^{w_c}(\tau) \tag{7.4.1}$$

for d a positive divisor of N and $\tau \in \mathbb{H}$. Here $\mu(n)$ denotes the usual Moebius function, which is $(-1)^k$ if n is squarefree with k prime divisors, and vanishes otherwise. Further, let $D(N)$ be the finite group given by the set of positive divisors of N together with the operation $c \star d := cd/(c,d)^2$ for $c, d\,|\,N$. Then $D(N)$ has a natural identification with the orthogonal group $O(L'/L)$ via the isomorphism

$$D(N) \to O(L'/L),\ d \mapsto w_d,$$

where $w_d \in O(L'/L)$ is given by (7.2.3). It can be shown that the characters of the group $D(N)$ are exactly given by the functions $c \mapsto \mu((c,d))$ for $d\,|\,N$. Finally, we let $E(N)$ be the subgroup of $D(N)$ given by the positive divisors d of N with $\mu(d) = 1$.

Lemma 7.4.1. *The set of theta functions θ_d with $d \in E(N) \setminus \{1\}$ forms an orthogonal basis of the space of cusp forms $S_{1/2,L}$, and*

$$(\theta_d, \theta_d) = \frac{2\pi\sigma_0(N)\sigma_1(N/d)\varphi(d)}{3\sqrt{N}}.$$

for $d \in E(N) \setminus \{1\}$. Here $\sigma_k(n) := \sum_{d|n} d^k$ is the usual divisor sum, and $\varphi(n)$ denotes Euler's totient function.

Proof. Using (7.2.4) we find that

$$(\theta_d, \theta_e)^{\mathrm{reg}} = \sum_{b,c\,|\,N} \mu((c,d))\mu((b,e))(\theta^{w_c}, \theta^{w_b})^{\mathrm{reg}} = \sum_{c\,|\,N} \mu((c, d\star e)) \cdot \sum_{a\,|\,N} \mu((a,e))(\theta^{w_a}, \theta)^{\mathrm{reg}}$$

for $d, e \in E(N)$. As $d \star e = 1$ if and only $d = e$, we have

$$\sum_{c\,|\,N} \mu((c, d\star e)) = \begin{cases} \sigma_0(N), & \text{if } d = e, \\ 0, & \text{if } d \neq e, \end{cases}$$

by the orthogonality relations for characters of finite groups. Therefore, the theta functions $\theta_d(\tau)$ for $d \in E(N)$ are pairwise orthogonal, and if $d = e$ we obtain

$$(\theta_d, \theta_d)^{\mathrm{reg}} = \sigma_0(N) \sum_{c\,|\,N} \mu((c,d))(\theta^{w_c}, \theta)^{\mathrm{reg}} = \frac{\pi\sigma_0(N)}{3\sqrt{N}} \sum_{c\,|\,N} \mu((c,d))(N/c + c),$$

where we again refer to [BS17, Example 5.6] for the computation of the inner product $(\theta^{w_c}, \theta)^{\mathrm{reg}}$. Since $\mu((c,d)) = \mu((N/c, d))$ for $c \,|\, N$ and $d \in E(N)$, we find

$$\sum_{c\,|\,N} \mu((c,d))(N/c + c) = 2N \sum_{c\,|\,N} \frac{\mu((c,d))}{c}.$$

In the latter sum we split c as $c = ab$ with $a \,|\, d$ and $b \,|\, \frac{N}{d}$, giving

$$\sum_{c\,|\,N} \frac{\mu((c,d))}{c} = \left(\sum_{a\,|\,d} \frac{\mu(a)}{a} \right) \cdot \left(\sum_{b\,|\,\frac{N}{d}} \frac{1}{b} \right) = \frac{\varphi(d)}{d} \cdot \frac{d}{N} \sigma_1(N/d),$$

where we have used the well-known identity $\varphi(n)/n = \sum_{d\,|\,n} \mu(d)/d$. Putting everything back together we obtain the claimed formulas for the inner products $(\theta_d, \theta_d)^{\mathrm{reg}}$.

It remains to prove that the set of theta function θ_d with $d \in E(N) \setminus \{1\}$ forms a basis of $S_{1/2,L}$. Let $d \in E(N)$. Firstly, we note that by (7.3.2) the $(n, n^2/4N)$'th Fourier coefficient of θ_d is given by

$$a_{\theta_d}(n, n^2/4N) = \sum_{c\,|\,N} \mu((c,d)) a_{\theta^{w_c}}(n, n^2/4N) = \mu((n,d)) + \mu((N/n, d)) = 2\mu((n,d))$$

for $n \,|\, N$, which shows that the θ_d with $d \in E(N)$ are all linearly independent, as the character functions $c \mapsto \mu((c,d))$ of the group $D(N)$ are. Moreover, we know that the set of unary theta functions θ^{w_c} with $c\,|\,N$ modulo the relation $c \sim N/c$ forms a basis of the space $M_{1/2,L}$ (see Lemma 7.2.1). Thus, since

$$|D(N)/\sim| = \frac{\sigma_0(N)}{2} = |E(N)|\,,$$

the linearly independent functions θ_d with $d \in E(N)$ need to generate the whole space $M_{1/2,L}$. Finally, noting that

$$a_{\theta_d}(0,0) = \sum_{c\,|\,N} \mu((c,d)) = \begin{cases} \sigma_0(N), & \text{if } d = 1,\\ 0, & \text{if } d \neq 1,\end{cases}$$

for $d \in E(N)$, we find that $\theta_d \in S_{1/2,L}$ for all $d \in E(N)\setminus\{1\}$. This proves the lemma. $\square$

Before we use the orthogonal basis of $S_{1/2,L}$ constructed in the previous lemma to compute the Borcherds product of $U_{\beta,m}(\tau)$, we quickly deduce the following well-known dimension formula. Here we emphasize that we always assume that N is squarefree.

