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Reflective Automorphic Forms and Siegel Theta Series for Niemeier Lattices

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von
Moritz Christopher Dittmann, M.Sc.
aus Frankfurt am Main

Referent:	Prof. Dr. Nils Scheithauer
1. Korreferent:	Prof. Dr. Martin Möller
2. Korreferent:	Prof. Riccardo Salvati Manni, PhD
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Zusammenfassung

Im ersten Teil dieser Dissertation untersuchen wir holomorphe automorphe Formen mit speziellen Divisoren zu Untergruppen von orthogonalen Gruppen, sogenannte reflektive automorphe Formen. Diesen liegt ein gerades Gitter von Signatur $(n, 2)$ zu Grunde. Hat dieses Gitter L quadratfreie Stufe N und spaltet es das Gitter $II_{1,1} \oplus II_{1,1}(N)$ ab, so zeigen wir, dass es bis auf Isomorphie nur endlich viele Möglichkeiten für L gibt.

Anschließend klassifizieren wir auf diesen Gittern die stark-reflektiven automorphen Formen von singulärem Gewicht (d.h. die reflektiven automorphen Formen von singulärem Gewicht mit einfachen Nullstellen) für den Diskriminantenkern. Mit einer bekannten Konstruktion erhält man soche automorphe Formen aus speziellen Automorphismen des Leech-Gitters und unser Klassifikationsresultat zeigt, dass es im Wesentlichen keine weiteren gibt.

Verlangt man nicht mehr, dass L das Gitter $II_{1,1} \oplus II_{1,1}(N)$ abspaltet, so gibt es weitere Beispiele für stark-reflektive automorphe Formen. Bekannt sind solche im Fall, dass die Stufe N prim ist. In dieser Arbeit konstruieren wir nun auch Beispiele für den Fall $N = 6$. Anschließend berechnen wir für eines dieser neuen Beispiele die Fourier-Entwicklung in allen Spitzen. Wir sehen, dass, anders als in den Beispielen bei denen N prim ist, keine dieser Entwicklungen zu einem Element der Automorphismengruppe des Leech-Gitters korrespondiert.

Im zweiten Teil dieser Arbeit bestimmen wir mit Hilfe von Thetareihen zu Niemeier-Gittern eine Basis des Raums der Siegelschen Spitzenformen von Grad 6 und Gewicht 14. Dazu berechnen wir mit Unterstützung eines Computers Fourierkoeffizienten dieser Theta-Reihen.

Als Folgerung erhalten wir, dass die Kodaira-Dimension des Modulraumes $\mathcal{A}_6$ der prinzipal-polarisierten abelschen Varietäten von Dimension 6 nicht negativ ist.

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Introduction

This thesis consists of two independent parts. The first part is concerned with the classification of reflective automorphic forms, while in the second part a certain space of Siegel cusp forms is constructed in order to show that the Kodaira dimension of the Siegel modular variety $\mathcal{A}_6$ is non-negative.

Reflective Automorphic Forms

Let $n \geq 3$ and let L be an even lattice of signature $(n, 2)$, i.e. a free $\mathbb{Z}$-module of finite rank with a non-degenerate $\mathbb{Z}$-valued quadratic form q such that any Gram matrix of the associated symmetric bilinear form $(\cdot, \cdot)$ has n positive and two negative eigenvalues over the real numbers $\mathbb{R}$. The set of oriented 2-dimensional negative definite subspaces of $V = L \otimes_{\mathbb{Z}} \mathbb{R}$ can be identified with the set

$$\mathcal{K} = \{[Z] \in \mathbb{P}(V \otimes_{\mathbb{R}} \mathbb{C}) : (Z, Z) = 0, (Z, \overline{Z}) < 0\},$$

where

$$\mathbb{P}(V \otimes_{\mathbb{R}} \mathbb{C}) = ((V \otimes_{\mathbb{R}} \mathbb{C}) \setminus \{0\}) / \mathbb{C}^*$$

is the projective space associated to $V \otimes_{\mathbb{R}} \mathbb{C}$ and the bilinear form on L is extended to a bilinear form on $V \otimes_{\mathbb{R}} \mathbb{C}$. The set $\mathcal{K}$ has two connected components, corresponding to the two possible choices of the orientation on the 2-dimensional negative definite subspaces of V. In the following $\mathcal{K}^+$ will be one of the two components and the subgroup $O(L)^+$ of the orthogonal group $O(L)$ of L consists of those isometries in $O(L)$ that stabilize $\mathcal{K}^+$ as a set. These are the elements that keep the orientation on the 2-dimensional negative definite subspaces of V.

An automorphic form of weight $k \in \mathbb{Z}$ is a meromorphic function $\Psi \colon \tilde{\mathcal{K}}^+ \to \mathbb{C}$ on the affine cone $\tilde{\mathcal{K}}^+$ over $\mathcal{K}^+$ that is homogeneous of degree $-k$ and such that

$$\Psi(\sigma(Z)) = \chi(\sigma)\Psi(Z)$$

for all σ in a finite index subgroup Γ of $O(L)^+$ and some character χ of Γ.

If the Witt rank of $L \otimes_{\mathbb{Z}} \mathbb{Q}$ is 2, then non-constant holomorphic automorphic forms do not exist if the weight is smaller than $s = (n-2)/2$, which is why this is called the singular weight.

Automorphic Products

The dual lattice L' of L is given by

$$L' = \{x \in L \otimes_{\mathbb{Z}} \mathbb{Q} : (x, y) \in \mathbb{Z} \text{ for all } y \in L\}.$$

The lattice L is a sublattice of L' and the quotient $D = L'/L$ is a finite abelian group. If n is even, then the Weil representation ρ_D is a unitary representation of $\mathrm{SL}_2(\mathbb{Z})$ on the group algebra $\mathbb{C}[D]$. A weakly holomorphic modular form of weight k for ρ_D is a holomorphic function $F\colon \mathbb{H} \to \mathbb{C}[D]$ with

$$F\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k \rho_D\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)^{-1} F(\tau)$$

for all $\tau \in \mathbb{H}$ and all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and that in addition satisfies the following growth condition at ∞: The function F has a Fourier expansion

$$F(\tau) = \sum_{\gamma \in D} \sum_{n \in \mathbb{Q}} a_\gamma(n) q^n \mathfrak{e}_\gamma,$$

where $q = e^{2\pi i \tau}$ and $\mathfrak{e}_\gamma$ is the standard basis vector of $\mathbb{C}[D]$ corresponding to $\gamma \in D$. The growth condition then states that for every $\gamma \in D$ the coefficient $a_\gamma(n)$ is zero for all but finitely many negative n.

Elements $\sigma \in O(L)$ act on the set of weakly holomorphic modular forms of weight k for ρ_D as follows: Every $\sigma \in O(L)$ is also in $O(L')$ and therefore defines an element of $\mathrm{Aut}(D)$, which acts by permuting the components (with respect to the standard basis) of F.

If $k = 1 - n/2$ and $a_\gamma(n)$ is an integer for all $\gamma \in D$ and all negative n, then Borcherds [Bor98] showed how one can associate to F an automorphic form $\Psi(F)$ of weight $a_0(0)/2$ for the group $O(L, F)^+$, which is defined by

$$O(L, F)^+ = \{\sigma \in O(L)^+ : \sigma(F) = F\}.$$

The function $\Psi(F)$ has infinite product expansions near norm 0 vectors of L; hence automorphic forms of this form are called automorphic products or Borcherds products. The zeros and poles of $\Psi(F)$ lie on sets of the form

$$\sum_{\substack{\lambda \in \mu + L \\ q(\lambda) = n}} \lambda^\perp \tag{0.1}$$

for elements $\mu \in D$ and $n \in \mathbb{Q}_{>0}$ and the multiplicities are determined by the coefficients of F at negative rational numbers.

Conversely, Bruinier [Bru14] showed that if the lattice L splits $I\!I_{1,1} \oplus I\!I_{1,1}(m)$, i.e. can be written as

$$L = K \oplus I\!I_{1,1} \oplus I\!I_{1,1}(m)$$

for a lattice K and a non-zero integer m, then every automorphic form for the kernel Γ_L of the map $O(L)^+ \to \mathrm{Aut}(D)$ whose divisor is a linear combination of sets as in (0.1) is up to a non-zero constant factor equal to an automorphic product.

Reflective Automorphic Forms

A holomorphic automorphic form Ψ for a finite-index subgroup $\Gamma \subset O(L)^+$ is called reflective if its zeros are of the form $\lambda^\perp$ for elements $\lambda \in L$ of positive norm such that the reflection in $\lambda^\perp$ is in $O(L)$. If in addition all zeros are simple, then Ψ is called strongly-reflective.

A prominent example of a strongly-reflective automorphic form is the function Φ_{12}. It is the automorphic product corresponding to the inverse of the modular discriminant

$$\Delta\colon \mathbb{H} \to \mathbb{C}, \Delta(\tau) = q \prod_{n\geq 1}(1-q^n)^{24} = q - 24q^2 + 252q^3 + \cdots$$

on the unique even unimodular lattice $II_{26,2}$ of signature $(26,2)$. The infinite product expansion of Φ_{12} at a cusp is given by

$$\Phi_{12}(Z) = e(-(\rho, Z)) \prod_{\alpha \in II_{25,1}^+} (1 - e(-(\alpha, Z)))^{[1/\Delta](-(\alpha,\alpha)/2)} \quad (e(z) = e^{2\pi i z}),$$

where ρ is the so-called Weyl vector, which is a certain primitive norm 0 vector in $II_{25,1}$ (the unique even unimodular lattice of signature $(25,1)$), $II_{25,1}^+$ is the subset of $II_{25,1}$ consisting of those elements that are either multiples of ρ or have negative inner product with ρ and $[1/\Delta](m)$ denotes the Fourier coefficient of $1/\Delta$ at q^m. Scheithauer [Sch17] showed that the automorphic product Φ_{12} is the only automorphic product of singular weight on an even unimodular lattice. The fact that it has singular weight allows the computation of its Fourier expansion. It is given by

$$\begin{aligned}\Phi_{12}(Z) &= e(-(\rho, Z)) \prod_{\alpha \in II_{25,1}^+} (1 - e(-(\alpha, Z)))^{[1/\Delta](-(\alpha,\alpha)/2)} \\ &= \sum_{g \in G} \det(g) \Delta(-(g\rho, Z)),\end{aligned}$$

where G is the reflection group of $II_{25,1}$, i.e. the subgroup of $O(II_{25,1})$ that is generated by reflections in hyperplanes orthogonal to norm 2 vectors of $II_{25,1}$.

This is the denominator identity of the fake monster algebra, which is an infinite-dimensional Lie algebra constructed by Borcherds [Bor90]. There are more cases of this connection with infinite-dimensional Lie algebras, which is why reflective automorphic forms are important for their classification (see e.g. [GN02]).

There are other, more geometric, applications of reflective automorphic forms. For example, Gritsenko and Hulek [GH14] showed that the existence of a strongly-reflective automorphic form of weight $k > n$ implies that the quasi-projective variety $\Gamma\backslash\mathcal{K}^+$ is uniruled. Another geometric application is due to Borcherds [Bor96], who used a reflective automorphic product for the lattice

$$L = E_8(2) \oplus II_{1,1} \oplus II_{1,1}(2)$$

to show that the moduli space of Enriques surfaces is a quasi-affine variety. This reflective automorphic product is also related to the theory of infinite-dimensional Lie algebras, as it is a twisted denominator function of the fake monster algebra.

Finiteness Results for Reflective Automorphic Forms

Reflective automorphic forms are rare. For example, Ma ([Ma17],[Ma18]) has shown that there are only finitely many lattices (here and in the following this always means up to isomorphisms) that carry a 2-reflective automorphic form (i.e. a reflective automorphic form whose zeros are of the form $\lambda^\perp$ for elements $\lambda \in L$ of norm 2) if $n \geq 7$ and that up to scaling there are only finitely many lattices that carry a strongly-reflective automorphic form if $n \geq 4$.

In this thesis we prove the following new result in Chapter 3, which is also published in [Dit18]:

Theorem 1 (Theorem 3.25)**.** *Let L be an even lattice of signature $(n, 2)$, $n \geq 4$ and squarefree level N that splits $II_{1,1} \oplus II_{1,1}(N)$. Suppose there is a non-constant reflective automorphic form $\Psi \colon \tilde{\mathcal{K}}^+ \to \mathbb{C}$. Then L must be one of finitely many lattices.*

This is proved by using Bruinier's converse theorem to reduce to the case of automorphic products and then showing that n and N are bounded if there is a weakly-holomorphic modular form F of weight $1 - n/2$ for ρ_D such that $\Psi(F)$ is reflective. The bounds on n and N are explicit, so our result is effective.

Classification of Strongly-Reflective Automorphic Forms of Singular Weight

The Leech lattice Λ is the unique even positive definite unimodular lattice of rank 24 without vectors of norm 2. Its automorphism group is the Conway group Co_0 and has order

$$2^{22} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23 = 8315553613086720000 \approx 8.3 \times 10^{18}.$$

Scheithauer [Sch04] used elements $g \in O(\Lambda)$ of squarefree level N with non-trivial fixpoint lattice $\Lambda^g \subset \Lambda$ to constructed strongly-reflective automorphic products of singular weight for the lattice $L = \Lambda^g \oplus II_{1,1} \oplus II_{1,1}(N)$. In Chapter 4 we prove that under the condition that L has squarefree level N and splits $II_{1,1} \oplus II_{1,1}(N)$ these are essentially the only strongly-reflective automorphic forms of singular weight for Γ_L. This is the second main result in [Dit18].

Theorem 2 (Corollary 4.7)**.** *Let L be an even lattice of signature $(n, 2), n \geq 4$ and squarefree level N that splits $II_{1,1} \oplus II_{1,1}(N)$ and Ψ a strongly-reflective automorphic form of singular weight for the discriminant kernel Γ_L. Then the function Ψ can be identified with one of the functions from this construction.*

The idea of the proof is the following: Because of Theorem 1 we only need to consider finitely many lattices. Since Theorem 1 is effective, we have bounds on the number n and the level N of L. In the remaining cases we can assume that Ψ is a strongly-reflective automorphic product $\Psi(F)$ for some weakly holomorphic modular form F of weight $1 - n/2$ because of Bruinier's converse theorem. The fact that $\Psi(F)$ is strongly-reflective of singular weight then translates to conditions on the Fourier coefficients of F. We work out these conditions explicitly to prove our result.

If L does not split $II_{1,1}$, then Scheithauer [Sch17] gave additional examples of strongly-reflective automorphic products of singular weight if the level N of L is a prime number.

In Chapter 4 we construct new examples of strongly-reflective automorphic products of singular weight on lattices of level 6. In one case we compute the Fourier expansions of the automorphic product at all primitive norm 0 vectors of L to see that it is not related to the Leech lattice:

Theorem 3 (Theorem 4.34)**.** *None of the shapes of the eta quotients occuring in the Fourier expansions of this strongly-reflective automorphic product is the cycle shape of an element of $O(\Lambda)$.*

This is a phenomenon that does not occur in the cases where the level N is prime.

Siegel Modular Forms

The second part of this thesis can be found in Chapter 5 and is based on joint work with Riccardo Salvati Manni and Nils Scheithauer.

Let $n \in \mathbb{Z}_{>0}$. The Siegel upper half-plane $\mathbb{H}_n$ is defined by

$$\mathbb{H}_n = \{Z \in \mathbb{C}^{n\times n} : Z^T = Z,\ \mathrm{Im}(Z) > 0\},$$

where Z^T denotes the transpose of Z and $\mathrm{Im}(Z) > 0$ means that $\mathrm{Im}(Z)$ is a positive definite matrix. The symplectic group $\mathrm{Sp}_{2n}(\mathbb{Z})$ consists of those matrices M in $\mathrm{GL}_{2n}(\mathbb{Z})$ that satisfy

$$M^T \Omega M = \Omega,$$

where Ω is the skew-symmetric matrix

$$\Omega = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$$

and I_n denotes the $n \times n$ identity matrix. For $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_{2n}(\mathbb{Z})$ and $Z \in \mathbb{H}_n$ we define MZ to be $(AZ+B)(CZ+D)^{-1}$. This defines a group action of $\mathrm{Sp}_{2n}(\mathbb{Z})$ on $\mathbb{H}_n$.

A Siegel modular form of degree n and weight $k \in \mathbb{Z}$ is a holomorphic function f from $\mathbb{H}_n$ to the complex numbers with

$$f(MZ) = \det(CZ+D)^k f(Z)$$

for all $Z \in \mathbb{H}_n$ and all $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_{2n}(\mathbb{Z})$ that additionally satisfies the following growth condition at ∞: Because of the transformation behaviour, the function f has a Fourier expansion of the form

$$f(Z) = \sum_{\substack{T \in \mathbb{Z}^{n\times n} \\ T \text{ even}}} a(T) e^{\pi i \,\mathrm{tr}(TZ)} \quad (\mathrm{tr} = \text{trace}),$$

where a matrix $T \in \mathbb{Z}^{n\times n}$ is called even if it is symmetric and its diagonal entries are even. The growth condition at ∞ then states that $a(T) = 0$ unless T is positive semi-definite. If $n > 1$, then this condition is automatically satisfied by the Koecher principle [Koe54]. A cusp form is a Siegel modular form whose Fourier coefficients $a(T)$ are zero unless T is positive definite.

Theta Series

Given an even positive definite unimodular lattice L of rank m, one can define the theta series

$$\Theta_L(Z) = \sum_{x \in L^n} e^{\pi i tr(T(x)Z)},$$

where $T(x)$ is the matrix given by $((x_i, x_j))_{1\leq i,j\leq n}$ for $x = (x_1, \dots, x_n)$. This is a Siegel modular form of degree n and weight $m/2$. For a positive semi-definite matrix T its Fourier coefficient $a(T)$ is given by

$$a(T) = |\{x \in L^n : T(x) = T\}|.$$

If $m > 2n$ and $y = (y_1, \dots, y_n) \in (L \otimes_{\mathbb{Z}} \mathbb{C})^n$ satisfies $T(y) = 0$ and $T(y, \overline{y}) > 0$ (where $T(y, \overline{y})$ is defined to be the matrix $((y_i, \overline{y_j}))_{1\leq i,j\leq n}$), one can define the more general theta series

$$\Theta_{L,y,k} = \sum_{x \in L^n} \det(T(x, y))^k e^{\pi i \operatorname{tr}(T(x)Z)}$$

for $k \in \mathbb{Z}_{>0}$. This is a cusp form of degree n and weight $m/2 + k$ and for a positive definite matrix T its Fourier coefficient $a(T)$ is given by

$$a(T) = \sum_{x \in L^n} \det(T(x, y))^k. \tag{0.2}$$

We construct a basis of the space $S_{6,14}$ of Siegel cusp forms of degree 6 and weight 14 consisting of such theta series for even positive definite unimodular lattices L of rank 24 and $k = 2$. This is done by using that $\dim S_{6,14} = 9$, which is a result of Taïbi [Taï17], and finding nine linearly independent theta series. To proof that they are linearly independent, we use a computer to calculate enough Fourier coefficients using formula (0.2).

The Kodaira Dimension

Let X be a smooth projective algebraic variety of dimension n. The canonical bundle ω_X of X is the n-th exterior power of the cotangent bundle Ω_X, i.e.

$$\omega_X = \bigwedge^n \Omega_X.$$

The (geometric) genus of X is the dimension of the space $H^0(X, \omega_X)$ of global sections of ω_X. More generally, one defines the d-th plurigenus of X to be $P_d(X) = \dim H^0(X, \omega_X^d)$ for $d \in \mathbb{Z}_{\geq 0}$. If $P_d(X) > 0$, then there is a rational map, called the d-canonical map, from X to $\mathbb{P}^m$ defined by

$$x \mapsto [\sigma_0(x) : \cdots : \sigma_m(x)],$$

where $\sigma_0, \dots, \sigma_m$ is a basis of $H^0(X, \omega_X^d)$. If ω_X^d is very ample, then this defines an embedding of X into the projective space $\mathbb{P}^m$.

Putting all the spaces $H^0(X, \omega_X^d)$ for $d \geq 0$ together defines the canonical ring

$$R(X, \omega_X) = \bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(X, \omega_X^d).$$

The variety $\mathrm{Proj}(R(X, \omega_X))$ is called the canonical model of X. By a deep result of Birkar, Cascini, Hacon and McKernan [BCHM10], the canonical ring is finitely generated and hence, the canonical model is a projective variety. It is isomorphic to the image of the d-canonical map for large d.

The dimension of $\mathrm{Proj}(R(X, \omega_X))$ is called the Kodaira dimension $\kappa(X)$ of X. If $P_d(X) = 0$ for all $d > 0$, then $\mathrm{Proj}(R(X, \omega_X)) = \emptyset$ and the Kodaira dimension depends on the definition of the dimension of the empty set; both $-\infty$ and -1 are used in the literature. The Kodaira dimension is bounded by the dimension of X and is equal to the smallest number κ such that $P_d(X)/d^\kappa$ is bounded as long as $P_d(X) \neq 0$ for some $d > 0$. Both the canonical ring and the Kodaira dimension only depend on the birational class of X.

The Kodaira dimension is useful to divide algebraic varieties into different classes. For example, if X is a smooth projective curve over $\mathbb{C}$, then $\kappa(X)$ is negative if the genus $P_1(X)$ is 0, $\kappa(X) = 0$ if $P_1(X) = 1$ and $\kappa(X) = 1$ if $P_1(X) \geq 2$. Comparing this with the uniformization theorem for real surfaces, one sees that negative Kodaira dimension corresponds to positive curvature, $\kappa(X) = 0$ corresponds to flatness and maximal Kodaira dimension corresponds to negative curvature. Most curves have Kodaira dimension $\kappa(X) = 1$ (the moduli space of curves of genus g has dimension 0 if $g = 0$, 1 if $g = 1$ and $3g - 3$ if $g \geq 2$) and it is expected that also in higher dimensions most smooth projective varieties have maximal Kodaira dimension; therefore, these are said to be of general type.

The Siegel Modular Variety $\mathcal{A}_n$

An abelian variety over the complex numbers $\mathbb{C}$ is an n-dimensional torus $\mathbb{C}^n/L$ that can be embedded into some projective space $\mathbb{P}^N$. For $n = 1$ every torus $\mathbb{C}/L$ can be embedded into a projective space and $\mathbb{C}/L$ is called an elliptic curve. If $n > 1$ there is such an embedding if and only if L admits a polarization, which means that there is a positive definite hermitian form H on $\mathbb{C}^n$ such that its imaginary part is integer-valued on L. If L has a $\mathbb{Z}$-basis such that the Gram matrix of $\mathrm{Im}(H)$ with respect to this basis is given by

$$\begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix},$$

then H is called a principal polarization and a principally polarized abelian variety of dimension n is a pair (A, H), where A is a torus $\mathbb{C}^n/L$ and H is a principal polarization of L.

The isomorphism classes of principally polarized abelian varieties are parametrized by the quotient $\mathcal{A}_n = \mathrm{Sp}_{2n}(\mathbb{Z})\backslash\mathbb{H}_n$, which has the structure of a normal quasi-projective variety of dimension $n(n+1)/2$ and is called Siegel modular variety. The Kodaira dimension of $\mathcal{A}_n$ is defined to be the Kodaira dimension of any smooth projective variety birational to $\mathcal{A}_n$. Such a variety exists as one can take the closure of $\mathcal{A}_n$ and then resolve the singularities.

The Kodaira dimension of $\mathcal{A}_n$ has been known for all $n \neq 6$ for over 30 years: Tai [Tai82] proved that $\mathcal{A}_n$ is of general type for $n \geq 9$ and his result was soon improved to $n \geq 8$ by Freitag [Fre83] and to $n \geq 7$ by Mumford [Mum83]. That $\mathcal{A}_5$ is unirational (and hence has negative Kodaira dimension) was shown by Donagi [Don84] and independently by Mori and Mukai [MM83] and Verra [Ver84]. The unirationality of $\mathcal{A}_4$ is due to [Cle83] and that $\mathcal{A}_n$ has negative Kodaira dimension is classical for $n \leq 3$ (see e.g. the discussion in [HS02], Section II.2).

In this thesis we prove the first partial result on the Kodaira dimension of the missing case $\mathcal{A}_6$:

Theorem 4 (Theorem 5.35)**.** *The Kodaira dimension of $\mathcal{A}_6$ is non-negative.*

We explain the idea of the proof: Let S be the set of points whose stabilizer in $\mathrm{Sp}_{2n}(\mathbb{Z})$ is not equal to $\{\pm I_{2n}\}$ and let $\mathbb{H}_n^\circ$ be the complement of S in $\mathbb{H}_n$. Then $\mathbb{H}_n^\circ$ is an open subset of $\mathbb{H}_n$ and the restrictions of Siegel modular forms of degree n and weight $n+1$ to $\mathbb{H}_n^\circ$ are in bijection with the global sections of the canonical bundle $\omega_{\mathcal{A}^\circ}$ of $\mathcal{A}^\circ = \mathrm{Sp}_{2n}(\mathbb{Z})\backslash\mathbb{H}_n^\circ$. More generally, the space $M_{n,(n+1)k}$ of Siegel modular forms of degree n and weight $(n+1)k$ is in bijection with the space of global sections of $\omega_{\mathcal{A}_n^\circ}^k$. By a result of Tai [Tai82], an element f in $M_{n,(n+1)k}$ can be extended to an element of $H^0(\overline{\mathcal{A}_n}, \omega_{\overline{\mathcal{A}_n}}^k)$ for a smooth projective model $\overline{\mathcal{A}_n}$ of $\mathcal{A}_n$ if the vanishing order of f at the cusp ∞ is at least k and $n \geq 5$. The basis of $S_{6,14}$ that we construct contains an element that has vanishing order 2 at ∞ and therefore defines an element of $H^0(\overline{\mathcal{A}_6}, \omega_{\overline{\mathcal{A}_6}}^2)$, which implies that $P_2(\overline{\mathcal{A}_6}) > 0$; therefore, the Kodaira dimension of $\mathcal{A}_6$ must be non-negative.

1. Lattices and Discriminant Forms

In this chapter we give a brief introduction on lattices and discriminant forms. Good references are [CS99] and [Nik80].

1.1. Lattices

Definition 1.1. A *lattice* is a free $\mathbb{Z}$-module of finite rank with a non-degenerate symmetric $\mathbb{Z}$-bilinear form $(\cdot,\cdot)\colon L \times L \to \mathbb{Q}$. The *norm* of an element $x \in L$ is (x, x). A *sublattice* of a lattice L is a submodule $M \subset L$ with bilinear form given by restricting the bilinear form on L to M.

For a lattice L, we denote the quadratic form $L \to \mathbb{Q}, x \mapsto (x, x)/2$ by q. The bilinear form can be recovered from q via the polarization identity

$$(x, y) = q(x + y) - q(x) - q(y).$$

Definition 1.2. An *isomorphism* between lattices L_1 and L_2 with bilinear forms $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ is an invertible linear map $\varphi\colon L_1 \to L_2$ such that $(\varphi(x), \varphi(y))_2 = (x, y)_1$ for all $x, y \in L_1$. The group of automorphisms of a lattice L is called the *orthogonal group* of L and is denoted by $O(L)$.

Definition 1.3. Let L be a lattice and let $B = (x_1, \ldots, x_n)$ be a basis of L. The matrix

$$\begin{pmatrix} (x_1, x_1) & \cdots & (x_1, x_n) \\ \vdots & \ddots & \vdots \\ (x_n, x_1) & \cdots & (x_n, x_n) \end{pmatrix}$$

is called the *Gram matrix* of L with respect to the basis B.

The Gram matrix is a symmetric matrix with rational coefficients and hence has real eigenvalues. We implicitly make use of this in the following definition:

Definition 1.4. Let L be a lattice and let B be a basis of L with corresponding Gram matrix G. We denote the number of positive eigenvalues of G by b^+ and the number of negative eigenvalues by b^-. We say that L has *signature* (b^+, b^-). Moreover, L is called *positive definite* if $b^- = 0$ and *negative definite* if $b^+ = 0$. In all other cases L is called *indefinite*.

All these notions are independent of the choice of the basis B. The rank of L equals $b^+ + b^-$ because the bilinear form on L is non-degenerate.

Remark 1.5. Let L_1 and L_2 be lattices with bilinear forms $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$. We define a non-degenerate bilinear form on the $\mathbb{Z}$-module $L_1 \oplus L_2$ by

$$(x_1 + x_2, y_1 + y_2) = (x_1, y_1)_1 + (x_2, y_2)_2$$

for $x_1, y_1 \in L_1$ and $x_2, y_2 \in L_2$. This turns $L_1 \oplus L_2$ into a lattice.

Definition 1.6. Let L and L_1 be lattices. We say that L *splits* L_1 if there is a lattice L_2 such that $L \cong L_1 \oplus L_2$.

Definition 1.7. Let L be a lattice with bilinear form $(\cdot,\cdot)$ and let $a \in \mathbb{Q} \setminus \{0\}$. Then we define a new non-degenerate bilinear form $(\cdot,\cdot)_a$ on L by $(x,y)_a = a(x,y)$ for all $x, y \in L$. This defines a new lattice, which is called the *rescaling* of L by a and denoted by $L(a)$.

The Dual Lattice

We also need the notion of the dual lattice:

Definition 1.8. Let L be a lattice. The bilinear form $(\cdot,\cdot)$ on L can be uniquely extended to $L_\mathbb{Q} = L \otimes_\mathbb{Z} \mathbb{Q}$. The *dual lattice* L' of L is given by

$$L' = \{x \in L_\mathbb{Q} : (x,y) \in \mathbb{Z} \text{ for all } y \in L\}$$

with the bilinear form given by $(\cdot,\cdot)$.

Remark 1.9. Let $B = \{b_1, \ldots, b_n\}$ be a $\mathbb{Z}$-basis of L. Then B is also a $\mathbb{Q}$-basis of $L_\mathbb{Q}$ and there exists a $\mathbb{Q}$-basis $B' = \{b_1', \ldots, b_n'\}$ of $L_\mathbb{Q}$ which is dual to B with respect to the bilinear form $(\cdot,\cdot)$, i.e. $(b_i, b_j') = \delta_{ij}$ (where δ_{ij} is the Kronecker delta). Then B' is a $\mathbb{Z}$-basis of L'. The Gram matrix of L' with respect to B' is the inverse of the Gram matrix of L with respect to B.

Proposition 1.10. *Let L be a lattice. Then $(L')' = L$ and $O(L) = O(L')$.*

Proof. It is obvious that $(L')' = L$. Moreover every element of $O(L)$ can be extended to an isometry of $L_\mathbb{Q}$. We can therefore identify $O(L)$ with a subgroup of the group of isometries $O(L_\mathbb{Q})$ of $L_\mathbb{Q}$, namely $O(L) = \{\varphi \in O(L_\mathbb{Q}) : \varphi(L) = L\}$.

It remains to show that $O(L)$ fixes L'. Let $\varphi \in O(L)$ and $y \in L'$. Then

$$(x, \varphi(y)) = (\varphi^{-1}(x), y) \in \mathbb{Z}$$

for all $x \in L$, which proves that $\varphi(y) \in L'$. □

Definition 1.11. A lattice L is called

- *integral*, if $(L, L) \in \mathbb{Z}$,
- *even*, if $q(L) \in \mathbb{Z}$,
- *unimodular*, if $L' = L$.

The proofs of the statements in the following remarks can be easily derived from the last definition.

Remark 1.12. The following are equivalent:

1. L is integral.

2. The Gram matrix of L with respect to any basis has integral entries.

3. $L \subset L'$.

Remark 1.13. Even lattices are integral and an integral lattice is even if and only if the Gram matrix with respect to any basis is even (a symmetric integer matrix is called even if its diagonal entries are even).

Remark 1.14. A unimodular lattice is integral and an integral lattice is unimodular if and only if its Gram matrix with respect to any basis is unimodular (a square integer matrix is called unimodular if it has determinant ± 1).

Definition 1.15. Let L be an integral lattice. The *level* of L is the smallest positive integer N such that $Nq(x) \in \mathbb{Z}$ for all $x \in L'$, i.e. it is the smallest positive integer N such that $L'(N)$ is even.

Isotropic Vectors and Cusps

Definition 1.16. Let L be a lattice. An element $x \in L$ is called *primitive* if

$$\mathbb{Q}x \cap L = \mathbb{Z}x.$$

Definition 1.17. Let L be a lattice. A subspace $V \subset L_{\mathbb{Q}}$ is called *isotropic* if the restriction of the bilinear form to V is 0 and a non-zero vector $z \in L_{\mathbb{Q}}$ is called *isotropic* if it has norm 0, i.e. if $q(z) = 0$. A primitive isotropic element $z \in L$ is called a *cusp*. The *level* of a cusp $z \in L$ is the unique $l \in \mathbb{Z}_{>0}$ with $(z, L) = l\mathbb{Z}$.

Proposition 1.18 (Meyer's theorem, see e.g. [Ser73], Chapter IV, Corollary 3.2)**.** *Let L be an indefinite lattice of rank at least* 5. *Then L contains an isotropic element.*

Remark 1.19. Let $z \in L$ be a cusp of level l. There is an element $z' \in L'$ with $(z, z') = 1$. Let m be the exponent of L'/L, i.e. the smallest positive integer m with $mL' \subset L$. Then $mz' \in L$ and $(z, mz') = m$, so $l \mid m$.

Roots and Root Lattices

We need to define roots, which are important in the context of reflective automorphic forms (see Section 3.2).

Definition 1.20. Let L be a lattice and $x \in L$ be a primitive element with $q(x) > 0$. Let $\sigma_x \in O(L_{\mathbb{Q}})$ be the reflection through the hyperplane orthogonal to x, i.e.

$$\sigma_x(y) = y - 2\frac{(x, y)}{(x, x)}x.$$

If $\sigma_x(L) = L$, then x is called a *root* of L.

Proposition 1.21 ([Sch06], Proposition 2.1)**.** *Let L be an even lattice of level N and let $x \in L$ be a root. Let $k = q(x)$. Then k is a positive divisor of N and x is in kL'.*

If the level of L is squarefree, then this condition is also sufficient.

Proposition 1.22 ([Sch06], Proposition 2.2)**.** *Let L be an even lattice of squarefree level N and let k be a positive divisor of N. If $x \in L \cap kL'$ satisfies $q(x) = k$, then x is a root.*

Proposition 1.23. *Let L be a lattice and let Φ be the set of roots of L. Then the following statements are true.*

1. *Let $x \in \Phi$. Then $\pm x$ are the only multiples of x in Φ.*
2. *For all $x, y \in \Phi$, the element $\sigma_x(y)$ is in Φ.*
3. *For all $x \in \Phi$ and $y \in L$, the number $2\frac{(x,y)}{(x,x)}$ is an integer.*

Proof.

1. This follows from the primitivity of roots.
2. Let $z = \sigma_x(y)$. Then σ_z is in $O(L)$ because $\sigma_z = \sigma_x \sigma_y \sigma_x$.
3. Since both y and $\sigma_x(y)$ are in L, so is their difference. Therefore, $2\frac{(x,y)}{(x,x)}x$ is in L and the primitivity of L then requires $2\frac{(x,y)}{(x,x)}$ to be an integer.

□

We now restrict to the case of positive definite lattices.

Definition 1.24. A *root lattice* is a positive definite lattice spanned by its roots.

Example 1.25. The following are examples of root lattices (for the definition of the root systems A_n, D_n, E_8, E_7 and E_6 see Appendix A).

1. Let $n \geq 1$ be an integer. The lattice A_n is defined to be the $\mathbb{Z}$-span of the root system A_n. A basis of A_n is given by the vectors $e_{i+1} - e_i$, $i = 1, \ldots, n$. The resulting Gram matrix is

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \ddots & 0 \\ 0 & 0 & -1 & 2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix},$$

which we call the standard Gram matrix of A_n.

2. Let $n \geq 4$ be an integer. The lattice D_n is defined to be the $\mathbb{Z}$-span of the root system D_n. A basis of D_n is given by the vectors $e_{i+1} - e_i$, $i = 1, \ldots, n-1$ and $-e_{n-1} - e_n$. The resulting Gram matrix is

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & 2 & -1 & 0 & 0 \\ 0 & \ddots & -1 & 2 & -1 & -1 \\ \vdots & \ddots & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix},$$

   which we call the standard Gram matrix of D_n.

3. The lattice E_8 is defined to be the $\mathbb{Z}$-span of the root system E_8. A basis of E_8 is given by the vectors $e_{i+1} - e_i$, $i = 1, \ldots, 7$ and $\frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 - e_6 - e_7 - e_8)$. The resulting Gram matrix is

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix},$$

   which we call the standard Gram matrix of E_8.

4. The lattice E_7 is defined to be the $\mathbb{Z}$-span of the root system E_7. A basis of E_7 is given by the vectors $e_{i+1} - e_i$, $i = 1, \ldots, 6$ and $\frac{1}{2}(e_1 + e_2 + e_3 + e_4 - e_5 - e_6 - e_7 - e_8)$. The resulting Gram matrix is

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix},$$

   which we call the standard Gram matrix of E_7.

5. The lattice E_6 is defined to be the $\mathbb{Z}$-span of the root system E_6. A basis of E_6 is given by the vectors $e_{i+1} - e_i$, $i = 2, \ldots, 6$ and $\frac{1}{2}(e_1 + e_2 + e_3 + e_4 - e_5 - e_6 - e_7 - e_8)$.

The resulting Gram matrix is

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix},$$

which we call the standard Gram matrix of E_6.

Remark 1.26. If $L = \bigoplus L_i$ with each L_i as in the previous example, then the block diagonal matrix with the blocks being the standard Gram matrices of the summands L_i is obviously a Gram matrix of L. We call this a *standard Gram matrix* of L (this is only unique up to permutations of the blocks because we can change the order of the summands L_i).

We finish this subsection with the classification of the root lattices.

Proposition 1.27. *If L is a positive definite lattice, then the set of roots is finite.*

Proof. By replacing L with $L(a)$ for a suitable positive integer a, we can assume that L is even (note that the set of roots of L coincides with that of $L(a)$ because $O(L) = O(L(a))$). We can then use Proposition 1.21 to see that the values of the quadratic form q are bounded on the set of roots. This implies that the set of roots is finite because a positive definite lattice has only finitely many elements of fixed norm. □

Propositions 1.23 and 1.27 imply that the set of roots Φ of a positive definite lattice is a reduced root system in the subspace $E \subset L \otimes_{\mathbb{Z}} \mathbb{R}$ spanned by Φ (see Appendix A for the definition of root systems and their classification). This can be used to determine all root lattices:

Proposition 1.28. *Let L be a root lattice. Then $L = \bigoplus_{j=1}^{n} L_j(a_j)$ with each L_j being a lattice from Example 1.25 and $a_j \in \mathbb{Q}_{>0}$.*

Proof. Let Φ be the set of roots of L. We have seen that Φ is a root system. If $\Phi = \Phi_1 \cup \Phi_2$ with orthogonal subsets Φ_1 and Φ_2, then L is the direct sum of the sublattices spanned by Φ_1 and Φ_2. We can therefore assume that Φ is irreducible. The lattices generated by the root systems A_n, D_n, E_6, E_7 and E_8 are the lattices A_n, D_n E_6, E_7 and E_8. The lattices generated by the root systems B_n, C_n, F_4 and G_2 are isomorphic to the lattices $A_1^n(1/2)$, D_n, $D_4(2)$ and A_2. The assertion then follows from the fact that an isomorphism between two irreducible root systems Φ_1 and Φ_2 is an isometry if the shortest vectors in Φ_1 and Φ_2 have the same norm. □

Corollary 1.29. *Let L be an integral positive definite lattice generated by norm 2 vectors. Then L is a direct sum of lattices from Example 1.25.*

Proof. It is easy to see that L must be even. Then the vectors of norm 2 are roots by Proposition 1.22. Therefore L is a root lattice and hence a direct sum of rescalings of lattices from Example 1.25, i.e. $L = \bigoplus_{j=1}^{n} L_j(a_j)$ with each L_j being a lattice from Example 1.25 and $a_j \in \mathbb{Q}_{>0}$. If $a_j < 1$ for some j, then $L_j(a_j)$ and hence L is not integral. If $a_j > 1$ for some j, then $L_j(a_j)$ and hence L is not generated by norm 2 vectors. Therefore $a_j = 1$ for all j. □

Definition 1.30. A 2-*root lattice* is an integral lattice spanned by its norm 2 vectors.

1.2. Even Unimodular Lattices

We now briefly mention some results on even unimodular lattices. We first note that these do not exist for arbitrary signatures:

Proposition 1.31 (see e.g. [Nik80], Theorem 1.1.1)**.** *Let L be an even unimodular lattice of signature (b^+, b^-). Then $b^+ - b^-$ is divisible by 8.*

1.2.1. Niemeier Lattices

If L is an even positive definite unimodular lattice, then its rank m must be divisible by 8 because of Proposition 1.31. The classification of such lattices up to isomorphisms seems to be hopeless for $m \geq 32$, as the number of isomorphism classes increases rapidly (this can be seen from the Minkowski-Siegel mass formula ([CS99], Chapter 16)); e.g. for $m = 32$ there are more than a billion isomorphism classes (see [Kin03]). On the other hand the classification is very simple for $m = 8$ and $m = 16$ (see for example [Wit41]): In rank 8 the only such lattice is E_8, while in rank 16 there are two such lattices, namely $E_8 \oplus E_8$ and

$$D_{16}^{+} = D_{16} \cup (1/2, \ldots, 1/2) + D_{16}.$$

The classification for $m = 24$ was obtained by Niemeier [Nie73], which is why the even positive definite unimodular lattices of rank 24 are called Niemeier lattices. Venkov later gave a simplified proof ([Ven78]; for an English version see Chapter 18 of [CS99]). It turns out that there are exactly 24 isomorphism classes and that the isomorphism class of a Niemeier lattice Λ is uniquely determined by the set of roots, which are exactly the norm 2 vectors by Propositions 1.21 and 1.22. They form one of the following root systems:

$$\begin{aligned}
&\emptyset,\\
&\mathsf{A}_1^{24}, \mathsf{A}_2^{12}, \mathsf{A}_3^{8}, \mathsf{A}_4^{6}, \mathsf{A}_6^{4}, \mathsf{A}_8^{3}, \mathsf{A}_{12}^{2}, \mathsf{A}_{24},\\
&\mathsf{D}_4^{6}, \mathsf{D}_6^{4}, \mathsf{D}_8^{3}, \mathsf{D}_{12}^{2}, \mathsf{D}_{24},\\
&\mathsf{E}_6^{4}, \mathsf{E}_8^{3},\\
&\mathsf{A}_5^{4}\mathsf{D}_4, \mathsf{A}_7^{2}\mathsf{D}_5^{2}, \mathsf{A}_9^{2}\mathsf{D}_6, \mathsf{A}_{15}\mathsf{D}_9, \mathsf{A}_{17}\mathsf{E}_7, \mathsf{D}_{10}\mathsf{E}_7^{2}, \mathsf{D}_{16}\mathsf{E}_8,\\
&\mathsf{A}_{11}\mathsf{D}_7\mathsf{E}_6.
\end{aligned}$$

If Φ is one of these root systems, then we denote the corresponding Niemeier lattice by $N(\Phi)$. The lattice $N(\emptyset)$ is the Leech lattice and is constructed in the next subsection.

The other 23 Niemeier lattices can be constructed from their respective root systems as follows: The root system Φ spans a sublattice L of $N(\Phi)$ of rank 24, so

$$L \subset N(\Phi) = N(\Phi)' \subset L'$$

and $N(\Phi)$ can be obtained from L by adding certain vectors from L', the so-called *glue vectors*.

Example 1.32. The Niemeier lattice $N(A_1^{24})$ is constructed as follows. Let L be the root lattice A_1^{24}. Then $L = \bigoplus_{i=1}^{24} L_i$ where $L_i = A_1$ for all $i \in \{1, \ldots, 24\}$, i.e. each L_i is spanned by a single element x_i of norm 2. The dual lattice of L is given by

$$L' = \bigoplus_{i=1}^{24} L_i'$$

and L_i' is spanned by $w_i = x_i/2$. The glue vectors are of the form $\sum_{i=1}^{24} \lambda_i w_i$, where $(\lambda_1, \ldots, \lambda_{24}) \in \{0,1\}^{24}$ is an element of the Golay code $\mathcal{G}$, which is defined in the next subsection. Together with L these span $N(A_1^{24})$.

Example 1.33. The lattice E_8^3 is already unimodular, so $N(E_8^3) = E_8^3$ and we don't need to add any glue vectors.

The glue vectors for the remaining Niemeier lattices with non-empty root system can be found in Chapter 16 of [CS99].

1.2.2. The Leech Lattice

The Leech lattice Λ is the Niemeier lattice $N(\emptyset)$, i.e. it is the unique Niemeier lattice without norm 2 vectors. It was discovered by Leech [Lee67]. In order to construct it we first introduce the Golay code.

Definition 1.34. A *linear binary code* of length n and rank k is a k-dimensional linear subspace $\mathcal{C}$ of $\mathbb{F}_2^n$. Elements of $\mathcal{C}$ are called *codewords*. The *weight* of a codeword is the number of coordinates (with respect to the standard basis) different from zero and the *distance* between two codewords is the number of coordinates in which they differ. An *automorphism* of $\mathcal{C}$ is a permutation of the coordinates that stabilizes $\mathcal{C}$ (as a set, not necessarily pointwise).

Definition 1.35. The *Golay code* $\mathcal{G}$ is a linear binary code of length 24 and rank 12 such that any two distinct elements of $\mathcal{G}$ have distance at least 8.

Remark 1.36. Any two Golay codes are isomorphic, i.e. related by a permutation of coordinates. It therefore makes sense to speak of *the* Golay code.

Remark 1.37. The Golay code is an error-correcting code and therefore also useful in data transmission.

One can (and we always do) take $\mathcal{G}$ to be the subspace generated by the rows of the matrix

$$\left(\begin{array}{cccccccccccc|cccccccccccc}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&1&1&1&0&1\\
0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&1&1&0&0&1&1&1&1&0\\
0&0&1&0&0&0&0&0&0&0&0&0&1&1&0&0&1&1&0&0&0&1&1&1\\
0&0&0&1&0&0&0&0&0&0&0&0&0&1&1&0&0&1&1&0&1&0&1&1\\
0&0&0&0&1&0&0&0&0&0&0&0&0&1&1&1&1&0&0&1&0&0&1&1\\
0&0&0&0&0&1&0&0&0&0&0&0&1&0&1&1&1&1&0&0&1&0&0&1\\
0&0&0&0&0&0&1&0&0&0&0&0&1&1&0&1&0&1&1&0&1&1&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&1&1&1&0&0&0&1&1&0&1&1&0\\
0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&1&0&1&1&1&0&0&1&1\\
0&0&0&0&0&0&0&0&0&1&0&0&1&1&0&0&1&0&1&1&1&0&0&1\\
0&0&0&0&0&0&0&0&0&0&1&0&0&1&1&0&1&1&0&1&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&1&1&1&1&0&0&1&1&0
\end{array}\right).$$

Elements in $\mathcal{G}$ of weight 8 are called *octads* and elements of weight 12 are called *dodecads*. There are 759 octads and 2576 dodecads in $\mathcal{G}$.

The automorphism group of $\mathcal{G}$ is the Mathieu group M_{24}, which is transitive on both octads and dodecads. For our choice of $\mathcal{G}$ it is shown in [Con93] that M_{24} is generated by the permutations

$$\begin{aligned}
g &= (1,14)(4,5)(6,7)(8,9)(11,24)(16,17)(18,19)(20,21) \text{ and} \\
h &= (1,2,3,4)(5,6,7,8)(9,10,11,12)(13,14,15,16)(17,18,19,20)(21,22,23,24).
\end{aligned} \tag{1.1}$$

The group M_{24} is one of the sporadic groups and has order

$$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 = 244823040 \approx 2.4 \times 10^8.$$

The Leech lattice Λ then consists of all vectors $x = (x_i)_{1 \le i \le 24} \in \mathbb{Z}^{24}$ for which there is a Golay codeword $(c_i)_{1 \le i \le 24} \in \mathcal{G}$ and an element $m \in \mathbb{F}_2$ such that

1. $x_i = m \mod 2$ for all $i \in \{1, \dots, 24\}$,
2. $x_i = m + 2c_i \mod 4$ for all $i \in \{1, \dots, 24\}$ and
3. $\sum_{i=1}^{24} x_i = 4m \mod 8$.

