The Helmholtz-Hodge Decomposition in Lebesgue Spaces on Exterior Domains and Evolution Equations on the Whole Real Time Axis

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Deutsche Zusammenfassung

Wir befassen uns mit zwei verschiedenen Themen, deren besondere zugrundeliegenden Schwierigkeiten mit der Theorie partieller Differentialgleichungen auf unbeschränkten Gebieten zusammenhängen. Dies ist erstens die Helmholtz-Hodge-Zerlegung von Vektorfeldern in Lebesgue-Räumen auf dreidimensionalen Außenraumgebieten. Das zweite Thema ist eine abstrakte Theorie für Evolutionsgleichungen auf der gesamten reellen Zeitachse mit Anwendung auf parabolische Differentialgleichungen in unbeschränkten Gebieten. Insbesondere zeigen wir die Existenz bezüglich der Zeit periodischer und fastperiodischer Lösungen.

Bei der Helmholtz-Hodge-Zerlegung handelt es sich um die Zerlegung eines Vektorfeldes in ein Gradientenfeld, ein Rotationsfeld und ein harmonisches Vektorfeld, für welche jeweils unterschiedliche Randbedingungen vorgeschrieben sind. Während die Existenz und Eindeutigkeit dieser Zerlegung in Lebesgue-Räumen auf beschränkten Gebieten bereits gut untersucht ist, so fehlt bislang eine Betrachtung in ungewichteten Lebesgue-Räumen auf Außenraumgebieten. Wir werden hier diese Lücke schließen und die Existenz und Eindeutigkeit der Zerlegung für verschiedene Randbedingungen zeigen, beziehungsweise gegebenenfalls widerlegen. Dabei spielt auch die Integrationsordnung der Lebesgue-Räume eine wesentliche Rolle. Einen wichtigen Eckpfeiler bildet im hier gewählten Ansatz die Lösbarkeitstheorie zu einem System schwacher Poisson-Gleichungen mit partiellen Dirichlet-Randbedingungen. Diese wird dazu benutzt das Vektorpotential für das Rotationsfeld zu konstruieren. Ein Bestandteil dieser Theorie ist die Charakterisierung homogener Sobolevräume und harmonischer Vektorfelder mit den entsprechenden Randbedingungen.

Das zweite Hauptthema sind Lösungen für abstrakte Evolutionsgleichungen, welche auf der gesamten reellen Zeitachse existieren. Speziell sind wir an in der Zeit beschränkten, periodischen und fast periodischen Lösungen interessiert. Der Fokus ist hierbei zweigeteilt. Als erstes untersuchen wir die Existenz und Stabilität von milden Lösungen für abstrakte parabolische Gleichungen mit Werten in reellen Interpolationsräumen. Als Grundlage dienen dabei polynomielle Abklingbedingungen an die zugehörige Halbgruppe oder Evolutionsfamilie. Die abstrakte Theorie wird danach auf semilineare parabolische Gleichungen und die Navier-Stokes-Gleichungen in Außenraumgebieten mit Werten in schwachen Lebesgue-Räumen und homogenen Sobolev-Räumen angewandt. Als zweites weisen wir mit ähnlichen Methoden maximale stetige Regularität parabolischer Gleichungen in stetigen Interpolationsräumen nach. Dies erlaubt es uns periodische und fast periodische starke Lösungen für quasilineare Gleichungen zu konstruieren.

Contents

Introduction

One topic in the theory of partial differential equations which is known to involve particular difficulties is the solution theory of differential equations on unbounded domains. Compared to the theory on bounded domains, one is usually confronted with a lack of compactness, weaker regularity and worse asymptotical properties in space and time. Here we consider two different topics, which are at least heavily influenced by differential equations on unbounded domains. The first one is the Helmholtz-Hodge decomposition in Lebesgue spaces on exterior domains. The second one is an abstract approach to evolution equations on the whole real time axis, which has its origin in works on the Navier-Stokes equations, again in exterior domains. This is a convenient setting for the construction of periodic and almost periodic solutions.

Helmholtz-Hodge Decomposition The first main topic is the Helmholtz-Hodge decomposition of vector fields in Lebesgue spaces on three-dimensional exterior domains. In the three-dimensional whole space $\mathbb{R}^3$, it was shown by Helmholtz [Hel58], that a smooth vector field $u \in C_c^\infty(\mathbb{R}^3)$ can be decomposed into a gradient field and a rotation field via

$$u = \nabla(\operatorname{div} F * u) - \operatorname{rot}(\operatorname{rot} F * u),$$

where F is the fundamental solution to the Laplace equation. Today, it is known that this decomposition can be extended to a direct decomposition of the Lebesgue space $L^p(\mathbb{R}^3)$ with $p \in (1, \infty)$. In contrast to the whole space, such a decomposition into a gradient field and a rotation field is in general not unique anymore on three-dimensional domains. Depending on the applications, different kinds of boundary conditions for the components and additional summands can be added.

In fluid mechanics, the most predominant decomposition is the Helmholtz decomposition: For each $u \in L^p(\Omega)$, there is a unique u_σ and a unique π (up to constants) such that

$$u = u_\sigma + \nabla\pi,$$

where

$$u_\sigma \in L^p_\sigma(\Omega) = \overline{\{v \in C_c^\infty(\Omega) : \operatorname{div} v = 0\}}^{L^p(\Omega)},$$
$$\pi \in \dot{\mathbf{H}}^1_p(\Omega) = \{\pi \in L^p_{loc}(\overline{\Omega}) : \nabla\pi \in L^p(\Omega)\}.$$

This decomposition includes implicitly that the elements of $L^p_\sigma(\Omega)$ vanish in the normal direction at the boundary $\partial\Omega$. For $p = 2$, the validity of this decomposition was shown by [Wey40] for arbitrary domains $\Omega \subset \mathbb{R}^3$. The case $p \in (1, \infty)$ has been treated by [FM77] for smooth bounded domains and [Miy82] for smooth exterior domains. But also other domains, like the half-space ([McC81]), aperture domains ([FS96]) or periodic unbounded domains ([Bab16]) have been considered.

In the context of electrodynamics, more refined decompositions are of interest. This includes

$$u = \operatorname{rot} w + h + \nabla\pi,$$

where

$$w \in \dot{\mathbf{H}}^{1,N}_p(\Omega) = \{w \in \dot{\mathbf{H}}^1_p(\Omega) : w \times n = 0 \text{ on } \partial\Omega\},$$
$$h \in L^p_{T,har}(\Omega) = \{h \in L^p(\Omega) : \operatorname{div} h = 0, \operatorname{rot} h = 0, h \cdot n = 0 \text{ on } \partial\Omega\},$$
$$\pi \in \dot{\mathbf{H}}^1_p(\Omega),$$

which is a more detailed version of the Helmholtz decomposition above. We will call this decomposition the *first Helmholtz-Hodge decomposition*. Similarly, decompositions of the form

$$u = \operatorname{rot} w + h + \nabla\pi,$$

where

$$w \in \dot{\mathbf{H}}^{1,T}_p(\Omega) = \{w \in \dot{\mathbf{H}}^1_p(\Omega) : w \cdot n = 0 \text{ on } \partial\Omega\},$$
$$h \in L^p_{N,har}(\Omega) = \{h \in L^p(\Omega) : \operatorname{div} h = 0, \operatorname{rot} h = 0, h \times n = 0 \text{ on } \partial\Omega\},$$
$$\pi \in \hat{H}^{1,0}_p(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\nabla\cdot\|_{L^p(\Omega)}},$$

have been considered. This kind of decomposition will be called *second Helmholtz-Hodge decomposition*. The interest in the harmonic vector fields $L^p_{T,har}(\Omega)$ and $L^p_{N,har}(\Omega)$ comes from the fact, that they describe the kernels of some second-order Maxwell operators. For other relations of these refined decompositions, for example to the construction of vector potentials and the theory of compensated compactness, see [Sch18] and the references therein. Usually, the first and second Helmholtz-Hodge decomposition are considered in the more general context of alternating differential forms, where they can be seen in a unified way as the same decomposition for differential forms of different order. The most common name for this decomposition seems to be the *Hodge decomposition*, although the name Hodge is sometimes accompanied or exchanged by at least one of the names Helmholtz, Weyl, de Rham, Kodaira or Leray.

In the case of bounded domains (or manifolds) D, the Hodge decomposition is rather well understood. For an overview on methods for $L^2(D)$ see [AM04]. A more general result including mixed boundary conditions and anisotropic media is given by [BPS16]. Other classical results are [Pic84], [Sch95] and [ABDG98]. For $p \neq 2$, the decompositions have been considered in [Sch95], [KY09], [AS11] for smooth domains and in [MM16] for Lipschitz domains. The case of exterior domains Ω has been treated less exhaustively. Picard [Pic81] considered the classical decomposition of $L^2(\Omega)$, Pauly [Pau08] and Schwarz [Sch95] treated weighted Lebesgue and Sobolev spaces.

Here, the Helmholtz-Hodge decompositions will be investigated in unweighted Lebesgue spaces $L^p(\Omega)$ with $p \in (1, \infty)$ for exterior domains with smooth boundary. Compared to the case of smooth bounded domains, where both decompositions have been shown to be valid for any $p \in (1, \infty)$, partially different observations will be made here. Regarding the first Helmholtz-Hodge decomposition, we will prove the following:

Theorem. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with smooth boundary and $p \in (1, \infty)$. Then the first Helmholtz-Hodge decomposition exists in $L^p(\Omega)$ and is direct.*

Moreover, the space of harmonic vector fields $L^p_{T,har}(\Omega)$ will be shown to be finite dimensional and independent of p. This remains in accordance with the situation in bounded domains. In contrast, we will get the following result regarding the second decomposition:

Theorem. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with smooth boundary and $p \in (1, \infty)$.*

1. *If $p \in (3/2, 3)$, then the second Helmholtz-Hodge decomposition exists in $L^p(\Omega)$ and is direct.*

2. *If $p \in [3, \infty)$, then the second Helmholtz-Hodge decomposition exists in $L^p(\Omega)$, but it is not unique.*

3. *If $p \in (1, 3/2]$, then the second Helmholtz-Hodge decomposition fails in $L^p(\Omega)$.*

Furthermore, we will see that the space of harmonic functions $L^p_{N,har}(\Omega)$ is one dimension smaller for $p \in (1, 3/2]$ than for $p \in (3/2, \infty)$. This is a remarkable different phenomenon, which is caused by the decay of these harmonic vector fields at infinity. The failure of the second Helmholtz-Hodge decomposition for $p \in (1, 3/2]$ can be circumvented, if one chooses a larger homogeneous Sobolev space for the scalar potential π. However, this comes at the cost that π does not decay anymore in general at infinity.

The proof follows roughly the arguments of [KY09] adjusted to exterior domains. The scalar potentials π will be constructed by solving either a weak Neumann problem

$$(\nabla\pi, \nabla\phi) = (u, \nabla\phi) \quad \text{for all } \phi \in \dot{\mathbf{H}}^1_{p'}(\Omega)$$

for the first Helmholtz-Hodge decomposition or a weak Dirichlet problem

$$(\nabla\pi, \nabla\phi) = (u, \nabla\phi) \quad \text{for all } \phi \in \hat{H}^{1,0}_{p'}(\Omega)$$

for the second decomposition. These problems have already been well-studied in the literature. For the construction of the vector potentials w, we will solve

$$\begin{aligned} (\operatorname{rot} w, \operatorname{rot}\phi) &= (u, \operatorname{rot}\phi) \quad \text{for all } \phi \in \dot{\mathbf{H}}^{1,N}_{p'}(\Omega), \\ \operatorname{div} w &= 0 \end{aligned} \tag{1}$$

with $w \in \dot{\mathbf{H}}^{1,N}_p(\Omega)$ as well as the analogous equation with vanishing normal boundary conditions instead of vanishing tangential boundary conditions. These problems have not been considered so far in exterior domains. The most similar result seems to be [AKST04] about the related resolvent problem for one of the boundary conditions. Thus, a complete solution theory for these problems complemented by a thorough treatment of the homogeneous Sobolev spaces $\dot{\mathbf{H}}^{1,T}_p(\Omega)$ and $\dot{\mathbf{H}}^{1,N}_p(\Omega)$ will be established here. One key point will be that a vector field $w \in \dot{\mathbf{H}}^{1,N}_p(\Omega)$ solves (1) if and only if it is a solution to

$$(\operatorname{div} w, \operatorname{div}\phi) + (\operatorname{rot} w, \operatorname{rot}\phi) = (u, \operatorname{rot}\phi) \quad \text{for all } \phi \in \dot{\mathbf{H}}^{1,N}_{p'}(\Omega).$$

We will solve this problem by means of a cut-off argument, which is much more convenient to apply here than for problem (1) as we do not have to deal with inhomogeneous divergence conditions. Another main part of the solution theory is the characterization of the kernel of the bilinear form on the left-hand side, which will turn out to be in general a non-trivial finite dimensional space of harmonic vector fields independent of p.

Evolution Equations on the Whole Real Time Axis The second main topic of this thesis is the investigation of abstract Cauchy problems

$$u'(t) - Au(t) = f(t), \quad t \in \mathbb{R},$$

on the whole real time axis. In particular, we are interested in the existence of time periodic and almost periodic solutions to partial differential equations under the assumption of time periodic or almost periodic exterior forces f. Most of the classical works on time periodic solutions treat such kinds of problems by means of the corresponding initial value problem. Serrin [Ser59] makes use of the asymptotic properties of the solutions to the Navier-Stokes equations in bounded domains to construct an initial value, which gives rise to a time periodic solution. This approach has been reused in different contexts (e.g. [Mat78], [Val83]). A different method has been employed by Arendt and Bu [AB02], who reformulated the time periodic problem as a problem on the torus and constructed solutions with the help of vector valued Fourier multipliers. Similarly, Kyed [Kye12] (see also [KS17]) constructs time periodic solutions to the Stokes equation on $\mathbb{R}^d$ by reformulating the problem as a problem on an abelian group and making use of Fourier analysis in this setting.

The main Ansatz here will rely on an adjustment of Duhamel's formula

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} f(s) \, \mathrm{d}s$$

on the positive real axis to the whole real axis by

$$u(t) = \int_{-\infty}^t e^{(t-s)A} f(s) \, \mathrm{d}s.$$

Such an approach has already been used for the construction of mild solutions in [MS03] under the assumption of exponential dichotomy of the semigroup, in [KN96] and [Yam00] in the context of the Navier-Stokes equations and in [GHN16] in an abstract setting of real interpolation spaces for time periodic autonomous problems. The main obstruction in the latter of these references is the lack of exponential decay of the semigroups, which is the main difficulty in order to show the boundedness of the integral of the generalized Duhamel's formula. Instead, the well known L^p-L^q estimates of parabolic problems and some abstract generalization of them are used in order to achieve suitable integrability. Here, we continue the abstract approach of [GHN16] and generalize it in different directions. In Section 3.1, we extend their results to exterior forces which are almost periodic or decaying in time. We see in Subsection 3.1.2, that in the time periodic setting, this approach can be extended to non-autonomous systems. Following the method of [Yam00], we also show the stability of these solutions. Finally, the abstract results are applied to semilinear parabolic equations in unbounded domains with values in weak Lebesgue spaces as well as the Navier-Stokes equations in homogeneous Besov spaces.

In Subsection 3.2.1, we modify the method used before in order to show maximal continuous regularity on the whole real time axis for generators of exponentially stable analytic semigroups in suitable continuous interpolation spaces. More precisely, the main result reads as follows:

Theorem. *Let E be a Banach space, $A\colon D(A) \subset E \to E$ be the generator of an exponentially stable analytic C_0-semigroup and $\theta \in (0,1)$. Then for each $f \in \mathrm{BUC}(\mathbb{R}; (E, D(A))^0_{\theta,\infty})$ there is a unique*

$$u \in \mathrm{BUC}^1(\mathbb{R}; (E, D(A))^0_{\theta,\infty}) \cap \mathrm{BUC}(\mathbb{R}; (D(A), D(A^2))^0_{\theta,\infty})$$

to the equation

$$u'(t) - Au(t) = f(t), \quad t \in \mathbb{R}.$$

While the setting resembles the one used in [DPG79], [Ang90] and [CS01] for the initial value problem, it differs in view of the treatment of maximal continuous regularity on the whole real axis. Again, we will make use of this setting to show the existence of time periodic and almost periodic solutions, but this time for quasilinear equations whose nonlinear terms possess suitable Lipschitz properties.

Some parts of this thesis have already been published or are in the process of publishing.

The existence of almost periodic mild solutions to evolution equations from Subsection 3.1.1 and its application to parabolic equations in weak Lebesgue spaces and the Navier-Stokes equations in homogeneous Besov spaces from Subsection 3.1.4 are contained in [HNS17].

The characterization of Dirichlet fields in Lebesgue spaces on exterior domains given in Subsection 2.4.3 is a part of [HKS+18], although with a slightly different proof.

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1 Notation and Preliminaries

In this chapter, we introduce the basic notation and recall some well known results. The set of natural numbers is denoted by $\mathbb{N} = \{0, 1, 2, \dots\}$. The set of integers, real numbers and complex numbers is denoted respectively by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$. For $d \in \mathbb{N}$ with $d \geq 2$, the d-dimensional half plane is defined as $\mathbb{R}^d_+ = \{x = (x_1, \dots, x_d) \in \mathbb{R}^d : x_d > 0\}$. Given a domain $\Omega \subset \mathbb{R}^3$, we set $\overline{\Omega}$ to be its closure, Ω^C to be its complement and $\partial\Omega$ to be its boundary. We say that Ω has C^k-boundary, if it satisfies the usual definition given in [AF03, Definition 4.10]. Supposed that $\partial\Omega$ is at least C^1, the exterior normal vector at a point $x \in \partial\Omega$ is denoted by $n = n(x)$. For two vectors $x, y \in \mathbb{C}^d$, the scalar product and the tensor product are defined by

$$x \cdot y = \sum_{k=1}^{d} x_k \bar{y}_k \quad \text{and} \quad (x \otimes y)_{ij} = x_i y_j, \quad 1 \leq i, j \leq d.$$

For $d = 3$, the cross-product is given by

$$x \times y = \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}.$$

The modulus of a vector $x \in \mathbb{C}^d$ is given by $|x| = (x \cdot x)^{1/2}$. The first $d - 1$ entries are sometimes bundled as x', i.e. $x = (x_1, \dots, x_d) = (x', x_d)$.

Let X be a normed vector space. The norm of an element $x \in X$ is denoted by $\|x\|_X$. For $R > 0$ and $x \in X$, we will write

$$\begin{aligned} B(x, R) &= \{y \in X : \|x - y\|_X < R\}, \\ \overline{B(x, R)} &= \{y \in X : \|x - y\|_X \leq R\}. \end{aligned}$$

For two normed vector spaces X, Y, we define $\mathcal{L}(X, Y)$ to be the space of bounded linear operators from X to Y. The domain of a linear operator A defined on a subset of X is denoted by $D(A)$. Its range will be called $R(A)$. For a linear operator $A\colon X \supseteq D(A) \to X$, the resolvent set will be denoted by $\rho(A)$. For each $\lambda \in \rho(A)$, we set $R(\lambda, A) := (\lambda - A)^{-1}$. The adjoint of A is denoted by A'. In the other direction, the pre-adjoint is labelled by $A^\flat$ as long as it is well defined. In particular, it holds $(A')^\flat = A$.

Given some $p \in [1,\infty]$, a normed vector space X and a (not necessarily open) domain $\Omega \subset \mathbb{R}^d$, the usual X-valued Lebesgue spaces are denoted by $L^p(\Omega; X)$. If $X = \mathbb{R}$, we will just write $L^p(\Omega)$. By $L^p_{loc}(\Omega)$ we denote the set of all measurable functions $f\colon \Omega \to \mathbb{R}$ which are in $L^p(K)$ for each compact $K \subset \Omega$. For two measurable functions $f, g\colon \Omega \to \mathbb{C}$, we formally set

$$(f, g) := \int_\Omega f(x)\overline{g(x)}\,\mathrm{d}x.$$

For $k \in \mathbb{N}$ and open $\Omega \subset \mathbb{R}^d$, we write $H^k_p(\Omega)$ for the inhomogeneous Sobolev space of order k. For $p, q \in [1,\infty]$ and $s \in \mathbb{R}$, the inhomogeneous Besov space is denoted by $B^s_{p,q}(\Omega)$. The homogeneous Sobolev space is defined as

$$\dot{\mathbf{H}}^1_p(\Omega) := \{u \in L^p_{loc}(\overline{\Omega}) : \nabla u \in L^p(\Omega)\}.$$

For any $p \in [1,\infty]$ the dual exponent p' is determined by the relation $\frac{1}{p} + \frac{1}{p'} = 1$.

Given a Banach space X, we denote its dual space by X'. The duality pairing of some $x \in X$ and $x' \in X'$ will be written as $\langle x, x'\rangle_{X,X'}$. If the spaces are clear in the context, we will drop the subscript and just write $\langle x, x'\rangle$.

1.1 Interpolation of Function Spaces

Let X_0 and X_1 be quasi-normed vector spaces. The pair (X_0, X_1) is called an interpolation couple, if X_0 and X_1 are embedded into a common topological Hausdorff space $\mathcal{V}$. For an interpolation couple X_0, X_1, we can consider its intersection $X_0 \cap X_1$ equipped with the quasi-norm

$$\|x\|_{X_0 \cap X_1} := \|x\|_{X_0} + \|x\|_{X_1}$$

and its sum $X_0 + X_1$ equipped with the quasi-norm

$$\|x\|_{X_0+X_1} := \inf\{\|x_0\|_{X_0} + \|x_1\|_{X_1} : x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\}.$$

A vector space X is called an interpolation space of an interpolation couple (X_0, X_1), if $X_0 \cap X_1 \subseteq X \subseteq X_0 + X_1$ together with continuous embeddings.

1.1.1 Complex Interpolation Spaces

Regarding complex interpolation spaces, we follow the definition of [Lun09, Section 2.1]. Other classical references are the books of Bergh and Löfström [BL76] and Triebel [Tri78]. Let (X_0, X_1) be an interpolation couple of complex Banach spaces. Define the strip

$$S := \{x + \mathrm{i}y \in \mathbb{C} : 0 \le x \le 1\}.$$

The set $\mathcal{F}(X_0, X_1)$ is the space of all $f\colon S \to X_0 + X_1$ having the following properties:

1. f is continuous on S and analytic in the interior of S.
2. $t \mapsto f(\mathrm{i}t) \in C(\mathbb{R}; X_0)$, $t \mapsto f(1+\mathrm{i}t) \in C(\mathbb{R}; X_1)$ and
$$\|f\|_{\mathcal{F}(X_0,X_1)} := \max\left\{\sup_{t\in\mathbb{R}} \|f(\mathrm{i}t)\|_{X_0}, \sup_{t\in\mathbb{R}} \|f(1+\mathrm{i}t)\|_{X_1}\right\} < \infty.$$

Definition 1.1.1. *Let $\theta \in [0,1]$ and (X_0, X_1) be an interpolation couple of complex Banach spaces. The complex interpolation space $[X_0, X_1]_\theta$ is defined as*

$$[X_0, X_1]_\theta := \{f(\theta) : f \in \mathcal{F}(X_0, X_1)\}$$

equipped with the norm

$$\|x\|_{[X_0,X_1]_\theta} := \inf\{\|f\|_{\mathcal{F}(X_0,X_1)} : f(\theta) = x\}.$$

The complex interpolation spaces $[X_0, X_1]_\theta$ are known to be interpolation spaces of exact type θ. That means:

Proposition 1.1.2. *Let (X_0, X_1) and (Y_0, Y_1) be interpolation couples of complex Banach spaces, $T \in \mathcal{L}(X_0, Y_0) \cap \mathcal{L}(X_1, Y_1)$ and $\theta \in (0,1)$. Then it holds $T \in \mathcal{L}([X_0, X_1]_\theta, [Y_0, Y_1]_\theta)$ and*

$$\|T\|_{\mathcal{L}([X_0,X_1]_\theta,[Y_0,Y_1]_\theta)} \leq \|T\|^{1-\theta}_{\mathcal{L}(X_0,Y_0)} \|T\|^{\theta}_{\mathcal{L}(X_0,Y_0)}.$$

1.1.2 Real Interpolation Spaces

Concerning real interpolation, we refer mainly to the books of Bergh and Löfström [BL76], especially Section 3.11 therein with respect to interpolation of quasi-normed vector spaces, and Lunardi [Lun09]. Basis for the real interpolation method is the K-functional. Let (X_0, X_1) be an interpolation couple of quasi-normed vector spaces and define

$$K(t, x, X_0, X_1) := \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1}, x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\}$$

for $x \in X_0 + X_1$ and $t \in [0, \infty)$. Usually, the order of the spaces X_0 and X_1 is clear, thus we will shortly write $K(t, x) = K(t, x, X_0, X_1)$. For $\theta \in (0,1)$ and $q \in [1, \infty]$, the real interpolation spaces are defined as

$$(X_0, X_1)_{\theta,q} := \{x \in X_0 + X_1 : \|x\|_{(X_0,X_1)_{\theta,q}} < \infty\},$$

where

$$\|x\|_{(X_0,X_1)_{\theta,q}} := \left(\int_0^\infty [t^{-\theta} K(t,x)]^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}},$$

if $q < \infty$ and

$$\|x\|_{(X_0,X_1)_{\theta,\infty}} := \sup_{t \in (0,\infty)} t^{-\theta} K(t,x).$$

Furthermore, for Banach spaces X_0 and X_1, the continuous interpolation spaces are given by

$$(X_0,X_1)^0_{\theta,\infty} := \{x \in (X_0,X_1)_{\theta,\infty} : \lim_{t\to 0} t^{-\theta} K(t,x) = \lim_{t\to\infty} t^{-\theta} K(t,x) = 0\}.$$

We gather some basic properties of these spaces.

Proposition 1.1.3. *Let (X_0, X_1) be an interpolation couple of quasi-normed vector spaces, $\theta \in (0,1)$ and $q \in [1,\infty]$.*

1. *The real interpolation spaces $(X_0,X_1)_{\theta,q}$ are quasi-normed vector spaces. If X_0 and X_1 are complete, then their real and continuous interpolation spaces are complete, too.*
2. *If X_0 and X_1 are normed vector spaces, then the spaces $(X_0,X_1)_{\theta,q}$ are normed, too.*
3. *If $q < \infty$, then $X_0 \cap X_1$ is dense in $(X_0,X_1)_{\theta,q}$. If additionally X_0 or X_1 is separable and $q < \infty$, then $(X_0,X_1)_{\theta,q}$ is separable, too.*

Like complex interpolation spaces, the real interpolation spaces are of exact type θ:

Proposition 1.1.4. *Let (X_0, X_1) and (Y_0, Y_1) be interpolation couples of quasi-normed spaces. Let $T \in \mathcal{L}(X_0,Y_0) \cap \mathcal{L}(X_1,Y_1)$ as well as $\theta \in (0,1)$ and $q \in [1,\infty]$. Then it holds that $T \in \mathcal{L}((X_0,X_1)_{\theta,q},(Y_0,Y_1)_{\theta,q})$ with the norm estimate*

$$\|T\|_{\mathcal{L}((X_0,X_1)_{\theta,q},(Y_0,Y_1)_{\theta,q})} \leq \|T\|^{1-\theta}_{\mathcal{L}(X_0,Y_0)} \|T\|^{\theta}_{\mathcal{L}(X_1,Y_1)}.$$

The relation of real and continuous interpolation is as follows (see [Lun09, Proposition 1.17]):

Proposition 1.1.5. *Let (X_0, X_1) be an interpolation couple of Banach spaces. Then $(X_0,X_1)^0_{\theta,\infty}$ coincides with the closure of $X_0 \cap X_1$ in $(X_0,X_1)_{\theta,\infty}$. It follows that $(X_0,X_1)^0_{\theta,\infty}$ is a Banach space, too.*

One advantage of real interpolation is that it can also be applied to sublinear operators. Given two real quasi-normed vector spaces X and Y, an operator $T\colon X \to Y$ is called sublinear, if

$$\begin{aligned} \|T(\lambda x)\|_Y &= |\lambda| \|T(x)\|_Y && \text{for all } x \in X, \lambda \in \mathbb{R}, \\ \|T(x_0 + x_1)\|_Y &\leq M(\|T(x_0)\|_Y + \|T(x_1)\|_Y) && \text{for all } x_0, x_1 \in X, \end{aligned}$$

where $M \geq 0$ is independent of x_0 and x_1. The constant M will be called quasi-norm of T. For the next property, see [BL76, Theorem 3.11.2 and the remark on p. 41].

Proposition 1.1.6. *Let (X_0, X_1) and (Y_0, Y_1) be interpolation couples of quasi-normed spaces. Let T be defined on $X_0 + X_1$ such that $T\colon X_0 \to Y_0$ as well as $T\colon X_1 \to Y_1$ are sublinear with quasi-norm M_0 and M_1 respectively. Then for any $\theta \in (0,1)$ and $q \in [1, \infty]$ it holds $T\colon (X_0, X_1)_{\theta,q} \to (Y_0, Y_1)_{\theta,q}$ with quasi-norm M bounded by*

$$M \leq M_0^{1-\theta} M_1^{\theta}.$$

Let (X_0, X_1) be an interpolation couple of Banach spaces. A normed vector space E satisfying $X_0 \cap X_1 \subseteq E \subseteq X_0 + X_1$ is said to be of type J_θ with $\theta \in [0,1]$, if there exists a $c > 0$ such that

$$\|x\|_E \leq c \|x\|_{X_0}^{1-\theta} \|x\|_{X_1}^{\theta} \quad \text{for all } x \in X_0 \cap X_1.$$

The space E is of type K_θ, if there is a constant $k > 0$ such that

$$K(t, x) \leq k t^{\theta} \|x\|_E \quad \text{for all } x \in E.$$

Due to [Lun09, Corollary 1.24], it holds that X_i is of type J_i and of type K_i, for $i \in \{0,1\}$. For $\theta \in (0,1)$ and $q \in [1, \infty]$, one has that $(X_0, X_1)_{\theta,q}$ and $(X_0, X_1)^0_{\theta,\infty}$ are of type J_θ and of type K_θ. Thus, the general reiteration theorem [Lun09, Theorem 1.23] yields:

Proposition 1.1.7. *Let (X_0, X_1) be an interpolation couple of Banach spaces, $\theta_0, \theta_1, \theta \in (0,1)$, $q_0, q_1, q \in [1, \infty]$. Then it holds that*

$$\begin{aligned} ((X_0, X_1)_{\theta_0,q_0}, (X_0, X_1)_{\theta_1,q_1})_{\theta,q} &= (X_0, X_1)_{(1-\theta)\theta_0 + \theta\theta_1, q}, \\ ((X_0, X_1)^0_{\theta_0,\infty}, (X_0, X_1)^0_{\theta_1,\infty})_{\theta,q} &= (X_0, X_1)_{(1-\theta)\theta_0 + \theta\theta_1, q}, \\ (X, (X_0, X_1)_{\theta_1,q_1})_{\theta,q} &= (X_0, X_1)_{\theta\theta_1, q}. \end{aligned}$$

The dual spaces of real interpolation spaces are given by the following relation:

Proposition 1.1.8. *Let (X_0, X_1) be an interpolation couple of Banach spaces such that $X_0 \cap X_1$ is dense in X_0 and X_1. Suppose that $\theta \in (0,1)$ and $q \in [1,\infty)$. Then it holds that*

$$((X_0, X_1)_{\theta,q})' = (X_0', X_1')_{\theta,q'} \quad \text{with } 1 = \tfrac{1}{q} + \tfrac{1}{q'},$$
$$((X_0, X_1)^0_{\theta,\infty})' = (X_0, X_1)_{\theta,1}.$$

We close this section by relating the limits of a sequence that converges in different interpolation spaces.

Lemma 1.1.9. *Let (Z_1, Z_2) be an interpolation couple of Banach spaces such that $Z_1 \cap Z_2$ is dense in Z_1 and Z_2. Let $\theta, \tilde{\theta} \in (0,1)$. Suppose that $(x_n)_{n\in\mathbb{N}} \subset (Z_1', Z_2')_{\theta,\infty} \cap (Z_1', Z_2')_{\tilde{\theta},\infty}$ satisfies*

$$x_n \to x \quad \text{in the norm-topology of } (Z_1', Z_2')_{\theta,\infty}, \tag{1.1}$$
$$x_n \to y \quad \text{in the weak-*-topology of } (Z_1', Z_2')_{\tilde{\theta},\infty}. \tag{1.2}$$

Then $x = y$.

Proof. By Proposition 1.1.8, it holds $(Z_1', Z_2')_{\theta,\infty} = ((Z_1, Z_2)_{\theta,1})'$ as well as $(Z_1', Z_2')_{\tilde{\theta},\infty} = ((Z_1, Z_2)_{\tilde{\theta},\infty})'$. It follows from (1.1), that for any $\psi \in Z_1 \cap Z_2$, we have

$$\langle x_n, \psi \rangle \to \langle x, \psi \rangle.$$

By (1.2), we obtain for any $\psi \in Z_1 \cap Z_2$ that

$$\langle x_n, \psi \rangle \to \langle x, \psi \rangle.$$

Due to the density of $Z_1 \cap Z_2$ in $(Z_1, Z_2)_{\theta,1}$ and $(Z_1, Z_2)_{\tilde{\theta},1}$, this implies $x = y$. $\square$

1.2 Function Spaces

Let $U \subseteq \mathbb{R}^d$ be open, X be a normed vector space and $k \in \mathbb{N}$. We define

$$C^k(U; X) := \{f \colon U \to \mathbb{R} : f \text{ is } k\text{-times continuously differentiable}\}.$$

For $X = \mathbb{R}^d$, we put $C^k(U; \mathbb{R}^d) = C^k(U)$, if d is clear in the context. The set $C^k(\overline{U}; X)$ is the set of all $f \in C^k(U; X)$, for which all of its derivatives up to order k can be continuously extended to $\overline{U}$. For $k = \infty$, we set

$$C^\infty(U; X) := \bigcap_{k=0}^{\infty} C^k(U; X) \quad \text{and} \quad C^\infty(\overline{U}; X) := \bigcap_{k=0}^{\infty} C^k(\overline{U}; X).$$

We add a subscript c to the spaces defined before, i.e. $C_c^k(U;X)$ and so on, if its elements are supposed to have compact support.

The closure of $C_c^k(U;X)$ and $C_c^k(\overline{U};X)$ with respect to the sup norm is denoted by $C_0^k(U;X)$ and $C_0^k(\overline{U};X)$ respectively. The space of bounded uniformly continuous functions from U to X will be denoted by $\mathrm{BUC}(U;X)$. As usual, this space is equipped with the sup norm.

Function spaces on the real axis will play a special role. Let X be a Banach space. For $T > 0$, the space of T-periodic functions is defined as

$$P_T(\mathbb{R};X) := \{f\colon \mathbb{R} \to X : f(\cdot + T) - f(\cdot) = 0\}.$$

A generalisation of periodicity is the property of being almost periodic:

Definition 1.2.1. *A bounded function $f\colon \mathbb{R} \to X$ is called (uniformly) almost periodic, if for each sequence $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}$, there is a subsequence $(t_{n_k})_{k\in\mathbb{N}}$ such that $(f(\cdot + t_{n_k}))_{k\in\mathbb{N}}$ is convergent with respect to the sup norm.*

We use $\mathrm{UAP}(\mathbb{R};X)$ for the space of X-valued almost periodic functions. It is known, that $\mathrm{UAP}(\mathbb{R};X)$ is a closed subspace of $\mathrm{BUC}(\mathbb{R};X)$, see for example [LZ82, Property 2 and 3 on pp. 2,3]. Furthermore, $\mathrm{UAP}(\mathbb{R};\mathbb{R})$ is closed under multiplication, see [LZ82, Property 6, p. 6]. The proof therein can be extended to Banach space valued almost periodic functions as long as the multiplication is well defined. It was shown in [ABHN11, (4.36) and Proposition 4.7.1], that $f \in C_0(\mathbb{R};X) \cap \mathrm{UAP}(\mathbb{R};X)$ implies $f = 0$. Thus, we can define the set of asymptotically almost periodic functions via the direct sum

$$\mathrm{AAP}(\mathbb{R};X) = \mathrm{UAP}(\mathbb{R};X) \oplus C_0(\mathbb{R};X).$$

1.2.1 Traces in Sobolev Spaces

Let $d \in \mathbb{N}$, $d \geq 2$. Let $\Omega \subset \mathbb{R}^d$ be a C^1-domain. It is well known that the operator $\gamma\colon C^1(\overline{\Omega}) \to C^1(\partial\Omega), u \mapsto u|_{\partial\Omega}$ can be continuously extended to an operator from $H_p^1(\Omega)$ to $B_{p,p}^{1-(1/p)}(\partial\Omega) = W^{1-1/p,p}(\partial\Omega)$. Analogous results hold for spaces of higher orders. More precisely, it holds:

Theorem 1.2.2 ([AF03, Theorem 7.39])**.** *Let $1 < p < \infty$, $k \in \mathbb{N} \setminus \{0\}$ and $u\colon \mathbb{R}^d \to \mathbb{R}$ be measurable. Then the following conditions are equivalent:*

1. *There is a $U \in H_p^k(\mathbb{R}^{d+1})$ such that $\gamma U = u$.*
2. *$u \in B_{p,p}^{k-(1/p)}(\mathbb{R}^d)$.*

By [AF03, Remark 7.45] the theorem above remains valid with $\Omega \subset \mathbb{R}^{d+1}$ in place of $\mathbb{R}^{d+1}$ and $\partial\Omega$ in place of $\mathbb{R}^d$ as long as the boundary $\partial\Omega$ is sufficiently smooth, e.g. strongly Lipschitz if $k = 1$.

Conversely, there exists a bounded and linear right inverse of the trace operator.

Lemma 1.2.3. *Let $\Omega \subset \mathbb{R}^3$ be a domain with compact C^∞-boundary and $1 < p < \infty$. Then there is a bounded linear operator*

$$F_p \colon W^{1-1/p,p}(\partial\Omega) \to H^1_p(\mathbb{R}^3)$$

which is a right inverse of the trace operator. This operator is consistent with respect to p.

Proof. For a fixed p, the existence of this operator was shown in [Mar87, Theorem 2]. More precisely, it was shown therein that the domain of the extension operator $\mathcal{E}$ constructed in [JW84, Theorem 3, p.197] coincides with the usual Sobolev-Slobodeckij space (or Besov space) $W^{1-1/p}(\partial\Omega)$, if the boundary $\partial\Omega$ is sufficiently smooth. As the extension operator $\mathcal{E}$ from [JW84] does not change, if the smoothness parameter $1-1/p$ remains between two integers, we also have the consistency of that operator. $\square$

Partial traces for vector fields, like tangential and normal traces, can be defined under less smoothness conditions than the classical traces via integration by parts. Let $\Omega \subset \mathbb{R}^3$ be a C^1-domain. For $\psi, \phi \in C^1_c(\overline{\Omega})$ it holds

$$\int_D \psi \nabla\phi + \int_D (\operatorname{div}\psi)\phi = \int_{\partial D} (\psi \cdot n)\phi, \tag{1.3}$$

$$\int_D \psi \operatorname{rot}\phi = \int_D (\operatorname{rot}\psi)\phi + \int_{\partial D} (\psi \times n) \cdot \phi. \tag{1.4}$$

Define

$$\begin{aligned} E^p_{div}(D) &:= \{u \in L^p(D) : \operatorname{div} u \in L^p(D)\}, \\ E^p_{rot}(D) &:= \{u \in L^p(D) : \operatorname{rot} u \in L^p(D)\}. \end{aligned}$$

Using (1.3) and (1.4), it is known, that one can extend the trace operators $u \mapsto (u \cdot n)$ and $u \mapsto (u \times n)$ from smooth functions to $E^p_{div}(\Omega)$ and $E^p_{rot}(\Omega)$. In particular, one has

$$\begin{aligned} \gamma_n \colon E^p_{div}(D) &\to W^{1-1/p',p'}(\partial D)', \quad \gamma_n u = u \cdot n, \\ \tau_n \colon E^p_{rot}(D) &\to W^{1-1/p',p'}(\partial D)', \quad \tau_n u = u \times n. \end{aligned}$$

in the sense, that

$$\langle \gamma_n u, \phi \rangle_{\partial\Omega} = (u, \nabla\phi) + (\operatorname{div} u, \phi) \qquad \text{for all } \phi \in H^1_{p'}(\Omega), \tag{1.5}$$

$$\langle \tau_n u, \phi \rangle_{\partial\Omega} = (u, \operatorname{rot}\phi) - (\operatorname{rot} u, \phi) \qquad \text{for all } \phi \in H^1_{p'}(\Omega). \tag{1.6}$$

In case of unbounded domains, it is of course sufficient to require only local the integrability conditions $u, \operatorname{div} u \in L^p_{loc}(\overline{\Omega})$ or $u, \operatorname{rot} u \in L^p_{loc}(\overline{\Omega})$ in order to define the normal and tangential traces respectively.

It is well known, that the space $H^{1,0}_p(\Omega)$ can be defined equivalently as the closure of $C^\infty_c(\Omega)$ in $H^1_p(\Omega)$ or as the set of elements of $H^{1,0}_p(\Omega)$ with zero trace. We will show that a similar result holds for the zero tangential and zero normal boundary conditions as long as Ω has a smooth boundary. Let $\Omega \subset \mathbb{R}^3$ be a domain with C^k-boundary and $m \in \mathbb{N} \cup \{\infty\}$. Define

$$\begin{aligned} C^{m,T}_c(\overline{\Omega}) &:= \{u \in C^m_c(\overline{\Omega}) : u \cdot n = 0\}, \\ H^{1,T}_p(\Omega) &:= \{u \in H^1_p(\Omega) : u \cdot n = 0\}, \\ C^{m,N}_c(\overline{\Omega}) &:= \{u \in C^m_c(\overline{\Omega}) : u \times n = 0\}, \\ H^{1,N}_p(\Omega) &:= \{u \in H^1_p(\Omega) : u \times n = 0\}. \end{aligned}$$

Lemma 1.2.4. *1. The space $C^{\infty,T}_c(\overline{\mathbb{R}^3_+})$ is dense in $H^{1,T}_p(\mathbb{R}^3_+)$.*

2. The space $C^{\infty,N}_c(\overline{\mathbb{R}^3_+})$ is dense in $H^{1,N}_p(\mathbb{R}^3_+)$.

Proof. In the half space, the boundary conditions simplify to

$$u \cdot n = -u_3 = 0 \quad \text{and} \quad u \times n = (-u_2, u_1, 0) = 0.$$

Hence, the spaces $H^{1,T}_p(\mathbb{R}^3_+)$ and $H^{1,N}_p(\mathbb{R}^3_+)$ coincide with

$$\begin{aligned} H^{1,T}_p(\mathbb{R}^3_+) &= H^1_p(\mathbb{R}^3_+) \times H^1_p(\mathbb{R}^3_+) \times H^{1,0}_p(\mathbb{R}^3_+), \\ H^{1,N}_p(\mathbb{R}^3_+) &= H^{1,0}_p(\mathbb{R}^3_+) \times H^{1,0}_p(\mathbb{R}^3_+) \times H^1_p(\mathbb{R}^3_+), \end{aligned}$$

and we have the inclusions

$$\begin{aligned} C^{\infty,T}_c(\overline{\mathbb{R}^3_+}) &\supset C^\infty_c(\overline{\mathbb{R}^3_+}) \times C^\infty_c(\overline{\mathbb{R}^3_+}) \times C^\infty_c(\mathbb{R}^3_+), \\ C^{\infty,N}_c(\overline{\mathbb{R}^3_+}) &\supset C^\infty_c(\mathbb{R}^3_+) \times C^\infty_c(\mathbb{R}^3_+) \times C^\infty_c(\overline{\mathbb{R}^3_+}). \end{aligned}$$

The density of $C^\infty_c(\mathbb{R}^3_+)$ in $H^{1,0}_p(\mathbb{R}^3_+)$ and $C^\infty_c(\overline{\mathbb{R}^3_+})$ in $H^1_p(\mathbb{R}^3_+)$ yield the desired result. $\square$

By suitable transformations, this lemma can be extended to bent half spaces. Let $\omega \in C^k(\mathbb{R}^2)$ with $k \in \mathbb{N} \setminus \{0\}$. Set

$$\mathbb{R}^3_\omega := \{x \in \mathbb{R}^3 : x_3 > \omega(x_1, x_2)\}.$$

A (not normalized) exterior perpendicular vector ν of $\mathbb{R}^3_\omega$ at some point $x = (x_1, x_2, \omega(x_1, x_2))$ is given by

$$\nu(x_1, x_2) = (\partial_1\omega, \partial_2\omega, -1).$$

Lemma 1.2.5. *There is a C^{k-1}-diffeomorphism from $H^1_p(\mathbb{R}^3_+)$ to $H^1_p(\mathbb{R}^3_\omega)$ that maps $C^{k-1,B}_c(\overline{\mathbb{R}^3_+})$ onto $C^{k-1,B}_c(\overline{\mathbb{R}^3_\omega})$.*

Proof. Define $\Phi\colon C^\infty_c(\overline{\mathbb{R}^3_+}) \to C^\infty_c(\overline{\mathbb{R}^3_\omega})$ via

$$(\Phi u)(x) = Q(x')u(x', x_3 - \omega(x')),$$

where

$$Q(x') = \begin{pmatrix} 1 + (\partial_2\omega)^2 & -\partial_1\omega\partial_2\omega & \partial_1\omega \\ -\partial_1\omega\partial_2\omega & 1 + (\partial_1\omega)^2 & \partial_2\omega \\ \partial_1\omega & \partial_2\omega & -1 \end{pmatrix}.$$

Due to the regularity of ω, it is easy to see, that Φ maps continuously from $H^1_p(\mathbb{R}^3_+)$ to $H^1_p(\mathbb{R}^3_\omega)$ and that it maps $C^{k-1}_c(\overline{\mathbb{R}^3_+})$ to $C^{k-1}_c(\overline{\mathbb{R}^3_\omega})$. Furthermore, an elementary calculation shows, that vanishing tangential boundary conditions and vanishing normal boundary conditions are preserved by Φ. The inverse of Φ is given by

$$(\Phi^{-1}v)(x) = Q^{-1}(x')v(x', x_3 + \omega(x')),$$

where

$$Q^{-1}(x') = \frac{1}{1 + (\partial_1\omega)^2 + (\partial_2\omega)^2} \begin{pmatrix} 1 & 0 & \partial_1\omega \\ 0 & 1 & \partial_2\omega \\ \partial_1\omega & \partial_2\omega & -1 \end{pmatrix},$$

which is as smooth as Φ itself. □

By means of a standard localization argument and Lemma 1.2.5, we get the following property:

Proposition 1.2.6. *Let $\Omega \subset \mathbb{R}^3$ be a domain with compact C^{k+1}-boundary. Then the set $C^{k,B}_c(\overline{\Omega})$ is dense in $H^{1,B}_p(\Omega)$.*

1.2.2 Lorentz Spaces

We collect some standard properties of Lorentz spaces. For proofs and details we refer for example to [AF03, pp. 221-228] and [Gra08, pp. 44-71]. Throughout this section, let $\Omega \subset \mathbb{R}^d$ be a measurable set. Given a measurable function $u\colon \Omega \to \mathbb{R}$, we define its distribution function as

$$d_u\colon [0,\infty) \to \mathbb{R}, \quad d_u(t) := |\{x \in \Omega : |u(x)| > t\}|.$$

The nonincreasing rearrangement of u is defined as

$$u^*\colon [0,\infty) \to \mathbb{R}, \quad u^*(s) := \inf\{t \in [0,\infty) : d_u(t) \leq s\}.$$

Using the nonincreasing rearrangement, the definition of the Lorentz spaces is as follows:

Definition 1.2.7. *Let $p, q \in (0, \infty]$. The Lorentz space $L^{p,q}(\Omega)$ is defined as*

$$L^{p,q}(\Omega) := \{f\colon \Omega \to \mathbb{R} : f \text{ measurable}, [f]_{p,q} < \infty\},$$

where

$$[f]_{p,q} := \begin{cases} \left(\int_0^\infty \left(t^{1/p} f^*(t)\right)^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} & \text{for } q < \infty, \\ \sup_{t>0} t^{1/p} f^*(t) & \text{for } q = \infty. \end{cases}$$

We collect some of their basic properties:

Proposition 1.2.8. 1. *For $p, q \in (0,\infty]$, the space $L^{p,q}(\Omega)$ is a complete quasi-normed vector space.*

2. *For $p \in (1,\infty]$, $q \in [1,\infty]$, there is a norm $\|\cdot\|_{L^{p,q}(\Omega)}$, which is equivalent to the quasi-norm $[\,\cdot\,]_{p,q}$ on $L^{p,q}(\Omega)$.*

3. *For $p \in [1,\infty]$, it holds $L^{p,p}(\Omega) = L^p(\Omega)$.*

4. *Let $p_0, q_0, p_1, q_1, q \in (0,\infty]$ and $\theta \in (0,1)$. Then it holds*

$$L^{p,q}(\Omega) = (L^{p_0,q_0}(\Omega), L^{p_1,q_1}(\Omega))_{\theta,q},$$

where $p \in (0,\infty]$ is given by $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

For the second parameter q being equal to ∞, these spaces do coincide with the weak Lebesgue spaces.

Definition 1.2.9. *For $p \in (0, \infty]$, the weak Lebesgue space is defined as*

$$L^p_\omega(\Omega) := \{f \colon \Omega \to \mathbb{R} : f \text{ measurable}, [f]_{p,\omega} < \infty\},$$

where

$$[u]_{p,\omega} := \sup\{t d_u(t)^{1/p} : t > 0\}.$$

Proposition 1.2.10. *Let $p \in (0, \infty]$. Then $L^{p,\infty}(\Omega) = L^p_\omega(\Omega)$.*

Using the quasi-norm $[\,\cdot\,]_{p,\omega}$, it is a standard computation to show that functions of the kind $|x|^{-\alpha}$ lie in $L^{p,\infty}(\Omega)$ for the right choice of parameters:

Proposition 1.2.11. *Let $d \in \mathbb{N} \setminus \{0\}$ and $\alpha \in (0, \infty]$. Then it holds $x \mapsto |x|^{-\alpha} \in L^{\alpha d,\infty}(\mathbb{R}^d)$.*

We conclude this section by noting that the classical Hölder and Young inequalities remain valid in Lorentz spaces.

Proposition 1.2.12.

1. *Let $p_0, q_0, p_1, q_1, p, q \in (0, \infty]$ satisfy $\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1}$ and $\frac{1}{q} = \frac{1}{q_0} + \frac{1}{q_1}$. Then for any $f \in L^{p_0,q_0}(\Omega)$ and $g \in L^{p_1,q_1}(\Omega)$ it holds $fg \in L^{p,q}(\Omega)$ and there is a constant $C > 0$ independent of f and g such that*

$$\|fg\|_{L^{p,q}(\Omega)} \le C\|f\|_{L^{p_0,q_0}(\Omega)}\|g\|_{L^{p_1,q_1}(\Omega)}.$$

2. *Let $p_0, p_1, p \in (1, \infty)$, $q_0, q_1, q \in (0, \infty]$ satisfy $1 + \frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1}$ and $\frac{1}{q} = \frac{1}{q_0} + \frac{1}{q_1}$. Then for any $f \in L^{p_0,q_0}(\mathbb{R}^d)$ and $g \in L^{p_1,q_1}(\mathbb{R}^d)$, it holds $f * g \in L^{p,q}(\mathbb{R}^d)$ and there is a constant $C > 0$ independent of f and g such that*

$$\|f * g\|_{L^{p,q}(\mathbb{R}^d)} \le C\|f\|_{L^{p_0,q_0}(\mathbb{R}^d)}\|g\|_{L^{p_1,q_1}(\mathbb{R}^d)}.$$

1.2.3 Spaces of Solenoidal Vector Fields

Here, we introduce the classical solenoidal subspaces of Lebesgue spaces and their generalisations in Lorentz spaces.

Definition 1.2.13. *Let $\Omega \subset \mathbb{R}^3$ be a domain with compact C^2-boundary. For $p \in (1, \infty)$, the spaces $L^p_\sigma(\Omega)$ are defined by*

$$L^p_\sigma(\Omega) := \{u \in L^p(\Omega) : \operatorname{div} u = 0 \text{ in } \Omega,\ u \cdot n = 0 \text{ on } \partial\Omega\}.$$

The following properties of $L^p_\sigma(\Omega)$ are well known (see for example [FM77], [Miy82]): The spaces $L^p_\sigma(\Omega)$ above coincide with the closure of $\{u \in C_c^\infty(\Omega) : \operatorname{div} u = 0\}$ in $L^p(\Omega)$. Moreover, they have a closed complement in $L^p(\Omega)$ and it holds $(L^p_\sigma(\Omega))' = L^{p'}_\sigma(\Omega)$. The bounded projection $\mathbb{P}$ from $L^p(\Omega)$ onto $L^p_\sigma(\Omega)$ will be called Helmholtz projection.

By real interpolation, the definition above can be extended to Lorentz spaces. This has been done in [BM95]. For $p \in (1,\infty)$ and $q \in [1,\infty]$, define

$$L^{p,q}_\sigma(\Omega) := (L^{p_1}_\sigma(\Omega), L^{p_2}_\sigma(\Omega))_{\theta,q},$$

where $1 < p_1 < p < p_2 < \infty$ and $\theta \in (0,1)$ satisfy

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}.$$

Similar to their counterparts in Lebesgue spaces, it holds, that

$$L^{p,q}_\sigma(\Omega) := \{u \in L^{p,q}(\Omega) : \operatorname{div} u = 0 \text{ in } \Omega,\ u \cdot n = 0 \text{ on } \partial\Omega\}$$

and for $q < \infty$, we have

$$(L^{p,q}_\sigma(\Omega))' = L^{p',q'}_\sigma(\Omega).$$

1.2.4 Homogeneous Besov Spaces

In the literature, there can be found different definitions for homogeneous Besov spaces. We will follow the introduction of [BCD11]. Throughout this section, the dimension $d \in \mathbb{N}$ is arbitrary, but fixed. We start by introducing the dyadic decomposition in Fourier space. Let $\chi \in C^\infty(\mathbb{R}^d;\mathbb{R})$ be a cut-off function with $\operatorname{supp}\chi \subseteq B(0,4/3)$, $0 \le \chi \le 1$ and $\chi = 1$ on $B(0,3/4)$. Furthermore, set $\phi(\xi) := \chi(\xi/2) - \chi(\xi)$ and $h := \mathcal{F}\phi$. We can now define the Littlewood-Paley decomposition $(\dot\Delta_j)_{j\in\mathbb{Z}}$ as

$$\dot\Delta_j u(x) = 2^{jd} \int_{\mathbb{R}^d} h(2^j y) u(x-y)\,\mathrm{d}y = (\mathcal{F}^{-1}\phi(2^{-j}\cdot)\mathcal{F}u)(x).$$

It will also be convenient to have the operators

$$\dot S_j u = \sum_{j' \le j-1} \dot\Delta_{j'} u$$

for any $j \in \mathbb{Z}$ at hand. We will use them right now to introduce the set $\mathcal{S}'_h$ of all tempered distributions $u \in \mathcal{S}'$ such that $\lim_{j\to-\infty} \dot S_j u = 0$.

The homogeneous Besov spaces can now be defined in the following way.

Definition 1.2.14. *Let $s \in \mathbb{R}$ and $p, q \in [1, \infty]$. Define*

$$\dot{B}^s_{p,q}(\mathbb{R}^d) := \{u \in \mathcal{S}'_h : \|u\|_{\dot{B}^s_{p,q}} < \infty\},$$

where

$$\|u\|_{\dot{B}^s_{p,q}} = \left(\sum_{j \in \mathbb{Z}} 2^{sqj} \|\dot{\Delta}_j u\|^q_{L^p}\right)^{1/q} \qquad \text{for } q < \infty,$$
$$\text{and} \quad \|u\|_{\dot{B}^s_{p,q}} = \sup_{j \in \mathbb{Z}} 2^{sj} \|\dot{\Delta}_j u\|_{L^p} \qquad \text{for } q = \infty.$$

These spaces are complete for suitable parameters (see [BCD11, Theorem 2.25]).

Proposition 1.2.15. *Let $s_1, s_2 \in \mathbb{R}$ and $p_1, p_2, q_1, q_2 \in [1, \infty]$. Assume that s_1, p_1 and q_1 satisfy*

$$s_1 < \frac{d}{p_1} \qquad \text{or} \qquad s_1 = \frac{d}{p_1},\ q_1 = 1.$$

Then the intersection $\dot{B}^{s_1}_{p_1,q_1}(\mathbb{R}^d) \cap \dot{B}^{s_2}_{p_2,q_2}(\mathbb{R}^d)$ is complete and admits the Fatou property: If $(u_n)_{n \in \mathbb{N}} \subset \dot{B}^{s_1}_{p_1,q_1} \cap \dot{B}^{s_2}_{p_2,q_2}$ is a bounded sequence, then there is a $u \in \mathcal{S}'$ such that

$$\lim_{n \to \infty} u_n = u \qquad \text{in } \mathcal{S}'$$

and there is a $C > 0$ independent of u_n and u such that

$$\|u\|_{\dot{B}^{s_k}_{p_k,q_k}} \le C \liminf \|u_n\|_{\dot{B}^{s_k}_{p_k,q_k}}, \quad k \in \{1, 2\}.$$

Inequalities of Sobolev type are available in homogeneous Besov spaces.

Lemma 1.2.16 ([BCD11] Proposition 2.20)**.** *Let $s \in \mathbb{R}$, $q \in [1, \infty]$ and $p_1, p_2 \in [1, \infty]$ such that $p_1 \le p_2$. Then there is a constant $C > 0$ such that for all $u \in \dot{B}^s_{p_1,q}(\mathbb{R}^d)$ it holds*

$$\|u\|_{\dot{B}^{s-d\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}_{p_2,q}} \le C \|u\|_{\dot{B}^s_{p_1,q}}.$$

In order to deal with nonlinearities, we make use of the Bony decomposition. Formally, a product of two tempered distributions u and v can be written as

$$uv = \sum_{j, j' \in \mathbb{Z}} \dot{\Delta}_j u \dot{\Delta}_{j'} v.$$

The Bony decomposition is the separation of the sum above into

$$uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u,v), \tag{1.7}$$

where

$$\begin{aligned} \dot{T}_f g &= \sum_{j' \le j-1, j, j' \in \mathbb{Z}} \dot{S}_{j-1} f \dot{\Delta}_{j'} g, \\ \dot{R}(f,g) &= \sum_{|j-j'| \le 1, j, j' \in \mathbb{Z}} \dot{\Delta}_j f \dot{\Delta}_{j'} g, \end{aligned}$$

The operator $\dot{T}$ can be estimated in the following way.

Lemma 1.2.17 ([BCD11], Theorem 2.47)**.** *Let $s \in \mathbb{R}$, $t \in (-\infty, 0)$ and $p, r_1, r_2 \in [1,\infty]$. Then there is a constant $C > 0$ such that*

$$\|\dot{T}_f g\|_{\dot{B}^{s+t}_{p,r}} \le C \|f\|_{\dot{B}^t_{\infty,r_1}} \|g\|_{\dot{B}^s_{p,r_2}}, \qquad \textit{where } \frac{1}{r} = \min\left(1, \frac{1}{r_1} + \frac{1}{r_2}\right).$$

Similar estimates hold for $\dot{R}$:

Lemma 1.2.18 ([BCD11], Theorem 2.52)**.** *Let $s_1, s_2 \in \mathbb{R}$, $p, p_1, p_2, r, r_1, r_2 \in [1,\infty]$ such that $s_1 + s_2 > 0$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$. Then there is a constant $C > 0$ such that*

$$\|\dot{R}(f,g)\|_{\dot{B}^{s_1+s_2}_{p,r}} \le C \|f\|_{\dot{B}^{s_1}_{p_1,r_1}} \|g\|_{\dot{B}^{s_2}_{p_2,r_2}}.$$

A special consequence of these two lemmata is the next estimate for the product of two functions in homogeneous Besov spaces.

Corollary 1.2.19. *For $\sigma \in (1,\infty)$, $u \in \dot{B}^{d/p-1}_{p,\infty}(\mathbb{R}^d)$, $v \in \dot{B}^{\sigma}_{p,\infty}(\mathbb{R}^d)$ it holds*

$$\|uv\|_{\dot{B}^{\sigma-1}_{p,\infty}(\mathbb{R}^d)} \le \|u\|_{\dot{B}^{d/p-1}_{p,\infty}(\mathbb{R}^d)} \|uv\|_{\dot{B}^{\sigma}_{p,\infty}(\mathbb{R}^d)}$$

Proof. Because of the Bony decomposition, it suffices to consider $\dot{T}_u v$, $\dot{T}_v u$ and $\dot{R}(u,v)$. Due to Lemma 1.2.17 and Lemma 1.2.16, we have

$$\begin{aligned} \|\dot{T_u} v\|_{\dot{B}^{\sigma-1}_{p,\infty}(\mathbb{R}^d)} &\le \|u\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^d)} \|v\|_{\dot{B}^{\sigma}_{p,\infty}(\mathbb{R}^d)} \\ &\le \|u\|_{\dot{B}^{d/p-1}_{p,\infty}(\mathbb{R}^d)} \|v\|_{\dot{B}^{\sigma}_{p,\infty}(\mathbb{R}^d)}. \end{aligned}$$

as well as

$$\begin{aligned} &\|\dot{T_v} u\|_{\dot{B}^{\sigma-1}_{p,\infty}(\mathbb{R}^d)} \\ &\le \|v\|_{\dot{B}^{\sigma-d/p}_{\infty,\infty}(\mathbb{R}^d)} \|u\|_{\dot{B}^{d/p-1}_{p,\infty}(\mathbb{R}^d)} \\ &\le \|v\|_{\dot{B}^{\sigma}_{p,\infty}(\mathbb{R}^d)} \|u\|_{\dot{B}^{d/p-1}_{p,\infty}(\mathbb{R}^d)}. \end{aligned}$$

Furthermore, using Lemma 1.2.18 and Lemma 1.2.16, it holds

$$\begin{aligned}\|\dot{R}(u,v)\|_{\dot{B}^{\sigma-1}_{p,\infty}(\mathbb{R}^d)} &\leq \|u\|_{\dot{B}^{d/p-1}_{p,\infty}(\mathbb{R}^d)}\|v\|_{\dot{B}^{\sigma-d/p}_{\infty,\infty}(\mathbb{R}^d)} \\ &\leq \|u\|_{\dot{B}^{d/p-1}_{p,\infty}(\mathbb{R}^d)}\|v\|_{\dot{B}^{\sigma}_{p,\infty}(\mathbb{R}^d)}.\end{aligned}$$

□

1.3 Semigroups and Evolution Families

This section is devoted to standard properties of operator semigroups and evolution families. As standard references, we refer to [ABHN11] or [EN00].

1.3.1 Strongly Continuous Semigroups

Throughout this section, X is a Banach space.

Definition 1.3.1. *An operator family $(T(t))_{t\geq 0} \subset \mathcal{L}(X,X)$ is called a semigroup, if*

1. $T(0) = \mathrm{id}$,
2. $T(t+s) = T(t)T(s)$ *for all* $t,s \in [0,\infty)$.

If additionally $t \mapsto T(t)x$ lies in $C([0,\infty),X)$ for each $x \in X$, we call $(T(t))_{t\geq 0}$ a strongly continuous semigroup or a C_0-semigroup.

The growth bound $\omega(T)$ of a semigroup $(T(t))_{t\geq 0}$ is defined as

$$\omega(T) = \inf\{\omega \in \mathbb{R} : \exists M_\omega \geq 1, \forall t \geq 0 : \|T(t)\|_{\mathcal{L}(X,X)} \leq M_\omega e^{\omega t}\}.$$

We say that a semigroup $(T(t))_{t\geq 0}$ is

- *bounded*, if $\sup_{t\geq 0} \|T(t)\|_{\mathcal{L}(X,X)} \leq C$ for some $C > 0$.
- *contractive*, if $\sup_{t\geq 0} \|T(t)\|_{\mathcal{L}(X,X)} \leq 1$.
- *exponentially stable*, if $\omega(T) < 0$.

We do associate an operator A to a C_0-semigroup. Define

$$D(A) := \{x \in X : \lim_{t\to 0} \frac{1}{t}(T(t) - \mathrm{id})x \text{ exists.}\}$$

and set

$$Ax := \lim_{t\to 0} \frac{1}{t}(T(t) - \mathrm{id})x \quad \text{ for each } x \in D(A).$$

The operator A will be called generator of $(T(t))_{t\geq 0}$. It is always uniquely determined, closed and densely defined and commutes with the semigroup on $D(A)$. Given a generator A of a strongly continuous semigroup, the corresponding semigroup will be usually denoted by $(e^{tA})_{t\geq 0}$.

Given a C_0-semigroup $(e^{tA})_{t\geq 0}$ with generator A, the adjoint family $((e^{tA})')_{t\geq 0}$ is again a semigroup, but not necessarily strongly continuous. However, due to the adjoint relation, it is easy to see that the adjoint semigroup is weak-*-continuous. By this, we mean

$$\lim_{t\to t_0} \langle x, (e^{tA})^* x'\rangle = \langle x, (e^{t_0 A})^* x'\rangle$$

for each $x \in X$, $x' \in X'$ and $t_0 \geq 0$. Using the weak-*-topology, it is possible to associate an operator to the adjoint semigroup. We set

$$D(A^\sigma) := \{x' \in X' : \lim_{t\to 0} \frac{1}{t}((e^{tA})'x' - x') \text{ exists in the weak-*-topology}\}$$

and define

$$A^\sigma x' := \text{weak-*-} \lim_{t\to 0} \frac{1}{t}((e^{tA})'x' - x')$$

for each $x' \in D(A^\sigma)$. It can be shown that A^σ coincides with A'. See for example [EN00, Exercise 2.8]. For this reason, we will also call A' the generator of $((e^{tA})')_{t\geq 0}$ and set $e^{tA'} := (e^{tA})'$.

1.3.2 Analytic Semigroups

For $\delta \in (0, \pi]$, define the sector

$$\Sigma_\delta := \{\lambda \in \mathbb{C} : |\arg \lambda| < \delta\} \setminus \{0\}.$$

Definition 1.3.2. *Let X be a Banach space and $\theta \in (0, \pi/2]$. A semigroup $(T(t))_{t\geq 0} \subset \mathcal{L}(X, X)$ is called a bounded analytic semigroup of angle θ, if there is bounded analytic extension of T to $\Sigma_{\theta'}$ for all $\theta' \in (0, \theta)$.*

If A is the generator of an analytic semigroup in X, we will write $A \in \mathrm{Hol}(X)$. If we also need information about the growth bound of $(e^{tA})_{t\geq 0}$, we will also write $A \in \mathrm{Hol}_\omega(X)$, where $\omega = \omega(T)$. The most important property of analytic semigroups for us is the following:

Proposition 1.3.3. *Let A be the generator of a bounded analytic semigroup $(e^{zA})_{z\in\Sigma_\delta\cup\{0\}}$ for some $\delta \in (0,\pi/2]$. Then it holds that $e^{tA}x \subset D(A)$ for each $t \in (0,\infty)$ and $x \in X$, and there is a constant $M > 0$ such that*

$$\sup_{t\in(0,\infty)} \|tAe^{tA}\|_{\mathcal{L}(X,X)} \leq M.$$

1.3.3 Evolution Families

Let X be a Banach space and $I \subseteq \mathbb{R}$ be a non-trivial interval. An operator family $(U(t,s))_{t,s\in I,t\geq s} \subset \mathcal{L}(X,X)$ is called an *evolution family*, if

$$\begin{aligned} &U(t,t) = \mathrm{id} &&\text{for all } t \in I,\\ &U(t,s) = U(t,r)U(r,s) &&\text{for all } t,r,s \in I, t \geq r \geq s. \end{aligned}$$

The family $(U(t,s))_{t,s\in I,t\geq s}$ is called strongly continuous, if the map

$$\{(\tau,\sigma) \in I^2 : \tau \geq \sigma\} \ni (t,s) \mapsto U(t,s)x$$

is continuous in X for each $x \in X$.

Let $(A(t))_{t\in I}$ be a family of possibly unbounded linear operators with domains $D(A(t)) \subseteq X$ and $s \in I$. A homogeneous non-autonomous abstract Cauchy problem is given by

$$\begin{aligned} u'(t) - A(t)u(t) &= 0, \quad t \geq s, t \in I\\ u(s) &= a. \end{aligned} \tag{1.8}$$

A strongly continuous evolution family $(U(t,s))_{t,s\in I,t\geq s}$ is said to solve (1.8), if there is a dense subspaces $X_r \subseteq X$ for each $r \in I$ such that

$$U(t,s)X_r \subseteq X_t \subseteq D(A(t)) \quad \text{for all } t \geq r, \quad t,r \in I$$

and the map $\{\tau \in I : \tau \geq s\} \ni t \mapsto U(t,s)a$ is a solution of (1.8) for each $s \in I$ and $a \in X_s$.

We will not make use of this very definition of an evolution family solving a non-autonomous Cauchy problem. Instead, we will only say, that an evolution family is associated to a non-autonomous Cauchy problem. This may be in the way given above, but it may also be any other reasonable way.

1.4 Neumann Problem in Exterior Domains

We assemble some higher order estimates for the Neumann problem in exterior domains. These are well known, but difficult to find in the literature. We will show the estimates by a cut-off argument in order to reduce the situation to Neumann problems on a bounded domain and the whole space. The Neumann problem we are concerned with is given by

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega, \\ \partial_n u &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{1.9}$$

with an exterior domain $\Omega \subset \mathbb{R}^3$.

The existence and uniqueness of weak solutions to (1.9) has been shown in [Miy82, Proof of Proposition 1.5]:

Lemma 1.4.1. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^2-boundary and $p \in (1,\infty)$. Then for each $g \in W^{-1/p,p}(\partial\Omega)$ with $\langle g, 1\rangle_{\partial\Omega} = 0$ there is a solution $u \in \dot{\mathbf{H}}^1_p(\Omega)$ to (1.9), which is unique up to constant functions. Moreover, there is a constant $C = C(p,\Omega)$ such that*

$$\|\nabla u\|_{L^p(\Omega)} \leq C\|g\|_{W^{-1/p,p}(\partial\Omega)}.$$

Let $R > 0$ be such that $\partial\Omega \subset B(0,R)$ and let $\eta \in C_c^\infty(\Omega)$ be a cut-off function such that $\eta = 1$ in $\Omega \cap B(0,R)$, $\eta = 0$ in $B(0,R+1)^C$ and $0 \leq \eta \leq 1$. Formally, given a solution u to (1.9), the functions $u_1 := \eta u$ and $u_2 := (1-\eta)u$ solve

$$\begin{aligned} -\Delta u_1 &= -2\nabla u \nabla \eta - u\Delta\eta \quad \text{in } \Omega \cap B(0,R), \\ \partial_n u_1 &= g \quad \text{on } \partial\Omega, \\ \partial_n u_1 &= 0 \quad \text{on } \partial B(0,R) \end{aligned} \tag{1.10}$$

as well as

$$-\Delta u_2 = 2\nabla u \nabla \eta + u\Delta\eta. \tag{1.11}$$

Thus, the question of regularity of solutions to (1.9) reduces to (1.10) and (1.11). On the whole space, we get the following higher regularity result.