Corollary 7.4.2. *We have*

$$\dim(S_{1/2,L}) = \begin{cases} \sigma_0(N)/2 - 1, & \text{if } N > 1,\\ 0, & \text{if } N = 1.\end{cases}$$

In particular, the space $S_{1/2,L}$ is trivial if and only if N is either 1 or a prime.

Proof. The statement of the corollary follows directly from Lemma 7.4.1 and the fact that $|D(N)| = 2|E(N)|$ for $N > 1$. $\square$

Proposition 7.4.3. *For $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m > 0$ the Borcherds product associated to $U_{\beta,m}(\tau)$ is given by*

$$\Psi(z; U_{\beta,m}) = \prod_{d\in E(N)\setminus\{1\}} \left(\prod_{c\,|\,N} \eta(cz)^{\mu((c,d))} \right)^{c_{\beta,m}(d)} \tag{7.4.2}$$

for $z \in \mathbb{H}$. Here $E(N)$ is the set of positive divisors d of N with $\mu(d) = 1$, and the exponents $c_{\beta,m}(d)$ are given as follows:

(i) If $4Nm$ is not a square, or if $4Nm = n^2$ with $n \in \mathbb{N}$ such that $N/(n,N)$ is 1 or a prime, then $c_{\beta,m}(d) = 0$ for all $d \in E(N)\setminus\{1\}$. In this case the right-hand side of (7.4.2) is simply 1.

(ii) If $4Nm = n^2$ with $n \in \mathbb{N}$ such that $N/(n,N)$ is neither 1 nor a prime, then the exponents $c_{\beta,m}(d)$ are given by

$$c_{\beta,m}(d) = \begin{cases} -\dfrac{24n\,\mu((f,d))}{\sigma_0(N/(n,N))\sigma_1(N/d)\varphi(d)}, & \text{if } (n,d) = 1,\\ 0, & \text{if } (n,d) > 1,\end{cases}$$

for $d \in E(N)\setminus\{1\}$. Here we choose $f\,|\,N$ such that $w_f(n) = \beta$. Moreover, $\mu(n)$ denotes the Moebius function, $\sigma_k(n)$ the usual divisor sum, and $\varphi(n)$ Euler's totient function. In this case, the right-hand side of (7.4.2) is non-trivial.

Proof. By part (a) of Theorem 6.3.5 $U_{\beta,m}(\tau)$ is a cusp form of weight 1/2 for ρ_L. Using the orthogonal basis given in Lemma 7.4.1 we can thus write

$$U_{\beta,m}(\tau) = \sum_{d\in E(N)\setminus\{1\}} \frac{(U_{\beta,m},\theta_d)}{(\theta_d,\theta_d)}\theta_d(\tau) = \sum_{d\in E(N)\setminus\{1\}} \frac{(U_{\beta,m},\theta_d)}{(\theta_d,\theta_d)} \sum_{c\,|\,N} \mu((c,d))\theta^{w_c}(\tau).$$

Furthermore, using the multiplicativity of Borcherds products and Lemma 7.3.2 we thus obtain

$$\Psi(z;U_{\beta,m}) = \prod_{d\in E(N)\setminus\{1\}} \left(\prod_{c\,|\,N} \eta(cz)^{\mu((c,d))}\right)^{c_{\beta,m}(d)} \tag{7.4.3}$$

with

$$c_{\beta,m}(d) := 2\,\frac{(U_{\beta,m},\theta_d)}{(\theta_d,\theta_d)}.$$

In the following we investigate the exponents $c_{\beta,m}(d)$. Let $d\in E(N)\setminus\{1\}$. Since $U_{\beta,m}(\tau)$ satisfies the inner product formula (6.3.28) we find that

$$(U_{\beta,m},\theta_d) = \sum_{c\,|\,N} \mu((c,d))(U_{\beta,m},\theta^{w_c}) = -8\pi\sqrt{m}\sum_{c\,|\,N}\mu((c,d))a_{\theta^{w_c}}(\beta,m). \tag{7.4.4}$$

We recall that the Fourier coefficients of the unary theta functions θ^{w_c} for $c\,|\,N$ are supported on indices of the form $(\gamma, n^2/4N)$ with $n\in\mathbb{Z}$. Thus, we have $a_{\theta^{w_c}}(\beta,m)=0$ for all c dividing N if $4Nm$ is not a square. In this case the inner product $(U_{\beta,m},\theta_d)$ vanishes. In fact, we have $c_{\beta,m}(d)=0$ for all $d\in E(N)\setminus\{1\}$ if $4Nm$ is not a sqaure.