The quadratic form on Λ is given by $q(x) = \frac{1}{8} \sum_{i=1}^{24} x_i^2$. There are 196560 vectors of norm 4 in Λ, namely the following:

1. Any vector $x = (x_i)_{1 \le i \le 24}$ with $x_i = 0$ for 22 coordinates and $x_i = \pm 4$ for the other two coordinates is an element of Λ. This gives $\binom{24}{2} \cdot 2^2 = 1104$ vectors of norm 4.
2. Let $c = (c_i)_{1 \le i \le 24} \in \mathcal{G}$ be an octad and let $x = (x_i)_{1 \le i \le 24}$ be a vector with $x_i = \pm 2$ if $c_i = 1$ and $x_i = 0$ otherwise. Then x is in Λ if and only if the number of negative coordinates is even. This gives $759 \cdot 2^7 = 97152$ vectors of norm 4.

3. Let $x = (x_i)_{1 \le i \le 24}$ be a vector such that $x_i = \pm 3$ for one coordinate and $x_i = \pm 1$ for the other 23 coordinates. We define $c = (c_i)_{1 \le i \le 24} \in \mathbb{F}_2^{24}$ by $c_i = 1$ if $x_i = 1 \mod 4$ and $c_i = 0$ if $x_i = 3 \mod 4$. Then $x \in \Lambda$ if and only if $c \in \mathcal{G}$. This gives $24 \cdot 2^{12} = 98304$ vectors of norm 4.

The automorphism group of Λ was determined by Conway [Con69], which is why it is now called the Conway group Co_0. Its order is

$$2^{22} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23 = 8315553613086720000 \approx 8.3 \times 10^{18}$$

and all of its elements (when considered as 24×24-matrices) have determinant 1. It is not a simple group; only after taking the quotient by its center $\{\pm 1\}$ one obtains another sporadic group, denoted by Co_1.

The automorphism group M_{24} of $\mathcal{G}$ acts on the Leech lattice by permutations of the coordinates and is hence a subgroup of Co_0. Moreover, any codeword $c = (c_i)_{1 \le i \le 24}$ in $\mathcal{G}$ defines an involution ε_c of Λ by mapping $x = (x_i)_{1 \le i \le 24}$ to $((-1)^{c_i} x_i)_{1 \le i \le 24}$. Since $|\mathcal{G}| = 2^{12}$, these involutions define an elementary abelian subgroup $E \subset \mathrm{Co}_0$ of order 2^{12} isomorphic to the additive group of $\mathcal{G}$. The product $N = EM_{24}$ of the subgroups E and M_{24} is a maximal subgroup of Co_0.

1.2.3. Indefinite Even Unimodular Lattices

The simplest example for an indefinite even unimodular lattice is given by a lattice of rank 2 with Gram matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We denote this lattice by $II_{1,1}$ and call it *hyperbolic plane*. Then every indefinite even unimodular lattice can be built from E_8 and $II_{1,1}$. More precisely, if it has signature (b^+, b^-), then it is isomorphic to a unique direct sum of copies of E_8 and $II_{1,1}$ if $b^+ \ge b^-$ and to a unique direct sum of copies of $E_8(-1)$ and $II_{1,1}$ if $b^+ < b^-$ (this follows from Proposition 1.31 and the fact that there is at most one lattice of genus $II_{b^+,b^-}(1)$; see Remark 1.67).

1.3. Discriminant Forms

In this section we define discriminant forms which are an important invariant of even lattices, but we first need to define finite quadratic forms:

Definition 1.38. Let D be a finite abelian group. A map $Q\colon D \to \mathbb{Q}/\mathbb{Z}$ is called a *finite quadratic form* if

1. $Q(a\gamma) = a^2 Q(\gamma)$ for all $a \in \mathbb{Z}$ and $\gamma \in D$,
2. the map $(\cdot,\cdot)\colon D \times D \to \mathbb{Q}/\mathbb{Z}$ defined by $(\delta,\gamma) = Q(\delta+\gamma) - Q(\delta) - Q(\gamma)$ is $\mathbb{Z}$-bilinear.

The map $(\cdot,\cdot)\colon D \times D \to \mathbb{Q}/\mathbb{Z}$ is called the *associated bilinear form* and $Q(\gamma)$ is called the *norm* of γ.

Definition 1.39. Let D be a finite abelian group and $Q\colon D \to \mathbb{Q}/\mathbb{Z}$ a finite quadratic form with associated bilinear form $(\cdot,\cdot)$.

1. Two elements $\delta, \gamma \in D$ are called *orthogonal* if $(\delta, \gamma) = 0 + \mathbb{Z}$.
2. Let $S \subset D$ be a subset. The *orthogonal complement* of S is the subgroup
$$S^{\perp} = \{\gamma \in D : (\gamma, \delta) = 0 + \mathbb{Z} \text{ for all } \delta \in S\} \subset D.$$
3. An element $\gamma \in D$ is called *isotropic* if $Q(\gamma) = 0 + \mathbb{Z}$.
4. A subgroup $H \subset D$ is called *isotropic* if all elements of S are isotropic.
5. The finite quadratic form Q is called non-degenerate if $D^{\perp} = \{0\}$.

Remark 1.40. If H is an isotropic subgroup, then $H \subset H^{\perp}$.

Definition 1.41. A *discriminant form* is a finite abelian group D with a non-degenerate finite quadratic form $Q\colon D \to \mathbb{Q}/\mathbb{Z}$. An *isomorphism* between discriminant forms D_1 and D_2 with finite quadratic forms Q_1 and Q_2 is a group isomorphism $\varphi\colon D_1 \to D_2$ such that $Q_2(\varphi(\gamma)) = Q_1(\gamma)$ for all $\gamma \in D_1$.

Remark 1.42. Let D_1 and D_2 be discriminant forms with finite quadratic forms Q_1 and Q_2. Let $D = D_1 \oplus D_2$ and define $Q\colon D \to \mathbb{Q}/\mathbb{Z}$ by $Q(\gamma_1 + \gamma_2) = Q_1(\gamma_1) + Q_2(\gamma_2)$ for $\gamma_1 \in D_1$ and $\gamma_2 \in D_2$. Then Q is a non-degenerate quadratic form and gives D the structure of a discriminant form.

Definition 1.43. A discriminant form D is called *decomposable* if there are non-trivial discriminant forms D_1, D_2 such that $D \cong D_1 \oplus D_2$. Otherwise D is called *indecomposable*.

Jordan Decomposition

Definition 1.44. We define the following discriminant forms (here and in the rest of this thesis $\left(\frac{\cdot}{\cdot}\right)$ denotes the Kronecker symbol):

1. Let $q > 1$ be a power of an odd prime p, $\epsilon \in \{-1, +1\}$ and let $D = \mathbb{Z}/q\mathbb{Z}$. Let γ be a generator of D and define the finite quadratic form Q by $Q(\gamma) = a/q + \mathbb{Z}$ where a is an integer satisfying $\left(\frac{2a}{p}\right) = \epsilon_2$. These discriminant forms are denoted by q^{ϵ} and are called the *indecomposable p-adic Jordan components* of order q.
2. Let $q > 1$ be a power of 2, $\epsilon \in \{-1, 1\}$ and let $D = \mathbb{Z}/q\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$. Let γ, δ be generators of D and define the finite quadratic form Q by $Q(\gamma + \delta) = 1/q + \mathbb{Z}$ and $Q(\gamma) = Q(\delta) = 0 + \mathbb{Z}$ if $\epsilon = +1$ and $Q(\gamma) = Q(\delta) = 1/q + \mathbb{Z}$ if $\epsilon = -1$. These discriminant forms are denoted by $q_{II}^{\epsilon 2}$ and are called the *indecomposable even 2-adic Jordan components* of order q.
3. Let $q > 1$ be a power of 2, $\epsilon \in \{-1, 1\}$ and let $D = \mathbb{Z}/q\mathbb{Z}$. Let γ be a generator of D and define the finite quadratic form Q by $Q(\gamma) = t/(2q) + \mathbb{Z}$ where $t \in \mathbb{Z}/8\mathbb{Z}$ with $\left(\frac{t}{2}\right) = \epsilon$. These discriminant forms are denoted by q_t^{ϵ} and are called the *indecomposable odd 2-adic Jordan components* of order q.

Remark 1.45. It is easy to see that a different choice of the generator(s) of D and the elements a and t leads to an isomorphic discriminant form.

Proposition 1.46 ([CS99], Section 15.7)**.** *The indecomposable Jordan components are indecomposable in the sense of Definition 1.43 and every indecomposable discriminant form is isomorphic to an indecomposable Jordan component.*

By taking direct sums of indecomposable Jordan components for a fixed prime power q one obtains the Jordan components, which are defined as follows:

Definition 1.47. We define the following discriminant forms:

1. Let $q > 1$ be a power of an odd prime p and let n be a positive integer. Let $q^{\epsilon_1}, \ldots, q^{\epsilon_n}$ be n indecomposable p-adic Jordan components of order q with $\epsilon_i = \pm 1$ and let $\epsilon = \prod \epsilon_i$. Then we define $q^{\epsilon n} = \bigoplus q^{\epsilon_i}$. These discriminant forms are called the *p-adic Jordan components* of rank n.

2. Let $q > 1$ be a power of 2 and let n be a positive integer. Let $q_{II}^{\epsilon_1 2}, \ldots, q_{II}^{\epsilon_n 2}$ be n indecomposable even 2-adic Jordan components of order q with $\epsilon_i = \pm 1$ and let $\epsilon = \prod \epsilon_i$. Then we define $q_{II}^{\epsilon 2n} = \bigoplus q_{II}^{\epsilon_i 2}$. These discriminant forms are called the *even 2-adic Jordan components* of rank $2n$.

3. Let $q > 1$ be a power of 2 and let n be a positive integer. Let $q_{t_1}^{\epsilon_1}, \ldots, q_{t_n}^{\epsilon_2}$ be n indecomposable odd 2-adic Jordan components of order q with $\epsilon_i = \pm 1, t_i \in \mathbb{Z}/8\mathbb{Z}$ and $\left(\frac{t_i}{2}\right) = \epsilon_i$. Let $\epsilon = \prod \epsilon_i$ and $t = \sum t_i$. Then we define $q_t^{\epsilon n} = \bigoplus q_{t_i}^{\epsilon_i}$. These discriminant forms are called the *odd 2-adic Jordan components* of rank n.

Remark 1.48. The decomposition of a Jordan component into indecomposable Jordan components is not unique: For example 3^{+2} is isomorphic to both $3^{+1} \oplus 3^{+1}$ and $3^{-1} \oplus 3^{-1}$. However, the Jordan components are well-defined as any two direct sums of indecomposable Jordan components that lead to the same symbol are isomorphic.

Special Subgroups and Subsets

Definition 1.49. Let D be a discriminant form and c an integer. We define the subgroups D_c and D^c of D by

$$D_c = \{\gamma \in D : c\gamma = 0\}$$

and

$$D^c = \{\gamma \in D : \gamma = c\delta \text{ for some } \delta \in D\},$$

i.e. D_c consists of the elements of order dividing c and D^c consists of the elements that are c-th powers.

Remark 1.50. D_c is the orthogonal complement of D^c and vice versa and there is an exact sequence

$$0 \to D_c \to D \to D^c \to 0$$

with the second arrow given by inclusion and the third arrow given by taking c-th powers.

Definition 1.51. Let D be a discriminant form and let $c \in \mathbb{Z}_{>0}$ and $x \in \mathbb{Q}$. Then we define the subset

$$D_{c,x} = \{\gamma \in D : \operatorname{ord}(\gamma) = c \text{ and } Q(\gamma) = x + \mathbb{Z}\}.$$

Remark 1.52. The subgroup D_c is in general not the union of the subsets $D_{c,x}$ for all $x \in \mathbb{Q}$ because elements in $D_{c,x}$ have order c, whereas elements in D_c can have smaller orders.

Proposition 1.53 ([Sch15], Proposition 5.1)**.** *Let D be a discriminant form of squarefree level N. Then* $\operatorname{Aut}(D)$ *acts transitively on $D_{c,x}$ for all $c \in \mathbb{Z}_{>0}$ and $x \in \mathbb{Q}$.*

For a given discriminant form D of squarefree level, we later need to know the number of elements of a given norm in the subgroups D_c.

Definition 1.54. Let p be a positive prime and $j \in \mathbb{Z}/p\mathbb{Z}$. We write $N(p^{\epsilon n}, j)$ for the number of elements of norm $j/p + \mathbb{Z}$ in the discriminant form $D = p^{\epsilon n}$.

Proposition 1.55 ([Sch06], Proposition 3.1)**.** *The number of elements in $2_{II}^{\epsilon n}$ of norm $j/2 + \mathbb{Z}$ is*

$$N(2_{II}^{\epsilon n}, j) = \begin{cases} 2^{n-1} - \epsilon 2^{(n-2)/2} & \text{if } j \neq 0, \\ 2^{n-1} + \epsilon 2^{(n-2)/2} & \text{if } j = 0. \end{cases}$$

Proposition 1.56 ([Sch06], Proposition 3.2)**.** *Let p be an odd prime. Then the number of elements in $p^{\epsilon n}$ of norm $j/p + \mathbb{Z}$ is given by*

$$N(p^{\epsilon n}, j) = \begin{cases} p^{n-1} - \epsilon \left(\frac{-1}{p}\right)^{n/2} p^{(n-2)/2} & \text{if } n \text{ is even and } j \neq 0, \\ p^{n-1} + \epsilon \left(\frac{-1}{p}\right)^{n/2} \left(p^{n/2} - p^{(n-2)/2}\right) & \text{if } n \text{ is even and } j = 0, \\ p^{n-1} + \epsilon \left(\frac{-1}{p}\right)^{(n-1)/2} \left(\frac{2}{p}\right) \left(\frac{j}{p}\right) p^{(n-1)/2} & \text{if } n \text{ is odd and } j \neq 0, \\ p^{n-1} & \text{if } n \text{ is odd and } j = 0. \end{cases}$$

Proposition 1.57 ([Sch06], Proposition 3.3)**.** *Let D be a discriminant form of squarefree level. Let $c \mid N$. Then the number of elements in D_c of norm $j/c + \mathbb{Z}$ is given by*

$$N(D_c, j) = \prod_{p \mid c} N(p^{\epsilon_p n_p}, cj/p).$$

The Discriminant Form of a Lattice

Let L be an integral lattice. Then $D = L'/L$ is a finite abelian group of order $|\det(G)|$, where G is any Gram matrix of L. If L is even, then the map

$$Q \colon D \to \mathbb{Q}/\mathbb{Z}, \quad Q(x + L) = q(x) + \mathbb{Z} \tag{1.2}$$

is well-defined. It is a non-degenerate finite quadratic form.

Definition 1.58. Let L be an even lattice. The *discriminant form associated to* L is the finite abelian group $D = L'/L$ with the finite quadratic form Q defined in (1.2).

Remark 1.59. We warn the reader that reducing the norm (x, x) of an element $x \in L$ modulo $\mathbb{Z}$ does not give the norm of $x + L$ in L'/L. To obtain the norm in L'/L one must instead reduce $q(x)$ which is half the norm of x.

In fact every discriminant form appears as the discriminant form associated to a lattice:

Proposition 1.60 ([Nik80], Theorem 1.3.2)**.** *Let D be a discriminant form. Then there is an even lattice L such that $D \cong L'/L$ as discriminant forms.*

To define the signature of a discriminant form we need the following proposition:

Proposition 1.61 (Milgram's formula)**.** *Let L be an even lattice of signature (b^+, b^-). Then*

$$\sum_{x+L \in L'/L} e(q(x)) = \sqrt{|L'/L|}e((b^+ - b^-)/8) \quad (e(z) = e^{2\pi i z}).$$

Proof. See for example [MH73], Appendix 4. □

Because of Propositions 1.60 and 1.61 the following is well-defined:

Definition 1.62. Let D be a discriminant form and let L be an even lattice of signature (b^+, b^-) such that $L'/L \cong D$. We define the *signature* $\operatorname{sign}(D) \in \mathbb{Z}/8\mathbb{Z}$ to be the residue class of $b^+ - b^-$ in $\mathbb{Z}/8\mathbb{Z}$.

We also define the level of a discriminant form:

Definition 1.63. Let D be a discriminant form. The *level* of D is the smallest positive integer with $NQ(\gamma) \in \mathbb{Z}$ for all $\gamma \in D$.

Remark 1.64. With this definition the level of an even lattice L and its discriminant form L'/L are the same.

Remark 1.65. If the level of a discriminant form D is squarefree, then it coincides with the exponent of D, i.e. the smallest positive integer a with $a\gamma = 0$ for all $\gamma \in D$. This can for example be seen by verifying it for all indecomposable Jordan components of prime level.

Definition 1.66. Let L be an even lattice of signature (b^+, b^-) with discriminant form D. Then we say that L has *genus* $II_{b^+,b^-}(D)$. If L is unimodular, in which case D is trivial, we just write II_{b^+,b^-}.

Remark 1.67. In general there can be different non-isomorphic lattices with the same genus, albeit their number is finite (see e.g. [Cas78], Section 9.4, Corollary 1). For example the 24 Niemeier lattices all have genus $II_{24,0}$. However, if L is an indefinite even lattice of rank n, then Corollary 15.22 in [CS99] shows that it is isomorphic to all lattices with the same genus if $|L'/L| < 5^{\binom{n}{2}}$. In this case we also use the symbol of the genus when we actually mean the unique lattice of this genus. We have already used this notation for the hyperbolic plane in Subsection 1.2.3.

Whether there exists an even lattice with given genus $II_{b^+,b^-}(D)$ is answered in [Nik80], Theorem 1.10.1 and [CS99], Chapter 15. We explain it for those cases where the level N of D is squarefree. Then $D = \bigoplus_{p|N} D_p$ and each D_p is a p-adic Jordan component of order p (for $p = 2$ it is an even 2-adic Jordan component).

Proposition 1.68 ([Nik80], Theorem 1.10.1; [CS99], Chapter 15)**.** *Let b^+ and b^- be non-negative integers and D a discriminant form of squarefree level. There is an even lattice of genus $II_{b^+,b^-}(D)$ if and only if*

1. $b^+ - b^- = \operatorname{sign}(D) \mod 8$ *and*
2. *for each prime p the p-rank n_p of the p-adic Jordan component D_p is at most $r = b^+ + b^-$. If $n_p = r$, then*

$$\epsilon_p = \left(\frac{(-1)^{b^-} \cdot |D^p|}{p} \right).$$

Two immediate consequences that we will later make use of are the following:

Corollary 1.69. *Let r and N be positive integers with N squarefree. Then the number of isomorphism classes of even lattices of rank r and level N is finite.*

Proof. If D is the discriminant form of such a lattice, then the p-ranks of the Jordan components of D are bounded by r because of the previous proposition. Therefore, only finitely many discriminant forms can occur and the result follows because every genus only contains finitely many non-isomorphic lattices. □

Corollary 1.70. *Let L be an even lattice of signature (b^+, b^-) and squarefree level N and let $D = L'/L$. Suppose L splits $II_{1,1}$. Then $D = \prod_{p|N} p^{\epsilon_p n_p}$ with $n_p \leq b^+ + b^- - 2$. If $n_p = b^+ + b^- - 2$, then*

$$\epsilon_p = \left(\frac{(-1)^{b^- -1} \cdot |D^p|}{p} \right).$$

Proof. By assumption, $L = K \oplus II_{1,1}$. The discriminant form of K equals that of L because $II_{1,1}$ is unimodular. Therefore K has genus $II_{b^+-1,b^--1}(D)$ and the result follows from Proposition 1.68. □

Later we will be interested in lattices that split $II_{1,1}(N)$. We therefore state the following proposition, which describes the discriminant form of $II_{1,1}(N)$:

Proposition 1.71. *Let N be a squarefree positive integer, $L = II_{1,1}(N)$ and $D = L'/L$. Then $D = \prod_{p|N} p^{\epsilon_p n_p}$ with $n_p = 2$ and $\epsilon_p = \left(\frac{-1}{p}\right)$ for all $p \mid N$.*

Proof. The dual lattice L' of L is given by $L' = \left(\frac{1}{N}\mathbb{Z}\right)^2$. Therefore, $D = (\mathbb{Z}/N\mathbb{Z})^2$, which proves that $n_p = 2$ for all $p \mid N$. We fix a prime $p \mid N$. Then $\gamma = (1/p, 0) + L \in D$ is an isotropic element of order p. In particular $D_p = p^{\epsilon_p 2}$ has non-trivial isotropic elements. By Propositions 1.55 and 1.56 this is only possible if ϵ_p is as claimed. □

2. Modular Forms for Congruence Subgroups of $\mathrm{SL}_2(\mathbb{Z})$

In this chapter we introduce modular forms, both scalar-valued and vector-valued. These will later be needed to construct and classify automorphic forms. The first two sections deal with congruence subgroups and scalar-valued modular forms for these. There is plenty of literature available (e.g. [CS99] or [Miy89] to just name two) and most of the material in these two sections is well-known. The third section covers the Weil representation and vector-valued modular forms for it. The main references are the articles [Sch06], [Sch09] and [Sch15] by Scheithauer.

2.1. Congruence Subgroups

In the following $\mathrm{SL}_2(\mathbb{Z})$ will be the group of integer matrices of determinant 1. It is generated by the matrices

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Definition 2.1. For a positive integer N we define the subgroups

$$\Gamma_0(N) = \left\{ M \in \mathrm{SL}_2(\mathbb{Z}) : M = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N \right\},$$
$$\Gamma_1(N) = \left\{ M \in \mathrm{SL}_2(\mathbb{Z}) : M = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod N \right\},$$
$$\Gamma(N) = \left\{ M \in \mathrm{SL}_2(\mathbb{Z}) : M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}$$

of $\mathrm{SL}_2(\mathbb{Z})$. Here $*$ means an arbitrary entry. The group $\Gamma(N)$ is called the *principal congruence subgroup of level N*.

Definition 2.2. A subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ containing $\Gamma(N)$ for some positive integer N is called a *congruence subgroup*.

Remark 2.3. Since $\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N)$, all of these groups are congruence subgroups.

The group $\Gamma(N)$ is the kernel of the map from $\mathrm{SL}_2(\mathbb{Z})$ to $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ given by reducing the entries modulo N. Therefore, every congruence subgroup has finite index in $\mathrm{SL}_2(\mathbb{Z})$. In the case of the groups $\Gamma(N)$, $\Gamma_1(N)$ and $\Gamma_0(N)$ it is given by the following proposition:

Proposition 2.4 ([DS05], Section 1.2)**.** *For a positive integer N the indices of the groups $\Gamma(N)$, $\Gamma_1(N)$ and $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbb{Z})$ are given by*

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N)] = N^3 \prod_{p|N} \left(1 - \frac{1}{p^2}\right),$$
$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(N)] = N^2 \prod_{p|N} \left(1 - \frac{1}{p^2}\right),$$
$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p|N} \left(1 + \frac{1}{p}\right),$$

where the products are over all positive primes dividing N.

Definition 2.5. Every congruence subgroup Γ acts on $\mathbb{Q} \cup \{\infty\}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{x}{y} = \frac{ax + by}{cx + dy},$$

where we interpret $\frac{x}{0}$ as ∞ for every $x \neq 0$. The orbits under this action are called the *cusps* of Γ. Their number is denoted by $\varepsilon_\infty(\Gamma)$.

Remark 2.6. The action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{Q} \cup \{\infty\}$ is transitive. Since a congruence subgroup Γ has finite index in $\mathrm{SL}_2(\mathbb{Z})$, the number of cusps of Γ is finite.

For the congruence subgroups from Definition 2.1, the numbers ε_∞ are as follows:

Proposition 2.7. *Let N be a positive integer. Then*

$$\varepsilon_\infty(\Gamma(N)) = \begin{cases} \frac{1}{2} N^2 \prod_{p|N} \left(1 - \frac{1}{p^2}\right) & \text{if } N > 2, \\ 3 & \text{if } N = 2, \end{cases}$$
$$\varepsilon_\infty(\Gamma_1(N)) = \begin{cases} 2 & \text{if } N = 2, \\ 3 & \text{if } N = 4, \\ \frac{1}{2} \sum_{d|N} \phi(d)\phi(N/d) & \text{if } N = 3 \text{ or } N > 4, \end{cases}$$
$$\varepsilon_\infty(\Gamma_0(N)) = \sum_{d|N} \phi((d, N/d)).$$

Here the sums are over all positive divisors of N, the product is over the positive primes dividing N and ϕ is the Euler totient function.

For $\Gamma_1(N)$ and $\Gamma_0(N)$ one can use the following two propositions to determine whether two elements of $\mathbb{Q} \cup \{\infty\}$ represent the same cusp:

Proposition 2.8 ([DS05], Proposition 3.8.3)**.** *Let N be a positive integer and let $s = a/c$ and $s' = a'/c'$ with $(a, c) = (a', c') = 1$ be in $\mathbb{Q} \cup \{\infty\}$. Then there exists a matrix $M \in \Gamma_1(N)$ with $Ms = s'$ if and only if there exists an integer j such that*

$$\begin{pmatrix} a' \\ c' \end{pmatrix} = \pm \begin{pmatrix} a + jc \\ c \end{pmatrix} \mod N.$$

Proposition 2.9 ([DS05], Proposition 3.8.3)**.** *Let N be a positive integer and let $s = a/c$ and $s' = a'/c'$ with $(a,c) = (a',c') = 1$ be in $\mathbb{Q} \cup \{\infty\}$. Then there exists a matrix $M \in \Gamma_0(N)$ with $Ms = s'$ if and only if there exist integers j and y with $(y, N) = 1$ such that*

$$\begin{pmatrix} ya' \\ c' \end{pmatrix} = \begin{pmatrix} a + jc \\ yc \end{pmatrix} \mod N.$$

Corollary 2.10. *If N is squarefree, then a set of representatives for the cusps of $\Gamma_0(N)$ is given by $\{1/c : c \mid N, c > 0\}$.*

We finish this section by looking at the stabilizer groups of elements in $\mathbb{Q} \cup \{\infty\}$:

Definition 2.11. Let Γ be a congruence subgroup and let s be in $\mathbb{Q}\cup\{\infty\}$. We let $\Gamma_s \subset \Gamma$ be the stabilizer of s. The *width* of s with respect to Γ is defined as the index of the group $\pm\Gamma_s$ in $\mathrm{SL}_2(\mathbb{Z})_s$. The *width* of a cusp of Γ is defined as the width of any representative (which is independent of the representative because the respective stabilizer subgroups are conjugate).

Proposition 2.12 (see [Sch15], Section 3)**.** *Let N be a positive integer and $s = a/c$ with $(a,c) = 1$ in $\mathbb{Q} \cup \{\infty\}$. Then the width of s with respect to $\Gamma_0(N)$ is $t = N/(c^2, N)$ and the stabilizer of s under the action of $\Gamma_0(N)$ on $\mathbb{Q} \cup \{\infty\}$ is given by*

$$\Gamma_0(N)_{a/c} = \{\pm T_{a/c}^n : n \in \mathbb{Z}\},$$

where

$$T_{a/c} = MT^tM^{-1} = \begin{pmatrix} 1 - act & a^2t \\ -c^2t & 1 + act \end{pmatrix}$$

for any matrix $M \in \mathrm{SL}_2(\mathbb{Z})$ with $M\infty = a/c$.

Proposition 2.13 (see [Sch15], Section 3)**.** *Let N be a positive integer and $s = a/c$ with $(a,c) = 1$ in $\mathbb{Q} \cup \{\infty\}$. Then the width of s with respect to $\Gamma_1(N)$ is given by $t = 1$ if $N = 4$ and $c = 2 \mod 4$ and $t = N/(c,N)$ otherwise. The stabilizer of s under the action of $\Gamma_1(N)$ on $\mathbb{Q} \cup \{\infty\}$ is given by*

$$\Gamma_1(N)_{a/c} = \begin{cases} \{(-T_{a/c})^n : n \in \mathbb{Z}\} & \text{if } N = 4 \text{ and } c = 2 \mod 4, \\ \{T_{a/c}^n : n \in \mathbb{Z}\} & \text{otherwise,} \end{cases}$$

where $T_{a/c}$ is as in Proposition 2.12.

2.2. Scalar-Valued Modular Forms

We let $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the *upper half-plane* and

$$\mathrm{GL}_2(\mathbb{Q})^+ = \left\{M \in \mathbb{Q}^{2\times 2} : \det(M) > 0\right\}.$$

Definition 2.14. The map

$$\left(M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau\right) \mapsto M\tau = \frac{a\tau + b}{c\tau + d} \tag{2.1}$$

defines a group action of $\mathrm{GL}_2^+(\mathbb{Q})$ on $\mathbb{H}$. For a function $f\colon \mathbb{H} \to \mathbb{C}$, an integer k and a matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q})^+$ we define the function $f|_k M\colon \mathbb{H} \to \mathbb{C}$ by

$$f|_k M(\tau) = \det(M)^{k/2}(c\tau + d)^{-k} f(M\tau).$$

Remark 2.15. If M_1 and M_2 are in Γ, then $(f|_k M_1)|_k M_2 = f|_k(M_1 M_2)$.

Suppose f is a meromorphic function on $\mathbb{H}$ with

$$f|_k M(\tau) = \chi(M) f(\tau) \tag{2.2}$$

for all $\tau \in \mathbb{H}$ and all $M \in \Gamma$, where Γ is a congruence subgroup and $\chi\colon \Gamma \to \mathbb{C}^*$ is a character such that $\ker(\chi)$ has finite index in Γ. Let $s \in \mathbb{Q} \cup \{\infty\}$ and $M_s \in \mathrm{SL}_2(\mathbb{Z})$ with $M_s\infty = s$. Then there is a positive integer n such that $M_s T^n M_s^{-1} \in \ker(\chi)$ because Γ has finite index in $\mathrm{SL}_2(\mathbb{Z})$ and $\ker(\chi)$ has finite index in Γ. But then (2.2) implies that

$$f|_k M_s(\tau) = f|_k M_s(\tau + n)$$

for all $\tau \in \mathbb{H}$, i.e. the function $g = f|_k M_s$ is n-periodic. It therefore has a Fourier expansion of the form

$$g(\tau) = f|_k M_s(\tau) = \sum_{m \in \frac{1}{n}\mathbb{Z}} a_m q^m \quad (q^m = e^{2\pi i m\tau}).$$

Remark 2.16. For an n-periodic function g and a rational number m we will often write $[g](m)$ for the Fourier coefficient a_m of g. If m is not in $\frac{1}{n}\mathbb{Z}$ this is defined to be 0.

Definition 2.17. We say that f is *meromorphic at s* if there exists a rational number m_0 such that $[f|_k M_s](m) = 0$ for all $m < m_0$. We call f *holomorphic at s* if we can choose $m_0 \geq 0$ and we say that f *vanishes at s* if we can choose $m_0 > 0$.

Definition 2.18. If f is meromorphic at s, we define the *order of f at s* to be the largest rational number m_0 such that $f|_k M_s(m) = 0$ for all $m < m_0$.

Remark 2.19. It is not difficult to see that these definitions are independent of the choice of M_s.

We can now define modular forms:

Definition 2.20. Let Γ be a congruence subgroup, $\chi\colon \Gamma \to \mathbb{C}^*$ a character such that $\ker(\chi)$ has finite index in Γ and $k \in \mathbb{Z}$. A holomorphic function $f\colon \mathbb{H} \to \mathbb{C}$ such that

1. $f|_k M = \chi(M) f$ for all $M \in \Gamma$ and
2. f is meromorphic at all $s \in \mathbb{Q} \cup \{\infty\}$

is called a *weakly holomorphic modular form of weight k and character χ for* Γ. If f is holomorphic at all $s \in \mathbb{Q} \cup \{\infty\}$, we say that f is a *holomorphic modular form* and if f vanishes at all $s \in \mathbb{Q} \cup \{\infty\}$, we say that f is a *cusp form*. The spaces of weakly holomorphic and holomorphic modular forms of weight k and character χ for Γ are denoted by $M_k^!(\Gamma, \chi)$ and $M_k(\Gamma, \chi)$. The space of cusp forms is denoted by $S_k(\Gamma, \chi)$. If χ is trivial, then we just write $M_k^!(\Gamma)$, $M_k(\Gamma)$ and $S_k(\Gamma)$.

Remark 2.21. Whether a function satisfying the first condition in Definition 2.20 is meromorphic (resp. holomorphic or vanishing) at $s \in \mathbb{Q} \cup \{\infty\}$ only depends on the orbit of s under the action of Γ, i.e. on the cusp represented by s. We can therefore define the notion of meromorphy, holomorphy and vanishing of f at a cusp and the second condition in Definition 2.20 is equivalent to f being meromorphic (resp. being holomorphic or vanishing) at all cusps.

Remark 2.22. The spaces $M_k(\Gamma, \chi)$ and $S_k(\Gamma, \chi)$ are finite-dimensional and they are trivial if $k < 0$.

In this thesis we will only need the congruence subgroups $\Gamma_0(N)$ and $\Gamma_1(N)$ for $N \geq 1$. A character χ of the multiplicative group $(\mathbb{Z}/N\mathbb{Z})^*$ for some $N \geq 1$ induces a character of $\Gamma_0(N)$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \chi(d).$$

Similarly, a character ψ of the additive group $\mathbb{Z}/N\mathbb{Z}$ induces a character of $\Gamma_1(N)$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \psi(b).$$

We also denote these induced characters by χ and ψ and note that their kernels have finite index in $\Gamma_0(N)$ and $\Gamma_1(N)$.

Remark 2.23. Let Γ be a congruence subgroup and let f be a non-zero element of $M_k^!(\Gamma, \chi)$ for some integer k and some character χ. For $s \in \mathbb{Q} \cup \{\infty\}$ and $\tau \in \mathbb{H}$, the orders of f at s and τ equal the orders of f at Ms and $M\tau$ for all $M \in \Gamma$.

Therefore, the equation in the following proposition is well-defined:

Proposition 2.24. *Let Γ be a congruence subgroup, $k \in \mathbb{Z}$ and χ a character of Γ such that* $\ker(\chi)$ *has finite index in Γ. Let $f \in M_k^!(\Gamma, \chi)$ be non-zero. Then*

$$\sum_{s \in \Gamma \backslash (\mathbb{Q} \cup \{\infty\})} [\mathrm{SL}_2(\mathbb{Z})_s : \pm\Gamma_s] \operatorname{ord}_s(f) + \sum_{\tau \in \Gamma \backslash \mathbb{H}} \frac{2}{|\pm\Gamma_\tau|} \operatorname{ord}_\tau(f) = \frac{k}{12}[\mathrm{SL}_2(\mathbb{Z}) : \pm\Gamma]. \tag{2.3}$$

Proof. If f is a holomorphic modular form and χ is trivial, then this is Theorem 4.1 in Appendix I of [HBJ94]. To get rid of the character χ we replace f by a sufficiently high power of f and to obtain a holomorphic modular form we multiply f with a sufficiently high power of $\eta(\tau)^{24}$, which is a cusp form of weight 12 with trivial character for $\mathrm{SL}_2(\mathbb{Z})$. □

2.2.1. Newforms for $\Gamma_0(N)$

Throughout this subsection N and k are integers with $N \geq 1$.

Let M be a positive divisor of N. Since residue classes modulo N give rise to residue classes modulo M, a character χ of $(\mathbb{Z}/M\mathbb{Z})^*$ induces a character of $(\mathbb{Z}/N\mathbb{Z})^*$, which we also denote by χ. Moreover, $\Gamma_0(N) \subset \Gamma_0(M)$ and hence $S_k(\Gamma_0(M), \chi)$ is a subspace of $S_k(\Gamma_0(N), \chi)$. We can also define another inclusion of $S_k(\Gamma_0(M), \chi)$ into $S_k(\Gamma_0(N), \chi)$, namely the following:

Proposition 2.25 ([Miy89], Lemma 4.6.1)**.** *Let $f \in S_k(\Gamma_0(M), \chi)$ and let $l = N/M$ and $\delta_l = \left(\begin{smallmatrix} l & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then the function*

$$f|_k\delta_l(\tau) = l^{k/2} f(l\tau),$$

is in $S_k(\Gamma_0(N), \chi)$.

Definition 2.26. Let $\chi\colon (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}$ be a character of conductor m. The space $S_k^{\mathrm{old}}(\Gamma_0(N), \chi)$ of *oldforms* of weight k and character χ for $\Gamma_0(N)$ is the subspace of $S_k(\Gamma_0(N), \chi)$ spanned by the functions

$$\bigcup_{\substack{M|N \\ m|M \\ M<N}} \bigcup_{l|N/M} \{f|_k\delta_l : f \in S_k(\Gamma_0(M), \chi)\}.$$

Definition 2.27. Let Γ be a congruence subgroup and let χ be a character such that $\ker(\chi)$ has finite index in Γ. The *Petersson inner product* $(\cdot,\cdot)\colon S_k(\Gamma,\chi) \times S_k(\Gamma,\chi) \to \mathbb{C}$ is given by

$$(f, g) = \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \pm\Gamma]} \int_{X(\Gamma)} f(\tau)\overline{g(\tau)}\,\mathrm{Im}(\tau)^k d\mu(\tau),$$

where $\int_{X(\Gamma)}$ is defined as in Section 5.4 in [DS05] and

$$d\mu(\tau) = \frac{dxdy}{y^2}, \quad \tau = x + iy \in \mathbb{H}.$$

Definition 2.28. Let $\chi\colon (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}$ be a character. The space $S_k^{\mathrm{new}}(\Gamma_0(N), \chi)$ of *newforms* of weight k and character χ for $\Gamma_0(N)$ is the orthogonal complement of $S_k^{\mathrm{old}}(\Gamma_0(N), \chi)$ in $S_k(\Gamma_0(N), \chi)$ with respect to the Petersson inner product.

Definition 2.29. Let χ be a character of $(\mathbb{Z}/N\mathbb{Z})^*$, $n \in \mathbb{Z}$ and

$$f(\tau) = \sum_{m=1}^{\infty} a_m q^m \in S_k(\Gamma_0(N), \chi).$$

We define a function $T_n f$ on $\mathbb{H}$ by

$$(T_n f)(\tau) = \sum_{m=1}^{\infty} b_m q^m$$

with

$$b_m = \sum_{d|(m,n)} \chi(d) d^{k-1} a_{mn/d^2}.$$

One can show that $T_n f$ is in $S_k(\Gamma_0(N), \chi)$ (see e.g. [DS05], Chapter 5) and we therefore obtain endomorphisms T_n of $S_k(\Gamma_0(N), \chi)$. These are called *Hecke operators.*

Definition 2.30. Let N be a positive integer and let χ be a character of $(\mathbb{Z}/N\mathbb{Z})^*$. An element

$$f = \sum_{m=1}^{\infty} a_m q^m \in S_k^{\text{new}}(\Gamma_0(N), \chi)$$

is called a *primitive form* if it is a simultaneous eigenform for all Hecke operators T_n and if $a_1 = 1$.

Proposition 2.31 (see [DS05], Theorem 5.8.2)**.** *The set of primitive forms is an orthogonal basis of $S_k^{\text{new}}(\Gamma_0(N), \chi)$.*

We also recall the definition of Aktin-Lehner involutions:

Definition 2.32. Let χ be a character of $(\mathbb{Z}/N\mathbb{Z})^*$. For a positive divisor $M \mid N$ with $(M, N/M) = 1$ we choose integers $n_1, \ldots, n_4$ such that

$$W_M^N = \begin{pmatrix} Mn_1 & n_2 \\ Nn_3 & Mn_4 \end{pmatrix}$$

has determinant M. Then $f \mapsto f|_k W_M^N$ defines a linear map from $S_k(\Gamma_0(N), \chi)$ to $S_k(\Gamma_0(N), {}^M\chi)$ for a character ${}^M\chi$ of $(\mathbb{Z}/N\mathbb{Z})^*$. These are called *Atkin-Lehner involutions* and map the subspace $S_k^{\text{new}}(\Gamma_0(N), \chi)$ to $S_k^{\text{new}}(\Gamma_0(N), {}^M\chi)$. Different choices of $n_1, \ldots, n_4$ are related by an element of $\Gamma_0(N)$ and the corresponding maps $f \to f|_k W_M^N$ thus only differ by a non-zero constant factor.

Remark 2.33. A proof for the assertions in the definition as well as the definition of the character ${}^M\chi$ can be found in [Asa76], Sections 1.1. and 1.2. If χ is a real character (i.e. $\chi(n)$ is -1 or 1 for all $n \in (\mathbb{Z}/N\mathbb{Z})^*$) then ${}^M\chi = \chi$.

Remark 2.34. As the name Atkin-Lehner involution indicates, we have

$$(f|_k W_M^N)|_k W_M^N = f$$

for all $f \in S_k(\Gamma_0(N), \chi)$.

For the rest of this subsection we assume that N is squarefree. Then every positive divisor m of N satisfies $(m, N/m) = 1$ and we can uniquely write a character χ of $(\mathbb{Z}/N\mathbb{Z})^*$ as

$$\chi = \chi_m \cdot \chi_{m'} \quad (m' = N/m)$$

for characters χ_m of $(\mathbb{Z}/m\mathbb{Z})^*$ and $\chi_{m'}$ of $(\mathbb{Z}/m'\mathbb{Z})^*$ (this follows from the Chinese remainder theorem, which implies that $(\mathbb{Z}/N\mathbb{Z})^* \cong (\mathbb{Z}/m\mathbb{Z})^* \times (\mathbb{Z}/m'\mathbb{Z})^*$).

Proposition 2.35 ([Asa76], Theorems 1 and 2)**.** *Let χ be a character of $(\mathbb{Z}/N\mathbb{Z})^*$, M be a positive divisor of N and let*

$$f(\tau) = \sum_{n=1}^{\infty} a_n q^n$$

be a primitive form in $S_k^{\mathrm{new}}(\Gamma_0(N), \chi)$. Then the Fourier expansion of $f|_k W_M^N$ is given by

$$f|_k W_M^N = \lambda \sum_{n=1}^{\infty} a_n^{(M)} q^n$$

where $a_n^{(M)}$ is defined by

$$\begin{cases} a_n^{(M)} = \overline{\chi_M}(n) a_n & \text{if } (n, M) = 1, \\ a_n^{(M)} = \chi_{M'}(n) \overline{a_n} & \text{if } (n, M') = 1, \\ a_{nm}^{(M)} = a_n^{(M)} a_m^{(M)} & \text{if } (n, m) = 1, \end{cases}$$

and λ is given by

$$\lambda = \chi(Mn_4 - M'n_3) \prod_{p|M} \chi_p(M/p)\lambda_p \quad (M' = N/M)$$

with

$$\lambda_p = \begin{cases} G(\chi_p) p^{-k/2} \overline{a_p} & \text{if } \chi_p \text{ is primitive,} \\ -p^{1-k/2} \overline{a_p} & \text{if } \chi_p \text{ is principal.} \end{cases}$$

Here the product runs over all positive primes dividing M and $G(\chi_p)$ is the Gauss sum

$$G(\chi_p) = \sum_{h=1}^{p-1} \chi_p(h) e(h/p).$$

Remark 2.36. One can choose W_M^N to be of the form

$$\omega_M^N = \begin{pmatrix} Mn_1 & n_2 \\ -N & M \end{pmatrix}.$$

In this case

$$\lambda = \chi_M(M')\chi_{M'}(M) \prod_{p|M} \chi_p(M/p)\lambda_p.$$

2.2.2. Fourier Expansions of Cusp Forms for $\Gamma_0(N)$

We continue to let k be an integer.

Suppose we are given a cusp form

$$f(\tau) = \sum_{n=1}^{\infty} a_n q^n \in S_k(\Gamma_0(M), \chi)$$

for a squarefree integer M and a character χ of $(\mathbb{Z}/M\mathbb{Z})^*$. Recall that then $g = f|_k\delta_l$ is also an element of $S_k(\Gamma_0(lM),\chi)$ for all positive integers l. If l is squarefree and coprime to M, then lM is also squarefree and the Fourier expansions of the Atkin-Lehner involutions of f are closely related to the Fourier expansions of $g|_kM_c$ where M_c is a matrix with $M_c = 1/c$ for a squarefree integer c.

Proposition 2.37. *Let M be a positive squarefree integer and $\chi\colon (\mathbb{Z}/M\mathbb{Z})^* \to \mathbb{C}$ a character. Let f be a cusp form in $S_k(\Gamma_0(M),\chi)$, l and c positive squarefree integers with $(l,M) = 1$ and $g = f|_k\delta_l$. Define $m_c = (c,M)$, $m'_c = M/m_c$, $l_c = (l,c)$ and $l'_c = l/l_c$ and let*

$$M_c = \begin{pmatrix} 1 & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

with $d = 0 \mod m'_c l'_c$. Then

$$g|_kM_c(\tau) = \chi_{m_c}^{-1}(m'_cl'_c)\chi_{m'_c}(-c/(l_cm_c))(l_c/(m'_cl'_c))^{k/2} f|_k\omega^M_{m'_c}(l_c\tau/(l'_cm'_c)).$$

Proof. Recall that $\omega^M_{m'_c} = \begin{pmatrix} m'_cn_1 & n_2 \\ -M & m'_c \end{pmatrix}$ with $n_1, n_2 \in \mathbb{Z}$ and $\det(\omega^M_{m'_c}) = m'_c$. Define

$$A = \begin{pmatrix} l'_cm'_c + bm_cl_c & n_1bl_c - n_2l'_c \\ cm'_c/l_c + dm_c/l'_c & n_1d/l'_c - n_2c/l_c \end{pmatrix}.$$

Then $A \in \Gamma_0(M)$ and

$$\delta_lM_c = \frac{1}{m'_c}A\omega^M_{m'_c}\begin{pmatrix} l_c & 0 \\ 0 & m'_cl'_c \end{pmatrix}.$$

This implies that

$$g|_kM_c(\tau) = \chi(n_1d/l'_c - n_2c/l_c)(l_c/(m'_cl'_c))^{k/2} f|_k\omega^M_{m'_c}(l_c\tau/(l'_cm'_c)),$$

so it only remains to compute $\chi(n_1d/l'_c - n_2c/l_c)$. This can be decomposed as

$$\chi(n_1d/l'_c - n_2c/l_c) = \chi_{m_c}(n_1d/l'_c - n_2c/l_c)\chi_{m'_c}(n_1d/l'_c - n_2c/l_c).$$

Since l is coprime to M, both l_c and l'_c are invertible modulo m_c and m'_c. Therefore

$$\chi_{m_c}(n_1d/l'_c - n_2c/l_c) = \chi_{m_c}(n_1d)\chi_{m_c}^{-1}(l'_c)$$

and

$$\chi_{m'_c}(n_1d/l'_c - n_2c/l_c) = \chi_{m'_c}(-n_2c)\chi_{m'_c}^{-1}(l_c).$$

To finish the proof we observe that $d = 1 \mod c$, $n_1 = {m'_c}^{-1} \mod m_c$ and that $n_2 = m_c^{-1} \mod m'_c$. $\square$

Later, for a given squarefree positive integer N and a character $\chi\colon (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}$, we need to know the Fourier expansions $f_j|_kM_c$ for all positive $c \mid N$ for a basis $(f_1,\ldots,f_r)$ of $S_k(\Gamma_0(N),\chi)$. These can now be computed as follows: Let m be the conductor of χ. The set

$$B = \bigcup_{\substack{M|N \\ m|M}} \bigcup_{l|N/M} \{f|_k\delta_l : f \text{ is a primitive form in } S_k^{\mathrm{new}}(\Gamma_0(M),\chi)\}$$

is a basis of $S_k(\Gamma_0(N), \chi)$ because of Proposition 2.31. But for every $f \in B$, the Fourier expansion of $f|_k M_c$ can then be computed with Propositions 2.37 and 2.35 once we know the Fourier expansion of f. The Fourier expansions of a basis of f can be obtained by computing the eigenvalues of the Hecke operators which is usually done using so-called *modular symbols*. We will not go into further details because for those cases that are relevant to us this is implemented in many computer algebra systems (e.g. in SageMath [Sag15], which we have used, one can simply use the `Newforms` command to obtain the first Fourier coefficients of all primitive forms in a specified space of cusp forms).