Lemma 1.4.2. *Let $p \in (1,\infty)$ and $f \in L^p(\mathbb{R}^d)$. Then for each $j \in \{1,\dots,d\}$ there is a unique solution $u \in \dot{\mathbf{H}}^1_p(\mathbb{R}^d)$ such that*

$$-\Delta u = \frac{\partial}{\partial x_k} f$$

in the sense of distributions. This u can be estimated by

$$\|\nabla u\|_{L^p(\mathbb{R}^d)} \leq C\|f\|_{L^p(\mathbb{R}^d)}, \tag{1.12}$$

with $C = C(p,d) > 0$ being independent of f. If additionally $\nabla f \in L^r(\mathbb{R}^d)$ for some $r \in (1,\infty)$, then $\nabla^2 u \in L^r(\mathbb{R}^d)$ and there is a $C = C(r,d) > 0$ such that

$$\|\nabla^2 u\|_{L^r(\mathbb{R}^d)} \leq C\|\nabla f\|_{L^r(\mathbb{R}^d)}. \tag{1.13}$$

Proof. The existence and uniqueness of $u \in \dot{\mathbf{H}}^1_p(\mathbb{R}^d)$ together with (1.12) was shown in [KY98, Lemma 2.4]. More precisely, it was shown therein, that the solution operator $T\colon L^p(\mathbb{R}^d) \to \dot{\mathbf{H}}^1_p(\mathbb{R}^d)$, $f \mapsto u$ can be represented as the Fourier multiplier

$$T\phi = \mathcal{F}\frac{-\mathrm{i}\xi_k}{|\xi|^2}\mathcal{F}\phi$$

for any $\phi \in C_c^\infty(\mathbb{R}^d)$. As Fourier multipliers commute with derivatives, we get

$$\nabla(T\phi) = T(\nabla\phi)$$

for any $\phi \in C_c^\infty(\mathbb{R}^d)$. Thus we can make use of the boundedness of T from $L^r(\mathbb{R}^d)$ to $\dot{\mathbf{H}}^1_r(\mathbb{R}^d)$ to get

$$\|\nabla^2 T\phi\|_{L^r(\mathbb{R}^d)} = \|\nabla T(\nabla\phi)\|_{L^r(\mathbb{R}^d)} \leq C\|\nabla\phi\|_{L^r(\mathbb{R}^d)}.$$

By a mollifier argument, there is for any $f \in L^p(\mathbb{R}^d)$ with $\nabla f \in L^r(\mathbb{R}^d)$ a sequence $f_n \in C_c^\infty(\mathbb{R}^d)$ such that $f_n \to f$ in $L^p(\mathbb{R}^d)$ and $\nabla f_n \to \nabla f$ in $L^r(\mathbb{R}^d)$. Thus (1.13) holds for any $f \in L^p(\mathbb{R}^3)$ with $\nabla f \in L^r(\mathbb{R}^3)$. □

In order to handle (1.11) , we can directly make use of [Ama93, Example 9.4, Remark 9.5], which covers equations of the form

$$\begin{aligned} -\Delta u &= f \quad \text{in } D, \\ \partial_n u &= g \quad \text{on } \partial D \end{aligned} \tag{1.14}$$

for bounded domains $D \subset \mathbb{R}^3$.

Lemma 1.4.3. *Let $D \subset \mathbb{R}^3$ be a bounded domain with C^2-boundary and $r \in (1, \infty)$. Then the following assertions hold:*

1. *The problem (1.14) has for each $f \in H_r^{-1}(D)$ and $g \in W^{-1/r,r}(\partial D)$ a unique weak solution $u \in H^1_r(D)$ if and only if*
$$\langle f, 1\rangle_D + \langle g, 1\rangle_{\partial D} = 0. \tag{1.15}$$
Moreover, there is some $C = C(r, D)$ such that
$$\|u\|_{H^1_r(D)} \leq C\big(\|f\|_{H_r^{-1}(D)} + \|g\|_{W^{-1,r}(\partial D)}\big).$$

2. *The problem (1.14) has for each $f \in L^r(D)$ and $g \in W^{1-1/r,r}(\partial D)$ a unique strong solution $u \in H^2_r(D)$ if and only if*
$$\int_D f(x)\,\mathrm{d}x + \int_{\partial D} g(x)\,\mathrm{d}S = 0. \tag{1.16}$$
Moreover, there is some $C = C(r, D)$ such that
$$\|u\|_{H^2_r(D)} \leq C\big(\|f\|_{L^r(D)} + \|g\|_{W^{1-1,r}(\partial D)}\big).$$

Combining these two results with the cut-off method yields the following result on an exterior domain.

Proposition 1.4.4. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^2-boundary, $p \in (1,\infty)$ and $r \in (1,p]$ such that $r \geq 3p/(3+p)$. Then for each $g \in W^{-1/p,p}(\partial\Omega) \cap W^{1-1/p,p}(\partial\Omega)$ with $\langle 1, g\rangle_{\partial\Omega} = 0$ there is a unique solution $u \in \dot{\mathbf{H}}^1_p(\Omega)$ to (1.9) with $\nabla^2 u \in L^r(\Omega)$.*

Proof. Due to Lemma 1.4.1, there is a unique weak solution $u \in \dot{\mathbf{H}}^1_p(\Omega)$ to (1.9), which satisfies $\|\nabla u\|_{L^p(\Omega)} \leq C\|g\|_{W^{-1/p,p}(\partial\Omega)}$. Cutting off this solution yields weak solutions $u_1 \in H^1_p(\Omega \cap B(0,R+1))$ and $u_2 \in \dot{\mathbf{H}}^1_p(\mathbb{R}^3)$ to (1.10) and (1.11) respectively. Note that $2\nabla u \nabla\eta + u\Delta\eta \in L^r(\Omega) \cap L^p(\Omega)$ because of $u \in L^p_{loc}(\overline{\Omega})$, the compact support of $\nabla\phi$ and $\Delta\phi$, as well as $r \leq p$. Furthermore, the existence of the weak solution u_1 to (1.10) implies, that the compatibility condition (1.15) is satisfied for the data g and $f - 2\nabla u\nabla\eta - u\Delta\eta$ given here. Because of their regularity, (1.16) has to be satisfied, too. Thus, Lemma 1.4.3 implies the existence of a unique strong solution $v_1 \in H^2_r(\Omega\cap B(0,R))$ to (1.10). By the Sobolev embedding, it holds $H^2_r(\Omega \cap B(0,R)) \subset H^1_p(\Omega \cap B(0,R))$. As strong solutions are weak solutions, this yields $v_1 = u_1$. Similarly, Lemma 1.4.2 yields the existence of a strong solution $v_2 \in \dot{\mathbf{H}}^1_p(\mathbb{R}^3)$ with $\nabla^2 u \in L^r(\mathbb{R}^3)$ to (1.11). Again, by the uniqueness of the weak solutions in $\dot{\mathbf{H}}^1_p(\mathbb{R}^3)$ to (1.11), it has to hold $v_2 = u_2$. This implies $\nabla^2 u = \nabla^2 u_1 + \nabla^2 u_2 \in L^r(\Omega)$. □

Remark 1.4.5. *In particular, Proposition 1.4.4 covers the cases $p = r$ and $p = r^* = \frac{3r}{3-r}$.*

2 Helmholtz-Hodge Decomposition in Exterior Domains

This chapter is devoted to the Helmholtz-Hodge decomposition in $L^p(\Omega)$ for smooth exterior domains $\Omega \subset \mathbb{R}^3$ and $p \in (1, \infty)$. The first main step of the proof will be a solution theory of a system of Poisson problems with vanishing tangential or vanishing normal boundary conditions. This will be used in order to construct suitable vector potentials needed for the decomposition. Beforehand, we establish a profound understanding of homogeneous Sobolev spaces of vector fields with the relevant boundary conditions in exterior domains. Based on the solution theory for the weak Poisson problem, as well as the theory about the weak Neumann and weak Dirichlet problem, we are then in the position to verify the existence and failure of the decompositions.

2.1 Homogeneous Sobolev Spaces

In this section, we are going to introduce the function spaces, that we will use throughout the whole chapter. These will be homogeneous Sobolev spaces of first order with a partial Dirichlet boundary condition. More precisely, the spaces we consider will consist of vector fields u with either vanishing normal component $u \cdot n = 0$ or vanishing tangential component $u \times n = 0$ at the boundary and will be normed by $\|\nabla \cdot\|_{L^p}$. It will turn out, that these vector fields share in most parts the same properties as functions with homogeneous Dirichlet boundary conditions on bounded and exterior domains, which have been considered in [SS96]. In particular, we will show Poincaré and Sobolev type inequalities, the existence of extension operators to $\mathbb{R}^3$, the density of smooth vector fields, and describe the behaviour at infinity in exterior domains depending the the integration parameter p. As one of our main concerns later on will be the bilinear form $a(u, v) = (\operatorname{div} u, \operatorname{div} v) + (\operatorname{rot} u, \operatorname{rot} v)$, we state some equivalent norms to $\|\nabla \cdot\|_{L^p}$, that include the divergence and the rotation instead of the gradient. Each of the stated properties will become important later on as technical tools.

Assume that $\Omega \subset \mathbb{R}^3$ is either a bounded domain or an exterior domain having C^∞-boundary. Let $1 < p < \infty$. Define the homogeneous Sobolev space of vector fields with vanishing normal component at the boundary by

$$\dot{\mathbf{H}}_p^{1,T}(\Omega) := \{u \in L^p_{loc}(\overline{\Omega}) : \nabla u \in L^p(\Omega); u \cdot n = 0 \text{ on } \partial\Omega\}.$$

Due to the boundary condition, every constant function in $\dot{\mathbf{H}}_p^{1,T}(\Omega)$ has to be zero. Therefore, the term $\|u\|_{\dot{\mathbf{H}}_p^{1,T}(\Omega)} := \|\nabla u\|_{L^p(\Omega)}$ defines a norm on $\dot{\mathbf{H}}_p^{1,T}(\Omega)$. It is clear, that the set of smooth and compactly supported vector fields

$$C_c^{k,T}(\overline{\Omega}) = \{u \in C_c^k(\overline{\Omega}) : u \cdot n = 0\}$$

is contained in $\dot{\mathbf{H}}_p^{1,T}(\Omega)$. The closure of $C_c^{\infty,T}(\overline{\Omega})$ in $\dot{\mathbf{H}}_p^{1,T}(\Omega)$ will be denoted by $\hat{H}_p^{1,T}(\Omega)$.

In the same way, we can consider vector fields with vanishing tangential component, or in different words, vector fields that are parallel to the normal at the boundary. Define

$$\dot{\mathbf{H}}_p^{1,N}(\Omega) := \{u \in L^p_{loc}(\overline{\Omega}) : \nabla u \in L^p(\Omega); u \times n = 0 \text{ on } \partial\Omega\}$$

equipped with the norm $\|\cdot\|_{\dot{\mathbf{H}}_p^{1,N}(\Omega)} := \|\nabla \cdot\|_{L^p(\Omega)}$. This is indeed a norm, as the only constant function in these spaces again has to be zero. Smooth and compactly supported vector fields with vanishing tangential component are denoted by

$$C_c^{k,N}(\overline{\Omega}) = \{u \in C_c^k(\overline{\Omega}) : u \times n = 0\}.$$

The closure of $C_c^{\infty,N}(\overline{\Omega})$ in $\dot{\mathbf{H}}_p^{1,N}(\Omega)$ will be called $\hat{H}_p^{1,N}(\Omega)$.

Several results (and proofs) in this section hold for both of the spaces $\dot{\mathbf{H}}_p^{1,T}(\Omega)$ and $\dot{\mathbf{H}}_p^{1,N}(\Omega)$. In these cases we will shortly write $\dot{\mathbf{H}}_p^{1,B}(\Omega)$. In the same way, we will use the notation $\hat{H}_p^{1,B}(\Omega)$ and $C_c^{k,B}(\overline{\Omega})$. One could also say $B \in \{T, N\}$.

In the case of $\Omega = \mathbb{R}^d$, it is clear that $\|\nabla \cdot\|_{L^p(\mathbb{R}^d)}$ does not define a norm on the set $\{u \in L^p_{loc}(\mathbb{R}^d) : \nabla u \in L^p(\mathbb{R}^d)\}$. Because of this, we consider in this case equivalence classes up to constant functions. In order to be precise, set

$$\dot{\mathbf{H}}_p^1(\mathbb{R}^d) := \{[u] : u \in L^p_{loc}(\mathbb{R}^d), \nabla u \in L^p(\mathbb{R}^d)\},$$

where $[u]$ denotes the equivalence class of u modulo constant functions. These spaces (and variations of them) have already been treated extensively throughout the literature. We refer for example to [SS96] or [DHHR11, II.12.2]. It is known, that these spaces are Banach spaces for $1 \le p \le \infty$ and reflexive for $1 < p < \infty$. An important property for us will be the density of smooth functions with compact support (see [SS96, Lemma 1.1]).

Lemma 2.1.1. *Let $d \in \mathbb{N}$, $d \ge 2$ and $1 \le p < \infty$. Then $C_c^\infty(\mathbb{R}^d)$ is dense in $\dot{\mathbf{H}}_p^1(\mathbb{R}^d)$.*

Basic Properties

We start by collecting some basic properties of the spaces defined above in bounded and exterior domains. The starting point is an abstract Poincaré type inequality, which will be proved by means of the lemma of Peetre-Tartar:

Lemma 2.1.2 ([GR86, Chapter I, Theorem 2.1]). *Let E_1, E_2 and E_3 be Banach spaces. Let $A \in \mathcal{L}(E_1, E_2)$ be bounded and $B \in \mathcal{L}(E_1, E_3)$ be compact. Assume that there are constants $c, C > 0$ such that for each $u \in E_1$ we have*

$$c\|u\|_{E_1} \leq \|Au\|_{E_2} + \|Bu\|_{E_3} \leq C\|u\|_{E_1}.$$

Then the dimension of the kernel of A is finite, the range of A is a closed subspace of E_2 and $A\colon E_1/\ker A \to R(A)$ is an isomorphism.

In case of subspaces of inhomogeneous Sobolev spaces which contain only the zero as constant function, this yields the following result:

Lemma 2.1.3. *Let $1 < p < \infty$ and $D \subset \mathbb{R}^3$ be a bounded domain with C^∞-boundary. Assume that W is a closed subspace of $H^1_p(D)$ such that the kernel of ∇ is trivial. Then there is a constant $C = C(D, p) > 0$ such that*

$$\frac{1}{C}\|\nabla u\|_{L^p(D)} \leq \|u\|_{H^1_p(D)} \leq C\|\nabla u\|_{L^p(D)}$$

for each $u \in W$.

Proof. We note that, the embedding $H^1_p(D) \hookrightarrow L^p(D)$ is compact by the theorem of Rellich-Kondrachov. Hence, we can apply Lemma 2.1.2 with $E_1 = W$, $E_2 = L^p(D)$, $E_3 = L^p(D)$ and $A = \nabla$, $B = \mathrm{id}$, which yields the desired estimate. □

This lemma has applications to homogeneous Sobolev spaces on bounded and unbounded domains. Beforehand, we would like to introduce the abbreviation

$$\Omega_R := \Omega \cap B(0, R),$$

where $R > 0$ might be arbitrary.

Proposition 2.1.4. *Let $1 < p < \infty$.*

1. *If $D \subset \mathbb{R}^3$ is a bounded domain with C^∞-boundary, then the homogeneous norm $\|\nabla \cdot\|_{L^p(D)}$ and the inhomogeneous norm $\|\nabla \cdot\|_{L^p(D)} + \|\cdot\|_{L^p(D)}$ are equivalent on $\dot{\mathbf{H}}^{1,B}_p(D)$.*

2. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary and $R > 0$ be such that $\partial\Omega \subset B(0, R)$. Then there is a constant $C = C(\Omega, R, p) > 0$ such that for each $u \in \dot{\mathbf{H}}^{1,B}_p(\Omega)$ it holds*

$$\|u\|_{L^p(\Omega_R)} \leq C\|\nabla u\|_{L^p(\Omega_R)} \leq C\|\nabla u\|_{L^p(\Omega)}.$$

Moreover, the embedding $\dot{\mathbf{H}}^{1,B}_p(\Omega) \hookrightarrow L^p(\Omega_R)$ is compact.

Proof. We start with the first claim. Define the spaces $H^{1,T}_p(D) = \{u \in H^1_p(D) : u \cdot n = 0 \text{ on } \partial D\}$ and $H^{1,N}_p(D) = \{u \in H^1_p(D) : u \times n = 0 \text{ on } \partial D\}$ and abbreviate them by $H^{1,B}_p(D)$. These spaces are closed subspaces of $H^1_p(D)$ by the continuity of the trace operator. As the kernel of the gradient is trivial in these spaces, we can apply Lemma 2.1.3 and get

$$\frac{1}{C}\|\nabla u\|_{L^p(D)} \leq \|u\|_{H^1_p(D)} \leq C\|\nabla u\|_{L^p(D)}$$

for all $u \in H^{1,B}_p(D)$ with $C > 0$ independent of u. By the definition of $\dot{\mathbf{H}}^{1,B}_p(D)$ and the boundedness of D, we know that $\dot{\mathbf{H}}^{1,B}_p(D) \subseteq H^{1,B}_p(D)$. Hence, the first claim follows.

For the second claim, we introduce the spaces $H^{1,(T)}_p(\Omega_R) := \{u \in H^1_p(\Omega_R) : u \cdot n = 0 \text{ on } \partial\Omega\}$ and $H^{1,(N)}_p(\Omega_R) := \{u \in H^1_p(\Omega_R) : u \times n = 0 \text{ on } \partial\Omega\}$ and abbreviate them by $H^{1,(B)}_p(\Omega_R)$. Similar as for the first claim, the spaces $H^{1,(B)}_p(\Omega_R)$ are closed subspaces of $H^1_p(\Omega_R)$ and the kernel of the gradient is trivial therein. Hence, by Lemma 2.1.3, there is a constant $C = C(\Omega, R, p) > 0$ such that

$$\|u\|_{L^p(\Omega_R)} \leq \|u\|_{H^1_p(\Omega_R)} \leq C\|\nabla u\|_{L^p(\Omega_R)} \tag{2.1}$$

for all $u \in H^{1,(B)}_p(\Omega_R)$. Given a function $u \in \dot{\mathbf{H}}^{1,B}_p(\Omega)$, the restriction of u to Ω_R lies in $H^{1,(B)}_p(\Omega_R)$. Therefore, (2.1) can be applied to the restriction, which is the desired estimate. As we have actually shown the embedding $\dot{\mathbf{H}}^{1,B}_p(\Omega) \hookrightarrow H^1_p(\Omega_R)$, the claim on the compact embedding follows by the theorem of Rellich-Kondrachov. □

It is important, that $R > 0$ cannot be chosen arbitrarily small in the second part of Proposition 2.1.4. Consider for example the case, that $\partial\Omega \cap B(0, R) = (\mathbb{R}^2 \times \{0\}) \cap B(0, R)$ for a suitable exterior domain $\Omega \subset \mathbb{R}^3$ and some $R > 0$. Then one can easily construct a vector field $v \in \dot{\mathbf{H}}^{1,T}_p(\Omega)$ such that $v|_{\Omega\cap B(0,R)} = e_1 = (1, 0, 0)$. That means, the estimate $\|v\|_{L^p(\Omega_R)} \leq C\|\nabla v\|_{L^p(\Omega_R)}$ cannot hold. Of course the weaker estimate $\|v\|_{L^p(\Omega_R)} \leq C\|\nabla v\|_{L^p(\Omega)}$ remains valid.

Having the Poincaré inequality at hand, we are able to show that the spaces $\dot{\mathbf{H}}^{1,B}_p(\Omega)$ are Banach spaces.

Proposition 2.1.5. *Let $1 < p < \infty$ and $\Omega \subset \mathbb{R}^3$ be a bounded or an exterior domain with C^∞-boundary. Then the spaces $\dot{\mathbf{H}}_p^{1,B}(\Omega)$ are complete.*

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\dot{\mathbf{H}}_p^{1,B}(\Omega)$. By Proposition 2.1.4, the restrictions $(u_n|_{\Omega_R})_{n\in\mathbb{N}}$ are Cauchy sequences in $H_p^1(\Omega_R)$ for all $R > 0$ with $\partial\Omega \subset B(0,R)$. Therefore, there are $u^R \in H_p^1(\Omega_R)$ such that $u_n|_{\Omega_R} \to u^R$ for $n \to \infty$ in $H_p^1(\Omega_R)$. Up to a set of measure zero, we moreover have $u^R = u^{R+r}|_{\Omega_R}$ for all $r \geq 0$. Hence, there are $u \in L^p_{loc}(\overline{\Omega})$ and $p \in L^p(\Omega)$ with

$$\begin{aligned} u_n &\to u \text{ in } L^p_{loc}(\overline{\Omega}), \\ \nabla u_n &\to p \text{ in } L^p(\Omega). \end{aligned}$$

Due to the local convergence in $H_p^1(\Omega_R)$ of $(u_n)_{n\in\mathbb{N}}$, it holds $\nabla u = p$. Finally, the continuity of the trace operator implies $u \cdot n = 0$ or $u \times n = 0$ respectively on $\partial\Omega$. Hence, $u \in \dot{\mathbf{H}}_p^{1,B}(\Omega)$. □

Another important property is the density of smooth functions in $\dot{H}_p^{1,B}(\Omega)$.

Corollary 2.1.6. *Let $D \subset \mathbb{R}^3$ be a bounded domain with C^∞-boundary and $1 < p < \infty$. Then $C^{\infty,B}(\overline{D})$ is dense in $\dot{\mathbf{H}}_p^{1,B}(D)$. In particular $\dot{\mathbf{H}}_p^{1,B}(D) = \hat{H}_p^{1,B}(D)$.*

Proof. It follows by Proposition 2.1.4, that the inhomogeneous norm $\|\cdot\|_{H_p^1(D)}$ and the homogeneous norm $\|\nabla\cdot\|_{L^p(D)}$ are equivalent on $\dot{\mathbf{H}}_p^{1,B}(D)$. Thus, Proposition 1.2.6 yields the claim. □

Remark 2.1.7. *We will see later on in Proposition 2.1.14 that in the case of exterior domains $\Omega \subset \mathbb{R}^3$ the set $C_c^{\infty,B}(\Omega)$ is in general not dense in $\dot{\mathbf{H}}_p^{1,B}(\Omega)$ for all $p \in (1,\infty)$.*

We now construct extension and restriction operators for the homogeneous Sobolev spaces. At first, we will consider bounded domains.

Proposition 2.1.8. *Let $D \subset \mathbb{R}^3$ be a bounded domain with C^∞-boundary and $1 < p < \infty$.*

1. *There is a bounded linear operator $E^B \colon \dot{\mathbf{H}}_p^{1,B}(D) \to H_p^1(\mathbb{R}^3)$, that is consistent for all p.*

2. *There is a bounded linear operator $R^B \colon H_p^1(\mathbb{R}^3) \to \dot{\mathbf{H}}_p^{1,B}(D)$, which is a left inverse of the extension operator E^B from above. This operator is consistent with respect to p.*

Proof. 1. Due to Proposition 2.1.4, we know that $\dot{\mathbf{H}}_p^{1,B}(D) \subset H_p^1(D)$. Thus, the restriction of the classical Sobolev extension operator $E\colon H_p^1(D) \to H_p^1(\mathbb{R}^3)$ to $\dot{\mathbf{H}}_p^{1,B}(D)$ has the desired properties.

2. By the smoothness of the boundary ∂D, the trace operator $\gamma\colon H_p^1(\mathbb{R}^3) \to W^{1-1/p,p}(\partial D)$ is bounded and well defined. Of course, it is also consistent with respect to p. Moreover, the exterior normal vector $n\colon \partial D \to \mathbb{R}^3$ lies in $W^{1,\infty}(\partial D)$. Thus, the function $u \mapsto (\gamma u \cdot n)n$ maps from $H_p^1(\mathbb{R}^3)$ to $W^{1-1/p,p}(\partial D)$. It is known, that γ has a bounded linear right inverse $F\colon W^{1-1/p,p}(\partial D) \to H_p^1(\mathbb{R}^3)$, which does not depend on p (see Lemma 1.2.3). Define

$$R^T\colon H_p^1(\mathbb{R}^3) \to H_p^1(D), \quad u \mapsto [u - F((\gamma u \cdot n)n)]|_D.$$

By the considerations above, this operator is linear, bounded and consistent with respect to p. Furthermore, for an arbitrary $u \in H_p^1(\mathbb{R}^3)$, we have

$$(R^T u) \cdot n = u \cdot n - (u \cdot n)n \cdot n = u \cdot n - (u \cdot n) = 0 \quad \text{on } \partial D,$$

which implies $R^T u \in \dot{\mathbf{H}}_p^{1,T}(D)$. Finally, for an arbitrary $u \in \dot{\mathbf{H}}_p^{1,T}(D)$, we get

$$R^T E^T u = [E^T u - F((\gamma E^T u \cdot n)n)]|_D = [E^T u - F(0)]|_D = [E^T u]|_D = u.$$

Therefore, R^T is a left inverse of E^T on $\dot{\mathbf{H}}_p^{1,T}(D)$.

The case of vanishing tangential components works similarly. Using the same operators as above, define

$$\begin{aligned} R^N\colon & H_p^k(\mathbb{R}^3) \to H_p^k(D), \\ & u \mapsto [u - F(\gamma u - (\gamma u \cdot n)n)]|_D = [u - F((1 - n \otimes n)\gamma u)]|_D. \end{aligned}$$

By the same reasons as for R^T, the operator R^N is linear, bounded and consistent with respect to p. Regarding the boundary conditions, we have

$$(R^N u) \times n = u \times n - u \times n + ((u \cdot n)n) \times n = (u \cdot n)(n \times n) = 0$$

on ∂D for each $u \in H_p^1(\mathbb{R}^3)$. Hence, R^N maps from $H_p^1(\mathbb{R}^3)$ into $\dot{\mathbf{H}}_p^{1,N}(D)$. In order to see that R^N is a left inverse of E^N, we note that for each $u \in \dot{\mathbf{H}}_p^{1,N}(D)$, we have $u = (u \cdot n)n$ on the boundary of D. Thus

$$R^N E^N u = [E^N u - F(\gamma E^N u - (\gamma E^N u \cdot n)n)]|_D = [E^N u - F(0)]|_D = u$$

for all $u \in \dot{\mathbf{H}}_p^{1,N}(D)$. $\square$

For exterior domains, an extension operator cannot map to the inhomogeneous Sobolev space $H^1_p(\mathbb{R}^3)$, but we still get an extension that is in $H^1_p(B(0,R))$ for any $R > 0$.

Proposition 2.1.9. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary and $p \in (1,\infty)$. Then there is a linear operator $E^B \colon \dot{\mathbf{H}}^{1,B}_p(\Omega) \to \dot{\mathbf{H}}^1_p(\mathbb{R}^3)$ such that $E^B u|_\Omega = u$, $\|\nabla E^B u\|_{L^p(\mathbb{R}^3)} \leq C(\Omega,p)\|\nabla u\|_{L^p(\Omega)}$, $E^B u \in L^p_{loc}(\mathbb{R}^3)$ and $\|E^B u\|_{H^1_p(B(0,R))} \leq C(\Omega,R,p)\|\nabla u\|_{L^p(\Omega)}$ for each $R > 0$.*

Proof. Let $u \in \dot{\mathbf{H}}^{1,B}_p(\Omega)$ and $R > 0$ be such that $\partial\Omega \subset B(0,R)$. By Proposition 2.1.4, we have $u|_{\Omega_R} \in H^1_p(\Omega_R)$ with $\|u\|_{H^1_p(\Omega_R)} \leq C\|\nabla u\|_{L^p(\Omega)}$. Using the usual extension operator for Sobolev spaces, we can extend $u|_{\Omega_R}$ to some function $E^B u$ defined on $B(0,R)$. This extension can be bounded by $\|E^B u\|_{H^1_p(B(0,R))} \leq C\|u\|_{H^1_p(\Omega_R)} \leq C\|\nabla u\|_{L^p(\Omega_R)}$ due to Proposition 2.1.4. Setting $E^B u = u$ on $B(0,R)^C$ yields the desired extension on $\mathbb{R}^3$. $\square$

Next, we will state a counterpart to the classical Sobolev inequality in exterior domains.

Proposition 2.1.10. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary and $1 < p < 3$.*

1. *There is a constant $C = C(\Omega,p) > 0$ such that for all $f \in \hat{H}^{1,B}_p(\Omega)$ we have*

$$\|f\|_{L^q(\Omega)} \leq C\|\nabla f\|_{L^p(\Omega)},$$

   where $\frac{1}{q} = \frac{1}{p} - \frac{1}{3}$.

2. *For any $u \in \dot{\mathbf{H}}^{1,B}_p(\Omega)$, there is a unique vector $v_u \in \mathbb{R}^3$ such that $u - v_u \in L^q(\Omega)$ and*

$$\|u - v_u\|_{L^q(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)},$$

   where $\frac{1}{q} = \frac{1}{p} - \frac{1}{3}$ and $C = C(\Omega,p) > 0$.

Proof. It was shown in [CWZ94, Theorem 1], that the first estimate above is true for all $f \in W^{1,p}(\Omega)$, where $p \in (1,3)$, and therefore for all $f \in C^{\infty,B}_c(\overline{\Omega})$. By a density argument, we can extend this to all $f \in \hat{H}^{1,B}_p(\Omega)$ with $p \in (1,3)$.

Regarding the second claim, we follow [SS96, Theorem 2.13] and reduce the situation to the whole space $\mathbb{R}^3$. Let $u \in \dot{\mathbf{H}}^{1,B}_p(\Omega)$ be arbitrary and $\tilde{u} = E^B u$ its extension to $\mathbb{R}^3$ in the sense of Proposition 2.1.9. Then, using Lemma 2.1.1, there is a sequence $(u_j)_{j\in\mathbb{N}} \subset C^\infty_c(\mathbb{R}^3)$ such that $\|\nabla\tilde{u} - \nabla u_j\|_{L^p(\mathbb{R}^3)} \to 0$ for

$j \to \infty$. Employing the classical Sobolev embedding, the sequence $(u_j)_{j\in\mathbb{N}}$ is a Cauchy-sequence in $L^q(\mathbb{R}^3)$ with some limit $u^* \in L^q(\mathbb{R}^3)$ that fulfils $\|u^*\|_{L^q(\mathbb{R}^3)} \le C\|\nabla\tilde{u}\|_{L^p(\mathbb{R}^3)}$. We show that $\nabla\tilde{u} = \nabla u^*$. Let $\phi \in C_c^\infty(\mathbb{R}^3)$. Then we have

$$\begin{aligned}\int_{\mathbb{R}^3} (\tilde{u} - u^*)\partial_k\phi &= -\int_{\mathbb{R}^3} (\partial_k\tilde{u})\phi - \lim_{j\to\infty}\int_{\mathbb{R}^3} u_j\partial_k\phi \\ &= -\int_{\mathbb{R}^3} (\partial_k\tilde{u})\phi + \lim_{j\to\infty}\int_{\mathbb{R}^3} (\partial_k u_j)\phi = 0,\end{aligned}$$

as $\partial_k u_j \to \partial_k\tilde{u}$ for $j \to \infty$ in the sense of distributions. Hence, there is a vector $v_u \in \mathbb{R}^3$ such that $\tilde{u} - u^* = v_u$, i.e. $\tilde{u} - v_u = u^* \in L^q(\mathbb{R}^3)$. This implies

$$\|u - v_u\|_{L^q(\Omega)} \le \|\tilde{u} - v_u\|_{L^q(\mathbb{R}^3)} = \|u^*\|_{L^q(\mathbb{R}^3)} \le C\|\nabla\tilde{u}\|_{L^p(\mathbb{R}^3)} \le C\|\nabla u\|_{L^p(\Omega)}.$$

It remains to show the uniqueness of v_u. Assume that there are $v_1, v_2 \in \mathbb{R}^3$ such that $u - v_1, u - v_2 \in L^q(\Omega)$. Then we have $v_2 - v_1 = u - v_1 - (u - v_2) \in L^q(\Omega)$ and consequently $v_1 = v_2$. $\square$

We can also show some kind of inverse statement to Proposition 2.1.10 *1.*, namely $\dot{\mathbf{H}}_p^{1,B}(\Omega) \cap L^q(\Omega) = \hat{H}_p^{1,B}(\Omega)$, if $\frac{1}{q} = \frac{1}{p} - \frac{1}{3}$. This was shown in [SS96, Theorem 2.8] for Dirichlet boundary conditions.

Proposition 2.1.11. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary and $1 < p < 3$. Let $q \in \mathbb{R}$ be such that $\frac{1}{q} = \frac{1}{p} - \frac{1}{3}$. Suppose $u \in \dot{\mathbf{H}}_p^{1,B}(\Omega)$. Then $u \in \hat{H}_p^{1,B}(\Omega)$ if and only if $u \in L^q(\Omega)$.*

As a preparation for the proof, we need the next lemma:

Lemma 2.1.12. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary and $1 < p < \infty$. Then it holds $H_p^1(\Omega) \cap \dot{\mathbf{H}}_p^{1,B}(\Omega) \subset \hat{H}_p^{1,B}(\Omega)$.*

Proof. Let $u \in H_p^{1,B}(\Omega) := H_p^1(\Omega) \cap \dot{\mathbf{H}}_p^{1,B}(\Omega)$ and $\eta \in C_c^\infty(\mathbb{R}^3)$ be a cut-off function such that $0 \le \eta \le 1$, $\eta(x) = 1$ for $|x| < 1$ and $\eta(x) = 0$ for $|x| > 2$. Set $\eta_k(x) := \eta(\frac{1}{k}x)$. Let $R > 0$ be such that $\partial\Omega \subset B(0,R)$ and consider only $k \in \mathbb{N}$ with $k > R$. For each such k, we regard $\eta_k u$ as an element of $\dot{\mathbf{H}}_p^{1,B}(\Omega_{2k+1})$. Because of Corollary 2.1.6, there are $u_k \in C^{\infty,B}(\overline{\Omega_{2k+1}})$ such that $\|\eta_k u - u_k\|_{H_p^1(\Omega_{2k+1})} < 1/k$. As $\eta_k u$ is zero in a neighbourhood of $\partial B(0, 2k+1)$, we may assume that this is the case for u_k, too. Thus, $u_k \in C_c^{\infty,B}(\overline{\Omega})$. It holds

$$\begin{aligned}\|u - u_k\|_{H_p^1(\Omega)} &\le \|u - \eta_k u\|_{H_p^1(\Omega)} + \|\eta_k u - u_k\|_{H_p^1(\Omega)} \\ &\le \|u - \eta_k u\|_{H_p^1(\Omega)} + \|\eta_k u - u_k\|_{H_p^1(\Omega_{2k+1})} \\ &\to 0\end{aligned}$$

for $k \to \infty$. This implies the convergence of u_k to u in $\dot{\mathbf{H}}_p^{1,B}(\Omega)$, which means $u \in \hat{H}_p^{1,B}(\Omega)$. $\square$

Proof of Proposition 2.1.11. The inclusion $\hat{H}^{1,B}_p(\Omega) \subset L^q(\Omega)$ was shown in Proposition 2.1.10. It remains to show the inclusion $\dot{\mathbf{H}}^{1,B}_p(\Omega) \cap L^q(\Omega) \subseteq \hat{H}^{1,B}_p(\Omega)$. Let $u \in \dot{\mathbf{H}}^{1,B}_p(\Omega) \cap L^q(\Omega)$ be arbitrary. Let $\rho \in C^\infty_c(\mathbb{R}^3)$ be such that $0 \le \rho \le 1$, $\rho(x) = 1$ for $|x| \le 1$ and $\rho(x) = 0$ for $|x| \ge 2$. Set $\rho_k(x) := \rho(\frac{1}{k}x)$, where $k \in \mathbb{N}$ fulfils $\partial\Omega \subset B(0,k)$. Then we have $\operatorname{supp}(\nabla\rho_k) \subset B(0,2k) \setminus B(0,k) =: A_k$ and $\|\nabla\rho_k\|_{L^\infty(\mathbb{R}^3)} \le \frac{1}{k}\|\nabla\rho\|_{L^\infty(\mathbb{R}^3)}$. It follows from Proposition 2.1.4, that $\rho_k u \in H^1_p(\Omega)$. We show, that $\rho_k u \to u$ in $\dot{\mathbf{H}}^{1,B}_p(\Omega)$. It holds

$$\|\nabla(u - \rho_k u)\|_{L^p(\Omega)} \le \|\nabla u - \rho_k \nabla u\|_{L^p(\Omega)} + \|u\nabla\rho_k\|_{L^p(\Omega)}.$$

The first summand on the right-hand side converges to zero for $k \to \infty$ by dominated convergence. Regarding the second summand, we remark, that $\frac{1}{p} = \frac{1}{q} + \frac{1}{3}$. Therefore, Hölder's inequality yields

$$\begin{aligned}
\|u\nabla\rho_k\|_{L^p(\Omega)} &= \|u\nabla\rho_k\|_{L^p(A_k)} \le \|u\|_{L^q(A_k)}\|\nabla\rho_k\|_{L^3(A_k)} \\
&\le \|u\|_{L^q(A_k)}\|\nabla\rho_k\|_{L^\infty(\mathbb{R}^3)}|A_k|^{1/3} \\
&\le C\|u\|_{L^q(A_k)}\frac{1}{k}(k^3)^{1/3} \le C\|u\|_{L^q(A_k)}.
\end{aligned}$$

As $\|u\|_{L^q(A_k)} \to 0$ for $k \to \infty$, this means $\rho_k u \to u$ with respect to $\|\nabla \cdot\|_{L^p(\Omega)}$. Applying Lemma 2.1.12 implies therefore $u \in \hat{H}^{1,B}_p(\Omega)$. □

Corollary 2.1.13. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary and $1 < p < 3$. Suppose $u \in \dot{\mathbf{H}}^{1,B}_p(\Omega)$. Then $u \in \hat{H}^{1,B}_p(\Omega)$ if and only if the vector $v_u \in \mathbb{R}^3$ from Proposition 2.1.10 equals zero.*

Proof. If $u \in \hat{H}^{1,B}_p(\Omega)$, we have $u \in L^q(\Omega)$ with $\frac{1}{q} = \frac{1}{p} - \frac{1}{3}$ by Proposition 2.1.10 and therefore $v_u = 0$. On the other hand, if $v_u = 0$, i.e. $u - 0 \in L^q(\Omega)$, then $u \in \hat{H}^{1,B}_p(\Omega)$ by Proposition 2.1.11. □

We can use the last corollary and propositions to characterize the gap between $\dot{\mathbf{H}}^{1,B}_p(\Omega)$ and $\hat{H}^{1,B}_p(\Omega)$ in the case of exterior domains and $1 < p < 3$.

Proposition 2.1.14. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary and $1 < p < 3$. Let $w_1, w_2, w_3 \in \dot{\mathbf{H}}^{1,B}_p(\Omega)$ and $v_1, v_2, v_3 \in \mathbb{R}^3$ be three linearly independent vectors such that $w_k - v_k \in \hat{H}^{1,B}_p(\Omega)$ for $k \in \{1,2,3\}$. Then it holds*

$$\dot{\mathbf{H}}^{1,B}_p(\Omega) = \hat{H}^{1,B}_p(\Omega) \oplus \mathit{span}\{w_1, w_2, w_3\}$$

and the projections to the respective subspaces are continuous.

Remark 2.1.15. *We note that there are always three vector fields w_1, w_2 and w_3 as they are supposed in Proposition 2.1.14. Let $\phi \in C^\infty(\mathbb{R}^3)$ be such that $\phi(x) = 1$ for $|x| \geq R+1$ and $\phi(x) = 0$ for $|x| \leq R$, where $R > 0$ fulfils $\partial\Omega \subset B(0,R)$. Then the functions $w_k := \phi v_k$ satisfy the requirements independently of p.*

Proof of Proposition 2.1.14. Let $u \in \dot{\mathbf{H}}^{1,B}_p(\Omega)$. Then, by Proposition 2.1.10, there is a unique vector $v_u \in \mathbb{R}^3$ such that $u - v_u \in L^q(\Omega)$ with $\frac{1}{q} = \frac{1}{p} - \frac{1}{3}$. Moreover, there are unique $\alpha_k \in \mathbb{R}$ such that $v_u = \sum_{k=1}^3 \alpha_k v_k$. Setting

$$u_0 := u - \sum_{k=1}^{3} \alpha_k w_k,$$

we get $u_0 \in \dot{\mathbf{H}}^{1,B}_p(\Omega) \cap L^q(\Omega)$ and therefore $u_0 \in \hat{H}^{1,B}_p(\Omega)$ by Corollary 2.1.13.

The existence of continuous projections onto $\hat{H}^{1,B}_p(\Omega)$ and $\operatorname{span}\{w_1, w_2, w_3\}$ follow from the finite dimension of the latter space. □

Having a description of the relation between $\dot{\mathbf{H}}^{1,B}_p(\Omega)$ and $\hat{H}^{1,B}_p(\Omega)$ for exterior domains and $1 < p < 3$, we change over to the case $3 \leq p < \infty$. As in the case of bounded domains (compare to Corollary 2.1.6), there is no difference between $\dot{\mathbf{H}}^{1,B}_p(\Omega)$ and $\hat{H}^{1,B}_p(\Omega)$. This is already known for Dirichlet boundary conditions, as it can be found in [SS96, Theorem 2.7]. We will make use of that statement for the situation here.

Proposition 2.1.16. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary and $3 \leq p < \infty$. Then $\dot{\mathbf{H}}^{1,B}_p(\Omega) = \hat{H}^{1,B}_p(\Omega)$.*

Proof. We will apply a cut-off argument. Let $R > 0$ be such that $\partial\Omega \subset B(0,R)$ and let $\eta \in C_c^\infty(\mathbb{R}^3)$ be such that $0 \leq \eta \leq 1$, $\eta = 1$ on $B(0,R)$ and $\eta = 0$ on $B(0,R+1)^C$. Set $A := \overline{B(0,R)}^C$. Define the homogeneous Sobolev space with Dirichlet boundary conditions by

$$\dot{\mathbf{H}}^{1,0}_p(A) := \{u \in L^p_{loc}(\overline{A}) : \nabla u \in L^p(A), u = 0 \text{ on } \partial A\}$$

equipped with the norm $\|u\|_{\dot{\mathbf{H}}^{1,0}_p(A)} := \|\nabla u\|_{L^p(A)}$. Let $u \in \dot{\mathbf{H}}^{1,B}_p(\Omega)$ be arbitrary and define $u_1 := \eta u$, $u_2 := (1-\eta)u$. Note that $u_1 \in \dot{\mathbf{H}}^{1,B}_p(\Omega_{R+1})$ and $u_2 \in \dot{\mathbf{H}}^{1,0}_p(A)$. By Corollary 2.1.6, there is a sequence $(u_1^n)_{n\in\mathbb{N}} \subset C_c^{\infty,B}(\overline{\Omega_{R+1}})$ such that $u_1^n \to u_1$ in $\dot{\mathbf{H}}^{1,B}_p(\Omega_{R+1})$. We may assume that $u_1^n = 0$ on $B(0,R+1)^C$. Using [SS96, Theorem 2.7], there is a sequence $(u_2^n)_{n\in\mathbb{N}} \subset C_c^\infty(A)$ such that

$u_2^n \to u_2$ in $\dot{\mathbf{H}}_p^{1,0}(A)$. Note that $u^n := u_1^n + u_2^n \in C_c^{\infty,B}(\overline{\Omega})$. Furthermore, we have

$$\begin{aligned}\|\nabla u - \nabla u^n\|_{L^p(\Omega)} &\leq \|\nabla u_1 - \nabla u_1^n\|_{L^p(\Omega)} + \|\nabla u_2 - \nabla u_2^n\|_{L^p(\Omega)} \\ &= \|\nabla u_1 - \nabla u_1^n\|_{L^p(\Omega_{R+1})} + \|\nabla u_2 - \nabla u_2^n\|_{L^p(A)} \\ &\to 0\end{aligned}$$

for $n \to \infty$. Therefore $u^n \to u$ in $\dot{\mathbf{H}}_p^{1,B}(\Omega)$, which implies $u \in \hat{H}_p^{1,B}(\Omega)$. □

With the knowledge of the difference between $\dot{\mathbf{H}}_p^{1,B}(\Omega)$ and $\hat{H}_p^{1,B}(\Omega)$, we can construct a set, which is dense in $\dot{\mathbf{H}}_p^{1,B}(\Omega)$ for any $p \in (1,\infty)$.

Corollary 2.1.17. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain or an exterior domain with C^∞-boundary and $1 < p < \infty$. Then there is a set $S \subset \bigcap_{q\in(1,\infty)} \dot{\mathbf{H}}_q^{1,B}(D)$, which is dense in $\dot{\mathbf{H}}_p^{1,B}(D)$. In particular $\dot{\mathbf{H}}_p^{1,B}(\Omega) \cap \dot{\mathbf{H}}_q^{1,B}(\Omega)$ is dense in $\dot{\mathbf{H}}_r^{1,B}(\Omega)$ for any $1 < p, q, r < \infty$.*

Proof. For bounded Ω, this follows from Corollary 2.1.6 with $S = C^{\infty,B}(\overline{\Omega})$. Hence, we only have to consider Ω being an exterior domain. Let w_1, w_2, w_3 be as in Remark 2.1.15. Then the set $S := C_c^{\infty,B}(\overline{\Omega}) + \operatorname{span}\{w_1, w_2, w_3\}$ is dense in $\dot{\mathbf{H}}_p^{1,B}(\Omega)$ for any p. Indeed, we clearly have $S \subset \dot{\mathbf{H}}_p^{1,B}(\Omega)$. For $3 \leq p < \infty$, the set $C_c^{\infty,B}(\overline{\Omega})$ is dense in $\dot{\mathbf{H}}_p^{1,B}(\Omega)$ by Proposition 2.1.16. Therefore, S has to be dense, too. For $1 < p < 3$, we have by definition, that $C_c^{\infty,B}(\overline{\Omega})$ is dense in $\hat{H}_p^{1,B}(\Omega)$. In view of the decomposition $\dot{\mathbf{H}}_p^{1,B}(\Omega) = \hat{H}_p^{1,B}(\Omega) \oplus \operatorname{span}\{w_1, w_2, w_3\}$ from Proposition 2.1.14, this implies the density of S in $\dot{\mathbf{H}}_p^{1,B}(\Omega)$. □

The next property can be seen as a consistency result in homogeneous Sobolev spaces. The proof follows [SS96, Theorem 2.12].

Proposition 2.1.18. *Let $\Omega \subset \mathbb{R}^3$ be a bounded or an exterior domain with C^∞-boundary and $1 < p < \infty$, $1 < q < \infty$. If $u \in \dot{\mathbf{H}}_p^{1,B}(\Omega)$ and $\nabla u \in L^q(\Omega)$, then $u \in \dot{\mathbf{H}}_q^{1,B}(\Omega)$.*

Proof. By the definition of $\dot{\mathbf{H}}_q^{1,B}(\Omega)$, we only have to show $u \in L^q_{loc}(\overline{\Omega})$. We will show by a bootstrap argument, that for any $R > 0$ with $\partial\Omega \subset B(0,R)$, it holds $u \in L^q(\Omega_R)$. Note that $\dot{\mathbf{H}}_p^{1,B}(\Omega) \hookrightarrow H_p^1(\Omega_R) \hookrightarrow H_r^1(\Omega_R)$ for some $r \in (1, \min\{p, 3/2\})$ due to Proposition 2.1.4 and Hölder's inequality. Therefore, we may always assume $u \in H_r^1(\Omega_R)$ for a fixed $r \in (1, 3/2)$. We start with the case $1 < q \leq r^*$, where $r^* := \frac{3r}{3-r}$ denotes the Sobolev exponent of r. By Proposition 2.1.4 and the Sobolev embedding, we get directly

$$u \in H_r^1(\Omega_R) \hookrightarrow L^{r^*}(\Omega_R) \hookrightarrow L^q(\Omega_R).$$

Now suppose $r^* < q \leq r^{**}$. This makes sense, as $3/2 < r^* < 3$. By the interpolation inequality, it holds $\|\nabla u\|_{L^{r^*}(\Omega)} \leq \|\nabla u\|_{L^r(\Omega)}^{1-\alpha} \|\nabla u\|_{L^q(\Omega)}^{\alpha}$ for some suitable $\alpha \in (0,1)$. Hence, the first case with $q = r^*$ yields $u \in H^1_{r^*}(\Omega_R)$. Employing the Sobolev embedding, this now implies

$$u \in H^1_{r^*}(\Omega_R) \hookrightarrow L^{r^{**}}(\Omega_R) \hookrightarrow L^q(\Omega_R).$$

Finally, assume $r^{**} < q < \infty$ and note that $r^{**} > 3$. By the same arguments as in the second second step, we get $u \in H^1_{r^{**}}(\Omega_R)$. Hence,

$$u \in H^1_{r^{**}}(\Omega_R) \hookrightarrow L^\infty(\Omega_R) \hookrightarrow L^q(\Omega_R).$$

Thus, $u \in H^1_q(\Omega_R)$ for any sufficiently large $R > 0$, which implies $u \in L^q_{loc}(\overline{\Omega})$. □

Equivalent Norms I

We head over to describe some equivalent norms on $\dot{\mathbf{H}}^{1,B}_p(\Omega)$, which incorporate the divergence and the rotation instead of the gradient. Due to the existence of non-trivial harmonic vector fields, it is generally not possible to consider norms, that only contain the divergence and the rotation alone, as it was in the case of the gradient. Hence, there will always appear some different additional term.

We begin with the consideration of bounded domains.

Proposition 2.1.19. *Let $D \subset \mathbb{R}^3$ be a bounded domain with C^∞-boundary and $1 < p < \infty$.*

1. *The following norms are equivalent on $\dot{\mathbf{H}}^{1,B}_p(D)$:*

$$\begin{aligned} \|\nabla \cdot\|_{L^p(D)} &\sim \|\nabla \cdot\|_{L^p(D)} + \|\cdot\|_{L^p(D)} \\ &\sim \|\operatorname{div} \cdot\|_{L^p(D)} + \|\operatorname{rot} \cdot\|_{L^p(D)} + \|\cdot\|_{L^p(D)}. \end{aligned}$$

2. *If $u \in L^p(D)$ fulfils $\operatorname{div} u, \operatorname{rot} u \in L^p(D)$ and $u \cdot n = 0$ or $u \times n = 0$ then $u \in \dot{\mathbf{H}}^{1,T}_p(D)$ or $\dot{\mathbf{H}}^{1,N}_p(D)$, respectively.*

Proof. The equivalence of $\|\nabla \cdot\|_{L^p(D)}$ and $\|\nabla \cdot\|_{L^p(D)} + \|\cdot\|_{L^p(D)}$ was shown in Proposition 2.1.4. The estimate $\|\nabla \cdot\|_{L^p(D)} + \|\cdot\|_{L^q(D)} \leq C[\|\operatorname{div} \cdot\|_{L^p(D)} + \|\operatorname{rot} \cdot\|_{L^p(D)} + \|\cdot\|_{L^p(D)}]$ is a consequence of [KY09, Theorem 2.4 (i)]. The converse inequality is clear.

In order to see the second statement, we introduce for the proof the spaces

$$\begin{aligned} \mathcal{H}^{1,T}_p(D) &:= \{u \in L^p(D) : \operatorname{div} u, \operatorname{rot} u \in L^p(D), u \cdot n = 0 \text{ on } \partial D\}, \\ \mathcal{H}^{1,N}_p(D) &:= \{u \in L^p(D) : \operatorname{div} u, \operatorname{rot} u \in L^p(D), u \times n = 0 \text{ on } \partial D\}, \end{aligned}$$

equipped with the norm

$$\|\cdot\|_{\mathcal{H}^1_p(D)} = \|\cdot\|_{L^p(D)} + \|\operatorname{div}\cdot\|_{L^p(D)} + \|\operatorname{rot}\cdot\|_{L^p(D)}.$$

It was shown in [AS11, Lemma 3.2 and Lemma 3.5], that $H^1_p(D) \cap \mathcal{H}^{1,B}_p(D)$ is dense in $\mathcal{H}^{1,B}_p(D)$. We note that the proofs therein do not use any geometric assumption but smoothness of the boundary. Using an approximation argument and the first statement, we get consequently $\mathcal{H}^{1,B}_p(D) \subseteq \dot{\mathbf{H}}^{1,B}_p(D)$. $\square$

In order to show an analogous result for exterior domains, we will apply a cut-off argument and make use of respective results in bounded domains and the whole space. The proof of the next lemma follows [FHZ13, Proposition 2.5].

Lemma 2.1.20. *Let $1 < p < \infty$ and $u \in L^p_{loc}(\mathbb{R}^3)$ such that $\nabla u \in L^p(\mathbb{R}^3)$. Then there is a constant $C > 0$ independent of u such that*

$$\|\nabla u\|_{L^p(\mathbb{R}^3)} \leq C[\|\operatorname{div} u\|_{L^p(\mathbb{R}^3)} + \|\operatorname{rot} u\|_{L^p(\mathbb{R}^3)}]. \tag{2.2}$$

Furthermore, if some $u \in L^p_{loc}(\mathbb{R}^3)$ fulfils $\operatorname{div} u, \operatorname{rot} u \in L^p(\mathbb{R}^3)$, then $\nabla u \in L^p(\mathbb{R}^3)$ and (2.2) holds.

Proof. By Lemma 2.1.1, there is a sequence $(u_n)_{n\in\mathbb{N}} \subset C^\infty_c(\mathbb{R}^3)$ such that $\|\nabla u^n - \nabla u\|_{L^p(\mathbb{R}^3)} \to 0$ for $n \to \infty$. We note that $|\operatorname{rot} u^n(x)| = \frac{1}{2}|\nabla u^n(x) - (\nabla u^n(x))^T|$. Right here, it will be convenient to work with the second expression for notational reasons. We will derive (2.2) with the help of some Laplace equation. It holds

$$\Delta u^n_j = (\nabla \operatorname{div} u^n - \operatorname{rot}\operatorname{rot} u^n)_j = \partial_j \operatorname{div} u^n + \sum_{i=1}^3 \partial_i(\nabla u^n - (\nabla u^n)^T)_{ij}$$

That means

$$u^n_j = -\partial_j(-\Delta)^{-1}\operatorname{div} u^n - \sum_{k=1}^3 \partial_i(-\Delta)^{-1}(\nabla u^n - (\nabla u^n)^T)_{ij}.$$

It follows, that each partial derivative of u^n can be computed by

$$\partial_k u^n_j = -R_j R_k(\operatorname{div} u^n) - \sum_{i=1}^3 R_i R_k(\nabla u^n - (\nabla u^n)^T)_{ij}, \tag{2.3}$$

where $R_l = \mathcal{F}^{-1}\frac{\mathrm{i}\xi_l}{|\xi|}\mathcal{F}$ denotes the Riesz transform, which is known to be bounded on $L^p(\mathbb{R}^3)$. Hence, we get the desired estimate for smooth functions. Taking $n \to \infty$, we get (2.2). Conversely, employing a mollifier argument, each $u \in$

$L^p_{loc}(\mathbb{R}^3)$ with $\operatorname{div} u \in L^p(\mathbb{R}^3)$ and $\operatorname{rot} u \in L^p(\mathbb{R}^3)$ can be approximated by smooth functions $u^n \in C^\infty(\mathbb{R}^3)$ in the sense, that $\operatorname{div} u_n \to \operatorname{div} u$ and $\operatorname{rot} u_n \to \operatorname{rot} u$ in $L^p(\mathbb{R}^3)$. Hence, one can make use of the representation (2.3) to show $\nabla u \in L^p(\mathbb{R}^3)$ with the desired estimate. □

Proposition 2.1.21. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary and $1 < p < \infty$. Furthermore, let $R > 0$ be such that $\partial\Omega \subset B(0, R-1)$. Then the following norms are equivalent on $\dot{\mathbf{H}}^{1,B}_p(\Omega)$:*

$$\begin{aligned}\|\nabla \cdot\|_{L^p(\Omega)} &\sim \|\nabla \cdot\|_{L^p(\Omega)} + \|\cdot\|_{L^p(\Omega_R)} \\ &\sim \|\operatorname{div} \cdot\|_{L^p(\Omega)} + \|\operatorname{rot} \cdot\|_{L^p(\Omega)} + \|\cdot\|_{L^p(\Omega_R)}.\end{aligned}$$

Moreover, if some $u \in L^p_{loc}(D)$ fulfils $\operatorname{div} u, \operatorname{rot} u \in L^p(\Omega)$ and $u \cdot n = 0$ or $u \times n = 0$, then $u \in \dot{\mathbf{H}}^{1,T}_p(\Omega)$ or $u \in \dot{\mathbf{H}}^{1,N}_p(\Omega)$, respectively.

Proof. The first equivalence follows directly from Proposition 2.1.4. The estimate "$\geq$" in the second equivalence is simple. The converse inequality will be shown by a cut-off argument. Let $\eta \in C_c^\infty(\mathbb{R}^3)$ be such that $0 \leq \eta \leq 1$, $\eta = 1$ on $B(0, R-1)$ and $\eta = 0$ on $B(0,R)^C$. Moreover, let $u \in \dot{\mathbf{H}}^{1,B}_p(\Omega)$ and set $u_1 := \eta u$, $u_2 := (1-\eta)u$. We will consider u_1 as a function in $\dot{\mathbf{H}}^{1,B}_p(\Omega_R)$ and u_2 as a function in $\dot{\mathbf{H}}^1_p(\mathbb{R}^3)$. Using Proposition 2.1.19 and Lemma 2.1.20, we get

$$\begin{aligned}&\|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega_R)} \\ &\leq C\|\nabla u\|_{L^p(\Omega)} \\ &\leq C\|\nabla u_1\|_{L^p(\Omega)} + \|\nabla u_2\|_{L^p(\Omega)} \\ &\leq C\|\nabla u_1\|_{L^p(\Omega_R)} + \|\nabla u_2\|_{L^p(\mathbb{R}^3)} \\ &\leq C[\|\operatorname{div} u_1\|_{L^p(\Omega_R)} + \|\operatorname{rot} u_1\|_{L^p(\Omega_R)} + \|u_1\|_{L^p(\Omega_R)} \\ &\qquad + \|\operatorname{rot} u_2\|_{L^p(\mathbb{R}^3)} + \|\operatorname{div} u_2\|_{L^p(\mathbb{R}^3)}].\end{aligned}$$

Using $\operatorname{supp} \nabla\eta \in A := B(0,R) \setminus B(0,R-1)$ yields

$$\begin{aligned}&\leq C[\|\nabla\eta\|_{L^\infty(A)}\|u\|_{L^p(\Omega_R)} + \|\eta\|_{L^\infty(\Omega)}\|\operatorname{div} u\|_{L^p(\Omega)} \\ &\quad + \|\nabla\eta\|_{L^\infty(A)}\|u\|_{L^p(\Omega_R)} + \|\eta\|_{L^\infty(\Omega)}\|\operatorname{rot} u\|_{L^p(\Omega)} \\ &\quad + \|\eta\|_{L^\infty(\Omega)}\|u\|_{L^p(\Omega_R)} \\ &\quad + \|\nabla(1-\eta)\|_{L^\infty(A)}\|u\|_{L^p(\Omega_R)} + \|(1-\eta)\|_{L^\infty(\Omega)}\|\operatorname{div} u\|_{L^p(\Omega)} \\ &\quad + \|\nabla(1-\eta)\|_{L^\infty(A)}\|u\|_{L^p(\Omega_R)} + \|(1-\eta)\|_{L^\infty(\Omega)}\|\operatorname{rot} u\|_{L^p(\Omega)}] \\ &\leq C[\|\operatorname{div} u\|_{L^p(\Omega)} + \|\operatorname{rot} u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega_R)}].\end{aligned}$$

The claim, that $u \in L^p_{loc}(\overline{\Omega})$, $\operatorname{div} u, \operatorname{rot} u \in L^p(\Omega)$ and $u \cdot n = 0$ or $u \times n = 0$ imply $u \in \dot{\mathbf{H}}^{1,T}_p(\Omega)$ or $u \in \dot{\mathbf{H}}^{1,N}_p(\Omega)$, respectively, can be shown by an analogous cut-off procedure together with the respective parts of Proposition 2.1.19 and Lemma 2.1.20. □

Complex Interpolation

For Sobolev spaces on domains $\Omega \subset \mathbb{R}^d$ with smooth and compact boundary, it is known, that their complex interpolation spaces are again Sobolev spaces. That means for $1 < p < q < \infty$ and $\theta \in (0,1)$, we have

$$[H_p^1(\Omega), H_q^1(\Omega)]_\theta = H_r^1(\Omega), \quad \text{where } \frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}. \tag{2.4}$$

We will show, that this remains true for the spaces $\dot{\mathbf{H}}_p^{1,B}(D)$ defined on bounded domains $D \subset \mathbb{R}^3$ with smooth boundary. We will do so by employing the method of retractions and coretractions.

Proposition 2.1.22. *Let $D \subset \mathbb{R}^3$ be a bounded domain with C^∞-boundary and $p, q \in (1,\infty)$, $\theta \in (0,1)$. Then it holds*

$$[\dot{\mathbf{H}}_p^{1,B}(D), \dot{\mathbf{H}}_q^{1,B}(D)]_\theta = \dot{\mathbf{H}}_r^{1,B}(D), \quad \textit{where } \frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}.$$

Proof. Because of $\dot{\mathbf{H}}_s^{1,B}(D) \subset H_s^1(D)$ for any $s \in (1,\infty)$, the continuity of the trace operators $u \mapsto u \cdot n$ and $u \mapsto u \times n$, as well as (2.4) we get

$$[\dot{\mathbf{H}}_p^{1,B}(D), \dot{\mathbf{H}}_q^{1,B}(D)]_\theta \subseteq \dot{\mathbf{H}}_r^{1,B}(D). \tag{2.5}$$

In order to see the converse inclusion, we will make use of the extension operators $E_s^B = E^B \colon \dot{\mathbf{H}}_s^{1,B}(\Omega) \to H_s^1(\mathbb{R}^3)$ and restriction operators $R_s^B = R^B \colon H_s^1(\mathbb{R}^3) \to \dot{\mathbf{H}}_s^{1,B}(D)$, which are given in Proposition 2.1.8. For the interpolated operators, we will similarly use the notation $E_\theta^B \colon [\dot{\mathbf{H}}_p^{1,B}(D), \dot{\mathbf{H}}_q^{1,B}(D)]_\theta \to [H_p^1(\mathbb{R}^3), H_q^1(\mathbb{R}^3)]_\theta$ and $R_\theta^B \colon [H_p^1(\mathbb{R}^3), H_q^1(\mathbb{R}^3)]_\theta \to [\dot{\mathbf{H}}_p^{1,B}(D), \dot{\mathbf{H}}_q^{1,B}(D)]_\theta$. By construction, the operators R_p^B, R_q^B and R_r^B as well as E_p^B, E_q^B and E_r^B are consistent. Thus R_θ^B and E_θ^B can be lined up there, too. As $H_r^1(\mathbb{R}^3) = [H_p^1(\mathbb{R}^3), H_q^1(\mathbb{R}^3)]_\theta$, the operators R_r^B and R_θ^B have to coincide algebraically. Because of the surjectivity of R_r^B, this means $\dot{\mathbf{H}}_r^{1,B}(D) \subseteq [\dot{\mathbf{H}}_p^{1,B}(D), \dot{\mathbf{H}}_q^{1,B}(D)]_\theta$ algebraically. Together with (2.5), that implies $\dot{\mathbf{H}}_r^{1,B}(D) = [\dot{\mathbf{H}}_p^{1,B}(D), \dot{\mathbf{H}}_q^{1,B}(D)]_\theta$ algebraically. Employing this equality, we can see, that E_r^B and E_θ^B do coincide algebraically, too. We recall, that each of these operators are continuous with respect to the corresponding topologies. That means the operators

$$\begin{aligned} I_1 &: [\dot{\mathbf{H}}_p^{1,B}(D), \dot{\mathbf{H}}_q^{1,B}(D)]_\theta \to \dot{\mathbf{H}}_r^{1,B}(D), \quad I_1 = R_r^B \circ E_\theta^B, \\ I_2 &: \dot{\mathbf{H}}_r^{1,B}(D) \to [\dot{\mathbf{H}}_p^{1,B}(D), \dot{\mathbf{H}}_q^{1,B}(D)]_\theta, \quad I_2 = R_\theta^B \circ E_r^B, \end{aligned}$$

are both continuous and algebraically identities. Thus $[\dot{\mathbf{H}}_p^{1,B}(D), \dot{\mathbf{H}}_q^{1,B}(D)]_\theta$ and $\dot{\mathbf{H}}_r^{1,B}(D)$ are also topologically equivalent. □

In the case of bounded domains D, it was possible to reduce the interpolation problem to the inhomogeneous Sobolev spaces $H^1_p(\mathbb{R}^3)$. For exterior domains Ω, this is not possible any more, as inhomogeneous and homogeneous Sobolev spaces with the respective boundary conditions are distinct in this case. This makes it more difficult to define the retraction R^B. In the space $H^1_p(\mathbb{R}^3)$, the trace operator $u \mapsto u|_{\partial D}$ is classically well defined. On $\dot{\mathbf{H}}^1_p(\mathbb{R}^3)$, this is not any more the case, as that space does consist of equivalence classes up to constant functions. We are going to circumvent this problem, by restricting ourselves to certain representatives of these classes, which have a vanishing mean value on a suitable ball. Adjusting the restrictions and extensions to these spaces, we can transfer the method for bounded domains above to exterior domains. Define for $p \in (1, \infty)$ and $R > 0$ the space

$$\dot{\mathbf{H}}^{1,m}_p(\mathbb{R}^3) := \dot{\mathbf{H}}^{1,m,R}_p(\mathbb{R}^3) := \{u \in L^p_{loc}(\mathbb{R}^3) : \nabla u \in L^p(\mathbb{R}^3), \int_{B(0,R)} u(x)\,\mathrm{d}x = 0\}$$

equipped with the norm $\|\nabla \cdot\|_{L^p(\mathbb{R}^3)}$. It is easy to see, that the map $u \mapsto [u]$ from $\dot{\mathbf{H}}^{1,m}_p(\mathbb{R}^3)$ to $\dot{\mathbf{H}}^1_p(\mathbb{R}^3)$ is an isometry and bijective. Thus, for any $p, q \in (1, \infty)$ and $\theta \in (0, 1)$, it follows from

$$[\dot{\mathbf{H}}^1_p(\mathbb{R}^d), \dot{\mathbf{H}}^1_q(\mathbb{R}^d)]_\theta = \dot{\mathbf{H}}^1_r(\mathbb{R}^d), \quad \text{where } \frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}, \tag{2.6}$$

which was shown in [KS92, Lemma 2.3], that

$$[\dot{\mathbf{H}}^{1,m}_p(\mathbb{R}^d), \dot{\mathbf{H}}^{1,m}_q(\mathbb{R}^d)]_\theta = \dot{\mathbf{H}}^{1,m}_r(\mathbb{R}^d), \quad \text{where } \frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}. \tag{2.7}$$

With these spaces at hand, we can now construct suitable retractions and coretractions for $\dot{\mathbf{H}}^{1,B}_p(\Omega)$ in exterior domains.

Lemma 2.1.23. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary and $1 < p < \infty$. Suppose that $R > 0$ fulfils $\partial\Omega \subset B(0, R)$. Then there are continuous and bounded linear operators*

$$\tilde{E}^B : \dot{\mathbf{H}}^{1,B}_p(\Omega) \to \dot{\mathbf{H}}^{1,m,R}_p(\mathbb{R}^3),$$
$$R^B : \dot{\mathbf{H}}^{1,m,R}_p(\mathbb{R}^3) \to \dot{\mathbf{H}}^{1,B}_p(\Omega),$$

such that $R^B\tilde{E}^B$ equals the identity on $\dot{\mathbf{H}}^{1,B}_p(\Omega)$. These operators are consistent with respect to p.

Proof. We start by constructing the restrictions R^T and R^N. Recall, that by Lemma 1.2.3 there is a bounded linear operator $F : W^{1-1/p,p}(\partial\Omega) \to H^1_p(\Omega)$, which is a right inverse for the trace operator $\gamma\colon H^1_p(\Omega) \to W^{1-1/p,p}(\partial\Omega)$. By the definition of $\dot{\mathbf{H}}^{1,m}_p(\Omega)$ and Poincaré's inequality, we have

$$\|\gamma u\|_{W^{1-1/p,p}(\partial\Omega)} \leq C\|u\|_{H^1_p(B(0,R))} \leq C\|\nabla u\|_{L^p(B(0,R))} \leq C\|\nabla u\|_{L^p(\mathbb{R}^3)}.$$

As additionally the map $n \mapsto n(x)$ is an element of $W^{1,\infty}(\partial\Omega)$, the operators

$$\begin{aligned} R^T &\colon \dot{\mathbf{H}}^{1,m,R}_p(\mathbb{R}^3) \to \dot{\mathbf{H}}^{1,T}_p(\Omega), \quad u \mapsto [u - F((\gamma u \cdot n)n)]|_\Omega, \\ R^N &\colon \dot{\mathbf{H}}^{1,m,R}_p(\mathbb{R}^3) \to \dot{\mathbf{H}}^{1,N}_p(\Omega), \quad u \mapsto [u - F(\gamma u - (\gamma u \cdot n)n)]|_\Omega \end{aligned}$$

are well defined. Indeed, the boundary conditions of $R^B u$ can be checked in the same way as in the proof of Proposition 2.1.8.