Let $4Nm=n^2$ for some $n\in\mathbb{N}$. Then we find $f\,|\,N$ such that $w_f(n)=\beta$ as $n^2\equiv\beta^2 \bmod 2N$, yielding $U^{w_f}_{n,n^2/4N}=U_{\beta,m}$. Thus, we can write (7.4.4) as

$$(U_{\beta,m},\theta_d) = \sum_{c\,|\,N}\mu((c,d))(U_{n,n^2/4N},\theta^{w_{c\star f}}) = \mu((f,d))\sum_{c\,|\,N}\mu((c,d))(U_{n,n^2/4N},\theta^{w_c}).$$

Now, using again the inner product formula (6.3.28) we get

$$(U_{\beta,m},\theta_d) = -\frac{4\pi n\,\mu((f,d))}{\sqrt{N}}\sum_{c\,|\,N}\mu((c,d))a_{\theta^{w_c}}(n,n^2/4N).$$

Here the latter Fourier coefficients have already been computed in (7.3.2), yielding

$$(U_{\beta,m},\theta_d) = -\frac{8\pi n\,\mu((f,d))}{\sqrt{N}}\sum_{c\,|\,(n,N)}\mu((c,d)).$$

Writing $c\,|\,(n,N)$ as $c=ab$ with $a\,|\,(n,d)$ and $b\,|\,(n,N/d)$ we further obtain

$$\sum_{c\,|\,(n,N)}\mu((c,d)) = \sigma_0((n,N/d))\sum_{a\,|\,(n,d)}\mu((a,d)) = \sigma_0((n,N/d))\sum_{a\,|\,(n,d)}\mu((a,(n,d))).$$

Here the remaining sum over the values of the character $a \mapsto \mu((a,(n,d))$ of the group $D((n,d))$ vanishes if $(n,d) > 1$, implying $c_{\beta,m}(d) = 0$. Conversely, if $(n,d) = 1$ then the above sum over the divisors of (n,d) is simply 1, showing that

$$(U_{\beta,m}, \theta_d) = -\frac{8\pi n\,\mu((f,d))\sigma_0((n,N))}{\sqrt{N}}. \tag{7.4.5}$$

Here we have also used that $(n, N/d) = (n,N)$ if $(n,d) = 1$. Recalling that the norm (θ_d, θ_d) was already computed in Lemma 7.4.1 we finally obtain

$$c_{\beta,m}(d) = -\frac{24n\,\mu((f,d))}{\sigma_0(N/(n,N))\sigma_1(N/d)\varphi(d)},$$

where we have used that $\sigma_0(N)/\sigma_0((n,N)) = \sigma_0(N/(n,N))$. In particular, if $4Nm$ is a square the coefficients $c_{\beta,m}(d)$ can only vanish simultaneously if $(n,d) > 1$ for all $d \in E(N) \setminus \{1\}$, which is exactly the case if (n,N) misses at most one prime factor of N, i.e., if $N/(n,N)$ is 1 or a prime.

It remains to note that the right-hand side of (7.4.2) is non-trivial if $4Nm = n^2$ is a square and $N/(n,N)$ is neither 1 nor a prime. Using that an eta quotient is uniquely determined by its exponents, one can check that for $d \in E(N) \setminus \{1\}$ the eta quotients

$$\prod_{c\,|\,N} \eta(cz)^{\mu((c,d))}$$

are linearly independent (in the multiplicative sense). Therefore, the product in (7.4.3) is trivial (in the multiplicative sense) if and only if the exponents $c_{\beta,m}(d)$ vanish simultaneously for all $d \in E(N) \setminus \{1\}$, proving the claimed statement. □

As in the parabolic case we can now use Corollary 7.2.4 to obtain an averaged Kronecker limit formula for hyperoblic Eisenstein series of level N.

Theorem 7.4.4. *For $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m > 0$ the averaged hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{\beta,m}(z,s)$ has the Laurent expansion*

$$E^{\mathrm{hyp}}_{\beta,m}(z,s) = -\sum_{d \in E(N)\setminus\{1\}} c_{\beta,m}(d) \log\left(\prod_{c\,|\,N} |\eta(cz)|^{\mu((c,d))}\right) \cdot s + O(s^2)$$

at $s = 0$, for $z \in \mathbb{H}$. Here the coefficients $c_{\beta,m}(d)$ are defined as in Proposition 7.4.3. In particular, the linear term of the above Laurent expansion is non-vanishing if and only if $4Nm = n^2$ for some $n \in \mathbb{N}$ such that $N/(n,N)$ is neither 1 nor a prime.

Proof. This is a direct application of Proposition 7.4.3 to part (a) of Corollary 7.2.4. □

We recall that given $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m > 0$ the corresponding averaged hyperbolic Eisenstein series $E^{\mathrm{hyp}}_{\beta,m}(z,s)$ is defined by

$$E^{\mathrm{hyp}}_{\beta,m}(z,s) = \sum_{Q \in \mathcal{Q}_{\beta,4Nm}/\Gamma_0(N)} E^{\mathrm{hyp}}_{c_Q}(z,s).$$

By Corollary 2.3.2 the Heegner geodesics c_Q with $Q \in \mathcal{Q}_{\beta,4Nm}$ are all infinite in the modular curve $X_0(N)$ exactly if $4Nm$ is a square, and closed otherwise. Moreover, by (2.6.5) the hyperbolic Eisenstein series $E_c^{\mathrm{hyp}}(z,s)$ has a double zero at $s=0$ if c is closed. Hence, also the averaged hyperbolic Eisenstein series $E_{\beta,m}^{\mathrm{hyp}}(z,s)$ needs to have a double zero at $s=0$ if $4Nm$ is not a square, which is confirmed by the previous Theorem.

From a different point of view, the vanishing of the linear term of the Laurent expansion of the averaged hyperbolic Eisenstein series $E_{\beta,m}^{\mathrm{hyp}}(z,s)$ at $s=0$ can also be explained by the vanishing of the corresponding Poincaré series $U_{\beta,m}(\tau,s)$ at $s=0$, i.e., of the cusp form $U_{\beta,m}(\tau)$. In the following we investigate this behaviour.

