On the primitive equations and the hydrostatic Stokes operator

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Deutsche Zusammenfassung

Gegenstand dieser Arbeit sind die so genannten *primitive equations* auf dem zylindrischen Gebiet $\Omega = G \times (-h, 0)$ mit horizontalem Querschnitt $G = (0,1)^2$ und vertikalem Höhenparameter $h > 0$. Dabei handelt es sich um ein System partieller Differentialgleichungen in der Form

$$\begin{aligned} \partial_t v - \Delta v + (u \cdot \nabla)v + \nabla_H \pi &= f \quad \text{in } \Omega \times (0,T), \\ \partial_z \pi &= 0 \quad \text{in } \Omega \times (0,T), \\ \operatorname{div} u &= 0 \quad \text{in } \Omega \times (0,T), \\ v(0) &= a \quad \text{in } \Omega, \end{aligned}$$

mit den Randbedingungen

$$\begin{array}{lll} \partial_z v = 0 \text{ oder } v = 0, & w = 0 & \text{auf } G \times \{0\} \times (0,T), \\ \partial_z v = 0 \text{ oder } v = 0, & w = 0 & \text{auf } G \times \{-h\} \times (0,T), \\ v, w, \pi \text{ periodisch} & & \text{auf } \partial G \times (-h,0) \times (0,T). \end{array}$$

Dabei ist $a \colon \Omega \to \mathbb{R}^2$ ein gegebener Anfangszustand und $T \in (0, \infty]$ eine beliebige Zeit. Die unbekannten dieses Systems sind die Geschwindigkeit des Fluids

$$u = (v, w) \colon \Omega \times (0,T) \to \mathbb{R}^2 \times \mathbb{R}$$

und der Oberflächendruck $\pi \colon G \to \mathbb{R}$. Dabei ist die vertikale Geschwindigkeit w durch die horizontale Geschwindigkeit v vollständig bestimmt.

Das Symbol $\nabla_H = (\partial_x, \partial_y)^T$ bezeichnet den Gradienten in den horizontalen Variablen $(x, y) \in G$ und ∂_z die partielle Ableitung in der vertikalen Variable $z \in (-h, 0)$, wohingegen $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ den Laplace-Operator, $\nabla = (\partial_x, \partial_y, \partial_z)^T$ den Gradienten und $\operatorname{div} f = \partial_x f_1 + \partial_y f_2 + \partial_z f_3$ die Divergenz in drei Raumdimensionen bezeichnen.

Diese Gleichungen beschreiben ein System der Geophysik wie den Ozean oder die Atmosphäre und approximieren die Navier-Stokes-Gleichungen für das Strömungsverhalten inkompressibler Fluide unter der Annahme eines hydrostatischen Gleichgewichts.

Die Linearisierung dieses Problems wird *hydrostatische Stokes-Gleichungen* genannt. Diese sind durch das System von partiellen Differentialgleichungen

$$\begin{aligned} \partial_t v - Av &= \mathbb{P} f \quad \text{in } \Omega \times (0,T), \\ \operatorname{div}_H \overline{v} &= 0 \quad \text{in } \Omega \times (0,T), \\ v(0) &= a \quad \text{in } \Omega \end{aligned}$$

gegeben. Dabei werden $A := \mathbb{P}\Delta$ der *hydrostatische Stokes-Operator* und $\mathbb{P}$ die *hydrostatische Helmholtz-Projektion* genannt. Weiterhin bezeichnet $\operatorname{div}_H f = \partial_x f_1 + \partial_y f_2$ die

Divergenz in den horizontalen Variablen und

$$\overline{f} = \frac{1}{h} \int_{-h}^{0} f(\cdot, z)\, dz$$

den vertikalen Mittelwert.

Wir betrachten die *primitive equations* und die hydrostatischen Stokes-Gleichungen im Rahmen von L^p-Räumen. Dabei liegt unser Interesse auf ihrer globalen Wohlgestelltheit für möglichst große Klassen von Anfangswerten.

Für $p, q \in (1, \infty)$ zeigen wir in Abschnitt 4.1, dass das linearisierte Problem die Eigenschaft der maximalen L^q-L^p-Regularität besitzt und beweisen damit in Abschnitt 5.1 die Existenz einer eindeutigen, globalen, starken Lösung für Anfangswerte der Klasse $a \in B^{2/p}_{p,q}$ mit $1/p + 1/q \leq 1$. Dabei ist es nicht nötig anzunehmen, dass der Anfangswert a klein ist.

Diese Ergebnisse basieren auf der Erkenntnis, dass sich der hydrostatische Stokes-Operator vom Laplace-Operator Δ nur um einen Term niedrigerer Ordnung unterscheidet. Dabei ist die Störung explizit gegeben durch

$$Av = \Delta v + Bv, \quad Bv = \frac{1}{h}(1 - Q) \left(\partial_z v\big|_{G\times\{0\}} - \partial_z v\big|_{G\times\{-h\}} \right), \quad \operatorname{div}_H \overline{v} = 0,$$

wobei Q die Helmholtz-Projektion auf dem zwei-dimensionalen Torus bezeichnet.

Wir entwickeln außerdem eine Theorie in Räumen vom L^∞-Typ. Aufgrund der anisotropen Struktur des nichtlinearen Terms $(u \cdot \nabla)v = (v \cdot \nabla_H)v + w\partial_z v$ verwenden wir dazu anisotrope L^∞-L^p-Räume von Funktionen welche sich in den horizontalen Variablen wie L^∞-Funktionen und in der vertikalen Variable wie L^p-Funktionen verhalten. Dabei unterscheiden wir verschiedene Randbedingungen für v. In Abschnitt 4.2 und 5.2 setzen wir den Fall von reinen Neumann-Randbegingungen auf dem oberen und unteren Rand voraus, d.h.,

$$\partial_z v\big|_{G\times\{0\}} = \partial_z v\big|_{G\times\{-h\}} = 0.$$

Dies hat zur Konsequenz, dass der Störterm Bv verschwindet und $A = \Delta$ lediglich die Einschränkung des Laplace-Operators ist. Dies vereinfacht das Problem stark und erlaubt es uns die Wohlgestelltheit der *primitive equations* im Fall $f = 0$ für Anfangswerte der Klasse

$$a \in BUC(G; L^p(-h, 0)), \quad p \in [1, \infty],$$

zu beweisen. In Abschnitt 4.3 und 5.3 gehen wir dann von den Randbedingungen

$$\partial_z v\big|_{G\times\{0\}} = v\big|_{G\times\{-h\}} = 0$$

aus. Daher müssen wir uns den Schwierigkeiten stellen die durch die Unbeschränktheit der Projektionen Q und $\mathbb{P}$ auf L^∞-Räumen entstehen. Trotz dieser Komplikationen beweisen wir die Existenz einer eindeutigen, globalen, starken Lösung der *primitive equations* für Anfangsdaten der Klasse

$$a \in BUC(G; L^p(-h,0)), \quad p \in (3,\infty).$$

Diese Arbeit ist wie folgt strukturiert. Nachdem wir in Kapitel 2 die mathematischen Grundlagen einführen welche zum Verständnis der darauffolgenden Kapitel notwendig sind, präsentieren wir in Kapitel 3 eine Reihe von Eigenschaften des Laplace-Operators und von Wärmeleitungshalbgruppen. In Kapitel 4 wenden wir diese dann auf das linearisierte Problem an, sowohl in L^p-Räumen für $p \in (1,\infty)$ in Abschnitt 4.1, als auch in anisotropen L^∞-L^p-Räumen in Abschnitt 4.2 und 4.3. Schließlich wenden wir diese Ergebnisse in Kapitel 5 auf das nichtlineare Problem an.

Die hier vorgestellten Ergebnisse sind das Resultat einer Zusammenarbeit mit Yoshikazu Giga, Matthias Hieber, Amru Hussein und Takahito Kashiwabara. Sie wurden in [38–41] veröffentlicht, bzw. zur Veröffentlichung eingereicht.

1 Introduction

The primitive equations for the ocean and atmosphere are a model for the movement of a viscous fluid in a large scale three-dimensional setting where the underlying domain is much wider than it is high. The equations describing this model are derived from the Navier-Stokes equations using the assumption of a hydrostatic balance. This assumption can be rigorously justified by a scaling argument, see [66].

The study of these equations through analytical means was commenced by Lions, Temam, and Wang in their series of papers [68–70], where they introduced the equations describing models for the ocean and atmosphere.

In this work we consider the model under the assumptions that physical quantities like temperature are constant, which reduces the model to the equations describing the conservation of momentum and mass of the fluid, explicitly given by the system of partial differential equations

$$\begin{aligned} \partial_t v - \Delta v + (u \cdot \nabla) v + \nabla_H \pi &= 0 \quad \text{in } \Omega \times (0, \infty), \\ \partial_z \pi &= 0 \quad \text{in } \Omega \times (0, \infty), \\ \operatorname{div} u &= 0 \quad \text{in } \Omega \times (0, \infty), \\ v(0) &= a \quad \text{in } \Omega. \end{aligned}$$

The setting is that of a cylindrical domain $\Omega := G \times (-h, 0)$ with $G := (0,1)^2$ and height parameter $h > 0$. The unknown quantities are the vector field describing the velocity of the fluid $u = (v, w)$ with horizontal and vertical components $v = (v_1, v_2)$ and w, respectively, as well as the surface pressure π, which is related to the full pressure $\mathcal{P}$ via the relation $\pi = \mathcal{P} + z\tau_0$, where $z \in (-h, 0)$ denotes the vertical variable and τ_0 the constant temperature. The initial data a for the horizontal velocity is a given vector field. Denoting the horizontal variables by $(x, y) \in G$, the symbol $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ denotes the three-dimensional Laplace operator and

$$\nabla = (\partial_x, \partial_y, \partial_z)^T, \quad \operatorname{div} u = \partial_x v_1 + \partial_y v_2 + \partial_z w$$

are the gradient and divergence, whereas $\nabla_H = (\partial_x, \partial_y)^T$ denotes the gradient in horizontal variables only. Dividing the boundary of Ω into the top, bottom, and lateral parts

$$\Gamma_u := \{0\} \times G, \quad \Gamma_b := \{-h\} \times G, \quad \Gamma_l := [-h, 0] \times \partial G,$$

we will consider the boundary conditions

$$\begin{aligned} &\partial_z v = 0 \text{ or } v = 0, \quad w = 0 && \text{on } \Gamma_u \times (0, \infty), \\ &\partial_z v = 0 \text{ or } v = 0, \quad w = 0 && \text{on } \Gamma_b \times (0, \infty), \\ &v, w, \pi \text{ periodic} && \text{on } \Gamma_l \times (0, \infty). \end{aligned}$$

Our notion of periodicity on the boundary is made precise in Section 2.4.5.

The first established results for the well-posedness of the primitive equations concerned L^2-type spaces. Lions, Temam, and Wang proved in [68–70] that, given initial data $a \in L^2$, there exists a global weak solution. The question of the uniqueness for L^2-data in three dimensions remains unanswered until today, see [59, 65, 76, 88] for recent developments. For the two-dimensional problem, the uniqueness of weak solutions was proven for continuous initial data in [59] and for initial data $a \in L^2$ also satisfying $\partial_z a \in L^2$ in [15], whereas the existence of global strong solutions was established in [76].

In [92–94], Ziane began the investigation of the three-dimensional linearized problem, establishing H^2-regularity for the solution of the resolvent problem.

The first result concerning the existence of strong solutions to the three-dimensional primitive equations was established by Guillén-González, Masmoudi, and Rodríguez-Bellido in [46], who utilized the result of Ziane to prove the existence of a local strong solution for initial data $a \in H^1$.

This result was improved significantly in [19] by Cao and Titi who were able to prove the existence of a unique, global strong solution without requiring a smallness condition for the initial data $a \in H^1$. Note that for the three-dimensional Navier-Stokes equations, this remains a famous open question. In their approach, they decomposed the horizontal velocity v into its vertical average and the remainder, i.e.,

$$v = \overline{v} + \tilde{v}, \quad \overline{v} := \frac{1}{h}\int_{-h}^{0} v(\cdot, z)\, dz, \quad \tilde{v} := v - \overline{v}, \tag{1.0.1}$$

and established L^∞-L^6-estimates for the remainder term $\tilde{v}$ in order to obtain an *a priori* H^1-estimate for the solution.

There are also a number of papers investigating the primitive equations for partial viscosity and diffusion, establishing global well-posedness for initial data $a \in H^2$, see [20] by Cao and Titi as well as [16–18] by Cao, Li, and Titi. Other recent results can be found in the survey article [67].

A more general L^p-theory was established in [49] by Hieber and Kashiwabara. Under the assumption that Neumann and Dirichlet boundary conditions are imposed on Γ_u and Γ_b, respectively, they proved the L^p-well-posedness of the linearized problem for $p \in (1, \infty)$ and the existence of a unique, global strong solution to the primitive equations for $p \in [6/5, \infty)$.

For the linearized problem, they introduced the *hydrostatic Helmholtz projection*, denoted by $\mathbb{P}$, which annihilates the horizontal pressure gradient and is bounded on $L^p(\Omega)^2$ for $p \in (1, \infty)$. They then introduced the *hydrostatic Stokes operator* $A_p := \mathbb{P}\Delta$ and proved that $-A_p$ is a sectorial operator of spectral angle 0 with bounded inverse and that A_p generates an exponentially decaying analytic semigroup. This operator is analogous to the Stokes operator in structure and role for the Navier-Stokes equations. The same relation holds between the projection $\mathbb{P}$ and the Helmholtz projection.

For the full nonlinear problem they adapted the approach of Fujita and Kato for the Navier-Stokes equations, see [29, 30], and constructed a unique, local, strong solution to the primitive equations by means of an iteration scheme. They considered arbitrarily large initial data a belonging to a closed subspace of the Bessel-potential space $H^{2/p,p}(\Omega)^2$ for $p \in (1,\infty)$, characterized as a complex interpolation space. Using the smoothing effect of the analytic semigroup, they showed that the local solution satisfies $v \in C((0,T^*];D(A_p))$ for some time $T^* > 0$ and established an *a priori* H^2-estimate. Using the embeddings $D(A_p) \hookrightarrow H^{2,p}(\Omega)^2 \hookrightarrow H^1(\Omega)^2$ for $p \in [6/5,\infty)$, they then obtained the existence of a unique, global strong solution.

This work goes beyond the results of [49] in several ways. The primary interest of our investigation lies in proving the strong, global well-posedness of the primitive equations for a large class of initial values. For this purpose, we establish new properties for the linearized problem in the L^p-setting for $p \in (1,\infty)$, allowing us to approach the nonlinear problem in the L^p-setting with new and powerful tools. We also consider function spaces of L^∞- and L^1-type, both for the linearized problem in Chapter 4 and the full nonlinear problem in Chapter 5.

We begin by extending the previously established L^p-theory for $p \in (1,\infty)$. For the linear problem, we rewrite the hydrostatic Stokes operator as a perturbation of the Laplace operator of the form

$$A_p v = \Delta v + Bv, \quad Bv = \frac{1}{h}(1-Q)\left(\partial_z v\big|_{\Gamma_u} - \partial_z v\big|_{\Gamma_b}\right), \quad \mathbb{P}v = v, \tag{1.0.2}$$

where Q denotes the Helmholtz projection on the two-dimensional torus.

Observe that this representation yields that if Neumann boundary conditions are imposed on both the top and bottom part of the boundary $\Gamma_u \cup \Gamma_b$, then we have $A_p = \Delta$, i.e., the Laplace operator Δ and hydrostatic Stokes projection $\mathbb{P}$ commute and the hydrostatic Stokes operator is simply the restriction of the Laplace operator onto the range of $\mathbb{P}$.

This choice of pure Neumann boundary conditions on $\Gamma_u \cup \Gamma_b$ was considered in [19] by Cao and Titi, as well as [16–18, 22, 59], whereas the mixed Neumann and Dirichlet boundary conditions

$$\partial_z v\big|_{\Gamma_u} = v\big|_{\Gamma_b} = 0$$

considered by Hieber and Kashiwabara were also chosen by Kukavica and Ziane in [60, 61] where they proved the strong, global well-posedness of the primitive equations for arbitrarily large initial data belonging to H^1 for this choice of boundary conditions.

We show in Section 4.1 that the negative hydrostatic Stokes operator $-A_p$ admits a bounded $\mathcal{H}^\infty$-calculus of angle 0. This allows for a variety of further corollaries, such as the property of maximal L^q-L^p-regularity for all $p, q \in (1,\infty)$, the characterization of domains of fractional powers, as well as L^p-L^q-smoothing estimates for the semigroup generated by A_p. In this context, we will distinguish between pure Neumann and (mixed) Dirichlet boundary conditions only in order to deal with the fact that 0 is an eigenvalue of the hydrostatic Stokes operator in the former case.

For the Stokes operator, maximal L^p-L^p-regularity was first obtained in [85] by Solonnikov. For maximal L^q-L^p-regularity, see [35, 36] by Giga as well as [42, 43] by Giga and Sohr. More details can be found in the survey article [50]. Note that the property of maximal regularity also has many applications to nonlinear problems. For example, it has been applied to problems on domains whose boundaries are not constant over time, see, e.g., [80] for the free boundary value problem of the Navier-Stokes equations or [32, 52] for the case of a fluid interacting with a rigid structure.

We similarly apply the property of maximal L^q-L^p-regularity for the hydrostatic Stokes operator to the full nonlinear problem. In Section 5.1, we develop a new proof of strong, global well-posedness for the primitive equations for initial data belonging to a suitable closed subspace of the Besov-space $B^{2(\mu-1/q)}_{p,q}$ for $p,q \in (1,\infty)$ with $1/p+1/q \leq \mu \leq 1$, which arises from real interpolation. This is achieved by proving the existence of a local solution via a fixed-point argument in spaces of maximal regularity with time-weights, the relevant theory for which has been developed by Prüss and Simonett in [79].

Observe that the critical choice of $\mu = 1/p+1/q$ corresponds to the minimal degree of differentiability $2(\mu-1/q) = 2/p$, which is the same degree of differentiability required for the result of Hieber and Kashiwabara in [49]. However, these spaces require a lower degree of integrability since the choice $q \geq \max\{2,p\}$ yields that $H^{2/p,p} \subset B^{2/p}_{p,q}$ and thus we obtain a larger class of admissible initial data. Furthermore, we explicitly note that we do not require the norm of our data to be small.

We even consider the case of non-vanishing external forces and prove that, for $t > 0$, the solution becomes infinitely continuously differentiable, and even real analytic, if the given external force has this property as well.

We also prove well-posedness for initial data without requiring any differentiability. Since we have global, strong well-posedness in the L^p-setting for $p \in (1,\infty)$ and the required degree of differentiability of $2/p$ vanishes in the limit $p \to \infty$, we turn to spaces of L^∞-type as a natural next step. In Sections 4.2 and 4.3, we establish a theory for the linearized problem in spaces of this type. Motivated by the anisotropic nature of the nonlinear term

$$(u \cdot \nabla)v = (v \cdot \nabla_H)v + w\partial_z v, \quad w(x,y,z) = -\int_{-h}^{z} \mathrm{div}_H v(x,y,\xi)\,d\xi,$$

we consider anisotropic spaces of functions belonging to L^∞ with respect to the horizontal variables x, y, and to L^p with respect to the vertical variable z. These spaces are denoted by $L^\infty_H L^p_z$ and introduced in Section 2.4.2. We prove a number of semigroup estimates involving derivatives of the hydrostatic Stokes semigroup in the anisotropic L^∞-L^p-spaces. In Section 4.2 we also consider fractional horizontal and vertical derivatives.

The primary difference between these sections is the choice of boundary conditions. Since imposing Neumann boundary conditions on both the top and bottom part of the boundary reduces the hydrostatic Stokes operator to a restriction of the Laplace operator onto an invariant subspace without a perturbation term, we are able to develop an L^∞-L^p-theory in Section 4.2 by applying the properties of heat semigroups that we present in Chapter 3. For this purpose, we need to take special care of the fact that the Riesz transform is unbounded on L^∞ by utilizing the smoothing properties of the heat semigroup. In Section 4.2 we also consider the case where the horizontal domain G is replaced by the whole space $\mathbb{R}^2$. Since horizontal periodicity is preserved, the case of periodic boundary conditions on G is obtained as a corollary.

In Section 4.3 we then assume that Dirichlet boundary conditions are imposed. As a consequence, we need to deal with the fact that the hydrostatic Stokes operator is a proper perturbation of the Laplace operator. This complicates the analysis of the problem, as can be seen in the difference between the arguments we utilize in these two sections.

Denoting the semigroup generated by A_p by S, the presence of the perturbation term B in (1.0.2) significantly complicates the proof of the parabolic decay estimates of the type

$$\begin{aligned} t^{1/2}\|\partial_i S(t)\mathbb{P}f\|_{L^\infty_H L^p_z} &\leq Ce^{t\beta}\|f\|_{L^\infty_H L^p_z}, \\ t^{1/2}\|S(t)\mathbb{P}\partial_i f\|_{L^\infty_H L^p_z} &\leq Ce^{t\beta}\|f\|_{L^\infty_H L^p_z}, \\ t\|\partial_i S(t)\mathbb{P}\partial_j f\|_{L^\infty_H L^p_z} &\leq Ce^{t\beta}\|f\|_{L^\infty_H L^p_z}, \end{aligned} \tag{1.0.3}$$

for $\partial_i, \partial_j \in \{\partial_x, \partial_y, \partial_z\}$, and particularly for the case of the vertical derivative ∂_z. This is due to the fact that both the hydrostatic Stokes projection $\mathbb{P}$ and the two-dimensional Helmholtz projection Q fail to be bounded with respect to the L^∞-norm. This problem also arises when dealing with the Stokes semigroup in L^∞-type spaces, compare [1, 2]. As a result, the methods we apply are only sufficient to prove the estimates in (1.0.3) for the range $p \in (3, \infty)$, whereas in the case of Neumann boundary conditions on $\Gamma_u \cup \Gamma_b$ we obtain these estimates for the whole range $p \in [1, \infty]$. Note that in the case of $L^p(\Omega)^2$ for $p \in (1, \infty)$, the boundedness of $\mathbb{P}$ prevents such complications when establishing analogous estimates.

In Sections 5.2 and 5.3 we then apply these results to the full primitive equations. Like Hieber and Kashiwabara, we construct a unique local solution via an iteration scheme in the vein of the classical approach to the Navier-Stokes problem by Fujita and Kato. We also take notes from the later approaches by Kato [56] and Giga [37].

In the case where Neumann boundary conditions are imposed on $\Gamma_u \cup \Gamma_b$, this method is successful for initial data of the form $a = a_1 + a_2$, where

$$a_1 \in BUC(\mathbb{R}^2; L^p(-h, 0)), \quad a_2 \in L^\infty(\mathbb{R}^2; L^p(-h, 0)), \quad \|a_2\|_{L^\infty_H L^p_z} \leq C(1 + \|a_1\|_{L^\infty_H L^p_z})^{-1},$$

for arbitrary $p \in [1, \infty]$. Here, BUC denotes the space of bounded, uniformly continuous functions and $C > 0$ is an absolute constant independent of a. We then obtain global, strong well-posedness under the assumption that a is horizontally periodic as in [49]. Since the vertical interval $(-h, 0)$ has finite measure, we have the chain of embeddings

$$L^\infty(\Omega) \hookrightarrow L^\infty(G; L^q(-h, 0)) \hookrightarrow L^\infty(G; L^p(-h, 0)) \hookrightarrow L^\infty(G; L^1(-h, 0))$$

whenever $1 \leq p \leq q \leq \infty$. In particular, we obtain well-posedness for the critical case $p = 1$ corresponding to the least amount of regularity on the scale of L^p-spaces. This means that we obtain global, strong well-posedness for the primitive equations without requiring the initial data to possess differentiability in any of the spatial variables $(x, y, z) \in \Omega$, or boundedness in the vertical variable $z \in (-h, 0)$. Observe that these L^∞-L^1-spaces are invariant under the rescaling

$$v_\lambda(t, x, y, z) := \lambda v(\lambda^2 t, \lambda x, \lambda y, \lambda z), \quad \lambda > 0,$$

i.e., that one has $v \in L^\infty(\mathbb{R}^2; L^1(-h, 0))$ if and only if $v_\lambda \in L^\infty(\mathbb{R}^2; L^1(-h\lambda^{-1}, 0))$ with equal norms. It further holds that v is a solution to the primitive equations if and only if v_λ is a solution to the rescaled primitive equations. This feature is shared with the Navier-Stokes equations, compare [12, 34, 64].

The smallness assumption on a_2 is due to the discontinuity of the semigroup S on $L^\infty_H L^p_z$, which we remedy via the parabolic estimate

$$\limsup_{t\to 0} t^{1/2} \|\nabla S(t) a_2\|_{L^\infty_H L^p_z} \le C \|a_2\|_{L^\infty_H L^p_z},$$

whereas the additional assumption of uniform continuity in the horizontal variables yields that the left-hand side vanishes for a_1 instead of a_2. Thus, no assumption of smallness is required for a_1 or the sum $a = a_1 + a_2$. A similar, but not directly comparable, result was established by Li and Titi in [65] under the assumption that a_1 is continuous or belongs to $\{a \in L^6 : \partial_z a \in L^2\}$, with an upper bound for a_2 depending on the L^4-norm of a_1.

In the presence of Dirichlet boundary conditions, we obtain analogous results for the range $p \in (3, \infty)$ only. This is due to the important role played by the semigroup estimates in (1.0.3) which we likewise establish for these values of p only. However, the upper bound for the norm of the discontinuous part a_2 is an absolute constant not depending on a_1. This is obtained by taking a reference solution v_{ref} to the primitive equations, the initial value of which is a smooth approximation of a_1, and then performing the iteration procedure for the auxiliary function $v - v_{\mathrm{ref}}$.

This work is structured as follows. We begin by introducing relevant concepts from functional analysis in Chapter 2. In Chapter 3 we then present a number of properties of the Laplace operator and heat semigroups that we will apply to the linearized problem. We will present comprehensive proofs of these properties for the sake of staying self-contained. In Chapter 4 we then cover the hydrostatic Stokes operator in L^p for $p \in (1, \infty)$, before turning to the $L^\infty_H L^p_z$-theory of the hydrostatic Stokes semigroup with pure Neumann boundary conditions on the layer domain $\mathbb{R}^2 \times (-h, 0)$ and with mixed Neumann and Dirichlet boundary conditions on Ω, respectively. Finally, in Chapter 5, we then apply these results to the primitive equations in these respective settings.

Note that the results presented here have been obtained in joint work with Yoshikazu Giga, Matthias Hieber, Amru Hussein, and Takahito Kashiwabara. They were published, or submitted for publication, in [38–41].

2 Preliminaries

This chapter provides an overview over basic notation as well as concepts from functional analysis which we will encounter during the later chapters of this work.

2.1 Basic notation

We denote the set of natural numbers $\{0, 1, 2, \ldots\}$ by $\mathbb{N}$, the set of integers by $\mathbb{Z}$, the real numbers by $\mathbb{R}$, and the complex numbers by $\mathbb{C}$. Given $x \in \mathbb{C}^d$ we denote its Euclidean norm by

$$|x| := \left(\sum_{k=1}^{d} x_k x_k^* \right)^{1/2},$$

where * denotes complex conjugation.

Given a normed vector space X, its norm is denoted by $\|\cdot\|_X$ and the space of bounded linear mappings $L\colon X \to X$ is denoted by $\mathcal{L}(X)$.

If (Ω, Σ, μ) is a measure space and f is a measurable function, we do not distinguish between f and the equivalence class of functions that agree with f almost everywhere, i.e., everywhere except for at most a set $A \in \Sigma$ such that $\mu(A) = 0$. If we write that such f satisfies an equation on A, we mean that f satisfies the equation almost everywhere.

Spaces of functions $f\colon U \to X$ for some set U will be denoted in the form $E(U; X)$, where the symbol E is a placeholder. In the scalar-valued case we will simply write

$$E(U) := E(U; \mathbb{K}), \quad \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}.$$

For example, the space of continuous functions $f\colon U \to X$ is denoted by $C(U; X)$. The subspaces of bounded continuous and bounded uniformly continuous functions are denoted by $C_b(U; X)$ and $BUC(U; X)$, respectively.

2.2 Fréchet derivatives and analytic functions

Definition 2.2.1. *Let X and Y be Banach spaces and $U \subset X$ be an open subset with $x_0 \in U$.*

1. A function $f\colon U \to Y$ is called *Fréchet differentiable* in x_0 if there exists a bounded linear operator $A\colon X \to Y$ such that

$$\lim_{u \to 0} \frac{\|f(x_0 + u) - f(x) - Au\|_X}{\|u\|_X} = 0.$$

In this case the operator $(Df)(x_0) := A$ is called the *Fréchet derivative* of f at x_0. If it exists for all $x \in U$, then the mapping $x \mapsto Df(x)$ is called the Fréchet derivative of f. Higher-order derivatives are denoted by $D^{n+1}f := D(D^n f)$, $n \in \mathbb{N}$.

2. Let $n \in \mathbb{N}$. A mapping

$$m\colon X^n \ni (x_1, \ldots, x_n) \mapsto m(x_1, \ldots, x_n) \in Y$$

is called *multilinear* if it is linear in each variable x_k, $1 \leq k \leq n$. A multilinear mapping is called *symmetric* if it satisfies

$$m(x_1, \ldots, x_n) = m(x_{\rho(1)}, \ldots, x_{\rho(n)})$$

for all permutations ρ of the set $\{1, \ldots, n\}$, and *bounded* if

$$\|m\| := \sup\{\|m(x_1, \ldots, x_n)\|_Y : (x_1, \ldots, x_n) \in X^n, \|x_k\|_X \leq 1 \text{ for all } 1 \leq k \leq n\}$$

is finite. The set of bounded multilinear mappings $m\colon X^n \to Y$ is denoted by $BM(X^n, Y)$. For $x, x_0 \in X$ we further use the notation

$$m_n(x - x_0)^n := m_n(x - x_0, \ldots, x - x_0).$$

3. A function $f\colon U \to Y$ is called *analytic* in x_0, if there exists a neighborhood $V \subset U$ of x_0 and a sequence of symmetric, bounded multilinear mappings $(m_n)_{n\in\mathbb{N}}$ satisfying $m_n \in BM(X^n, Y)$ for all $n \in \mathbb{N}$ such that

$$f(x) = \sum_{n=0}^{\infty} m_n(x - x_0)^n$$

for all $x \in V$ and

$$\sup\{r^n \|m_n\| : n \in \mathbb{N}\} < \infty$$

for some $r > 0$. If this holds for all $x_0 \in U$, then f is called analytic on U.

2.3 Interpolation

In the following, we give a brief overview of the theory of interpolation spaces provided in [13, 72, 90].

Definition 2.3.1.

1. Let $(X, \|\cdot\|_X)$ be a complex Banach space and (Z, τ) be a complex Hausdorff topological vector space. Then the space X *embeds into* Z if the identity mapping is a well-defined continuous linear operator from X into Z. This property is denoted by $X \hookrightarrow Z$.

2. Let $(X_0, \|\cdot\|_{X_0})$ and $(X_1, \|\cdot\|_{X_1})$ be complex Banach spaces. Then the couple (X_0, X_1) is called *compatible* if there exists a Hausdorff topological vector space (Z, τ) such that X_0 and X_1 both embed into Z.

3. Let (X_0, X_1) be a compatible couple. A Hausdorff topological vector space X is called an *intermediate space between* X_0 *and* X_1 if

$$X_0 \cap X_1 \hookrightarrow X \hookrightarrow X_0 + X_1,$$

where the intersection and sum of X_0 and X_1 are equipped with the norms

$$\|x\|_{X_0 \cap X_1} := \max\{\|x\|_{X_0}, \|x\|_{X_1}\},$$
$$\|x\|_{X_0 + X_1} := \inf\{\|x_0\|_{X_0} + \|x_1\|_{X_1} : x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\}.$$

4. Let (X_0, X_1) and (Y_0, Y_1) be compatible couples with intermediate Banach spaces X and Y, respectively. Then (X, Y) is called an *interpolation pair* if every linear operator $A\colon X_0 + X_1 \to Y_0 + Y_1$ which continuously maps both X_0 into Y_0 and X_1 into Y_1 also continuously maps X into Y. If in addition there exists $\theta \in (0,1)$ and a constant $C > 0$ such that

$$\|A\|_{\mathcal{L}(X,Y)} \le C \|A\|_{\mathcal{L}(X_0,Y_0)}^{1-\theta} \|A\|_{\mathcal{L}(X_1,Y_1)}^{\theta}$$

for all such A, then (X, Y) is *of exponent* θ. If it further holds that $C = 1$, it is called *exact*. Here

$$\|A\|_{\mathcal{L}(X,Y)} := \sup\{\|Ax\|_Y : \|x\|_X \le 1, x \in X\}$$

denotes the operator norm.

The functions spaces between which we will interpolate will be related to Lebesgue and Sobolev spaces. One of the most famous results for interpolation spaces is the following theorem.

Theorem 2.3.2 (Riesz-Thorin theorem, see, e.g., [13])**.** *Let* $(\Omega_0, \Sigma_0, \mu_0)$ *and* $(\Omega_1, \Sigma_1, \mu_1)$ *be* σ*-finite measure spaces. Then, given parameters* $p_0, p_1, q_0, q_1 \in [1, \infty]$ *and* $\theta \in (0,1)$*, the couple*

$$(L^{p_\theta}(\Omega_0, \Sigma_0, \mu_0), L^{q_\theta}(\Omega_1, \Sigma_1, \mu_1)),$$

with auxiliary parameters

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

is an exact interpolation pair for

$$(L^{p_0}(\Omega_0, \Sigma_0, \mu_0), L^{q_0}(\Omega_0, \Sigma_0, \mu_0)) \quad \textit{and} \quad (L^{p_1}(\Omega_1, \Sigma_1, \mu_1), L^{q_1}(\Omega_1, \Sigma_1, \mu_1)).$$

Remark 2.3.3. Let (Ω, Σ, μ) be a measure space, $p_0, p_1 \in [1, \infty]$, and $\theta \in (0,1)$. Given p_θ as above, it follows from Hölder's inequality that

$$\|f\|_{L^{p_\theta}(\Omega,\Sigma,\mu)} \leq \|f\|_{L^{p_0}(\Omega,\Sigma,\mu)}^{1-\theta} \|f\|_{L^{p_1}(\Omega,\Sigma,\mu)}^{\theta}$$

for all $f \in L^{p_0}(\Omega, \Sigma, \mu) \cap L^{p_1}(\Omega, \Sigma, \mu)$. This inequality states the logarithmic convexity of L^p-norms and is a key element of the proof of the Riesz-Thorin theorem. In fact, given a compatible couple (X_0, X_1) and a Banach space X_θ such that $(\mathbb{C}, X_\theta)$ is an interpolation pair of exponent $\theta \in (0,1)$ for $(\mathbb{C}, \mathbb{C})$ and (X_0, X_1), it holds that

$$\|x\|_{X_\theta} \leq C \|x\|_{X_0}^{1-\theta} \|x\|_{X_1}^{\theta}, \tag{2.3.1}$$

for some constant $C > 0$ and all $x \in X_0 \cap X_1$. To verify this we adapt the proof of [90, Theorem 1.3.3 (g)] by considering the operator

$$A_x \colon \mathbb{C} \ni \lambda \mapsto \lambda x \in X_0 + X_1.$$

The claim then follows from $\|A_x\|_{\mathcal{L}(\mathbb{C}, X_j)} = \|x\|_{X_j}$ for $j \in \{0, \theta, 1\}$. For this reason, estimates of the form (2.3.1) are also called *interpolation inequalities*.

Interpolation couples for compatible couples of Banach spaces can be constructed in a number of ways. The two most well-known methods are those of real and complex interpolation. Following [90], we introduce them as follows.

Definition 2.3.4. Let (X_0, X_1) be a compatible couple, $\theta \in (0,1)$, and $q \in [1, \infty]$. Then the mapping

$$K(t,x) := \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\}, \quad t > 0, x \in X_0 + X_1,$$

is called the K-functional and the space

$$(X_0, X_1)_{\theta,q} := \{x \in X_0 + X_1 : \|x\|_{(X_0,X_1)_{\theta,q}} < \infty\},$$

with norm

$$\|x\|_{(X_0,X_1)_{\theta,q}} := \begin{cases} \left(\int_0^\infty \left[t^{-\theta} K(t,x)\right]^q \frac{dt}{t}\right)^{1/q}, & q \in [1, \infty), \\ \sup\{t^{-\theta} K(t,x) : t \in (0, \infty)\}, & q = \infty, \end{cases}$$

is called the *real interpolation space* between X_0 and X_1 with parameters θ and q.

Definition 2.3.5. Let (X_0, X_1) be a compatible couple, $\theta \in (0,1)$, and $\gamma \in \mathbb{R}$. Further, let $S := \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$ and denote by $F(X_0, X_1, \gamma)$ the space of all functions $f \colon \overline{S} \to X_0 + X_1$ such that

(i) f is continuous on $\overline{S}$ and analytic on S,

(ii) the mapping $\overline{S} \ni z \mapsto e^{-|\gamma| \operatorname{Im} z} f(z) \in X_0 + X_1$ is bounded,

(iii) the mappings

$$\mathbb{R} \ni t \mapsto f(it) \in X_0 \quad \text{and} \quad \mathbb{R} \ni t \mapsto f(1+it) \in X_1$$

are well-defined and continuous.

Then the space

$$[X_0, X_1]_\theta := \{x \in X_0 + X_1 : f(\theta) = x \text{ for some } f \in F(X_0, X_1, \gamma)\},$$

is called the *complex interpolation space* with parameter θ and its norm is given by

$$\|x\|_{[X_0,X_1]_{\theta,\gamma}} := \inf_{\substack{f \in F(X_0,X_1,\gamma),\\ x=f(\theta)}} \max_{j=0,1} \sup_{t\in\mathbb{R}} e^{-|\gamma|t} \|f(j+it)\|_{X_j}.$$

Remark 2.3.6. By [90, Theorem 1.9.2], any choice of the parameter $\gamma \in \mathbb{R}$ leads to the same space with equivalent norms. The symbol $\|\cdot\|_{[X_0,X_1]_\theta}$ thus simply denotes one of many equivalent norms, unless the interpolation space is identified with a space possessing a canonical norm.

Proposition 2.3.7. *Given two compatible couples* (X_0, X_1) *and* (Y_0, Y_1) *and setting*

$$X := (X_0, X_1)_{\theta,q}, \quad Y := (Y_0, Y_1)_{\theta,q},$$

for $\theta \in (0, 1)$ *and arbitrary* $q \in [1, \infty]$*, one has that* (X, Y) *is an interpolation pair of exponent* θ*. The same is true using complex interpolation.*

For a detailed look into the theory of interpolation spaces, see [90].

2.4 Function spaces

Throughout this chapter, let (Ω, Σ, μ) be a measure space and X a Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. When dealing with measurable functions $f : \Omega \to X$, we will always assume that X is equipped with the Borel-σ-Algebra $\mathcal{B}(X)$ and treat it interchangeably with the measurable space $(X, \mathcal{B}(X))$. Many types of function spaces commonly encountered in functional analysis can be treated both in the cases of scalar, as well as vector-valued functions. In this section we present an overview of the spaces we will be working with during the later chapters.

2.4.1 The Bochner integral

Here, we provide an introduction to the theory of Lebesgue-Bochner integrals as in [11, Chapter I, Section 1.1] and [91, Chapter V.4-5].

Definition 2.4.1. A function $s \colon \Omega \to X$ is called *simple* if

(i) s is measurable, i.e., if for all $A \in \mathcal{B}(X)$ one has that $s^{-1}(A) \in \Sigma$,

(ii) s only takes finitely many values,

(iii) for any $0 \neq x \in X$, one has $\mu(s^{-1}(\{x\})) < \infty$.

Given a simple function s, there exist $n \in \mathbb{N}$ as well as $A_i \in \Sigma$ and $x_i \in X$ for $1 \leq i \leq n$ such that $A_i \cap A_j = \emptyset$ for $i \neq j$ and $s = \Sigma_{i=1}^{n} x_i \chi_{A_i}$. Here χ_A denotes the characteristic function of the set A, i.e.,

$$\chi_A(\omega) = \begin{cases} 1, & \omega \in A, \\ 0, & \omega \notin A. \end{cases}$$

For such a function, the expression

$$\int_\Omega s \, d\mu := \int_\Omega s(\omega) \, d\mu(\omega) := \sum_{i=1}^{n} \mu(A_i) x_i \in X$$

is well-defined. This definition can then be extended in the following way.

Definition 2.4.2. A function $f \colon \Omega \to X$ is called *Bochner integrable* (w.r.t. μ), if there exists a sequence of simple functions $(f_n)_{n \in \mathbb{N}}$ with $f_n \colon \Omega \to X$ for all $n \in \mathbb{N}$ such that

$$\lim_{n \to \infty} \int_\Omega \|f_n(\omega) - f(\omega)\|_X \, d\mu(\omega) = 0.$$

In this case, for any $A \in \Sigma$, the limit

$$\int_A f \, d\mu := \lim_{n \to \infty} \int_\Omega \chi_A f_n \, d\mu \in X$$

exists, does not depend on the approximating sequence $(f_n)_{n \in \mathbb{N}}$, and is called the *Bochner integral of f over A.*

Remark 2.4.3. Note that we will primarily be dealing with the case where $\Omega \subset \mathbb{R}^d$ is equipped with the Lebesgue-Borel-measure on the trace σ-Algebra

$$\Sigma = \{A \cap \Omega : A \in \mathcal{B}(\mathbb{R}^d)\}.$$

In this case we will write $\int_\Omega f(x) \, dx := \int_\Omega f(x) \, d\mu(x)$ and treat (Ω, Σ, μ) as interchangeable with Ω.

For the purpose of applications, the approximation condition of this definition is somewhat unwieldy to verify. This can be relieved in the following way.

Definition 2.4.4. A function $f \colon \Omega \to X$ is called

(i) *strongly measurable* (w.r.t. μ), if there exists a sequence of simple functions $(f_n)_{n \in \mathbb{N}}$ such that for μ-almost all $\omega \in \Omega$ one has

$$f(\omega) = \lim_{n \to \infty} f_n(\omega),$$

i.e., if $\lim_{n \to \infty} \|f_n(\omega) - f(\omega)\|_X = 0$ for all $\omega \in \Omega \setminus N$ for some set $N \in \Sigma$ such that $\mu(N) = 0$,

(ii) *weakly measurable*, if the composition $\varphi \circ f \colon \Omega \to \mathbb{K}$ is measurable for any continuous linear functional $\varphi \colon X \to \mathbb{K}$,

(iii) *almost separably-valued* (w.r.t. μ), if the set $\{f(\omega) : \omega \in \Omega \setminus N\}$ is separable for some set $N \in \Sigma$ such that $\mu(N) = 0$.

Remark 2.4.5. It is straightforward to see the following.

- If f is Bochner integrable, then f is strongly measurable.
- If f is (strongly) measurable, then it is weakly measurable.
- If there exists a sequence of compact sets $(K_n)_{n\in\mathbb{N}}$ such that $\Omega = \cup_{n\in\mathbb{N}} K_n$ and $f \colon \Omega \to X$ is a continuous function, then its range $\{f(\omega) : \omega \in \Omega\}$ is separable.

We further have the following equivalences.

Proposition 2.4.6.

1. *A function $f \colon \Omega \to X$ is strongly measurable if and only if it is weakly measurable and almost separably-valued, see [77].*

2. *A strongly measurable function $f \colon \Omega \to X$ is Bochner integrable if and only if the mapping $\Omega \ni \omega \mapsto \|f(\omega)\|_X \in [0, \infty)$ is integrable, see [14]. In this case one has*

$$\left\| \int_\Omega f \, d\mu \right\|_X \leq \int_\Omega \|f\|_X \, d\mu.$$

The vector-valued L^p-spaces are then defined as follows.

Definition 2.4.7. Let $p \in [1, \infty]$. The space of X-valued L^p-functions on (Ω, Σ, μ) is given by

$$L^p(\Omega, \Sigma, \mu; X) := \{f \colon \Omega \to X : f \text{ strongly measurable }, \|f\|_{L^p(\Omega,\Sigma,\mu;X)} < \infty\}$$

with the norm

$$\|f\|_{L^p(\Omega,\Sigma,\mu;X)} := \begin{cases} \inf\{C > 0 : \mu(\{\omega \in \Omega : \|f(\omega)\|_X > C\}) > 0\}, & p = \infty, \\ \left(\int_\Omega \|f\|_X^p \, d\mu\right)^{1/p}, & p \in [1, \infty). \end{cases}$$

We will write $L^p(\Omega; X)$ whenever $\Omega \subset \mathbb{R}^d$ is equipped with the Lebesgue-Borel-measure. Note that since X is a Banach space, so is $L^p(\Omega, \Sigma, \mu; X)$.

2.4.2 Anisotropic L^q-L^p-spaces

Consider two nonempty measurable sets $U_1 \subset \mathbb{R}^{d_1}$, $U_2 \subset \mathbb{R}^{d_2}$ and their product

$$U := U_1 \times U_2 \subset \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}.$$

Definition 2.4.8. Given $p, q \in [1, \infty]$ the *anisotropic L^q-L^p-space* is given by

$$L^q(U_1; L^p(U_2)) := \{f\colon U \to \mathbb{C} \text{ measurable} : \|f\|_{L^q_H L^p_z(U)} < \infty\},$$

equipped with the norm

$$\|f\|_{L^q(U_1;L^p(U_2))} := \begin{cases} \left(\int_{U_1} \|f(x', \cdot)\|^q_{L^p(U_2)}\, dx'\right)^{1/q}, & q \in [1, \infty), \\ \operatorname{ess\,sup}_{x' \in U_1} \|f(x', \cdot)\|_{L^p(U_2)}, & q = \infty. \end{cases}$$

Then $L^q(U_1; L^p(U_2))$ is a Banach space for all $p, q \in [1, \infty]$. Since we are interested specifically in cylindrical domains $U \subset \mathbb{R}^3$ of the form $U = U' \times U_3 \subset \mathbb{R}^2 \times \mathbb{R}$, we will also write

$$L^q_H L^p_z(U) := L^q(U'; L^p(U_3)).$$

In many ways these spaces behave as one would expect from the isotropic case $p = q$. In the following we give some examples of properties which we will utilize.

Given a domain U, let $C^\infty_c(U)$ denote the space of smooth functions $f\colon U \to \mathbb{C}$ with compact support and $C_0(\mathbb{R}^d)$ the space of continuous functions $f\colon \mathbb{R}^d \to \mathbb{C}$ vanishing at infinity. Then $C^\infty_c(\mathbb{R}^3)$ is dense in $L^q_H L^p_z(\mathbb{R}^3)$ whenever $p, q \in [1, \infty)$ as well as

$$\overline{C^\infty_c(\mathbb{R}^3)}^{\|\cdot\|_{L^\infty_H L^p_z}} = C_0(\mathbb{R}^2; L^p(\mathbb{R})), \quad \overline{C^\infty_c(\mathbb{R}^3)}^{\|\cdot\|_{L^q_H L^\infty_z}} = L^q(\mathbb{R}^2; C_0(\mathbb{R})), \tag{2.4.1}$$

via a vector-valued version of the Stone-Weierstrass theorem, see [58, Theorem 1]. Assuming $q_1 \geq q_2$ and that $U' \subset \mathbb{R}^2$ has finite measure, then one has

$$L^{q_1}_H L^p_z(U) \hookrightarrow L^{q_2}_H L^p_z(U).$$

Similarly, if $p_1 \geq p_2$ and $U_3 \subset \mathbb{R}$ has finite measure, then one has

$$L^q_H L^{p_1}_z(U) \hookrightarrow L^q_H L^{p_2}_z(U).$$

More details are presented in [49, Section 5].

Two important estimates in these types of spaces are the anisotropic versions of the Hölder and Young's inequality. Recall that given $f, g\colon \mathbb{R}^d \to \mathbb{C}$, their *convolution* is given by

$$(f * g)(x) := \int_{\mathbb{R}} f(x - y) g(y)\, dy = \int_{\mathbb{R}} g(x - y) f(y)\, dy, \quad x \in \mathbb{R}^d.$$

Then we have the following.

Lemma 2.4.9.

1. *Let* $p, p_1, p_2, q, q_1, q_2 \in [1, \infty]$ *with* $1/p_1 + 1/p_2 = 1/p$ *and* $1/q_1 + 1/q_2 = 1/q$. *Then for all* $f \in L^{q_1}_H L^{p_1}_z(U)$ *and* $g \in L^{q_2}_H L^{p_2}_z(U)$ *it holds that* $fg \in L^q_H L^p_z(U)$ *with*

$$\|fg\|_{L^q_H L^p_z(U)} \leq \|f\|_{L^{q_1}_H L^{p_1}_z(U)} \|g\|_{L^{q_2}_H L^{p_2}_z(U)}.$$

2. *For any* $p, q \in [1, \infty]$ *one has*

$$\|g * f\|_{L^q_H L^p_z(\mathbb{R}^3)} \leq \|g\|_{L^1(\mathbb{R}^3)} \|f\|_{L^q_H L^p_z(\mathbb{R}^3)}$$

for all $f \in L^q_H L^p_z(\mathbb{R}^3)$ *and* $g \in L^1(\mathbb{R}^3)$.

The first estimate is obtained by applying the Hölder inequality separably in horizontal and vertical variables, respectively. For the second estimate see [45, Theorem 3.1].

2.4.3 Vector-valued tempered distributions

We now introduce tempered distributions and the Fourier transform for the vector-valued case as in [4, Chapter 3, Section III.4.1 and III.4.2].

Recall that the space of *Schwartz functions* is given by

$$\mathcal{S}(\mathbb{R}^d) := \{\varphi \in C^\infty(\mathbb{R}^d; \mathbb{K}) : \sup_{x \in \mathbb{R}^d} (1 + |x|^2)^{s/2} |\partial^\alpha \varphi(x)| < \infty \text{ for all } s \in \mathbb{R}, \alpha \in \mathbb{N}^d\}$$

where $\partial^\alpha = \partial^{\alpha_1}_{x_1} \ldots \partial^{\alpha_d}_{x_d}$ for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$.

Definition 2.4.10.

1. The space of *X-valued tempered distributions* is then defined as

$$\mathcal{S}'(\mathbb{R}^d; X) := \{T \colon \mathcal{S}(\mathbb{R}^d) \to X : T \text{ is linear and continuous}\}$$

where T is continuous if and only if for any sequence $(\varphi_n)_{n \in \mathbb{N}}$ and φ in $\mathcal{S}(\mathbb{R}^d)$ satisfying the condition

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}^d} (1 + |x|^2)^{s/2} |\partial^\alpha \varphi_n(x) - \partial^\alpha \varphi(x)| = 0$$

for all $s \in \mathbb{R}$ and $\alpha \in \mathbb{N}^d$, it holds that $\lim_{n \to \infty} \|T\varphi_n - T\varphi\|_X = 0$.