In order to construct the extension operator $\tilde{E}^B$, we need to make sure, that the mean value in $B(0,R)$ of the extended function vanishes. We will accomplish this with the help of a corrector function. Let $G \subset \Omega^C$ be open and nonempty. Fix a function $\eta \in C^\infty_c(\mathbb{R}^3)$, which is supported in G and which fulfils $\int_{B(0,R)} \eta(x)\,\mathrm{d}x = 1$. Furthermore, let $E^B\colon \dot{\mathbf{H}}^{1,B}_p(\Omega) \to \dot{\mathbf{H}}^1_p(\mathbb{R}^3)$ be the extension operator given by Proposition 2.1.9. Define

$$\tilde{E}^B\colon \dot{\mathbf{H}}^{1,B}_p(\Omega) \to \dot{\mathbf{H}}^{1,m,R}_p(\mathbb{R}^3), \quad u \mapsto E^B u - \eta\frac{1}{|B(0,R)|}\int_{B(0,R)} E^B u(x)\,\mathrm{d}x.$$

Because of the properties of E^B, we have

$$\begin{aligned} \|\nabla\tilde{E}^B u\|_{L^p(\mathbb{R}^3)} &\leq \|\nabla E^B u\|_{L^p(\mathbb{R}^3)} + \|\nabla\eta\frac{1}{|B(0,R)|}\int_{B(0,R)} E^B u(x)\,\mathrm{d}x\|_{L^p(\mathbb{R}^3)} \\ &\leq C\|\nabla u\|_{L^p(\Omega)} + \|\nabla\eta\frac{1}{|B(0,R)|}\int_{B(0,R)} E^B u(x)\,\mathrm{d}x\|_{L^p(G)} \\ &\leq C\|\nabla u\|_{L^p(\Omega)} + \|\nabla\eta\|_{L^p(G)}\|1\|_{L^{p'}(B(0,R))}\|E^B u\|_{L^p(B(0,R))} \\ &\leq C\|\nabla u\|_{L^p(\Omega)}. \end{aligned}$$

Thus, the operator is indeed well defined and bounded. Note that $\gamma(\tilde{E}^B u) = \gamma(u)$, as η is not supported in a neighbourhood of $\partial\Omega$. By the same computations as in the proof of Proposition 2.1.8, one can show, that $R^B\tilde{E}^B u = u$ for all $u \in \dot{\mathbf{H}}^{1,B}_p(\Omega)$. □

Exchanging the retractions and coretractions used in the proof of Proposition 2.1.22 by the ones given in Lemma 2.1.23 and making use of (2.7) instead of (2.4), we can conclude the following:

Proposition 2.1.24. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary and $1 < p, q < \infty$. Then we have for each $0 < \theta < 1$, that*

$$[\dot{\mathbf{H}}^{1,B}_p(\Omega), \dot{\mathbf{H}}^{1,B}_q(\Omega)]_\theta = \dot{\mathbf{H}}^{1,B}_r(\Omega), \qquad \textit{where } \frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}.$$

2.2 A Weak Poisson Problem

This section is about problems of the kind

$$(2.8) \quad a(u,\phi) := (\operatorname{div} u, \operatorname{div}\phi) + (\operatorname{rot} u, \operatorname{rot}\phi) = \langle f, \phi\rangle \quad \text{for all } \phi \in \dot{\mathbf{H}}^{1,B}_{p'}(\Omega),$$

where $f \in (\dot{\mathbf{H}}^{1,B}_{p'}(\Omega))'$ and a solution u is supposed to be found in $\dot{\mathbf{H}}^{1,B}_{p}(\Omega)$. The main difficulty of these problems is the existence of non-trivial harmonic vector fields. Here, a vector field h is called harmonic, if $\operatorname{div} h = 0$ and $\operatorname{rot} h = 0$. These vector fields cause a lack of uniqueness of solutions for any right-hand side, as well as a lack of existence of solutions for right-hand sides, that do not vanish on harmonic vector fields. For this reason, we will at first characterize the subspace of harmonic vector fields in homogeneous Sobolev spaces and then solve problem (2.8). It will turn out, that the subset of harmonic vector fields will always be of finite dimension in our setting and does not depend on p. The complement of that space will be a suitable environment to solve (2.8).

In the literature, harmonic vector fields h that fulfil $h \times n = 0$ are frequently called Dirichlet fields. Similarly, harmonic vector h fields that satisfy $h \cdot n = 0$ are called Neumann fields. We will occasionally adopt this notion.

2.2.1 Harmonic Vector Fields in Bounded Domains

We collect some results on harmonic vector fields on bounded domains D, which are defined as

$$\dot{\mathbf{H}}^{1,B,har}_{p}(D) := \{h \in \dot{\mathbf{H}}^{1,B}_{p}(D) : \operatorname{div} h = 0, \operatorname{rot} h = 0\}.$$

We will use these properties later on to gain another equivalent norm on $\dot{\mathbf{H}}^{1,B}_{p}(D)$.

Proposition 2.2.1. *Let $D \subset \mathbb{R}^3$ be a bounded domain with C^∞-boundary and $p \in (1,\infty)$. Then $\dot{\mathbf{H}}^{1,B,har}_{p}(D)$ is finite dimensional and independent of p. Furthermore, there is a direct complement $\tilde{\mathbf{H}}^{1,B}_{p}(D)$ of $\dot{\mathbf{H}}^{1,B,har}_{p}(D)$, i.e.*

$$(2.9) \qquad \dot{\mathbf{H}}^{1,B}_{p}(D) = \tilde{\mathbf{H}}^{1,B}_{p}(D) \oplus \dot{\mathbf{H}}^{1,B,har}_{p}(D).$$

Remark 2.2.2. *Because of the independence of $\dot{\mathbf{H}}^{1,B,har}_{p}(D)$, we will usually drop the parameter p and just write $\dot{\mathbf{H}}^{1,B,har}(D)$. Furthermore, we will frequently make use of orthogonal bases of $\dot{\mathbf{H}}^{1,B,har}(D)$. By this, we mean an orthogonal basis of $\dot{\mathbf{H}}^{1,B,har}_{2}(D)$ with respect to the scalar product $(\nabla\cdot, \nabla\cdot)$.*

Proof of Proposition 2.2.1. The characterisations of the spaces $\dot{\mathbf{H}}_p^{1,T,har}(D)$ and $\dot{\mathbf{H}}_p^{1,N,har}(D)$ have been shown in [KY09, Theorem 2.1]. As finite dimensional subspaces are always complemented, the direct decompositions are an immediate consequence of this. □

Due to the invariance of $\dot{\mathbf{H}}_p^{1,B,har}(D)$ with respect to p, we can define a consistent projection onto the harmonic vector fields.

Definition 2.2.3. *Let $D \subset \mathbb{R}^3$ be a bounded domain with C^∞-boundary and let $(\phi_j)_{j=1}^N$ be an orthonormal basis of $\dot{H}^{1,B,har}(D)$. Define the projection P from $\dot{\mathbf{H}}_2^{1,B}(D)$ onto $\dot{\mathbf{H}}_2^{1,B,har}(D)$ by*

$$Pu = \sum_{n=1}^{N} (\nabla u, \nabla \phi_j)\phi_j.$$

Proposition 2.2.4. *Let $D \subset \mathbb{R}^3$ be a bounded domain with C^∞-boundary. Let $p \in (1, \infty)$. Then the projection P from Definition 2.2.3 can be extended to a continuous projection from $\dot{\mathbf{H}}_p^{1,B}(D)$ onto $\dot{\mathbf{H}}_p^{1,B,har}(D)$.*

Proof. Using the invariance of $\dot{\mathbf{H}}^{1,B,har}(D)$ with respect to p and Hölder's inequality, we get

$$\|\nabla Pu\|_{L^p(D)} \leq \sum_{j=1}^{N} \|\nabla u\|_{L^p(D)} \|\nabla \phi_j\|_{L^{p'}(D)} \|\nabla \phi_j\|_{L^p(D)} \leq C\|\nabla u\|_{L^p(D)}$$

for any $u \in \dot{H}_p^{1,B}(D)$. Hence, P is continuous on that space. The projection property easily carries over, too. □

A convenient consequence of the proposition above is that smooth functions with compact support are a subset of $\tilde{\mathbf{H}}_p^{1,B}(D)$.

Corollary 2.2.5. *Let $D \subset \mathbb{R}^3$ be a bounded domain with C^∞-boundary and $1 < p < \infty$. Then $C_c^\infty(D) \subset \tilde{\mathbf{H}}_p^{1,B}(D)$.*

Proof. Let $\psi \in C_c^\infty(D)$. Using the projection P onto $\dot{\mathbf{H}}^{1,B,har}(D)$ given in Proposition 2.2.4, we get using integration by parts that

$$P\psi = \sum_{n=1}^{N} (\nabla \psi, \nabla \phi_j)\phi_j = -\sum_{n=1}^{N} (\psi, \Delta \phi_j)\phi_j = 0.$$

Hence, $(1 - P)\psi = \psi$, which implies $\psi \in \tilde{\mathbf{H}}_p^{1,B}(D)$. □

Proposition 2.2.6. *Let $D \subset \mathbb{R}^3$ be a bounded domain with C^∞-boundary and $p \in (1, \infty)$. Then the following expressions are equivalent norms on $\dot{\mathbf{H}}_p^{1,B}(D)$:*

$$\|\nabla \cdot \|_{L^p(D)} \sim \| \operatorname{div} \cdot \|_{L^p(D)} + \| \operatorname{rot} \cdot \|_{L^p(D)} + \sum_{j=1}^{N} |(\nabla \cdot, \nabla \phi_j)|, \tag{2.10}$$

where $(\phi_k)_{k=1}^N$ is an orthonormal basis of $\dot{\mathbf{H}}^{1,B,har}(D)$.

Proof. We make use of the decomposition of Proposition 2.2.1 and Lemma 2.1.2. We know from Proposition 2.1.19, that

$$\|\nabla \cdot \|_{L^p(D)} \sim \| \operatorname{div} \cdot \|_{L^p(D)} + \| \operatorname{rot} \cdot \|_{L^p(D)} + \| \cdot \|_{L^p(D)}$$

on $\dot{\mathbf{H}}_p^{1,B}(D)$. Let $E_1 = \tilde{\mathbf{H}}_p^{1,B}(D)$, $E_2 = L^p(D) \times L^p(D)$ and $E_3 = L^p(D)$. Additionally, set $A\colon E_1 \to E_2, u \mapsto (\operatorname{div} u, \operatorname{rot} u)$ and $B\colon E_1 \to E_3, u \mapsto u$. Then Lemma 2.1.2 implies that $A\colon E_1/\ker A \to R(A)$ is an isomorphism. Due to $\ker A = \dot{\mathbf{H}}^{1,B,har}(D)$ and the direct decomposition (2.9), this means

$$\|\nabla \cdot \|_{L^p(D)} \sim \| \operatorname{div} \cdot \|_{L^p(D)} + \| \operatorname{rot} \cdot \|_{L^p(D)}$$

on $\tilde{\mathbf{H}}_p^{1,B}(D)$. Concerning $\dot{\mathbf{H}}^{1,B,har}(D)$, it is easy to see, that $\sum_{j=1}^N |(\nabla \cdot, \nabla \phi_j)|$ is a norm for it. As a result of the finite dimension of $\dot{\mathbf{H}}^{1,B,har}(D)$, this norm is equivalent to $\|\nabla \cdot \|_{L^p(D)}$. Given an arbitrary $u \in \dot{\mathbf{H}}_p^{1,B}(D)$, there are unique $\tilde{u} \in \tilde{\mathbf{H}}_p^{1,B}(D)$ and $u^h \in \dot{\mathbf{H}}^{1,B,har}(D)$ such that $u = \tilde{u} + u^h$. Estimating $\tilde{u}$ and u^h with the respective norms above yields the desired estimates. □

2.2.2 The Weak Poisson Problem in Bounded Domains

Throughout this subsection, D is a bounded domain in $\mathbb{R}^3$ with C^∞-boundary. We want to recall at this place, that the homogeneous norm $\|\nabla \cdot \|_{L^p(D)}$ and the inhomogeneous norm $\|\nabla \cdot \|_{L^p(D)} + \| \cdot \|_{L^p(D)}$ are equivalent on $\dot{\mathbf{H}}_p^{1,B}(D)$ by Proposition 2.1.4. We will show the following main result:

Lemma 2.2.7. *Let $1 < p < \infty$, $D \subset \mathbb{R}^3$ be a domain with C^∞-boundary.*

1. *For each $f \in (\tilde{\mathbf{H}}_{p'}^{1,B}(D))'$, there is a unique $u \in \tilde{\mathbf{H}}_p^{1,B}(D)$ that solves*

$$a(u, \phi) := (\operatorname{div} u, \operatorname{div} \phi) + (\operatorname{rot} u, \operatorname{rot} \phi) = \langle f, \phi \rangle \tag{2.11}$$

for all $\phi \in \tilde{\mathbf{H}}_{p'}^{1,B}(D)$. This solution is subject to the estimate

$$\|\nabla u\|_{L^p(D)} \le C \|f\|_{(\tilde{\mathbf{H}}_{p'}^{1,B}(D))'}, \tag{2.12}$$

where $C > 0$ is independent of f.

2. *If $v \in \dot{\mathbf{H}}^{1,B}_p(D)$ is a solution to (2.11), then it is subject to the estimates*

$$\|\nabla v\|_{L^p(D)} \leq C\big[\|f\|_{(\tilde{\mathbf{H}}^{1,B}_{p'}(D))'} + \sum_{j=1}^{N} |(\nabla v, \nabla \phi_j)|\big], \tag{2.13}$$

and

$$\|\nabla v\|_{L^p(D)} \leq C\big[\|f\|_{(\tilde{\mathbf{H}}^{1,B}_{p'}(D))'} + \|v\|_{L^q(D)}\big], \tag{2.14}$$

where $C > 0$ is independent of f and $(\phi_j)_{j=1}^N$ is an orthonormal basis of $\dot{\mathbf{H}}^{1,B,har}(D)$.

In order to show Lemma 2.2.7, we will make use of the generalised version of the Lax-Milgram Lemma given by Kozono and Yanagisawa [KY13]. Beforehand, we have to show some variational inequality involving the bilinear form $a(\cdot,\cdot)$ on $\dot{\mathbf{H}}^{1,B}_p(D) \times \dot{\mathbf{H}}^{1,B}_{p'}(D)$. Hereby, we will rely on results and techniques from [KY09]. Our main tool is their following statement from [KY09, Lemma 3.1].

Lemma 2.2.8. *Let $1 < p < \infty$. Then there is a constant $C = C(p, D) > 0$ such that*

$$\|\nabla u\|_{L^p(D)} + \|u\|_{L^p(D)} \leq C \sup_{\phi \in C^{\infty,B}(\overline{D})\setminus\{0\}} \frac{|(\nabla u, \nabla \phi)|}{\|\phi\|_{H^1_{p'}(D)}} \tag{2.15}$$

for all $u \in \dot{\mathbf{H}}^{1,B}_p(D)$. Moreover, if $u \in \dot{\mathbf{H}}^{1,B}_q(D)$ for some $1 < q < \infty$ fulfils (2.15), then $u \in \dot{\mathbf{H}}^{1,B}_p(D)$.

At first, we will show the uniqueness of solutions to (2.11) in $\tilde{\mathbf{H}}^{1,B}_p(D)$.

Lemma 2.2.9. *Let $1 < p < \infty$ and $u \in \dot{\mathbf{H}}^{1,B}_p(D)$ such that*

$$a(u, \phi) = 0 \tag{2.16}$$

for all $\phi \in C^{\infty,B}(\overline{D})$. Then $u \in \dot{\mathbf{H}}^{1,B,har}(D)$.

Proof. We start by showing that $u \in \dot{\mathbf{H}}^{1,B}_2(D)$. For the moment, we have to distinguish slightly between $\dot{\mathbf{H}}^{1,T}_p(D)$ and $\dot{\mathbf{H}}^{1,N}_p(D)$. At first, we will consider the case of vector fields with vanishing normal component. Because D is bounded, we have $\dot{\mathbf{H}}^{1,T}_p \hookrightarrow \dot{\mathbf{H}}^{1,T}_2(D)$ for all $p \geq 2$. It is therefore sufficient to consider $1 < p < 2$. Let $u \in \dot{\mathbf{H}}^{1,T}_p(D)$ for $1 < p < 2$. We start with equality (3.64) from [KY09], which reads as

$$(\nabla u, \nabla \phi) = a(u, \phi) - \int_{\partial D} u \cdot (\phi \cdot \nabla n + \phi \times \operatorname{rot} n)\, \mathrm{d}S \tag{2.17}$$

for all $\phi \in C^{\infty,T}(\overline{D})$. Here, ∇n and $\operatorname{rot} n$ have to be understand as the gradient and rotation of a smooth extension of n to a neighbourhood of $\partial\Omega$. Due to our assumption on u, we have here

$$(\nabla u, \nabla\phi) = -\int_{\partial D} u \cdot (\phi \cdot \nabla n + \phi \times \operatorname{rot} n) \, \mathrm{d}S. \tag{2.18}$$

We will estimate the right-hand side by means of Sobolev embeddings and the trace theorem. Set $r = p^* = 3p/(3-p)$ and choose $1 < q < \infty$ such that $\frac{1}{q} = \frac{1}{p} - (1 - \frac{1}{p})/2$. We then have

$$W^{1-1/p,p}(\partial D) \hookrightarrow L^q(\partial D).$$

Moreover, it holds $\frac{1}{q'} = \frac{1}{r'} - (1 - \frac{1}{r'})/2$, which means

$$W^{1-1/r',r'}(\partial D) \hookrightarrow L^{q'}(\partial D).$$

Hence, we can estimate (2.18) by

$$\begin{aligned} |(\nabla u, \nabla\phi)| &\le \|u\|_{L^q(\partial D)} \|\phi\|_{L^{q'}(\partial D)} \\ &\le C\|u\|_{W^{1-1/p,p}(\partial D)} \|\phi\|_{W^{1-1/r',r'}(\partial D)} \\ &\le C\|u\|_{H^1_p(D)} \|\phi\|_{H^1_{r'}(D)}. \end{aligned}$$

Furthermore, we have by $p' = (r')^*$, that

$$|(u, \phi)| \le \|u\|_{L^p(D)} \|\phi\|_{L^{p'}(D)} \le C\|u\|_{L^p(D)} \|\phi\|_{H^{1,r'}(D)}.$$

Combining the last two estimates yields

$$\sup_{\phi \in C^{\infty,T}(\overline{D}) \setminus \{0\}} \frac{|(\nabla u, \nabla\phi) + (u, \phi)|}{\|\phi\|_{H^1_{r'}(D)}} \le C\|u\|_{H^{1,T}_p(D)}.$$

Lemma 2.2.8 does now imply that $u \in \dot{\mathbf{H}}^{1,T}_r(D)$. If $r \ge 2$, we additionally get $u \in \dot{\mathbf{H}}^{1,T}_2(D)$. If $r < 2$, one has to repeat the procedure once again with $r_2 = r^* = p^{**}$.

In a similar way, the case of vanishing tangential components can be treated. For $u \in \dot{\mathbf{H}}^{1,N}_p(D)$ and $\phi \in C^{\infty,N}(\overline{D})$, the counterpart to (2.17) reads as

$$(\nabla u, \nabla\phi) = a(u, \phi) - \int_{\partial D} u \cdot (\phi \cdot \nabla n - \phi \operatorname{div} n) \, \mathrm{d}S, \tag{2.19}$$

which was shown in [KY09, (3.67)]. Again, ∇n and $\operatorname{div} n$ are the gradient and divergence of a smooth extension of n to a neighbourhood of $\partial\Omega$. As the terms inside the integral are of the same kind as in (2.17), one can estimate it in the same way. Hence, the same treatment yields $u \in \dot{\mathbf{H}}_2^{1,N}(D)$.

By the continuity of $a(\cdot,\cdot)$ on $\dot{\mathbf{H}}_2^{1,B}(D) \times \dot{\mathbf{H}}_2^{1,B}(D)$ and the density of $C^{\infty,B}(\overline{D})$ in $\dot{\mathbf{H}}_p^{1,B}(\Omega)$ given in Corollary 2.1.6, the equation (2.16) can be extended to all $\phi \in \dot{\mathbf{H}}_2^{1,B}(D)$. Choose $\phi = u$. This implies

$$0 = a(u,u) = \|\operatorname{div} u\|_{L^2(D)}^2 + \|\operatorname{rot} u\|_{L^2(D)}^2,$$

which means $\operatorname{div} u = 0$ and $\operatorname{rot} u = 0$. □

We can now show, that the norm of a function in $\dot{\mathbf{H}}_p^{1,B}(D)$ can be estimated with the help of the bilinear form $a(\cdot,\cdot)$.

Lemma 2.2.10. *Let $1 < p < \infty$. Then there is a constant $C(p,D) > 0$ such that for all $u \in \dot{\mathbf{H}}_p^{1,B}(D)$, it holds*

$$\|u\|_{H_p^1(D)} \le C \sup_{\phi \in C^{\infty,B}(\overline{D})\setminus\{0\}} \frac{|a(u,\phi)|}{\|\phi\|_{H_{p'}^1(D)}} + \sum_{j=1}^N |(\nabla u, \nabla \phi_j)|. \tag{2.20}$$

Here, $(\phi_j)_{j=1}^N$ is an orthonormal basis of $\dot{\mathbf{H}}^{1,B,har}(D)$.

Proof. At first, we will show the weaker estimate

$$\|u\|_{H_p^1(D)} \le C \sup_{\phi \in C^{\infty,T}(\overline{D})\setminus\{0\}} \frac{|a(u,\phi)|}{\|\phi\|_{H_{p'}^1(D)}} + C\|u\|_{L^p(D)} \tag{2.21}$$

for all $u \in \dot{\mathbf{H}}_p^{1,B}(D)$. Using (2.17) and

$$\|f\|_{L^r(\partial D)} \le \epsilon \|\nabla f\|_{L^r(D)} + C_\epsilon \|f\|_{L^r(D)}, \quad 1 < r < \infty,\ f \in \dot{H}_p^1(D), \tag{2.22}$$

where $\epsilon > 0$ can be chosen arbitrarily, we get

$$|(\nabla u, \nabla \phi)| \le |a(u,\phi)| + \left(\epsilon \|\nabla u\|_{L^p(D)} + C_\epsilon \|u\|_{L^p(D)}\right) \|\phi\|_{H_{p'}^1(D)} \tag{2.23}$$

for all $\phi \in C^{\infty,T}(\overline{D})$ and $u \in \dot{\mathbf{H}}_p^{1,T}(D)$. Likewise, equation (2.19) and (2.22) imply (2.23) for all $\phi \in C^{\infty,N}(\overline{D})$ and $u \in \dot{\mathbf{H}}_p^{1,N}(D)$. Hence, this inequality holds for all $\phi \in C^{\infty,B}(\overline{D})$ and $u \in \dot{\mathbf{H}}_p^{1,B}(D)$. We can now proceed without distinguishing between the boundary conditions. Let $u \in \dot{\mathbf{H}}_p^{1,B}(D)$. Together with Lemma 2.2.8, the estimate (2.23) yields

$$\begin{aligned}
&\|\nabla u\|_{L^p(D)} + \|u\|_{L^p(D)} \\
&\leq C \sup_{\phi \in C^{\infty,B}(\overline{D})\setminus\{0\}} \frac{|(\nabla u, \nabla\phi)|}{\|\phi\|_{H^1_{p'}(D)}} \\
&\leq C \sup_{\phi \in C^{\infty,B}(\overline{D})\setminus\{0\}} \frac{|a(u,\phi)| + \big(\epsilon\|\nabla u\|_{L^p(D)} + C_\epsilon\|u\|_{L^p(D)}\big)\,\|\phi\|_{H^1_{p'}(D)}}{\|\phi\|_{H^{1,p'}(D)}} \\
&\leq C\Big[\sup_{\phi \in C^{\infty,B}(\overline{D})\setminus\{0\}} \frac{|a(u,\phi)|}{\|\phi\|_{H^1_{p'}(D)}} + \epsilon\|\nabla u\|_{L^p(D)} + C_\epsilon\|u\|_{L^p(D)}\Big].
\end{aligned}$$

Choosing ϵ small enough, we can absorb the term $\|\nabla u\|_{L^p(D)}$ on the right-hand side, which ends up in (2.21).

We can now show (2.20) by contraposition and suppose that it is not true. Then there is a sequence $(u_j)_{j\in\mathbb{N}} \subset \dot{\mathbf{H}}^{1,B}_p(D)$ such that

$$\|\nabla u_j\|_{L^p(D)} + \|u_j\|_{L^p(D)} = 1, \tag{2.24}$$

$$\epsilon_j := \sup_{\phi \in C^{\infty,B}(\overline{D})\setminus\{0\}} \frac{|a(u_j,\phi)|}{\|\phi\|_{H^1_{p'}(D)}} \to 0 \quad \text{for } j \to \infty, \tag{2.25}$$

$$\sum_{k=1}^{N} |(\nabla u_j, \nabla\phi_k)| \to 0 \quad \text{for } j \to \infty. \tag{2.26}$$

Due to the first condition, we may assume that there is a $v \in \dot{\mathbf{H}}^{1,B}_p(D)$ such that

$$\begin{aligned}
u_j &\to v \quad \text{strongly in } L^p(D), \\
\nabla u_j &\to \nabla v \text{ weakly in } L^p(D).
\end{aligned}$$

The weak convergence of the gradients together with (2.25) imply that $a(v,\phi) = 0$ for all $\phi \in C^{\infty,B}(\overline{D})$. Lemma 2.2.9 therefore yields $v \in \dot{H}^{1,B,har}(D)$. Using (2.26), we additionally have $(\nabla v, \nabla\phi_k) = 0$ for all $k = \{1,\dots,N\}$. As $(\phi_k)_k$ is an orthonormal basis of $\dot{\mathbf{H}}^{1,B,har}(D)$, this means $v = 0$. Together with (2.21), we get

$$\|u_j\|_{H^1_p(D)} \leq C(\epsilon_j + \|u_j\|_{L^p(D)}) \to 0$$

for $j \to \infty$, which contradicts (2.24). This completes the proof. □

Let us now cite a simplified version of [KY13, Theorem 1.1], that is sufficient for our purpose.

Lemma 2.2.11. *Let X, Y be reflexive Banach spaces and $b(\cdot,\cdot)$ be a bilinear form defined on $X \times Y$. Assume that there are constants $M, C > 0$ such that*

$$|b(u,\phi)| \leq M\|u\|_X\|\phi\|_Y, \tag{2.27}$$

and

$$\begin{aligned} \|u\|_X &\leq C \sup_{\phi \in Y\setminus\{0\}} \frac{|b(u,\phi)|}{\|\phi\|_Y} \quad \text{for all } \phi \in Y, \\ \|\phi\|_Y &\leq C \sup_{u \in X\setminus\{0\}} \frac{|b(u,\phi)|}{\|u\|_X} \quad \text{for all } u \in X. \end{aligned} \tag{2.28}$$

Then for each $f \in Y'$, there is a $u \in X$ such that

$$b(u,\phi) = \langle f, \phi\rangle \quad \text{for all } \phi \in Y.$$

Furthermore, there is a constant $c > 0$ that does not depend on f such that

$$\|u\|_X \leq c\|f\|_{Y'}.$$

We are now in the position to give a proof of the main result of this section.

Proof of Lemma 2.2.7. In Lemma 2.2.11, we choose $X := \tilde{\mathbf{H}}_p^{1,B}(D)$, $Y := \tilde{\mathbf{H}}_{p'}^{1,B}(D)$ and $b(\cdot,\cdot) := a(\cdot,\cdot)$. It is clear, that a fulfils the boundedness assumption (2.27). By the density of $C^{\infty,B}(\overline{D})$ in $\dot{\mathbf{H}}_p^{1,B}(D)$ and $\dot{\mathbf{H}}_{p'}^{1,B}(D)$ and Lemma 2.2.10, we have

$$\|u\|_{H_p^1(D)} \leq C\Big[\sup_{\phi \in \dot{\mathbf{H}}_{p'}^{1,B}(D)\setminus\{0\}} \frac{|a(u,\phi)|}{\|\phi\|_{H_{p'}^1(D)}} + \sum_{j=1}^N |(\nabla u, \nabla \phi_j)|\Big]$$

for all $u \in \dot{\mathbf{H}}_p^{1,B}(D)$. As $(\nabla u, \nabla\phi_j) = 0$ for all $u \in \tilde{\mathbf{H}}_p^{1,B}$ and $j \in \{1, \ldots, N\}$, this implies

$$\|u\|_{H_p^1(D)} \leq C \sup_{\phi \in \tilde{\mathbf{H}}_{p'}^{1,B}(D)\setminus\{0\}} \frac{|a(u,\phi)|}{\|\phi\|_{H_{p'}^1(D)}}$$

for all $u \in \tilde{\mathbf{H}}_p^{1,B}(D)$. It is clear, that the roles of p and p' in these considerations can be exchanged without a problem. Hence, the variational inequalities (2.28) are both satisfied. Lemma 2.2.11 now yields a solution of (2.11) that fulfils estimate (2.12). The uniqueness of the solution in $\tilde{\mathbf{H}}_p^{1,B}(D)$ follows by Lemma 2.2.9.

It remains to prove (2.13) and (2.14). Let $v \in \dot{\mathbf{H}}_p^{1,B}(D)$ be a solution to (2.11). Due to the direct decomposition (2.9), there are unique $\tilde{v} \in \tilde{\mathbf{H}}_p^{1,B}(D)$ and $v^h \in \dot{\mathbf{H}}^{1,B,har}(D)$ such that $v = \tilde{v} + v^h$. As $\tilde{v}$ is the unique solution in $\tilde{\mathbf{H}}_p^{1,B}(D)$ to (2.11), we can estimate it with the help of (2.12). The harmonic part v^h can be estimated by using Proposition 2.2.6. Combining these estimates yields (2.13). The estimate (2.14) follows from that one by

$$\sum_{k=1}^{N} |(\nabla v, \nabla \phi_k)| = \sum_{k=1}^{N} |(\nabla v^h, \nabla \phi_k)| \leq C \sum_{k=1}^{N} |(v^h, \phi_k)| \leq C\|v\|_{L^p(D)}.$$

Here, we have used, that $\sum_{k=1}^{N} |(\nabla \cdot, \nabla \phi_k)|$ and $\sum_{k=1}^{N} |(\cdot, \phi_k)|$ are equivalent norms on $\dot{\mathbf{H}}^{1,B,har}(D)$ as it is a finite dimensional space. □

2.2.3 The Weak Poisson Problem in the Whole Space

In this subsection, we will solve the weak Poisson problem on the whole space. In this case, the problem reduces to the weak Laplace equation, of which the treatment is standard in the literature. Opposed to the case of bounded domains, we do not need to characterize the set of harmonic vector fields in $\dot{\mathbf{H}}_p^1(\mathbb{R}^3)$ beforehand. However, we will see by Liouville's theorem for harmonic functions, that this set consists only of the equivalence class of constant functions $[0]$. The following result can be found in [KY98, Lemma 2.4].

Lemma 2.2.12. *Let $1 < p < \infty$. For each $f \in L^p(\mathbb{R}^d)$ and each $j = 1, \ldots, d$, there exists a unique $u \in \dot{\mathbf{H}}_p^1(\mathbb{R}^d)$ such that*

$$-\Delta u = \frac{\partial}{\partial x_j} f \tag{2.29}$$

in the sense of distributions. This solution satisfies

$$\|\nabla u\|_p \leq C(d,p)\|f\|_p.$$

In order to use this Lemma to solve the Laplace equation on the whole space with the right-hand side being a functional on a homogeneous space, we need the following statement.

Lemma 2.2.13. *Let $1 < p < \infty$. Then for every $f \in \dot{\mathbf{H}}_{p'}^1(\mathbb{R}^d)'$, there is a vector function $F \in L^p(\mathbb{R}^d)$ such that $\operatorname{div} F = f$ in the sense that*

$$\langle f, \phi \rangle = -(F, \nabla \phi) \tag{2.30}$$

for all $\phi \in \dot{\mathbf{H}}_{p'}^1(\mathbb{R}^d)$. This F can be chosen such that

$$\|F\|_{L^p(\mathbb{R}^d)} \leq C(d,p)\|f\|_{\dot{\mathbf{H}}_{p'}^1(\mathbb{R}^d)'}.$$

Proof. This Lemma is a special case of [KY98, Lemma 2.2] on the whole space $\mathbb{R}^d$. □

These two lemmata allow us to treat the existence and uniqueness of weak solutions to the Poisson equation in the whole space. The proof is a slight variation of [KY98, Lemma 2.5].

Lemma 2.2.14. *Let $1 < p < \infty$. Then for every $f \in \dot{\mathbf{H}}^1_{p'}(\mathbb{R}^d)'$, there is a unique $u \in \dot{\mathbf{H}}^1_p(\mathbb{R}^d)$ such that*

$$(\nabla u, \nabla \phi) = \langle f, \phi \rangle \tag{2.31}$$

for all $\phi \in C_c^\infty(\mathbb{R}^d)$. For $d = 3$, the equation (2.31) *is equivalent to*

$$a(u, \phi) := (\operatorname{div} u, \operatorname{div} \phi) + (\operatorname{rot} u, \operatorname{rot} \phi) = \langle f, \phi \rangle$$

for all $\phi \in C_c^\infty(\mathbb{R}^3)$. The solution u fulfils the estimate

$$\|\nabla u\|_p \le C(d,p) \|f\|_{\dot{\mathbf{H}}^1_{p'}(\mathbb{R}^d)'}.$$

Proof. Consider the equation

$$-\langle \Delta u, \phi \rangle = \langle f, \phi \rangle \quad \forall \phi \in C_c^\infty(\mathbb{R}^d). \tag{2.32}$$

By Lemma 2.2.13 there is a matrix valued function $F \in L^p$ such that $\operatorname{div} F = f$ and

$$\|F\| \le C(d,p) \|f\|_{\dot{\mathbf{H}}^1_{p'}(\mathbb{R}^d)'}. \tag{2.33}$$

Hence, (2.32) reduces to

$$-\Delta u = \operatorname{div} F$$

in the sense of distributions. This equation has a unique solution $u \in \dot{\mathbf{H}}^1_p(\mathbb{R}^d)$ due to Lemma 2.2.12 and this solution can be estimated using (2.33) by

$$\|\nabla u\|_p \le C \|F\|_p \le C(d,p) \|f\|_{\dot{H}^1_{p',q'}(\mathbb{R}^d)'}.$$

For $d = 3$, we can employ $\Delta = \operatorname{div} \nabla$ and $\Delta = \nabla \operatorname{div} - \operatorname{rot} \operatorname{rot}$ in order to see that u fulfils (2.31). This completes the proof of the existence of a solution.

In order to prove uniqueness, assume that there is another solution $v \in \dot{H}^1_p(\mathbb{R}^d)$ of (2.31). Then $w := u - v$ is a distributional solution to

$$-\Delta w = 0$$

on $\mathbb{R}^d$. Hence, $h := \nabla w$ is harmonic, too. The inclusion $h \in L^p(\mathbb{R}^d) \subset L^1_{loc}(\mathbb{R}^d)$ and Weyl's lemma imply $h \in C^\infty(\mathbb{R}^d)$. Thus, we can apply the mean value property of harmonic functions to h. For $x \neq 0$, we get that

$$\begin{aligned} |h(x)| &\leq \frac{1}{|B(x,|x|)|} \int_{B(x,|x|)} |h(y)|\,\mathrm{d}y \\ &\leq |B(x,|x|)|^{-1/p} \|h\|_{L^p(\mathbb{R}^d)} \\ &= C(d,p)|x|^{-d/p} \|h\|_{L^p(\mathbb{R}^d)}. \end{aligned}$$

Thus, h is bounded away from zero. Together with $h \in C^\infty(\mathbb{R}^d)$, this means h is bounded on $\mathbb{R}^d$. Applying Liouville's theorem for harmonic functions yields $h = 0$. Thus, w is constant, which completes the proof. □

Using Lemma 2.2.14, we can show, that there are no harmonic vector fields in $\dot{\mathbf{H}}^{1,B}_p(\mathbb{R}^3)$ besides the constant functions.

Corollary 2.2.15. *Let $1 < p < \infty$ and $u \in \dot{\mathbf{H}}^1_p(\mathbb{R}^3)$ be harmonic. Then $u = [0]$.*

Proof. As u is harmonic, it fulfils $a(u, \phi) = 0$ for all $\phi \in C^\infty_c(\mathbb{R}^3)$. By Lemma 2.2.14, it is the only solution to that equation in $\dot{\mathbf{H}}^1_p(\mathbb{R}^3)$. Therefore, it has to coincide with a constant function, as these solve that equation. □

The solutions given by Lemma 2.2.14 are consistent, as the next lemma shows.

Lemma 2.2.16. *Let $1 < p_0, p_1 < \infty$. Let $f \in \dot{\mathbf{H}}^1_{p_0'}(\mathbb{R}^d)'$ and $u \in \dot{\mathbf{H}}^1_{p_0}(\mathbb{R}^d)$ be a solution to (2.31). If additionally $f \in \dot{\mathbf{H}}^1_{p_1'}(\mathbb{R}^d)'$, then $\nabla u \in L^{p_1}(\mathbb{R}^d)$.*

Proof. By Lemma 2.2.14, there is a unique solution $v \in \dot{\mathbf{H}}^1_{p_1}(\mathbb{R}^d)$ of (2.31). Consider the difference $w = u - v$. Then $\Delta w = 0$ and therefore also $\Delta(\nabla w) = 0$ in the distributional sense. As w is also locally integrable, Weyl's lemma implies $\nabla w \in C^\infty(\mathbb{R}^d)$. By the same arguments as in the proof of the uniqueness in Lemma 2.2.14, we get $\nabla w = 0$, which implies $\nabla u \in L^{p_1}(\mathbb{R}^d)$. □

2.2.4 Harmonic Vector Fields in Exterior Domains

In this subsection, we characterize the subset of harmonic vector fields

$$\dot{\mathbf{H}}^{1,B,har}_p(\Omega) := \{h \in \dot{\mathbf{H}}^{1,B}_p(\Omega) : \operatorname{div} h = 0, \operatorname{rot} h = 0\}.$$

for exterior domains Ω. We are mainly concerned with its cardinality and its dependence on the integrability parameter p. It will turn out that as in the case of bounded domains, it is a finite dimensional space and independent of p. The first of these properties is a rather simple consequence of Proposition 2.1.21:

Proposition 2.2.17. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary. Then the set $\dot{\mathbf{H}}_p^{1,B,har}(\Omega)$ is of finite dimension for each $1 < p < \infty$.*

Proof. We will show, that the unit ball in $\dot{\mathbf{H}}_p^{1,B,har}(\Omega)$ is sequentially compact. Let $(u_n)_{n\in\mathbb{N}} \subset \dot{\mathbf{H}}_p^{1,B,har}(\Omega)$ be a sequence with $\|\nabla u_n\|_{L^p(\Omega)} = 1$ for all $n \in \mathbb{N}$. Then, by Proposition 2.1.4, there is a subsequence of $(u_n)_{n\in\mathbb{N}}$ that is convergent in $L^p(\Omega_R)$, where $R > 0$ fulfils $\partial\Omega \subset B(0,R)$. Due to Proposition 2.1.21, the expressions $\|\nabla \cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{L^p(\Omega_R)}$ are equivalent norms on $\dot{\mathbf{H}}_p^{1,B,har}(\Omega)$. Thus $(u_n)_{n\in\mathbb{N}}$ has even a convergent subsequence in $\dot{\mathbf{H}}_p^{1,B}(\Omega)$. This completes the proof. □

The second main result of this section reads as follows:

Theorem 2.2.18. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary. Then it holds*

$$\dot{\mathbf{H}}_p^{1,B,har}(\Omega) = \dot{\mathbf{H}}_2^{1,B,har}(\Omega)$$

for all $1 < p < \infty$.

The proof of this property is more involved. Let us briefly sketch the strategy. We begin with the proof of the inclusion $\dot{\mathbf{H}}_p^{1,B,har}(\Omega) \subseteq \dot{\mathbf{H}}_2^{1,B,har}(\Omega)$ by showing that solutions of the equation

$$(\operatorname{div} h, \operatorname{rot} \phi) + (\operatorname{rot} h, \operatorname{rot} \phi) = 0 \quad \text{ for all } \phi \in C_c^{\infty,B}(\overline{\Omega})$$

are in $\dot{\mathbf{H}}_2^{1,B,har}(\Omega)$. The converse inclusion will be established by constructing harmonic vector and scalar potentials of harmonic vector fields, whose asymptotic properties can be investigated by means of their Laurent expansion at infinity. Suitable decay properties of the harmonic vector fields themselves can then be deduced by the expansion at infinity, too.

As a part of the proof, we will frequently consider harmonic vector fields in Lebesgue spaces. These will be denoted in the following way.

Definition 2.2.19. *For $1 < p < \infty$, define*

$$\begin{aligned} L^p_{T,har}(\Omega) &:= \{u \in L^p(\Omega) : \operatorname{div} u = 0, \operatorname{rot} u = 0, u \cdot n = 0\}, \\ L^p_{N,har}(\Omega) &:= \{u \in L^p(\Omega) : \operatorname{div} u = 0, \operatorname{rot} u = 0, u \times n = 0\}. \end{aligned}$$

Note that the trace conditions are indeed well defined by (1.5) and (1.6). We show that harmonic functions in Lebesgue spaces are always smooth functions up to the boundary. Therefore, the question of integrability will only depend on their decay at infinity. For the proof we need the following technical lemma taken from [KY09, Lemma 4.5].

Lemma 2.2.20. *Let $D \subset \mathbb{R}^3$ be a bounded domain with C^∞-boundary, $s \geq 2$ and $1 < p < \infty$. Assume that $u \in L^p(D)$ fulfils $\operatorname{div} u \in W^{s-1,p}(D)$, $\operatorname{rot} u \in W^{s-1,p}(D)$ and $u \cdot n \in W^{s-\frac{1}{p},p}(\partial D)$. Then $u \in W^{s,p}(D)$ and*

$$\begin{aligned}&\|u\|_{W^{s,p}(D)}\\ &\quad \leq C\big[\|\operatorname{div} u\|_{W^{s-1,p}(D)} + \|\operatorname{rot} u\|_{W^{s-1,p}(D)} + \|u\|_{L^p(D)} + \|u \cdot n\|_{W^{s-\frac{1}{p},p}(\partial D)}\big].\end{aligned}$$

In the same way, if $v \in L^p(D)$ fulfils $\operatorname{div} v \in W^{s-1,p}(D)$, $\operatorname{rot} v \in W^{s-1,p}(D)$ and $v \times n \in W^{s-\frac{1}{p},p}(\partial D)$, then $v \in W^{s,p}(D)$ and

$$\begin{aligned}&\|v\|_{W^{s,p}(D)}\\ &\quad \leq C\big[\|\operatorname{div} v\|_{W^{s-1,p}(D)} + \|\operatorname{rot} v\|_{W^{s-1,p}(D)} + \|v\|_{L^p(D)} + \|v \times n\|_{W^{s-\frac{1}{p},p}(\partial D)}\big].\end{aligned}$$

We can now proof the smoothness of harmonic vector fields.

Lemma 2.2.21. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary and $1 < p < \infty$. Let $h, k \in L^p(\Omega)$ be such that $\operatorname{div} h = 0$, $\operatorname{rot} h = 0$ and $h \cdot n = 0$ as well as $\operatorname{div} k = 0$, $\operatorname{rot} k = 0$ and $k \times n = 0$. Then $h, k \in C^\infty(\overline{\Omega})$.*

Proof. We only proof the claim containing the vector field h, as the proof for k works exactly the same. As the components of h are harmonic functions and locally integrable, we know by Weyl's Lemma, that $h \in C^\infty(\Omega)$. Therefore, it remains to check the smoothness at the boundary of Ω. We will achieve this by a cut-off argument. Let $R > 0$ be such that $\partial\Omega \subset B(0, R-1)$ and $\eta \in C_c^\infty(\mathbb{R}^3)$ be such that $0 \leq \eta \leq 1$, $\eta(x) = 1$ for $|x| \leq R-1$ and $\eta(x) = 0$ for $|x| \geq R$. Set $h_1 := \eta h$ and $h_2 := (1-\eta)h$ and consider h_1 as a function defined on Ω_R and h_2 as a function on $\mathbb{R}^3$. Due to the smoothness of h and η in a neighbourhood of the cut-off region $B(0,R) \setminus B(0,R-1)$, the functions h_1 and h_2 are smooth there, too. Hence, $h_2 \in C^\infty(\mathbb{R}^3)$. Regarding h_1, we will make use of Lemma 2.2.20. Clearly, $h_1 \in L^p(\Omega_R)$ and h_1 fulfils $h_1 \cdot n = 0$ on $\partial\Omega_R$, i.e. $h_1 \cdot n \in W^{1-\frac{s}{p},p}(\partial\Omega_R)$ for all $s \geq 0$. Near $\partial\Omega$, the divergence and rotation of h_1 are zero. In the remaining part of Ω_R, these are smooth functions up to $\partial B(0,R)$ due to the smoothness of h_1 there. Hence, $\operatorname{div} h_1, \operatorname{rot} h_1 \in W^{s,p}(\Omega_R)$ for all $s \geq 0$. Therefore Lemma 2.2.20 implies $h_1 \in W^{s,p}(\Omega_R)$ for all $s \geq 0$, which yields $h_1 \in C^\infty(\overline{\Omega_R})$. Hence, $h = h_1 + h_2 \in C^\infty(\overline{\Omega})$. $\square$

We now establish the inclusion $\dot{\mathbf{H}}_p^{1,B,har}(\Omega) \subseteq \dot{\mathbf{H}}_2^{1,B,har}(\Omega)$. We will do that by making use of the next consistency result, which is more general than necessary right now. However, it will become useful later on when we investigate the Poisson problem in exterior domains.

Lemma 2.2.22. *Let* $p_0 \in (1,\infty)$, $p_1 \in (3/2,\infty)$ *and* $f \in \dot{\mathbf{H}}^{1,B}_{p_0'}(\Omega)' \cap \dot{\mathbf{H}}^{1,B}_{p_1'}(\Omega)'$. *Assume that* $w \in \dot{\mathbf{H}}^{1,B}_{p_0}(\Omega)$ *fulfils*

$$a(w,\phi) = \langle f, \phi\rangle \tag{2.34}$$

for all $\phi \in C_c^{\infty,B}(\overline{\Omega})$. *Then* $w \in \dot{\mathbf{H}}^{1,B}_{p_1}(\Omega)$.

Proof. We will split the problem by a cut-off argument to the case of the whole space $\mathbb{R}^3$ and of a bounded domain. Let $R > 0$ be such that $\partial\Omega \subset B(0, R-2)$. Let $\psi_1 \in C_c^\infty(\mathbb{R}^3)$ be a cut-off function with $\psi_1 = 1$ on $B(0, R-1)$ and $\psi_1 = 0$ on $B(0,R)^C$. Set $\psi_2 = 1 - \psi_1$. We consider the sub-problems

$$a(\psi_1 w, \phi_1) = \langle f_1, \phi_1\rangle \quad \text{for all } \phi_1 \in C^{\infty,B}(\overline{\Omega_{R+2}}), \tag{S_1}$$

and

$$a(\psi_2 w, \phi_2) = \langle f_2, \phi_2\rangle \quad \text{for all } \phi_2 \in C_c^{\infty}(\mathbb{R}^3), \tag{S_2}$$

where

$$f_k = \psi_k f - 2\nabla\psi_k\nabla w - \Delta\psi_k w, \quad \in \{1,2\}.$$

For notational convenience, set

$$\Omega_1 = \Omega_{R+1} \quad \text{and } \Omega_2 = \mathbb{R}^3.$$

We will regard $\psi_1 w$ as an element of $\dot{\mathbf{H}}^{1,B}_p(\Omega_1)$ and $\psi_2 w$ as an element of $\dot{\mathbf{H}}^1_p(\mathbb{R}^3)$.

At first, we verify, that $f_1 \in (\dot{\mathbf{H}}^{1,B}_{p_1'}(\Omega_1))'$ and $f_2 \in (\dot{\mathbf{H}}^{1}_{p_1'}(\Omega_2))'$. Let $\phi_1 \in C^{\infty,B}(\overline{\Omega_1})$. Then we can can consider $\psi_1\phi_1$ as an element of $C_c^{\infty,B}(\overline{\Omega})$ and estimate its norm by

$$\begin{aligned}
\|\nabla(\psi_1\phi_1)\|_{L^{p_1'}(\Omega)} &= \|\nabla(\psi_1\phi_1)\|_{L^{p_1'}(\Omega_1)} \\
&\le \|(\nabla\psi_1)\phi_1\|_{L^{p_1'}(\Omega_1)} + \|\psi_1\nabla\phi_1\|_{L^{p_1'}(\Omega_1)} \\
&\le C\|\phi_1\|_{L^{p_1'}(\Omega_1)} + C\|\nabla\phi_1\|_{L^{p_1'}(\Omega_1)} \\
&\le C\|\nabla\phi_1\|_{L^{p_1'}(\Omega_1)},
\end{aligned}$$

where we have used Proposition 2.1.4 in the last line. Therefore, we have

$$\begin{aligned}
|\langle\psi_1 f, \phi_1\rangle| &= |\langle f, \psi_1\phi_k\rangle| \\
&\le \|f\|_{(\dot{\mathbf{H}}^{1,B}_{p_1'}(\Omega))'}\|\nabla(\psi_1\phi_1)\|_{L^{p_1'}(\Omega)} \\
&\le C\|f\|_{(\dot{\mathbf{H}}^{1,B}_{p_1'}(\Omega))'}\|\nabla\phi_1\|_{L^{p_1'}(\Omega_1)}.
\end{aligned} \tag{2.35}$$

As $C^{\infty,B}(\overline{\Omega_1})$ is dense in $\dot{\mathbf{H}}^{1,B}_{p_1'}(\Omega_1)$, this means $\psi_1 f \in (\dot{\mathbf{H}}^{1,B}_{p_1'}(\Omega_1))'$. In the whole space, it is sufficient to consider $\phi_2 \in C_c^\infty(\Omega_2)$ because of the density of these functions in $\dot{\mathbf{H}}^1_{p_1}(\Omega_2)$. Here, we note that $p_1' < 3$. Hence, its Sobolev exponent $\gamma := p_1^* := \frac{3p_1}{3-p_1} \in (3/2, \infty)$ is well defined. In a similar fashion as before, we do now get

$$\begin{aligned}
\|\nabla(\psi_2\phi_2)\|_{L^{p_1'}(\Omega)} &= \|\nabla(\psi_2\phi_2)\|_{L^{p_1'}(\Omega_2)} \\
&\leq \|(\nabla\psi_2)\phi_2\|_{L^{p_1'}(B(0,R+1))} + \|\psi_2\nabla\phi_2\|_{L^{p_1'}(\Omega_2)} \\
&\leq C\|\phi_2\|_{L^{p_1'}(B(0,R+1))} + C\|\nabla\phi_2\|_{L^{p_1'}(\Omega_2)} \\
&\leq C\|\phi_2\|_{L^{\gamma}(B(0,R+1))} + C\|\nabla\phi_2\|_{L^{p_1'}(\Omega_2)} \\
&\leq C\|\nabla\phi_2\|_{L^{p_1'}(\Omega_2)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\langle\psi_2 f, \phi_2\rangle| &= |\langle f, \psi_2\phi_2\rangle| \\
&\leq \|f\|_{(\dot{\mathbf{H}}^{1,B}_{p_1'}(\Omega))'}\|\nabla(\psi_1\phi_1)\|_{L^{p_1'}(\Omega)} \\
&\leq C\|f\|_{(\dot{\mathbf{H}}^{1,B}_{p_1'}(\Omega))'}\|\nabla\phi_1\|_{L^{p_1'}(\Omega_1)},
\end{aligned} \tag{2.36}$$

which implies $\psi_2 f \in (\dot{\mathbf{H}}^1_{p_1'}(\Omega_2))'$. In order to deal with the other summands of f_1 and f_2, we have to consider multiple cases. Let ϕ_1 and ϕ_2 be as before. At first, we consider the case

$$\frac{1}{p_0} - \frac{1}{3} \leq \frac{1}{p_1} < \frac{2}{3}. \tag{2.37}$$

Let $s \in (1,\infty)$ be given by $\frac{1}{p_1} = \frac{1}{s} - \frac{1}{3}$. Because of the boundedness of Ω_1, $p_0' \leq s'$, the classical Sobolev embedding and Proposition 2.1.4, we have

$$\|\phi_1\|_{L^{p_0'}(\Omega_{R+2})} \leq C\|\phi_1\|_{L^{s'}(\Omega_{R+2})} \leq C\|\phi_1\|_{H^1_{p_1'}(\Omega_{R+2})} \leq C\|\nabla\phi_1\|_{L^{p_1'}(\Omega_1)}.$$

By the Sobolev inequality, we similarly get

$$\|\phi_2\|_{L^{p_0'}(\Omega_{R+2})} \leq C\|\phi_2\|_{L^{s'}(\Omega_{R+2})} \leq C\|\nabla\phi_2\|_{L^{p_1'}(\Omega_2)}.$$

Because of the density of $C^{\infty,B}(\overline{\Omega_1})$ in $\dot{\mathbf{H}}^{1,B}_p(\Omega_1)$ and the density of $C_c^\infty(\Omega_2)$ in $\dot{\mathbf{H}}^1_p(\Omega_2)$, this implies the embeddings

$$\dot{\mathbf{H}}^1_{p_1'}(\Omega_1) \hookrightarrow L^{p_0'}(\Omega_{R+2}) \quad \text{and} \quad \dot{\mathbf{H}}^1_{p_1'}(\Omega_2) \hookrightarrow L^{p_0'}(\Omega_{R+2})$$

and their dual counterparts

$$L^{p_0}(\Omega_{R+2}) \hookrightarrow \left(\dot{\mathbf{H}}^1_{p_1'}(\Omega_1)\right)' \quad \text{and} \quad L^{p_0}(\Omega_{R+2}) \hookrightarrow \left(\dot{\mathbf{H}}^1_{p_1'}(\Omega_2)\right)'.$$

Coming back to f_k, it is clear, that $\nabla\psi_k \nabla w \in L^{p_0}(\Omega_{R+2})$. Using Proposition 2.1.4, the same holds for $(\Delta\psi_k)w$. Hence, we can apply the embedding above and the established statements on $\psi_1 f$ and $\psi_2 f$ to gain $f_1 \in \dot{\mathbf{H}}^{1,B}_{p_1'}(\Omega_1)'$ and $f_2 \in \dot{\mathbf{H}}^{1,B}_{p_1'}(\Omega_2)'$.

Now, we make sure, that w_1 and w_2 are actually solutions to (S_1) and (S_2). We begin with the latter equation on the whole space. Let $\phi_2 \in C_c^\infty(\Omega_2)$. As we do not have to deal with any boundaries, we have in the sense of distributions

$$\begin{aligned} a(w_2, \phi_2) &= -\langle \Delta w_2, \phi_2 \rangle \\ &= -\langle \psi_2 \Delta w, \phi_2 \rangle - 2\langle \nabla\psi_2 \nabla w, \phi_2 \rangle - \langle w \Delta\psi_2, \phi_2 \rangle \\ &= -\langle \Delta w, \psi_2 \phi_2 \rangle - 2\langle \nabla\psi_2 \nabla w, \phi_2 \rangle - \langle w \Delta\psi_2, \phi_2 \rangle. \end{aligned}$$

We can consider $\psi_2\phi_2$ as an element of $C_c^\infty(\Omega)$, which implies

$$-\langle \Delta w, \psi_2\phi_2 \rangle = a(w, \psi_2\phi_2) = \langle f, \psi_2\phi_2 \rangle = \langle \psi_2 f, \phi_2 \rangle.$$

Therefore

$$a(w_2, \phi_2) = \langle \psi_2 f, \phi_2 \rangle - 2\langle \nabla\psi_2 \nabla w, \phi_2 \rangle - \langle w \Delta\psi_2, \phi_2 \rangle = \langle f_2, \phi_2 \rangle.$$

For the equation on the bounded domain Ω_1, we are going to decompose a given function $\phi_1 \in C^{\infty,B}(\overline{\Omega_1})$ into three parts in order to separate the computations in the cut-off area from the ones near the boundary. Let $\phi_1^1, \phi_1^2, \phi_1^3 \in C^{\infty,B}(\overline{\Omega_1})$ be such that

$$\begin{aligned} \phi_1 &= \phi_1^1 + \phi_1^2 + \phi_1^3, \\ \operatorname{supp}\phi_1^1 &\subset B(0, R-1), \\ \operatorname{supp}\phi_1^2 &\subset B(0, R+1) \setminus B(0, R-2), \\ \operatorname{supp}\phi_1^3 &\subset B(0, R)^C. \end{aligned}$$

Then we have

$$a(w_1, \phi_1) = a(\psi_1 w, \phi_1^1) + a(\psi_1 w, \phi_1^2) + a(\psi_1 w, \phi_1^3).$$

We investigate each of the summands on the right-hand side on its own. Because of $\psi_1 = 1$ on $\operatorname{supp}\phi_1^1$, we have $a(\psi_1 w, \phi_1^1) = a(w, \phi_1^1)$. We can consider ϕ_1^1 as an element of $C_c^{\infty,B}(\overline{\Omega})$. As a result, we get $a(w, \phi_1^1) = \langle f, \phi_1^1 \rangle$. Again, because of $\psi_1 = 1$ on $\operatorname{supp}\phi_1^1$, we obtain

$$a(\psi_1 w, \phi_1^1) = \langle f, \phi_1^1 \rangle - 2\langle \nabla\psi_1 \nabla w, \phi_1^1 \rangle - \langle w \Delta\psi_1, \phi_1^1 \rangle = \langle f_1, \phi_1^1 \rangle.$$

Regarding $a(\psi_1 w, \phi_1^2)$, we can argue almost as in the whole space case. As ϕ_1^2 is compactly supported in the interior of Ω, we get

$$\begin{aligned} a(w_1, \phi_1^2) &= -\langle \Delta w_1, \phi_1 \rangle \\ &= -\langle \psi_1 \Delta w, \phi_1^2 \rangle - 2\langle \nabla \psi_1 \nabla w, \phi_1^2 \rangle - \langle w \Delta \psi, \phi_1^2 \rangle \\ &= -\langle \Delta w, \psi_1 \phi_1^2 \rangle - 2\langle \nabla \psi_1 \nabla w, \phi_1^2 \rangle - \langle w \Delta \psi, \phi_1^2 \rangle. \end{aligned}$$

Considering $\psi_1 \phi_1^2$ as an element of $C_c^\infty(\Omega)$ yields

$$-\langle \Delta w, \psi_1 \phi_1^2 \rangle = a(w, \psi_1 \phi_1^2) = \langle f, \psi_1 \phi_1^2 \rangle = \langle \psi_1 f, \phi_1^2 \rangle$$

and therefore $a(\psi_1 w, \phi_1^2) = \langle f_1, \phi_1^2 \rangle$. For the last term, we get $a(\psi_1 w, \phi_1^3) = 0$, because of the disjoint supports of ψ_1 and ϕ_1^3. Finally, we have $\langle f_1, \phi_1^3 \rangle = 0$, because of the same reason. Summing everything up, we have

$$a(\psi_1 w, \phi_1^1) + a(\psi_1 w, \phi_1^2) + a(\psi_1 w, \phi_1^3) = \langle f_1, \phi_1^1 \rangle + \langle f_1, \phi_1^2 \rangle + \langle f_1, \phi_1^3 \rangle,$$

which implies

$$a(w_1, \phi_1) = \langle f_1, \phi_1 \rangle.$$

We can now make use of the consistency results in bounded domains and $\mathbb{R}^3$. Due to Lemma 2.2.16, we get that $\nabla(\psi_2 w) \in L^{p_1}(\mathbb{R}^3)$, and due to Lemma 2.2.7, that $\psi_1 w \in \dot{\mathbf{H}}^{1,B}_{p_1}(\Omega_1)$. Combining these statements yields $\nabla w \in L^{p_1}(\Omega)$. Together with Proposition 2.1.18, this implies $w \in \dot{\mathbf{H}}^{1,B}_{p_1}(\Omega)$.

The other cases can be treated by a bootstrap argument. Let

$$\frac{1}{p_0} - \frac{2}{3} \le \frac{1}{p_1} < \frac{1}{p_0} - \frac{1}{3} = \frac{1}{p_0^*}.$$

Note that for $\frac{1}{p_0^*} := \frac{1}{p_0} - \frac{1}{3}$, we have $p_1' \le (p_0^*)' < p_0'$. That means $\dot{\mathbf{H}}^{1,B}_{(p_1)'}(\Omega_k)' \cap \dot{\mathbf{H}}^{1,B}_{(p_0)'}(\Omega_k)' \hookrightarrow \dot{\mathbf{H}}^{1,B}_{(p_0^*)'}(\Omega_k)'$ by Proposition 2.1.24. Thus, we have $f \in \dot{\mathbf{H}}^{1,B}_{(p_0^*)'}(\Omega_k)'$. Therefore, we can apply the case proven above with p_0^* at the position of p_1 and p_0 being just itself and get $w \in \dot{\mathbf{H}}^{1,B}_{p_0^*}(\Omega)$. In the same way, we can now apply that case with p_0^* at the position of p_0 and and p_1 being itself and the new case is proven. Repeating this argumentation once more for $\frac{1}{p_0} - \frac{3}{3} < 0 < \frac{1}{p_1} < \frac{1}{p_0} - \frac{2}{3}$ yields $w \in \dot{\mathbf{H}}^{1,B}_{p_1}(\Omega)$ in the full range of parameters. □

As each harmonic vector field in $\dot{\mathbf{H}}^{1,B,har}_p(\Omega)$ fulfils (2.34) for $f = 0$, we directly get:

Corollary 2.2.23. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary. Then $\dot{\mathbf{H}}^{1,B,har}_p(\Omega) \subseteq \dot{\mathbf{H}}^{1,B,har}_2(\Omega)$ holds for $p \in (1, \infty)$.*

For the converse inclusion, we start by showing the existence of vector and scalar potentials of harmonic vector fields. First, we show that Neumann fields in $L^p(\Omega)$ admit a vector potential.

Lemma 2.2.24. *Let $1 < p < \infty$ and $h \in L^p_{T,har}(\Omega)$. Then there is a vector field $v \in L^p_{loc}(\overline{\Omega})$ such that $\nabla v \in L^p(\Omega)$, $\operatorname{div} v = 0$, $\operatorname{rot} v = h$ and $\Delta v = 0$.*

Proof. Set $\tilde{h} := h$ in Ω and $\tilde{h} := 0$ in Ω^C. Then $\operatorname{div} \tilde{h} = 0$. Indeed, for $\phi \in C_c^\infty(\mathbb{R}^3)$, we have

$$\begin{aligned} -\langle \operatorname{div} \tilde{h}, \phi \rangle &= (\tilde{h}, \nabla \phi) \\ &= \int_\Omega \tilde{h} \nabla \phi + \int_{\Omega^C} \tilde{h} \nabla \phi \\ &= \int_\Omega h \nabla \phi + 0 \\ &= \int_{\partial \Omega} \phi h \cdot n - \int_\Omega \phi \operatorname{div} h \\ &= 0. \end{aligned}$$

We define

$$\tilde{v} := \operatorname{rot}(E * \tilde{h}),$$

where E is the fundamental solution of the Laplace equation. We can see directly that $\operatorname{div} \tilde{v} = \operatorname{div} \operatorname{rot}(E * \tilde{h}) = 0$. Furthermore, we have

$$\begin{aligned} \operatorname{rot} \tilde{v} = \operatorname{rot} \operatorname{rot}(E * \tilde{h}) &= \nabla \operatorname{div}(E * \tilde{h}) - \Delta(E * \tilde{h}) \\ &= \nabla(E * \operatorname{div} \tilde{h}) + \tilde{h} = \tilde{h}. \end{aligned}$$

Using the Calderon-Zygmund inequality, we get

$$\begin{aligned} \|\nabla \tilde{v}\|_{L^p(\mathbb{R}^3)} &= \|\nabla \operatorname{rot}(E * \tilde{h})\|_{L^p(\mathbb{R}^3)} \\ &\leq C \|\Delta(E * \tilde{h})\|_{L^p(\mathbb{R}^3)} = C \|\tilde{h}\|_{L^p(\mathbb{R}^3)} = C \|h\|_{L^p(\Omega)}. \end{aligned}$$

Hence, $\nabla \tilde{v} \in L^p(\mathbb{R}^3)$. Making use of [AG94, Proposition 2.10], this additionally implies $\tilde{v} \in L^p_{loc}(\mathbb{R}^3)$. Setting $v := \tilde{v}|_\Omega$, we have constructed a vector field such that $v \in L^p_{loc}(\overline{\Omega})$, $\nabla v \in L^p(\Omega)$, $\operatorname{div} v = 0$ and $\operatorname{rot} v = h$. Combining the last two properties, we moreover have

$$\Delta v = \nabla \operatorname{div} v - \operatorname{rot} \operatorname{rot} v = 0 - \operatorname{rot} h = 0.$$

□

The corresponding claim for Dirichlet fields reads as follows.

Lemma 2.2.25. *Let $1 < p < \infty$ and $h \in L^p_{N,har}(\Omega)$. Then there is a scalar field $v \in L^p_{loc}(\overline{\Omega})$ such that $\nabla v \in L^p(\Omega)$, $\nabla p = h$ and $\Delta v = 0$.*

Proof. Set $\tilde{h} := h$ in Ω and $\tilde{h} := 0$ in Ω^C. Then for any $\phi \in C_c^\infty(\mathbb{R}^3)$, we have

$$\begin{aligned}\langle \operatorname{rot} \tilde{h}, \phi \rangle = (\tilde{h}, \operatorname{rot} \phi) &= \int_\Omega \tilde{h} \operatorname{rot} \phi + \int_{\Omega^C} \tilde{h} \operatorname{rot} \phi = \int_\Omega h \operatorname{rot} \phi + 0 \\ &= \int_\Omega \operatorname{rot} h \phi + \int_{\partial\Omega} (h \times n) \cdot \phi = 0.\end{aligned}$$

Hence, $\operatorname{rot} \tilde{\phi} = 0$. We set

$$\tilde{v} := -\operatorname{div}(E * \tilde{h}),$$

where E is the fundamental solution of the Laplace equation. Then

$$\begin{aligned}\nabla \tilde{v} &= -\nabla \operatorname{div}(E * \tilde{h}) = -\Delta(E * \tilde{h}) - \operatorname{rot}\operatorname{rot}(E * \tilde{h}) \\ &= \tilde{h} - \operatorname{rot}(E * (\operatorname{rot} \tilde{h})) = \tilde{h}.\end{aligned}$$

By the Calderon-Zygmund inequality, we get

$$\begin{aligned}\|\nabla \tilde{v}\|_{L^p(\mathbb{R}^3)} &= \|\nabla \operatorname{div}(E * \tilde{h})\|_{L^p(\mathbb{R}^3)} \\ &\leq C \|\Delta(E * \tilde{h})\|_{L^p(\mathbb{R}^3)} = C \|\tilde{h}\|_{L^p(\mathbb{R}^3)} = C \|h\|_{L^p(\Omega)},\end{aligned}$$

i.e. $\nabla \tilde{v} \in L^p(\mathbb{R}^3)$. Employing [AG94, Proposition 2.10], this implies $\tilde{v} \in L^p_{loc}(\mathbb{R}^3)$. Setting now $v := \tilde{v}|_\Omega$, we get $v \in L^p_{loc}(\overline{\Omega})$, $\nabla v \in L^p(\Omega)$ and $\nabla v = h$. Finally, we also have $\Delta v = \operatorname{div} \nabla v = \operatorname{div} h = 0$. $\square$

Remark 2.2.26. *During the construction of the scalar potential above, we actually only made use of the conditions $h \in L^p(\Omega)$, $h \times n = 0$ and $\operatorname{rot} h = 0$. The property $\operatorname{div} h = 0$ was only used for showing that the potential is a harmonic function.*

In order to determine the behaviour of the potentials gained above at infinity, we need the Sobolev-Morrey embedding.

Lemma 2.2.27. *Let $2 \leq d \in \mathbb{N}$, $d < p < \infty$ and $w \colon \mathbb{R}^d \to \mathbb{R}$ be a continuously differentiable function (not necessarily bounded) such that $\nabla w \in L^p(\mathbb{R}^d)$. Then there is a constant $C = C(p, d) > 0$ such that*

$$|w(x) - w(y)| \leq C |x - y|^{1 - \frac{d}{p}} \|\nabla w\|_{L^p(\mathbb{R}^3)}.$$

Although this inequality is generally not stated explicitly in this form, it can be usually found in proofs of the classical Sobolev-Morrey embedding, see for example [Eva98, Theorem 5.6.4].

Assuming additionally that w is harmonic, we can conclude with the help of the Laurent series that w behaves asymptotically like a constant function at infinity.

Lemma 2.2.28. *Let $2 \leq d \in \mathbb{N}$, $d < p < \infty$ and $w \in L^p_{loc}(\overline{\Omega})$ be a harmonic function with $\nabla w \in L^p(\Omega)$. Then there is a $v_0 \in \mathbb{R}$ such that $w(x) \to v_0$ for $|x| \to \infty$.*

Proof. At first, we will show, that $|w|$ grows at most like a root function at infinity. Let $R > 0$ be such that $\partial\Omega \subset B(0, R-2)$ and $\eta \in C^\infty(\mathbb{R}^3)$ be a cut-off function with $0 \leq \eta \leq 1$, $\eta = 0$ in $B(0, R-1)$ and $\eta = 1$ in $B(0, R)^C$. Set $u := \eta w$. By Weyl's lemma, we have $w \in C^\infty(\Omega)$ and therefore $u \in C^\infty(\Omega)$. Furthermore, we have $\nabla u \in L^p(\Omega)$ and $w = 0$ in a neighbourhood of $\partial\Omega$. Thus, we can consider u as an element of $C^\infty(\mathbb{R}^d)$ by setting $u(x) = 0$ for $x \in \Omega^C$. Without loss of generality, we may assume that $0 \notin \Omega$. Let $x \in \Omega$ be arbitrary. Making use of Lemma 2.2.27 yields

$$|u(x)| = |u(x) - u(0)| \leq C|x|^{1-\frac{d}{p}}.$$

Because of $u = w$ on $B(0, R)^C$, this implies

$$|w(x)| \leq C|x|^\beta \quad \text{with } \beta = 1 - \frac{d}{p} \in (0,1) \text{ for } |x| > R. \tag{2.38}$$

We will now make use of the expansion of harmonic functions at infinity as Laurent series. By [ABR01, 10.1 Laurent Series], there are harmonic polynomials p_m, q_m, which are homogeneous of order m, such that

$$w(x) = \sum_{m=0}^{\infty} p_m(x) + \sum_{m=0}^{\infty} \frac{q_m(x)}{|x|^{2m+1}} \quad \text{for all } x \in B(0, R)^C.$$

If there was an $m \geq 1$ with $p_m \neq 0$, then $|u(x)|$ would grow at least linearly for $|x| \to \infty$. This would contradict (2.38). Hence, $p_m = 0$ for all $m \geq 1$. As p_0 is a constant function and $\sum_{m=0}^{\infty} q_m(x)/|x|^{2m+1} \to 0$ for $|x| \to \infty$, we get the desired asymptotic behaviour of w. $\square$

It is now rather easy to estimate the decay of gradients of harmonic vector fields at infinity.