Let q be a positive divisor of N. In order to distinguish lattices, we write $L_N := L$ for our usual lattice of level $4N$, and $L_{N/q}$ for the corresponding lattice of signature $(2,1)$ and level $4N/q$ (compare Section 4.3). Similarly, we write q_N and $q_{N/q}$ for the corresponding quadratic forms, and ρ_N and $\rho_{N/q}$ for the corresponding Weil representations. We also put additional superscripts at some places to emphasize the underlying lattice.

Since the space $M_{1/2,L_{N/q}}$ is isomorphic to the space $J^*_{1,N/q}$ of skew-holomorphic Jacobi forms of weight 1 and index N/q (compare for example [EZ85], Theorem 5.7), we can carry over the operator V_q defined in Section 4 of [EZ85] from the theory of Jacobi forms to the present setting of vector valued modular form for the Weil representation. More precisely, we obtain an operator

$$V_q\colon M_{1/2,L_{N/q}} \to M_{1/2,L_N},$$

acting on the Fourier expansion

$$F(\tau) = \sum_{\gamma\in L'_{N/q}/L_{N/q}} \sum_{\substack{n\in\mathbb{Z}+q_{N/q}(\gamma)\\ n\geq 0}} a_F(\gamma,n)e(n\tau)\mathfrak{e}_\gamma.$$

by

$$V_q(F)(\tau) := \sum_{\gamma\in L'_N/L_N} \sum_{\substack{n\in\mathbb{Z}+q_N(\gamma)\\ n\geq 0}} \left(\sum_{a\,|\,(n-q_N(\gamma),\gamma,q)} a_F(\gamma/a, qn/a^2)\right) e(n\tau)\mathfrak{e}_\gamma,$$

for $\tau\in\mathbb{H}$.

Proposition 7.4.5. *Let $\beta\in L'/L$ and $m\in\mathbb{Z}+q(\beta)$ with $4Nm=n^2$ for some $n\in\mathbb{N}$. Further, let $q:=(n,N)$. Then*

$$U^N_{\beta,m} = \frac{q}{\sigma_1(q)}\; V_q\Big(U^{N/q}_{\beta/q,m/q}\Big).$$

Here $U^N_{\beta,m}(\tau)$ and $U^{N/q}_{\beta/q,m/q}(\tau)$ are the unique cusp forms of weight $1/2$ for ρ_N and $\rho_{N/q}$ satisfying the inner product formula given in (6.3.28), *respectively.*

Proof. Let d be a divisor of N/q. One can show that the operator V_q commutes with the d'th Atkin Lehner involution, i.e., that

$$\big(V_q(F)\big)^{w_d^N} = V_q\Big(f^{w_d^{N/q}}\Big) \tag{7.4.6}$$

for $F \in M_{1/2,L_{N/q}}$. Here $w_d^{N/q} \in O(L'_{N/q}/L_{N/q})$ and $w_d^N \in O(L'_N/L_N)$ denote the Atkin-Lehner involutions corresponding to the divisor d of N/q and N, respectively. Comparing Fourier expansions we further find that

$$V_q(\theta^{N/q}) = \sum_{c\,|\,q} (\theta^N)^{w_c^N}, \tag{7.4.7}$$

where $\theta^{N/q}$ and θ^N denote the unary theta functions defined in (7.2.1) of weight 1/2 for $\rho_{N/q}$ and ρ_N, respectively. Combining (7.4.6) and (7.4.7) we can deduce that

$$V_q(\theta_d^{N/q}) = \theta_d^N$$

for $d \in E(N/q)$, where $\theta_d^{N/q}$ and θ_d^N denote the theta functions defined in (7.4.1) of weight 1/2 for $\rho_{N/q}$ and ρ_N, respectively. Hence

$$\big(V_q(\theta_d^{N/q}), \theta_e^N\big) = \begin{cases} (\theta_d^N, \theta_d^N), & \text{if } d = e, \\ 0, & \text{if } d \neq e, \end{cases}$$

for $d \in E(N/q) \setminus \{1\}$ and $e \in E(N) \setminus \{1\}$ as the functions θ_d^N are orthogonal to each other by Lemma 7.4.1. On the other hand, we also have

$$\big(\theta_d^{N/q}, \theta_e^{N/q}\big) = \begin{cases} (\theta_d^{N/q}, \theta_d^{N/q}), & \text{if } d = e, \\ 0, & \text{if } d \neq e, \end{cases}$$

for $d, e \in E(N/q) \setminus \{1\}$. Comparing the formulas for the norms of the theta functions $\theta_d^{N/q}$ and θ_d^N given in Lemma 7.4.1 we thus obtain

$$V_q^*(\theta_d^N) = \begin{cases} \dfrac{\sigma_0(q)\sigma_1(q)}{\sqrt{q}}\, \theta_d^{N/q}, & \text{if } (d,q) = 1, \\ 0, & \text{if } (d,q) > 1, \end{cases}$$

for $d \in E(N) \setminus \{1\}$. Here V_q^* denotes the adjoint operator of V_q, which maps modular forms of weight 1/2 for ρ_N to modular forms of weight 1/2 for $\rho_{N/q}$.