2. An X-valued tempered distribution T is called *regular* if there exists a strongly measurable function $f \colon \mathbb{R}^d \to X$ such that for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$ one has

$$T\varphi = \int_{\mathbb{R}^d} f(x) \varphi(x) \, dx.$$

In this case one also writes $T = [f]$ or even $T = f$.

3. The *derivative of a tempered distribution* (w.r.t. a multi-index $\alpha \in \mathbb{N}^d$) is defined via
$$(\partial^\alpha T)\varphi := (-1)^{|\alpha|} T \partial^\alpha \varphi$$
and one writes $\partial^\alpha f = g$ whenever $\partial^\alpha [f] = [g]$. In this case g is called the *weak derivative* of f (w.r.t. the multi-index α).

4. The *Fourier transform* on $\mathcal{S}'(\mathbb{R}^d; X)$ is defined via $(\mathcal{F}T)\varphi := T(\mathcal{F}\varphi)$, where $\mathcal{F}\varphi$ is the scalar-valued Fourier transform of φ given by
$$(\mathcal{F}\varphi)(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\xi x} \varphi(x)\, dx.$$

5. Let $m \in C^\infty(\mathbb{R}^d; \mathbb{K})$ be such that the mapping $\varphi \mapsto m\varphi$ leaves $\mathcal{S}(\mathbb{R}^d)$ invariant. Then the multiplication of m and $T \in \mathcal{S}'(\mathbb{R}^d; X)$ is defined by $(mT)(\varphi) := T(m\varphi)$.

Remark 2.4.11. Many properties known from the scalar-valued case carry over to the vector-valued case. For example, the notation in point 2 is justified, since $[f] = [g]$ implies that $f = g$ almost everywhere. For point 3, it is straightforward to verify that whenever the partial derivative exists in the classical sense, it also exists in the weak sense and the two agree (up to a set of measure zero). In point 4, the mapping $T \mapsto \hat{T}$ is an isomorphism of $\mathcal{S}'(\mathbb{R}^d; X)$, see [4, Section III.4.2].

2.4.4 Sobolev, Bessel potential, Besov and Triebel-Lizorkin spaces

Like the Lesbesgue-spaces $L^p(\Omega, \Sigma, \mu)$, the Sobolev spaces $W^{k,p}(\Omega, \Sigma, \mu)$ can likewise be generalized to the vector valued case. Here we also introduce the vector-valued Sobolev-Slobodeckij, Besov, and Triebel-Lizorkin spaces as in [89].

Due to their prominence in functional analysis, we cannot hope to present a full picture of their properties. For this purpose we refer to [89, 90]. Of particular interest are the embedding and interpolation properties, see, e.g., [89, Section 2.3.2, 2.7.1, 3.3.1] and [89, Section 2.4 and 3.3.6], respectively.

Definition 2.4.12.

1. For $k \in \mathbb{N}$ and $p \in [1, \infty]$, the *Sobolev space* of k-times weakly differentiable, X-valued, L^p-integrable functions on $\mathbb{R}^d$ is defined via
$$W^{k,p}(\mathbb{R}^d; X) := \{f \in \mathcal{S}'(\mathbb{R}^d; X) : \partial^\alpha f \in L^p(\mathbb{R}^d; X) \text{ for all } \alpha \in \mathbb{N}^d : 0 \leq |\alpha| \leq k\}$$
and its norm is given by
$$\|f\|_{W^{k,p}(\mathbb{R}^d;X)} := \sum_{\substack{\alpha \in \mathbb{N}^d: \\ 0 \leq |\alpha| \leq k}} \|\partial^\alpha f\|_{L^p(\mathbb{R}^d;X)}.$$

2. For $p \in [1,\infty)$ as well as $s = k+\theta$ with $k \in \mathbb{N}$ and $\theta \in (0,1)$, the *Sobolev-Slobodeckij space* is defined via

$$W^{s,p}(\mathbb{R}^d;X) := \{f \in W^{k,p}(\mathbb{R}^d;X) : [\partial^\alpha f]_{\theta,p} < \infty \text{ for all } \alpha \in \mathbb{N}^d, 0 \leq |\alpha| \leq k\},$$

where

$$[g]_{\theta,p} := \left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\|g(x)-g(y)\|_X^p}{|x-y|^{\theta p+d}}\,dx\,dy\right)^{1/p},$$

and its norm is given by

$$\|f\|_{W^{s,p}(\Omega;X)} := \|f\|_{W^{k,p}(\Omega;X)} + \sum_{\substack{\alpha\in\mathbb{N}^d:\\ 0\leq|\alpha|\leq k}} [\partial^\alpha f]_{\theta,p}.$$

Definition 2.4.13. Let $p \in [1,\infty)$ as well as $s \in \mathbb{R}$ and $m_s(x) := (1+|x|^2)^{s/2}$. Then the X-valued *Bessel potential space* is defined via

$$H^{s,p}(\mathbb{R}^d;X) := \{f \in \mathcal{S}'(\mathbb{R}^d;X) : \mathcal{F}^{-1}(m_s\mathcal{F}f) \in L^p(\mathbb{R}^d;X)\}$$

and its norm is given by

$$\|f\|_{H^{s,p}(\mathbb{R}^d;X)} := \|\mathcal{F}^{-1}(m_s\mathcal{F}f)\|_{L^p(\mathbb{R}^d;X)}.$$

In the case $p = 2$ one also writes $H^s(\mathbb{R}^d;X)$ instead of $H^{s,p}(\mathbb{R}^d;X)$.

Remark 2.4.14. The spaces $W^{k,p}(\mathbb{R}^d;X)$, $W^{s,p}(\mathbb{R}^d;X)$ and $H^{s,p}(\mathbb{R}^d;X)$ are Banach spaces and one has $H^{k,p}(\mathbb{R}^d) = W^{k,p}(\mathbb{R}^d)$ whenever $k \in \mathbb{N}$. However, the identity $H^{s,p}(\mathbb{R}^d) = W^{s,p}(\mathbb{R}^d)$ does not hold for general $s > 0$, see [89, Section 2.2.2, Remark 1].

Following [89] we introduce Besov and Triebel-Lizorkin spaces as follows.

Definition 2.4.15. Let $\varphi = (\varphi_n)_{n\in\mathbb{N}}$ be a sequence of functions belonging to $\mathcal{S}(\mathbb{R}^d)$ such that

(i) the supports of these functions satisfy

$$\begin{aligned} \operatorname{supp}\varphi_0 &\subset \{x \in \mathbb{R}^d : |x| \leq 2\},\\ \operatorname{supp}\varphi_n &\subset \{x \in \mathbb{R}^d : 2^{n-1} \leq |x| \leq 2^{n+1}\}, \quad n \geq 1, \end{aligned}$$

(ii) for every multi-index $\alpha \in \mathbb{N}^d$ there exists a constant $C_\alpha > 0$ such that

$$2^{|\alpha|n}\|\partial^\alpha\varphi_n\|_{L^\infty(\mathbb{R}^d)} \leq C_\alpha \quad \text{for all } n \in \mathbb{N},$$

(iii) for every $x \in \mathbb{R}^d$ one has $\sum_{n=0}^\infty \varphi_n(x) = 1$.

Then, given parameters $s \in \mathbb{R}$ and $p, q \in [1, \infty]$, the X-valued *Besov space* is defined as

$$B^s_{p,q}(\mathbb{R}^d; X) := \{f \in \mathcal{S}'(\mathbb{R}^d; X) : (2^{sn} \|\mathcal{F}^{-1}\varphi_n \mathcal{F} f\|_{L^p(\mathbb{R}^d;X)})_{n\in\mathbb{N}} \in l^q(\mathbb{N})\}$$

and its norm is given by

$$\|f\|_{B^s_{p,q,\varphi}(\mathbb{R}^d;X)} := \|(2^{sn} \|\mathcal{F}^{-1}\varphi_n \mathcal{F} f\|_{L^p(\mathbb{R}^d;X)})_{n\in\mathbb{N}}\|_{l^q(\mathbb{N})}.$$

Furthermore, the X-valued *Triebel-Lizorkin space* is defined as

$$F^s_{p,q}(\mathbb{R}^d; X) := \{f \in \mathcal{S}'(\mathbb{R}^d; X) : x \mapsto 2^{sn} \|(\mathcal{F}^{-1}\varphi_n \mathcal{F} f)(x)\|_{l^q(\mathbb{N};X)} \in L^p(\mathbb{R}^d)\}$$

and its norm is given by

$$\|f\|_{F^s_{p,q,\varphi}(\mathbb{R}^d;X)} := \|x \mapsto 2^{sn} \|(\mathcal{F}^{-1}\varphi_n \mathcal{F} f)(x)\|_{l^q(\mathbb{N};X)}\|_{L^p(\mathbb{R}^d)}.$$

Remark 2.4.16. In this definition, for any two such sequences $\varphi^{(1)}$ and $\varphi^{(2)}$, the induced norms $\|\cdot\|_{E^s_{p,q,\varphi^{(1)}}(\mathbb{R}^d;X)}$ and $\|\cdot\|_{E^s_{p,q,\varphi^{(2)}}(\mathbb{R}^d;X)}$ are equivalent for $E \in \{B, F\}$ and so the space does not depend on the choice of φ, see [89, Section 2.3.2]. Since there is no canonical choice of φ, the notations $\|\cdot\|_{B^s_{p,q}(\mathbb{R}^d)}$ and $\|\cdot\|_{F^s_{p,q}(\mathbb{R}^d)}$, when used, only refer to equivalence classes of norms. However, a canonical equivalent norm can be found in a number of cases. In the case $p \in (1, \infty)$ and $q = 2$ one has by [89, Section 2.3.5] that

$$F^s_{p,2}(\mathbb{R}^d) = H^{s,p}(\mathbb{R}^d), \quad s \geq 0,$$

whereas for $p = q \in (1, \infty)$ it holds by [89, Section 2.2.2, Remark 3, Section 2.3.2 Proposition 2.(iii) and Section 2.3.5] that

$$B^s_{p,p}(\mathbb{R}^d) = F^s_{p,p}(\mathbb{R}^d) = W^{s,p}(\mathbb{R}^d), \quad s \in [0, \infty) \setminus \mathbb{N},$$

and in the case $p = q = \infty$ one has by [89, Section 2.3.5] that

$$B^\theta_{\infty,\infty}(\mathbb{R}^d) = C^\theta(\mathbb{R}^d) = \{f \in C(\mathbb{R}^d) : \|f\|_{C^\theta(\mathbb{R}^d)} < \infty\}, \quad \theta \in (0, 1),$$

where the latter denotes the X-valued *Hölder space*, i.e., the space of bounded Hölder continuous functions of exponent $\theta \in (0, 1)$, equipped with the norm

$$\|f\|_{C^\theta(\mathbb{R}^d)} := \|f\|_\infty + \sup_{\substack{x,y\in\mathbb{R}^d \\ x\neq y}} \frac{\|f(x) - f(y)\|_X}{|x - y|^\theta}.$$

One can even characterize the whole range of $B^s_{\infty,\infty}(\mathbb{R}^d)$ for $s > 0$ using Zygmund spaces, see [89, Theorem 2.5.7]. In all of these cases the norms of the left-hand and right-hand side spaces are equivalent and so the norm of the latter can be taken as a canonical representative.

Remark 2.4.17. By varying the parameters s and k, and thus the corresponding spaces, one obtains spaces that serve as scales of differentiability in the following sense.

It is obvious from the definitions that the Sobolev and Sobolev-Slobodeckij spaces that the mapping

$$f \mapsto \sum_{\substack{\alpha \in \mathbb{N}^d: \\ |\alpha| \leq k}} \|\partial^\alpha f\|_{W^{s-k,p}(\mathbb{R}^d)}, \quad 0 \leq k \leq s,$$

defines an equivalent norm on $W^{s,p}(\mathbb{R}^d)$. Seeing how these spaces are included in the Besov and Triebel-Lizorkin scales, it is reasonable to ask whether or not the norms of these spaces behave similarly w.r.t. derivatives. The answer to this is affirmative, see [89, Theorem 2.3.8]. In particular one has that the mapping $f \mapsto \partial^\alpha f$ for $|\alpha| = k$ maps $B^s_{p,q}(\mathbb{R}^d)$ and $F^s_{p,q}(\mathbb{R}^d)$ into $B^{s-k}_{p,q}(\mathbb{R}^d)$ and $F^{s-k}_{p,q}(\mathbb{R}^d)$, respectively.

We now turn to the issue of functions spaces on domains $\Omega \subset \mathbb{R}^d$.

Definition 2.4.18. For a domain $\Omega \subset \mathbb{R}^d$ and $E \in \{W^{k,p}, W^{s,p}, H^{s,p}, B^s_{p,q}, F^s_{p,q}\}$, where $s \geq 0, k \in \mathbb{N}$, and $p, q \in [1, \infty]$, the corresponding space of functions $f \colon \Omega \to X$ is given via

$$E(\Omega; X) := \{f\big|_\Omega : f \in E(\mathbb{R}^d; X)\},$$

and equipped with the norm

$$\|f\|_{E(\Omega;X)} := \inf\{\|g\|_{E(\mathbb{R}^d;X)} : g \in E(\mathbb{R}^d; X), f = g\big|_\Omega\}.$$

A detailed study of these spaces can be found in [89, Chapter 3]. In particular, they are again Banach spaces.

Remark 2.4.19. The definition above is not the only possible way of defining these spaces. Given a domain $\Omega \subset \mathbb{R}^d$, consider a function $f \in L^p(\Omega)$ such that for any $\alpha \in \mathbb{N}^d$ such that $0 \leq |\alpha| \leq k$ there exists $g_\alpha \in L^p(\Omega)$ satisfying

$$\int_\Omega f(x) \partial^\alpha \varphi(x)\, dx = (-1)^{|\alpha|} \int_\Omega g_\alpha(x) \varphi(x)\, dx$$

for all $\varphi \in C^\infty_c(\Omega)$, i.e., for all smooth functions such that the support of φ is a compact subset of Ω. It is reasonable to ask whether or not this is sufficient or necessary for $f \in W^{k,p}(\Omega)$ with $\partial^\alpha f = g_\alpha$. Whereas one easily observes that this is a necessary condition, it is, however, not sufficient. As an example, consider the case of a slit circle domain

$$\Omega := \{x = (x_1, x_2) \in \mathbb{R}^2 : |x| < 1\} \setminus \{(x_1, 0) \in \mathbb{R}^2 : x_1 \in [0, 1)\}$$

and a smooth function $f \colon \Omega \to \mathbb{R}$ such that for all $x = (x_1, x_2) \in \Omega$ with $x_1 > 1/2$ one has

$$f(x_1, x_2) = \begin{cases} 1, & x_2 > 0 \\ 0, & x_2 < 0. \end{cases}$$

If f were an element of $W^{1,p}(\Omega)$, there would exist a function $g \in W^{1,p}(\mathbb{R}^2)$ such that $g\big|_\Omega = f$. However, by [89, Section 2.7.1], any such g would be a continuous function on

$\mathbb{R}^2$ for $p \in (2, \infty)$ and f clearly does not have a continuous extension onto the open unit disc.

As this example illustrates, we could identify the spaces $W^{1,p}(\Omega)$ and $W^{1,p}(B_1(0))$ with each other since the external definition via restrictions implies good behavior at the boundary of the underlying domain, whereas a definition that only takes into account functions supported in Ω does not require such a thing.

However, this problem can be avoided by only considering domains with sufficiently regular boundary. In particular, if Ω is a bounded Lipschitz domain, then any such f does admit an extension belonging to $W^{1,p}(\mathbb{R}^2)$ and thus belongs to $W^{1,p}(\Omega)$, see [86, Chapter VI, Theorem 5].

While the fact that $W^{s,p}(\mathbb{R}^d) = B^s_{p,p}(\mathbb{R}^d) = F^s_{p,p}(\mathbb{R}^d)$ for $s \in (0, \infty) \setminus \mathbb{N}$ allows for a similar descriptions of function $f \in W^{s,p}(\Omega)$, there generally is no simple description for the space $F^s_{p,q}(\Omega)$ when $p \neq q$ that only uses the properties on Ω. However, this is different for $B^s_{p,q}(\Omega)$, see [89, Theorem 2.5.7].

2.4.5 Periodic spaces

Let $G := (0, 1)$ and $\Omega := G \times (-h, 0)$ for some $h > 0$. Following [49] and [41], we define spaces of functions that are periodic in the variables pertaining to G.

Given $m \in \mathbb{N}$, a function $f : \overline{\Omega} \to \mathbb{C}$ is *periodic of order m on Γ_l* if for all $k \in \mathbb{N}$ with $0 \leq k \leq m$, as well as $x, y \in (0, 1)$ and $z \in (-h, 0)$ it holds that

$$\frac{\partial^k f}{\partial x^k}(0, y, z) = \frac{\partial^k f}{\partial x^k}(1, y, z) \quad \text{and} \quad \frac{\partial^k f}{\partial y^k}(x, 0, z) = \frac{\partial^k f}{\partial y^k}(x, 1, z).$$

For a function $f : \overline{G} \to \mathbb{C}$, periodicity of order m on ∂G is defined analogously. Given the spaces

$$\begin{aligned} C^\infty_{\text{per}}(\overline{\Omega}) &:= \{f \in C^\infty(\overline{\Omega}) : f \text{ is periodic on } \Gamma_l \text{ for any order } m \in \mathbb{N}\}, \\ C^\infty_{\text{per}}(\overline{G}) &:= \{f \in C^\infty(\overline{G}) : f \text{ is periodic on } \partial G \text{ for any order } m \in \mathbb{N}\}, \end{aligned}$$

the periodic Besov and Bessel potential spaces for $s > 0$ and $p, q \in (1, \infty)$ are defined as

$$B^s_{p,q,\text{per}}(\Omega) := \overline{C^\infty_{\text{per}}(\overline{\Omega})}^{\|\cdot\|_{B^s_{p,q}(\Omega)}}, \quad H^{s,p}_{\text{per}}(\Omega) := \overline{C^\infty_{\text{per}}(\overline{\Omega})}^{\|\cdot\|_{H^{s,p}(\Omega)}}, \tag{2.4.2}$$

while $B^s_{p,q,\text{per}}(G)$ and $H^{s,p}_{\text{per}}(G)$ are defined analogously.

2.5 Operator semigroups and generators

Let X be a Banach space over $\mathbb{C}$, $D(A) \subset X$ a subspace, and $A : D(A) \to X$ a linear operator. We will introduce the notions of semigroups and generators with an interest in applications to initial values problems of the form

$$\partial_t u(t) - Au(t) = 0, \quad t > 0, \quad u(0) = x, \tag{2.5.1}$$

where $x \in X$ is a given initial value and $u\colon [0,\infty) \to X$ is an unknown solution. Note that it is not uncommon to find discussions of these problems where the term $-Au(t)$ is instead written as $Au(t)$, resulting in different conventions for some of the classes of operators we will introduce. Here, we primarily followed [26], but also refer to [11,71,91] for more details.

2.5.1 Semigroups

Definition 2.5.1.

1. An *operator semigroup* is a family of operators $(S(t))_{t\geq 0} \subset \mathcal{L}(X)$ or parametrization $S\colon [0,\infty) \to \mathcal{L}(X)$ such that

$$S(t_1 + t_2) = S(t_1)S(t_2), \quad t_1, t_2 \geq 0, \quad S(0) = 1.$$

   Here 1 denotes the identity mapping on X.

2. A semigroup is called *bounded* if $(S(t))_{t\geq 0}$ is uniformly bounded in $\mathcal{L}(X)$, and *exponentially stable* if there exist constants $C, \beta > 0$ such that the estimate

$$\|S(t)x\|_X \leq Ce^{-\beta t}\|x\|_X$$

   holds for all $t > 0$ and $x \in X$.

3. A semigroup is called *strongly continuous* if the mapping $S\colon [0,\infty) \to \mathcal{L}(X)$ is strongly continuous, i.e., if the orbit mappings $[0,\infty) \ni t \mapsto S(t)x \in X$ are continuous for all $x \in X$.

4. Given a semigroup S, its *generator* is given by the mapping

$$A\colon D(A) \to X, \quad Ax := \lim_{t\searrow 0} t^{-1}(S(t)x - x),$$

   defined on the domain

$$D(A) := \{x \in X : \lim_{t\searrow 0} t^{-1}(S(t)x - x) \text{ exists}\}.$$

Given a semigroup S on X with generator A, the mapping $[0,\infty) \ni t \mapsto S(t)x \in X$ is continuous if and only if x belongs to the closure of $D(A)$ in X. In particular, a semigroup is strongly continuous if and only if A is *densely defined*, i.e., if $D(A)$ is dense in X, see [71, Proposition 2.1.4]. One further has that A is bounded if and only if $D(A) = X$.

The class of operators that generate strongly continuous semigroups is well-understood and characterized by the Hille-Yosida theorem. One further has that a semigroup is uniquely determined by its generator.

For $\theta \in (0,\pi)$ we denote the sector in the complex plane with opening angle θ by

$$\Sigma_\theta := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg(\lambda)| < \theta\}.$$

While from a philosophical perspective it may seem that semigroups defined on $[0, \infty)$ are sufficient as the solution operators to problems of the form (2.5.1) as one would be primarily interested in the behavior of solutions in real time, the question of whether or not such solution operators may be extended onto such a complex sector is of interest in the mathematical analysis of such problems.

Definition 2.5.2. Let $\theta \in (0, \pi/2]$. A family of operators $(S(\tau))_{\tau \in \Sigma_\theta \cup \{0\}} \subset \mathcal{L}(X)$ or parametrization $S\colon \Sigma_\theta \cup \{0\} \to \mathcal{L}(X)$ is called an *analytic semigroup* of angle θ if one has

$$S(\tau_1 + \tau_2) = S(\tau_1)S(\tau_2), \quad \tau_1, \tau_2 \in \Sigma_\theta \cup \{0\}, \quad S(0) = 1, \tag{2.5.2}$$

and the mapping $S\colon \Sigma_\theta \to \mathcal{L}(X)$ is analytic. It is called *bounded analytic* if the family $\{S(\tau) : \tau \in \Sigma_\psi\}$ is uniformly bounded in $\mathcal{L}(X)$ for all $\psi \in [0, \theta)$.

2.5.2 Sectorial operators

Given a mapping $A\colon D(A) \to X$, we take $D(A)$ to be equipped with the *graph norm* given by

$$\|x\|_A := \|x\|_X + \|Ax\|_X, \quad x \in D(A).$$

We denote by

$$\rho(A) := \{\lambda \in \mathbb{C} : \lambda - A\colon D(A) \to X \text{ has continuous inverse}\}$$

the *resolvent set* of A and its *resolvent* by $(\lambda - A)^{-1}\colon X \to D(A)$ for $\lambda \in \rho(A)$. Note that due to the embedding $D(A) \hookrightarrow X$ one also has that $(\lambda - A)^{-1} \in \mathcal{L}(X)$ whenever $\lambda \in \rho(A)$ and that A is closed whenever $\rho(A) \neq \emptyset$.

Definition 2.5.3. An operator A is called a *sectorial operator*, if $D(A)$ and $R(A)$ are both dense in X and there exists an angle $\phi \in (0, \pi)$ such that $\Sigma_{\pi-\phi} \subset \rho(-A)$ and the family of operators

$$\{\lambda(\lambda + A)^{-1} : \lambda \in \Sigma_{\pi-\phi}\}$$

is uniformly bounded in $\mathcal{L}(X)$. The *spectral angle of* A, denoted by ϕ_A, is defined as the infimum of all $\phi \in (0, \pi)$ for which this holds. We also write $\mathcal{S}(X)$ for the set of sectorial operators on X.

For more details on sectorial operators, see [26, Chapter 1] or [71, Chapter 2]. In particular, any $A \in \mathcal{S}(X)$ is injective, see [26, Proposition 1.2].

2.5.3 Dunford Calculus

Definition 2.5.4. Let A be a sectorial operator and $\phi_A < \psi < \phi < \pi$. Further let $f\colon \Sigma_\phi \to \mathbb{C}$ be a suitable bounded holomorphic function. Then the mapping

$$f \mapsto f(A) := \frac{1}{2\pi i} \int_{\Gamma_\psi} f(\lambda)(\lambda + A)^{-1}\, d\lambda, \quad \Gamma_\psi = (\infty, 0]e^{i\psi} \cup [0, \infty)e^{-i\psi}, \tag{2.5.3}$$

is called the *Dunford calculus* of A.

Remark 2.5.5.

1. Since the mapping $\rho(-A) \ni \lambda \mapsto (\lambda + A)^{-1} \in \mathcal{L}(X)$ is analytic, any choice of angles ϕ and ψ in the definitions above yields the same right-hand sides.

2. Given a sectorial operator A, it is not straightforward to see for which class of bounded holomorphic functions f the definition of $f(A)$ yields a bounded linear operator, compare Definition 2.6.1. However, it is known that for general sectorial operators A, one has that $f(A)$ defines a bounded linear operator on X whenever f exhibits suitable behavior at $z = 0$ and as z goes to infinity, compare, e.g., [26, Chapter 1.4 and 2.4], particularly the definition of the space $\mathcal{H}_0^\infty(\Sigma_\phi)$.

Proposition 2.5.6.

1. *In the case $\phi_A < \pi/2$ one may define*
$$S_A(t) := \frac{1}{2\pi i} \int_{\Gamma_{\psi,\varepsilon}} e^{-t\lambda}(\lambda + A)^{-1}\, d\lambda, \quad t > 0, \tag{2.5.4}$$
for auxiliary parameters $\psi \in (\phi_A, \pi/2)$, $\varepsilon > 0$, and
$$\Gamma_{\psi,\varepsilon} := (\infty, \varepsilon]e^{i\psi} \cup \varepsilon e^{i[\psi,-\psi]} \cup [\varepsilon, \infty)e^{-i\psi}.$$
Then the mapping $(0, \infty) \ni t \mapsto S_A(t) \in \mathcal{L}(X)$ has an analytic extension onto the sector $\Sigma_{\pi/2-\phi_A}$. Further setting $S(0) = 1$ it holds that S is an analytic semigroup of angle $\pi/2 - \phi_A$ generated by $-A$.

2. *By [71, Proposition 2.1.1] it holds that the semigroup is uniformly bounded on $[0, \infty)$. If there instead exists $\nu \in \mathbb{R}$ such that $\nu + A$ is sectorial, then $-A$ generates the semigroup given by $S_A(z) := e^{\nu z} S_{\nu+A}(z)$ and it holds that $\|S_A(t)\|_{\mathcal{L}(X)} \le Ce^{\nu t}$ for all $t > 0$ and some constant $C > 0$.*

3. *Conversely, one may characterize the resolvent of A as the Laplace transform of its generated semigroup, i.e., one has*
$$(\lambda + A)^{-1} = \int_0^\infty e^{-\lambda t} S_A(t)\, dt,$$
see, e.g., [71, Lemma 2.1.6].

4. *The representation* (2.5.4) *further allows us to translate resolvent estimates into semigroup estimates in the following way. Let B and C be operators such that*
$$\{|\lambda|^\alpha B(\lambda + A)^{-1}C : \lambda \in \Sigma_{\pi-\phi}\}$$
is a uniformly bounded family of well-defined operators on X for some $\phi \in (\phi_A, \pi/2)$ and $\alpha \in (0, 1)$. Then an elementary calculation yields that the mapping
$$(0, \infty) \ni t \mapsto t^{1-\alpha} B S_A(t) C \in \mathcal{L}(X)$$
is uniformly bounded as well.

2.6 Bounded and $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus

Given $\phi \in (0, \pi)$, consider the space

$$\mathcal{H}^\infty(\Sigma_\phi) := \{f \colon \Sigma_\phi \to \mathbb{C} : f \text{ bounded and holomorphic}\}.$$

As previously mentioned in Remark 2.5.5, the integral formula of the Dunford calculus (2.5.3) does not necessarily yield a bounded operator for arbitrary functions $f \in \mathcal{H}^\infty(\Sigma_\phi)$ and the space of functions such that $f(A) \in \mathcal{L}(X)$ typically does not admit an explicit characterization. This motivates the following definition.

Definition 2.6.1. Let A be a sectorial operator on a Banach space X with spectral angle ϕ_A. Then A admits a *bounded* $\mathcal{H}^\infty$*-calculus* if there exists $\phi \in (\phi_A, \pi)$ such that the Dunford calculus (2.5.3) admits an extension to a bounded linear mapping

$$\mathcal{H}^\infty(\Sigma_\phi) \ni f \mapsto f(A) \in \mathcal{L}(X).$$

The infimum of all such angles $\phi > \phi_A$ is denoted by ϕ_A^∞ and called the $\mathcal{H}^\infty$*-angle of* A. The set of sectorial operators admitting a bounded $\mathcal{H}^\infty$-calculus on X is in turn denoted by $\mathcal{H}^\infty(X)$.

Remark 2.6.2. For practical purposes, the class of operators $\mathcal{H}^\infty(X)$ admits the following useful characterization. A sectorial operator $A \in S(X)$ admits a bounded $\mathcal{H}^\infty$-calculus of $\mathcal{H}^\infty$-angle ϕ_A^∞ on X if and only if for all $\phi \in (\phi_A^\infty, \pi)$ there exists a constant $C_\phi > 0$ such that

$$\|f(A)\|_{\mathcal{L}(X)} \leq C_\phi \|f\|_{L^\infty(\Sigma_\phi)}$$

for all $f \in \mathcal{H}_0^\infty(\Sigma_\phi)$. For the definition of the space $\mathcal{H}_0^\infty(\Sigma_\phi)$ see [26, Chapter 2.4].

Given a family of bounded operators on some Banach space X, the property of uniform boundedness is obviously of significance from the perspective of functional analysis. However, on general Banach spaces X, it turns out that uniform boundedness is often not sufficient to derive properties that one can indeed derive on spaces such as $\mathbb{R}^d$ and $\mathbb{C}^d$, compare, e.g., [26, Theorem 4.4]. It turns out that for these purposes, one requires an even stronger property, namely that of $\mathcal{R}$-boundedness. Due to its particularly technical nature, we omit a definition here and simply refer to the literature, see, e.g., [26, Definition 3.1].

Definition 2.6.3. Let $A \in \mathcal{H}^\infty(X)$. If there exists $\phi > \phi_A^\infty$ such that the set

$$\{f(A) : f \in \mathcal{H}^\infty(\Sigma_\phi), \|f\|_{L^\infty(\Sigma_\phi)} \leq 1\}$$

is $\mathcal{R}$*-bounded* in $\mathcal{L}(X)$, then A admits an $\mathcal{R}$*-bounded* $\mathcal{H}^\infty$*-calculus*. The infimum of all such angles $\phi > \phi_A^\infty$ is called the $\mathcal{R}\mathcal{H}^\infty$*-angle of* A and denoted by $\phi_A^{\mathcal{R}\infty}$. The set of operators $A \in \mathcal{H}^\infty(X)$ admitting an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus is in turn denoted by $\mathcal{R}\mathcal{H}^\infty(X)$.

Due to its complicated definition, one would typically like to try to avoid having to verify the $\mathcal{R}$-boundedness of a family of operators. Fortunately, there are classes of Banach spaces where this difficulty is alleviated. In particular, whenever the Banach space X satisfies the property (α), one has $\mathcal{H}^\infty(X) = \mathcal{R}\mathcal{H}^\infty(X)$ and $\phi_A^\infty = \phi_A^{\mathcal{R}\infty}$ for all $A \in \mathcal{H}(X)$, see, e.g., [55, Theorem 5.3.1]. For the definition of the property (α) see [62, Property 4.9]. When X is a Hilbert space one even has that any uniformly bounded set of operators is also $\mathcal{R}$-bounded, but this is not generally the case, even when X has property (α).

2.6.1 Bounded imaginary powers

Consider a sectorial operator A and let A^z for suitable $z \in \mathbb{C}$ be defined via the extended functional calculus presented in [26, Chapter 2] with domain $D(A^z)$.

Definition 2.6.4. An operator $A \in \mathcal{S}(X)$ has *bounded imaginary powers* if it holds that $A^{is} \in \mathcal{L}(X)$ for all $s \in \mathbb{R}$ and the family of operators

$$\{A^{is} : s \in [-1, 1]\}$$

is uniformly bounded in $\mathcal{L}(X)$. The set of operators on X with bounded imaginary powers is denoted by $\mathcal{BIP}(X)$.

This property is useful as it allows one to characterize domains of fractional powers. Given $A \in \mathcal{BIP}(X)$ and taking $\vartheta \in (0, 1)$ and equipping $D(A^\vartheta)$ with the norm

$$\|x\|_\vartheta := \|x\|_X + \|A^\vartheta x\|_X$$

one has that

$$D(A^\vartheta) = [X, D(A)]_\vartheta, \quad \vartheta \in (0, 1)$$

with equivalent norms, see [26, Theorem 2.5].

2.7 Maximal regularity

2.7.1 Definition and basic properties

Definition 2.7.1. Let X be a Banach space, $A\colon D(A) \to X$ a closed operator, and $p \in [1, \infty]$. Then A has *maximal L^p-regularity* if for all $f \in L^p(\mathbb{R}_+; X)$, $\mathbb{R}_+ := (0, \infty)$, the Cauchy problem

$$\partial_t u - Au = f \text{ on } \mathbb{R}_+, \quad u(0) = 0, \tag{2.7.1}$$

has a unique solution $u \in H^{1,p}(\mathbb{R}_+; X)$ such that $u(t) \in D(A)$ for almost all $t > 0$ and $Au \in L^p(\mathbb{R}_+; X)$. The set of operators with maximal L^p-regularity on X is denoted by $\mathcal{M}_p(X)$.

Remark 2.7.2.

1. Given $p, q \in (1, \infty)$ one has
$$\mathcal{M}_1(X) \subset \mathcal{M}_p(X) = \mathcal{M}_q(X),$$
i.e., if A has maximal L^p-regularity for some $p \in [1, \infty]$, then A also has maximal L^q-regularity for all $q \in (1, \infty)$, see [84].

2. If A has maximal L^p-regularity on X, then it generates a strongly continuous, bounded analytic semigroup on X, see, e.g., [81, Section 10], and if X is a Hilbert apace, then the reverse holds true as well, see [24].

3. It holds that if every operator that generates a strongly continuous bounded analytic semigroup on X has maximal L^p-regularity on X, then X is isomorphic to a Hilbert space. Abstract examples of operators that generate a strongly continuous, bounded analytic semigroup on a non-Hilbert space X but do not have maximal L^p-regularity on X have also been constructed, see [54]. Whether or not there are differential operators of this type remains an open question.

4. Given $A \in \mathcal{M}_p(X)$ one may also consider the problem (2.7.1) with inhomogeneous initial data
$$u(0) = u_0 \in (X, D(A))_{1-1/p,p},$$
where $(\cdot, \cdot)_{\vartheta,p}$ denotes the real interpolation functor, and by the closed graph theorem there exists a constant $C > 0$ such that
$$\|u\|_{H^{1,p}(\mathbb{R}_+;X)} + \|Au\|_{L^p(\mathbb{R}_+;X)} \leq C \left(\|f\|_{L^p(\mathbb{R}_+;X} + \|u_0\|_{(X,D(A))_{1-1/p,p}} \right),$$
see, e.g., [6] or [27].

5. If $A - \lambda$ has maximal regularity for some $\lambda > 0$, then A has the property of maximal regularity when only finite intervals $(0, T)$ for $T \in (0, \infty)$ are considered instead of $\mathbb{R}_+$ and in these instances we will still say that A has maximal regularity.

The property of maximal regularity is related to that of $\mathcal{R}$-boundedness, see [26, Theorem 4.4]. On one hand, whenever A has maximal L^p-regularity for some $p \in (1, \infty)$, then one has $i\mathbb{R} \subset \rho(A)$ and the family of operators
$$\{A(is - A)^{-1} : s \in \mathbb{R}\}$$
is $\mathcal{R}$-bounded in $\mathcal{L}(X)$. In particular, it holds that $0 \in \rho(A)$. If in addition X is a so-called *UMD space* and $-A$ is sectorial with spectral angle $\phi_{-A} < \pi/2$, then A has maximal L^p-regularity for all $p \in (1, \infty)$ if and only if the set
$$\{A(\lambda - A)^{-1} : \lambda \in \Sigma_{\pi-\theta}\}$$
is $\mathcal{R}$-bounded in $\mathcal{L}(X)$ for some $\theta \in (0, \pi/2)$. For a definition of the class of UMD spaces, see [3, Section 4.4]. In particular, if $-A \in \mathcal{RH}^\infty(X)$ satisfies $\phi^{\mathcal{R}\infty}_{-A} < \pi/2$, then one has $A \in \mathcal{M}_p(X)$ whenever X is a UMD space. If in addition X also has property (α), then one even has $A \in \mathcal{M}_p(X)$ for all $p \in (1, \infty)$ whenever $-A \in \mathcal{H}^\infty(X)$ with $\phi^\infty_{-A} < \pi/2$. By [55, Theorem 5.3] it even holds that $-A \in \mathcal{H}^\infty(A)$ implies that the resolvent set $\{A(\lambda - A)^{-1} : \lambda \in \Sigma_{\pi-\theta}\}$ is $\mathcal{R}$-bounded in $\mathcal{L}(X)$ for all $\theta > \phi^\infty_{-A}$ whenever X is a UMD space.

2.7.2 Time-weights

Consider the spaces

$$\begin{aligned} \mathbb{E}_0(\mathbb{R}_+) &:= L^p(\mathbb{R}_+;X), \\ \mathbb{E}_1(\mathbb{R}_+) &:= H^{1,p}(\mathbb{R}_+;X)\cap L^p(\mathbb{R}_+;D(A)), \\ X_{\gamma,p} &:= (X,D(A))_{1-1/p,p}. \end{aligned} \tag{2.7.2}$$

Then A has maximal L^p-regularity if and only if the mapping

$$\mathbb{E}_1(\mathbb{R}_+)\ni u\mapsto (\partial_t u - Au, u(0))\in \mathbb{E}_0(\mathbb{R}_+)\times X_{\gamma,p}$$

is an isomorphism between Banach spaces. This perspective also allows us to consider the Cauchy-Problem (2.7.1) in spaces with *time-weights*. In the following, we give a short introduction to the theory developed in [79].

Let $p\in(1,\infty)$, $\mu\in[0,1]$, and $k\in\mathbb{N}$. For an interval $I\subset[0,\infty)$, we denote the space of all functions $f\colon I\to X$ such that $f\in L^1(K;X)$ for all compact subsets $K\subset I$ by $L^1_{\mathrm{loc}}(I;X)$.

Then the *time-weighted L^p-space* and the corresponding Bessel-potential spaces are given by

$$\begin{aligned} L^p_\mu(I;X) &:= \{f\in L^1_{\mathrm{loc}}(I;X) : [t\mapsto t^{1-\mu}f(t)]\in L^p(I;X)\}, \\ H^{1,p}_\mu(I;X) &:= \{f\in L^p_\mu(I;X)\cap H^{1,1}_{\mathrm{loc}}(I;X) : \partial_t f\in L^p_\mu(I;X)\}, \\ H^{k+1,p}_\mu(I;X) &:= \{f\in L^p_\mu(I;X)\cap H^{1,1}_{\mathrm{loc}}(I;X) : \partial_t f\in H^{k,p}_\mu(I;X)\}. \end{aligned} \tag{2.7.3}$$

These spaces are Banach spaces when equipped with the respective norms

$$\|f\|_{L^p_\mu(I;X)} := \left(\int_I \|t^{1-\mu}f(t)\|_X^p\,dt\right)^{1/p}, \quad \|f\|_{H^{k,p}_\mu(I;X)} := \sum_{i=0}^k \|\partial_t^i f\|_{L^p_\mu(I;X)}.$$

Analogously to (2.7.2), one further defines

$$\mathbb{E}_{0,\mu}(I) := L^p_\mu(I;X), \quad \mathbb{E}_{1,\mu}(I) := H^{1,p}_\mu(I;X)\cap L^p_\mu(I;D(A)). \tag{2.7.4}$$

This then leads to the following definition.

Definition 2.7.3. Let $p\in(1,\infty)$ and $\mu\in[0,1]$. An operator A has *maximal L^p_μ-regularity* if for all $f\in\mathbb{E}_{0,\mu}(\mathbb{R}_+)$ the problem (2.7.1) has a unique solution $u\in\mathbb{E}_{1,\mu}(\mathbb{R}_+)$. The set of operators possessing maximal L^p_μ-regularity is denoted by $\mathcal{M}_{p,\mu}(X)$.

Remark 2.7.4.

1. Obviously the choice $\mu=1$ means that no time-weights are considered and so one trivially has $\mathcal{M}_{p,1}(X)=\mathcal{M}_p(X)$. Moreover, by [79, Theorem 2.4] it even holds that

   $$\mathcal{M}_{p,\mu}(X)=\mathcal{M}_p(X)$$

   whenever $1/p<\mu\le 1$.

2. The more general framework of time-weighted L^p-spaces for parameters $\mu \in [0,1)$ allow us to consider the problem (2.7.1) for data with non-integrable singularities at $t = 0$. A useful property of these time-weights is the smoothing effect when we consider the functions away from the potential singularity $t = 0$, since for all $\delta > 0$ it holds that
$$H^{1,p}_{\mu}(0,T;X) \cap L^p_{\mu}(0,T;D(A)) \hookrightarrow H^{1,p}(\delta,T;X) \cap L^p(\delta,T;D(A)).$$
3. As for the unweighted case, one can even consider initial data $u(0) = u_0$ belonging to the space
$$X_{\gamma,\mu,p} := (X, D(A))_{\mu-1/p,p}$$
whenever $1/p < \mu \leq 1$ by [79, Theorem 3.2]. In this case it even holds that
$$\mathbb{E}_{1,\mu} \hookrightarrow BUC([0,\infty); X_{\gamma,\mu,p}) \cap C((0,\infty); X_{\gamma,p})$$
by [79, Theorem 3.1]. In particular, it holds that $\mathbb{E}_1 \hookrightarrow BUC([0,\infty); X_{\gamma,p})$.

3 The Laplace operator in L^p and L^q-L^p-spaces

Recall that given a function $u\colon \Omega \to \mathbb{C}$ for some domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$, the *Laplace operator* is given by

$$\Delta u := \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} u = \sum_{i=1}^{d} \partial_{x_i}^2 u.$$

As the prototypical second order elliptic differential operator, there is a wide variety of literature and applicable results for its realizations in various partial differential equations, see, e.g., [3, 8, 26, 28, 44, 74, 75, 89], as well as the references therein. The corresponding parabolic initial value problem

$$\partial_t u(t,x) - \Delta u(t,x) = 0, \quad u(0,x) = u_0(x),$$

is called a *heat equation*. Whenever the problem admits unique solution for a particular class of initial values, the corresponding family of solution operators

$$u_0 \mapsto S(t)u_0 := u(t,\cdot), \quad t \geq 0$$

is called a *heat semigroup*.

The intention of this chapter is to present a foundation on which we may build during the following chapter, which is devoted to the study of the hydrostatic Stokes operator and semigroup. For this purpose, we will discuss the Laplace operator and its corresponding heat semigroup in various settings. Some of the results we establish are well-known, but we nevertheless include them here for the sake of staying self-contained. In Section 3.1 we will consider the heat semigroup on the whole space and show a combination of pointwise and norm estimates for various types of derivatives. Section 3.2 is devoted to the study of the resolvent problem for the Laplace operator on the whole space in anisotropic L^q-L^p-spaces. Concerning the issue of bounded domains, Section 3.3 covers the case where the underlying domain is an interval. We establish estimates in L^p-spaces for the case of periodic, as well as a combination of Neumann and Dirichlet boundary conditions. In Sections 3.4 and 3.5 we consider the case of a cylindrical domain with mixed Neumann, Dirichlet and periodic boundary conditions in L^p and anisotropic L^q-L^p-spaces, respectively.

These results have previously been published in [41, Section 3 and 4], [40, Section 3 and 4], [38, Section 4], and [39, Section 5].

3.1 Pointwise and L^p-estimates for the heat semigroup on the whole space

Probably the most famous parabolic partial differential equation is the heat equation on the whole space $\mathbb{R}^d$ for some dimension $d \geq 1$, i.e., the problem

$$\begin{aligned} \partial_t u(t,x) - \Delta u(t,x) &= 0, \qquad x \in \mathbb{R}^d,\ t>0, \\ u(0,x) &= u_0(x), \quad x \in \mathbb{R}^d. \end{aligned}$$

The heat semigroup corresponding to this problem is explicitly given by

$$S(t)u_0 = e^{t\Delta}u_0 := G_t * u_0, \quad G_t(x) = (4\pi t)^{-d/2}\exp\left(-|x|^2/4t\right), \quad x \in \mathbb{R}^d, \quad t>0,$$

and the functions G_t for $t>0$ are called *Gaussian kernels*. We begin by providing a pointwise estimate for their derivatives.

Lemma 3.1.1. *Let $d \geq 1$ and $\alpha \in \mathbb{N}^d$. Then there exists a constant $C = C_{d,\alpha} > 0$ such that for all $t > 0$ it holds that*

$$|\partial^\alpha G_t| \leq C t^{-|\alpha|/2} G_{2t}.$$

Proof. Let $t>0$ and $x \in \mathbb{R}^d$ as well as $1 \leq i \leq d$. Then we have that

$$\partial_i G_t(x) = t^{-1/2}(4\pi t)^{-d/2}\left(-\frac{x_i}{2t^{1/2}}\right)\exp\left(-|x|^2/4t\right)$$

and since

$$\frac{|x|}{2t^{1/2}}\exp\left(-|x|^2/8t\right) \leq C_0 := \sup\{a\exp\left(-a^2/2\right) : a>0\} < \infty,$$

we obtain

$$|\partial_i G_t| \leq 2^{d/2} C_0 t^{-1/2} G_{2t}.$$

Estimates for higher order terms are obtained analogously. □

Remark 3.1.2. Our approaches to the cases of the whole space $\mathbb{R}^d$ and bounded domains $\Omega \subset \mathbb{R}^d$ are very different. While pointwise estimates for kernels are a very powerful tool, we will only be using them for the case of the whole space where the study of these kernels is most straightforward. While a theory of kernels for heat semigroups on domains exists, compare [75, Chapter 6], it is not needed for our purposes. For more details concerning the case of the whole space we refer to [23].

The fractional powers of the negative Laplace operator $(-\Delta)^{\alpha/2}$ for $\alpha > 0$ can be defined via the extended functional calculus for sectorial operators presented in [26, Chapter 2], but they are also subject to a variety of equivalent definitions that allow the operator to be represented through a number different formulas, see, e.g., [63].

On the range of $(-\Delta)^{\alpha/2}$, the inverse mapping can in turn be given via the formula

$$(-\Delta)^{-\alpha/2} f = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty s^{\alpha/2-1}(G_s * f)\, ds, \quad \alpha > 0, \tag{3.1.1}$$

where $\Gamma(x) = \int_0^\infty e^{-s} s^{x-1}\, ds$ is the Gamma function, see, e.g., [4, Chapter III, Theorem 4.6.6].

We now establish estimates for the heat semigroup on the whole space $\mathbb{R}^d$ for a general dimension $d \geq 1$ involving fractional powers of the Laplace operator as well as the Riesz transform. We denote the Riesz transform in the i-th direction, $1 \leq i \leq d$, by $R_i := \partial_i(-\Delta)^{-1/2}$.

Lemma 3.1.3. *Let $d \geq 1$, $\alpha \in [0, 2]$, and $\beta \in (0, 2]$. Then there exist families of uniformly integrable functions $H_t^\alpha, \tilde{H}_t^\beta, \breve{H}_t : \mathbb{R}^d \to (0, \infty)$, $t > 0$ such that*

$$t^{\alpha/2}|e^{t\Delta}(-\Delta)^{\alpha/2} f| \leq H_t^\alpha * |f|, \tag{1}$$

$$t^{\beta/2}|e^{t\Delta} R_i R_j (-\Delta)^{\beta/2} f| \leq \tilde{H}_t^\beta * |f|, \tag{2}$$

$$t^{1/2}|e^{t\Delta} R_i R_j \partial_k f| \leq \breve{H}_t * |f|, \tag{3}$$

for all $t > 0$, $1 \leq i, j, k \leq d$, and $f \in L^p(\mathbb{R}^d)$ for some $p \in [1, \infty]$. In particular, there exist constants $C_{d,\alpha}, C_{d,\beta}, C_d > 0$ such that

$$t^{\alpha/2}\|e^{t\Delta}(-\Delta)^{\alpha/2} f\|_{L^p(\mathbb{R}^d)} \leq C_{d,\alpha}\|f\|_{L^p(\mathbb{R}^d)}, \tag{a}$$

$$t^{\beta/2}\|e^{t\Delta} R_i R_j (-\Delta)^{\beta/2} f\|_{L^p(\mathbb{R}^d)} \leq C_{d,\beta}\|f\|_{L^p(\mathbb{R}^d)}, \tag{b}$$

$$t^{1/2}\|e^{t\Delta} R_i R_j \partial_k f\|_{L^p(\mathbb{R}^d)} \leq C_d\|f\|_{L^p(\mathbb{R}^d)}, \tag{c}$$

for all $t > 0$ and $p \in [1, \infty]$.

Remark 3.1.4. Note that although the Riesz transform is unbounded on $L^1(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$, due to the smoothing effect of the heat semigroup, the compositions of operators $e^{t\Delta} R_i R_j (-\Delta)^{\alpha/2}$ and $\partial_k e^{t\Delta} R_i R_j$ nevertheless define bounded operators for $t > 0$.