Lemma 2.2.29. *Let $p \in (3, \infty)$ and $h \in L^p_{B,har}(\Omega)$. Then $\nabla h \in L^q(\Omega)$ for all $q \in (1, \infty)$.*

Proof. We only consider the case of Neumann fields. The proof for Dirichlet fields works exactly the same. By Lemma 2.2.24, there is a vector potential $v \in L^p_{loc}(\overline{\Omega})$ of h such that $\nabla v \in L^p(\Omega)$, $\operatorname{rot} v = h$ and $\Delta v = 0$. Due to Lemma 2.2.28, there is a $v_0 \in \mathbb{R}$ such that $v(x) \to v_0$ for $|x| \to \infty$. We may assume that $v_0 = 0$. By the harmonicity of v, there are harmonic polynomials p_m and q_m, which are homogeneous of order m such that

$$v(x) = \sum_{m=0}^{\infty} p_m(x) + \sum_{m=0}^{\infty} \frac{q_m(x)}{|x|^{2m-1}} \quad \text{for all } x \in B(0,R)^C. \tag{2.39}$$

Here $R > 0$ is such that $\partial\Omega \subset B(0,R)$. As $v(x) \to 0$ for $|x| \to \infty$, we get $p_m = 0$ for all $m \in \mathbb{N}$. Thus $|v(x)| \leq C|x|^{-1}$ for $|x| \to \infty$. As the sum in (2.39) can be differentiated piecewise, we also have $|\nabla^2 v(x)| \leq C|x|^{-3}$ for $|x| \to \infty$. This means $\nabla h = \nabla \operatorname{rot} v \in L^q(B(0,R)^C)$ for all $1 < q < \infty$. Because of $h \in C^\infty(\overline{\Omega})$, which has been shown in Lemma 2.2.21, this implies $\nabla h \in L^q(\Omega)$ for all $1 < q < \infty$. □

By means of the Sobolev inequality, we are now in the position to show that $\hat{H}^{1,B,har}_2(\Omega) := \dot{\mathbf{H}}^{1,B,har}_2(\Omega) \cap \hat{H}^{1,B}_2(\Omega)$ is contained in $\dot{\mathbf{H}}^{1,B,har}_p(\Omega)$ for any $1 < p < \infty$.

Lemma 2.2.30. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary. Then $\hat{H}^{1,B,har}_2(\Omega) \subseteq \dot{\mathbf{H}}^{1,B,har}_p(\Omega)$ for all $1 < p < \infty$.*

Proof. Let $h \in \hat{H}^{1,B,har}_2(\Omega)$. Then, by Proposition 2.1.10, we get $h \in L^6_{B,har}(\Omega)$. With Lemma 2.2.29, this implies $\nabla h \in L^p(\Omega)$ for any $p \in (1,\infty)$. Due to Proposition 2.1.18, this means $h \in \dot{\mathbf{H}}^{1,B}_p(\Omega)$ and therefore $h \in \dot{\mathbf{H}}^{1,B,har}_p(\Omega)$ for any $1 < p < \infty$. □

As $\dot{\mathbf{H}}^{1,B}_2(\Omega)$ is strictly larger than $\hat{H}^{1,B}_2(\Omega)$ (see Proposition 2.1.14), we have to ask, whether there are additional harmonic vector fields in $\dot{\mathbf{H}}^{1,B}_2(\Omega)$. The next lemma answers this question positively. It even states, that the difference between these two spaces can be filled solely with harmonic vector fields.

Lemma 2.2.31. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary and $1 < p < 3$. Then there are harmonic vector fields $h_1, h_2, h_3 \in \bigcap_{q\in(1,\infty)} \dot{\mathbf{H}}^{1,B}_q(\Omega)$ such that*

$$\dot{\mathbf{H}}^{1,B}_p(\Omega) = \hat{H}^{1,B}_p(\Omega) \oplus span(\{h_1, h_2, h_3\}).$$

Proof. Let $v_1, v_2, v_3 \in \mathbb{R}^3$ be linearly independent. We have to show, that the vector fields w_1, w_2, w_3 in Proposition 2.1.14 can be chosen to be harmonic. We start with the case of vector fields with vanishing normal components. We note that v_k considered as a constant function on Ω fulfils $\operatorname{div} v_k = 0$ and $\operatorname{rot} v_k = 0$. However, it can not fulfil $w_k \times n = 0$ everywhere on $\partial\Omega$. We will correct this by adding a suitable harmonic vector field. Consider the equation

$$(2.40)\qquad \begin{aligned} \Delta q_k &= 0 && \text{in } \Omega,\\ \frac{\partial q_k}{\partial n} &= v_k \cdot n && \text{on } \partial\Omega,\\ q_k(x) &\to 0 && \text{for } |x| \to \infty. \end{aligned}$$

We remark, that $v_k \cdot n$ is a smooth function on $\partial\Omega$ and that

$$\int_{\partial\Omega} v_k \cdot n \,\mathrm{d}S = -\int_{\Omega^C} \operatorname{div} v_k = 0.$$

Hence, the equation (2.40) is well-posed. Because of Proposition 1.4.4, the solution q_k satisfies $\nabla q_k, \nabla^2 q_k \in L^q(\Omega)$ for all $q \in (1,\infty)$. Setting $w_k := v_k - \nabla q_k$, it is a simple computation, that $\operatorname{div} w_k = 0$, $\operatorname{rot} w_k = 0$ in Ω, $w_k \cdot n = 0$ on $\partial\Omega$ and $w_k \in \dot{\mathbf{H}}^{1,T}_p(\Omega)$. Moreover, we also have $w_k(x) \to v_k$ for $|x| \to \infty$. Hence, w_k admits each of the desired properties.

The case of vector fields with vanishing tangential components works similarly. Let $v_1, v_2, v_3 \in \mathbb{R}^3$ be linearly independent. Considering these vectors as constant functions on Ω, we can see, that they are harmonic but do not fulfil the desired boundary conditions. We correct them with the help of the differential equations

$$(2.41)\qquad \begin{aligned} \Delta q_k &= 0 && \text{in } \Omega,\\ q_k(x) &= v_k \cdot x && \text{on } \partial\Omega,\\ q_k(x) &\to 0 && \text{for } |x| \to \infty. \end{aligned}$$

As $v_k \cdot x$ is a smooth function on $\partial\Omega$, [SV04, Theorem 3.2] implies the existence of a unique solution to this problem, which fulfils for all $q \in (1,\infty)$ and $R > 0$ that $q_k \in H^2_q(\Omega_R)$ and $\nabla^2 q_k \in L^q(\Omega)$. We note that $v_k = \nabla(v_k \cdot x)$, and set $w_k(x) := \nabla(v_k \cdot x - q_k(x))$. One can easily see, that $\operatorname{div} w_k = 0$ and $\operatorname{rot} w_k = 0$. Furthermore, since $v_k \cdot x - q_k(x)$ is constantly zero on $\partial\Omega$, we get $w_k \times n = 0$ on $\partial\Omega$. Therefore, w_k is a harmonic vector field in $\dot{\mathbf{H}}^{1,N}_p(\Omega)$. As $w_k(x) \to v_k$ for $|x| \to \infty$, we can see that w_k has all the desired properties. $\square$

Remark 2.2.32. *If one does not ask for a direct decomposition in Lemma 2.2.31, one can cover the full range of $p \in (1,\infty)$. In other words, for each $p \in (1,\infty)$, it holds*

$$\dot{\mathbf{H}}^{1,B}_p(\Omega) = \hat{H}^{1,B}_p(\Omega) + \mathit{span}(\{h_1, h_2, h_3\})$$

with h_1, h_2 and h_3 as in Lemma 2.2.31.

We can now extend Lemma 2.2.30 to nondecaying vector fields.

Lemma 2.2.33. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary. Then $\dot{\mathbf{H}}_2^{1,B,har}(\Omega) \subseteq \dot{\mathbf{H}}_p^{1,B,har}(\Omega)$ for all $1 < p < \infty$.*

Proof. Let $h \in \dot{\mathbf{H}}_2^{1,B,har}(\Omega)$. We know that $\dot{H}_2^{1,B}(\Omega)$ can be directly decomposed into $\hat{H}_2^{1,B}(\Omega)$ and $\operatorname{span}(\{h_1, h_2, h_3\})$, where h_1, h_2, h_3 are chosen as in Lemma 2.2.31. Hence, there are unique $\hat{h} \in \hat{H}_2^{1,B}(\Omega)$ and $k \in \operatorname{span}(\{h_1, h_2, h_3\})$ such that $h = \hat{h} + k$. As k and h are harmonic, the same is the case for $\hat{h}$. This means $\hat{h} \in \hat{H}_2^{1,B,har}(\Omega)$. Due to Lemma 2.2.30, we get $\hat{h} \in \dot{\mathbf{H}}_p^{1,B,har}(\Omega)$ for any $1 < p < \infty$. As we also have $h_1, h_2, h_3 \in \dot{\mathbf{H}}_p^{1,B}(\Omega)$ for any $1 < p < \infty$, the same is true for k. Thus $h \in \dot{\mathbf{H}}_p^{1,B,har}(\Omega)$, too. □

Proof of Theorem 2.2.18. We only have to combine Lemma 2.2.33 and Corollary 2.2.23. □

As a result of the independence of $\dot{\mathbf{H}}_p^{1,B,har}(\Omega)$ in terms of p, we can define as in the case of bounded domains (see Proposition 2.2.4) a consistent and bounded projection from $\dot{\mathbf{H}}_p^{1,B}(\Omega)$ onto $\dot{\mathbf{H}}^{1,B,har}(\Omega)$.

Definition 2.2.34. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary. Let $P\colon \dot{\mathbf{H}}_2^{1,B}(\Omega) \to \dot{\mathbf{H}}_2^{1,B}(\Omega)$ be the orthogonal projection onto the harmonic vector fields given by*

$$Pu = \sum_{k=1}^{N} (\nabla u, \nabla \phi_j)\phi_j,$$

where $(\phi_j)_{j=1}^N$ is an orthonormal basis of $\dot{\mathbf{H}}_2^{1,B,har}(\Omega)$.

Proposition 2.2.35. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary. Then the projection P can be extended to a continuous projection from $\dot{\mathbf{H}}_p^{1,B}(\Omega)$ onto $\dot{\mathbf{H}}_p^{1,B,har}(\Omega)$.*

Proof. Let $u \in \dot{\mathbf{H}}_p^{1,B}(\Omega)$. Because of Theorem 2.2.18, we know that $\phi_j \in \dot{\mathbf{H}}_{p'}^{1,B}(\Omega)$ for all $j = 1, \ldots, N$. Therefore Hölder's inequality yields

$$\|\nabla Pu\|_{L^p(\Omega)} \leq \sum_{j=1}^{N} \|\nabla u\|_{L^p(\Omega)} \|\nabla \phi_j\|_{L^{p'}(\Omega)} \|\nabla \phi_j\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}.$$

Thus, P can be extended to $\dot{\mathbf{H}}_p^{1,B}(\Omega)$ for all $p \in (1, \infty)$. It is easy to see, that the property $P^2 = P$ remains valid for these extension. □

We collect some more properties, which we have already shown for bounded domains.

Corollary 2.2.36. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary and $1 < p < \infty$. Then there is a linear subspace $\tilde{\mathbf{H}}_p^{1,B}(\Omega)$ of $\dot{\mathbf{H}}_p^{1,B}(\Omega)$ such that*

$$\dot{\mathbf{H}}_p^{1,B}(\Omega) = \tilde{\mathbf{H}}_p^{1,B}(\Omega) \oplus \dot{\mathbf{H}}_p^{1,B,har}(\Omega).$$

Furthermore, $C_c^\infty(\Omega) \subset \tilde{\mathbf{H}}_p^{1,B}(\Omega)$.

Proof. The direct decomposition follows by the finite dimension of $\dot{\mathbf{H}}_p^{1,B,har}(\Omega)$. In order to see $C_c^\infty(\Omega) \subset \dot{\mathbf{H}}_p^{1,B}(\Omega)$, let $\psi \in C_c^\infty(\Omega)$ and P be the projection from Proposition 2.2.35. Then we have

$$P\phi = \sum_{j=1}^{N} (\nabla\psi, \nabla\phi_j)\phi_j = -\sum_{j=1}^{N} (\psi, \Delta\phi_j)\phi_j = 0.$$

Hence, $(1 - P)\phi = \phi$ which implies $\phi \in \tilde{\mathbf{H}}_p^{1,B}(\Omega)$. □

Remark 2.2.37. *Due to the independence of p of the space $\dot{\mathbf{H}}_p^{1,B,har}(\Omega)$, we will occasionally drop the subscript p and just write $\dot{\mathbf{H}}^{1,B,har}(\Omega)$.*

Proposition 2.2.38. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary and $1 < p < \infty$. Then the following norms are equivalent on $\dot{\mathbf{H}}_p^{1,B}(\Omega)$:*

$$\|\nabla \cdot\|_{L^p(\Omega)} \sim \|\operatorname{div} \cdot\|_{L^p(\Omega)} + \|\operatorname{rot} \cdot\|_{L^p(\Omega)} + \sum_{j=1}^{N} |(\nabla\cdot, \nabla\phi_j)|,$$

where $(\phi_k)_{k=1}^N$ is an orthonormal basis of $\dot{\mathbf{H}}^{1,B,har}(\Omega)$. Furthermore, on $\tilde{\mathbf{H}}_p^{1,B}(\Omega)$, it holds

$$\|\nabla \cdot\|_{L^p(\Omega)} \sim \|\operatorname{div} \cdot\|_{L^p(\Omega)} + \|\operatorname{rot} \cdot\|_{L^p(\Omega)}.$$

Proof. The proof uses the same arguments as the proof of Proposition 2.2.6 for bounded domains. □

2.2.5 The Weak Poisson Problem in Exterior Domains

In this section, we will solve the weak Poisson problem in exterior domains. Main arguments of the proof will be functional analytic ones. In order to make this clear, we need the following definition: Let $1 < p < \infty$ and recall that the space of harmonic vector fields $\dot{\mathbf{H}}_p^{1,B,har}(\Omega)$ has a direct complement $\tilde{\mathbf{H}}_p^{1,B}(\Omega)$ in

$\dot{\mathbf{H}}_p^{1,B}(\Omega)$, compare Corollary 2.2.36. Define the weak Laplacian with vanishing normal component by

$$S_p^T : \dot{H}_p^{1,T}(\Omega) \to \dot{H}_{p'}^{1,T}(\Omega)', \quad w \mapsto -\Delta w,$$

where $-\Delta w$ has to be understood as

$$\langle S_p^T w, \phi\rangle = (\operatorname{div} w, \operatorname{div} \phi) + (\operatorname{rot} w, \operatorname{rot} \phi), \quad \phi \in \dot{\mathbf{H}}_{p'}^{1,T}(\Omega).$$

We will also consider some kind of restriction of S_p^T, namely

$$\tilde{S}_p^T : \tilde{\mathbf{H}}_p^{1,T}(\Omega) \to \tilde{\mathbf{H}}_{p'}^{1,T}(\Omega)', \quad w \mapsto -\Delta w$$

defined by

$$\langle \tilde{S}_p^T w, \phi\rangle = (\operatorname{div} w, \operatorname{div} \phi) + (\operatorname{rot} w, \operatorname{rot} \phi), \quad \phi \in \tilde{\mathbf{H}}_{p'}^{1,T}(\Omega).$$

Analogously, define the weak Laplacian with vanishing tangential component by

$$\begin{aligned} S_p^N &: \dot{\mathbf{H}}_p^{1,N}(\Omega) \to \dot{\mathbf{H}}_{p'}^{1,N}(\Omega)', \quad w \mapsto -\Delta w, \\ \tilde{S}_p^N &: \tilde{\mathbf{H}}_p^{1,N}(\Omega) \to \tilde{\mathbf{H}}_{p'}^{1,N}(\Omega)', \quad w \mapsto -\Delta w, \end{aligned}$$

where $-\Delta w$ has to be understood as

$$\begin{aligned} \langle S_p^N w, \phi\rangle &= (\operatorname{div} w, \operatorname{div} \phi) + (\operatorname{rot} w, \operatorname{rot} \phi), \quad \phi \in \dot{\mathbf{H}}_{p'}^{1,N}(\Omega), \\ \langle \tilde{S}_p^N w, \phi\rangle &= (\operatorname{div} w, \operatorname{div} \phi) + (\operatorname{rot} w, \operatorname{rot} \phi), \quad \phi \in \tilde{\mathbf{H}}_{p'}^{1,N}(\Omega). \end{aligned}$$

We will make use of the abbreviating notation S_p^B for statements, that are true for S_p^T and S_p^N. The same will be done with $\tilde{S}_p^B$.

The strategy of this section is as follows: By means of a cut-off argument, we will show the consistency with respect to p and a preliminary norm estimate of solutions to the equation $S_p^B u = f$. Due to the consistency, we will then be able to see that the kernel of S_p^B coincides with $\dot{\mathbf{H}}_p^{1,B,har}(\Omega)$ and that $\tilde{S}_p^B$ is injective. Using the duality relation $(\tilde{S}_p^B)' = \tilde{S}_{p'}^B$ and the closed range theorem, we then get the main theorem of this section:

Theorem 2.2.39. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary. Then, for $p \in (1, \infty)$, the operator $\tilde{S}_p^B$ is continuously invertible. In particular, for each $f \in \tilde{\mathbf{H}}_{p'}^{1,B}(\Omega)'$, there is a unique $u \in \tilde{\mathbf{H}}_p^{1,B}(\Omega)$ such that*

$$a(u, \phi) = \langle f, \phi\rangle \quad \textit{for all } \phi \in \tilde{\mathbf{H}}_{p'}^{1,B}(\Omega). \tag{2.42}$$

We start by showing a first norm estimate for solutions of (2.42). We will employ the following lemma, which is stated in [KY98, Lemma 2.3] for Dirichlet zero boundary conditions. However, the proof does not make any use of this condition.

Lemma 2.2.40. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary. Let Ω_0 be a subdomaim of Ω such that $\overline{\Omega_0} \subset \Omega$. If $1 < p < \infty$, then there is a constant $C = C(\Omega, \Omega_0, p) > 0$ such that for all $f \in (\hat{H}^{1,B}_{p'}(\Omega))'$ with $\operatorname{supp}(f) \subseteq \Omega_0$, it holds $(f \in \dot{\mathbf{H}}^{1,B}_{p'}(\mathbb{R}^3))'$ together with*

$$\|f\|_{\dot{\mathbf{H}}^{1,B}_{p'}(\mathbb{R}^3)'} \leq C\|f\|_{\dot{\mathbf{H}}^{1,B}_{p'}(\Omega)'}.$$

The norm estimate on exterior domains now reads as follows:

Lemma 2.2.41. *Let $R > 0$ be such that $\partial\Omega \subset B(0, R-2)$. Let $1 < p < \infty$ and $f \in \dot{\mathbf{H}}^{1,B}_{p'}(\Omega)'$. Assume that $w \in \dot{\mathbf{H}}^{1,B}_{p}(\Omega)$ satisfies*

$$a(w, \phi) = \langle f, \phi\rangle \quad \text{for all } \phi \in C_c^{\infty,B}(\overline{\Omega}).$$

Then there is a constant $C = C(p, R) > 0$ such that

$$\|\nabla w\|_{L^p(\Omega)} \leq C\big(\|f\|_{\dot{\mathbf{H}}^{1,B}_{p'}(\Omega)'} + \|w\|_{L^p(\Omega_{R+1})}\big).$$

Proof. Let ψ_1, ψ_2, Ω_1, Ω_2 as well as (S_1) and (S_2) as in the proof of Lemma 2.2.22. By Proposition 2.2.7 and Proposition 2.2.14, we have

$$\|\nabla(\psi_1 w)\|_{L^p(\Omega_1)} \leq C[\|f_1\|_{\dot{H}^{1,T}_{p'}(\Omega_1)'} + \|w\|_{L^p(\Omega_{R+1})}], \tag{2.43}$$

and

$$\|\nabla(\psi_2 w)\|_{L^p(\Omega_2)} \leq C\|f_2\|_{\dot{H}^{1,T}_{p'}(\Omega_2)'} \tag{2.44}$$

Therefore, we have to estimate f_1 and f_2 in the respective norms. We will handle each of their summands on their own and start with $\psi_k f$. Let $\phi_1 \in C^{\infty,B}(\overline{\Omega_1})$ and $\phi_2 \in C_c^\infty(\Omega_2)$. Recall, that $\operatorname{supp}\psi_1 \subset B(0,R)$. We have

$$\begin{aligned}
|\langle\psi_1 f, \phi_1\rangle| &= |\langle f, \psi_1\phi_1\rangle| \\
&\leq C\|f\|_{\dot{\mathbf{H}}^{1,B}_{p'}(\Omega)'}\|\nabla(\psi_1\phi_1)\|_{L^{p'}(\Omega)} \\
&\leq C\|f\|_{\dot{\mathbf{H}}^{1,B}_{p'}(\Omega)'}\big[\|(\nabla\psi_1)\phi_1\|_{L^{p'}(\Omega_1)} + \|\psi_1\nabla\phi_1\|_{L^{p'}(\Omega_1)}\big] \\
&\leq C\|f\|_{\dot{\mathbf{H}}^{1,B}_{p'}(\Omega)'}\big[\|\nabla\psi\|_{L^\infty(\Omega)}\|\phi\|_{L^{p'}(\Omega_1)} + \|\psi\|_{L^\infty(\Omega)}\|\nabla\phi\|_{L^{p'}(\Omega_1)}\big] \\
&\leq C\|f\|_{\dot{\mathbf{H}}^{1,B}_{p'}(\Omega)'}\|\nabla\phi\|_{L^{p'}(\Omega_1)},
\end{aligned}$$

where we have used Proposition 2.1.4 in the last line. By the density of $C^{\infty,B}(\overline{\Omega_1})$ in $\dot{\mathbf{H}}^{1,B}_{p'}(\Omega_1)$ (see Corollary 2.1.6), this implies $\psi_1 f \in \dot{\mathbf{H}}^{1,B}_{p'}(\Omega_1)'$ with

$$\|\psi_1 f\|_{\dot{\mathbf{H}}^{1,B}_{p'}(\Omega_1)'} \leq C\|f\|_{\dot{\mathbf{H}}^{1,B}_{p'}(\Omega)'}.$$

Because of $\operatorname{supp}(\psi_2 f) \subset B(0, R-1)^C$, we can apply Lemma 2.2.40 and get

$$\|\psi_2 f\|_{\dot{\mathbf{H}}^{1}_{p'}(\Omega_2)'} \leq C\|\psi_2 f\|_{\dot{\mathbf{H}}^{1,B}_{p'}(\Omega_2)'} \leq C\|f\|_{\dot{\mathbf{H}}^{1,B}_{p'}(\Omega)'}.$$

In order to estimate the other summands of f_k, recall that the supports of $\nabla\psi_k$ and $\Delta\psi_k$ are contained in Ω_R. Therefore, we have

$$\begin{aligned}
|\langle \nabla\psi_1 \nabla w, \phi_1\rangle| &= |\langle w, \operatorname{div}(\nabla\psi_1\phi_1)\rangle| \\
&\leq \|w\|_{L^p(\Omega_R)}\|\nabla(\nabla\psi_1\phi_1)\|_{L^{p'}(\Omega_R)} \\
&\leq \|w\|_{L^p(\Omega_R)}\left(\|\nabla^2\psi_1\phi_1\|_{L^{p'}(\Omega_R)} + \|\nabla\psi_1\nabla\phi_1\|_{L^{p'}(\Omega_R)}\right) \\
&\leq C\|w\|_{L^p(\Omega_R)}\|\nabla\phi_1\|_{L^{p'}(\Omega_1)},
\end{aligned}$$

as well as

$$\begin{aligned}
|\langle \Delta\psi_1 w, \phi_1\rangle| &\leq |\langle w, \Delta\psi_1\phi_1\rangle| \\
&\leq C\|w\|_{L^p(\Omega_R)}\|\phi_1\|_{L^{p'}(\Omega_R)} \\
&\leq C\|w\|_{L^p(\Omega_R)}\|\nabla\phi_1\|_{L^{p'}(\Omega_1)},
\end{aligned}$$

where we have used Proposition 2.1.4 in the last inequalities respectively. Therefore we have

$$\|f_1\|_{\dot{\mathbf{H}}^{1,B}_{p'}(\Omega_1)'} \leq C(\|f\|_{\dot{\mathbf{H}}^{1,B}_{p'}(\Omega)'} + \|w\|_{L^p(\Omega_{R+1})}). \tag{2.45}$$

In order to obtain an analogous estimate for $k = 2$, we show that $\nabla\psi_2\nabla w \in \hat{H}^{1,B}_{p'}(\Omega)'$, $(\Delta\psi_2)w \in \hat{H}^{1,B}_{p'}(\Omega)'$ with norms bounded by $\|w\|_{L^p(\Omega_R)}$. By Lemma 2.2.40, we then obtain $f_2 \in \dot{\mathbf{H}}^{1,B}_{p'}(\mathbb{R}^3)'$. Let $\phi \in C^{\infty,B}_c(\overline{\Omega})$. Then

$$\begin{aligned}
|\langle \Delta\psi_2 w, \phi\rangle| = |\langle w, \Delta\psi_2\phi\rangle| &\leq \|w\|_{L^p(\Omega_R)}\|\Delta\psi_2\phi\|_{L^{p'}(\Omega_R)} \\
&\leq C\|w\|_{L^p(\Omega_R)}\|\nabla\phi\|_{L^{p'}(\Omega)}
\end{aligned}$$

and

$$|\langle \nabla\psi_2\nabla w, \phi\rangle| = |\langle w, \operatorname{div}(\nabla\psi_2\phi)\rangle| \leq \|w\|_{L^p(\Omega)}\|\nabla\phi\|_{L^{p'}(\Omega)},$$

where we have again used Proposition 2.1.4. Due to the density of smooth functions in $\hat{H}^{1,B}_{p'}(\Omega)$ this implies

$$\|\nabla\psi_2\nabla w\|_{\hat{H}^{1,B}_{p'}(\Omega)'}, \|(\Delta\psi_2)w\|_{\hat{H}^{1,B}_{p'}(\Omega)} \leq \|w\|_{L^p(\Omega_R)}.$$

Therefore, we have

$$\|f_2\|_{(\dot{\mathbf{H}}^1_{p'}(\Omega_2))'} \leq C(\|f\|_{\dot{\mathbf{H}}^{1,B}_{p'}(\Omega)'} + \|w\|_{L^p(\Omega_{R+1})}).$$

Combining this estimate and (2.45) with (2.44) and (2.43) yields the desired result. □

A rather simple consequence is the analogous result for the operator $\tilde{S}^B_p$.

Lemma 2.2.42. *Let $R > 0$ be such that $\partial\Omega \subset B(0, R-2)$. Let $1 < p < \infty$, $f \in \tilde{\mathbf{H}}^{1,B}_{p'}(\Omega)'$. Assume that $w \in \tilde{\mathbf{H}}^{1,B}_p(\Omega)$ fulfils*

$$a(w, \phi) = \langle f, \phi\rangle \quad \text{for all } \phi \in \tilde{\mathbf{H}}^{1,B}_{p'}(\Omega).$$

Then there is a constant $C = C(p, R) > 0$ such that

$$\|\nabla w\|_{L^p(\Omega)} \leq C\big(\|f\|_{\tilde{\mathbf{H}}^{1,B}_{p'}(\Omega)'} + \|w\|_{L^p(\Omega_{R+1})}\big).$$

Proof. We will reduce the situation here to the one of Lemma 2.2.41. Recall that we have by Corollary 2.2.36 that $\dot{\mathbf{H}}^{1,B}_p(\Omega)$ can be directly decomposed into $\tilde{\mathbf{H}}^{1,B}_p(\Omega) \oplus \dot{\mathbf{H}}^{1,B,har}(\Omega)$. Let $g \in \dot{\mathbf{H}}^{1,B}_{p'}(\Omega)'$ be the zero extension of f. Using the projection P from Proposition 2.2.35, that means

$$\langle g, \phi\rangle = \langle f, (1-P)\phi\rangle$$

for all $\phi \in \dot{\mathbf{H}}^{1,B}_{p'}(\Omega)$. Consider w as an element of $\dot{\mathbf{H}}^{1,B}_p(\Omega)$. Then w is a solution to

$$a(w, \phi) = \langle g, \phi\rangle \quad \text{for all } \phi \in \dot{H}^{1,B}_{p'}(\Omega).$$

Hence, we can apply Lemma 2.2.41, which yields

$$\|\nabla w\|_{L^p(\Omega)} \leq C[\|g\|_{\dot{\mathbf{H}}^{1,B}_{p'}(\Omega)'} + \|w\|_{L^p(\Omega_{R+1})}].$$

As $\|g\|_{\dot{\mathbf{H}}^{1,B}_{p'}(\Omega)'} = \|f\|_{\tilde{\mathbf{H}}^{1,B}_{p'}(\Omega)'}$, this completes the proof. □

We use the preceding lemmata to conclude first structural properties of the kernels and ranges of S^B_p and $\tilde{S}^B_p$.

Lemma 2.2.43. *Let $1 < p < \infty$ and $\Omega \subset \mathbb{R}^3$ be an exterior domain with smooth boundary.*

1. *It holds* $\ker(S_p) = \dot{\mathbf{H}}^{1,B,har}(\Omega)$.

2. *There is a constant $C = C(p,\Omega) > 0$ such that for all $w \in \tilde{\mathbf{H}}_p^{1,B}(\Omega)$, it holds*

$$\|\nabla w\|_{L^p(\Omega)} \leq C\|S_p^B w\|_{\dot{H}_{p'}^{1,B}(\Omega)'}. \tag{2.46}$$

3. *The range $R(S_p)$ is a closed subspaces of $\dot{\mathbf{H}}_{p'}^{1,B}(\Omega)'$.*

4. *The operator $\tilde{S}_p^B$ is injective and its range is a closed subspace of $\tilde{\mathbf{H}}_{p'}^{1,B}(\Omega)'$.*

Proof. Let $u \in \ker(S_p^B)$. This means, we have

$$(\operatorname{div} u, \operatorname{div} \phi) + (\operatorname{rot} u, \operatorname{rot} \phi) = 0 \tag{2.47}$$

for every $\phi \in \dot{\mathbf{H}}_{p'}^{1,B}(\Omega)$. Due to Lemma 2.2.22, we get that $u \in \dot{\mathbf{H}}_2^{1,B}(\Omega)$. Recall, that $\dot{\mathbf{H}}_p^{1,B}(\Omega) \cap \dot{\mathbf{H}}_2^{1,B}(\Omega)$ is dense in $\dot{\mathbf{H}}_2^{1,B}(\Omega)$ by Corollary 2.1.17. Therefore, the equality (2.47) is also valid for all $\phi \in \dot{\mathbf{H}}_2^{1,B}(\Omega)$. This allows us to choose $\phi = u$, which yields

$$\|\operatorname{div} u\|_{L^2(\Omega)}^2 + \|\operatorname{rot} u\|_{L^2(\Omega)}^2 = 0.$$

Thus, $u \in \dot{\mathbf{H}}^{1,B,har}(\Omega)$.

We can now show (2.46) by contraposition. Assume that (2.46) is not true. Then there is a sequence $(w_j)_{j\in\mathbb{N}} \subset \tilde{\mathbf{H}}_p^{1,B}(\Omega)$ such that

$$\begin{aligned} &\|\nabla w_j\|_{L^p(\Omega)} = 1, \\ &\|f_j\|_{\dot{\mathbf{H}}_{p'}^{1,B}(\Omega)'} \to 0 \quad \text{for } j \to \infty, \end{aligned}$$

where $f_j = S_p^B w_j$. By Proposition 2.1.4, the embedding $\dot{\mathbf{H}}_p^{1,B}(\Omega) \hookrightarrow L^p(\Omega_R)$ is compact. Hence, there is a subsequence of $(w_j)_{j\in\mathbb{N}}$ that converges in $L^p(\Omega_R)$. We do not rename that subsequence. In view of the estimate given by Lemma 2.2.41, this already means that $(w_j)_{j\in\mathbb{N}}$ is a convergent sequence in $\dot{\mathbf{H}}_p^{1,B}(\Omega)$ by the Cauchy criterion. We denote its limit by $w \in \tilde{\mathbf{H}}_p^{1,B}(\Omega)$. Using the boundedness of S_p^B, we get that

$$\|S_p w\|_{\dot{\mathbf{H}}_{p'}^{1,B}(\Omega)'} = \lim_{j\to\infty} \|S_p w_j\|_{\dot{\mathbf{H}}_{p'}^{1,B}(\Omega)'} = \lim_{j\to\infty} \|f_j\|_{\dot{\mathbf{H}}_{p'}^{1,B}(\Omega)'} = 0.$$

Hence, $w \in \ker(S_p^B) \cap \tilde{\mathbf{H}}_p^{1,B}(\Omega)$ and due to the direct decomposition $\dot{\mathbf{H}}_p^{1,B}(\Omega) = \ker(S_p^B) \oplus \tilde{\mathbf{H}}_p^{1,B}(\Omega)$, this means $w = 0$. But this contradicts $w_j \to w$ in $\dot{\mathbf{H}}_p^{1,B}(\Omega)$ and $\|\nabla w_j\|_{L^p(\Omega)} = 1$. Therefore, (2.46) is valid and the second claim is proven.

The closedness of the range of S_p^B follows from the estimate (2.46). Let $(f_j)_{j\in\mathbb{N}} \subset R(S_p^B)$ be a Cauchy sequence with limit $f \in \dot{\mathbf{H}}_{p'}^{1,B}(\Omega)'$. We have to show, that $f \in R(S_p^B)$. For each $j \in \mathbb{N}$, there is a unique $w_j \in \tilde{\mathbf{H}}_p^{1,B}(\Omega)$ such that $S_p^B w_j = f_j$. Due to (2.46) and the linearity of S_p^B, we have

$$\|\nabla(w_j - w_k)\|_{L^p(\Omega)} \leq C\|f_j - f_k\|_{\dot{\mathbf{H}}_{p'}^{1,B}(\Omega)'}$$

for any $j, k \in \mathbb{N}$. Therefore, $(w_j)_{j\in\mathbb{N}}$ is a Cauchy sequence in $\dot{\mathbf{H}}_p^{1,B}(\Omega)$ with some limit w. By the continuity of S_p^B, it holds

$$S_p^B w = \lim_{j\to\infty} S_p^B w_j = \lim_{j\to\infty} f_j = f.$$

Hence, $f \in R(S_p^B)$.

We do now show that $\ker(\tilde{S}_p^B) = \ker(S_p^B) \cap \tilde{\mathbf{H}}_p^{1,B}(\Omega)$. The inclusion $\ker(\tilde{S}_p^B) \supseteq \ker(S_p^B) \cap \tilde{\mathbf{H}}_p^{1,B}(\Omega)$ is simple. For the other one, fix $u \in \ker(\tilde{S}_p^B)$ and let $\phi \in \dot{\mathbf{H}}_{p'}^{1,B}(\Omega)$ be arbitrary. Decompose $\phi = \tilde{\phi} + \phi^h$, where $\tilde{\phi} \in \tilde{\mathbf{H}}_{p'}^{1,B}(\Omega)$ and $\phi^h \in \dot{\mathbf{H}}^{1,B,har}(\Omega)$. Then we have

$$a(u, \phi) = a(u, \tilde{\phi}) + a(u, \phi^h).$$

The first summand on the right-hand side is zero as $u \in \ker(\tilde{S}_p^B)$, the second one is zero because ϕ^h is harmonic. Hence, $u \in \ker(S_p^B) \cap \tilde{\mathbf{H}}_p^{1,B}(\Omega)$. Because of $\ker(S_p^B) = \dot{\mathbf{H}}^{1,B,har}(\Omega)$, this yields $\ker(\tilde{S}_p^B) = \{0\}$ and therefore the injectivity of $\tilde{S}_p^B$.

The closedness of the range of $\tilde{S}_p^B$ can be shown in the same way as the closedness of the range of S_p^B but using Lemma 2.2.42 instead of Lemma 2.2.41. $\square$

Proof of Theorem 2.2.39. We have seen in Lemma 2.2.43, that $\tilde{S}_p^B$ is injective for any $p \in (1, \infty)$. The surjectivity of $\tilde{S}_p^B$ follows from the injectivity of $\tilde{S}_{p'}^B$ via the closed range theorem, again Lemma 2.2.43, and the relation

$$(\tilde{S}_p^B)' = \tilde{S}_{p'}^B.$$

Therefore, $\tilde{S}_p^B$ is bijective. The continuity of its inverse follows by the closed graph theorem. $\square$

Theorem 2.2.39 and the characterization of the kernel of S_p^B given by Lemma 2.2.43 imply the following existence result for S_p^B:

Corollary 2.2.44. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary and $p \in (1, \infty)$. Then, for a given $f \in \dot{\mathbf{H}}_{p'}^{1,B}(\Omega)'$, there is a $u \in \dot{\mathbf{H}}_p^{1,B}(\Omega)$ such that*

$$a(u, \phi) = \langle f, \phi \rangle \quad \textit{for all } \phi \in \dot{\mathbf{H}}_{p'}^{1,B}(\Omega)$$

if and only if $\langle f, h\rangle = 0$ *for all* $h \in \dot{\mathbf{H}}^{1,B,har}(\Omega)$. *Moreover, the solution is unique up to vector fields from* $\dot{\mathbf{H}}^{1,B,har}(\Omega)$ *and the unique solution* $w \in \tilde{\mathbf{H}}^{1,B}_p(\Omega)$ *satisfies*

$$\|\nabla w\|_{L^p(\Omega)} \le C\|f\|_{\dot{\mathbf{H}}^{1,B}_{p'}(\Omega)'}$$

for some $C > 0$ *being independent of* f.

2.2.6 Divergence Free Solutions

In view of the Helmholtz-Hodge decomposition, it will be necessary to find for each $u \in L^p(\Omega)$ a solution $w \in \dot{\mathbf{H}}^{1,B}_p(\Omega)$ to

$$a(w, \phi) = (u, \operatorname{rot}\phi) \quad \text{for all } \phi \in \dot{\mathbf{H}}^{1,B}_{p'}(\Omega), \tag{2.48}$$

which is solenoidal. In this section, we will show that solutions to that problem given by Corollary 2.2.44 are automatically divergence free. The strategy to do that is as follows. We will first consider the Hilbert space case $p = 2$. By the lemma of Lax-Milgram, we will show, that there is a unique solution to

$$\begin{aligned}(\operatorname{rot} w, \operatorname{rot}\phi) &= (u, \operatorname{rot}\phi),\\ \operatorname{div} w &= 0\end{aligned}$$

for all ϕ in a suitable subspace of $\dot{\mathbf{H}}^{1,B}_2(\Omega)$. With the help of a Helmholtz type decomposition in $\dot{\mathbf{H}}^{1,T}_2(\Omega)$ and $\dot{\mathbf{H}}^{1,N}_2(\Omega)$, we will then see, that this solution coincides with the one of (2.48). The case of general $p \in (1, \infty)$ can afterwards be established by some approximation argument.

We begin by showing a decomposition of vector fields from $\dot{\mathbf{H}}^{1,T}_2(\Omega)$ and $\dot{\mathbf{H}}^{1,N}_2(\Omega)$ into a divergence-free and a rotation-free part. Define the spaces

$$\begin{aligned}\hat{H}^{1,B,\sigma}_2(\Omega) &:= \{u \in \hat{H}^{1,B}_2(\Omega) : \operatorname{div} u = 0\},\\ \dot{\mathbf{H}}^{1,B,\sigma}_2(\Omega) &:= \{u \in \dot{\mathbf{H}}^{1,B}_2(\Omega) : \operatorname{div} u = 0\}.\end{aligned}$$

Lemma 2.2.45. *Let* $\Omega \subset \mathbb{R}^3$ *be an exterior domain with* C^∞*-boundary. Let* $\phi \in \dot{\mathbf{H}}^{1,T}_2(\Omega)$. *Then there is a divergence-free vector field* $\phi_\sigma \in \hat{H}^{1,T,\sigma}_2(\Omega)$ *and a rotation-free vector field* $\phi_{rot} \in \dot{\mathbf{H}}^{1,T}_2(\Omega)$ *such that* $\phi = \phi_\sigma + \phi_{rot}$.

Proof. We start by considering vector fields, which decay at infinity. Let $\phi \in \hat{H}^{1,T}_2(\Omega)$. We make use of the classical Helmholtz decomposition in Lebesgue spaces in exterior domains, as it was proven in [Miy82]. Note that ϕ is an element of $L^6(\Omega)$ with $\|\phi\|_{L^6(\Omega)} \le C\|\nabla\phi\|_{L^2(\Omega)}$ by Proposition 2.1.10. Consider the problems

$$\Delta p_1 = \operatorname{div}\tilde{\phi} \quad \text{in } \mathbb{R}^3, \tag{2.49}$$

and

$$(2.50)\qquad \begin{aligned} \Delta p_2 &= 0 && \text{in } \Omega, \\ \partial_n p_2 &= (\phi - \nabla p_1)\cdot n && \text{on } \partial\Omega, \\ |p_2(x)| &\to 0 && \text{for } |x|\to\infty. \end{aligned}$$

Here $\tilde{\phi} = E\phi$ is an extension of ϕ to $\mathbb{R}^3$ in the sense of Proposition 2.1.9. Both of these problems are uniquely solvable with $p_1 \in L^6_{loc}(\mathbb{R}^3)$, $\nabla p_1 \in L^6(\mathbb{R}^3)$, $\nabla^2 p_1 \in L^2(\mathbb{R}^3)$ and $p_2 \in L^6_{loc}(\Omega)$, $\nabla p_2 \in L^6(\Omega)$, $\nabla^2 p_2 \in L^2(\Omega)$. See Lemma 1.4.2 and Proposition 1.4.4. Set $p := p_1|_\Omega + p_2$. Then we directly have $\nabla p \in L^6(\Omega)$ and $\nabla^2 p \in L^2(\Omega)$. Regarding the boundary condition of ∇p, we have

$$\nabla p \cdot n = \nabla(p_1 + p_2)\cdot n = \nabla(p_1)\cdot n + (\phi - \nabla p_1)\cdot n = \phi\cdot n = 0.$$

Hence, $\nabla p \in \hat{H}^{1,T}_2(\Omega)$. Setting now $\phi_\sigma := \phi - \nabla p$, we can see easily, that $\phi_\sigma \in \hat{H}^{1,T,\sigma}_2(\Omega)$. For general $u \in \dot{\mathbf{H}}^{1,T}_2(\Omega)$, we make use of the direct decomposition $\dot{\mathbf{H}}^{1,T}_2(\Omega) = \hat{H}^{1,T}_2(\Omega) \oplus \operatorname{span}(\{h_1, h_2, h_3\})$ as it is given in Lemma 2.2.31. By this decomposition, there are a unique $\hat{u} \in \hat{H}^{1,T}_2(\Omega)$ and a $u^h \in \operatorname{span}(\{h_1, h_2, h_3\})$ such that $u = \hat{u} + u^h$. We have already seen, that there are $\hat{u}_\sigma \in \hat{H}^{1,T,\sigma}_2(\Omega)$ and $\nabla\hat{p} \in \hat{H}^{1,T}_2(\Omega)$ such that $\hat{u} = \hat{u}_\sigma + \nabla\hat{p}$. Therefore, $u = \hat{u}_\sigma + (\nabla\hat{p} + u^h) =: u_\sigma + u_{rot}$ is the desired decomposition of u into a divergence-free and a rotation-free part. $\square$

The next lemma is the analogous statement for vector fields with vanishing tangential component. It is proven similarly.

Lemma 2.2.46. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary. Then for any $\phi \in \dot{\mathbf{H}}^{1,N}_2(\Omega)$ there are a divergence-free $\phi_\sigma \in \hat{H}^{1,N}_2(\Omega)$ and a rotation-free $\phi_{rot} \in \dot{\mathbf{H}}^{1,N}_2(\Omega)$ such that $\phi = \phi_\sigma + \phi_{rot}$.*

Proof. Let $\phi \in \hat{H}^{1,N}_2(\Omega)$. Consider the problems

$$(2.51)\qquad \Delta p_1 = \operatorname{div}\tilde{\phi} \quad \text{in } \mathbb{R}^3,$$

and

$$(2.52)\qquad \begin{aligned} \Delta p_2 &= 0 && \text{in } \Omega, \\ p_2 &= p_1 && \text{on } \partial\Omega, \\ |p_2(x)| &\to 0 && \text{for } |x|\to\infty, \end{aligned}$$

where $\tilde{\phi}$ is an extension of ϕ to $\mathbb{R}^3$ in the sense of Proposition 2.1.9. It is known, that there is a unique solution p_1 to (2.51) with $p_1 \in L^6_{loc}(\mathbb{R}^3)$, $\nabla p_1 \in L^6(\mathbb{R}^3)$, $\nabla^2 p_1 \in L^2(\mathbb{R}^3)$. Note that $p_1|_{\partial\Omega} \in W^{3/2,2}(\partial\Omega) \cap W^{5/6,6}(\partial\Omega)$. Hence, by [SV04, Theorem 3.2], there is a unique solution $p_2 \in L^2_{loc}(\overline{\Omega})$ with $\nabla^2 p_2 \in L^2(\Omega)$ and $|p_2(x)| \sim 1/|x|$ for $|x| \to \infty$. Making use of the Laurent expansion of p_2 at infinity, that implies $|\nabla p_2(x)| \sim 1/|x|^2$ and therefore in particular $\nabla p_2 \in L^6(\Omega)$. Set $p := p_1|_\Omega - p_2$. Then $\nabla p_1 \in L^6(\Omega)$ and $\nabla^2 p \in L^2(\Omega)$. Furthermore $p(x) = 0$ for all $x \in \partial\Omega$. Hence, $\nabla p \times n = 0$, which implies $\nabla p \in \hat{H}^{1,N}_2(\Omega)$. Setting $\phi_\sigma = \phi - \nabla p$, it is easy to see, that $\phi_\sigma \in \hat{H}^{1,N,\sigma}_2(\Omega)$. We do now consider general $\phi \in \dot{\mathbf{H}}^{1,N}_2(\Omega)$. Using the decomposition $\dot{\mathbf{H}}^{1,N}_2(\Omega) = \hat{H}^{1,N}_2(\Omega) \oplus \operatorname{span}(\{h_1, h_2, h_3\})$ as it is given in Lemma 2.2.31, we get a unique $\hat{\phi} \in \hat{H}^{1,N}_2(\Omega)$ and a unique harmonic $h \in \operatorname{span}(\{h_1, h_2, h_3\})$ such that $\phi = \hat{\phi} + h$. We have already shown, that $\hat{\phi}$ can be decomposed into $\hat{\phi} = \hat{\phi}_\sigma + \nabla\hat{p}$ with $\hat{\phi}_\sigma \in \hat{H}^{1,N,\sigma}_2(\Omega)$ and $\nabla\hat{p} \in \hat{H}^{1,N}_2(\Omega)$. This means $\phi = \phi_\sigma + \phi_{rot} := \hat{\phi}_\sigma + (\nabla\hat{p} + h)$, where ϕ_{rot} is rotation-free. □

For the weak Poisson problem, it has been useful to work in a space without harmonic vector fields. This remains valid here, too. Thus we introduce the space $\dot{\mathbf{H}}^{1,B,\sigma}_2(\Omega)$, by the direct decomposition

$$\dot{\mathbf{H}}^{1,B,\sigma}_2(\Omega) = \tilde{\mathbf{H}}^{1,B,\sigma}_2(\Omega) \oplus \dot{\mathbf{H}}^{1,B,har}(\Omega).$$

Lemma 2.2.47. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary. Let $u \in L^2(\Omega)$. Then there is a unique $w \in \tilde{\mathbf{H}}^{1,B,\sigma}_2(\Omega)$ such that*

$$(\operatorname{rot} w, \operatorname{rot}\phi) = (u, \operatorname{rot}\phi) \quad \text{for all } \phi \in \tilde{\mathbf{H}}^{1,B,\sigma}_2(\Omega). \tag{2.53}$$

This solution w also solves

$$a(w,\phi) = (u, \operatorname{rot}\phi) \quad \text{for all } \phi \in \dot{\mathbf{H}}^{1,B}_2(\Omega). \tag{2.54}$$

Proof. It is easy to see that $(\operatorname{rot}\cdot, \operatorname{rot}\cdot)$ is a continuous bilinear form on $\tilde{\mathbf{H}}^{1,B,\sigma}_2(\Omega) \times \tilde{\mathbf{H}}^{1,B,\sigma}_2(\Omega)$. Furthermore, it follows from Proposition 2.2.38, that $\|\operatorname{rot}\cdot\|_{L^2(\Omega)}$ is a norm on $\tilde{\mathbf{H}}^{1,B,\sigma}_2(\Omega)$ that is equivalent to $\|\nabla\cdot\|_{L^2(\Omega)}$. Therefore, $(\operatorname{rot}\cdot, \operatorname{rot}\cdot)$ is also coercive on $\tilde{\mathbf{H}}^{1,B,\sigma}_2(\Omega)$. Finally, the map $\phi \mapsto (u, \operatorname{rot}\phi)$ is a bounded functional on $\tilde{\mathbf{H}}^{1,B,\sigma}_2(\Omega)$ by Hölder's inequality. Thus, we get the unique solvability of (2.53) by the lemma of Lax-Milgram.

Let $\phi \in \dot{\mathbf{H}}^{1,B}_2(\Omega)$ be arbitrary. Then, by Lemma 2.2.45 and Lemma 2.2.46, there are $\phi_\sigma \in \hat{H}^{1,B}_2(\Omega)$ and $\phi_{rot} \in \dot{\mathbf{H}}^{1,B}_2(\Omega)$ with $\operatorname{rot}\phi_{rot} = 0$ such that $\phi = \phi_\sigma + \phi_{rot}$. As w is a solution to (2.53), we get

$$\begin{aligned} a(w,\phi) &= (\operatorname{div} w, \operatorname{div}\phi) + (\operatorname{rot} w, \operatorname{rot}\phi) \\ &= (\operatorname{rot} w, \operatorname{rot}\phi) \\ &= (\operatorname{rot} w, \operatorname{rot}\phi_\sigma) + (\operatorname{rot} w, \operatorname{rot}\phi_{rot}) \\ &= (u, \operatorname{rot}\phi_\sigma) + 0 \\ &= (u, \operatorname{rot}\phi_\sigma) + (u, \operatorname{rot}\phi_{\mathrm{rot}}) \\ &= (u, \operatorname{rot}\phi). \end{aligned}$$

Thus, w is also a solution to (2.54). $\square$

We already know from Corollary 2.2.44, that there is a unique solution $v \in \tilde{\mathbf{H}}_2^{1,B}(\Omega)$ to (2.54). By the uniqueness, it has to coincide with the solution w given by Lemma 2.2.47. Hence, Lemma 2.2.47 has the following consequence.

Corollary 2.2.48. *Let $w \in \tilde{\mathbf{H}}_2^{1,B}(\Omega)$ be the solution to*

$$a(w, \phi) = (u, \operatorname{rot} \phi) \quad \textit{for all } \phi \in \tilde{\mathbf{H}}_2^{1,B}(\Omega).$$

Then $\operatorname{div} w = 0$.

We can directly generalise Lemma 2.2.47 to general Lebesgue spaces by an approximation argument.

Theorem 2.2.49. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary. Let $1 < p < \infty$ and $u \in L^p(\Omega)$. Then there is a unique $w \in \tilde{\mathbf{H}}_p^{1,B}(\Omega)$ that solves*

$$\begin{aligned} (\operatorname{rot} w, \operatorname{rot} \phi) &= (u, \operatorname{rot} \phi) \quad \textit{for all } \phi \in \dot{\mathbf{H}}_{p'}^{1,B}(\Omega), \\ \operatorname{div} w &= 0. \end{aligned} \tag{2.55}$$

This solution is subject to the estimate

$$\|\nabla w\|_{L^p(\Omega)} \leq C \|u\|_{L^p(\Omega)}, \tag{2.56}$$

where $C = C(\Omega, p) > 0$. If $v \in \dot{\mathbf{H}}_p^{1,B}(\Omega)$ is another solution to (2.55)*, then $w - v \in \dot{\mathbf{H}}^{1,B,har}(\Omega)$.*

Proof. Let $u \in L^p(\Omega)$ and $(u_n)_{n\in\mathbb{N}} \subset L^p(\Omega) \cap L^2(\Omega)$ be a sequence, that converges to u in $L^p(\Omega)$. Due to Hölder's inequality, the maps $\phi \mapsto (u, \operatorname{rot} \phi)$ and $\phi \mapsto (u_n, \operatorname{rot} \phi)$ are bounded linear functionals on $\dot{\mathbf{H}}_{p'}^{1,B}(\Omega)$. Thus, by Corollary 2.2.44, there is a unique solution $w \in \tilde{\mathbf{H}}_p^{1,B}(\Omega)$ to

$$a(w, \phi) = (u, \operatorname{rot} \phi) \quad \text{for all } \phi \in \dot{\mathbf{H}}_{p'}^{1,B}(\Omega). \tag{2.57}$$

This solution satisfies

$$\|\nabla w\|_{L^p(\Omega)} \leq C \|(u, \operatorname{rot} \cdot)\|_{\dot{\mathbf{H}}_{p'}^{1,B}(\Omega)'} \leq C \|u\|_{L^p(\Omega)}.$$

By the same reason, there is a unique solution $w_n \in \tilde{\mathbf{H}}_p^{1,B}(\Omega)$ to

$$a(w_n, \phi) = (u_n, \operatorname{rot} \phi) \quad \text{for all } \phi \in \dot{\mathbf{H}}_{p'}^{1,B}(\Omega) \tag{2.58}$$

for each $n \in \mathbb{N}$. We now use that $\phi \mapsto (u_n, \operatorname{rot}\phi)$ is a bounded linear functional on $\dot{\mathbf{H}}_2^{1,B}(\Omega)$. Due to the consistency of problem (2.58) stated in Lemma 2.2.22, we get that $w_n \in \tilde{\mathbf{H}}_2^{1,B}(\Omega)$. Hence, Corollary 2.2.48 implies $\operatorname{div} w_n = 0$. As the solution operator to (2.57) is continuous from $\tilde{\mathbf{H}}_{p'}^{1,B}(\Omega)'$ to $\tilde{\mathbf{H}}_p^{1,B}(\Omega)$, we have $w_n \to w$ in $\dot{\mathbf{H}}_p^{1,B}(\Omega)$. This does also imply

$$\operatorname{div} w = \lim_{n\to\infty} \operatorname{div} w_n = 0.$$

Therefore, w is also a solution to (2.55). It remains to prove the uniqueness of solutions to (2.55) in $\tilde{\mathbf{H}}_p^{1,B}(\Omega)$. We note that every solution to (2.55) is a solution to (2.57). Because the solutions to the latter problem are unique up to vector fields from $\dot{\mathbf{H}}^{1,B,har}(\Omega)$ due to Corollary 2.2.44, we get the desired result. □

2.3 First Helmholtz-Hodge Decomposition

We are now in the position to prove the first Helmholtz-Hodge decomposition for vector fields in exterior domains, which is a refinement of the classical Helmholtz decomposition. We recall the definition of the space

$$\dot{\mathbf{H}}_p^1(\Omega) = \{q \in L^p_{loc}(\overline{\Omega}) : \nabla q \in L^p(\Omega)\}$$

equipped with the seminorm $\|\nabla \cdot\|_{L^p(\Omega)}$.

Theorem 2.3.1. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary and $1 < p < \infty$. Then for any $u \in L^p(\Omega)$ there are $h \in L^p_{T,har}(\Omega)$, $w \in \dot{\mathbf{H}}_p^{1,N}(\Omega)$ with $\operatorname{div} w = 0$, $\pi \in \dot{\mathbf{H}}_p^1(\Omega)$ such that*

$$u = h + \operatorname{rot} w + \nabla\pi.$$

The operators $Q_p, R_p, S_p \colon L^p(\Omega) \to L^p(\Omega)$ defined by

$$Q_p u = h, \quad R_p u = \operatorname{rot} w, \quad S_p u = \nabla\pi$$

are uniquely determined bounded linear projections. Moreover, it holds that

$$(Q_p)' = Q_{p'}, \quad (R_p)' = R_{p'}, \quad (S_p)' = S_{p'}.$$

Remark 2.3.2. *Concerning the vector potential w, we will actually construct it to be in $\tilde{\mathbf{H}}_p^{1,N}(\Omega)$. In view of Lemma 2.2.31 and Proposition 2.1.16, this implies $w \in \hat{H}_p^{1,N}(\Omega)$.*

Proof. It is known by [Miy82, Proposition 1.5 and its proof], that the weak Neumann problem

$$(\nabla\pi, \nabla\phi) = (u, \nabla\phi) \quad \text{for all } \phi \in \dot{\mathbf{H}}^1_{p'}(\Omega)$$

has a unique solution $\pi \in \dot{\mathbf{H}}^1_p(\Omega)$ satisfying

$$\|\nabla\pi\|_{L^p(\Omega)} \leq C\|u\|_{L^p(\Omega)}.$$

Moreover, it holds $u-\nabla\pi \in L^p_\sigma(\Omega)$. This implies in particular that $(u-\nabla\pi)\cdot n = 0$. We have seen in Theorem 2.2.49, that there is a unique solution $w \in \tilde{\mathbf{H}}^{1,N}_p(\Omega)$ to

$$\begin{aligned} (\operatorname{rot} w, \operatorname{rot}\phi) &= (u, \operatorname{rot}\phi) \quad \text{for all } \phi \in \dot{\mathbf{H}}^{1,N}_{p'}(\Omega), \\ \operatorname{div} w &= 0, \end{aligned} \tag{2.59}$$

together with the estimate

$$\|\nabla w\|_{L^p(\Omega)} \leq C\|u\|_{L^p(\Omega)}.$$

Defining

$$h := u - \operatorname{rot} w - \nabla\pi,$$

we will show that $h \in L^p_{T,har}(\Omega)$. It is clear, that $h \in L^p(\Omega)$. For any $\phi \in C^\infty_c(\Omega)$, we have

$$\begin{aligned} (h, \nabla\phi) &= (u, \nabla\phi) - (\operatorname{rot} w, \nabla\phi) - (\nabla\pi, \nabla\phi) \\ &= (u, \nabla\phi) - (w, \operatorname{rot}\nabla\phi) - (u, \nabla\phi) = 0 \end{aligned}$$

as well as

$$\begin{aligned} (h, \operatorname{rot}\phi) &= (u, \operatorname{rot}\phi) - (\operatorname{rot} w, \operatorname{rot}\phi) - (\nabla\pi, \operatorname{rot}\phi) \\ &= (u, \operatorname{rot}\phi) - (u, \operatorname{rot}\phi) + (\pi, \operatorname{div}\operatorname{rot}\phi) = 0. \end{aligned}$$

Thus, $\operatorname{div} h = 0$ and $\operatorname{rot} h = 0$ in the sense of distributions. It remains to show that $h \cdot n = 0$. We know already, that $(u - \nabla\pi) \cdot n = 0$. Hence, we only need to prove $\operatorname{rot} w \cdot n = 0$. This is equivalent to verifying

$$\langle \operatorname{rot} w \cdot n, \phi\rangle = 0 \quad \text{for all } \phi \in H^1_{p'}(\Omega).$$

Because of

$$\langle \operatorname{rot} w \cdot n, \phi\rangle_{\partial\Omega} = (\operatorname{rot} w, \nabla\phi) + (\operatorname{div}\operatorname{rot} w, \phi) = (\operatorname{rot} w, \nabla\phi),$$

this is the same as

$$(\operatorname{rot} w, \nabla\phi) = 0 \quad \text{for all } \phi \in H^1_{p'}(\Omega).$$

Due to (1.6), we obtain

$$(\operatorname{rot} w, \nabla\phi) = (w, \operatorname{rot} \nabla\phi) - \langle w \times n, \nabla\phi\rangle_{\partial\Omega} = 0,$$

for all $\phi \in C_c^\infty(\overline{\Omega})$, since $w \times n = 0$ and $\operatorname{rot} \nabla\phi = 0$. By density, this is also true for all $\phi \in W^{1,p'}(\Omega)$. This yields $h \in L^p_{T,har}(\Omega)$. The norm estimate for h follows by

$$\|h\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega)} + \|\operatorname{rot} w\|_{L^p(\Omega)} + \|\nabla\pi\|_{L^p(\Omega)} \leq C\|u\|_{L^p(\Omega)}.$$

We now show the uniqueness of h, $\operatorname{rot} w$ and $\nabla\pi$. Let $\bar{h} \in L^p_{T,har}(\Omega)$, $\bar{w} \in \dot{\mathbf{H}}^{1,N}_p(\Omega)$ with $\operatorname{div} \bar{w} = 0$, $\bar{\pi} \in \dot{\mathbf{H}}^1_p(\Omega)$ be another decomposition of u in the sense, that $u = \bar{h} + \operatorname{rot} \bar{w} + \nabla\bar{\pi}$. For reasons of clarity and comprehensibility, most of the computations here will be fetched up in the subsequent Lemma 2.3.3. At first, we will show that $\bar{w}$ is a solution to

$$\begin{aligned} (\operatorname{rot} \bar{w}, \operatorname{rot} \phi) &= (u, \operatorname{rot} \phi), \\ \operatorname{div} \bar{w} &= 0 \end{aligned} \tag{2.60}$$

for all $\phi \in \dot{\mathbf{H}}^{1,N}_{p'}(\Omega)$. The divergence condition is satisfied by assumption. Using the decomposition of u, we get

$$(\operatorname{rot} \bar{w}, \operatorname{rot} \phi) = (u, \operatorname{rot} \phi) - (\bar{h}, \operatorname{rot} \phi) - (\nabla\bar{\pi}, \operatorname{rot} \phi).$$

On the right-hand side, the second term is zero by (2.62) and the third one by (2.63), which will be shown in the next lemma. Thus $\bar{w}$ is indeed a solution of (2.60). As the solutions to this equation are unique up to Dirichlet fields by Theorem 2.2.49, we conclude that $\operatorname{rot} w = \operatorname{rot} \bar{w}$. A similar argumentation can also be employed to $\bar{\pi}$. We show that

$$(\nabla\bar{\pi}, \nabla\phi) = (u, \nabla\phi) \quad \text{for all } \phi \in \dot{\mathbf{H}}^1_{p'}(\Omega). \tag{2.61}$$

For any $\phi \in \dot{\mathbf{H}}^1_{p'}(\Omega)$, it holds

$$(\nabla\bar{\pi}, \nabla\phi) = (u, \nabla\phi) - (\bar{h}, \nabla\phi) - (\operatorname{rot} \bar{w}, \nabla\phi).$$

By (2.64), the second summand on the right-hand side is zero, and by (2.63), the last one equals zero, too. Hence, $\bar{\pi}$ is indeed a solution to (2.61). Because of the uniqueness of solutions to this problem (up to constants), we get $\nabla\bar{\pi} = \nabla\pi$. It follows directly, that

$$\bar{h} = u - \operatorname{rot}\bar{w} - \nabla\bar{\pi} = u - \operatorname{rot} w - \nabla\pi = h.$$

This implies the uniqueness of the decomposition. It follows from that uniqueness, that the operators Q_p, R_p, S_p and P_p are projections.

It remains to show the duality relations of the operators Q_p, R_p and S_p. Let $u \in L^p(\Omega)$ and $u' \in L^{p'}(\Omega)$, and consider their decompositions

$$\begin{aligned} u &= h + \operatorname{rot} w + \nabla\pi = Q_p u + R_p u + S_p u, \\ u' &= h' + \operatorname{rot} w' + \nabla\pi' = Q_{p'} u + R_{p'} u + S_{p'} u. \end{aligned}$$

Then we get

$$\begin{aligned} (R_p u, u') &= (\operatorname{rot} w, u') \\ &= (\operatorname{rot} w, h') + (\operatorname{rot} w, \operatorname{rot} w') + (\operatorname{rot} w, \nabla\pi') \\ &= (\operatorname{rot} w, \operatorname{rot} w') \\ &= (h, \operatorname{rot} w') + (\operatorname{rot} w, \operatorname{rot} w') + (\nabla\pi, \operatorname{rot} w') \\ &= (u, R_{p'} u'). \end{aligned}$$

Here, the third and forth equality follow by (2.62) and (2.63). Therefore $(R_p)' = R_{p'}$. The relations $(S_p)' = S_{p'}$ and $(Q_p)' = Q_{p'}$ follow by analogous arguments. □

In order to complete the last proof, we have to verify the next technical equalities.

Lemma 2.3.3. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary and $1 < p < \infty$. Then the following equalities hold:*

(2.62) $$(h, \operatorname{rot}\phi) = 0 \quad \textit{for all} \quad h \in L^p_{T,har}(\Omega), \phi \in \dot{\mathbf{H}}^{1,N}_{p'}(\Omega),$$

(2.63) $$(\nabla\pi, \operatorname{rot}\phi) = 0 \quad \textit{for all} \quad \pi \in \dot{\mathbf{H}}^1_p(\Omega), \phi \in \dot{\mathbf{H}}^{1,N}_{p'}(\Omega),$$

(2.64) $$(h, \nabla\phi) = 0 \quad \textit{for all} \quad h \in L^p_{T,har}(\Omega), \phi \in \dot{\mathbf{H}}^1_{p'}(\Omega).$$

Proof. 1. Because of Lemma 2.2.31 and Remark 2.2.32, we can decompose ϕ into $\hat{\phi} + \phi^h$, where $\phi^h \in \dot{\mathbf{H}}^{1,N,har}_{p'}(\Omega)$ and $\hat{\phi} \in \hat{H}^{1,N}_{p'}(\Omega)$. For any $\psi \in C_c^{\infty,N}(\overline{\Omega})$, we have

$$(h, \operatorname{rot}\psi) = (\operatorname{rot} h, \psi) - \langle h, \psi \times n\rangle_{\partial\Omega} = 0.$$

By approximation, the same holds for $\hat{\phi}$ instead of ψ. Therefore, we get

$$(h, \operatorname{rot}\phi) = (h, \operatorname{rot}\hat{\phi}) + (h, \operatorname{rot}\phi^h) = 0.$$

2. Due to [Sim90, Theorem 2.4], there is a sequence $(\pi_n)_{n\in\mathbb{N}} \subset C_c^\infty(\overline{\Omega})$ such that $\|\nabla\pi - \nabla\pi_n\|_{L^p(\Omega)} \to 0$ for $n \to \infty$. Therefore, we have
$$(\nabla\pi, \operatorname{rot}\phi) = \lim_{n\to\infty}(\nabla\pi_n, \operatorname{rot}\phi) = \lim_{n\to\infty}(\operatorname{rot}\nabla\pi_n, \phi) - \langle\nabla\pi, \phi\times n\rangle_{\partial\Omega} = 0.$$

3. Because of [Sim90, Theorem 2.4], there is a sequence $(\phi_n)_{n\in\mathbb{N}} \subset C_c^\infty(\overline{\Omega})$ such that $\|\nabla\phi - \nabla\phi_n\|_{L^{p'}(\Omega)} \to 0$ for $n \to \infty$. That implies
$$(h, \nabla\phi) = \lim_{n\to\infty}(h, \nabla\phi_n) = \lim_{n\to\infty}\langle h\cdot n, \phi_n\rangle_{\partial\Omega} - (\operatorname{div} h, \phi_n) = 0.$$

□

Most results on the Helmholtz-Hodge decomposition do not require the vector potential to be divergence free, although this is a convenient extra information. Using the decomposition above, we can regain this more classical formulation. Beforehand we need the following preparatory result:

Lemma 2.3.4. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary and $p \in (1,\infty)$. Then for each $v \in \dot{\mathbf{H}}_p^{1,N}(\Omega)$, there is a $w \in \dot{\mathbf{H}}_p^{1,N}(\Omega)$ such that* $\operatorname{rot} w = \operatorname{rot} v$, $\operatorname{div} w = 0$ *and $\|\nabla w\|_{L^p(\Omega)} \leq C\|\nabla v\|_{L^p(\Omega)}$, where $C > 0$ is independent of v.*

Proof. Because of Theorem 2.2.49, there is a unique $w \in \tilde{\mathbf{H}}_p^{1,N}(\Omega)$, that solves
$$\begin{aligned}(\operatorname{rot} w, \operatorname{rot}\phi) &= (\operatorname{rot} v, \operatorname{rot}\phi) \quad \text{for all } \phi \in \dot{\mathbf{H}}_{p'}^{1,N}(\Omega),\\ \operatorname{div} w &= 0.\end{aligned}$$

This solutions can be estimated by $\|\nabla w\|_{L^p(\Omega)} \leq C\|\nabla v\|_{L^p(\Omega)}$ with $C > 0$ being independent of v. We show that $\operatorname{rot} w = \operatorname{rot} v$. Let $\psi \in C_c^\infty(\Omega)$ be arbitrary. Then, by Theorem 2.3.1, there are $w_\psi \in \dot{\mathbf{H}}_{p'}^{1,N}(\Omega)$, $h_\psi \in L_{T,har}^{p'}(\Omega)$ and $\pi_\psi \in \dot{\mathbf{H}}_{p'}^{1}(\Omega)$ such that
$$\psi = \operatorname{rot} w_\psi + h_\psi + \nabla\pi_\psi.$$

Therefore we have
$$\begin{aligned}(\operatorname{rot} w, \psi) &= (\operatorname{rot} w, \operatorname{rot} w_\psi) + (\operatorname{rot} w, h_\psi) + (\operatorname{rot} w, \nabla\pi_\psi)\\ &= (\operatorname{rot} w, \operatorname{rot} w_\psi)\\ &= (\operatorname{rot} v, \operatorname{rot} w_\psi)\\ &= (\operatorname{rot} v, \operatorname{rot} w_\psi) + (\operatorname{rot} v, h_\psi) + (\operatorname{rot} v, \nabla\pi_\psi)\\ &= (\operatorname{rot} v, \psi),\end{aligned}$$

where we have used (2.62) and (2.63) in the third and fifth equality. As $\psi \in C_c^\infty(\Omega)$ was arbitrary, this completes the proof. □

Combining this lemma with Theorem 2.3.1 yields:

Corollary 2.3.5. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary and $p \in (1,\infty)$. Then it holds that*

$$L^p(\Omega) = \operatorname{rot} \dot{\mathbf{H}}_p^{1,N}(\Omega) \oplus L^p_{T,har}(\Omega) \oplus \nabla \dot{\mathbf{H}}_p^1(\Omega).$$

We close this section by showing, that $L^p_{T,har}(\Omega)$ is independent of p.