Now let $d \in E(N) \setminus \{1\}$. Then

(7.4.8)

$$\big(V_q(U^{N/q}_{\beta/q,m/q}), \theta_d^N\big) = \big(U^{N/q}_{\beta/q,m/q}, V_q^*(\theta_d^N)\big) = \begin{cases} \dfrac{\sigma_0(q)\sigma_1(q)}{\sqrt{q}}\, \big(U^{N/q}_{\beta/q,m/q}, \theta_d^{N/q}\big), & \text{if } (d,q) = 1, \\ 0, & \text{if } (d,q) > 1. \end{cases}$$

Suppose that $(d,q) = 1$. The inner product appearing in this case has already been computed in (7.4.5), giving

$$\big(U^{N/q}_{\beta/q,m/q}, \theta_d^{N/q}\big) = \begin{cases} -\dfrac{8\pi\tilde{n}\, \mu((\tilde{f},d))\sigma_0((\tilde{n},N/q))}{\sqrt{N/q}}, & \text{if } (\tilde{n},d) = 1, \\ 0, & \text{if } (\tilde{n},d) > 1. \end{cases}$$

Here $\tilde{n} = n/q$ as $4(N/q)(m/q) = (n/q)^2$ and q is by definition a divisor of n. Moreover, given $f \mid N$ with $w_f(n) = \beta$ we can choose $\tilde{f} := f/q$ such that $w_{\tilde{f}}(\tilde{n}) = \beta/q$, and assuming that $(d, q) = 1$ we also have $(\tilde{f}, d) = (f, d)$. Hence, using again the formula (7.4.5) we obtain

$$(7.4.9) \qquad \big(U^{N/q}_{\beta/q,m/q}, \theta^{N/q}_d\big) = -\frac{8\pi n\, \mu((f,d))\sigma_0((n/q, N/q))}{\sqrt{qN}} = \frac{1}{\sigma_0(q)\sqrt{q}}\,\big(U^N_{\beta,m}, \theta^N_d\big).$$

for $(d, q) = 1$. Here we have also used that $\sigma_0((n/q, N/q))\sigma_0(q) = \sigma_0((n, N))$, and that $(n, d) = 1$ if and only if $(q, d) = 1$ and $(\tilde{n}, d) = 1$. Comparing (7.4.8) and (7.4.9), and recalling that the functions θ^N_d with $d \in E(N) \setminus \{1\}$ form a basis of the space $S_{1/2,L_N}$ we obtain the claimed statement. □

Now the previous proposition yields a different proof for the possible vanishing of the cusp form $U_{\beta,m}(\tau)$, which we already noted in the course of the proof of Proposition 7.4.3. However, the following alternative proof shows that the vanishing is in fact inevitable, as it is based on the basic facts that Fourier coefficients of modular forms of weight 1/2 for ρ_L are supported on squares, and that the space $S_{1/2,L}$ is trivial if $N = 1$ or if N is a prime.

Corollary 7.4.6. *Let $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m > 0$ such that one of the following two conditions is satisfied:*

(i) Either $4Nm$ is not a square, or

(ii) $4Nm = n^2$ with $n \in \mathbb{N}$ such that $N/(n, N)$ is 1 or a prime.

Then the cusp form $U_{\beta,m}(\tau)$ vanishes identically.

Proof. For (i) it suffices to recall that

$$(F, U_{\beta,m}) = -8\pi\sqrt{m}\, a_F(\beta, m)$$

for all $F \in S_{1/2,L}$ by the inner product formula (6.3.28), and that $a_F(\beta, m) = 0$ if $4Nm$ is not a square as the space $M_{1/2,L}$ is generated by the unary theta functions θ^{w_d} with $d \mid N$. Let now $4Nm = n^2$ with $n \in \mathbb{N}$ and set $q := (n, N)$. Then

$$U_{\beta,m} = \frac{q}{\sigma_1(q)}\, V_q(F)$$

for some $F \in S_{1/2,\rho_{N/q}}$ by Proposition 7.4.5, and if N/q is 1 or a prime then F needs to vanish identically, since the corresponding space of cusp forms $S_{1/2,\rho_{N/q}}$ is trivial in these cases (see for example Corollary 7.4.2), forcing $U_{\beta,m}$ to vanish, too. □

7.5 Elliptic Kronecker limit formulas

Finally, let $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m < 0$. We study the Borcherds product of the harmonic Maass form $U_{\beta,m}(\tau)$, which by Theorem 6.3.5 is the unique harmonic Maass form of weight 1/2 for ρ_L, which maps to a cusp form under $\xi_{1/2}$, which has principal part $e(m\tau)(\mathfrak{e}_\beta + \mathfrak{e}_{-\beta})$, and which is orthogonal to cusp forms. However, for general squarefree N we can only describe its Borcherds product in terms of its roots and poles on the modular curve $X_0(N)$. In particular, it may transform with a character of infinite order.

Proposition 7.5.1. *For $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m < 0$ the Borcherds product*

$$\Psi_{\beta,m}(z) := \Psi(z; U_{\beta,m})$$

is a weakly holomorphic modular form of weight 0*, level* N *and some unitary character of possibly infinite order. Further,* $\Psi_{\beta,m}(z)$ *satisfies the following properties:*

(i) The roots of $\Psi_{\beta,m}(z)$ *in* $\mathbb{H}$ *are located at the Heegner points* τ_Q *for* $Q \in \mathcal{Q}_{\beta,4Nm}$*, and the order of these roots is* 2 *if* $\beta = -\beta$ *in* L'/L *and* 1 *otherwise.*

(ii) The order of $\Psi_{\beta,m}(z)$ *at the cusp* $1/d$ *with* $d \,|\, N$ *is given by*

$$\operatorname{ord}_{1/d}(\Psi_{\beta,m}) = -\frac{H_N(\beta,m)}{\sigma_0(N)},$$

where $\sigma_0(N) = \sum_{c\,|\,N} 1$ *and*

$$H_N(\beta,m) := \sum_{Q \in \mathcal{Q}_{\beta,4Nm}/\Gamma_0(N)} \frac{2}{\operatorname{ord}(\tau_Q)}$$

is the Hurwitz class number.