2.2.3. Eta Quotients

The Dedekind eta function $\eta\colon \mathbb{H} \to \mathbb{C}$ is defined by

$$\eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n).$$

It is holomorphic on $\mathbb{H}$ with no zeros and satisfies

$$\eta(\tau + 1) = \eta(\tau), \quad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\eta(\tau),$$

where $\sqrt{\cdot}$ is such that $-\pi/2 < \arg\sqrt{\cdot} \leq \pi/2$. More generally, due to Rademacher (see the last Theorem of Chapter 9 in [Rad73]), for a matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $c > 0$ it satisfies

$$\eta(M\tau) = \varepsilon(M)\sqrt{c\tau + d}\eta(\tau)$$

with

$$\varepsilon(M) = \begin{cases} \left(\frac{d}{c}\right) e((-3c + bd(1 - c^2) + c(a + d))/24) & \text{if } c \text{ is odd}, \\ \left(\frac{c}{d}\right) e((3d - 3 + ac(1 - d^2) + d(b - c))/24) & \text{if } c \text{ is even}. \end{cases}$$

To build eta quotients we also need to consider the functions $\eta_k(\tau) = \eta(k\tau)$ for positive integers k. They satisfy the following transformation formula:

Proposition 2.38 ([Sch09], Proposition 6.2)**.** *Let* $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ *with* $c > 0$. *Let* $r, s, t \in \mathbb{Z}$ *with* $r, t > 0$ *and*

$$rt = k, \quad r \mid c, \quad k \mid (dr - cs).$$

Then

$$\eta_k(M\tau) = \varepsilon\left(\begin{pmatrix} at & br - as \\ c/r & (dr - cs)/k \end{pmatrix}\right) \frac{1}{\sqrt{t}} \sqrt{c\tau + d}\,\eta\left(\frac{r\tau + s}{t}\right).$$

Definition 2.39. Let $k_1, \ldots, k_n$ be distinct positive integers and $r_1, \ldots, r_n$ be arbitrary integers. We define the *eta quotient* $\eta_{k_1^{r_1} \cdots k_n^{r_n}}$ by

$$\eta_{k_1^{r_1} \cdots k_n^{r_n}}(\tau) = \prod_{j=1}^{n} \eta_{k_j}^{r_j}(\tau).$$

The *cycle shape* of $\eta_{k_1^{r_1} \cdots k_n^{r_n}}$ is the symbol $k_1^{r_1} \cdots k_n^{r_n}$.

Since η has neither poles nor zeros on $\mathbb{H}$, this defines a holomorphic function on $\mathbb{H}$. For suitable choices of the integers k_j and r_j this is a weakly holomorphic modular form:

Proposition 2.40 ([Bor00], Theorem 6.2)**.** *Suppose we are given positive integers N and m with $4 \nmid N$ and integers r_n for $n \mid N$ with $m/\prod_{n|N} n^{r_n}$ a rational square. Suppose that $\sum_{n|N} r_n n \in 24\mathbb{Z}$ and $N \sum_{n|N} r_n/n \in 24\mathbb{Z}$. Then the eta quotient*

$$\prod_{n|N} \eta(n\tau)^{r_n}$$

is a weakly holomorphic modular form of weight $k = \sum_{n|N} r_n/2$ for $\Gamma_0(N)$ with character χ_m given by

$$\chi_m\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \left(\frac{d}{m}\right).$$

Remark 2.41. It can happen that $k \notin \mathbb{Z}$. In this case the eta quotient is a weakly holomorphic modular form of half-integral weight. They will not appear in this thesis as they cannot have squarefree level.

Similarly, we can also build weakly holomorphic modular forms for $\Gamma_1(N)$:

Proposition 2.42 ([DHS15], Proposition 5.1)**.** *Let N be a positive integer. Take integers r_n for $n \mid N$ such that $\frac{N}{24}\sum_{n|N} r_n n$ and $\frac{N}{24}\sum_{n|N} r_n/n$ are integers and $\sum_{n|N} r_n$ is even. Then the eta quotient*

$$\prod_{n|N} \eta(n\tau)^{r_n}$$

is a weakly holomorphic modular form for $\Gamma_1(N)$ of weight $k = \sum_{n|N} r_n/2$ and character

$$\chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = e\left(\frac{b}{24}\sum_{n|N} r_n n\right).$$

We finish this section by showing how one can check whether an eta quotient is a cusp form:

Proposition 2.43. *Suppose N is squarefree. Then the eta quotient from the last proposition is a cusp form if and only if*

$$\sum_{n|N} \frac{(n,c)}{(n,c')} r_n > 0 \quad (c' = N/c)$$

for all $c \mid N$.

Proof. Let $a/c \in \mathbb{Q} \cup \{\infty\}$ and $c' = N/(N,c)$. We can suppose that $c > 0$, because every cusp of $\Gamma_1(N)$ is equivalent to one with $c > 0$. We choose b and d such that the matrix $M_{a/c} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is in $\mathrm{SL}_2(\mathbb{Z})$ and $c' \mid d$. We want to apply Proposition 2.38 to compute $\eta_n(M_{a/c}\tau)$ and note that we can choose $r = (c,n)$, $t = (c',n)$ and $s = 0$. Therefore,

$$\eta_n(M_{a/c}\tau) = \lambda\sqrt{c\tau + d}\,\eta\left(\frac{(n,c)}{(n,c')}\tau\right)$$

for some nonzero complex number λ. It follows that the smallest power of q occuring in the Fourier expansion of

$$\prod_{n|N} \eta(n\tau)^{r_n}|_k M_{a/c}$$

is

$$\frac{1}{24}\sum_{n|N}\frac{(n,c)}{(n,c')}r_n,$$

so that the eta quotient vanishes at a/c if and only if this sum is positive. Since (n,c) and (n,c') only depend on (c,N) rather than c it suffices to consider those c that divide N. □

2.3. Modular Forms for the Weil Representation

In this section D is always a discriminant form of even signature with finite quadratic form Q and bilinear form $(\cdot,\cdot)$.

2.3.1. The Weil Representation

Let $\mathbb{C}[D]$ be the group algebra of D, i.e. the $\mathbb{C}$-vector space of formal linear combinations

$$\sum_{\gamma\in D} a_\gamma \mathfrak{e}_\gamma$$

with $a_\gamma \in \mathbb{C}$, basis $\{\mathfrak{e}_\gamma : \gamma \in D\}$ and multiplication defined by $\mathfrak{e}_\beta\mathfrak{e}_\gamma = \mathfrak{e}_{\beta+\gamma}$. We also define a scalar product (which is linear in the first and antilinear in the second argument) on $\mathbb{C}[D]$ by $(\mathfrak{e}_\beta, \mathfrak{e}_\gamma) = \delta_{\beta\gamma}$.

Definition 2.44. Let $T = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ be the usual generators of $\mathrm{SL}_2(\mathbb{Z})$. We define

$$\begin{aligned}\rho_D(T)\mathfrak{e}_\gamma &= e(-Q(\gamma))\mathfrak{e}_\gamma,\\ \rho_D(S)\mathfrak{e}_\gamma &= \frac{e(\mathrm{sign}(D)/8)}{\sqrt{|D|}}\sum_{\beta\in D} e((\beta,\gamma))\mathfrak{e}_\beta.\end{aligned}$$

This defines a unitary representation $\rho_D\colon \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}(\mathbb{C}[D])$ of $\mathrm{SL}_2(\mathbb{Z})$, called the *Weil representation* of D.

Remark 2.45. To check that ρ_D is well-defined one needs to calculate that

$$\rho_D(S)^2 = (\rho_D(S)\rho_D(T))^3 = \rho_D(-I_2),$$

where I_2 is the 2×2 identity matrix; this is an easy exercise. Checking that the defined representation is unitary is another short calculation we omit.

Remark 2.46. If D has odd signature, one can still define ρ_D as above. However, this only defines a projective representation. To be more precise, it gives a well-defined map into $\mathrm{GL}(\mathbb{C}[D])/\pm 1$, which is why one has to work with the metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$ instead of $\mathrm{SL}_2(\mathbb{Z})$ if one wants ρ_D to be a representation. The action can still be computed but the formulas become much more tedious. We omit the details because the discriminant forms we work with in this thesis have squarefree level and therefore even signature. The reader who is interested in the formulas for the general case should consult [Str13].

Remark 2.47. We also need the dual representation ρ_D^* of ρ_D. Since ρ_D is unitary, ρ_D^* is given by the complex conjugate of ρ_D, i.e $\rho_D^* = \overline{\rho_D}$. The formulas for ρ_D therefore immediately give similar formulas for ρ_D^*. We call ρ_D^* the *dual Weil representation.*

An automorphism σ of D can be extended linearly to an automorphism of $\mathbb{C}[D]$.

Proposition 2.48. *The Weil representation commutes with* $\mathrm{Aut}(D)$*, i.e.*

$$\rho_D(M)\mathfrak{e}_{\sigma(\gamma)} = \sigma(\rho_D(M)\mathfrak{e}_\gamma)$$

for all $M \in \mathrm{SL}_2(\mathbb{Z})$ *and* $\sigma \in \mathrm{Aut}(D)$*.*

Proof. It suffices to prove this for the generators T and S of $\mathrm{SL}_2(\mathbb{Z})$. For T this is obvious and for S we note that

$$\begin{aligned}\rho_D(S)\mathfrak{e}_{\sigma(\gamma)} &= \frac{e(\mathrm{sign}(D)/8)}{\sqrt{|D|}} \sum_{\beta\in D} e((\beta,\sigma(\gamma)))\mathfrak{e}_\beta \\ &= \frac{e(\mathrm{sign}(D)/8)}{\sqrt{|D|}} \sum_{\beta\in D} e((\sigma^{-1}(\beta),\gamma))\mathfrak{e}_\beta \\ &= \frac{e(\mathrm{sign}(D)/8)}{\sqrt{|D|}} \sum_{\beta\in D} e((\beta,\gamma))\mathfrak{e}_{\sigma(\beta)}.\end{aligned}$$

□

In the rest of this subsection we state several formulas for the action of various matrices in $\mathrm{SL}_2(\mathbb{Z})$ in the Weil representation.

Proposition 2.49 ([Sch09], Proposition 4.3)**.** *Let N be a positive integer such that the level of D divides N. Then $\Gamma(N)$ acts trivially in the Weil representation.*

Definition 2.50. Let D have level N. We define the character $\chi_D\colon (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}$ by

$$\chi_D(a) = \left(\frac{a}{|D|}\right) e((a-1)\,\mathrm{sign}(D_2)/8).$$

Remark 2.51. If N is squarefree, then $\chi_D(a) = \left(\frac{a}{|D|}\right)$.

Proposition 2.52 ([Sch09], Proposition 4.5)**.** *Let N be a positive integer such that the level of D divides N. Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Then*

$$\rho_D(M)\mathfrak{e}_\gamma = \chi_D(M)e(-bdQ(\gamma))\mathfrak{e}_{d\gamma}.$$

The formula for the action of a general elements $M \in \mathrm{SL}_2(\mathbb{Z})$ has been derived by Scheithauer (see [Sch09], Theorem 4.7). Since it is quite lengthy, we only state it for discriminant forms of squarefree level:

Theorem 2.53 ([Sch06], Theorem 6.3)**.** *Let D be a discriminant form of squarefree level and let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Then M acts in the Weil representation as*

$$\rho_D(M)\mathfrak{e}_\gamma = \xi(M)\frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{\beta \in D^c} e(-aQ_c(\beta))e(-b(\beta,\gamma))e(-bdQ(\gamma))\mathfrak{e}_{d\gamma+\beta},$$

where

$$\xi(M) = e(\mathrm{sign}(D)/8)\left(\frac{d}{|D_c|}\right)\left(\frac{c}{|D^c|}\right)\prod_{p|c} e(-\mathrm{sign}(D_p)/8)$$

with the product over all positive primes dividing c and $Q_c(\beta)$ is defined as $c \cdot Q(\mu)$, where μ is an element in D with $\beta = c\mu$.

2.3.2. Modular Forms for the Weil Representation

Assume that $F = \sum_{\gamma \in D} F_\gamma \colon \mathbb{H} \to \mathbb{C}[D]$ is a holomorphic function (i.e. every F_γ is holomorphic) with

$$F(T\tau) = \rho_D(T)F(\tau)$$

for $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Then

$$F_\gamma(\tau + 1) = e(-Q(\gamma))F_\gamma(\tau),$$

which implies that the function $e(Q(\gamma)\tau)F_\gamma(\tau)$ is 1-periodic. Thus, F_γ has a Fourier expansion of the form

$$F_\gamma(\tau) = \sum_{n \in \mathbb{Z} - Q(\gamma)} a_\gamma(n)q^n.$$

If $F = \sum_{\gamma \in D} F_\gamma \colon \mathbb{H} \to \mathbb{C}[D]$ is a holomorphic function with

$$F(T\tau) = \rho_D^*(T)F(\tau),$$

then a similar argument shows that each F_γ has a Fourier expansion of the form

$$F_\gamma(\tau) = \sum_{n \in \mathbb{Z} + Q(\gamma)} a_\gamma(n)q^n.$$

Definition 2.54. Let $F = \sum_{\gamma \in D} F_\gamma \mathfrak{e}_\gamma \colon \mathbb{H} \to \mathbb{C}[D]$ be a holomorphic function and k an integer. We say that F is a *weakly holomorphic modular form of weight k for ρ_D (resp. ρ_D^*)* if

1. $F(M\tau) = (c\tau + d)^k \rho(M)F$ for $\rho = \rho_D$ (resp. $\rho = \rho_D^*$) for all $M \in \mathrm{SL}_2(\mathbb{Z})$.

2. F is meromorphic at ∞, meaning that every F_γ has a Fourier expansion

$$F_\gamma(\tau) = \sum_{n\in\mathbb{Z}-Q(\gamma)} a_\gamma(n)q^n \quad \left(\text{resp. } F_\gamma(\tau) = \sum_{n\in\mathbb{Z}+Q(\gamma)} a_\gamma(n)q^n\right)$$

with $a_\gamma(n) = 0$ for all but finitely many negative n.

We say that F is a *holomorphic modular form of weight k for ρ_D (resp. ρ_D^*)* if $a_\gamma(n) = 0$ for all $\gamma \in D$ and $n < 0$ and a *cusp form of weight k for ρ_D (resp. ρ_D^*)* if $a_\gamma(n) = 0$ for all $\gamma \in D$ and $n \leq 0$.

Remark 2.55. We denote the sets of weakly holomorphic modular forms, holomorphic modular forms and cusp forms of weight k for ρ_D by $M_k^!(\rho_D), M_k(\rho_D)$ and $S_k(\rho_D)$ (and similarly for ρ_D^*). Note that all of these sets are $\mathbb{C}$-vector spaces.

Remark 2.56. The matrix $S^2 = -I_2$ acts in the Weil representation by sending $\mathfrak{e}_\gamma$ to $e(\operatorname{sign}(D)/4)\mathfrak{e}_{-\gamma}$. Therefore, if $F = \sum_{\gamma\in D} F_\gamma \mathfrak{e}_\gamma$ is in $M_k^!(\rho_D)$, then

$$F_\gamma = (-1)^k e(\operatorname{sign}(D)/4)F_{-\gamma}.$$

Proposition 2.57. *Let $F = \sum_{\gamma\in D} F_\gamma \mathfrak{e}_\gamma$ be in $M_k^!(\rho_D)$, $M_k(\rho_D)$ or $S_k(\rho_D)$. Then so is $\sigma(F) = \sum_{\gamma\in D} F_\gamma \mathfrak{e}_{\sigma(\gamma)}$.*

Proof. This follows immediately from Proposition 2.48. □

Definition 2.58. A weakly holomorphic modular form F for ρ_D is called *symmetric* if $\sigma(F) = F$ for all $\sigma \in \operatorname{Aut}(D)$.

The components of a modular form for ρ_D are scalar-valued modular forms by the following proposition:

Proposition 2.59. *Let N be the level of D and $k \in \mathbb{Z}$. Let $F = \sum_{\gamma\in D} F_\gamma \mathfrak{e}_\gamma$ be in $M_k^!(\rho_D)$. Then $F_\gamma \in M_k^!(\Gamma_1(N), \chi_\gamma)$, where χ_γ is the character of $\mathbb{Z}/N\mathbb{Z}$ defined by $\chi_\gamma(b) = e(-bQ(\gamma))$. Moreover, if F is a holomorphic modular form (resp. a cusp form), then so is F_γ.*

Proof. Since F is holomorphic on $\mathbb{H}$, so is every F_γ. The transformation formula for F_γ follows from the one for F and Proposition 2.52. The last assertion of the proposition is true because $F_\gamma|_k M$ is a linear combination of the components F_β ($\beta \in D$) for all $M \in \operatorname{SL}_2(\mathbb{Z})$. □

Similarly one can see that the component F_0 is a modular form for the larger group $\Gamma_0(N)$:

Proposition 2.60. *Let k be an integer and let $F = \sum_{\gamma\in D} F_\gamma \mathfrak{e}_\gamma$ be in $M_k^!(\rho_D), M_k(\rho_D)$ or $S_k(\rho_D)$. Then $F_0 \in M_k^!(\Gamma_0(N), \chi_D), M_k(\Gamma_0(N), \chi_D)$ or $S_k(\Gamma_0(N), \chi_D)$ where N is the level of D.*

Corollary 2.61. *The spaces $M_k(\rho_D)$ have finite dimension for all k. If $k < 0$, then $M_k(\rho_D) = \{0\}$.*

Definition 2.62. Let k be an integer. For an element $F \in M_k^!(\rho_D)$ with

$$F(\tau) = \sum_{\gamma \in D} \sum_{n \in \mathbb{Z} - Q(\gamma)} a_\gamma(n) q^n \mathfrak{e}_\gamma$$

we define the *principal part* of F to be the Fourier polynomial

$$F(\tau) = \sum_{\gamma \in D} \sum_{\substack{n \in \mathbb{Z} - Q(\gamma) \\ n<0}} a_\gamma(n) q^n \mathfrak{e}_\gamma.$$

Remark 2.63. Because of Corollary 2.61 an element $F \in M_k^!(\rho_D)$ is uniquely determined by its principal part if k is a negative integer.

Because of Proposition 2.59 we can compute $F_\gamma|_k M$ for matrices $M \in \Gamma_1(N)$. The following proposition gives a formula for $F_\gamma|_k M$ for arbitrary $M \in \mathrm{SL}_2(\mathbb{Z})$ if the level of D is squarefree:

Proposition 2.64 ([Sch06], Theorem 6.3)**.** *Let k be an integer, assume that D has squarefree level and let $F = \sum_{\gamma \in D} F_\gamma \mathfrak{e}_\gamma$ be in $M_k^!(\rho_D)$. Then*

$$F_\gamma|_k M(\tau) = \xi(M) \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{\beta \in D^c} e(-dQ_c(\beta)) e(-b(\beta,\gamma)) e(-abQ(\gamma)) F_{a\gamma+\beta}$$

for all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Here ξ is the root of unity from Theorem 2.53.

Proof. We have $F_\gamma(\tau) = (F(\tau), \mathfrak{e}_\gamma)$ and hence

$$\begin{aligned}
F_\gamma|_k M(\tau) &= (c\tau + d)^{-k} (F(M\tau), \mathfrak{e}_\gamma) \\
&= (\rho_D(M) F(\tau), \mathfrak{e}_\gamma) \\
&= \sum_{\alpha \in D} F_\alpha(\tau) (\rho_D(M) \mathfrak{e}_\alpha, \mathfrak{e}_\gamma) \\
&= \sum_{\alpha \in D} F_\alpha(\tau) (\mathfrak{e}_\alpha, \rho_D(M^{-1}) \mathfrak{e}_\gamma) \\
&= \sum_{\substack{\alpha \in D \\ \beta \in D^c}} F_\alpha(\tau) \left(\mathfrak{e}_\alpha, \xi(M^{-1}) \frac{\sqrt{|D_c|}}{\sqrt{|D|}} e(dQ_c(\beta)) e(b(\beta,\gamma)) e(abQ(\gamma)) \mathfrak{e}_{a\gamma+\beta} \right) \\
&= \overline{\xi}(M^{-1}) \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{\beta \in D^c} e(-dQ_c(\beta)) e(-b(\beta,\gamma)) e(-abQ(\gamma)) F_{a\gamma+\beta},
\end{aligned}$$

where we have used that ρ_D is unitary and Theorem 2.53. It therefore remains to show that $\overline{\xi}(M^{-1}) = \xi(M)$. The root of unity $\overline{\xi}(M^{-1})$ is given by

$$e(-\operatorname{sign}(D)/8) \left(\frac{d}{|D_c|} \right) \left(\frac{-c}{|D^c|} \right) \prod_{p|c} e(\operatorname{sign}(D_p)/8)$$

and hence the quotient of $\overline{\chi}(M^{-1})$ and $\chi(M)$ is given by

$$e(-\operatorname{sign}(D)/4)\left(\frac{-1}{|D^c|}\right)\prod_{p|c} e(\operatorname{sign}(D_p)/4). \tag{2.4}$$

Since N is squarefree, the set D^c is equal to $D_{c'}$ where $c' = N/c$ and we can therefore write (2.4) as

$$e(-\operatorname{sign}(D_{c'})/4)\left(\frac{-1}{|D_{c'}|}\right)$$

which is equal to 1 (see [Sch06], Section 6). □

2.3.3. Maps between Spaces of Modular Forms

We have seen that the components of a modular form F for ρ_D are scalar-valued modular forms. Conversely, one can use scalar-valued modular forms to obtain modular forms for ρ_D as follows:

Proposition 2.65 ([Sch15], Theorem 3.1)**.** *Let k be an integer, N the level of D, H an isotropic subset of D which is invariant under $(\mathbb{Z}/N\mathbb{Z})^*$ as a set and $f \in M_k^!(\Gamma_0(N), \chi_D)$. Then*

$$F_{D,\Gamma_0(N),f,H} = \sum_{M\in\Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})} \sum_{\gamma\in H} f|_k M \rho_D(M^{-1})\mathfrak{e}_\gamma \tag{2.5}$$

is an element of $M_k^!(\rho_D)$ which is invariant under the automorphisms of D that stabilize H as a set. Moreover, if f is a holomorphic modular form (resp. a cusp form), then so is $F_{D,\Gamma_0(N),f,H}$.

We will only use this result for $H = \{0\}$, in which case the functions $F_{D,\Gamma_0(N),f,H}$ are examples of symmetric functions. If the level of D is squarefree, then these are the only ones:

Proposition 2.66 ([Sch15], Corollary 5.5)**.** *Suppose the level N of D is squarefree and let F be a symmetric weakly holomorphic modular form of weight $k \in \mathbb{Z}$ for ρ_D. Then there exists a weakly holomorphic modular form $f \in M_k^!(\Gamma_0(N), \chi_D)$ with $F = F_{D,\Gamma_0(N),f,0}$. If F is in $M_k(\rho_D)$ (resp. $S_k(\rho_D)$), then we can choose $f \in M_k(\Gamma_0(N), \chi_D)$ (resp. $S_k(\Gamma_0(N), \chi_D)$).*

We summarize Section 6 of [Sch06] to show how $F_{D,\Gamma_0(N),f,0}$ can be computed if N is squarefree. By Corollary 2.10, a set of representatives for the cusps of $\Gamma_0(N)$ is given by the elements $1/c$ for c a positive divisor of N. Given such a c, we choose a matrix

$$M_c = \begin{pmatrix} 1 & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

with $d = 0 \mod c'$, where $c' = N/c$ and we let

$$f_c = f|_k M_c.$$

The width of the cusp corresponding to $1/c$ is c' and $\chi_D(T_{1/c}) = 1$ (see Proposition 2.12). Consequently, the function f_c is c'-periodic and has a Fourier expansion in integral powers of $q^{1/c'}$. Then

$$F_{D,\Gamma_0(N),f,0}(\tau) = \sum_{c|N} \sum_{\mu \in D_{c'}} \xi(M_c^{-1}) \frac{\sqrt{|D_c|}}{\sqrt{|D|}} c' f_c(\tau)_{-Q(\mu)} \mathfrak{e}_\mu \tag{2.6}$$

where $f_c(\tau)_{-Q(\mu)}$ is defined as follows: The function f_c has a Fourier expansion

$$f_c(\tau) = \sum_{n \in \frac{1}{c'}\mathbb{Z}} a(n) q^n.$$

We then let

$$f_c(\tau)_{-Q(\mu)} = \sum_{n \in \mathbb{Z} - Q(\mu)} a(n) q^n,$$

i.e. we drop all Fourier coefficients with not in the set $\mathbb{Z} - Q(\mu)$.

There is a similar map from modular forms for $\Gamma_1(N)$:

Proposition 2.67 ([Sch15], Theorem 3.1). *Let k be an integer, N the level of D and $f \in M_k^!(\Gamma_1(N), \chi_\gamma)$. Then*

$$F_{D,\Gamma_1(N),f,\gamma} = \sum_{M \in \Gamma_1(N) \backslash \mathrm{SL}_2(\mathbb{Z})} f|_k M \rho_D(M^{-1}) \mathfrak{e}_\gamma$$

is an element of $M_k^!(\rho_D)$ which is invariant under the automorphisms of D that fix γ. Moreover, if f is a holomorphic modular form (resp. a cusp form), then so is $F_{D,\Gamma_1(N),f,\gamma}$.

If N is squarefree, then the image of a scalar-valued modular form f under this map can be computed as follows (see [Sch15], Theorem 3.7, which also gives a formula if N is not squarefree): Let P be a set of representatives for the cusps of $\Gamma_1(N)$. For every $a/c \in P$ with $(a,c) = 1$ we choose a matrix

$$M_{a/c} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $d = 0 \mod c'$ ($c' = N/(N,c)$). Then $F = \sum_{a/c \in P} F_{a/c}$ with

$$F_{a/c}(\tau) = \xi(M_{a/c}^{-1}) \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{\mu \in a\gamma + D^c} e(b(\mu,\gamma)) e(-abQ(\gamma)) \\ c'(f|_k M_{a/c}(\tau))_{-Q(\mu)} \left(\mathfrak{e}_\mu + (-1)^k e(\mathrm{sign}(D)/4) \mathfrak{e}_{-\mu} \right). \tag{2.7}$$

We also define a map between two spaces of vector-valued modular forms:

Proposition 2.68 ([Sch15], Theorem 4.1)**.** *Let k be an integer and $H \subset D$ an isotropic subgroup. Then $D_H = H^\perp/H$ is a discriminant form. Let $F = \sum_{\gamma+H\in D_H} F_{\gamma+H}\mathfrak{e}_{\gamma+H}$ be in $M_k^!(\rho_{D_H})$, $M_k(\rho_{D_H})$ or $S_k(\rho_{D_H})$. Then*

$$\hat{F} = \sum_{\gamma\in H^\perp} F_{\gamma+H}\mathfrak{e}_\gamma$$

is in $M_k^!(\rho_D)$, $M_k(\rho_D)$ or $S_k(\rho_D)$.

Definition 2.69. Let k be an integer and $F \in M_k^!(\rho_D)$. If there is an isotropic subgroup $H \subset D$ such that $F = \hat{G}$ for an element $G \in M_k^!(\rho_{D_H})$, we say that F is the *lift of G on H*.

2.3.4. Eisenstein Series

In this subsection we summarize the results of [Sch06], Section 7.

Let N be the level of D, $\gamma \in D$ an element of norm $Q(\gamma) = 0 + \mathbb{Z}$ and $k \geq 3$. Then

$$E_{k,\gamma}(\tau) = \frac{1}{4}\sum_{M\in \mathrm{SL}_2(\mathbb{Z})_\infty\backslash \mathrm{SL}_2(\mathbb{Z})} 1|_k M\rho_D^*(M^{-1})\mathfrak{e}_\gamma$$

is a holomorphic modular form of weight k for ρ_D^*. In the following we only consider the case $\gamma = 0$. By Proposition 2.52 a matrix $M \in \Gamma_0(N)$ acts by multiplication with $\chi_D(M)$ on $\mathfrak{e}_0$. Therefore (and because $\mathrm{SL}_2(\mathbb{Z})_\infty = \{\pm T^n : n \in \mathbb{Z}\} = \Gamma_0(N)_\infty$), the Eisenstein series $E_{k,0}$ is equal to

$$\begin{aligned} E_{k,0}(\tau) &= \frac{1}{4}\sum_{M\in\Gamma_0(N)_\infty\backslash\Gamma_0(N)}\ \sum_{N\in\Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})} 1|_k(MN)\rho_D^*((MN)^{-1})\mathfrak{e}_0 \\ &= \frac{1}{2}\sum_{N\in\Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})} E_{k,\chi_D^{-1}}|_k N(\tau)\rho_D^*(N^{-1})\mathfrak{e}_0 \end{aligned} \tag{2.8}$$

with

$$E_{k,\chi_D^{-1}} = \frac{1}{2}\sum_{M\in\Gamma_0(N)_\infty\backslash\Gamma_0(N)} \chi_D(M)1|_k M.$$

The function $E_{k,\chi_D^{-1}}$ is an Eisenstein series in $M_k(\Gamma_0(N), \chi_D^{-1})$ and formulas for the Fourier coefficients of $E_{k,\chi_D^{-1}}|_k N$ are known (see e.g. [DS05], Chapter 4 or (for squarefree N) [Sch06], Section 5). Comparing (2.8) with (2.5) we see that

$$E_{k,0} = \frac{1}{2}F^*_{D,\Gamma_0(N),E_{k,\chi_D^{-1}},0},$$

where $F^*_{D,\Gamma_0(N),f,0}$ is the analogue of $F_{D,\Gamma_0(N),f,0}$ from Proposition 2.65 for ρ_D^*. The Fourier expansions of the components of $E_{k,0}$ can therefore be computed from those of $E_{k,\chi_D^{-1}}$ with (2.6). This was done in the case of squarefree N by Scheithauer in [Sch06], Section 7. To summarize the result we introduce some further notation: For a Dirichlet character χ we let ψ be the associated primitive character and $L(s,\chi)$ the L-series associated to χ, i.e.

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

We denote by $B_{k,\chi}$ the generalized Bernoulli numbers associated to the Dirichlet character χ of modulus m, defined by

$$\sum_{a=1}^{m} \frac{\chi(a)te^{at}}{e^{mt}-1} = \sum_{k=0}^{\infty} B_{k,\psi}\frac{t^k}{k!}.$$

Finally, if $c \mid N$, we write χ_c and $\chi_{c'}$ for the unique characters of modulus c and c' such that $\chi_c\chi_{c'}$ (and similarly for ψ).

Proposition 2.70 ([Sch06], Theorem 7.1)**.** *Suppose $\chi_D(-1) = (-1)^k$. Let $\gamma \in D$ and $n > 0$ with $Q(\gamma) = n + \mathbb{Z}$. Let $E = E_{k,0}$, $\chi = \chi_D$ and m be the conductor of χ. For a positive integer c we let $m_c = (m,c)$ and $c' = N/(N,c)$. Then*

$$[E_\gamma](n) = A \sum_{\substack{c|N \\ c'\gamma=0}} \psi_c(N/m_c)\psi_{c'}(-c)\frac{\psi_c(2)}{\psi(2)}\frac{\varepsilon_c}{\varepsilon_N}\frac{\sqrt{m_c|D_c|}}{\sqrt{m|D|}}c'a_{k,\chi,c}(c'n)$$

with

$$A = -\frac{L(k,\psi)}{L(k,\chi)}\frac{m^k}{N^k}\frac{2k}{B_{k,\psi}},$$

$$a_{k,\chi,c}(n) = \sum_{d|n} \chi_c'(n/d)\psi_c(d)d^{k-1} \prod_{\substack{p|c/m_c \\ p|n}} (p-1) \prod_{\substack{p|c/m_c \\ p\nmid n}} (-1)$$

and

$$\varepsilon_c = \prod_{p|c/m_c} \epsilon_p \left(\frac{-1}{p}\right)^{n_p/2} \prod_{p|m_c} \epsilon_p \left(\frac{m_c/p}{p}\right)\left(\frac{-1}{p}\right)^{(n_p+1)/2}.$$

Proposition 2.71 ([Sch06], Theorem 7.2)**.** *Let $E = E_{k,0}$. Then the constant term of E_0 is*

$$[E_0](0) = 1.$$

For all $\gamma \in D$ with $Q(\gamma) = 0 + \mathbb{Z}$, the constant term of E_γ is zero if $\gamma \neq 0$.

2.3.5. Reduction to Sublattices

Suppose L is a lattice of signature (b^+, b^-) with $b^+ - b^-$ even. Let $z \in L$ be a cusp of level l, $z' \in L'$ with $(z,z') = 1$ and let $K = L \cap z'^\perp \cap z^\perp$. Then

$$L \otimes_{\mathbb{Z}} \mathbb{Q} = (K \otimes_{\mathbb{Z}} \mathbb{Q}) \oplus \mathbb{Q}z' \oplus \mathbb{Q}z,$$

so K has signature $(b^+ - 1, b^- - 1)$. We define the sublattice L'_0 of L' by

$$L'_0 = \{\lambda \in L' : (\lambda, z) = 0 \mod l\}.$$

Then L is obviously a sublattice of L'_0. We choose an element $\zeta \in L$ with $(z, \zeta) = l$, which allows us to write the $\mathbb{Z}$-module L as

$$L = K \oplus \mathbb{Z}z \oplus \mathbb{Z}\zeta \tag{2.9}$$

(see [Bru02], Proposition 2.2). This is of course not a direct sum of lattices as z and ζ are not orthogonal. For $v \in L \otimes_{\mathbb{Z}} \mathbb{R}$ we let v_K be the image of v under the orthogonal projection from $L \otimes_{\mathbb{Z}} \mathbb{R}$ to $K \otimes_{\mathbb{Z}} \mathbb{R}$. Note that if $v \in L'$, then $v_K \in K'$. There is a projection p from L'_0 to K' given by

$$p(\lambda) = \lambda_K - \frac{(\lambda, z)}{l}\zeta_K.$$

The map p acts as the identity on K and maps z and ζ to 0. Therefore $p(L) = K$ and p induces a surjective map $p\colon L'_0/L \to K'/K$.

Remark 2.72. If z' satisfies $lz' \in L$, then we can choose $\zeta = lz'$, in which case $p\colon L'_0 \to K'$ is just the orthogonal projection. By the following proposition there always exists such a z' if the level of L is squarefree:

Proposition 2.73. *If the level of L is squarefree, then we can always choose the vector z' such that z' has norm 0 and $lz' \in L$. Then the sublattice of L generated by z and lz' is isomorphic to $II_{1,1}(l)$.*

Proof. Let N be the level of L and let z' be any vector in L' with $(z, z') = 1$. The exponent of $D = L'/L$ is N and hence $l \mid N$. Let $l' = N/l$. The discriminant form D decomposes into the direct sum of D_l and $D_{l'}$, so there are elements w_l and $w_{l'}$ in L' with $z' = w_l + w_{l'}$, $lw_l \in L$ and $l'w_{l'} \in L$. Then

$$(z, l'w_{l'}) \in l'\mathbb{Z} \cap l\mathbb{Z} = N\mathbb{Z},$$

so $(z, w_{l'}) \in l\mathbb{Z}$. It follows that $(z, w_l) = 1 \mod l$ and by adding a suitable element of L to w_l we can assume that $(z, w_l) = 1$. By replacing z' with w_l we can hence assume that lz' is in L.

We observe that $q(z')$ is an element of $\frac{1}{N}\mathbb{Z}$ (because L has level N) but also of $\frac{1}{l^2}\mathbb{Z}$ because $l^2 q(z') = q(lz') \in \mathbb{Z}$. Therefore, $n = lq(z')$ is in $\frac{1}{l'}\mathbb{Z} \cap \frac{1}{l}\mathbb{Z} = \mathbb{Z}$. Let $w = lz' - nz$. Then $q(w) = 0$, $(w, z) = l$ and w/l is in L' (because z has level l). The element $x = w/l$ is thus an element of the form we are looking for. The lattice generated by z and $w = lx$ is obviously isomorphic to $II_{1,1}(l)$. □

Definition 2.74. For a modular form F of weight $k \in \mathbb{Z}$ for $\rho_{L'/L}$ we define a function $F_K\colon \mathbb{H} \to \mathbb{C}[K'/K]$ by

$$F_K = \sum_{\beta \in K'/K} \sum_{\substack{\gamma \in L'_0/L \\ p(\gamma) = \beta}} F_\gamma \mathfrak{e}_\beta$$

and call F_K the *reduction of F to K*.

Proposition 2.75 ([Bor98], Theorem 5.3). *The reduction* $F_K\colon \mathbb{H} \to \mathbb{C}[K'/K]$ *of* F *is in* $M^!_k(\rho_{L'/L})$.

Remark 2.76. Given $\beta \in K'/K$, a set of representatives for $\gamma \in L'_0/L$ with $p(\gamma) = \beta$ is given by the elements $\gamma = \beta - (\beta, \zeta)z/l + bz/l$ where b runs through a set of representatives for $\mathbb{Z}/l\mathbb{Z}$. Let $\tilde{\zeta}$ be a different choice of ζ, i.e. another element in L with $(z, \tilde{\zeta}) = l$. Then $\tilde{\zeta} = \alpha + mz + \zeta$ for some $\alpha \in K$ and $m \in \mathbb{Z}$ because of (2.9) and

$$(\beta, \tilde{\zeta}) = (\beta, \alpha) + (\beta, \zeta) \in (\beta, \zeta) + \mathbb{Z}.$$

Therefore F_K does not depend on the choice of ζ.

Remark 2.77. Let $\tilde{z}'$ be a different choice of z' and let $\tilde{K} = L \cap \tilde{z}'^{\perp} \cap z^{\perp}$. Then

$$K \to \tilde{K}, \quad \lambda \mapsto \lambda - (\lambda, \tilde{z}')z$$

defines an isomorphism from K to $\tilde{K}$ under which the reductions F_K and $F_{\tilde{K}}$ coincide. We can therefore also say that F_K does not depend on the choice of z'.

3. Reflective Automorphic Forms and Products

In this chapter we define automorphic forms. These are functions with a certain transformation behaviour under a subgroup of the orthogonal group of an even lattice L of signature $(n, 2)$. In the first section we define these functions and explain how modular forms for the Weil representation can be used to obtain automorphic forms. All of this is well-known and more details can be found in [Bru02]. In the second section we investigate holomorphic automorphic forms with special divisors, so-called reflective automorphic forms, and obtain an effective finiteness result.

In this chapter L is always an even lattice of signature $(n, 2)$, $n \geq 3$ with bilinear form $(\cdot, \cdot)$ and quadratic form q.

3.1. Automorphic Forms and Products

We let $V = L \otimes_{\mathbb{Z}} \mathbb{R}$ and $V(\mathbb{C}) = V \otimes_{\mathbb{R}} \mathbb{C}$ and extend $(\cdot, \cdot)$ to a bilinear form on both V and $V(\mathbb{C})$. Let $\mathbb{P}(V(\mathbb{C}))$ be the projective space associated to $V(\mathbb{C})$. We write $[Z_L]$ for the image of $Z_L \in V(\mathbb{C}) \setminus \{0\}$ under the canonical projection

$$V(\mathbb{C}) \setminus \{0\} \to \mathbb{P}(V(\mathbb{C}))$$

and consider the zero-quadric

$$\mathcal{N} = \{[Z_L] \in \mathbb{P}(V(\mathbb{C})) : (Z_L, Z_L) = 0\}$$

as well as its open subset

$$\mathcal{K} = \{[Z_L] \in \mathcal{N} : (Z_L, \overline{Z_L}) < 0\}.$$

Lemma 3.1. *Let $Z_L = X_L + iY_L \in V(\mathbb{C})$ with $X_L, Y_L \in V$. Then $[Z_L] \in \mathcal{K}$ if and only if $X_L \perp Y_L$ and $q(X_L) = q(Y_L) < 0$.*

Proof.

$$(Z_L, Z_L) = (X_L + iY_L, X_L + iY_L) = 2q(X_L) + 2i(X_L, Y_L) - 2q(Y_L),$$

so that $[Z_L] \in \mathcal{N}$ is equivalent to $X_L \perp Y_L$ and $q(X_L) = q(Y_L)$. Moreover,

$$(Z_L, \overline{Z_L}) = (X_L + iY_L, X_L - iY_L) = 2q(X_L) + 2q(Y_L),$$

which shows that $[Z_L] \in \mathcal{K}$ if and only if $q(X_L) = q(Y_L) < 0$. □

The set $\mathcal{K}$ has two connected components, which are interchanged by $[Z_L] \mapsto [\overline{Z_L}]$. We fix one of these components and denote it by $\mathcal{K}^+$.

Remark 3.2. Lemma 3.1 can be used to see that the map $[Z_L] \mapsto \mathbb{R}X_L + \mathbb{R}Y_L$ defines a bijection from $\mathcal{K}^+$ to the set of 2-dimensional negative definite subspaces of V. Interchanging the connected components then corresponds to changing the orientation on the 2-dimensional negative definite subspaces of V.

The orthogonal group $O(L)$ acts naturally on $\mathcal{K}$ and we let $O(L)^+$ be the subgroup of $O(L)$ that stabilizes $\mathcal{K}^+$ as a set. Since $O(L) = O(L')$ by Proposition 1.10, we get a natural map $O(L)^+ \to \mathrm{Aut}(D)$ for $D = L'/L$. The kernel of this map is called *discriminant kernel* and is denoted by Γ_L. It has finite index in $O(L)^+$ because $\mathrm{Aut}(D)$ is a finite group.

Remark 3.3. With Remark 3.2 we see that the subgroup $O(L)^+$ consists of those isometries in $O(L)$ that fix the orientation on 2-dimensional negative definite subspaces of V. In particular, we see that reflections σ_λ for roots λ of L are in $O(L)^+$ because roots have positive norm.

We let $\tilde{\mathcal{K}}^+ = \{Z_L \in V(\mathbb{C}) \setminus \{0\} : [Z_L] \in \mathcal{K}^+\}$ be the affine cone over $\mathcal{K}^+$.

Definition 3.4. Let $\Gamma \subset O(L)^+$ be a subgroup of finite index, $\chi\colon \Gamma \to \mathbb{C}^*$ a unitary character and $k \in \mathbb{Z}$. A meromorphic function $\Psi\colon \tilde{\mathcal{K}}^+ \to \mathbb{C}$ is called an *automorphic form of weight k and character χ for Γ* if

1. $\Psi(tZ_L) = t^{-k}Z_L$ for all $Z_L \in \tilde{\mathcal{K}}^+$ and $t \in \mathbb{C}^*$ and
2. $\Psi(\sigma(Z_L)) = \chi(\sigma)\Psi(Z_L)$ for all $\sigma \in \Gamma$ and $Z_L \in \tilde{\mathcal{K}}^+$.

If in addition Ψ is holomorphic, then we call Ψ a *holomorphic automorphic form.*

Remark 3.5. One can also define automorphic forms if $n < 3$. However, one should then require that Ψ is meromorphic (holomorphic for holomorphic automorphic forms) at the boundary of $\mathcal{K}^+$. If $n \geq 3$ this follows automatically from the Koecher principle (see [Koe54]).

Remark 3.6. Since $n \geq 3$, all characters have finite order (see Remark 3.19 in [Bru02]).

Proposition 3.7. *If the Witt rank (i.e. the dimension of a maximal isotropic subspace) of $L \otimes_{\mathbb{Z}} \mathbb{Q}$ is 2, then there are no non-constant holomorphic automorphic forms of weight less than $s = (n-2)/2$.*

Proof. See e.g. [Bun01], Satz 3.1.19 or [Bor95], Corollary 3.3. □

This is the reason for the following definition:

Definition 3.8. The weight $s = (n-2)/2$ is called the *singular weight.*

Remark 3.9. Note that the Witt rank is at most 2 because L has signature $(n, 2)$. Moreover, the Witt rank is automatically 2 if $n \geq 5$ because every indefinite lattice of rank at least 5 contains an isotropic vector by Proposition 1.18.

We can associate to a holomorphic automorphic form Ψ for the discriminant kernel Γ_L a Fourier expansion, which will depend on the choice of a cusp $z \in L$ as follows: Choose a vector $z' \in L'$ with $(z, z') = 1$ and let K be the lattice $L \cap z'^{\perp} \cap z^{\perp}$. We consider the set

$$\mathcal{H} = \{X + iY \in K \otimes_{\mathbb{Z}} \mathbb{C} : q(Y) < 0\}$$

and for $Z \in \mathcal{H}$ we write

$$Z_L = Z - (q(Z) + q(z'))z + z' \in V(\mathbb{C}).$$

An easy calculation shows that $[Z_L] \in \mathcal{K}$. Moreover, the map

$$\psi \colon \mathcal{H} \to \mathcal{K}, \quad Z \mapsto [Z_L]$$

is biholomorphic (see [BGHZ08], Part 2, Lemma 2.18). We write $\mathcal{H}^+$ for the preimage of $\mathcal{K}^+$. This is of the form

$$\mathcal{H}^+ = K \otimes_{\mathbb{Z}} \mathbb{R} + iC$$

where C is one of the two connected components of $\{Y \in K \otimes_{\mathbb{Z}} \mathbb{R} : q(Y) < 0\}$. We call C the *positive cone*. The action of $O(L)^+$ on $\mathcal{K}^+$ then induces an action on $\mathcal{H}^+$ via ψ. This action is no longer linear.

For an automorphic form Ψ we define the function Ψ_z on $\mathcal{H}^+$ by $\Psi_z(Z) = \Psi(Z_L)$.

Proposition 3.10. *Let Ψ be a holomorphic automorphic form for the discriminant kernel Γ_L and $z \in L$ a cusp. Then there exists a vector $\rho \in K \otimes_{\mathbb{Z}} \mathbb{Q}$ such that Ψ_z has a Fourier expansion*

$$\Psi_z(Z) = \sum_{\lambda \in \rho + K'} a(\lambda) e(-(\lambda, Z))$$

with $a(\lambda) \in \mathbb{C}$ for all $Z \in \mathcal{H}^+$. Moreover, $a(\lambda)$ is 0 for all λ that do not lie in the closure $\overline{C}$ of C.

Remark 3.11. If the Witt rank of $L \otimes_{\mathbb{Z}} \mathbb{Q}$ is 2 and Ψ has singular weight $(n-2)/2$, then $a(\lambda) = 0$ unless λ is isotropic.

One way to obtain automorphic forms is described in the following theorem:

Theorem 3.12 ([Bor98], Theorem 13.3 and [Bru02], Theorem 3.22)**.** *Let $D = L'/L$ and F a weakly holomorphic modular form of weight $1 - n/2$ for ρ_D with integral coefficients $[F_\gamma](m)$ for all $m < 0$ and $[F_0](0) \in 2\mathbb{Z}$. Then there is a meromorphic function $\Psi(F) \colon \mathcal{K}^+ \to \mathbb{C}$ with the following properties:*

1. *$\Psi(F)$ is an automorphic form of weight $[F_0](0)/2$ for the group*

$$O(L, F)^+ = \{\sigma \in O(L)^+ : \sigma(F) = F\}$$

with respect to some unitary character $\chi \colon O(L, F)^+ \to \mathbb{C}^*$.

2. *The only zeros or poles of* Ψ *are of the form* $\lambda^\perp$ *for* $\lambda \in L$ *with* $q(\lambda) > 0$ *and are zeros of order*

$$\sum_{\substack{0<x\in\mathbb{Q}\\ x\lambda\in L'}} [F_{x\lambda+L}](-q(x\lambda)) \tag{3.1}$$

or poles if this number is negative.

3. *Let* $z \in L$ *be a cusp and* W *a Weyl chamber of* $K = L \cap z^\perp \cap z'^\perp$. *Let* m_0 *be the smallest rational number such that* $[F_\gamma](m_0) \neq 0$ *for some* $\gamma \in D$. *On*

$$\{Z = X + iY \in \mathcal{H}^+ : Z \text{ is not a pole of } \Psi_z(F) \text{ and } (Y,Y) < m_0\}$$

the restriction $\Psi_z(F)$ *of* $\Psi(F)$ *has an infinite product expansion which is some constant times*

$$e(-(Z,\rho(W))) \prod_{\substack{\lambda\in K'\\ (\lambda,W)<0}} \prod_{\substack{\delta\in L_0'/L\\ p(\delta)=\lambda}} (1 - e(-(\lambda,Z) + (\delta,z')))^{[F_\delta](-q(\lambda))}.$$

Remark 3.13. The theorem is also valid if $[F_0](0) \notin 2\mathbb{Z}$, in which case $\Psi(F)$ has rational weight and χ no longer needs to be a character. We have omitted this case because one would have to work with covers of $\mathcal{K}^+$ in order to define automorphic forms of rational weight. Note, however, that every automorphic form has half-integral weight as was shown in [Hil07], so $[F_0](0)$ will always be an integer.

The third property of $\Psi(F)$ in Theorem 3.12 justifies the following definition:

Definition 3.14. Automorphic forms of the form $\Psi(F)$ for some $F \in M^!_{1-n/2}(\rho_D)$ are called *automorphic products*.