Proposition 2.3.6. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary. Then for any $p \in (1,\infty)$, it holds*

$$L^p_{T,har}(\Omega) = L^2_{T,har}(\Omega)$$

Proof. Let $p \in (1,\infty)$ and $h \in L^p_{T,har}(\Omega)$. Because of Proposition 2.2.38, we have $h \in \dot{\mathbf{H}}_p^{1,T,har}(\Omega)$. On the one hand, this implies, that $L^p_{T,har}(\Omega)$ of finite dimension. On the other hand, Theorem 2.2.18 implies $h \in \dot{\mathbf{H}}_r^{1,T}(\Omega)$ for any $r \in (1,\infty)$. Choosing $r \in (1,3)$ and making use of the Sobolev inequality from Proposition 2.1.10, we get $h - v \in L^{r^*}_{T,har}(\Omega)$ with $\frac{1}{r^*} = \frac{1}{r} - \frac{1}{3} \in (3/2,\infty)$ and some $v \in \mathbb{R}^3$. Because of $h \in L^p_{T,har}(\Omega)$, the vector v has to be zero. Thus, we have the inclusion

$$L^p_{N,har}(\Omega) \subseteq L^q_{N,har}(\Omega) \tag{2.65}$$

for any $p \in (1,\infty)$ and $q \in (3/2,\infty)$. Let Q_q be the projection from $L^q(\Omega)$ onto $L^q_{T,har}(\Omega)$ given in Theorem 2.3.1. Because of the duality $(Q_q)' = (Q'_q)$, we know that $L^q_{T,har}(\Omega)$ and $L^{q'}_{T,har}(\Omega)$ have the same dimension for any $p \in (1,\infty)$. Combining this with (2.65) yields $L^p_{T,har}(\Omega) = L^2_{T,har}(\Omega)$ for any $p \in (1,\infty)$. $\square$

2.4 Second Helmholtz-Hodge Decomposition

We now consider the second Helmholtz-Hodge decomposition. In the case of a bounded domain $D \subset \mathbb{R}^3$, it has been shown in [KY09], that for any $u \in L^p(D)$ and $1 < p < \infty$, there are $h \in L^p_{N,har}(D)$, $w \in \dot{\mathbf{H}}_p^{1,T}(D)$ with $\operatorname{div} w = 0$ and $\pi \in \dot{\mathbf{H}}_p^1(D)$ with $\pi|_{\partial\Omega} = 0$ such that

$$u = h + \operatorname{rot} w + \nabla \pi. \tag{2.66}$$

Furthermore, they have shown that any other decomposition $u = \bar{h} + \operatorname{rot} \bar{w} + \nabla \bar{\pi}$, with $\bar{h}$, $\bar{w}$ and $\bar{\pi}$ from the respective spaces, fulfils $h = \bar{h}$, $\operatorname{rot} w = \operatorname{rot} \bar{w}$ and $\nabla \pi = \nabla \bar{\pi}$. We will see that for exterior domains, the situation is far more involved, as it will also depend on the order of integration p and the very choice of the function space for the potentials. For $p \in (3/2, 3)$ we will come upon the same situation as in bounded domains, for $p \in [3, \infty)$ a lack of uniqueness of the decomposition will occur, and for $p \in (1, 3/2]$ the analogous decomposition will fail at all. More precisely, we will find a vector field v which is not included in $L^p_{N,har}(\Omega) + \nabla \hat{H}^{1,0}_p(\Omega) + \operatorname{rot} \dot{\mathbf{H}}^{1,T}_p(\Omega)$ for $p \in (1, 3/2]$. However, this vector field v is contained in $\nabla \dot{\mathbf{H}}^{1,0}_p(\Omega)$. Even more, exchanging $\nabla \hat{H}^{1,0}_p(\Omega)$ by $\nabla \dot{\mathbf{H}}^{1,0}_p(\Omega)$ will allow us to show the decomposition also for $p \in (1, 3/2]$.

2.4.1 A Simplified Decomposition

We are going to consider at first a less detailed decomposition into the rotation of a vector field and a gradient field, both of which satisfy certain boundary conditions. This is still possible without any restriction on the order of integration p. Beforehand, we introduce a function space, which will be used in that decomposition and throughout this section. Let $\Omega \subset \mathbb{R}^3$ be a domain with smooth boundary. Assume that $\partial\Omega$ consists of finitely many connected components $\Gamma_1, \ldots, \Gamma_K$. For $1 < p < \infty$, we define the set

$$\dot{\mathbf{H}}^{1,c}_p(\Omega) := \{u \in L^p_{loc}(\overline{\Omega}) : \nabla u \in L^p(\Omega), u|_{\Gamma_i} = c_i \ \forall i \in \{1, \ldots, K\}, c_i \in \mathbb{R}\}.$$

Note that $u|_{\Gamma_i}$ is constant, if and only if $\nabla u \times n = 0$.

Proposition 2.4.1. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary and $1 < p < \infty$. Then for any $u \in L^p(\Omega)$ there are $w \in \dot{\mathbf{H}}^{1,T}_p(\Omega)$ with $\operatorname{div} w = 0$ and $\pi \in \dot{\mathbf{H}}^{1,c}_p(\Omega)$ such that*

$$u = \nabla \pi + \operatorname{rot} w. \tag{2.67}$$

Furthermore, the maps $u \mapsto \nabla\pi$ and $u \mapsto \operatorname{rot} w$ with w and π as above define uniquely determined bounded linear projections.

Proof. Let $w \in \tilde{\mathbf{H}}^{1,T}_p(\Omega)$ be the unique solution to

$$\begin{aligned} (\operatorname{rot} w, \operatorname{rot} \phi) &= (u, \operatorname{rot} \phi) \qquad \text{for all } \phi \in \tilde{\mathbf{H}}^{1,T}_{p'}(\Omega), \\ \operatorname{div} w &= 0, \end{aligned}$$

which is given by Theorem 2.2.49. Set $g := u - \operatorname{rot} w$. Then there is a scalar potential $\pi \in \dot{\mathbf{H}}^{1,c}_p(\Omega)$ such that $g = \nabla\pi$. We are going to construct this potential by Fourier methods. First we show that $\operatorname{rot} g = 0$ holds in the sense of distributions. Indeed, for $\phi \in C^\infty_c(\Omega)$, we have

$$(g, \operatorname{rot}\phi) = (u, \operatorname{rot}\phi) - (\operatorname{rot} w, \operatorname{rot}\phi) = 0.$$

Next, we prove $g \times n = 0$ on $\partial\Omega$. For any $\phi \in C_c^\infty(\overline{\Omega})$, we get

$$\langle g \times n, \phi \rangle_{\partial\Omega} = (g, \operatorname{rot} \phi) - (\operatorname{rot} g, \phi) = (g, \operatorname{rot} \phi) = (u, \operatorname{rot} \phi) - (\operatorname{rot} w, \operatorname{rot} \phi).$$

Applying the Helmholtz decomposition in $H^1_{p'}(\Omega)$ to ϕ yields the existence of some $\phi_\sigma \in W^{1,p'}(\Omega) \cap L^{p'}_\sigma(\Omega) \subset \dot{\mathbf{H}}^{1,T}_{p'}(\Omega)$ and $q \in \dot{\mathbf{H}}^1_{p'}(\Omega)$ such that $u = u_\sigma + \nabla q$. Therefore, we can continue by

$$\begin{aligned}\langle g \times n, \phi \rangle_{\partial\Omega} &= (u, \operatorname{rot} \phi_\sigma) - (\operatorname{rot} w, \operatorname{rot} \phi_\sigma) + (u, \operatorname{rot} \nabla q) - (\operatorname{rot} w, \operatorname{rot} \nabla q) \\ &= 0.\end{aligned}$$

By density, we can extend this equality to all $\phi \in W^{1,p'}(\Omega)$. Hence, $g \times n = 0$. By the same arguments as in the proof of Lemma 2.2.25 (see also Remark 2.2.26), this implies, that there is a $\pi \in \dot{\mathbf{H}}^1_{p'}(\Omega)$ such that $g = \nabla\pi$. Due to $\nabla\pi \times n = g \times n = 0$, we additionally have $\pi \in \dot{\mathbf{H}}^{1,c}_p(\Omega)$. It is easy to see that the operator $u \mapsto \operatorname{rot} w$ given here is a bounded projection. Hence, the operator $u \mapsto \nabla\pi$ has to be a bounded projection, too.

In order to see uniqueness of the decomposition, let $\bar{w} \in \dot{\mathbf{H}}^{1,T}_p(\Omega)$, $\operatorname{div} \bar{w} = 0$ and $\bar{\pi} \in \dot{\mathbf{H}}^{1,c}_p(\Omega)$ be such that $u = \nabla\bar{\pi} + \operatorname{rot} \bar{w}$. We show, that $\operatorname{rot} \bar{w} = \operatorname{rot} w$, where w is the vector field constructed above. We will do so, by showing that $\bar{w}$ solves

$$\begin{aligned}(\operatorname{rot} \bar{w}, \operatorname{rot} \phi) &= (u, \operatorname{rot} \phi) \quad \text{for all } \phi \in \tilde{\mathbf{H}}^{1,T}_{p'}(\Omega), \\ \operatorname{div} w &= 0.\end{aligned} \tag{2.68}$$

For any $\phi \in \dot{\mathbf{H}}^{1,T}_{p'}(\Omega)$, it holds that

$$(\operatorname{rot} \bar{w}, \operatorname{rot} \phi) = (u, \operatorname{rot} \phi) - (\nabla\bar{\pi}, \operatorname{rot} \phi).$$

Because of Lemma 2.2.31 and Remark 2.2.32, there are $\hat{\phi} \in \hat{H}^{1,T}_{p'}(\Omega)$ and $\phi^h \in \dot{\mathbf{H}}^{1,T,har}_{p'}(\Omega)$ such that $\phi = \hat{\phi} + \phi^h$. Thus, we have

$$\begin{aligned}(\nabla\bar{\pi}, \operatorname{rot} \phi) &= (\nabla\bar{\pi}, \operatorname{rot} \hat{\phi}) + (\nabla\bar{\pi}, \operatorname{rot} \phi^h) \\ &= (\operatorname{rot} \nabla\bar{\pi}, \hat{\phi}) + \langle \nabla\bar{\pi} \times n, \hat{\phi} \rangle_{\partial\Omega} + 0 = 0.\end{aligned}$$

Hence, $\bar{w}$ is a solution to (2.68). As solutions to that problem are unique up to Neumann fields, we get $\operatorname{rot} w = \operatorname{rot} \bar{w}$. It follows directly, that $\nabla\pi = \nabla\bar{\pi}$. This completes the proof. □

2.4.2 Main Results

Depending on the integrability parameter p, the decomposition from Proposition 2.4.1 can (or has to) be refined in different ways. Before stating the main result, we introduce homogeneous Sobolev spaces with vanishing boundary values.

Definition 2.4.2. *Let $\Omega \subset \mathbb{R}^d$ be a domain with C^∞-boundary. Set*

$$\dot{\mathbf{H}}_p^{1,0}(\Omega) := \{q \in L^p_{loc}(\overline{\Omega};\mathbb{R}) : \nabla q \in L^p(\Omega), q = 0 \text{ on } \partial\Omega\},$$

equipped with the norm $\|q\|_{\dot{\mathbf{H}}_p^{1,0}(\Omega)} = \|\nabla q\|_{L^p(\Omega)}$. Furthermore, define

$$\hat{H}_p^{1,0}(\Omega) := \overline{C_c^\infty(\Omega)}^{\dot{\mathbf{H}}_p^{1,0}(\Omega)}.$$

The second Helmholtz-Hodge decomposition reads as follows:

Theorem 2.4.3. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary, $p \in (1,\infty)$ and $u \in L^p(\Omega)$.*

1. *If $p \in (1, 3/2]$, then there are $h \in L^p_{N,har}(\Omega)$, $w \in \dot{\mathbf{H}}_p^{1,T}(\Omega)$, $\pi \in \hat{H}_p^{1,0}(\Omega)$ and $q \in \dot{\mathbf{H}}_p^{1,c}(\Omega)$ such that*

$$u = \operatorname{rot} w + h + \nabla q + \nabla \pi.$$

   Furthermore, the operators $u \mapsto h$, $u \mapsto \nabla q$, $u \mapsto \nabla\pi$ and $u \mapsto \operatorname{rot} w$ can be chosen to be bounded linear projections. The function q is independent of u up to some scalar multiple and globally constant along the boundary.

2. *If $p \in (3/2, 3)$, then there are $h \in L^p_{N,har}(\Omega)$, $w \in \dot{\mathbf{H}}_p^{1,T}(\Omega)$ with $\operatorname{div} w = 0$, $\pi \in \hat{H}_p^{1,0}(\Omega)$ such that*

$$u = \operatorname{rot} w + h + \nabla\pi.$$

   The operators $Q_p, R_p, S_p \colon L^p(\Omega) \to L^p(\Omega)$ defined by

$$Q_p u := h, \quad R_p u := \operatorname{rot} w, \quad S_p u := \nabla \pi$$

   are uniquely determined bounded linear projections and

$$(Q_p)' = Q_{p'}, \quad (R_p)' = R_{p'}, \quad (S_p)' = S_{p'}.$$

3. *If $p \in [3, \infty)$, then there are $h \in L^p_{N,har}(\Omega)$, $w \in \dot{\mathbf{H}}^{1,T}_p(\Omega)$ with* $\operatorname{div} w = 0$, *$\pi \in \hat{H}^{1,0}_p(\Omega)$ such that*

$$u = \operatorname{rot} w + h + \nabla\pi.$$

The maps $u \mapsto h$, $u \mapsto \operatorname{rot} w$ and $u \mapsto \nabla\pi$ can be chosen to be bounded linear projections in $L^p(\Omega)$. Moreover, the space $L^p_{N,har}(\Omega) \cap \nabla\hat{H}^{1,0}_p(\Omega)$ is one dimensional.

Remark 2.4.4. *The decomposition in the first part for $p \in (1, 3/2]$ is not unique as $\hat{H}^{1,0}_p(\Omega) \subset \dot{\mathbf{H}}^{1,c}_p(\Omega)$. Furthermore, the summand ∇q cannot be dropped in general as we will see in Lemma 2.4.20.*

The failure of the classical decomposition into three summands in the case $p \in (1, 3/2]$ can be corrected by choosing $\dot{\mathbf{H}}^{1,0}_p(\Omega)$ as the space of the scalar potentials instead of $\hat{H}^{1,0}_p(\Omega)$. More precisely, we will show:

Corollary 2.4.5. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary, $p \in (1, 3/2]$ and $u \in L^p(\Omega)$. Then there are $w \in \dot{\mathbf{H}}^{1,T}_p(\Omega)$ with* $\operatorname{div} w = 0$, *$h \in L^p_{N,har}(\Omega)$ and $\pi \in \dot{\mathbf{H}}^{1,0}_p(\Omega)$ such that*

$$u = \operatorname{rot} w + h + \nabla\pi.$$

Moreover, the operators $u \mapsto \operatorname{rot} w$, $u \mapsto h$ and $u \mapsto \nabla\pi$ are uniquely determined bounded linear operators.

The proofs of each part of Theorem 2.4.3 and Corollary 2.4.5 require knowledge about the weak Dirichlet problem in exterior domains. The question of unique solvability of this equation in homogeneous Sobolev spaces has been extensively considered in [Sim90] and [SS96]. We collect some of the main results of these works.

The weak Dirichlet problem can be formulated as follows: Given a functional $f \in (\hat{H}^{1,0}_{p'}(\Omega))'$, find a (unique) solution $\pi \in \hat{H}^{1,0}_p(\Omega)$ to

$$(\nabla\pi, \nabla\phi) = \langle f, \phi\rangle \quad \text{for all} \quad \phi \in \hat{H}^{1,0}_{p'}(\Omega).$$

As for the weak Poisson problem, it is important to characterize the kernel of the bilinear form on the left-hand side, which will be denoted by

$$\hat{H}^{1,0,har}_p(\Omega) := \{q_{har} \in \hat{H}^{1,0}_p(\Omega) : \Delta q_{har} = 0\}.$$

Depending on the parameter p, it has the following form (see [Sim90, Theorem 7.1]):

Lemma 2.4.6. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary. If $p \in (1,3)$, it holds $\hat{H}_p^{1,0,har}(\Omega) := \{0\}$. If $p \in [3,\infty)$, then $\hat{H}_p^{1,0,har}(\Omega)$ is one-dimensional. It is spanned by a function $q_{har} \in C^\infty(\mathbb{R}^3) \cap C(\overline{\Omega})$. Given some $R > 0$ such that $\partial\Omega \subset B(0,R)$, this function has the representation*

$$q_{har}(x) = a + \frac{b}{|x|} + \frac{1}{|x|^2} u\left(\frac{x}{|x|^2}\right) \quad (x \in B(0,R)^C)$$

with $a, b \neq 0$ and some harmonic function $u\colon B(0,1/R) \to \mathbb{R}$ satisfying $u(0) = 0$.

For $p \in [3,\infty)$, the lemma above implies the existence of a closed subspace $\tilde{H}_p^{1,0}(\Omega)$ of $\hat{H}_p^{1,0}(\Omega)$ such that

$$\hat{H}_p^{1,0}(\Omega) = \tilde{H}_p^{1,0}(\Omega) \oplus \hat{H}_p^{1,0,har}(\Omega).$$

For notational simplicity, set $\tilde{H}_p^{1,0}(\Omega) := \hat{H}_p^{1,0}(\Omega)$, if $p \in (1,3)$. We would like to emphasize, that for $p \in [3,\infty)$, the inclusion $C_c^\infty(\Omega) \subseteq \tilde{H}_p^{1,0}(\Omega)$ is wrong. Using this notation, the existence result for the weak Dirichlet problem in [Sim90, Theorem 7.3] reads as follows:

Lemma 2.4.7. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary and $1 < p < \infty$.*

1. *Suppose that $f \in \tilde{H}_{p'}^{1,0}(\Omega)'$. Then there is a unique $q \in \tilde{H}_p^{1,0}(\Omega)$, that solves*

   $$(\nabla q, \nabla \phi) = \langle f, \phi \rangle \quad \text{for all } \phi \in \tilde{H}_{p'}^{1,0}(\Omega). \tag{2.69}$$

   Furthermore, there is a constant $C = C(p,\Omega) > 0$ such that

   $$\|\nabla q\|_{L^p(\Omega)} \leq C \|f\|_{\tilde{H}_{p'}^{1,0}(\Omega)'}.$$

2. *For $p \in (1,3/2]$, the problem*

   $$(\nabla q, \nabla \phi) = \langle f, \phi \rangle \quad \text{for all } \phi \in \hat{H}_{p'}^{1,0}(\Omega) \tag{2.70}$$

   has a unique solution $q \in \hat{H}_p^{1,0}(\Omega)$ for a given $f \in (\hat{H}_{p'}^{1,0}(\Omega))'$, if and only if $f(q_{har}) = 0$ for all $q_{har} \in \hat{H}_{p'}^{1,0,har}(\Omega)$.

3. *For $p \in [3,\infty)$, the problem (2.70) has a solution $q \in \hat{H}_p^{1,0}(\Omega)$ for any given $f \in (\hat{H}_{p'}^{1,0}(\Omega))'$. However, this solution is only unique in $\tilde{H}_p^{1,0}(\Omega)$.*

We would like to highlight, that for $p \in (3/2, 3)$, one can exchange $\tilde{H}^{1,0}_p(\Omega)$ and $\tilde{H}^{1,0}_{p'}(\Omega)$ by $\hat{H}^{1,0}_p(\Omega)$ and $\hat{H}^{1,0}_{p'}(\Omega)$ respectively in the lemma above, as these spaces do coincide in the given range of parameters. This is the main reason, why the proof of Theorem 2.4.3 (2) can be done in almost the same manner as the one of Theorem 2.3.1.

Proof of Theorem 2.4.3 (2). Let $u \in L^p(\Omega)$ be arbitrary. Define $\pi \in \hat{H}^{1,0}_p(\Omega)$ to be the unique solution to

$$(\nabla \pi, \nabla \phi) = (u, \nabla \phi) \quad \text{for all } \phi \in \hat{H}^{1,0}_{p'}(\Omega) \tag{2.71}$$

given by Lemma 2.4.7. Furthermore, let $w \in \tilde{\mathbf{H}}^{1,T}_p(\Omega)$ be the unique solution to

$$\begin{aligned} (\operatorname{rot} w, \operatorname{rot} \phi) &= (u, \operatorname{rot} \phi) \quad \text{for all } \phi \in \dot{\mathbf{H}}^{1,T}_{p'}(\Omega), \\ \operatorname{div} w &= 0, \end{aligned}$$

which is given by Theorem 2.2.49. Then the vector field $h := u - \operatorname{rot} \phi - \nabla \pi$ lies in $L^p_{N,har}(\Omega)$. Indeed, it is clear that $h \in L^p(\Omega)$. We check, that $\operatorname{div} h = 0$ and $\operatorname{rot} h = 0$. Let $\phi \in C^\infty_c(\Omega)$. Then

$$\begin{aligned} (h, \nabla \phi) &= (u, \nabla \phi) - (\operatorname{rot} w, \nabla \phi) - (\nabla \pi, \nabla \phi) \\ &= (u, \nabla \phi) - (w, \operatorname{rot} \nabla \phi) - (u, \nabla \phi) = 0, \end{aligned}$$

as well as

$$\begin{aligned} (h, \operatorname{rot} \phi) &= (u, \operatorname{rot} \phi) - (\operatorname{rot} w, \operatorname{rot} \phi) - (\nabla \pi, \operatorname{rot} \phi) \\ &= (u, \operatorname{rot} \phi) - (u, \operatorname{rot} \phi) + (\pi, \operatorname{div} \operatorname{rot} \phi) = 0. \end{aligned}$$

It remains to verify the boundary conditions of h. Using (1.6), we get

$$\begin{aligned} \langle h \times n, \phi \rangle &= (\operatorname{rot} h, \phi) + (h, \operatorname{rot} \phi) = (h, \operatorname{rot} \phi) \\ &= (u, \operatorname{rot} \phi) - (\operatorname{rot} w, \operatorname{rot} \phi) - (\nabla \pi, \operatorname{rot} \phi) \end{aligned} \tag{2.72}$$

for each $\phi \in H^1_{p'}(\Omega)$. Hence, it is sufficient to show $(u, \operatorname{rot} \phi) - (\operatorname{rot} w, \operatorname{rot} \phi) - (\nabla \pi, \operatorname{rot} \phi) = 0$ for all $\phi \in H^1_{p'}(\Omega)$. At first, we note that

$$(\nabla \pi, \operatorname{rot} \phi) = \langle \pi, \operatorname{rot} \phi \cdot n \rangle - (\pi, \operatorname{div} \operatorname{rot} \phi) = 0$$

for each $\phi \in C^\infty_c(\overline{\Omega})$. By approximation, the same is true for all ϕ in $H^1_{p'}(\Omega)$. Regarding the other summands, we recall that the classical Helmholtz projection is bounded on $H^1_{p'}(\Omega)$. Thus, for each $\phi \in H^1_{p'}(\Omega)$ there are $\phi_\sigma \in H^1_{p'}(\Omega) \cap L^{p'}_\sigma(\Omega) \subset \dot{H}^{1,T}_p(\Omega)$ and $q \in L^p_{loc}(\overline{\Omega})$ with $\nabla q \in H^1_{p'}(\Omega)$ such that

$$\phi = \phi_\sigma + \nabla q.$$

Employing this decomposition yields

$$\begin{aligned}
&(u, \operatorname{rot}\phi) - (\operatorname{rot} w, \operatorname{rot}\phi) \\
&= (u, \operatorname{rot}\phi_\sigma) - (\operatorname{rot} w, \operatorname{rot}\phi_\sigma) + (u, \operatorname{rot}\nabla q) - (\operatorname{rot} w, \operatorname{rot}\nabla q) \\
&= (u, \operatorname{rot}\phi_\sigma) - (u, \operatorname{rot}\phi_\sigma) + 0 + 0 = 0.
\end{aligned}$$

Hence, $h \times n = 0$, which implies $h \in L^p_{N,har}(\Omega)$. As the maps $u \mapsto \operatorname{rot} w$ and $u \mapsto \nabla\pi$ are bounded from $L^p(\Omega)$ to $L^p(\Omega)$, the map $u \mapsto h$ has to be bounded, too.

We do now show, that any other decomposition $u = \bar{h} + \operatorname{rot}\bar{w} + \nabla\bar{\pi}$ with $\bar{h} \in L^p_{N,har}(\Omega)$, $\bar{w} \in \dot{\mathbf{H}}^{1,T}_p(\Omega)$, $\operatorname{div}\bar{w} = 0$, $\bar{\pi} \in \tilde{\mathbf{H}}^{1,0}_p(\Omega)$ does coincide with the one constructed above. We start with $\bar{\pi}$ and show, that it is a solution to

$$(\nabla\bar{\pi}, \nabla\phi) = (u, \nabla\phi) \quad \text{for all } \phi \in \hat{H}^{1,0}_{p'}(\Omega). \tag{2.73}$$

We have

$$(\nabla\bar{\pi}, \nabla\phi) = (u, \nabla\phi) - (\operatorname{rot}\bar{w}, \nabla\phi) - (\bar{h}, \nabla\phi).$$

Here, the second summand on the right-hand side equals zero because of (2.76), and the third one equals zero because of (2.77), which will both be proven subsequently in Lemma 2.4.21. Thus, $\bar{\pi}$ is a solution to (2.73). By the uniqueness of solutions to that problem, as stated in Lemma 2.4.7, we obtain $\nabla\pi = \nabla\bar{\pi}$. In a similar way, we can show, that $\bar{w}$ is a solution to

$$\begin{aligned}
(\operatorname{rot}\bar{w}, \operatorname{rot}\phi) &= (u, \operatorname{rot}\phi) \quad \text{for all } \phi \in \dot{\mathbf{H}}^{1,T}_{p'}(\Omega), \\
\operatorname{div}\bar{w} &= 0.
\end{aligned} \tag{2.74}$$

Indeed, we have

$$(\operatorname{rot}\bar{w}, \operatorname{rot}\phi) = (u, \operatorname{rot}\phi) - (\bar{h}, \operatorname{rot}\phi) - (\nabla\bar{\pi}, \operatorname{rot}\phi).$$

Here, we can conclude $(\bar{h}, \operatorname{rot}\phi) = 0$ because of (2.75) and $(\nabla\bar{\pi}, \operatorname{rot}\phi) = 0$ because of (2.76). Thus $\bar{w}$ is a solution to (2.74). By means of Theorem 2.2.49, solutions to that problem are unique up to Neumann fields, which implies $\operatorname{rot}\bar{w} = \operatorname{rot} w$. Finally, we have

$$\bar{h} = u - \operatorname{rot}\bar{w} - \nabla\bar{\pi} = u - \operatorname{rot} w - \nabla\pi = h.$$

The uniqueness of the decomposition does also imply, that Q_p, R_p and S_p are projections.

It remains to show duality relations of Q_p, R_p and S_p. Let $u \in L^p(\Omega)$ and $u' \in L^{p'}(\Omega)$ with their respective decompositions

$$\begin{aligned} u &= h + \operatorname{rot} w + \nabla\pi = Q_p u + R_p u + S_p u, \\ u' &= h' + \operatorname{rot} w' + \nabla\pi' = Q_{p'} u' + R_{p'} u' + S_{p'} u'. \end{aligned}$$

Then we have

$$\begin{aligned} (Q_p u, u') &= (h, h') + (h, \operatorname{rot} w') + (h, \nabla\pi') \\ &= (h, h') \\ &= (h, h') + (\operatorname{rot} w, h') + (\nabla\pi, h') \\ &= (u, Q_{p'} u'). \end{aligned}$$

Here, the second and third equality follow both from (2.75) and (2.77). Thus $Q_p' = Q_{p'}$. The equalities $R_p' = R_{p'}$ and $S_p' = S_{p'}$ can be treated similarly. $\square$

For completing the proof above, it remains to supplement the proofs of the following equalities.

Lemma 2.4.8. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary and $1 < p < \infty$. Then the following equalities hold:*

$$(h, \operatorname{rot}\phi) = 0 \quad \textit{for all} \quad h \in L^p_{N,har}(\Omega), \phi \in \dot{\mathbf{H}}^{1,T}_{p'}(\Omega), \tag{2.75}$$

$$(\nabla\pi, \operatorname{rot}\phi) = 0 \quad \textit{for all} \quad \pi \in \hat{H}^{1,0}_p(\Omega), \phi \in \dot{\mathbf{H}}^{1,T}_{p'}(\Omega), \tag{2.76}$$

$$(h, \nabla\phi) = 0 \quad \textit{for all} \quad h \in L^p_{N,har}(\Omega), \phi \in \hat{H}^{1,0}_{p'}(\Omega). \tag{2.77}$$

Proof. 1. Because of Lemma 2.2.31, we can decompose ϕ into $\hat{\phi} + \phi^h$, where $\phi^h \in \dot{\mathbf{H}}^{1,T,har}_{p'}(\Omega)$ and $\hat{\phi} \in \hat{H}^{1,T}_{p'}(\Omega)$. For any $\psi \in C^{\infty,T}_c(\overline{\Omega})$, it holds

$$(h, \operatorname{rot}\psi) = (\operatorname{rot} h, \psi) + \langle h \times n, \psi \rangle_{\partial\Omega} = 0.$$

By approximation, this can be extended to $\hat{\phi}$ instead of ψ. This implies

$$(h, \operatorname{rot}\phi) = (h, \operatorname{rot}\hat{\phi}) + (h, \operatorname{rot}\phi^h) = 0.$$

2. By definition of $\hat{H}^{1,0}_p(\Omega)$, there is a sequence $(\pi_n)_{n\in\mathbb{N}} \subset C^\infty_c(\Omega)$ such that $\|\nabla\pi - \nabla\pi_n\|_{L^p(\Omega)} \to 0$ for $n \to \infty$. Therefore, we get

$$(\nabla\pi, \operatorname{rot}\phi) = \lim_{n\to\infty} (\nabla\pi_n, \operatorname{rot}\phi) = \lim_{n\to\infty} (\operatorname{rot}\nabla\pi_n, \phi) = 0.$$

3. By definition of $\hat{H}^{1,0}_{p'}(\Omega)$, there is a sequence $(\phi_n)_{n\in\mathbb{N}} \subset C^\infty_c(\Omega)$ such that $\|\nabla\phi - \nabla\phi_n\|_{L^{p'}(\Omega)} \to 0$ for $n \to \infty$. Hence,

$$(h, \operatorname{rot}\phi) = \lim_{n\to\infty} (h, \operatorname{rot}\phi_n) = \lim_{n\to\infty} (\operatorname{rot} h, \phi_n) = 0.$$

□

The proof of Theorem 2.4.3 (2) relied on the fact, that the weak Dirichlet problem is well-posed in $\hat{H}^{1,0}_p(\Omega)$ and that the space of test functions contains $C^\infty_c(\Omega)$. The former property has been important to show the uniqueness of the decomposition, the latter one has allowed us to show, that $\operatorname{div} h = 0$. For $p \in (1, 3/2] \cup [3, \infty)$, the weak Dirichlet problem is not well-posed anymore in $\hat{H}^{1,0}_p(\Omega)$ and the specific form of the functional $\phi \mapsto (u, \nabla\phi)$ on the right-hand side has no impact on this. However, this is not a problem due to the method used, but an intrinsic problem of the decomposition.

2.4.3 Intermezzo on Dirichlet Fields in $L^p(\Omega)$

It is known, that the size of the space of harmonic vector fields in bounded and exterior domains depends on the topology of the underlying domain Ω. In the case of Dirichlet fields, it depends on the second Betti number of Ω. In the particular case of three-dimensional domains, this coincides with the number of connected components of the boundary. This remains true in the case of exterior domains, but there will also appear a dependence on the order of integration p. We will illustrate this by an explicit example. Consider the exterior of the unit ball $\Omega = \overline{B(0,1)}^C$. It is a simple computation, that $h(x) := \frac{x}{|x|^3} = -\nabla\frac{1}{|x|}$ satisfies $\operatorname{div} h = 0$, $\operatorname{rot} h = 0$ and $h \times n = 0$. In other words, it is a Dirichlet field. If we ask for its integrability, we can observe that $h \in L^p(\Omega)$ for $p \in (3/2, \infty)$ but $h \notin L^q(\Omega)$ for $q \in (1, 3/2]$. This stands in contrast to the space of Neumann fields $L^p_{T,har}(\Omega)$, which does not depend on p as we have seen in Proposition 2.3.6. Another point of interest is the relation of $L^p_{N,har}(\Omega)$ to $\hat{H}^{1,0,har}_p(\Omega)$. It is easy to see, that $q_{har} := 1 - \frac{1}{|x|}$ is an element of $\hat{H}^{1,0,har}_p(\Omega)$ for $p \in [3, \infty)$. But it does also hold $\nabla q_{har} = h$. Hence, the intersection $L^p_{N,har}(\Omega) \cap \nabla\hat{H}^{1,0,har}_p(\Omega)$ is not trivial for $p \in [3, \infty)$. We will see in this section, that both of the observations just described generalise to any exterior domain with the same range of parameters.

In order to determine the set $L^p_{N,har}(\Omega)$ properly, we need to carry out a more refined analysis than in the case of $\dot{\mathbf{H}}^{1,N,har}_p(\Omega)$ in Subsection 2.2.4. In the proof of Theorem 2.2.18, we have only proven that a harmonic vector field h in Ω, that vanishes at infinity, decays like $1/|x|^2$ for $|x| \to \infty$. This is not enough to decide, whether h is in $L^p(\Omega)$ for $1 < p \leq 3/2$. In this subsection, we will see that there are Dirichlet fields which decay exactly like $1/|x|^2$ for $|x| \to \infty$. Thus, $L^p_{N,har}(\Omega)$ is actually smaller for $p \in (1, 3/2]$ than for $p \in (3/2, \infty)$. More precisely, we will show:

Proposition 2.4.9. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary. Denote the number of connected components of $\partial\Omega$ by L. If $p \in (1, 3/2]$, then $L^p_{N,har}$ is of dimension $L - 1$. If $p \in (3/2, \infty)$, then $L^p_{N,har}(\Omega)$ has dimension L. Furthermore, it holds $L^p_{N,har}(\Omega) \subseteq L^q_{N,har}(\Omega)$ for each $1 < p < \infty$ and $q \in (3/2, \infty)$.*

Remark 2.4.10. *If $\partial\Omega$ has just one connected component, Proposition 2.4.9 implies, that $L^p_{N,har}(\Omega) = \{0\}$ for all $p \in (1, 3/2]$.*

Throughout this subsection, we assume that $\Omega \subset \mathbb{R}^3$ is an exterior domain with C^∞-boundary. We denote by $\Gamma_1, \dots, \Gamma_L$ the connected components of $\partial\Omega$. That means

$$\begin{aligned}
&\Gamma_i \text{ is a smooth, compact, two dimensional and orientable} \\
&\quad \text{manifold without boundary for each } i = 1, \dots, L, \\
&\Gamma_i \cap \Gamma_j = \emptyset \quad \text{for} \quad i \neq j, \\
&\bigcup_{j=1}^{L} \Gamma_j = \partial\Omega.
\end{aligned}$$

The strategy of the proof of the proposition above can be sketched as follows. We will show that Dirichlet fields admit scalar potentials, which are solutions to a Laplace equation with Dirichlet boundary conditions. Making use of the maximum principle and the Laurent expansion of harmonic vector fields, we will be able to determine the decay of harmonic vector fields sufficiently well to fully characterize $L^p_{N,har}(\Omega)$. Beforehand, we begin with a first consistency result of Dirichlet fields in Lebesgue spaces.

Lemma 2.4.11. *Let $p \in (1, \infty)$ and $h \in L^p_{N,har}(\Omega)$. Then $h \in L^q_{N,har}(\Omega)$ for any $q \in (3/2, \infty)$.*

Proof. Let $h \in L^p_{N,har}(\Omega)$ and fix some $q \in (3/2, \infty)$. Because of Proposition 2.1.21, we have $h \in H^1_p(\Omega)$ and therefore $h \in \dot{\mathbf{H}}^{1,N,har}_p(\Omega)$. Therefore, Theorem 2.2.18 implies $h \in \dot{\mathbf{H}}^{1,N}_r(\Omega)$ for any $r \in (1, \infty)$. This is especially the case for r fulfilling $\frac{1}{r} + \frac{1}{3} = \frac{1}{q}$. Hence, because of Proposition 2.1.10, there is a $v \in \mathbb{R}^3$ such that $h - v \in L^q(\Omega)$. Due to $v = h - (h - v) \in L^p(\Omega) + L^q(\Omega)$, we get $v = 0$, thus $h \in L^q(\Omega)$. $\square$

As a consequence of Lemma 2.4.11, it is sufficient to know $L^p_{N,har}(\Omega)$ for one $p \in (3/2, \infty)$ to get to know it for all $p \in (3/2, \infty)$. It will turn out to be convenient to consider parameters smaller than three. We will show that in this case, the harmonic functions admit a scalar potential that solves a certain Laplace equation including some decay condition at infinity. The latter condition is a consequence of the Sobolev inequality, which is not available for $p \geq 3$. More precisely, we have:

Lemma 2.4.12. *Let $h \in L^p_{N,har}(\Omega)$ with $1 < p < 3$. Then there is a scalar potential $v \in L^p_{loc}(\overline{\Omega})$ such that $\nabla v = h$. Moreover, v can be chosen in such a way, that it is a solution to*

$$\begin{aligned} \Delta v &= 0 \quad && \text{in } \Omega, \\ v &= c_i && \text{on } \Gamma_i,\ i \in \{1, \dots, L\} \\ v(x) &\to 0 && \text{for } |x| \to \infty, \end{aligned} \tag{2.78}$$

where $c_1, \dots, c_L \in \mathbb{R}$ are some constants.

For the proof of Lemma 2.4.12, we need a Sobolev inequality on exterior domains without assuming any boundary conditions at all. In this sense, it is a generalisation of Proposition 2.1.10. The proof is almost the same.

Lemma 2.4.13. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary and $1 < p < 3$. Suppose that $f \in L^p_{loc}(\overline{\Omega})$ and $\nabla f \in L^p(\Omega)$. Then there is a unique constant $C > 0$ independent of f and a number $d \in \mathbb{R}$ such that*

$$\|f - d\|_{L^q(\Omega)} \leq C \|\nabla f\|_{L^p(\Omega)},$$

where $\frac{1}{q} = \frac{1}{p} - \frac{1}{3}$.

Proof. We reduce the situation to the whole space $\mathbb{R}^3$. Let $R > 0$ be such that $\partial\Omega \subset B(0, R)$. By our assumptions, $f \in H^1_p(\Omega_R)$. By the classical Poincaré inequality, there is a constant $C > 0$, such that

$$\|f - m\|_{L^p(\Omega_R)} \leq C \|\nabla f\|_{L^p(\Omega_R)},$$

where $m = |\Omega_R|^{-1} \int_{\Omega_R} f(x)\, \mathrm{d}x$. We may assume that $m = 0$, which implies

$$\|f\|_{H^1_p(\Omega_R)} \leq C \|\nabla f\|_{L^p(\Omega_R)} \leq C \|\nabla f\|_{L^p(\Omega)}.$$

Therefore, there is an extension operator that extends f to some function $\tilde{f}$ defined on $\mathbb{R}^3$, which satisfies $\tilde{f} \in L^p_{loc}(\mathbb{R}^3)$ and $\|\nabla \tilde{f}\|_{L^p(\mathbb{R}^3)} \leq C \|\nabla f\|_{L^p(\Omega)}$. Here $C > 0$ is independent of f. Using Lemma 2.1.1, there is a sequence

$(f_j)_{j\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^3)$ such that $\|\nabla\tilde{f} - \nabla f_j\|_{L^p(\mathbb{R}^3)} \to 0$ for $j \to \infty$. Employing the classical Sobolev embedding, the sequence $(f_j)_{j\in\mathbb{N}}$ is a Cauchy sequence in $L^q(\mathbb{R}^3)$ with some limit $f^* \in L^q(\mathbb{R}^3)$ that fulfils $\|f^*\|_{L^q(\mathbb{R}^3)} \leq C\|\nabla\tilde{f}\|_{L^p(\mathbb{R}^3)}$. We show, that $\nabla\tilde{f} = \nabla f^*$. Let $\phi \in C_c^\infty(\mathbb{R}^3)$. Then we have

$$\begin{aligned}\int_{\mathbb{R}^3}(\tilde{f} - f^*)\partial_k\phi &= -\int_{\mathbb{R}^3}(\partial_k\tilde{f})\phi - \lim_{j\to\infty}\int_{\mathbb{R}^3} f_j\partial_k\phi \\ &= -\int_{\mathbb{R}^3}(\partial_k\tilde{f})\phi + \lim_{j\to\infty}\int_{\mathbb{R}^3}(\partial_k f_j)\phi = 0,\end{aligned}$$

as $\partial_k f_j \to \partial_k\tilde{f}$ for $j \to \infty$ in $L^p(\mathbb{R}^3)$. Hence, there is a real number d such that $\tilde{f} - d = f^* \in L^q(\mathbb{R}^3)$. This implies

$$\|f - d\|_{L^q(\Omega)} \leq \|\tilde{f} - d\|_{L^q(\mathbb{R}^3)} = \|f^*\|_{L^q(\mathbb{R}^3)} \leq C\|\nabla\tilde{f}\|_{L^p(\mathbb{R}^3)} \leq C\|\nabla f\|_{L^p(\Omega)}.$$

It remains to show the uniqueness of d. Assume that there are $d_1, d_2 \in \mathbb{R}$ such that $f - d_1, f - d_2 \in L^q(\Omega)$. Then we have $d_2 - d_1 = f - d_1 - (f - d_2) \in L^q(\Omega)$ and consequently $d_1 = d_2$. $\square$

Proof of Lemma 2.4.12. The existence of a potential $v \in L^p_{loc}(\overline{\Omega})$ with $\nabla v = h$ and $\Delta v = 0$ was shown in Lemma 2.2.25. Because of $\nabla v \times n = h \times n = 0$, it follows that v is constant on each connected component of $\partial\Omega$. Due to Lemma 2.4.13, there is a $v_0 \in \mathbb{R}$ such that $v - v_0 \in L^q(\Omega)$, where $\frac{1}{q} = \frac{1}{p} - \frac{1}{3}$. Without loss of generality, we may assume that $v_0 = 0$, because a shift by v_0 does not affect the other collected properties of v. This proves the lemma. $\square$

Except for the integrability condition, the reverse statement of Lemma 2.4.12 is also true:

Lemma 2.4.14. *Let v be a solution to* (2.78)*. Then $h := \nabla v$ satisfies* $\operatorname{div} h = 0$*,* $\operatorname{rot} h = 0$ *and* $h \times n = 0$*.*

Proof. It holds $\operatorname{div} h = \operatorname{div}\nabla v = \Delta v = 0$ as well as $\operatorname{rot} h = \operatorname{rot}\nabla v = 0$. The property $h \times n = \nabla v \times n = 0$ on $\partial\Omega$ follows from v being locally constant there. $\square$

In view of the integrability of solutions to (2.78), it will be convenient to consider only non-negative boundary values. This will allow us to make use of the maximum principle, which will be crucial in order to derive a precise estimate of the solutions at infinity. Due to their linearity, it is easy to see, that the problems (2.78) are solvable for any $c_1, \ldots, c_L \in \mathbb{R}$ if and only if they are solvable for all $c_1, \ldots, c_L \geq 0$. Regarding these latter boundary values, we have the following result:

Lemma 2.4.15. *Let $c_1, \dots, c_L \geq 0$ be such that $\sum_{k=1}^{L} c_k > 1$. Then the equation* (2.78) *has a unique classical solution v. Let $R > 0$ satisfy $\partial\Omega \subset B(0, R)$. Then there is a constant $c > 0$ such that*

$$\frac{1}{c|x|} \leq v(x) \leq c\frac{1}{|x|},$$
$$\frac{1}{c|x|^2} \leq |\nabla v(x)| \leq c\frac{1}{|x|^2}$$

for all $x \in B(0, R)^C$.

Proof. The existence and uniqueness part are classical results. The same holds for the estimate from above. Both of them can be established by the method of expanding domains and the maximum principle. Only the estimate from below, does not seem to be stated in the literature, although, it can be done along the same lines. For the sake of completeness, we will prove that part.

We will construct a harmonic function, that is pointwise smaller than v outside a sufficiently large ball. Regarding this function, the decay at infinity will be easily seen by an explicit formula. Let $R > 0$ be such that $\partial\Omega \subset B(0, R)$. By Weyl's lemma, we know that v is continuous on $\partial B(0, R)$. As a non-constant harmonic function cannot attain its maximum and minimum in the interior of an open domain, we get that $0 < v < M := \max\{c_k : k = 1, \dots, L\}$ in Ω. Hence, there is an $m \in \mathbb{R}$ with $0 < m < \min\{v(x) : x \in \partial B(0, R)\}$. Consider the problems

$$\begin{aligned} \Delta w_n &= 0 \quad \text{in } \overline{B(0, R)}^C, \\ w_n(x) &= m \quad \text{on } \partial B(0, R), \\ w_n(x) &= 0 \quad \text{on } \partial B(0, n), \end{aligned} \tag{2.79}$$

where $n \in \mathbb{N}$ and $n > R$. For each n in the prescribed range, this problem has a unique solution by the classical theory of the Laplace equation. Using the maximum principle, we can see that $w_n(x)$ is a pointwise monotonically increasing sequence that is bounded from above by m. Hence, it converges pointwise to some function w on $\overline{B_R(0)}^C$. Using the mean value theorem, we additionally get, that the pointwise limit is a harmonic function on $\overline{B(0, R)}^C$. By the uniqueness of the solution to (2.79) with $n = \infty$, the function w has to coincide with $x \mapsto mR/|x|$. Moreover, we have $w_n = m < v$ on $\partial B(0, R)$ and $0 = w_n < v < M$ on $\partial B(0, n)$. Hence, we have $w_n(x) < v(x)$ for each $x \in \Omega \cap B(0, n)$ by the maximum principle. This implies $w \leq v$, which yields $1/c|x| < v(x) < c/|x|$ on $B(0, R)^C$. The exact estimate of ∇v does follow by the Laurent expansion of v on $B(0, R)^C$.

□

By suitable linear combinations, it is easy to get upper estimates for the decay of general solutions to (2.78). Note, however, that the estimates from below might fail in general due to cancellations.

Lemma 2.4.16. *Let $c_1, \ldots, c_L \in \mathbb{R}$ and v be a solution to* (2.78)*. Let $R > 0$ be such that $\partial\Omega \subset B(0,R)$. Then there is a constant $c > 0$ such that*

$$|v(x)| \leq c\frac{1}{|x|},$$
$$|\nabla v(x)| \leq c\frac{1}{|x|^2}.$$

Proof of Proposition 2.4.9. At first, we will estimate the dimension of $L^p_{N,har}(\Omega)$ from above. Let $p \in (1,3)$ and $h \in L^p_{N,har}(\Omega)$. Then, by Lemma 2.4.12, there is a scalar potential v of h that solves (2.78). As the space of solutions to that equation is L dimensional, we get $\dim L^p_{N,har}(\Omega) \leq L$. Regarding $p \in [3,\infty)$, we know by Lemma 2.4.11 that $L^p_{N,har}(\Omega) = L^2_{N,har}(\Omega)$. Due to the previous case, that implies $\dim L^p_{N,har}(\Omega) \leq L$ for this case, too.

In order to determine the dimension of $L^p_{N,har}(\Omega)$ precisely, we investigate the integrability of solutions to (2.78). This step will be divided into three different cases. We start with the case $p \in (3/2, 3)$. Let v be a solution to (2.78). By Lemma 2.4.14, we know that $h := \nabla v$ is a Dirichlet field. Because of Lemma 2.4.16, we furthermore get $|h(x)| \leq c/|x|^2$ for $|x| \to \infty$. This implies, that h lies in $L^p(B(0,R)^C)$ for some sufficiently large $R > 0$. Since $h \in C^\infty(\overline{\Omega})$ by Lemma 2.2.21, we even have $h \in L^p(\Omega)$. As the solution space to (2.78) is L-dimensional, we get $\dim L^p_{N,har}(\Omega) \geq L$. Together with the upper estimate for the dimension, this implies $\dim L^p_{N,har}(\Omega) = L$.

Let now $p \in [3,\infty)$. Because of Lemma 2.4.11, the spaces $L^p_{N,har}(\Omega)$ and $L^2_{N,har}(\Omega)$ do coincide. Hence, we can make use of the case considered before.

It remains to investigate $p \in (1, 3/2]$. By means of Lemma 2.4.15, the gradient of the solution v_1 to (2.78) with $c_1, \ldots, c_L = 1$ behaves like $1/|x|^2$ at infinity. This implies, that ∇v_1 cannot be an element of $L^p(\Omega)$. Thus, $\dim L^p_{N,har}(\Omega)$ has to be strictly smaller than L. We show that $L^p_{N,har}(\Omega)$ is actually just one dimension smaller. Extend v_1 by $v_2, \ldots, v_L$ to a basis of the solution space to (2.78). Because of [ABR01, 10.1 Laurent Series], there are harmonic polynomials p^i_m for each $i = 1, \ldots, L$, which are homogeneous of degree m such that

$$v_i(x) = \sum_{m=0}^{\infty} \frac{p^i_m(x)}{|x|^{2m+1}} = \frac{p^i_0}{|x|} + O(|x|^{-2}) \qquad \text{for } x \in B(0,R)^C.$$

Note that p^i_0 is just a constant for each $i \in \{1, \ldots, L\}$ and that $p^1_0 \neq 0$.

Let again $h \in L^p_{N,har}(\Omega)$. Setting

$$w_i := \sum_{k=1}^{K} \alpha_k^i v_i, \quad i \in \{2, \dots L\}$$

with

$$\alpha_1^i = \frac{p_0^i}{p_0^1}, \quad \alpha_i^i = -1, \quad \alpha_k^i = 0 \quad \text{for } k \in \{1, \dots, L\} \setminus \{1, i\},$$

we can see, that each w_i has a Laurent expansion of the form

$$w_i(x) = \sum_{m=1}^{\infty} \frac{q_m^i(x)}{|x|^{2m+1}}$$

with some harmonic polynomials q_m^i being homogeneous of degree m. This implies $\nabla w_i \sim \frac{1}{|x|^3}$ for $|x| \to \infty$ and therefore $\nabla w_i \in L^r(B(0,R)^C)$ for any $r \in (1, \infty)$. As ∇w_i is a Dirichlet field and therefore smooth up to the boundary, we get $\nabla w_i \in L^p_{N,har}(\Omega)$. Thus, we have

$$\operatorname{span}\{\nabla w_i : i = 2, \dots, L\} \subseteq L^p_{N,har}(\Omega) \subsetneq \operatorname{span}\{\nabla v_i : i = 1, \dots, L\}.$$

Since $\operatorname{span}\{\nabla w_i : i = 2, \dots, L\}$ has the maximal dimension for a proper subspace of $\operatorname{span}\{\nabla v_i : i = 1, \dots, L\}$, it holds

$$\operatorname{span}\{\nabla w_i : i = 2, \dots, L\} = L^p_{N,har}(\Omega),$$

as well as $\dim L^p_{N,har}(\Omega) = L - 1$.

$\square$

Remark 2.4.17. *For $p \in (1, 3/2]$, we note that each scalar potential w of a given $h \in L^p_{N,har}(\Omega) \setminus \{0\}$ has some strictly positive and some strictly negative boundary values. This is a consequence of Lemma 2.4.15.*

Knowing the decay at infinity of elements of $L^p_{N,har}(\Omega)$ for $p \in (1, 3/2]$, we can also deduce some information about the behaviour of these vector fields at the boundary.

Lemma 2.4.18. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary, $p \in (1, 3/2]$ and $h \in L^p_{N,har}(\Omega)$. Then it holds*

$$\int_{\partial\Omega} h \cdot n \, \mathrm{d}S = 0. \tag{2.80}$$

Proof. For each $R > 0$ with $\partial\Omega \subset B(0,R)$, we get

$$(2.81) \qquad 0 = \int_{\Omega_R} \operatorname{div} h \, \mathrm{d}x = \int_{\partial\Omega} h \cdot n \, \mathrm{d}S + \int_{\partial B(0,R)} h \cdot n \, \mathrm{d}S.$$

In the proof of Proposition 2.4.9, we have seen that $|h(x)|$ behaves like $1/|x|^3$ at infinity. It follows, that

$$\Big| \int_{\partial B(0,R)} h \cdot n \, \mathrm{d}S \Big| \le |\partial B(0,R)| \|h\|_{L^\infty(\partial B(0,R))} \le CR^2 \frac{1}{R^3} \to 0$$

for $R \to \infty$. Together with (2.81), this implies (2.80). □

We close this section by establishing the relation between the spaces $L^p_{N,har}(\Omega)$ and $\hat{H}^{1,0,har}_p(\Omega)$.

Proposition 2.4.19. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary and $p \in (1,\infty)$. Then $\nabla \hat{H}^{1,0,har}_p(\Omega) \subseteq L^p_{N,har}(\Omega)$.*

Proof. For $p \in (1,3)$, the inclusion is trivial since $\hat{H}^{1,0,har}_p(\Omega) = \{0\}$ by Lemma 2.4.6. Therefore, consider $p \in [3,\infty)$. Let $q_{har} \in \hat{H}^{1,0,har}_p(\Omega)$ and set $h := \nabla q_{har}$. Then it holds $\operatorname{rot} h = \operatorname{rot} \nabla q_{har} = 0$, $\operatorname{div} h = \operatorname{div} \nabla q_{har} = \Delta q_{har} = 0$ as well as $h \times n = \nabla q_{har} \times n = 0$ because q_{har} is constant on $\partial\Omega$. As $h \in L^p(\Omega)$, this implies $h \in L^p_{N,har}(\Omega)$. □

2.4.4 Continuation of the Proof of the Main Results

Having proper knowledge of $L^p_{N,har}(\Omega)$ at hand, we can now complete the proof of Theorem 2.4.3. We begin with the third claim therein.

Proof of Theorem 2.4.3 (3). Let $u \in L^p(\Omega)$. Define $w \in \tilde{\mathbf{H}}^{1,T}_p(\Omega)$ to be the unique solution to

$$\begin{aligned} (\operatorname{rot} w, \operatorname{rot} \phi) &= (u, \operatorname{rot} \phi) \quad \text{for all } \phi \in \tilde{\mathbf{H}}^{1,T}_{p'}(\Omega), \\ \operatorname{div} w &= 0 \end{aligned}$$

given by Theorem 2.2.49. Moreover, set $\pi \in \tilde{H}^{1,0}_p(\Omega)$ to be the unique solution to

$$(\nabla\pi, \nabla\phi) = (u, \nabla\phi) \quad \text{for all } \phi \in \tilde{H}^{1,0}_{p'}(\Omega),$$

see Lemma 2.4.7. Note that $\tilde{H}^{1,0}_{p'}(\Omega) = \hat{H}^{1,0}_{p'}(\Omega)$ in this case. By the same arguments as in the proof of Theorem 2.4.3 (2), one can see, that $h := u -$

$\operatorname{rot} w - \nabla\pi$ lies in $L^p_{N,har}(\Omega)$. Hence, the existence of the decomposition is shown. However, the decomposition is not unique because of Proposition 2.4.19. It remains to show, that $u \mapsto \operatorname{rot} w$, $u \mapsto \nabla\pi$ and $u \mapsto h$ with w, π and h as given above are bounded linear projections. The boundedness of these operators follows from the norm estimates in Theorem 2.2.49 and Lemma 2.4.7. It is clear, that the unique solution $v \in \tilde{\mathbf{H}}^{1,T}_p(\Omega)$ to

$$\begin{aligned}(\operatorname{rot} v, \operatorname{rot}\phi) &= (\operatorname{rot} w, \operatorname{rot}\phi) \qquad \text{for all } \phi \in \tilde{\mathbf{H}}^{1,T}_{p'}(\Omega),\\ \operatorname{div} w &= 0\end{aligned}$$

has to be w itself. Thus, $u \mapsto \operatorname{rot} w$ is a projection. The same argumentation works for $u \mapsto \nabla\pi$, too. Hence, $u \mapsto h$ has to be a projection as well. □

As a preliminary for the proof of Theorem 2.4.3 (1), we show that $L^p(\Omega)$ cannot be spanned by $L^p_{N,har}(\Omega)$, $\nabla\hat{H}^{1,0}_p(\Omega)$ and $\operatorname{rot}\dot{\mathbf{H}}^{1,T}_p(\Omega)$ for $p \in (1, 3/2]$. The idea behind this is as follows: In view of Proposition 2.4.1, the validity of the second Helmholtz-Hodge decomposition reduces to the question, whether $\nabla\dot{\mathbf{H}}^{1,c}_p(\Omega)$ can be (directly) decomposed into $L^p_{N,har}(\Omega) + \nabla\hat{H}^{1,0}_p(\Omega)$. Given some $\eta \in \dot{\mathbf{H}}^{1,c}_p(\Omega)$, it is therefore necessary to find some $h \in L^p_{N,har}(\Omega)$ with scalar potential v such that $\eta - v \in \hat{H}^{1,0}_p(\Omega)$. For $p \in (3/2, \infty)$ this is possible, since for any possible kind of boundary values of η, there is a harmonic vector field, whose scalar potential admits the right kind of boundary values. However, for $p \in (1, 3/2]$, the space $L^p_{N,har}(\Omega)$ is one dimension smaller than in the previous case. This generally causes the lack of a harmonic vector field $h \in L^p_{N,har}(\Omega)$ with scalar potential v such that $\eta - v \in \hat{H}^{1,0}_p(\Omega)$. Thus the main reason for our choice of η to disprove the second Helmholtz-Hodge decomposition are its boundary values.

Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with smooth boundary and $R > 0$ such that $\partial\Omega \subset B(0, R)$. Let $\eta \in C^\infty_c(\overline{\Omega})$ be such that

$$\begin{aligned}\eta(x) &= 1 \qquad \text{for } |x| \le R,\\ \eta(x) &= 0 \qquad \text{for } |x| \ge R+1.\end{aligned} \tag{2.82}$$

Then we have:

Lemma 2.4.20. *Let $\eta \in C^\infty_c(\overline{\Omega})$ have the same properties as in (2.82). Then for any $h \in L^p_{N,har}(\Omega)$, $\pi \in \hat{H}^{1,0}_p(\Omega)$ and $w \in \dot{\mathbf{H}}^{1,T}_p(\Omega)$, it holds*

$$\nabla\eta \neq h + \operatorname{rot} w + \nabla\pi. \tag{2.83}$$

Proof. First we make use of the simplified decomposition from Proposition 2.4.1 in order to show that the term $\operatorname{rot} w$ in (2.83) has to be zero. Recall that the vector potential $w \in \dot{\mathbf{H}}_p^{1,T}(\Omega)$ can be constructed as the solution to

$$\begin{aligned} (\operatorname{rot} w, \operatorname{rot} \phi) &= (\nabla\eta, \operatorname{rot} \phi) \qquad \text{for all } \phi \in \tilde{\mathbf{H}}_{p'}^{1,T}(\Omega), \\ \operatorname{div} w &= 0, \end{aligned}$$

which exists and is unique in $\tilde{\mathbf{H}}_p^{1,T}(\Omega)$ due to Theorem 2.2.49. However, as $\nabla\eta \in C_c^\infty(\Omega)$ is supported away from $\partial\Omega$, it holds

$$(\nabla\eta, \operatorname{rot} \phi) = (\operatorname{rot} \nabla\eta, \phi) = 0$$

for any $\phi \in \tilde{\mathbf{H}}_{p'}^{1,T}(\Omega)$. That implies $w = 0$. Because of the uniqueness of the vector potentials up to harmonic vector fields, this means $\operatorname{rot} w = 0$ for any vector potential $w \in \dot{\mathbf{H}}_p^{1,T}(\Omega)$ of $\nabla\eta$. Hence, it remains to show that $\nabla\eta$ cannot be written as $h + \nabla\pi$ with $h \in L^p_{N,har}(\Omega)$ and $\pi \in \hat{H}_p^{1,0}(\Omega)$. Because of Lemma 2.4.12, each $h \in L^p_{N,har}(\Omega)$ has a scalar potential v which is locally constant along $\partial\Omega$. As η and any $\pi \in \hat{H}_p^{1,0}(\Omega)$ are globally constant on $\partial\Omega$, it follows that v needs to be globally constant on $\partial\Omega$, too. However, we have seen in Remark 2.4.17 that for $p \in (1, 3/2]$, only $v = 0$ is a possible scalar potential of h that is globally constant along the boundary. Hence, $h = 0$. Clearly, any $\pi \in \hat{H}_p^{1,0}(\Omega)$ vanishes along $\partial\Omega$ and at infinity. As η only vanishes at infinity but not along $\partial\Omega$, there are no $\pi \in \hat{H}_p^{1,0}(\Omega)$ and $c \in \mathbb{R}$ such that $\eta = \pi + c$. This is a contradiction to $\nabla\eta = \nabla\pi$, which completes the proof. $\square$

It is a natural question to ask, whether there are in some sense more vector fields besides $\nabla\eta$ that cannot be decomposed in the sense of the second Helmholtz-Hodge decomposition. The answer will turn out to be no. That means only a one-dimensional space is missing for the second Helmholtz-Hodge decomposition to be valid for $p \in (1, 3/2]$. In the following considerations, we will add that space in order to prove a generalised decomposition. More precisely, we will add a one-dimensional space to the solution space of the weak Dirichlet problem in order to make it well-posed. The rest then goes along the same lines of the proofs of the other decompositions.

At first, we will modify the scalar field η from Lemma 2.4.20 to be a nicely behaving addition to the weak Dirichlet problem, which does not interact with the other projection operators we will use later on.

Lemma 2.4.21. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with C^∞-boundary and*

$p \in (1, 3/2]$. *Then there is a function* $q \in \dot{\mathbf{H}}^{1,c}_p(\Omega)$ *such that*

$$(\nabla q, \nabla \phi) = 0 \quad \textit{for all } \phi \in \tilde{H}^{1,0}_{p'}(\Omega), \tag{2.84}$$

$$(\nabla q, \operatorname{rot} \phi) = 0 \quad \textit{for all } \phi \in \dot{\mathbf{H}}^{1,T}_{p'}(\Omega), \tag{2.85}$$

$$(\nabla q, h) = 0 \quad \textit{for all } h \in L^p_{N,har}(\Omega), \tag{2.86}$$

and for $q_{har} \in \hat{H}^{1,0,har}_{p'}(\Omega) \setminus \{0\}$, *it holds*

$$(\nabla q, \nabla q_{har}) \neq 0. \tag{2.87}$$

This function q *is not contained in* $L^p_{N,har}(\Omega) + \nabla \hat{H}^{1,0}_p(\Omega) + \operatorname{rot} \dot{\mathbf{H}}^{1,T}_p(\Omega)$ *and globally constant on* $\partial\Omega$.

Proof. Let $\eta \in C^\infty_c(\overline{\Omega})$ be as in (2.82). It is clear, that $\nabla \eta \in L^p(\Omega)$ for any $p \in (1, \infty)$. Let $\alpha \in \hat{H}^{1,0}_p(\Omega)$ be the unique solution to

$$(\nabla \alpha, \nabla \phi) = (\nabla \eta, \nabla \phi) \quad \text{for all } \phi \in \tilde{H}^{1,0}_p(\Omega),$$

compare to Lemma 2.4.7. Define

$$q := \eta - \alpha.$$

Then for any $\phi \in \tilde{H}^{1,0}_{p'}(\Omega)$, it holds

$$(\nabla q, \nabla \phi) = (\nabla \eta, \nabla \phi) - (\nabla \alpha, \nabla \phi) = 0.$$

Thus, (2.84) is satisfied. For $\phi \in \dot{\mathbf{H}}^{1,T}_{p'}(\Omega)$, we have

$$(\nabla q, \operatorname{rot} \phi) = (\nabla \eta, \operatorname{rot} \phi) - (\nabla \alpha, \operatorname{rot} \phi) = (\operatorname{rot} \nabla \eta, \phi) - (\nabla \alpha, \operatorname{rot} \phi) = 0,$$

where we used (2.76) in the last equality. Hence, (2.85) is fulfilled. Given some $h \in L^p_{N,har}(\Omega)$, we get

$$\begin{aligned}
(\nabla q, h) &= (\nabla \eta, h) - (\nabla \alpha, h) \\
&= -(\eta, \operatorname{div} h) + \langle \eta, h \cdot n \rangle_{\partial\Omega} - (\nabla \alpha, h) \\
&= 0 + \langle 1, h \cdot n \rangle_{\partial\Omega} - 0 \\
&= \int_{\partial\Omega} h \cdot n \, \mathrm{d}S.
\end{aligned}$$

Here we have made use of (2.77) and the harmonicity of h in the third equality. Because of Lemma 2.4.18, the boundary integral vanishes. Therefore, (2.86) is satisfied, too. The inequality (2.87) is more involved. Let $q_{har} \in \hat{H}^{1,0,har}_{p'}(\Omega) \setminus \{0\}$. Then it holds

$$(\nabla q, \nabla q_{har}) = (\nabla \eta, \nabla q_{har}) - (\nabla \alpha, \nabla q_{har}) = (\nabla \eta, \nabla q_{har}),$$

because of (2.77) and $\nabla q_{har} \in L^{p'}_{N,har}(\Omega)$ by Proposition 2.4.19. We will make use of some representations of harmonic functions, which assume that the harmonic function vanishes at infinity. In view of Lemma 2.4.7, we may assume that $q_{har}(x) - a =: k(x) \to 0$ for $|x| \to \infty$ for some $a \in \mathbb{R} \setminus \{0\}$. For simplicity, we assume that a is positive. As k is harmonic, the maximum principle implies $-a < k(x) < 0$ for all $x \in \Omega$. Thus, we have $\partial_n k(x) \le 0$ for all $x \in \partial\Omega$. We show that $\partial_n k(x)$ is not constantly zero on $\partial\Omega$ by contraposition. So assume that $\partial_n k = 0$ on $\partial\Omega$. Because of the harmonicity of k and the integral representation [GT01, (2.18)], it holds

$$k(x) = \int_{\partial\Omega} k(y) \frac{\partial E(x-y)}{\partial n(y)} - E(x-y) \frac{\partial k(y)}{\partial n(y)} \, \mathrm{d}y \quad \text{for all } x \in \Omega,$$

where E is the fundamental solution to the Laplace equation in $\mathbb{R}^3$. Because of $\partial_n k = 0$ and $k = -a$ on $\partial\Omega$, we even get

$$k(x) = -a \int_{\partial\Omega} \frac{\partial E(x-y)}{\partial n(y)} \, \mathrm{d}y \quad \text{for all } x \in \Omega.$$

As $\nabla E(x) \sim \frac{1}{|x|^2}$ for $|x| \to \infty$, that means $|k(x)| \sim \frac{1}{|x|^2}$ at infinity. This contradicts the representation of $q_{har} = k - a$ given in Lemma 2.4.6. Thus $\partial_n q_{har}$ is not constantly zero on $\partial\Omega$. Together with the non-positivity of $\partial_n q_{har}$ and (2.86), this implies $(\nabla \eta, \nabla q_{har}) \neq 0$ and therefore

$$(\nabla q, \nabla q_{har}) \neq 0.$$

As $\eta = 1$ and $\alpha = 0$ on $\partial\Omega$, it holds $q \equiv 1$ globally on $\partial\Omega$. Finally, as $\nabla \eta$ is not contained in $L^p_{N,har}(\Omega) + \operatorname{rot} \dot{\mathbf{H}}^{1,T}_p(\Omega) + \nabla \hat{H}^{1,0}_p(\Omega)$ due to Lemma 2.4.20, the same has to be the case for ∇q. □

The function q constructed above allows us to solve the weak Dirichlet problem for the proof of Helmholtz-Hodge decomposition in a suitably generalised sense. The rest of the proof of Theorem 2.4.3 (1) is similar to the other parts.

Proof of Theorem 2.4.3 (1). Making use of Lemma 2.4.21, we can now solve the weak Dirichlet problem in a suitable sense. By Lemma 2.4.7, there is a unique solution $\pi \in \tilde{H}^{1,0}_p(\Omega) = \hat{H}^{1,0}_p(\Omega)$ of the problem

$$(\nabla \pi, \nabla \phi) = (u, \nabla \phi) \quad \text{for all } \phi \in \tilde{H}^{1,0}_{p'}(\Omega).$$

Let $q \in \dot{\mathbf{H}}^{1,c}_p(\Omega)$ be the function given by Lemma 2.4.21. Then there is a unique $a \in \mathbb{R}$ such that

$$(a\nabla q, \nabla q_{har}) = (u, \nabla q_{har}) \text{ for all } q_{har} \in \hat{H}^{1,0,har}_{p'}(\Omega).$$

Employing (2.84), we get

$$(\nabla(\pi + aq), \nabla\phi) = (u, \nabla\phi) \quad \text{for all } \phi \in \hat{H}^{1,0}_{p'}(\Omega). \tag{2.88}$$

Set $h := u - a\nabla q - \nabla\pi - \operatorname{rot} w$, where $w \in \tilde{\mathbf{H}}^{1,T}_p(\Omega)$ is the unique solution to

$$\begin{aligned}(\operatorname{rot} w, \operatorname{rot}\phi) &= (u, \operatorname{rot}\phi) \quad \text{for all } \phi \in \dot{\mathbf{H}}^{1,T}_{p'}(\Omega),\\ \operatorname{div} w &= 0,\end{aligned}$$

which is given by Theorem 2.2.49. We show, that $h \in L^p_{N,har}(\Omega)$. Let $\phi \in C^\infty_c(\mathbb{R}^3)$. Then

$$\begin{aligned}(h, \nabla\phi) &= (u, \nabla\phi) - (a\nabla q, \nabla\phi) - (\nabla\pi, \nabla\phi) - (\operatorname{rot} w, \nabla\phi)\\ &= 0 - (w, \operatorname{rot}\nabla\phi)\\ &= 0,\end{aligned}$$

where we have used (2.88) and integration by parts. Similarly

$$\begin{aligned}(h, \operatorname{rot}\phi) &= (u, \operatorname{rot}\phi) - (\operatorname{rot} w, \operatorname{rot}\phi) - (a\nabla q, \operatorname{rot}\phi) - (\nabla\pi, \operatorname{rot}\phi)\\ &= 0 + (aq, \operatorname{div}\operatorname{rot}\phi) + (\pi, \operatorname{div}\operatorname{rot}\phi)\\ &= 0.\end{aligned}$$

Hence, h is a harmonic vector field. Regarding the boundary condition, it is sufficient to show

$$\langle h \times n, \phi\rangle_{\partial\Omega} = 0 \quad \text{for all } \phi \in H^1_{p'}(\Omega). \tag{2.89}$$

For any $\phi \in C^\infty_c(\overline{\Omega})$, it holds

$$\begin{aligned}\langle h \times n, \phi\rangle_{\partial\Omega} &= (\operatorname{rot} h, \phi) - (h, \operatorname{rot}\phi) = (h, \operatorname{rot}\phi)\\ &= (u, \operatorname{rot}\phi) - (\operatorname{rot} w, \operatorname{rot}\phi) - (a\nabla q, \operatorname{rot}\phi) - (\nabla\pi, \operatorname{rot}\phi).\end{aligned}$$

For the first two summands on the right-hand side, we apply of the Helmholtz decomposition in $H^1_{p'}(\Omega)$ and decompose ϕ into

$$\phi = \phi_\sigma + \nabla\varpi$$

with $\phi_\sigma \in H^1_{p'}(\Omega) \cap L^{p'}_\sigma(\Omega) \subset \dot{\mathbf{H}}^{1,T}_{p'}(\Omega)$, $\varpi \in \dot{\mathbf{H}}^1_{p'}(\Omega)$ and $\nabla^2\varpi \in L^{p'}(\Omega)$. This yields

$$\begin{aligned}&(u, \operatorname{rot}\phi) - (\operatorname{rot} w, \operatorname{rot}\phi)\\ &\quad = (u, \operatorname{rot}\phi_\sigma) - (\operatorname{rot} w, \operatorname{rot}\phi_\sigma) + (u, \operatorname{rot}\nabla\varpi) - (\operatorname{rot} w, \operatorname{rot}\nabla\varpi)\\ &\quad = 0.\end{aligned}$$

Concerning the other two summands, we have

$$(a\nabla q, \operatorname{rot}\phi) + (\nabla\pi, \operatorname{rot}\phi) = -(\operatorname{rot}\nabla(aq+\pi), \phi) + \langle \nabla(aq+\pi) \times n, \phi\rangle_{\partial\Omega} = 0.$$

Here, we have used $\nabla q \times n = \nabla\pi \times n = 0$, which follows from q and π being constant on $\partial\Omega$. Hence, we have $\langle h \times n, \phi\rangle_{\partial\Omega} = 0$ for all $\phi \in C^\infty_c(\overline{\Omega})$. By the density of $C^\infty_c(\overline{\Omega})$ in $H^1_{p'}(\Omega)$, this implies (2.89). Hence, we can conclude that $h \in L^p_{N,har}(\Omega)$. $\square$

Using Theorem 2.4.3 and the characterisation of $L^p_{N,har}(\Omega)$, we can show Corollary 2.4.5.