(iii) The leading coefficient in the Fourier expansion of $\Psi_{\beta,m}(z)$ *at* ∞ *is* 1*.*

Proof. The proof is a direct application of Theorem 7.2.2 to the harmonic Maass form $U_{\beta,m}(\tau)$. In particular, the Borcherds product $\Psi_{\beta,m}(z)$ is well-defined since the holomorphic Fourier coefficients of $U_{\beta,m}(\tau)$ are real (see Corollary 6.3.3), and since the principal part of $U_{\beta,m}(\tau)$, namely $e(m\tau)(\mathfrak{e}_\beta + \mathfrak{e}_{-\beta})$ (compare Theorem 6.3.5), is integral.

Now part (i) of Theorem 7.2.2 yields, that $\Psi_{\beta,m}(z)$ is a weakly meromorphic modular form of weight $a^+_{P_{\beta,m}}(0,0) = 0$, level N and some unitary character. Further, the product expansion of $\Psi_{\beta,m}(z)$ directly implies part (iii) of the present proposition, and since the principal part of $U_{\beta,m}(\tau)$ is given by $e(m\tau)(\mathfrak{e}_\beta + \mathfrak{e}_{-\beta})$ we find

$$\frac{1}{2}\sum_{\gamma \in L'/L}\sum_{\substack{n \in \mathbb{Z}+q(\gamma)\\ n<0}} a^+_{U_{\beta,m}}(\gamma,n) \sum_{Q \in \mathcal{Q}_{\gamma,4Nn}} (\tau_Q) = \frac{1}{2}\sum_{Q \in \mathcal{Q}_{\beta,4Nm}} \Big((\tau_Q) + (\tau_{-Q})\Big) = \sum_{Q \in \mathcal{Q}_{\beta,4Nm}} (\tau_Q).$$

Thus, part (ii) of Theorem 7.2.2 gives property (i) of the present statement. In particular, we find that $\Psi_{\beta,m}(z)$ is holomorphic on $\mathbb{H}$, i.e., $\Psi_{\beta,m}(z)$ is indeed a weakly holomorphic modular form. So it remains to prove property (ii).

Let d be a positive divisor of N. By part (iii) of Theorem 7.2.2 and the formula (7.2.5) the order of $\Psi_{\beta,m}(z)$ at the cusp $1/d$ is given by

$$\operatorname{ord}_{1/d}(\Psi_{\beta,m}) = \frac{\sqrt{N}}{8\pi}\,(U_{\beta,m}, \theta^{w_d})^{\mathrm{reg}}.$$

As the constant coefficient of θ^{w_d} is 1, the difference $\theta^{w_d} - \theta_1/\sigma_0(N)$ is a cusp form, where $\theta_1 = \sum_{c\,|\,N} \theta^{w_c}$ as in (7.4.1). Hence we obtain

$$\operatorname{ord}_{1/d}(\Psi_{\beta,m}) = \frac{\sqrt{N}}{8\pi\sigma_0(N)}\,(U_{\beta,m}, \theta_1)^{\mathrm{reg}}, \tag{7.5.1}$$

since $U_{\beta,m}$ is orthogonal to cusp forms by Theorem 6.3.5.

In order to compute the latter inner product, we introduce Zagier's non-holomorphic Eisenstein series $E_{3/2}(\tau)$ of weight 3/2, which was first studied by Zagier for level $N = 1$ in [Zag75a], and later generalized to arbitrary level N by Bruinier and Funke in [BF06] (see Remark 4.6, (i) of their work). The Eisenstein series $E_{3/2}(\tau)$ is a harmonic Maass form of weight 3/2 for the dual Weil representaion ρ_L^* with holomorphic part given by

$$E_{3/2}^+(\tau) = \sum_{\gamma\in L'/L} \sum_{\substack{n\in\mathbb{Z}-q(\gamma)\\ n\geq 0}} H_N(\gamma,-n)e(n\tau)\mathfrak{e}_\gamma.$$

Here $H_N(\gamma, n)$ is the Hurwitz class number, which has been defined in the proposition for $\gamma \in L'/L$ and $n \in \mathbb{Z} + q(\gamma)$ with $n \leq 0$, and for $n = \gamma = 0$ we set $H_N(0,0) := -\sigma_1(N)/6$. Furthermore, the Eisenstein series $E_{3/2}(\tau)$ is orthogonal to cusp forms with respect to the inner product given in (3.4.4), and its image under the differential operator $\xi_{3/2}$ is given by

$$\xi_{3/2}E_{3/2}(\tau) = -\frac{\sqrt{N}}{4\pi}\theta_1(\tau).$$

Here the latter identity can be checked using the Fourier expansion of the Eisenstein series $E_{3/2}(\tau)$ given in Remark 4.6 (i) of [BF06]. Therefore, using (7.5.1) and Stokes' theorem (applied as in the proof of Proposition 3.5 in [BF04]) we obtain

$$\begin{aligned}\mathrm{ord}_{1/d}(\Psi_{\beta,m}) &= -\frac{1}{2\sigma_0(N)}\left(U_{\beta,m}, \xi_{3/2}E_{3/2}\right)^{\mathrm{reg}} \\ &= \frac{1}{2\sigma_0(N)}\left(E_{3/2}, \xi_{1/2}U_{\beta,m}\right)^{\mathrm{reg}} - \frac{1}{2\sigma_0(N)} \sum_{\gamma\in L'/L} \sum_{\substack{n\in\mathbb{Z}+q(\gamma)\\ n\leq 0}} a^+_{U_{\beta,m}}(\gamma,n)a^+_{E_{3/2}}(\gamma,-n).\end{aligned}$$