We want to explain the notation in the third property of $\Psi(F)$ in Theorem 3.12 in more detail. We let l be the level of the cusp z and define L_0' and p as in Subsection 2.3.5. For $\gamma \in K'/K$ and $n > 0$ we define the *Heegner divisor* $H(\gamma, n)$ on the positive cone $C \subset K \otimes_{\mathbb{Z}} \mathbb{R}$ to be

$$H(\gamma,n) = \bigcup_{\substack{\lambda\in\gamma+K\\ q(\lambda)=n}} \lambda^\perp.$$

The *Weyl chambers* of K are the connected components of the complement of

$$\bigcup_{\gamma\in L_0'/L} \bigcup_{\substack{n\in\mathbb{Z}-Q(\gamma)\\ n>0\\ [F_\gamma](-n)\neq 0}} H(p(\gamma), n)$$

in C and the *Weyl group* G is the group generated by the reflections at the walls between two adjacent Weyl chambers. It acts simply transitive on the Weyl chambers. The vector $\rho(W)$ is the so-called *Weyl vector* of W. If the closure of W contains a cusp of K then $\rho(W)$ can be computed as follows ([Bor98], Theorem 10.4): Let F_K be the reduction of F

to K, $w \in K$ a cusp contained in the closure of W and w' a vector in K' with $(w, w') = 1$. Let $\tilde{K} = K \cap w^\perp \cap w'^\perp$ and let m be the level of w. Then $\rho(W) = \rho - \rho_{w'} w' + \rho_w w$, where

$$\rho = -\frac{1}{2} \sum_{\substack{\lambda \in \tilde{K}' \\ (\lambda, W) < 0}} [(F_K)_{\lambda + K}](-q(\lambda))\lambda,$$

$$\rho_{w'} = \text{ constant term of } \sum_{\gamma \in \tilde{K}'/\tilde{K}} E_2(\tau)\theta_\gamma(\tau)(F_{\tilde{K}})_\gamma(\tau)/24,$$

$$\rho_w = -\rho_{w'} q(w') + \frac{1}{4} \sum_{\lambda \in \tilde{K}'} \sum_{\substack{\delta \in K_0'/K \\ p(\delta) = \lambda + \tilde{K}}} [(F_K)_\delta](-q(\lambda)) B_2((\delta, w')).$$

Here K_0' and $p\colon K_0'/K \to \tilde{K}'/\tilde{K}$ are as defined in Subsection 2.3.5, $F_{\tilde{K}}$ is the reduction of F_K to $\tilde{K}$ and $\theta_\gamma\colon \mathbb{H} \to \mathbb{C}$ is defined by

$$\theta_\gamma(\tau) = \sum_{\lambda \in \gamma + \tilde{K}} e^{2\pi i q(\lambda)\tau}.$$

Note that θ_γ converges because $\tilde{K}$ is positive definite. Finally, B_2 is the 1-periodic function defined by

$$B_2(x) = x^2 - x + \frac{1}{6}$$

for $0 \leq x < 1$ (i.e. B_2 is the second Bernoulli polynomial for $0 \leq x < 1$).

There is also a converse to Theorem 3.12 which is due to Bruinier:

Proposition 3.15 ([Bru14], Theorem 1.2)**.** *Let $L \cong M \oplus II_{1,1}(N) \oplus II_{1,1}$ for a positive definite even lattice M of rank $n - 2 \geq 1$ and some positive integer N. Then every automorphic form Ψ with respect to Γ_L whose divisor is a linear combination of divisors of the form*

$$\sum_{\substack{\lambda \in \mu + L \\ q(\lambda) = n}} \lambda^\perp \quad (\mu \in L'/L, n \in \mathbb{Q}_{>0})$$

is up to some constant factor equal to $\Psi(F)$ for some $F \in M^!_{1-n/2}(\rho_{L'/L})$.

Suppose $L = K \oplus II_{1,1}(m)$ for some positive integer m, let $M \subset K$ be a sublattice of finite index and let $N = M \oplus II_{1,1}(m) \subset L$. Then $H = L/N \subset L'/N \subset N'/N$ is an isotropic subgroup of the discriminant form N'/N and $H^\perp$ is given by L'/N. Therefore, $H^\perp/H$ is isomorphic to L'/L. As seen in Proposition 2.68, a modular form $F_L \in M^!_{1-n/2}(\rho_{L'/L})$ induces a modular form $F_N = \hat{F_L} \in M^!_{1-n/2}(\rho_{N'/N})$. We assume that the Fourier coefficients of $[(F_L)_\gamma](j)$ are integers for all $\gamma \in L'/L$ and $j < 0$ and that $[(F_L)_0](0) \in 2\mathbb{Z}$, so that there exists an automorphic product $\Psi(F_L)$. Then the same is obviously true for F_N. Because N is a sublattice of L of finite index, the sets $\mathcal{K}^+$ associated to L and N can be identified.

Proposition 3.16 ([Sch17], Proposition 3.4)**.** *Under this identification the automorphic products $\Psi(F_L)$ and $\Psi(F_N)$ coincide as functions on $\tilde{\mathcal{K}}^+$.*

3.2. Reflective Automorphic Forms

We now consider automorphic forms with special divisors.

Definition 3.17. Let $\Psi\colon \tilde{\mathcal{K}}^+ \to \mathbb{C}$ be a holomorphic automorphic form. We call Ψ *reflective* if all of its zeros are of the form $\lambda^\perp$ for roots $\lambda \in L$. If in addition all zeros are simple, then we say that Ψ is *strongly-reflective*.

The following proposition states that modular forms for $\rho_{L'/L}$ whose principal parts have a certain shape lift to (strongly)-reflective automorphic products $\Psi(F)$:

Proposition 3.18 (see [Sch06], Section 9)**.** *Assume that L has squarefree level and let $D = L'/L$ be its discriminant form. Suppose $F \in M^!_{1-n/2}(\rho_D)$ satisfies the following:*

1. *If $\gamma \in D_{m,1/m}$ for a positive integer m, then the Fourier expansion of F_γ is of the form $F_\gamma = c_{\gamma,-1/m}q^{-1/m} + O(1)$ with $c_{\gamma,-1/m} \in \mathbb{Z}_{>0}$.*
2. *F_γ is holomorphic at ∞ for all other $\gamma \in D$.*

Then the automorphic product $\Psi(F)$ is reflective. Moreover, if $c_{\gamma,-1/m} \leq 1$ for all $\gamma \in D$ and $m \in \mathbb{Z}_{>0}$, then $\Psi(F)$ is strongly-reflective.

Proof. The coefficients $[F_\gamma](j)$ for $\gamma \in D$ and $j < 0$ are obviously non-negative integers, so that $\Psi(F)$ exists and has no poles by (3.1). It is therefore holomorphic.

Let $\lambda \in L$ be a primitive vector of positive norm and suppose that $\Psi(F)$ has a zero at $\lambda^\perp$. Then

$$\sum_{\substack{0<x\in\mathbb{Q}\\ x\lambda\in L'}} [F_{x\lambda+L}](-x^2q(\lambda)) > 0$$

by the formula (3.1) for the divisor of $\Psi(F)$. Thus, there is some $x \in \mathbb{Q}$ such that $[F_{x\lambda+L}](-x^2q(\lambda)) > 0$. Let $m = 1/x$. We show that m is a positive integer dividing the level N of L: From the shape of F we know that $x^2q(\lambda) = 1/k$ for some positive $k \mid N$, so $1/x^2 = kq(\lambda)$ and m must be an integer with $k \mid m^2$. Let $\gamma = x\lambda + L \in D$. Then $m\gamma = 0 + L$, so γ has order dividing m. The primitivity of λ then implies that the order of γ is exactly m and the shape of F then forces $x^2q(\lambda) = 1/m$, i.e. $m = k$. We have therefore shown that m is a positive divisor of N. It follows that λ is in $L \cap mL'$ and $q(\lambda) = m$, so that λ is a root by Proposition 1.22.

It remains to show that $\Psi(F)$ is strongly-reflective if $c_{\gamma,-1/m} \leq 1$ for all $\gamma \in D$ and $m \in \mathbb{Z}_{>0}$, i.e. we have to show that if $\lambda \in L$ is a root, then

$$\sum_{\substack{0<x\in\mathbb{Q}\\ x\lambda\in L'}} [F_{x\lambda+L}](-x^2q(\lambda))$$

is 0 or 1. Suppose this sum is non-zero. We have already seen that then the only non-zero summand is for $x = 1/m$, where $m = q(\lambda)$. Therefore,

$$\sum_{\substack{0<x\in\mathbb{Q}\\ x\lambda\in L'}} [F_{x\lambda+L}](-x^2q(\lambda)) = [F_{\lambda/m+L}](-1/m) = c_{\lambda/m+L,-1/m} = 1.$$

□

The converse of the previous proposition is also true if L splits $II_{1,1}$ as is shown in the following proposition:

Proposition 3.19. *Suppose L has squarefree level and that it splits $II_{1,1}$. Let $D = L'/L$ be its discriminant form. Then an automorphic product $\Psi(F)$ is reflective if and only if the modular form $F \in M^!_{1-n/2}(\rho_D)$ satisfies the following:*

1. *If $\gamma \in D_{m,1/m}$ for some positive integer m, then the Fourier expansion of F_γ is of the form $F_\gamma = c_{\gamma,-1/m}q^{-1/m} + O(1)$ with $c_{\gamma,-1/m} \in \mathbb{Z}_{>0}$.*

2. *F_γ is holomorphic at ∞ for all other $\gamma \in D$.*

Moreover, $\Psi(F)$ is strongly-reflective if and only if $c_{\gamma,-1/m} \leq 1$ for all $\gamma \in D$ and all $m \in \mathbb{Z}_{>0}$.

Proof. We only need to show that if $\Psi(F)$ is (strongly-)reflective, then F is as claimed because the other implication is the statement of the previous proposition.

Write $L = K \oplus II_{1,1}$ and note that $L' = K' \oplus II_{1,1}$ and $D \cong K'/K$. Let $\gamma \in D$ and $x < 0$ such that $[F_\gamma](x) \neq 0$ and note that then $Q(\gamma) = -x + \mathbb{Z}$. Let k be the order of γ. We have to show that $x = 1/k$ and that $[F_\gamma](x) > 0$.

Let m be the largest positive integer with $[F_{m\gamma}](m^2x) \neq 0$ and choose $\kappa \in K'$ with $\kappa + K = m\gamma$. The hyperbolic plane $II_{1,1}$ contains primitive vectors μ with $q(\mu) = j$ for any $j \in \mathbb{Z}$ (take for example the element $(1, j) \in II_{1,1}$), so we can add a suitable element in $II_{1,1}$ to κ to obtain a primitive vector $\lambda \in L'$ with $q(\lambda) = -m^2x$. Let $\tilde{k} = k/(m, k)$ be the order of $\lambda + L$ in D. Then $\tilde{k}\lambda$ is primitive in L and the order of $\Psi(F)$ at $(\tilde{k}\lambda)^\perp$ is equal to

$$\sum_{j\in\mathbb{Z}_{>0}} [F_{j\lambda+L}](-j^2q(\lambda)) = \sum_{j\in\mathbb{Z}_{>0}} [F_{jm\gamma}](j^2m^2x).$$

By the maximality of m, this last sum is equal to $[F_{m\gamma}](m^2x)$ and since $\Psi(F)$ is assumed to be reflective, it follows that $\tilde{k}\lambda$ is a root of L and that $[F_{m\gamma}](m^2x) \in \mathbb{Z}_{>0}$ (and $[F_{m\gamma}](m^2x) = 1$ if $\Psi(F)$ is strongly-reflective).

To complete the proof, we show that $m = 1$ and $x = 1/k$. Because $\tilde{k}\lambda$ is a root, $q(\tilde{k}\lambda) = -\tilde{k}^2m^2x$ is equal to some positive divisor a of the level N of L and $\tilde{k}\lambda/a$ must be in L'. Since λ is primitive in L', it follows that $\tilde{k}/a \in \mathbb{Z}_{>0}$, i.e. $\tilde{k} = ab$ for some $b \in \mathbb{Z}_{>0}$. Then

$$x = -\frac{1}{ab^2m^2}.$$

But $Q(\gamma) = -x + \mathbb{Z}$, so the denominator of x must be squarefree because N is squarefree. This is only possible if $m = b = 1$, so $a = \tilde{k} = k$ and $x = 1/k$. □

The following definition is motivated by Proposition 3.19:

Definition 3.20. Let $D = L'/L$. In the case that L has squarefree level, a modular form $F \in M^!_{1-n/2}(\rho_D)$ is called *semi-reflective* if it satisfies the following:

1. If $\gamma \in D_{m,1/m}$ for some positive integer m, then the Fourier expansion of F_γ is of the form $F_\gamma = c_{\gamma,-1/m}q^{-1/m} + O(1)$ with $c_{\gamma,-1/m} \in \mathbb{C}$.

2. F_γ is holomorphic at ∞ for all other $\gamma \in D$.

If in addition all $c_{\gamma,-1/m}$ are in $\mathbb{Z}_{>0}$, then we say that F is *reflective* and if all $c_{\gamma,-1/m}$ are in $\{0,1\}$, then we say that F is *strongly-reflective*.

Remark 3.21. By Proposition 3.18 every (strongly-)reflective modular form F lifts to a (strongly)-reflective automorphic product $\Psi(F)$ and Proposition 3.19 shows that if L splits $II_{1,1}$, then the converse also holds. The notion of semi-reflectivity was added because it has the advantage that the set of semi-reflective modular forms for L is a complex vector space.

For the rest of this section we assume that L has squarefree level N (and hence even signature) and splits $II_{1,1}(N)$ and that F is a semi-reflective modular form with $F_0 \neq 0$ on $D = L'/L$. We replace F by

$$\frac{1}{|\operatorname{Aut}(D)|} \sum_{\sigma \in \operatorname{Aut}(D)} \sigma(F), \tag{3.2}$$

which is symmetric, semi-reflective and non-vanishing (because $F_0 \neq 0$). By Proposition 2.66 there is a non-zero $f \in M^!_{1-n/2}(\Gamma_0(N), \chi_D)$ such that $F = F_{D,\Gamma_0(N),f,0}$. For a positive divisor c of N we define the matrices M_c as in Subsection 2.3.3. The next lemma gives bounds on the pole orders of $f|_{1-n/2}M_c$:

Lemma 3.22. *Let c be a positive divisor of N, $c' = N/c$ and $k = 1 - n/2$. Then $f|_k M_c = O(q^{-1/c'})$.*

Proof. Suppose there is some positive $c \mid N$ such that $f|_k M_c \neq O(q^{-1/c'})$. We can assume that c is the smallest positive divisor of N with this property, so $f|_k M_{\tilde{c}} = O(q^{-1/\tilde{c}'})$ for all positive $\tilde{c} \mid N$ with $\tilde{c} < c$. Let a be the smallest positive integer such that $f|_k M_c = O(q^{-a/c'})$. Then $a > 1$ and there is an element $\gamma \in D$ of order c' with $Q(\gamma) = a/c' + \mathbb{Z}$ because L splits $II_{1,1}(N)$ (take for example the element $(1/c', aj/c') + \mathbb{Z}^2$ in $II_{1,1}(N)'/II_{1,1}(N)$, where $j \in \mathbb{Z}$ is such that $jc = 1 \mod c'$). By (2.6) the component F_γ of F is given by

$$\begin{aligned} F_\gamma(\tau) &= \sum_{d|c} \xi(M_d^{-1}) \frac{\sqrt{|D_d|}}{\sqrt{|D|}} d'(f|_k M_d(\tau))_{-Q(\gamma)} \\ &= \xi(M_c^{-1}) \frac{\sqrt{|D_c|}}{\sqrt{|D|}} c'(f|_k M_d(\tau))_{-Q(\gamma)} + \sum_{\substack{d|c \\ d<c}} \xi(M_d^{-1}) \frac{\sqrt{|D_d|}}{\sqrt{|D|}} d'(f|_k M_d(\tau))_{-Q(\gamma)}. \end{aligned}$$

For $d < c$ we have $(f|_k M_d)_{-Q(\gamma)} = O(q^{-1/d'})$ because $f|_k M_d = O(q^{-1/d'})$ by the minimality of c, whereas the Fourier expansion of $(f|_k M_c)_{-Q(\gamma)}$ starts with a non-zero multiple of $q^{-a/c'}$. Since $1/d' < 1/c'$ for $d < c$, it follows that the Fourier expansion of F_γ starts with a non-zero multiple of $q^{-a/c'}$, which is a contradiction to the semi-reflectivity of F. □

Lemma 3.23. *Let N be a squarefree integer, $k \in \mathbb{Z}_{<0}$ and $f \in M_k^!(\Gamma_0(N), \chi) \setminus \{0\}$ for some character χ for $\Gamma_0(N)$ such that $\ker(\chi)$ has finite index in $\Gamma_0(N)$. Assume that $f|_k M_c = O(q^{-1/c'})$ for all $c \mid N$. Then*

$$\prod_{\substack{p|N \\ p \text{ prime}}} (p+1) \leq -2^{\omega(N)} \frac{12}{k}$$

where $\omega(N)$ is the number of primes dividing N.

Proof. Proposition 2.24 yields

$$\sum_{s \in \Gamma_0(N) \backslash (\mathbb{Q} \cup \{\infty\})} [\mathrm{SL}_2(\mathbb{Z})_s : \pm\Gamma_0(N)_s] \operatorname{ord}_s(f) \leq \frac{k}{12} [\mathrm{SL}_2(\mathbb{Z}) : \pm\Gamma_0(N)].$$

A set of representatives for $\Gamma_0(N)\backslash(\mathbb{Q} \cup \{\infty\})$ is given by the rational numbers $1/c$ for the positive divisors $c \mid N$. The width of the cusp corresponding to $1/c$ is given by c' and the order of f at $1/c$ is bounded from below by $-1/c'$ by our assumption on f. Moreover, the index of $\Gamma_0(N)$ (which equals $\pm\Gamma_0(N)$ because $-I_2$ is in $\Gamma_0(N)$) in $\mathrm{SL}_2(\mathbb{Z})$ is given by

$$\prod_{\substack{p|N \\ p \text{ prime}}} (p+1)$$

by Proposition 2.4. Therefore

$$\sum_{\substack{c|N \\ c>0}} c' \cdot \frac{1}{c'} \leq \frac{k}{12} \prod_{\substack{p|N \\ p \text{ prime}}} (p+1),$$

from which the claim follows because the number of positive divisors of N is $2^{\omega(N)}$.

□

The inequality from the previous lemma can only be satisfied if k is at least -12 and $\omega(N)$ is at most 3. For fixed k and $\omega(N)$, the level N is then bounded by the values in the following table (with "-" meaning that there is no possible N).

Table 3.1.: Bound on N depending on k and $\omega(N)$

		k											
		-1	-2	-3	-4	-5	-6	-7	-8	-9	-10	-11	-12
$\omega(N)$	0	1	1	1	1	1	1	1	1	1	1	1	1
	1	23	11	7	5	3	3	2	2	-	-	-	-
	2	35	15	6	6	-	-	-	-	-	-	-	-
	3	42	-	-	-	-	-	-	-	-	-	-	-

Lemmas 3.22 and 3.23 immediately lead to the following theorem:

Theorem 3.24. *Let L be an even lattice of signature $(n, 2)$, $n \geq 4$ and squarefree level N that splits $II_{1,1}(N)$. Suppose there is a semi-reflective modular form F with $F_0 \neq 0$ on $D = L'/L$. Then depending on $k = 1-n/2$ and $\omega(N)$ the level N is bounded by the values in Table 3.1. In particular L must be one of finitely many lattices (up to isomorphisms).*

Proof. After symmetrizing F as in (3.2) we can assume that F is symmetric. Then $F = F_{D,\Gamma_0(N),f,0}$ for a non-zero weakly holomorphic modular form f of weight k and character χ_D. We then apply Lemmas 3.22 and 3.23 to f.

In particular, the rank $n + 2 = 4 - 2k$ and the level N of L are bounded. The last assertion of the theorem then follows because there are only finitely many isomorphism classes of lattices of fixed rank and level (see Corollary 1.69). □

This has the following consequence for the existence of reflective automorphic forms:

Theorem 3.25. *Let L be an even lattice of signature $(n, 2)$, $n \geq 4$ and squarefree level N that splits $II_{1,1} \oplus II_{1,1}(N)$. Suppose there is a non-constant reflective automorphic form $\Psi\colon \tilde{\mathcal{K}}^+ \to \mathbb{C}$. Then depending on $k = 1 - n/2$ and $\omega(N)$ the level N is bounded by the values in Table 3.1. In particular L must be one of finitely many lattices (up to isomorphisms).*

Proof. The function Ψ is an automorphic form for some subgroup $\Gamma \subset O(L)^+$ of finite index with some character. Since the character has finite order (see Remark 3.6) we can replace Ψ with some power of Ψ to get rid of the character. Let $\sigma_1, \ldots, \sigma_l$ be representatives for the cosets of Γ in $O(L)^+$ and replace Ψ with the function

$$\Psi(Z) = \prod_{i=1}^{l} \Psi(\sigma_i(Z)).$$

This is then a reflective automorphic form for $O(L)^+$ and in particular for the discriminant kernel Γ_L. We can therefore use Proposition 3.15 to see that it is up to a non-zero constant factor equal to an automorphic product $\Psi(F)$ for a modular form F of weight k on D. Then F must be reflective by Proposition 3.19 and $F_0 \neq 0$ because $[F_0](0) \neq 0$ as $\Psi(F)$ is non-constant. We are therefore in the situation of Theorem 3.24. □

4. Strongly-Reflective Automorphic Forms of Singular Weight

This chapter is concerned with the classification of strongly-reflective automorphic forms for the discriminant kernel on lattices L of squarefree level N. In the first section we assume that the lattice L splits $I\!I_{1,1} \oplus I\!I_{1,1}(N)$. For these lattices we give a complete classification and we see that all strongly-reflective automorphic forms arise from a construction involving the Leech lattice. In the second section we construct new strongly-reflective automorphic forms on lattices that do not split $I\!I_{1,1}$ and in the third section we compute the Fourier expansions of one of these automorphic forms at all cusps to see that it is not related to the Leech lattice.

4.1. Lattices Splitting Two Hyperbolic Planes

In this section L is always an even lattice of signature $(n, 2)$, $n \geq 4$ and squarefree level N such that L splits $I\!I_{1,1} \oplus I\!I_{1,1}(N)$.

4.1.1. The Symmetric Case

Strongly-reflective automorphic products of singular weight arise in the following way, which was described by Scheithauer (see [Sch04] and [Sch06]):

Let g be an element of order n in the automorphism group Co_0 of the Leech lattice Λ. Then g is also an orthogonal automorphism of the vector space $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ and hence has a characteristic polynomial that can be written in the form

$$\prod_{k|n} (x^k - 1)^{b_k}$$

with integers b_k. Note that the highest power of x occuring in the product is given by $x^{\sum_{k|n} kb_k}$, so that $\sum_{k|n} kb_k$ must be equal to 24. We associate the eta quotient

$$\eta_g(\tau) = \prod_{k|n} \eta(k\tau)^{b_k}$$

to g and let N be the smallest positive integer such that 24 divides $N \sum_{k|n} b_k/k$. We call N the level of g. Then we can use Proposition 2.40 to see that η_g is a weakly-holomorphic modular form of weight $r = \sum_{k|n} b_k/2$ for $\Gamma_0(N)$, possibly with a character. Let Λ^g be the sublattice of Λ that consists of those vectors that are fixed by g. If Λ^g is non-trivial and N is squarefree, then we let

$$L = \Lambda^g \oplus I\!I_{1,1} \oplus I\!I_{1,1}(N)$$

and $D = L'/L$ be the discriminant form of L. It turns out that the character of η_g is χ_D, so we can apply Proposition 2.65 to $f_g = 1/\eta_g$ and obtain the symmetric vector-valued modular form $F_g = F_{D,\Gamma_0(N),f_g,0}$. Explicit calculations show that F_g is a valid input for the Borcherds lift (i.e. it satisfies the asssumptions of Theorem 3.12) and that $\Psi_g = \Psi(F_g)$ is a strongly-reflective automorphic form of singular weight. The Fourier expansions of Ψ_g at the cusps can be computed because Ψ_g has singular weight and the nonzero Fourier coefficients hence correspond to isotropic vectors. These calculations are similar to the ones in Section 4.3 and one obtains the following:

Proposition 4.1 ([Sch04], Theorem 9.8). *Let $z \in L$ be a cusp of level $m \mid N$, define $m' = N/m$ and let $\eta_{g,m'} = \eta_g|_r W^N_{m'}$ and $f_{g,m'} = 1/\eta_{g,m'}$. Suppose that $f_{g,m'}$ has a pole at ∞. We can write $L = \Lambda^g \oplus I\!I_{1,1}(m') \oplus I\!I_{1,1}(m)$ and we let $K = \Lambda^g \oplus I\!I_{1,1}(m')$. Then the Fourier expansion of Ψ_g at the cusp z is up to a non-zero constant given by*

$$(\Psi_g)_z(Z) = \sum_{w \in G} \det(w)\eta_{g,m'}(-(w\rho, Z)),$$

where ρ is the Weyl vector and G is the Weyl group (see Section 3.1).

Proposition 4.2 ([Sch04], Theorem 9.10). *Let $z \in L$ be a cusp of level $m \mid N$, $m' = N/m$ and let $\eta_{g,m'} = \eta_g|_r W^N_{m'}$ and $f_{g,m'} = 1/\eta_{g,m'}$. Suppose that $f_{g,m'}$ is holomorphic at ∞. We can write $L = \Lambda^g \oplus I\!I_{1,1}(m') \oplus I\!I_{1,1}(m)$ and we let $K = \Lambda^g \oplus I\!I_{1,1}(m')$. Then the Fourier expansion of Ψ_g at the cusp z is up to a non-zero constant given by*

$$(\Psi_g)_z(Z) = 1 + \sum_{\lambda \in K'} c(\lambda)e(-(\lambda, Z))$$

where $c(\lambda)$ is the coefficient of q^n in $a\eta_{g,m'}$ if λ is n times a primitive isotropic vector in the positive cone and 0 otherwise. The constant a is such that the constant Fourier coefficient of $a\eta_{g,m'}$ is 1.

There is also the following different point of view on this: Let $m \mid N$. As in [Que97] and [Sch04] we define the lattice

$$W_m(L) = \sqrt{m}\left(L' \cap \frac{1}{m}L\right).$$

This is isomorphic to $W_m(\Lambda^g) \oplus I\!I_{1,1}(m) \oplus I\!I_{1,1}(m')$, $m' = N/m$ and also has level N. Then a suitable multiple of $f_{g,m} = f_g|_r W^N_m$ can be lifted first to a symmetric vector-valued modular form on $W_m(L)$ and then to a strongly-reflective automorphic product $\Psi_{g,m}$ of singular weight. The Fourier expansion of Ψ_g at a cusp of level m' is then up to a constant given by the Fourier expansion of $\Psi_{f,g}$ at a cusp of level N.

The following table summarizes the strongly-reflective automorphic forms $\Psi_{g,m}$ of singular weight that are obtained in this way:

Table 4.1.:

L	f	L	f
$II_{4,2}(2_{II}^{+2}3^{+3})$	$\eta_{1^{-1}2^{2}3^{3}6^{-6}}$	$II_{6,2}(2_{II}^{-6}5^{-5})$	$\eta_{1^{2}2^{-3}5^{-2}10^{-1}}$
$II_{4,2}(2_{II}^{+2}3^{+3})$	$-3\eta_{1^{3}2^{-6}3^{-1}6^{2}}$	$II_{6,2}(2_{II}^{+2}5^{+5})$	$\eta_{1^{-3}2^{2}5^{-1}10^{-2}}$
$II_{4,2}(2_{II}^{-4}3^{-3})$	$-2\eta_{1^{2}2^{-1}3^{-6}6^{3}}$	$II_{6,2}(2_{II}^{+2}5^{+3})$	$\eta_{1^{-1}2^{-2}5^{-3}10^{2}}$
$II_{4,2}(2_{II}^{-4}3^{-3})$	$6\eta_{1^{-6}2^{3}3^{2}6^{-1}}$	$II_{6,2}(11^{-4})$	$\eta_{1^{-2}11^{-2}}$
$II_{4,2}(2_{II}^{+4}7^{-3})$	$\eta_{1^{1}2^{-2}7^{1}14^{-2}}$	$II_{6,2}(2_{II}^{+4}7^{-4})$	$\eta_{1^{-1}2^{-1}7^{-1}14^{-1}}$
$II_{4,2}(2_{II}^{+2}7^{-3})$	$2\eta_{1^{-2}2^{1}7^{-2}14^{1}}$	$II_{6,2}(3^{+4}5^{-4})$	$\eta_{1^{-1}3^{-1}5^{-1}15^{-1}}$
$II_{4,2}(3^{+3}5^{-3})$	$\eta_{1^{-2}3^{1}5^{1}15^{-2}}$	$II_{8,2}(3^{-7})$	$\eta_{1^{3}3^{-9}}$
$II_{4,2}(3^{-3}5^{+3})$	$-\eta_{1^{1}3^{-2}5^{-2}15^{1}}$	$II_{8,2}(3^{-3})$	$9\eta_{1^{-9}3^{3}}$
$II_{4,2}(23^{-3})$	$\eta_{1^{-1}23^{-1}}$	$II_{8,2}(2_{II}^{-8}3^{+3})$	$\eta_{1^{-4}2^{-1}3^{4}6^{-5}}$
$II_{4,2}(2_{II}^{+4}3^{-3}5^{+3})$	$\eta_{1^{-1}3^{1}5^{1}6^{-1}10^{-1}15^{-1}}$	$II_{8,2}(2_{II}^{-8}3^{+7})$	$\eta_{1^{4}2^{-5}3^{-4}6^{-1}}$
$II_{4,2}(2_{II}^{+4}3^{+3}5^{-3})$	$\eta_{1^{1}2^{-1}3^{-1}5^{-1}15^{1}30^{-1}}$	$II_{8,2}(2_{II}^{+2}3^{-7})$	$\eta_{1^{-5}2^{4}3^{-1}6^{-4}}$
$II_{4,2}(2_{II}^{+2}3^{+3}5^{-3})$	$\eta_{2^{-1}3^{-1}5^{-1}6^{1}10^{1}30^{-1}}$	$II_{8,2}(2_{II}^{+2}3^{-3})$	$\eta_{1^{-1}2^{-4}3^{-5}6^{4}}$
$II_{4,2}(2_{II}^{+2}3^{-3}5^{+3})$	$\eta_{1^{-1}2^{1}6^{-1}10^{-1}15^{-1}30^{1}}$	$II_{8,2}(7^{-5})$	$\eta_{1^{-3}7^{-3}}$
$II_{6,2}(5^{+5})$	$\eta_{1^{1}5^{-5}}$	$II_{10,2}(2_{II}^{+10})$	$\eta_{1^{8}2^{-16}}$
$II_{6,2}(5^{+3})$	$5\eta_{1^{-5}5^{1}}$	$II_{10,2}(2_{II}^{+2})$	$16\eta_{1^{-16}2^{8}}$
$II_{6,2}(2_{II}^{+6}3^{-4})$	$\eta_{1^{2}2^{-4}3^{2}6^{-4}}$	$II_{10,2}(5^{+6})$	$5\eta_{1^{-4}5^{-4}}$
$II_{6,2}(2_{II}^{+2}3^{-4})$	$4\eta_{1^{-4}2^{2}3^{-4}6^{2}}$	$II_{10,2}(2_{II}^{+6}3^{-6})$	$\eta_{1^{-2}2^{-2}3^{-2}6^{-2}}$
$II_{6,2}(2_{II}^{-4}3^{-6})$	$\eta_{1^{1}2^{1}3^{-3}6^{-3}}$	$II_{14,2}(3^{-8})$	$\eta_{1^{-6}3^{-6}}$
$II_{6,2}(2_{II}^{-4}3^{-2})$	$3\eta_{1^{-3}2^{-3}3^{1}6^{1}}$	$II_{18,2}(2_{II}^{+10})$	$\eta_{1^{-8}2^{-8}}$
$II_{6,2}(2_{II}^{-6}5^{-3})$	$\eta_{1^{-2}2^{-1}5^{2}10^{-3}}$	$II_{26,2}$	$\eta_{1^{-24}}$

In the rest of this section we prove the following converse theorem:

Theorem 4.3. *Let L be an even lattice of signature $(n,2)$, $n \geq 4$ and squarefree level N that splits $II_{1,1} \oplus II_{1,1}(N)$ and let F be a symmetric strongly-reflective modular form with $[F_0](0) = n-2$ on $D = L'/L$. Then L is one of the lattices from Table 4.1 and $F = F_{D,\Gamma_0(N),f,0}$ for the corresponding f.*

Proof. The vector-valued modular form F is determined by its principal part, so we only need to show that the principal parts of the functions $F_{D,\Gamma_0(N),f,0}$ for $D = L'/L$ with L and f from Table 4.1 are the only ones that can occur. This is done as follows: We already know that L is one of finitely many lattices because of Theorem 3.24 and we have the bounds on N and $k = 1 - n/2$ from Table 3.1. The remaining cases can be looked at separately. The principal part of F is determined by the Fourier coefficients $c_{\gamma,-1/m} = [F_\gamma](-1/m)$ for $\gamma \in D_{m,1/m}$ for positive integers $m \mid N$ because these are the only non-vanishing coefficients of the principal part by the definition of reflectivity. Since

F is assumed to be strongly-reflective, we also know that $c_{\gamma,-1/m} \in \{0,1\}$. Moreover, as $\operatorname{Aut}(D)$ acts transitively on $D_{m,1/m}$ (see Proposition 1.53), we know that

$$c_{\gamma,-1/m} = c_{\beta,-1/m}$$

for all $\beta, \gamma \in D_{m,1/m}$ with fixed m. We can therefore define $c_m = c_{\gamma,-1/m}$ for any $\gamma \in D_{m,1/m}$ (such a γ exists because L splits $II_{1,1}(N)$).

Since F is symmetric, there is some $f \in M_k^!(\Gamma_0(N), \chi_D)$ such that $F = F_{D,\Gamma_0(N),f,0}$. Let $d \mid N$, $d' = N/d$ and let M_d be a matrix as in Subsection 2.3.3. We can apply Lemma 3.22 to see that $f|_k M_d \in O(q^{-1/d'})$. Let a_d be the coefficient of $f|_k M_d$ at $q^{-1/d'}$. From the formula

$$F_\gamma = \sum_{l|d'} \xi(M_l)^{-1} \frac{\sqrt{|D_l|}}{\sqrt{|D|}} l'(f|_k M_l)_{-Q(\gamma)}$$

for the $\Gamma_0(N)$-lift (see (2.6)), we find that

$$c_d = \xi(M_{d'}^{-1}) \frac{\sqrt{|D_{d'}|}}{\sqrt{|D|}} d a_{d'} \tag{4.1}$$

because the summands for $l < d'$ give no contribution to c_d, similarly as in the proof of Lemma 3.22. Equation (4.1) is equivalent to

$$a_d = \xi(M_d) \frac{\sqrt{|D|}}{\sqrt{|D_d|}} \frac{c_{d'}}{d'}.$$

Now let $k' = 1 + n/2$ and let $g \in S_{k'}(\Gamma_0(N), \chi_D)$. Then the product fg is a weakly holomorphic modular form of weight 2 and trivial character for $\Gamma_0(N)$. Therefore,

$$h = \sum_{M \in \Gamma_0(N) \backslash \mathrm{SL}_2(\mathbb{Z})} (fg)|_2 M \tag{4.2}$$

is a weakly holomorphic modular form of weight 2 for $\mathrm{SL}_2(\mathbb{Z})$. Such functions can be identified with meromorphic differentials on the modular curve $X(1)$ (see e.g. [DS05], Section 3.3). Since h is weakly holomorphic this differential is holomorphic everywhere except possibly at the cusp ∞. The residue theorem then shows that the residue at ∞ must vanish and this residue is up to a non-zero constant given by the constant term in the Fourier expansion of h. The vanishing of this coefficient then gives conditions on the numbers a_d and c_d as follows:

Let P be a set of representatives for the cusps of $\Gamma_0(N)$. Then

$$h = \sum_{M \in \Gamma_0(N) \backslash \mathrm{SL}_2(\mathbb{Z})} (fg)|_2 M = \sum_{s \in P} \sum_{\substack{M \in \Gamma_0(N) \backslash \mathrm{SL}_2(\mathbb{Z}) \\ M\infty = s}} (fg)|_2 M.$$

We can take P to be $P = \{1/d : d \mid N\}$ and a set of representatives for the cosets of $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbb{Z})$ that send ∞ to $1/d$ is given by $M_d T^j$ where $j = 0, \ldots, t_d - 1$ and $t_d = d'$ is the width of $1/d$ (all of this follows easily from Proposition 2.12). Therefore,

$$h = \sum_{d|N} \sum_{j=0}^{d'-1} (fg)|_2 M_d T^j.$$

To obtain the constant Fourier coefficient of h we must therefore add the constant Fourier coefficients of the functions $(fg)|_2 M_d T^j$. But the constant coefficient of $(fg)|_2 M_d T^j$ is equal to that of $(fg)|_2 M_d$ for all j, so the constant coefficient of h is given by

$$\sum_{d|N} d' \cdot [(fg)|_2 M_d](0) = \sum_{d|N} d' \sum_{\substack{\alpha \in \frac{1}{d'}\mathbb{Z} \\ \alpha > 0}} [f|_k M_d](-\alpha)[g|_{k'} M_d](\alpha)$$
$$= \sum_{d|N} d'[f|_k M_d](-1/d')[g|_{k'} M_d](1/d'),$$

where in the last step we have used that $f|_k M_d \in O(q^{-1/d'})$. Letting b_d be the coefficient of $g|_{k'} M_d$ at $q^{1/d'}$, we have thus shown that

$$0 = \sum_{d|N} d' a_d b_d = \sum_{d|N} \xi(M_d) \frac{\sqrt{|D|}}{\sqrt{|D_d|}} c_{d'} b_d. \tag{4.3}$$

We have described how to compute the coefficients b_d for a basis of $S_{k'}(\Gamma_0(N), \chi_D)$ in Subsection 2.2.2. The resulting conditions are described in Appendix B.

So far we have not used that we know the coefficient $[F_0](0)$. In order to change this, we let E be the Eisenstein series of weight k' for the dual Weil representation ρ_D^* as described in Subsection 2.3.4. We let

$$h(\tau) = \sum_{\gamma \in D} E_\gamma(\tau) F_\gamma(\tau),$$

i.e. $h(\tau)$ is the inner product of $E(\tau)$ and $F(\tau)$ in $\mathbb{C}[D]$. By the definition of the dual representation, h is a modular form of weight 2 for $\mathrm{SL}_2(\mathbb{Z})$. With the same argument as before the constant coefficient of h must vanish. But this coefficient is given by

$$[F_0](0)[E_0](0) + \sum_{d|N} c_d E_d N(D_d, 1)$$

where $E_d = [E_\gamma](1/d)$ for any $\gamma \in D_{d,1/d}$ (this is well-defined because E is symmetric) and $N(D_d, 1)$ is as in Proposition 1.57. Inserting $[F_0](0) = n - 2$, $[E_0](0) = 1$ and the values for E_d given in Proposition 2.70 yields

$$\frac{k}{k-2} \frac{1}{B_{k,\psi}} \frac{L(k,\psi)}{L(k,\chi)} \frac{m^k}{N^k} \sum_{cd|N} \varepsilon_{c,d} c_d N(D_d, 1) \frac{\sqrt{m_c |D_c|}}{\sqrt{n|D|}} \frac{N^k}{c^k d^{k-1}} = 1, \tag{4.4}$$

where

$$\varepsilon_{c,d} = \psi_c(N^2/(cdm_c)) \psi_{c'}(-c) \frac{\psi_c(2)}{\psi(2)} \frac{\varepsilon_c}{\varepsilon_N} \prod_{\substack{p | c/m_c \\ p | n}} (p-1) \prod_{\substack{p | c/m_c \\ p \nmid n}} (-1)$$

and the remaining notation is as in Proposition 2.70.

Computing and solving the equations from cusp forms and the Eisenstein condition in all of the remaining cases completes the proof. We have included a detailed computation for one case in Appendix B. □

Remark 4.4. That the constant coefficient of the function h in (4.2) vanishes for all $g \in S_{k'}(\Gamma_0(N), \chi_D)$ is not only a necessary condition for the existence of f (and hence F) but also sufficient (this can for example be seen by Serre duality; see [Bor99], Theorem 3.1). With the method described in the proof one can therefore obtain all symmetric strongly-reflective modular forms F without the restriction that $[F_0](0) = n-2$ by solving the linear equations (4.3) for all cusp forms. If one no longer requires the numbers c_m to be in $\{0, 1\}$ but in $\mathbb{C}$ one can classify all symmetric semi-reflective modular forms. This is done in Appendix B.

4.1.2. The Non-Symmetric Case

We now suppose that the strongly-reflective modular form F with $[F_0](0) = n - 2$ is not symmetric. We can no longer define c_d by $c_{\gamma,-1/d}$ since this is not independent of the choice of $\gamma \in D_{d,1/d}$. Instead, we let

$$\tilde{c}_d = \sum_{\gamma \in D_{d,1/d}} c_{\gamma,-1/d}.$$

Of course the numbers $\tilde{c}_d$ no longer determine the principal part of F. However, they determine the principal part of the symmetrization

$$G = \frac{1}{|\operatorname{Aut}(D)|} \sum_{\sigma \in \operatorname{Aut}(D)} \sigma(F),$$

which is no longer reflective but still semi-reflective. Note that $[G_0](0) = [F_0](0) = n - 2$ and that

$$[G_\gamma](-1/d) = \frac{1}{N(D_d, 1)} \tilde{c}_d$$

for $d \mid N$ and $\gamma \in D_{d,1/d}$. If we let $c_d = \tilde{c}_d / N(D_d, 1)$, then all the equations from the proof of Theorem 4.3 must still hold as these did not use that F was strongly-reflective but only that it was semi-reflective. The only difference is that we no longer have $c_d \in \{0, 1\}$. However, we know that $\tilde{c}_d = c_d N(D_d, 1)$ is an integer satisfying $0 \le \tilde{c}_d \le N(D_d, 1)$ and that $\tilde{c}_d$ is even if $d > 2$ because $F_\gamma = F_{-\gamma}$ for all $\gamma \in D$. We can now prove the following result:

Theorem 4.5. *Let L be an even lattice of squarefree level N and signature $(n, 2)$, $n \ge 4$ such that L splits $II_{1,1} \oplus II_{1,1}(N)$ and let F be a strongly-reflective modular form on $D = L'/L$ with $[F_0](0) = n - 2$. If F is not symmetric, then L and the numbers $\tilde{c}_d$ are one of the following.*

n	N	L	
10	2	$II_{10,2}(2_{II}^{+4})$	$\tilde{c}_1 = 0, \tilde{c}_2 = 2$
		$II_{10,2}(2_{II}^{+6})$	$\tilde{c}_1 = 0, \tilde{c}_2 = 4$
		$II_{10,2}(2_{II}^{+8})$	$\tilde{c}_1 = 0, \tilde{c}_2 = 8$
		$II_{10,2}(2_{II}^{+10})$	$\tilde{c}_1 = 0, \tilde{c}_2 = 16$

8	3	$II_{8,2}(3^{+5})$	$\tilde{c}_1 = 0, \tilde{c}_3 = 18$
		$II_{8,2}(3^{-7})$	$\tilde{c}_1 = 0, \tilde{c}_3 = 54$
6	3	$II_{6,2}(3^{-4})$	$\tilde{c}_1 = 0, \tilde{c}_3 = 4$
	5	$II_{6,2}(5^{+5})$	$\tilde{c}_1 = 0, \tilde{c}_5 = 100$
	6	$II_{6,2}(2_{II}^{-4}3^{+4})$	$\tilde{c}_1 = \tilde{c}_2 = 0, \tilde{c}_3 = 6, \tilde{c}_6 = 60$
		$II_{6,2}(2_{II}^{-4}3^{-6})$	$\tilde{c}_1 = \tilde{c}_2 = 0, \tilde{c}_3 = 18, \tilde{c}_6 = 180$
		$II_{6,2}(2_{II}^{+4}3^{-4})$	$\tilde{c}_1 = 0, \tilde{c}_2 = 2, \tilde{c}_3 = 0, \tilde{c}_6 = 60$
		$II_{6,2}(2_{II}^{+6}3^{-4})$	$\tilde{c}_1 = 0, \tilde{c}_2 = 4, \tilde{c}_3 = 0, \tilde{c}_6 = 120$
4	14	$II_{4,2}(2_{II}^{+4}7^{-3})$	$\tilde{c}_1 = 0, \tilde{c}_2 = 2, \tilde{c}_7 = 0, \tilde{c}_{14} = 112$

Proof. Note that $N \neq 1$ because F is not symmetric. If N is prime, the assertion is part of [Sch17], Theorem 6.27. In the other cases the conditions coming from cusp forms for $\Gamma_0(N)$ and the Eisenstein series leave the following non-symmetric possibilities:

n	N	L	
8	6	$II_{8,2}(2_{II}^{+4}3^{-3})$	$\tilde{c}_1 = 0, \tilde{c}_2 = 3, \tilde{c}_3 = 6, \tilde{c}_6 = 36$
		$II_{8,2}(2_{II}^{+6}3^{-3})$	$\tilde{c}_1 = 0, \tilde{c}_2 = 7, \tilde{c}_3 = 6, \tilde{c}_6 = 84$
		$II_{8,2}(2_{II}^{+4}3^{-7})$	$\tilde{c}_1 = 1, \tilde{c}_2 = 3, \tilde{c}_3 = 0, \tilde{c}_6 = 2268$
		$II_{8,2}(2_{II}^{+6}3^{-7})$	$\tilde{c}_1 = 1, \tilde{c}_2 = 7, \tilde{c}_3 = 0, \tilde{c}_6 = 5292$
6	6	$II_{6,2}(2_{II}^{-4}3^{+4})$	$\tilde{c}_1 = \tilde{c}_2 = 0, \tilde{c}_3 = 6, \tilde{c}_6 = 60$
			$\tilde{c}_1 = 0, \tilde{c}_2 = 1, \tilde{c}_3 = 6, \tilde{c}_6 = 36$
			$\tilde{c}_1 = 0, \tilde{c}_2 = 2, \tilde{c}_3 = 6, \tilde{c}_6 = 12$
		$II_{6,2}(2_{II}^{-4}3^{-6})$	$\tilde{c}_1 = \tilde{c}_2 = 0, \tilde{c}_3 = 18, \tilde{c}_6 = 180$
			$\tilde{c}_1 = 0, \tilde{c}_2 = 2, \tilde{c}_3 = 24, \tilde{c}_6 = 84$
		$II_{6,2}(2_{II}^{+4}3^{-4})$	$\tilde{c}_1 = 0, \tilde{c}_2 = 2, \tilde{c}_3 = 0, \tilde{c}_6 = 60$
			$\tilde{c}_1 = 0, \tilde{c}_2 = \tilde{c}_3 = 2, \tilde{c}_6 = 48$
			$\tilde{c}_1 = 0, \tilde{c}_2 = 2, \tilde{c}_3 = 4, \tilde{c}_6 = 36$
			$\tilde{c}_1 = 0, \tilde{c}_2 = 2, \tilde{c}_3 = 6, \tilde{c}_6 = 24$
			$\tilde{c}_1 = 0, \tilde{c}_2 = 2, \tilde{c}_3 = 8, \tilde{c}_6 = 12$
			$\tilde{c}_1 = 0, \tilde{c}_2 = 2, \tilde{c}_3 = 10, \tilde{c}_6 = 0$
		$II_{6,2}(2_{II}^{+6}3^{-4})$	$\tilde{c}_1 = 0, \tilde{c}_2 = 4, \tilde{c}_3 = 0, \tilde{c}_6 = 120$
			$\tilde{c}_1 = 0, \tilde{c}_2 = 5, \tilde{c}_3 = 6, \tilde{c}_6 = 66$
			$\tilde{c}_1 = 0, \tilde{c}_2 = 6, \tilde{c}_3 = 12, \tilde{c}_6 = 12$
	10	$II_{6,2}(2_{II}^{+4}5^{+3})$	$\tilde{c}_1 = 0, \tilde{c}_2 = 3, \tilde{c}_5 = 20, \tilde{c}_{10} = 90$
		$II_{6,2}(2_{II}^{+4}5^{+5})$	$\tilde{c}_1 = 0, \tilde{c}_2 = 5, \tilde{c}_5 = 200, \tilde{c}_{10} = 650$
			$\tilde{c}_1 = 1, \tilde{c}_2 = 3, \tilde{c}_5 = 0, \tilde{c}_{10} = 1950$

4	6	$II_{4,2}(2_{II}^{-4}3^{-3})$	$\tilde{c}_1 = 0, \tilde{c}_2 = 1, \tilde{c}_3 = \tilde{c}_6 = 0$
			$\tilde{c}_1 = \tilde{c}_2 = \tilde{c}_3 = 0, \tilde{c}_6 = 12$
			$\tilde{c}_1 = \tilde{c}_2 = 0, \tilde{c}_3 = 2, \tilde{c}_6 = 8$
			$\tilde{c}_1 = \tilde{c}_2 = 0, \tilde{c}_3 = \tilde{c}_6 = 4$
	14	$II_{4,2}(2_{II}^{+4}7^{-3})$	$\tilde{c}_1 = 0, \tilde{c}_2 = 2, \tilde{c}_7 = 0, \tilde{c}_{14} = 112$
	15	$II_{4,2}(3^{+3}5^{-3})$	$\tilde{c}_1 = 0, \tilde{c}_3 = 4, \tilde{c}_5 = 10, \tilde{c}_{15} = 0$
	30	$II_{4,2}(2_{II}^{+4}3^{-3}5^{+3})$	$\tilde{c}_1 = 0, \tilde{c}_2 = 3, \tilde{c}_3 = 6, \tilde{c}_5 = 20, \tilde{c}_6 = 36,$ $\tilde{c}_{10} = 90, \tilde{c}_{15} = 0, \tilde{c}_{30} = 360$
		$II_{4,2}(2_{II}^{+4}3^{+3}5^{-3})$	$\tilde{c}_1 = \tilde{c}_2 = 1, \tilde{c}_3 = 8, \tilde{c}_5 = 20, \tilde{c}_6 = 30,$ $\tilde{c}_{10} = 100, \tilde{c}_{15} = 120, \tilde{c}_{30} = 360$
			$\tilde{c}_1 = 1, \tilde{c}_2 = 2, \tilde{c}_3 = 4, \tilde{c}_5 = 10, \tilde{c}_6 = 24,$ $\tilde{c}_{10} = 80, \tilde{c}_{15} = 120, \tilde{c}_{30} = 720$
			$\tilde{c}_1 = 1, \tilde{c}_2 = 3, \tilde{c}_3 = \tilde{c}_5 = 0, \tilde{c}_6 = 18,$ $\tilde{c}_{10} = 60, \tilde{c}_{15} = 120, \tilde{c}_{30} = 1080$

For these remaining cases we proceed as follows: Let $\gamma \in D$. Then F_γ is an element of $M_k^!(\Gamma_1(N), \chi_\gamma)$. If g is a cusp form of weight $k' = 1 + n/2$ and character $\overline{\chi_\gamma}$ for $\Gamma_1(N)$, then (similarly to h in (4.2))

$$h = \sum_{M \in \Gamma_1(N) \backslash \mathrm{SL}_2(\mathbb{Z})} (F_\gamma g)|_2 M$$

is a weakly holomorphic modular form of weight 2 for $\mathrm{SL}_2(\mathbb{Z})$, which we call the pairing of F_γ with g. As described in the proof of Theorem 4.3, the constant term in its Fourier expansion must vanish. If we let P be a set of representatives for the cusps of $\Gamma_1(N)$, then we can write h as

$$h = \sum_{s \in P} \sum_{\substack{M \in \Gamma_1(N) \backslash \mathrm{SL}_2(\mathbb{Z}) \\ M\infty = s}} (F_\gamma g)|_2 M.$$

Note that $N > 2$ in our cases. It then follows easily from Proposition 2.13 that a set of representatives for the cosets of $\Gamma_1(N)$ in $\mathrm{SL}_2(\mathbb{Z})$ mapping ∞ to the cusp a/c is given by $\pm M_{a/c} T^j$ where $M_{a/c}$ is any matrix in $\mathrm{SL}_2(\mathbb{Z})$ mapping ∞ to a/c and $j = 0, \ldots, t_c - 1$, where $t_c = N/(c, N)$ is the width of a/c. Therefore,

$$h = 2 \sum_{a/c \in P} \sum_{j=0}^{t_c - 1} (F_\gamma g)|_2 M_{a/c} T^j.$$

The constant coefficient of $(F_\gamma g)|_2 M_{a/c} T^j$ equals that of $(F_\gamma g)|_2 M_{a/c}$, so the constant coefficient of h is given by the constant coefficient of

$$2 \sum_{a/c \in P} t_c (F_\gamma|_k M_{a/c})(g|_{k'} M_{a/c}). \tag{4.5}$$

We see that we need the Fourier expansions of $F_\gamma|_k M_{a/c}$ and of $g|_{k'} M_{a/c}$ for all $a/c \in P$. For F_γ these can be computed using Proposition 2.64. In the following g is always an eta quotient, so its Fourier expansions can be computed with Proposition 2.38.