Proof of Corollary 2.4.5. Because of Theorem 2.4.3 (1), there are $w \in \dot{\mathbf{H}}^{1,T}_p(\Omega)$ with $\operatorname{div} w = 0$, $h \in L^p_{N,har}(\Omega)$, $\hat{\pi} \in \hat{H}^{1,0}_p(\Omega)$ and $q \in \dot{\mathbf{H}}^1_p(\Omega)$ such that

$$u = \operatorname{rot} w + h + \nabla\hat{\pi} + \nabla q.$$

Furthermore, q is globally constant along $\partial\Omega$. Hence, there is a scalar $a \in \mathbb{R}$ such that $q - a = 0$ on $\partial\Omega$. This implies $q - a \in \dot{\mathbf{H}}^{1,0}_p(\Omega)$ and thus $\pi := \hat{\pi} + q - a \in \dot{\mathbf{H}}^{1,0}_p(\Omega)$. Clearly, it holds

$$u = \operatorname{rot} w + h + \nabla\pi.$$

It remains to show the uniqueness of the decomposition. The existence of corresponding bounded linear projections then follows immediately. Let $u = \operatorname{rot}\bar{w} + \bar{h} + \nabla\bar{\pi}$ with $\bar{w} \in \dot{\mathbf{H}}^{1,T}_p(\Omega)$, $\operatorname{div}\bar{w} = 0$, $\bar{h} \in L^p_{N,har}(\Omega)$ and $\bar{\pi} \in \dot{\mathbf{H}}^{1,0}_p(\Omega)$ be another decomposition of u. It is a consequence of Proposition 2.4.1 that $\operatorname{rot} w = \operatorname{rot}\bar{w}$. In order to treat the other summands, we show that $\nabla\dot{\mathbf{H}}^{1,0}_p(\Omega) \cap L^p_{N,har}(\Omega) = \{0\}$. Let $k \in L^p_{N,har}(\Omega) \setminus \{0\}$ be arbitrary. By Lemma 2.4.12 there is some $v \in \dot{\mathbf{H}}^{1,c}_p(\Omega)$ such that $\nabla v = k$. Moreover v cannot be globally constant along $\partial\Omega$ because of Remark 2.4.17. Thus there is no $c \in \mathbb{R}$ such that $v - c$ is constantly zero on $\partial\Omega$. As scalar potentials are unique up to constant functions, this implies $k \notin \nabla\dot{\mathbf{H}}^{1,0}_p(\Omega)$ and therefore $\nabla\dot{\mathbf{H}}^{1,0}_p(\Omega) \cap L^p_{N,har}(\Omega) = \{0\}$. It directly follows that $h = \bar{h}$ and $\nabla\pi = \nabla\bar{\pi}$. $\square$

3 Evolution Equations on the Whole Real Time Axis

In this chapter, we will construct bounded solutions to abstract partial differential equations on the whole real time axis. The main motivation for this setting is the construction of time periodic and almost periodic solutions for given periodic or almost periodic exterior forces. While in the first part, we focus on the construction of mild solutions for semilinear equations with application to equations on unbounded domains, the second part will be about strong solutions and quasilinear equations.

3.1 Mild Solutions

This section is about the investigation of mild solutions on the whole real time axis. We bring together the abstract setting of [GHN16] with the regularity and stability results of [Yam00]. The new generalisations will be shown to be applicable to various parabolic equations in distinct functional settings. Besides the applications with values in weak Lebesgue spaces, which are the only ones that have been considered before, the abstract theory will also be applied to the Navier-Stokes equations in homogeneous Besov spaces.

3.1.1 Autonomous Equations

This section is devoted to the investigation of abstract Cauchy problems of the form

$$u'(t) - Au(t) = Bf(t), \quad t \in \mathbb{R}, \tag{3.1}$$

where A and B are linear operators. We will consider generalised mild solutions to this problem. For the initial value problem

$$\begin{aligned} u'(t) - Au(t) &= Bf(t), \quad t > 0, \\ u(0) &= a, \end{aligned}$$

mild solutions are formally given by Duhamel's formula

$$u(t) = e^{tA}a + \int_0^t e^{(t-s)A} Bf(s)\,\mathrm{d}s.$$

We can modify this formula to the whole real axis by dropping the initial value term and extending the integral to the interval $(-\infty, t)$, i.e. one can formally define a mild solution to (3.1) by

$$u(t) = \int_{-\infty}^t e^{(t-s)A} Bf(s)\,\mathrm{d}s. \tag{3.2}$$

In the case of B being the identity and $(e^{tA})_{t\geq 0}$ being an exponentially stable semigroup on some Banach space Y, it is easy to see that the integral in (3.2) is well defined for any $f \in L^\infty(\mathbb{R}; Y)$. Furthermore, one can use this representation formula of the solution to transfer properties of f to the solution u. For example, if f is T-periodic for some $T > 0$, one easily gets

$$\begin{aligned} u(t) &= \int_{-\infty}^t e^{(t-s)A} Bf(s)\,\mathrm{d}s = \int_{-\infty}^{t+T} e^{(t-(r-T))A} Bf(r-T)\,\mathrm{d}r \\ &= \int_{-\infty}^{t+T} e^{(t+T-r)A} Bf(r)\,\mathrm{d}r = u(t+T). \end{aligned}$$

Hence, the function u is also T-periodic.

However, this notion of solution also contains some differences and difficulties compared to the classical notion of mild solutions for initial value problems. Consider for example the equation

$$u'(t) - \mathrm{i}u(t) = 0, \quad t \in \mathbb{R},$$

on the Banach space $X = \mathbb{C}$. Then there are infinitely many classical solutions to this equation, namely all functions of the kind $u(t) = xe^{\mathrm{i}t}$ with $x \in \mathbb{C}$. But the only solution given by formula (3.2) is $u(t) = 0$. This means, in contrast to Duhamel's formula for the initial value problem, not every classical solution to (3.1) is given by (3.2). Secondly, we necessarily have to deal with integrals over unbounded intervals. Later on, we want to apply the theory here to problems, where the semigroup $(e^{tA})_{t\geq 0}$ is merely bounded. This is for example the case, if zero is contained in the spectrum of A. Under this circumstance, the integral in (3.2) is generally not well defined. Even under our upcoming assumptions, this integral will not exist in the usual sense of Bochner integrals. Nevertheless, we will be able to show, that the integral in (3.2) exists in the weak-*-sense. More precisely, we will consider the following notion of solution.

Definition 3.1.1. *Let Z be a Banach space and $Y := Z'$. A function $u \in L^\infty(\mathbb{R}; Y)$ is called a mild solution to* (3.1) *if for each $\psi \in Z$*

$$\langle u(t), \psi\rangle_{Y,Z} = \int_{-\infty}^{t} \langle e^{(t-s)A} Bf(r), \psi\rangle_{Y,Z}\, \mathrm{d}r. \tag{3.3}$$

Due to this notion of solution, we will make frequently use of duality arguments and therefore of 'pre-adjoints' of the operators A, B and e^{tA}. We will denote them by a superscript $\flat$. Hence, we have for example $(A^\flat)' = A$.

We will now formulate and motivate our upcoming general assumptions. These can be seen as abstract versions of the well-known L^p-L^q-estimates of the heat semigroup, which we would like to recall now. Consider the realization of the Laplace operator in Lebesgue spaces on $\mathbb{R}^d$, that means the Laplacian with domain $D(\Delta) = W^{2,p}(\mathbb{R}^d) \subset L^p(\Omega)$. It is known, that Δ is the generator of a strongly continuous semigroup for all $1 < p < \infty$. Furthermore, for any $1 < p \le q < \infty$ there is a constant $C > 0$ such that

$$\|e^{t\Delta} u\|_{L^q(\mathbb{R}^d)} \le C t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p(\mathbb{R}^d)}$$

for all $t > 0$ and $u \in L^p(\mathbb{R}^d)$. If $d \ge 3$, the parameters p and q can be chosen in such a way, that the decay exponent $\frac{d}{2}(\frac{1}{p} - \frac{1}{q})$ is strictly larger than one. And, of course, strictly smaller than one is possible, too. We will require such kind of estimates in order to show the existence of mild solutions for abstract Cauchy problems.

Assumptions 3.1.2 (ACPEx). *Let Z_1, Z_2 be an interpolation couple of Banach spaces. Assume that $Z_1 \cap Z_2$ is dense in Z_1 and Z_2. Let $A^\flat$ be the generator of consistent semigroups on Z_1 and Z_2 and let $B^\flat$ be a linear operator defined on the range of $e^{tA^\flat}$, which maps to some normed vector space V. Suppose that there are constants $N > 0$ and $\alpha_1 > 1 > \alpha_2 \ge 0$, such that*

$$\|B^\flat e^{tA^\flat} \psi\|_V \le N t^{-\alpha_i} \|\psi\|_{Z_i}, \quad t > 0, \phi \in Z_i, i \in \{1, 2\}. \tag{3.4}$$

Additionally to these assumptions, we will make use of some reappearing relying on these hypotheses.

Notation 3.1.3. *Let $\theta \in (0,1)$ be such that $1 = (1-\theta)\alpha_1 + \theta\alpha_2$ and set $Y_1 := Z_1'$, $Y_2 := Z_2'$, $Y := (Y_1, Y_2)_{\theta,\infty}$ and $X := V'$.*

By Proposition 1.2.11, the map $t \mapsto t^{-\alpha}$ lies in $L^{1/\alpha,\infty}(\mathbb{R}_+; \mathbb{R})$ for any $\alpha \in [0, \infty)$. In view of (3.4), that means $t \mapsto \|B^\flat e^{tA^\flat} \psi\|_V$ lies in $L^{1/\alpha_1,\infty}(\mathbb{R}_+; \mathbb{R}) \cap L^{1/\alpha_2,\infty}(\mathbb{R}_+; \mathbb{R})$ for any $\psi \in Z_1 \cap Z_2$. Because of $\alpha_1 > 1 > \alpha_2 \ge 0$, $L^1(\mathbb{R}_+; \mathbb{R})$ is contained in $L^{1/\alpha_1,\infty}(\mathbb{R}_+; \mathbb{R}) \cap L^{1/\alpha_2,\infty}(\mathbb{R}_+; \mathbb{R})$. Thus, $t \mapsto \|B^\flat e^{tA^\flat} \psi\|_V$ is an element of $L^1(\mathbb{R}_+;\mathbb{R})$. Additionally making use of the boundedness of f, this gives a rough sketch, how these assumptions can be used to show, that (3.3) is finite. We do now make this rigorous in the following existence theorem.

Theorem 3.1.4. *Let the Assumptions 3.1.2 (ACPEx) be valid and let θ, Y and X be as in Notation 3.1.3. Suppose that $f \in L^\infty(\mathbb{R};X)$. Then problem* (3.1) *has a mild solution $u \in L^\infty(\mathbb{R};Y)$. Furthermore, there is a constant $M > 0$ such that*

$$\|u(t)\|_Y \leq M\|f\|_{L^\infty(\mathbb{R};X)}, \quad t \in \mathbb{R}. \tag{3.5}$$

Proof. We have to show that the integral in equation (3.3) is well defined for Y and $Z := (Z_1, Z_2)_{\theta,1}$. By the density of $Z_1 \cap Z_2$ in Z_1 and Z_2, this is indeed a dual pairing by Proposition 1.1.8. For $\psi \in Z$ we have

$$\begin{aligned}
&\Big|\int_{-\infty}^t \langle e^{(t-s)A} Bf(s), \psi\rangle_{Y,Z}\,\mathrm{d}s\Big| \\
&\leq \int_{-\infty}^t |\langle f(s), B^\flat e^{(t-s)A^\flat}\psi\rangle_{X,V}|\,\mathrm{d}s \\
&\leq \int_{-\infty}^t \|f(s)\|_X \|B^\flat e^{(t-s)A^\flat}\psi\|_V\,\mathrm{d}s \\
&\leq \|f\|_{L^\infty(\mathbb{R};X)} \int_{-\infty}^t \|B^\flat e^{(t-s)A^\flat}\psi\|_V\,\mathrm{d}s \\
&= \|f\|_{L^\infty(\mathbb{R};X)} \int_0^\infty \|B^\flat e^{rA^\flat}\psi\|_V\,\mathrm{d}r.
\end{aligned} \tag{3.6}$$

Due to (3.4), the sublinear operator

$$\begin{aligned}
T\colon Z_1 + Z_2 &\to L^{1/\alpha_1,\infty}(\mathbb{R}_+;\mathbb{R}) + L^{1/\alpha_2,\infty}(\mathbb{R}_+;\mathbb{R}), \\
\phi_1 + \phi_2 &\mapsto \|B^\flat e^{\cdot A^\flat}\phi_1\|_V + \|B^\flat e^{\cdot A^\flat}\phi_2\|_V
\end{aligned}$$

is well defined with some bounds

$$\|T\phi_1\|_{L^{1/\alpha_1,\infty}} \leq M\|\phi_1\|_{Z_1} \quad \text{and} \quad \|T\phi_2\|_{L^{1/\alpha_2,\infty}} \leq M\|\phi_2\|_{Z_2}.$$

Here, $M > 0$ only depends on N and α_i. By the interpolation theorem of Marcinkiewicz, T maps from $(Z_1, Z_2)_{\theta,1} = Z$ to $(L^{1/\alpha_1,\infty}, L^{1/\alpha_2,\infty})_{\theta,1} = L^1$ together with the bound

$$\|T\psi\|_{L^1} \leq M\|\psi\|_Z.$$

Plugging this inequality into (3.6) yields

$$\begin{aligned}
&\Big| \int_{-\infty}^{t} \langle e^{(t-s)A} B f(s), \psi \rangle_{Y,Z} \, \mathrm{d}s \Big| \\
&\leq \|f\|_{L^\infty(\mathbb{R},X)} \int_0^\infty \|B^\flat e^{rA^\flat} \psi\|_V \, \mathrm{d}r \\
&\leq \|f\|_{L^\infty(\mathbb{R},X)} M \|\psi\|_Z.
\end{aligned}$$

Taking the supremum over all $\psi \in Z$ with norm one, this implies the assertion. □

As we will frequently use the theorem above, we want to give the solution operator therein its own name.

Definition 3.1.5. *Theorem 3.1.4 above yields a solution operator, that maps a right-hand side f to a mild solution u of* (3.1)*. This linear and continuous operator will be denoted by $S\colon L^\infty(\mathbb{R}; X) \to L^\infty(\mathbb{R}; Y)$.*

We want to highlight the following shift property of the operator S for any $t, s \in \mathbb{R}$:

$$\begin{aligned}
&S(f(\cdot))(t+s) \\
&= \int_{-\infty}^{t+s} e^{(t+s-r)A} B f(r) \, \mathrm{d}r \\
&= \int_{-\infty}^{t} e^{(t+s-(\tau+s))A} B f(\tau+s) \, \mathrm{d}\tau \\
&= \int_{-\infty}^{t} e^{(t-\tau)A} B f(\tau+s) \, \mathrm{d}\tau \\
&= S(f(\cdot + s))(t).
\end{aligned}$$

This means, if we shift the exterior force by some time s, the corresponding solutions gets shifted in the same way. Above, we have dropped the dual pairing of (3.3) for notational simplicity. We will occasionally do this in the future, if we do not make any computations using the dual pairing. The shift property comes in handy, when showing regularity results as well asymptotical properties of mild solutions. We start by showing the former one. We do consider two different conditions, which allow us to verify continuity and Hölder continuity of mild solutions. One of these conditions requires the exterior force to be uniformly continuous, the other one demands the semigroup to be bounded analytic.

Proposition 3.1.6. *If additionally to the requirements of Theorem 3.1.4 the function f lies in* $\mathrm{BUC}(\mathbb{R}; X)$*, then the solution u lies in* $\mathrm{BUC}(\mathbb{R}; Y)$.

Proof. Let $\epsilon > 0$. Then there is a $\delta > 0$ such that for each $t \in \mathbb{R}$ and $\tau \in \mathbb{R}$ with $|\tau| < \delta$ we have $\|f(t) - f(t+\delta)\|_X < \epsilon/M$. Furthermore, we know by the shift property of S that

$$u(t+\tau) = S(f(\cdot))(t+\tau) = S(f(\cdot+\tau))(t).$$

Together with the linearity of S, this implies

$$u(t) - u(t+\tau) = S(f(\cdot))(t) - S(f(\cdot+\tau))(t) = S(f(\cdot) - f(\cdot+\tau))(t).$$

Hence, we get

$$\|u(t) - u(t+\tau)\|_Y \leq M\|f(\cdot) - f(\cdot+\tau)\|_{L^\infty(\mathbb{R};X)} < \epsilon,$$

for all $t \in \mathbb{R}$ and $|\tau| < \delta$. Thus, u is uniformly continuous. □

The second regularity condition requires bounded analyticity of the semigroup generated by $A^\flat$, but no further assumption on f besides being bounded. It is a generalisation of [Yam00, Theorem 1.4].

Proposition 3.1.7. *Assume that the requirements of Theorem 3.1.4 are fulfilled. Let additionally $A^\flat$ be the generator of a bounded analytic semigroup in Z_1 and Z_2. Then there is a constant $C > 0$ independent of f such that the mild solution u of* (3.1) *admits*

$$\|u(t) - u(s)\|_{Y_+} \leq C\|f\|_{L^\infty(\mathbb{R};X)}|t-s|^{1-\alpha_+} \quad \text{for all } t, s \in \mathbb{R},\ t > s,$$

where $Y_+ = (Y_1, Y_2)_{\theta_+,1}$, $\theta_+ \in (\theta, 1)$ and $\alpha_+ = (1-\theta_+)\alpha_1 + \theta_+\alpha_2$.

Before proving this proposition, we would like to describe the decay of $B^\flat e^{tA^\flat} : (Z_1, Z_2)_{\eta,q} \to V$ in dependence of η. Due to real interpolation and Assumption 3.1.2 (ACPEx), we get

$$\|B^\flat e^{tA^\flat}\psi\|_V \leq Ct^{-\alpha(\eta)}\|\psi\|_{(Z_1,Z_2)_{\eta,q}},$$

where $\alpha(\eta) = (1-\eta)\alpha_1 + \eta\alpha_2$. Thus, $\alpha(\cdot)$ is an affine linear function with, $\alpha(0) = \alpha_1$, $\alpha(1) = \alpha_2$ and $\alpha(\theta) = 1$, where θ is as in Theorem 3.1.4. That means for $\eta \in (\theta, 1)$, it holds $\alpha(\eta) \in (0, 1)$.

Proof. We will show the Hölder continuity by a duality argument. Let $Z_+^0 := (Z_1, Z_2)^0_{\theta_+,\infty}$. Note that by Proposition 1.1.8 we have $(Z_+^0)' = Y_+$, and that the embedding $Z_+^0 \hookrightarrow Y_+'$ is an isometry. Let $\phi \in Z_1 \cap Z_2$ and $-\infty < s < t < \infty$. We have

$$
\begin{aligned}
&|\langle u(t) - u(s), \phi\rangle| \\
&= |\int_{-\infty}^{t} \langle e^{(t-r)A} Bf(r), \phi\rangle \,\mathrm{d}r - \int_{-\infty}^{s} \langle e^{(s-r)A} Bf(r), \phi\rangle \,\mathrm{d}r| \\
&\leq |\int_{-\infty}^{s} \langle (e^{(t-r)A} - e^{(s-r)A}) Bf(r), \phi\rangle \,\mathrm{d}r| + |\int_{s}^{t} \langle e^{(t-r)A} Bf(r), \phi\rangle \,\mathrm{d}r| \\
&=: I_1 + I_2.
\end{aligned}
$$

We estimate the last two summands separately. Due to the estimate (3.4), we get by real interpolation

$$
\|B^\flat e^{tA^\flat} \psi\|_V \leq C t^{-\alpha_+} \|\psi\|_{Z_+^0}.
$$

Note that $\alpha_+ \in (0, 1)$. Therefore we have

$$
\begin{aligned}
I_2 &\leq \int_s^t |\langle f(r), B^\flat e^{(t-r)A^\flat} \phi\rangle| \,\mathrm{d}r \\
&= \int_0^{t-s} |\langle f(r+s), B^\flat e^{(t-s-r)A^\flat} \phi\rangle| \,\mathrm{d}r \\
&\leq \|f\|_{L^\infty(\mathbb{R};X)} \int_0^{t-s} \|B^\flat e^{\tau A^\flat} \phi\|_V \,\mathrm{d}\tau \\
&\leq C \|f\|_{L^\infty(\mathbb{R};X)} \int_0^{t-s} \tau^{-\alpha_+} \|\phi\|_{Z_+^0} \,\mathrm{d}\tau \\
&\leq C \|f\|_{L^\infty(\mathbb{R};X)} |t-s|^{1-\alpha_+} \|\phi\|_{Z_+^0}.
\end{aligned}
$$

Regarding I_1, we get

$$
\begin{aligned}
I_1 &\leq \|f\|_{L^\infty(\mathbb{R};X)} \int_{-\infty}^{s} \|B^\flat (e^{(t-r)A^\flat} - e^{(s-r)A^\flat}) \phi\|_V \,\mathrm{d}r \\
&= \|f\|_{L^\infty(\mathbb{R};X)} \int_0^\infty \|B^\flat (e^{(t-s+\tau)A^\flat} - e^{\tau A^\flat}) \phi\|_V \,\mathrm{d}\tau \\
&\leq \|f\|_{L^\infty(\mathbb{R};X)} \Big[\int_0^{t-s} \|B^\flat (e^{(t-s+\tau)A^\flat} - e^{\tau A^\flat}) \phi\|_V \,\mathrm{d}\tau \\
&\qquad + \int_{t-s}^{\infty} \|B^\flat (e^{(t-s+\tau)A^\flat} - e^{\tau A^\flat}) \phi\|_V \,\mathrm{d}\tau \Big] \\
&=: \|f\|_{L^\infty(\mathbb{R};X)} \left[I_3 + I_4 \right].
\end{aligned}
$$

Using the boundedness of $(e^{tA^\flat})_{t\geq 0}$, we can estimate I_3 in the same way as I_2:

$$\begin{aligned}
I_3 &= \int_0^{t-s} \|B^\flat e^{\tau A^\flat}(e^{(t-s)A^\flat} - \mathrm{id})\phi\|_V \,\mathrm{d}\tau \\
&\leq \int_0^{t-s} \tau^{-\alpha_+} \|(e^{(t-s)A^\flat} - \mathrm{id})\phi\|_{Z_+^0} \,\mathrm{d}\tau \\
&\leq C|t-s|^{1-\alpha_+}\|\phi\|_{Z_+^0}.
\end{aligned}$$

In order to estimate I_4, we will make use of the analyticity of $(e^{tA^\flat})_{t\geq 0}$:

$$\begin{aligned}
I_4 &= \int_{t-s}^{\infty} \|B^\flat e^{(\tau/2)A^\flat}(e^{(t-s)A^\flat} - \mathrm{id})e^{(\tau/2)A^\flat}\phi\|_V \,\mathrm{d}\tau \\
&\leq C\int_{t-s}^{\infty} \tau^{-\alpha_+}\|(e^{(t-s)A^\flat} - \mathrm{id})\mathrm{e}^{(\tau/2)A^\flat}\phi\|_{Z_+^0} \,\mathrm{d}\tau \\
&\leq C\int_{t-s}^{\infty} \tau^{-\alpha_+}\Big\| \int_0^{t-s} A^\flat e^{(\sigma+(\tau/2))A^\flat}\phi \;\mathrm{d}\sigma\Big\|_{Z_+^0} \mathrm{d}\tau \\
&\leq C\int_{t-s}^{\infty} \tau^{-\alpha_+}\int_0^{t-s} \frac{1}{\sigma+\tau/2}\|\phi\|_{Z_+^0} \,\mathrm{d}\sigma \,\mathrm{d}\tau \\
&\leq C\int_{t-s}^{\infty} \tau^{-\alpha_+}\int_0^{t-s} \frac{2}{\tau}\|\phi\|_{Z_+^0} \,\mathrm{d}\sigma \,\mathrm{d}\tau \\
&\leq C|t-s|\int_{t-s}^{\infty} \tau^{-\alpha_+-1}\|\phi\|_{Z_+^0} \,\mathrm{d}\tau \\
&\leq C|t-s|^{1-\alpha_+}\|\phi\|_{Z_+^0}.
\end{aligned}$$

Combining the estimates for $I_1, \ldots, I_4$ yields

$$|\langle u(t) - u(s), \phi\rangle| \leq C\|f\|_{L^\infty(\mathbb{R};X)}|t-s|^{1-\alpha_+}\|\phi\|_{Z_+^0}.$$

By the density of $Z_1 \cap Z_2 \in Z_+^0$, this inequality can be extended to all $\phi \in Z_+^0$. Taking the supremum over all $\phi \in Z_+^0$ with $\|\phi\|_{Z_+^0} = 1$ now yields the desired result. □

We will now show that certain asymptotical properties of the exterior force are conveyed to the mild solution.

Theorem 3.1.8. *Assume that the requirements of Theorem 3.1.4 are valid and let $u \in L^\infty(\mathbb{R}; Y)$ be the mild solution to (3.1) given by that theorem.*

1. *If $f \in C_0(\mathbb{R}; X)$, then $u \in C_0(\mathbb{R}; Y)$.*
2. *Let $T > 0$. If $f \in P_T(\mathbb{R}; X)$, then $u \in P_T(\mathbb{R}; Y)$.*

3. *If* $f \in \mathrm{UAP}(\mathbb{R}; X)$*, then* $u \in \mathrm{UAP}(\mathbb{R}; Y)$*.*

4. *If* $f \in \mathrm{AAP}(\mathbb{R}; X)$*, then* $u \in \mathrm{AAP}(\mathbb{R}; Y)$*.*

Proof. For the first statement, it is enough to consider exterior forces in the space $C_c(\mathbb{R}; X)$. The rest follows then by density and the continuity of the solution operator S. Hence, let $f \in C_c(\mathbb{R}; X)$ and let $a, b \in \mathbb{R}$ be such that supp f $\subset (a, b)$. For $t < a$ one directly gets

$$u(t) = \int_{-\infty}^{t} e^{(t-s)A} B f(s) \, \mathrm{d}s = 0.$$

Now, let $t > b$. By (3.4) and real interpolation, we have $\|e^{rA} B\psi\|_Y \leq C r^{-1} \|\psi\|_X$ for all $r > 0$ and all $\psi \in X$. It follows that

$$\begin{aligned} \|u(t)\|_Y &\leq \int_{-\infty}^{t} \|e^{(t-s)A} B f(s)\|_Y \, \mathrm{d}s \\ &\leq \int_{a}^{b} \|e^{(t-s)A} B f(s)\|_Y \, \mathrm{d}s \\ &\leq C \int_{a}^{b} (t-b)^{-1} \|f(s)\|_X \, \mathrm{d}s \\ &\leq C(b-a) \|f\|_{L^\infty(\mathbb{R}; X)} (t-b)^{-1}. \end{aligned}$$

Hence, $\|u(t)\|_Y \to 0$ for $t \to \infty$. The continuity of u follows by the embedding $C_0(\mathbb{R}; X) \hookrightarrow \mathrm{BUC}(\mathbb{R}; X)$ and Proposition 3.1.6. That means $u \in C_0(\mathbb{R}; Y)$.

The second assertion can be seen by the shift property of the solution operator S, namely

$$u(t+T) = S(f(\cdot))(t+T) = S(f(\cdot + T))(t) = S(f(\cdot))(t) = u(t).$$

Regarding the third statement, let $(t_n)_{n \in \mathbb{N}}$ be a sequence with values in $\mathbb{R}$. We have to show, that $(u(\cdot + t_n))_{n \in \mathbb{N}}$ has a convergent subsequence in $L^\infty(\mathbb{R}; Y)$. By the almost periodicity of f, there is a subsequence $(t_{n_k})_{k \in \mathbb{N}}$ of $(t_n)_{n \in \mathbb{N}}$ such that $(f(\cdot + t_{n_k}))_{k \in \mathbb{N}}$ is convergent in $L^\infty(\mathbb{R}; X)$ and therefore a Cauchy sequence therein. Furthermore, due to the linearity and boundedness of S, we get

$$\|u(\cdot + u_{n_k}) - u(\cdot + u_{n_l})\|_{L^\infty(\mathbb{R}; Y)} \leq M \|f(\cdot + u_{n_k}) - f(\cdot + u_{n_l})\|_{L^\infty(\mathbb{R}; X)}$$

for any $k, l \in \mathbb{N}$. Hence, $(u(\cdot + t_{n_k}))_{k \in \mathbb{N}}$ is a Cauchy sequence in $L^\infty(\mathbb{R}; Y)$ and therefore convergent therein.

The last claim follows by the first and third assertion and the direct decomposition $\mathrm{AAP}(\mathbb{R}; X) = \mathrm{UAP}(\mathbb{R}; X) \oplus C_0(\mathbb{R}; X)$. $\square$

Semilinear Autonomous Equations

Having developed the linear theory, we can now construct mild solutions to semilinear equations by a standard fixed point argument.

Generally, we will now consider semilinear equations of the form

$$u'(t) - Au(t) = G(u)(t), \quad t \in \mathbb{R}. \tag{3.7}$$

Mild solutions to these equations will be defined consistently to the linear case:

Definition 3.1.9. *Let Z be a Banach space and $Y := Z'$. A function $u \in L^\infty(\mathbb{R}; Y)$ is called a mild solution to* (3.7) *if for each $\psi \in Z$*

$$\langle u(t), \psi \rangle_{Y,Z} = \int_{-\infty}^{t} \langle e^{(t-s)A} BG(u)(s), \psi \rangle_{Y,Z} \, \mathrm{d}s.$$

Verifying the existence of mild solutions to these semilinear equations will be a rather short application of the fixed point theorem of Banach. We assume that the non-linear term G of the semilinear abstract Cauchy problem satisfies the following Lipschitz condition:

Assumptions 3.1.10 (sACPEx). *Let G map from $L^\infty(\mathbb{R}; Y)$ to $L^\infty(\mathbb{R}; X)$. Suppose that there are constants $L, R > 0$ and a function $u_0 \in L^\infty(\mathbb{R}; Y)$ such that*

$$\|G(v_1) - G(v_2)\|_{L^\infty(\mathbb{R};X)} \leq L \|v_1 - v_2\|_{L^\infty(\mathbb{R};Y)} \tag{3.8}$$

for all $v_1, v_2 \in \overline{B(u_0, R)} \subset L^\infty(\mathbb{R}, Y)$ and

$$ML < 1, \quad R(1 - LM) \geq M \|G(u_0)\|_{L^\infty(\mathbb{R};X)} + \|u_0\|_{L^\infty(\mathbb{R};Y)}. \tag{3.9}$$

Here, M is the same constant as in Theorem 3.1.4.

Under these requirements, we get the following result.

Theorem 3.1.11. *Suppose that the Assumptions 3.1.2 (ACPEx) and Assumptions 3.1.10 (sACPEx) are valid and let X and Y be as in Notation 3.1.3.*

- *Then there is a unique mild solution $u \in \overline{B(u_0, R)} \subset L^\infty(\mathbb{R}; Y)$ to* (3.7).
- *If $G\colon \mathrm{BUC}(\mathbb{R}; Y) \to \mathrm{BUC}(\mathbb{R}; X)$, then $u \in \mathrm{BUC}(\mathbb{R}; Y)$.*
- *If $G\colon C_0(\mathbb{R}; Y) \to C_0(\mathbb{R}; X)$, then $u \in C_0(\mathbb{R}; Y)$.*
- *If $G\colon P_T(\mathbb{R}; Y) \to P_T(\mathbb{R}; X)$ for some $T > 0$, then $u \in P_T(\mathbb{R}; Y)$.*

- *If* $G\colon \operatorname{UAP}(\mathbb{R};Y) \to \operatorname{UAP}(\mathbb{R};X)$*, then* $u \in \operatorname{UAP}(\mathbb{R};Y)$.
- *If* $G\colon \operatorname{AAP}(\mathbb{R};Y) \to \operatorname{AAP}(\mathbb{R};X)$*, then* $u \in \operatorname{AAP}(\mathbb{R};Y)$.

Proof. Consider the auxiliary problem

$$u'(t) - Au(t) = G(v)(t), \quad t \in \mathbb{R}. \tag{3.10}$$

Due to the mapping properties of G and Theorem 3.1.4, there is for each $v \in L^\infty(\mathbb{R};Y)$ a unique mild solution u to (3.10). We denote the solution operator by $\Phi(v) := u$. It it easy to see, that some $u \in L^\infty(\mathbb{R};Y)$ is a mild solution to (3.7) if and only if it is a fixed point of Φ. We will show that Φ admits a fixed point under the given assumptions by the Banach fixed point theorem. First we check, that Φ maps $\overline{B(u_0,R)} \subset L^\infty(\mathbb{R};Y)$ into itself. Given some v in that ball, we get

$$\begin{aligned}
\|u_0 - \Phi(v)\|_{L^\infty(\mathbb{R};Y)} &\le \|u_0 - \Phi(u_0) + \Phi(u_0) - \Phi(v)\|_{L^\infty(\mathbb{R};Y)} \\
&\le \|u_0\|_{L^\infty(\mathbb{R};Y)} + M\|G(u_0)\|_{L^\infty(\mathbb{R};X)} + \|\Phi(u_0) - \Phi(v)\|_{L^\infty(\mathbb{R};Y)} \\
&\le R(1-LM) + LMR \\
&= R,
\end{aligned}$$

where we have used the assumptions on L and R. Regarding the contraction property, we get due to the Lipschitz condition on G for any $v_1, v_2 \in \overline{B(u_0,R)} \subset L^\infty(\mathbb{R};Y)$ that

$$\|\Phi(v_1) - \Phi(v_2)\|_{L^\infty(\mathbb{R};Y)} \le M\|G(v_1) - G(v_2)\|_{L^\infty(\mathbb{R};X)} \le LM\|v_1 - v_2\|_{L^\infty(\mathbb{R};Y)}.$$

As $LM < 1$, this means that Φ is a contraction map and the first part of the theorem is shown.

It remains to show the asymptotic properties of u under the extra assumptions on f. Due to Proposition 3.1.6, we can exchange $L^\infty(\mathbb{R};X)$ and $L^\infty(\mathbb{R};Y)$ by $\operatorname{BUC}(\mathbb{R};X)$ and $\operatorname{BUC}(\mathbb{R};Y)$ in the proof above. Thus, we get a fixed point $u \in \operatorname{BUC}(\mathbb{R};Y)$. The other cases work the same way by employing Theorem 3.1.8. □

3.1.2 Non-autonomous Case

We change over to the case of non-autonomous Cauchy problems. This forces us to leave behind the use of semigroups for more general evolution families. The lack of the semigroup property will cause us to leave out the regularity results of the autonomous case and to restrict ourselves to the study of time periodic solutions. Apart from that, the methods and arguments remain almost the same with evolution families in place of semigroups.

As before, we start by setting up our notion of mild solution and stating our general assumptions. Note that the autonomous case will be contained therein. Throughout this section, we will consider the problem

$$u'(t) - A(t)u(t) = Bf(t), \quad t \in \mathbb{R}. \tag{3.11}$$

Mild solutions to this equation will be defined as follows:

Definition 3.1.12. *Let Z be a Banach space and $Y := Z'$. A function $u \in L^\infty(\mathbb{R}; Y)$ is called a mild solution to* (3.11) *if for each $\psi \in Z$*

$$\langle u(t), \psi \rangle = \int_{-\infty}^{t} \langle U(t,s)Bf(r), \psi \rangle_{Y,Z} \, \mathrm{d}r, \tag{3.12}$$

where $(U(t,s))_{t \geq s}$ is an evolution family associated to the homogeneous equation

$$u'(t) = A(t)u(t), \quad t \in \mathbb{R}.$$

The counterpart to Assumptions 3.1.2 (ACPEx) reads as follows:

Assumptions 3.1.13 (nACPEx). *Let Z_1, Z_2 be an interpolation couple of Banach spaces. Assume that $Z_1 \cap Z_2$ is dense in Z_1 and Z_2. Suppose that the homogeneous equation*

$$u'(t) = A^\flat(t)u(t), \quad t \in \mathbb{R}$$

can be associated in Z_1 and Z_2 to a consistent evolution family $(U^\flat(t,s))_{t \geq s}$. Let $B^\flat$ be a linear operator defined on the range of $U^\flat(t,s)$ for any $-\infty < s \leq t < \infty$ with values in some normed vector space V. Assume that there are constants $N > 0$ and $\alpha_1 > 1 > \alpha_2 \geq 0$, which are independent of t and s such that

$$\|B^\flat U^\flat(t,s)\psi\|_V \leq N(t-s)^{-\alpha_i}\|\psi\|_{Z_i}, \quad t > s, \quad \psi \in Z_i, i \in \{1,2\}. \tag{3.13}$$

The rest of the notation will be the same as in the preceding section. We repeat them for convenience:

Notation 3.1.14. *Let $\theta \in (0,1)$ satisfy $1 = (1-\theta)\alpha_1 + \theta\alpha_2$. Define $Y_1 := Z_1'$, $Y_2 := Z_2'$, $Y := (Y_1, Y_2)_{\theta,\infty}$ and $X := V'$.*

The existence of mild solutions to (3.11) works in the same way as in Theorem 3.1.4.

Theorem 3.1.15. *Suppose that the Assumptions 3.1.13 (nACPEx) are satisfied. Let X and Y be as in Notation 3.1.14. Then problem* (3.11) *has a mild solution $u \in L^\infty(\mathbb{R}; Y)$ for any right-hand side $f \in L^\infty(\mathbb{R}; X)$. Furthermore, there is a constant $M > 0$ such that*

$$\|u(t)\|_Y \leq M\|f\|_{L^\infty(\mathbb{R};Y)}, \quad t \in \mathbb{R}. \tag{3.14}$$

Proof. We have to show that the integral in equation (3.12) is bounded for the pair Y and $Z := (Z_1, Z_2)_{\theta,1}$. By the density of $Z_1 \cap Z_2$ in Z_1 and Z_2, this is indeed a dual pairing due to the duality theorem of real interpolation 1.1.8. For $\psi \in Z$ we have

$$\begin{aligned}
&\left|\int_{-\infty}^{t} \langle U(t,s)Bf(s), \psi\rangle_{Y,Z}\,\mathrm{d}s\right| \\
&\leq \int_{-\infty}^{t} |\langle f(s), B^\flat U^\flat(t,s)\psi\rangle_{X,V}|\,\mathrm{d}s \\
&\leq \int_{-\infty}^{t} \|f(s)\|_X \|B^\flat U^\flat(t,s)\psi\|_V\,\mathrm{d}s \qquad (3.15)\\
&\leq \|f\|_{L^\infty(\mathbb{R},X)} \int_{-\infty}^{t} \|B^\flat U^\flat(t,s)\psi\|_V\,\mathrm{d}s \\
&= \|f\|_{L^\infty(\mathbb{R},X)} \int_{0}^{\infty} \|B^\flat U^\flat(t,t-r)\psi\|_V\,\mathrm{d}r.
\end{aligned}$$

It follows from (3.13), that the sublinear operator

$$\begin{aligned}
T_t\colon Z_1 + Z_2 &\to L^{1/\alpha_1,\infty}(\mathbb{R}_+;\mathbb{R}) + L^{1/\alpha_2,\infty}(\mathbb{R}_+;\mathbb{R}),\\
\phi_1 + \phi_2 &\mapsto \|B^\flat U(t,t-\cdot)^\flat\phi_1\|_V + \|B^\flat U(t,t-\cdot)^\flat\phi_2\|_V
\end{aligned}$$

is well defined and there is some constant $M > 0$ such that

$$\|T_t\phi_1\|_{L^{1/\alpha_1,\infty}} \leq M\|\phi_1\|_{Z_1} \quad \text{and} \quad \|T\phi_2\|_{L^{1/\alpha_2,\infty}} \leq M\|\phi_2\|_{Z_2}.$$

This constant $M > 0$ only depends on N and α_i but not on t. By real interpolation, T_t maps from $(Z_1, Z_2)_{\theta,1} = Z$ to $(L^{1/\alpha_1,\infty}, L^{1/\alpha_2,\infty})_{\theta,1} = L^1$ with the estimate

$$\|T_t\psi\|_{L^1} \leq M\|\psi\|_Z.$$

Plugging this inequality into (3.15) yields the theorem. □

In the autonomous case, the solution operator S for the mild solution has the convenient shift property $S(f(\cdot + s))(t) = S(f(\cdot))(t+s)$. However, in the non-autonomous case, this is generally not true any more. Indeed, this can be already seen in the context of ordinary differential equations. Suppose that $A(\cdot) \in \mathrm{BUC}^\infty(\mathbb{R};\mathbb{R})$, $1 \leq A(\cdot) \leq 2$, $A(t) = 1$ for $t \in (-\infty, -1)$ and $A(t) = 2$ for $t \in (1, \infty)$. We can consider $A(\cdot)$ as a family of bounded operators on $\mathbb{R}$. Then the associated evolution family $(U(t,s))_{t\geq s}$ is given by

$$U(t,s) = \exp\left[\int_s^t A(r)\,\mathrm{d}r\right].$$

Given the right-hand side $f(t) = 1$ and some $s \in \mathbb{R}$, it is now easy to see, that

$$\begin{aligned} \int_{-\infty}^{t} U(t,r) f(r+s)\,\mathrm{d}r &= \int_{-\infty}^{t} U(t,r)\,\mathrm{d}r \\ &\neq \int_{-\infty}^{t+s} U(t+s,r)\,\mathrm{d}r = \int_{-\infty}^{t+s} U(t+s,r) f(r)\,\mathrm{d}r \end{aligned}$$

for $t < -1$ and $t + s > 1$. As our proofs before heavily relied on this general shift property, we are only able to transfer the considerations of time periodic solutions.

Theorem 3.1.16. *Assume that the requirements of Theorem 3.1.15 are satisfied. Let $u \in L^\infty(\mathbb{R}; Y)$ be the mild solution to* (3.11) *given by Theorem 3.1.15 and let $T > 0$. If $f(\cdot)$ and $A(\cdot)$ are T-periodic, then u is T-periodic, too.*

Proof. We note that $A(t+T) = A(t)$ for any $t \in \mathbb{R}$ implies $U(t+T, s+T) = U(t,s)$ for any $-\infty < s \leq t < \infty$. Therefore, we get

$$\begin{aligned} u(t) = \int_{-\infty}^{t} U(t,s) B f(s)\,\mathrm{d}s &= \int_{-\infty}^{t+T} U(t, r-T) B f(r-T)\,\mathrm{d}r \\ &= \int_{-\infty}^{t+T} U(t+T, r) B f(r)\,\mathrm{d}r = u(t+T), \end{aligned}$$

which is the desired result. $\square$

Non-autonomous Semilinear Equations

We now change over to general semilinear equations of the form

$$u'(t) - A(t)u(t) = G(u)(t), \quad t \in \mathbb{R}. \tag{3.16}$$

Again, the definition of mild solutions does not deviate from the other ones:

Definition 3.1.17. *Let Z be a Banach space and $Y := Z'$. A function $u \in L^\infty(\mathbb{R}; Y)$ is called a mild solution to* (3.16) *if for each $\psi \in Z$*

$$\langle u(t), \psi \rangle = \int_{-\infty}^{t} \langle U(t,s) B G(u)(r), \psi \rangle_{Y,Z}\,\mathrm{d}r, \tag{3.17}$$

where $(U(t,s))_{t \geq s}$ is an evolution family associated to the homogeneous equation

$$u'(t) = A(t)u(t), \quad t \in \mathbb{R}.$$

The assumptions on the semilinear term are the same as in the autonomous case, but containing a reference to Theorem 3.1.15 instead of Theorem 3.1.4:

Assumptions 3.1.18 (snACPEx). *Let G map from $L^\infty(\mathbb{R};Y)$ to $L^\infty(\mathbb{R};X)$. Suppose that there are constants $L, R > 0$ and a function $u_c \in L^\infty(\mathbb{R};Y)$ such that*

$$\|G(v_1) - G(v_2)\|_{L^\infty(\mathbb{R};X)} \leq L\|v_1 - v_2\|_{L^\infty(\mathbb{R};Y)}$$

for all $v_1, v_2 \in \overline{B(u_c, R)} \subset L^\infty(\mathbb{R}, Y)$ and

$$ML < 1, \quad R(1 - LM) \geq M\|G(u_c)\|_{L^\infty(\mathbb{R};X)} + \|u_0\|_{L^\infty(\mathbb{R};Y)}.$$

Here, M is the same constant as in Theorem 3.1.15.

Copying the proof of the autonomous case, using Theorem 3.1.15 and Theorem 3.1.16 instead of Theorem 3.1.4 and Theorem 3.1.8, we get the following result on existence of mild solutions to (3.16).

Theorem 3.1.19. *Suppose that Assumptions 3.1.13 (nACPEx) and Assumptions 3.1.18 (snACPEx) are satisfied, where X and Y are as in Notation 3.1.14. Then there is a unique mild solution $u \in L^\infty(\mathbb{R};Y)$ in $\overline{B(u_0, R)}$ to* (3.7). *If f lies additionally in $P_T(\mathbb{R}, X)$ for some $T > 0$, then $u \in P_T(\mathbb{R}, Y)$.*

3.1.3 Stability

In this section, we will turn our attention to the classical situation of initial value problems in the same setting as before. Our main concern is to describe the asymptotic behaviour of the difference of two solutions to the same semilinear Cauchy problem with different initial values. Therefore, we will at first show counterparts to Theorem 3.1.15 and Theorem 3.1.19 for the corresponding initial value problems.

As usual, we will start with the linear equation

$$\begin{aligned} u'(t) - A(t)u(t) &= Bf(t), \quad t > 0, \\ u(0) &= a. \end{aligned} \tag{3.18}$$

Mild solutions to these equations are defined analogously to the case of equations on the whole real axis. Only the term for the initial value is new.

Definition 3.1.20. *Let Z be a Banach space and $Y := Z'$. A function $u \in L^\infty(\mathbb{R}_+;Y)$ is called a mild solution of the initial value problem* (3.18) *if for each $\phi \in Z$ and $t > 0$ it fulfils*

$$\langle u(t), \phi\rangle_{Y,Z} = \langle U(t,0)u_0, \phi\rangle_{Y,Z} + \int_0^t \langle U(t,s)Bf(s), \phi\rangle_{Y,Z}\, \mathrm{d}s.$$

Here $(U(t,s))_{t\geq s}$ *is the evolution family associated to the homogeneous problems* $u'(t) = A(t)u(t)$.

Establishing the existence of mild solutions for the initial value problem does require almost the same assumptions as for the corresponding problem on the whole real axis. We only have to add the boundedness of the relevant evolution family, which is now necessary due to the appearance of the initial value term.

Assumptions 3.1.21 (IVPEx). *Let* Z_1, Z_2 *be an interpolation couple of Banach spaces. Assume that* $Z_1 \cap Z_2$ *is dense in* Z_1 *and* Z_2. *Suppose that the homogeneous equation*

$$u'(t) = A^\flat(t)u(t), \quad t \in \mathbb{R} \tag{3.19}$$

can be associated in Z_1 *and* Z_2 *to a consistent and bounded evolution family* $(U^\flat(t,s))_{t\geq s}$. *Let* $B^\flat$ *be a linear operator defined on the range of* $U^\flat(t,s)$ *for any* $-\infty < s \leq t < \infty$ *with values in some normed vector space* V. *Assume that there are constants* $N > 0$ *and* $\alpha_1 > 1 > \alpha_2 \geq 0$, *which are independent of* t *and* s, *such that*

$$\|B^\flat U^\flat(t,s)\psi\|_V \leq N(t-s)^{-\alpha_i}\|\psi\|_{Z_i}, \quad t > s, \quad \psi \in Z_i, i \in \{1,2\}. \tag{3.20}$$

As in the section before, we fix some notation:

Notation 3.1.22. *Let* $\theta \in (0,1)$ *fulfil* $1 = (1-\theta)\alpha_1 + \theta\alpha_2$. *Set* $X := V'$ *and* $Y := (Z_1', V')_{\theta,\infty}$.

The counterpart to Theorem 3.1.15 reads now as follows:

Theorem 3.1.23. *Let the Assumptions 3.1.21 (IVPEx) be valid and let* X *and* Y *be as in Notation 3.1.22. Then for each* $f \in L^\infty(\mathbb{R}_+;X)$ *and* $a \in Y$, *there is a mild solution* $u \in L^\infty(\mathbb{R}_+;Y)$ *of* (3.18). *Furthermore, there are constants* $M_0, M > 0$ *independent of* f *and* a *such that*

$$\|u\|_{L^\infty(\mathbb{R}_+;Y)} \leq M_0\|a\|_Y + M\|f\|_{L^\infty(\mathbb{R}_+;X)}. \tag{3.21}$$

Proof. Just as in the proof of Theorem 3.1.15, we show the existence by a duality argument. Let $Z := (Z_1, V)_{\theta,1}$. Note that $Z' = Y$ due to Proposition 1.1.8. For any $\phi \in Z$ we have

$$|\langle u(t), \phi\rangle_{Y,Z}| \leq |\langle U(t,0)a, \phi\rangle_{Y,Z}| + \int_0^t |\langle U(t,s)Bf(s), \phi\rangle_{Y,Z}|\,\mathrm{d}s.$$

The integral term can be estimated along the same lines as in the proof of Theorem 3.1.15. The boundedness of the initial value term follows by the boundedness of the evolution family $(U^\flat(t,s))_{t\geq s}$ in Z and duality. $\square$

As before, the case of semilinear equations can be treated by a fixed point argument under some Lipschitz condition on the non-linearity. We consider equations of the kind

$$\begin{aligned} u'(t) - A(t)u(t) &= G(u)(t), \quad t \in \mathbb{R}, \\ u(0) &= a. \end{aligned} \tag{3.22}$$

Just as in the case of the problem on the whole real axis, we say that a function $u \in L^\infty(\mathbb{R}; Y)$ is a mild solution to (3.22), if it satisfies

$$\langle u(t), \phi \rangle_{Y,Z} = \langle U(t,0)a, \phi \rangle_{Y,Z} + \int_0^t \langle U(t,s)BG(u)(s), \phi \rangle_{Y,Z} \, \mathrm{d}s$$

for all $\phi \in Z$. The non-linearity G is expected to fulfil the following conditions:

Assumptions 3.1.24 (sIVPEx). *Let G map from $L^\infty(\mathbb{R}; Y)$ to $L^\infty(\mathbb{R}; X)$. Suppose that there are constants $L, R, R_0 > 0$ and a function $u_c \in L^\infty(\mathbb{R}; Y)$ such that*

$$\|G(v_1) - G(v_2)\|_{L^\infty(\mathbb{R};X)} \leq L \|v_1 - v_2\|_{L^\infty(\mathbb{R};Y)} \tag{3.23}$$

for all $v_1, v_2 \in \overline{B(u_c, R)} \subset L^\infty(\mathbb{R}, Y)$ and

$$ML < 1, \quad R(1 - LM) \geq M_0 R_0 + M \|G(u_c)\|_{L^\infty(\mathbb{R};X)} + \|u_c\|_{L^\infty(\mathbb{R};Y)}.$$

Here, M and M_0 are the same constants as in Theorem 3.1.23.

The proof of the existence of mild solutions differs from Theorem 3.1.19 only by the point, that we have to mind the initial value term.

Theorem 3.1.25. *Let Assumptions 3.1.21 (IVPEx) as well as Assumptions 3.1.24 (sIVPEx) be valid and let X and Y be as in Notation 3.1.22. Suppose that $a \in Y$ satisfies $\|a\|_Y \leq R_0$. Then there is a mild solution $u \in \overline{B(u_c, R)} \subset L^\infty(\mathbb{R}; Y)$ of (3.22), which is unique in that ball.*

Proof. We will employ a fixed point argument. Consider the auxiliary problem

$$\begin{aligned} u'(t) - A(t)u(t) &= G(v)(t), \quad t > 0, \\ u(0) &= a, \end{aligned} \tag{3.24}$$

where $v \in L^\infty(\mathbb{R}_+; Y)$. By our assumptions, the right-hand side $G(v)$ lies in $L^\infty(\mathbb{R}_+; Y)$ for any $v \in L^\infty(\mathbb{R}_+; Y)$. Thus, by Theorem 3.1.23, there is a unique mild solution $u \in L^\infty(\mathbb{R}_+; Y)$ to (3.24). We denote the solution operator by $u = \Phi(v)$. Note that a function $u \in L^\infty(\mathbb{R}_+; Y)$ is a solution to (3.22) if and only if it is a fixed point of Φ. By employing the Banach fixed point theorem, we will show that Φ admits a fixed point in $\overline{B(u_c, R)} \subset L^\infty(\mathbb{R}; Y)$, which is the desired result. We start by showing that Φ leaves $\overline{B(u_c, R)}$ invariant. Let $v \in \overline{B(u_c, R)} \subset L^\infty(\mathbb{R}; Y)$ be arbitrary. Then it holds

$$\begin{aligned}
&\|\Phi(v) - u_c\|_{L^\infty(\mathbb{R}_+;Y)} \\
&\le \|\Phi(v) - \Phi(u_c)\|_{L^\infty(\mathbb{R}_+;Y)} + \|\Phi(u_c)\|_{L^\infty(\mathbb{R}_+;Y)} + \|u_c\|_{L^\infty(\mathbb{R}_+;Y)} \\
&\le M_0\|0\|_Y + M\|G(v) - G(u_c)\|_{L^\infty(\mathbb{R}_+;X)} \\
&\quad + M_0\|a\|_Y + M\|G(u_c)\|_{L^\infty(\mathbb{R}_+;X)} + \|u_c\|_{L^\infty(\mathbb{R}_+;Y)} \\
&\le ML\|v - u_c\|_{L^\infty(\mathbb{R}_+;Y)} + R(1 - LM) \\
&= R.
\end{aligned}$$

Furthermore, for arbitrary $v_1, v_2 \in \overline{B(u_c, R)} \subset L^\infty(\mathbb{R}_+; Y)$, we get

$$\begin{aligned}
\|\Phi(v_1) - \Phi(v_2)\|_{L^\infty(\mathbb{R}_+;Y)} &\le M_0\|0\|_Y + M\|G(v_1) - G(v_2)\|_{L^\infty(\mathbb{R}_+;Y)} \\
&\le ML\|v_1 - v_2\|_{L^\infty(\mathbb{R}_+;Y)}.
\end{aligned}$$

As $ML < 1$ by our assumptions, this yields that Φ is a contraction map on $\overline{B(u_c, R)}$. Hence, we can apply the Banach fixed point theorem, which completes the proof. □

Remark 3.1.26. *We would like to emphasize, that the proof above yields explicit approximating sequences $(u_n)_{n\in\mathbb{N}} \subset L^\infty(\mathbb{R}_+; Y)$ of mild solutions to the semilinear problem. A suitable one is given by*

$$\begin{aligned}
u_0 &= u_c, \\
u_{n+1} &= \Phi(u_n) \quad \textit{for} \quad n > 0.
\end{aligned}$$

Having the existence of mild solutions to the semilinear equation (3.22) at hand, we can now take a step towards the stability of mild solutions. The proof can be sketched as follows: When comparing two solutions u and v of (3.22) to two different initial values a and b, we have to investigate the solution to the equation

$$\begin{aligned}
w'(t) - A(t)w(t) &= G(u)(t) - G(v)(t), \quad t > 0, \\
w(0) &= a - b,
\end{aligned}$$

where $w = u - v$. Due to Remark 3.1.26, the solutions u and v can be constructed by some recursively defined sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$. Because of $u_0 = v_0 = u_c$, we get for the difference $w_1 := u_1 - v_1$, that it is a solution to

$$\begin{aligned}
w_1(t) - A(t)w_1(t) &= 0, \qquad t > 0, \\
w_1(0) &= a - b.
\end{aligned}$$

Obviously, the right-hand side above admits high regularity. We will consider it as an element of some vector space weighted in time and having values in some interpolation space $\tilde{X} \neq X$. By adding some decay assumptions to the evolution family $(U^\flat(t,s))_{t\geq s}$ and adjusting the arguments for the existence of mild solutions, we will see that the solution w_1 lies additionally in some weighted space with values in some real interpolation space $\tilde{Y} \neq Y$. These kind of estimates will turn out to survive the approximation procedure such that we get a weighted estimate for w with values in $\tilde{Y}$. This one can then be regarded as asymptotical stability of the solutions.

Following this sketch, we have to introduce two basic tools. One of them are time-weighted Lebesgue spaces, the other one are additional decay estimates for the evolution family corresponding to (3.19). Let $\gamma > 0$ and X be a Banach space. Define

$$L^\infty_\gamma(\mathbb{R}_+; X) := \{f\colon \mathbb{R}_+ \to X : t \mapsto t^\gamma f(t) \in L^\infty(\mathbb{R}; X)\}$$

equipped with the norm

$$\|f\|_{L^\infty_\gamma(\mathbb{R}_+;X)} = \|t \mapsto t^\gamma f(t)\|_{L^\infty(\mathbb{R};X)}.$$

Assumptions 3.1.27 (IVPStab). *Let $\tilde{Z}_1$, $\tilde{Z}_2$ be an interpolation couple of separable Banach spaces. Assume that $\tilde{Z}_1 \cap \tilde{Z}_2$ is dense in $\tilde{Z}_1$ and $\tilde{Z}_2$. Assume that $B^\flat U^\flat(t,s)$ has values in some normed vector space $\tilde{V}$ and that there are constants $N > 0$ and $\tilde{\alpha}_1 > 1 > \tilde{\alpha}_2 \geq 0$, which are independent of t and s such that*

$$\|B^\flat U^\flat(t,s)\psi\|_{\tilde{V}} \leq N(t-s)^{-\tilde{\alpha}_i}\|\psi\|_{\tilde{Z}_i}, \quad t > s, \quad \psi \in Z_i, i \in \{1,2\}. \tag{3.25}$$

Furthermore suppose that there is a constant $C > 0$ such that

$$\|U^\flat(t,s)\phi\|_Z \leq Ct^{-\gamma}\|\psi\|_{\tilde{Z}} \tag{3.26}$$

for some $\gamma > 0$.

For convenience, we also introduce some counterpart to Notation 3.1.22.

Notation 3.1.28. *Define $\tilde{\theta} \in (0,1)$ by $1 = (1-\tilde{\theta})\tilde{\alpha}_1 + \tilde{\theta}\tilde{\alpha}_2$ and set $\tilde{X} := \tilde{V}'$ and $\tilde{Y} := (\tilde{Z}_1', \tilde{Z}_2{}')_{\tilde{\theta},\infty}$.*

We can now start by considering the linear initial value problem with right-hand side having values in some weighted Lebesgue space. This will be crucial for the refined investigation of the fixed point procedure for the semilinear equations.

Lemma 3.1.29. *Let Assumptions 3.1.21 (IVPEx) as well as Assumptions 3.1.27 (IVPStab) be valid. Let $\tilde{X}$ and $\tilde{Y}$ be as in Notation 3.1.28. Suppose that $q \in L^\infty(\mathbb{R}_+; X) \cap L^\infty_\gamma(\mathbb{R}_+; \tilde{X})$ and $w_0 \in Y$. Furthermore, let w be the mild solution to*

$$\begin{aligned} w'(t) - A(t)w(t) &= Bq(t), \\ w(0) &= w_0 \end{aligned}$$

given by Theorem 3.1.23. Then $w \in L^\infty(\mathbb{R}_+; Y) \cap L^\infty_\gamma(\mathbb{R}_+; \tilde{Y})$. Additionally, there are constants $K_0, K > 0$ independent of q and w_0 such that

$$\|w\|_{L^\infty_\gamma(\mathbb{R}_+;\tilde{Y})} \leq K_0 \|w_0\|_Y + K \|q\|_{L^\infty_\gamma(\mathbb{R}_+;\tilde{X})}.$$

Proof. The property $w \in L^\infty(\mathbb{R}_+; Y)$ is included in Theorem 3.1.23. Hence, we only have to show $w \in L^\infty_\gamma(\mathbb{R}_+; \tilde{Y})$. By the definition of the mild solution, we have

$$\|w\|_{\tilde{Y}} \leq \|U(t,0)w_0\|_{\tilde{Y}} + \| \int_0^t U(t,s)Bq(s)\,\mathrm{d}s\|_{\tilde{Y}}.$$

The first summand on the right-hand side can be bounded with the dual estimate of (3.26) by

$$\|U(t,0)w_0\|_{\tilde{Y}} \leq Ct^{-\gamma}\|w_0\|_Y. \tag{3.27}$$

The second summand will be split up into

$$\begin{aligned} \| \int_0^t U(t,s)Bq(s)\,\mathrm{d}s\|_{\tilde{Y}} &\leq \| \int_0^{t/2} U(t,s)Bq(s)\,\mathrm{d}s\|_{\tilde{Y}} + \| \int_{t/2}^t U(t,s)Bq(s)\,\mathrm{d}s\|_{\tilde{Y}} \\ &=: I_1 + I_2. \end{aligned}$$

We will show both estimates by a duality argument. Let $\phi \in \tilde{Z} := (\tilde{Z}_1, \tilde{Z}_2)_{\tilde{\theta},1}$. Because of Proposition 1.1.8, it holds $\tilde{Z}' = \tilde{Y}$. Note that the estimates (3.20) imply $\|B^\flat U^\flat(t,s)\psi\|_V \leq C(t-s)^{-1}\|\psi\|_Z$ by interpolation. Together with (3.26), this yields

$$\begin{aligned} I_1 &\leq | \int_0^{t/2} |\langle U(t,s)Bq(s), \phi\rangle_{\tilde{Y},\tilde{Z}}|\,\mathrm{d}s \\ &\leq \int_0^{t/2} \|q(s)\|_{\tilde{X}} \|B^\flat U^\flat(t,s)\phi\|_{\tilde{V}}\,\mathrm{d}s \\ &\leq \|q\|_{L^\infty(\mathbb{R}_+;\tilde{X})} \int_0^{t/2} C(t-s)^{-\gamma-1}\|\phi\|_{\tilde{Z}}\,\mathrm{d}s \\ &\leq C\|q\|_{L^\infty(\mathbb{R}_+;\tilde{X})} t^{-\gamma}\|\phi\|_{\tilde{Z}}. \end{aligned}$$

Regarding I_2, we get

$$\begin{aligned} I_2 &\le \int_{t/2}^{t} |\langle U(t,s)Bq(s), \phi\rangle_{\tilde{Y},\tilde{Z}}| \,\mathrm{d}s \\ &\le \int_{t/2}^{t} \|q(s)\|_{\tilde{X}} \|B^\flat U^\flat(t,s)\phi\|_{\tilde{V}} \,\mathrm{d}s \\ &\le Ct^{-\gamma} \|q\|_{L^\infty_\gamma(\mathbb{R}_+;\tilde{X})} \int_{t/2}^{t} \|B^\flat U^\flat(t,s)\phi\|_{\tilde{V}} \,\mathrm{d}s \\ &\le Ct^{-\gamma} \|q\|_{L^\infty_\gamma(\mathbb{R}_+;\tilde{X})} \int_{-\infty}^{0} \|B^\flat U^\flat(t,t-r)\phi\|_{\tilde{V}} \,\mathrm{d}r. \end{aligned}$$

The integral on the right-hand side will now be estimated from above by the same techniques used in the proof of Theorem 3.1.15. Consider the sublinear operator

$$T_t : \phi \mapsto \|B^\flat U^\flat(t, t-\cdot)\phi\|_{\tilde{V}}.$$

Due to (3.25), it holds

$$T_t : \tilde{Z}_i \to L^{1/\tilde{\alpha}_i,\infty}(\mathbb{R}_+;\mathbb{R}), \quad i \in \{1,2\}$$

with some norm estimate

$$\|T_t\phi\|_{L^{1/\tilde{\alpha}_i,\infty}(\mathbb{R}_+;\mathbb{R})} \le C\|\phi\|_{\tilde{Z}_i},$$

where $C > 0$ is independent of ϕ and t. Using the interpolation theorem of Marcinkiewicz yields

$$T_t : \tilde{Z} = (\tilde{Z}_1, \tilde{Z}_2)_{\tilde{\theta},1} \to (L^{1/\tilde{\alpha}_1,\infty}, L^{1/\tilde{\alpha}_2,\infty})_{\tilde{\theta},1} = L^1$$

with the norm estimate

$$\|T_t\phi\|_{L^1} \le C\|\phi\|_{\tilde{Z}}.$$

Plugging this into the estimate of I_2 gives

$$I_2 \le Ct^{-\gamma}\|q\|_{L^\infty_\gamma(\mathbb{R}_+;\tilde{X})}\|\phi\|_{\tilde{Z}}.$$

Combining the estimates for I_1 and I_2 and taking the supremum over all $\phi \in \tilde{Z}$ with $\|\phi\|_{\tilde{Z}} = 1$ yields

$$\|w(t)\|_{\tilde{Y}} \le Ct^{-\gamma}\|q\|_{L^\infty_\gamma(\mathbb{R}_+;\tilde{X})},$$

which is the desired result. □

The proof of the stability of mild solutions does now rely on the combination of the last lemma with the approximation procedure as it is given in Remark 3.1.26 as well as additional Lipschitz estimates of the non-linearity.

Assumptions 3.1.30 (sIVPStab). *Let the function G be as in Assumptions 3.1.24 (sIVPEx) and let $\tilde{X}$ and $\tilde{Y}$ be as in Notation 3.1.28. Suppose that there is some $\tilde{L} > 0$ such that for any $v_1, v_2 \in \overline{B(u_c, R)} \subset L^\infty(\mathbb{R}_+; Y)$ with $v_1 - v_2 \in L^\infty(\mathbb{R}_+; \tilde{Y})$*

$$\|G(v_1) - G(v_2)\|_{L^\infty_\gamma(\mathbb{R}_+;\tilde{X})} \leq \tilde{L}\|v_1 - v_2\|_{L^\infty_\gamma(\mathbb{R};\tilde{Y})}. \tag{3.28}$$

This constant $\tilde{L}$ is demanded to satisfy $K\tilde{L} < 1$, where K is the constant appearing in 3.1.29.

We like to remark that the two conditions (3.23) and (3.28) on the nonlinearity are usually just two different instances of the same kind of inequality. For example in applications where the spaces $X, \tilde{X}, Y$ and $\tilde{Y}$ are Lorentz spaces, and the nonlinearity is some multiplication, they rely both on Hölder's inequality, but for different parameters.

Theorem 3.1.31. *Let Assumptions 3.1.21 (IVPEx)and 3.1.27 (IVPStab) be valid. Suppose that the non-linearity G fulfils Assumptions 3.1.24 (sIVPEx) and Assumptions 3.1.30 (sIVPStab). Let $a, b \in Y$ satisfy $\|a\|_Y$, $\|b\|_Y \leq R_0$. Then there are mild solutions $u, v \in L^\infty(\mathbb{R}_+; Y)$ to (3.22) with the respective initial values a and b and there is a constant $C > 0$ such that*

$$\|u(t) - v(t)\|_{\tilde{Y}} \leq Ct^{-\gamma}$$

for almost all $t > 0$.