Proof of Lemma 3.1.3. For (1), using the smoothing effect of $e^{t\Delta}$ for $t > 0$ we have

$$e^{t\Delta}(-\Delta)^{\alpha/2} f = (-\Delta)^{-(1-\alpha/2)}(-\Delta) e^{t\Delta} f$$

and so via the representation (3.1.1) we obtain

$$e^{t\Delta}(-\Delta)^{\alpha/2} f = \frac{1}{\Gamma(1-\alpha/2)} \int_0^\infty s^{-\alpha/2}(-\Delta G_{s+t}) * f\, ds.$$

Via Lemma 3.1.1 it thus follows that

$$\begin{aligned} |e^{t\Delta}(-\Delta)^{\alpha/2} f| &\leq \frac{C}{\Gamma(1-\alpha/2)} \int_0^\infty s^{-\alpha/2}(s+t)^{-1} G_{2(s+t)} * |f|\, ds \\ &= \frac{C}{\Gamma(1-\alpha/2)} \int_0^\infty (tu)^{-\alpha/2}(tu+t)^{-1} G_{2(tu+t)} * |f| t\, du \\ &= \frac{C}{\Gamma(1-\alpha/2)} t^{-\alpha/2} \int_0^\infty u^{-\alpha/2}(u+1)^{-1} G_{2t(u+1)} * |f|\, du. \end{aligned}$$

We set

$$H_t^\alpha := \frac{C}{\Gamma(1-\alpha/2)} \int_0^\infty u^{-\alpha/2}(u+1)^{-1} G_{2t(u+1)}\, du$$

and observe that

$$\|H_t^\alpha\|_1 = \frac{C}{\Gamma(1-\alpha/2)} \int_0^\infty u^{-\alpha/2}(u+1)^{-1}\, du = C_\alpha^1 < \infty, \quad \alpha \in (0,2).$$

For $\alpha \in \{0,2\}$ we set $H_t^0 := G_t$ and $H_t^2 := G_{2t}$ and apply Lemma 3.1.1. This yields estimate (1). In order to prove estimate (2) we observe that

$$e^{t\Delta} R_i R_j (-\Delta)^{\beta/2} f = (-\Delta)^{-(1-\beta/2)} \partial_i \partial_j e^{t\Delta} f, \quad 1 \le i,j \le d.$$

The case $\beta = 2$ then follows from Lemma 3.1.1 by setting $\tilde{H}_t^2 := G_{2t}$, whereas for $\beta \in (0,2)$ we have

$$e^{t\Delta} R_i R_j (-\Delta)^{\beta/2} f = \frac{1}{\Gamma(1-\beta/2)} \int_0^\infty s^{-\beta/2} (\partial_i \partial_j G_{s+t}) * f\, ds$$

and thus the same argument used to derive (1) applies. For (3) we write

$$e^{t\Delta} R_i R_j \partial_k f = (-\Delta)^{-1} \partial_i \partial_j \partial_k e^{t\Delta} f = \int_0^\infty \partial_i \partial_j \partial_k G_{s+t} * f\, ds$$

and since by Lemma 3.1.1 we have $|\partial_i \partial_j \partial_k G_{s+t}| \le C(s+t)^{-3/2} G_{2(s+t)}$ for $s,t > 0$ we may set

$$\breve{H}_t := \int_0^\infty (u+1)^{-3/2} G_{2t(u+1)}\, du$$

which satisfies $\|\breve{H}_t\|_1 \le \int_0^\infty (u+1)^{-3/2}\, du = C < \infty$, yielding the estimate (3). Estimates (a)-(c) then follow from estimates (1)-(3) and Young's inequality. This completes the proof. □

We will also require that estimate (c) from Lemma 3.1.3 holds when we consider the complex extension of the heat semigroup. For this purpose we utilize the theory of Fourier multipliers. Note that we could have also proven Lemma 3.1.3 via similar methods, but chose not since the restriction onto the real half-life $[0,\infty)$ allows us to proceed with simpler arguments.

Lemma 3.1.5. *Let $d \ge 1$, $p \in [1,\infty]$ and $\theta \in (0,\pi/2)$. Then there exists a constant $C = C_{\theta,d} > 0$ such that for all $1 \le i,j,k \le d$, $\tau \in \Sigma_\theta$, and $f \in L^p(\mathbb{R}^2)$ it holds that*

$$|\tau|^{1/2} \|\partial_k e^{\tau\Delta} R_i R_j f\|_{L^p(\mathbb{R}^d)} \le C \|f\|_{L^p(\mathbb{R}^d)}.$$

Proof. Observe that for $1 \le i,j,k \le d$, the operator $|\tau|^{1/2} \partial_k e^{\tau\Delta} R_i R_j$ is the Fourier multiplier corresponding to the symbol

$$m_{\tau,i,j,k}(\xi) = \begin{cases} |\tau|^{1/2} \xi_i \xi_j \xi_k |\xi|^{-2} \exp\left(-\tau|\xi|^2\right), & \xi \in \mathbb{R}^2 \setminus \{0\}, \\ 0, \quad \xi = 0. \end{cases}$$

It is well known that Fourier multiplier properties are invariant under rescaling of symbols. By replacing ξ with $|\tau|^{-1/2}\xi$ we may assume without loss of generality that $|\tau| = 1$.

We will show that each such symbol $m = m_{\tau,i,j,k}$ satisfies $m = \hat{g}$ where $g \in L^1(\mathbb{R}^2)$ satisfies $\|g\|_{L^1(\mathbb{R}^2)} \leq C_\theta$, which then implies our estimate via Young's inequality. For this purpose we take a smooth cut-off function $\varphi \in C_c^\infty(\mathbb{R}^2)$ satisfying $\varphi(\xi) = 1$ for $|\xi| \leq 2$ and consider the decomposition $m = (1-\varphi)m + \varphi m$. We then respectively apply [11, Proposition 8.2.3 and Lemma 8.2.4] to these terms, yielding the desired result. First, observe that

$$m \in C(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2 \setminus \{0\})$$

for each of the symbols above. We now verify the condition

$$\max_{|\alpha| \leq J} \sup_{\xi \in \mathbb{R}^2 \setminus \{0\}} |\xi|^{|\alpha|+\delta} |D^\alpha m(\xi)| < C < \infty, \tag{3.1.2}$$

for $J := \min\{k \in \mathbb{N} : k > \lceil d/2 \rceil\}$ and some $\delta \in (0,1)$. For this purpose, we split the symbol m into the factors

$$m_1(\xi) := \xi_i \xi_j |\xi|^{-2}, \quad m_2(\xi) := \xi_k \exp\left(-\tau|\xi|^2\right),$$

and observe that the factor m_1 is homogeneous of order 0 and thus an elementary calculation yields

$$\sup_{\xi \in \mathbb{R}^2 \setminus \{0\}} |\xi|^\alpha \left|D^\alpha m_1(\xi)\right| < C_\alpha < \infty,$$

for every multi-index $\alpha \in \mathbb{N}^2$ and a constant $C_\alpha > 0$. By $|\tau| = 1$ we obtain

$$\left|D^\alpha \xi_k \exp\left(-\tau|\xi|^2\right)\right| \leq |P_\alpha(\xi)| \exp\left(-\cos(\psi)|\xi|^2\right) \leq |P_\alpha(\xi)| \exp\left(-\cos(\theta)|\xi|^2\right),$$

where P_α is a polynomial and $\psi = \arg(\tau)$, yielding

$$\sup_{\xi \in \mathbb{R}^2 \setminus \{0\}} |\xi|^{\alpha+\delta} \left|D^\alpha m_2(\xi)\right| \leq C_{\alpha,\delta,\theta} < \infty$$

for any $\delta \in (0,1)$. Thus (3.1.2) is satisfied and so we may apply [11, Proposition 8.2.3] to the term $(1-\varphi)m$. We now show that we further have

$$|\xi|^{|\alpha|} |D^\alpha m(\xi)| \leq C_\alpha |\xi|, \quad |\xi| \leq 1, \xi \neq 0. \tag{3.1.3}$$

For this purpose we again divide the symbol m into the factors

$$m_3(\xi) := \xi_j \xi_k \xi_j |\xi|^{-2}, \quad m_4(\xi) := \exp\left(-\tau|\xi|^2\right).$$

Again using homogeneity we have

$$|\xi|^{|\alpha|} \left| D^\alpha \frac{\xi_j \xi_k \xi_j}{|\xi|^2} \right| \leq C_\alpha |\xi|, \quad |\xi| \leq 1, \xi \neq 0$$

for any $\alpha \in \mathbb{N}^d$, whereas proceeding as above we obtain

$$\left|D^\alpha \exp\left(-\tau|\xi|^2\right)\right| \leq |P_\alpha(\xi)| \exp\left(-\cos(\psi)|\xi|^2\right) \leq |P_\alpha(\xi)| \exp\left(-\cos(\theta)|\xi|^2\right),$$

and thus

$$|\xi|^{|\alpha|} \left|D^\alpha \exp\left(-\tau|\xi|^2\right)\right| \leq C_{\alpha,\theta}, \quad \xi \in \mathbb{R}^2.$$

It follows that condition (3.1.3) is satisfied and so applying [11, Lemma 8.2.4] to the term φm yields the desired result. $\square$

3.2 L^q-L^p-estimates on the whole space

3.2.1 The resolvent problem

We now turn to the resolvent problem for the three-dimensional Laplacian on the whole space, given by

$$\lambda v - \Delta v = f \text{ on } \mathbb{R}^3. \tag{3.2.1}$$

We will also be considering the case where the right-hand side is given as a derivative, i.e., when the problem is of the form

$$\lambda w - \Delta w = \partial_i f \text{ on } \mathbb{R}^3 \tag{3.2.2}$$

for $\partial_i \in \{\partial_x, \partial_y, \partial_z\}$. It is well-known that the Green function of problem (3.2.1) is given by

$$K_\lambda(x) := \frac{1}{4\pi} \frac{e^{-\lambda^{1/2}|x|}}{|x|}, \quad x \in \mathbb{R}^3 \setminus \{0\},$$

i.e., one has $(\lambda - \Delta)K_\lambda = \delta_0$, where $\delta_0(\phi) := \phi(0)$ for $\phi \in C_c^\infty(\mathbb{R}^3)$, and thus

$$(\lambda - \Delta)(K_\lambda * f) = \delta_0 * f = f, \quad (\lambda - \Delta)(\partial_i K_\lambda * f) = \partial_i(\delta_0 * f) = \partial_i f,$$

in the sense of distributions whenever f is bounded with compact support. The following lemma establishes an estimate in the anisotropic L^q-L^p-spaces from Definition 2.4.8.

Lemma 3.2.1. *Let $\lambda \in \Sigma_\theta$ for some $\theta \in (0, \pi)$ and assume that either*

(i) $q, p \in [1, \infty)$ *and* $f \in L^q_H L^p_z(\mathbb{R}^3)$,

(ii) $q = \infty$, $p \in [1, \infty)$ *and* $f \in L^\infty_H L^p_z(\mathbb{R}^3)$ *has compact support in the horizontal directions,*

(iii) $q \in [1, \infty)$, $p = \infty$ *and* $f \in L^q_H L^\infty_z(\mathbb{R}^3)$ *has compact support in the vertical direction,*

(iv) $q = p = \infty$ *and* $f \in L^\infty(\mathbb{R}^3)$ *has compact support.*

Then the functions

$$v := K_\lambda * f, \quad w := \partial_i K_\lambda * f,$$

are the unique solutions to the problems (3.2.1) *and* (3.2.2), *respectively. There further exists a constant $C = C_\theta > 0$ such that*

$$|\lambda| \cdot \|v\|_{L^q_H L^p_z(\mathbb{R}^3)} + |\lambda|^{1/2} \|\nabla v\|_{L^q_H L^p_z(\mathbb{R}^3)} + \|\Delta v\|_{L^q_H L^p_z(\mathbb{R}^3)} \le C \|f\|_{L^q_H L^p_z(\mathbb{R}^3)} \tag{3.2.3}$$

and

$$|\lambda|^{1/2} \|w\|_{L^q_H L^p_z(\mathbb{R}^3)} \le C \|f\|_{L^q_H L^p_z(\mathbb{R}^3)}. \tag{3.2.4}$$

Proof. We may assume without loss of generality that f is bounded with compact support, since in the case $q, p \in [1, \infty)$ we have that $C_c^\infty(\mathbb{R}^3)$ is dense in $L^q_H L^p_z(\mathbb{R}^3)$, whereas in the cases (ii), (iii) or (iv) the additional assumption on the support of f yields the same result. It thus suffices to prove the estimates (3.2.3) and (3.2.4), which we will do via uniform estimates for K_λ and $\partial_i K_\lambda$ in $L^1(\mathbb{R}^3)$. For this purpose let $\psi := \arg(\lambda) \in (-\theta, \theta)$. Using spherical coordinates and the fact that K_λ is radially symmetric we obtain

$$\int_{\mathbb{R}^3} |K_\lambda(x)|\, dx = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{-|\lambda|^{1/2}\cos(\psi/2)|x|}}{|x|}\, dx \;= \int_0^\infty r e^{-|\lambda|^{1/2}\cos(\psi/2) r}\, dr$$

whereas for its derivatives we have

$$\int_{\mathbb{R}^3} |\partial_i K_\lambda(x)|\, dx \leq \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(1+|\lambda|^{1/2}|x|) e^{-|\lambda|^{1/2}|x|}}{|x|^2}\, dx = \int_0^\infty (1+|\lambda|^{1/2} r) e^{-|\lambda|^{1/2}\cos(\psi/2) r}\, dr.$$

The formulas

$$\int_0^\infty r e^{-ar}\, dr = a^{-2}, \qquad \int_0^\infty e^{-ar}\, dr = a^{-1}, \quad a > 0,$$

then yield

$$|\lambda| \cdot \|K_\lambda\|_{L^1(\mathbb{R}^3)} = \cos(\psi/2)^{-2}, \quad |\lambda|^{1/2} \|\nabla K_\lambda\|_{L^1(\mathbb{R}^3)} \leq \cos(\psi/2)^{-1} + \cos(\psi/2)^{-2},$$

and since $\cos(\psi/2)^{-1} \leq \cos(\theta/2)^{-1} < \infty$ we find that the estimates hold for

$$C_\theta := 1 + \cos(\theta/2)^{-1} + 2\cos(\theta/2)^{-2}$$

via the anisotropic version of Young's inequality for convolutions stated in Lemma 2.4.9.2. □

Remark 3.2.2. Observe that since continuity is preserved, the statement of the lemma remains true if the space $L^q_H L^p_z(\mathbb{R}^3)$ for $p, q = \infty$ is replaced by $C_c(\mathbb{R}^2; L^p(\mathbb{R}))$ or $L^q(\mathbb{R}^2; C_c(\mathbb{R}))$.

3.2.2 An interpolation inequality for fractional powers

We now consider a cylindrical domain of the form $\mathbb{R}^d \times U$ for some domain $U \subset \mathbb{R}^m$, as well as the Laplace operator in the first d, i.e., the differential operator

$$\Delta_d := \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}.$$

Further setting $\nabla_d := (\partial_1, ..., \partial_d)$, the following lemma establishes an interpolation inequality for this operator's negative fractional powers on anisotropic L^q-L^p-spaces.

Lemma 3.2.3. *Let $d \geq 1$ and $\alpha \in (0, 1]$ as well as $q, p \in [1, \infty]$. Then there exists a constant $C = C_{d,\alpha} > 0$ such that for all $f \in L^q(\mathbb{R}^d; L^p(U))$ with $\nabla_d f \in L^q(\mathbb{R}^d; L^p(U))$ it holds that*

$$\left\|\nabla_d(-\Delta_d)^{-(1-\alpha)/2} f\right\|_{L^q(\mathbb{R}^d; L^p(U))} \leq C \|f\|^{1-\alpha}_{L^q(\mathbb{R}^d; L^p(U))} \|\nabla_H f\|^{\alpha}_{L^q(\mathbb{R}^d; L^p(U))}.$$

Remark 3.2.4. Given $\alpha \in (0,1]$, the isotropic estimate

$$\|\nabla(-\Delta)^{-(1-\alpha)/2} f\|_{L^p(\mathbb{R}^d)} \leq C_{d,\alpha} \|f\|_{L^p(\mathbb{R}^d)}^{1-\alpha} \|\nabla f\|_{L^p(\mathbb{R}^d)}^{\alpha},$$

is well-known to hold for all $p \in [1,\infty]$. The case $\alpha = 0$ is well-known to hold for all $p \in (1,\infty)$ but not for $p \in \{1,\infty\}$ since the Riesz transform fails to be bounded on $L^1(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$. For the purpose of this work, the case $d = 2$ and $q = \infty$ is of particular interest since we will eventually apply it to obtain an estimate for the primitive equations in L^∞-L^p-spaces on a layer domain.

Proof of Lemma 3.2.3. The case $\alpha = 1$ is trivial. Given $f\colon \mathbb{R}^d \times U \to \mathbb{R}$ we use the fact that the negative fractional powers of $-\Delta_d$ are given by

$$(-\Delta_d)^{-(1-\alpha)/2} f = \frac{1}{\Gamma((1-\alpha)/2)} \int_0^\infty s^{-(1+\alpha)/2} (G_s *_d f)\, ds, \quad \alpha \in (0,1),$$

compare (3.1.1). Here G_t denotes the Gaussian kernel on $\mathbb{R}^d$ and $*_d$ denotes convolution in the variables $(x_1, ..., x_d) \in \mathbb{R}^d$. Using

$$\nabla_d (G_s *_d f) = (\nabla_d G_s) *_d f = G_s *_d \nabla_d f$$

it holds for arbitrary $\mu \in (0,\infty)$ that

$$|\nabla_d(-\Delta_d)^{-(1-\alpha)/2} f| \leq C_\alpha \left(\int_0^\mu s^{-(1+\alpha)/2} |G_s *_d \nabla_d f|\, ds + \int_\mu^\infty s^{-(1+\alpha)/2} |(\nabla_d G_s) *_d f|\, ds \right).$$

Observe that for $f = 0$ or $\nabla_d f = 0$ we thus have $\nabla_d(-\Delta_d)^{-(1-\alpha)/2} f = 0$ and the estimate is trivial. Otherwise we set

$$\mu := \|f\|_{L^q(\mathbb{R}^d;L^p(U))}^2 \|\nabla_H f\|_{L^q(\mathbb{R}^d;L^p(U))}^{-2} \in (0,\infty). \tag{3.2.5}$$

Denoting the first d variables by $x := (x_1, ..., x_d)$, we apply the Minkowski inequality to obtain

$$\begin{aligned}\|(\nabla_d(-\Delta_d)^{-(1-\alpha)/2} f)(x,\cdot)\|_{L^p(U)} \leq{}& C_\alpha \int_0^\mu s^{-(1+\alpha)/2} \|(G_s *_d \nabla_d f)(x,\cdot)\|_{L^p(U)}\, ds \\ &+ C_\alpha \int_\mu^\infty s^{-(1+\alpha)/2} \|(\nabla_d G_s *_d f)(x,\cdot)\|_{L^p(U)}\, ds.\end{aligned}$$

Another application of the Minkowski inequality together with Lemma 3.1.1 yields

$$\begin{aligned}\|(G_s *_d \nabla_d f)(x,\cdot)\|_{L^p(U)} &\leq \int_{\mathbb{R}^d} G_s(x-y) \|\nabla_d f(y,\cdot)\|_{L^p(U)}\, dy, \\ \|(\nabla_d G_s *_d f)(x,\cdot)\|_{L^p(U)} &\leq C s^{-1/2} \int_{\mathbb{R}^d} G_{2s}(x-y) \|f(y,\cdot)\|_{L^p(U)}\, dy,\end{aligned}$$

and by applying it a third time we obtain

$$\|\nabla_d(-\Delta_d)^{-(1-\alpha)/2}f\|_{L^q(\mathbb{R}^d;L^p(U))} \leq C_\alpha\left(\int_{\mathbb{R}^d}\left[\int_0^\mu s^{-(1+\alpha)/2}G_s(x-y)\,ds\right]\|\nabla_d f(y,\cdot)\|_{L^p(U)}\,dy,\right)^{1/q}$$
$$+ C_\alpha\left(\int_{\mathbb{R}^d}\left[\int_\mu^\infty s^{-(1+\alpha/2)}G_{2s}(x-y)\,ds\right]\|f(y,\cdot)\|_{L^p(U)}\,dy,\right)^{1/q}.$$

Using the formulas

$$\int_0^\mu s^{-(1+\alpha)/2}\,ds = \frac{\mu^{(1-\alpha)/2}}{(1-\alpha)/2}, \quad \int_\mu^\infty s^{-(1+\alpha/2)}\,ds = \frac{\mu^{-\alpha/2}}{\alpha/2},$$

Young's inequality for convolutions yields

$$\|\nabla_d(-\Delta_d)^{-\alpha/2}f\|_{L^q(\mathbb{R}^d;L^p(U))} \leq C_\alpha\frac{\mu^{(1-\alpha)/2}}{(1-\alpha)/2}\|\nabla_d f\|_{L^q(\mathbb{R}^d;L^p(U))}$$
$$+ C_\alpha\frac{\mu^{-\alpha/2}}{\alpha/2}\|f\|_{L^q(\mathbb{R}^d;L^p(U))}.$$

and so plugging in the value of μ as in (3.2.5) then yields the desired result. □

3.3 L^p-estimates for heat semigroups on intervals

3.3.1 Existence, contractivity, derivatives and analyticity

We now consider various heat equations on a one-dimensional interval, beginning with the case of periodic boundary conditions on a nonempty interval (a,b), i.e.,

$$\begin{cases} \partial_t u(t,z) - \partial_z^2 u(t,z) &= 0 & t\in(0,\infty),\ z\in(a,b), \\ u(t,a) &= u(t,b), & t\in(0,\infty), \\ u(0,z) &= u_0(z) & z\in(a,b). \end{cases} \tag{3.3.1}$$

Note that we chose to denote the spatial variable by z and write $\partial_z f := df/dz$ since we will eventually apply these results to the issue of vertical estimates involving the primitive equations.

Denoting the solution mappings $u_0 \mapsto u(t,\cdot)$ for $t \geq 0$ by $S_{\text{per}}(t)$, the following lemma establishes that these operators are well-defined and admit suitable L^p-estimates as well as an analytic extension.

Lemma 3.3.1.

(i) Let $p\in[1,\infty]$. Then for all $u_0 \in L^p(a,b)$ the problem (3.3.1) *has a unique solution $u \in C((0,\infty);L^p(a,b))$. There further exists an absolute constant $C>0$ such that*

$$\|u(t,\cdot)\|_{L^p(a,b)} \leq \|u_0\|_{L^p(a,b)}, \tag{3.3.2}$$
$$t^{1/2}\|\partial_z u(t,\cdot)\|_{L^p(a,b)} \leq C\|u_0\|_{L^p(a,b)}, \tag{3.3.3}$$

for all $t>0$ and $p\in[1,\infty]$.

(ii) *The mapping* $S_{per}\colon [0,\infty) \ni t \mapsto u(t,\cdot) \in L^p(a,b)$ *defines an operator semigroup that is strongly continuous for* $p \in [1,\infty)$*.*

(iii) *The semigroup* S_{per} *admits a bounded analytic extension of angle* $\pi/2$*.*

(iv) *There further exists a constant* $C > 0$ *such that*

$$t^{1/2}\|S_{per}(t)f\|_{L^\infty(a,b)} \leq C\|f\|_{L^1(a,b)}$$

for all $f \in L^1(a,b)$ *with* $\int_a^b f(z)\,dz = 0$*.*

Proof. Let G_t denote the one-dimensional Gaussian kernel for $t > 0$. Given $u_0 \in L^p(a,b)$, we further denote its periodic extension of period $b-a$ onto $\mathbb{R}$ by $E_{\text{per}}u_0$. Since $C_c^\infty(a,b)$ is dense in $L^p(a,b)$ for $p \in [1,\infty)$ we may assume without loss of generality that $u_0 \in L^\infty(a,b)$. Then the dominated convergence theorem and an elementary calculation yield that

$$(G_t * E_{\text{per}}u_0)(z) = \int_a^b K_t(z-s)u_0(s)\,ds, \quad t > 0, z \in (a,b) \tag{3.3.4}$$

with the periodic kernel

$$K_t(z) := \sum_{k=-\infty}^{\infty} G_t(z + k(b-a)), \quad z \in (a,b).$$

Since G_t is positive we further have that K_t is positive as well and satisfies

$$\int_a^b K_t(z)\,dz = \int_{-\infty}^{\infty} G_t(z)\,dz = 1.$$

Now consider the one-dimensional torus $\mathbb{T} := \mathbb{R}/(b-a)\mathbb{Z}$. Since $E_{\text{per}}u_0$ and K_t are periodic functions of period $b-a$, we may identify them with functions on the torus via

$$[f](z + (b-a)\mathbb{Z}) := f(z),$$

and identify the right-hand side of (3.3.4) with the convolution on the torus, which we denote by $[K_t] *_{\mathbb{T}} [u_0]$. Applying Young's inequality for convolutions on $\mathbb{T}$ then yields that the function given by

$$u(t,z) := S_{\text{per}}(t)u_0 := ([K_t] *_{\mathbb{T}} [E_{\text{per}}u_0])(z), \quad t > 0, z \in \mathbb{T},$$

satisfies the estimate

$$\|u(\cdot,t)\|_{L^p(\mathbb{T})} \leq \|K_t\|_{L^1(a,b)}\|f\|_{L^p(a,b)} = \|f\|_{L^p(a,b)}, \quad t > 0.$$

By (3.3.4) we also have $E_{\text{per}}u(\cdot,t) = G_t * E_{\text{per}}u_0$, i.e., $E_{\text{per}}u$ solves the heat equation on the the whole space $\mathbb{R}$ with initial data $E_{\text{per}}u_0$ and so u is the solution to (3.3.1). Furthermore, any solution to (3.3.1) can be periodically extended this way and by the uniqueness of solutions on the whole space we have that u is unique as well. We now turn to derivatives of the solution. Using Lemma 3.1.1 we obtain

$$|\partial_z u(t,z)| = |\partial_z (G_t * E_{\mathrm{per}} u_0)(z)| \leq C t^{-1/2} \int_a^b K_{2t}(z-s) u_0(s)\, ds$$

so an analogous argument yields the estimate

$$\|\partial_z u(\cdot,t)\|_{L^p(\mathbb{T})} \leq C t^{-1/2} \|f\|_{L^p(a,b)}.$$

This concludes the proof of (i).

For (ii), we observe that the semigroup property follows from the uniqueness of solutions and that the strong continuity follows from the strong continuity of the heat semigroup on $\mathbb{R}$.

In order to prove (iii), we note that the analytic extension of the heat semigroup on $\mathbb{R}$ is given by $e^{\tau\Delta} f = G_\tau * f$, where the complex Gaussian kernel is in turn given by

$$G_\tau \colon \mathbb{R} \to (0,\infty), \quad G_\tau(x) = (4\pi\tau)^{-1/2} \exp\left(-x^2/4\tau\right), \quad \operatorname{Re}\tau > 0.$$

An elementary calculation then shows that

$$|G_\tau| = \cos(\psi)^{-1/2} G_{|\tau|/\cos(\psi)}, \quad \psi = \arg(\tau),$$

yielding $\|G_\tau\|_{L^1(\mathbb{R})} = \cos(\psi)^{-1/2} \leq \cos(\theta)^{-1/2} < \infty$ whenever $\tau \in \Sigma_\theta$ for $\theta \in (0,\pi/2)$. For the estimate in (iv), we observe that $\int_{\mathbb{T}} [f](z)\, dz = 0$ yields

$$K_t *_{\mathbb{T}} [f] = (K_t - k_0) *_T [f]$$

for all constants $k_0 \in \mathbb{C}$. Furthermore, by writing K_t as a Fourier series with coefficients $(\hat{K}_t(k))_{k\in\mathbb{Z}}$, a straightforward calculation yields that $\|K_t - \hat{K}_t(0)\|_{L^\infty(\mathbb{T})} \leq C t^{-1/2}$ for all $t > 0$ and an absolute constant $C > 0$. This yields the desired result. □

We now consider the heat equation

$$\begin{cases} \partial_t u - \partial_z^2 u &= 0 \quad \text{on } (-h,0)\times(0,\infty), \\ \quad u(0) &= u_0 \quad \text{on } (-h,0), \end{cases} \tag{3.3.5}$$

for the boundary conditions

$$\begin{array}{llll} \text{(N)} & \partial_z u = 0 \quad \text{on } \{0\}\times(0,\infty), & \partial_z u = 0 \text{ on } & \{-h\}\times(0,\infty), \\ \text{(ND)} & \partial_z u = 0 \quad \text{on } \{0\}\times(0,\infty), & u = 0 \text{ on } & \{-h\}\times(0,\infty), \\ \text{(D)} & u = 0 \quad \text{on } \{0\}\times(0,\infty), & u = 0 \text{ on } & \{-h\}\times(0,\infty), \end{array}$$

Denoting the corresponding heat semigroups by S_N, S_{ND}, and S_D, we then have the following.

Lemma 3.3.2. *Let $p \in [1,\infty]$ and $S_* \in \{S_N, S_{ND}, S_D\}$. Then we have the following.*

(i) The family of operators $(S_*(t))_{t\geq 0}$ is a well-defined contraction semigroup on $L^p(-h,0)$, i.e., it holds that

$$\|S_*(t)f\|_{L^p(-h,0)} \leq \|f\|_{L^p(-h,0)},$$

for all $t \geq 0$ and $f \in L^p(-h,0)$.

(ii) There exists a constant $C > 0$ such that

$$t^{1/2}\|\partial_z S_*(t)f\|_{L^p(-h,0)} \leq C\|f\|_{L^p(-h,0)},$$

for all $t > 0$ and $f \in L^p(-h,0)$.

(iii) If $p \in [1,\infty)$, then S_* is strongly continuous on $L^p(-h,0)$.

(iv) These semigroup S_* admits a bounded analytic extension of angle $\pi/2$.

(v) There exists a constant $C > 0$ such that the semigroup S_N admits the L^1-L^p-smoothing estimate

$$t^{(1-1/p)/2}\|S_N(t)f\|_{L^p(-h,0)} \leq C\|f\|_{L^1(-h,0)}$$

for all $f \in L^p(-h,0)$ with $\int_{-h}^0 f(z)\,dz = 0$. The same estimate holds for S_{ND} and S_D for all $f \in L^1(-h,0)$.

(vi) Every constant function f is a fixed point of $S_N(t)$ for all $t \geq 0$.

Proof. We begin with (i) and (ii) and first consider the case $S_* = S_N$. For this purpose we take $f \in L^p(-h,0)$ and construct an extension onto $(-h,3h)$ by applying an even reflection at $z=0$ and then again at $z=h$. Denoting this extension by $E_{\text{even}}f$ we may then apply the estimates (3.3.2) and (3.3.3) for $a=-h$ and $b=3h$ to obtain

$$\begin{aligned}\|S_{\text{per}}(t)E_{\text{even}}f\|_{L^p(\mathbb{T})} &\leq \|E_{\text{even}}f\|_{L^p(-h,3h)} = 4^{1/p}\|f\|_{L^p(-h,0)},\\ t^{1/2}\|\partial_z S_{\text{per}}(t)E_{\text{even}}f\|_{L^p(\mathbb{T})} &\leq C\|E_{\text{even}}f\|_{L^p(-h,3h)} = C4^{1/p}\|f\|_{L^p(-h,0)}.\end{aligned}$$

Using elementary methods one further verifies that by identifying $S_{\text{per}}(t)E_{\text{even}}f$ with a function on $(-h,3h)$, one obtains a function that is even w.r.t. $z=h$. Furthermore, its restriction to $(-h,h)$ is again even w.r.t. $z=0$. This yields that the restriction of $S_{\text{per}}(t)E_{\text{even}}f$ onto $(-h,0)$ is a solution to the problem (3.3.5) and thus

$$S_N(t)f = S_{\text{per}}(t)E_{\text{even}}f\big|_{(-h,0)}.$$

We further have

$$4^{1/p}\|S_N(t)f\|_{L^p(-h,0)} = \|S_{\text{per}}(t)E_{\text{even}}f\|_{L^p(-h,3h)} \leq \|E_{\text{even}}f\|_{L^p(-h,3h)} = 4^{1/p}\|f\|_{L^p(-h,0)},$$

and analogously

$$4^{1/p}\|\partial_z S_N(t)f\|_{L^p(-h,0)} \leq 4^{1/p}Ct^{-1/2}\|f\|_{L^p(-h,0)}.$$

This yields the desired estimates for S_N. The cases $S_* = S_{ND}$ and $S_* = S_D$ are obtained analogously if one extends f onto $(-h, 3h)$ by applying odd reflections at the boundary where Dirichlet boundary conditions are imposed, instead of even ones. This yields (i) and (ii). Point (iii) follows from the strong continuity of S_{per} by Lemma 3.3.1.(iii). Estimate (iv) follows using the same arguments as (i) via Lemma 3.3.1.(ii). For (v) we use that $\int_{-h}^{0} f(z)\,dz = 0$ implies that $\int_{-h}^{3h}(E_{\text{even}}f)(z)\,dz = 0$ and so by Lemma 3.3.1.(iv) it follows that

$$\|S_N(t)f\|_{L^\infty(-h,0)} \leq Ct^{-1/2}\|f\|_{L^1(-h,0)},$$

whereas for $S_* \in \{S_{ND}, S_D\}$, we apply the Poincaré inequality to obtain

$$\|S_*(t)f\|_{L^\infty(-h,0)} \leq C\|\partial_z S_*(t)f\|_{L^1(-h,0)} \leq Ct^{-1/2}\|f\|_{L^1(-h,0)}$$

and so the interpolation-inequality

$$\begin{aligned}
\|f\|_{L^p(-h,0)} &= \left(\int_{-h}^{0} |f(z)|^p\,dz\right)^{1/p} \\
&\leq \left(\int_{-h}^{0} |f(z)|\,dz\right)^{1/p} \left(\|f\|_{L^\infty(-h,0)}^{p-1}\right)^{1/p} \\
&\leq \|f\|_{L^1(-h,0)}^{1/p}\|f\|_{L^\infty(-h,0)}^{1-1/p}
\end{aligned}$$

yields

$$\|S_*(t)f\|_{L^p(-h,0)} \leq \|S_*(t)f\|_{L^1(-h,0)}^{1/p}\|S_*(t)f\|_{L^\infty(-h,0)}^{1-1/p} \leq Ct^{-(1-1/p)/2}\|f\|_{L^1(-h,0)}$$

for $S_* \in \{S_N, S_{ND}, S_D\}$. Point (vi) follows from the fact that if $u_0 \in \mathbb{C}$ is a constant, then $u(t) = u_0$ solves (3.3.5) with boundary conditions (N). This completes the proof. □

3.3.2 Fractional derivatives

We now consider fractional derivatives. For this purpose we introduce the following concepts and notations, compare [51, Chapter 10].

Definition 3.3.3. Let $\alpha \geq 0$. Then the *Riemann-Liouville integral* of a locally integrable function $f\colon (-h, 0) \to \mathbb{C}$ is given by

$$\left(I^\alpha f\right)(z) = \frac{1}{\Gamma(\alpha)}\int_{-h}^{z}(z-s)^{\alpha-1}f(s)\,ds, \quad z \in [-h, 0], \quad \alpha > 0, \tag{3.3.6}$$

where $\Gamma(z) = \int_0^\infty e^{-s}s^{z-1}\,ds$ is the Gamma function, and $I_0 f := f$.

Remark 3.3.4.

1. The Riemann-Liouville integral can be written in the convolution form

$$I^\alpha f = \frac{z_0^{\alpha-1}}{\Gamma(\alpha)} * f_0,$$

   where z_0 and f_0 respectively denote the extensions by zero of the identity mapping on $(0, h)$ and f on $(-h, 0)$ onto $\mathbb{R}$.

2. It is straightforward to verify that whenever $\alpha \in \mathbb{N}$, then I^α is the α-times integration of f and one has the semigroup property $I^{\alpha_1} I^{\alpha_2} f = I^{\alpha_1+\alpha_2} f$ for all $\alpha_1, \alpha_2 \geq 0$.

Definition 3.3.5. Given a function $f \in W^{1,p}(-h,0)$ and $\alpha \in (0,1)$, the *Caputo derivative* of order α of f is defined via

$$\partial_z^\alpha f := I^{1-\alpha}(\partial_z f). \tag{3.3.7}$$

The following lemma establishes that this operator is well-defined and admits a suitable interpolation inequality.

Lemma 3.3.6. *Let $p \in [1,\infty]$, $\alpha \in (0,1)$, and $\mu \in (0,h]$. Then for every function $f \in W^{1,p}(-h,0)$, it holds that*

$$\|\partial_z^\alpha f\|_{L^p(-h,-h+\mu)} \leq \frac{\mu^{1-\alpha}}{\Gamma(2-\alpha)} \|\partial_z f\|_{L^p(-h,-h+\mu)}. \tag{i}$$

If in addition it holds that $f(-h) = 0$, one further has

$$\|\partial_z^\alpha f\|_{L^p(-h,0)} \leq \frac{2}{\Gamma(2-\alpha)} \|f\|_{L^p(-h,0)}^{1-\alpha} \|\partial_z f\|_{L^p(-h,0)}^{\alpha}. \tag{ii}$$

Proof. In order to prove the inequality (i), we take $\mu \in (0,h]$ and consider the auxiliary functions

$$g_1(z) := \frac{z^{-\alpha}}{\Gamma(1-\alpha)} \chi_{(0,\mu)}(z), \quad g_2(z) := \partial_z f(z) \chi_{(-h,-h+\mu)}(z), \quad z \in \mathbb{R},$$

where χ_A denotes the characteristic function of $A \subset \mathbb{R}$. Then one has $g_1 * g_2 = \partial_z^\alpha f$ on $(-h,-h+\mu)$, yielding

$$\begin{aligned}
\|\partial_z^\alpha f\|_{L^p(-h,-h+\mu)} &\leq \|g_1 * g_2\|_{L^p(\mathbb{R})} \leq \|g_1\|_{L^1(\mathbb{R})} \|g_2\|_{L^p(\mathbb{R})} \\
&\leq \frac{1}{\Gamma(1-\alpha)} \left(\int_0^\mu z^{-\alpha} \, dz \right) \|\partial_z f\|_{L^p(-h,-h+\mu)} \\
&= \frac{\mu^{1-\alpha}}{\Gamma(2-\alpha)} \|\partial_z f\|_{L^p(-h,-h+\mu)},
\end{aligned}$$

where we respectively used Young's inequality in the second, as well as

$$\int_0^\mu z^{-\alpha} \, dz = \frac{\mu^{1-\alpha}}{1-\alpha}, \quad x\Gamma(x) = \Gamma(x+1)$$

in the third step. Concerning the interpolation inequality (ii), observe that by the assumption $f(-h) = 0$ the Poincaré inequality yields

$$\|f\|_{L^p(-h,0)} \leq h \|\partial_z f\|_{L^p(-h,0)}$$

and thus we may assume without loss of generality that neither factor on the right-hand side is zero. We may further define the auxiliary parameter

$$\mu := \|f\|_{L^p(-h,0)}/\|\partial_z f\|_{L^p(-h,0)}$$

and assume that it satisfies $\mu \in (0,h]$. Due to (i), it remains to estimate $\|\partial_z^\alpha f\|_{L^p(-h+\mu,0)}$. For this purpose we observe that integration by parts yields

$$\begin{aligned}(\partial_z^\alpha f)(z) &= \frac{1}{\Gamma(1-\alpha)}\int_{-h}^{z-\mu}(z-s)^{-\alpha}\partial_s f(s)\,ds + \frac{1}{\Gamma(1-\alpha)}\int_{z-\mu}^{z}(z-s)^{-\alpha}\partial_s f(s)\,ds \\ &= \frac{\alpha}{\Gamma(1-\alpha)}\int_{-h}^{z-\mu}(z-s)^{-\alpha-1}f(s)\,ds + \frac{\mu^{-\alpha}}{\Gamma(1-\alpha)}f(z-\mu) \\ &+ \frac{1}{\Gamma(1-\alpha)}\int_{z-\mu}^{z}(z-s)^{-\alpha}\partial_s f(s)\,ds,\end{aligned}$$

where we used $f(-h)=0$. Introducing further auxiliary functions

$$\begin{aligned} g_3(z) &:= \tfrac{\alpha}{\Gamma(1-\alpha)} z^{-\alpha-1}\chi_{(\mu,\infty)}(z), \quad & g_4(z) &:= f(z)\chi_{(-h,0)}(z), \\ g_5(z) &:= \tfrac{1}{\Gamma(1-\alpha)} z^{-\alpha}\chi_{(0,\mu)}(z), & g_6(z) &:= \partial_z f(z)\chi_{(-h,0)}(z),\end{aligned}$$

we observe that for $z \in (-h+\mu, 0)$ we have

$$(\partial_z^\alpha f)(z) = (g_3 * g_4)(z) + \frac{\mu^{-\alpha}}{\Gamma(1-\alpha)}f(z-\mu) + (g_5 * g_6)(z)$$

and so by Young's inequality we then have

$$\begin{aligned}\|\partial_z^\alpha f\|_{L^p(-h+\mu,0)} &\le \|g_3\|_{L^1(\mathbb{R})}\|g_4\|_{L^p(\mathbb{R})} + \frac{\mu^{-\alpha}}{\Gamma(1-\alpha)}\|f\|_{L^p(-h,-\mu)} + \|g_5\|_{L^1(\mathbb{R})}\|g_6\|_{L^p(\mathbb{R})} \\ &\le \frac{\mu^{-\alpha}}{\Gamma(1-\alpha)}\|f\|_{L^p(-h,0)} + \frac{\mu^{-\alpha}}{\Gamma(1-\alpha)}\|f\|_{L^p(-h,0)} + \frac{\mu^{1-\alpha}}{\Gamma(2-\alpha)}\|\partial_z f\|_{L^p(-h,0)}.\end{aligned}$$

Combining this estimate with (i) and using $\Gamma(2-\alpha) \le \Gamma(1-\alpha)$ for $\alpha \in (0,1)$, we obtain

$$\|\partial_z^\alpha f\|_{L^p(-h,0)} \le \frac{2\mu^{-\alpha}}{\Gamma(2-\alpha)}\|f\|_{L^p(-h,0)} + \frac{2\mu^{1-\alpha}}{\Gamma(2-\alpha)}\|\partial_z f\|_{L^p(-h,0)}.$$

Plugging in the value $\mu = \|f\|_{L^p(-h,0)}/\|\partial_z f\|_{L^p(-h,0)}$ then yields the desired estimate. □

We now derive a semigroup estimate involving fractional derivatives.

Lemma 3.3.7. *Let $\alpha \in [0,1]$ and $S_* \in \{S_N, S_{ND}, S_D\}$. Then there exists a constant $C = C_\alpha > 0$ such that for all $f \in L^p(-h,0)$ for some $p \in [1,\infty]$ satisfying $(I^\alpha f)(0) = 0$ one has that*

$$\|S_*(t)\partial_z I^\alpha f\|_{L^p(-h,0)} \le C t^{-(1-\alpha)/2}\|f\|_{L^p(-h,0)}, \quad t > 0.$$

For $\alpha \in \{0,1\}$ the estimate even holds true for all $f \in L^p(-h,0)$.

Proof. In the case $\alpha = 1$ we have $\partial_z I^1 f = f$, so the desired estimate was already obtained in Lemma 3.3.2 and is even valid without the assumption $(I^1 f)(0) = 0$. We now consider the case $\alpha \in (0,1)$. By a duality argument we have

$$\|S_*(t)\partial_z I^\alpha f\|_{L^p(-h,0)} = \sup\{|\langle S_*(t)\partial_z I^\alpha f, \varphi\rangle| : \varphi \in C_c^\infty(-h,0), \|\varphi\|_q = 1\}$$

where we used the notation $\langle f, g\rangle := \int_{-h}^0 f(z)g(z)\,dz$ and $1/p + 1/q = 1$. We write

$$\langle S_*(t)\partial_z I^\alpha f, \varphi\rangle = \langle \partial_z I^\alpha f, S_*(t)\varphi\rangle = -\langle I^\alpha f, \partial_z S_*(t)\varphi\rangle$$

where in the second step we used $(I^\alpha f)(0) = 0$ by assumption and $(I^\alpha f)(-h) = 0$ by definition. The validity of the first step follows from the structure of the heat semigroups detailed in the proof of Lemma 3.3.1 and 3.3.2 as well as

$$\int_{\mathbb{T}} (K_t *_{\mathbb{T}} f_1)(z) f_2(z)\,dz = \int_{\mathbb{T}} f_1(z)(K_t *_{\mathbb{T}} f_2)(z)\,dz.$$

By further setting

$$(\overline{I}^\alpha g)(z) := \frac{1}{\Gamma(\alpha)} \int_z^0 (s-z)^{\alpha-1} g(s)\,ds, \quad z \in [-h,0],\, \alpha \in (0,1),$$

an elementary calculation then shows that $\langle I^\alpha f, g\rangle = \langle f, \overline{I}^\alpha g\rangle$ for all functions f, g and thus we obtain

$$\langle S_*(t)\partial_z I^\alpha f, \varphi\rangle = -\langle f, \overline{I}^\alpha \partial_z S_*(t)\varphi\rangle.$$

Due to the structural similarities between $\overline{I}^\alpha$ and I^α, the proof of Lemma 3.3.6 can be modified to obtain the analogous estimate

$$\|\overline{I}^\alpha \partial_z g\|_q \le C_\alpha \|g\|_q^\alpha \|\partial_z g\|_q^{1-\alpha}, \quad \alpha \in (0,1)$$

for all $g \in W^{1,p}(-h,0)$ such that $\partial_z g(0) = 0$. Since we have $(\partial_z S_*(t)\varphi)(0) = 0$ due to our choice of boundary conditions, we obtain

$$\begin{aligned}
\left\|\overline{I}_{z_1}^\alpha \partial_z S_*(t)\varphi\right\|_q &\le C_\alpha \|S_*(t)\varphi\|_q^\alpha \|\partial_z S_*(t)\varphi\|_q^{1-\alpha} \\
&\le C_\alpha \|\varphi\|_q^\alpha \left(Ct^{-1/2}\|\varphi\|_q\right)^{1-\alpha} \\
&= C_\alpha C^{1-\alpha} t^{-(1-\alpha)/2}, \quad t > 0,
\end{aligned}$$

where we used Lemma 3.3.2 and $\|\varphi\|_q = 1$. This then yields the desired estimate. The case $\alpha = 0$ follows analogously using $I^0 f = f$ and setting $\overline{I}^0 f := f$. □

3.4 L^p-theory on cylindrical domains

We now consider the cylindrical domain

$$\Omega := G \times (-h,0), \quad G := (0,1)^2, \quad h > 0. \tag{3.4.1}$$

In the following, we denote horizontal variables by $(x, y) \in G$ and vertical ones by $z \in (-h, 0)$. We decompose the boundary of Ω into the upper, lateral and bottom parts

$$\Gamma_u := G \times \{0\}, \quad \Gamma_l := \partial G \times (-h, 0), \quad \Gamma_b := G \times \{-h\}, \tag{3.4.2}$$

and consider the mixed boundary conditions

$$\begin{aligned} v \text{ periodic} \quad & \text{on } \Gamma_l \times (0, \infty), \\ v = 0 \quad & \text{on } \Gamma_D \times (0, \infty), \\ \partial_z v = 0 \quad & \text{on } \Gamma_N \times (0, \infty). \end{aligned} \tag{3.4.3}$$

Here

$$\Gamma_D \in \{\emptyset, \Gamma_u, \Gamma_b, \Gamma_u \cup \Gamma_b\}, \quad \Gamma_N = (\Gamma_u \cup \Gamma_b) \setminus \Gamma_D \tag{3.4.4}$$

denote the parts of the boundary on which we impose homogeneous Dirichlet and Neumann boundary conditions, respectively.

Recall the definition of periodic Bessel potential spaces from (2.4.2). In this section we consider the L^p-realization of the Laplace operator on Ω with boundary conditions (3.4.3). For this purpose we define Δ_p on $L^p(\Omega)$ via

$$\Delta_p v := \Delta v, \quad D(\Delta_p) := \{v \in H^{2,p}_{per}(\Omega) : \partial_z v\big|_{\Gamma_N} = 0, v\big|_{\Gamma_D} = 0\}. \tag{3.4.5}$$

This is an example of an elliptic operator on a cylindrical domain with mixed boundary conditions. An in-depth study of this kind of operators was provided in [74, Section 6 - 8], which, in particular, established the following result.

Lemma 3.4.1. *Let $p \in (1, \infty)$ and $\mu > 0$. Then the operator $-\Delta_p + \mu$ admits an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus on $L^p(\Omega)$ of angle $\phi^{\mathcal{R}\infty}_{-\Delta_p+\mu} = 0$. If $\Gamma_D \neq \emptyset$ this also holds true for $\mu = 0$ and Δ_p has a bounded inverse.*

Proof. We first consider the Laplacian on the three-dimensional torus, corresponding to periodic boundary conditions in all variables. For this purpose let $H^{2,p}(\mathbb{T}^3)$ denote the space of periodic functions belonging to $H^{2,p}((0, 2\pi)^3)$ with periodic derivatives and define $\Delta_{p,\mathbb{T}^3}$ on $L^p(\mathbb{T}^3)$ via

$$\Delta_{p,\mathbb{T}^3} u = \Delta u, \quad D(\Delta_{p,\mathbb{T}^3}) = H^{2,p}(\mathbb{T}^3).$$

Classically, the formal symbol of $-\Delta$ is given by $|\xi|^2$ for $\xi \in \mathbb{R}^3$. This is a parameter elliptic symbol of ellipticity angle zero according to [74, Definition 6.4]. Going from the whole space $\mathbb{R}^3$ to the torus $\mathbb{T}^3$ the dual group $\mathbb{R}^3$ is replaced by $\mathbb{Z}^3$ and so, taking the restriction of the classical symbol onto $\mathbb{Z}^3$, compare [74, Definition 7.4], one has that $-\Delta_{p,\mathbb{T}^3}$ has the discrete symbol $P(k) = |k|^2$, $k \in \mathbb{Z}^3$. Given $\mu > 0$ it is clear that $(P(k) + \mu)^{-1}$ exists for all $k \in \mathbb{Z}^3$ and that the set $\{P(k)(P(k) + \mu)^{-1} : k \in \mathbb{Z}^3\}$ is bounded in $\mathbb{C}$. It thus follows from [74, Definition 7.13] and [74, Theorem 7.15] that the operator $-\Delta_{p,\mathbb{T}^3} + \mu$ for $\mu > 0$ is a closed operator on $L^p(\mathbb{T}^3)$ with bounded inverse.

Since $L^p(\mathbb{T}^3)$ has property (α) it further follows from [74, Proposition 7.25] that $-\Delta_{p,\mathbb{T}^3} + \mu$ has an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus of angle $\phi^{\mathcal{R}\infty}_{-\Delta_{p,\mathbb{T}^3}+\mu} \le \phi_P = 0$.

Replacing $\mathbb{T}^3$ with $\Omega^* := G \times (-2h, 2h)$ where periodicity holds with periods 1 and $4h$ in horizontal and vertical variables, respectively, a similar argument yields the same result for the analogous operator $\Delta_{\Omega^*,p}$. Denote by E the operator extending from $(-h, 0)$ to $(-2h, 2h)$ by applying odd and even reflections at Γ_D and Γ_N, respectively. Then E maps $L^p(\Omega)$ into $L^p(\Omega^*)$ and $D(\Delta_p)$ into $D(\Delta_{\Omega^*,p})$. One also has for $f \in L^p(\Omega)$ and $\lambda \in \rho(\Delta_{\Omega^*,p})$ that

$$v := (\lambda - \Delta_{\Omega^*,p})^{-1} Ef\big|_\Omega$$

belongs to $D(\Delta_p)$ and satisfies $(\lambda - \Delta_p)v = f$, compare the proof of [74, Proposition 7.16]. As a result, Δ_p inherits the properties of $\Delta_{\Omega^*,p}$ with $\rho(\Delta_{\Omega^*,p}) \subset \rho(\Delta_p)$ and so one has $-\Delta_p + \mu \in \mathcal{RH}^\infty(L^p(\Omega))$ with $\phi^{\mathcal{R}\infty}_{-\Delta_p+\mu} = 0$ for $\mu > 0$.