The remaining regularized inner product vanishes, since the Eisenstein series $E_{3/2}(\tau)$ is orthogonal to cusp forms, and $U_{\beta,m}$ is mapped to a cusp form by $\xi_{1/2}$ (see Theorem 6.3.5). Hence, as the principal part of $U_{\beta,m}$ is given by $e(m\tau)(\mathfrak{e}_\beta + \mathfrak{e}_{-\beta})$, we are left with

$$\mathrm{ord}_{1/d}(\Psi_{\beta,m}) = -\frac{1}{2\sigma_0(N)}\Big(a^+_{E_{3/2}}(\beta,-m) + a^+_{E_{3/2}}(-\beta,-m)\Big) = -\frac{1}{\sigma_0(N)}H_N(\beta,m),$$

where we have also used that $H_N(\beta,m) = H_N(-\beta,m)$. This finishes the proof. $\square$

Remark 7.5.2. Let $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m < 0$. Since the harmonic Maass form $U_{\beta,m}(\tau)$ is orthogonal to cusp forms (compare Theorem 6.3.5), part (a) of Proposition 7.2.3 yields that $\Psi_{\beta,m}(z)$ transforms with a character of finite order if and only if the holomorphic Fourier coefficients $a^+_{U_{\beta,m}}(n, n^2/4N)$ are rational for all $n \in \mathbb{N}$. By Corollary 6.3.3 these are given by

$$a^+_{U_{\beta,m}}(n,n^2/4N) = \frac{2\sqrt{2N}}{n}\sum_{j=0}^{\infty}\frac{(-4\pi^2mn^2/N)^j}{(2j)!}Z(j;\beta,m,n,n^2/4N)$$

for $n \in \mathbb{N}$. There does not seem to be a reason, why these coefficients should be rational. Hence, we expect that in general $\Psi_{\beta,m}(z)$ transforms with a character of infinite order.

Next, we state the general averaged Kronecker limit formula for elliptic Eisenstein series of level N. Afterwards, we treat a certain special case, where the Borcherds product $\Psi_{\beta,m}(z)$ can be given explicitly.

Theorem 7.5.3. *For $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m < 0$ the averaged elliptic Eisenstein series $E^{\mathrm{ell}}_{\beta,m}(z,s)$ has the Laurent expansion*

$$E^{\mathrm{ell}}_{\beta,m}(z,s) = -\log\left|\Psi_{\beta,m}(z)\right| \cdot s + O(s^2)$$

at $s = 0$, for $z \in \mathbb{H} \setminus H_{\beta,4Nm}$, where $\Psi_{\beta,m}(z)$ is the weakly holomorphic modular form of weight 0, level N and some unitary character given in Proposition 7.5.1.

Proof. This is a direct application of Proposition 7.5.1 to part (c) of Corollary 7.2.4. Here the term $\log|\Psi_{\beta,m}(z)|$ has singularities at the zeros of the Borcherds product $\Psi_{\beta,m}(z)$, namely at the Heegner points τ_Q with $Q \in \mathcal{Q}_{\beta,4Nm}$. $\square$

Eventually, we recall that there are finitely many squarefree N such that the corresponding modular curve $X_0(N)$ has genus 0, namely the integers

$$N = 1, 2, 3, 5, 6, 7, 10, 13.$$

Let N be one of these integers. Using the Riemann-Roch theorem for the modular curve $X_0(N)$ one finds that given a cusp p and a point $w \in \mathbb{H}$ there exists a corresponding *Hauptmodul* $j_{N,p,w}(z)$, which is the unique generator of the function field $M_0^!(\Gamma_0(N))$ satisfying the following properties:

(i) The poles and zeros of $j_{N,p,w}$ on $X_0(N)$ are completely determined by the divisor $(w) - (p)$, i.e., $j_{N,p,w}$ is holomorphic up to a simple pole at the cusp p and vanishes exactly at the distinguished point w, modulo $\Gamma_0(N)$.

(ii) The Hauptmodul $j_{N,p,w}$ is normalized in the sense that the leading coefficient in the Fourier expansion of $j_{N,p,w}$ at ∞ is 1.

In particular, the Hauptmodul $j_{N,p,w}$ is holomorphic and non-vanishing at the cusps different from p, and the induced map $j_{N,p,w} \colon X_0(N) \to \mathbb{C} \cup \{\infty\}$ is a bijection of compact Riemann surfaces of genus 0.