We now apply this procedure to the remaining cases. We use the following notation: For a positive divisor d of N we let

$$M_d = \{\gamma \in D_{d,1/d} : c_{\gamma,-1/d} = 1\}.$$

Furthermore, for $\gamma \in D$ we define $a_\gamma^d = |M_d \cap \gamma^\perp|$. If e is another positive divisor of N, then we let $N_\gamma^{d,e}$ be the number of elements in M_e that map to γ under the natural projection $D = D_d \oplus D^d \to D_d$.

- If $n = 8$ and $N = 6$, we let $\beta \in D_{2,1/2}$ and

$$g_1(\tau) = \eta(\tau)^6\eta(2\tau)^3\eta(3\tau)^2\eta(6\tau)^{-1}.$$

This is a cusp form of weight $k' = 5$ and character $\overline{\chi_\beta} = \chi_\beta$ for $\Gamma_1(6)$ by Propositions 2.42 and 2.43. The group $\Gamma_1(6)$ has four cusps, represented by $s = 1/6, 1/3, 1/2$ and $1/1$. We choose the matrices

$$M_{1/6} = \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}, \quad M_{1/3} = \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}, \quad M_{1/2} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}, \quad M_{1/1} = \begin{pmatrix} 1 & 5 \\ 1 & 6 \end{pmatrix}.$$

With Proposition 2.64 we can compute

$$\begin{aligned}
F_\beta|_{-3}M_{1/6} &= F_\beta = c_{\beta,-1/2}q^{-1/2} + O(q^{1/2}),\\
F_\beta|_{-3}M_{1/3} &= \xi(M_{1/3})\frac{\sqrt{|D_3|}}{\sqrt{|D|}}\sum_{\mu\in D^3} e(-(\beta,\mu))e(-Q(\beta))F_{\beta+\mu}\\
&= -\epsilon_2\frac{1}{\sqrt{|D_2|}}\sum_{\mu\in D_2} e((\beta,\mu))F_\mu\\
&= -\epsilon_2\frac{1}{\sqrt{|D_2|}}\left(\tilde{c}_1q^{-1} + (a_\beta^2 - (\tilde{c}_2 - a_\beta^2))q^{-1/2}\right) + O(1)\\
&= -\epsilon_2\frac{1}{\sqrt{|D_2|}}\left(\tilde{c}_1q^{-1} + \left(2a_\beta^2 - \tilde{c}_2\right)q^{-1/2}\right) + O(1),\\
F_\beta|_{-3}M_{1/2} &= \xi(M_{1/2})\frac{\sqrt{|D_2|}}{\sqrt{|D|}}\sum_{\mu\in D^2} e(-(\beta,\mu))e(-Q(\beta))F_{\beta+\mu}\\
&= -i\epsilon_2\frac{1}{\sqrt{|D_3|}}\sum_{\mu\in\beta+D_3} F_\mu\\
&= -i\epsilon_2\frac{1}{\sqrt{|D_3|}}\left(c_{\beta,-1/2}q^{-1/2} + N_\beta^{2,6}q^{-1/6}\right) + O(q^{1/6}),\\
F_\beta|_{-3}M_{1/1} &= \xi(M_{1/1})\frac{\sqrt{|D_1|}}{\sqrt{|D|}}\sum_{\mu\in D^1} e((\beta,\mu))e(-Q(\beta))F_{\beta+\mu}
\end{aligned}$$

$$
\begin{aligned}
&= i\frac{1}{\sqrt{|D|}}\sum_{\mu\in D} e((\beta,\mu))F_\mu \\
&= i\frac{1}{\sqrt{|D|}}\left(\tilde{c}_1 q^{-1} + (2a_\beta^2 - \tilde{c}_2)q^{1/2} + \tilde{c}_3 q^{-1/3} + (2a_\beta^6 - \tilde{c}_6)q^{-1/6}\right) + O(1).
\end{aligned}
$$

The Fourier expansions of g_1 at the cusps can be computed using Proposition 2.38 and are given by

$$
\begin{aligned}
g_1|_5 M_{1/6} &= g_1 = \eta(\tau)^6\eta(2\tau)^3\eta(3\tau)^2\eta(6\tau)^{-1} \\
&= q^{1/2} + O(q^{3/2}), \\
g_1|_5 M_{1/3} &= -\frac{1}{2}\eta(\tau)^6\eta(\tau/2)^3\eta(3\tau)^2\eta(3\tau/2)^{-1} \\
&= -\frac{1}{2}q^{1/2} + \frac{3}{2}q + O(q^{3/2}), \\
g_1|_5 M_{1/2} &= -i\frac{\sqrt{3}}{3}\eta(\tau)^6\eta(2\tau)^3\eta(\tau/3)^2\eta(2\tau/3)^{-1} \\
&= -i\frac{\sqrt{3}}{3}\left(q^{1/2} - 2q^{5/6}\right) + O(q^{3/2}), \\
g_1|_5 M_{1/1} &= i\frac{\sqrt{3}}{6}\eta(\tau)^6\eta(\tau/2)^3\eta(\tau/3)^2\eta(\tau/6)^{-1} \\
&= i\frac{\sqrt{3}}{6}\left(q^{1/3} + q^{1/2} - 2q^{5/6} - 3q\right) + O(q^{4/3}).
\end{aligned}
$$

Inserting these expansions into (4.5) we find that the constant coefficient of the pairing of F_β with h_1 is up to a non-zero constant given by

$$
\left(1 - \epsilon_2\frac{\sqrt{3}}{\sqrt{|D_3|}}\right)c_{\beta,-1/2} + \left(\epsilon_2\frac{1}{\sqrt{|D_2|}} - \frac{\sqrt{3}}{\sqrt{|D|}}\right)\left(-3\tilde{c}_1 + 2a_\beta^2 - \tilde{c}_2\right) - \frac{\sqrt{3}}{\sqrt{|D|}}\tilde{c}_3.
$$

Since this coefficient must vanish we obtain

$$
a_\beta^2 = \frac{3}{2}\tilde{c}_1 + \frac{1}{2}\tilde{c}_2 - \epsilon_2\frac{\sqrt{|D_2|}}{2}c_{\beta,-1/2} - \frac{1}{2}\left(1 - \epsilon_2\frac{\sqrt{|D_3|}}{\sqrt{3}}\right)^{-1}\tilde{c}_3. \tag{4.6}
$$

If we choose $\beta \in M_2$ (which we can do because $\tilde{c}_2 \neq 0$ in the remaining cases), then we obtain $a_\beta^2 = 1$ for all remaining lattices with $n = 8$ and $N = 6$.

Another cusp form of weight 5 and character χ_β for $\Gamma_1(6)$ is given by

$$
g_2(\tau) = \eta(\tau)^7\eta(2\tau)^{-2}\eta(3\tau)^7\eta(6\tau)^{-2}.
$$

The Fourier expansions of g_2 at the cups can be computed similarly to those of g_1. The equation resulting from pairing F_β with g_2 is then given by

$$
0 = \left(1 - \epsilon_2\frac{7}{3\sqrt{3|D_3|}}\right)c_{\beta,-1/2} - \left(\epsilon_2\frac{8}{\sqrt{|D_2|}} + \frac{8}{\sqrt{3|D|}}\right)\tilde{c}_1 + \epsilon_2\frac{1}{3\sqrt{3|D_3|}}N_\beta^{2,6}
$$

$$- \frac{16}{3\sqrt{3|D|}}(2a_\beta^2 - \tilde{c}_2) - \frac{8}{3\sqrt{3|D|}}\tilde{c}_3.$$

Solving for $N_\beta^{2,6}$ yields

$$N_\beta^{2,6} = \left(7 - \epsilon_2 3\sqrt{3|D_3|}\right) c_{\beta,-1/2} + \frac{24}{\sqrt{|D_2|}} \left(\epsilon_2 + \sqrt{3|D_3|}\right) \tilde{c}_1 + \epsilon_2 \frac{8}{\sqrt{|D_2|}} \tilde{c}_3 + \epsilon_2 \frac{16}{\sqrt{|D_2|}} \left(2a_\beta^2 - \tilde{c}_2\right).$$

Letting $\beta \in M_2$ and inserting $a_\beta^2 = 1$ and the other values gives a negative value for $N_\beta^{2,6}$ in the cases where $n_3 = 3$, which is of course impossible. For the lattice $L_1 = II_{8,2}(2_{II}^{+4}3^{-7})$ we obtain $N_\beta^{2,6} = 252$ and for $L_2 = II_{8,2}(2_{II}^{+6}3^{-7})$ we obtain $N_\beta^{2,6} = 0$.
We continue by letting $\alpha \in D_{6,1/6}$ and

$$g_3(\tau) = \eta(\tau)^5\eta(2\tau)^4\eta(3\tau)^5\eta(6\tau)^{-4}.$$

Then g_3 is a cusp form of weight 5 and character $\overline{\chi_\alpha}$ for $\Gamma_1(6)$ by Propositions 2.42 and 2.43 and we can compute the pairing of F_α with g_3 in the same way as the pairings we have seen before. The resulting condition is

$$0 = c_{\alpha,-1/6} + \epsilon_2 \frac{\sqrt{3}}{\sqrt{|D_3|}} c_{3\alpha,-1/2} - \frac{48\sqrt{3}}{\sqrt{|D|}}\tilde{c}_1 - \frac{8\sqrt{3}}{\sqrt{|D|}}(2a_\alpha^2 - \tilde{c}_2) - \frac{2\sqrt{3}}{\sqrt{|D|}} \left(\frac{3}{2}a_\alpha^3 - \tilde{c}_3\right). \tag{4.7}$$

For both L_1 and L_2 there exists an $\alpha \in M_6$ with $3\alpha \notin M_2$ because

$$\sum_{\beta \in M_2} N_\beta^{2,6} < \tilde{c}_6.$$

For such an α we can compute a_α^2 using (4.6) because $a_\alpha^2 = a_{3\alpha}^2$. We obtain

$$a_\alpha^2 = \frac{3 + \tilde{c}_2}{2}$$

for both L_1 and L_2. Inserting this and the other known values into (4.7) shows that

$$0 = 1 - \frac{72\sqrt{3}}{\sqrt{|D|}},$$

i.e. $|D| = 2^6 \cdot 3^5$, which is not the case. This completes the proof that the combination $n = 8$ and $N = 6$ cannot occur.

- If $n = 6$ and $N = 6$, we let $\beta \in D_{2,1/2}$ and

$$g_1(\tau) = \eta(\tau)^9\eta(2\tau)^{-3}\eta(3\tau)\eta(6\tau).$$

Pairing F_β with g_1 yields

$$0 = \left(1 + \epsilon_2 \frac{1}{\sqrt{|D_3|}}\right)\left(c_{\beta,-1/2} + \epsilon_2 \frac{12}{\sqrt{|D_2|}}\tilde{c}_1 + \epsilon_2 \frac{4}{\sqrt{|D_2|}}\left(2a_\beta^2 - \tilde{c}_2\right)\right) - \epsilon_2 \frac{1}{\sqrt{|D_3|}} N_\beta^{2,6} - \frac{4}{\sqrt{|D|}}\tilde{c}_3. \tag{4.8}$$

We describe how this can be used to show that the case $L = II_{6,2}(2_{II}^{-4}3^{+4})$ with $\tilde{c}_2 = 1$ cannot occur: There is an element, say β, in $D_{2,1/2} \setminus M_2$ because

$$\tilde{c}_2 = 1 < 10 = N(D_2, 1).$$

Then a_β^2 is 1 or 0 depending on whether β is orthogonal to the element in M_2 or not. If it is 0 then $N_\beta^{2,6}$ must be -2 by (4.8). This is impossible; hence $a_\beta^2 = 1$, in which case (4.8) yields $N_\beta^{2,6} = 14$. There are $N(D_2, 1) - \tilde{c}_2 = 9$ elements in $D_{2,1/2} \setminus M_2$. It follows that the number $\tilde{c}_6$ of elements in M_6 is at least $9 \cdot 14 = 126$, which is a contradiction.

With a similar argument one can also eliminate all cases except those stated in the proposition and the case $L = II_{6,2}(2_{II}^{+6}3^{-4})$ with $\tilde{c}_2 = 5$. In this case we let $\beta \in M_2$. Then (4.8) gives

$$N_\beta^{2,6} = 10a_\beta^2 - 18. \tag{4.9}$$

We can compute the pairing of F_β with

$$g_2(\tau) = \eta(\tau)^4 \eta(2\tau)^{-2} \eta(6\tau)^6$$

and obtain

$$0 = \left(\epsilon_2 + \frac{1}{\sqrt{|D_3|}}\right)\left(\frac{1}{\sqrt{|D_2|}}\tilde{c}_1 + \frac{1}{2\sqrt{|D_2|}}(2a_\beta^2 - \tilde{c}_2)\right) - \epsilon_2 \frac{1}{9\sqrt{|D_3|}} N_\beta^{2,6} + \frac{1}{18\sqrt{|D|}}\left(2a_\beta^6 - \tilde{c}_6\right) - \frac{1}{3\sqrt{|D|}}\tilde{c}_3. \tag{4.10}$$

Inserting the values for L we see that

$$180a_\beta^2 - 16N_\beta^{2,6} + 2a_\beta^6 = 552$$

and with (4.9) this becomes

$$10a_\beta^2 + a_\beta^6 = 132,$$

which is impossible because $a_\beta^2 \leq \tilde{c}_2 = 5$ and $a_\beta^6 \leq \tilde{c}_6 = 66$.

- If $n = 6$ and $N = 10$, we let $\beta \in D_{2,1/2}$ and

$$g(\tau) = \eta(\tau)^7 \eta(5\tau),$$

which is in $S_4(\Gamma_1(N), \chi_\beta)$ by Propositions 2.42 and 2.43. The congruence subgroup $\Gamma_1(10)$ has eight cusps, represented by $1/1, 1/2, 1/3, 1/4, 1/5, 2/5, 1/10$ and $3/10$. Possible choices for $M_{a/c}$ are therefore

$$M_{1/1} = \begin{pmatrix} 1 & 9 \\ 1 & 10 \end{pmatrix}, M_{1/2} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}, M_{1/3} = \begin{pmatrix} 1 & 3 \\ 3 & 10 \end{pmatrix}, M_{1/4} = \begin{pmatrix} 1 & 1 \\ 4 & 5 \end{pmatrix},$$
$$M_{1/5} = \begin{pmatrix} 1 & 1 \\ 5 & 6 \end{pmatrix}, M_{2/5} = \begin{pmatrix} 2 & 3 \\ 5 & 8 \end{pmatrix}, M_{1/10} = \begin{pmatrix} 1 & 0 \\ 10 & 1 \end{pmatrix}, M_{3/10} = \begin{pmatrix} 3 & 2 \\ 10 & 7 \end{pmatrix}.$$

As before we can compute the Fourier expansions of $F_\beta|_{-2}M_{a/c}$ and $g|_4 M_{a/c}$ for these matrices to calculate the condition coming from pairing F_β with g. This condition turns out to be

$$0 = \left(2 - \epsilon_2 \frac{2\sqrt{5}}{\sqrt{|D_5|}}\right) c_{\beta,-1/2} + \left(\frac{4\sqrt{5}}{\sqrt{|D|}} - \epsilon_2 \frac{4}{\sqrt{|D_2|}}\right)(2a_\beta^2 - \tilde{c}_2).$$

Therefore

$$a_\beta^2 = \epsilon_2 \frac{\sqrt{|D_2|}}{4} c_{\beta,-1/2} + \frac{1}{2}\tilde{c}_2.$$

In the cases that we are considering we have $\tilde{c}_2 \neq N(D_2, 1)$, so we can assume that β is not in M_2. Then $a_\beta^2 = \tilde{c}_2/2$, so $\tilde{c}_2$ must be even. But $\tilde{c}_2$ is odd in all remaining cases with $n = 6$ and $N = 10$, so these cannot occur.

- If $n = 4$ and $N = 6$, then the only possible lattice is $L = II_{4,2}(2_{II}^{-4}3^{-3})$. We let $\beta \in D_{2,1/2}$ and

$$g_1(\tau) = \eta(\tau)^3 \eta(3\tau)^3.$$

Then g_1 is is in $S_3(\Gamma_1(6), \chi_\beta)$ and we can compute the pairing of F_β with g_1 in the same way as before. The resulting condition is

$$0 = \frac{4}{3}c_{\beta,-1/2} + \frac{4}{3}a_\beta^2 - \frac{2}{3}\tilde{c}_2 - \frac{1}{9}N_\beta^{2,6} - \frac{1}{9}a_\beta^6 + \frac{1}{18}\tilde{c}_6. \tag{4.11}$$

If $\tilde{c}_2 = 1$, then $\tilde{c}_6 = 0$ and therefore a_β^6 and $N_\beta^{2,6}$ must also be 0. Then

$$a_\beta^2 = \frac{1}{2} - c_{\beta,-1/2} \notin \mathbb{Z},$$

which is not possible. Therefore, $\tilde{c}_2 = 0$ and consequently also $a_\beta^2 = c_{\beta,-1/2} = 0$.

  - If $\tilde{c}_3 = \tilde{c}_6 = 4$, then

  $$N_\beta^{2,6} = 2 - a_\beta^6$$

  by (4.11). Since $N_\beta^{2,6}$ cannot be larger than a_β^6 (because every element in M_6 that projects to β under $D = D_2 \oplus D_3 \to D_2$ is orthogonal to β) we must have $N_\beta^{2,6} \leq 1$. But $N_\beta^{2,6}$ must be even (if $\alpha \in M_6$ projects to β, then so does $-\alpha$, which must also be in M_6 because $F_\alpha = F_{-\alpha}$). Therefore $N_\beta^{2,6} = 0$. This must hold for every $\beta \in D_{2,1/2}$, which is only possible if $\tilde{c}_6 = 0$. This is a contradiction.

- If $\tilde{c}_3 = 2$ and $\tilde{c}_6 = 8$, then we let $\gamma \in M_3$. Note that then $M_3 = \{\pm\gamma\}$, so $M_3 \cap \gamma^\perp = \emptyset$, i.e. $a^3_\gamma = 0$. The function

$$g_2(\tau) = \eta(\tau)^2\eta(2\tau)^3\eta(3\tau)^2\eta(6\tau)^{-1}$$

is in $S_3(\Gamma_1(6), \overline{\chi_\gamma})$ and pairing F_γ with g_2 yields

$$\begin{aligned} 0 &= \frac{5}{4}c_{\gamma,-1/3} - \frac{5}{8}a^3_\gamma + \frac{5}{24}\tilde{c}_3 - \frac{1}{8}a^6_\gamma + \frac{1}{24}\tilde{c}_6 \\ &= 2 - \frac{1}{8}a^6_\gamma, \end{aligned}$$

so $a^6_\gamma = 16$. This is impossible because a^6_γ cannot be larger than $\tilde{c}_6$.

- If $\tilde{c}_3 = 0$ and $\tilde{c}_6 = 12$, then (4.11) simplifies to

$$N^{2,6}_\beta = 6 - a^6_\beta.$$

Since $N^{2,6}_\beta \le a^6_\beta$ and even as described before, it must be 0 or 2. There must therefore be six elements $\beta_1, \ldots, \beta_6 \in D_{2,1/2}$ with $N^{2,6}_{\beta_i} = 2$ and $a^6_{\beta_i} = 4$. Moreover, each of these elements is orthogonal to itself and exactly one of the others because

$$a^6_\beta = \sum_{\gamma \in D_{2,1/2} \cap \beta^\perp} N^{2,6}_\gamma$$

for every $\beta \in D_{2,1/2}$.

We continue by letting $\alpha \in D_{6,1/6}$ and

$$g_3(\tau) = \eta(\tau)^{10}\eta(2\tau)^{-3}\eta(3\tau)^{-2}\eta(6\tau),$$

which Propositions 2.42 and 2.43 show to be in $S_3(\Gamma_1(6), \overline{\chi_\alpha})$. Pairing F_α with g_3 yields

$$0 = c_{\alpha,-1/6} + \frac{3}{2}N^\perp_{3\alpha} - 1 - 2M^\perp_{4\alpha} + N^{3,6}_{4\alpha}, \tag{4.12}$$

where $N^\perp_{3\alpha}$ is the number of elements in $M_6 \cap (4\alpha)^\perp$ that map to 3α under the natural projection $D = D_2 \oplus D_3 \to D_2$ and $M^\perp_{4\alpha}$ is the number of elements in $M_6 \cap (3\alpha)^\perp$ that map to 4α under $D = D_2 \oplus D_3 \to D_3$.

Suppose there is an element $\delta \in D_{3,2/2}$ such that $\delta + \beta \notin M_6$ for all $\beta \in D_{2,1/2}$, i.e. such that $N^{3,6}_\delta = 0$. Then we let $\alpha_1 = \delta + \beta_1$, with β_1 as above. Note that $4\alpha_1 = \delta$, so both $c_{\alpha_1,-1/6}$ and $N^{3,6}_{4\alpha_1}$ are 0. The same is true for $M^\perp_{4\alpha_1}$, so $N^\perp_{3\alpha_1} = 2/3$ by (4.12), which is of course impossible.

There is therefore no $\delta \in D_{3,2/3}$ with $N^{3,6}_\delta = 0$. Since

$$|D_{3,2/3}| = N(D_3, 2) = 12 = \tilde{c}_6,$$

we must have $N^{3,6}_\delta = 1$ for all $\delta \in D_{3,2/3}$; hence there is exactly one $\beta \in D_{2,1/2}$ with $\beta + \delta \in M_6$ for every $\delta \in D_{3,2/3}$. We let $\alpha \in M_6$. Then $N^{3,6}_{4\alpha} = 1$ as just explained, i.e. there is exactly one element in M_6 that projects to 4α under $D \to D_3$. This element is α, which is orthogonal to 3α, so $M^\perp_{4\alpha} = 1$. Then again $N^\perp_{3\alpha} = 2/3$ by (4.12), again giving a contradiction.

- If $n = 4$ and $N = 15$, then $L = II_{4,2}(3^{+3}5^{-5})$ and $\tilde{c}_3 = 4$, so there exists an element γ in M_3. We let

$$g(\tau) = \eta(3\tau)\eta(5\tau)^7\eta(15\tau)^{-2} \in S_3(\Gamma_1(15), \overline{\chi_\gamma}).$$

There are 16 classes of cusps for $\Gamma_1(15)$, represented by

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \frac{1}{7}, \frac{1}{15}, \frac{2}{15}, \frac{4}{15}, \frac{7}{15}$$

and possible choices for $M_{a/c}$ are therefore

$$\begin{aligned}
M_{1/1} &= \begin{pmatrix} 1 & 14 \\ 1 & 15 \end{pmatrix}, M_{1/2} = \begin{pmatrix} 1 & 7 \\ 2 & 15 \end{pmatrix}, M_{1/3} = \begin{pmatrix} 1 & 3 \\ 3 & 10 \end{pmatrix}, M_{2/3} = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}, \\
M_{1/4} &= \begin{pmatrix} 1 & 11 \\ 4 & 45 \end{pmatrix}, M_{1/5} = \begin{pmatrix} 1 & 1 \\ 5 & 6 \end{pmatrix}, M_{2/5} = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}, M_{3/5} = \begin{pmatrix} 3 & 7 \\ 5 & 12 \end{pmatrix}, \\
M_{4/5} &= \begin{pmatrix} 4 & 7 \\ 5 & 9 \end{pmatrix}, M_{1/6} = \begin{pmatrix} 1 & 4 \\ 6 & 25 \end{pmatrix}, M_{5/6} = \begin{pmatrix} 5 & 4 \\ 6 & 5 \end{pmatrix}, M_{1/7} = \begin{pmatrix} 1 & 2 \\ 7 & 15 \end{pmatrix}, \\
M_{1/15} &= \begin{pmatrix} 1 & 0 \\ 15 & 1 \end{pmatrix}, M_{2/15} = \begin{pmatrix} 2 & 1 \\ 15 & 8 \end{pmatrix}, M_{4/15} = \begin{pmatrix} 4 & 1 \\ 15 & 4 \end{pmatrix}, M_{7/15} = \begin{pmatrix} 7 & 6 \\ 15 & 13 \end{pmatrix}.
\end{aligned}$$

As before we compute the Fourier expansions of $F_\gamma|_{-1}M_{a/c}$ and $g|_3 M_{a/c}$ and the pairing of F_γ with g. The resulting condition is

$$\begin{aligned}
0 =& 4 - \epsilon_5 \frac{28}{5\sqrt{5|D_5|}} - \frac{216}{5\sqrt{15|D|}} \left(\frac{3}{2} a_\gamma^3 - \frac{1}{2}\tilde{c}_3 \right) - \frac{36}{\sqrt{15|D|}} \tilde{c}_5 \\
&- \frac{36}{5\sqrt{15|D|}} \left(\frac{3}{2} a_\gamma^{15} - \frac{1}{2}\tilde{c}_{15} \right),
\end{aligned}$$

which in our case simplifies to

$$0 = \frac{376}{125} - \frac{36}{125} a_\gamma^3,$$

which is impossible because a_γ^3 must be an integer.

- For $n = 4$ and $N = 30$ we let $\beta \in D_{2,1/2}$ and

$$g(\tau) = \eta(\tau)^3\eta(2\tau)^{-1}\eta(5\tau)^2\eta(6\tau)\eta(10\tau)\eta(15\tau)\eta(30\tau)^{-1} \in S_3(\Gamma_1(30), \chi_\beta).$$

There are 32 classes of cusps for $\Gamma_1(30)$, so the computation of the Fourier expansions of F_β and g becomes quite tedious. In the end the pairing of F_β and g leads to

$$0 = \left(4 - \epsilon_5 \frac{8\sqrt{5}}{5\sqrt{|D_5|}} + \frac{4\sqrt{15}}{5\sqrt{|D_{15}|}} \right) c_{\beta,-1/2} - \left(\epsilon_5 \frac{8\sqrt{5}}{5\sqrt{|D_{10}|}} + \frac{8\sqrt{15}}{5\sqrt{|D|}} \right) \tilde{c}_5 + \frac{16\sqrt{15}}{5\sqrt{|D|}} \tilde{c}_3$$

$$- \left(\frac{8}{\sqrt{|D_2|}} - \epsilon_5 \frac{8\sqrt{3}}{\sqrt{|D_6|}} + \epsilon_5 \frac{8\sqrt{5}}{\sqrt{|D_{10}|}} + \frac{24\sqrt{15}}{5\sqrt{|D|}} \right) \tilde{c}_1 - \frac{8\sqrt{15}}{5\sqrt{|D|}} \left(2a_\beta^6 - \tilde{c}_6\right)$$
$$+ \left(\frac{8\sqrt{15}}{5\sqrt{|D|}} - \epsilon_5 \frac{24\sqrt{5}}{5\sqrt{|D_{10}|}} \right) \left(2a_\beta^2 - \tilde{c}_2\right) - \frac{4\sqrt{15}}{5\sqrt{|D_{15}|}} N_\beta^{2,6} + \frac{4\sqrt{15}}{5\sqrt{|D_{15}|}} N_\beta^{2,10}.$$

Suppose $L = II_{4,2}(2_{II}^{+4}3^{-3}5^{+3})$. Since $\tilde{c}_2 = 3 \neq 0$, we can let $\beta \in M_2$. Then the lengthy expression simplifies to

$$a_\beta^2 = \frac{33}{4} + \frac{1}{8} N_\beta^{2,10} - \frac{1}{8} N_\beta^{2,6} - \frac{1}{8} a_\beta^6. \tag{4.13}$$

Note that $N_\beta^{2,6}$ is at most equal to the number of elements in $D_{3,2/3}$, i.e.

$$N_\beta^{2,6} \leq N(D_3, 2) = 12.$$

Moreover, $a_\beta^6 \leq \tilde{c}_6 = 36$. Therefore, (4.13) shows that a_β^2 is at least 3. Since $a_\beta^2 \leq \tilde{c}_2 = 3$, we must in fact have $a_\beta^2 = 3$. This holds for any $\beta \in M_2$, so the three elements in M_2 must be pairwise orthogonal. However, one easily checks that it is not possible to choose three pairwise orthogonal elements γ with $\mathrm{ord}(\gamma) = 2$ and $Q(\gamma) = 1/2 + \mathbb{Z}$ in 2_{II}^{+4}. Therefore, this case cannot occur. A similar argument also works in the other remaining cases with $N = 30$. □

Theorem 4.6. *Let L be an even lattice of squarefree level N and signature $(n, 2)$, $n \geq 4$ such that L splits $II_{1,1} \oplus II_{1,1}(N)$. Let F be a strongly-reflective modular form on $D = L'/L$ with $[F_0](0) = n - 2$. If F is not symmetric, then there exists an isotropic subgroup $H \subset D$ such that F is the lift $\hat{F}_H$ of a symmetric strongly-reflective modular form F_H for $D_H = H^\perp/H$ on H. The function F_H satisfies $[(F_H)_0](0) = n - 2$.*

Proof. We prove this separately for the cases from Theorem 4.5. If the level N is prime, then this follows from [Sch17], Theorem 6.27, so we can assume that N is 6 or 14.

- Suppose the level N is 6. If $\tilde{c}_2 > 0$, then $L = II_{6,2}(2_{II}^{+n_2}3^{-4})$ for $n_2 = 4$ or $n_2 = 6$. Let $\beta \in M_2$. Since $\tilde{c}_1 = \tilde{c}_3 = 0$ and $\tilde{c}_2 = 2^{n_2/2-1}$, (4.8) states that

  $$N_\beta^{2,6} = \frac{80}{2^{n_2/2}} a_\beta^2 - 10.$$

  Furthermore, we also have

  $$N_\beta^{2,6} = \frac{90}{2^{n_2/2}} a_\beta^2 + \frac{1}{2^{n_2/2}} a_\beta^6 - 30$$

  because of (4.10). These two conditions together yield

  $$N_\beta^{2,6} = 150 - \frac{8}{2^{n_2/2}} a_\beta^6 \geq 30$$

because $a_\beta^6 \le \tilde{c}_6 = 30\cdot 2^{n_2/2-1}$. As there are only $N(D_3,2) = 30$ elements in $D_{3,2/3}$, it follows that $N_\beta^{2,6} = 30$ and therefore $a_\beta^2 = \tilde{c}_2$ and $a_\beta^6 = \tilde{c}_6$, i.e. the elements in M_2 are pairwise orthogonal and

$$M_6 = M_2 + D_{3,2/3}. \tag{4.14}$$

Let H be the set of isotropic elements in D_2 that are orthogonal to M_2, i.e.

$$H = D_{2,0} \cap M_2^\perp.$$

Let $\beta \in M_2$ and $\mu \in H$. Then $\beta+\mu$ is obviously in $D_{2,1/2}$, but it is also in M_2; otherwise (4.8) gives $N_{\beta+\mu}^{2,6} = 20$, which contradicts (4.14). Therefore $\beta + H \subset M_2$. But the other inclusion $M_2 \subset \beta + H$ also holds: Let $\beta' \in M_2$. Then $\beta'-\beta$ is isotropic (because β' and β are orthogonal) and in $M_2^\perp$ (because β' and β are). Now that we have shown that $M_2 = \beta + H$ we prove that H is a group: Let $\mu_1, \mu_2 \in H$ and write $\mu_1 = \beta_1 - \beta$ and $\mu_2 = \beta_2 - \beta$ with $\beta_1, \beta_2 \in M_2$. Then $\mu_1 - \mu_2 = \beta_1 - \beta_2$ is isotropic and orthogonal to M_2 and hence in H. On $D_H = H^\perp/H = 2_{II}^{+2}3^{-4}$ there is a symmetric strongly-reflective modular form F_H with $[(F_H)_0](0) = 4$ (see Table 4.1), which can be lifted on H. It is easy to see that the principal parts of $\hat{F}_H$ and F coincide, so $F = \hat{F}_H$.

If $\tilde{c}_3 > 0$, then $L = II_{6,2}(2_{II}^{-4}3^{\epsilon_3 n_3})$ where $\epsilon_3 n_3 = +4$ or -6, $\tilde{c}_3 = 2\cdot 3^{n_3/2-1}$ and $\tilde{c}_6 = 20\cdot 3^{n_3/2-1}$. Let $\gamma \in M_3$. The function $\eta(\tau)^8$ is in $S_4(\Gamma_1(6), \overline{\chi_\gamma})$ by Propositions 2.42 and 2.43 and the pairing of F_γ with $\eta(\tau)^8$ results in the condition

$$0 = \left(1 + \epsilon_2 \frac{2}{\sqrt{|D_2|}}\right) c_{\gamma,-1/3} - \left(\epsilon_2 \frac{3}{\sqrt{|D_3|}} + \frac{6}{\sqrt{|D|}}\right)\left(\frac{3}{2}a_\gamma^3 - \frac{1}{2}\tilde{c}_3\right), \tag{4.15}$$

which in our case simplifies to $a_\gamma^3 = 0$, i.e. no two elements in M_3 are orthogonal. Next we compute the pairing of F_γ with

$$\eta(\tau)^{-1}\eta(2\tau)^9\eta(3\tau)^3\eta(6\tau)^{-3} \in S_4(\Gamma_1(6), \overline{\chi_\gamma})$$

and obtain

$$\begin{aligned} 0 = &\left(1 - \epsilon_2 \frac{1}{4\sqrt{|D_2|}}\right) c_{\gamma,-1/3} + \left(\frac{9}{\sqrt{|D|}} - \epsilon_2 \frac{9}{\sqrt{|D_3|}}\right)\tilde{c}_1 - \frac{9}{4\sqrt{|D|}}\left(\frac{3}{2}a_\gamma^3 - \frac{1}{2}\tilde{c}_3\right) \\ &- \frac{3}{4\sqrt{|D|}}\left(\frac{3}{2}a_\gamma^6 - \frac{1}{2}\tilde{c}_6\right) - \frac{9}{2\sqrt{|D_2|}}\tilde{c}_2, \end{aligned}$$

which shows that in our case $a_\gamma^6 = \tilde{c}_6$, i.e $M_3 \perp M_6$. We continue by letting $\delta \in D_{3,2/3}$ and pairing F_δ with

$$\eta(\tau)^9\eta(2\tau)^{-1}\eta(3\tau)^{-3}\eta(6\tau)^3 \in S_4(\Gamma_1(6), \overline{\chi_\delta}).$$

This yields

$$\begin{aligned}0 = &\left(\epsilon_2 - \frac{1}{\sqrt{|D_2|}}\right)\left(\frac{18}{\sqrt{|D_3|}}\tilde{c}_1 + \frac{3}{\sqrt{|D_3|}}\left(\frac{3}{2}a_\delta^3 - \frac{1}{2}\tilde{c}_3\right)\right)\\ &- \epsilon_2\frac{1}{\sqrt{|D_2|}}N_\delta^{3,6} + \frac{9}{\sqrt{|D|}}\tilde{c}_2.\end{aligned} \tag{4.16}$$

If $N_\delta^{3,6} \neq 0$, then $a_\delta^3 = \tilde{c}_3$ because $M_3 \perp M_6$. In this case (4.16) shows that $N_\delta^{3,6} = 10$. This proves that there are $\tilde{c}_6/10 = \tilde{c}_3$ elements $\delta \in D_{3,2/3}$ with $N_\delta^{3,6} = 10$ and that $N_\delta^{3,6} = 0$ for the remaining $\delta \in D_{3,2/3}$. We let H be the set of isotropic elements in $D_3 \cap M_3^\perp$ and let $\gamma \in M_3$ and $\mu \in H$. Then $\gamma + \mu$ is obviously in $D_{3,1/3}$ and $M_3 \cap (\gamma+\mu)^\perp = \emptyset$. Applying (4.15) to $\gamma + \mu$ shows that this is only possible if $\gamma + \mu \in M_3$. Therefore, $M_3 + H \subset M_3$. We now want to show that this is actually an equality of sets: Let $\gamma_1, \gamma_2 \in M_3$. Since these are not orthogonal, exactly one of $\gamma_1 + \gamma_2$ and $\gamma_1 - \gamma_2$ is isotropic. By replacing γ_2 with $-\gamma_2$ if necessary, we can assume that $\mu = \gamma_1 - \gamma_2$ is isotropic. The pairing of μ with

$$\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^2 \in S_4(\Gamma_1(6))$$

can be computed as usual and gives

$$\begin{aligned}0 = &\left(\epsilon_2\frac{1}{\sqrt{|D_3|}} - \frac{1}{\sqrt{|D|}}\right)\tilde{c}_1 + \left(\frac{1}{3\sqrt{|D|}} - \epsilon_2\frac{1}{3\sqrt{|D_3|}}\right)\left(\frac{3}{2}a_\mu^3 - \frac{1}{2}\tilde{c}_3\right)\\ &- \frac{1}{6\sqrt{|D|}}\left(\frac{3}{2}a_\mu^6 - \frac{1}{2}\tilde{c}_6\right) + \frac{1}{2\sqrt{|D_2|}}\tilde{c}_2,\end{aligned}$$

which, using that $\mu \in M_6^\perp$, results in $a_\mu^3 = \tilde{c}_3$, i.e. $\mu \in M_3^\perp$. This proves that $\mu \in H$ and that $M_3 = \pm\gamma + H$ for any $\gamma \in M_3$. Next we show that

$$M_6 = \pm\delta + D_{2,1/2} + H,$$

where δ is any element in $D_{3,2/3}$ with $N_\delta^{3,6} = 10$: Note that both sides have the same size because $|D_{2,1/2}| = N(D_2, 1) = 10$ and $|H| = \frac{1}{2}\tilde{c}_3 = 3^{n_3/2-1}$. It therefore suffices to show that the right hand side is contained in the left. We let S be the set of elements $\delta \in D_{3,2/3}$ with $N_\delta^{3,6} = 10$. Since $|D_{2,1/2}| = N_\delta^{3,6}$, we only have to show that $\pm\delta + H \subset S$. Let $\mu \in H$. We have seen that $M_3 = \pm\gamma + H$ for $\gamma \in M_3$, so every element in H is the sum of two elements in M_3. Since M_3 is orthogonal to M_6, this shows that $H \subset M_6^\perp$, so in particular $\mu \in M_6^\perp$. Therefore, μ and δ are orthogonal and $Q(\pm\delta + \mu) = 2/3 + \mathbb{Z}$ for all $\delta \in S$. By definition, μ is also orthogonal to M_3, so that $\pm\delta + \mu$ is orthogonal to M_3. We can then use (4.16) to see that $N_{\pm\delta+\mu}^{3,6} = 10$, so $\pm\delta + \mu$ is in S, which is what we wanted to show. Finally we prove that H is a group: Let $\mu_1, \mu_2 \in H$. Then obviously $\mu_1 - \mu_2 \in M_3^\perp$, so we only have to show that $\mu_1 - \mu_2$ is isotropic. Suppose it is not. Then $Q(\mu_1 - \mu_2)$ is $1/3 + \mathbb{Z}$ or $2/3 + \mathbb{Z}$. By replacing μ_2 with $-\mu_2$ if necessary, we can assume that $Q(\mu_1 - \mu_2) = 2/3 + \mathbb{Z}$.

Then (4.16) shows that μ is in S and therefore not in $M_6^\perp$, which is impossible because $H \subset M_6^\perp$. Therefore $\mu_1 - \mu_2$ is isotropic and hence in H and H is indeed a group. That F must then be a lift of a symmetric strongly-reflective modular form on $D_H = H^\perp/H$ can be seen as in the case with $\tilde{c}_2 > 0$.

- If $N = 14$, then $L = II_{4,2}(2_{II}^{+4}7^{-3})$ and $\tilde{c}_2 = 2$, so we can let $\beta \in M_2$. The pairing of F_β with

$$g_1(\tau) = \eta(\tau)^5\eta(7\tau) \in S_3(\Gamma_1(14), \chi_\beta)$$

is computed as usual and yields

$$0 = 3 - 6\frac{1}{\sqrt{|D_2|}}(2a_\beta^2 - \tilde{c}_2) - 3\frac{\sqrt{7}}{\sqrt{|D_7|}} + 6\frac{\sqrt{7}}{\sqrt{|D|}}(2a_\beta^2 - \tilde{c}_2),$$

which in our case simplifies to $a_\beta^2 = 2$, i.e. the two elements in M_2 are orthogonal. We can also pair F_β with

$$g_2(\tau) = \eta(\tau)^{-2}\eta(2\tau)^7\eta(7\tau)^2\eta(14\tau)^{-1} \in S_3(\Gamma_1(14), \chi_\beta)$$

and obtain

$$\begin{aligned} 0 = &\left(3 + \frac{3\sqrt{7}}{\sqrt{|D_7|}}\right)c_{\beta,-1/2} - \left(\frac{3}{4\sqrt{|D_2|}} + \frac{3\sqrt{7}}{4\sqrt{|D|}}\right)\left(2a_\beta^2 - \tilde{c}_2\right) - \frac{3\sqrt{7}}{4\sqrt{|D|}}\tilde{c}_7 \\ &+ \left(\frac{21}{4\sqrt{|D_2|}} + \frac{21\sqrt{7}}{4\sqrt{|D|}}\right)\tilde{c}_1 - \frac{3\sqrt{7}}{4\sqrt{|D|}}\left(2a_\beta^{14} - \tilde{c}_{14}\right). \end{aligned}$$

Inserting the known values yields $a_\beta^{14} = 112 = \tilde{c}_{14}$, i.e. β is orthogonal to M_{14}. Since this holds for both $\beta \in M_2$, we obtain $M_2 \perp M_{14}$. There are exactly 112 elements in $D_{14,1/14}$ that are orthogonal to M_2, namely those in $M_2 + D_{7,4/7}$. Let $H = \{0, \beta_1 - \beta_2\}$, where β_1 and β_2 are the two elements in M_2. Then $M_2 = \beta_1 + H$ and $M_{14} = \beta_1 + H + D_{7,4/7}$. As in the previous cases, it then follows that F is the lift of the unique symmetric strongly-reflective modular form F_H with $[(F_H)_0](0) = 2$ on $D_H = H^\perp/H$. □

This has the following consequence for strongly-reflective automorphic forms of singular weight:

Corollary 4.7. *Let L be an even lattice of signature $(n,2), n \geq 4$ and squarefree level N that splits $II_{1,1} \oplus II_{1,1}(N)$ and Ψ a strongly-reflective automorphic form of singular weight for the discriminant kernel Γ_L. Then the function Ψ can be identified with $\Psi(F_{D,\Gamma_0(N),f,0})$ with f one of the functions from Table 4.1. In particular, Ψ can be realized as the theta lift of a symmetric form F.*

Proof. By Proposition 3.15, the function Ψ is of the form $\Psi = \Psi(F)$ for some vector-valued modular form F, which must be strongly-reflective because of Proposition 3.19. For symmetric F the result follows from Theorem 4.3. If F is not symmetric, then Theorem 4.6 shows that F is the lift of a symmetric strongly-reflective form F_H on some smaller discriminant form $D_H = H^\perp/H$. In all cases D_H can be realized as N'/N for some suitable sublattice $N \subset L$ as described at the end of Section 3.1 and Ψ can therefore be identified with $\Psi(F_H)$ by Proposition 3.16. □

4.2. New Strongly-Reflective Automorphic Forms of Singular Weight

In the previous section we have seen that if the lattice L has squarefree level N and splits $II_{1,1} \oplus II_{1,1}(N)$, then as functions on the corresponding set $\tilde{\mathcal{K}}^+$ all strongly-reflective automorphic forms of singular weight for the discriminant kernel Γ_L can be constructed from the Leech lattice as described at the beginning of the previous section. If L is no longer required to split $II_{1,1} \oplus II_{1,1}(N)$, then there are examples of strongly-reflective automorphic products of singular weight that are not of this form. If N is prime, such examples can be found in [Sch17].

In this section we construct strongly-reflective modular forms F with $[F_0](0) = n-2$. In all cases the underlying lattice L is an even lattice of signature $(n, 2)$, with $4 \leq n \leq 8$ and level 6 that splits $II_{1,1}(6)$ but not $II_{1,1}$. The corresponding automorphic product $\Psi(F)$ is then a strongly-reflective automorphic form of singular weight. To the best knowledge of the author these do not yet appear in the literature.

4.2.1. n=8

If $n = 8$, then we build one new strongly-reflective automorphic form of singular weight. For the construction we need the following lemma:

Lemma 4.8. *Let D be the discriminant form with Jordan symbol 2_{II}^{-10}. Up to automorphisms of D there is exactly one set $S \subset A_2$ with $|S| = 66$ and*

$$\left|S \cap \gamma^\perp\right| = \begin{cases} 46 & \text{if } \gamma \in S, \\ 30 & \text{if } \gamma \in A_2 \setminus S, \\ 34 & \text{if } Q(\gamma) = 0 + \mathbb{Z} \text{ and } \gamma \neq 0. \end{cases}$$

Proof. The discriminant form D is the orthogonal sum of five copies of 2_{II}^{-2}. Let β_i and β_i' with

$$Q(\beta_i) = Q(\beta_i') = (\beta_i, \beta_i') = 1/2 + \mathbb{Z}$$

be generators of the i-th copy. We then let S be the smallest set that contains the elements

$$\beta_1, \beta_1', \beta_2, \beta_2', \beta_1 + \beta_2 + \beta_3, \beta_4, \beta_4', \beta_5, \beta_5', \beta_1 + \beta_3' + \beta_4, \beta_3 + \beta_4 + \beta_5$$

and that is closed under taking sums of two non-orthogonal elements. It is easy to explicitly determine all elements of S and to check that S has the desired properties.