The case of mixed Dirichlet-Neumann type boundary conditions on cube domains for even operators was also considered, see [74, (7.5)] and the following discussion of boundary conditions. The corresponding results concerning an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus was given in [74, Proposition 7.26], yielding $-\Delta_p + \mu \in \mathcal{RH}^\infty(L^p(\Omega))$ with $\phi^{\mathcal{R}\infty}_{-\Delta_p+\mu} = 0$ whenever $\mu > 0$ and if one even has $\Gamma_D \neq \emptyset$ then this also holds true for $\mu = 0$. The claim $0 \in \rho(\Delta_p)$ is obtained via [74, Proposition 7.23]. Note that although our case of mixed periodic-Dirichlet-Neumann boundary conditions is not explicitly considered, it can be incorporated into the framework via the extension above and restriction argument, compare the proof of [74, Proposition 7.16]. □

It follows from Lemma 3.4.1 that the domains of the fractional powers of the Laplacian can be characterized using complex interpolation spaces. While exact characterizations are known for Besov and Triebel-Lizorkin spaces on the whole space $\mathbb{R}^d$, see [89, Section 2.4.2], the issue is more difficult on domains when one considers spaces subject to boundary conditions. In [48, Section 4] an argument of Amann, based on retracts between spaces on the whole space $\mathbb{R}^3$ and spaces on domains with boundary conditions, together with a localization procedure, was used to establish the interpolation result [3, Theorem 5.2]. The same principle may also be applied here, yielding the following result.

Lemma 3.4.2. *Let $1 < p < \infty$, $\mu > 0$, and $\vartheta \in [0, 1]$ with $\vartheta \notin \{1/2p, 1/2 + 1/2p\}$. Then it holds that*

$$D((\mu - \Delta_p)^\vartheta) = \begin{cases} \{v \in H^{2\vartheta,p}_{per}(\Omega) : \partial_z v\big|_{\Gamma_N} = 0,\ v\big|_{\Gamma_D} = 0\}, & 1 + 1/p < 2\vartheta \le 2, \\ \{v \in H^{2\vartheta,p}_{per}(\Omega) : v\big|_{\Gamma_D} = 0\}, & 1/p < 2\vartheta < 1 + 1/p, \\ H^{2\vartheta,p}_{per}(\Omega), & 0 \le 2\vartheta < 1/p. \end{cases}$$

If $\Gamma_D \neq \emptyset$ this also holds true even for $\mu = 0$.

Corollary 3.4.3. *Let $\Gamma_D \neq \emptyset$. Then $\partial_i(-\Delta_p)^{-1/2}$ and $(-\Delta_p)^{-1/2}\partial_i$ are bounded on $L^p(\Omega)$ for $p \in (1, \infty)$ and $\partial_i \in \{\partial_x, \partial_y, \partial_z\}$. In particular, for $\theta \in (0, \pi)$ there exists a* *constant $C = C_{\Omega,\theta,p} > 0$ such that for all $\lambda \in \Sigma_\theta$ it holds that*

$$\begin{aligned} |\lambda|^{1/2}\|\partial_j(\lambda - \Delta_p)^{-1} f\|_{L^p(\Omega)} &\leq C\|f\|_{L^p(\Omega)}, \\ |\lambda|^{1/2}\|(\lambda - \Delta_p)^{-1}\partial_j f\|_{L^p(\Omega)} &\leq C\|f\|_{L^p(\Omega)}, \\ \|\partial_j(\lambda - \Delta)^{-1}\partial_i f\|_{L^p(\Omega)} &\leq C\|f\|_{L^p(\Omega)}. \end{aligned} \tag{3.4.6}$$

Proof. The operator $(-\Delta_p)^{-1/2}$ maps $L^p(\Omega)$ into $H^{1,p}(\Omega)$ by Lemma 3.4.2 and thus $\partial_i(-\Delta_p)^{-1/2}$ defines a bounded linear operator on $L^p(\Omega)$. Elementary calculations show that the adjoint of Δ_p is given by Δ_q where $1/p + 1/q = 1$ and thus $(-\Delta_p)^{-1/2}\partial_i$ is the adjoint of $\partial_i(-\Delta_q)^{-1/2}$ and bounded as well. Since $-\Delta_p$ admits a bounded $\mathcal{H}^\infty$-calculus of of angle $\phi^\infty_{-\Delta_p} = 0$, the families of operators

$$\{|\lambda|^{1/2}(-\Delta_p)^{1/2}(\lambda - \Delta_p)^{-1} : \lambda \in \Sigma_\theta\}, \quad \{(-\Delta_p)(\lambda - \Delta_p)^{-1} : \lambda \in \Sigma_\theta\},$$

are uniformly bounded on $L^p(\Omega)$ for all $\theta \in (0, \pi)$. The resolvent estimates then follow from

$$\begin{aligned} |\lambda|^{1/2}\partial_i(\lambda - \Delta_p)^{-1} &= \partial_i(-\Delta_p)^{-1/2}|\lambda|^{1/2}(-\Delta_p)^{1/2}(\lambda - \Delta_p)^{-1}, \\ |\lambda|^{1/2}(\lambda - \Delta_p)^{-1}\partial_i &= |\lambda|^{1/2}(-\Delta_p)^{1/2}(\lambda - \Delta_p)^{-1}(-\Delta_p)^{-1/2}\partial_i, \\ \partial_i(\lambda - \Delta_p)^{-1}\partial_j &= \partial_i(-\Delta_p)^{-1/2}(-\Delta_p)(\lambda - \Delta_p)^{-1}(-\Delta_p)^{-1/2}\partial_j. \end{aligned}$$

This concludes the proof. □

We further establish higher-order smoothing properties for the resolvent of the Laplace operator in this setting.

Lemma 3.4.4. *Let $p \in (1, \infty)$ and $s, \mu > 0$. Then $(\mu - \Delta_p)^{-1}$ maps $H^{s,p}_{per}(\Omega)$ into $H^{2+s,p}_{per}(\Omega)$ and there exists a constant $C = C_{p,\mu,s} > 0$ such that*

$$\|(\mu - \Delta_p)^{-1} f\|_{H^{2+s,p}(\Omega)} \leq C\|f\|_{H^{s,p}(\Omega)}$$

for all $f \in H^{s,p}_{per}(\Omega)$. If $\Gamma_D \neq \emptyset$ then this even holds for $\mu = 0$.

Proof. First suppose that $\Gamma_D = \emptyset$ and $\mu > 0$. Since $H^{s,p}_{\text{per}}(\Omega) \hookrightarrow L^p(\Omega)$, the function $v := (\mu - \Delta_p)^{-1} f \in D(A_p)$ exists. Denote by E_{per} the operator extending periodically from $G = (0, 1)^2$ onto $G' = (-1, 2)^2$ and observe that it maps $H^{s,p}(\Omega)$ into $H^{s,p}(\Omega')$ with $\Omega' = G' \times (-h, 0)$. We now decompose v into an upper and bottom part. For this purpose we consider horizontal and vertical cut-off functions $\phi \in C^\infty_c(\mathbb{R}^2)$ and $\psi \in C^\infty([-h, 0])$ satisfying

$$\begin{aligned} &\phi = 1 \text{ on } [-1/2, 3/2]^2, && \operatorname{supp}\phi \subset G', \\ &\psi = 1 \text{ on } [-h/3, 0], && \operatorname{supp}\psi \subset [-h/2, 0]. \end{aligned}$$

We then set $\chi_u(x, y, z) := \phi(x, y)\psi(z)$, $\chi_b(x, y, z) := \phi(x, y)(1 - \psi(z))$ and consider the functions

$$v_u := \chi_u E_{\text{per}} v, \quad v_b := \chi_b E_{\text{per}} v$$

on $\mathbb{R}^2\times(-\infty,0)$ and $\mathbb{R}^2\times(-h,\infty)$. Clearly $v_u+v_b\big|_\Omega=v$. Observe that they respectively satisfy

$$(\mu-\Delta)v_u=F_u \quad \text{on} \quad \mathbb{R}^2\times(-\infty,0), \quad (\mu-\Delta)v_b=F_b \quad \text{on} \quad \mathbb{R}^2\times(-h,\infty),$$

with right-hand sides

$$F_i:=\chi_i E_{\text{per}} f-2\nabla\chi_i\cdot\nabla(E_{\text{per}}v)-(\Delta\chi_i)E_{\text{per}}v, \quad i\in\{u,b\},$$

and boundary conditions

$$\partial_z v_u=0 \text{ on } \mathbb{R}^2\times\{0\}, \quad \partial_z v_b=0 \text{ on } \mathbb{R}^2\times\{-h\}.$$

By applying a translation and rotation, we may assume that these equations hold on the half-space domain $\mathbb{R}^3_+:=\mathbb{R}^2\times(0,\infty)$. It then follows from the well-known results for elliptic regularity of the Laplace operator with Neumann or Dirichlet boundary conditions on $\mathbb{R}^3_+$, see, e.g., [89, Section 4.2], that $F_i\in H^{r,p}(\mathbb{R}^3_+)$ implies $v_i\in H^{2+r,p}(\mathbb{R}^3_+)$ for all $r\geq 0$. Since $v\in H^{2,p}(\Omega)$ we have for $i\in\{u,b\}$ that $v_i\in H^{2,p}(\mathbb{R}^+_+)$ and thus $F_i\in H^{r_0,p}(\mathbb{R}^3_+)$ with $r_0:=\min\{s,1\}$. By iteration this argument, we obtain $v_i\in H^{2+r_n,p}(\mathbb{R}^3_+)$ and $F_i\in H^{r_n,p}(\mathbb{R}^3_+)$ with

$$r_{n+1}=\min\{s,1+r_n\}.$$

Since this sequence either terminates for $r_n=s$ or increases by 1, we obtain $v_i\in H^{2+s,p}(\mathbb{R}^3_+)$ and thus $v\in H^{2+s,p}(\Omega)$. Since $\mu-\Delta_p$ is a bounded mapping from $H^{2+s,p}(\Omega)\cap D(\Delta_p)$ into $H^{s,p}(\Omega)$ the resolvent estimate follows from the bounded inverse theorem. In the case where $\Gamma_N=\emptyset$, we may argue as above, merely replacing the boundary conditions for v_u and v_b. If we instead have both $\Gamma_D\neq\emptyset$ and $\Gamma_N\neq\emptyset$, we apply an even reflection at Γ_N and argue as in the case $\Gamma_N=\emptyset$ on the new vertical interval $(-h,h)$. □

3.5 L^q-L^p-estimates for the resolvent on cylindrical domains

In this section, building on the results of Lemma 3.4.1 and Corollary 3.4.3, we establish estimates for the Laplace operator in anisotropic L^q-L^p-spaces in the case $\Gamma_D\neq\emptyset$. For this purpose we investigate the resolvent problems

$$\lambda v-\Delta v=f \text{ on } \Omega, \tag{3.5.1}$$
$$\lambda w-\Delta w=\partial_i f \text{ on } \Omega, \tag{3.5.2}$$

for $\partial_i\in\{\partial_x,\partial_y,\partial_z\}$, where on the boundary we assume one of

$$\partial_z v=0 \text{ on } \Gamma_u, \quad v \text{ periodic on } \Gamma_l, \quad v=0 \text{ on } \Gamma_b, \tag{ND}$$
$$v=0 \text{ on } \Gamma_u, \quad v \text{ periodic on } \Gamma_l, \quad v=0 \text{ on } \Gamma_b. \tag{DD}$$

The following lemma establishes a resolvent estimate in $L^q_H L^p_z(\Omega)$. Its proof employs scaling arguments also utilized in the Masuda-Stewart method, compare, e.g., [87].

Lemma 3.5.1. *Let $\lambda \in \Sigma_\theta$ for some $\theta \in (0,\pi)$ as well as $f \in L^q_H L^p_z(\Omega)$ for $q \in [1,\infty]$ and $p \in [1,\infty)$. Then there exist constants $C = C_{\Omega,\theta} > 0$ and $\lambda_0 > 0$ such that for $|\lambda| \geq \lambda_0$ the problems* (3.5.1) *and* (3.5.2) *with boundary conditions (ND) or (DD) have unique solutions $v, w \in L^q_H L^p_z(\Omega)$, respectively, which further satisfy the estimates*

$$|\lambda| \cdot \|v\|_{L^q_H L^p_z(\Omega)} + |\lambda|^{1/2} \|\nabla v\|_{L^q_H L^p_z(\Omega)} + \|\Delta v\|_{L^q_H L^p_z(\Omega)} \leq C \|f\|_{L^q_H L^p_z(\Omega)}, \tag{3.5.3}$$

$$|\lambda|^{1/2} \|w\|_{L^q_H L^p_z(\Omega)} \leq C \|f\|_{L^q_H L^p_z(\Omega)}. \tag{3.5.4}$$

In the case $q = \infty$ and $p \in (2,\infty)$ one can chose $\lambda_0 = 0$ with a constant $C = C_{\theta,p} > 0$.

Remark 3.5.2. The constraint $p > 2$ for $q = \infty$ is due to fact that the proof makes use of the embedding $W^{1,p}(G) \hookrightarrow L^\infty(G)$.

Proof of Lemma 3.5.1. We begin with an approximation argument. Observe that

(i) in the case $1 \leq q, p < \infty$ we have that $C^\infty_{\text{per}}([0,1]^2; C^\infty_c(-h,0))$ is a dense subspace of $L^q_H L^p_z(\Omega)$,

(ii) $L^\infty(G; C^\infty_c(-h,0))$ is dense in $L^\infty_H L^p_z(\Omega)$ since $C^\infty_c(-h,0)$ is dense in $L^p(-h,0)$.

In either case we may assume without loss of generality that $f = 0$ on $\Gamma_u \cup \Gamma_b$ and $f \in L^\infty(\Omega)$. Since $L^\infty(\Omega) \hookrightarrow L^4(\Omega)$, we now apply the fact that $-\Delta_4$ is sectorial of angle 0 by Lemma 3.4.1as well the estimate (3.4.6) to obtain solutions to the problems (3.5.1) and (3.5.2), respectively belonging to the spaces

$$v \in W^{2,4}(\Omega) \hookrightarrow W^{1,\infty}(\Omega), \quad w \in W^{1,4}(\Omega) \hookrightarrow L^\infty(\Omega).$$

It thus remains to prove the estimates (3.5.3) and (3.5.4). For this purpose we utilize the extension operator

$$E = E^{\text{even,odd}}_z \circ E^{\text{per}}_H.$$

Here E^{per}_H denotes the periodic extension operator from G to $\mathbb{R}^2$ in horizontal direction and $E^{\text{even,odd}}_z$ denotes the operator extending from $(-h,0)$ to $(-2h,h)$ by applying an odd reflection at $z = -h$ and an even reflection at $z = 0$ in the case of (ND) or an odd reflection in the case of (DD). Further consider a family of cut-off functions $\chi_r \in C^\infty_c(\mathbb{R}^3)$ for $r \in (0,\infty)$ given via

$$\chi_r(x,y,z) = \varphi_r(x,y)\psi_r(z), \quad \varphi_r \in C^\infty_c(\mathbb{R}^2), \psi_r \in C^\infty_c(\mathbb{R}),$$

such that the horizontal and vertical parts satisfy

$$\begin{aligned} \varphi_r &= 1 \quad \text{on } [-1/4, 5/4]^2, \quad \varphi_r = 0 \quad \text{on } \left((-\infty, -r-1/4] \cup [5/4+r, \infty)\right)^2, \\ \psi_r &= 1 \quad \text{on } [-5h/4, h/4], \quad \psi_r = 0 \quad \text{on } (-\infty, -r-5h/4] \cup [h/4+r, \infty), \end{aligned}$$

as well as

$$\|\varphi_r\|_\infty + \|\psi_r\|_\infty + r\left(\|\nabla_H \varphi_r\|_\infty + \|\partial_z \psi_r\|_\infty\right) + r^2\left(\|\Delta_H \varphi_r\|_\infty + \|\partial_z^2 \psi_r\|_\infty\right) \leq M$$

for a constant $M > 0$ and all $r > 0$. We may now extend v from Ω to the whole space $\mathbb{R}^3$ by setting

$$u(x, y, z) := \chi_r(x, y, z)(Ev)(x, y, z)$$

where $r > 0$ will be chosen later on. Since E commutes with derivatives of v, it follows that if v is a solution to (3.5.1), then u is a solution to

$$\lambda u - \Delta u = F \text{ on } \mathbb{R}^3, \quad F := \chi_r Ef - 2(\nabla\chi_r) \cdot E(\nabla v) - (\Delta\chi_r)Ev.$$

By $f \in L^\infty(\Omega)$ and $v \in W^{1,\infty}(\Omega)$ we also that $F \in L^\infty(\mathbb{R}^3)$ and that F has compact support. By Lemma 3.2.1 we may apply the estimate (3.2.3) to obtain

$$|\lambda| \cdot \|u\|_{L^q_H L^p_z(\mathbb{R}^3)} + |\lambda|^{1/2}\|\nabla u\|_{L^q_H L^p_z(\mathbb{R}^3)} \leq C_\theta \|F\|_{L^q_H L^p_z(\mathbb{R}^3)}.$$

Assume that $0 < 4r < 3\min\{1, h\}$. Then χ_r is supported on $(-1, 2)^2 \times (-2h, h)$ and $\chi_r = 1$ on Ω, yielding the estimate

$$\begin{aligned} \|\chi_r Ef\|_{L^q_H L^p_z(\mathbb{R}^3)} &\leq 27M^2\|f\|_{L^q_H L^p_z(\Omega)}, \\ \|(\nabla\chi_r) \cdot E(\nabla v)\|_{L^q_H L^p_z(\mathbb{R}^3)} &\leq 27M^2 r^{-1}\|\nabla v\|_{L^q_H L^p_z(\Omega)}, \\ \|(\Delta\chi_r)Ev\|_{L^q_H L^p_z(\mathbb{R}^3)} &\leq 27M^2 r^{-2}\|v\|_{L^q_H L^p_z(\Omega)}. \end{aligned}$$

We now set $r = \eta|\lambda|^{-1/2}$ for some $\eta > 0$ to obtain

$$\|F\|_{L^q_H L^p_z(\mathbb{R}^3)} \leq 27M^2 \left(\|f\|_{L^q_H L^p_z(\Omega)} + 2\eta^{-1}|\lambda|^{1/2}\|\nabla v\|_{L^q_H L^p_z(\Omega)} + \eta^{-2}|\lambda| \cdot \|v\|_{L^q_H L^p_z(\Omega)}\right)$$

and $\eta > 0$ so large that

$$54C_\theta M^2\eta^{-1} < 1/2, \quad 27C_\theta M^2\eta^{-2} < 1/2$$

and $\lambda_0 > 0$ so large that $4\eta\lambda_0^{-1/2} < 3\min\{1, h\}$. Since u is an extension of v this yields

$$|\lambda| \cdot \|v\|_{L^q_H L^p_z(\Omega)} + |\lambda|^{1/2}\|\nabla v\|_{L^q_H L^p_z(\Omega)} \leq 54C_\theta M^2\|f\|_{L^q_H L^p_z(\Omega)}, \quad |\lambda| \geq \lambda_0$$

and this conclude the proof of estimate (3.5.3). We further have that if $\partial_i \in \{\partial_x, \partial_y\}$, then $\partial_i v$ solves the problem (3.5.2) and thus the estimate (3.5.4) follows from (3.5.3). In order to prove (3.5.4) for the case $\partial_i = \partial_z$, we observe that if w is a solution to the problem (3.5.2), then the extension u, constructed as above for w instead of v, solves the problem

$$\lambda u - \Delta u = G \text{ on } \mathbb{R}^3, \quad G := \chi_r E(\partial_z f) - 2(\nabla\chi_r) \cdot E(\nabla w) - (\Delta\chi_r)Ew.$$

We rewrite the right-hand side terms as

$$-2(\nabla\chi_r) \cdot E(\nabla w) - (\Delta\chi_r)Ew = -2\mathrm{div}(\nabla\chi_r Ew) + (\Delta\chi_r)Ew,$$

and since $f = 0$ on $\Gamma_u \cup \Gamma_b$ we also have

$$\chi_r E(\partial_z f) = \partial_z(\chi_r sEf) - (\partial_z\chi_r)sEf$$

where

$$s(z) = \begin{cases} 1, & z \in (-2h, 0), \\ -1, & x \in (0, h), \end{cases}$$

for the boundary conditions (ND) and $s = 1$ for the boundary conditions (DD). This yields

$$\lambda u - \Delta u = \partial_z G_1 + \mathrm{div}_H G_2 + G_3 \quad \text{on } \mathbb{R}^3,$$

with right-hand sides

$$G_1 := \chi_r s E f, \quad G_2 := -2(\nabla \chi_r) E w, \quad G_3 := -(\partial_z \chi_r) s E f + (\Delta \chi_r) E w.$$

As above we have that these terms are bounded with compact support so by Lemma 3.2.1 we have

$$|\lambda|^{1/2} \|u\|_{L^q_H L^p_z(\mathbb{R}^3)} \leq C_\theta \left(\|G_1\|_{L^q_H L^p_z(\mathbb{R}^3)} + \|G_2\|_{L^q_H L^p_z(\mathbb{R}^3)} + |\lambda|^{-1/2} \|G_3\|_{L^q_H L^p_z(\mathbb{R}^3)} \right).$$

We may estimate the right-hand sides in the same way we did previously, yielding

$$\begin{aligned} \|G_1\|_{L^q_H L^p_z(\mathbb{R}^3)} &\leq 27 M^2 \|f\|_{L^q_H L^p_z(\Omega)} \\ \|G_2\|_{L^q_H L^p_z(\mathbb{R}^3)} &\leq 54 M^2 \eta^{-1} |\lambda|^{1/2} \|w\|_{L^\infty_H L^p_z(\Omega)}, \\ \|G_3\|_{L^\infty_H L^p_z(\mathbb{R}^3)} &\leq 27 M^2 \eta^{-1} |\lambda|^{1/2} \|f\|_{L^\infty_H L^p_z(\Omega)} + 27 M^2 \eta^{-2} |\lambda| \cdot \|w\|_{L^\infty_H L^p_z(\Omega)}, \end{aligned}$$

and given our assumptions on η and λ_0 we obtain estimate (3.5.4).

We now show that the results are valid for the entire range of $\lambda \in \Sigma_\theta$ if we have $q = \infty$ and $p \in (2, \infty)$. For this purpose we take $\lambda \in \Sigma_\theta$ with $0 < |\lambda| < \lambda_0$ and consider the auxiliary parameter $\lambda_1 := \frac{\lambda_0}{|\lambda|} \lambda$. For the first problem, we utilize the embedding $f \in L^\infty_H L^p_z(\Omega) \hookrightarrow L^p(\Omega)$ which together with the fact that $-\Delta_p$ is sectorial of angle 0 and estimate (3.4.6) yields

$$|\lambda| \cdot \|v\|_{L^p(\Omega)} + |\lambda|^{1/2} \|\nabla v\|_{L^p(\Omega)} + \|\Delta v\|_{L^p(\Omega)} \leq C_{\theta,p} \|f\|_{L^p(\Omega)}.$$

Since $|\lambda_1| = \lambda_0$ we may rewrite the problem into $\lambda_1 v - \Delta v = f + (\lambda_1 - \lambda) v$ and apply estimate (3.5.3), yielding

$$|\lambda_1| \cdot \|v\|_{L^\infty_H L^p_z(\Omega)} + |\lambda_1|^{1/2} |\nabla v|_{L^\infty_H L^p_z(\Omega)} + \|\Delta v\|_{L^\infty_H L^p_z(\Omega)} \leq C_{\theta,p} \left(\|f + (\lambda_1 - \lambda) v\|_{L^\infty_H L^p_z(\Omega)} \right).$$

We further estimate the right-hand side via $|\lambda_1 - \lambda| < \lambda_0$ as well as

$$\begin{aligned} \|v\|_{L^\infty_H L^p_z(\Omega)} &\leq C_p \|v\|_{W^{1,p}_H L^p_z(\Omega)} \\ &\leq C_p \|v\|_{W^{2,p}(\Omega)} \\ &\leq C_p \|\Delta v\|_{L^p(\Omega)} \\ &\leq C_p \|f\|_{L^p(\Omega)} \\ &\leq C_p \|f\|_{L^\infty_H L^p_z(\Omega)}, \end{aligned}$$

where in the second step we also used the Sobolev embedding $W^{2,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ for $p \in (3/2, \infty)$ and in the third step we used the fact that $\Delta_p \colon D(\Delta_p) \hookrightarrow L^p(\Omega)$ is invertible and bounded when $D(\Delta_p)$ is equipped with the $W^{2,p}$-norm. The fact that $|\lambda| < \lambda_0 = |\lambda_1|$ then implies that estimate (3.5.3) is valid for the full range $\lambda \in \Sigma_\theta$ with a constant $C = C_{\theta,p} > 0$.

For the second problem we analogously obtain

$$|\lambda|^{1/2} \cdot \|w\|_{L^p(\Omega)} + \|\nabla w\|_{L^p(\Omega)} \leq C_{\theta,p}\|f\|_{L^p(\Omega)}$$

via estimate (3.4.6) together with

$$|\lambda_1|^{1/2} \cdot \|w\|_{L^\infty_H L^p_z(\Omega)} + \|\nabla w\|_{L^\infty_H L^p_z(\Omega)} \leq C_\theta \left(\|f\|_{L^\infty_H L^p_z(\Omega)} + |\lambda_1|^{-1/2}|\lambda_1 - \lambda| \cdot \|w\|_{L^\infty_H L^p_z(\Omega)}\right).$$

Here we can estimate $|\lambda_1|^{-1/2}|\lambda_1 - \lambda| \leq \lambda_0^{1/2}$ as well as

$$\begin{aligned} \|w\|_{L^\infty_H L^p_z(\Omega)} &\leq C_p\|w\|_{W^{1,p}_H L^p_z(\Omega)} \\ &\leq C_p\|w\|_{W^{1,p}(\Omega)} \\ &\leq C_p\|\nabla w\|_{L^p(\Omega)} \\ &\leq C_p\|f\|_{L^p(\Omega)} \\ &\leq C_p\|f\|_{L^\infty_H L^p_z(\Omega)}, \end{aligned}$$

where we used the Sobolev embedding $W^{1,p}(G) \hookrightarrow L^\infty(G)$ for $p \in (2, \infty)$ in the second step and the Poincaré inequality

$$\|w\|_{L^p(\Omega)} \leq C_p\|\partial_z w\|_{L^p(\Omega)} \leq C_p\|\nabla w\|_{L^p(\Omega)}, \quad w\big|_{\Gamma_b} = 0,$$

in the third step. The claim then follows as above. □

4 The hydrostatic Stokes operator

We now move on from the theory of Laplace operators and heat equations to our primary interest. We continue to consider the cylindrical domain

$$\Omega = G \times (-h, 0), \quad G = (0,1)^2, \quad h > 0.$$

Recall that the primitive equations, as given in [68–70], assuming temperature and salinity are constant, are formulated in the form

$$\begin{aligned} \partial_t v - \Delta v + (u \cdot \nabla) v + \nabla_H \pi &= f && \text{in } \Omega \times (0,\infty), \\ \partial_z \pi &= 0 && \text{in } \Omega \times (0,\infty), \\ \operatorname{div} u &= 0 && \text{in } \Omega \times (0,\infty), \\ v(0) &= a && \text{in } \Omega. \end{aligned} \tag{4.0.1}$$

Recall that $\nabla_H f := (\partial_x f, \partial_y f)^T$ denotes the gradient in horizontal variables only, $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ is the full Laplace operator, $\pi \colon G \to \mathbb{R}$ denotes the surface pressure, v and w respectively denote the horizontal and vertical parts of the full velocity vector field

$$u = (v, w) \colon \Omega \times (0,\infty) \to \mathbb{R}^2 \times \mathbb{R},$$

whereas f and a are given external force and initial horizontal velocity, respectively. As in Section 3.4, compare (3.4.2) through (3.4.4), we decompose the boundary of Ω into

$$\Gamma_u := G \times \{0\}, \quad \Gamma_l := \partial G \times (-h, 0), \quad \Gamma_b := G \times \{-h\},$$

and consider the boundary conditions

$$\begin{aligned} v, \pi \text{ periodic} \quad & \text{on } \Gamma_l \times (0,\infty), \\ v = w = 0 \quad & \text{on } \Gamma_D \times (0,\infty), \\ \partial_z v = w = 0 \quad & \text{on } \Gamma_N \times (0,\infty), \end{aligned}$$

where

$$\Gamma_D \in \{\emptyset, \Gamma_u, \Gamma_b, \Gamma_u \cup \Gamma_b\}, \quad \Gamma_N = (\Gamma_u \cup \Gamma_b) \setminus \Gamma_D.$$

Observe that the condition $\operatorname{div} u = 0$ together with the fact that w vanishes on Γ_b implies that w is determined by v via the relation

$$w(x,y,z) = -\int_{-h}^{z} \operatorname{div}_H v(x,y,\xi)\, d\xi.$$

Since w also vanishes on Γ_u, this further yields the condition

$$\mathrm{div}_H \overline{v} = 0$$

where

$$\overline{f} := \frac{1}{h} \int_{-h}^{0} f(\cdot, z)\, dz, \quad \mathrm{div}_H f := \partial_x f_1 + \partial_y f_2$$

respectively denote the vertical average and divergence in horizontal variables only.

The focus of this chapter is on the linearized version of the primitive equations (4.0.1), called the *hydrostatic Stokes equations*, given by

$$\partial_t v - \Delta v + \nabla_H \pi = f, \quad \mathrm{div}\,_H \overline{v} = 0, \quad v(0) = a \tag{4.0.2}$$

with boundary conditions (3.4.3) for v and π, compare [49]. The choice of the name *hydrostatic Stokes equations* is due to the assumption of a hydrostatic balance when deriving the full primitive equations from the Navier-Stokes equations, compare, e.g., [68–70].

As with the Stokes equation, one can eliminate the pressure gradient term via the use of a projection, called the *hydrostatic Helmholtz projection*, given by the mapping

$$\mathbb{P} f := f - \nabla_H \pi$$

where $\nabla_H \pi$ is given as the weak solution of the problem

$$\Delta_H \pi = \mathrm{div}_H \overline{f} \text{ on } G, \quad \pi \text{ periodic on } \partial G, \tag{4.0.3}$$

compare [49, Proposition 4.2 f.]. As the fact that this problem is given on the two-dimensional domain G with periodic boundary conditions suggests, the hydrostatic Helmholtz projection $\mathbb{P}$ is closely related to the two-dimensional *Helmholtz projection* with periodic boundary conditions, denoted by Q, via the relation

$$(1 - \mathbb{P}) f = (1 - Q) \overline{f},$$

where the operator $1-Q$ is the solution operator $\overline{f} \mapsto \nabla_H \pi$ for the weak problem (4.0.3). It can be formally understood as $-\nabla_H (-\Delta_H)^{-1} \mathrm{div}_H$, where $\Delta_H = \partial_x^2 + \partial_y^2$ denotes the Laplace operator on the two-dimensional torus with periodic boundary conditions, which is justified by the fact that it is the closure of the latter operator in $L^p(G)^2$ for $p \in (1, \infty)$.

By applying this projection to (4.0.2), one obtains the equivalent Cauchy problem

$$\partial_t v - Av = \mathbb{P} f, \quad v(0) = a, \tag{4.0.4}$$

where the operator $A := \mathbb{P}\Delta$ is called the *hydrostatic Stokes operator*.

An L^p-theory for this operator was established by Hieber and Kashiwabara in [49] for $p \in (1, \infty)$ by splitting (4.0.2) into a three-dimensional heat equation and a two-dimensional Stokes equation. Denoting the range of the hydrostatic Helmholtz projection by

$$L^p_{\overline{\sigma}}(\Omega) := \mathbb{P} L^p(\Omega)^2, \quad p \in (1, \infty),$$

which takes an analogous role as the solenoidal L^p-vector fields L^p_σ for the Stokes and Navier-Stokes equations, and the $L^p_{\overline{\sigma}}(\Omega)$-realization of the hydrostatic Stokes operator by

$$A_p v := \mathbb{P}\Delta v, \quad D(A_p) := \{v \in H^{2,p}_{\mathrm{per}}(\Omega)^2 : \mathrm{div}_H \overline{v} = 0, \partial_z v = 0 \text{ on } \Gamma_u, v = 0 \text{ on } \Gamma_b\},$$

where the periodic function space $H^{2,p}_{\mathrm{per}}(\Omega)$ is as defined in (2.4.2), one then has the following.

Proposition 4.0.1. *[49, Proposition 4.2-4.4] Let $p \in (1,\infty)$. Then the following holds.*

1. *The mapping $\mathbb{P}$ defines a continuous projection on $L^p(\Omega)^2$ with*

$$L^p_{\overline{\sigma}}(\Omega) = \overline{\{v \in C^\infty_{per}(\overline{\Omega})^2 : div_H \overline{v} = 0\}}^{\|\cdot\|_{L^p(\Omega)^2}},$$
$$(1-\mathbb{P})L^p(\Omega)^2 = (1-Q)L^p(G)^2 = \{\nabla_H \pi : \pi \in H^{1,p}_{per}(G)\}.$$

2. *The hydrostatic Stokes operator A_p generates a strongly continuous, exponentially stable, bounded analytic semigroup of angle $\pi/2$ on $L^p_{\overline{\sigma}}(\Omega)$.*

Note that in particular, one also has that Q is bounded on $L^p(G)^2$ with

$$(1-\mathbb{P})L^p(\Omega)^2 = (1-Q)L^p(G)^2.$$

The semigroup generated by A_p is called the *hydrostatic Stokes semigroup* and henceforth simply denoted by

$$S(\tau) := e^{\tau A} := e^{\tau A_p}, \quad \mathrm{Re}\,\tau > 0.$$

This chapter is devoted to extending this result and in large part rests on the following observation. By solving the problem (4.0.4) for v, it is possible to recover the pressure gradient in the following way. Applying the vertical average and horizontal divergence to (4.0.2) yields the weak problem

$$\Delta_H \pi = \mathrm{div}_H \left(\overline{f} + \frac{1}{h}\partial_z v\big|_{\Gamma_u} - \frac{1}{h}\partial_z v\big|_{\Gamma_b} \right), \quad \pi \text{ periodic on } \partial G \tag{4.0.5}$$

and by solving this problem for $\nabla_H \pi$ one obtains the representation

$$\nabla_H \pi = (1-Q)\overline{f} - Bv, \quad Bv := \frac{1}{h}(1-Q)\partial_z v\big|_{\Gamma_b} - \frac{1}{h}(1-Q)\partial_z v\big|_{\Gamma_u}. \tag{4.0.6}$$

Plugging this representation back into (4.0.2) then yields the equivalent problem

$$\partial_t v - (\Delta + B)v = \mathbb{P}f, \quad \mathrm{div}_H \overline{v} = 0, \quad v(0) = a \tag{4.0.7}$$

with boundary conditions (3.4.3) for v. The representation $A = \Delta + B$ then allows us to study the hydrostatic Stokes operator using perturbation methods. We will denote the $L^p(\Omega)^2$-realization of $\Delta + B$ by $\mathcal{A}_p$ to distinguish it from its restriction onto $L^p_{\overline{\sigma}}(\Omega)$, the hydrostatic Stokes operator A_p.

This chapter is structured as follows. In Section 4.1 we will first extend the results on L^p-spaces for $p \in (1, \infty)$ and also give new proofs of previously established results, excluding the invertibility of A_p. Using perturbation arguments we will prove that, for $p \in (1, \infty)$, the operator A_p possesses an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus and thus bounded imaginary powers as well as maximal L^q-regularity for $q \in (1, \infty)$. We will also prove that the hydrostatic Stokes semigroup S has L^p-L^q-smoothing properties for $1 < p \leq q < \infty$, as well as higher-order smoothing properties for the resolvent mapping.

In Sections 4.2 and 4.3 we will turn to the case $p = \infty$. For this purpose, we will make use of anisotropic L^p-spaces which we will define in Section 4.3. Since the representation $A = \Delta + B$ with B as in (4.0.6) makes it clear that imposing Neumann boundary conditions on both Γ_u and Γ_b leads to the special case $A = \Delta$, we will consider this case separately in Section 4.2 and the case where Dirichlet conditions are imposed in 4.3. The focus of these sections is on smoothing properties of the hydrostatic Stokes semigroup in the anisotropic L^∞-L^p-spaces.

Throughout this chapter, we will repeatedly utilize the results of Chapter 3, which we have established to serve as a foundation for this study of the hydrostatic Stokes operator and semigroup.

4.1 L^p-theory using perturbation arguments

While the L^p-theory for the hydrostatic Stokes operator developed in [49] assumes that the boundary conditions

$$\partial_z v = 0 \text{ on } \Gamma_u, \quad v \text{ periodic on } \Gamma_l, \quad v = 0 \text{ on } \Gamma_b \tag{4.1.1}$$

are imposed, we will instead consider the more general case

$$A_p v := \mathbb{P}\Delta v, \quad D(A_p) := \{v \in H^{2,p}_{\text{per}}(\Omega)^2 : \text{div}_H \overline{v} = 0, \partial_z v = 0 \text{ on } \Gamma_N, v = 0 \text{ on } \Gamma_D\}.$$

Here Γ_N and Γ_D are as described in (3.4.4), meaning that one either has only Neumann boundary conditions on both the top and bottom part of the boundary, or only Dirichlet boundary conditions, or that Neumann boundary conditions are imposed on one of Γ_u or Γ_b with Dirichlet boundary conditions on the other. In this section there is little need to distinguish between these cases, with the only relevant factor being whether or not one Dirichlet conditions are imposed, i.e., whether or not one has $\Gamma_D \neq \emptyset$ or $\Gamma_D = \emptyset$. In the former case, the result of [49] that $-A_p$ is invertible and sectorial with spectral angle $\phi_{-A_p} = 0$ continues to hold true. However, in the latter case it holds that $\mathbb{P}$ and Δ commute, compare (4.0.6), and thus one has that $A_p v = \Delta v$ for all $v \in D(A_p)$. This implies that 0 is an eigenvalue of A_p and thus $-A_p$ is not injective and not sectorial. This can be alleviated by considering the operator $-A_p + \mu$, $\mu > 0$, instead. The results of this section were previously published in [38] and [41].

As previously stated, the following is based on the observation that whenever it holds that $\text{div}_H \overline{v} = 0$, the hydrostatic Stokes operator $A = \mathbb{P}\Delta$ can be written as

$$Av = \Delta v + Bv, \quad Bv := \frac{1}{h}(1-Q)\partial_z v\big|_{\Gamma_b} - \frac{1}{h}(1-Q)\partial_z v\big|_{\Gamma_u}.$$

Since the operator $1 - Q$ can be understood as $\nabla_H \Delta^{-1} \text{div}_H$, the perturbation term B is of lower order.

4.1.1 Main results and corollaries

The main result of this section is the following.

Theorem 4.1.1. *Let $p \in (1,\infty)$ and $\mu > 0$. Then the operator $-A_p + \mu$ has a bounded $\mathcal{H}^\infty$-calculus on $L^p_{\overline{\sigma}}(\Omega)$ with angle $\phi^\infty_{-A_p+\mu} = 0$. In the case $\Gamma_D \neq \emptyset$ this result is true even for $\mu = 0$.*

To illustrate the strength of this result, we give a number of corollaries. The first invokes the notion of $\mathcal{R}$-boundedness. Since L^p-spaces with values in a Hilbert space have the property (α), the subspace $L^p_{\overline{\sigma}}(\Omega) \subset L^p(\Omega)^2$ has it as well, see [62, Remark 4.10]. This yields the following.

Corollary 4.1.2. *Let $p \in (1,\infty)$ and $\mu > 0$. Then the operator $-A_p + \mu$ has an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus on $L^p_{\overline{\sigma}}(\Omega)$ with angle $\phi^{\mathcal{R}\infty}_{-A_p+\mu} = 0$. In the case $\Gamma_D \neq \emptyset$ this result is true even for $\mu = 0$.*

Since $L^p_{\overline{\sigma}}(\Omega)$ is also a UMD space, see [4, Theorem 4.5.2], this allows for a further corollary, namely that of maximal regularity.

Corollary 4.1.3. *Let $p, q \in (1,\infty)$ and $\mu > 0$. Then the operator $A_p - \mu$ has maximal L^q-regularity. In the case $\Gamma_D \neq \emptyset$ this result is true even for $\mu = 0$.*

Note that even in the event where $\Gamma_D = \emptyset$, A_p still has the maximal regularity property if one considers this problem on a finite time interval $(0, T)$ with $T \in (0,\infty)$ instead of $\mathbb{R}_+$.

The property of L^q-maximal regularity allows one to consider the Cauchy problem

$$\partial_t v - A_p v = f, \quad v(0) = v_0 \tag{4.1.2}$$

with $f \in L^q(\mathbb{R}_+; L^p_{\overline{\sigma}}(\Omega))$ and initial data

$$X_{\gamma,p,q} := (L^p_{\overline{\sigma}}(\Omega), D(A_p))_{1-1/q,q}, \quad p, q \in (1,\infty). \tag{4.1.3}$$

For the purpose of applications, it is natural to investigate an exact characterization of these spaces. The same method as for Lemma 3.4.2 yields the following result, compare also [6, Section 4] for real interpolation spaces.

Corollary 4.1.4. *Let $p, q \in (1,\infty)$ and $\vartheta \in [0,1] \setminus \{1/2p, 1/2 + 1/2p\}$. Then, in the respective cases*

$$\text{(i) } 1 + 1/p < 2\vartheta \leq 2, \qquad \text{(ii) } 1/p < 2\vartheta < 1 + 1/p, \qquad \text{(iii) } 0 < 2\vartheta < 1/p,$$

the space $(L^p_{\overline{\sigma}}(\Omega), D(A_p))_{\vartheta,q}$ is given by

$$\{v \in B^{2\vartheta}_{p,q,per}(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega) : \partial_z v\,|_{\Gamma_N} = 0, v\,|_{\Gamma_D} = 0\}, \tag{i}$$
$$\{v \in B^{2\vartheta}_{p,q,per}(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega) : v\,|_{\Gamma_D} = 0\}, \tag{ii}$$
$$B^{2\vartheta}_{p,q,per}(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega). \tag{iii}$$

Since the existence of a bounded $\mathcal{H}^\infty$-calculus implies that the operator has bounded imaginary powers, we are also able to use complex interpolation in order to characterize domains of fractional powers, see [26, Theorem 2.5], yielding

$$D((-A_p)^\vartheta) = [L^p_{\overline{\sigma}}(\Omega), D(A_p)]_\vartheta \quad \vartheta \in [0,1].$$

These complex interpolation spaces were previously featured in [49, Lemma 4.6 (a)] where they played the role of admissible initial values for the primitive equations (4.0.1). While an exact characterization was provided for $\vartheta = 1/2$, for general $\vartheta \in [0,1]$ they were only treated as closed subspaces of $H^{2\vartheta,p}(\Omega)^2$. The same argument as for Lemma 3.4.2 and Corollary 4.1.4 allows us to characterize the general case.

Corollary 4.1.5. *Let $p \in (1,\infty)$ and $\vartheta \in [0,1] \setminus \{1/2p, 1/2+1/2p\}$ as well as $\Gamma_D \neq \emptyset$. Then the space $D((-A_p)^\vartheta) = [L^p_{\overline{\sigma}}(\Omega), D(A_p)]_\vartheta$ is given by*

$$\{v \in H^{2\vartheta,p}_{per}(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega) : \partial_z v \mid_{\Gamma_N} = 0, v \mid_{\Gamma_D} = 0\}, \tag{i}$$
$$\{v \in H^{2\vartheta,p}_{per}(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega) : v \mid_{\Gamma_D} = 0\}, \tag{ii}$$
$$H^{2\vartheta,p}_{per}(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega), \tag{iii}$$

with cases (i)-(iii) as in Corollary 4.1.4. In the case $\Gamma_D = \emptyset$ the same result holds with $-A_p + \mu$ with $\mu > 0$ instead of $-A_p$.

The conditions for ϑ as well as p and q in the previous two corollaries are closely tied to mapping properties of the trace operator. Denote the degree of differentiability of the right-hand side space, i.e., $2-2/q$ in Corollary 4.1.4 or 2ϑ in Corollary 4.1.4, by s. In the case where $s < 1/p$, the mapping $v \mapsto v\big|_{\partial\Omega}$ fails to be bounded from $H^{s,p}$ into L^p and so boundary conditions cannot be meaningfully defined or expected. However, in the case $s > 1/p$ the mapping is bounded and so is $v \mapsto \partial_z v\big|_{\partial\Omega}$ for $s > 1+1/p$, so boundary conditions for v its derivatives can be defined and expected. The cases where $s = 1/p$ or $s = 1+1/p$ remain open questions.

If one considers the case $\vartheta = 1/2$, one can define a hydrostatic analogue to the Riesz transformation and obtain their L^p-boundedness, compare Corollary 3.4.3.

Corollary 4.1.6. *Let $p \in (1,\infty)$ and $\Gamma_D \neq \emptyset$. Then the operators*

$$\partial_i(-A_p)^{-1/2}\mathbb{P}, \quad (-A_p)^{-1/2}\mathbb{P}\partial_i, \quad \partial_i \in \{\partial_x, \partial_y, \partial_z\},$$

are well-defined and bounded on $L^p(\Omega)^2$. In particular, for all $p \in (1,\infty)$ and $\theta \in (0,\pi)$ there exists a constant $C = C_{\Omega,\theta,p} > 0$ such that for all $\lambda \in \Sigma_\theta$ and $f \in L^p(\Omega)^2$ as well as $\partial_i, \partial_j \in \{\partial_x, \partial_y, \partial_z\}$ it holds that

$$|\lambda|^{1/2}\|\partial_i(\lambda - A_p)^{-1}\mathbb{P}f\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)},$$
$$|\lambda|^{1/2}\|(\lambda - A_p)^{-1}\mathbb{P}\partial_i f\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)},$$
$$\|\partial_i(\lambda - A_p)^{-1}\mathbb{P}\partial_j f\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}.$$

We also show that the hydrostatic Stokes semigroup S has L^p-L^q-smoothing properties and that the resolvent mapping has higher-order smoothing properties. Both of these results are typical for elliptic second order differential operators.

Theorem 4.1.7. *Let $1 < p \leq q < \infty$ and $\Gamma_D \neq \emptyset$. Then there exists a constant $C = C_{\Omega,p,q} > 0$ such that*

$$\|S(t)\mathbb{P}f\|_{L^q(\Omega)} \leq Ct^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{L^p(\Omega)},$$
$$\|\partial_i S(t)\mathbb{P}f\|_{L^q(\Omega)} \leq Ct^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}\|f\|_{L^p(\Omega)},$$
$$\|S(t)\mathbb{P}\partial_i f\|_{L^q(\Omega)} \leq Ct^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}\|f\|_{L^p(\Omega)}$$

where $\partial_i \in \{\partial_x, \partial_y, \partial_z\}$, for all $t > 0$ and $f \in L^p(\Omega)^2$.

Lemma 4.1.8. *Let $p \in (1, \infty)$ and $s > 0$. Then*

$$\mu(\mu - A_p)^{-1} \colon H^{s,p}_{per}(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega) \to H^{2+s,p}_{per}(\Omega) \cap L^p_{\overline{\sigma}}(\Omega), \quad \mu > 0,$$

is a well-defined family of uniformly bounded linear operators, i.e., $(\mu - A_p)^{-1}$ maps $H^{s,p}_{per}(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega)$ into $H^{2+s,p}_{per}(\Omega) \cap L^p_{\overline{\sigma}}(\Omega)$ for all $\mu > 0$ and there exists a constant $C = C_{p,s} > 0$ such that

$$\mu \|(\mu - A_p)^{-1} f\|_{H^{2+s,p}(\Omega)} \leq C \|f\|_{H^{s,p}(\Omega)}$$

for all $f \in H^{s,p}_{per} \cap L^p_{\overline{\sigma}}(\Omega)$. If in addition we have $\Gamma_D \neq \emptyset$, then this also holds for A_p^{-1}.

In particular, the spectrum of A_p does not depend on $p \in (1, \infty)$, consists of countably many negative eigenvalues of finite multiplicity, and all eigenvectors belong to $C^\infty(\overline{\Omega})^2$.

Remark 4.1.9. Our proof of Theorem 4.1.1 utilizes the fact that $-A_p$ is sectorial of angle 0 with $0 \in \rho(A_p)$ for $\Gamma_D \neq \emptyset$, which was established in [49]. However, we are also able to derive this property from the intermediate result (4.1.7) and the mapping properties of the resolvent. This allows us to stay self-contained.

4.1.2 Proof of main results

Before we turn to prove our main Theorem 4.1.1, we prove a lemma that establishes results in the case of our cylindrical domain Ω, that are well-known in the case of full and half space domains.

Lemma 4.1.10.

1. *There exists a continuous and linear extension operator*

$$E \colon L^p(\Omega) \to L^p(\mathbb{R}^3)$$

that also continuously maps $H^{s,p}(\Omega)$ into $H^{s,p}(\mathbb{R}^3)$ for all $p \in (1, \infty)$ and $s \geq 0$. In particular, one has $[L^p(\Omega), H^{2,p}(\Omega)]_\vartheta = H^{2\vartheta,p}(\Omega)$, for all $\vartheta \in [0, 1]$.

2. *Let $\Gamma \in \{\Gamma_u, \Gamma_b\}$. Then the trace operator $H^{s+1/p,p}(\Omega) \ni v \mapsto v\big|_\Gamma \in B^s_{p,p}(G)$ is bounded for all $p \in (1, \infty)$ and $s > 0$.*

Proof.

1. Given a function $f \colon \Omega \to \mathbb{C}$, we decompose it into an odd and even part with respect to the vertical variable at $z = -h/2$ and then extend these parts by applying odd and even reflections at $z = 0$ and $z = -h$, respectively, yielding a function on $(0, 1)^2 \times \mathbb{R}$. Repeating the same procedure in the horizontal variables with respect to $x, y = 1/2$ then yields a function on $\mathbb{R}^3$. We then multiply with a smooth cut-off function $\varphi \in C_c^\infty(\mathbb{R}^3)$ such that $\varphi = 1$ on a neighborhood of $\overline{\Omega}$ and define the newly obtained function as Ef. The mapping properties of the mapping $f \mapsto Ef$ then follow via [74, Theorem 2.14].

Since the restriction operator given by

$$R\colon L^p(\mathbb{R}^3) \to L^p(\Omega), \quad Rv := v\big|_{\Omega}$$

defines a retraction for E, the interpolation result follows via [90, Theorem 1.2.4].

2. The fact that the trace operator is a bounded mapping from $H^{s+1/p,p}$ into $B^s_{p,p}$ for $s > 0$ is established in [89, Theorem 2.7.2] for the case where the underlying domain is a half space. Given $f \in H^{s+1/p,p}(\Omega)$, one can then simply consider the restrictions of $Ef \in H^{s+1/p,p}(\mathbb{R}^3)$ onto $\mathbb{R}^2 \times (-\infty, 0)$ and $\mathbb{R}^2 \times (-h, \infty)$, respectively, and apply the result on the half space to obtain the desired result. □

We are now able to prove the main result of this section.