Corollary 7.5.4. *Let $\beta \in L'/L$ and $m \in \mathbb{Z} + q(\beta)$ with $m < 0$. Further, let N be a squarefree positive integer such that the group $\Gamma_0(N)$ has genus 0. Then the averaged elliptic Eisenstein series $E^{\mathrm{ell}}_{\beta,m}(z,s)$ has the Laurent expansion*

$$E^{\mathrm{ell}}_{\beta,m}(z,s) = -\frac{1}{\sigma_0(N)}\left(\sum_{Q \in \mathcal{Q}_{\beta,4Nm}/\Gamma_0(N)} \sum_{d\,|\,N} \log\left(\left|j_{N,1/d,\tau_Q}(z)\right|^{2/\operatorname{ord}(\tau_Q)}\right)\right) \cdot s + O(s^2)$$

at $s = 0$, for $z \in \mathbb{H} \setminus H_{\beta,4Nm}$.

Proof. Let $Q \in \mathcal{Q}_{\beta,4Nm}$ and $d \mid N$. By the above considerations the function $j_{N,1/d,\tau_Q}$ is a weakly holomorphic modular form of weight 0 and level N, which is holomorphic up to a simple pole at the cusp $1/d$, and which vanishes exactly at the point τ_Q, modulo $\Gamma_0(N)$. Hence, by the well-known valence formula of weight 0 (which is essentially a reformulation of the Riemann-Roch theorem for the modular curve $X_0(N)$) we have

$$0 = \sum_{z \in \Gamma_0(N)\backslash\mathbb{H}} \frac{2\operatorname{ord}_z(j_{N,1/d,\tau_Q})}{\operatorname{ord}(z)} + \sum_{c \mid N} \operatorname{ord}_{1/c}(j_{N,1/d,\tau_Q}) = \frac{2\operatorname{ord}_{\tau_Q}(j_{N,1/d,\tau_Q})}{\operatorname{ord}(\tau_Q)} - 1,$$

giving $\operatorname{ord}_{\tau_Q}(j_{N,1/d,\tau_Q}) = \operatorname{ord}(\tau_Q)/2$. Next, we set

$$f(z) := \left(\prod_{Q \in \mathcal{Q}_{\beta,4Nm}/\Gamma_0(N)} \prod_{d \mid N} j_{N,1/d,\tau_Q}(z)^{2/\operatorname{ord}(\tau_Q)} \right)^{1/\sigma_0(N)}$$

for $z \in \mathbb{H}$. By construction, f is a weakly holomorphic modular form of weight 0 and level N, which vanishes exactly at the Heegner points τ_Q with $Q \in \mathcal{Q}_{\beta,4Nm}$ with corresponding orders of vanishing given by

$$\operatorname{ord}_{\tau_Q}(f) = \frac{1}{\sigma_0(N)} \sum_{\substack{Q' \in \mathcal{Q}_{\beta,4Nm}/\Gamma_0(N) \\ \tau_Q = \tau_{Q'}}} \frac{2}{\operatorname{ord}(\tau_Q)} \sum_{d \mid N} \operatorname{ord}_{\tau_Q}(j_{N,1/d,\tau_Q}) = \begin{cases} 2, & \text{if } \beta = -\beta, \\ 1, & \text{if } \beta \neq -\beta. \end{cases}$$

Here the sum over quadratic forms $Q' \in \mathcal{Q}_{\beta,4Nm}/\Gamma_0(N)$ with $\tau_Q = \tau_{Q'}$ either only runs over Q itself (if $\beta \neq -\beta$), or over the two elements Q and $-Q$ (if $\beta = -\beta$). Moreover, the order of f at a cusp $1/c$ with $c \mid N$ is clearly given by

$$\operatorname{ord}_{1/c}(f) = \frac{1}{\sigma_0(N)} \sum_{Q \in \mathcal{Q}_{\beta,4Nm}/\Gamma_0(N)} \frac{2}{\operatorname{ord}(\tau_Q)} \sum_{d \mid N} \operatorname{ord}_{1/c}(j_{N,1/d,\tau_Q}) = -\frac{H_N(\beta,m)}{\sigma_0(N)},$$

as $\operatorname{ord}_{1/c}(j_{N,1/d,\tau_Q}) = -\delta_{c,d}$. Since the leading coefficient in the Fourier expansion of f at ∞ is clearly 1, as the Hauptmoduls $j_{N,1/d,\tau_Q}(z)$ are normalized, the function f is a weakly holomorphic modular form of weight 0 and level N, which satisfies the properties (ii) and (iii) of Proposition 7.5.1. Therefore, part (b) of Proposition 7.2.3 implies that $f = \Psi_{\beta,m}$, and the claimed Laurent expansion follows from Theorem 7.5.3. $\square$

Remark 7.5.5. Let $N = 1$ and let j be the usual modular j-function (see Section 2.6.2). Then $j_{1,\infty,w}(z) := j(z) - j(w)$ is a Hauptmodule in the above sense, and given $\beta \in \mathbb{Z}/2\mathbb{Z}$ and $m \in \mathbb{Z} + \beta^2/4$ the Kronecker limit formula given in Corollary 7.5.4 can be written as

$$E^{\mathrm{ell}}_{\beta,m}(z,s) = -\left(\sum_{Q \in \mathcal{Q}_{\beta,4m}/\operatorname{SL}_2(\mathbb{Z})} \log\left(\left| j(z) - j(\tau_Q) \right|^{2/\operatorname{ord}(\tau_Q)} \right) \right) \cdot s + O(s^2)$$

as $s \to 0$, for $z \in \mathbb{H} \setminus H_{\beta,4m}$. This is an averaged version of the elliptic Kronecker limit formula given in equation (2.6.7).

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