We now want to show the uniqueness of S up to automorphisms of D: Suppose S_1 and S_2 are two sets with the properties stated in the proposition and let $\gamma_1 \in S_1$. Since S_1 has exactly 46 elements orthogonal to γ_1, there must be a $\gamma_2 \in S_1$ with $(\gamma_1, \gamma_2) = 1/2 + \mathbb{Z}$. Then γ_1, γ_2 span a discriminant form isomorphic to 2_{II}^{-2} and their orthogonal complement A in D is isomorphic to 2_{II}^{+8}. With the same argument there exist $\gamma_1', \gamma_2' \in S_2$ with $(\gamma_1', \gamma_2') = 1/2 + \mathbb{Z}$ and orthogonal complement A' isomorphic to 2_{II}^{+8}. Let σ be any isomorphism from A to A'. We can extend σ to an automorphism of D by defining

$\sigma(\gamma_1) = \gamma_1'$ and $\sigma(\gamma_2) = \gamma_2'$. We can therefore assume that $\gamma_1 = \gamma_1'$ and $\gamma_2 = \gamma_2'$. Then of course $A = A'$. For each pair $a = (a_1, a_2) \in \mathbb{F}_2^2$ we denote by x_a the number of elements $\beta \in S_1$ which project to $a_1\gamma_1 + a_2\gamma_2$ under the projection

$$\begin{aligned} D = \langle \gamma_1, \gamma_2 \rangle \oplus A &\to \langle \gamma_1, \gamma_2 \rangle \\ (\alpha, \beta) &\mapsto \alpha. \end{aligned}$$

An element $\beta \in D$ is orthogonal to γ_1 if and only if it projects to 0 or to γ_1. As γ_1 is orthogonal to exactly 46 elements in S_1, it follows that

$$x_{(0,0)} + x_{(1,0)} = 46.$$

By replacing γ_1 with $0, \gamma_2$ and $\gamma_1 + \gamma_2$, one similarly obtains

$$\begin{aligned} x_{(0,0)} + x_{(0,1)} + x_{(1,0)} + x_{(1,1)} &= 66 \\ x_{(0,0)} + x_{(0,1)} &= 46 \\ x_{(0,0)} + x_{(1,1)} &= \begin{cases} 46 & \text{if } \gamma_1 + \gamma_2 \in S_1, \\ 30 & \text{otherwise.} \end{cases} \end{aligned}$$

The only solution to these four equations is

$$\big(x_{(0,0)}, x_{(0,1)}, x_{(1,0)}, x_{(1,1)}\big) = \begin{cases} (36, 10, 10, 10) & \text{if } \gamma_1 + \gamma_2 \in S_1, \\ (28, 18, 18, 2) & \text{otherwise.} \end{cases}$$

We see that $S_1 \cap A$ has at least 28 elements. It is not possible to find 28 pairwise orthogonal elements γ with $\mathrm{ord}(\gamma) = 2$ and $Q(\gamma) = 1/2 + \mathbb{Z}$ in 2_{II}^{+8}. Therefore, there are elements γ_3 and γ_4 with $(\gamma_3, \gamma_4) = 1/2 + \mathbb{Z}$ in $S_1 \cap A$. The orthogonal complement B of $\{\gamma_3, \gamma_4\}$ in A is then isomorphic to 2_{II}^{-6}. Replacing S_1 by S_2, we also obtain γ_3' and γ_4' with $(\gamma_3', \gamma_4') = 1/2 + \mathbb{Z}$ in $S_2 \cap A$ with orthogonal complement B' in A. With the argument already used for γ_1 and γ_2, we can assume that $\gamma_3 = \gamma_3'$, $\gamma_4 = \gamma_4'$ and $B = B'$. The set B is the orthogonal complement of $\{\gamma_1, \ldots, \gamma_4\}$ in D. For each 4-tuple $a = (a_1, a_2, a_3, a_4) \in \mathbb{F}_2^4$ we now denote by x_a the number of elements $\beta \in S_1$ that project to $a_1\gamma_1 + \ldots + a_4\gamma_4$ under the projection

$$\begin{aligned} D = \langle \gamma_1, \ldots, \gamma_4 \rangle \oplus B &\to \langle \gamma_1, \ldots, \gamma_4 \rangle \\ (\alpha, \beta) &\mapsto \alpha. \end{aligned}$$

This time we obtain 16 linear equations from the cardinalities of the sets $S_1 \cap \gamma^\perp$ for $\gamma \in \langle \gamma_1 \ldots, \gamma_4 \rangle$, with the right hand side depending on which of $\gamma_1 + \gamma_2$ and $\gamma_3 + \gamma_4$ are in S_1. The unique solution of these equations is a vector of non-negative integers only if both $\gamma_1 + \gamma_2$ and $\gamma_3 + \gamma_4$ are in S_1. In this case the solution is given by

$$x_a = x_{(a_1,a_2,a_3,a_4)} = \begin{cases} 15 & \text{if } a = (0,0,0,0), \\ 7 & \text{if } Q(a_1\gamma_1 + \ldots + a_4\gamma_4) = 1/2 + \mathbb{Z}, \\ 1 & \text{otherwise.} \end{cases} \tag{4.17}$$

We remark that, as γ_1 and γ_2 were arbitrary non-orthogonal elements in S_1 and $\gamma_1+\gamma_2$ must be in S_1, the set S_1 must be closed under taking the sum of two non-orthogonal elements.

We observe that (4.17) implies that there is exactly one element $\mu \in S_1$ that projects to $\gamma_1+\gamma_3$. We let $\gamma_5 = \gamma_1+\gamma_3+\mu \in B$ and choose an element $\gamma_6 \in B$ such that $Q(\gamma_6) = (\gamma_5,\gamma_6) = 1/2+\mathbb{Z}$. We do the same for S_2, obtain elements γ_5' and γ_6' and can assume that $\gamma_5 = \gamma_5'$ and $\gamma_6 = \gamma_6'$ with the argument that we have already used for γ_1 and γ_2 and for γ_3 and γ_4. Since S_1 and S_2 are closed under sums to two non-orthogonal elements, all elements of the form $\mu_1+\mu_2+\gamma_5$ with $\mu_1 \in \{\gamma_1,\gamma_2,\gamma_1+\gamma_2\}$ and $\mu_2 \in \{\gamma_3,\gamma_4,\gamma_3+\gamma_4\}$ must be in both S_1 and S_2. This gives nine elements in $S_1 \cap S_2$. Together with the six elements $\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_1+\gamma_2,\gamma_3+\gamma_4$ we have found 15 elements that are in $S_1\cap S_2$. Also note that neither $\gamma_1+\gamma_3+\gamma_6$ nor $\gamma_1+\gamma_3+\gamma_5+\gamma_6$ are in S_1 or in S_2, as there is only one element in S_1 and S_2 that projects to $\gamma_1+\gamma_3$, namely $\mu = \gamma_1+\gamma_3+\gamma_5$. Using that S_1 and S_2 are closed under sums of two non-orthogonal elements we find that also none of $\gamma_6, \gamma_5+\gamma_6, \mu_1+\mu_2+\gamma_6$ and $\mu_1+\mu_2+\gamma_5+\gamma_6$ (with μ_1 and μ_2 as before) are in S_1 or S_2.

Let C be the orthogonal complement of $\{\gamma_1,\dots,\gamma_6\}$ in D. Then C is isomorphic to 2_{II}^{+4}. For each 6-tuple $a=(a_1,\dots,a_6)\in\mathbb{F}_2^6$ we let x_a be the number of elements $\beta \in S_1$ that project to $a_1\gamma_1+\ldots+a_6\gamma_6$ under the projection

$$\begin{aligned} D = \langle\gamma_1,\dots,\gamma_6\rangle \oplus C &\to \langle\gamma_1,\dots,\gamma_6\rangle \\ (\alpha,\beta) &\mapsto \alpha. \end{aligned}$$

This time we get 64 linear equations with the right hand side depending on whether γ_5 is in S_1 or not. The solution has non-negative integral coordinates if and only if $\gamma_5 \notin S_1$. In this case

$$x_a = x_{(a_1,\dots,a_6)} = \begin{cases} 6 & \text{if } a=(0,\dots,0), \\ 3 & \text{if } a_6 = 1 \text{ and } Q(a_1\gamma_1+\ldots+a_6\gamma_6) = 0+\mathbb{Z}, \\ 9 & \text{if } a = (0,0,0,0,1,0), \\ 1 & \text{if } a_6=0, a\neq(0,0,0,0,1,0) \text{ and} \\ & \quad Q(a_1\gamma_1+\ldots+a_6\gamma_6) = 1/2+\mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.18}$$

Of course, replacing S_1 by S_2 gives the same solution for x_a. There are three elements in S_1 that project to $\gamma_1+\gamma_6$. Let μ be one of them and let $\gamma_7 = \gamma_1+\gamma_6+\mu \in C$. Similarly we let μ' be one of the three elements in S_2 that projects to $\gamma_1+\gamma_6$ and let $\gamma_7' = \gamma_1+\gamma_6+\mu' \in C$. Let γ_8 and γ_8' be elements in C with

$$Q(\gamma_8) = Q(\gamma_8') = (\gamma_7,\gamma_8) = (\gamma_7',\gamma_8') = 1/2+\mathbb{Z}.$$

As before, by applying an appropriate automorphism of C we can assume that $\gamma_7=\gamma_7'$ and $\gamma_8 = \gamma_8'$. Note that there are six elements γ with $Q(\gamma) = 1/2+\mathbb{Z}$ in C and that all of them must be in $S_1\cap S_2$ by (4.18). The elements γ_7, γ_8 and $\gamma_7+\gamma_8$ are such elements.

Since S_1 is closed under sums of two non-orthogonal elements, the three elements in S_1 that project to $\gamma_1 + \gamma_6$ are

$$\gamma_1 + \gamma_6 + \gamma_7, \gamma_1 + \gamma_6 + \gamma_8 \text{ and } \gamma_1 + \gamma_6 + \gamma_7 + \gamma_8$$

and the same is true for S_2. The discriminant form C contains 10 elements α of norm $0 + \mathbb{Z}$. For nine of them $\gamma_5 + \alpha$ is in S_1 because of (4.18). Since we already know that $\gamma_5 \notin S_1$, the only such α with $\gamma_5 + \alpha \notin S_1$ is $\alpha = 0$ and the same is true if we replace S_1 by S_2. The smallest set S containing all of the elements in $S_1 \cap S_2$ that we have found so far and that is also closed under sums of two non-orthogonal elements contains 66 elements, so $S = S_1 = S_2$. □

Proposition 4.9. *Let $L = II_{8,2}(2_{II}^{-10}3^{-5})$ and $D = L'/L$. Let $S \subset D_2$ be a subset as in the previous lemma and let*

$$g_1 = \eta(\tau)^4\eta(2\tau)^{-5}\eta(3\tau)^{-4}\eta(6\tau)^{-1},$$
$$g_2 = \eta(\tau)^{-1}\eta(2\tau)^{-4}\eta(3\tau)^{-5}\eta(6\tau)^4,$$
$$f_1 = \eta(\tau)^{-3}\eta(3\tau)^{-3}.$$

Then

$$F = F_{D,\Gamma_0(6),g_1,0} + 6F_{D,\Gamma_0(6),g_2,0} + \frac{1}{6}\sum_{\gamma \in S} F_{D,\Gamma_1(6),f_1,\gamma}$$

is a strongly-reflective modular form with $[F_0](0) = 6$ on D. Its principal part is given by

$$M_1 = \{0\}, \quad M_2 = S, \quad M_3 = D_{3,1/3}, \quad M_6 = M_2 + D_{3,2/3}.$$

Proof. The functions g_1 and g_2 are weakly holomorphic modular forms of weight -3 and character $\chi_D = \left(\frac{\cdot}{3}\right)$ for $\Gamma_0(6)$ by Propositions 2.40 and 2.43 and f_1 is a weakly holomorphic modular form of weight -3 and character $\chi(b) = e(b/2)$ for $\Gamma_1(6)$ by Propositions 2.42 and 2.43. Both $\Gamma_0(6)$ and $\Gamma_1(6)$ have four cusps, represented by $1/6, 1/3, 1/2$ and $1/1$. We let

$$M_6 = \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}, M_3 = \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}, M_2 = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}, M_1 = \begin{pmatrix} 1 & 5 \\ 1 & 6 \end{pmatrix}$$

and compute the Fourier expansions of $g_1|_{-3}M_c$, $g_2|_{-3}M_c$ and $f_1|_{-3}M_c$ for all $c \mid N$ using Proposition 2.38 and obtain

$$g_1|_{-3}M_6 = \eta(\tau)^4\eta(2\tau)^{-5}\eta(3\tau)^{-4}\eta(6\tau)^{-1} = q^{-1} - 4 + O(q),$$
$$g_1|_{-3}M_3 = -8\eta(\tau)^4\eta(\tau/2)^{-5}\eta(3\tau)^{-4}\eta(3\tau/2)^{-1} = -8q^{-1/2} + 40 + O(q^{1/2}),$$
$$g_1|_{-3}M_2 = -9\sqrt{3}i\eta(\tau)^4\eta(2\tau)^{-5}\eta(\tau/3)^{-4}\eta(2\tau/3)^{-1} = -9\sqrt{3}i\left(q^{-1/3} + 4\right) + O(q^{1/3}),$$
$$g_1|_{-3}M_1 = -72\sqrt{3}i\eta(\tau)^4\eta(\tau/2)^{-5}\eta(\tau/3)^{-4}\eta(\tau/6)^{-1} = -72\sqrt{3}i + O(q^{1/6}),$$

$$g_2|_{-3}M_6 = \eta(\tau)^{-1}\eta(2\tau)^{-4}\eta(3\tau)^{-5}\eta(6\tau)^4 = 1 + O(q),$$

$$g_2|_{-3}M_3 = \eta(\tau)^{-1}\eta(\tau/2)^{-4}\eta(3\tau)^{-5}\eta(3\tau/2)^4 = q^{-1/2} + 4 + O(q^{1/2}),$$
$$g_2|_{-3}M_2 = \sqrt{3}i\eta(\tau)^{-1}\eta(2\tau)^{-4}\eta(\tau/3)^{-5}\eta(2\tau/3)^4 = \sqrt{3}i\left(q^{-1/3} + 5\right) + O(q^{1/3}),$$
$$g_2|_{-3}M_1 = -\sqrt{3}i\eta(\tau)^{-1}\eta(\tau/2)^{-4}\eta(\tau/3)^{-5}\eta(\tau/6)^4 = -\sqrt{3}i\left(q^{-1/6} - 4\right) + O(q^{1/6}),$$

$$f_1|_{-3}M_6 = \eta(\tau)^{-3}\eta(3\tau)^{-3} = q^{-1/2} + O(q^{1/2}),$$
$$f_1|_{-3}M_3 = \eta(\tau)^{-3}\eta(3\tau)^{-3} = q^{-1/2} + O(q^{1/2}),$$
$$f_1|_{-3}M_2 = 3\sqrt{3}i\eta(\tau)^{-3}\eta(\tau/3)^{-3} = 3\sqrt{3}iq^{-1/6} + O(q^{1/6}),$$
$$f_1|_{-3}M_1 = 3\sqrt{3}i\eta(\tau)^{-3}\eta(\tau/3)^{-3} = 3\sqrt{3}iq^{-1/6} + O(q^{1/6}).$$

With these Fourier expansions at hand we can now compute $F_{D,\Gamma_0(6),g_1,0}$, $F_{D,\Gamma_0(6),g_2,0}$ and $F_{D,\Gamma_1(6)f_1,\gamma}$ for $\gamma \in D_{2,1/2}$ using the formulas (2.6) and (2.7).

We explain the computation of $F_{D,\Gamma_1(6),f_1,\gamma}$ in more detail: To simplify notation, let $G = F_{D,\Gamma_1(6),f_1,\gamma}$. As explained in Subsection 2.3.3, we can write

$$G = G_6 + G_3 + G_2 + G_1$$

with

$$G_c = \xi(M_c^{-1})\frac{\sqrt{|D_c|}}{\sqrt{|D|}}c' \sum_{\mu\in\gamma+D_{c'}} e(b(\mu,\gamma))e(-bQ(\gamma))(f_1|_{-3}M_c)_{-Q(\mu)}(\tau)\left(\mathfrak{e}_\mu + \mathfrak{e}_{-\mu}\right).$$

The root of unity $\xi(M_c^{-1})$ (see Theorem 2.53) is given by

$$\xi(M_c^{-1}) = \begin{cases} 1 & \text{if } c = 6, \\ -1 & \text{if } c = 3, \\ i & \text{if } c = 2 \text{ or } c = 1. \end{cases}$$

Therefore,

$$G_6(\tau) = \left(\eta(\tau)^{-3}\eta(3\tau)^{-3}\right)_{1/2}\left(\mathfrak{e}_\gamma + \mathfrak{e}_{-\gamma}\right).$$

But note that $\gamma = -\gamma$ and that $\eta(\tau)^{-3}\eta(3\tau)^{-3}$ has a Fourier expansion in odd powers of $q^{1/2}$ and that therefore

$$\left(\eta(\tau)^{-3}\eta(3\tau)^{-3}\right)_{1/2} = \eta(\tau)^{-3}\eta(3\tau)^{-3},$$

so that

$$G_6(\tau) = 2\eta(\tau)^{-3}\eta(3\tau)^{-3}\mathfrak{e}_\gamma.$$

Similarly,

$$G_3(\tau) = \frac{1}{16}\sum_{\mu\in D_2} e((\mu,\gamma))\left(\eta(\tau)^{-3}\eta(3\tau)^{-3}\right)_{-Q(\mu)}\left(\mathfrak{e}_\mu + \mathfrak{e}_{-\mu}\right)$$

$$= \frac{1}{8} \sum_{\mu \in D_{2,1/2}} e((\mu,\gamma))\eta(\tau)^{-3}\eta(3\tau)^{-3}\mathfrak{e}_\mu.$$

In the same way we obtain

$$G_2(\tau) = 2 \sum_{\mu \in \gamma + D_3} (\eta(\tau)^{-3}\eta(\tau/3)^{-3})_{-Q(\mu)}\mathfrak{e}_\mu$$

and

$$G_1(\tau) = \frac{1}{8} \sum_{\mu \in D_{2,1/2}+D_3} \left(\eta(\tau)^{-3}\eta(\tau/3)^{-3}\right)_{-Q(\mu)} \cdot \mathfrak{e}_\mu$$

For an element $\mu \in D$ we can now compute the Fourier expansions of G_μ by collecting all the corresponding summands from G_1, G_2, G_3 and G_6. We obtain

$$G_\mu = \begin{cases} \frac{17}{8}q^{-1/2} + O(q^{1/2}) & \text{if } \mu = \gamma, \\ \frac{1}{8}e(\mu\gamma)q^{-1/2} + O(q^{1/2}) & \text{if } \mu \in D_{2,1/2} \setminus \{\gamma\}, \\ \frac{17}{8}q^{-1/6} + O(q^{5/6}) & \text{if } \mu \in \gamma + D_{3,2/3}, \\ \frac{1}{8}e(\mu\gamma)q^{-1/6} + O(q^{5/6}) & \text{if } \mu \in D_{2,1/2} \setminus \{\gamma\} + D_{3,2/3}, \\ O(q^{1/6}) & \text{otherwise.} \end{cases}$$

Now that we have computed $F_{D,\Gamma_1(6),f_1,\gamma}$, it is easy to compute

$$H = \sum_{\gamma \in S} F_{D,\Gamma_1(6),f_1,\gamma}.$$

For example, if $\mu \in S$, then μ is orthogonal to 45 elements in $S \setminus \{\mu\}$ and is not orthogonal to 20 elements in S. Therefore,

$$\begin{aligned} H_\mu &= \frac{17}{8}q^{-1/2} + 45 \cdot \frac{1}{8}q^{-1/2} + 20 \cdot \frac{-1}{8}q^{-1/2} + O(q^{1/2}) \\ &= \frac{21}{4}q^{-1/2} + O(q^{1/2}). \end{aligned}$$

For the other elements $\mu \in D$ we can compute H_μ in the same way and obtain

$$H_\mu = \begin{cases} \frac{21}{4}q^{-1/2} + O(q^{1/2}) & \text{if } \mu \in S, \\ -\frac{3}{4}q^{-1/2} + O(q^{1/2}) & \text{if } \mu \in D_{2,1/2} \setminus S, \\ \frac{21}{4}q^{-1/6} + O(q^{5/6}) & \text{if } \mu \in S + D_{3,2/3}, \\ -\frac{3}{4}q^{-1/6} + O(q^{5/6}) & \text{if } \mu \in \left(D_{2,1/2} \setminus S\right) + D_{3,2/3}, \\ O(q^{1/6}) & \text{otherwise.} \end{cases}$$

The components of $F_{D,\Gamma_0(6),g_1,0}$ and $F_{D,\Gamma_0(6),g_2,0}$ can be computed similarly using (2.6). One obtains

$$(F_{D,\Gamma_0(6),g_1,0})_\mu = \begin{cases} q^{-1}+12+O(q) & \text{if } \mu = 0, \\ 4+O(q) & \text{if } \mu \in D_{2,0}, \\ \frac{1}{2}q^{-1/2}+O(q^{1/2}) & \text{if } \mu \in D_{2,1/2}, \\ \frac{27}{2}+O(q) & \text{if } \mu \in D_{3,0}, \\ 3q^{-1/3}+O(q^{2/3}) & \text{if } \mu \in D_{3,1/3}, \\ \frac{3}{2}+O(q) & \text{if } \mu \in D_{6,0}, \\ O(q^{1/6}) & \text{otherwise} \end{cases}$$

and

$$(F_{D,\Gamma_0(6),g_2,0})_\mu = \begin{cases} -1+O(q) & \text{if } \mu = 0, \\ -\frac{1}{3}+O(q) & \text{if } \mu \in D_{2,0}, \\ -\frac{1}{16}q^{-1/2}+O(q^{1/2}) & \text{if } \mu \in D_{2,1/2}, \\ -\frac{7}{4}+O(q) & \text{if } \mu \in D_{3,0}, \\ -\frac{1}{3}q^{-1/3}+O(q^{2/3}) & \text{if } \mu \in D_{3,1/3}, \\ -\frac{1}{12}+O(q) & \text{if } \mu \in D_{6,0}, \\ \frac{1}{48}q^{-1/6}+O(q^{5/6}) & \text{if } \mu \in D_{6,1/6}, \\ O(q^{1/6}) & \text{otherwise.} \end{cases}$$

Therefore, the components F_μ of

$$F = F_{D,\Gamma_0(6),g_1,0} + 6F_{D,\Gamma_0(6),g_2,0} + \frac{1}{6}\sum_{\gamma\in S} F_{D,\Gamma_1(6),f_1,\gamma}$$

are given by

$$F_\mu = \begin{cases} q^{-1}+6+O(q) & \text{if } \mu = 0, \\ 2+O(q) & \text{if } \mu \in D_{2,0}, \\ q^{-1/2}+O(q^{1/2}) & \text{if } \mu \in S, \\ 3+O(q) & \text{if } \mu \in D_{3,0}, \\ q^{-1/3}+O(q^{2/3}) & \text{if } \mu \in D_{3,1/3}, \\ 1+O(q) & \text{if } \mu \in D_{6,0}, \\ q^{-1/6}+O(q^{5/6}) & \text{if } \mu \in S + D_{3,2/3}, \\ O(q^{1/6}) & \text{otherwise.} \end{cases}$$

In particular, F is strongly-reflective with principal part as claimed and $[F_0](0) = 6$. □

4.2.2. n=6

In this subsection we need the following eta quotients:

$$g_1(\tau) = \eta(\tau)^{-3}\eta(2\tau)^{-3}\eta(3\tau)\eta(6\tau),$$

$$
\begin{aligned}
g_2(\tau) &= \eta(\tau)^{-4}\eta(2\tau)^{2}\eta(3\tau)^{-4}\eta(6\tau)^{2},\\
g_3(\tau) &= \eta(\tau)^{-7}\eta(2\tau)^{5}\eta(3\tau)^{5}\eta(6\tau)^{-7},\\
g_4(\tau) &= \eta(\tau)\eta(2\tau)\eta(3\tau)^{-3}\eta(6\tau)^{-3},\\
f_1(\tau) &= \eta(\tau)^{3}\eta(2\tau)^{-3}\eta(3\tau)^{-5}\eta(6\tau),\\
f_2(\tau) &= \eta(\tau)^{-5}\eta(2\tau)\eta(3\tau)^{3}\eta(6\tau)^{-3},\\
f_3(\tau) &= \eta(\tau)^{-4}\eta(2\tau)^{-2}\eta(3\tau)^{4}\eta(6\tau)^{-2}.
\end{aligned}
$$

The following propositions can be proven similarly to Proposition 4.9:

Proposition 4.10. *Let $L = II_{6,2}(2_{II}^{-6}3^{-2})$ and $D = L'/L$. Let $\gamma \in D_{2,1/2}$. Then*

$$F = 3F_{D,\Gamma_0(6),g_1,0} + \frac{1}{2}F_{D,\Gamma_1(6),f_2,\gamma}$$

is a strongly-reflective modular form with $[F_0](0) = 4$ on D. Its principal part is given by

$$M_1 = \emptyset, \quad M_2 = \{\gamma\}, \quad M_3 = D_{3,1/3}, \quad M_6 = \gamma^{\perp} \cap \big(D_{2,1/2} \setminus \{\gamma\}\big) + D_{3,2/3}.$$

Proposition 4.11. *Let $L = II_{6,2}(2_{II}^{-6}3^{-6})$ and $D = L'/L$. Let $\gamma \in D_{2,1/2}$. Then*

$$F = F_{D,\Gamma_0(6),g_4,0} - \frac{1}{2}F_{D,\Gamma_1(6),f_1,\gamma}$$

is a strongly-reflective modular form with $[F_0](0) = 4$ on D. Its principal part is given by

$$M_1 = \{0\},\ M_2 = \gamma^{\perp} \cap \big(D_{2,1/2} \setminus \{\gamma\}\big),\ M_3 = \emptyset,\ M_6 = \gamma + D_{3,2/3}.$$

Proposition 4.12. *Let $L = II_{6,2}(2_{II}^{+2}3^{+6})$ and $D = L'/L$. Let $\gamma_1, \ldots, \gamma_6$ be six pairwise orthogonal elements in $D_{3,1/3}$. Then*

$$F = 4F_{D,\Gamma_0(6),g_2,0} + \frac{3}{4}\sum_{i=1}^{6} F_{D,\Gamma_1(6),f_3,\gamma_i}$$

is a strongly-reflective modular form with $[F_0](0) = 4$ on D. Its principal part is given by

$$M_1 = \emptyset,\ M_2 = D_{2,1/2},\ M_3 = \pm\{\gamma_1, \ldots, \gamma_6\},\ M_6 = D_{2,1/2} + \{\pm\gamma_i \pm \gamma_j : 1 \le i < j \le 6\}.$$

Proposition 4.13. *Let $L = II_{6,2}(2_{II}^{+6}3^{+6})$ and $D = L'/L$. Let $\gamma_1, \ldots, \gamma_6$ be six pairwise orthogonal elements in $D_{3,2/3}$. Then*

$$F = F_{D,\Gamma_0(6),g_3,0} + \frac{3}{4}\sum_{\substack{i,j=1\\ i<j}}^{6} F_{D,\Gamma_1(6),f_3,\gamma_i+\gamma_j} + \frac{3}{4}\sum_{\substack{i,j=1\\ i<j}}^{6} F_{D,\Gamma_1(6),f_3,\gamma_i-\gamma_j}$$

is a strongly-reflective modular form with $[F_0](0) = 4$ on D. Its principal part is given by

$$M_1 = \{0\},\ M_2 = \emptyset,\ M_3 = \{\pm\gamma_i \pm \gamma_j : 1 \le i < j \le 6\},\ M_6 = D_{2,1/2} \pm \{\gamma_1, \ldots, \gamma_6\}.$$

4.2.3. n=4

For $n = 4$ we need the following eta quotients:

$$\begin{aligned} g_1(\tau) &= \eta(\tau)^{-6}\eta(2\tau)^3\eta(3\tau)^2\eta(6\tau)^{-1}, \\ f_1(\tau) &= \eta(2\tau)^3\eta(3\tau)^{-4}\eta(6\tau)^{-1}, \\ f_2(\tau) &= \eta(\tau)\eta(2\tau)^{-2}\eta(3\tau)\eta(6\tau)^{-2}, \\ f_3(\tau) &= \eta(\tau)^4\eta(2\tau)^{-7}\eta(3\tau)^{-4}\eta(6\tau)^5, \\ f_4(\tau) &= \eta(\tau)^2\eta(2\tau)^{-5}\eta(3\tau)^2\eta(6\tau)^{-1}, \\ f_5(\tau) &= \eta(\tau)^2\eta(2\tau)^{-3}\eta(3\tau)^{-2}\eta(6\tau), \\ f_6(\tau) &= \eta(\tau)^{-7}\eta(2\tau)^4\eta(3\tau)^5\eta(6\tau)^{-4}. \end{aligned}$$

Again, the proofs of the following propositions are similar to the proof of Proposition 4.9:

Proposition 4.14. *Let $L = II_{4,2}(2_{II}^{+4}3^{+3})$ and $D = L'/L$. Let β_1 and β_2 be two orthogonal elements in $D_{2,1/2}$. Then*

$$F = \frac{1}{3}F_{D,\Gamma_1(6),f_1,\beta_1} + \frac{1}{3}F_{D,\Gamma_1(6),f_1,\beta_2} - \frac{1}{2}F_{D,\Gamma_1(6),f_2,\beta_2}$$

is a strongly-reflective modular form with $[F_0](0) = 2$ on D. Its principal part is given by

$$M_1 = \emptyset,\ M_2 = \{\beta_1\},\ M_3 = \emptyset,\ M_6 = \{\beta_2\} + D_{3,2/3}.$$

Proposition 4.15. *Let $L = II_{4,2}(2_{II}^{-6}3^{-3})$ and $D = L'/L$. Let $\beta \in D_{2,1/2}$. Then*

$$F = \frac{1}{2}F_{D,\Gamma_1(6),f_2,\beta}$$

is a strongly-reflective modular form with $[F_0](0) = 2$ on D. Its principal part is given by

$$M_1 = \emptyset,\ M_2 = \{\beta\},\ M_3 = \emptyset,\ M_6 = \{\beta\} + D_{3,2/3}.$$

Proposition 4.16. *Let $L = II_{4,2}(2_{II}^{+2}3^{-5})$ and $D = L'/L$. Let $\delta_1, \dots, \delta_4 \in D_{3,2/3}$ and $\gamma \in D_{3,1/3}$ such that these five elements are pairwise orthogonal. Then*

$$F = \frac{1}{2}F_{D,\Gamma_1(6),f_4,\gamma} - \frac{1}{2}\sum_{i=1}^{4} F_{D,\Gamma_1(6),f_3,\delta_i}$$

is a strongly-reflective modular form with $[F_0](0) = 2$ on D. Its principal part is given by

$$M_1 = \emptyset,\ M_2 = \emptyset,\ M_3 = \{\pm\gamma\},\ M_6 = D_{2,1/2} \pm \{\delta_1, \dots, \delta_4\}.$$

Proposition 4.17. *Let $L = II_{4,2}(2_{II}^{-4}3^{+5})$ and $D = L'/L$. Let $\gamma_1, \dots, \gamma_4 \in D_{3,1/3}$ and $\delta \in D_{3,2/3}$ such that these five elements are pairwise orthogonal. Then*

$$F = F_{D,\Gamma_1(6),f_3,\delta} + \frac{1}{2}\sum_{i=1}^{4} F_{D,\Gamma_1(6),f_4,\gamma_i}$$

is a strongly-reflective modular form with $[F_0](0) = 2$ on D. Its principal part is given by

$$M_1 = \emptyset,\ M_2 = \emptyset,\ M_3 = \pm\{\gamma_1, \dots, \gamma_4\},\ M_6 = D_{2,1/2} + \{\pm\delta\}.$$

Proposition 4.18. *Let $L = II_{4,2}(2_{II}^{-6}3^{+5})$ and $D = L'/L$. Let $\gamma_1, \ldots, \gamma_4 \in D_{3,1/3}$ and $\delta \in D_{3,2/3}$ such that these five elements are pairwise orthogonal. Define*

$$\begin{aligned}\gamma_5 &= \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4,\\ \gamma_6 &= \gamma_1 - \gamma_2 + \gamma_3 - \gamma_4,\\ \gamma_7 &= \gamma_1 + \gamma_2 - \gamma_3 - \gamma_4,\\ \gamma_8 &= \gamma_1 - \gamma_2 - \gamma_3 + \gamma_4.\end{aligned}$$

Then

$$F = 2F_{D,\Gamma_1(6),f_3,\delta} + \frac{1}{2}\sum_{i=1}^{8} F_{D,\Gamma_1(6),f_4,\gamma_i}$$

is a strongly-reflective vector-valued modular form with $[F_0](0) = 2$ on D. Its principal part is given by

$$M_1 = \emptyset,\ M_2 = \emptyset,\ M_3 = \pm\{\gamma_1, \ldots, \gamma_8\},\ M_6 = D_{2,1/2} + \{\pm\delta\}.$$

Proposition 4.19. *Let $L = II_{4,2}(2_{II}^{-6}3^{+5})$ and $D = L'/L$. Let $\gamma_1, \ldots, \gamma_4 \in D_{3,1/3}$ and $\delta \in D_{3,2/3}$ such that these five elements are pairwise orthogonal and let $\beta \in D_{2,1/2}$. Then*

$$F = -F_{D,\Gamma_1(6),f_5,\beta+\delta} + \frac{1}{2}F_{D,\Gamma_1(6),f_2,\beta} + F_{D,\Gamma_1(6),f_3,\delta} + \frac{1}{2}\sum_{i=1}^{4} F_{D,\Gamma_1(6),f_4,\gamma_i}$$

is a strongly-reflective vector-valued modular form with $[F_0](0) = 2$ on D. We define sets X and Y by $X = D_{3,2/3} \cap \delta^\perp$ and $Y = \left(D_{2,1/2} \cap \beta^\perp\right) \setminus \{\beta\}$. Then the principal part of F is given by

$$M_1 = \emptyset,\ M_2 = \{\beta\},\ M_3 = \pm\{\gamma_1 \ldots, \gamma_4\},\ M_6 = (\{\beta\} + X) \cup (Y + \{\pm\delta\}).$$

For the construction of the last strongly-reflective modular form in this section we need two lemmas:

Lemma 4.20. *Let D be the discriminant form with Jordan symbol 3^{-5} and $\gamma \in D_{3,1/3}$. The set $S = D_{3,2/3} \cap \gamma^\perp$ contains 24 elements and can be written as*

$$S = S_1 \cup S_2 \cup S_3,$$

where each S_i is of the form

$$S_i = \{\pm\delta_1, \ldots, \pm\delta_4\}$$

with $\delta_i \perp \delta_j$ for $i \neq j$. Moreover, there are 48 elements in $D_{3,2/3}$ that are not orthogonal to γ.

Proof. Note that $D \cap \gamma^\perp$ is isomorphic to the discriminant form with Jordan symbol 3^{+4}. This has 24 elements γ with $\mathrm{ord}(\gamma) = 3$ and $Q(\gamma) = 2/3 + \mathbb{Z}$ by Proposition 1.56 and is the orthogonal sum of four copies of 3^{+1}, each of which is generated by an element γ with

$\mathrm{ord}(\gamma) = 3$ and $Q(\gamma) = 2/3 + \mathbb{Z}$. Let $\delta_1, \dots, \delta_4$ be the generators of these four copies. We can then take

$$\begin{aligned}
S_1 &= \pm\{\delta_1, \dots, \delta_4\}, \\
S_2 &= \pm\{\delta_1 + \delta_2 + \delta_3 + \delta_4, \delta_1 + \delta_2 - \delta_3 - \delta_4, \delta_1 - \delta_2 + \delta_3 - \delta_4, \delta_1 - \delta_2 - \delta_3 + \delta_4\}, \\
S_3 &= \pm\{\delta_1 + \delta_2 + \delta_3 - \delta_4, \delta_1 + \delta_2 - \delta_3 + \delta_4, \delta_1 - \delta_2 + \delta_3 + \delta_4, \delta_1 - \delta_2 - \delta_3 - \delta_4\}.
\end{aligned}$$

Finally, $D_{3,2/3}$ has 72 elements by Proposition 1.56, so $72 - 24 = 48$ of them are not orthogonal to γ. □

Lemma 4.21. *Let D be the discriminant form with Jordan symbol 2_{II}^{+4} and let $\beta \in D_{2,1/2}$. Then $|D_{2,1/2} \cap \beta^\perp| = 4$.*

Proof. The proof is similar to the one of the previous lemma. □

Proposition 4.22. *Let $L = II_{4,2}(2_{II}^{+4}3^{-5})$ and $D = L'/L$. Let $\gamma \in D_{3,1/3}$ and $\beta \in D_{2,1/2}$. Write*

$$D_{3,2/3} \cap \gamma^\perp = S_1 \cup S_2 \cup S_3$$

as in Lemma 4.20 and let X be the set of elements in $D_{3,2/3}$ that are not orthogonal to γ. Let β_1, β_2 and β_3 be the three elements in $D_{2,1/2}$ that are orthogonal to but different from β and let Y be the set of elements in $D_{2,1/2}$ that are not orthogonal to β. Then

$$\begin{aligned}
F =& F_{D,\Gamma_1(6),f_6,\gamma} + \frac{1}{2} F_{D,\Gamma_1(6),f_2,\beta} + 10 F_{D,\Gamma_0(6),g_1,0} - \frac{1}{36} \sum_{\rho \in X} F_{D,\Gamma_1(6),f_5,\beta+\rho} \\
& - \frac{5}{72} \sum_{\mu \in Y} \sum_{\rho \in X} F_{D,\Gamma_1(6),f_5,\mu+\rho} + \frac{1}{6} \sum_{i=1}^{3} \sum_{\delta \in S_i} F_{D,\Gamma_1(6),f_5,\beta_i+\delta}
\end{aligned}$$

is a strongly-reflective modular form with $[F_0](0) = 2$ on D. Its principal part is given by

$$M_1 = \emptyset,\ M_2 = \{\beta\},\ M_3 = \pm\{\gamma\},\ M_6 = \{\beta\} + \left(D_{3,2/3} \cap \gamma^\perp\right).$$

4.3. Fourier Expansions of $\Psi(F)$

We want to compute the Fourier expansions of the automorphic product $\Psi(F)$ for the vector-valued modular form F from Proposition 4.10 at the cusps of L. The lattice L has genus $II_{6,2}(2_{II}^{-6}3^{-2})$ and is isomorphic to $D_4 \oplus II_{1,1}(2) \oplus II_{1,1}(6)$ because there is only one isomorphism class in this genus.

4.3.1. Orbits of Cusps

We first prove that the set of cusps in L has four orbits under the action of $O(L, F)^+$. In the following the lattice K will always be equal to either $D_4 \oplus II_{1,1}(2)$ or $D_4 \oplus II_{1,1}(6)$ and we then identify L with $K \oplus II_{1,1}(6)$ or $L = K \oplus II_{1,1}(2)$. In both cases elements of $O(K)$ can be extended to elements of $O(L)$ by defining their action on the orthogonal complement of K in L to be trivial. We can therefore identify $O(K)$ with a subgroup of $O(L)$ and we define $O(K)^+$ to be $O(K) \cap O(L)^+$.

Lemma 4.23. *Let $K = D_4 \oplus II_{1,1}(2)$ and let $D = K'/K$. Then $O(K)^+$ acts transitively on $D_{2,1/2}$ and on $D_{2,0}$.*

Proof. From the definition of D_4 and $II_{1,1}(2)$, we see that the lattice K is given by

$$K = \left\{ (x_1, \dots, x_6) \in \mathbb{Z}^6 : \sum_{i=1}^{4} x_i \in 2\mathbb{Z} \right\} \tag{4.19}$$

with quadratic form

$$(x_1, \dots, x_6) \mapsto \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2 + \frac{1}{2}x_4^2 + 2x_5x_6.$$

The dual lattice K' is given by

$$\left\{ (x_1, \dots, x_6) \in \frac{1}{2}\mathbb{Z}^6 : (x_1, \dots, x_4) \in \mathbb{Z}^4 \text{ or } (x_1, \dots, x_4) \in \left(\frac{1}{2} + \mathbb{Z}\right)^4 \right\}. \tag{4.20}$$

Since D_4 has genus $II_{4,0}(2_{II}^{-2})$ and $II_{1,1}(2)$ has genus $II_{1,1}(2_{II}^{+2})$, it follows that $D = 2_{II}^{-4}$. Generators of D are given by

$$\begin{aligned}
\beta_1 &= (1, 0, 0, 0, 0, 0) + K, \\
\beta_2 &= (1/2, 1/2, 1/2, 1/2, 0, 0) + K, \\
\gamma_1 &= (0, 0, 0, 0, 1/2, 0) + K, \\
\gamma_2 &= (0, 0, 0, 0, 0, 1/2) + K.
\end{aligned}$$

The set $D_{2,1/2}$ has ten elements (use Proposition 1.55) and is given by

$$\{\beta_1, \beta_2, \beta_1 + \beta_2, \gamma_1 + \gamma_2, \beta_1 + \gamma_1, \beta_1 + \gamma_2, \beta_2 + \gamma_1, \beta_2 + \gamma_2, \beta_1 + \beta_2 + \gamma_1, \beta_1 + \beta_2 + \gamma_2\}$$

and the set $D_{2,0}$ has five elements and is given by

$$\{\gamma_1, \gamma_2, \beta_1 + \gamma_1 + \gamma_2, \beta_2 + \gamma_1 + \gamma_2, \beta_1 + \beta_2 + \gamma_1 + \gamma_2\}.$$

Let

$$\begin{aligned}
\alpha_1 &= (-1, 1, 1, 1, 0, 0), \\
\alpha_2 &= (1, 1, 1, 1, 0, 0), \\
\alpha_3 &= (0, 0, 0, 0, 1, 1), \\
\alpha_4 &= (2, 0, 0, 0, 1, 0).
\end{aligned}$$

Using Proposition 1.22, we see that these elements are roots of K, so the reflections σ_{α_i} are elements of $O(K)^+$ (here we use Remark 3.3). We can calculate the action of σ_{α_i} on the elements in $D_{2,1/2}$ and see that the subgroup $\langle \sigma_{\alpha_1}, \dots, \sigma_{\alpha_4} \rangle \subset O(K)^+$ acts transitively on $D_{2,1/2}$ and on $D_{2,0}$. ☐

Lemma 4.24. *Let $K = D_4 \oplus II_{1,1}(6)$ and let $D = K'/K$. In both cases $O(K)^+$ acts transitively on $D_{2,1/2}$ and on $D_{2,0}$.*

Proof. This can be proven similarly to the previous lemma. □

Lemma 4.25. *Let $K = D_4 \oplus II_{1,1}(2)$ and let $D = K'/K$. Let $\gamma \in D_{2,1/2} \cup D_{2,0}$. Then there is a cusp $z \in K$ such that z has level* 2 *and $(z, \gamma) = 1 + 2\mathbb{Z}$.*

Proof. We write the elements of K and K' as in (4.19) and (4.20). Because of Lemma 4.23, we can assume that γ is equal to $(0, 0, 0, 0, 1/2, 1/2) + K$ (if $\gamma \in D_{2,1/2}$) or to $(0, 0, 0, 0, 1/2, 0) + K$ (if $\gamma \in D_{2,0}$). Then we can choose $z = (0, 0, 0, 0, 0, 1)$. □

Lemma 4.26. *Let $K = D_4 \oplus II_{1,1}(6)$ and let $D = K'/K$. Let $\gamma \in D_{2,1/2} \cup D_{2,0}$. Then there is a cusp $z \in K$ such that z has level* 6 *and $(z, \gamma) = 3 + 6\mathbb{Z}$*

Proof. The proof is similar to the proof of the previous lemma. □

Lemma 4.27. *Let z_1 and z_2 be cusps of level* 6 *in L. Then there is an element $\sigma \in O(L)^+$ with $\sigma(z_1) = z_2$.*

Proof. By Proposition 2.73 there exist cusps $\zeta_1, \zeta_2 \in L$ of level 6 with $(\zeta_1, z_1) = 6$ and $(\zeta_2, z_2) = 6$. Then the sublattice spanned by ζ_i and z_i $(i = 1, 2)$ is isomorphic to $II_{1,1}(6)$ and

$$K_1 \oplus II_{1,1}(6) = L = K_2 \oplus II_{1,1}(6)$$

with $K_1 = L \cap \zeta_1^\perp \cap z_1^\perp$ and $K_2 = L \cap \zeta_2^\perp \cap z_2^\perp$ by (2.9). Then K_1 and K_2 are both lattices of genus $II_{5,1}(2_{II}^{-4})$. Since this genus only contains one isomorphism class the lattices K_1 and K_2 must be isomorphic. We let σ be any isomorphism from K_1 to K_2 and extend it to an automorphism of L by mapping z_1 to z_2 and ζ_1 to ζ_2. If this is in $O(L)^+$, then we are done; otherwise we compose it with the map τ that acts by multiplication with -1 on K_2 and as the identity on $II_{1,1}(6)$. Then τ is not in $O(L)^+$ (this can be seen from Remark 3.3) and therefore $\tau \circ \sigma \in O(L)^+$. □

Lemma 4.28. *Let z_1 and z_2 be cusps of level* 2 *in L. Then there is an element $\sigma \in O(L)^+$ with $\sigma(z_1) = z_2$.*

Proof. The proof is similar to the proof of the previous proposition. □

Proposition 4.29. *Let $D = L'/L$ and γ and F as in Proposition 4.10. Then the set of cusps in L has four orbits under the action of $O(L, F)^+$, namely*

$$\begin{aligned} O_1 &= \{z \in L \text{ primitive } : (z, z) = 0, (z, L) = 6\mathbb{Z} \text{ and } (z, \gamma) = 0 + 6\mathbb{Z}\}, \\ O_2 &= \{z \in L \text{ primitive } : (z, z) = 0, (z, L) = 6\mathbb{Z} \text{ and } (z, \gamma) = 3 + 6\mathbb{Z}\}, \\ O_3 &= \{z \in L \text{ primitive } : (z, z) = 0, (z, L) = 2\mathbb{Z} \text{ and } (z, \gamma) = 0 + 2\mathbb{Z}\}, \\ O_4 &= \{z \in L \text{ primitive } : (z, z) = 0, (z, L) = 2\mathbb{Z} \text{ and } (z, \gamma) = 1 + 2\mathbb{Z}\}. \end{aligned}$$

Proof. We remark that an element in $O(L)^+$ is in $O(L, F)^+$ if and only if it fixes the element $\gamma \in D_{2,1/2}$.

Let $z \in L$ be a cusp of level l. By Proposition 2.73 there exists a cusp $\zeta \in L$ of level l with $(\zeta, z) = l$. Then

$$L = K \oplus II_{1,1}(l)$$

with $K = L \cap \zeta^\perp \cap z^\perp$ by (2.9). Since L neither splits $II_{1,1}(1)$ nor $II_{1,1}(3)$, we must have $l = 2$ or $l = 6$.

Suppose $l = 6$ and $(z, \gamma) = 0 + 6\mathbb{Z}$. Then K has genus $II_{5,1}(2_{II}^{-4})$ and is therefore isomorphic to $D_4 \oplus II_{1,1}(2)$. Write $D' = K'/K \oplus II_{1,1}(6)'/II_{1,1}(6)$.

We show that we can assume that $(\zeta, \gamma) = 0 + 6\mathbb{Z}$: Suppose $(\zeta, \gamma) = 3 + 6\mathbb{Z}$. Then $\gamma = \gamma_1 + \gamma_2$ for an element $\gamma_1 \in K'/K$ with $Q(\gamma_1) = 1/2 + \mathbb{Z}$ and $\gamma_2 = z/2 + II_{1,1}(6)$ in $II_{1,1}(6)'/II_{1,1}(6)$. We use Lemma 4.25 to obtain a cusp $x \in K$ of level 2 that satisfies $(x, \gamma_1) = 1 + 2\mathbb{Z}$ and replace ζ by $\zeta + 3x$. We can therefore assume that $(\zeta, \gamma) = 0 + 6\mathbb{Z}$, in which case the element γ is in K'/K. That the elements in O_1 are in the same orbit under the action of $O(L, F)^+$ now follows from Lemmas 4.23 and 4.27.

With a similar argument one can also show that for each $i \in \{2, \dots, 4\}$ the elements in O_i are in the same orbit. To complete the proof we observe that there is obviously no element in $O(L, F)^+$ that maps an element of O_i to O_j unless $i = j$. $\square$

4.3.2. Product and Fourier Expansions of $\Psi(F)$

We can now compute the Fourier expansions of $\Psi = \Psi(F)$ at the cusps of L. We use the notation from Subsection 2.3.5, i.e. given a cusp $z \in L$ of level l, we choose an element $\zeta \in L \cap lL'$ with $(\zeta, z) = l$ and let $K = L \cap z^\perp \cap \zeta^\perp$. We also define L_0' and $p\colon L_0'/L_0 \to K'/K$ as in Subsection 2.3.5.