Proof of Theorem 4.1.1. We take

$$Bv = \frac{1}{h}(1-Q)\partial_z v\big|_{\Gamma_b} - \frac{1}{h}(1-Q)\partial_z v\big|_{\Gamma_u}$$

as in (4.0.6) and consider the operators $B_p\colon D(B_p) \to L^p(\Omega)^2$ given by

$$B_p v := Bv, \quad D(B_p) := H^{1+1/p+s,p}(\Omega)^2, \quad s \in (0, 1-1/p), \tag{4.1.4}$$

as well as $\mathcal{A}_p\colon D(\mathcal{A}_p) \to L^p(\Omega)^2$ defined via

$$\mathcal{A}_p := \Delta_p + B_p, \quad D(\mathcal{A}_p) := D(\Delta_p), \tag{4.1.5}$$

where Δ_p is the $L^p(\Omega)^2$-realization of the operator defined in (3.4.5). We first observe that $B_p\colon D(B_p) \to L^p(\Omega)^2$ is bounded. In detail, one has that the mappings

$$D(B_p) \ni v \mapsto \partial_z v \in H^{1/p+s,p}(\Omega)^2,$$
$$H^{1/p+s,p}(\Omega)^2 \ni \partial_z v \mapsto \partial_z v\big|_{\Gamma_u}, \partial_z v\big|_{\Gamma_b} \in B^s_{p,p}(G)^2,$$

are bounded, see Lemma 4.1.10 for the trace term. Since it further holds that

$$B^s_{p,p}(G)^2 \cong W^{s,p}(G)^2 \hookrightarrow L^p(G)^2,$$

where $W^{s,p}(G)$ denotes the Sobolev-Slobodeckij space on G of order s, and the projection Q is bounded on $L^p(G)^2$, the claim follows.

Observe that by (4.0.5) through (4.0.7), we have that $\mathcal{A}_p$ is an extension of A_p. It further holds that, whenever $\lambda \in \rho(\mathcal{A}_p)$, we have that $v = (\lambda - \mathcal{A}_p)^{-1}\mathbb{P}f$ is the unique solution to the problem

$$\lambda v - \Delta v + \nabla_H \pi = f, \quad \partial_z \pi = 0, \quad \mathrm{div}_H \overline{v} = 0,$$

with boundary conditions (3.4.3). In particular it holds that $(\lambda - \mathcal{A}_p)^{-1}$ maps $L^p_{\overline{\sigma}}(\Omega)$ into itself as well as $\rho(\mathcal{A}_p) \subset \rho(A_p)$.

Perturbations of operators possessing a bounded $\mathcal{H}^\infty$-calculus have been studied by many others, compare, e.g., [8, 25, 26, 53, 62, 81]. Here we will utilize the results of [62, Proposition 13.1] and [53, Proposition 6.10].

Note that we have $-\Delta_p \in \mathcal{H}^\infty(L^p(\Omega)^2)$ with $\phi^\infty_{-\Delta_p} = 0$ as well as $0 \in \rho(\Delta_p)$ whenever $\Gamma_D \neq \emptyset$ by Lemma 3.4.1. For $v \in D(\Delta_p) \subset D(B_p)$ we also have

$$\|B_p v\|_{L^p(\Omega)} \leq C\|v\|_{H^{1+1/p+s,p}(\Omega)} \leq C\|(-\Delta_p)^{1-\delta} v\|_{L^p(\Omega)},$$

where $s \in (0, 1-1/p)$ is arbitrary and $\delta \in (0,1)$ is chosen in such a way that we have $2\delta + s < 1 - 1/p$. Here we used the boundedness of B_p in the first step, and Lemma 3.4.2 as well as $0 \in \rho(\Delta_p)$ in the second step.

It now follows from [62, Proposition 13.1] that for an arbitrarily small angle $\phi \in (0, \pi)$ there exists sufficiently large $\mu = \mu_\phi \geq 0$ such that the translated perturbation $-\mathcal{A}_p + \mu$ satisfies

$$-\mathcal{A}_p + \mu \in \mathcal{H}^\infty(L^p(\Omega)^2), \quad \phi^\infty_{-\mathcal{A}_p+\mu} \leq \phi. \tag{4.1.6}$$

This property is retained under the restriction on the invariant subspace $L^p_{\overline{\sigma}}(\Omega)$, yielding

$$-A_p + \mu \in \mathcal{H}^\infty(L^p_{\overline{\sigma}}(\Omega)), \quad \phi^\infty_{-A_p+\mu} \leq \phi. \tag{4.1.7}$$

We now show that $-A_p - \varepsilon$ is sectorial with spectral angle 0 whenever $\varepsilon > 0$ is sufficiently small. Since it was established in [49, Section 3 and 4] that the operator $-A_p$ is invertible and sectorial with spectral angle $\phi_{-A_p} = 0$ whenever $\Gamma_D \neq \emptyset$, it follows that $\sigma(-A_p)$ is contained in the interval (δ, ∞) for some $\delta > 0$. Taking $\varepsilon > 0$ such that $2\varepsilon < \delta$ we thus have for all $\phi \in (0, \pi)$ that $\Sigma_{\pi-\phi} \subset \rho(A_p + \varepsilon)$ as well as

$$\lambda(\lambda - \varepsilon - A_p)^{-1} = \lambda(\lambda - A_p)^{-1}\left(1 + \varepsilon(\lambda - \varepsilon - A_p)^{-1}\right), \quad \lambda \in \Sigma_{\pi-\phi}$$

by an elementary calculation. Since the family of operators $\{\lambda(\lambda - A_p)^{-1} : \lambda \in \Sigma_{\pi-\phi}\}$ is uniformly bounded on $L^p_{\overline{\sigma}}(\Omega)$ by the sectoriality of $-A_p$, it remains to show that $\{\varepsilon(\lambda - \varepsilon - A_p)^{-1} : \lambda \in \Sigma_{\pi-\phi}\}$ is uniformly bounded. Consider arbitrary $f \in L^p_{\overline{\sigma}}(\Omega)$ and $\lambda \in \Sigma_{\pi-\phi}$. Taking an angle $\psi \in (0, \pi)$ such that

$$\{\lambda - \varepsilon : \lambda \in \Sigma_{\pi-\phi}\} \subset \Sigma_{\pi-\psi} \cup \overline{B_{2\varepsilon}(0)} \subset \rho(A_p),$$

we distinguish between the two following cases.

(i) If we have $|\lambda - \varepsilon| \leq 2\varepsilon$, we use the fact that the resolvent mapping $\lambda \mapsto (\lambda - A_p)^{-1}$ is analytic on $\rho(A_p)$ and thus bounded on $\overline{B_{2\varepsilon}(0)}$, yielding

$$\|\varepsilon(\lambda - \varepsilon - A_p)^{-1} f\|_{L^p(\Omega)} \leq C\|f\|_{L^p},$$

where $C > 0$ is a constant only depending on $\varepsilon > 0$ and $p \in (1, \infty)$.

(ii) If we have $|\lambda - \varepsilon| > 2\varepsilon$, then it holds that

$$\|\varepsilon(\lambda - \varepsilon - A_p)^{-1} f\|_{L^p(\Omega)} \leq C_\psi \frac{\varepsilon}{|\lambda - \varepsilon|}\|f\|_{L^p(\Omega)} \leq \frac{1}{2} C_\psi \|f\|_{L^p(\Omega)}.$$

It follows that $-A_p - \varepsilon$ is sectorial with spectral angle 0 and so by (4.1.7) and [53, Proposition 6.10] we conclude that

$$-A_p \in \mathcal{H}^\infty(L^p_{\overline{\sigma}}(\Omega)), \quad \phi^\infty_{-A_p} = 0.$$

In the case $\Gamma_D = \emptyset$ we have that $Bv = 0$ for all $v \in D(\Delta_p)$ and thus $\mathcal{A}_p = \Delta_p$, so the property (4.1.6) was already obtained in Lemma 3.4.1 for arbitrary $\mu > 0$ which is inherited by $-A_p + \mu$ via the same restriction argument. □

The arguments through which corollaries 4.1.2 through 4.1.5 are obtained are all straightforward as previously stated. We now turn to derivatives of the resolvent.

Proof of Corollary 4.1.6. By Corollary 4.1.5 the operator $\partial_i(-A_p)^{-1/2}$ is bounded from $L^p_{\overline{\sigma}}(\Omega)$ into $L^p(\Omega)^2$ for $\partial_i \in \{\partial_x, \partial_y, \partial_z\}$ and thus $\partial_i(-A_p)^{-1/2}\mathbb{P}$ is bounded on $L^p(\Omega)^2$ for $p \in (1, \infty)$. Now suppose that $\partial_i \in \{\partial_x, \partial_y\}$ is a horizontal derivative. Since it commutes with both the horizontal divergence div_H and the vertical average $\bar{\cdot}$, the space $L^p_{\overline{\sigma}}(\Omega)$ is left invariant under $\partial_i(-A_p)^{-1/2}$ for $\partial_i \in \{\partial_x, \partial_y\}$. By [49, Remark 4.5], the adjoint operator of A_p is given by A_q where $1/p + 1/q = 1$ and so it follows that $(-A_p)^{-1/2}\partial_i$ is likewise bounded on $L^p_{\overline{\sigma}}(\Omega)$ for $p \in (1, \infty)$ for $\partial_i \in \{\partial_x, \partial_y\}$ and so $(-A_p)^{-1/2}\partial_i\mathbb{P}$ is bounded on $L^p(\Omega)^2$. Since ∂_i also commutes with $\mathbb{P}$, we obtain the L^p-boundedness of $(-A_p)^{-1/2}\mathbb{P}\partial_i$ for $\partial_i \in \{\partial_x, \partial_y\}$.

In the case $\partial_i = \partial_z$ we have that $\partial_z(-A)^{-1/2}$ maps $L^p_{\overline{\sigma}}(\Omega)$ into $L^p(\Omega)^2$ and thus $(-A_p)^{-1/2}\partial_z$ is a bounded mapping from $L^p(\Omega)^2$ into $L^p_{\overline{\sigma}}(\Omega)$. Since smooth functions with compact support are dense in $L^p(\Omega)^2$, we may assume without loss of generality that

$$\overline{\partial_z f} = f\big|_{\Gamma_u} - f\big|_{\Gamma_b} = 0, \quad f \in L^p(\Omega)^2,$$

and thus it follows that $\mathbb{P}\partial_z f = \partial_z f$ and $(-A_p)^{-1/2}\mathbb{P}\partial_z = (-A_p)^{-1/2}\partial_z$ is bounded on $L^p(\Omega)^2$. The resolvent estimates then follow from the fact that the families of operators

$$\{|\lambda|^{1/2}(-A_p)^{1/2}(\lambda - A_p)^{-1} : \lambda \in \Sigma_\theta\}, \quad \{(-A_p)(\lambda - A_p)^{-1} : \lambda \in \Sigma_\theta\},$$

are uniformly bounded on $L^p_{\overline{\sigma}}(\Omega)$ for all $\theta \in (0, \pi)$, together with

$$\begin{aligned}
|\lambda|^{1/2}\partial_i(\lambda - A_p)^{-1}\mathbb{P} &= \partial_i(-A_p)^{-1/2}|\lambda|^{1/2}(-A_p)^{1/2}(\lambda - A_p)^{-1}\mathbb{P},\\
|\lambda|^{1/2}(\lambda - A_p)^{-1}\mathbb{P}\partial_i &= |\lambda|^{1/2}(-A_p)^{1/2}(\lambda - A_p)^{-1}(-A_p)^{-1/2}\mathbb{P}\partial_i,\\
\partial_i(\lambda - A_p)^{-1}\mathbb{P}\partial_j &= \partial_i(-A_p)^{-1/2}(-A_p)(\lambda - A_p)^{-1}(-A_p)^{-1/2}\mathbb{P}\partial_j.
\end{aligned}$$

This concludes the proof. □

We now prove the L^p-L^q-smoothing properties for the hydrostatic Stokes semigroup.

Proof of Theorem 4.1.7. Using the result of Lemma 4.1.10 we may proceed analogously to the proof of [33, Proposition 3.1] for $n = 3$. Since S is bounded analytic on $L^p_{\overline{\sigma}}(\Omega)$ there exists a constant $C = C_{\Omega,p} > 0$ such that

$$\|S(t)\mathbb{P}f\|_{L^p(\Omega)} + t\|A_pS(t)\mathbb{P}f\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)},$$

for all $t > 0$, and since A_p has a bounded inverse we have $\|v\|_{H^{2,p}(\Omega)} \leq C\|A_p v\|_{L^p(\Omega)}$ for all $v \in D(A_p)$. Lemma 4.1.10 then yields the estimate

$$\|S(t)\mathbb{P}f\|_{H^{2\vartheta,p}(\Omega)} \leq Ct^{-\vartheta}\|f\|_{L^p(\Omega)}, \quad \vartheta \in [0,1],$$

Setting $\alpha := 3(1/p - 1/q)$ we have the embedding $H^{\alpha,p}(\Omega) \hookrightarrow L^q(\Omega)$. Assume now that $\alpha \leq 2$. Then the first inequality follows from

$$\begin{aligned}\|S(t)\mathbb{P}f\|_{L^q(\Omega)^2} &\leq C\|S(t)\mathbb{P}f\|_{H^{\alpha,p}(\Omega)} \\ &\leq Ct^{-\alpha/2}\|f\|_{L^p(\Omega)} \\ &= Ct^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{L^p(\Omega)}, \quad t > 0.\end{aligned}$$

In the case $\alpha > 2$ we have $p < 2 < q$ and so the estimate follows via

$$\|S(t)\mathbb{P}f\|_{L^q(\Omega)} \leq Ct^{-\frac{3}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}\|S(t/2)\mathbb{P}f\|_{L^2(\Omega)} \leq Ct^{-\frac{3}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}t^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)}\|f\|_{L^p(\Omega)}.$$

The remaining inequalities are obtained analogously using Corollary 4.1.5 and 4.1.6 by writing

$$\partial_i = \partial_i(-A_p)^{-1/2}(-A_p)^{1/2}, \quad \mathbb{P}\partial_i = (-A_p)^{1/2}(-A_p)^{-1/2}\mathbb{P}\partial_i$$

as well as $S(t)\mathbb{P} = \mathbb{P}S(t)\mathbb{P}$, compare the proof of Corollary 4.1.6 above. □

It remains to prove the elliptic regularity of the hydrostatic Stokes operator. For this purpose we also require the following lemma.

Lemma 4.1.11. *Let $p \in (1,\infty]$ and $s > 0$. Then the two-dimensional Helmholtz projection with periodic boundary conditions Q is bounded on $B^s_{p,p,per}(G)^2$.*

Proof. Recall that the operator $1 - Q$ is given by $f \mapsto \nabla_H \pi$ where

$$\Delta_H \pi = \mathrm{div}_H f, \quad \pi \text{ periodic on } \partial G.$$

We identify G with periodic boundary conditions with the two-dimensional torus $\mathbb{T}^2$. Then $1 - Q$ agrees with the Fourier multiplier with the discrete symbol

$$m(k) = k \otimes k|k|^{-2}, \quad k \in \mathbb{Z}^2 \setminus \{0\},$$

where we used the notation $x \otimes x := (x_i x_j)_{1\leq i,j\leq 2}$ for $x \in \mathbb{C}^2$. On the whole-space $\mathbb{R}^2$, the Fourier multiplier with the symbol

$$m(\xi) = \xi \otimes \xi|\xi|^{-2}, \quad \xi \in \mathbb{R}^2 \setminus \{0\},$$

is bounded on $B^s_{p,p}(\mathbb{R}^2)^2$ by the theory of Fourier multipliers on Besov spaces, see, e.g., [5, Theorem 6.2], including the case $p = \infty$ for $s > 0$. These arguments can then be adapted to the case of the torus, compare, e.g., [47, Proposition 4.5]. □

Proof of Lemma 4.1.8. As in the proof of Theorem 4.1.1 we have that

$$B_p : H^{s,p}_{\mathrm{per}}(\Omega)^2 \ni v \mapsto \frac{1}{h}(1-Q)\partial_z v\big|_{\Gamma_b} - \frac{1}{h}(1-Q)\partial_z v\big|_{\Gamma_u} \in H^{s-1-1/p-\varepsilon}_{\mathrm{per}}(\Omega)^2$$

defines a bounded linear operator for all $s > 1 + 1/p + \varepsilon$ and $\varepsilon > 0$. In detail, we have that the mappings

$$H^{s,p}_{\mathrm{per}}(\Omega)^2 \ni v \mapsto \partial_z v \in H^{s-1,p}_{\mathrm{per}}(\Omega)^2,$$
$$H^{s-1,p}_{\mathrm{per}}(\Omega)^2 \ni \partial_z v \mapsto \partial_z v\big|_{\Gamma_u}, \partial_z v\big|_{\Gamma_b} \in B^{s-1-1/p,p}_{p,p,\mathrm{per}}(G)^2$$

are bounded, whereas $1 - Q$ is bounded on $B^{s-1-1/p,p}_{p,p,\mathrm{per}}(G)^2$ by Lemma 4.1.11. The embedding

$$B^{s-1-1/p,p}_{p,p,\mathrm{per}}(G)^2 \hookrightarrow F^{s-1-1/p-\varepsilon,p}_{p,2,\mathrm{per}}(G)^2 = H^{s-1-1/p-\varepsilon,p}_{\mathrm{per}}(G)^2$$

then yields the boundedness. We set $v := (\mu - A_p)^{-1} f \in D(A_p)$. Then it holds that $(\mu - \Delta_p)v = f + B_p v$ and so by Lemma 3.4.4 we have for all $r > 0$ that $v \in H^{2+r,p}(\Omega)$ if $f + B_p v \in H^{r,p}(\Omega)$. Since $v \in H^{2,p}_{\mathrm{per}}(\Omega)^2$ we have $f + B_p v \in H^{r_0,p}_{\mathrm{per}}(\Omega)^2$ where

$$r_0 = \min\{s, \delta\}, \quad \delta = 1 - 1/p - \varepsilon, \quad \varepsilon \in (0, 1 - 1/p).$$

Iterating this argument yields $v \in H^{2+r_n,p}_{\mathrm{per}}(\Omega)^2$ and $f + B_p v \in H^{r_n,p}_{\mathrm{per}}(\Omega)^2$ for a recursive sequence $(r_n)_{n\in\mathbb{N}}$ given by

$$r_{n+1} := \min\{s, r_n + \delta\}.$$

Since this sequence either increases by $\delta > 0$ or terminates at $r_n = s$, we thus obtain $v \in H^{2+s,p}_{\mathrm{per}}(G)$ after finitely many steps.

Due to the compactness of the embedding $H^{2,p}(\Omega) \hookrightarrow L^p(\Omega)$, see [89, Section 4.3.2, Remark 1], we have that $D(A_p) \hookrightarrow L^p_{\overline{\sigma}}(\Omega)$ is compact as well and thus $(\lambda - A_p)^{-1}$ is a compact mapping for all $\lambda \in \rho(A_p)$, see [57, Chapter III, Theorem 6.29]. We further have $\rho(A_p) \neq \emptyset$ by (4.1.7). This implies that $\sigma(A_p)$ consists of only a discrete sequence of eigenvalues of finite multiplicity.

If $v \in D(A_p)$ is an eigenvector with eigenvalue λ, we then have $A_p v = \lambda v \in H^{2,p}_{\mathrm{per}}(\Omega)^2$ and thus $v \in H^{2n,p}(\Omega)^2$ for all $n \in \mathbb{N}$ by induction. Sobolev embedding theory then implies that $v \in C^\infty(\overline{\Omega})^2$.

Now observe that $\mathrm{div}_H \overline{v} = 0$ implies that $\int_\Omega \nabla_H \pi \cdot v^* \, d\mu = 0$ for all $\pi \in W^{1,p}_{\mathrm{per}}(G)$. Since $Av = \Delta v + Bv$ and the perturbation term is of the form $Bv = \nabla_H \pi$ for some such $\pi \in W^{1,p}_{\mathrm{per}}(G)$, it follows that

$$\int_\Omega Av \cdot v^* \, d\mu = \int_\Omega \Delta v \cdot v^* \, d\mu = -\int_\Omega |\nabla v|^2 \, d\mu \leq 0,$$

and thus $Av = \lambda v$ for $v \neq 0$ implies that $\lambda \leq 0$. If in addition it holds that $\Gamma_D \neq \emptyset$, then we further have $\lambda \neq 0$. This completes the proof. □

Remark 4.1.12. We deliberately chose to present a proof based on bootstrapping arguments to highlight the close ties between the hydrostatic Stokes operator and the Laplace operator, as well as the applicability of the theory of elliptic operators. A different proof of this result can be performed using the concept of Banach scales, see [4, Chapter V.1].

4.2 L^∞-L^p-theory for Neumann boundary conditions

In this section, we will deviate from 4.1 in several ways. On the one hand, we will consider the unbounded layer domain $\mathbb{L} = \mathbb{R}^2 \times (-h, 0)$ without boundary conditions in the horizontal variables. On the other hand, we will impose pure Neumann boundary conditions on $\partial\mathbb{L}$, meaning that here we will consider the hydrostatic Stokes equations (4.0.2) in the form

$$\left\{\begin{array}{rcll} \partial_t v - \Delta v + \nabla_H \pi &=& f & \text{in } \mathbb{L} \times (0,T), \\ \operatorname{div}_H \overline{v} &=& 0 & \text{in } \mathbb{L} \times (0,T), \\ \partial_z v &=& 0 & \text{on } \partial\mathbb{L} \times (0,T), \\ v(0) &=& a & \text{in } \mathbb{L}. \end{array}\right. \tag{4.2.1}$$

Finally, we will be moving from L^p-spaces for $p \in (1, \infty)$ to spaces with a norm resembling that of L^∞ and L^1. For this purpose we will be making use of the anisotropic L^p-spaces defined in Section 2.4.2. The results of this sections have been previously published in [40, Section 2-5].

Let $p \in [1, \infty]$ and consider the space $L^\infty_H L^p_z(\mathbb{L}) = L^\infty(\mathbb{R}^2; L^p(-h, 0))$ as defined in Section 2.4.2, as well as its closed subspace

$$L^{\infty,p}_{\overline{\sigma}}(\mathbb{L}) := \left\{ v \in L^\infty_H L^p_z(\mathbb{L})^2 : \int_{\mathbb{R}^2} \overline{v}\, \nabla_H \varphi \, dx = 0 \text{ for all } \varphi \in \widehat{W}^{1,1}(\mathbb{R}^2) \right\}. \tag{4.2.2}$$

Here $\widehat{W}^{1,1}(\mathbb{R}^2) = \{\varphi \in L^1_{loc}(\mathbb{R}^2) : \nabla_H \varphi \in L^1(\mathbb{R}^2)\}$ is a homogeneous Sobolev space, meaning that $L^{\infty,p}_{\overline{\sigma}}(\mathbb{L})$ is the space of all $v \in L^\infty_H L^p_z(\mathbb{L})^2$ such that $\operatorname{div}_H \overline{v} = 0$ in the sense of distributions.

In this setting, there is again a hydrostatic Helmholtz projection, again denoted by $\mathbb{P}$, given by the mapping

$$\mathbb{P} f := f - \nabla_H \pi, \quad \Delta_H \pi = \operatorname{div}_H \overline{f} \text{ on } \mathbb{R}^2,$$

compare (4.0.3). Observing that the solution operator of the weak problem above is related to the Riesz transform, we find that

$$\mathbb{P} f = f + (R \otimes R)\overline{f}, \quad R \otimes R := (R_i R_j)_{1 \le i,j \le 2}, \tag{4.2.3}$$

where $R_i = \partial_i (-\Delta_H)^{-1/2}$ for $i = 1, 2$ denotes the two-dimensional Riesz transforms in the horizontal variables. Note that since the Riesz transforms fail to be bounded on $L^p(\mathbb{R}^2)$ for $p = 1, \infty$, the projection $\mathbb{P}$ is likewise unbounded.

By applying $\mathbb{P}$ to the problem (4.2.1) one again obtains a Cauchy problem for v, compare (4.0.7). However, since Neumann boundary conditions are imposed on both the top and bottom part of the boundary in (4.2.1), the new problem is simply given by

$$\left\{\begin{array}{rcll} \partial_t v - \Delta v &=& \mathbb{P} f & \text{in } \mathbb{L} \times (0,T), \\ \operatorname{div}_H \overline{v} &=& 0 & \text{in } \mathbb{L} \times (0,T), \\ \partial_z v &=& 0 & \text{on } \partial\mathbb{L} \times (0,T), \\ v(0) &=& a & \text{in } \mathbb{L}, \end{array}\right. \tag{4.2.4}$$

since we have $Av = \mathbb{P}\Delta v = \Delta v$, compare (4.0.6). In particular, the heat semigroup generated by Δ with Neumann boundary conditions on $\mathbb{L}$ leaves the space $L^{\infty,p}_{\overline{\sigma}}(\mathbb{L})$ invariant and the hydrostatic Stokes semigroup is simply given by the restriction of the heat semigroup. Given $f = 0$, the solution to (4.2.4) is given via

$$v(t) = (S_H(t) \otimes S_N(t))\, a = S_H(t)S_N(t)a, \quad t \geq 0,$$

where S_H denotes the heat semigroup on $\mathbb{R}^2$ given by the convolution with the two-dimensional Gaussian kernel G_t and S_N is the vertical heat semigroup on $(-h, 0)$ from Lemma 3.3.2. The tensor notation means that the operators are applied successively, commute, and preserve product structures, i.e., if we have

$$a(x, y, z) = (a_H \otimes a_z)(x, y, z) := a_H(x, y)a_z(z) \tag{4.2.5}$$

for functions $a_h \colon \mathbb{R}^2 \to \mathbb{C}^2$, $a_z \colon (-h, 0) \to \mathbb{C}$, then

$$(S_H(t) \otimes S_N(t))\, a = S_H(t)a_H \otimes S_N(t)a_z.$$

We will not be distinguishing between the heat semigroup on $L^\infty_H L^p_z(\mathbb{L})$ and the hydrostatic Stokes semigroup on $L^{\infty,p}_{\overline{\sigma}}(\mathbb{L})$ and simply denote both via

$$S(t) := S_H(t) \otimes S_N(t) := S_H(t)S_N(t), \quad t \geq 0. \tag{4.2.6}$$

Since these semigroups operate in different variables, the tensor product is simply the composition of these operators, applied first in the vertical variable and then in the horizontal one, yielding the representation

$$(S(t)f)(x, y, z) = \int_{\mathbb{R}^2} G_t(x - x', y - y')(S_N(t)f)(x', y', z)\, d(x', y') \tag{4.2.7}$$

for all $(x, y, z) \in \mathbb{L}$. Since S_H and S_N are contraction semigroups on $L^\infty(\mathbb{R}^2)$ and $L^p(-h, 0)$, respectively, see Lemma 3.3.2, it follows that S is a contraction semigroup $L^\infty_H L^p_z(\mathbb{L})$. However, since the two-dimensional heat semigroup fails to be strongly continuous on $L^\infty(\mathbb{R}^2)$, S is not strongly continuous on $L^{\infty,p}_{\overline{\sigma}}(\mathbb{L})$. However, it is strongly continuous, and even bounded analytic, on the space $BUC(\mathbb{R}^2; L^p(-h, 0))^2$, as well as its invariant subspace

$$X^{\infty,p}_{\overline{\sigma}}(\mathbb{L}) := BUC(\mathbb{R}^2; L^p(-h, 0))^2 \cap L^{\infty,p}_{\overline{\sigma}}(\mathbb{L}). \quad p \in [1, \infty). \tag{4.2.8}$$

The results of this sections are similar to Theorem 4.1.7 for the case $p = q$. Here we will consider more general estimates, namely ones involving fractional derivatives. Due to the tensor structure of the semigroup, we distinguish between those in horizontal and vertical direction, using fractional powers of the horizontal Laplace operator $-\Delta_H$, compare (3.1.1), as well as Caputo derivatives defined in (3.3.7).

Due to the fact that the hydrostatic Stokes semigroup is merely the restriction of the heat semigroup and allows for the representation (4.2.7), the proof of the following theorem only requires estimates for the heat-semigroups S_H and S_N. Using the shorthand notation $\|\cdot\|_{\infty,p} := \|\cdot\|_{L^\infty_H L^p_z(\mathbb{L})}$, we have the following.

Theorem 4.2.1. *Let $p \in [1,\infty]$ and $\alpha \in (0,1)$. Then the following holds.*

(a) The hydrostatic Stokes semigroup S is a contraction semigroup on $L^\infty_H L^p_z(\mathbb{L})^2$ with invariant subspaces $L^{\infty,p}_{\overline{\sigma}}(\mathbb{L})$ and $X^{\infty,p}_{\overline{\sigma}}(\mathbb{L})$. If $p \in [1,\infty)$, it is strongly continuous on $BUC(\mathbb{R}^2; L^p(-h,0))^2$.

(b) The operator $S(t)$ maps $L^\infty_H L^1_z(\mathbb{L})^2$ into $L^\infty_H L^p_z(\mathbb{L})^2$ for all $t > 0$ and there exists a constant $C > 0$ such that for all $f \in L^\infty_H L^1_z(\mathbb{L})^2$ and $t > 0$ it holds that

$$\|S(t)f\|_{\infty,p} \leq C\left(1 + t^{-(1-1/p)/2}\right)\|f\|_{\infty,1}$$

for all $t > 0$.

(c) There exists a constant $C = C_\alpha > 0$, such that for all $f \in L^\infty_H L^p_z(\mathbb{L})^2$ and $t > 0$ it holds that

$$\|\nabla S(t)f\|_{\infty,p} \leq Ct^{-1/2}\|f\|_{\infty,p}, \tag{i}$$
$$\|S(t)\nabla_H \cdot f\|_{\infty,p} \leq Ct^{-1/2}\|f\|_{\infty,p}, \tag{ii}$$
$$\|S(t)\partial_z f\|_{\infty,p} \leq Ct^{-1/2}\|f\|_{\infty,p}, \tag{iii}$$
$$\left\|S(t)\mathbb{P}(-\Delta_H)^{\alpha/2} f\right\|_{\infty,p} \leq Ct^{-\alpha/2}\|f\|_{\infty,p}, \tag{iv}$$
$$\|S(t)\mathbb{P}\nabla_H \cdot f\|_{\infty,p} \leq Ct^{-1/2}\|f\|_{\infty,p}, \tag{v}$$

as well as

$$\|S(t)\partial_z I^\alpha f\|_{\infty,p} \leq Ct^{-(1-\alpha)/2}\|f\|_{\infty,p} \tag{vi}$$

whenever $(I^\alpha f)(0) = 0$.

(d) If $p \in [1,\infty)$, then for any $f \in BUC(\mathbb{R}^2; L^p(-h,0))^2$ it holds that

$$\lim_{t\to 0+} t^{1/2}\|\nabla S(t)f\|_{\infty,p} = 0.$$

Proof. In the following we use the notation $\|\cdot\|_p := \|\cdot\|_{L^p(-h,0)}$ for simplicity. For (a), the contraction property follows from the fact that S_N is a contraction semigroup on $L^p(-h,0)$ by Lemma 3.3.2 and the fact that $G_t > 0$ and $\|G_t\|_{L^1(\mathbb{R}^2)} = 1$ for all $t > 0$, yielding

$$\begin{aligned}
\|(S(t)f)(x,y,\cdot)\|_{L^p(-h,0)} &= \|(S_H(t)S_N(t)f)(x',y',\cdot)\|_p \\
&= \left\| \int_{\mathbb{R}^2} G_t(x-x',y-y')(S_N(t)f)(x',y',\cdot)\,d(x',y') \right\|_p \\
&\leq \int_{\mathbb{R}^2} G_t(x-x',y-y')\,\|(S_N(t)f)(x',y',\cdot)\|_p\,d(x',y') \\
&\leq \int_{\mathbb{R}^2} G_t(x-x',y-y')\,\|f(x',y',\cdot)\|_p\,d(x',y') \\
&\leq \|G_t\|_{L^1(\mathbb{R}^2)}\|f\|_{\infty,p} \\
&= \|f\|_{\infty,p}
\end{aligned}$$

for all $(x, y) \in \mathbb{R}^2$ and thus $\|S(t)f\|_{\infty,p} \le \|f\|_{\infty,p}$ for all $t > 0$. Here we also used the Minkowski inequality in the third step.

The strong continuity for $p \in [1, \infty)$ follows from the fact that S_H and S_N are strongly continuous on $BUC(\mathbb{R}^2)^2$ and $L^p(-h, 0)$, respectively, and thus S is strongly continuous on the set

$$\{f \otimes g : f \in BUC(\mathbb{R}^2)^2, g \in L^p(-h, 0)\},$$

the linear hull of which is dense in $BUC(\mathbb{R}^2; L^p(-h, 0))^2$. Here we used the notation from (4.2.5). The space $BUC(\mathbb{R}^2; L^p(-h, 0))^2$ is invariant since S_H preserves continuity and $L^{\infty,p}_{\overline{\sigma}}(\mathbb{L})$ is invariant since Lemma 3.3.2.(vi) implies that $\overline{S(t)f} = S_H\overline{f}$ and S_H commutes with div_H. Thus, $X^{\infty,p}_{\overline{\sigma}}(\mathbb{L})$ is invariant as well. For (b), we similarly have

$$S(t)\overline{f} = S_H(t)S_N(t)\overline{f} = S_H(t)I^1 f,$$

where $I^1 f := \int_{-h}^0 f(\cdot, z)\, dz$ denotes the vertical integral. Since S_H is contractive on $L^\infty(\mathbb{R}^2)$ it follows that

$$\|S(t)I^1 f\|_{\infty,p} \le \|I^1 f\|_{L^\infty(\mathbb{R}^2)} \le \|f\|_{\infty,1}.$$

We now set $g := f - I^1 f$ and recall from the proof of (a) that

$$\begin{aligned}
\|(S(t)g)(x, y, \cdot)\|_{L^p(-h,0)} &\le \int_{\mathbb{R}^2} G_t(x - x', y - y')\, \|(S_N(t)g)(x', y', \cdot)\|_p \, d(x', y') \\
&\le Ct^{-(1-1/p)/2} \int_{\mathbb{R}^2} G_t(x - x', y - y')\, \|g(x', y', \cdot)\|_1 \, d(x', y') \\
&\le Ct^{-(1-1/p)/2} \|g\|_{\infty,1},
\end{aligned}$$

where we used the L^1-L^p-smoothing of S_N from Lemma 3.3.2.(v) in the second step. The estimate then follows from $\|g\|_{\infty,1} \le C\|f\|_{\infty,1}$. Estimate (i) in (c) follows from $\|G_t\|_1 = 1$ for all $t > 0$ and

$$\begin{aligned}
\|(\partial_z S(t)f)(x, y, \cdot)\|_p &= \|S_H(t)\partial_z S_N(t)f(x, y, \cdot)\|_p \\
&= \left\| \int_{\mathbb{R}^2} G_t(x - x', y - y')(\partial_z S_N(t)f)(x', y', \cdot)\, d(x', y') \right\|_p \\
&\le \int_{\mathbb{R}^2} G_t(x - x', y - y')\|(\partial_z S_N(t)f)(x', y', \cdot)\|_p \, d(x', y') \\
&\le Ct^{-1/2} \int_{\mathbb{R}^2} G_t(x - x', y - y')\|f(x', y', \cdot)\|_p \, d(x', y'), \\
&\le Ct^{-1/2}\|f\|_{\infty,p},
\end{aligned}$$

where we again used the Minkowski inequality in the third and Lemma 3.3.2.(ii) in the fourth step, as well as

$$\begin{aligned}
\|(\partial_i S(t)f)(x, y, \cdot)\|_p &= \|\partial_i S_H(t)S_N(t)f(x, y, \cdot)\|_p \\
&\le Ct^{-1/2} \int_{\mathbb{R}^2} G_{2t}(x - x', y - y')\|f(x', y', \cdot)\|_p \, d(x', y')
\end{aligned}$$

for $\partial_i \in \{\partial_x, \partial_y\}$ via Lemma 3.1.3 and the contraction property for the vertical semigroup. Estimate (ii) follows from (i) via $S(t)\partial_i f = \partial_i S(t)f$, whereas estimates (iii) and (vi) follow from Lemma 3.3.7 via

$$\begin{aligned}\|(S(t)\partial_z I^\alpha f)(x,y,\cdot)\|_p &\le \int_{\mathbb{R}^2} G_t(x-x',y-y')\|(S_N(t)\partial_z I^\alpha f)(x',y',\cdot)\|_p \, d(x',y')\\ &\le Ct^{-(1-\alpha)/2}\int_{\mathbb{R}^2} G_t(x-x',y-y')\|f(x',y',\cdot)\|_p \, d(x',y')\\ &\le Ct^{-(1-\alpha)/2}\|f\|_{\infty,p}.\end{aligned}$$

In order to obtain estimate (iv) we use (4.2.3) to write

$$S(t)\mathbb{P}(-\Delta_H)^{\alpha/2} f = S(t)(-\Delta_H)^{\alpha/2} f + S(t)(R\otimes R)(-\Delta_H)^{\alpha/2}\overline{f}$$

and by the contraction property of the vertical semigroup and estimate (1) and (2) from Lemma 3.1.3 we have using $S(t) = S_H(t)S_N(t) = S_N(t)S_H(t)$ that

$$\begin{aligned}\|S(t)(-\Delta_H)^{\alpha/2} f(x,y,\cdot)\|_p &\le \|S_H(t)(-\Delta_H)^{\alpha/2} f(x,y,\cdot)\|_p\\ &\le t^{-\alpha/2}\left\|\int_{\mathbb{R}^2} H_t^\alpha(x-x',y-y')|f(x',y',\cdot)| \, d(x',y')\right\|_p\\ &\le t^{-\alpha/2}\int_{\mathbb{R}^2} H_t^\alpha(x-x',y-y')\left\|f(x',y',\cdot)\right\|_p \, d(x',y')\\ &\le C_\alpha t^{-\alpha/2}\|f\|_{\infty,p},\end{aligned}$$

as well as

$$\begin{aligned}\|S(t)(R\otimes R)(-\Delta_H)^{\alpha/2}\overline{f}(x,y)\|_p &\le \|S_H(t)(R\otimes R)(-\Delta_H)^{\alpha/2}\overline{f}(x,y)\|_p\\ &= h^{1/p}|S_H(t)(R\otimes R)(-\Delta_H)^{\alpha/2}\overline{f}(x,y)|\\ &\le h^{1/p}t^{-\alpha/2}(\tilde{H}_t^\alpha *_H \overline{f})(x,y).\end{aligned}$$

The estimate then follows from $\|H_t^\alpha\|_1 + \|\tilde{H}_t^\alpha\|_1 \le C_\alpha$ for all $t > 0$. Estimate (v) is obtained analogously via estimates (ii) and (3) from Lemma 3.1.3. Finally, for (d), we take $\varepsilon > 0$ and use the fact that $C_c^\infty(-h,0)$ is dense in $L^p(-h,0)$ and that functions belonging to $BUC(\mathbb{R}^2)$ can be uniformly approximated by smooth functions. This allows us to approximate $f \in BUC(\mathbb{R}^2; L^p(-h,0))^2$ by

$$g \in C^\infty(\mathbb{R}^2; C_c^\infty(-h,0))^2, \quad \|f-g\|_{\infty,p} \le \frac{\varepsilon}{2C}, \quad \|\nabla g\|_{\infty,p} < \infty,$$

where $C > 0$ is as in estimate (i). This yields

$$t^{1/2}\|\nabla S(t)f\|_{\infty,p} \le \frac{\varepsilon}{2} + t^{1/2}\|\nabla S(t)g\|_{\infty,p}.$$

Since ∇_H commutes with the two-dimensional heat semigroup S_H as well as the vertical semigroup S_N, we have $\nabla_H S(t)g = S(t)\nabla_H g$ and due to $\partial_z S_N(t) = S_D(t)\partial_z$ for S_D as in Lemma 3.3.2 we further have $\partial_z S(t)g = S_H(t)S_D(t)\partial_z g$. This yields

$$\|\nabla S(t)g\|_{\infty,p} \le \|\nabla g\|_{\infty,p},$$

which implies the desired result. □

4.3 L^∞-L^p-theory for Dirichlet boundary conditions

In this section we will again turn to the setting considered in Sections 3.4, 3.5, and 4.1, i.e., that of a cylindrical domain $\Omega = G \times (-h, 0)$ with $G = (0,1)^2$ and $h > 0$ and boundary conditions

$$\begin{aligned} v, \pi \text{ periodic} \quad &\text{on } \Gamma_l \times (0,\infty), \\ v = 0 \quad &\text{on } \Gamma_D \times (0,\infty), \\ \partial_z v = 0 \quad &\text{on } \Gamma_N \times (0,\infty), \end{aligned}$$

with $\Gamma_D \neq \emptyset$. Here we choose the case considered in [49], i.e.,

$$\partial_z v = 0 \text{ on } \Gamma_u, \quad v \text{ periodic on } \Gamma_l, \quad v = 0 \text{ on } \Gamma_b. \tag{4.3.1}$$

Whereas in Section 4.1 we discussed a more general case, the proofs presented in this section are tailored specifically to this choice of boundary conditions and while the arguments involved can be adapted to cover the cases $\Gamma_D = \Gamma_u$ or $\Gamma_D = \Gamma_u \cup \Gamma_b$, we chose to omit these details for the sake of brevity. The results of this section have been previously published in [39, Section 6 and 7].

As a result of imposing homogeneous Dirichlet boundary conditions, it follows from (4.0.6) that one generally has

$$Av = \mathbb{P}\Delta v = \Delta v + Bv \neq \Delta v, \quad \mathrm{div}_H \overline{v} = 0.$$

This has the consequence that the arguments of this sections are closer to those of Section 4.1 and [49] than to those of Section 4.2.

The primary focus of this section is the investigation of resolvent estimates similar to the ones stated in Corollary 4.1.6 in spaces equipped with the norm of $L^\infty_H L^p_z(\Omega)$ for the range $3 < p < \infty$ which then translate into semigroup estimates analogous to the ones stated in Theorem 4.2.1. Recall the operator $\mathcal{A}_p$ defined in (4.1.5) for $p \in (1,\infty)$ as

$$\mathcal{A}_p v := \Delta v + Bv, \quad D(\mathcal{A}_p) := \big\{ v \in H^{2,p}_{per}(\Omega)^2 : \partial_z v\big|_{\Gamma_u} = 0, v\big|_{\Gamma_b} = 0 \big\}.$$

In order to deal with $L^\infty_H L^p_z$-type spaces, we make the following observation.

Lemma 4.3.1. *Let $p \in (1,\infty)$ and $\lambda \in \rho(\mathcal{A}_p)$. Then $L^\infty_H L^p_z(\Omega)^2$ and $L^\infty_H L^p_z(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega)$ are invariant under the resolvent mapping $(\lambda - \mathcal{A}_p)^{-1}$.*

Proof. Given $f \in L^\infty_H L^p_z(\Omega)^2 \hookrightarrow L^p(\Omega)^2$ and $\lambda \in \rho(\mathcal{A}_p)$ we have

$$(\lambda - \mathcal{A}_p)^{-1} f \in D(\mathcal{A}_p) \hookrightarrow H^{2,p}_{\mathrm{per}}(\Omega)^2 \hookrightarrow H^{2,p}_{\mathrm{per}}(G; L^p(-h,0))^2 \hookrightarrow L^\infty_H L^p_z(\Omega)^2, \quad p \in (1,\infty),$$

where we used the Sobolev embedding $H^{2,p}(G) \hookrightarrow L^\infty(G)$ in the second step. Thus $L^\infty_H L^p_z(\Omega)^2$ is an invariant subspace of $(\lambda - \mathcal{A}_p)^{-1}$. Further recall from the proof of Theorem 4.1.1 that $L^p_{\overline{\sigma}}(\Omega)$ is also an invariant subspace of $(\lambda - \mathcal{A}_p)^{-1}$. Thus, the claim follows. □

This allows us to define the $L^\infty_H L^p_z(\Omega)^2$-realization of A via

$$\mathcal{A}_{\infty,p} v := \Delta v + Bv, \quad D(\mathcal{A}_{\infty,p}) := \{v \in D(\mathcal{A}_p) : Av \in L^\infty_H L^p_z(\Omega)^2\}. \tag{4.3.2}$$

Using the notation

$$L^{\infty,p}_{\overline{\sigma}}(\Omega) := L^\infty_H L^p_z(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega) \tag{4.3.3}$$

we define the $L^{\infty,p}_{\overline{\sigma}}(\Omega)$-realization of A via

$$A_{\infty,p} v := \Delta v + Bv, \quad D(A_{\infty,p}) := D(\mathcal{A}_{\infty,p}) \cap L^p_{\overline{\sigma}}(\Omega). \tag{4.3.4}$$

Since $p \in (3, \infty)$ yields that

$$D(\mathcal{A}_p) \hookrightarrow H^{2,p}(\Omega) \hookrightarrow C^{1,1-3/p}(\overline{\Omega}),$$

see, e.g., [28, Chapter 5.6, Theorem 4 and 5], the operators $\mathcal{A}_{\infty,p}$ and $A_{\infty,p}$ are not densely defined. However, we will show that the $L^\infty_H L^p_z(\Omega)^2$-closures of their domains are given by the spaces $C_{\mathrm{per}}([0,1]^2; L^p(-h,0))^2$ and

$$X^{\infty,p}_{\overline{\sigma}}(\Omega) := C_{\mathrm{per}}([0,1]^2; L^p(-h,0))^2 \cap L^p_{\overline{\sigma}}(\Omega), \tag{4.3.5}$$

respectively. Observe that, due to the smoothing properties of the resolvent of and semigroup generated by $\mathcal{A}_p$, these are invariant subspaces of these mapping and thus the same holds true for $\mathcal{A}_{\infty,p}$ and $A_{\infty,p}$.

The main result of this section is the following.

Theorem 4.3.2. *Let $p \in (3, \infty)$.*

(a) The operator $\mathcal{A}_{\infty,p}$ generates an analytic semigroup S that is strongly continuous on $C_{per}([0,1]^2; L^p(-h,0))^2$. In particular, there exist constants $C > 0$, $\beta \in \mathbb{R}$ such that

$$\|S(t)f\|_{L^\infty_H L^p_z(\Omega)} \le Ce^{\beta t}\|f\|_{L^\infty_H L^p_z(\Omega)} \tag{i}$$

holds for all $t > 0$ and $f \in L^\infty_H L^p_z(\Omega)^2$.

(b) For $\partial_i, \partial_j \in \{\partial_x, \partial_y, \partial_z\}$ it further holds that

$$t^{1/2}\|\partial_i S(t)f\|_{L^\infty_H L^p_z(\Omega)} \le Ce^{\beta t}\|f\|_{L^\infty_H L^p_z(\Omega)}, \tag{ii}$$
$$t^{1/2}\|\partial_j S(t)\mathbb{P}f\|_{L^\infty_H L^p_z(\Omega)} \le Ce^{\beta t}\|f\|_{L^\infty_H L^p_z(\Omega)}, \tag{iii}$$
$$t^{1/2}\|S(t)\mathbb{P}\partial_j f\|_{L^\infty_H L^p_z(\Omega)} \le Ce^{\beta t}\|f\|_{L^\infty_H L^p_z(\Omega)}, \tag{iv}$$
$$t\|\partial_i S(t)\mathbb{P}\partial_j f\|_{L^\infty_H L^p_z(\Omega)} \le Ce^{\beta t}\|f\|_{L^\infty_H L^p_z(\Omega)} \tag{v}$$

for all $t > 0$ and $f \in L^\infty_H L^p_z(\Omega)^2$.

(c) The restriction of S to $L^{\infty,p}_{\overline{\sigma}}(\Omega)$ is an exponentially stable, bounded analytic semigroup of angle $\pi/2$ generated by $A_{\infty,p}$ and strongly continuous on $X^{\infty,p}_{\overline{\sigma}}(\Omega)$.

(d) For all $v \in C_{per}([0,1]^2; L^p(-h,0))^2$ we have

$$\lim_{t\to 0+} t^{1/2} \|\nabla S(t)v\|_{L^\infty_H L^p_z(\Omega)} = 0.$$

Remark 4.3.3. The condition $3 < p < \infty$ arises due to a number of factors, compare Remark 4.3.15. Whereas in the proof of Theorem 4.1.1 we estimated the perturbation term $Bv = \frac{1}{h}(1-Q)\partial_z v\big|_{\Gamma_b}$ in $L^p(\Omega)^2$ by using the fact that the Helmholtz projection Q with periodic boundary conditions defines a bounded mapping on $L^2(G)^2$ for $p \in (1,\infty)$, we make up for the fact that Q fails to be bounded on $L^\infty(G)^2$ by using the Sobolev embedding

$$H^{2,p}(\Omega) \hookrightarrow C^{1,1-3/p}(\overline{\Omega})$$

for $p > 3$ and the fact that Q is bounded on spaces of Hölder continuous functions $C^{0,\alpha}_{\text{per}}([0,1]^2)^2 = C^{0,\alpha}(\mathbb{T}^2)^2 = B^{\alpha}_{\infty,\infty}(\mathbb{T}^2)^2$ for $\alpha > 0$ by Lemma 4.1.8. The restriction $p < \infty$ arises out of constraints for local L^∞-L^p estimates for Q, compare Remark 4.3.13.

We will divide Theorem 4.3.2 into several smaller pieces that we individually prove over the course of this section, see Lemma 4.3.4 and its corollaries, as well as Lemma 4.3.9, 4.3.14, and 4.3.20. Rather than proving the semigroup estimates directly, we will establish a number of resolvent estimates for $\mathcal{A}_{\infty,p}$ and $A_{\infty,p}$. These then imply the semigroup estimates using the Dunford functional calculus, see Proposition 2.5.6.

This section is structured as follows. First, we provide estimates for the terms

$$(\lambda - \mathcal{A}_{\infty,p})^{-1} \quad \text{and} \quad \nabla(\lambda - \mathcal{A}_{\infty,p})^{-1}$$

on $L^\infty_H L^p_z(\Omega)^2$ by again treating the term B as a perturbation of lower order in $L^\infty_H L^p_z(\Omega)^2$ and utilizing the $L^\infty_H L^p_z(\Omega)$-estimates for the resolvent of the Laplace operator established in Section 3.5. These results are sufficient to establish the claims of Theorem 4.3.2 that do not involve the hydrostatic Helmholtz projection $\mathbb{P}$.

In the second step, we turn to the term $\nabla_H(\lambda - A_{\infty,p})^{-1}\mathbb{P}$ and prove an analogous estimate using the L^p-theory for the heat semigroups on the whole space $\mathbb{R}^2$ and vertical interval $(-h,0)$ established in Sections 3.1 and 3.3.

Third and fourth, we respectively consider the terms

$$\partial_z(\lambda - A_{\infty,p})^{-1}\mathbb{P} \quad \text{and} \quad (\lambda - A_{\infty,p})^{-1}\mathbb{P}\partial_z.$$

This is the most technically involved part of this section. Our primary tools are scaling arguments as in the Masuda-Stewart method we previously utilized in the proof of Lemma 3.5.1. However, the problems considered in these steps are notably more complicated.

4.3.1 First resolvent estimates in L^∞-L^p-spaces

With the results of Chapter 3 and the L^p-theory established in [49] and Section 4.1, we are already in a position to prove a number of mapping properties for the hydrostatic Stokes operator on $L^\infty_H L^p_z$-type spaces.