Proof. The existence of the mild solutions u and v is due to Theorem 3.1.25. Due to remark 3.1.26, these solutions can be approximated by the sequences $(u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}} \subseteq \overline{B(u_c, R)} \subset L^\infty(\mathbb{R}; X)$, which are given by $u_0 = v_0 = u_c$ for $n = 0$ and as solutions to

$$\begin{aligned} u_n'(t) - A(t)u_n(t) &= G(u_{n-1})(t), \quad t > 0, \\ u_n(0) &= a, \end{aligned}$$

as well as

$$\begin{aligned} v_n'(t) - A(t)v_n(t) &= G(v_{n-1})(t), \quad t > 0, \\ v_n(0) &= b, \end{aligned}$$

for $n \geq 1$ respectively. Set $w_n := u_n - v_n$. Then we get that w_n is a mild solution to

$$\begin{aligned} w_n'(t) - A(t)w_n(t) &= G(u_{n-1})(t) - G(v_{n-1})(t), \\ w(0) &= a - b. \end{aligned}$$

For $n = 1$, the right-hand side of the first line above equals zero. By means of Lemma 3.1.29, this implies for $n = 1$, that there are constants $K_0, K > 0$ such that

$$\|w_1\|_{L^\infty_\gamma(\mathbb{R}_+;\tilde{Y})} \leq K_0\|a-b\|_Y + K\|0\|_{L^\infty_\gamma(\mathbb{R}_+;\tilde{X})}.$$

Iterating this argument and making use of (3.28) yields

$$\begin{aligned} \|w_n\|_{L^\infty(\mathbb{R}_+;Y)} &\leq K_0\|a-b\|_Y + K\|G(u_{n-1}) - G(v_{n-1})\|_{L^\infty_\gamma(\mathbb{R}_+;\tilde{X})} \\ &\leq K_0\|a-b\|_Y + K\tilde{L}\|w_{n-1}\|_{L^\infty_\gamma(\mathbb{R}_+;\tilde{X})} \end{aligned}$$

for all $n \geq 1$. Thus, employing the condition $K\tilde{L} < 1$ and an induction argument now imply

$$\|w_n\|_{L^\infty_\gamma(\mathbb{R}_+;\tilde{Y})} \leq \frac{1}{1-K\tilde{L}} K_0\|a-b\|_Y \tag{3.29}$$

for all $n \in \mathbb{N}$. We have to make sure that this already implies $w \in L^\infty_\gamma(\mathbb{R}_+;\tilde{Y})$. Up to some subsequence, we may assume that $w_n(t) \to w(t)$ in Y for almost every $t \in \mathbb{R}_+$. Let us fix such a time t. Due to (3.29), it holds $\|w_n(t)\|_{\tilde{Y}} \leq Ct^{-\gamma}$ for some C independent of t. As $\tilde{Y}$ is the dual of the separable space $\tilde{Z} = (\tilde{Z}_1, \tilde{Z}_2)_{\tilde{\theta},1}$, we may assume that $w_n(t)$ converges in the weak-*-topology of $\tilde{Y}$ to some $\tilde{w}(t)$ with $\|\tilde{w}(t)\|_{\tilde{Y}} \leq Ct^{-\gamma}$ up to some subsequence. By Lemma 1.1.9, we already get $\tilde{w}(t) = w(t)$. Thus, $w(t) \in \tilde{Y}$ with the estimate $\|w(t)\|_{\tilde{Y}} \leq Ct^{-\gamma}$. This completes the proof. □

Remark 3.1.32. *The separability of $\tilde{Z}_1$ and $\tilde{Z}_2$ was only needed in order to show that the sequence $(w_n(t))_{n\in\mathbb{N}}$ is weakly-*-convergent in $\tilde{Y}$ (up to some subsequence). This requirement can be replaced by any other condition, which allows us to conclude from $w_n(t) \to w(t)$ in Y and $\|w_n\|_{\tilde{Y}} \leq Ct^{-\gamma}$, that $w \in \tilde{Y}$ with $\|w\| \leq Ct^{-\gamma}$.*

We like to note that one generally cannot expect asymptotical stability in the solution space Y itself. This was already noted in [Yam00, Remark 1.4] for the Navier-Stokes equations in $\mathbb{R}^3$ even without the presence of an external force.

3.1.4 Applications

We will now apply the abstract theory to specific equations. These include heat equations, Navier-Stokes equations in rotating exterior domains and Ornstein-Uhlenbeck equations. The most prominent setting are Lorentz spaces. This is the same one as in [Yam00], which has been the prototype for the approach here. But we will also investigate the Navier-Stokes equations in homogeneous Besov spaces. Whereas the examples in Lorentz spaces rely on L^p-L^q-type estimates, the example in Besov spaces depends on the analyticity of the Stokes semigroup (or heat semigroup) in these spaces. These properties cannot be reduced in general to a common denominator.

Semigroups with L^p-L^q-Smoothing

We start by describing a general setting in Lorentz spaces that is satisfied by several parabolic equations. Let $2 < d \in \mathbb{N}$ and $\Omega \subseteq \mathbb{R}^d$. Suppose that L is the generator of a semigroup on $L^{p,r}(\Omega)$ for $1 < p < \infty$, $1 \le r \le \infty$, which is strongly continuous in the case of $r < \infty$. Furthermore, we assume that there is a constant $C > 0$ such that

$$\|e^{tL}\phi\|_{L^{q,r}(\Omega)} \le Ct^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}\|\phi\|_{L^{p,r}(\Omega)}, \quad t > 0, \phi \in L^{p,r}(\Omega), \tag{3.30}$$

for all $1 < p \le q < \infty$ and $r \in [1,\infty]$. This is a suitable foundation in view of the Assumptions 3.1.2 (ACPEx).

Theorem 3.1.33. *Let $2 < d \in \mathbb{N}$, $1 < p < d/2$. Let $f \in L^\infty(\mathbb{R}; L^{p,\infty}(\Omega))$. Then the equation*

$$u'(t) - Lu(t) = f(t), \quad t \in \mathbb{R} \tag{3.31}$$

has a unique mild solution $u \in L^\infty(\mathbb{R}; L^{q,\infty}(\Omega))$, where $q = \frac{pd}{d-2p}$. Additionally, there is an $M > 0$ such that

$$\|u\|_{L^\infty(\mathbb{R};L^{q,\infty}(\Omega))} \le M\|f\|_{L^\infty(\mathbb{R};L^{p,\infty}(\Omega))}.$$

If additionally

- *$f \in \mathrm{BUC}(\mathbb{R}; L^{p,\infty}(\Omega))$, then $u \in \mathrm{BUC}(\mathbb{R}; L^{q,\infty}(\Omega))$.*
- *$f \in C_0(\mathbb{R}; L^{p,\infty}(\Omega))$, then $u \in C_0(\mathbb{R}; L^{q,\infty}(\Omega))$.*
- *$f \in P_T(\mathbb{R}; L^{p,\infty}(\Omega))$ for some $T > 0$, then $u \in P_T(\mathbb{R}; L^{q,\infty}(\Omega))$.*
- *$f \in \mathrm{UAP}(\mathbb{R}; L^{p,\infty}(\Omega))$, then $u \in \mathrm{UAP}(\mathbb{R}; L^{q,\infty}(\Omega))$.*

- *$f \in \mathrm{AAP}(\mathbb{R}; L^{p,\infty}(\Omega))$, then $u \in \mathrm{AAP}(\mathbb{R}; L^{q,\infty}(\Omega))$.*

Proof. We set up all necessary spaces and variables in order to apply Theorem 3.1.4. Let

$$V = L^{p',1}(\Omega), \quad Z_1 = L^{q_1',1}(\Omega), \quad Z_2 = L^{q_2',1}(\Omega)$$

with $p \leq q_2 < q < q_1 < \infty$ and let $B = \mathrm{id}$. Clearly, Z_1 and Z_2 is an interpolation couple of Banach spaces and it holds that $Z_1 \cap Z_2$ is dense in Z_1 and Z_2. Due to (3.30), the estimates (3.3) are satisfied with $\alpha_i := \frac{d}{2}(\frac{1}{q_i'} - \frac{1}{p'})$. Hence, the Assumptions 3.1.2 (ACPEx) are fulfilled. Therefore, Theorem 3.1.4 yields for each $f \in L^\infty(\mathbb{R}; V') = L^\infty(\mathbb{R}; L^{p,\infty}(\Omega))$ the existence of a mild solution $u \in L^\infty(\mathbb{R}; (Z_1', Z_2')_{\theta,\infty})$ to (3.31), with θ satisfying $(1-\theta)\alpha_1 + \theta\alpha_2 = 1$. Furthermore, it gives the boundedness of the solution operator. The space $(Z_1', Z_2')_{\theta,\infty}$ equals $L^{q,\infty}(\Omega)$. Indeed, this is a consequence of

$$\begin{aligned} \frac{1-\theta}{q_1} + \frac{\theta}{q_2} &= (1-\theta)\left(\frac{1}{p} - \frac{2\alpha_1}{d}\right) + \theta\left(\frac{1}{p} - \frac{2\alpha_2}{d}\right) \\ &= \frac{1}{p} - \frac{2}{d}((1-\theta)\alpha_1 + \theta\alpha_2) \\ &= \frac{d-2p}{dp} = \frac{1}{q}. \end{aligned}$$

This yields the assertion of the existence of mild solutions to (3.31) in the desired spaces. The assertions on regularity and asymptotics of that solution follow by Proposition 3.1.6 and Theorem 3.1.8. □

Remark 3.1.34. *The restrictions on the parameters p and d in the statement above cannot be removed in the approach used here. For $p \geq d/2$, one gets for all $q \geq p$ by* (3.30) *that*

$$\|e^{tL}\phi\|_{L^{p',1}(\Omega)} \leq Ct^{-\alpha}\|\phi\|_{L^{q',1}(\Omega)}$$

with $\alpha \leq 1$. As the interpolation argument of Theorem 3.1.4 requires at least one estimate with some decay being stronger than t^{-1}, we cannot apply our abstract theory here. Thus, we always need $p < d/2$. Furthermore, we always need to have $p > 1$ in order to have the estimates (3.30) *available. Due to the condition $p < d/2$, we therefore also need the restriction $d \geq 3$.*

We change over to a semilinear case with some polynomial nonlinearity. Consider the equation

$$u'(t) - Lu(t) = f(t) + u^m(t), \quad t > 0, \tag{3.32}$$

where $m \in \mathbb{N}$ and $m > 1$.

Theorem 3.1.35. *Let $2 < d \in \mathbb{N}$, $1 < m \in \mathbb{N}$ and assume that $\frac{d(m-1)}{2m} > 1$. Set $p := \frac{d(m-1)}{2m}$ and $q := \frac{d(m-1)}{2}$. Then there are constants $\delta, R_0 > 0$ such that for each $0 < R < R_0$ and $f \in L^\infty(\mathbb{R}; L^{p,\infty}(\Omega))$ with*

$$\|f\|_{L^\infty(\mathbb{R};L^{p,\infty}(\Omega))} \leq R\delta$$

there exists a unique mild solution $u \in L^\infty(\mathbb{R}; L^{q,\infty}(\Omega))$ to (3.32) in the ball $\overline{B(0,R)} \subset L^\infty(\mathbb{R}; L^{q,\infty}(\Omega))$. Furthermore, if

- *$f \in \mathrm{BUC}(\mathbb{R}; L^{p,\infty}(\Omega))$, then $u \in \mathrm{BUC}(\mathbb{R}; L^{q,\infty}(\Omega))$.*
- *$f \in C_0(\mathbb{R}; L^{p,\infty}(\Omega))$, then $u \in C_0(\mathbb{R}; L^{q,\infty}(\Omega))$.*
- *$f \in P_T(\mathbb{R}; L^{p,\infty}(\Omega))$ for some $T > 0$, then $u \in P_T(\mathbb{R}; L^{q,\infty}(\Omega))$.*
- *$f \in \mathrm{UAP}(\mathbb{R}; L^{p,\infty}(\Omega))$, then $u \in \mathrm{UAP}(\mathbb{R}; L^{q,\infty}(\Omega))$.*
- *$f \in \mathrm{AAP}(\mathbb{R}; L^{p,\infty}(\Omega))$, then $u \in \mathrm{AAP}(\mathbb{R}; L^{q,\infty}(\Omega))$.*

Proof. We will apply Theorem 3.1.11. Note that the parameters p and q here are special cases of the ones in Theorem 3.1.33. Hence, the Assumptions 3.1.2 (ACPEx) are fulfilled for $V = L^{p',1}(\Omega)$, $Z_1 = L^{q_1',1}(\Omega)$ and $Z_2 = L^{q_2',1}(\Omega)$, where $p \leq q_2 < \frac{d(m-1)}{2} < q_1$. Moreover, it was shown that $(Z_1', Z_2')_{\theta,\infty} = L^{q,\infty}(\Omega)$. Hence, it only remains to verify the estimates on the nonlinear term as they are given in Assumptions 3.1.10 (sACPEx) with $X = L^{p,\infty}(\Omega)$ and $Y = L^{q,\infty}(\Omega)$. Set $G(v)(t) := v_m(t) + f(t)$. By Hölder's inequality, it holds for each $v \in L^\infty(\mathbb{R}; L^{q,\infty}(\Omega))$, that

$$\begin{aligned}
\|G(v)\|_{L^\infty(\mathbb{R};L^{p,\infty}(\Omega))} &\leq \|v^m\|_{L^\infty(\mathbb{R};L^{p,\infty}(\Omega))} + \|f\|_{L^\infty(\mathbb{R};L^{p,\infty}(\Omega))} \\
&\leq \|v\|^m_{L^\infty(\mathbb{R};L^{mp,\infty}(\Omega))} + \|f\|_{L^\infty(\mathbb{R};L^{p,\infty}(\Omega))} \\
&= \|v\|^m_{L^\infty(\mathbb{R};L^{q,\infty}(\Omega))} + \|f\|_{L^\infty(\mathbb{R};L^{p,\infty}(\Omega))} \\
&< \infty.
\end{aligned}$$

Thus, $G\colon L^\infty(\mathbb{R}; Y) \to L^\infty(\mathbb{R}; X)$ is well defined. Let $0 < R < (mM)^{-\frac{1}{m-1}} = R_0$, set $L = mR^{m-1}$ and let f satisfy $\|f\|_{L^\infty(\mathbb{R};X)} < R(1-LM)/M =: R\delta$. Here, M is the constant given in Theorem 3.1.33. Then we have

$$ML = MmR^{m-1} < 1$$

as well as

$$R(1 - LM) \geq M\|f\|_{L^\infty(\mathbb{R};X)}.$$

Moreover, it holds for $v_1, v_2 \in \overline{B(0,R)} \subset L^\infty(\mathbb{R};Y)$, that

$$\begin{aligned}
&\|G(v_1) - G(v_2)\|_{L^\infty(\mathbb{R};X)}\\
&= \|v_1^m - v_2^m\|_{L^\infty(\mathbb{R};X)}\\
&= \|(v_1 - v_2)\sum_{j=0}^{m-1} v_1^j v_2^{m-1-j}\|_{L^\infty(\mathbb{R};L^{p,\infty}(\Omega))}\\
&\le \|v_1 - v_2\|_{L^\infty(\mathbb{R};L^{mp,\infty}(\Omega))}\|\sum_{j=0}^{m-1} v_1^j v_2^{m-1-j}\|_{L^\infty(\mathbb{R};L^{mp/(m-1),\infty}(\Omega))}\\
&\le \|v_1 - v_2\|_{L^\infty(\mathbb{R};L^{mp,\infty}(\Omega))}\sum_{j=0}^{m-1} \|v_1^j v_2^{m-1-j}\|_{L^\infty(\mathbb{R};L^{mp/(m-1),\infty}(\Omega))}\\
&\le \|v_1 - v_2\|_{L^\infty(\mathbb{R};L^{mp,\infty}(\Omega))} mR^{m-1}\\
&= L\|v_1 - v_2\|_{L^\infty(\mathbb{R};L^{q,\infty}(\Omega))}.
\end{aligned}$$

Thus, we can apply Theorem 3.1.15 and get the existence of a mild solution $u \in \overline{B(0,R)} \subset L^\infty(\mathbb{R};Y)$, which is unique in that ball. If f lies additionally in $C_0(\mathbb{R};X)$, one can easily see that that G maps from $C_0(\mathbb{R};Y)$ to $C_0(\mathbb{R};X)$. Thus, the remaining part of the theorem follows by the additional statement of Theorem 3.1.15. The others cases work the same way. □

In order to investigate the stability of global solutions to (3.32), we introduce the corresponding initial value problem

$$\begin{aligned} u'(t) - Lu(t) &= f(t) + u^m(t), \quad t > 0, \\ u(0) &= u_0. \end{aligned} \tag{3.33}$$

The whole investigation is essentially an application of Theorem 3.1.31.

Theorem 3.1.36. *Let $2 < d \in \mathbb{N}$, $1 < m \in \mathbb{N}$ and assume that $\frac{d(m-1)}{2m} > 1$. Set $p := \frac{d(m-1)}{2m}$ and $q := \frac{d(m-1)}{2}$. Then there are constants $\delta_1, \delta_2, R > 0$ such that if $f \in L^\infty(\mathbb{R}_+; L^{p,\infty}(\Omega))$ and $a, b \in L^{q,\infty}(\Omega)$ satisfy*

$$\|f\|_{L^\infty(\mathbb{R}_+;L^{p,\infty}(\Omega))} \le \delta_1 \quad \textit{and} \quad \|a\|_{L^{q,\infty}(\Omega)}, \|a\|_{L^{q,\infty}(\Omega)} \le \delta_2,$$

then there are mild solutions $u, v \in \overline{B(0,R)} \subset L^\infty(\mathbb{R}_+; L^{q,\infty}(\Omega))$ to (3.33) with initial value $u_0 = a$ and $u_0 = b$ respectively and right-hand side f. Moreover, for each $\tilde{q} > q$, there is a constant $C > 0$ such that

$$\|u(t) - v(t)\|_{L^{\tilde{q},\infty}} \le Ct^{-\frac{d}{2}(\frac{1}{q} - \frac{1}{\tilde{q}})}, \quad t > 0.$$

Proof. At first, we show, that the Assumptions 3.1.21 (IVPEx) are fulfilled for

$$V := L^{p',1}(\Omega), \quad Z_1 := L^{q_1',1}(\Omega), \quad Z_2 := L^{q_2',1}(\Omega),$$

with $p \le q_2 < q < q_1 < \infty$, and $B = \mathrm{id}$. We already know that L generates a consistent and bounded C_0-semigroup on Z_1 and Z_2, which satisfies (3.20) with $\alpha_i = \frac{d}{2}(\frac{1}{q_i'} - \frac{1}{p'})$ due to (3.30). Clearly, Z_1 and Z_2 is an interpolation couple of Banach spaces and $Z_1 \cap Z_2$ is dense in Z_1 and Z_2. Hence, the Assumptions 3.1.21 (IVPEx) are fulfilled. Note that for $\theta \in (0,1)$ satisfying $1 = (1-\theta)\alpha_1 + \theta\alpha_2$, it holds $(Z_1', Z_2')_{\theta,\infty} = L^{q,\infty}(\Omega)$. In the same way, one can check Assumptions 3.1.27 (IVPStab) for

$$\tilde{V} = L^{\tilde{p}',1}(\Omega), \quad \tilde{Z}_1 = L^{\tilde{q}_1',1}(\Omega), \quad \tilde{Z}_2 = L^{\tilde{q}_2',1}(\Omega),$$

where $\tilde{p} = \frac{d\tilde{q}}{2\tilde{q}+d}$ and $\tilde{p} \le \tilde{q}_1 < \tilde{q} < \tilde{q}_2 < \infty$. We only have to additionally note that $\tilde{Z}_1$ and $\tilde{Z}_2$ are separable, and that (3.26) is satisfied with $\gamma = \frac{d}{2}(\frac{1}{\tilde{q}'} - \frac{1}{q'})$. As before, for $\tilde{\theta} \in (0,1)$ satisfying $1 = (1-\tilde{\theta})\tilde{\alpha}_1 + \tilde{\theta}\tilde{\alpha}_2$, it holds $(\tilde{Z}_1', \tilde{Z}_2')_{\theta,\infty} = L^{\tilde{q},\infty}(\Omega)$. It remains to show the Lipschitz conditions for the nonlinear part for the spaces

$$X = L^{p,\infty}(\Omega), \quad Y = L^{q,\infty}(\Omega), \quad \tilde{X} = L^{\tilde{p},\infty}(\Omega), \quad \tilde{Y} = L^{\tilde{q},\infty}(\Omega).$$

Let $G(v)(t) := v^m(t) + f(t)$. Using Hölder's inequality, we get for each $v \in L^\infty(\mathbb{R}_+; Y)$, that

$$\begin{aligned}
\|G(v)\|_{L^\infty(\mathbb{R};L^{p,\infty}(\Omega))} &\le \|v^m\|_{L^\infty(\mathbb{R};L^{p,\infty}(\Omega))} + \|f\|_{L^\infty(\mathbb{R};L^{p,\infty}(\Omega))} \\
&\le \|v\|^m_{L^\infty(\mathbb{R};L^{mp,\infty}(\Omega))} + \|f\|_{L^\infty(\mathbb{R};L^{p,\infty}(\Omega))} \\
&= \|v\|^m_{L^\infty(\mathbb{R};L^{q,\infty}(\Omega))} + \|f\|_{L^\infty(\mathbb{R};L^{p,\infty}(\Omega))} \\
&< \infty.
\end{aligned}$$

Hence, G maps form $L^\infty(\mathbb{R}_+; Y)$ to $L^\infty(\mathbb{R}_+; X)$. Recall the constants M and K, which appear in Theorem 3.1.23 and Lemma 3.1.29. Let $0 < R < (mM)^{-\frac{1}{m-1}}$, $L = \tilde{L} = mR^{m-1}$ and assume that $v_1, v_2 \in \overline{B(0,R)} \subset L^\infty(\mathbb{R}_+; Y)$. Then it holds

$$
\begin{aligned}
&\|G(v_1) - G(v_2)\|_{L^\infty(\mathbb{R}_+;X)} = \|v_1^m - v_2^m\|_{L^\infty(\mathbb{R}_+;X)} \\
&= \|(v_1 - v_2) \sum_{j=0}^{m-1} v_1^j v_2^{m-1-f}\|_{L^\infty(\mathbb{R}_+;L^{p,\infty}(\Omega))} \\
&\le \|v_1 - v_2\|_{L^\infty(\mathbb{R}_+;L^{mp,\infty}(\Omega))} \|\sum_{j=0}^{m-1} v_1^j v_2^{m-1-j}\|_{L^\infty(\mathbb{R}_+;L^{mp/(m-1),\infty}(\Omega))} \\
&\le \|v_1 - v_2\|_{L^\infty(\mathbb{R}_+;L^{mp,\infty}(\Omega))} \sum_{j=0}^{m-1} \|v_1^j v_2^{m-1-j}\|_{L^\infty(\mathbb{R}_+;L^{mp/(m-1),\infty}(\Omega))} \\
&\le \|v_1 - v_2\|_{L^\infty(\mathbb{R}_+;L^{mp,\infty}(\Omega))} mR^{m-1} \\
&= L\|v_1 - v_2\|_{L^\infty(\mathbb{R}_+;L^{q,\infty}(\Omega))}.
\end{aligned}
$$

If additionally $v_1 - v_2 \in L^\infty_\gamma(\mathbb{R}_+;\tilde{Y})$, we get by similar arguments that

$$
\begin{aligned}
&t^\gamma \|G(v_1)(t) - G(v_2)(t)\|_{\tilde{X}} \\
&= t^\gamma \|v_1^m(t) - v_2^m(t)\|_{L^{\tilde{p},\infty}(\Omega)} \\
&= t^\gamma \|(v_1(t) - v_2(t)) \sum_{j=0}^{m-1} v_1^j(t) v_2^{m-1-j}(t)\|_{L^{\tilde{p},\infty}(\Omega)} \\
&\le t^\gamma \|v_1(t) - v_2(t)\|_{L^{\tilde{q},\infty}(\Omega)} \|\sum_{j=0}^{m-1} v_1^j(t) v_2^{m-1-j}(t)\|_{L^{q/(m-1),\infty}(\Omega)} \\
&\le t^\gamma \|v_1(t) - v_2(t)\|_{L^{\tilde{q},\infty}(\Omega)} \sum_{j=0}^{m-1} \|v_1^j(t) v_2^{m-1-j}(t)\|_{L^{q/(m-1),\infty}(\Omega)} \\
&\le t^\gamma \|v_(t)1 - v_2(t)\|_{L^{\tilde{q},\infty}(\Omega)} mR^{m-1} \\
&= \tilde{L} t^\gamma \|v_1(t) - v_2(t)\|_{\tilde{Y}},
\end{aligned}
$$

in other words

$$
\|G(v_1) - G(v_2)\|_{L^\infty_\gamma(\mathbb{R}_+;\tilde{X})} \le \tilde{L}\|v_1 - v_2\|_{L^\infty_\gamma(\mathbb{R}_+;\tilde{Y})}.
$$

Furthermore, we have for $\|f\|_{L^\infty(\mathbb{R}_+;Y)}$ and $R_0 > 0$ sufficiently small that

$$
\begin{aligned}
ML &< 1, \\
K\tilde{L} &< 1, \\
R(1 - LM) &\ge M\|f\|_{L^\infty(\mathbb{R}_+;Y)} + \tilde{M}R_0.
\end{aligned}
$$

Thus, we can apply Theorem 3.1.31, if $\|a\|_Y$, $\|b\|_Y \le R_0$, which yields the desired result. □

We now give some examples of operators and semigroups, which fit into the setting described above.

Uniformly Elliptic Operators Let $d \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^d$ be a domain. Consider the differential equation

$$u'(t,x) - \mathcal{A}u(t,x) = f(t,x), \quad t > 0, x \in \Omega,$$
$$u(t,x) = 0, \quad t > 0, x \in \partial\Omega,$$

where

$$\mathcal{A}u(t,x) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}[a_{ij}(x)\frac{\partial}{\partial x_j}u](t,x).$$

We suppose that there are constants $0 < \mu \leq M < \infty$ such that for each $x \in \Omega$ we have $\mu \leq a_{ij}(x) \leq M$. Moreover, we assume the matrix (a_{ij}) to be real valued and symmetric. For $1 < p < \infty$, we define the realization L of $\mathcal{A}$ in $L^p(\Omega)$ by

$$D(A) := \{f \in W_0^{1,p}(\Omega) : Af \in L^p(\Omega)\},$$
$$Af := \mathcal{A}f.$$

It is a classical result (see for example [Dav89, Theorem 3.2.7]) that A is the generator of a C_0-semigroup on $L^p(\Omega)$, which is given by an integral kernel $k\colon (0,\infty) \times \Omega \times \Omega$ via

$$e^{tA}f(x) = \int_\Omega k(t,x,y)f(y)\,\mathrm{d}y.$$

Hence, the semigroup e^{tA} is consistent for any $p, q \in (1,\infty)$. In this sense, it was legitimate to leave out the parameter p in A. The integral kernel k is subject to the Gaussian estimate

$$k(t,x,y) \leq ct^{-\frac{d}{2}}e^{-\frac{a|x-y|^2}{t}}$$

with some constants $a, c > 0$. By Young's inequality, this implies that for some suitable $C = C(p,q,d)$, one gets

$$\|e^{tA}\phi\|_{L^q(\Omega)} \leq Ct^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}\|\phi\|_{L^p(\Omega)}, \quad t > 0, \phi \in L^p(\Omega).$$

By real interpolation, we can extend the results above to Lorentz spaces. More specifically, for $1 < p < \infty$ and $1 \leq r \leq \infty$, the semigroup $(e^{tA})_{t\geq 0}$

extends to a semigroup on $L^{p,r}(\Omega)$, which is strongly continuous if $r < \infty$ and weak-*-continuous if $r = \infty$. Furthermore, it is subject to the estimate

$$\|e^{tA}\phi\|_{L^{q,r}(\Omega)} \leq Ct^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}\|\phi\|_{L^{p,r}(\Omega)}, \quad t > 0, \phi \in L^{p,r}(\Omega),$$

for any $1 < p \leq q < \infty$ and $r \in [1,\infty]$. Hence, the operator A fulfils all the properties of the operator L in Subsection 3.1.4, which implies the following result:

Corollary 3.1.37. *Let $2 < d \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^d$ be a domain and define the operator A as above. Then the Theorems 3.1.33, 3.1.35 and 3.1.36 are true with $L = A$.*

Using the special case of the heat semigroup on the whole space, we give an example where the integrals

$$\text{(3.34)} \qquad \int_{-\infty}^{t} e^{(t-s)A}Bf(s)\,\mathrm{d}s \quad \text{and} \quad \int_{0}^{t} e^{(t-s)A}Bf(s)\,\mathrm{d}s$$

are generally not well-defined in the classical sense. This justifies the Definitions 3.1.1 and 3.1.20. In order to work with as less parameters as possible, we will consider the special equation

$$\text{(3.35)} \qquad u'(t) - \Delta u(t) = f(t), \quad t \in \mathbb{R}$$

on $\mathbb{R}^5$ with a right-hand side $f \in L^\infty(\mathbb{R}; L^{2,\infty}(\mathbb{R}^5))$. We have shown in Theorem 3.1.33, that there is a unique mild solution $u \in L^\infty(\mathbb{R}; L^{10,\infty}(\mathbb{R}^5))$ to (3.35). However, we are able to give a special right-hand side such that the integral (3.34) does not exist. We are going to use the following technical result on the action of the heat semigroup on homogeneous functions.

Lemma 3.1.38. *Let $1 \leq d \in \mathbb{N}$, $a \in (0,\infty)$ and $\Gamma : \mathbb{R}^d \to \mathbb{R}$ be homogeneous of order $-a$ almost everywhere, i.e. $\Gamma(\lambda x) = \lambda^{-a}\Gamma(x)$ for almost every $x \in \mathbb{R}^d$ and each $\lambda > 0$. Then*

$$e^{t\Delta}\Gamma(x) = \lambda^a e^{\lambda^2 t\Delta}\Gamma(\lambda x).$$

Proof. We make use of the representation of the heat semigroup as convolution with the heat kernel $G_t(x) = \frac{1}{(4\pi t)^{d/2}}e^{|x|^2/4t}$. The assertions follows by substitution:

$$\begin{aligned}
e^{t\Delta}\Gamma(x) = G_t * \Gamma(x) &= \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} \Gamma(y)\,\mathrm{d}y \\
&= \frac{1}{(4\pi t)^{d/2}} \frac{\lambda^d}{\lambda^d} \int_{\mathbb{R}^d} e^{-\frac{\lambda^2|x-y|^2}{4\lambda^2 t}} \frac{\lambda^a}{\lambda^a} \Gamma(y)\,\mathrm{d}y \\
&= \frac{\lambda^d}{(4\pi\lambda^2 t)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|\lambda x-\lambda y|^2}{4\lambda^2 t}} \lambda^a \Gamma(\lambda y)\,\mathrm{d}y \\
&= \frac{1}{(4\pi\lambda^2 t)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|\lambda x-z|^2}{4\lambda^2 t}} \Gamma(z) \lambda^a\,\mathrm{d}z \\
&= \lambda^a G_{\lambda^2 t} * \Gamma(\lambda x) \\
&= \lambda^a e^{\lambda^2 t\Delta} \Gamma(\lambda x). \qquad \square
\end{aligned}$$

We now specifically choose $f(t,x) = g(x) = |x|^{-5/2}$. As $x \mapsto |x|^{-\alpha} \in L^{d/\alpha,\infty}(\mathbb{R}^d)$ for any $\alpha > 0$ by Proposition 1.2.11, we have $g \in L^{2,\infty}(\mathbb{R}^5)$. Hence, Corollary 3.1.37 implies the existence of a unique mild solution $u \in L^\infty(\mathbb{R}; L^{10,\infty}(\mathbb{R}^5))$ of (3.35). On the other hand, we note that g is homogeneous of degree $-5/2$. Therefore, Lemma 3.1.38 with $\lambda = 1/\sqrt{t}$ yields that

$$\|e^{t\Delta} g(\cdot)\|_{L^{p,\infty}} = \sqrt{t}^{-\frac{5}{2}} \|e^{1\Delta} g(t^{-\frac{1}{2}} \cdot)\|_{L^{p,\infty}} = t^{\frac{5}{2p}-\frac{5}{4}} \|e^{1\Delta} g(\cdot)\|_{L^{p,\infty}}.$$

For $p = 10$, this implies

$$\|e^{t\Delta} g\|_{L^{10,\infty}} = Ct^{-1},$$

where $C > 0$ is some constant independent of t. Hence, we have

$$\int_{-\infty}^{t} \|e^{(t-s)\Delta} f(s)\|_{L^{10,\infty}(\mathbb{R}^5)}\,\mathrm{d}s = C \int_{-\infty}^{t} (t-s)^{-1}\,\mathrm{d}s = \infty.$$

This means the left integral in (3.34) does not exist in the sense of Bochner integrals, although it is well defined in the weak-*-sense given in Definition 3.1.1. By an analogous calculation, this is also the case for the right integral.

Heat Equation with Neumann Boundary Let $3 \le d \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^d$ be an exterior domain with Lipschitz boundary. We define the Laplacian with zero Neumann boundary via form methods. Consider the form

$$a(f,g) := \int_\Omega \nabla f(x) \nabla g(x)\,\mathrm{d}x, \quad f, g \in W^{1,2}(\Omega).$$

There is a unique non-positive self-adjoint operator Δ_N associated to that form a in the following sense: We define the domain of Δ_N by

$$D(\Delta_N) := \big\{ f \in W^{1,2}(\Omega) | \exists g \in L^2(\Omega), \\ a(f,h) = -\int_\Omega g(x)h(x)\,\mathrm{d}x \text{ for all } h \in W^{1,2}(\Omega) \big\}$$

and set $\Delta_N f = g$, where g is the same as in the definition of $D(\Delta_N)$. For Ω with smooth boundary, Δ_N is known to be the realization of the Laplacian with zero Neumann boundary condition in $L^2(\Omega)$. Furthermore, it is known that Δ_N is the generator of a semigroup $e^{t\Delta_N}$ on $L^2(\Omega)$ that is given by an integral kernel $k\colon (0,\infty) \times \Omega \times \Omega$ via

$$e^{tA} f(x) = \int_\Omega k(t,x,y) f(y)\,\mathrm{d}y.$$

It was shown in [CWZ94, Theorem 2] that this kernel is subject to Gaussian estimates

$$k(t,x,y) \leq \frac{c}{t^{d/2}} e^{-\frac{|x-y|^2}{Ct}}$$

for some $c, C > 0$. This implies the following properties of $e^{t\Delta_N}$:

Lemma 3.1.39. *The semigroup $e^{t\Delta_N}$ can be extrapolated to a bounded and strongly continuous semigroup on $L^{p,r}(\Omega)$ for any $1 < p < \infty$ and $1 \leq r < \infty$. Additionally, there are constants $C = C(p,q) > 0$ such that*

$$\|e^{t\Delta_N}\phi\|_{L^{q,r}(\Omega)} \leq C t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \|\phi\|_{L^{p,r}(\Omega)}, \quad t > 0, \phi \in L^p(\Omega),$$

for any $1 < p \leq q < \infty$ and $r \in [1,\infty)$.

As these are the estimates from Subsection 3.1.4, we also have the following consequences:

Corollary 3.1.40. *Let $2 < d \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^d$ be an exterior domain with Lipschitz-boundary. Then the Theorems 3.1.33, 3.1.35 and 3.1.36 are true with $L = \Delta_N$.*

Linear Ginzburg-Landau Equation Let $d \in \mathbb{N}$ and $\Omega \subseteq \mathbb{R}^d$ be either the whole space $\mathbb{R}^d$, the half space $\mathbb{R}^d_+$ or a domain with compact C^2-boundary. Consider the linearised Ginzburg-Landau equation

$$u'(t,x) - (\lambda + \mathrm{i}\beta)\Delta u(t,x) = f(t,x), \quad t \in \mathbb{R}, x \in \Omega,$$

where $\lambda > 0$ and $\beta \in \mathbb{R}$ are some fixed constants. For $1 < p < \infty$ define the realization G of $(\lambda + \mathrm{i}\beta)\Delta$ in $L^p(\Omega)$ to be

$$\begin{aligned} D(G) &:= W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \\ Gf &:= (\lambda + \mathrm{i}\beta)\Delta f. \end{aligned}$$

It was shown in [SYY16, Proposition 2.1] that G generates a bounded strongly continuous semigroup with the same L^p-L^q-decay as the heat semigroup. More specifically, we have:

Lemma 3.1.41. *Let $1 < p < \infty$, $\lambda > 0$ and $\beta \in \mathbb{R}$. Then G generates a bounded analytic C_0-semigroup on $L^p(\Omega)$ which is given independently of p by some integral kernel $k\colon (0,\infty) \times \Omega \times \Omega$ via*

$$e^{tG} f(x) = \int_\Omega k(t,x,y) f(y)\,\mathrm{d}y.$$

Moreover, for any $1 < p \le q < \infty$ there is a constant $C = C(p,q,d) > 0$ such that

$$\|e^{tG}\phi\|_{L^q(\Omega)} \le C t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \|\phi\|_{L^p(\Omega)}, \quad t > 0. \tag{3.36}$$

By real interpolation, this Lemma can be extended to Lorentz spaces and we get, as before, the following result:

Corollary 3.1.42. *Let $2 < d \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^d$ be either $\mathbb{R}^d$, $\mathbb{R}^d_+$ or a domain with compact C^2-boundary. Then the Theorems 3.1.33, 3.1.35 and 3.1.36 are true with $L = G$.*

Navier-Stokes Equations in Besov Spaces

Let $1 < d \in \mathbb{N}$. Consider the Stokes equations on the whole space

$$\begin{aligned} u'(t,x) - \Delta u(t,x) + \pi(t,x) = f(t,x), &\quad t \in \mathbb{R}, x \in \mathbb{R}^d, \\ \operatorname{div} u(t,x) = 0, &\quad t \in \mathbb{R}, x \in \mathbb{R}^d. \end{aligned}$$

We are going to consider these equations in the setting of homogeneous Besov spaces. For a definition of these spaces, see section 1.2.4. Applying the Helmholtz projection to the Stokes equations reduces the problem to

$$u'(t) - \Delta u(t) = \mathbb{P} f(t), \quad t \in \mathbb{R}. \tag{3.37}$$

Here, we have used that $\mathbb{P}$ and Δ commute on $\mathbb{R}^d$. Hence, the investigation reduces to the one of the heat semigroup in homogeneous Besov spaces.

Lemma 3.1.43. *Let $1 < p < \infty$, $1 \le q \le \infty$ and $\sigma \le \tau$. Then there is a constant $C = C(p, q, d) > 0$ such that*

$$\|e^{-t\Delta} f\|_{\dot{B}^{\tau}_{p,q}(\mathbb{R}^d)} \le Ct^{-\frac{\tau-\sigma}{2}} \|f\|_{\dot{B}^{\sigma}_{p,q}(\mathbb{R}^d)}, \quad t > 0. \tag{3.38}$$

Proof. Let $\mathcal{C}$ be an annulus in $\mathbb{R}^d$. By Lemma 2.4 from [BCD11], there are $c, C > 0$ such that for any $t, \lambda > 0$ and $u \in L^p(\mathbb{R}^d)$ with $\operatorname{supp} u \subset \lambda\mathcal{C}$, we have

$$\|e^{t\Delta} u\|_{L^p} \le Ce^{-ct\lambda^2} \|u\|_{L^p}. \tag{3.39}$$

We will apply this estimate to the Littlewood-Paley decomposition of f. Let $\phi \in C_c^\infty(\mathbb{R}^d)$ be the cut-off function from the Littlewood-Paley decomposition $(\dot{\Delta}_j)_{j\in\mathbb{Z}}$ in Subsection 1.2.4 and let $\mathcal{C}$ be such that $\operatorname{supp} \phi \subset \mathcal{C}$. We then have

$$\operatorname{supp} \mathcal{F}\dot{\Delta}_j f \subset 2^j\mathcal{C}$$

for any $j \in \mathbb{Z}$. Using that the Fourier multipliers $e^{t\Delta}$ and $\dot{\Delta}_j$ commute, we obtain by (3.39) that

$$\begin{aligned} 2^{j\tau}\|\dot{\Delta}_j e^{t\Delta} f\|_{L^p} &= 2^{j\tau}\|e^{-t\Delta}\dot{\Delta}_j f\|_{L^p} \\ &\le C2^{j\tau}e^{-ct2^{2j}}\|\dot{\Delta}_j f\|_{L^p} \\ &= Ce^{-ct2^{2j}}t^{\frac{\tau-\sigma}{2}}2^{j(\tau-\sigma)}t^{-\frac{\tau-\sigma}{2}}2^{j\sigma}\|\dot{\Delta}_j f\|_{L^p}. \end{aligned}$$

Due to Lemma 2.35 in [BCD11], the factor $e^{-ct2^{2j}}t^{\frac{\tau-\sigma}{2}}2^{j(\tau-\sigma)}$ can be bounded independently of t and j by some constant $C > 0$. Therefore, we get

$$2^{j\tau}\|\dot{\Delta}_j e^{-t\Delta} f\|_{L^p} \le Ct^{-\frac{\tau-\sigma}{2}}2^{j\sigma}\|\dot{\Delta}_j f\|_{L^p}.$$

The assertion follows now by taking the $\ell^q(\mathbb{Z})$-norm on both sides with respect to j. □

The estimate (3.38) fits well to Assumptions 3.1.2 (ACPEx).

Theorem 3.1.44. *Let $1 < d \in \mathbb{N}$, $1 < p < \infty$ and $s \in \mathbb{R}$ with $s < d/p$. Let $f \in L^\infty(\mathbb{R}; \dot{B}^{s-2}_{p,\infty}(\mathbb{R}^d))$. Then there is a unique mild solution $u \in L^\infty(\mathbb{R}; \dot{B}^{s}_{p,\infty}(\mathbb{R}^d))$ to (3.37). Furthermore, there is a constant $M > 0$ independent of f such that*

$$\|u\|_{L^\infty(\mathbb{R};\dot{B}^s_{p,\infty})} \le M\|f\|_{L^\infty(\mathbb{R};\dot{B}^{s-2}_{p,\infty})}.$$

Proof. Let $\delta \in (0, 2)$ fulfil $s + \delta < d/p$ and set $Z_1 := \dot{B}^{-s-\delta}_{p',1}$, $Z_2 := \dot{B}^{-s+\delta}_{p',1}$ and $V := \dot{B}^{-s+2}_{p',1}$. We show that the Assumptions 3.1.2 (ACPEx) are satisfied here for $A = \Delta$ and $B = \mathbb{P}$. By Lemma 3.1.43 and the boundedness of $\mathbb{P}$ on V we get

$$\|\mathbb{P}e^{t\Delta}\phi\|_V \leq Ct^{-\alpha_i}\|\phi\|_{Z_i}, \quad t > 0, \phi \in Z_i,$$

where $\alpha_1 = 1 + \delta/2$ and $\alpha_2 = 1 - \delta/2$. Note that we have $f \in L^\infty(\mathbb{R}; V')$ and that for $\theta := 1/2$ we get $1 = (1-\theta)\alpha_1 + \theta\alpha_2$. Hence, Theorem 3.1.4 yields a mild solution $u \in L^\infty(\mathbb{R}; Y)$ of (3.37), where

$$Y = (Z_1', Z_2')_{\theta,\infty} = (\dot{B}^{s+\delta}_{p,\infty}, \dot{B}^{s-\delta}_{p,\infty})_{\theta,\infty} = \dot{B}^s_{p,\infty}.$$

The desired norm estimate is also given by the same theorem. □

We now change over to the full Navier-Stokes equations

$$u'(t) - \Delta u(t) = \mathbb{P}[f(t) - \operatorname{div}(u(t) \otimes u(t))], \quad t \in \mathbb{R}. \tag{3.40}$$

Compared to the linear Stokes equations, we do now have to make sure that the function spaces fit well to the nonlinearity. This will cause some restrictions to the integrability and smoothness parameters.

Theorem 3.1.45. *Let $s \in (0, \infty)$, $3 \leq d \in \mathbb{N}$ and $p \in (1, d)$ be such that $s = \frac{d}{p} - 1$. Then there are constants $R_0, \delta > 0$ such that for each $0 < R < R_0$, $f \in L^\infty(\mathbb{R}; \dot{B}^{s-2}_{p,\infty}(\mathbb{R}^d))$ with $\|f\|_{L^\infty(\mathbb{R}; \dot{B}^{s-2}_{p,\infty}(\mathbb{R}^d))} \leq R\delta$, there is a unique mild solution $u \in \overline{B(0,R)} \subset L^\infty(\mathbb{R}; \dot{B}^s_{p,\infty}(\mathbb{R}^d))$ to (3.40). Moreover, if*

- *$f \in \mathrm{BUC}(\mathbb{R}; \dot{B}^{s-2}_{p,\infty}(\mathbb{R}^d))$, then $u \in \mathrm{BUC}(\mathbb{R}; \dot{B}^s_{p,\infty}(\mathbb{R}^d))$.*
- *$C_0(\mathbb{R}; \dot{B}^{s-2}_{p,\infty}(\mathbb{R}^d))$, then $u \in C_0(\mathbb{R}; \dot{B}^s_{p,\infty}(\mathbb{R}^d))$.*
- *$P_T(\mathbb{R}; \dot{B}^{s-2}_{p,\infty}(\mathbb{R}^d))$ for some $T > 0$, then $u \in P_T(\mathbb{R}; \dot{B}^s_{p,\infty}(\mathbb{R}^d))$.*
- *$\mathrm{UAP}(\mathbb{R}; X)$, then $u \in \mathrm{UAP}(\mathbb{R}; \dot{B}^s_{p,\infty}(\mathbb{R}^d))$.*
- *$\mathrm{AAP}(\mathbb{R}; \dot{B}^{s-2}_{p,\infty}(\mathbb{R}^d))$, then $u \in \mathrm{AAP}(\mathbb{R}; \dot{B}^s_{p,\infty}(\mathbb{R}^d))$.*

Proof. We will apply Theorem 3.1.11 with

$$Z_1 := \dot{B}^{-s-\delta}_{p',1}(\mathbb{R}^d), \quad Z_2 := \dot{B}^{-s+\delta}_{p',1}(\mathbb{R}^d) \quad \text{and} \quad V := \dot{B}^{-s+2}_{p',1}(\mathbb{R}^d),$$

where $\delta \in (0, 2)$ fulfils $s + \delta < d/p$, as well as $A = \Delta$ and $B = \mathbb{P}$. As the spaces here are a special case of the ones considered in Theorem 3.1.44, the Assumptions 3.1.2 (ACPEx) are satisfied. Hence, we only have to check the Assumptions 3.1.10 (sACPEx) with

$$X = V' = \dot{B}^{s-2}_{p,\infty}(\mathbb{R}^d) \quad \text{and } Y = (Z_1', Z_2')_{\theta,\infty} = \dot{B}^s_{p,\infty}(\mathbb{R}^d)$$

and $G(v)(t) := f(t) - \operatorname{div}(v(t) \otimes v(t))$. Corollary 1.2.19 with $\sigma = d/p$ implies that $G(v) \in L^\infty(\mathbb{R}; X)$ for each $v \in L^\infty(\mathbb{R}; Y)$ with the estimate

$$\begin{aligned}\|G(v)\|_{L^\infty(\mathbb{R};X)} &\le \|f\|_{L^\infty(\mathbb{R};X)} + \|\operatorname{div}(v \otimes v)\|_{L^\infty(\mathbb{R};\dot{B}^{s-2}_{p,\infty}(\mathbb{R}^d))}\\ &\le \|f\|_{L^\infty(\mathbb{R};X)} + C\|v \otimes v\|_{L^\infty(\mathbb{R};\dot{B}^{s-1}_{p,\infty}(\mathbb{R}^d))}\\ &\le \|f\|_{L^\infty(\mathbb{R};X)} + C\|v\|^2_{L^\infty(\mathbb{R};Y)}.\end{aligned}$$

Similarly, let $R > 0$ and let $v_1, v_2 \in \overline{B(0,R)} \subset L^\infty(\mathbb{R}; Y)$. Then it holds

$$\begin{aligned}\|G(v_1) - G(v_2)\|_{L^\infty(\mathbb{R};Y)} &\le \|\operatorname{div}(v_1 \otimes v_1 - v_2 \otimes v_2)\|_{L^\infty(\mathbb{R};X)}\\ &\le C\|v_1 \otimes v_1 - v_2 \otimes v_2\|_{L^\infty(\mathbb{R};\dot{B}^{s-1}_{p,\infty}(\mathbb{R}^d))}\\ &\le C\|(v_1 - v_2) \otimes v_1 - (v_1 - v_2) \otimes v_2\|_{L^\infty(\mathbb{R};\dot{B}^{s-1}_{p,\infty}(\mathbb{R}^d))}\\ &\le C(\|v_1\|_{L^\infty(\mathbb{R};Y)} + \|v_2\|_{L^\infty(\mathbb{R};Y)})\|v_1 - v_2\|_{L^\infty(\mathbb{R};Y)}\\ &\le 2C_0R.\end{aligned}$$

Now let $R < 1/(2C_0M)$ and set $L := 2C_0R$. If $\|f\|_{L^\infty(\mathbb{R}:\dot{B}^{s-2}_{p,\infty}(\mathbb{R}^d))} \le R(1 - LM)/M$, we can see by the estimate above that the requirements of (3.8) and (3.9) are satisfied. Thus, we can apply Theorem 3.1.15 and get the existence of a unique mild solution $u \in \overline{B(0,R)} \subset L^\infty(\mathbb{R}; Y)$. If additionally $f \in C_0(\mathbb{R}; X)$ one can see easily that G maps from $C_0(\mathbb{R}; Y)$ to $C_0(\mathbb{R}; X)$. Therefore, one obtains that the solution u lies in $C_0(\mathbb{R}; Y)$ by the second part of Theorem 3.1.15. The other cases work the same way. □

The setting here is also suitable for a stability analysis. The corresponding initial value problem to (3.40) reads as

$$\begin{aligned}u'(t) - \Delta u(t) &= \mathbb{P}[f(t) - \operatorname{div}(u(t) \otimes u(t))], \quad t > 0,\\ u(0) &= u_0.\end{aligned} \tag{3.41}$$

Similar to the whole real time axis, we have the following result:

Theorem 3.1.46. *Let $s \in (0,\infty)$, $3 \le d \in \mathbb{N}$, $p \in (1,d)$ such that $s = \frac{d}{p} - 1$. Then there are constants $\delta_1, \delta_2, R > 0$ such that if $f \in L^\infty(\mathbb{R}; \dot{B}^{s-2}_{p,\infty}(\mathbb{R}^d))$ and $a, b \in \dot{B}^s_{p,\infty}(\mathbb{R}^d))$ satisfy*

$$\|f\|_{L^\infty(\mathbb{R};\dot{B}^{s-2}_{p,\infty}(\mathbb{R}^d))} \le \delta_1, \quad \|a\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^d)}, \|b\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^d)} \le \delta_2,$$

then there are unique mild solutions $u, v \in \overline{B(0,R)} \subset L^\infty(\mathbb{R}; \dot{B}^s_{p,\infty}(\mathbb{R}^d))$ to (3.41) with initial value $u_0 = a$ and $u_0 = b$ respectively . Moreover, for any $\sigma \in (s, d/p)$ there is a constant $C > 0$ such that

$$\|u(t) - v(t)\|_{\dot{B}^\sigma_{p,\infty}(\mathbb{R}^d)} \le Ct^{\frac{\sigma - s}{2}}.$$

Proof. Let $\delta \in (0,2)$ satisfy $s+\delta < d/p$ and set

$$Z_1 := \dot{B}^{-s-\delta}_{p,1}(\mathbb{R}^d), \quad Z_2 := \dot{B}^{-s+\delta}_{p,1}(\mathbb{R}^d), \quad V := \dot{B}^{-s+2}_{p,1}(\mathbb{R}^d)$$

as well as $A := \Delta$ and $B := \mathbb{P}$. We have already seen in the proof of Theorem 3.1.45, that the Assumptions 3.1.2 (ACPEx) are satisfied in this setting. As the heat semigroup is bounded on Z_1 and Z_2, the Assumptions 3.1.21 (IVPEx) are satisfied, too. Concerning Assumptions 3.1.27 (IVPStab), let

$$\tilde{Z}_1 := \dot{B}^{-\sigma-\delta}_{p,1}(\mathbb{R}^d), \quad \tilde{Z}_2 := \dot{B}^{-\sigma+\delta}_{p,1}(\mathbb{R}^d), \quad \tilde{V} := \dot{B}^{-\sigma+2}_{p,1}(\mathbb{R}^d).$$

By the same arguments as for the previous case, most of the Assumptions 3.1.21 (IVPEx) can be shown, too. We only have to additionally note that (3.26) is satisfied due to (3.38) with $\gamma = \frac{\sigma-s}{2}$. In view of Remark 3.1.32, the separability condition can be exchanged by the Fatou property of $Z' \cap \tilde{Z}'$, see Proposition 1.2.15. It remains to show the Lipschitz conditions of Assumption 3.1.24 (sIVPEx) and Assumption 3.1.30 (sIVPStab) for

$$X := \dot{B}^{s-2}_{p,\infty}(\mathbb{R}^d), \quad Y := \dot{B}^{s}_{p,\infty}(\mathbb{R}^d), \quad \tilde{X} := \dot{B}^{\sigma-2}_{p,\infty}(\mathbb{R}^d), \quad \tilde{Y} := \dot{B}^{\sigma}_{p,\infty}(\mathbb{R}^d)'$$

and the nonlinearity $G(u)(t) = f(t) - \operatorname{div}(u(t) \otimes (t))$. Let $R > 0$ and $u, v \in \overline{B(0,R)} \subset L^\infty(\mathbb{R}_+; Y)$. By means of Corollary 1.2.19, it holds

$$\begin{aligned}
\|G(v_1) - G(v_2)\|_{L^\infty(\mathbb{R}_+;X)} &\le \|\operatorname{div}(v_1 \otimes v_1 - v_2 \otimes v_2)\|_{L^\infty(\mathbb{R}_+;X)} \\
&\le C\|v_1 \otimes v_1 - v_2 \otimes v_2\|_{L^\infty(\mathbb{R}_+;\dot{B}^{s-1}_{p,\infty}(\mathbb{R}^d))} \\
&\le C\|(v_1 - v_2) \otimes v_1 - (v_1 - v_2) \otimes v_2\|_{L^\infty(\mathbb{R}_+;\dot{B}^{s-1}_{p,\infty}(\mathbb{R}^d))} \\
&\le C(\|v_1\|_{L^\infty(\mathbb{R}_+;Y)} + \|v_2\|_{L^\infty(\mathbb{R}_+;Y)})\|v_1 - v_2\|_{L^\infty(\mathbb{R}_+;Y)} \\
&\le 2C_0R.
\end{aligned}$$

If additionally $v_1 - v_2 \in L^\infty(\mathbb{R}_+; \tilde{Z})$, we get

$$\begin{aligned}
&\|G(v_1) - G(v_2)\|_{L^\infty(\mathbb{R}_+;\tilde{X})} \\
&= \|\operatorname{div}(v_1 \otimes v_1 - v_2 \otimes v_2)\|_{L^\infty(\mathbb{R}_+;\dot{B}^{\sigma-2}_{p,\infty}(\mathbb{R}^d))} \\
&\le C\|(v_1 - v_2) \otimes v_1 - (v_1 - v_2) \otimes v_2\|_{L^\infty(\mathbb{R}_+;\dot{B}^{\sigma-1}_{p,\infty}(\mathbb{R}^d))} \\
&\le C(\|v_1\|_{L^\infty(\mathbb{R}_+;\dot{B}^{s}_{p,\infty}(\mathbb{R}^d))} + \|v_2\|_{L^\infty(\mathbb{R}_+;\dot{B}^{s}_{p,\infty}(\mathbb{R}^d))})\|v_1 - v_2\|_{L^\infty(\mathbb{R}_+;\dot{B}^{\sigma}_{p,\infty}(\mathbb{R}^d))} \\
&\le 2C_1R\|v_1 - v_2\|_{L^\infty(\mathbb{R}_+;\tilde{Y})}.
\end{aligned}$$

Hence, for $R < \min\{\frac{1}{2C_0M}, \frac{1}{2C_1K}\}$, $L = 2C_0R$, $\tilde{L} = 2C_1R$

$$\|f\|_{L^\infty(\mathbb{R};\dot{B}^{s-2}_{p,\infty}(\mathbb{R}^d))} + M_0R_0 < R(1 - LM),$$

where M, M_0 and K are the constants from Theorem 3.1.23 and Lemma 3.1.29, the Assumptions 3.1.24 (sIVPEx) and Assumptions 3.1.30 (sIVPStab) are satisfied. An application of Theorem 3.1.31 yields the desired result. □

Navier-Stokes Equations with Rotating Effect

Consider the Navier-Stokes equations around a rotating body. Fixing a reference frame on the rotating body, these are given by

$$\begin{aligned} v'(t,x) - \Delta v(t,x) - (\omega \times x) \cdot \nabla v(t,x) + \omega \times v(t,x) + \nabla p(t,x) & \\ = \bar{f}(t,x) - \operatorname{div}(v(t,x) \otimes v(t,x)), & \qquad x \in \Omega, \\ \operatorname{div} v(t,x) = 0, & \qquad x \in \Omega, \\ v(t,x) = \omega \times x, & \qquad x \in \partial\Omega, \\ v(t,x) \to 0, & \qquad \text{for } |x| \to \infty, \\ v(0,x) = w_0(x), & \qquad x \in \Omega, \end{aligned}$$

where $t \in \mathbb{R}$ and $\omega \in \mathbb{R}$ is the angular velocity of the rotation of the body Ω^C. For a derivation of these equations, see [His99]. Due to the non-homogeneous boundary conditions, we are not able to deal with this problem via semigroup methods. In order to bypass this issue, we are going to consider perturbations of stationary solutions to the equations above. Hence, let u_s and p_s be a solution to

$$\begin{aligned} -\Delta u_s(x) - (\omega \times x) \cdot \nabla u_s(x) + \omega \times u_s(x) + \nabla p_s(x) & \\ = f_s(x) - \operatorname{div}(u_s(t,x) \otimes u_s(t,x)), & \qquad x \in \Omega, \\ \operatorname{div} u_s(x) = 0, & \qquad x \in \Omega, \\ u_s(x) = \omega \times x, & \qquad t > 0, x \in \partial\Omega, \\ u_s(x) \to 0, & \qquad \text{for } |x| \to \infty, \end{aligned}$$

where f_s is some stationary exterior force. Setting $u(t,x) = v(t,x) - u_s(x)$, $\pi(t,x) = p(t,x) - p_x(x)$ and $f(t,x) = \bar{f}(t,x) - f_s(x)$, we get a solution to

(3.42)

$$\begin{aligned} u'(t,x) - \Delta u(t,x) - (\omega \times x) \cdot \nabla u(t,x) + \omega \times u(t,x) + \nabla \pi(t,x) & \\ = f(t,x) - \operatorname{div}(\mathcal{G}(t,x)), & \qquad x \in \Omega, \\ \operatorname{div} v(t,x) = 0, & \qquad x \in \Omega, \\ v(t,x) = 0, & \qquad x \in \partial\Omega, \\ v(t,x) \to 0, & \qquad \text{for } |x| \to \infty, \\ v(0,x) = w_0(x), & \qquad x \in \Omega \end{aligned}$$

with $t > 0$ and

$$\mathcal{G}(t,x) = \operatorname{div}(u_s(t,x) \otimes u(t,x) + u(t,x) \otimes u_s(t,x) + u(t,x) \otimes u(t,x)).$$

We are going to deal with this problem in the context of Lorentz spaces. It was shown by Farwig and Hishida [FH07] that for $\omega \in \mathbb{R}$ and $f_s \in \dot{H}^{-1}_{3/2,\infty}(\Omega)$ with sufficiently small norms there is a unique weak solution u_s to the stationary Navier-Stokes equations and this solution is subject to the estimate

$$\|u_s\|_{L^{3,\infty}(\Omega)} \leq C(|\omega| + \|f_s\|_{\dot{H}^{-1}_{3/2,\infty}(\Omega)}) \tag{3.43}$$

with $C > 0$ being independent of f_s. We will show the existence of mild solutions to perturbations of the stationary case in an analogous setting.

In order to describe our setting, we formally apply the Helmholtz projection to (3.42) and assume the exterior force to be of the form $f = \operatorname{div} F$, which yields

$$\begin{aligned} u'(t,x) - \mathbb{P}[\Delta u(t,x) + (\omega \times x) \cdot \nabla u(t,x) - \omega \times u(t,x)]& \\ = \mathbb{P}\operatorname{div}(G(t,x)),& \qquad t > 0, x \in \Omega, \\ v(t,x) = 0,& \qquad t > 0, x \in \partial\Omega, \\ v(t,x) \to 0,& \qquad \text{for } |x| \to \infty, t > 0, \\ v(0,x) = w_0(x),& \qquad x \in \Omega \end{aligned}$$

with

$$G(t,x) := F(t,x) - u_s \otimes u(t,x) - u(t,x) \otimes u_s(x) - u(t,x) \otimes u(t,x).$$

We realize the linear part on the left-hand side of the first equation in Lebesgue spaces via

$$\begin{aligned} D(\mathcal{L}) &:= \{u \in L^p_\sigma(\Omega) \cap W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) | (\omega \times u) \cdot \nabla u \in L^p(\Omega)\}, \\ \mathcal{L}u &:= -\mathbb{P}[\Delta u + (\omega \times x)\nabla u - \omega \times u], \quad u \in D(\mathcal{L}). \end{aligned}$$

With this operator the projected problem can be rewritten as

$$u'(t) - \mathcal{L}u(t) = \mathbb{P}\operatorname{div}(G(t)), \quad t \in \mathbb{R}. \tag{3.44}$$

We will at first consider the linearised equation

$$u'(t) - \mathcal{L}u(t) = \mathbb{P}\operatorname{div} F(t), \quad t \in \mathbb{R}. \tag{3.45}$$

It was shown in [HS09, Theorem 1.1] that $\mathcal{L}$ generates a bounded (but not analytic) C_0-semigroup in $L^p_\sigma(\Omega)$ for any $p \in (1, \infty)$ that can be extended to Lorentz spaces with the following properties:

Lemma 3.1.47. *Let $a_0 > 0$ and $|\omega| = |a| \leq a_0$. Let $p \in (1, \infty)$ and $r \in [1, \infty)$. Then $\mathcal{L}$ generates a bounded C_0 semigroup in $L^{p,r}_\sigma(\Omega)$. For any $p \leq q < \infty$, this semigroup is subject to the estimate*

$$\|e^{t\mathcal{L}}\phi\|_{L^{q,r}} \leq Ct^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})}\|\phi\|_{L^{p,r}}, \quad t > 0,$$

for any $\phi \in L^{p,r}_\sigma(\Omega)$, where $C = C(a_0, p, q, r) > 0$. Moreover, for any $1 < p \leq q \leq 3$ there is a constant $C = C(a_0, p, q, r) > 0$ such that

$$\|\nabla e^{t\mathcal{L}}\phi\|_{L^{q,r}} \leq Ct^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})}\|\phi\|_{L^{p,r}}, \quad t > 0,$$

for any $\phi \in L^{p,r}_\sigma(\Omega)$.

Using the gradient estimates, we are able to prove the following theorem:

Theorem 3.1.48. *Let $1 < p < 3$ and $F \in L^\infty(\mathbb{R}; L^{p,\infty}(\Omega))$. Then the Stokes problem (3.45) has a unique mild solution $u \in L^\infty(\mathbb{R}; L^{q,\infty}_\sigma(\Omega))$, where $\frac{1}{q} = \frac{1}{p} - \frac{1}{3}$. Furthermore, there is a constant $M > 0$ independent of F such that*

$$\|u\|_{L^\infty(\mathbb{R};L^{q,\infty}(\Omega))} \leq M\|F\|_{L^\infty(\mathbb{R};L^{p,\infty}(\Omega))}. \tag{3.46}$$

If additionally

- *$F \in C_0(\mathbb{R}; L^{p,\infty}(\Omega))$, then $u \in C_0(\mathbb{R}; L^{q,\infty}_\sigma(\Omega))$.*
- *$F \in \mathrm{BUC}(\mathbb{R}; L^{p,\infty}(\Omega))$, then $u \in \mathrm{BUC}(\mathbb{R}; L^{q,\infty}_\sigma(\Omega))$.*
- *$F \in P_T(\mathbb{R}; L^{p,\infty}(\Omega))$ for some $T > 0$, then $u \in P_T(\mathbb{R}; L^{q,\infty}_\sigma(\Omega))$.*
- *$F \in \mathrm{AAP}(\mathbb{R}; L^{p,\infty}(\Omega))$, then $u \in \mathrm{AAP}(\mathbb{R}; L^{q,\infty}_\sigma(\Omega))$.*
- *$F \in \mathrm{UAP}(\mathbb{R}; L^{p,\infty}(\Omega))$, then $u \in \mathrm{UAP}(\mathbb{R}; L^{q,\infty}_\sigma(\Omega))$.*

Proof. Let $p \leq q_2 < q < q_1 < \infty$. We set $V := L^{p',1}(\Omega)$, $Z_1 := L^{q_1',1}_\sigma(\Omega)$ and $Z_2 = L^{q_2',1}_\sigma(\Omega)$ and choose $A := \mathcal{L}$ and $B = \mathbb{P}\operatorname{div}$. We remark that $V' = L^{p,\infty}(\Omega)$ and that $(e^{t\mathcal{L}}\mathbb{P}\operatorname{div})' = \nabla e^{t\mathcal{L}}$. By Lemma 3.1.47 we have

$$\|\nabla e^{t\mathcal{L}}\phi\|_V \leq Ct^{-\alpha_i}\|\phi\|_{Z_i}, \quad t > 0, \phi \in Z_i,$$

where $\alpha_i = \frac{1}{2} + \frac{3}{2}(\frac{1}{q_i'} - \frac{1}{p'})$ and therefore $\alpha_1 > 1 > \alpha_2 > 0$. As Z_1 and Z_2 is an interpolation couple of Banach spaces and $Z_1 \cap Z_2$ is dense in Z_1 and Z_2, the Assumptions 3.1.2 (ACPEx) are fulfilled. Choose $\theta \in (0, 1)$ such that $1 = (1-\theta)\alpha_1 + \theta\alpha_2$ and note that this is equivalent to $\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$. Theorem 3.1.4 implies the existence of a mild solution $u \in L^\infty(\mathbb{R}; Y)$ to (3.45), where

$$Y = (Z_1, Z_2)'_{\theta,1} = (L^{q_1',1}_\sigma, L^{q_2',1}_\sigma)'_{\theta,1} = (L^{q_1,\infty}_\sigma, L^{q_2,\infty}_\sigma)_{\theta,\infty} = L^{q,\infty}_\sigma,$$

together with the norm estimate (3.46). This completes the proof of the first statement.

It remains to prove the statements about the asymptotic behaviour of the solution. If additionally $F \in \mathrm{BUC}(\mathbb{R}; X)$, the second part of theorem here follows by Proposition 3.1.6. The other cases are a consequence of Theorem 3.1.8. □

We do now consider the full Navier-Stokes system (3.44).

Theorem 3.1.49.