Proposition 4.30. *Let $z \in L$ be cusp of level 6 with $(z, \gamma) = 0 + 6\mathbb{Z}$. Then the Fourier expansion of Ψ at z is up to a constant given by*

$$\begin{aligned}\Psi_z(Z) &= e(-(\rho, Z)) \prod_{\substack{\lambda \in K' \\ (\lambda, W) < 0}} \prod_{a \in \mathbb{Z}/6\mathbb{Z}} (1 - e(a/6)e(-(\lambda, Z)))^{[F_{\lambda + az/6 + L}](-q(\lambda))} \\ &= \sum_{g \in G} \det(g) \eta_{2^5 4^{-1} 6^{-3} 12^3}(-(g\rho, Z)),\end{aligned}$$

where ρ is a norm 0 vector in K' with $(\rho, \gamma) = 1/2 + \mathbb{Z}$ and G is the Weyl group as defined in Section 3.1.

Proof. We write $L = D_4 \oplus II_{1,1}(2) \oplus II_{1,1}(6)$ as

$$L = \left\{ (x_1, \dots, x_8) \in \mathbb{Z}^8 : \sum_{i=1}^{4} x_i \in 2\mathbb{Z} \right\}$$

with quadratic form

$$(x_1, \dots, x_8) \mapsto \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2 + \frac{1}{2}x_4^2 + 2x_5x_6 + 6x_7x_8.$$

Then L' is given by

$$L' = \left\{ (x_1, \dots, x_8) \in \frac{1}{6}\mathbb{Z}^8 : x_5, x_6 \in \frac{1}{2}\mathbb{Z}, (x_1, \dots, x_4) \in \mathbb{Z}^4 \cup \left(\frac{1}{2} + \mathbb{Z}^4 \right) \right\}.$$

We can assume that $\gamma = (1, 0, 0, 0, 1/2, 0, 1/2, 0) + L \in D$ because $\mathrm{Aut}(D)$ acts transitively on $D_{2,1/2}$ and that $z = (0, 0, 0, 0, 0, 0, 1, 0)$ because of Lemma 4.23. We then choose $z' = (0, 0, 0, 0, 0, 0, 0, 1/6)$ and $\zeta = 6z'$, so that

$$K = \{(x_1, \dots, x_8) \in L : x_7 = x_8 = 0\}.$$

The lattice L_0' is then given by

$$L_0' = \{x \in L' : x_8 \in \mathbb{Z}\}$$

and the map $p\colon L_0' \to K'$ is

$$(x_1, \dots, x_8) \mapsto (x_1, \dots, x_6, 0, 0).$$

Let $\mu \in K'/K$ be the image of $\gamma + L$ under p, i.e.

$$\mu = (1, 0, 0, 0, 1/2, 0, 0, 0) + K.$$

The set of vectors of negative norm in $K \otimes_{\mathbb{Z}} \mathbb{R}$ has two connected components, given by $x_6 < 0$ and $x_6 > 0$. We let

$$C = \{x = (x_1, \dots, x_8) \in \mathbb{R}^8 : x_7 = x_8 = 0, q(x) < 0, x_6 < 0\}$$

and call it the positive cone. Recall that the Weyl chambers are the connected components of the complement of

$$\bigcup_{\beta \in L_0'/L} \bigcup_{\substack{n \in \mathbb{Z} - Q(\beta) \\ n > 0 \\ [F_\beta](-n) \neq 0}} \bigcup_{\substack{\lambda \in p(\beta) + K \\ q(\lambda) = n}} \{x \in C : (x, \lambda) = 0\}$$

in C. The only pairs $(\beta, n) \in L'/L \times \mathbb{Q}_{>0}$ with $[F_\beta](-n) \neq 0$ are $(\gamma, 1/2)$, $(\rho, 1/3)$ and $(\delta, 1/6)$ for $\rho \in D_{3,1/3}$ and certain $\delta \in D_{6,1/3}$. But K' has no elements λ with $q(\lambda) = 1/3$ or $q(\lambda) = 1/6$ because K has level 2. Therefore, the Weyl chambers are the connected components of the complement of

$$\bigcup_{\substack{\lambda \in \mu + K \\ q(\lambda) = 1/2}} \{x \in C : (x, \lambda) = 0\}$$

in C and the Weyl group G is generated by the reflections

$$\sigma_\lambda(x) = x - 2(\lambda, x)\lambda$$

for those $\lambda \in K'$ that satisfy $\lambda \in \mu + K$ and $q(\lambda) = 1/2$. We remark that G is a subgroup of $O(K)^+$ because 2λ is a root of K by Proposition 1.22 whenever λ is an element in K' with $q(\lambda) = 1/2$.

We let $w = (0, 0, 0, 0, 0, -1, 0, 0) \in K$ and note that w is in the closure $\overline{C}$ of C. We let W be a Weyl chamber whose closure contains w and compute the corresponding Weyl vector ρ. Before we do this, we compute the restriction F_K of F. The Fourier expansion of F can be computed as in the proof of Proposition 4.9. One obtains

$$F_\beta = \begin{cases} 4 + O(q) & \text{if } \beta = 0, \\ 2 + O(q) & \text{if } \beta \in D_{2,0} \cap \gamma^\perp, \\ q^{-1/2} & \text{if } \beta = \gamma, \\ q^{-1/3} & \text{if } \beta \in D_{3,1/3}, \\ 1 + O(q) & \text{if } \beta \in D_{6,0} \cap \gamma^\perp, \\ -2 + O(q) & \text{if } \beta \in D_{6,0} \text{ and } (\gamma, \beta) \neq 0 + \mathbb{Z}, \\ q^{-1/6} + O(q^{5/6}) & \text{if } \beta \in D_{6,1/6} \cap \gamma^\perp \text{ and } 3\beta \neq \gamma, \\ O(q^{1/6}) & \text{otherwise.} \end{cases} \tag{4.21}$$

The components of the restriction F_K of F are therefore

$$(F_K)_\beta = \begin{cases} 8 + O(q) & \text{if } \beta = 0 \text{ or } \beta \in D_{2,0} \cap \mu^\perp, \\ -8 + O(q) & \text{if } \beta \in D_{2,0} \text{ and } (\beta, \mu) \neq 0 + \mathbb{Z}, \\ q^{-1/2} + O(q^{1/2}) & \text{if } \beta = \mu, \\ O(q^{1/2}) & \text{otherwise.} \end{cases}$$

We let

$$w' = (0, 0, 0, 0, -1/2, 0, 0, 0) \in K'$$

and $\tilde{K} = K \cap w^\perp \cap w'^\perp$, so

$$\tilde{K} = \{(x_1, \ldots, x_8) \in L : x_5 = x_6 = x_7 = x_8 = 0\}.$$

We let $F_{\tilde{K}}$ be the restriction of F_K to $\tilde{K}$. Then $F_{\tilde{K}} = 0$ because its principal part must vanish as $\mu \notin K_0'/K$. Therefore, using the formula from Section 3.1, we see that the Weyl vector ρ is given by

$$\rho = \frac{1}{4} \sum_{\lambda \in \tilde{K}'} \sum_{a \in \mathbb{Z}/2\mathbb{Z}} (F_K)_{\lambda + aw/2 + K}(-q(\lambda^2)) B_2(a/2) w.$$

The lattice $\tilde{K}$ is positive definite, so $q(\lambda) \geq 0$ and $q(\lambda) = 0$ if and only if $\lambda = 0$. By our choice of w, none of the elements $\lambda + aw/2 + K$ with $\lambda \in \tilde{K}'$ and $a \in \mathbb{Z}/2\mathbb{Z}$ equals μ, so the only contribution to the sum comes from $\lambda = 0$. Therefore,

$$\begin{aligned} \rho &= \frac{1}{4} \left((F_K)_0(0) B_2(0) + (F_K)_{w/2+K}(0) B_2(1/2) \right) w \\ &= \frac{1}{4} \left(8 \cdot \frac{1}{6} - 8 \cdot \frac{-1}{12} \right) w \\ &= \frac{1}{2} w. \end{aligned}$$

To compute the Fourier expansion of Ψ_z, we recall that the expansion is of the form

$$\Psi_z(Z) = \sum_{\substack{\lambda \in \rho + K' \\ \lambda \in \overline{C}}} c(\lambda) e(-(\lambda, Z))$$

and that $c(\lambda) = 0$ if $q(\lambda) \neq 0$ because Ψ has singular weight. Let $\alpha \in K'$ be an element with $q(\alpha) = 1/2$ and $\alpha + K = \mu$. Then α is also in L' and the reflection

$$\sigma_\alpha(x) = x - 2(\alpha, x)\alpha$$

is in $O(L)^+$. It maps K to K and it is easy to see that it maps the coset $\gamma \in L'/L$ to itself and hence is in $O(L,F)^+$. It follows that

$$\Psi_z(\sigma_\alpha(Z)) = \Psi(\sigma_\alpha(Z)_L) = \Psi(\sigma_\alpha(Z_L)) = \chi(\sigma_\alpha)\Psi(Z) = \chi(\sigma_\alpha)\Psi_z(Z)$$

because Ψ is an automorphic form for $O(L,F)^+$. The function Ψ has a simple zero at $\alpha^\perp$ and therefore $\chi(\sigma_\alpha) = -1$ (see the Remark after Theorem 13.3 in [Bor98]). Since G is generated by such σ_α, it follows that

$$\Psi_z(g(Z)) = \det(g)\Psi_z(Z)$$

for all $g \in G$. This implies that

$$c(g(\lambda)) = \det(g)c(\lambda)$$

and in particular, $c(\lambda) = 0$ if λ is on the wall between two adjacent Weyl chambers. The Weyl group G acts transitively on the Weyl chambers; therefore,

$$\Psi_z(Z) = \sum_{g \in G} \det(g) \sum_{\substack{\lambda \in \rho + K' \\ \lambda \in \overline{W}}} c(\lambda) e(-(g(\lambda), Z)).$$

We want to determine $c(\lambda)$, so let $\lambda \in \rho + K'$ be an element with $\lambda \in \overline{W}$ and $q(\lambda) = 0$. As in the proof of Theorem 6 in [DHS15] one can show that $c(\lambda) = 0$ unless $\lambda = n\rho$ for a positive integer n. Suppose $n\rho = \sum \lambda_i$ with $\lambda_i \in K'$ and $(\lambda_i, W) < 0$ and that $[F_{\lambda_i + az/6+L}](-q(\lambda_i)) \neq 0$ for some $a \in \mathbb{Z}/6\mathbb{Z}$, i.e. the λ_i contribute to $c(n\rho)$ in the product expansion of Ψ_z. Then all λ_i must be positive integral multiples of ρ (this can be seen in the same way as in the proof of Theorem 6 in [DHS15]). Therefore,

$$\sum_{\substack{\lambda \in \rho + K' \\ \lambda \in \overline{W}}} c(\lambda) e(-(\lambda, Z)) = e(-(\rho, Z)) \prod_{m>0} \prod_{a \in \mathbb{Z}/6\mathbb{Z}} (1 - e(a/6)e(-(m\rho, Z)))^{[F_{m\rho+az/6+L}](0)}. \tag{4.22}$$

We can read off the coefficients $[F_{m\rho+az/6+L}](0)$ from (4.21) and find that

$$[F_{m\rho+az/6+L}](0) = \begin{cases} 4 & \text{if } 2 \mid m \text{ and } a = 0 + 6\mathbb{Z}, \\ 1 & \text{if } 2 \mid m \text{ and } a = \pm 1 + 6\mathbb{Z}, \\ 2 & \text{if } 2 \mid m \text{ and } a = 3 + 6\mathbb{Z}, \\ -2 & \text{if } 2 \nmid m \text{ and } a \in \pm\{1 + 6\mathbb{Z}, 2 + 6\mathbb{Z}\}, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the sum in (4.22) equals

$$\begin{aligned}
&e(-(\rho,Z))\prod_{\substack{m>0\\2|m}}(1-e(-(m\rho,Z)))^4(1-e(1/6)e(-(m\rho,Z)))\times\\
&\qquad\times(1-e(1/2)e(-(m\rho,Z)))^2(1-e(5/6)e(-(m\rho,Z)))\times\\
&\qquad\times\prod_{\substack{m>0\\2\nmid m}}(1-e(1/6)e(-(m\rho,Z)))^{-2}(1-e(1/3)e(-(m\rho,Z)))^{-2}\times\\
&\qquad\times(1-e(2/3)e(-(m\rho,Z)))^{-2}(1-e(5/6)e(-(m\rho,Z)))^{-2}\\
=\;&e(-(\rho,Z))\prod_{\substack{m>0\\2|m}}(1-e(-(6m\rho,Z)))(1-e(-(3m\rho,Z)))^{-1}\times\\
&\qquad\times(1-e(-(2m\rho,Z)))(1-e(-(m\rho,Z)))^3\times\\
&\qquad\times\prod_{\substack{m>0\\2\nmid m}}(1-e(-(6m\rho,Z)))^{-2}(1-e(-(2m\rho,Z)))^2\\
=\;&e(-(\rho,Z))\prod_{m>0}(1-e(-(12m\rho,Z)))(1-e(-(6m\rho,Z)))^{-1}\times\\
&\qquad\times(1-e(-(4m\rho,Z)))(1-e(-(2m\rho,Z)))^3\times\\
&\qquad\times\prod_{m>0}(1-e(-(6m\rho,Z)))^{-2}(1-e(-(2m\rho,Z)))^2\times\\
&\qquad\times(1-e(-(12m\rho,Z)))^2(1-e(-(4m\rho,Z)))^{-2}\\
=\;&e(-(\rho,Z))\prod_{m>0}(1-e(-2m\rho,Z)))^5(1-e(-4m\rho,Z)))^{-1}\times\\
&\qquad\times(1-e(-6m\rho,Z)))^{-3}(1-e(-12m\rho,Z)))^3
\end{aligned}$$

This is equal to $\eta_{2^5 4^{-1} 6^{-3} 12^3}(-(\rho,Z))$ and the Fourier expansion of Ψ_z is therefore as claimed. ☐

Proposition 4.31. *Let $z\in L$ be a cusp of level 6 with $(z,\gamma)\neq 0+6\mathbb{Z}$ and define K'^{+} by $K'^{+}=(K'\cap\overline{C})\setminus\{0\}$. Then the Fourier expansion of Ψ at z is up to a constant given by*

$$\begin{aligned}
\Psi_z(Z)&=\prod_{\lambda\in K'^{+}}\prod_{a\in\mathbb{Z}/6\mathbb{Z}}(1-e(a/6)e(-(\lambda,Z)))^{[F_{\lambda+az/6+L}](-q(\lambda))}\\
&=1+\sum_{\lambda\in K'^{+}}c(\lambda)e(-(\lambda,Z)),
\end{aligned}$$

where $c(\lambda)=0$ unless $\lambda=n\mu$ for a primitive isotropic $\mu\in K'^{+}$, in which case $c(\lambda)$ equals the coefficient at q^n in

$$\begin{cases}\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^{-2} & \text{if } \mu\in K,\\ \eta(\tau)^{-1}\eta(2\tau)^5\eta(3\tau)^3\eta(6\tau)^{-3} & \text{if } \mu\notin K.\end{cases}$$

Proof. We let $z = (0,0,0,0,0,0,0,1)$, $z' = (0,0,0,0,0,0,1/6,0)$ and $\zeta = 6z'$. Then K is as in the proof of the previous proposition and

$$L'_0 = \{x \in L' : x_7 \in \mathbb{Z}\}.$$

Then F_γ is holomorphic for all $\gamma \in L'_0/L \subset L'/L$ and therefore the restriction F_K of F is zero and the set

$$\bigcup_{\beta \in L'_0/L} \bigcup_{\substack{n \in \mathbb{Z}-\beta^2/2 \\ n>0 \\ [F_\beta](-n)\neq 0}} \bigcup_{\substack{\lambda \in p(\beta)+K \\ q(\lambda)=n}} \{x \in C : (x,\lambda) = 0\}$$

is empty. There is therefore only one Weyl chamber, namely C itself and the Weyl vector ρ is zero. It is easy to see that the set of $\lambda \in K'$ with $(\lambda, C) < 0$ is given by K'^+. Therefore, Ψ_z has the given product expansion.

To compute the Fourier expansion, we first rewrite the product expansion as

$$\Psi_z(Z) = \prod_{\substack{\lambda \in K'^+ \\ \lambda \text{ primitive}}} \prod_{m>0} \prod_{a \in \mathbb{Z}/6\mathbb{Z}} (1 - e(a/6)e(-(m\lambda, Z)))^{[F_{m\lambda+az/6+L}](-q(m\lambda))}$$

and remark that Ψ_z has a Fourier expansion of the form

$$\Psi_z(Z) = 1 + \sum_{\lambda \in K'^+} c(\lambda) e(-(\lambda, Z))$$

with $c(\lambda) = 0$ unless $q(\lambda) = 0$ because Ψ has singular weight. So let $\lambda \in K'^+$ with $q(\lambda) = 0$. Suppose $\lambda = \sum \lambda_i$ with $\lambda_i \in K'^+$ and that $[F_{\lambda_i+az/6+L}](-q(\lambda_i)) \neq 0$ for some $a \in \mathbb{Z}/6\mathbb{Z}$, i.e. the λ_i contribute to $c(\lambda)$ in the product expansion of Ψ_z. Write $\lambda = n\mu$ with $n > 0$ and $\mu \in K'$ primitive. Then $(\lambda_i, \mu) \leq 0$ because both λ_i and μ are in K'^+ and $\sum(\lambda_i, \mu) = n(\mu, \mu) = 0$, so $(\lambda_i, \mu) = 0$ and λ_i must be a multiple of μ. Therefore, the coefficient $c(n\mu)$ equals the coefficient of

$$\prod_{m>0} \prod_{a \in \mathbb{Z}/6\mathbb{Z}} (1 - e(a/6)e(-(m\mu, Z)))^{[F_{m\mu+az/6+L}](0)} \tag{4.23}$$

at $e(-(n\mu, Z))$.

If $\mu \in K$, then $m\mu + az/6 + L = az/6 + L$, so the product equals

$$\begin{aligned} &\prod_{m>0} (1 - e(-(m\mu, Z)))^4 (1 - e(1/6)e(-(m\mu, Z)))^{-2} (1 - e(5/6)e(-(m\mu, Z)))^{-2} \\ = &\prod_{m>0} (1 - e(-(m\mu, Z)))^2 e(-(2m\mu, Z)))^2 (1 - e(-(3m\mu, Z)))^2 (1 - e(-(6m\mu, Z)))^{-2} \end{aligned}$$

in this case. This can be written as $\eta_{1^2 2^2 3^2 6^{-2}}(-(\rho, Z))$.

If $\mu \notin K$, then $m\mu + az/6 + L = az/6 + L$ if $2 \mid m$; otherwise it equals $\mu + az/6 + L$. Therefore, (4.23) can be written as

$$\prod_{\substack{m>0 \\ 2|m}} (1 - e(-(m\mu, Z)))^4 (1 - e(1/6)e(-(m\mu, Z)))^{-2} (1 - e(5/6)e(-(m\mu, Z)))^{-2} \times$$

$$\times \prod_{\substack{m>0\\ 2\nmid m}} (1-e(-(m\mu,Z)))^2(1-e(1/6)e(-(m\mu,Z)))^{-2}(1-e(1/3)e(-(m\mu,Z)))\times$$
$$\times(1-e(2/3)e(-(m\mu,Z)))(1-e(5/6)e(-(m\mu,Z)))^{-2}$$
$$= \prod_{m>0} (1-e(-(12m\mu,Z)))^{-2}e(-(6m\mu,Z)))^2(1-e(-(4m\mu,Z)))^2(1-e(-(2m\mu,Z)))^2\times$$
$$\times \prod_{m>0} (1-e(-(6m\mu,Z)))^{-2}e(-(3m\mu,Z)))^3e(-(2m\mu,Z)))^2(1-e(-(m\mu,Z)))^{-1}$$
$$\times(1-e(-(12m\mu,Z)))^2e(-(6m\mu,Z)))^{-3}e(-(4m\mu,Z)))^{-2}(1-e(-(2m\mu,Z)))$$
$$= \prod_{m>0} (1-e(-(m\mu,Z)))^{-1}(1-e(-(2m\mu,Z)))^5\times$$
$$\times(1-e(-(3m\mu,Z)))^3(1-e(-(6m\mu,Z)))^{-3}$$

in this case. This can be written as $\eta_{1^{-1}2^5 3^3 6^{-3}}(-(\mu,Z))$ and finishes the proof. □

Similarly, we can also compute the expansion of Ψ at the other cusps in L and obtain the following:

Proposition 4.32. *Let $z \in L$ be a cusp of level 2 with $(z,\gamma) = 0 + 2\mathbb{Z}$. Then the expansion of Ψ at z is up to a constant given by*

$$\Psi_z(Z) = e(-(\rho,Z)) \prod_{\substack{\lambda\in K'\\ (\lambda,W)<0}} \prod_{a\in\mathbb{Z}/2\mathbb{Z}} (1-e(a/2)e(-(\lambda,Z)))^{[F_{\lambda+az/2+L}](-q(\lambda))}$$
$$= \sum_{g\in G} \det(g)\eta_{2^{-3}4^3 6^5 12^{-1}}(-(g\rho,Z)),$$

where ρ is a norm 0 vector in $K' \cap \frac{1}{6}K$ with $(\rho,\gamma) = 1/2+\mathbb{Z}$ and G is the Weyl group as defined in Section 3.1.

Proposition 4.33. *Let $z \in L$ be a cusp of level 2 with $(z,\gamma) \neq 0 + 2\mathbb{Z}$. Then the expansion of Ψ at z is up to a constant given by*

$$\Psi_z(Z) = e(-(\rho,Z)) \prod_{\substack{\lambda\in K'\\ (\lambda,W)<0}} \prod_{a\in\mathbb{Z}/2\mathbb{Z}} (1-e(a/2)e(-(\lambda,Z)))^{[F_{\lambda+az/2+L}](-q(\lambda))}$$
$$= \sum_{g\in G} \det(g)\eta_{1^3 2^{-3}3^{-1}6^5}(-(g\rho,Z)),$$

where ρ is a norm 0 vector in $K' \cap \frac{1}{6}K$ and G is the Weyl group as defined in Section 3.1.

4.4. Final Remarks

For the automorphic products Ψ obtained from Table 4.1, the Fourier expansion at a suitable cusp is of the form $\sum_{w\in G}\det(w)\eta_g(-(w\rho,Z))$ (see Proposition 4.1) where g is the element in Co_0 that was used to construct Ψ. We have proven that if L has squarefree level N and splits $II_{1,1} \oplus II_{1,1}(N)$, these are essentially the only strongly-reflective automorphic forms of singular weight for the discriminant kernel that can occur. If we no longer assume that L splits $II_{1,1}$, then there are additional examples. For $N = 6$ these are constructed in the previous section, while for N prime they are stated in [Sch17]. If N is prime, then the Fourier expansions of these new examples at a suitable cusp are still related to Co_0 in the sense that the shape of the eta quotient occuring there is the cycle shape of an element of Co_0 (see [Sch17]). If $N = 6$, the situation is different:

Theorem 4.34. *None of the shapes of the eta quotients occuring in the Fourier expansions of the strongly-reflective automorphic product Ψ from the previous section is the cycle shape of an element of $O(\Lambda) = \mathrm{Co}_0$.*

Proof. The cycle shape of an element g in $O(\Lambda)$ can be calculated from the traces of powers of g (see [Sch04], Section 5). To obtain these traces we note that $O(\Lambda)$ naturally acts on $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}^{24}$ and hence has a 24-dimensional representation. This is irreducible and the traces are given by the character of this representation. We therefore only need to look up the values in a character table of Co_0. Such a character table can e.g. be obtained with GAP; just use the command `Display(CharacterTable("2.Co1"))`. □

5. The Space of Siegel Cusp Forms of Degree 6 and Weight 14

This chapter is based on joint work with Riccardo Salvati Manni and Nils Scheithauer. The former of the two suggested to consider a theta series of the form

$$\Theta_{S,F}(Z) = \sum_{G\in\mathbb{Z}^{24\times 6}} \det(FS^{1/2}G)^2 e^{\pi i\,\mathrm{tr}(G^T SGZ)},$$

where S is a Gram matrix of the Leech lattice Λ, $S^{1/2}$ is the unique positive definite matrix T with $T^2 = S$ and F is a complex 6×24 matrix such that $FF^T = 0$ and $\overline{F}F^T$ is positive definite. Such a function is a Siegel cusp form of degree 6 and weight 14. Its vanishing order at ∞ is at least 2 because Λ has no vectors of norm 2. By a result of Tai (see Proposition 5.34), it therefore defines a global section of the second power of the canonical bundle on a smooth projective model of $\mathcal{A}_6$. An immediate consequence of this is that the Kodaira dimension of $\mathcal{A}_6$ is non-negative if $\Theta_{S,F}$ is non-zero.

Showing that $\Theta_{S,F}$ is non-zero is difficult. The Fourier coefficients of $\Theta_{S,F}$ correspond to 6-dimensional sublattices of Λ and it suffices to show that the Fourier coefficient corresponding to one 6-dimensional sublattice L of Λ is non-zero. This is still a difficult task, since it requires to compute a sum that runs over all 6-dimensional sublattices of Λ that are isomorphic to L and there is a large number of these for every choice of L. In Section 5.2 we explain an algorithm with which we succeed in computing two coefficients. One of these coefficients is non-zero, which means that $\Theta_{S,F}$ does not vanish everywhere and that the Kodaira dimension of $\mathcal{A}_6$ is non-negative, as explained above.

We also compute a basis of the space of cusp forms of degree 6 and weight 14 to show that the cusp form with vanishing order 2 is unique up to non-zero constant factors.

5.1. Siegel Modular Forms

We give a brief introduction into Siegel modular forms, focusing on the results we need in the following section. We encourage the reader to consult [Fre83] should he want to go into more depth.

Definition 5.1. The *Siegel upper half-plane* $\mathbb{H}_n$ is defined by

$$\mathbb{H}_n = \left\{ Z\in\mathbb{C}^{n\times n} : Z = Z^T, \mathrm{Im}(Z) > 0 \right\},$$

Remark 5.2. For a square matrix Z we write $Z > 0$ (resp. $Z \geq 0$) if Z is positive definite (resp. positive semi-definite). Therefore, $\mathbb{H}_n$ consists of the symmetric complex $n\times n$ matrices whose imaginary part is positive definite.

Definition 5.3. The *symplectic group* $\mathrm{Sp}_{2n}(\mathbb{Z})$ is the group of matrices $M \in \mathrm{GL}_{2n}(\mathbb{Z})$ that satisfy

$$M^T \Omega M = \Omega,$$

where Ω is the skew-symmetric matrix

$$\Omega = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}.$$

Let $Z \in \mathbb{H}_n$ and $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_{2n}(\mathbb{Z})$. Then one can show that $CZ + D$ is invertible and that $(AZ + B)(CZ + D)^{-1}$ is again in $\mathbb{H}_n$. Therefore, the following definition makes sense:

Definition 5.4. For

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{Z})$$

and $Z \in \mathbb{H}_n$, we let

$$MZ = (AZ + B)(CZ + D)^{-1}.$$

This defines a group action of $\mathrm{Sp}_{2n}(\mathbb{Z})$ on $\mathbb{H}_n$.

Remark 5.5. If $n = 1$, then $\mathbb{H}_n$ is just the usual upper half-plane $\mathbb{H}$ and $\mathrm{Sp}_{2n}(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})$, so that the action we have just defined is the action by linear fractional transformations seen in (2.1) (restricted to the subgroup $\mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{GL}_2(\mathbb{Q})^+$).

Definition 5.6. A holomorphic function $f\colon \mathbb{H}_n \to \mathbb{C}$ is called a *Siegel modular form* of degree n and weight $k \in \mathbb{Z}$ if

1. $f(MZ) = \det(CZ + D)^k f(Z)$ for all $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_{2n}(\mathbb{Z})$ and
2. f is bounded in the domain

$$\{Z \in \mathbb{H}_n : \mathrm{Im}(Z) - Y_0 \geq 0\}$$

for all positive definite $Y_0 \in \mathbb{R}^{n\times n}$.

Remark 5.7. If $n > 1$, then we can drop the second condition as it is automatically satisfied by the Koecher principle (see [Koe54]).

Recall that a symmetric matrix in $\mathbb{Z}^{n\times n}$ is called even if all diagonal entries are even. Every Siegel modular form of degree n and weight k has a Fourier expansion of the form

$$f(Z) = \sum_{\substack{T \in \mathbb{Z}^{n\times n} \\ T \text{ even} \\ T \geq 0}} a(T) e^{\pi i \,\mathrm{tr}(TZ)} \quad (\mathrm{tr} = \text{trace})$$

with complex Fourier coefficients $a(T)$. These coefficients satisfy

$$a(U^T T U) = \det(U)^k a(T) \tag{5.1}$$

for all $U \in \mathrm{GL}_n(\mathbb{Z})$.

Definition 5.8. For an even positive semi-definite matrix $T \in \mathbb{Z}^{n\times n}$ we define

$$m(T) = \min_{x\in\mathbb{Z}^n\setminus\{0\}} \frac{1}{2}x^T T x,$$

so $m(T)$ is the smallest integer that is represented non-trivially by the quadratic form with Gram matrix T.

Definition 5.9. Let

$$f(Z) = \sum_{\substack{T\in\mathbb{Z}^{n\times n}\\ T \text{ even}\\ T\geq 0}} a(T)e^{\pi i \operatorname{tr}(TZ)}$$

be a Siegel modular form of degree n and weight k. We define the *vanishing order* m of f at ∞ to be

$$m = \min_{\substack{T\in\mathbb{Z}^{n\times n}, T \text{ even}\\ T\geq 0\\ a(T)\neq 0}} m(T)$$

and we say that f is a *cusp form* if $m \geq 1$.

Remark 5.10. Note that the cusp forms are exactly the modular forms that satisfy

$$a(T) \neq 0 \Rightarrow T > 0,$$

i.e. those whose Fourier expansion is supported on positive definite matrices.

The space of Siegel modular forms of degree n and weight k is denoted by $M_{n,k}$ and $S_{n,k}$ denotes the subspace of Siegel cusp forms. All of these spaces are finite-dimensional. For $k < 0$ they are trivial.

Theta Series

Examples of Siegel modular forms are theta series, the most basic of which are as in the following proposition:

Proposition 5.11. *Let Λ be an even positive definite unimodular lattice of rank m with bilinear form $(\cdot,\cdot)$. Then the Siegel theta series $\Theta_\Lambda\colon \mathbb{H}_n \to \mathbb{C}$ defined by*

$$\Theta_\Lambda(Z) = \sum_{x\in\Lambda^n} e^{\pi i \operatorname{tr}(T(x)Z)},$$

where

$$T(x) = ((x_i, x_j))_{1\leq i,j\leq n} \quad (x = (x_1, \ldots, x_n))$$

is a Siegel modular form of degree n and weight $m/2$. Its Fourier expansion is given by

$$\Theta_\Lambda(Z) = \sum_{\substack{T\in\mathbb{Z}^{n\times n}\\ T \text{ even}\\ T\geq 0}} a(T)e^{\pi i \operatorname{tr}(TZ)},$$

where

$$a(T) = \left|\{x \in \Lambda^n : T(x) = T\}\right|.$$

Remark 5.12. The finiteness of $a(T)$ follows easily from the positive definiteness of Λ.

Proposition 5.13 ([Böc83], Theorem III)**.** *Let n and k be positive integers such that $k > 2n$ and $4 \mid k$. Then every Siegel modular form of degree n and weight k is a linear combination of Siegel theta series Θ_Λ for even positive definite unimodular lattices Λ of rank $2k$.*

To define more general theta series we need to define the notion of a harmonic form:

Definition 5.14. Let $P\colon \mathbb{C}^{m\times n} \to \mathbb{C}$ be a polynomial in the entries of a complex $m \times n$ matrix. We call P a *harmonic form* of weight k if

1. $P(XA) = \det(A)^k P(X)$ for all $A \in \mathbb{C}^{n\times n}$, and
2. $\Delta P = \sum_{\mu,\nu} \frac{\partial^2}{(\partial x_{\mu,\nu})^2} P = 0$.

Remark 5.15 ([Maa80], Satz 2). If $m \geq 2n$, then the space of harmonic forms P of weight k is spanned by the polynomials of the form

$$P_L(X) = \det(LX)^k,$$

where L is a complex $n \times m$ matrix such that $LL^T = 0$ and $\overline{L}L^T > 0$.

Proposition 5.16. *Let $S \in \mathbb{Z}^{m\times m}$ be an even positive definite unimodular matrix and let P be a harmonic form of weight k. Then*

$$\Theta_{S,P}(Z) = \sum_{G\in\mathbb{Z}^{m\times n}} P(S^{1/2}G)e^{\pi i \operatorname{tr}(G^T SGZ)}$$

is a Siegel cusp form of degree n and weight $m/2 + k$.

The analogue of Proposition 5.13 is the following:

Proposition 5.17 ([Böc85], Satz 16)**.** *Let n, m and k be positive integers with $8 \mid m$ and $m/2 > 2n$. Then every Siegel cusp form of degree n and weight $m/2 + k$ is a linear combination of Siegel theta series of the form $\Theta_{S,P}$ for even positive definite unimodular matrices S and harmonic forms P of weight k.*

Remark 5.18. Note that we have formulated Proposition 5.11 in terms of lattices, whereas we have formulated Proposition 5.16 in terms of matrices, i.e. we have chosen a basis of the lattice. We now also give a description of $\Theta_{S,P}$ in terms of lattices if P is of the form P_L as in Remark 5.15: Let Λ be an even positive definite unimodular lattice of rank m with bilinear form $(\cdot,\cdot)$ and Gram matrix S and let $\tilde{L} = LS^{-1/2}$. Then

$$\begin{aligned}\Theta_{S,P_L} &= \sum_{G\in\mathbb{Z}^{m\times n}} P_L(S^{1/2}G)e^{\pi i \operatorname{tr}(G^T SGZ)}\\ &= \sum_{G\in\mathbb{Z}^{m\times n}} \det(LS^{1/2}G)^k e^{\pi i \operatorname{tr}(G^T SGZ)}\\ &= \sum_{G\in\mathbb{Z}^{m\times n}} \det(\tilde{L}SG)^k e^{\pi i \operatorname{tr}(G^T SGZ)}\end{aligned}$$

and the conditions $LL^T = 0$ and $\overline{L}L^T > 0$ are equivalent to $\tilde{L}S\tilde{L}^T = 0$ and $\overline{\tilde{L}}S\tilde{L}^T > 0$. Let $\Lambda_{\mathbb{C}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ and extend the bilinear form $(\cdot, \cdot)$ to a bilinear form on $\Lambda_{\mathbb{C}}$. For two tuples $w = (w_1, \ldots, w_n)$ and $z = (z_1, \ldots, z_n)$ in $\Lambda_{\mathbb{C}}^n$ we write

$$T(w, z) = ((w_i, z_j))_{1 \leq i,j \leq n}$$

and $T(w) = T(w, w)$. The rows of $\tilde{L}$ determine elements $y_1, \ldots, y_n$ in $\Lambda_{\mathbb{C}}$ and the conditions on $\tilde{L}$ are equivalent to $T(y) = 0$ and $T(y, \overline{y}) > 0$, where $y = (y_1, \ldots, y_n)$. Moreover, Θ_{S,P_L} is obviously the same as

$$\sum_{x \in \Lambda^n} \det(T(x, y))^k e^{\pi i \operatorname{tr}(T(x)Z)},$$

which we denote by $\Theta_{\Lambda,y,k}$.

Remark 5.19. The Fourier expansion of $\Theta_{\Lambda,y,k}$ is easily seen to be given by

$$\Theta_{\Lambda,y,k}(Z) = \sum_{\substack{T \in \mathbb{Z}^{n \times n} \\ T \text{ even} \\ T \geq 0}} a(T) e^{\pi i \operatorname{tr}(TZ)}$$

with

$$a(T) = \sum_{\substack{x \in \Lambda^n \\ T(x) = T}} \det(T(x, y))^k.$$

5.2. The Space of Cusp Forms of Degree 6 and Weight 14

In this section we determine a basis of the space $S_{6,14}$ of Siegel cusp forms of degree 6 and weight 14. In order to do this we first note the following:

Proposition 5.20 (see [Taï17], Table 3)**.** *The dimension of the space of Siegel cusp forms of degree* 6 *and weight* 14 *is* 9.

In the following we specify nine theta series of the form $\Theta_{\Lambda,y,2}$ with Λ a Niemeier lattice and $y \in (\Lambda \otimes_{\mathbb{Z}} \mathbb{C})^6$ satisfying $T(y) = 0$ and $T(y, \overline{y}) > 0$ and show that they are linearly independent by calculating enough Fourier coefficients with a standard computer (with 64 GB of RAM). We do not know beforehand that there exists a basis of this form because Proposition 5.17 cannot be applied since $24/2$ is not larger than $2 \cdot 6$.

Theorem 5.21. *Let* Φ *be one of the root systems*

$$\mathsf{A}_6^4, \mathsf{A}_4^6, \mathsf{A}_3^8, \mathsf{A}_2^{12}, \mathsf{D}_6^4, \mathsf{D}_4^6, \mathsf{E}_6^4.$$

Let $N(\Phi)$ *be the corresponding Niemeier lattice and let* $e = (e_1, \ldots, e_{24}) \in N(\Phi)^{24}$ *be such that* $T(e)$ *is the standard Gram matrix (see Remark 1.26) of the* 2*-root lattice spanned by* Φ*. For* $j = 1, \ldots, 6$ *we let* $b_j = e_j + ie_{12+j}$ *and we define vectors* $y_1, \ldots, y_6$ *in* $N(\Phi) \otimes_{\mathbb{Z}} \mathbb{C}$ *depending on* Φ *as follows:*

- *If* $\Phi = \mathsf{A}_6^4$, *then* $y_j = \sum_{k=1}^{j} kb_k$ *for all* $j \in \{1, \ldots, 6\}$.

- *If* $\Phi = \mathsf{A}_4^6$, *then*

$$y_j = \begin{cases} \sum_{k=1}^{j} kb_k & \text{if } 1 \leq j \leq 4, \\ \sum_{k=5}^{j} (k-4)b_k & \text{if } 4 < j \leq 6. \end{cases}$$

- *If* $\Phi = \mathsf{A}_3^8$, *then*

$$y_j = \begin{cases} \sum_{k=1}^{j} ke_k + ike_{3+3j+k} & \text{if } 1 \leq j \leq 3, \\ \sum_{k=4}^{j} (k-3)e_k + i(k-3)e_{3j+k} & \text{if } 3 < j \leq 6. \end{cases}$$

- *If* $\Phi = \mathsf{A}_2^{12}$, *then*

$$y_j = \begin{cases} \sum_{k=1}^{j} kb_k & \text{if } 1 \leq j \leq 2, \\ \sum_{k=3}^{j} (k-2)b_k & \text{if } 2 < j \leq 4, \\ \sum_{k=5}^{j} (k-4)b_k & \text{if } 4 < j \leq 6. \end{cases}$$

- *If* $\Phi = \mathsf{D}_6^4$, *then*

$$y_j = \begin{cases} \sum_{k=1}^{j} kb_k & \text{if } 1 \leq j \leq 5, \\ 2b_5 + 3b_6 + \sum_{k=1}^{4} kb_k & \text{if } j = 6. \end{cases}$$

- *If* $\Phi = \mathsf{D}_4^6$, *then*

$$y_j = \begin{cases} \sum_{k=1}^{j} kb_k & \text{if } 1 \leq j \leq 3, \\ b_3 + 2b_4 + \sum_{k=1}^{2} kb_k & \text{if } j = 4, \\ \sum_{k=5}^{j} (k-4)b_k & \text{if } 4 < j \leq 6. \end{cases}$$

- *If* $\Phi = \mathsf{E}_6^4$, *then*

$$y_j = \begin{cases} \sum_{k=1}^{j} kb_k & \text{if } 1 \leq j \leq 5, \\ b_1 + 2b_2 + 3b_3 + 2b_4 + b_5 + b_6 & \text{if } j = 6. \end{cases}$$

We then let $y = (y_1, \ldots, y_6)$. *Then* $\Theta_{N(\Phi),y,2}$ *is a Siegel cusp form of degree* 6 *and weight* 14. *After multiplying* $\Theta_{N(\Phi),y,2}$ *with a non-zero constant the coefficients* $a(T)$ *where* T *is the Gram matrix of a* 6*-dimensional* 2*-root lattice* K *are given in the following table:*

	Φ: A_6^4	A_4^6	A_3^8	A_2^{12}	D_6^4	D_4^6	E_6^4
K: A_6	1	0	0	0	0	0	0
D_6	0	0	0	0	1	0	0
E_6	0	0	0	0	0	0	1
A_5A_1	−12	0	0	0	−30	0	−20
D_5A_1	0	0	0	0	−10	0	−32
A_4A_2	30	1	0	0	120	0	240
$A_4A_1^2$	40	−2	0	0	260	0	320
D_4A_2	0	0	0	0	72	1	192
$D_4A_1^2$	0	0	0	0	48	−2	384
A_3^2	−32	0	1	0	−192	0	−432
$A_3A_2A_1$	−96	−8	−6	0	−276	−6	−480
$A_3A_1^3$	576	48	20	0	672	36	192
A_2^3	648	54	0	1	1296	36	900
$A_2^2A_1^2$	−432	−36	72	−2	−1080	0	−1152
$A_2A_1^4$	−1152	96	−480	12	−3456	−152	768
A_1^6	−11520	−2880	4000	−120	3840	240	−46080

Proof. That $\Theta_{N(\Phi),y,2}$ is a Siegel cusp form of degree 6 and weight 14 follows from Proposition 5.16 because $T(y) = 0$ and $T(y, \overline{y}) > 0$ (as one easily checks).

To compute the Fourier coefficients we first remark that if T and S are Gram matrices of the same 6-dimensional lattice, then $a(T) = a(S)$ by (5.1). We therefore only need to compute $a(T)$ when T is a standard Gram matrix of a 6-dimensional 2-root lattice K. Recall that

$$a(T) = \sum_{\substack{x \in N(\Phi)^6 \\ T(x)=T}} \det(T(x, y))^2.$$

Since K is generated by norm 2 vectors, it follows that every x in the above sum must be in the sublattice $L \subset N(\Phi)$ that is generated by the norm 2 vectors (i.e. the roots) in $N(\Phi)$. Therefore

$$a(T) = \sum_{\substack{x \in L^6 \\ T(x)=T}} \det(T(x, y))^2. \tag{5.2}$$

Since e is a basis of L and y is defined in terms of e, we do not have to work with the Niemeier lattice $N(\Phi)$ but only with the 2-root lattice L. Also note that (5.2) does not depend on the choice of e as any two choices of e are related by an automorphism of L.

We explain how we calculate $a(T)$ with a computer in the case that $\Phi = \mathsf{A}_6^4$. The other cases are similar.

Recall that the lattice A_6 is given by

$$A_6 = \left\{ (x_1, \ldots, x_7) \in \mathbb{Z}^7 : \sum_{i=1}^{7} x_i = 0 \right\}$$

with the standard quadratic form on $\mathbb{Z}^7$. Therefore,

$$L = A_6^4 = \left\{ (x_1, \ldots, x_{28}) \in \mathbb{Z}^{28} : \sum_{i=1}^{7} x_i = \sum_{i=8}^{14} x_i = \sum_{i=15}^{21} x_i = \sum_{i=22}^{28} x_i = 0 \right\}.$$

The vectors $y_1, \ldots, y_6$ are then elements of $\mathbb{C}^{28}$ and we write them as rows of a matrix $F \in \mathbb{C}^{6\times 28}$. We compute and store a list S of all norm 2 vectors in L (these are just the vectors in L with one coordinate equal to 1, one equal to -1 and all others equal to 0). We remark that L contains no sublattices isomorphic to D_4, D_5, D_6 and E_6. For the remaining irreducible 2-root lattices $M = A_n$, $n \leq 6$ of rank at most 6 we let T_M be the standard Gram matrix of these and compute the set S_M consisting of all elements $x = (x_1, \ldots, x_n)$ in L^n with $T(x) = T_M$ and such that $Fx_1, \ldots, Fx_n$ are linearly independent. This is done by first computing the set S_{A_1} (which is just given by those $x \in S$ with $Fx \neq 0$). For $n > 1$, S_{A_n} is computed by iterating over all elements $(x_1, \ldots, x_{n-1})$ of $S_{A_{n-1}}$ and all elements x_n in S_{A_1} and then checking whether $x = (x_1, \ldots, x_n)$ satisfies the conditions to be in S_{A_n}.

Having computed the set S_M for $M = A_n$, $n \leq 6$, we continue as follows: For each of the lattices A_n, $n \leq 6$ we implement $O(A_n)$ as the group of integral $n \times n$ matrices g that satisfy $g^T T_{A_n} g = T_{A_n}$. If we view elements x of S_{A_n} as $28 \times n$ matrices, then there is an obvious right action of $O(A_n)$ on L_{A_n}, namely the one given by the matrix product. We compute a set of representatives R_{A_n} for $S_{A_n}/O(A_n)$ as follows: First compute and store $O(A_n)$ (in SageMath this can be done by specifying generators and then using the `MatrixGroup` command). We then compute the orbits of the action in the most simple way: For each $x \in S_{A_n}$ that is not in one of the orbits that have already been computed, we iterate over all $g \in O(A_n)$, compute $\tilde{x} = xg$ and store that $\tilde{x}$ is in the same orbit as x.

Next, we consider 2-root lattices of the form $M = A_{n_1} \oplus A_{n_2}$ with $n_1 + n_2 \leq 6$ and $n_2 \leq n_1$ and let T_M be a standard Gram matrix of M. Let $n = n_1 + n_2$. We let S_M be the set of $x = (x_1, \ldots, x_n) \in L^n$ with $T(x) = T_M$ and such that $Fx_1, \ldots, Fx_n$ are linearly independent. Then S_M is a subset of $S_{A_{n_1}} \times S_{A_{n_2}}$ and $O(A_{n_1})$ and $O(A_{n_2})$ act on S_M by their action on $S_{A_{n_1}}$ and $S_{A_{n_2}}$. We can compute a set of representatives R'_M for $S_M/(O(A_{n_1}) \times O(A_{n_2}))$ by iterating over all elements $(x_1, \ldots, x_{n_1}) \in R_{A_{n_1}}$ and all elements $(x_{n_1+1}, \ldots, x_n)$ in $R_{A_{n_2}}$ and checking if $x = (x_1, \ldots, x_n)$ satisfies $T(x) = T_M$ and if $Fx_1, \ldots, Fx_n$ are linearly independent. If $n_1 \neq n_2$, then $O(M) = O(A_{n_1}) \times O(A_{n_2})$. In this case we set $R_M = R'_M$. If $n_1 = n_2$, then $O(M)$ is the semi-direct product of the symmetric group S_2 and $O(A_{n_1}) \times O(A_{n_2})$. In this case we let R_M be a set of representatives for R'_M/S_2 where S_2 acts by interchanging x_j with x_{j+n_1} for $j = 1, \ldots, n_1$. This can be computed similarly to all previously calculated sets of representatives. In all cases R_M is a set of representatives for $S_M/O(M)$.

Continuing in this way, we can compute a set of representatives for $S_M/O(M)$ where $M = A_{n_1} \oplus \ldots \oplus A_{n_k}$, $n_1 \geq n_2 \geq \ldots \geq n_k$, $n = n_1 + \ldots + n_k$, $n \leq 6$, T_M is the standard

Gram matrix of M, and

$$S_M = \{x = (x_1, \dots, x_n) \in L^n : T(x) = T_M, Fx_1, \dots, Fx_n \text{ linearly independent}\}.$$

If $n = 6$, we then compute

$$\begin{aligned} a(T_M) &= \sum_{\substack{x\in L^6 \\ T(x)=T_M}} \det(T(x,y))^2 = \sum_{\substack{x\in L^6 \\ T(x)=T_M}} \det(Fx)^2 \\ &= \sum_{x\in S_M} \det(Fx)^2 = \sum_{x\in R_M} \sum_{g\in O(M)} \det(Fxg)^2 \\ &= |O(M)| \sum_{x\in R_M} \det(Fx)^2, \end{aligned}$$

where we again identify $x \in L^6$ with a 28×6 matrix. □

Next, we consider the lattice $N(\mathsf{A}_1^{24})$. We use the notation from Example 1.32 and take the explicit form of the Golay code given in Subsection 1.2.2. For $j = 1, \dots, 6$ we define the vectors y_j by $y_j = x_j + ix_{6+j}$.