Lemma 4.3.4. *Let $p \in (3,\infty)$ and $\theta \in (0,\pi)$. Then there exists $\mu = \mu_\theta > 0$ such that*

$$\Sigma_\theta \cup \{0\} \subset \rho(\mathcal{A}_{\infty,p} - \mu)$$

and a constant $C = C_{\Omega,\theta,p} > 0$ such that for all $\lambda \in \Sigma_\theta$ and $f \in L^\infty_H L^p_z(\Omega)^2$ the unique solution to the problem $(\lambda+\mu)v - \mathcal{A}_{\infty,p} v = f$ satisfies

$$|\lambda| \cdot \|v\|_{L^\infty_H L^p_z(\Omega)} + |\lambda|^{1/2} \|\nabla v\|_{L^\infty_H L^p_z(\Omega)} + \|\mathcal{A}_{\infty,p} v\|_{L^\infty_H L^p_z(\Omega)} \le C \|f\|_{L^\infty_H L^p_z(\Omega)}.$$

The operator $A_{\infty,p}$ admits the same estimates for $\mu = 0$ and all $f \in L^{\infty,p}_{\overline{\sigma}}(\Omega)$.

Proof. Let $\phi = \pi - \theta$ and recall that in the proof of Theorem 4.1.1 we established the auxiliary result that there exists $\mu = \mu_\phi > 0$ such that $-\mathcal{A}_p + \mu$ admits a bounded $\mathcal{H}^\infty$-calculus on $L^p(\Omega)^2$ for $p \in (1,\infty)$ of angle $\phi^\infty_{-\mathcal{A}_p+\mu} \le \phi$ by (4.1.6). In particular, we have that $-\mathcal{A}_p + \mu$ is sectorial of angle $\phi_{-\mathcal{A}_p+\mu} \le \phi^\infty_{-\mathcal{A}_p+\mu} \le \phi$. In particular, we have that

$$v := (\lambda + \mu - \mathcal{A}_p)^{-1} f, \quad \lambda \in \Sigma_\theta, \quad f \in L^p(\Omega)^2$$

exists and satisfies the estimate

$$|\lambda| \cdot \|v\|_{L^p(\Omega)} + \|(\mathcal{A}_p - \mu)v\|_{L^p(\Omega)} \le C_{\Omega,\theta,p} \|f\|_{L^p(\Omega)}, \tag{4.3.6}$$

for some constant $C_{\theta,p} > 0$. We may further take $\mu > 0$ to be sufficiently large to obtain $0 \in \rho(\mathcal{A}_p - \mu)$. Given $f \in L^\infty_H L^p_z(\Omega)^2 \hookrightarrow L^p(\Omega)^2$ we have

$$v \in D(\mathcal{A}_p) \hookrightarrow H^{2,p}(\Omega) \hookrightarrow L^\infty(\Omega)^2 \hookrightarrow L^\infty_H L^p_z(\Omega)^2,$$

where we used $p \in (3/2,\infty)$ for the Sobolev embedding in the second step, and thus we also have $\mathcal{A}_p v = (\lambda+\mu)v - f \in L^\infty_H L^p_z(\Omega)^2$ which means that $(\lambda + \mu - \mathcal{A}_p)^{-1}$ maps $L^\infty_H L^p_z(\Omega)^2$ into $D(\mathcal{A}_{\infty,p})$. By $D(\mathcal{A}_{\infty,p}) \subset D(\mathcal{A}_p)$ we also have that $\lambda + \mu - \mathcal{A}_{\infty,p}$ is injective whenever $\lambda \in \rho(\mathcal{A}_p - \mu)$. This yields

$$\Sigma_\theta \subset \rho(\mathcal{A}_p - \mu) \subset \rho(\mathcal{A}_{\infty,p} - \mu).$$

In order to prove the resolvent estimates, we make use of the equivalence

$$(\lambda+\mu)v - \mathcal{A}_p v = f \iff \lambda v - \Delta_p v = f + B_p v - \mu v. \tag{4.3.7}$$

We may estimate the right-hand side terms in $L^\infty_H L^p_z(\Omega)$ via

$$\begin{aligned} \|Bv\|_{L^\infty_H L^p_z(\Omega)} &= h^{1/p} \|Bv\|_{L^\infty(G)} \\ &\le C_{\Omega,p} \|Bv\|_{C^{0,\alpha}([0,1]^2)} \\ &\le C_{\Omega,p} \|\partial_z v\big|_{\Gamma_b}\|_{C^{0,\alpha}([0,1]^2)} \\ &\le C_{\Omega,p} \|v\|_{C^{1,\alpha}(\overline{\Omega})}, \end{aligned}$$

where in the third step we used the fact that Q is continuous on $C^{0,\alpha}_{per}([0,1]^2)$ for $\alpha \in (0,1)$, compare Remark 4.3.3. The condition $p \in (3,\infty)$ then yields the embedding

$$H^{2,p}(\Omega) \hookrightarrow C^{1,1-3/p}(\overline{\Omega}),$$

and by estimate (4.3.6) and the fact that $\mathcal{A}_p - \mu$ has a bounded inverse we obtain

$$\|Bv\|_{L^\infty_H L^p_z(\Omega)} \le C_{\Omega,p}\|v\|_{H^{2,p}(\Omega)} \le C_{\Omega,p}\|(\mathcal{A}_p-\mu)v\|_{L^p(\Omega)} \le C_{\Omega,\theta,p}\|f\|_{L^p(\Omega)} \le C_{\Omega,\theta,p}\|f\|_{L^\infty_H L^p_z(\Omega)}$$

as well as

$$\|\mu v\|_{L^\infty_H L^p_z(\Omega)} \le \mu C_p \|v\|_{C^{1,1-3/p}(\overline{\Omega})} \le C_{\theta,p}\|f\|_{L^\infty_H L^p_z(\Omega)}.$$

Applying Lemma 3.5.1 to the second problem in (4.3.7) then yields the estimate

$$\begin{aligned}|\lambda|\cdot\|v\|_{L^\infty_H L^p_z(\Omega)} + |\lambda|^{1/2}\|\nabla v\|_{L^\infty_H L^p_z(\Omega)} + \|\Delta v\|_{L^\infty_H L^p_z(\Omega)} &\le C_\theta\|f + Bv - \mu v\|_{L^\infty_H L^p_z(\Omega)}\\ &\le C_{\theta,p}\|f\|_{L^\infty_H L^p_z(\Omega)},\end{aligned}$$

which together with

$$\|\mathcal{A}_{\infty,p}v\|_{L^\infty_H L^p_z(\Omega)} \le \|\Delta v\|_{L^\infty_H L^p_z(\Omega)} + \|Bv\|_{L^\infty_H L^p_z(\Omega)} \le C_{\theta,p}\|f\|_{L^\infty_H L^p_z(\Omega)}$$

yields the desired estimate. The estimate for $A_{\infty,p}$ for $\mu = 0$ is obtained analogously since we have $0 \in \rho(A_p)$ as well as $-A_p \in \mathcal{H}^\infty(L^p_{\overline{\sigma}}(\Omega))$ with $\phi^\infty_{-A_p} = 0$. □

From this we now obtain our first result about the hydrostatic Stokes semigroup in $L^\infty_H L^p_z(\Omega)^2$.

Corollary 4.3.5. *Let $p \in (3,\infty)$. Then the following holds.*

(i) The operator $\mathcal{A}_{\infty,p}$ generates an analytic semigroup S on $L^\infty_H L^p_z(\Omega)^2$ that is strongly continuous on $C_{per}([0,1]^2; L^p(-h,0))^2$ and there exist constants $C = C_{\Omega,p} > 0$ and $\beta \in \mathbb{R}$ such that

$$t^{1/2}\|\nabla S(t)f\|_{L^\infty_H L^p_z(\Omega)} \le Ce^{\beta t}\|f\|_{L^\infty_H L^p_z(\Omega)} \tag{4.3.8}$$

for all $t > 0$ and $f \in L^\infty_H L^p_z(\Omega)^2$.

(ii) The restriction of S onto $L^{\infty,p}_{\overline{\sigma}}(\Omega)$ defines an exponentially stable, bounded analytic semigroup of angle $\pi/2$ that is strongly continuous on $X^{\infty,p}_{\overline{\sigma}}(\Omega)$ and generated by $A_{\infty,p}$. In particular, the estimate (4.3.8) even holds for $\beta < 0$ whenever $f \in L^{\infty,p}_{\overline{\sigma}}(\Omega)$.

Proof. The resolvent estimates established in Lemma 4.3.4 imply that $\mu - \mathcal{A}_{\infty,p}$ is sectorial. Thus $\mathcal{A}_{\infty,p} - \mu$ generates a bounded analytic semigroup and $\mathcal{A}_{\infty,p}$ generates the semigroup

$$S(z) := e^{z\mu}e^{z(\mathcal{A}_{\infty,p}-\mu)}, \quad z \in \Sigma_\theta.$$

The semigroup estimate follows via the Dunford calculus. Since $A_{\infty,p}$ is the restriction of $\mathcal{A}_{\infty,p}$ onto $L^{\infty,p}_{\overline{\sigma}}(\Omega)$ and the latter is an invariant subspace under the resolvent, the restriction of S onto $L^{\infty,p}_{\overline{\sigma}}(\Omega)$ is generated by $A_{\infty,p}$. Since $-A_p$ is sectorial of angle 0 with $0 \in \rho(A_p) \subset \rho(A_{\infty,p})$, the semigroup generated by $A_{\infty,p}$ can be extended onto the sector $\Sigma_{\pi/2}$ and since $\rho(A_{\infty,p})$ is an open subset of $\mathbb{C}$, it follows that its spectral bound

$$\beta := \sup\{\mathrm{Re}(\lambda) : \lambda \in \sigma(A_{\infty,p})\}$$

is negative and thus S is exponentially stable on $L^{\infty,p}_{\overline{\sigma}}(\Omega)$.

It remains to show that $D(\mathcal{A}_{\infty,p})$ and $D(A_{\infty,p})$ are dense in $C_{\mathrm{per}}([0,1]^2; L^p(-h,0))^2$ and $X^{\infty,p}_{\overline{\sigma}}(\Omega)$, respectively. For this purpose, we note that the space

$$C^\infty_{\mathrm{per}}([0,1]^2; C^\infty_c(-h,0))^2$$

is contained in $D(\mathcal{A}_{\infty,p})$ and dense in $C_{\mathrm{per}}([0,1]^2; L^p(-h,0))^2$ since $C^\infty_{\mathrm{per}}([0,1]^2)$ and $C^\infty_c(-h,0)$ are dense in $C^\infty_c(-h,0)$ and $L^p(-h,0)$, respectively, compare [58]. Thus the semigroup generated by $\mathcal{A}_{\infty,p}$ is strongly continuous on $C_{\mathrm{per}}([0,1]^2; L^p(-h,0))^2$. Since it leaves $L^p_{\overline{\sigma}}(\Omega)$ invariant, the restriction of S onto $C_{\mathrm{per}}([0,1]^2; L^p(-h,0))^2 \cap L^p_{\overline{\sigma}}(\Omega) = X^{\infty,p}_{\overline{\sigma}}(\Omega)$ is also strongly continuous and $D(A_{\infty,p})$ is dense in $X^{\infty,p}_{\overline{\sigma}}(\Omega)$ as well. □

We also obtain another property that is typical for strongly continuous analytic semigroups.

Corollary 4.3.6. *Let $p \in (3,\infty)$. Then for all $v \in C_{per}([0,1]^2; L^p(-h,0))^2$ we have that*

$$\lim_{t\to 0+} t^{1/2}\|\nabla S(t)v\|_{L^\infty_H L^p_z(\Omega)} = 0.$$

Proof. Let $\varepsilon > 0$ and $v \in C_{\mathrm{per}}([0,1]^2; L^p(-h,0))^2$. Since $D(\mathcal{A}_{\infty,p})$ contains the dense subspace $C^\infty_{\mathrm{per}}([0,1]^2; C^\infty_c(-h,0))^2$, we may take $v' \in D(\mathcal{A}_{\infty,p})$ such that $\|v - v'\|_{L^\infty_H L^p_z(\Omega)} < \varepsilon/2C$, where $C > 0$ is the constant from estimate (4.3.8). We then have

$$t^{1/2}\|\nabla S(t)v\|_{L^\infty_H L^p_z(\Omega)} \leq \frac{\varepsilon}{2} + t^{1/2}\|\nabla S(t)v'\|_{L^\infty_H L^p_z(\Omega)}$$

where we may estimate the right-hand side via

$$\|\nabla S(t)v'\|_{L^\infty_H L^p_z(\Omega)} \leq C\|S(t)v'\|_{C^1(\overline{\Omega})} \leq C\|S(t)v'\|_{D(\mathcal{A}_p)} \leq C\|(\mathcal{A}_p - \mu)S(t)v'\|_{L^p(\Omega)}$$

where in the last step we used that $\mathcal{A}_p - \mu$ is invertible. This yields

$$t^{1/2}\|\nabla S(t)v'\|_{L^\infty_H L^p_z(\Omega))} \leq Ct^{1/2}\|S(t)(\mathcal{A}_p - \mu)v'\|_{L^p(\Omega)} \leq Ct^{1/2}e^{\beta t}\|(\mathcal{A}_p - \mu)v'\|_{L^p(\Omega)}$$

and since $v' \in D(\mathcal{A}_{\infty,p}) \subset D(\mathcal{A}_p)$ we have $\|(\mathcal{A}_p - \mu)v'\|_{L^p(\Omega)} < \infty$ and so the claim follows. □

4.3.2 The estimate for $\nabla_H(\lambda - A)^{-1}\mathbb{P}$

We now turn to the issue of the hydrostatic Stokes projection $\mathbb{P}$. Since it fails to be bounded with respect to the norm of $L^\infty_H L^p_z(\Omega)$, estimates involving it require us to adapt our approach in nontrivial ways. We begin with the issue of horizontal derivatives.

While the choice of boundary conditions (4.1.1) implies that the hydrostatic Stokes semigroup is not just the restriction of a heat semigroup, we may nevertheless use the tensor structure of the heat semigroup on the cylindrical domain $\Omega = G \times (-h, 0)$ to obtain information about the hydrostatic Stokes semigroup. Analogously to the representation (4.2.6), we have that the heat semigroup generated by the Laplace operator equipped with the boundary conditions (4.1.1) satisfies

$$e^{t\Delta_p}(f \otimes g) = (S_{\mathbb{T}^2}(t)f) \otimes (S_{ND}(t)g), \quad f\colon G \to \mathbb{R}^2, \quad g\colon (-h, 0) \to \mathbb{R}, \tag{4.3.9}$$

where $(f \otimes g)(x, y, z) := f(x, y)g(z)$ is an elementary tensor, $S_{\mathbb{T}^2}$ is the heat semigroup corresponding to the horizontal Laplace operator $\Delta_H = \partial_x^2 + \partial_y^2$ on G with periodic boundary conditions and S_{ND} is the one-dimensional heat semigroup from Lemma 3.3.2 with mixed Neumann and Dirichlet boundary conditions at $z = 0$ and $z = -h$, respectively. For the horizontal semigroup, recall that Q is the Helmholtz projection on G with periodic boundary conditions and given by $Qf = f - \nabla_H\pi$ where π is the weak solution to the problem

$$\Delta_H\pi = \mathrm{div}_H f \text{ on } G, \quad \pi \text{ periodic on } \partial G.$$

We now provide an estimate for the composition of Q and the heat semigroup on the two-dimensional torus.

Lemma 4.3.7. *Let $\theta \in (0, \pi/2)$. Then there exists a constant $C_\theta > 0$ such that for all $\tau \in \Sigma_\theta$ and $p \in [1, \infty]$ it holds that*

$$|\tau|^{1/2}\|\nabla_H S_{\mathbb{T}^2}(\tau)(1 - Q)f\|_{L^p(G)} \le C_\theta\|f\|_{L^\infty(G)}, \quad f \in L^p(G).$$

Remark 4.3.8. While the two-dimensional Helmholtz projector with periodic boundary conditions is unbounded on $L^\infty(G)$, the composition $\nabla_H S_{\mathbb{T}^2}(\tau)Q$, however, defines a bounded operator for $\mathrm{Re}\,\tau > 0$, compare Remark 3.1.4.

Proof of Lemma 4.3.7. Let Q_2 be the Helmholtz projection on $\mathbb{R}^2$, E_H^{per} the periodic extension operator from G onto $\mathbb{R}^2$ and $(e^{t\Delta_H})_{\mathrm{Re}\,\tau > 0}$ be the heat semigroup on $\mathbb{R}^2$. Then an elementary calculation yields that

$$E_H^{\mathrm{per}}(1 - Q)f = (1 - Q_2)E_H^{\mathrm{per}}f, \quad f\colon G \to \mathbb{R}^2$$

as well as

$$\begin{aligned} E_H^{\mathrm{per}}|\tau|^{1/2}\nabla_H S_{\mathbb{T}^2}(\tau)(1 - Q)f &= |\tau|^{1/2}\nabla_H e^{\tau\Delta_H} E_H^{\mathrm{per}}(1 - Q)f \\ &= |\tau|^{1/2}\nabla_H e^{\tau\Delta_H}(1 - Q_2)E_H^{\mathrm{per}}f. \end{aligned}$$

Since E_H^{per} is isometric w.r.t. the norm $\|\cdot\|_\infty$ and $(1 - Q_2) = (R_iR_j)_{1\le i,j\le 2}$, the claim then follows from Lemma 3.1.5 for $d = 2$. $\square$

From this, we now obtain an estimate for horizontal derivatives of the hydrostatic Stokes semigroup composed with the hydrostatic Stokes projection.

Lemma 4.3.9. *Let $p \in (3,\infty)$, $\theta \in (0,\pi)$, and $\partial_i \in \{\partial_x, \partial_y\}$. Then there exists a constant $C = C_{\Omega,\theta,p} > 0$ such that*

$$|\lambda|^{1/2} \|\partial_i(\lambda - A_{\infty,p})^{-1}\mathbb{P}f\|_{L^\infty_H L^p_z(\Omega)} + |\lambda|^{1/2}\|(\lambda - A_{\infty,p})^{-1}\mathbb{P}\partial_i f\|_{L^\infty_H L^p_z(\Omega)} \leq C\|f\|_{L^\infty_H L^p_z(\Omega)}$$

for all $\lambda \in \Sigma_\theta$ and $f \in L^\infty_H L^p_z(\Omega)^2$. In particular, there exist constants $C = C_p > 0$ and $\beta < 0$ such that

$$t^{1/2}\|\partial_i S(t)\mathbb{P}f\|_{L^\infty_H L^p_z(\Omega)} + t^{1/2}\|S(t)\mathbb{P}\partial_i f\|_{L^\infty_H L^p_z(\Omega)} \leq Ce^{\beta t}\|f\|_{L^\infty_H L^p_z(\Omega)}$$

for all $t > 0$ and $f \in L^\infty_H L^p_z(\Omega)^2$.

Proof. It suffices to show the resolvent estimate since the semigroup estimate follows via the Dunford integral calculus. Since an elementary calculation shows that $\partial_i \in \{\partial_x, \partial_y\}$ commutes with both A and $\mathbb{P}$ and therefore also commutes with $(\lambda - A)^{-1}$ and S, yielding $\partial_i S(t)\mathbb{P}f = S(t)\mathbb{P}\partial_i f$ and thus it further suffices to show the first estimate.

For this purpose let $f \in L^\infty_H L^p_z(\Omega)^2 \hookrightarrow L^p(\Omega)^2$. Then $\mathbb{P}f \in L^p_{\overline{\sigma}}(\Omega)$ implies that

$$v := (\lambda - A_p)^{-1}\mathbb{P}f \in D(A_p) \hookrightarrow L^{\infty,p}_{\overline{\sigma}}(\Omega)$$

exists and belongs to $D(A_{\infty,p})$ since $A_{\infty,p}v = \lambda v - f \in L^{\infty,p}_{\overline{\sigma}}(\Omega)$. In order to obtain the estimate

$$|\lambda|^{1/2}\|\nabla_H v\|_{L^\infty_H L^p_z(\Omega)} \leq C_{\theta,p}\|f\|_{L^\infty_H L^p_z(\Omega)}, \tag{4.3.10}$$

we observe that the hydrostatic Stokes projection $\mathbb{P}$ is given by

$$\mathbb{P}f = f - (1-Q)\overline{f}, \quad \overline{f} = \frac{1}{h}\int_{-h}^{0} f(\cdot, z)\,dz.$$

Proceeding analogously as for (4.3.7) we obtain

$$\lambda v - A_p v = \mathbb{P}f \iff \lambda v - \Delta_p v = f + B_p v - (1-Q)\overline{f}.$$

Since we already provided an estimate for $(\lambda - \Delta_p)^{-1}(f + B_p v)$ as part of the proof of Lemma 4.3.4 and the vertical average satisfies

$$\|\overline{f}\|_{L^\infty(G)} \leq h^{-1}\|f\|_{L^\infty_H L^1_z(\Omega)} \leq C_p\|f\|_{L^\infty_H L^p_z(\Omega)},$$

it suffices to show an estimate of the form

$$|\lambda|^{1/2}\|\nabla_H(\lambda - \Delta_p)^{-1}Q\overline{f}\|_{L^\infty_H L^p_z(\Omega)} \leq C_{\theta,p}\|\overline{f}\|_{L^\infty(G)}.$$

For this purpose, observe that since $(1-Q)\overline{f}$ does not depend on the vertical variable z, we may take $\lambda \in \Sigma_{\pi/2}$ and use [71, Lemma 2.1.6] to write

$$\begin{aligned}\nabla_H(\lambda-\Delta_p)^{-1}(1-Q)\overline{f} &= \int_0^\infty e^{-\lambda t}\nabla_H e^{t\Delta_p}(1-Q)\overline{f}\,dt\\ &= \int_0^\infty e^{-\lambda t}\left(\nabla_H S_{\mathbb{T}^2}(t)(1-Q)\overline{f}\right)\otimes\left(S_{ND}(t)1\right)\,dt,\end{aligned}$$

compare (4.3.9). By Lemma 4.3.7 and 3.3.2 we have for $\psi := \arg(\tau) < \pi/2$ that

$$\begin{aligned}\|\nabla_H S_{\mathbb{T}^2}(\tau)(1-Q)\overline{f}\|_{L^\infty(G)} &\le C_\psi|\tau|^{-1/2}\|\overline{f}\|_{L^\infty(G)},\\ \|S_{ND}(\tau)1\|_{L^p(-h,0)} &\le C_\psi.\end{aligned}$$

Now let $\phi := \arg(\lambda)$ and assume $|\phi| < \pi/2-\epsilon$. Then we obtain

$$\begin{aligned}|\lambda|^{1/2}\|\nabla_H(\lambda-\Delta_p)^{-1}Q\overline{f}\|_{L^\infty_H L^p_z(\Omega)} &\le C|\lambda|^{1/2}\left(\int_0^\infty e^{-|\lambda|\cos(\phi)t}t^{-1/2}\,dt\right)\|\overline{f}\|_{L^\infty(G)}\\ &= C\frac{\sqrt{\pi}}{\sqrt{\cos(\phi)}}\|\overline{f}\|_{L^\infty(G)}\\ &\le C\frac{\sqrt{\pi}}{\sqrt{\cos(\pi/2-\varepsilon)}}\|\overline{f}\|_{L^\infty(G)}.\end{aligned}$$

The case $\lambda \in \Sigma_\theta$ for $\theta \in [\pi/2,\pi)$ is obtained analogously if we replace the operators Δ_{per} and Δ_{ND} with $e^{i\psi}\Delta_{\mathrm{per}}$ and $e^{i\psi}\Delta_{ND}$ for $\psi = \theta-\pi/2 \in [0,\pi/2)$. This completes the proof. □

4.3.3 The estimate for $\partial_z(\lambda-A)^{-1}\mathbb{P}$

In this section we establish that the first part of Lemma 4.3.9 is also valid for the case of the vertical derivative ∂_z. However, whereas the arguments used in 4.3.1 and 4.3.2 can be modified to include the case $p=\infty$, here this is no longer possible and we are truly restricted to $3<p<\infty$.

We begin by establishing two auxiliary local estimates subject to scaling in the horizontal component. For this purpose, we employ the notations

$$(x',z) := (x,y,z), \quad B(x_0';r) := \{x' \in \mathbb{R}^2 : |x'-x_0'| < r\} \tag{4.3.11}$$

for points belonging to $\mathbb{R}^2\times\mathbb{R}$ and the two-dimensional ball of radius $r>0$ centered around x_0', respectively. The first takes the form of an anisotropic interpolation quality.

Lemma 4.3.10. *Let $p \in (2,\infty)$ and $q \in [1,\infty)$. Then there exists a constant $C = C_{\Omega,p,q} > 0$ such that for all functions $v \in H^{1,p}(B(x_0';r);L^q(-h,0))$ and all $r>0$, $x_0' \in \mathbb{R}^2$ it holds that*

$$\|v\|_{L^\infty(B(x_0';r);L^q(-h,0))} \le Cr^{-2/p}\left(\|v\|_{L^p(B(x_0';r);L^q(-h,0))} + r\|\nabla_H v\|_{L^p(B(x_0';r);L^q(-h,0))}\right).$$

Remark 4.3.11. It was proven in [71, Lemma 3.1.4] that the claim of this lemma holds if we replace the vertical space $L^q(-h,0)$ with the one-dimensional space $\mathbb{C}$. In the following proof we derive the vector-valued case from the scalar-valued one.

Proof of Lemma 4.3.10. Consider the auxiliary function

$$w(x') := \|v(x',\cdot)\|_{L^q(-h,0)}, \quad x' \in B(x_0';r)$$

and observe that

$$\|w\|_{L^\infty(B(x_0';r))} = \|v\|_{L^\infty(B(x_0';r));L^q(-h,0)}, \quad \|w\|_{L^p(B(x_0';r))} = \|v\|_{L^p(B(x_0';r));L^q(-h,0)}.$$

It further holds that w is weakly differentiable with $\partial_i w(x') = 0$ for $\partial_i \in \{\partial_x, \partial_y\}$ whenever $w(x') = 0$ and

$$\partial_i w(x') = \left(\int_{-h}^0 |v(x',z)|^q \, dz\right)^{1/q-1} \int_{-h}^0 |v(x',z)|^{q-2}(\partial_i v(x',z) \cdot v(x',z)) \, dz$$

otherwise. We estimate the right-hand side via

$$\begin{aligned} |\partial_i w(x')| &\leq \left(\int_{-h}^0 |v(x',z)|^q \, dz\right)^{1/q-1} \int_{-h}^0 |v(x',z)|^{q-1} |\partial_i v(x',z)| \, dz \\ &\leq \left(\int_{-h}^0 |\partial_i v(x',z)|^q \, dz\right)^{1/q}, \end{aligned}$$

where we used Hölder's inequality for the second integral. This yields

$$\|\nabla_H w\|_{L^p(B(x_0';r))} \leq \|\nabla_H v\|_{L^p(B(x_0';r);L^q(-h,0))}.$$

Since the interpolation inequality holds for the scalar-valued function w by [71, Lemma 3.1.4], it holds that

$$\|w\|_{L^\infty(B(x_0';r))} \leq Cr^{-2/p} \left(\|w\|_{L^p(B(x_0';r))} + r\|\nabla_H w\|_{L^p(B(x_0';r))}\right), \tag{4.3.12}$$

which implies the desired estimate. □

The next lemma allows us to overcome a problem arising from the fact that the operator $1 - Q$ fails to be bounded on $L^\infty(G)^2$ by establishing a local L^∞-L^p-estimate. Here we utilize scaling arguments similar to the Masuda-Stewart argument used in the proof of Lemma 3.5.1.

Lemma 4.3.12. *Let $p \in (1,\infty)$ and $x_0' \in G$. Then there exist constants $r_0 > 0$ and $C = C_{G,p} > 0$ such the weak solution of*

$$\Delta_H \pi = \operatorname{div}_H f \quad \text{in} \quad G, \qquad \pi \text{ periodic on } \partial G, \qquad \int_G \pi(x') \, dx' = 0, \tag{4.3.13}$$

satisfies the estimate

$$\|\nabla_H \pi\|_{L^p(B(x_0';r))} \leq Cr^{2/p}(1 + |\log r|)\|f\|_{L^\infty(G)}$$

for all $f \in L^\infty(G)^2$ and $0 < r < r_0$.

Proof. Note that since $f \in L^\infty(G)^2 \hookrightarrow L^p(G)^2$, the problem is well-posed. We extend both π and f periodically and consider the problem on the enlarged square-domain $G' := (-2,3)^2$. Furthermore, we take $0 < r_0 < 1/8$ to ensure the inclusion

$$B(x_0'; 4r_0) \subset (-1/2, 3/2)^2 \subset G' \tag{4.3.14}$$

and consider a cut-off function $\omega \in C_c^\infty(\mathbb{R}^2)$ such that

$$\omega = 1 \text{ on } [-1,2]^2, \quad \operatorname{supp}(\omega) \subset G', \quad \text{ and } \quad \|\nabla_H^k \omega\|_{L^\infty(\mathbb{R}^2)} \leq C \tag{4.3.15}$$

for some constant $C > 0$ and $k \in \{0,1,2\}$. Since π is the unique solution to (4.3.13), it follows that

$$\Delta_H(\omega\pi) = -(\nabla_H \omega) \cdot f - (\Delta_H \omega)\pi + \operatorname{div}_H\left(2(\nabla_H\omega)\pi + \omega f\right) \quad \text{ on } \mathbb{R}^2.$$

Recalling that $\Psi(x', y') := \frac{1}{2\pi}\log(|x'-y'|)$ is a Green's function for the operator Δ_H on $\mathbb{R}^2$, it follows from integration by parts that

$$\begin{aligned}(\omega\pi)(x') = &-\int_{\mathbb{R}^2} \Psi(x',y')\big[(\nabla_H\omega)\cdot f + (\Delta_H\omega)\pi\big](y')\, dy' \\ &- \int_{\mathbb{R}^2} (\nabla_{y'}\Psi)(x',y') \cdot \big[2(\nabla_{y'}\omega)\pi + \omega f\big](y')\, dy'.\end{aligned}$$

Given $x' \in B(x_0'; r) \subset (-1/2, 3/2)^2$ for $0 < r < r_0$ by (4.3.14), the fact that $\omega = 1$ on $[-1,2]^2$ yields the representation

$$\begin{aligned}\nabla_H \pi(x') = &-\int_{\mathbb{R}^2} (\nabla_{x'}\Psi)(x',y')\big[(\nabla_H\omega)\cdot f + (\Delta_H\omega)\pi\big](y')\, dy' \\ &-2\int_{\mathbb{R}^2} (\nabla_{x'}\nabla_{y'}\Psi)(x',y')\left[(\nabla_{y'}\omega)\pi\right](y')\, dy' \\ &- \int_{\mathbb{R}^2} (\nabla_{x'}\nabla_{y'}\Psi)(x',y')(\omega f)(y')\, dy'.\end{aligned} \tag{4.3.16}$$

We now estimate the terms on the right-hand side terms as follows. For the first term

$$I_1(x') := \int_{\mathbb{R}^2} (\nabla_{x'}\Psi)(x',y')\big[(\nabla_H\omega)\cdot f + (\Delta_H\omega)\pi\big](y')\, dy'$$

we utilize that the derivative of Ψ admits the estimate

$$|\nabla_{x'}\Psi(x',y')| \leq C|x'-y'|^{-1}, \quad x', y' \in \mathbb{R}^2,$$

together with the fact that derivatives of ω are supported on $[-2,3]^2 \setminus [-1,2]^2$ and bounded, yielding

$$|\nabla_{x'}\Psi(x',y')| \leq 2C, \quad x' \in (-1/2, 3/2)^2, \quad y' \in [-2,3]^2 \setminus [-1,2]^2.$$

Using Young's inequality we obtain the estimate

$$\|I_1\|_{L^p(B(x_0';r))} \le 2C\|1\|_{L^p(B(x_0';r))}(\|f\|_{L^1(G)} + \|\pi\|_{L^1(G)}).$$

For the terms on the right-hand side we further have $\|1\|_{L^p(B(x_0';r))} = Cr^{2/p}$ as well as $\|f\|_{L^1(G)} \le \|f\|_{L^\infty(G)}$ and

$$\|\pi\|_{L^1(G)} \le C\|\nabla_H \pi\|_{L^1(G)} \le C\|\nabla_H \pi\|_{L^2(G)} \le C\|f\|_{L^2(G)} \le C\|f\|_{L^\infty(G)},$$

where we used the Poincaré inequality for the case $\int_G \pi(x')\,dx' = 0$ and the fact that the solution mapping $f \mapsto \nabla_H \pi$ of problem (4.3.13) is bounded on $L^2(G)^2$. Combining these estimates yields

$$\|I_1\|_{L^p(B(x_0';r))} \le Cr^{2/p}\|f\|_{L^\infty(G)}.$$

For the term

$$I_2(x') := 2\int_{\mathbb{R}^2} (\nabla_{x'}\nabla_{y'}\Psi)(x',y')\,[(\nabla_{y'}\omega)\pi]\,(y')\,dy'$$

we proceed analogously, noting that second order derivatives of Ψ admit the estimate

$$|\nabla_{x'}\nabla_{y'}\Psi(x',y')| \le C|x'-y'|^{-2}, \quad x',y' \in \mathbb{R}^2, \tag{4.3.17}$$

yielding

$$\|I_2(x')\|_{L^p(B(x_0';r))} \le Cr^{2/p}\|\pi\|_{L^1(G)} \le Cr^{2/p}\|f\|_{L^\infty(G)}.$$

The integrals we estimated so far have shown good behavior at $r = 0$. However, the remaining term leads to the divergent, and thus problematic, logarithmic term. In order to estimate the remaining terms, we consider a family of cut-off functions $\chi_r \in C_c^\infty(\mathbb{R}^2)$ for $0 < r < r_0$, satisfying

$$\chi_r = 1 \text{ on } B(x_0';2r), \quad \operatorname{supp}(\chi_r) \subset B(x_0';4r), \quad \|\chi_r\|_{L^\infty(\mathbb{R}^2)} \le C, \tag{4.3.18}$$

for some constant $C > 0$ and all $0 < r < r_0$. Note that, in particular, we have $\omega\chi_r = \chi_r$ by (4.3.15). We now further decompose

$$\begin{aligned} -\int_{\mathbb{R}^2} (\nabla_{x'}\nabla_{y'}\Psi)(x',y')(\omega f)(y')\,dy' = &-\int_{\mathbb{R}^2} (\nabla_{x'}\nabla_{y'}\Psi)(x',y')(\chi_r f)(y')\,dy' \\ &-\int_{\mathbb{R}^2} (\nabla_{x'}\nabla_{y'}\Psi)(x',y')[\omega(1-\chi_r)f](y')\,dy'. \end{aligned}$$

Since the term

$$I_3(x') := -\int_{\mathbb{R}^2} (\nabla_{x'}\nabla_{y'}\Psi)(x',y')(\chi_r f)(y')\,dy'$$

satisfies $I_3 = \nabla_H u$ where u solves $\Delta_H u = -\operatorname{div}_H(\chi_r f)$ on $\mathbb{R}^2$, the Calderón–Zygmund inequality and (4.3.18) yield the estimate

$$\|I_3\|_{L^p(\mathbb{R}^2)} \le C\|\chi_r f\|_{L^p(\mathbb{R}^2)} \le C\|\chi_r\|_{L^p(B(x_0';4r))}\|f\|_{L^\infty(G')} \le Cr^{2/p}\|f\|_{L^\infty(G)}.$$

For the last remaining term

$$I_4(x') := -\int_{\mathbb{R}^2} (\nabla_{x'}\nabla_{y'}\Psi)(x',y')[\omega(1-\chi_r)f](y')\,dy'$$

we observe that

$$\operatorname{supp}(\omega(1-\chi_r)) = \operatorname{supp}(\omega-\chi_r) \subset \operatorname{supp}(\omega)\setminus B(x_0';2r) \subset \{r \le |x'-y'| \le 4\}$$

by (4.3.15) and (4.3.18) for $x' \in B(x_0';r)$. Using estimate (4.3.17), we thus obtain

$$\begin{aligned}\|I_4\|_{L^p(B(x_0';r))} &\le \|1\|_{L^p(B(x_0';r))}\left(\sup_{x'\in B(x_0';r)}\int_{r\le|x'-y'|\le 4} C|x'-y'|^{-2}\,dy'\right)\|\omega(1-\chi_r)F\|_{L^\infty(G')}\\ &\le Cr^{2/p}(1+|\log r|)\|F\|_{L^\infty(G)}.\end{aligned}$$

Combining the estimates for I_i, $1 \le i \le 4$, with the representation (4.3.16) yields the desired estimate. □

Remark 4.3.13. Observe that the Calderón–Zygmund inequality used to establish the estimate for I_3 is only valid in L^p for $p \in (1,\infty)$. Since the solution operator $f \mapsto \nabla_H\pi$ fails to be bounded on $L^\infty(G)^2$, the estimate fails for $p=\infty$.

We are now able to prove the following estimate. As with the two previous lemmas, the proof is reliant on scaling-arguments involving families of cut-off functions.

Lemma 4.3.14. *Let $p \in (3,\infty)$ and $\theta \in (0,\pi)$. Then there exists a constant $C = C_{\Omega,\theta,p} > 0$ such that for all $\lambda \in \Sigma_\theta$ and $f \in L^\infty_H L^p_z(\Omega)^2$ it holds that*

$$|\lambda|^{1/2}\|\partial_z(\lambda - A_{\infty,p})^{-1}\mathbb{P}f\|_{L^\infty_H L^p_z(\Omega)} \le C\|f\|_{L^\infty_H L^p_z(\Omega)}.$$

In particular, there exists a constant $C = C_{\Omega,p} > 0$ such that for all $t > 0$ it holds that

$$t^{1/2}\|\partial_z S(t)\mathbb{P}f\|_{L^\infty_H L^p_z(\Omega)} \le C\|f\|_{L^\infty_H L^p_z(\Omega)}.$$

Remark 4.3.15. The requirement $3 < p < \infty$ is due to the fact that Lemma 4.3.4 and 4.3.12 were not proven for the cases $1 \le p \le 3$ and $p = \infty$, respectively.

Proof of Lemma 4.3.14. Since $f \in L^\infty_H L^p_z(\Omega)^2 \hookrightarrow L^p(\Omega)^2$ and $\Sigma_\theta \subset \rho(A_p)$, it follows that

$$v := (\lambda - A_{\infty,p})^{-1}\mathbb{P}f = (\lambda - A_p)^{-1}\mathbb{P}f \in D(A_p)$$

for all $\lambda \in \Sigma_\theta$. We will now decompose v into a part for which estimates already exists and a remainder we will treat with Lemma 4.3.10 and 4.3.12. For this purpose, we recall the equivalence

$$\lambda v - Av = \mathbb{P}f \iff \lambda v - \Delta v + \nabla_H\pi = f,$$

with π as in (4.0.5). Using (4.0.6) and Lemma 3.5.1 in $L^\infty_H L^p_z(\Omega)^2$ we further have that the auxiliary function

$$v_1 := (\lambda - \Delta)^{-1}(f + Bv)$$

exists and satisfies $\lambda v_1 - \Delta v_1 + \nabla_H \pi_1 = f$ where the pressure is determined by

$$\Delta_H \pi_1 = \mathrm{div}_H \left(\frac{1}{h} \partial_z v \big|_{\Gamma_u} - \frac{1}{h} \partial_z v \big|_{\Gamma_b} \right), \quad \pi \text{ periodic on } \partial G.$$

By (4.0.5), the function $v_2 := v - v_1$ thus solves the problem

$$\lambda v_2 - \Delta v_2 + \nabla_H \pi_2 = 0, \quad \Delta_H \pi_2 = \mathrm{div}_H \overline{f}, \quad \pi \text{ periodic on } \partial G. \tag{4.3.19}$$

From Lemma 4.3.9 and 3.5.1 it follows that

$$\begin{aligned} |\lambda|^{1/2} \|\nabla_H v\|_{L^\infty_H L^p_z(\Omega)} &\leq C_{\Omega,\theta,p} \|f\|_{L^\infty_H L^p_z(\Omega)}, \\ |\lambda|^{1/2} \|\nabla v_1\|_{L^\infty_H L^p_z(\Omega)} &\leq C_{\Omega,\theta} \|f + Bv\|_{L^\infty_H L^p_z(\Omega)} \leq C_{\Omega,\theta,p} \|f\|_{L^\infty_H L^p_z(\Omega)}, \end{aligned} \tag{4.3.20}$$

for all $\lambda \in \Sigma_\theta$, $\theta \in (0, \pi)$. Here we used that the term Bv can be estimated in $L^\infty_H L^p_z(\Omega)^2$ via

$$\|Bv\|_{L^\infty_H L^p_z(\Omega)} \leq C_{\Omega,p} \|v\|_{H^{2,p}(\Omega)}$$

by the arguments as in the proof of Lemma 4.3.4 for $p \in (3, \infty)$. We further have

$$\|v\|_{H^{2,p}(\Omega)} \leq C_{\Omega,\theta,p} \|A_p v\|_{L^p(\Omega)} \leq C_{\Omega,\theta,p} \|\mathbb{P} f\|_{L^p(\Omega)} \leq C_{\Omega,\theta,p} \|f\|_{L^p(\Omega)}, \tag{4.3.21}$$

where we used that $0 \in \rho(A_p)$ in the first, the sectoriality of A_p in the second, and the L^p-boundedness of $\mathbb{P}$ in the third step. Combining the estimates in (4.3.20) then yields

$$|\lambda|^{1/2} \|\nabla_H v_2\|_{L^\infty_H L^p_z(\Omega)} \leq C_{\Omega,\theta,p} \|f\|_{L^\infty_H L^p_z(\Omega)}. \tag{4.3.22}$$

In order to estimate $\partial_z v_2$, we proceed as follows. First we apply a periodic extension to v_2, π_2, and f. By denoting their periodic extensions with the same symbols, we may assume that the equation (4.3.19) holds on the larger domains $\Omega' = G' \times (-h, 0)$, $G' = (-2, 3)^3$ with boundary conditions

$$\partial_z v_2 = 0 \text{ on } \Gamma'_u, \quad v_2 \text{ periodic on } \Gamma'_l, \quad v_2 = 0 \text{ on } \Gamma'_b, \quad \pi_2 \text{ periodic on } \partial G',$$

with $\Gamma'_u := G' \times \{0\}$, $\Gamma'_l := \partial G' \times [-h, 0]$, and $\Gamma'_b := G \times \{-h\}$. Following the proof of Lemma 3.5.1, we take $\eta > 1$ to be a sufficiently large parameter to be decided later on and take $\lambda_0 > 0$ such that

$$r_0 := \eta \, \lambda_0^{-1/2} < \min\{1/8, h/4\}, \tag{4.3.23}$$

and the condition for Lemma 3.5.1 is satisfied. Given $\lambda \in \Sigma_\theta$ for some $\theta \in (0, \pi)$, we will first consider the case where $|\lambda| > \lambda_0$ and then extend this result to the full range $\lambda \in \Sigma_\theta$ by a similar argument as in the proof of Lemma 3.5.1. For this purpose we consider the scaling-parameter $r := \eta |\lambda|^{-1/2} \in (0, r_0)$ and utilize two vertical cut-off functions $\alpha = \alpha_r, \beta = \beta_r \in C^\infty([-h, 0])$, such that

$$\begin{aligned} &\alpha = 0 \text{ on } [-h, -h + r], \quad \alpha = 1 \text{ on } [-h + 2r, 0], \quad \|\partial_z^k \alpha\|_{L^\infty(-h,0)} \leq C r^{-k}, \\ &\beta = 1 \text{ on } [-h, -h + 2r], \quad \beta = 0 \text{ on } [-h + 3r, 0], \quad \|\partial_z^k \beta\|_{L^\infty(-h,0)} \leq C r^{-k}, \end{aligned} \tag{4.3.24}$$

for an absolute constant $C > 0$ and all $k \in \{0, 1, 2\}$, $0 < r < r_0$. Since

$$[-h, -h + 2r] \cup [-h + 2r, 0] = [-h, 0],$$

we have

$$\|\partial_z v_2\|_{L^\infty_H L^p_z(\Omega)} \leq \|\partial_z(\alpha v_2)\|_{L^\infty_H L^p_z(\Omega)} + \|\partial_z(\beta v_2)\|_{L^\infty_H L^p_z(\Omega)}. \tag{4.3.25}$$

This allows us to separate the Neumann boundary condition at Γ'_u from the Dirichlet boundary condition at Γ'_b from one another by individually estimating the upper part αv_2 and the lower part βv_2. Observe that

$$\|f\|_{L^\infty_H L^p_z(\Omega)} = \sup_{x'_0 \in G} \|f\|_{L^\infty(B(x'_0,R);L^p_z)}, \quad 0 < R < r_0. \tag{4.3.26}$$

Given an arbitrary point $x'_0 \in G$, we further utilize a horizontal cut-off function $\chi = \chi_{r,x'_0} \in C^\infty_c(\mathbb{R}^2)$ satisfying

$$\chi = 1 \text{ in } \overline{B(x'_0; |\lambda|^{-1/2})}, \quad \operatorname{supp} \chi \subset B(x'_0; r), \quad \|\nabla^k_H \chi\|_{L^\infty(\mathbb{R}^2)} \leq C r^{-k}, \tag{4.3.27}$$

for $k = 0, 1, 2$, with an absolute constant $C > 0$ not depending on x'_0 or r. We now estimate the right-hand sides of (4.3.25) in the following way.