There are constants $\delta_1, \delta_2, R > 0$ such that for each $F \in L^\infty(\mathbb{R}; L^{3/2,\infty}(\Omega))$ with $\|F\|_{L^\infty(\mathbb{R};L^{3/2,\infty}(\Omega))} \le R\delta_1$ and $u_s \in L^{3,\infty}_\sigma(\Omega)$ with $\|u_s\|_{L^{3,\infty}(\Omega)} \le \delta_2$ there is a unique mild solution $u \in \overline{B(0,R)} \subset L^\infty(\mathbb{R}; L^{3,\infty}_\sigma(\Omega))$ to (3.44). If additionally

- *$F \in C_0(\mathbb{R}; L^{3/2,\infty}(\Omega))$, then $u \in C_0(\mathbb{R}; L^{3,\infty}_\sigma(\Omega))$.*
- *$F \in \mathrm{BUC}(\mathbb{R}; L^{3/2,\infty}(\Omega))$, then $u \in \mathrm{BUC}(\mathbb{R}; L^{3,\infty}_\sigma(\Omega))$.*
- *$F \in P_T(\mathbb{R}; L^{3/2,\infty}(\Omega))$ for some $T > 0$, then $u \in P_T(\mathbb{R}; L^{3,\infty}_\sigma(\Omega))$.*
- *$F \in \mathrm{AAP}(\mathbb{R}; L^{3/2,\infty}(\Omega))$, then $u \in \mathrm{AAP}(\mathbb{R}; L^{3,\infty}_\sigma(\Omega))$.*
- *$F \in \mathrm{UAP}(\mathbb{R}; L^{3/2,\infty}(\Omega))$, then $u \in \mathrm{UAP}(\mathbb{R}; L^{3,\infty}_\sigma(\Omega))$.*

Proof. We want to apply Theorem 3.1.15. We begin by setting up all involved spaces and variables. Let

$$V := L^{3,1}(\Omega), \quad Z_1 := L^{q_1',1}_\sigma(\Omega), \quad Z_2 := L^{q_2',1}_\sigma(\Omega),$$

where $q_1, q_2 \in \mathbb{R}$ satisfy $3/2 \le q_2 < 3 < q_1 < \infty$, and fix $A = \mathcal{L}$ and $B = \mathbb{P}\,\mathrm{div}$. Note that this is a special case of the situation in Theorem 3.1.48. Thus, Assumptions 3.1.2 (ACPEx) are satisfied. It remains to show the Lipschitz condition for the nonlinearity. Beforehand, we note that the relevant solution space was also computed in the proof of Theorem 3.1.48 as $L^{3,\infty}_\sigma(\Omega)$. We define $G(v)(t) := F(t) - u_s \otimes u(t) - u(t) \otimes u_s - u(t) \otimes u(t)$. It is easy to see by Hölder's inequality that G is well defined as a map from $L^{3,\infty}_\sigma(\Omega)$ to $L^{3/2,\infty}(\Omega)$. Let M be the constant from Theorem 3.1.48. We assume that $R > 0$ satisfies $R < 1/4M$ and set $L := 4R$. Then if $\|u_s\|_{L^{3,\infty}(\Omega)} \le R$ and $\|f\|_{L^\infty(\mathbb{R};L^{3/2,\infty}(\Omega))} \le R(1 - LM)/M =: R\delta_1$, the estimates (3.8) and (3.9) are fulfilled. Indeed, for any $v_1, v_2 \in \overline{B(0,R)} \subset L^\infty(\mathbb{R}; L^{3,\infty})$ it holds

$$\begin{aligned}
&\|G(v_1) - G(v_2)\|_{L^\infty(\mathbb{R};L^{3/2,\infty}(\Omega))} \\
&\leq \|u_s \otimes (v_1 - v_2) + (v_1 - v_2) \otimes u_s + v_1 \otimes v_1 - v_2 \otimes v_2\|_{L^\infty(\mathbb{R};L^{3/2,\infty}(\Omega))} \\
&\leq \|u_s \otimes (v_1 - v_2) + (v_1 - v_2) \otimes u_s \\
&\quad + (v_1 - v_2) \otimes v_1 - (v_2 - v_2) \otimes v_2\|_{L^\infty(\mathbb{R};L^{3/2,\infty}(\Omega))} \\
&\leq (2\|u_s\|_{L^\infty(\mathbb{R};L^{3,\infty}(\Omega))} + \|v_1\|_{L^\infty(\mathbb{R};L^{3,\infty}(\Omega))} + \|v_2\|_{L^\infty(\mathbb{R};L^{3,\infty}(\Omega))}) \\
&\quad \times \|v_1 - v_2\|_{L^\infty(\mathbb{R};L^{3,\infty}(\Omega))} \\
&\leq (\tfrac{2}{8M} + \tfrac{2}{4M})\|v_1 - v_2\|_{L^\infty(\mathbb{R};L^{3,\infty}(\Omega))} \\
&\leq 4R\|v_1 - v_2\|_{L^\infty(\mathbb{R};L^{3,\infty}(\Omega))} \\
&\leq L\|v_1 - v_2\|_{L^\infty(\mathbb{R};L^{3,\infty}(\Omega))}.
\end{aligned}$$

Thus, we can apply Theorem 3.1.15 and get the existence of a unique mild solution $u \in \overline{B(0,R)} \subset L^\infty(\mathbb{R}; L^{3,\infty}_\sigma(\Omega))$ to (3.44). If additionally F lies in $\in C_0(\mathbb{R}; L^{3/2,\infty}(\Omega))$, it is easy to see that G maps from $C_0(\mathbb{R}; L^{3,\infty}_\sigma(\Omega))$ to $C_0(\mathbb{R}; L^{3/2,\infty}(\Omega))$. By the second part of Theorem 3.1.15, it follows that the corresponding mild solution u lies in $C_0(\mathbb{R}; L^{3,\infty}_\sigma(\Omega))$. The other cases can be treated in the same way. □

We will now investigate the stability of mild solutions to the problem considered above. Therefore, we will introduce the initial value problem

$$\begin{aligned}
&u'(t) - \mathcal{L}u(t) \\
&\qquad = \mathbb{P}(\operatorname{div} F(t) - u_s \otimes u(t) - u(t) \otimes u_s - u(t) \otimes u(t)), \quad t > 0, \\
&u(0) = u_0.
\end{aligned} \tag{3.47}$$

Theorem 3.1.50. *There are constants $\delta_1, \delta_2, \delta_3, R > 0$ such that for each $F \in L^\infty(\mathbb{R}; L^{3/2,\infty}(\Omega))$, $a, b \in L^{3,\infty}_\sigma(\Omega)$ and $u_s \in L^{3,\infty}_\sigma(\Omega)$ with $\|F\|_{L^\infty(\mathbb{R};L^{3/2,\infty}(\Omega))} \leq \delta_1$, $\|a\|_{L^{3,\infty}(\Omega)}, \|b\|_{L^{3,\infty}(\Omega)} \leq \delta_2$ and $\|u_s\|_{L^{3,\infty}(\Omega)} \leq \delta_3$ there are unique mild solutions $u, v \in \overline{B(0,R)} \subset L^\infty(\mathbb{R}; L^{3,\infty}_\sigma(\Omega))$ to (3.47) with initial values $u_0 = a$ and $u_0 = b$ respectively. Furthermore, for each $\tilde{q} \in (3, \infty)$ there is a constant $C = C(\tilde{q}) > 0$ such that*

$$\|u(t) - v(t)\|_{L^{\tilde{q},\infty}(\Omega)} \leq Ct^{-\frac{3}{2}(\frac{1}{3} - \frac{1}{\tilde{q}})}, \quad t > 0. \tag{3.48}$$

Proof. We will apply Theorem 3.1.31 and have to check the Assumptions 3.1.21 (IVPEx) first. Let

$$V := L^{3,1}(\Omega), \quad Z_1 := L^{q_1',1}_\sigma(\Omega), \quad Z_2 := L^{q_2',1}_\sigma(\Omega),$$

where $3/2 \leq q_2 < 3 < q_1 < \infty$, and $B = \mathbb{P}\operatorname{div}$. It is clear, that Z_1 and Z_2 form an interpolation couple of Banach spaces and that $Z_1 \cap Z_2$ is dense in Z_1 and Z_2. Because of Lemma 3.1.47 we know that $\mathcal{L}$ generates bounded and consistent C_0-semigroups on Z_1 and Z_2 which satisfy (3.20) with $\alpha_i = \frac{1}{2} + \frac{3}{2}(\frac{1}{q_i'} - \frac{1}{p'})$. Thus, the Assumptions 3.1.21 (IVPEx) are satisfied. Concerning Assumptions 3.1.27 (IVPStab), let

$$\tilde{V} := L^{\tilde{p},1}(\Omega), \quad \tilde{Z}_1 := L_\sigma^{\tilde{q}_1',1}(\Omega), \quad \tilde{Z}_2 := L_\sigma^{\tilde{q}_2',1}(\Omega),$$

where $\tilde{p} = \frac{3\tilde{q}}{\tilde{q}+3}$ and $\tilde{p} \leq \tilde{q}_2 < \tilde{q} < \tilde{q}_1 < \infty$. The estimates (3.25) are satisfied again by Lemma 3.1.47 with $\alpha_i = \frac{1}{2} + \frac{3}{2}(\frac{1}{\tilde{q}_i'} - \frac{1}{\tilde{p}'})$. Furthermore, $\tilde{Z}_1$ and $\tilde{Z}_2$ is an interpolation couple of separable Banach spaces for which $\tilde{Z}_1 \cap \tilde{Z}_2$ is dense in $\tilde{Z}_1$ and $\tilde{Z}_2$. Finally, the estimate (3.26) is satisfied by Lemma 3.1.47 with $\gamma = \frac{3}{2}(\frac{1}{\tilde{q}'} - \frac{1}{q'})$. Hence, the Assumptions 3.1.27 (IVPStab) are fulfilled, too. We have to check the Lipschitz conditions on the nonlinearity. Note that for $\theta, \tilde{\theta} \in (0,1)$ satisfying $1 = (1-\theta)\alpha_1 + \theta\alpha_2$ and $1 = (1-\tilde{\theta})\tilde{\alpha}_1 + \tilde{\theta}\tilde{\alpha}_2$ it holds

$$Y := (Z_1', Z_2')_{\theta,\infty} = L_\sigma^{3,\infty}(\Omega) \quad \text{and} \quad \tilde{Y} := (\tilde{Z}_1', \tilde{Z}_2')_{\tilde{\theta},\infty} = L^{\tilde{q},\infty}(\Omega),$$

and that

$$X := V' = L^{3/2,\infty}(\Omega) \quad \text{and} \quad \tilde{X} := \tilde{V}' = L^{\tilde{p},\infty}(\Omega).$$

Define $G(v)(t) := F(t) - u_s \otimes u(t) - u(t) \otimes u_s - u(t) \otimes u(t)$. Due to Hölder's inequality, one can see easily that $G\colon L^\infty(\mathbb{R}; Y) \to L^\infty(\mathbb{R}; X)$ is well defined. Let $M_0, M, K > 0$ be the constants from Theorem 3.1.23 and Lemma 3.1.29. Let $R > 0$ satisfy $R < M/4$ and set $L := \tilde{L} := 4R$. Furthermore, we assume that $\|u_s\|_{L^{3,\infty}(\Omega)} \leq R$ and that f and some $R_0 > 0$ satisfy $M\|f\|_{L^\infty(\mathbb{R}_+; L^{3/2,\infty}(\Omega))} + M_0 R_0 \leq R(1 - LM)$. Then for any $v_1, v_2 \in \overline{B(0,R)} \subset L^\infty(\mathbb{R}_+; L_\sigma^{3,\infty}(\Omega))$ it holds

$$\begin{aligned}
&\|G(v_1) - G(v_2)\|_{L^\infty(\mathbb{R}; L^{3/2,\infty}(\Omega))} \\
&\leq \|u_s \otimes (v_1 - v_2) + (v_1 - v_2) \otimes u_s + v_1 \otimes v_1 - v_2 \otimes v_2\|_{L^\infty(\mathbb{R}; L^{3/2,\infty}(\Omega))} \\
&\leq \|u_s \otimes (v_1 - v_2) + (v_1 - v_2) \otimes u_s \\
&\quad + (v_1 - v_2) \otimes v_1 - (v_2 - v_2) \otimes v_2\|_{L^\infty(\mathbb{R}; L^{3/2,\infty}(\Omega))} \\
&\leq (2\|u_s\|_{L^\infty(\mathbb{R}; L^{3,\infty}(\Omega))} + \|v_1\|_{L^\infty(\mathbb{R}; L^{3,\infty}(\Omega))} \\
&\quad + \|v_2\|_{L^\infty(\mathbb{R}; L^{3,\infty}(\Omega))})\|v_1 - v_2\|_{L^\infty(\mathbb{R}; L^{3,\infty}(\Omega))} \\
&\leq (\tfrac{2}{8M} + \tfrac{2}{4M})\|v_1 - v_2\|_{L^\infty(\mathbb{R}; L^{3,\infty}(\Omega))} \\
&\leq 4R\|v_1 - v_2\|_{L^\infty(\mathbb{R}; L^{3,\infty}(\Omega))} \\
&\leq L\|v_1 - v_2\|_{L^\infty(\mathbb{R}; L^{3,\infty}(\Omega))}.
\end{aligned}$$

If additionally $v_1 - v_2 \in L^\infty(\mathbb{R}_+; L^{\tilde{q},\infty}(\Omega))$, we get

$$\begin{aligned}
&\|G(v_1) - G(v_2)\|_{L^\infty_\gamma(\mathbb{R};L^{\tilde{p},\infty}(\Omega))} \\
&\le \|u_s \otimes (v_1 - v_2) + (v_1 - v_2) \otimes u_s + v_1 \otimes v_1 - v_2 \otimes v_2\|_{L^\infty_\gamma(\mathbb{R};L^{\tilde{p},\infty}(\Omega))} \\
&\le \|u_s \otimes (v_1 - v_2) + (v_1 - v_2) \otimes u_s \\
&\quad + (v_1 - v_2) \otimes v_1 - (v_2 - v_2) \otimes v_2\|_{L^\infty_\gamma(\mathbb{R};L^{\tilde{p},\infty}(\Omega))} \\
&\le (2\|u_s\|_{L^\infty(\mathbb{R};L^{3,\infty}(\Omega))} + \|v_1\|_{L^\infty(\mathbb{R};L^{3,\infty}(\Omega))} \\
&\quad + \|v_2\|_{L^\infty(\mathbb{R};L^{3,\infty}(\Omega))})\|v_1 - v_2\|_{L^\infty_\gamma(\mathbb{R};L^{\tilde{q},\infty}(\Omega))} \\
&\le (\tfrac{2}{8M} + \tfrac{2}{4M})\|v_1 - v_2\|_{L^\infty_\gamma(\mathbb{R};L^{\tilde{q},\infty}(\Omega))} \\
&\le 4R\|v_1 - v_2\|_{L^\infty_\gamma(\mathbb{R};L^{\tilde{q},\infty}(\Omega))} \\
&\le \tilde{L}\|v_1 - v_2\|_{L^\infty_\gamma(\mathbb{R};L^{\tilde{q},\infty}(\Omega))}.
\end{aligned}$$

Thus, if $a, b \in L^{3,\infty}_\sigma(\Omega)$ satisfy $\|a\|_{L^{3,\infty}(\Omega)}, \|a\|_{L^{3,\infty}(\Omega)} \le R_0$, the Assumptions 3.1.24 (sIVPEx) and Assumptions 3.1.30 (sIVPStab) are fulfilled. Therefore, Theorem 3.1.31 yields the desired existence and stability of mild solutions. □

Remark 3.1.51. *The estimate* (3.48) *can be improved by an additional interpolation argument to*

$$\|u(t) - v(t)\|_{L^{\tilde{q},1}(\Omega)} \le Ct^{-\frac{3}{2}(\frac{1}{3}-\frac{1}{\tilde{q}})}, \quad t > 0.$$

In particular, if $f = 0$ and $a = 0$ we rediscover Theorem 1.3 (2) of [HS09].

Nonautonomous Parabolic Evolution Equations

This subsection is devoted to the nonautonomous parabolic equation

$$u'(t,x) - A(t)u(t,x) = f(t,x), \quad t \in \mathbb{R},\ x \in \mathbb{R}^d, \tag{3.49}$$

where

$$A(t)\phi(x) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_k}[a_{kj}(t,x)\frac{\partial}{\partial x_j}u](t,x).$$

We assume that $a \in L^\infty(\mathbb{R} \times \mathbb{R}^d)$ is pointwise symmetric and that there is a constant $\mu > 0$ such that

$$\sum_{i,j=1}^{d} a(t,x)\xi_i\xi_j \ge \mu|\xi|^2$$

for any $t \in \mathbb{R}$ and $x, \xi \in \mathbb{R}^d$. It is known that the corresponding initial value problem

$$\begin{aligned} u'(t,x) - A(t)u(t,x) &= f(t,x), \quad t > s \in \mathbb{R}, x \in \mathbb{R}^d, \\ u(s,x) &= u_s(x), \quad x \in \mathbb{R}^d, \end{aligned}$$

admits a unique bounded weak solution u which is given by some integral kernel k, i.e.

$$u(t,x) = \int_{\mathbb{R}^d} k(t,s,x,y) u_s(y) \, \mathrm{d}y.$$

Moreover, this kernel admits Gaussian bounds. Defining the operator family

$$U(t,s)f(x) := \int_{\mathbb{R}^d} k(t,s,x,y) f(y) \, \mathrm{d}y,$$

it was shown in [FS86, Theorem 1.6] and [RS99, Lemma 5.1] that this family can be extended to an evolution family on $L^p(\mathbb{R}^d)$. More precisely, we have:

Lemma 3.1.52. *Let $1 < p < \infty$. Then the operator family $(U(t,s))_{t \geq s}$ is an evolution family. Moreover, for each $1 < p \leq q < \infty$ there is a constant $C(p,q,d) > 0$ such that*

$$\|U(t,s)\phi\|_{L^q} \leq C(p,q,d)(t-s)^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \|\phi\|_{L^p}, \quad -\infty < s < t < \infty,$$

for any $\phi \in L^p(\mathbb{R}^d)$.

Using the approach from Section 3.1.4 with Theorem 3.1.15 instead of Theorem 3.1.4 yields the following existence result.

Theorem 3.1.53. *Let $2 < d \in \mathbb{N}$, $1 < p < d/2$. Let $f \in L^\infty(\mathbb{R}; L^{p,\infty}(\mathbb{R}^d))$. Then the equation (3.49) has a unique mild solution $u \in L^{q,\infty}(\mathbb{R}; L^{q,\infty}(\mathbb{R}; \mathbb{R}^d))$, where $q = \frac{pd}{d-2p}$. Additionally, there is an $M > 0$ such that*

$$\|u\|_{L^{q,\infty}(\mathbb{R}; L^{q,\infty}(\mathbb{R}^d))} \leq M \|f\|_{L^\infty(\mathbb{R}; L^{p,\infty}(\mathbb{R}^d))}.$$

In the same way one can consider the semilinear equation

$$u'(t,x) - A(t)u(t,x) = u^m(t) + f(t,x), \quad t \in \mathbb{R}, x \in \mathbb{R}^d, \tag{3.50}$$

for some given $1 < m \in \mathbb{N}$. In this case, we obtain the analogous counterpart to Theorem 3.1.35.

Theorem 3.1.54. *Let $2 < d \in \mathbb{N}$, $1 < m \in \mathbb{N}$ and assume that $\frac{d(m-1)}{2m} > 1$. Set $p := \frac{d(m-1)}{2m}$ and $q := \frac{d(m-1)}{2}$. Then there exists a constant $R_0 > 0$ such that for each $0 < R \leq R_0$, $f \in L^\infty(\mathbb{R}; L^{p,\infty}(\Omega))$ with*

$$\|f\|_{L^\infty(\mathbb{R};L^{p,\infty}(\Omega))} \leq \frac{R}{4M}$$

there exists a unique mild solution $u \in L^\infty(\mathbb{R}; L^{q,\infty}(\Omega))$ to (3.50) in the ball $\overline{B(0,R)} \subset L^\infty(\mathbb{R}; L^{q,\infty}(\Omega))$. If additionally $f \in P_T(\mathbb{R}; L^{p,\infty}(\Omega))$ for some $T > 0$, then $u \in P_T(\mathbb{R}; L^{q,\infty}(\Omega))$.

The stability results of Theorem 3.1.36 are valid here in the same way, too. The corresponding initial value problem to (3.50) reads as

$$\begin{aligned} u'(t,x) - A(t)u(t,x) &= u^m(t) + f(t,x), \quad t > 0,\ x \in \mathbb{R}^d, \\ u(0,x) &= u_o(x), \quad x \in \mathbb{R}^d. \end{aligned} \tag{3.51}$$

Theorem 3.1.55. *Let $2 < d \in \mathbb{N}$, $1 < m \in \mathbb{N}$ and assume that $\frac{d(m-1)}{2m} > 1$. Set $p := \frac{d(m-1)}{2m}$ and $q := \frac{d(m-1)}{2}$. Then there are constants $\delta_1, \delta_2, R > 0$ such that if $f \in L^\infty(\mathbb{R}_+; L^{p,\infty}(\Omega))$ and $a, b \in L^{q,\infty}(\Omega)$ satisfy*

$$\|f\|_{L^\infty(\mathbb{R}_+;L^{p,\infty}(\Omega))} \leq \delta_1 \quad \text{and} \quad \|a\|_{L^{q,\infty}(\Omega)}, \|a\|_{L^{q,\infty}(\Omega)} \leq \delta_2,$$

then there are mild solutions $u, v \in \overline{B(0,R)} \subset L^\infty(\mathbb{R}_+; L^{q,\infty}(\Omega))$ to (3.51) with initial value $u_0 = a$ and $u_0 = b$ respectively. Moreover, for each $\tilde{q} > q$, there is a constant $C > 0$ such that

$$\|u(t) - v(t)\|_{L^{\tilde{q},\infty}} \leq Ct^{-\frac{d}{2}(\frac{1}{q} - \frac{1}{\tilde{q}})}, \quad t > 0.$$

Non-autonomous Ornstein-Uhlenbeck Equations

Consider the non-autonomous problem

$$u(t,x) - \mathcal{L}(t)u(t,x) = f(t,x), \quad t \in \mathbb{R},$$

where

$$\mathcal{L}(t)f(x) = \frac{1}{2}\operatorname{Tr}(Q(t)Q^*(t)D_x^2 f(x)) + \langle M(t)x + c(t), D_x\phi(x)\rangle, \quad x \in \mathbb{R}^d, t \in \mathbb{R}.$$

We assume that $Q, M \colon \mathbb{R} \to \mathbb{R}^{d\times d}$ and $c : \mathbb{R} \to \mathbb{R}^d$ are bounded and Hölder continuous. Furthermore, we assume that there is a constant $c > 0$ such that

$$|Q(t)x| \geq c|x|, \quad t \in \mathbb{R}, x \in \mathbb{R}^d.$$

Finally, we suppose that $M(t)$ has only eigenvalues $\lambda(t)$ with $\mathrm{Re}(\lambda) \leq 0$ and that the geometric and algebraic multiplicity of those eigenvalues with $\mathrm{Re}(\lambda(t)) = 0$ do coincide. Define the L^p-realization $L(t)$ of $\mathcal{L}(t)$ by

$$\begin{aligned} D(L(t)) &:= \{u \in W^{2,p}(\mathbb{R}^d) : \langle M(t)x, D_x u(x)\rangle \in L^p(\mathbb{R}^d)\}, \\ L(t)u &:= \mathcal{L}(t)u, \quad u \in D(L(t)). \end{aligned}$$

Hence, the problem can be rewritten as

$$u'(t) - L(t)u(t) = f(t), \quad t \in \mathbb{R}. \tag{3.52}$$

The following result is due to [HR11, Proposition 2.1 and Proposition 2.4]:

Lemma 3.1.56. *Let $1 < p < \infty$. Then the equation*

$$u'(t) - L(t)u(t) = 0, \quad t \in \mathbb{R}$$

is well-posed in $L^p(\mathbb{R}^d)$ and the evolution family $(U(t,s))_{t\geq s}$ is given by an integral kernel $k\colon \mathbb{R} \times \mathbb{R}^d$ via

$$U(t,s)f(x) = \int_{\mathbb{R}^d} k(t,s,x,y)f(y)\,\mathrm{d}y.$$

Moreover, for any $1 < p \leq q < \infty$ the evolution family $(U(t,s))_{t\geq s}$ is subject to the estimate

$$\|U(t,s)\phi\|_{L^q} \leq C(p,q,d)(t-s)^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}\|\phi\|_{L^p}, \quad -\infty < s < t < \infty,$$

for any $\phi \in L^p$.

The estimates given here are of the same kind as in the last subsection. Therefore, we are able to get the following analogous statements for the linear problem (3.52), the semilinear problem

$$u'(t) - L(t)u(t) = u^m(t) + f(t), \quad t \in \mathbb{R} \tag{3.53}$$

and the initial value problem

$$\begin{aligned} u'(t) - L(t)u(t) &= u^m(t) + f(t), \quad t > 0, \\ u(0) &= u_0. \end{aligned} \tag{3.54}$$

Theorem 3.1.57. *The Theorems 3.1.53, 3.1.54 and 3.1.55 are valid for* (3.52), (3.53) *and* (3.54) *respectively, too.*

3.2 Strong Solutions

In this section, we are going to consider the solvability of quasilinear parabolic equations of the form

$$u'(t) - A(u(t))u(t) = N(t, u(t)), \quad t \in \mathbb{R}, \tag{3.55}$$

with $A(u)$ being a linear operator for each u and $N(t, u)$ being some nonlinear term. We will do so by means of maximal continuous regularity. More precisely, we require for a suitable Banach space X that for each $f \in \mathrm{BUC}(\mathbb{R}; X)$, there is a unique $u \in \mathrm{BUC}(\mathbb{R}; D(A(0)) \cap \mathrm{BUC}^1(\mathbb{R}; X)$ that solves

$$u'(t) - A(0)u(t) = f(t), \quad t \in \mathbb{R}.$$

Adapting the method used in Section 3.1, such an assumption turns out to be satisfied for $A(0)$ being the generator of an exponentially stable bounded analytic semigroup in some Banach space E, if one chooses X to be a continuous interpolation space of E and $D(A(0))$. This goes hand in hand with classical results for the corresponding initial value problem, see [DPG79], [Ang90], [CS01] or the monograph [Ama95, Section III.3]. The quasilinear problem will then be solved under some Lipschitz condition to the maps $u \mapsto A(u)$ and $u \mapsto N(\cdot, u)$ via a fixed point argument. As in the case of mild solutions in Section 3.1, having a solution on the whole real time axis will also allow us to give sufficient conditions for time periodic and almost periodic solutions to (3.55). In the last part of this chapter, we will apply a similar argument for the construction of time periodic solutions to (3.55) by means of maximal periodic L^p-regularity.

3.2.1 Maximal Continuous Regularity and Quasilinear Equations on the Whole Real Axis

Adapting the techniques from Theorem 3.1.4 for mild solutions on the whole real axis, it is possible to gain results regarding maximal continuous regularity on the whole real axis that resemble the classical results of Da Prato and Grisvard [DPG79] and Clement and Simonett [CS01] for the initial value problem. Here, maximal continuous regularity is defined as follows:

Definition 3.2.1. *Let X, Y be Banach spaces and $A\colon Y \subseteq X \to X$ be a densely defined closed operator. The pair (X, Y) has the property of maximal continuous regularity for A if the operator $\frac{d}{dt} - A$ is an isomorphism from $\mathrm{BUC}^1(\mathbb{R}; X) \cap \mathrm{BUC}(\mathbb{R}; Y)$ to $\mathrm{BUC}(\mathbb{R}; X)$.*

In other words, (X, Y) is a pair of maximal continuous regularity of A if for each $f \in \mathrm{BUC}(\mathbb{R}; X)$, there is a unique solution $u \in \mathrm{BUC}(\mathbb{R}; Y) \cap \mathrm{BUC}^1(\mathbb{R}; X)$

to the equation

$$u'(t) - Au(t) = f(t), \quad t \in \mathbb{R}.$$

Due to the closed graph theorem, this implies the existence of a constant $M > 0$ independent of f such that

$$\|u\|_{\mathrm{BUC}^1(\mathbb{R};X)\cap \mathrm{BUC}(\mathbb{R};Y)} \leq M\|f\|_{\mathrm{BUC}(\mathbb{R};X)}.$$

If X and Y are clear from the context, we will also just say that A admits maximal continuous regularity.

Note that we can regard the operator A in the definition above as an element of $\mathcal{L}(Y, X)$. Usually, it holds $Y = D(A)$. We introduce some notation for function spaces that will appear frequently when dealing with maximal continuous regularity. Let X be a Banach space and let $A\colon D(A) \subseteq X \to X$ be a closed linear operator. Assume that $(X, D(A))$ is a pair of maximal continuous regularity. The spaces appearing in the definition of maximal continuous regularity will be denoted by

$$\begin{aligned} \mathrm{MR}_c &:= \mathrm{BUC}^1(\mathbb{R}; X) \cap \mathrm{BUC}(\mathbb{R}; D(A)), \\ X_c &:= \mathrm{BUC}(\mathbb{R}; X). \end{aligned}$$

In the context here, we will only deal with linear operators A on some Banach space X that are generators of exponentially decaying bounded analytic semigroups. If we considered maximal regularity for the initial value problem, this would be no restriction, except for the decay, as maximal continuous regularity for the initial value problem already implies that A is the generator of a C_0-semigroup and that this semigroup is analytic (see [LS11, Theorem 2.2]). However, for maximal regularity on the whole real axis, such an implication cannot generally hold, as maximal continuous regularity for A implies the same for $-A$. Indeed, assume that A admits maximal continuous regularity and consider the equation

$$v'(t) + Av(t) = f(t), \quad t \in \mathbb{R}, \tag{3.56}$$

with some $f \in X_c$. Set $g(t) := -f(-t)$. Then the problem

$$u'(t) - Au(t) = g(t), \quad t \in \mathbb{R}$$

admits a unique solution $u \in \mathrm{MR}_c$. Setting $v(t) := u(-t)$, it is easy to see that v is a unique solution to (3.56).

In the context of the initial value problem, it is known that if (X, Y) is a pair of maximal continuous regularity for A, then $X = Y$ or X contains a closed subspace which is topologically linear isomorphic to the space of null sequences c_0. This has been shown in [Bai80]. Under our general assumptions, this result is also relevant for the setting here on the whole real time axis, as A having maximal continuous regularity and A generating an exponentially decaying semigroup imply that the corresponding initial value problem admits maximal continuous regularity on $\mathbb{R}_+$:

Proposition 3.2.2. *Let $(X, D(A))$ be a pair of maximal continuous regularity for a linear operator A. Suppose that A is the generator of an exponentially decaying C_0-semigroup. Then for each $f \in \mathrm{BUC}(\mathbb{R}_+; X)$ there is a unique solution $u \in \mathrm{BUC}^1(\mathbb{R}_+; X) \cap \mathrm{BUC}(\mathbb{R}_+; Y)$ to the problem*

$$\begin{aligned} v'(t) - Av(t) &= f(t), \quad t > 0, \\ v(0) &= 0. \end{aligned} \tag{3.57}$$

Proof. Let $\tilde{f} \in X_c$ be such that $\tilde{f}|_{\mathbb{R}_+} = f$ and $\|\tilde{f}\|_{X_c} = \|f\|_{\mathrm{BUC}(\mathbb{R}_+;X)}$. Then, by the maximal continuous regularity of A, there is a unique $u \in \mathrm{MR}_c$, which solves

$$u'(t) - Au(t) = \tilde{f}, \quad t \in \mathbb{R}.$$

As $u(0) \in D(A)$ and $(e^{tA})_{t\geq 0}$ is exponentially bounded, we also have $w := e^{\cdot A}u(0) \in \mathrm{MR}_c$. This implies $v := u - w \in \mathrm{MR}_c$. A simple computation shows that $v|_{\mathbb{R}_+}$ is a strong solution to (3.57). The uniqueness of that solution is clear by the uniqueness of mild solutions. □

We now describe a setting for which some operator A admits maximal continuous regularity. This setting coincides with the one in [Ang90] and [CS01]. Let E_0 be a Banach space and $A \in \mathrm{Hol}_\omega(E_0)$ for some $\omega \in (-\infty, 0)$. Note that the exponential decay of $(e^{tA})_{t\geq 0}$ implies that A is invertible. Hence, the spaces $E_1 := D(A)$ and $E_2 := D(A^2)$ equipped with the respective norms $\|A \cdot\|_{E_0}$ and $\|A^2 \cdot\|_{E_0}$ are well defined Banach spaces. Fix some $\theta \in (0, 1)$ and set

$$X := (E_0, E_1)^0_{\theta,\infty}, \quad Y := (E_1, E_2)^0_{\theta,\infty}.$$

Then we have the following result:

Theorem 3.2.3. *Let X, Y and A be as above. Then (X, Y) is a pair of maximal continuous regularity for A.*

The proof of the theorem above can be seen as a dual argument to the proof of Theorem 3.1.4. Before starting it, we note that $A \in \mathrm{Hol}_\omega(X)$ with domain $D(A) = Y$. See for example [Ama95, Section III.3.2].

Proof. Let $f \in \mathrm{BUC}(\mathbb{R}; X)$. We start by showing that there is a function $u \in \mathrm{BUC}(\mathbb{R}; Y)$ such that

$$\langle u(t), \phi \rangle = \int_{-\infty}^{t} \langle e^{(t-s)A} f(s), \phi \rangle \, \mathrm{d}s \quad \text{for all } \phi \in Y' \tag{3.58}$$

and that there is a constant $M > 0$ such that

$$\|u\|_{L^\infty(\mathbb{R};Y)} \leq M \|f\|_{L^\infty(\mathbb{R};X)}. \tag{3.59}$$

Here, we consider e^{tA} as an operator from X to Y and therefore $(e^{tA})'$ as an operator from Y' to X'. For a given $\phi \in Y'$, we have

$$\begin{aligned} \left| \int_{-\infty}^{t} \langle e^{(t-s)A} f(s), \phi \rangle \, \mathrm{d}s \right| &\leq \int_{-\infty}^{t} |\langle f(s), (e^{(t-s)A})' \phi \rangle| \, \mathrm{d}s \\ &\leq \int_{-\infty}^{t} \|f(s)\|_X \|(e^{(t-s)A})' \phi\|_{X'} \, \mathrm{d}s \\ &\leq \|f\|_{L^\infty(\mathbb{R};X)} \int_{-\infty}^{t} \|(e^{(t-s)A})' \phi\|_{X'} \, \mathrm{d}s \\ &\leq \|f\|_{L^\infty(\mathbb{R};X)} \int_{0}^{\infty} \|(e^{rA})' \phi\|_{X'} \, \mathrm{d}r. \end{aligned} \tag{3.60}$$

We will estimate the integral term by an interpolation argument. Let $0 < \theta_2 < \theta < \theta_1 < 1$ and set $Y_1 := (E_1, E_2)^0_{\theta_1,\infty}$, $Y_2 := (E_1, E_2)^0_{\theta_2,\infty}$ and $Z_1 := Y_1'$, $Z_2 := Y_2'$. We determine suitable norm estimates of $e^{tA} : X \to Y_i$. It holds that

$$\begin{aligned} \|e^{tA}\|_{X \to Y_i} &\leq \|e^{tA}\|^{1-\theta}_{\mathcal{L}(E_0,Y_i)} \|e^{tA}\|^{\theta}_{\mathcal{L}(E_1,Y_i)} \\ &\leq \big[\|e^{tA}\|^{1-\theta_i}_{\mathcal{L}(E_0,E_1)} \|e^{tA}\|^{\theta_i}_{\mathcal{L}(E_0,E_2)} \big]^{1-\theta} \big[\|e^{tA}\|^{1-\theta_i}_{\mathcal{L}(E_1,E_1)} \|e^{tA}\|^{\theta_i}_{\mathcal{L}(E_1,E_2)} \big]^{\theta} \\ &\leq C \big[t^{-(1-\theta_i)} t^{-2\theta_i} \big]^{1-\theta} t^{-\theta_i \theta} \\ &\leq C t^{-1+(\theta-\theta_i)}. \end{aligned}$$

Set $\alpha_i := 1 - (\theta - \theta_i)$. Note that $0 < \alpha_2 < 1 < \alpha_1$. By duality and the estimate above, this implies

$$\|(e^{tA})' \psi\|_{X'} \leq C t^{-\alpha_i} \|\psi\|_{Z_i'} \quad \text{for all } \psi \in Z_i', i \in \{1, 2\}. \tag{3.61}$$

Hence, the sublinear operator

$$\begin{aligned} T \colon Z_1 + Z_2 &\to L^{1/\alpha_1,\infty}(\mathbb{R}_+; \mathbb{R}) + L^{1/\alpha_2,\infty}(\mathbb{R}_+; \mathbb{R}), \\ \psi_1 + \psi_2 &\mapsto \|e^{\cdot A} \psi_1\|_{X'} + \|e^{\cdot A} \psi_2\|_{X'} \end{aligned}$$

is well defined and there is a constant $M > 0$ such that

$$\|T\psi_i\|_{L^{1/\alpha_i,\infty}} \le K\|\psi_i\|_{Z_i} \quad \text{for all } \psi_i \in Z_i, i \in \{1,2\}.$$

Set

$$\eta = \frac{\theta_1 - \theta}{\theta_1 - \theta_2}.$$

Then it holds $\alpha_1(1-\eta)+\alpha_2\eta = 1$. Thus, the interpolation theorem of Marcinkiewicz yields that the operator T maps from $Z = (Z_1, Z_2)_{\eta,1}$ to $(L^{1/\alpha_1,\infty}, L^{1/\alpha_2,\infty})_{\eta,1} = L^1$ and

$$\|T\psi\|_{L^1} \le K\|\psi\|_Z \quad \text{for all } \psi \in Z. \tag{3.62}$$

Note that $Z = Y'$. Indeed, using the reiteration theorem 1.1.7 we can see

$$Y = (E_1, E_2)^0_{\theta,\infty} = (E_1, E_2)^0_{(1-\eta)\theta_1+\eta\theta_2,\infty} = (Y_1, Y_2)^0_{\eta,\infty}.$$

Since $A \in \mathrm{Hol}_\omega(E_0)$, we know that $D(A^k)$ is dense in E_0 for any $k \in \mathbb{N}$. Hence, $Y_1 \cap Y_2$ is dense in Y_1 and in Y_2. By Lemma 1.1.8, this implies that

$$Y' = [(Y_1, Y_2)^0_{\eta,\infty}]' = (Y_1', Y_2')_{\eta,1} = (Z_1, Z_2)_{\eta,1} = Z.$$

Combining estimate (3.62) with (3.60) and taking the supremum over all $\phi \in Z$ yields that the function u in (3.58) is well defined with values in Y'' and satisfies the bound

$$\|u\|_{L^\infty(\mathbb{R};Y'')} \le M\|f\|_{L^\infty(\mathbb{R};X)}. \tag{3.63}$$

We make sure that u is uniformly continuous with values in Y''. Let us denote the operator $f \mapsto u$ by S. Then it holds for all $t, \tau \in \mathbb{R}$ and $\phi \in Z$ that

$$\begin{aligned} \langle S(f(\cdot))(t+\tau), \phi\rangle &= \langle u(t+\tau), \phi\rangle \\ &= \int_{-\infty}^{t+\tau} \langle e^{(t+\tau-s)A} f(s), \phi\rangle \,\mathrm{d}s \\ &= \int_{-\infty}^{t} \langle e^{(t+\tau-(r+\tau))A} f(r+\tau), \phi\rangle \,\mathrm{d}r \\ &= \langle S(f(\cdot+\tau))(t), \phi\rangle. \end{aligned} \tag{3.64}$$

Let now $\epsilon > 0$. As f is uniformly continuous there is a $\delta > 0$ such that $\|f(t+\tau) - f(t)\|_X < \epsilon$ for each $t \in \mathbb{R}$ and $|\tau| < \delta/M$. Together with (3.63) and (3.64), this implies

$$\begin{aligned} \|u(t) - u(t+\tau)\|_Y &= \|S(f(\cdot))(t) - S(f(\cdot))(t+\tau)\|_Y \\ &= \|S(f(\cdot))(t) - S(f(\cdot+\tau))(t)\|_Y \\ &\le M\|f(\cdot) - f(\cdot+\tau)\|_{L^\infty(\mathbb{R};X)} \\ &< \epsilon \end{aligned}$$

for all $|\tau| < \delta$. Thus we have $u \in \mathrm{BUC}(\mathbb{R}; Y'')$.

We still have to make sure that u actually takes values in Y. We will do so by an approximation argument. It is a consequence of the density of E_1 in E_0 that E_1 is dense in X. Therefore, it also holds, that $\mathrm{BUC}(\mathbb{R}; E_1)$ is dense in $\mathrm{BUC}(\mathbb{R}; X)$. Hence, there is a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathrm{BUC}(\mathbb{R}; E_1)$ that converges to f in $\mathrm{BUC}(\mathbb{R}; X)$. Let $u_n := S(f_n) \in \mathrm{BUC}(\mathbb{R}; Y'')$ be the corresponding mild solution to

$$u_n'(t) - Au_n(t) = f_n(t), \quad t \in \mathbb{R}. \tag{3.65}$$

Regarding u_n, it is easy to see that it has values in Y and not just in Y''. Indeed, as e^{tA} maps from E_1 to Y with norm bound

$$\|e^{tA}\|_{\mathcal{L}(E_1,Y)} \le Ct^{-\theta},$$

we obtain

$$\begin{aligned}
\|u_n(t)\|_Y &\le \int_{-\infty}^{t} \|e^{(t-s)A} f_n(s)\|_Y \,\mathrm{d}s \\
&\le \int_{t-1}^{t} \|e^{(t-s)A} f_n(s)\|_Y \,\mathrm{d}s + \int_{-\infty}^{t-1} \|e^{(t-s)A} f_n(s)\|_Y \,\mathrm{d}s \\
&\le c\int_{t-1}^{t} (t-s)^{-\theta} \|f(s)\|_{E_1} \,\mathrm{d}s + \int_{-\infty}^{t-1} \|e^{\frac{1}{2}(t-s)A} e^{\frac{1}{2}(t-s)A} f(s)\|_Y \,\mathrm{d}s \\
&\le c\int_{t-1}^{t} (t-s)^{-\theta} \|f(s)\|_{E_1} \,\mathrm{d}s + \int_{-\infty}^{t-1} e^{-\frac{\eta}{2}(t-s)} \big(\tfrac{t-s}{2}\big)^{-\theta} \|f(s)\|_{E_1} \,\mathrm{d}s \\
&< \infty,
\end{aligned}$$

where $\eta \in (\omega, 0)$. Because of the isometric embedding $Y \hookrightarrow Y''$, this implies $u_n \in \mathrm{BUC}(\mathbb{R}; Y)$ with the same norm as in $\mathrm{BUC}(\mathbb{R}; Y'')$. Since $\mathrm{BUC}(\mathbb{R}; Y)$ is a closed subspace of $\mathrm{BUC}(\mathbb{R}; Y'')$ and due to (3.63), we get $u \in \mathrm{BUC}(\mathbb{R}; Y)$ together with (3.59).

We now show that u is actually a strong solution of the problem

$$u(t) - Au(t) = f(t), \quad t \in \mathbb{R}. \tag{3.66}$$

Since the semigroup $(e^{tA})_{t\ge 0}$ is exponentially decaying, the integral

$$v(t) := \int_{-\infty}^{t} e^{(t-s)A} f(s) \,\mathrm{d}s$$

is well defined in X. Moreover, because of $Y \subseteq X$, we have $X' \subseteq Y'$. This implies for all $\phi \in X'$ that

$$\begin{aligned}\langle v(t), \phi \rangle_{X,X'} &= \langle \int_{-\infty}^{t} e^{(t-s)A} f(s)\,\mathrm{d}s, \phi \rangle_{X,X'} \\ &= \int_{-\infty}^{t} \langle e^{(t-s)A} f(s), \phi \rangle_{X,X'}\,\mathrm{d}s \\ &= \int_{-\infty}^{t} \langle e^{(t-s)A} f(s), \phi \rangle_{Y,Y'}\,\mathrm{d}s \\ &= \langle u(t), \phi \rangle_{Y,Y'}.\end{aligned}$$

Hence, $v(t) = u(t)$ in X for all $t \in \mathbb{R}$. Therefore, u can be represented in $\mathrm{BUC}(\mathbb{R}; X)$ by the classical Duhamel's formula. Indeed, let $t_0, t \in \mathbb{R}$ with $t_0 < t$. By the exponential decay of $(e^{tA})_{t\geq 0}$ it holds in X that

$$\begin{aligned}u(t) &= \int_{-\infty}^{t} e^{(t-s)A} f(s)\,\mathrm{d}s \\ &= \int_{-\infty}^{t_0} e^{(t-s)A} f(s)\,\mathrm{d}s + \int_{t_0}^{t} e^{(t-s)A} f(s)\,\mathrm{d}s \\ &= \int_{-\infty}^{t_0} e^{(t-t_0)A} e^{(t_0-s)A} f(s)\,\mathrm{d}s + \int_{t_0}^{t} e^{(t-s)A} f(s)\,\mathrm{d}s \qquad (3.67) \\ &= e^{(t-t_0)A} \int_{-\infty}^{t_0} e^{(t_0-s)A} f(s)\,\mathrm{d}s + \int_{t_0}^{t} e^{(t-s)A} f(s)\,\mathrm{d}s \\ &= e^{(t-t_0)A} u(t_0) + \int_{t_0}^{t} e^{(t-s)A} f(s)\,\mathrm{d}s.\end{aligned}$$

But as we already know that $u \in \mathrm{BUC}(\mathbb{R}; Y) = \mathrm{BUC}(\mathbb{R}; D(A))$, this implies $u' \in \mathrm{BUC}(\mathbb{R}; X)$ and that u is a strong solution to (3.66). Together with the estimate (3.59), this yields the maximal continuous regularity. $\square$

We now return to the abstract setting, as we will not need the specific case of Theorem 3.2.3. In the proof of that theorem, we constructed a solution that admits a variation of Duhamel's formula adjusted to the whole real axis. We will now show that any strong solution on the whole real axis admits such a representation, supposed that the semigroup $(e^{tA})_{t\geq 0}$ is exponentially stable:

Lemma 3.2.4. *Let X and Y be to Banach spaces and $A \in Hol_\omega(X)$ for some $\omega \in (-\infty, 0)$ be such that (X, Y) is a pair of maximal regularity for A. Let $f \in X_c$ and $u \in MR_c$ be a solution to*

$$u'(t) - Au(t) = f(t), \quad t \in \mathbb{R}. \qquad (3.68)$$

Furthermore, let v be a mild solution to that equation in the sense that

$$v(t) = \int_{-\infty}^{t} e^{(t-s)A} f(s)\, \mathrm{d}s$$

in X. Then $u = v$.

Proof. Let $t_0, t \in \mathbb{R}$ with $t_0 < t$. By the exponential decay of $(e^{tA})_{t\geq 0}$ it holds in X that

$$v(t) = e^{(t-t_0)A} v(t_0) + \int_{t_0}^{t} e^{(t-s)A} f(s)\, \mathrm{d}s,$$

which can be seen in the same way as in (3.67). Similarly, as u is a strong solution to (3.68), it can be written as

$$u(t) = e^{(t-t_0)A} u(t_0) + \int_{t_0}^{t} e^{(t-s)A} f(s)\, \mathrm{d}s.$$

Hence,

$$u(t) - v(t) = e^{(t-t_0)A}(u(t_0) - v(t_0)).$$

As $u \in \mathrm{MR}_c$, we know that $\|u(t)\|_X$ is uniformly bounded with respect to t. Due to the exponential decay of $(e^{tA})_{t\geq 0}$, the same can be said about $\|v(t)\|_X$. Thus, there exists some time independent constant $C > 0$ such that

$$\|u(t) - v(t)\|_X \leq C e^{-\eta(t-t_0)},$$

where $\eta \in (\omega, 0)$. Taking the limit $t_0 \to -\infty$ yields $u(t) = v(t)$. Since t was arbitrary, we obtain $u = v$. $\square$

The representation by Duhamel's formula allows us to deduce quite easily the decay at infinity of solutions, if the external forces also decay at infinity. It can also be used to do the same regarding periodicity in time, although maximal regularity would already be enough for said task.

Corollary 3.2.5. *Let X and Y be to Banach spaces and $A \in Hol_\omega(X)$ for some $\omega \in (-\infty, 0)$. Suppose that (X, Y) is a pair of maximal regularity for A. Let $f \in X_c$ and $u \in MR_c$ be a strong solution to*

$$u'(t) - Au(t) = f(t), \quad t \in \mathbb{R}. \tag{3.69}$$

If additionally $f \in C_0(\mathbb{R}; X)$, $P_T(\mathbb{R}; X)$, $\mathrm{UAP}(\mathbb{R}; X)$ or $\mathrm{AAP}(\mathbb{R}; X)$, then the same is the case for u, u' and Au.

Proof. Because of Lemma 3.2.4, we can express u via the formula

$$u(t) = \int_{-\infty}^{t} e^{(t-s)A} f(s)\,\mathrm{d}s.$$

If $f \in C_c(\mathbb{R}; X)$ with $\operatorname{supp} f \subset (a, b)$, where $a < b$ are some real numbers, then $u(t) = 0$ for all $t < a$. Additionally, for $t > b$ we get

$$\|u(t)\|_X = \| \int_a^b e^{(t-s)A} f(s)\,\mathrm{d}s\|_X \leq \|f\|_{X_c} \int_a^b \|e^{(t-s)A}\|_{\mathcal{L}(X,X)}\,\mathrm{d}s \to 0$$

for $t \to \infty$. Thus $u \in C_0(\mathbb{R}; X)$. Regarding Au, let $t_0 > b$ and $t > t_0$. Then it holds that

$$\begin{aligned}\|Au(t)\|_X &\leq \|Ae^{(t-t_0)A} u(t_0)\|_X + \|A \int_{t_0}^{t} e^{(t-s)A} f(s)\,\mathrm{d}s\|_X \\ &\leq Ct^{-(t-t_0)} \|u(t_0)\|_X + 0 \to 0\end{aligned}$$

for $t \to \infty$. Thus, we also have $Au \in C_0(\mathbb{R}; X)$. Because of $u' = f + Au$, this also means $u' \in C_0(\mathbb{R}; X)$. By approximation, this can be extended to $f \in C_0(\mathbb{R}; X)$. If $T > 0$ and $f \in P_T(\mathbb{R}; X)$, we get

$$\begin{aligned}u(t+T) &= \int_{-\infty}^{t+T} e^{(t+T-s)A} f(s)\,\mathrm{d}s \\ &= \int_{-\infty}^{t} e^{(t+T-(r+T))A} f(r+T)\,\mathrm{d}r \\ &= \int_{-\infty}^{t} e^{(t-r)A} f(r)\,\mathrm{d}r \\ &= u(t).\end{aligned}$$

Hence, u lies in $P_T(\mathbb{R}; X)$, too. It is clear, that this also implies $u', Au \in P_T(\mathbb{R}; X)$. Now let $f \in \mathrm{UAP}(\mathbb{R}; X)$ and let $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ be an arbitrary sequence. We have to show that $(u(\cdot + t_n))_{n\in\mathbb{N}}$ has a convergent subsequence in X_c. First, we note that a shift of the right-hand side f by some time s causes the solution u of (3.69) to shift in the same way. By the almost periodicity of f, there is a subsequence $(t_{n_k})_{k\in\mathbb{N}}$ of $(t_n)_{n\in\mathbb{N}}$ such that $(f(\cdot + t_{n_k}))_{k\in N}$ is a Cauchy sequence in X_c. By the linearity and boundedness of the solution operator to (3.58), as well as its mentioned shift property, we have

$$\|u(\cdot + t_{n_k}) - u(\cdot + t_{n_l})\|_{X_c} \leq M\|f(\cdot + t_{n_k}) - f(\cdot + t_{n_l})\|_{X_c}$$

for any $k, l \in \mathbb{N}$. This implies that $(u(\cdot + t_{n_k}))_{k\in\mathbb{N}}$ is a Cauchy sequence in X_c, too. Thus $u \in \mathrm{UAP}(\mathbb{R}; X)$. Due to [LZ82, Property 5, p.4], we directly obtain the almost periodicity of u'. Thus, we also have the almost periodicity of $Au = u' - f$. The remaining case of asymptotic almost periodicity follows by the direct decomposition $\mathrm{AAP}(\mathbb{R}; X) = \mathrm{UAP}(\mathbb{R}; X) \oplus C_0(\mathbb{R}; X)$ and the results for the respective summands. $\square$

Supposing suitable Lipschitz conditions on the nonlinearities, we will show the existence of strong solutions to the quasilinear equation

$$u'(t) - A(u(t))u(t) = N(t, u(t)), \quad t \in \mathbb{R}. \tag{3.70}$$

The precise assumptions read as follows:

Assumptions 3.2.6. *Let X be a Banach space.*

- *Let $A(\cdot)\colon D(A) \to \mathcal{L}(D(A), X)$ be a family of linear operators that are closed in X and have the same domain $D(A)$. Assume that $A(0)$ admits maximal continuous regularity for the pair $(X, D(A))$. Suppose that for each $R > 0$, there is a constant $L(R) > 0$ such that*

$$\|A(u) - A(v)\|_{\mathcal{L}(D(A),X)} \leq L\|u - v\|_{D(A)} \tag{3.71}$$

for all $u, v \in B(0, R) \subset D(A)$.

- *Let $N\colon \mathbb{R} \times D(A) \to X$ fulfil the following properties: For any $v \in MR_c$, the map $t \mapsto N(t, v(t))$ is an element of X_c and for each $R > 0$, there is a function $h_R \in L^\infty(\mathbb{R})$ such that*

$$\|N(t, u) - N(t, v)\|_X \leq h_R(t)\|u - v\|_{D(A)} \tag{3.72}$$

for all $u, v \in D(A)$ and all $t \in \mathbb{R}$.

Theorem 3.2.7. *Let X, $A(\cdot)$ and N be as in Assumption 3.2.6. Then there are $\delta_1, \delta_2, r > 0$ such that if $R > 0$ can be chosen to satisfy $\|h_R\|_{L^\infty} < \delta_1$ and $\|N(\cdot, 0)\|_{X_c} < \delta_2$, the equation (3.70) has a unique solution $u \in \overline{B(0, r)} \subset MR_c$.*

Proof. We are going to make use of a fixed point argument. Rewriting (3.70) as

$$u'(t) - A(0)u(t) = N(t, u(t)) - [A(0) - A(u(t))]u(t), \quad t \in \mathbb{R},$$

and exchanging u on the right-hand side by some $v \in \mathrm{MR}_c$ leads to

$$u'(t) - A(0)u(t) = N(t, v(t)) - [A(0) - A(v(t))]v(t), \quad t \in \mathbb{R}. \tag{3.73}$$

By our assumptions, for any $v \in \mathrm{MR}_c$, the right-hand side is an element of X_c. Since $A(0)$ admits maximal continuous regularity, there exists for any $v \in \mathrm{MR}_c$

a unique solution $u \in \mathrm{MR}_c$ to (3.73). We denote the solution operator by $\Phi(v) = u$. It is clear that a function $u \in \mathrm{MR}_c$ is a strong solution to (3.70) if and only if it is a fixed point of (3.73). We will show the existence of a fixed point of (3.73) for small data by the Banach fixed point theorem. Let $R > 0$ be arbitrary but fixed and $u, v \in \overline{B(0,R)} \subset \mathrm{MR}_c$. Then it holds that

$$
\begin{aligned}
\|\Phi(v)\|_{\mathrm{MR}_c} &\le M\big[\|N(\cdot,v(\cdot))\|_{X_c} + \|[A(0)-A(v)]v\|_{X_c}\big] \\
&\le M\|N(\cdot,v(\cdot)) - N(\cdot,0)\|_{X_c} + M\|N(\cdot,0)\|_{X_c} \\
&\quad + M\|A(0)-A(v(\cdot))\|_{\mathrm{BUC}(\mathbb{R};\mathcal{L}(D(A),X))}\|v\|_{\mathrm{MR}_c} \\
&\le M\|h_R\|_{L^\infty}\|v\|_{\mathrm{MR}_c} + M\|N(\cdot,0)\|_{X_c} + ML(R)\|v\|_{\mathrm{MR}_c}\|v\|_{\mathrm{MR}_c},
\end{aligned}
\tag{3.74}
$$

where we have used (3.71) and (3.72). Furthermore, for the difference of two solutions, we have

$$
\begin{aligned}
&\|\Phi(v)-\Phi(w)\|_{\mathrm{MR}_c} \\
&\le M\|N(\cdot,v(\cdot)) - N(\cdot,w(\cdot)) - [A(0)-A(v)]v + [A(0)-A(w)]w\|_{X_c} \\
&\le M\|h_R\|_{L^\infty}\|v-w\|_{\mathrm{MR}_c} \\
&\quad + M\|[A(0)-A(v)](v-w)\|_{X_c} + M\|[A(v)-A(w)]w\|_{X_c} \\
&\le M\|h_R\|_{L^\infty}\|v-w\|_{\mathrm{MR}_c} \\
&\quad + ML(R)\|v\|_{\mathrm{MR}_c}\|v-w\|_{\mathrm{MR}_c} + ML(R)\|v-w\|_{\mathrm{MR}_c}\|w\|_{\mathrm{MR}_c} \\
&\le M\big[\|h_R\|_{L^\infty} + L(R)\|v\|_{\mathrm{MR}_c} + L(R)\|w\|_{\mathrm{MR}_c}\big]\|v-w\|_{\mathrm{MR}_c}.
\end{aligned}
\tag{3.75}
$$

Assume now that $R > 0$ can be chosen such that

$$\|h_R\|_{L^\infty} \le \frac{1}{3M} =: \delta_1$$

and set

$$r := \min\left\{\frac{1}{4ML(R)}, R\right\}.$$

Suppose furthermore that

$$\|N(\cdot,0)\|_{X_c} \le \frac{1}{3M} r =: \delta_2.$$

Then $\Phi\colon \overline{B(0,r)} \to \overline{B(0,r)}$ is well defined and a contraction. Indeed, by (3.74), we get

$$\|\Phi(v)\|_{\mathrm{MR}_c} \le \frac{1}{3}r + \frac{1}{3}r + \frac{1}{4}r \le \frac{11}{12}r$$

and by (3.75), we obtain

$$\|\Phi(v) - \Phi(w)\| \leq \frac{1}{3}\|v - w\|_{\mathrm{MR}_c} + \frac{1}{2}\|v - w\|_{\mathrm{MR}_c} \leq \frac{5}{6}\|v - w\|_{\mathrm{MR}_c}$$

for all $v, w \in \overline{B(0,r)} \subset \mathrm{MR}_c$. Hence, Φ admits a unique fixed point in $\overline{B(0,r)} \subset \mathrm{MR}_c$, which is a solution to (3.70). □

We extend Corollary 3.2.5 to the quasilinear case. It will turn out that we only need to make some additional structural assumptions on the nonlinearity N. The mapping

$$K\colon v \mapsto [A(0) - A(v)]v$$

already behaves well for periodic and almost periodic functions under the Lipschitz condition of Theorem 3.2.7.

Lemma 3.2.8. *Let $A(\cdot)\colon D(A) \to \mathcal{L}(D(A), X)$ be as in Assumption 3.2.6. Let $v \in MR_c$. If v lies additionally in $C_0(\mathbb{R}; X)$ or $P_T(\mathbb{R}; X)$, then it also holds $K(v) \in C_0(\mathbb{R}; X)$ or $P_T(\mathbb{R}; X)$, respectively. If $v, Av, \in \mathrm{UAP}(\mathbb{R}; X)$ or $\mathrm{AAP}(\mathbb{R}; X)$, then $K(v) \in \mathrm{UAP}(\mathbb{R}; X)$ or $\mathrm{AAP}(\mathbb{R}; X)$, respectively.*

Proof. First, we show that the operator $K\colon \mathrm{MR}_c \to X_c$ defined by $K(v) = [A(0) - A(v)]v$ is continuous. Let $(v_n)_{n\in\mathbb{N}} \subset \mathrm{MR}_c$ be a convergent sequence with limit v and let $R > 0$ be such that $\|v_n\|_{\mathrm{MR}_c} \leq R$ for all $n \in \mathbb{N}$ .Then we have

$$\begin{aligned}
\|K(v_n) - K(v)\|_{X_c} &= \|[A(0) - A(v_n)]v_n - [A(0) - A(v)]v\|_{X_c} \\
&\leq \|[A(0) - A(v_n)](v_n - v)\|_{X_c} + \|[A(v_n) - A(v)]v\|_{X_c} \\
&\leq \|h_R\|_{L^\infty}\|v_n\|_{\mathrm{MR}_c}\|v_n - v\|_{\mathrm{MR}_c} \\
&\quad + \|h_R\|_{L^\infty}\|v_n - v\|_{\mathrm{MR}_c}\|v\|_{\mathrm{MR}_c} \\
&\leq 2\|h_R\|_{L^\infty}\|v_n - v\|_{\mathrm{MR}_c} \\
&\to 0
\end{aligned}$$

for $n \to \infty$. Thus $K(v_n) \to K(v)$ in X_c for $n \to \infty$, which implies the continuity of K. Suppose that $v \in \mathrm{MR}_c$. If v has compact support, it is clear that $K(v)$ has compact support, too. By approximation, this implies $K(v) \in C_0(\mathbb{R}; X)$ if v lies in $C_0(\mathbb{R}; X)$. If v is T-periodic for some $T > 0$, it holds that

$$\begin{aligned}
K(v)(t+T) &= [A(0) - A(v(t+T))]v(t+T) \\
&= [A(0) - A(v(t))]v(t) \\
&= K(v)(t),
\end{aligned}$$

i.e. $K(v)$ is also T-periodic. We now assume that $v, Av \in \mathrm{UAP}(\mathbb{R};X)$ or equivalently $v \in \mathrm{UAP}(\mathbb{R};D(A))$. Let $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ be an arbitrary sequence. Then there is a subsequence $(t_{n_k})_{k\in\mathbb{N}}$ such that $(v(\cdot + t_{n_k}))_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{UAP}(\mathbb{R};D(A))$. By means of the continuity of K, the sequence $(K(v(\cdot + t_{n_k})))_{k\in\mathbb{N}}$ has to be a Cauchy sequence in $\mathrm{BUC}(\mathbb{R};X)$, which means $K(v)$ is almost periodic with values in X. The case of asymptotically almost periodicity follows from the first and the third case, which we have already considered. □

The proof of time periodic and almost periodic solutions to (3.70) can now be kept rather short.

Theorem 3.2.9. *Let X and A be as in Assumption 3.2.6. Additionally, suppose that for all $v \in MR_c(\mathbb{R};X)$ with v, v', Av being in one of the space $C_0(\mathbb{R};X)$, $P_T(\mathbb{R};X)$, $\mathrm{UAP}(\mathbb{R};X)$ or $\mathrm{AAP}(\mathbb{R};X)$, it holds that $N(\cdot, v(\cdot)) \in C_0(\mathbb{R};X)$, $P_T(\mathbb{R};X)$, $\mathrm{UAP}(\mathbb{R};X)$ and $\mathrm{AAP}(\mathbb{R};X)$ respectively. Then the solution $u \in MR_c$ given by Theorem 3.2.7 fulfils $u, u', Au \in C_0(\mathbb{R};X)$, $P_T(\mathbb{R};X)$, $\mathrm{UAP}(\mathbb{R};X)$ and $\mathrm{AAP}(\mathbb{R};X)$ respectively, too.*

Proof. The proof works exactly the same way as the one of Theorem 3.2.7. We only remark that $C_0(\mathbb{R};X)$, $P_T(\mathbb{R};X)$, $\mathrm{UAP}(\mathbb{R};X)$ and $\mathrm{AAP}(\mathbb{R};X)$ are closed subspaces of $\mathrm{BUC}(\mathbb{R};X)$ and that the solution operator Φ leaves the spaces $C_0^1(\mathbb{R};X) \cap C_0(\mathbb{R};D(A))$ etc. invariant by our additional assumption as well as Corollary 3.2.5 and Lemma 3.2.8. □

3.2.2 Quasilinear Time Periodic Problems in L^p

We now investigate the existence of time periodic solutions to some quasilinear parabolic problems which are L^p-integrable in time. This works along the same lines as in the previous section. The key tool is the property of maximal periodic L^p-regularity, which we introduce now:

Definition 3.2.10. *Let X be a Banach space, $p \in (1,\infty)$, and $A\colon D(A) \subseteq X \to X$ be a linear operator. The operator A admits maximal periodic L^p-regularity, if for each $f \in L^p(0,2\pi;X)$ there is a unique solution $u \in H_p^1(0,2\pi;X) \cap L^p(0,2\pi;D(A))$ to the problem*

$$\begin{aligned} u'(t) - Au(t) &= f(t), \quad t \in (0,2\pi), \\ u(0) &= u(2\pi). \end{aligned}$$

By the closed graph theorem, the definition above implies the existence of a constant $M > 0$ independent of f such that

$$\|u\|_{H_p^1(0,2\pi;X)\cap L^p(0,2\pi;D(A))} \le M\|f\|_{L^p(0,2\pi;X)}.$$

Remark 3.2.11. *The restriction to the time period being 2π is no practical restriction, as any time period can be reduced to that case by some scaling in time.*

A comprising abstract theory regarding the characterization of maximal periodic regularity for $p \in (1,\infty)$ has been already established by Arendt and Bu in [AB02]. Similar to the theory for the initial value problem, they were able to fully describe the conditions for maximal periodic L^p-regularity in the case of UMD-spaces by the $\mathcal{R}$-boundedness of the resolvent of A in a suitable subset of $\mathbb{C}$. More precisely, they have shown the following.

Theorem 3.2.12. *[AB02, Theorem 2.3] Let $1 < p < \infty$ and X be a UMD-space and $A\colon D(A) \subseteq X \to X$ be a closed operator. Then the following conditions are equivalent:*

1. *A admits maximal periodic L^p-regularity.*
2. *$\mathrm{i}\mathbb{Z} \subset \rho(A)$ and $(kR(\mathrm{i}k, A))_{k\in\mathbb{Z}}$ is $\mathcal{R}$-bounded.*

Regarding applications, the property of maximal L^p-regularity of the initial value problem has already been proven for several examples. Thus, a rather convenient condition for maximal periodic L^p-regularity as a consequence of the maximal regularity for the initial value problem and some spectral property of $e^{2\pi A}$ is given by the following theorem:

Theorem 3.2.13 ([AB02, Theorem 5.1])**.** *Let A be the generator of a strongly continuous semigroup on a Banach space X and $1 < p < \infty$. Then the following conditions are equivalent:*

1. *A admits maximal periodic L^p-regularity.*
2. *$1 \in \rho(e^{2\pi A})$ and A admits maximal L^p-regularity for the initial value problem, i.e. for each $f \in L^p(0,2\pi;X)$ there is a unique solution $u \in H^1_p(0,2\pi;X) \cap L^p(0,2\pi;D(A))$ to*

$$\begin{aligned} u'(t) + Au(t) &= f(t), \quad t \in (0,2\pi), \\ u(0) &= 0. \end{aligned}$$

The last theorem implies that there is a suitable amount of operators at hand that admit maximal periodic L^p-regularity.

As we are only dealing with time-periodic solutions, we rewrite (3.55) as

$$\begin{aligned} u'(t) - A(u(t))u(t) &= N(t,u(t)), \quad t \in [0,2\pi], \\ u(0) &= u(2\pi). \end{aligned} \tag{3.76}$$

We assume that the domain of $A(u)$ is independent of u. Hence, it will just be denoted by $D(A)$. Furthermore, we assume that $D(A)$ is densely embedded into X. We introduce some notation for frequently occurring function spaces. Define

$$\begin{aligned} X_p &:= L^p(0, 2\pi; X), \\ \mathrm{MR}_p &:= H^1_p(0, 2\pi; X) \cap L^p(0, 2\pi; D(A)). \end{aligned}$$

It is known (see for example [Ama95, Theorem 4.10.2]) that the embedding

$$\mathrm{MR}_p \hookrightarrow \mathrm{BUC}([0, 2\pi]; Y)$$

is continuous, where

$$Y := (X, D(A))_{1-1/p,p}.$$

In order to deal with the quasilinear problem (3.76), we suppose some Lipschitz conditions on the operator family $A(\cdot)$ and the nonlinearity N:

Assumptions 3.2.14. *Let X be a Banach space.*

- *Let $A(\cdot)\colon Y \to \mathcal{L}(D(A), X)$ be a family of closed linear operators on X. Assume that $A(0)$ admits maximal periodic L^p-regularity. Suppose that for each $R > 0$ there is an $L = L(R) > 0$ such that*

  $$\|[A(x) - A(y)]z\|_{X_p} \le L(R)\|x - y\|_{MR_p}\|z\|_{MR_p} \tag{3.77}$$

  for each $x, y \in \overline{B(0, R)} \subset MR_p$ and $z \in MR_p$.

- *Let $N\colon MR_p \to X_p$ fulfil the following property: For each $R > 0$ there is a constant $C_R > 0$ such that*

  $$\|N(\cdot, u(\cdot)) - N(\cdot, v(\cdot))\|_{X_p} \le C_R\|u - v\|_{MR_p} \tag{3.78}$$

  for all $u, v \in \overline{B(0, R)} \subset MR_p$.

Remark 3.2.15. *A sufficient condition for* (3.77) *is the following: For each $R > 0$ there is an $L = L(R) > 0$ such that*

$$\|A(x) - A(y)\|_{\mathcal{L}(D(A),X)} \le L\|x - y\|_Y \tag{3.79}$$

for each $x, y \in \overline{B(0, R)} \subset Y$. This can be seen via the embedding $\mathrm{MR}_p \hookrightarrow \mathrm{BUC}([0, 2\pi], Y)$.

The counterpart to Theorem 3.2.7 reads as follows:

Theorem 3.2.16. *Let X, $A(\cdot)$ and N be as in Assumption 3.2.14. Then there are $\delta_1, \delta_2, r > 0$ such that if $R > 0$ can be chosen to satisfy $C_R < \delta_1$ and $\|N(\cdot, 0)\|_{X_p} < \delta_2$, there is a unique solution $u \in \overline{B(0,r)} \subset MR_p$ to (3.76).*

Proof. Consider the auxiliary problem

$$
\begin{aligned}
u'(t) - A(0)u(t) &= N(t, v(t)) - [A(0) - A(v(t))]v(t), \quad t \in (0, 2\pi), \\
u(0) &= u(2\pi),
\end{aligned}
\tag{3.80}
$$

where $v \in \mathrm{MR}_p$. Note that the right-hand side is an element of X_p for any $v \in \mathrm{MR}_p$. As $A(0)$ admits maximal periodic L^p-regularity, we obtain for each $v \in X_p$ a unique solution $u \in \mathrm{MR}_p$ to (3.80). Moreover, there is a constant $M > 0$ independent of v such that

$$\|u\|_{\mathrm{MR}_p} \leq M\|S(\cdot, v(\cdot)) - [A(0) - A(u(\cdot))]v(\cdot)\|_{X_p}.$$

We denote the corresponding solution operator $v \mapsto u$ by $\Phi\colon \mathrm{MR}_p \to \mathrm{MR}_p$. We show that Φ is a contraction map on a suitable ball around zero. Let $v, w \in \overline{B(0,R)} \subset \mathrm{MR}_p$, where $R > 0$ is arbitrary but fixed. Then it holds that

$$
\begin{aligned}
&\|\Phi(v) - \Phi(w)\|_{\mathrm{MR}_p} \\
&\leq M\|N(\cdot, v(\cdot)) - N(\cdot, w(\cdot)) - [A(0) - A(v)]v + [A(0) - A(w)]w\|_{X_p} \\
&\leq M\|N(\cdot, v(\cdot)) - N(\cdot, w(\cdot))\|_{X_p} \\
&\quad + M\|[A(0) - A(v)]v - [A(0) - A(v)]w - [A(v) - A(w)]w\|_{X_p} \\
&\leq M\|N(\cdot, v(\cdot)) - N(\cdot, w(\cdot))\|_{X_p} \\
&\quad + M\big(\|[A(0) - A(v)](v - w)\|_{X_p} + \|[A(v) - A(w)]w\|_{X_p}\big).
\end{aligned}
\tag{3.81}
$$

The first summand on the right-hand side can be estimated directly with (3.78) by

$$\|N(\cdot, v(\cdot)) - N(\cdot, w(\cdot))\|_{X_p} \leq C_R\|v - w\|_{\mathrm{MR}_p}.$$

The third summand on the right-hand side of (3.81) can be estimated by

$$\|[A(v) - A(w)]w\|_{X_p} \leq L(R)\|v - w\|_{\mathrm{MR}_p}\|w\|_{\mathrm{MR}_p}$$

because of (3.77). For the same reasons, we also get for the second summand on the right hand-side of (3.81)

$$\|[A(0) - A(v)](v - w)\|_{X_p} \leq L(R)\|v\|_{\mathrm{MR}_p}\|v - w\|_{\mathrm{MR}_p}.$$

Combining the last four estimates yields

$$
\begin{aligned}
&\|\Phi(v) - \Phi(w)\|_{\mathrm{MR}_p} \\
&\quad \leq MC_R\|v - w\|_{\mathrm{MR}_p} + ML(R)\big(\|v\|_{\mathrm{MR}_p} + \|w\|_{\mathrm{MR}_p}\big)\|v - w\|_{\mathrm{MR}_p}.
\end{aligned}
\tag{3.82}
$$

Additionally, we obtain by similar estimates

(3.83)
$$\begin{aligned}\|\Phi(v)\|_{\mathrm{MR}_p} &\leq M\big(\|N(\cdot,v(\cdot))-N(\cdot,0)+N(\cdot,0)\|_{X_p}+\|[A(0)-A(v)]v\|_{X_p}\big)\\ &\leq MC_R\|v\|_{\mathrm{MR}_p}+M\|N(\cdot,0)\|_{X_p}+ML(R)\|v\|_{\mathrm{MR}_p}\|v\|_{\mathrm{MR}_p}.\end{aligned}$$

Assume now the existence of an $R>0$ such that

$$C_R \leq \frac{1}{3M} =: \delta_1$$

and set

$$r := \min\left\{\frac{1}{4ML(R)}, R\right\}.$$

Suppose additionally that

$$\|N(\cdot,0)\|_{X_p} \leq \frac{1}{3M} r =: \delta_2.$$

Then Φ is a contraction map on $\overline{B(0,r)} \subset \mathrm{MR}_p$. Indeed, for any $v,w \in \overline{B(0,r)} \subset \mathrm{MR}_p$, it holds

$$\begin{aligned}\|\Phi(v)-\Phi(w)\|_{\mathrm{MR}_p} &\leq \frac{1}{3}\|v-w\|_{\mathrm{MR}_p}+\frac{1}{2}\|v-w\|_{\mathrm{MR}_p}\\ &\leq \frac{5}{6}\|v-w\|_{\mathrm{MR}_p},\end{aligned}$$

due to (3.82) and

$$\|\Phi(v)\|_{MR_p} \leq \frac{1}{3}\|v\|_{\mathrm{MR}_p}+\frac{1}{3}r+\frac{1}{4}\|v\|_{\mathrm{MR}_p} \leq \frac{11}{12}r \leq r,$$

due to (3.83). Hence, the Banach fixed point theorem yields the existence of a unique fixed point of Φ in $\overline{B(0,r)} \subset \mathrm{MR}_p$. This fixed point is the unique solution to (3.76) in that ball. □

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