Theorem 5.22. *The theta series $\Theta_{N(\mathsf{A}_1^{24}),y,2}$ is a Siegel cusp form of degree 6 and weight 14, the coefficient $a(T)$ vanishes if T is the Gram matrix of a 2-root lattice and*

$$a(T) = 36238786560$$

if T is a Gram matrix of $A_1(2) \oplus A_1^5$.

Proof. That $\Theta_{N(\mathsf{A}_1^{24}),y,2}$ is a Siegel cusp form of degree 6 and weight 14 follows from Proposition 5.16 and the choice of $y_1, \dots, y_6$.

The sublattice $L \subset N(\mathsf{A}_1^{24})$ that is generated by the norm 2 vectors in $N(\mathsf{A}_1^{24})$ contains no 2-root lattices except direct sums of A_1. Therefore, $a(T) = 0$ if T is the Gram matrix of a 2-root lattice different from A_1^6. If T is the standard Gram matrix of A_1^6, then we can compute $a(T)$ in the same way as in the proof of the previous proposition.

It remains to compute $a(T)$ for a Gram matrix T of $A_1(2) \oplus A_1^5$; a possible choice for T is $T = \operatorname{diag}(4, 2, 2, 2, 2, 2)$. This can almost be done as in the proof of the previous proposition, except that we also need to determine the set of vectors of norm 4 in $N(\mathsf{A}_1^{24})$. It is easy to see that the norm 4 vectors in $N(\mathsf{A}_1^{24})$ are the following:

1. Any sum of two orthogonal norm 2 vectors in L. This gives $\binom{24}{2} \cdot 2^2 = 1104$ vectors of norm 4.

2. Any vector $\sum_{i=1}^{24} \lambda_i w_i$ such that $\lambda_i \in \{-1, 0, 1\}$ and the reduction of $(\lambda_1, \dots, \lambda_{24})$ modulo 2 is an octad in the Golay code. This gives another $759 \cdot 2^8 = 194304$ elements of norm 4.

The calculation of $a(T)$ is now similar to the calculations in the proof of the previous proposition. □

From the Fourier coefficients we have calculated, we see that the eight theta series we have considered so far are linearly independent. The final element of our basis is the theta series

$$\Theta_{\Lambda,y,2}$$

for the Leech lattice Λ and a suitable choice of $y \in (\Lambda \otimes_{\mathbb{Z}} \mathbb{C})^6$. Recall that the Leech lattice has no vectors of norm 2, so the coefficient $a(T)$ will trivially be zero for all T that we have considered so far. It therefore suffices to prove that $\Theta_{\Lambda,y,2}$ is not the zero function to show that it can be added to the previous eight theta series to form a basis of $S_{6,14}$. The idea for proving $\Theta_{\Lambda,y,2} \neq 0$ is to show that there is a non-vanishing Fourier coefficient $a(T)$ for some T. Our choice for T is

$$T = \begin{pmatrix} 4 & -2 & 0 & 0 & 0 & 0 \\ -2 & 4 & -2 & 0 & 0 & 0 \\ 0 & -2 & 4 & -2 & 0 & -2 \\ 0 & 0 & -2 & 4 & -2 & 0 \\ 0 & 0 & 0 & -2 & 4 & 0 \\ 0 & 0 & -2 & 0 & 0 & 4 \end{pmatrix},$$

which is a Gram matrix of $E_6(2)$. We now encounter the problem that the number of sublattices in Λ that are isomorphic to $E_6(2)$ is very large, so $a(T)$ cannot be calculated as easily as the coefficients we have seen so far in this section. Instead, a more sophisticated algorithm to be described in the following is needed.

We let c be a dodecad in $\mathcal{G}$; in our computations we have taken

$$c = (1, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 1, 1, 0, 0, 1).$$

The subgroup of M_{24} that fixes c is isomorphic to the Mathieu group M_{12}, which is also one of the sporadic simple groups. Its order is $2^6 \cdot 3^3 \cdot 5 \cdot 11 = 95040$. We let g be an element in M_{12} with cycle type 2^{12}; in our computations we have taken

$$g = (1, 9)(2, 12)(3, 7)(4, 24)(5, 21)(6, 18)(8, 19)(10, 16)(11, 17)(13, 20)(14, 23)(15, 22).$$

Let $I \subset \{1, \dots, 24\}$ be the support of $c = (c_j)_{1 \leq j \leq 24}$, i.e. the set of those indices j for which $c_j = 1$. For each $j \in I$ the image $g(j)$ under g is also in I and we can therefore arrange the elements of I in pairs $(j, g(j))$, $j < g(j)$. For our choice of c and g we obtain the pairs

$$(1, 9), (4, 24), (5, 21), (8, 19), (10, 16), (13, 20).$$

We order the six pairs. Suppose the j-th pair is given by (a, b). We then define the vectors $y_j \in \Lambda \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}^{24}$ by $y_j = e_a + ie_b$, where $e_1, \dots, e_{24}$ is the standard basis of

$\mathbb{C}^{24}$. For our choice of c and g the vector y_j is the j-th row of the matrix

$$F = \begin{pmatrix} 1&0&0&0&0&0&0&0&i&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&i \\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&i&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&i&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&i&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&i&0&0&0&0 \end{pmatrix}.$$

Let $y = (y_1, \ldots, y_6) \in \Lambda_{\mathbb{C}}^6$. Recall that the bilinear form on Λ is given by

$$(x, w) = \frac{1}{8} \sum_{j=1}^{24} x_i w_i \quad (x = (x_1, \ldots, x_{24}), w = (w_1, \ldots, w_{24})).$$

Therefore, $T(y) = \frac{1}{8} F F^T = 0$ and $T(y, \overline{y}) = \frac{1}{8} F \overline{F}^T = \frac{1}{4} I_6$, so $\Theta_{\Lambda,y,2}$ is a cusp form of degree 6 and weight 14 by Proposition 5.16.

Let $N = EM_{24}$ be the maximal subgroup of Co_0 described in Subsection 1.2.2. We let H be the subgroup of N that consists of those elements $h \in N$ for which the matrix with rows $h(y_1), \ldots, h(y_6)$ equals the product of a complex 6×6-matrix of determinant ± 1 with the matrix with rows $y_1, \ldots, y_6$ (i.e. the matrix F for our choice of y). From our construction of y, it is easy to see that H is a subgroup of EM_{12} and even a subgroup of EH_1 where H_1 is the subgroup of M_{12} that permutes the six pairs above. The group H_1 is a maximal subgroup of order 240 of M_{12}. It is the centralizer of g in M_{12} and the group EH_1 has order $4096 \cdot 240 = 983040$.

On the computer we implement this as follows: M_{24} is the subgroup of S_{24} generated by the elements in (1.1). In SageMath we obtain M_{24} from these generators with the `PermutationGroup` command.

We then compute the subgroup M_{12}: Let $S = \emptyset$. Iterate over the elements $\pi \in M_{24}$ and check whether $\pi(c)$ equals c. If it does, then add π to S and compute the group generated by the elements in S (in SageMath we again use the `PermutationGroup` command). If it has order 95040, then S is a generating set for M_{12} and we are done. Otherwise, continue with the next element $\pi \in M_{24}$.

We then determine $H_1 \subset M_{12}$ in the same way, i.e. by iterating over elements in M_{12} and checking whether they commute with g or not.

To compute H we implement EH_1 as a matrix group (the elements in 2^{12} are diagonal matrices and the elements in H_1 are permutation matrices). Then the action of EH_1 on elements of the Leech lattice is given by matrix multiplication. We iterate over all elements h in EH_1 and compute Fh (whose rows will then be the elements $h(y_1), \ldots, h(y_6)$). We then check whether there is a matrix $M \in \mathbb{C}^{6\times 6}$ of determinant ± 1 with

$$Fh = MF \tag{5.3}$$

as follows: Suppose there is such a matrix M. We multiply (5.3) with $\overline{F}^T$ from the right to obtain

$$Fh\overline{F}^T = MF\overline{F}^T = 2M$$

because $\overline{F}F^T = 2I_6$. Therefore, if such a matrix M exists, it must be equal to

$$\frac{1}{2}Fh\overline{F}^T,$$

so we only need to check if

$$Fh = \frac{1}{2}FhF\overline{F}^T.$$

It turns out that $|H| = 30720$. We remark that

$$\det(Fhx)^2 = \det(Fx)^2$$

for every $h \in H$ and $x \in \Lambda^6$ (where we identify x with a 24×6 matrix) because

$$\det(Fhx)^2 = \det(MFx)^2 = \det(M)^2 \det(Fx)^2 = \det(Fx)^2$$

for a suitable complex 6×6 matrix M of determinant ± 1 by our definition of H.

We let EM_{12} and for $x \in \Lambda^6$, we let G_x be the stabilizer of x in G. We continue to identify elements $x \in \Lambda^6$ with 24×6 matrices. Then the coefficient $a(T)$ is given by

$$\begin{aligned}
a(T) &= \sum_{\substack{x \in \Lambda^6 \\ T(x) = T}} \det(T(x,y))^2 \\
&= \sum_{\substack{x \in \Lambda^6 \\ T(x) = T}} \det\left(\frac{1}{8}Fx\right)^2 \\
&= \frac{1}{2^{36}} \sum_{\substack{[x] \in G\backslash\Lambda^6 \\ T(x) = T}} \frac{1}{|G_x|} \sum_{a \in G} \det(Fax)^2 \\
&= \frac{1}{2^{36}} \sum_{\substack{[x] \in G\backslash\Lambda^6 \\ T(x) = T}} \frac{1}{|G_x|} \sum_{Ha \in H\backslash G} \sum_{h \in H} \det(Fhax)^2 \\
&= \frac{1}{2^{36}} \sum_{\substack{[x] \in G\backslash\Lambda^6 \\ T(x) = T}} \frac{1}{|G_x|} \sum_{Ha \in H\backslash G} \sum_{h \in H} \det(Fax)^2 \\
&= \frac{|H|}{2^{36}} \sum_{\substack{[x] \in G\backslash\Lambda^6 \\ T(x) = T}} \frac{1}{|G_x|} \sum_{Ha \in H\backslash G} \det(Fax)^2.
\end{aligned}$$

Remark 5.23. Of course we would prefer to write $a(T)$ as

$$a(T) = \frac{|H|}{2^{36}} \sum_{\substack{[x] \in H\backslash\Lambda^6 \\ T(x) = T}} \frac{1}{|H_x|} \det(Fx)^2,$$

where H_x is the stabilizer of x in H. The problem with this is that the set

$$\{[x] \in H\backslash\Lambda^6 : T(x) = T\}$$

is too large to be computed.

Both a set of representatives for

$$\{[x] \in G\backslash\Lambda^6 : T(x) = T\}$$

and for $H\backslash G$ can be obtained with a computer as we will see later. However, these sets have sizes 14622480 and 12672 so we would have to calculate $14622480 \cdot 12672 \approx 1.85 \cdot 10^{11}$ determinants of 6×6 matrices, which takes too long. To solve this problem we let W be the Weyl group of E_6, i.e. the subgroup of $O(E_6)$ that is generated by reflections through the hyperplanes orthogonal to the roots of E_6. It has order $2^7 \cdot 3^4 \cdot 5 = 51840$. Then every $w \in W$ is also an automorphism of $E_6(2)$ and W hence acts from the right on the set

$$L_6 = \{x \in \Lambda^6 : T(x) = T\}.$$

Viewing every $w \in W$ as a 6×6 matrix of determinant ± 1 and every $x \in \Lambda^6$ as a 24×6 matrix, this action is given by matrix multiplication. Then

$$\det(Fxw)^2 = \det(Fx)^2$$

for every $x \in L_6$ and every $w \in W$. We can therefore write

$$\begin{aligned} a(T) &= \frac{|H|}{2^{36}} \sum_{[x] \in G\backslash L_6} \frac{1}{|G_x|} \sum_{Ha \in H\backslash G} \det(Fax)^2 \\ &= \frac{|H|}{2^{36}} \sum_{[x] \in G\backslash L_6} \frac{|Gx|}{|G|} \sum_{Ha \in H\backslash G} \det(Fax)^2 \\ &= \frac{1}{2^{36} \cdot |G : H|} \sum_{[x] \in G\backslash L_6/W} |GxW| \sum_{Ha \in H\backslash G} \det(Fax)^2, \end{aligned}$$

which can be computed once we have a set of representatives for the double cosets in $G\backslash L_6/W$ as well as the sizes of these double cosets and a set of representatives for $H\backslash G$. We now describe how these can be computed. We use the notation

$$L_j = \{x \in \Lambda^j : T(x) = T_j\},$$

where T_j is the upper left $j \times j$ submatrix of T.

1. We store the set Λ_4 of norm 4 vectors in Λ.

2. We then compute a set of representatives R_1 for $G\backslash L_1$ and the stabilizer subgroups $G_x \subset G$ for every $x \in L_1$. Moreover, for every $z \in \Lambda_4$ we compute and store an element $g_{1,z} \in G$ that satisfies $g_{1,z}(z) \in R_1$. This can be done as follows: We let $S = \emptyset$. Start with any element x in Λ_4 and iterate over all $g \in G$. Compute $z = g(x)$. If $z = x$, then store that g is an element of G_x. If z is in S, then we continue with the next g. Otherwise we set $g_{1,z} = g^{-1}$ and add z to S. Having iterated over all $g \in G$ we add x to R_1 and repeat these steps with the next element in Λ_4 that is not in S until S contains all elements of Λ_4.

3. For every $x_1 \in R_1$ we compute a set of representatives R_{x_1} for $G_{x_1} \backslash L_{x_1}$, where
$$L_{x_1} = \{x_2 \in \Lambda_4 : T((x_1, x_2)) = T_2\},$$
as well as the stabilizer $G_x \subset G_{x_1}$ of $x = (x_1, x_2)$. Moreover, for every $x_1 \in R_1$ and every $z \in L_{x_1}$ we store an element $g_{2,x_1,z} \in G_{x_1}$ such that $g_{2,x_1,z}(z) \in R_{x_1}$. This can be done in a similar way as in the previous step. We note that
$$R_2 = \bigcup_{x_1 \in R_1} \bigcup_{x_2 \in R_{x_1}} (x_1, x_2)$$
is a set of representatives for $G \backslash L_2$ and that if $z = (z_1, z_2)$ is an arbitrary element in L_2, then we can quickly find the representative $x \in R_2$ for the orbit of z under G as follows: Let $g_1 = g_{1,z_1}$ and $x_1 = g_{1,z_1}(z_1)$. Then x_1 is in R_1 and $g_1(z) = (x_1, z_2')$ for some $z_2' \in L_{x_1}$. We then look up the element $g_2 = g_{2,x_1,z_2'} \in G_{x_1}$ and let $x_2 = g_{2,x_1,z_2'}(z_2')$. Then $g_2 g_1(z) = (x_1, x_2)$ is the representative in R_2 we were looking for.

4. We continue in the same way to obtain representatives R_j for $G \backslash L_j$ for all $j \in \mathbb{Z}$ with $2 \le j \le 6$ as well as the stabilizer $G_{x_j} \subset G$ for every $x_j \in R_j$. Moreover, for every $x = (x_1, \ldots, x_{j-1}) \in R_{j-1}$ and every $z \in \Lambda_4$ with
$$T(x') = T_j \quad (x' = (x_1, \ldots, x_{j-1}, z))$$
we compute an element $g_{j,x_{j-1},z} \in G_{x_{j-1}}$ with $g_{j,x_{j-1},z}(z) \in R_{x_{j-1}}$.

5. We let $S = \emptyset$ and start with any element x in R_6. We iterate over all $w \in W$ and let $z = w(x)$. If z is in S, then we continue with the next w. Otherwise, we write $z = (z_1, \ldots, z_6)$ and compute $g_{1,z_1}(z)$ to get an element $z' = (z_1', \ldots, z_6') \in L_6$ with $z_1' \in R_1$. We then compute $g_{2,z_1',z_2'}(z')$ to get an element $z'' = (z_1', z_2'', \ldots, z_6'') \in L_6$ with $(z_1', z_2'') \in R_2$. Continuing in this way, we finally obtain an element $\tilde{z} \in R_6$ that describes the same coset in $G \backslash L_6$ as z. Therefore, x and $\tilde{z}$ describe the same double coset in $G \backslash L_6 / W$. We append $\tilde{z}$ to S and add the size of the coset of $\tilde{z}$ in $G \backslash L_6$ (this is just the size of G divided by the size of the stabilizer of $G_{\tilde{z}}$) to the size of the double coset represented by x. Having iterated over all $w \in W$, we repeat these steps with the next element in R_6 that is not in S until S contains all elements of R_6. In the end we have a set of representatives for the double cosets in $G \backslash L_6 / W$ as well as the sizes of these double cosets. We remark that we obtain 875 double cosets.

6. A set of representatives for $H \backslash G$ can be obtained as follows: Note that the elements in EM_{24} can be viewed as signed permutations on a 24-element set. Therefore, both H and G are subgroups of the group of signed permutations of a 24-element set (this is also called the hyperoctahedral group as it is the symmetry group of a 24-dimensional cube. It is isomorphic to the Weyl groups of the root systems B_{24} and C_{24}). This group is isomorphic to the group of all permutations π of the 48-element set $S = \{\pm 1, \ldots, \pm 24\}$ that satisfy $\pi(-i) = -\pi(i)$ for all $i \in S$. We can

therefore view H and G as subgroups of S_{48}. We have implemented H and G in GAP [GAP14] by specifying generators in S_{48}. A set of representatives for $H\backslash G$ can then be obtained with the `RightCosets` command in GAP. The computer only needs seconds to do this.

It remains to compute $875 \cdot 12672$ determinants to obtain the coefficient $a(T)$. This is no problem and we finally obtain that

$$a(T) = 2^{13} \cdot 3^6 \cdot 5 = 29859840.$$

We remark that this is equal to $3 \cdot |W| \cdot \det(T)$. In particular we have the following:

Theorem 5.24. *Let Λ be the Leech lattice and $y \in (\Lambda \otimes_{\mathbb{Z}} \mathbb{C})^6$ as above. Then $\Theta_{\Lambda,y,2}$ does not vanish.*

Remark 5.25. Step 5 as described above takes several days on our computer. It is much faster to split this step into two steps: First define the subgroup W_1 of W to be the subgroup of those elements of W that fix the first basis vector of E_6. Then $|W_1| = 720$. We then first compute a set of representatives R_{6,W_1} for $G\backslash L_6/W_1$ together with the sizes of the double cosets in the same way as in Step 5 above. For every element x in R_6 we also store elements $g_x \in G$ and $w_x \in W$ such that $g_x x w_x$ is in R_{6,W_1}. Only then do we calculate a set of representatives for $G\backslash L_6/W$, namely by taking elements x in R_{6,W_1}, iterating over representatives $\tilde{w}$ of $W_1\backslash W$, computing $z = x\tilde{w}$ and using that $g_z z w_z$ is in $G\backslash L_6/W_1$.

Remark 5.26. In the same way we have also computed the Fourier coefficient $a(T)$ for the Gram matrix

$$T = \begin{pmatrix} 4 & 2 & -1 & 1 & -1 & -1 \\ 2 & 4 & 1 & 2 & -2 & -2 \\ -1 & 1 & 4 & 2 & -2 & 1 \\ 1 & 2 & 2 & 4 & -1 & -1 \\ -1 & -2 & -2 & -1 & 4 & 1 \\ -1 & -2 & 1 & -1 & 1 & 4 \end{pmatrix}$$

of $E_6'(3)$. In this case $G\backslash L_6$ has size 44632560 and $G\backslash L_6/W$ has size 1203. The Fourier coefficient $a(T)$ turns out to be 0.

We summarize the results of this section in the following theorem.

Theorem 5.27. *The cusp forms from Theorems 5.21, 5.22 and 5.24 form a basis of $S_{6,14}$.*

5.3. The Kodaira Dimension of the Siegel Modular Variety $\mathcal{A}_n$

The quotient $\mathcal{A}_n = \mathrm{Sp}_{2n}(\mathbb{Z})\backslash\mathbb{H}_n$ has the structure of a normal quasi-projective algebraic variety of dimension $n(n+1)/2$. It is the (coarse) moduli space of principally polarized abelian varieties. For more details on the geometry of $\mathcal{A}_n$ and its compactifications we recommed the survey article [HS02]. In this section we only deal with the Kodaira dimension.

Definition 5.28. Let X be a smooth variety of dimension n over an algebraically closed field k and let Ω_X be its cotangent bundle. Then $\omega_X = \bigwedge^n \Omega_X$ is called the *canonical bundle* of X.

Definition 5.29. Let X be a smooth projective variety over an algebraically closed field. For $d \in \mathbb{Z}_{\geq 0}$ the *d-th plurigenus* $P_d(X)$ of X is defined to be the dimension of the space of global sections of ω_X^d, i.e.

$$P_d(X) = \dim H^0(X, \omega_X^d).$$

Definition 5.30. Let X be a smooth projective variety over an algebraically closed field. The *Kodaira dimension* $\kappa(X)$ of X is defined to be $-\infty$ if $P_d(X) = 0$ for all $d > 0$. Otherwise, $\kappa(X)$ is defined to be the smallest $k \in \mathbb{Z}_{\geq 0}$ such that $P_d(X)/d^k$ is bounded. If $\kappa(X) = \dim(X)$, then we say that X is of *general type*.

Remark 5.31. The Kodaira dimension is bounded by $\dim(X)$.

Remark 5.32. The Kodaira dimension is a birational invariant, i.e. if X and Y are smooth projective varieties that are birational, then $\kappa(X) = \kappa(Y)$.

We also want to assign a Kodaira dimension to $\mathcal{A}_n$, which is neither projective nor smooth. We use the statement of the previous remark to define the Kodaira dimension of $\mathcal{A}_n$ to be the Kodaira dimension of any smooth projective variety $\overline{\mathcal{A}_n}$ birational to $\mathcal{A}_n$. Such a variety exists by the following proposition:

Proposition 5.33. *Let X be a variety over a field of characteristic* 0*. Then X is birational to a smooth projective variety.*

Proof. Every variety is birational to a projective one (take for example the projective closure of any open affine subset). Since X is a variety over a field of characteristic 0, one can then resolve the singularities, as was first proved by Hironaka in [Hir64a] and [Hir64b]. □

As already mentioned in the Introduction of this thesis, the Kodaira dimension of $\mathcal{A}_n$ has been known for all $n \neq 6$ for more than 30 years. Namely, $\mathcal{A}_n$ is of general type for $n \geq 7$ and the Siegel modular variety $\mathcal{A}_n$ has Kodaira dimension $\kappa(\mathcal{A}_n) = -\infty$ for $n \leq 5$.

We can use the theta series for the Leech lattice from the previous section to construct a non-zero global section of $\omega^2_{\overline{\mathcal{A}_n}}$ and obtain the first partial result on the Kodaira dimension of $\mathcal{A}_6$ as follows: We let $n \geq 3$ and write an element $Z \in \mathbb{H}_n$ as

$$Z = \begin{pmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & \ddots & \vdots \\ z_{1n} & \cdots & z_{nn} \end{pmatrix}.$$

Let $\mathbb{H}_n^\circ \subset \mathbb{H}_n$ be the subset of points whose stabilizer in $\mathrm{Sp}_{2n}(\mathbb{Z})$ is equal to $\{\pm I_{2n}\}$. Then $\mathbb{H}_n^\circ$ is an open subset of $\mathbb{H}_n$ and $\mathcal{A}_n^\circ = \mathrm{Sp}_{2n}(\mathbb{Z})\backslash\mathbb{H}_n^\circ$ is the smooth locus of $\mathcal{A}_n$ (see [Fre83], Hilfssatz III.5.15). Let f be a Siegel modular form of degree n and weight $(n+1)k$. Then the transformation property of f implies that

$$f(Z)(dz_{11} \wedge dz_{12} \wedge \cdots \wedge dz_{nn})^k$$

is a global section of $\omega^k_{\mathcal{A}_n^\circ}$. The following result is due to Tai [Tai82]:

Proposition 5.34 ([Fre83], Satz III.5.24)**.** *Suppose $n \geq 5$ and the vanishing order of f at ∞ is at least k. Then*

$$f(Z)(dz_{11} \wedge dz_{12} \wedge \cdots \wedge dz_{nn})^k$$

can be extended to a global section of $\omega^k_{\overline{\mathcal{A}_n}}$.

We can now use this result to prove the following theorem on the Kodaira dimension of $\mathcal{A}_6$:

Theorem 5.35. *The Kodaira dimension of $\mathcal{A}_6$ is non-negative.*

Proof. The theta series for the Leech lattice described in the previous section has weight 14 and is non-zero. Its vanishing order at ∞ is 2 because the Leech lattice has no norm 2 vectors, so it defines a non-zero section of $\omega^2_{\overline{\mathcal{A}_6}}$ by Proposition 5.34. In particular, $P_2(\overline{\mathcal{A}_6}) \neq 0$, so $\kappa(\mathcal{A}_6) = \kappa(\overline{\mathcal{A}_6}) \geq 0$. □

Appendices

A. Root Systems and Their Classification

We briefly summarize the definition and classification of root systems from [Hum72].

Definition A.1. Let E be a finite-dimensional euclidean space with inner product $(\cdot, \cdot)$. A finite subset $\Phi \subset E$ is called *root system* if it satisfies the following:

1. Φ spans E.
2. If α is in Φ, then the only multiples of α in Φ are $\pm\alpha$.
3. If α is in Φ, then the reflection σ_α stabilizes Φ as a set.
4. If α and β are in Φ, then $\langle \beta, \alpha \rangle = 2(\beta, \alpha)/(\alpha, \alpha)$ is an integer.

Definition A.2. Two root systems Φ_1 and Φ_2 in euclidean spaces E_1 and E_2 are called *isomorphic* if there exists an isomorphism $f \colon E_1 \to E_2$ of vector spaces (not necessarily an isometry) such that $f(\Phi_1) = \Phi_2$ and $\langle f(\beta), f(\alpha) \rangle = \langle \beta, \alpha \rangle$ for all $\alpha, \beta \in \Phi_1$.

Definition A.3. A non-empty root system Φ in a finite-dimensional euclidean space E is called *irreducible* if it cannot be written as $\Phi = \Phi_1 \cup \Phi_2$ with proper subsets Φ_1 and Φ_2 such that each element in Φ_1 is orthogonal to each element in Φ_2.

Then every root system Φ in a finite-dimensional euclidean space E can be uniquely decomposed as the union of irreducible root systems Φ_i in subspaces E_i of E such that E is the orthogonal direct sum of the subspaces E_i. Each irreducible root system is isomorphic to exactly one of the following (in all cases E will be a subspace of the euclidean space $\mathbb{R}^n$ with the standard inner product and the vectors $e_1, \dots, e_n$ denote the standard orthonormal basis vectors of $\mathbb{R}^n$):

- A_n $(n \geq 1)$: Let E be the n-dimensional subspace of $\mathbb{R}^{n+1}$ defined by
$$E = \left\{ (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i = 0 \right\}.$$
The root system A_n is then given by the vectors $e_i - e_j$ with $1 \leq i, j \leq n+1$.
- B_n $(n \geq 2)$: Let $E = \mathbb{R}^n$. The root system B_n is then given by the vectors $\pm e_i$ $(1 \leq i \leq n)$ and $\pm(e_i \pm e_j)$ $(1 \leq i < j \leq n)$.
- C_n $(n \geq 3)$: Let $E = \mathbb{R}^n$. The root system C_n is then given by the vectors $\pm 2e_i$ $(1 \leq i \leq n)$ and $\pm(e_i \pm e_j)$ $(1 \leq i < j \leq n)$.
- D_n $(n \geq 4)$: Let $E = \mathbb{R}^n$. The root system D_n is then given by the vectors $\pm(e_i \pm e_j)$ $(1 \leq i < j \leq n)$.

- E_8: Let $E = \mathbb{R}^8$. The root system E_8 is then given by the vectors $\pm(e_i \pm e_j)$ for $1 \leq i < j \leq n$ and the vectors of the form $\frac{1}{2}\sum_{i=1}^{8} \varepsilon_i e_i$ with $\varepsilon_i = \pm 1$ and $\prod_{i=1}^{8} \varepsilon_i = 1$.
- E_7: Let E be the be the subspace of $\mathbb{R}^8$ orthogonal to $e_1 + \cdots + e_8$. The root system E_7 is then the intersection of E_8 with E.
- E_6: Let E be the subspace of $\mathbb{R}^8$ orthogonal to $e_1 + \cdots + e_8$ and $e_1 + e_8$. The root system E_6 is then the intersection of E_8 with E.
- F_4: Let $E = \mathbb{R}^4$. The root system F_4 is then given by the vectors $\pm e_i$ $(1 \leq i \leq 4)$, $\pm(e_i \pm e_j)$ $(1 \leq i < j \leq 4)$ and $\pm\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)$.
- G_2: Let E be the subspace of $\mathbb{R}^2$ orthogonal to $e_1 + e_2 + e_3$. Then the root system G_2 is given by the set

$$\pm\{e_1 - e_2, e_2 - e_3, e_1 - e_3, 2e_1 - e_2 - e_3, 2e_2 - e_1 - e_3, 2e_3 - e_1 - e_2\}.$$

B. Conditions on Symmetric Semi-Reflective Modular Forms

We let L be an even lattice of signature $(n, 2)$, $n \geq 4$ and squarefree level N such that L splits $II_{1,1}(N)$ and let $D = L'/L$. Recall that a symmetric semi-reflective modular form is determined by the complex numbers $c_m = c_{\gamma,-1/m}$ for any $\gamma \in D_{m,1/m}$ for positive divisors m of N. In this appendix we want to answer the question for which choices of $(c_m)_{m|N}$ there exists a symmetric semi-reflective modular form with principal part described by $(c_m)_{m|N}$. We already know that there is no symmetric strongly-reflective modular form unless k (which is given by $k = 1 - n/2$) and N satisfy the bounds given in Table 3.1. We now consider the remaining cases.

All of the following propositions can be proved by computing the Fourier coefficients at all cusps for a basis of $S_{1+n/2}(\Gamma_0(N), \chi_D)$ and solving the resulting linear equations (4.3). As mentioned in Remark 4.4 these equations suffice to classify the symmetric semi-reflective modular forms.

Proposition B.1. *If $N = 19$, 21, 22, 33, 35 or 42, then there are no non-zero symmetric semi-reflective modular forms F on L. The same is true if*

1. *$n = 10$, $N = 5$ and n_5 is odd,*
2. *$n = 6$, $N = 15$ and n_5 is odd, or*
3. *$n = 4$, $N = 30$ and n_5 is even.*

Proof. In these cases the resulting system of equations only has the trivial solution. □

Proposition B.2. *For the following values of n and N there exists a symmetric semi-reflective form F with principal part $(c_m)_{m|N}$ for every choice of $(c_m)_{m|N}$.*

n	26	18	10	8	6	4
N	1	1	1, 2	3	2, 3	3, 6

The same is true if $n = 6$, $N = 5$ and n_5 is odd.

Proof. This follows from the fact that in these cases there is no cusp form of weight $1+n/2$ and character χ_D (note that χ_D is completely determined by the oddity condition in all cases) for $\Gamma_0(N)$. □

Proposition B.3. *In all other cases there is a symmetric semi-reflective vector-valued modular form F with principal part $(c_m)_{m|N}$ if and only if the numbers c_m satisfy the conditions given in the following table:*

n	N	conditions	Remarks
18	2	$2^{5-n_2/2}c_1 = c_2$	
14	2	$2^{4-n_2/2}c_1 = c_2$	
	3	$3^{4-n_3/2}c_1 = c_3$	
12	3	$3^{(7-n_3)/2}c_1 = c_3$	
10	3	$3^{3-n_3/2}c_1 = c_3$	
	5	$5^{3-n_5/2}c_1 = c_5$	if n_5 is even
	6	$2^{3-n_2/2}c_1 = \epsilon_2 c_2$, $3^{3-n_3/2}c_1 = \epsilon_2 c_3$, $2^{3-n_2/2}3^{3-n_3/2}c_1 = c_6$	
8	6	$3^{(7-n_3)/2}c_2 - 2^{4-n_2/2}c_3 = \epsilon_2 c_6$, $\epsilon_2 c_3 - 2^{n_2/2-1}c_6 = -3^{(7-n_3)/2}c_1$	
	7	$7^{(5-n_7)/2}c_1 = c_7$	
6	5	$5^{2-n_5/2}c_1 = c_5$	if n_5 is even
	6	$36c_1 + 9\epsilon_2 2^{n_2/2}c_2 - 4\epsilon_2 3^{n_3/2}c_3 = 2^{n_2/2}3^{n_3/2}c_6$	
	7	$7^{2-n_7/2}c_1 = c_7$	
	10	$5^{(5-n_5)/2}c_2 - 2^{3-n_2/2}c_5 = \epsilon_2 c_{10}$, $5^{(5-n_5)/2}c_1 + \epsilon_2 c_5 = 2^{n_2/2-1}c_{10}$	if n_5 is odd
		$2^{2-n_2/2}c_1 = \epsilon_2 c_2$, $5^{2-n_5/2}c_1 = \epsilon_2 c_5$, $2^{2-n_2/2}5^{2-n_5/2}c_1 = c_{10}$	if n_5 is even
	11	$11^{2-n_{11}/2}c_1 = c_{11}$	
	14	$2^{2-n_2/2}c_1 = \epsilon_2 c_2$, $7^{2-n_7/2}c_1 = \epsilon_2 c_7$, $2^{2-n_2/2}7^{2-n_7/2}c_1 = c_{14}$	
	15	$3^{2-n_3/2}c_1 = -\epsilon_5 c_3$, $5^{2-n_5/2}c_1 = -\epsilon_5 c_5$, $3^{2-n_3/2}5^{2-n_5/2}c_1 = c_{15}$	if n_5 is even
4	7	$7^{(3-n_7)/2}c_1 = c_7$	
	11	$11^{(3-n_{11})/2}c_1 = c_{11}$	
	14	$7^{(3-n_7)/2}c_1 = \epsilon_2 c_7$, $7^{(3-n_7)/2}c_2 = \epsilon_2 c_{14}$	
	15	$3^{(3-n_3)/2}5^{(3-n_5)/2}c_1 = c_{15}$, $5^{(3-n_5)/2}c_3 = 3^{(3-n_3)/2}c_5$	if n_5 is odd
		$5^{2-n_5/2}c_3 - 3^{(5-n_3)/2}c_5 = 2\epsilon_5 c_{15}$, $3^{(5-n_3)/2}c_1 - 2\epsilon_5 c_3 = 5^{n_5/2-1}c_{15}$	if n_5 is even
	23	$23^{(3-n_{23})/2}c_1 = c_{23}$	
	30	$3^{(3-n_3)/2}5^{(3-n_5)/2}c_2 = \epsilon_2 c_{30}$, $3^{(3-n_3)/2}5^{(3-n_5/2)}c_1 = \epsilon_2 c_{15}$, $\epsilon_5 5^{(n_5-3)/2}c_{10} = -\epsilon_2 2^{2-n_2/2}c_1 + c_2$, $\epsilon_2\epsilon_5 3^{(n_3-3)/2}c_6 = -\epsilon_2 2^{2-n_2/2}c_1 + c_2$, $\epsilon_5 5^{(n_5-3)/2}c_5 = -c_1 + \epsilon_2 2^{n_2/2-1}c_2$, $\epsilon_2\epsilon_5 3^{(n_3-3)/2}c_3 = -c_1 + \epsilon_2 2^{n_2/2-1}c_2$	if n_5 is odd

Example B.4. In this example we want to derive the condition for $n = 12$ and $N = 3$. We also show that there is no symmetric strongly-reflective modular form F in $M^!_k(\rho_D)$ with $[F_0](0) = n - 2 = 10$. We know that n_3 must be odd because the signature of D is $2 + 8\mathbb{Z}$ and consequently $\chi_D = \left(\frac{\cdot}{3}\right)$. The space $S_7(\Gamma_0(3), \chi_D)$ is 1-dimensional and is spanned by a primitive form g with Fourier expansion

$$g(\tau) = q - 27q^3 + 64q^4 + O(q^6).$$

The group $\Gamma_0(3)$ has two cusps, represented by $1/3$ and $1/1$. We let

$$M_3 = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \text{ and } M_1 = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}.$$

Then $M_3 \in \Gamma_0(3)$ and

$$g|_7 M_3(\tau) = g(\tau) = q - 27q^3 + 64q^4 + O(q^6),$$

so the coefficient $b_3 = [g|_7 M_3](1)$ is equal to 1. Using Proposition 2.37 we find that

$$g|_7 M_1(\tau) = \chi_D(-1) 3^{-7/2} g|_7 \omega_3^3(\tau/3).$$

With Proposition 2.35 we compute

$$g|_7 \omega_3^3(\tau) = \lambda \sum_{n=1}^{\infty} a_n^{(3)} q^n$$

with

$$\lambda = G(\chi_D) 3^{-7/2} \overline{a_3} = \sqrt{3} \cdot i \cdot 3^{-7/2} \cdot (-27) = -i.$$

It follows that

$$g|_7 M_1(\tau) = 3^{-7/2} \cdot i \sum_{n=1}^{\infty} a_n^{(3)} q^{n/3}$$

and $b_1 = [g|_7 M_1](1/3) = 3^{-7/2} i$. Inserting this into (4.3) we obtain the condition

$$\begin{aligned} 0 &= \xi(M_3) c_1 b_3 + \xi(M_1) \sqrt{|D|} c_3 b_1 \\ &= c_1 - 3^{(n_3 - 7)/2} c_3, \end{aligned}$$

which is the condition in the table on the previous page.

To show that there is no symmetric strongly-reflective modular form F with $[F_0](0) = 10$ we compute (4.4). We note that $\psi = \chi = \chi_D$ and that $B_{k,\psi} = 98/3$. To compute the signs $\varepsilon_{c,d}$ we remark that $\left(\frac{-1}{3}\right)^{(n_3+1)/2} = \epsilon_3$. Then

$$\varepsilon_{1,1} = \varepsilon_{1,3} = \varepsilon_{3,1} = 1$$

and (4.4) becomes

$$\frac{3}{70} \left(1 + \frac{3^6 \sqrt{3}}{\sqrt{|D|}} \right) c_1 + \frac{\sqrt{3}}{\sqrt{|D|}} N(D_3, 1) c_3 = 1. \tag{B.1}$$

If $c_1 = 0$, then $c_3 = 0$ by the equation from the cusp form. In this case $F = 0$ and $[F_0](0)$ is not equal to 10. Therefore $c_1 = 1$. In this case $c_3 = 3^{(n_3-7)/2}$. Since c_3 must be in $\{0, 1\}$, it follows that $n_3 = 7$ and consequently $|D| = 3^7$. Then $\epsilon_3 = \left(\frac{-1}{3}\right)^4 = 1$ and $N(D_3, 1) = 756$ because of Proposition 1.56. Inserting these values into (B.1) results in a contradiction.

Bibliography

[Asa76] Tetsuya Asai, *On the Fourier coefficients of automorphic forms at various cusps and some applications to Rankin's convolution*, J. Math. Soc. Japan **28** (1976), no. 1, 48–61.

[BCHM10] Caucher Birkar, Paolo Cascini, Christopher Hacon, and James McKernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. **23** (2010), no. 2, 405–468.

[BGHZ08] Jan H. Bruinier, Gerard van d. Geer, Günter Harder, and Don Zagier, *The 1-2-3 of Modular Forms*, first ed., Springer-Verlag Berlin Heidelberg, 2008.

[Böc83] Siegfried Böcherer, *Über die Fourier-Jacobi-Entwicklung Siegelscher Eisensteinreihen*, Math. Z. **183** (1983), no. 1, 21–46.

[Böc85] ———, *Über die Fourier-Jacobi-Entwicklung Siegelscher Eisensteinreihen II*, Math. Z. **189** (1985), no. 1, 81–110.

[Bor90] Richard E. Borcherds, *The monster Lie algebra*, Adv. Math. **83** (1990), no. 1, 30–47.

[Bor95] ———, *Automorphic forms on $O_{s+2,2}(R)$ and infinite products*, Invent. Math. **120** (1995), 161–213.

[Bor96] ———, *The moduli space of Enriques surfaces and the fake Monster Lie superalgebra*, Topology **35** (1996), no. 3, 699–710.

[Bor98] ———, *Automorphic forms with singularities on Grassmannians*, Invent. Math. **132** (1998), no. 3, 491–562.

[Bor99] ———, *The Gross-Kohnen-Zagier theorem in higher dimensions*, Duke Math. J. **97** (1999), no. 2, 219–233.

[Bor00] ———, *Reflection groups of Lorentzian lattices*, Duke Math. J. **104** (2000), no. 2, 319–366.

[Bru02] Jan H. Bruinier, *Borcherds Products on O(2,l) and Chern Classes of Heegner Divisors*, Springer-Verlag Berlin Heidelberg, 2002.

[Bru14] ———, *On the converse theorem for Borcherds products*, J. Algebra **397** (2014), 315–342.

[Bun01] Michael Bundschuh, *Über die Endlichkeit der Klassenzahl gerader Gitter der Signatur (n,2) mit einfachem Kontrollraum*, Dissertation, Ruprecht-Karls-Universität Heidelberg, 2001.

[Cas78] J. W. S. Cassels, *Rational quadratic forms*, London Mathematical Society Monographs, vol. 13, Academic Press, Inc., 1978.

[Cle83] C. Herbert Clemens, *Double solids*, Adv. in Math. **47** (1983), no. 2, 107–230.

[Con69] John H. Conway, *A group of order* $8,315,553,613,086,720,000$, Bull. London Math. Soc. **1** (1969), 79–88.

[Con93] Marston Conder, *Generating the Mathieu groups and associated Steiner systems*, Discrete Math. **112** (1993), no. 1, 41 – 47.

[CS99] John H. Conway and Neil J. A. Sloane, *Sphere Packings, Lattices and Groups*, third ed., Springer-Verlag New York, 1999.

[DHS15] Moritz Dittmann, Heike Hagemeier, and Markus Schwagenscheidt, *Automorphic products of singular weight for simple lattices*, Math. Z. **279** (2015), no. 1, 585–603.

[Dit18] Moritz Dittmann, *Reflective automorphic forms on lattices of squarefree level*, Trans. Amer. Math. Soc. (2018), to appear.

[Don84] Ron Donagi, *The Unirationality of* $\mathcal{A}_5$, Ann. of Math. (2) **119** (1984), no. 2, 269–307.

[DS05] Fred Diamond and Jerry Shurman, *A First Course in Modular Forms*, first ed., Springer-Verlag New York, 2005.

[Fre83] Eberhard Freitag, *Siegelsche Modulfunktionen*, Springer-Verlag Berlin Heidelberg, 1983.

[GAP14] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.7.5*, 2014.

[GH14] Valery A. Gritsenko and Klaus Hulek, *Uniruledness of orthogonal modular varieties*, J. Algebraic Geom. **23** (2014), no. 4, 711–725.

[GN02] Valery A. Gritsenko and V. V. Nikulin, *On classification of Lorentzian Kac-Moody algebras*, Russian Math. Surveys **57** (2002), no. 5, 921–979.

[HBJ94] Friedrich Hirzebruch, Thomas Berger, and Rainer Jung, *Manifolds and Modular Forms*, Vieweg+Teubner Verlag, 1994.

[Hil07] Richard Hill, *Weights of Modular Forms on* $\mathrm{SO}^+(2,l)$ *and Congruences Between Eisenstein Series and Cusp forms of Half-Integral Weight on* SL_2, Mathematika **54** (2007), no. 1–2, 47–58.

[Hir64a] Heisuke Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero: I*, Ann. of Math. (2) **79** (1964), 109–203.

[Hir64b] ——, *Resolution of singularities of an algebraic variety over a field of characteristic zero: II*, Ann. of Math. (2) **79** (1964), 205–326.

[HS02] Klaus Hulek and G. K. Sankaran, *The geometry of Siegel Modular Varieties*, Higher Dimensional Birational Geometry, Advanced Studies in Pure Mathematics, vol. 35, 2002, pp. 89–156.

[Hum72] James E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag New York, 1972.

[Kin03] Oliver D. King, *A mass formula for unimodular lattices with no roots*, Math. Comp. **72** (2003), 839–863.

[Koe54] Max Koecher, *Zur Theorie der Modulformen n-ten Grades. I.*, Math. Z. **59** (1954), 399–416.

[Lee67] John Leech, *Notes on sphere packings*, Canad. J. Math. **19** (1967), 251–267.

[Ma17] Shouhei Ma, *Finiteness of* 2*-reflective lattices of signature* $(2, n)$, Amer. J. Math. **139** (2017), no. 2, 513–524.

[Ma18] ——, *On the Kodaira dimension of orthogonal modular varieties*, Invent. Math. (2018).

[Maa80] Hans Maaß, *Harmonische Formen in einer Matrixvariablen*, Math. Ann. **252** (1980), 133–140.

[MH73] John Milnor and Dale Husemoller, *Symmetric Bilinear Forms*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 73, Springer-Verlag Berlin Heidelberg, 1973.

[Miy89] Toshitsune Miyake, *Modular Forms*, Springer-Verlag Berlin Heidelberg, 1989.

[MM83] Shigefumi Mori and Shigeru Mukai, *The uniruledness of the moduli space of curves of genus 11*, Algebraic Geometry, Lecture Notes in Mathematics, vol. 1016, Springer Berlin Heidelberg, 1983, pp. 334–353.

[Mum83] David Mumford, *On the Kodaira dimension of the Siegel modular variety*, Algebraic Geometry — Open Problems, Lecture Notes in Mathematics, vol. 997, Springer Berlin Heidelberg, 1983, pp. 348–375.

[Nie73] Hans-Volker Niemeier, *Definite quadratische Formen der Dimension* 24 *und Diskriminante* 1, J. Number Theory **5** (1973), no. 2, 142–178.

[Nik80] V. V. Nikulin, *Integral symmetric bilinear forms and some of their applications*, Math. USSR-Izv. **14** (1980), no. 1, 103–167.

[Que97] Heinz-Georg Quebbemann, *Atkin-Lehner eigenforms and strongly modular lattices*, Einseign. Math. (2) **43** (1997), 55–65.

[Rad73] Hans Rademacher, *Topics in Analytic Number Theory*, first ed., Springer-Verlag Berlin Heidelberg, 1973.

[Sag15] The Sage Developers, *SageMath, the Sage Mathematics Software System (Version 6.8)*, 2015, `http://www.sagemath.org`.

[Sch04] Nils R. Scheithauer, *Generalized Kac-Moody algebras, automorphic forms and Conway's group I*, Adv. Math. **183** (2004), no. 2, 240–270.

[Sch06] ———, *On the classification of automorphic products and generalized Kac-Moody algebras*, Invent. Math. **164** (2006), no. 3, 641–678.

[Sch09] ———, *The Weil representation of* $\mathrm{SL}_2(\mathbb{Z})$ *and some of its applications*, Int. Math. Res. Not. IMRN **8** (2009), 1488–1545.

[Sch15] ———, *Some constructions of modular forms for the Weil representation of* $\mathrm{SL}_2(\mathbb{Z})$, Nagoya Math. J. **220** (2015), 1–43.

[Sch17] ———, *Automorphic products of singular weight*, Compos. Math. **153** (2017), no. 9, 1855–1892.

[Ser73] Jean-Pierre Serre, *A Course in Arithmetic*, Springer-Verlag New York, 1973.

[Str13] Fredrik Strömberg, *Weil representations associated to finite quadratic modules*, Math. Z. **275** (2013), no. 1, 509–527.

[Tai82] Yung-Sheng Tai, *On the Kodaira Dimension of the Moduli Space of Abelian Varieties*, Invent. Math. **48** (1982), no. 3, 425–439.

[Taï17] Olivier Taïbi, *Dimensions of spaces of level one automorphic forms for split classical groups using the trace formula*, Ann. Sci. Éc. Norm. Supér. (4) **50** (2017), no. 2, 269–344.

[Ven78] Boris B. Venkov, *On the classification of integral even unimodular* 24-*dimensional quadratic forms*, Trudy Mat. Inst. Steklov. **148** (1978), 65–76.

[Ver84] Alessandro Verra, *A short proof of the unirationality of* $\mathcal{A}_5$, Nederl. Akad. Wetensch. Indag. Math. **87** (1984), no. 3, 339 – 355.

[Wit41] Ernst Witt, *Eine Identität zwischen Modulformen zweiten Grades*, Abh. Math. Sem. Univ. Hamburg **14** (1941), no. 1, 323–337.

Wissenschaftlicher Werdegang

2009	allgemeine Hochschulreife
2009-2013	Studium der Mathematik mit Nebenfach Informatik an der Technischen Universität Darmstadt
2012	Bachelor of Science in Mathematik
2013	Master of Science in Mathematik
seit 2013	Wissenschaftlicher Mitarbeiter am Fachbereich Mathematik der Technischen Universität Darmstadt
2018	Promotion in Mathematik