Step 1: We begin by establishing an estimate for $\partial_z(\alpha v_2)$. Consider the cylinder

$$\mathcal{C}(x'_0; |\lambda|^{-1/2}) := B(x'_0; |\lambda|^{-1/2}) \times (-h, 0).$$

Then we have $\mathcal{C}(x'_0; |\lambda|^{-1/2}) \subset \Omega'$ by (4.3.23) and it follows from Lemma 4.3.10 with radius $r = \eta|\lambda|^{-1/2} > 0$ and $q = p$ as well as (4.3.26) that

$$\begin{aligned} |\lambda|^{1/2} \|\partial_z(\alpha v_2)\|_{L^\infty_H L^p_z(\Omega)} \leq C_{\Omega,p} |\lambda|^{1/p} \sup_{x'_0 \in G} \Big(& |\lambda|^{1/2} \|\partial_z(\alpha v_2)\|_{L^p(\mathcal{C}(x'_0;|\lambda|^{-1/2}))} \\ & + \|\nabla_H \partial_z(\alpha v_2)\|_{L^p(\mathcal{C}(x'_0;|\lambda|^{-1/2}))} \Big). \end{aligned} \tag{4.3.28}$$

Since we have $\chi = 1$ on $\overline{B(x'_0; |\lambda|^{-1/2})}$ and (4.3.23) yields $\mathcal{C}(x'_0; |\lambda|^{-1/2}) \subset \Omega'$, it follows that

$$\begin{aligned} \|\partial_z(\alpha v)\|_{L^p(\mathcal{C}(x'_0;|\lambda|^{-1/2}))} &\leq \|\partial_z(\chi \alpha v_2)\|_{L^p(\Omega')}, \\ \|\nabla \partial_z(\alpha v_2)\|_{L^p(\mathcal{C}(x'_0;|\lambda|^{-1/2}))} &\leq \|\nabla[\chi \partial_z(\alpha v_2)]\|_{L^p(\Omega')}. \end{aligned} \tag{4.3.29}$$

Observe that $\chi \alpha v_2$ satisfies

$$\lambda(\chi \alpha v_2) - \Delta(\chi \alpha v_2) = -\chi \alpha \nabla_H \pi_2 - 2\nabla(\chi \alpha) \cdot \nabla v_2 - (\Delta(\chi \alpha)) v_2 \quad \text{on} \quad \Omega',$$

with boundary conditions

$$\partial_z(\chi \alpha v_2)|_{\Gamma'_u \cup \Gamma'_b} = 0, \quad \chi \alpha v_2 \text{ periodic on } \Gamma'_l.$$

We further differentiate with respect to z to obtain

$$\lambda(\chi\partial_z(\alpha v_2)) - \Delta(\chi\partial_z(\alpha v_2)) = F_1 + \partial_z F_2 \quad \text{on} \quad \Omega',$$

with right-hand sides

$$\begin{aligned} F_1 &:= -\chi(\partial_z\alpha)(\nabla_H\pi_2) - (\Delta_H\chi)(\partial_z\alpha)v_2 - (\Delta_H\chi)\alpha(\partial_z v_2), \\ F_2 &:= -2(\nabla_H\chi)\alpha\cdot(\nabla_H v_2) - 2\chi(\partial_z\alpha)(\partial_z v_2) - \chi(\partial_z^2\alpha)v_2, \end{aligned}$$

and boundary conditions

$$\chi\partial_z(\alpha v_2)\big|_{\Gamma'_u\cup\Gamma'_b} = 0, \quad \chi\partial_z(\alpha v_2) \text{ periodic on } \Gamma'_l.$$

Lemma 3.5.1 on Ω' for the case (DD) and $q = p$ then yields the estimate

$$|\lambda|^{1/2}\|\chi\partial_z(\alpha v_2)\|_{L^p(\Omega')} + \|\nabla[\chi\partial_z(\alpha v_2)]\|_{L^p(\Omega')} \le C_{\Omega',\theta}\big(|\lambda|^{-1/2}\|F_1\|_{L^p(\Omega')} + \|F_2\|_{L^p(\Omega')}\big). \tag{4.3.30}$$

We further employ the notations

$$\|\cdot\|_{L^p_{H,r}} := \|\cdot\|_{L^p(B(x'_0;r))}, \quad \|\cdot\|_{L^p_z} := \|\cdot\|_{L^p(-h,0)}.$$

By (4.3.24) and (4.3.27) we have that the cut-off functions satisfy the estimates

$$\|\chi\|_{L^p_{H,r}} \le C_p r^{2/p}, \quad \|\nabla_H\chi\|_{L^p_{H,r}} \le C_p r^{2/p-1}, \quad \|\Delta_H\chi\|_{L^p_{H,r}} \le C_p r^{2/p-2}$$

as well as

$$\|\partial_z\alpha\|_{L^p_z} \le C_p r^{1/p-1}, \quad \|\partial_z^2\alpha\|_{L^p_z} \le C_p r^{1/p-2}.$$

Via the anisotropic Hölder inequality from Lemma 2.4.9, we now provide estimates for the right-hand side terms of (4.3.30), beginning with F_1. Since π_2 satisfies

$$\Delta_H\pi_2 = \mathrm{div}_H\overline{f}, \quad \pi_2 \text{ periodic on } \partial G', \quad \int_{G'}\pi_2 = 0,$$

we may estimate $\nabla_H\pi_2$ via Lemma 4.3.12, yielding

$$\begin{aligned} \|\chi(\partial_z\alpha)(\nabla_H\pi_2)\|_{L^p(\Omega')} &\le \|\chi\|_{L^\infty(\mathbb{R}^2)}\|\partial_z\alpha\|_{L^p_z}\|\nabla_H\pi_2\|_{L^p_{H,r}} \\ &\le C_{G',p} r^{3/p-1}(1+|\log r|)\|f\|_{L^\infty_H L^p_z(\Omega)}. \end{aligned}$$

By applying the vertical Poincaré inequality

$$\|f\|_{L^\infty(G';L^p(-h,-h+\mu))} \le \mu\|\partial_z f\|_{L^\infty_H L^p_z(\Omega')}, \quad 0\le\mu\le h, \quad f\big|_{\Gamma'_b} = 0, \tag{4.3.31}$$

we further have

$$\begin{aligned} \|(\Delta_H\chi)(\partial_z\alpha)v_2\|_{L^p(\Omega')} &\le \|\Delta_H\chi\|_{L^p_{H,r}}\|\partial_z\alpha\|_\infty\|v_2\|_{L^\infty(G';L^p(-h,-h+2r))} \\ &\le C_p r^{2/p-2}\|\partial_z v_2\|_{L^\infty_H L^p_z(\Omega)}, \end{aligned}$$

and the remaining term in F_1 is estimated via

$$\begin{aligned}\|(\Delta_H\chi)\alpha(\partial_z v_2)\|_{L^p(\Omega')} &\leq \|\Delta_H\chi\|_{L^p_{H,r}}\|\alpha\|_{L^\infty(-h,0)}\|\partial_z v_2\|_{L^\infty_H L^p_z(\Omega')}\\ &\leq C_p r^{2/p-2}\|\partial_z v_2\|_{L^\infty_H L^p_z(\Omega)},\end{aligned}$$

where we used that $\partial_z v_2$ is periodic as well. For the first term in F_2, we note that the horizontal periodicity of $\nabla_H v_2$ together with estimate (4.3.22) implies that

$$\begin{aligned}\|(\nabla_H\chi)\alpha(\nabla_H v_2)\|_{L^p(\Omega')} &\leq \|\nabla_H\chi\|_{L^p_{H,r}}\|\alpha\|_{L^\infty(-h,0)}\|\nabla_H v_2\|_{L^\infty_H L^p_z(\Omega)}\\ &\leq C_{\Omega,\theta,p} r^{2/p-1}|\lambda|^{-1/2}\|f\|_{L^\infty_H L^p_z(\Omega)}.\end{aligned}$$

The second term is estimated via

$$\begin{aligned}\|\chi(\partial_z\alpha)(\partial_z v_2)\|_{L^p(\Omega')} &\leq \|\chi\|_{L^p_{H,r}}\|\partial_z\alpha\|_{L^\infty(-h,0)}\|\partial_z v_2\|_{L^\infty_H L^p_z(\Omega)}\\ &\leq C_p r^{2/p-1}\|\partial_z v_2\|_{L^\infty_H L^p_z(\Omega)},\end{aligned}$$

whereas for the last term we employ the vertical Poincaré inequality (4.3.31), yielding the estimate

$$\begin{aligned}\|\chi(\partial_z^2\alpha)v_2\|_{L^p(\Omega')} &\leq \|\chi\|_{L^p_{H,r}}\|\partial_z^2\alpha\|_{L^\infty(-h,0)}\|v_2\|_{L^\infty_H L^p_z(\Omega)}\\ &\leq C_p r^{2/p-1}\|\partial_z v_2\|_{L^\infty_H L^p_z(\Omega)}.\end{aligned}$$

We now estimate $\partial_z(\alpha v_2)$ as follows. By combining (4.3.28), (4.3.29), (4.3.30), the estimates for F_1 and F_2, and plugging in the value $r = \eta|\lambda|^{-1/2}$, we obtain

$$\begin{aligned}|\lambda|^{1/2}\|\partial_z(\alpha v_2)\|_{L^\infty_H L^p_z(\Omega)} &\leq C_{\Omega',\theta,p}\left(\eta^{2/p-2}+\eta^{3/p-2}|\lambda|^{-1/2p}+\eta^{2/p-1}r^{1/p}|\log(r)|\right)\|f\|_{L^\infty_H L^p_z(\Omega)}\\ &\quad + C_{\Omega',\theta,p}(\eta^{2/p-1}+\eta^{2/p-2})|\lambda|^{1/2}\|\partial_z v_2\|_{L^\infty_H L^p_z(\Omega)}\\ &\leq C_{\Omega',\theta,p}\eta^{2/p-1}\big(1+r^{1/p}|\log r|\big)\|f\|_{L^\infty_H L^p_z(\Omega)}\\ &\quad + C_{\Omega',\theta,p}(\eta^{2/p-1}+\eta^{2/p-2})|\lambda|^{1/2}\|\partial_z v_2\|_{L^\infty_H L^p_z(\Omega)}.\end{aligned} \tag{4.3.32}$$

Step 2: We now establish an estimate for $\partial_z(\beta v_2)$. While the arguments we employ here are very similar to those used in the previous step, we nevertheless present them in detail for the sake of completeness. Applying Lemma 4.3.10 as in Step 1 yields the estimate

$$\begin{aligned}|\lambda|^{1/2}\|\partial_z(\beta v_2)\|_{L^\infty_H L^p_z(\Omega)} \leq C_p|\lambda|^{1/p}\sup_{x_0'\in G}&\big(|\lambda|^{1/2}\|\partial_z(\beta v_2)\|_{L^p(\mathcal{C}(x_0';|\lambda|^{-1/2}))}\\ &+\|\nabla_H\partial_z(\beta v_2)\|_{L^p(\mathcal{C}(x_0';|\lambda|^{-1/2}))}\big).\end{aligned} \tag{4.3.33}$$

Using the same horizontal cut-off function χ as before, the same argument used to derive (4.3.29) yields

$$\begin{aligned}\|\partial_z(\beta v_2)\|_{L^p(\mathcal{C}(x_0';|\lambda|^{-1/2})} &\leq \|\nabla(\chi\beta v_2)\|_{L^p(\Omega')},\\ \|\nabla_H\partial_z(\beta v_2)\|_{L^p(\mathcal{C}(x_0';|\lambda|^{-1/2})} &\leq \|\nabla_H\partial_z(\chi\beta v_2)\|_{L^p(\Omega')}\\ &\leq \|\chi\beta v_2\|_{H^{2,p}(\Omega')}\\ &\leq C_{\Omega',p}\|\Delta(\chi\beta v_2)\|_{L^p(\Omega')},\end{aligned} \tag{4.3.34}$$

where we used the invertibility of the Laplace operator for $\Gamma_D \neq \emptyset$. We further have that $\chi\beta v_2$ satisfies

$$\lambda(\chi\beta v_2) - \Delta(\chi\beta v_2) = F_3 \quad \text{on} \quad \Omega',$$

with right-hand side

$$F_3 := -\chi\beta(\nabla_H \pi_2) - 2(\nabla_H \chi)\beta \cdot (\nabla_H v_2) - 2\chi(\partial_z\beta)(\partial_z v_2) - (\Delta_H \chi)\beta v_2 - 2\chi(\partial_z^2\beta)v_2,$$

and boundary conditions

$$\partial_z(\chi\beta v_2)|_{\Gamma'_u} = 0, \quad \chi\beta v_2\big|_{\Gamma'_b} = 0, \quad \chi\beta v_2 \text{ periodic on } \Gamma'_l.$$

We apply Lemma 3.5.1 for the case (ND) and $q = p$, yielding the estimate

$$|\lambda|^{1/2}\|\nabla(\chi\beta v_2)\|_{L^p(\Omega')} + \|\Delta(\chi\beta v_2)\|_{L^p(\Omega')} \leq C_{\Omega',\theta}\|F_3\|_{L^p(\Omega')}. \tag{4.3.35}$$

In order to estimate F_3 we first observe that (4.3.24) implies the estimates

$$\|\beta\|_{L^p_z} \leq C_p r^{1/p}, \quad \|\partial_z\beta\|_{L^p_z} \leq C_p r^{1/p-1}, \quad \|\partial_z^2\beta\|_{L^p_z} \leq C_p r^{1/p-2}.$$

Again applying Lemma 4.3.12 then yields

$$\begin{aligned}\|\chi\beta(\nabla_H\pi_2)\|_{L^p(\Omega')} &\leq \|\chi\|_{L^\infty(\mathbb{R}^2)}\|\beta\|_{L^p_z}\|\nabla_H\pi_2\|_{L^p_{H,r}} \\ &\leq C_{G',p} r^{3/p}(1 + |\log r|)\|f\|_{L^\infty_H L^p_z(\Omega)}.\end{aligned}$$

For the second term in F_3, we use estimate (4.3.22) to obtain

$$\begin{aligned}\|(\nabla_H\chi)\beta \cdot (\nabla_H v_2)\|_{L^p_{H,r}} &\leq \|\nabla_H\chi\|_{L^p_{H,r}}\|\beta\|_{L^\infty(-h,0)}\|\nabla_H v_2\|_{L^\infty_H L^p_z(\Omega)} \\ &\leq C_{\Omega',\theta,p} r^{2/p-1}|\lambda|^{-1/2}\|f\|_{L^\infty_H L^p_z(\Omega)},\end{aligned}$$

whereas for the third term we have

$$\begin{aligned}\|\chi(\partial_z\beta)(\partial_z v_2)\|_{L^p(\Omega')} &\leq \|\chi\|_{L^p_{H,r}}\|\partial_z\beta\|_{L^\infty(-h,0)}\|\partial_z v_2\|_{L^\infty_H L^p_z(\Omega)} \\ &\leq C_p r^{2/p-1}\|\partial_z v_2\|_{L^\infty_H L^p_z(\Omega)}.\end{aligned}$$

Using the vertical Poincaré inequality (4.3.31), we estimate the fourth term via

$$\begin{aligned}\|(\Delta_H\chi)\beta v_2\|_{L^p(\Omega')} &\leq \|\Delta_H\chi\|_{L^p_{H,r}}\|\beta\|_{L^\infty(-h,0)}\|v_2\|_{L^\infty(G;L^p(-h,-h+3r))} \\ &\leq C_p r^{2/p-1}\|\partial_z v_2\|_{L^\infty_H L^p_z(\Omega)},\end{aligned}$$

and the fifth via

$$\begin{aligned}\|\chi(\partial_z^2\beta)\, v_2\|_{L^p(\Omega')} &\leq \|\chi\|_{L^p_{H,r}}\|\partial_z^2\beta\|_{L^\infty(-h,0)}\|v_2\|_{L^\infty(G;L^p(-h,-h+3r))} \\ &\leq C_p r^{2/p-1}\|\partial_z v_2\|_{L^\infty_H L^p_z(\Omega)}.\end{aligned}$$

By combining these estimates with (4.3.33), (4.3.34), and (4.3.35), and further plugging in the value $r = \eta|\lambda|^{-1/2}$, we obtain

$$\begin{aligned}|\lambda|^{1/2}\|\partial_z(\beta v_2)\|_{L^\infty_H L^p_z(\Omega)} \leq C_{\Omega',\theta,p}\Big(\eta^{2/p-1} + \eta^{3/p}|\lambda|^{-1/2p}(1+|\log(\eta|\lambda|^{-1/2})|)\Big)\|f\|_{L^\infty_H L^p_z(\Omega)} \\ + C_{\Omega',\theta,p}\eta^{2/p-1}|\lambda|^{1/2}\|\partial_z v_2\|_{L^\infty_H L^p_z(\Omega)}.\end{aligned} \tag{4.3.36}$$

Step 3: The estimate for $\partial_z v_2$. By combining (4.3.25) with (4.3.32) and (4.3.36), as well as taking the parameter $\eta > 1$ to be sufficiently large, compare the proof of Lemma 3.5.1, we obtain the estimate

$$\begin{aligned}|\lambda|^{1/2}\|\partial_z v_2\|_{L^\infty_H L^p_z(\Omega)} \leq C_{\theta,p,\lambda_0}\Big(\eta^{2/p-1}(1 + r^{1/p}|\log(r)|) \\ +\eta^{3/p}|\lambda|^{-1/2p}(1+|\log(|\lambda|)|)\Big)\|f\|_{L^\infty_H L^p_z(\Omega)}.\end{aligned}$$

Since for any $r_0, \lambda_0 > 0$ and $p \in (1,\infty)$ it holds that

$$\sup_{0<r<r_0} r^{1/p}|\log(r)| = C_p < \infty, \qquad \sup_{|\lambda|>\lambda_0} |\lambda|^{-1/2p}(1+|\log(|\lambda|)|) = C_{p,\lambda_0} < \infty,$$

we obtain

$$|\lambda|^{1/2}\|\partial_z v_2\|_{L^\infty_H L^p_z(\Omega)} \leq C_{\Omega,\theta,p,\lambda_0}\|f\|_{L^\infty_H L^p_z(\Omega)}, \quad |\lambda| \geq \lambda_0,$$

and combining this with estimate (4.3.20) yields

$$|\lambda|^{1/2}\|\partial_z v\|_{L^\infty_H L^p_z(\Omega)} \leq C_{\Omega,\theta,p,\lambda_0}\|f\|_{L^\infty_H L^p_z(\Omega)}, \quad |\lambda| \geq \lambda_0.$$

In order to prove this estimate for the full range $\lambda \in \Sigma_\theta$, we observe that we have $v := (\lambda - \mathcal{A}_{\infty,p})^{-1}\mathbb{P}f \in D(A_p)$ since $\mathbb{P}f \in L^p_{\overline{\sigma}}(\Omega)$ and therefore

$$\|v\|_{L^\infty_H L^p_z(\Omega)} \leq C_p\|v\|_{H^{2,p}(\Omega)} \leq C_p\|f\|_{L^p_{\overline{\sigma}}(\Omega)} \leq C_p\|f\|_{L^\infty_H L^p_z(\Omega)},$$

compare (4.3.21). Proceeding as in the proof of Lemma 3.5.1 by setting $\lambda_1 := \frac{\lambda_0}{|\lambda|}\lambda$ for $\lambda \in \Sigma_\theta$ such that $0 < |\lambda| < \lambda_0$ and using the equivalence

$$\lambda v - \mathcal{A}_{\infty,p}v = \mathbb{P}f \iff \lambda_1 v - \mathcal{A}_{\infty,p}v = \mathbb{P}f + (\lambda_1 - \lambda)v,$$

as well as $|\lambda_1| = \lambda_0$ and $|\lambda_1 - \lambda| \leq \lambda_0$, we then obtain

$$\begin{aligned}|\lambda|^{1/2}\|\partial_z v\|_{L^\infty_H L^p_z(\Omega)} &\leq \lambda_0^{1/2}\|\partial_z v\|_{L^\infty_H L^p_z(\Omega)} \\ &\leq C_{\Omega,\theta,p}\left(\|f\|_{L^\infty_H L^p_z(\Omega)} + |\lambda_1 - \lambda|\cdot\|v\|_{L^\infty_H L^p_z(\Omega)}\right), \\ &\leq C_{\Omega,\theta,p}(1 + C_p\lambda_0)\|f\|_{L^\infty_H L^p_z(\Omega)}\end{aligned}$$

for all $\lambda \in \Sigma_\theta$ with $0 < |\lambda| < \lambda_0$. This completes the proof. □

4.3.4 The estimate for $(\lambda - A)^{-1}\mathbb{P}\partial_z$

In this section we complete the proof of Theorem 4.3.2 by establishing an estimate for $S(t)\mathbb{P}\partial_z$. For this purpose, we consider the resolvent problem

$$\lambda v - Av = \mathbb{P}\partial_z f \quad \text{on } \Omega, \tag{4.3.37}$$

with boundary conditions (4.3.1). By Corollary 4.1.6 we know that this problem is well-posed in L^p. In order to establish a suitable estimate in $L^\infty_H L^p_z(\Omega)$, we utilize a horizontal cut-off function $\delta \in C_c^\infty(\mathbb{R}^2)$ supported in $G' = (-2,3)^2$. Using the notation for horizontal variables and open balls from (4.3.11), we set

$$\delta_\varepsilon(x') := \frac{1}{\varepsilon^2}\delta\left(\frac{x'}{\varepsilon}\right), \quad \delta_{\varepsilon,x_0'}(x') := \delta_\varepsilon(x' - x_0'), \quad \varepsilon > 0,\ x_0' \in G,$$

and employ the decomposition

$$\begin{aligned} |v(x',z)|^p = &\int_{G'} \big(|v(x',z)|^p - |v(y',z)|^p\big)\delta_{\varepsilon,x_0'}(y')\,dy' \\ &+ \int_{G'} |v(y',z)|^p \delta_{\varepsilon,x_0'}(y')\,dy'. \end{aligned} \tag{4.3.38}$$

Here we identified v with its periodic extension onto $G' \times (-h,0)$. Our approach to estimating these right-hand side terms is also used to obtain L^∞-error estimates for the finite element method, compare, e.g., [83]. The following lemma establishes an estimate for the first term.

Lemma 4.3.16. *Let $\theta \in (0,\pi)$ and $\lambda \in \Sigma_\theta$ with $|\lambda| > 1$, as well as $p \in (2,\infty)$ and $f \in L^\infty_H L^p_z(\Omega)^2$. Further let $\delta \in C_c^\infty(\mathbb{R}^2)$ be a smooth non-negative function satisfying*

$$\mathrm{supp}\,(\delta) \subset B(0;1), \qquad \int_{\mathbb{R}^2} \delta(x')\,dx' = 1,$$

and let $\varepsilon > 0$ be such that $\varepsilon^{1-2/p} = |\lambda|^{-1/2}$. Then there exists a constant $C = C_{\Omega,\theta,p} > 0$ such that the function

$$v := (\lambda - A_p)^{-1}\mathbb{P}\partial_z f$$

satisfies the estimate

$$\left|\int_{-h}^0 \int_{G'} \left(|v(x',z)|^p - |v(y',z)|^p\right)\delta_{\varepsilon,x_0'}(y')\,dy'dz\right| \le C|\lambda|^{-1/2}\|f\|_{L^\infty_H L^p_z(\Omega)}\|v\|^{p-1}_{L^\infty_H L^p_z(\Omega)}$$

for all $x_0' \in G$ and $x' \in B(x_0';\varepsilon)$.

Proof. The condition $|\lambda| > 1$ yields $\varepsilon \in (0,1)$ and thus $B(x_0';\varepsilon) \subset G'$ for all $x_0' \in G$. Since the rescaled cut-off function $\delta_{\varepsilon,x_0'}$ satisfies

$$\int_{\mathbb{R}^2} \delta_{\varepsilon,x_0'}(y')\,dy' = 1, \quad \mathrm{supp}\,\delta_{\varepsilon,x_0'} \subset B(x_0';\varepsilon),$$

it follows from Young's inequality for convolutions that for all $x' \in B(x_0';\varepsilon)$ we have

$$\left|\int_{-h}^0\int_{G'} (|v(x',z)|^p - |v(y',z)|^p)\, \delta_{\varepsilon,x_0'}(y')\, dy'dz\right| \leq \sup_{y'\in B(x_0';\varepsilon)} \int_{-h}^0 \big||v(x',z)|^p - |v(y',z)|^p\big|\, dz.$$

Observe that given $a,b\geq 0$, one has the inequality

$$|a^p - b^p| \leq p\max\{a,b\}^{p-1}|a-b| \leq p(a+b)^{p-1}|a-b| \leq p2^{p-2}(a^{p-1}+b^{p-1})|a-b|,$$

where we used the fact that $p\in(2,\infty)$ implies that the function

$$[0,1]\ni x\mapsto (2x)^{p-1} + (2(1-x))^{p-1}$$

has its minimum at $x = 1/2$. This yields the estimate

$$\big||v(x',z)|^p - |v(y',z)|^p\big| \leq C_p(|v(x',z)|^{p-1} + |v(y',z)|^{p-1})\big|v(x',z) - v(y',z)\big|$$

for all $x',y'\in B(x_0';\varepsilon)$. By applying Hölder's inequality in the vertical variable z we thus obtain

$$\int_{-h}^0 \big||v(x',z)|^p - |v(y',z)|^p\big|\, dz \leq C_p\left(\|v(x',\cdot)\|_{L_z^p}^{p-1} + \|v(y',\cdot)\|_{L_z^p}^{p-1}\right)\|v(x',\cdot) - v(y',\cdot)\|_{L_z^p},$$

for $x',y'\in B(x_0';\varepsilon)$, where we used the shorthand notation $\|\cdot\|_{L_z^p} := \|\cdot\|_{L^p(-h,0)}$. The first factor on this right-hand side can be estimated via

$$\|v(x',\cdot)\|_{L_z^p}^{p-1} + \|v(y',\cdot)\|_{L_z^p}^{p-1} \leq 2\|v\|_{L_H^\infty L_z^p(\Omega)}^{p-1}, \quad x'\in G,\ y'\in B(x_0';\varepsilon),$$

where we used the horizontal periodicity of v. To estimate the second factor we observe that

$$\begin{aligned}\|v\|_{H^{1,p}(\Omega)} &\leq C_{\Omega,p}\|\nabla v\|_{L^p(\Omega)}\\ &= C_{\Omega,p}\|\nabla(\lambda - A_p)^{-1}\mathbb{P}\partial_z f\|_{L^p(\Omega)}\\ &\leq C_{\Omega,\theta,p}\|f\|_{L^p(\Omega)}\\ &\leq C_{\Omega,\theta,p}\|f\|_{L_H^\infty L_z^p(\Omega)}\end{aligned}\tag{4.3.39}$$

where we used the vertical Poincaré inequality $\|v\|_{L^p(\Omega)}\leq C_{\Omega,p}\|\partial_z v\|_{L^p}$ in the first, the definition of v in the second, Corollary 4.1.6 in the third, and the embedding $L_H^\infty L_z^p(\Omega)\hookrightarrow L^p(\Omega)$ in the last step. Applying the Sobolev embedding $H^{1,p}(G)\hookrightarrow C^{1-2/p}(\overline{G})$, see, e.g., [28, Chapter 5.6, Theorem 5], then yields that

$$\|v\|_{C^{1-2/p}(\overline{G};L^p(-h,0))} \leq C_{\Omega,p}\|v\|_{H^{1,p}(\Omega)},$$

and so, using (4.3.39) and $x',y'\in B(x_0';\varepsilon)$ with $\varepsilon^{1-2/p} = |\lambda|^{-1/2}$, we obtain

$$\begin{aligned}\|v(x',\cdot) - v(y',\cdot)\|_{L_z^p} &\leq |x'-y'|^{1-2/p}\|v\|_{C^{1-2/p}(\overline{G};L^p(-h,0))}\\ &\leq C_{\Omega,p}|x'-y'|^{1-2/p}\|v\|_{H^{1,p}(\Omega)}\\ &\leq C_{\Omega,\theta,p}|x'-y'|^{1-2/p}\|f\|_{L_H^\infty L_z^p(\Omega)}\\ &\leq C_{\Omega,\theta,p}|\lambda|^{-1/2}\|f\|_{L_H^\infty L_z^p(\Omega)}.\end{aligned}$$

The claim then follows via the combination of these estimates. □

In order to estimate the second right-hand side term of (4.3.38), we observe that

$$\int_{-h}^{0}\int_{G'} |v(y',z)|^p \delta_{\varepsilon,x_0'}(y')\,dy' = \langle v, \delta_{\varepsilon,x_0'}|v|^{p-2}v^*\rangle, \tag{4.3.40}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard inner product of $L^2(\Omega')$ with domain $\Omega' = G' \times (-h,0)$ for $G' = (-2,3)^2$ and v^* denotes the complex conjugate of v. We will establish an estimate for this right-hand side using a duality argument. For this purpose, we begin by providing the following anisotropic estimate.

Lemma 4.3.17. *Let $p,q \in (1,\infty)$ satisfy $1/p+1/q=1$ and $1 \le s \le q$. Then there exists a constant $C = C_{\Omega,q,s} > 0$ such that*

$$\|\delta_{\varepsilon,x_0'}|v|^{p-2}v^*\|_{L^s(\Omega')} \le C\|\delta_{\varepsilon,x_0'}|v|^{p-2}v^*\|_{L^s_H L^q_z(\Omega')} \le C\varepsilon^{2/s-2}\|\delta\|_{L^s(\mathbb{R}^2)}\|v\|^{p-1}_{L^\infty_H L^p_z(\Omega')},$$

for all $\delta \in C_c^\infty(\mathbb{R}^2)$ and $v \in L^\infty_H L^p_z(\Omega')^2$ as well as $\varepsilon \in (0,1)$, $x_0' \in G$.

Proof. The first estimate is due to $L^q(-h,0) \hookrightarrow L^s(-h,0)$ for $s \le q$. For the second estimate, we observe that $|\delta_{\varepsilon,x_0'}|v|^p|^q = \delta^q_{\varepsilon,x_0'}|v|^p$. Since δ does not depend on the vertical variable z, applying Hölder's inequality in the horizontal variable $x' \in G'$ yields

$$\begin{aligned}
\|\delta_{\varepsilon,x_0'}|v|^{p-2}v^*\|_{L^s_H L^q_z(\Omega')} &= \left[\int_{G'}\left(\int_{-h}^{0}\delta_\varepsilon(x'-x_0')^q|v(x',z)|^p\,dz\right)^{s/q}dx'\right]^{1/s}\\
&= \left[\int_{G'}\delta_\varepsilon(x'-x_0')^s\left(\int_{-h}^{0}|v(x',z)|^p\,dz\right)^{s/q}dx'\right]^{1/s}\\
&\le \left(\int_{G'}\delta_\varepsilon(x'-x_0')^s\,dx'\right)^{1/s}\|v\|^{p/q}_{L^\infty_H L^p_z(\Omega')}\\
&\le \varepsilon^{2/s-2}\|\delta\|_{L^s(\mathbb{R}^2)}\|v\|^{p-1}_{L^\infty_H L^p_z(\Omega')}.
\end{aligned}$$

Thus, the claim follows. □

We now provide a dual estimate for an auxiliary problem. The proof is similar to that of Lemma 4.3.14 as they both utilize a pair of vertical cut-off functions. However, we do not need to introduce a horizontal cut-off function in this instance, which simplifies the proof.

Lemma 4.3.18. *Let $p \in (2,\infty)$ and $1/p+1/q=1$ as well as $\theta \in (0,\pi)$. Then there exists $\lambda_0 > 0$ such that for all $\lambda \in \Sigma_\theta$ with $|\lambda| > \lambda_0$, $v \in L^\infty_H L^p(\Omega')^2$, $\delta \in C_c^\infty(\mathbb{R}^2)$, $\varepsilon \in (0,1)$, and $x_0' \in G$, the problem*

$$\begin{aligned}
\lambda^* w - \Delta w + \nabla_H \Pi &= \delta_{\varepsilon,x_0'}|v|^{p-2}v^* && \text{on } \Omega',\\
\partial_z \Pi &= 0 && \text{on } \Omega',\\
\operatorname{div}_H \bar{w} &= 0 && \text{on } G',
\end{aligned}$$

with boundary conditions

$$\partial_z w = 0 \text{ on } \Gamma'_u, \quad w, \Pi \text{ periodic on } \Gamma'_l, \quad w = 0 \text{ on } \Gamma'_b,$$

has a unique pair of solutions $(w, \Pi) \in H^{2,q}(\Omega') \times H^{1,q}(G')$ *with* $\int_{G'} \Pi(x')\, dx' = 0$, *and there exists a constant* $C = C_{\Omega',\theta,q,s,\delta} > 0$ *such that*

$$|\lambda|^{1/2} \|\partial_z w\|_{L^1_H L^q_z(\Omega')} \le C \left(1 + |\lambda|^{-1/2q} \varepsilon^{2/s-2}\right) \|v\|^{p-1}_{L^\infty_H L^p_z(\Omega')},$$

for all $s \in (1, q]$, $\varepsilon \in (0,1)$ *and* $\lambda \in \Sigma_\theta$ *with* $|\lambda| > \lambda_0$.

Remark 4.3.19. Since we only seek to establish an $L^1_H L^q_z$-estimate for the solution w and Lemma 4.3.17 states that the right-hand side belongs to $L^s_H L^q_z(\Omega')$ for all $1 \le s \le q$, the factor we obtain on the right-hand side can be scaled differently depending on our choice of the auxiliary parameter s.

Proof of Lemma 4.3.18. Observe that the problem is equivalent to

$$\lambda^* w - A_q w = \mathbb{P} \delta_{\varepsilon, x'_0} |v|^{p-2} v^*,$$

compare the discussion concerning (4.0.7). Since the right-hand side belongs to $L^q(\Omega')^2$ by Lemma 4.3.17, the L^q-theory for the hydrostatic Stokes operator on the extended domain Ω' yields that the problem is well-posed. It remains to prove the estimate for $\partial_z w$. For this purpose we take an auxiliary parameter $\eta > 1$ which will be chosen later and let $\lambda_0 > 0$ be so large that $\eta \lambda_0^{-1/2} < 1$ and the requirement of Lemma 3.5.1 are satisfied. Setting $r := \eta |\lambda|^{-1/2}$ for $\lambda \in \Sigma_\theta$ with $|\lambda| > \lambda_0$, we then consider two vertical cut-off functions $\alpha = \alpha_r, \beta = \beta_r \in C^\infty([-h, 0])$ satisfying

$$\begin{aligned} &\alpha = 0 \text{ on } [-h, -h + r], \quad \alpha = 1 \text{ on } [-h + 2r, 0], \quad \|\partial_z^k \alpha\|_{L^\infty(-h,0)} \le C r^{-k}, \\ &\beta = 1 \text{ on } [-h, -h + 2r], \quad \beta = 0 \text{ on } [-h + 3r, 0], \quad \|\partial_z^k \beta\|_{L^\infty(-h,0)} \le C r^{-k}, \end{aligned}$$

for an absolute constant $C > 0$ and $k \in \{0, 1, 2\}$, compare (4.3.24). Analogously as for (4.3.25), the fact that $[-h, -h + 2r] \cup [-h + 2r, 0] = [-h, 0]$ yields the estimate

$$\|\partial_z w\|_{L^1_H L^q_z(\Omega')} \le \|\partial_z(\alpha w)\|_{L^1_H L^q_z(\Omega')} + \|\partial_z(\beta w)\|_{L^1_H L^q_z(\Omega')}. \tag{4.3.41}$$

We now estimate the right-hand side terms individually.

Step 1: In order to estimate $\partial_z(\alpha w)$, we observe that αw satisfies the equation

$$\lambda^* \alpha w - \Delta(\alpha w) = \alpha \delta_{\varepsilon, x'_0} |v|^{p-2} v^* - \alpha(\nabla_H \Pi) - 2(\partial_z \alpha)(\partial_z w) - (\partial_z^2 \alpha) w \quad \text{on } \Omega',$$

with boundary conditions

$$\partial_z(\alpha w) = 0 \text{ on } \Gamma'_u \cup \Gamma'_b, \quad \alpha w \text{ periodic on } \Gamma'_l.$$

By differentiating with respect to the vertical variable z, we obtain the equation

$$\lambda^* \partial_z(\alpha w) - \Delta(\partial_z(\alpha w)) = \partial_z F_1 + F_2 \text{ on } \Omega',$$

with right-hand sides

$$F_1 := \alpha\delta_{\varepsilon,x_0'}|v|^{p-2}v^* - 2(\partial_z\alpha)(\partial_z w) - (\partial_z^2\alpha)w, \quad F_2 := -(\partial_z\alpha)(\nabla_H\Pi),$$

and boundary conditions

$$\partial_z(\alpha w) = 0 \text{ on } \Gamma_u' \cup \Gamma_b', \quad \partial_z(\alpha w) \text{ periodic on } \Gamma_l'.$$

It follows from Lemma 3.5.1 in $L^1_H L^q_z(\Omega')$ for the case (DD) that

$$|\lambda|^{1/2}\|\partial_z(\alpha w)\|_{L^1_H L^q_z(\Omega')} \leq C_{\Omega',\theta}\left(\|F_1\|_{L^1_H L^q_z(\Omega')} + |\lambda|^{-1/2}\|F_2\|_{L^1_H L^q_z(\Omega')}\right), \tag{4.3.42}$$

for all $\lambda \in \Sigma_\theta$ with $|\lambda| > \lambda_0$. We now estimate the terms on the right-hand side as follows. For the first term in F_1, we apply Lemma 4.3.17 for $s = 1$ to obtain

$$\begin{aligned}\|\alpha\delta_{\varepsilon,x_0'}|v|^{p-2}v^*\|_{L^1_H L^q_z(\Omega')} &\leq \|\alpha\|_{L^\infty(-h,0)}\|\delta_{\varepsilon,x_0'}|v|^{p-2}v^*\|_{L^1_H L^q_z(\Omega')} \\ &\leq C_{\Omega,q,\delta}\|v\|^{p-1}_{L^\infty_H L^p_z(\Omega')},\end{aligned}$$

whereas the second term is simply estimated via

$$\|(\partial_z\alpha)(\partial_z w)\|_{L^1_H L^q_z(\Omega')} \leq Cr^{-1}\|\partial_z w\|_{L^1_H L^q_z(\Omega')}.$$

For the third term we apply the vertical Poincaré inequality (4.3.31), yielding

$$\begin{aligned}\|(\partial_z^2\alpha)w\|_{L^1_H L^q_z(\Omega')} &\leq Cr^{-2}\|w\|_{L^1(G';L^q(-h,-h+2r))} \\ &\leq Cr^{-1}\|\partial_z w\|_{L^1_H L^q_z(\Omega')}.\end{aligned}$$

In order to estimate F_2, we observe that the horizontal pressure gradient admits the representation

$$\nabla_H\Pi = (1 - Q_{G'})\overline{\delta_{\varepsilon,x_0'}|v|^{p-2}v^*} - B_{G'}w,$$

where $Q_{G'}$ is the Helmholtz projection with periodic boundary conditions associated with $G' = (-2,3)^2$ as opposed to G, and $B_{G'}$ is the operator defined in (4.0.6) with $Q_{G'}$ instead of Q. This yields

$$\|\nabla_H\Pi\|_{L^s(G')} \leq \|(1 - Q_{G'})\overline{\delta_{\varepsilon,x_0'}|v|^{p-2}v^*}\|_{L^s(G')} + \|B_{G'}w\|_{L^s(G')}.$$

We estimate the first term via

$$\|(1 - Q_{G'})\overline{\delta_{\varepsilon,x_0'}|v|^{p-2}v^*}\|_{L^s(G')} \leq C_{G',s}\|\overline{\delta_{\varepsilon,x_0'}|v|^{p-2}v^*}\|_{L^s(G')} \leq C_{\Omega',s}\|\delta_{\varepsilon,x_0'}|v|^{p-2}v^*\|_{L^s(\Omega')},$$

where we used that $Q_{G'}$ is bounded on $L^s(G')$ in the first and the L^s-boundedness of the vertical average in the second step. For the second term, the same arguments used to derive (4.3.21) yield

$$\|B_{G'}w\|_{L^s(G')} \leq C_{\Omega',s}\|w\|_{H^{2,s}(\Omega')} \leq C_{\Omega',\theta,s}\|\delta_{\varepsilon,x_0'}|v|^{p-2}v^*\|_{L^s(\Omega')},$$

so, using the embedding $L^s(G') \hookrightarrow L^1(G')$ and Lemma 4.3.17, we obtain

$$\begin{aligned}\|F_2\|_{L^1_H L^q_z(\Omega')} &= \|(\partial_z \alpha)(\nabla_H \Pi)\|_{L^1_H L^q_z(\Omega')} \\ &\le C_s \|\partial_z \alpha\|_{L^q(-h,0)} \|\nabla_H \Pi\|_{L^s(G')} \\ &\le C_{\Omega',q,s} r^{1/q-1} \|\delta_{\varepsilon,x_0'} |v|^{p-2} v^*\|_{L^s(\Omega')} \\ &\le C_{\Omega',q,s,\delta} r^{1/q-1} \varepsilon^{2/s-2} \|v\|^{p-1}_{L^\infty_H L^p_z(\Omega')}.\end{aligned}$$

By combining the estimates for F_1 and F_2 together with (4.3.42) and using $r = \eta|\lambda|^{-1/2}$, it follows that

$$\begin{aligned}|\lambda|^{1/2} \|\partial_z(\alpha w)\|_{L^1_H L^q_z(\Omega')} &\le C r^{-1} \|\partial_z w\|_{L^1_H L^q_z(\Omega')} \\ &\quad + C\left(1 + |\lambda|^{-1/2} r^{1/q-1} \varepsilon^{2/s-2}\right) \|v\|^{p-1}_{L^\infty_H L^p_z(\Omega')} \\ &= C(1 + \eta^{1/q-1} |\lambda|^{-1/2q} \varepsilon^{2/s-2}) \|v\|^{p-1}_{L^\infty_H L^p_z(\Omega)} \\ &\quad + C\eta^{-1} |\lambda|^{1/2} \|\partial_z w\|_{L^1_H L^q_z(\Omega')}\end{aligned} \tag{4.3.43}$$

for a constant $C = C_{\Omega',\theta,q,s,\delta} > 0$.

Step 2: In order to provide an estimate for $\partial_z(\beta w)$, we observe that βw satisfies the equation

$$\lambda^* \beta w - \Delta(\beta w) = F_3 \text{ on } \Omega',$$

with right-hand side

$$F_3 := \beta \delta_{\varepsilon,x_0'} |v|^{p-2} v^* - \beta \nabla_H \Pi - 2(\partial_z \beta)(\partial_z w) - (\partial_z^2 \beta) w,$$

and boundary conditions

$$\partial_z(\beta w) = 0 \text{ on } \Gamma_u', \quad \beta w \text{ periodic on } \Gamma_l', \quad \beta w = 0 \text{ on } \Gamma_b'.$$

It follows from Lemma 3.5.1 in $L^1_H L^q_z(\Omega')$ for the case (ND) that

$$|\lambda|^{1/2} \|\partial_z(\beta w)\|_{L^1_H L^q_z(\Omega')} \le C_{\Omega',\theta} \|F_3\|_{L^1_H L^q_z(\Omega')}, \quad \lambda \in \Sigma_\theta,\ |\lambda| > \lambda_0.$$

We estimate F_3 via the same arguments used in Step 1, yielding

$$\|F_3\|_{L^1_H L^q_z(\Omega')} \le C r^{-1} \|\partial_z w\|_{L^1_H L^q_z(\Omega')} + C(1 + r^{1/q} \varepsilon^{2/s-2}) \|v\|^{p-1}_{L^\infty_H L^p_z(\Omega')},$$

for a constant $C = C_{\Omega',q,s,\delta} > 0$. Since $r = \eta|\lambda|^{-1/2}$, it thus follows that

$$\begin{aligned}|\lambda|^{1/2} \|\partial_z(\beta w)\|_{L^1_H L^q_z} &\le C_{\Omega',\theta,q,s,\delta} (1 + \eta^{1/q} |\lambda|^{-1/2q} \varepsilon^{2/s-2}) \|v\|^{p-1}_{L^\infty_H L^p_z} \\ &\quad + C_{\Omega',\theta,q,s,\delta} \eta^{-1} |\lambda|^{1/2} \|\partial_z w\|_{L^1_H L^q_z}.\end{aligned} \tag{4.3.44}$$

Step 3: The estimate for $\partial_z w$. By combining (4.3.41) with (4.3.43) and (4.3.44), and taking the parameter $\eta > 1$ to be sufficiently large, we obtain the estimate

$$\begin{aligned}|\lambda|^{1/2}\|\partial_z w\|_{L^1_H L^q_z} &\leq C_{\Omega',\theta,q,s,\delta}\left(1+[\eta^{1/q}++\eta^{1/q-1}]|\lambda|^{-1/2q}\varepsilon^{2/s-2}\right)\|v\|^{p-1}_{L^\infty_H L^p_z(\Omega)}\\ &\quad + \eta^{-1}C_{\Omega',\theta,q,s,\delta}|\lambda|^{1/2}\|\partial_z w\|_{L^1_H L^q_z}\\ &\leq C_{\Omega',\theta,q,s,\delta,\eta}(1+|\lambda|^{-1/2q}\varepsilon^{2/s-2})\|v\|^{p-1}_{L^\infty_H L^p_z(\Omega)}\\ &\quad + \frac{1}{2}|\lambda|^{1/2}\|\partial_z w\|_{L^1_H L^q_z}.\end{aligned}$$

This implies the desired result. □

We are now able to prove the last component of Theorem 4.3.2.

Lemma 4.3.20. *Let $p \in (3,\infty)$ and $\theta \in (0,\pi)$. Then there exists a constant $C = C_{\Omega,\theta,p} > 0$ such that for all $\lambda \in \Sigma_\theta$ and $f \in L^\infty_H L^p_z(\Omega)^2$ it holds that*

$$|\lambda|^{1/2}\|(\lambda - A_{\infty,p})^{-1}\mathbb{P}\partial_z f\|_{L^\infty_H L^p_z(\Omega)} \leq C\|f\|_{L^\infty_H L^p_z(\Omega)}.$$

In particular, there exists a constant $C = C_{\Omega,p} > 0$ such that for all $t > 0$ it holds that

$$t^{1/2}\|S(t)\mathbb{P}\partial_z f\|_{L^\infty_H L^p_z(\Omega)} \leq C\|f\|_{L^\infty_H L^p_z(\Omega)}.$$

Proof. Let $v := (\lambda - \mathcal{A}_{\infty,p})^{-1}\mathbb{P}\partial_z f$. Since v is periodic in the horizontal variables, we have

$$\|v\|^p_{L^\infty_H L^p_z(\Omega)} = \sup_{x_0'\in G}\ \sup_{x'\in B(x_0';\varepsilon)} \int_{-h}^0 |v(x',z)|^p\,dz, \quad 0<\varepsilon<1, \tag{4.3.45}$$

as well as

$$\begin{aligned}\int_{-h}^0 |v(x',z)|^p\,dz &= \int_{-h}^0\int_{G'} (|v(x',z)|^p - |v(y',z)|^p)\,\delta_{\varepsilon,x_0'}(y')\,dy'dz\\ &\quad + \langle v, \delta_{\varepsilon,x_0'}|v|^{p-2}v^*\rangle\end{aligned} \tag{4.3.46}$$

for some fixed δ as in Lemma 4.3.16, compare (4.3.38) and (4.3.40). By Lemma 4.3.16 we directly have

$$\left|\int_{-h}^0\int_{G'} (|v(x',z)|^p - |v(y',z)|^p)\,\delta_{\varepsilon,x_0'}(y')\,dy'dz\right| \leq C_{\Omega,\theta,p}|\lambda|^{-1/2}\|f\|_{L^\infty_H L^p_z(\Omega)}\|v\|^{p-1}_{L^\infty_H L^p_z(\Omega)} \tag{4.3.47}$$

for all $x_0' \in G$ and $x' \in B(x_0';\varepsilon)$ with $\varepsilon^{1-2/p} = |\lambda|^{-1/2}$. To estimate the second term, we write

$$\langle v, \delta_{\varepsilon,x_0'}|v|^{p-2}v^*\rangle = \langle v, \lambda^* w - \Delta w + \nabla_H \Pi\rangle$$

where w and Π are as in Lemma 4.3.18. Integration by parts yields

$$\langle v, \lambda^* w - \Delta w + \nabla_H \Pi\rangle = \langle \lambda v - \Delta v + \nabla_H \pi, w\rangle = \langle \partial_z f, w\rangle = -\langle f, \partial_z w\rangle,$$

where we used in the first step that $\mathrm{div}_H \overline{v} = \mathrm{div}_H \overline{w} = 0$ and $\partial_z \pi = \partial_z \Pi = 0$ yield

$$\langle v, \nabla_H \Pi\rangle = \langle \nabla_H \pi, w\rangle = 0,$$

as well as the equivalence

$$\lambda v - Av = \mathbb{P}\partial_z f \iff \lambda v - \Delta v + \nabla_H = \partial_z f, \quad \mathrm{div}_H \overline{v} = 0,$$

for a suitable horizontal pressure gradient $\nabla_H \pi$ in the second step, compare (4.0.5) through (4.0.7). We also identified f with its periodic extension onto $\Omega' = G' \times (-h, 0)$ for $G' = (-2, 3)^2$. In the last step we further used that we may assume without loss of generality that $f = 0$ on $\Gamma_u' \cup \Gamma_b$ since $C_c^\infty(-h, 0)$ is dense in $L^p(-h, 0)$ for $1 \le p < \infty$. Using the anisotropic Hölder inequality and Lemma 4.3.18 we then obtain

$$\begin{aligned} \left|\langle v, \delta_{\varepsilon, x_0'} |v|^{p-2} v^*\rangle\right| &= |\langle f, \partial_z w\rangle| \\ &\le \|f\|_{L_H^\infty L_z^p(\Omega)} \|\partial_z w\|_{L_H^1 L^q(\Omega')} \\ &\le C_{\Omega',\theta,q,s,\delta} \left(1 + |\lambda|^{-1/2q} \varepsilon^{2/s-2}\right) |\lambda|^{-1/2} \|f\|_{L_H^\infty L_z^p(\Omega)} \|v\|^{p-1}_{L_H^\infty L_z^p(\Omega')} \end{aligned} \tag{4.3.48}$$

for all $\lambda \in \Sigma_\theta$ with $|\lambda| > \lambda_0$, $\varepsilon \in (0, 1)$, and $x_0' \in G$. As for the first term, we now chose $\varepsilon > 0$ such that $\varepsilon^{1-2/p} = |\lambda|^{-1/2}$ for $\lambda \in \Sigma_\theta$ with $|\lambda| > \lambda_0 > 1$ and further set

$$s = \min\left\{\frac{4p}{3p+2}, q\right\} > 1, \quad p \in (2, \infty).$$

Then we have

$$\left(1 - \frac{1}{s}\right) \le \frac{p-2}{4p},$$

yielding

$$\begin{aligned} -\frac{1}{2q} + \left(1 - \frac{2}{p}\right)^{-1} \left(-\frac{1}{2}\right) \left(\frac{2}{s} - 2\right) &= -\frac{1}{2} + \frac{1}{2p} + \left(1 - \frac{1}{s}\right) \frac{p}{p-2} \\ &\le -\frac{1}{2} + \frac{1}{2p} + \frac{1}{4} \\ &= -\frac{1}{4} + \frac{1}{2p} < 0, \quad p \in (2, \infty). \end{aligned}$$

It follows that $1 + |\lambda|^{-1/2q} \varepsilon^{2/s-2} \le 2$ for $|\lambda| > 1$ and so combining estimates (4.3.45) through (4.3.48) we obtain

$$\|v\|^p_{L_H^\infty L_z^p(\Omega)} \le C_{\Omega,\theta,p,s,\delta} |\lambda|^{-1/2} \|f\|_{L_H^\infty L_z^p(\Omega)} \|v\|^{p-1}_{L_H^\infty L_z^p(\Omega)}, \quad \lambda \in \Sigma_\theta,\ |\lambda| > \lambda_0,$$

where we used the dependence of Ω' and q on Ω and p, respectively. In order to obtain the full range $\lambda \in \Sigma_\theta$, we proceed as in the proof of Lemma 3.5.1 by setting

$$\lambda_1 := \frac{\lambda_0}{|\lambda|}\lambda, \quad \lambda \in \Sigma_\theta,\ 0 < |\lambda| < \lambda_0,$$

and using the equivalence

$$\lambda v - Av = \mathbb{P}\partial_z f \iff \lambda_1 v - Av = \mathbb{P}\partial_z f + (\lambda_1 - \lambda)v.$$

Then $0 < |\lambda| < \lambda_0 = |\lambda_1|$ yields

$$|\lambda|^{1/2}\|v\|_{L^\infty_H L^p_z(\Omega)} \le \lambda_0 \|v\|_{L^\infty_H L^p_z(\Omega)} \le C_{\Omega,\theta,p}\left(\|f\|_{L^\infty_H L^p_z(\Omega)} + \lambda_0^{-1/2}|\lambda_1 - \lambda| \cdot \|v\|_{L^\infty_H L^p_z(\Omega)}\right).$$

We estimate the right-hand side via $\lambda_0^{-1/2}|\lambda_1 - \lambda| \le \lambda_0^{1/2}$ as well as

$$\begin{aligned}
\|v\|_{L^\infty_H L^p_z(\Omega)} &\le C_{G,p}\|v\|_{H^{1,p}(G;L^p(-h,0))}, \\
&\le C_{G,p}\|v\|_{H^{1,p}(\Omega)} \\
&\le C_{\Omega,p}\|\nabla v\|_{L^p(\Omega)} \\
&= C_{\Omega,p}\|\nabla(\lambda - A_p)^{-1}\mathbb{P}\partial_z f\|_{L^p(\Omega)} \\
&\le C_{\Omega,\theta,p}\|f\|_{L^p(\Omega)} \\
&\le C_{\Omega,\theta,p}\|f\|_{L^\infty_H L^p_z(\Omega)},
\end{aligned}$$

where we used the embedding $H^{1,p}(G) \hookrightarrow L^\infty(G)$ for $p \in (2,\infty)$ in the first step, the vertical Poincaré inequality

$$\|v\|_{L^p(\Omega)} \le C_{\Omega,p}\|\partial_z v\|_{L^p(\Omega)} \le C_{\Omega,p}\|\nabla v\|_{L^p(\Omega)}, \quad v\big|_{\Gamma_b} = 0,$$

in the third, the definition of v in the fourth, Corollary 4.1.6 in the fifth, and the embedding $L^\infty_H L^p_z(\Omega) \hookrightarrow L^p(\Omega)$ in the sixth step. This yields

$$|\lambda|^{1/2}\|v\|_{L^\infty_H L^p_z(\Omega)} \le C_{\Omega,\theta,p}\|f\|_{L^\infty_H L^p_z(\Omega)}$$

for all $\lambda \in \Sigma_\theta$, thus completing the proof. □

5 Well-posedness of the primitive equations

Recall from the discussion in the beginning of Chapter 4 that the primitive equations can be equivalently formulated as

$$\begin{array}{rcll} \partial_t v - \Delta v + (v \cdot \nabla_H)v + w\partial_z v + \nabla_H \pi & = & f & \text{in } \Omega \times (0,\infty), \\ \partial_z \pi & = & 0 & \text{in } \Omega \times (0,\infty), \\ \mathrm{div}_H \overline{v} & = & 0 & \text{in } G \times (0,\infty), \\ v(0) & = & a & \text{in } \Omega, \\ v, \pi & & \text{periodic} & \text{on } \Gamma_l \times (0,\infty), \\ v & = & 0 & \text{on } \Gamma_D \times (0,\infty), \\ \partial_z v & = & 0 & \text{on } \Gamma_N \times (0,\infty), \end{array} \tag{5.0.1}$$

where the vertical velocity w is determined by v via the relation

$$w(x,y,z) = -\int_{-h}^{z} \mathrm{div}_H v(x,y,\xi)\, d\xi, \tag{5.0.2}$$

and $\overline{v} = \frac{1}{h}\int_{-h}^{0} v(\cdot,z)\,dz$ denotes the vertical average.

Our starting point for the study of the well-posedness of the problem (5.0.1) is the main result of Hieber and Kashiwabara in [49]. Following their terminology, a solution to the primitive equations (v,π) is called *strong*, if

$$\begin{aligned} & v \in C^1((0,\infty; L^p(\Omega))^2 \cap C((0,\infty); H^{2,p}(\Omega)^2, \\ & \pi \in C((0,\infty); H^{1,p}(G) \cap L^p_0(G)), \end{aligned} \tag{5.0.3}$$

where $L^p_0(G) := \{\pi \in L^p(G) : \int_G \pi\, d\mu = 0\}$. Considering the boundary conditions,

$$\partial_z v = 0 \text{ on } \Gamma_u, \quad v, w \text{ periodic on } \Gamma_l, \quad v = 0 \text{ on } \Gamma_b, \tag{5.0.4}$$

they proved the following.

Theorem 5.0.1. *[49, Theorem 6.1] Let $f = 0$, $p \in [6/5,\infty)$, and $a \in D((-A_p)^{1/p})$. Then there exists a unique, strong global solution (v,π) to the primitive equations* (5.0.1) *with boundary conditions* (5.0.4)*. Moreover, the solution (v,π) decays exponentially, i.e., there exist constants $C,\alpha,\beta > 0$ such that*

$$\|\partial_t v(t)\|_{L^p(\Omega)} + \|v(t)\|_{H^{2,p}(\Omega)} + \|\pi(t)\|_{H^{1,p}(G)} \le Ct^{-\alpha}e^{-\beta t}, \quad t > 0. \tag{5.0.5}$$

In this chapter, we apply the theory for the linear problem established in Chapter 4 to extend this result in a number of ways. The results presented here have been previously published in [38–40].

In Section 5.1, we utilize the property of maximal L^q-L^p-regularity for the hydrostatic Stokes operator in time-weighted spaces, compare Section 2.7.2 and [79]. We obtain the existence of a unique global solution v for initial data

$$a \in (L^p_{\overline{\sigma}}(\Omega), D(A_p))_{\mu-1/q,q} \subset B^{2\mu-2/q}_{p,q,\mathrm{per}}(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega),$$

for parameters $p, q \in (1, \infty)$ and $\mu \in (0, 1]$ with $1/p+1/q \leq \mu \leq 1$. Recall that these real interpolation spaces were characterized in Corollary 4.1.4. We are even able to consider a given external force f where

$$\mathbb{P}f \in L^q_\mu(0, T; L^p_{\overline{\sigma}}(\Omega)).$$

The solution we obtain belongs to the natural solution space for maximal L^q-L^p-regularity, i.e., we have that

$$v \in H^{1,q}(0, T; L^p_{\overline{\sigma}}(\Omega)) \cap L^q(0, T; D(A_p)).$$

However, under the assumption of additional regularity in time and space variables for $\mathbb{P}f$ we also prove additional regularity for the solution. We even show that v is real analytic on $\Omega \times (0, T)$ for $T > 0$ if $\mathbb{P}f$ is real analytic as well, so in particular for the case $f = 0$. As in Section 3.4, we consider the general boundary conditions

$$\Gamma_D \in \{\emptyset, \Gamma_u, \Gamma_b, \Gamma_u \cup \Gamma_b\}, \quad \Gamma_N = (\Gamma_u \cup \Gamma_b) \setminus \Gamma_D.$$

In Section 5.2, we consider the case $\Gamma_N = \Gamma_u \cup \Gamma_b$ and $f = 0$. Given $p \in [1, \infty]$, we show well-posedness for $L^\infty_H L^p_z$-initial data of the form

$$a = a_1 + a_2,$$

where a_1 is continuous in the horizontal variables and a_2 is a small perturbation. In Section 5.3, we instead consider the boundary conditions (5.0.4) and the range $p \in (3, \infty)$.

Our approach in these two sections is based on the classical approach for the Navier-Stokes equations of Fujita and Kato, as well as Kato and Giga. First, we use an iteration scheme to construct a local mild solution, i.e., a function v satisfying

$$v(t) = S(t)a - \int_0^t S(t-s)\mathbb{P}\left((v(s) \cdot \nabla_H)v(s) + w(s)\partial_z v(s)\right) ds, \tag{5.0.6}$$

for some $T \in (0, \infty]$ and all $t \in (0, T)$. Here, S denotes the hydrostatic Stokes semigroup from Theorem 4.2.1 or 4.3.2, depending on the choice of boundary conditions. The semigroup estimates obtained in these theorems are the key element of this procedure. While in Section 5.2 we construct v directly, in Section 5.3 we instead decompose the initial data into $a = a_{\mathrm{ref}} + a_0$, where a_{ref} is smooth and a_0 is small. Introducing an auxiliary reference solution v_{ref} to the primitive equations with initial data $v_{\mathrm{ref}}(0) = a_{\mathrm{ref}}$, we then obtain the desired solution v by constructing the difference $V := v - v_{\mathrm{ref}}$. The smallness condition for the perturbation part a_2 arises since we require the left-hand side of the inequality

$$\limsup_{t\to 0} t^{1/2} \|\nabla S(t) a_2\|_{L^\infty_H L^p_z} \leq C \|a_2\|_{L^\infty_H L^p_z}$$

to be small, compare estimate (i) and (ii) in Theorem 4.2.1 and 4.3.2, respectively, whereas we have that the left-hand side vanishes for the horizontally continuous part a_1 instead of a_2 by point (d) of these theorems.

We then show that the solution regularizes for $t > 0$. This allows us to consider the new, more regular initial data $v(t_0)$ for arbitrary $t_0 > 0$ and obtain a global, strong solution on (t_0, ∞) using well-posedness in L^p-spaces for horizontally periodic data. By proving the uniqueness of mild solutions, we then obtain that v is strong for $t \in (0, \infty)$. The surface pressure π can then be reconstructed from v by solving the problem

$$\Delta_H \pi = -\mathrm{div}_H \overline{((v \cdot \nabla_H) v + w \partial_z v)} \tag{5.0.7}$$

on $\mathbb{R}^2$ and $G = (0,1)^2$ with periodic boundary conditions, respectively.

5.1 The maximal L^q-regularity approach

In this section, we consider the primitive equations (5.0.1) in the presence of a given external force $f : \Omega \times (0,T)$ for some $T > 0$, i.e.,

$$\begin{array}{rcll} \partial_t v - \Delta v + (v \cdot \nabla_H) v + w \partial_z v + \nabla_H \pi &=& f & \text{in } \Omega \times (0,T), \\ \partial_z \pi &=& 0 & \text{in } \Omega \times (0,T), \\ \mathrm{div}_H \overline{v} &=& 0 & \text{in } G \times (0,T), \\ v(0) &=& a & \text{in } \Omega, \\ v, \pi & & \text{periodic} & \text{on } \Gamma_l \times (0,\infty), \\ v &=& 0 & \text{on } \Gamma_D \times (0,\infty), \\ \partial_z v &=& 0 & \text{on } \Gamma_N \times (0,\infty). \end{array} \tag{5.1.1}$$

One of the big draws of the maximal-regularity approach is that it allows us to include such right-hand sides, assuming they belong to a suitable regularity class. The results of this section have been previously published in [38].

5.1.1 Main results

Recall that in Corollary 4.1.4 we explicitly characterized the real interpolation spaces $(L^p_{\overline{\sigma}}(\Omega), D(A_p))_{\vartheta,q}$, as well as the definition of the spaces $L^p_\mu(I;X)$ and $H^{k,p}_\mu(I;X)$ from (2.7.3). The first main result of this section is that of global well-posedness of the primitive equations

Theorem 5.1.1. *Let $T > 0$, $p, q \in (1,\infty)$ with $1/p + 1/q \leq 1$, and $\mu \in [1/p + 1/q, 1]$. Then for every $a \in (L^p_{\overline{\sigma}}(\Omega), D(A_p))_{\mu-1/q,q}$ and $f \colon \Omega \times (0,T) \to \mathbb{C}^2$ such that*

$$\mathbb{P} f \in H^{1,q}_\mu(0,T; L^p_{\overline{\sigma}}(\Omega)) \cap H^{1,2}(\delta, T; L^2_{\overline{\sigma}}(\Omega))$$

for sufficiently small $\delta \in (0,T)$, there exists a unique solution

$$v \in H^{1,q}_\mu(0,T; L^p_{\overline{\sigma}}(\Omega)) \cap L^q_\mu(0,T; D(A_p))$$

to the primitive equations (5.1.1).

Recall that, in [49], Hieber and Kashiwabara established well-posedness for initial data belonging to a closed subspace of $H^{2/p,p}_{\mathrm{per}}(\Omega)^2$ and observe that the choice of the critical time-weight $\mu := 1/p + 1/q \leq 1$ yields well-posedness for initial data belonging to a closed subspace of $B^{2/p}_{p,q,\mathrm{per}}(\Omega)^2$. Taking $q \geq \max\{2, p\}$, we then have the chain of inclusions

$$H^{2/p,p}_{\mathrm{per}}(\Omega)^2 = F^{2/p}_{p,2,\mathrm{per}}(\Omega)^2 \hookrightarrow F^{2/p}_{p,q,\mathrm{per}}(\Omega)^2 \hookrightarrow B^{2/p}_{p,q,\mathrm{per}}(\Omega)^2,$$

where we used [89, Section 2.3.2, Proposition 2]. This means that the maximal L^q-regularity approach allows us to improve upon their result for L^p-spaces by not only dealing with the existence of external forces, but we are also able to consider a larger set of initial data.

The case $p = q = 2$ with $\mu = 1$ is also of particular interest, since it corresponds to initial data

$$a \in (L^2_{\overline{\sigma}}(\Omega), D(A_2))_{1/2,2} \subset B^1_{2,2,\mathrm{per}}(\Omega)^2 = H^1_{\mathrm{per}}(\Omega)^2,$$

which is the same regularity class of initial data considered in [19,46]. We are further able to expand the L^2-theory by the following result for the time derivative. As in [49], we will use the global existence in the L^2-setting to prove the global existence in Theorem 5.1.1.

Lemma 5.1.2. *Let $T \in (0, \infty)$. Then for every $a \in \{H^1_{per}(\Omega)^2 : v\big|_{\Gamma_D} = 0\} \cap L^2_{\overline{\sigma}}(\Omega)$ and $f\colon \Omega \times (0,T) \to \mathbb{C}^2$ such that $\mathbb{P}f \in L^2(0,T; L^2_{\overline{\sigma}}(\Omega))$, there exists a unique solution*

$$v \in H^1(0,T; L^2_{\overline{\sigma}}(\Omega)) \cap L^2(0,T; D(A_2))$$

to the primitive equations (5.1.1)*. If in addition it holds that $t \cdot \partial_t \mathbb{P}f \in L^2(0,T; L^2_{\overline{\sigma}}(\Omega))$, then it further holds that*

$$t \cdot \partial_t v \in H^1(0,T; L^2_{\overline{\sigma}}(\Omega)) \cap L^2(0,T; D(A_2)).$$

Remark 5.1.3. The space

$$\{H^1_{\mathrm{per}}(\Omega)^2 : v\big|_{\Gamma_D} = 0\} \cap L^2_{\overline{\sigma}}(\Omega) = D((-A_2)^{1/2}) = (L^2_{\overline{\sigma}}(\Omega), D(A_2))_{1/2,2}$$

has been explicitly characterized via form methods in [49, Proposition 4.7].

It turns out that the solution even admits additional regularity under the assumption that f does so as well. For this purpose, let C^ω denote the set of real analytic functions. Then we have the following.

Theorem 5.1.4. *Let $T > 0$, $p, q \in (1, \infty)$ with $1/p + 1/q \leq 1$, and $\mu \in [1/p + 1/q, 1]$. Further let*

$$v \in H^{1,q}_\mu(0,T; L^p_{\overline{\sigma}}(\Omega)) \cap L^q_\mu(0,T; D(A_p))$$

be a solution to the primitive equations (5.1.1) *with data*

$$a \in (L^p_{\overline{\sigma}}(\Omega), D(A_p))_{\mu-1/q,q}, \quad \mathbb{P}f \in L^q_\mu(0,T; L^p_{\overline{\sigma}}(\Omega)).$$

Then the following holds.

(a) If $\mathbb{P}f \in H^{k,q}_\mu(0,T; L^p_{\overline{\sigma}}(\Omega))$ for some $k \in \mathbb{N}$, then v satisfies

$$v \in H^{k+1,q}_{loc}(0,T; L^p_{\overline{\sigma}}(\Omega)) \cap H^{k,q}_{loc}(0,T; D(A_p)) \cap C^k((0,T); (L^p_{\overline{\sigma}}(\Omega), D(A_p))_{1-1/q,q})$$

as well as

$$t^j \partial_t^j v \in H^{1,q}_\mu(0,T'; L^p_{\overline{\sigma}}(\Omega)) \cap L^q_\mu(0,T'; D(A_p))$$

for all $1 \leq j \leq k$ and $T' \in (0,T)$.

(b) If $\mathbb{P}f \in C^\infty((0,T); L^p_{\overline{\sigma}}(\Omega))$ or $\mathbb{P}f \in C^\omega((0,T); L^p_{\overline{\sigma}}(\Omega))$, then v satisfies

$$v \in C^\infty((0,T); D(A_p)), \quad v \in C^w((0,T); D(A_p)),$$

respectively.

(c) *If* $\mathbb{P}f \in C^\infty((0,T;C^\infty_{per}(\Omega))^2$ *or* $\mathbb{P}f \in C^\omega((0,T);C^\omega_{per}(\Omega))^2$*, then* v *satisfies*

$$v \in C^\infty((0,T);C^\infty_{per}(\Omega))^2, \quad v \in C^\omega((0,T);C^\omega_{per}(\Omega))^2,$$

respectively.

Let us shed some light on the assumption on the regularity of f in Theorem 5.1.1. In order to obtain local well-posedness, we only require the weaker condition

$$\mathbb{P}f \in L^q_\mu(0,T;L^p_{\overline{\sigma}}(\Omega)),$$

see Lemma 5.1.12. However, in order to extend the solution onto the entire time interval $(0,T)$, we require the additional regularity of the external force in order to use the results of Lemma 5.1.2 and Theorem 5.1.4 to obtain the global existence and regularity of the solution.

Remark 5.1.5. Hieber and Kashiwabara also proved maximal Hölder regularity for local solutions to the primitive equations, see [49, Remark 5.7, Proposition 5.8]. However, unlike Theorem 5.1.4, terms of higher order are not considered.

5.1.2 Maximal regularity theory for semilinear evolution equations

Before we are able to prove our main results, we give a brief overview on the maximal regularity approach to quasilinear evolution equations developed in [82].

Throughout this section, let X_0 and X_1 be complex Banach spaces such that X_1 continuously embeds into and is dense in X_0. Given a bounded linear operator A and a function F from X_1 into X_0, we consider the semilinear Cauchy problem

$$\partial_t - Av = F(v) + f \text{ on } (0,T), \quad u(0) = a, \tag{5.1.2}$$

for given a and $f\colon (0,T) \to X_0$. For brevity we use the notations

$$\mathbb{E}_{0,\mu}(0,T) := L^q_\mu(0,T;X_0), \quad \mathbb{E}_{1,\mu}(0,T) := H^{1,q}_\mu(0,T;X_0) \cap L^q_\mu(0,T;X_1),$$

as well as

$$X_{\gamma,\mu} := (X_0,X_1)_{\mu-1/q,q}, \quad \mu \in (1/q,1], \quad X_\vartheta := [X_0,X_1]_\vartheta, \quad \vartheta \in [0,1].$$

The following result is a special case of [82, Theorem 1.2] by Prüss and Wilke concerning the existence of local solutions for the case $f = 0$. However, the proof can be modified to include general right-hand sides, compare the proof of Lemma 5.1.13.

Theorem 5.1.6. *Assume that for* $q \in (1,\infty)$ *and* $\mu \in (1/q,1]$ *it holds that*

(H1) *the operator* A *has maximal* L^q*-regularity;*

(H2) *there exists* $\vartheta \in [0,1]$ *such that* $2\vartheta - (\mu - 1/q) \leq 1$ *and the mapping* $F\colon X_\vartheta \to X_0$ *is continuous and satisfies the local Lipschitz-estimate*

$$\|F(v_1) - F(v_2)\|_{X_0} \leq C_\vartheta \left(\|v_1\|_{X_\vartheta} + \|v_2\|_{X_\vartheta}\right) \|v_1 - v_2\|_{X_\vartheta}$$

for all $v_1, v_2 \in X_\vartheta$*;*

(S) X_0 *is a UMD space and it holds that*

$$H^{1,q}(\mathbb{R}; X_0) \cap L^q(\mathbb{R}; X_1) \hookrightarrow H^{1-\vartheta,q}(\mathbb{R}; X_\vartheta)$$

for all $\vartheta \in (0,1)$*.*

Then for all $a \in X_{\gamma,\mu}$ *and* $f \in \mathbb{E}_{0,\mu}(0,T)$ *there exists a time* $T' = T'(a,f) \in (0,T]$ *such that the problem* (5.1.2) *has a unique solution* $v \in \mathbb{E}_{1,\mu}(0,T')$*.*

The existence of global solutions has been proven by Prüss and Simonett in [80] under the assumption of of suitable *a priori* bounds. While they did not consider the issue of time-weights, the result nevertheless remains valid in this setting and the proof is the same.

Theorem 5.1.7. *[80, Theorem 5.7.1] Under the assumptions of Theorem 5.1.6, let*

$$T^* := \sup\{T' > 0 : \textit{the problem } (5.1.2) \textit{ has a solution } v \in \mathbb{E}_{1,\mu}(0,T')\}.$$

Then, if there exists $\overline{\mu} \in (\mu, 1]$ *such that the embedding* $X_{\gamma,\overline{\mu}} \hookrightarrow X_{\gamma,\mu}$ *is compact and there exists* $\delta \in (0,T)$ *such that* $v \in C_b([\delta, T^*); X_{\gamma,\overline{\mu}})$*, then the solution extends onto* $(0,T)$*, i.e., there exists a global solution.*

Finally, concerning the issue of additional regularity of the solution for sufficiently regular right-hand sides, the proof of Theorem 5.1.4 is based on the implicit function theorem, compare [7, Chapter VII.8] for a perspective on the vector-valued case.

This method originated with an argument of Masuda, who in [73] proved the analyticity of solutions to the Navier-Stokes equations in the spatial variables by introducing additional parameters. A general theory for quasilinear evolution equations was established by Angenent, see [9, 10]. Here, we take the approach of Prüss and Simonett, who included maximal regularity into this method, compare, e.g., [80, Section 9.4]. In order to include the time-dependent external force f, we make use of the following version of [21, Theorem 9.1], adapted to the setting of time-weighted spaces.

Theorem 5.1.8. *Given* $q \in (1,\infty)$*,* $\mu \in (1/q, 1]$*,* $k \in \mathbb{N}$*, let* $\mathbf{F}\colon X_1 \to X_0$ *be continuously differentiable and*

$$\mathcal{F}\colon \mathbb{E}_{1,\mu}(0,T) \to \mathbb{E}_{0,\mu}(0,T), \quad \mathcal{F}(v) := \mathbf{F} \circ v$$

be k-times continuously differentiable. Further suppose that $v \in \mathbb{E}_{1,\mu}(0,T)$ *and* $f \in H^{k,q}_\mu(0,T; X_0)$ *satisfy*

$$\partial_t v + \mathcal{F}(v) = f \;\; \textit{on } (0,T)$$

and that for every $T' \in (0,T)$ and $g \in \mathbb{E}_{0,\mu}(0,T)$ the problem

$$\partial_t u + \partial_v \mathcal{F}(v)u = g, \quad u(0) = 0,$$

has a unique solution $u \in \mathbb{E}_{1,\mu}(0,T)$. Then it holds that

$$v \in H^{k+1,q}_{loc}(0,T';X_0) \cap H^{k,q}_{loc}(0,T';X_1) \quad \text{as well as} \quad [t \mapsto t^j \partial_t^j v(t)] \in \mathbb{E}_{1,\mu}(0,T')$$

for all $1 \leq j \leq k$. If it additionally holds that $\mathcal{F}$ and f belong to C^∞ or C^ω, then it respectively holds that $v \in C^\infty((0,T);X_1)$ and $v \in C^\omega((0,T);X_1)$.

Remark 5.1.9. If we were to restrict ourselves to the case $f = 0$ then the result of [78, Theorem 5.1] would be sufficient for our applications. However, in order to cover the case of non-vanishing external forces we need the weaker assumptions of [21, Theorem 9.1].

5.1.3 Local well-posedness

From now on we use the notations $X_0 := L^p_{\overline{\sigma}}(\Omega)$, $X_1 := D(A_p)$, as well as

$$X^p_\vartheta := [L^p_{\overline{\sigma}}(\Omega), D(A_p)]_\vartheta, \quad X^{q,p}_{\gamma,\mu} := (L^p_{\overline{\sigma}}(\Omega), D(A_p))_{\mu-1/q,q},$$

and further set

$$\mathbb{E}^{q,p}_{0,\mu}(0,T) := L^q_\mu(0,T;L^p_{\overline{\sigma}}(\Omega)), \quad \mathbb{E}^{q,p}_{1,\mu}(0,T) := H^{1,q}_\mu(0,T;L^p_{\overline{\sigma}}(\Omega)) \cap L^q_\mu(0,T;D(A_p)).$$

We now consider the bilinear mapping

$$F(v_1,v_2) := -\mathbb{P}\left((v_1 \cdot \nabla_H)v_2 + w_1 \partial_z v_2\right),$$

where $w_i = w(v_i)$ as in (5.0.2), and further set $F(v) = F(v,v)$. These mappings are subject to the following estimates.

Lemma 5.1.10. *[49, Lemma 5.1] Let $p \in (1,\infty)$. Then the following holds:*

1. *There exists a constant $C = C(\Omega,p) > 0$ such that*

$$\|F(v_1,v_2)\|_{L^p(\Omega)} \leq C\|v_1\|_{H^{1+1/p,p}(\Omega)}\|v_2\|_{H^{1+1/p,p}(\Omega)}$$

   for all $v_1, v_2 \in H^{1+1/p,p}(\Omega)^2$.

2. *Let $\vartheta = \frac{1}{2}(1+1/p)$. Then there exists a constant $C = C(\Omega,p) > 0$ such that*

$$\|F(v_1) - F(v_2)\|_{L^p_{\overline{\sigma}}(\Omega)} \leq C\left(\|v_1\|_{X^p_\vartheta} + \|v_2\|_{X^p_\vartheta}\right)\|v_1 - v_2\|_{X^p_\vartheta}.$$

Remark 5.1.11. One can even show that F defines a continuous mapping

$$F\colon H^{s+1+1/p,p}(\Omega)^2 \times H^{s+1+1/p,p}(\Omega)^2 \to H^{s,p}(\Omega)^2, \quad s > 0,$$

and utilize the resolvent mapping property from Lemma 4.1.8 to obtain an alternative proof of additional regularity for solutions to the primitive equations, see [38, Lemma 6.1 and Remark 6.5] for details.

We are now able to prove local well-posedness of the primitive equations using maximal regularity.

Lemma 5.1.12. *Let $T > 0$, $p, q \in (1, \infty)$ with $1/p + 1/q \leq 1$, and $\mu \in [1/p + 1/q, 1]$. Then for all*

$$a \in X^{q,p}_{\gamma,\mu} \quad \text{and} \quad \mathbb{P}f \in \mathbb{E}^{q,p}_{0,\mu}(0,T)$$

there exists $T' = T'(a, f) \in (0, T]$ such that the primitive equations (5.1.1) *have a unique solution $v \in \mathbb{E}^{q,p}_{1,\mu}(0, T')$.*

Proof. It suffices to verify the conditions of Theorem 5.1.6. Condition (H1) holds for A_p on $(0, T)$ for $T \in (0, \infty)$ by Corollary 4.1.3, whereas (H2) for $X^p_\vartheta = [L^p_{\overline{\sigma}}(\Omega), D(A_p)]_\vartheta$ follows from Lemma 5.1.10 and Corollary 4.1.5. Following [82, Remark 1.1], we observe that (S) holds, even for general intervals $I \subset \mathbb{R}$, for $X_0 = L^p_{\overline{\sigma}}(\Omega)$ and $X_1 = D(A_p)$ since closed subspaces or $L^p(\Omega)$ are UMD spaces and the embedding is valid since $\lambda - A_p$ for $\lambda > 0$ has a bounded $\mathcal{H}^\infty$-calculus of angle $\phi^\infty_{-A_p} = 0$. □

5.1.4 Additional regularity

Lemma 5.1.13. *Let $T > 0$, $p, q \in (1, \infty)$ with $1/p + 1/q \leq 1$, and $\mu \in [1/p + 1/q, 1]$. Then the following holds.*

1. *For every $T > 0$, the mapping*

$$\mathcal{F} \colon \mathbb{E}^{q,p}_{1,\mu}(0,T) \to \mathbb{E}^{q,p}_{0,\mu}(0,T), \quad \mathcal{F}v := A_p v + F(v)$$

   is infinitely continuously differentiable and analytic with

$$\partial_v \mathcal{F}(v)u = A_p u + F(v, u) + F(u, v).$$

2. *For all $v \in \mathbb{E}^{q,p}_{1,\mu}(0,T)$ and $g \in \mathbb{E}^{q,p}_{0,\mu}(0,T)$ the problem*

$$\partial_t u - \partial_v \mathcal{F}(v)u = g, \quad u(0) = 0,$$

   has a unique solution $u \in \mathbb{E}^{q,p}_{1,\mu}(0,T)$ satisfying

$$\|u\|_{\mathbb{E}^{q,p}_{1,\mu}(0,T)} \leq B(v, g).$$

Proof.

In order to prove the first point, we observe that since $F(\cdot, \cdot)$ is bilinear, we have that

$$\mathcal{F}(v + u) - \mathcal{F}(h) = A_p u + F(v, u) + F(u, v) + F(u, u)$$

for all $v, u \in \mathbb{E}^{q,p}_{1,\mu}(0,T)$. We now show that

$$\lim_{\|u\|_{\mathbb{E}^{q,p}_{1,\mu}(0,T)} \to 0} \frac{\|F(u, u)\|_{\mathbb{E}^{q,p}_{0,\mu}(0,T)}}{\|u\|_{\mathbb{E}^{q,p}_{1,\mu}(0,T)}} = 0. \tag{5.1.3}$$

Following [82], we decompose $u = \tilde{u} + u^*$ into the auxiliary functions

$$u^*(t) := S(t)u(0), \quad \tilde{u}(t) := u(t) - u^*(t), \quad t \geq 0,$$

where S denotes the hydrostatic Stokes semigroup on $L^p_{\overline{\sigma}}(\Omega)$, which satisfy the initial conditions $u^*(0) = u(0)$ and $\tilde{u}(0) = 0$. By [80, Proposition 3.4.2] we then have $u^*, \tilde{u} \in \mathbb{E}^{q,p}_{1,\mu}(0,T)$ with

$$\|u^*\|_{\mathbb{E}^{q,p}_{1,\mu}(0,T)} \leq C_{q,\mu}\|u(0)\|_{X^{q,p}_{\gamma,\mu}} \leq C_{q,\mu,T}\|u\|_{\mathbb{E}^{q,p}_{1,\mu}(0,T)}, \tag{5.1.4}$$

and thus

$$\|\tilde{u}\|_{\mathbb{E}^{q,p}_{1,\mu}(0,T)} \leq \|u^*\|_{\mathbb{E}^{q,p}_{1,\mu}(0,T)} + \|u\|_{\mathbb{E}^{q,p}_{1,\mu}(0,T)} \leq C_{q,\mu,T}\|u\|_{\mathbb{E}^{q,p}_{1,\mu}(0,T)}. \tag{5.1.5}$$

We further consider the auxiliary parameters

$$\vartheta := (1 + \mu - 1/q)/2 \in (0,1), \quad \sigma := (1 + \mu)/2 \in (0,1].$$

Then it holds that $\vartheta - 1 + \sigma - 1/2q = \mu - 1/q > 0$ and so by [80, Proposition 3.4.3] and $X^p_\vartheta = D((-A_p)^\vartheta)$ we obtain $u^* \in L^{2q}_\sigma(0,T;X^p_\vartheta)$ with

$$\|u^*\|_{L^{2q}_\sigma(0,T;X^p_\vartheta)} \leq C_{q,\mu}\|u(0)\|_{X^{q,p}_{\gamma,\mu}} \leq C_{q,\mu,T}\|u\|_{\mathbb{E}^{q,p}_{1,\mu}(0,T)}, \tag{5.1.6}$$

whereas for $\tilde{u}$ we utilize the embeddings

$$\begin{aligned}\{u \in \mathbb{E}^{q,p}_{1,\mu}(0,T) : v(0) = 0\} &\hookrightarrow \{u \in H^{1-\vartheta,q}_\mu(0,T;X^p_\vartheta) : v(0) = 0\}\\ &\hookrightarrow \{u \in H^{1-\vartheta-1/2q,2q}_\mu(0,T;X^p_\vartheta) : v(0) = 0\}\\ &\hookrightarrow L^{2q}_\sigma(0,T;X^p_\vartheta),\end{aligned} \tag{5.1.7}$$

compare the proof of [82, Theorem 1.2], where the fractional time-weighted spaces are defined via complex interpolation as

$$H^{s,q}_\mu(0,T;X^p_\vartheta) := \left[L^p_\mu(0,T;X^p_\vartheta), H^{1,q}_\mu(0,T;X^p_\vartheta)\right]_s, \quad s \in (0,1),$$

compare [80, Chapter 3, Section 4.5] for more details. It follows from (5.1.5) that

$$\|\tilde{u}\|_{L^{2q}_\sigma(0,T;X^p_\vartheta)} \leq C_{q,\mu,T}\|\tilde{u}\|_{E^{q,p}_{1,\mu}(0,T)} \leq C_{q,\mu,T}\|u\|_{E^{q,p}_{1,\mu}(0,T)}. \tag{5.1.8}$$

Since Hölder's inequality yields

$$\|f_1 \cdot f_2\|_{L^q_\mu(0,T)} \leq \|f_1\|_{L^{2q}_\sigma(0,T)}\|f_2\|_{L^{2q}_\sigma(0,T)},$$

we obtain from Lemma 5.1.10 that

$$\begin{aligned}\|F(v_1,v_2)\|_{\mathbb{E}^{q,p}_{0,\mu}(0,T)} &\leq C\left(\int_0^T \left[t^{1-\mu}\|v_1(t)\|_{X^p_{1/2+1/2p}}\|v_2(t)\|_{X^p_{1/2+1/2p}}\right]^q dt\right)^{1/q}\\ &\leq C\|v_1\|_{L^{2q}_\sigma(0,T;X^p_\vartheta)}\|v_2\|_{L^{2q}_\sigma(0,T;X^p_\vartheta)},\end{aligned} \tag{5.1.9}$$

where we used that $\mu \geq 1/p + 1/q$ implies $\vartheta \geq 1/2 + 1/2p$ and thus $X^p_\vartheta \hookrightarrow X^p_{1/2+1/2p}$. This yields

$$\begin{aligned}\|F(u,u)\|_{\mathbb{E}^{q,p}_{0,\mu}(0,T)} &\leq \|F(\tilde{u},\tilde{u})\|_{\mathbb{E}^{q,p}_{0,\mu}(0,T)} + \|F(\tilde{u},u^*)\|_{\mathbb{E}^{q,p}_{0,\mu}(0,T)} + \|F(u^*,\tilde{u})\|_{\mathbb{E}^{q,p}_{0,\mu}(0,T)} \\ &\quad + \|F(u^*,u^*)\|_{\mathbb{E}^{q,p}_{0,\mu}(0,T)} \\ &\leq C\|\tilde{u}\|^2_{L^{2q}_\sigma(0,T;X^p_\vartheta)} + 2C\|\tilde{u}\|_{L^{2q}_\sigma(0,T;X^p_\vartheta)}\|u^*\|_{L^{2q}_\sigma(0,T;X^p_\vartheta)} \\ &\quad + C\|u^*\|^2_{L^{2q}_\sigma(0,T;X^p_\vartheta)} \\ &\leq C_{q,\mu,T}\|u\|^2_{\mathbb{E}^{q,p}_{1,\mu}(0,T)}.\end{aligned}$$

where we used that F is bilinear in the first step, as well as (5.1.6) and (5.1.8) in the third step. This yields (5.1.3) and thus

$$\partial_v \mathcal{F}(v) = A_p + F(v,\cdot) + F(\cdot,v)$$

exists, is linear, and, by the same arguments as above, continuous. In particular, we have that $\mathcal{F}$ is infinitely continuously differentiable as well as polynomial and therefore analytic.

In order to prove the second point, we now consider the problem

$$\partial_t u - \partial_v \mathcal{F}(v)u = g \text{ on } (0,T), \quad u(0) = u_0, \tag{5.1.10}$$

for given $v \in \mathbb{E}^{q,p}_{1,\mu}(0,T)$, $g \in \mathbb{E}^{q,p}_{0,\mu}(0,T)$ and $u_0 \in X^{q,p}_{\gamma,\mu}$. We again introduce an auxiliary function

$$U^*_0(t) := S(t)u_0 + \int_0^t S(t-s)g(s)\,ds, \quad t \geq 0.$$

Note that we consider general initial data $u_0 \neq 0$ since the solution is constructed iteratively. By the maximal L^q-regularity of A_p we have $U^*_0 \in \mathbb{E}^{q,p}_{1,\mu}(0,T)$ and by the embedding (5.1.7) and estimate (5.1.6), respectively applied to $U^*_0(t) - S(t)u_0$ and $S(t)u_0$, we further have $U^*_0 \in L^{2q}_\sigma(0,T;X_\vartheta)$. Now consider the mapping

$$\mathcal{T}_{u_0} \colon \mathbb{B}(u_0, U^*_0, R, T) \to \mathbb{E}^{q,p}_{1,\mu}(0,T), \quad \mathcal{T}_{u_0}u = w,$$

where w is the unique solution to the problem

$$\partial_t w - A_p w = g - F(v,u) - F(u,v) \text{ on } (0,T), \quad w(0) = u_0,$$

on the domain

$$\mathbb{B}(u_0, U^*_0, R, T) := \{u \in \mathbb{E}^{q,p}_{1,\mu}(0,T) : u(0) = u_0, \|u - U^*_0\|_{\mathbb{E}^{q,p}_{1,\mu}(0,T)} \leq R\}, \quad R > 0.$$

We now show that $\mathcal{T}_{u_0}$ leaves $\mathbb{B}(u_0, U^*_0, R, T)$ invariant and is contractive for a suitable choice of parameters $R, T > 0$. In order to verify the invariance, we observe that the solution w satisfies

$$\begin{aligned}\partial_t(w - U^*_0) - A_p(w - U^*_0) &= -F(v,u) - F(u,v) \text{ on } (0,T), \\ (w - U^*_0)(0) &= 0,\end{aligned}$$

and since A_p has maximal L^q-regularity we obtain for $u \in \mathbb{B}(u_0, U_0^*, R, T)$ that

$$\begin{aligned}
\|\mathcal{T}_{u_0}u - U_0^*\|_{E^{q,p}_{1,\mu}(0,T)} &\leq C_{q,p,\mu,T}\left(\|F(v,u)\|_{E^{q,p}_{0,\mu}(0,T)} + \|F(u,v)\|_{E^{q,p}_{0,\mu}(0,T)}\right), \\
&\leq C_{q,p,\mu,T}\|v\|_{L^{2q}_\sigma(0,T;X_\vartheta)}\left(\|u - U_0^*\|_{L^{2q}_\sigma(0,T;X_\vartheta)} + \|U_0^*\|_{L^{2q}_\sigma(0,T;X_\vartheta)}\right) \\
&\leq C_{q,p,\mu,T}\|v\|_{L^{2q}_\sigma(0,T;X_\vartheta)}\left(\|u - U_0^*\|_{\mathbb{E}^{q,p}_{1,\mu}(0,T)} + \|U_0^*\|_{L^{2q}_\sigma(0,T;X_\vartheta)}\right) \\
&\leq C_{q,p,\mu,T}\|v\|_{L^{2q}_\sigma(0,T;X_\vartheta)}\left(R + \|U_0^*\|_{L^{2q}_\sigma(0,T;X_\vartheta)}\right),
\end{aligned}$$

where we used the maximal regularity property on $(0,T)$ in the first, the estimate for F from (5.1.9) in the second, and the embeddings (5.1.7) in the third step.

In order to verify the contractivity, we observe that for $u_1, u_2 \in \mathbb{B}(u_0, U_0^*, R, T)$ we further have that $w_i := \mathcal{T}_{u_0}u_i$, $i \in \{1,2\}$, satisfy

$$\begin{aligned}
\partial_t(w_1 - w_2) - A_p(w_1 - w_2) &= -F(v, u_1 - u_2) - F(u_1 - u_2, v) \text{ on } (0,T), \\
(u_1 - u_2)(0) &= 0,
\end{aligned}$$

and so we analogously obtain

$$\begin{aligned}
\|\mathcal{T}_{u_0}u_1 - \mathcal{T}_{u_0}u_2\|_{E^{q,p}_{1,\mu}(0,T)} &\leq C_{q,p,\mu,T}\left(\|F(v,u_1 - u_2)\|_{E^{q,p}_{0,\mu}(0,T)} + \|F(u_1 - u_2, v)\|_{E^{q,p}_{0,\mu}(0,T)}\right) \\
&\leq C_{q,p,\mu,T}\|v\|_{L^{2q}_\sigma(0,T;X_\vartheta)}\|u_1 - u_2\|_{L^{2q}_\sigma(0,T;X_\vartheta)} \\
&\leq C_{q,p,\mu,T}\|v\|_{L^{2q}_\sigma(0,T;X_\vartheta)}\|u_1 - u_2\|_{\mathbb{E}^{q,p}_{1,\mu}(0,T)}.
\end{aligned}$$

We now utilize the fact that the solution mappings

$$E^{q,p}_{0,\mu}(0,T') \times X^{q,p}_{\gamma,\mu} \ni (f, u_0) \to u \in \mathbb{E}^{q,p}_{o,\mu}(0,T')$$

for the problem

$$\partial_t u - A_p u = f \text{ on } (0,T'), \quad u(0) = u_0,$$

are uniformly bounded for $T' \in (0,T]$, whereas the embedding constants of (5.1.7) are independent of $T > 0$, whereas for (5.1.4) they are not, compare [82]. This yields

$$\begin{aligned}
\|\mathcal{T}_{u_0}u - U_0^*\|_{E^{q,p}_{1,\mu}(0,T')} &\leq C_{q,p,\mu,T}\|v\|_{L^{2q}_\sigma(0,T';X_\vartheta)}\left(R + \|U_0^*\|_{L^{2q}_\sigma(0,T';X_\vartheta)}\right), \\
\|\mathcal{T}_{u_0}u_1 - \mathcal{T}_{u_0}u_2\|_{E^{q,p}_{1,\mu}(0,T')} &\leq C_{q,p,\mu,T}\|v\|_{L^{2q}_\sigma(0,T';X_\vartheta)}\|u_1 - u_2\|_{\mathbb{E}^{q,p}_{1,\mu}(0,T')}
\end{aligned}$$

for all $T' \in (0,T]$. Since the dominated convergence theorem implies that

$$\lim_{t \to 0+}\|f\|_{L^{2q}_\sigma(t_0,t_0+t;X_\vartheta)} = 0$$

for all $t_0 \in [0,T)$ and $f \in L^{2q}_\sigma(0,T;X_\vartheta)$, we obtain a finite sequence

$$0 = T_0 < T_1 < T_2 < \ldots < T_N = T$$

such that

$$C_{q,p,\mu,T}\|v\|_{L^{2q}_\sigma(T_n,T_{n+1};X_\vartheta)} \leq \frac{1}{4}$$

for $0 \leq n \leq N-1$ and further take $R_1 \geq \|U_0^*\|_{L^{2q}_\sigma(0,T_1;X_\vartheta)}$. This implies that $\mathcal{T}_{u_0}$ is a strict contraction on $\mathbb{B}(u_0, U_0^*, R_1, T_1)$ and leaves the set invariant. By the Banach fixed-point theorem we obtain that $\mathcal{T}_{u_0}$ has a unique fixed-point $u \in \mathbb{B}(u_0, U_0^*, R_1, T_1)$. In particular, the problem (5.1.10) has a unique solution on $(0, T_1)$, since any solution $U \in \mathbb{E}^{q,p}_{1,\mu}(0,T)$ satisfies

$$\begin{aligned} \partial_t(u-U) - A_p(u-U) &= -F(v, u-U - F(u-\tilde{u}, v) \text{ on } (0,T), \\ (u-U)(0) &= 0, \end{aligned}$$

and so by the same argument as above we have

$$\|u-U\|_{E^{q,p}_{1,\mu}(0,T_1)} \leq C_{q,p,\mu,T}\|v\|_{L^{2q}_\sigma(0,T_1;X_\vartheta)}\|u-U\|_{\mathbb{E}^{q,p}_{1,\mu}(0,T_1)} \leq \frac{1}{4}\|u-U\|_{\mathbb{E}^{q,p}_{1,\mu}(0,T_1)},$$

yielding $u = U$ on $(0, T_1)$. Since it holds that

$$H^{1,q}_\mu(t_0,t_1;L^p_{\overline{\sigma}}(\Omega)) \cap L^q_\mu(t_0,t_1;D(A_p)) \hookrightarrow C([t_0,t_1];X^{q,p}_{\gamma,\mu}), \quad t_0 < t_1,$$

the linearity of the problem allows us to iterate this argument for the new initial data $u(T_1) \in X^{q,p}_{\gamma,\mu}$ by taking a new radius

$$R_{n+1} \geq \|U_n^*\|_{L^{2q}_\sigma(0,T_{n+1}-T_n;X_\vartheta)},$$

where U_n^* is given by

$$U_n^*(t) := S(t)u(T_n) + \int_0^t S(t-s)g(T_n+t)\,ds.$$

Piecing together the solutions on $[T_n, T_{n+1}]$ for $0 \leq n \leq N-1$ then yields a solution $u \in \mathbb{E}^{q,p}_{1,\mu}(0,T)$. □

We are now in the position to prove our main result concerning the additional regularity of solutions to the primitive equations.

Proof of Theorem 5.1.4. Point (a) and (b) follow from Theorem 5.1.8 via Lemma 5.1.13. In order to obtain the additional regularity in the spatial variables $(x,y,z) \in \Omega$, we make use of the fact that the periodic boundary conditions effectively smoothen $\partial\Omega$, allowing us to proceed as in the setting of a layer domain and treat the horizontal spatial variables $(x,y) \in [0,1]^2$ together with the time variable $t \in (0,T)$. For this purpose we take parameters $\lambda \in (0,\infty)$ and $\eta \in \mathbb{R}^3$ and introduce the coordinate transformation

$$\Phi_{\lambda,\eta}\colon (0,\infty)\times\mathbb{R}^3 \to (0,\infty)\times\mathbb{R}^3, \quad \Phi_{\lambda,\eta}(t,\vec{x}) := (\lambda t, \vec{x} + t\eta),$$

where we denoted the spatial variables by $\vec{x} = (x,y,z)$. Since the functions we are considering are periodic in the horizontal variables, we identify $\Omega_{\text{per}} := \Omega \cup \Gamma_l$ with $S^1 \times S^1 \times (-h,0)$, where $S^1 = \mathbb{R}/\mathbb{Z}$ denotes the one-dimensional torus. Taking

$$\eta_H = (\eta_x, \eta_y, 0) \in S^1 \times S^1 \times \mathbb{R},$$

it then holds that Φ_{λ,η_H} is an isomorphism of $(0,\infty)\times\Omega_{\text{per}}$.

Now let $\varepsilon \in (0,1)$. For $u \in \mathbb{E}^{q,p}_{1,\mu}(0,T)$, we define $u_{\lambda,\eta_H} := u \circ \Phi_{\lambda,\eta_H}$ as well as the mapping

$$H\colon (1-\varepsilon, 1+\varepsilon) \times (-\varepsilon,\varepsilon)^2 \times \mathbb{E}^{q,p}_{1,\mu}(0,T) \to \mathbb{E}^{q,p}_{0,\mu}(0,T/(1+\varepsilon)) \times X^{q,p}_{\gamma,\mu},$$

via

$$H(\lambda,\eta_H,u) := \Big(\partial_t u_{\lambda,\eta_H} - \lambda\left(\mathcal{F}(u) + \mathbb{P}f\right)_{\lambda,\eta_H} - \eta_H \cdot \nabla u_{\lambda,\eta_H}, v(0) - a\Big),$$

where $\mathcal{F}$ is as in Lemma 5.1.13. Since the chain rule yields that

$$\partial_t u_{\lambda,\eta_H} = (\lambda \partial_t u + \eta_H \cdot \nabla u) \circ \Phi_{\lambda,\eta_H}, \quad \partial_i u_{\lambda,\eta_H} = (\partial_i v) \circ \Phi_{\lambda,\eta_H}, \quad \partial_i \in \{\partial_x, \partial_y, \partial_z\},$$

we obtain that $A_p u_{\lambda,\eta_H} = (A_p u)_{\lambda,\eta_H}$ and $F(u)_{\lambda,\eta_H} = F(u_\lambda, \eta_H)$ and thus v is a solution to the primitive equations (5.1.1) if and only if $H(1,0,v) = (0,0)$. Using similar arguments as in the proof of Lemma 5.1.13, since H is a second-order polynomial in v, we obtain that the operator $\partial_v H(1,0,v)$, given by

$$\partial_v H(1,0,v)u = (\partial_t u - \partial_v \mathcal{F}(v)u, u(0)),$$

is an isomorphism between $\mathbb{E}^{q,p}_{1,\mu}(0,T)$ and $\mathbb{E}^{q,p}_{0,\mu}(0,T) \times X^{q,p}_{\gamma,\mu}$. Following [78, Section 5] we then obtain that the mapping $(\lambda,\eta_H) \mapsto v(\lambda t, \vec{x} + t\eta_H)$ is analytic on a neighborhood of $(1,0)$ by the implicit function theorem and thus v is analytic in t and (x,y). The regularity for the pressure can then be obtained using (5.0.7). Since $w = w(v)$ vanishes on $\Gamma_u \cup \Gamma_b$, it holds that

$$\overline{w\partial_z v} = \overline{(-\partial_z w)v} = \overline{(\mathrm{div}_H v)v}$$

and thus the pressure is given via

$$\Delta_H \pi = -\mathrm{div}_H \overline{((v\cdot\nabla_H)v + (\mathrm{div}_H v)v)}, \quad \pi \text{ periodic on } \partial G.$$

The analyticity in $(t,x,y) \in (0,T) \times G$ for v then yields analyticity for the pressure. In order to obtain analyticity for v in the vertical variable $z \in (-h,0)$ we need to distinguish between the cases $\Gamma_D = \emptyset$ and $\Gamma_D \neq \emptyset$. If $\Gamma_N = \Gamma_u \cup \Gamma_b$, we may apply even extensions at both Γ_u and Γ_b to obtain a function that is periodic in all spatial variables. Since the surface pressure π does not depend on z and both w and $\partial_z v$ vanish at the boundary, applying this extension operator retains the structure of the primitive equations and thus an analogous argument yields the analyticity in z. The case $\Gamma_D \neq \emptyset$ is treated via a localization procedure as in [80, Section 9.4]. For this purpose, one rewrites the problem as

$$\partial_t v - \Delta v + (v \cdot \nabla_H)v + w\partial_z v = f - \nabla_H \pi$$

and since the right-hand side $f - \nabla_H \pi$ is analytic and the term $(v\cdot\nabla_H)v + w\partial_z v$ is bilinear, one may proceed analogously. □

We now establish a lemma concerning additional regularity for time derivative of solutions, which we will use to prove Lemma 5.1.2.

Lemma 5.1.14. *Under the assumptions of Lemma 5.1.12, if it further holds that*

$$t \cdot \partial_t \mathbb{P} f \in \mathbb{E}^{q,p}_{0,\mu}(0,T),$$

then the solution v also satisfies $t \cdot \partial_t v \in \mathbb{E}^{q,p}_{1,\mu}(0,T)$ and it holds that

$$\|t \cdot \partial_t v\|_{\mathbb{E}^{q,p}_{1,\mu}(0,T)} \leq B(q,p,\mu,T,v,f).$$

Proof. We take auxiliary parameters $T' \in (0,T)$ and $0 < \varepsilon < \frac{T-T'}{T'}$ and consider the map

$$G\colon (-\varepsilon,\varepsilon) \times \mathbb{E}^{q,p}_{1,\mu}(0,T') \to \mathbb{E}^{q,p}_{0,\mu}(0,T') \times X^{q,p}_{\gamma,\mu},$$

defined via

$$G(\lambda,u) := (\partial_t u - (1+\lambda)\mathcal{F}(u) - (1+\lambda)(\mathbb{P}f)_\lambda, u(0) - a)\,,$$

where $\mathcal{F}$ is as in Lemma 5.1.13 and $g_\lambda(t,\cdot) = g((1+\lambda)t,\cdot)$. Observe that we then have

$$(\partial_u G)(\lambda,u)w = (\partial_t w - (1+\lambda)\partial_v \mathcal{F}(u)w, w(0))\,,$$

Proceeding as in [21, Section 9.2], the implicit function theorem yields the existence of a function

$$g\colon (-\varepsilon',\varepsilon') \to \mathbb{E}^{q,p}_{1,\mu}(0,T') \quad \text{such that} \quad G(\lambda, g(\lambda)) = (0,0), \quad \lambda \in (-\varepsilon',\varepsilon')$$

for some $0 < \varepsilon' < \varepsilon$. By the uniqueness of solutions to the primitive equations in $\mathbb{E}^{q,p}_{1,\mu}(0,T')$ from Lemma 5.1.12 we then obtain that $g(\lambda) = v_\lambda$ on $(0,T')$ for all $\lambda \in (-\varepsilon',\varepsilon')$. Taking the implicit derivative with respect to the parameter λ, we obtain that

$$\begin{aligned}\partial_\lambda g_\lambda(0) = t \cdot \partial_t v &= -(\partial_v G)(0,v)^{-1}(\partial_\lambda G)(0,v) \\ &= -\left[w \mapsto (\partial_t w - \partial_v \mathcal{F}(v)w, w(0)\right]^{-1}(-\mathcal{F}(v) - \mathbb{P}f - t \cdot \partial_t \mathbb{P}f, 0)\,,\end{aligned}$$

where we used that

$$\partial_\lambda (1+\lambda)(\mathbb{P}f)_\lambda\big|_{\lambda=0} = \mathbb{P}f + t \cdot \partial_t \mathbb{P}f$$

by the product and chain rule. This means that the function $u := t \cdot \partial_t v$ is the unique solution to the problem

$$\partial_t u - \partial_v \mathcal{F}(v)u = A_p v + F(v) + \mathbb{P}f + t \cdot \partial_t \mathbb{P}f \text{ on } (0,T'), \quad u(0) = 0,$$

and since the right-hand side belongs to $\mathbb{E}^{q,p}_{0,\mu}(0,T')$ we obtain that $t \cdot \partial_t v \in \mathbb{E}^{q,p}_{1,\mu}(0,T')$ by Lemma 5.1.13. By further estimating

$$\begin{aligned}\|A_p v + F(v) + \mathbb{P}f + t \cdot \partial_t \mathbb{P}f\|_{\mathbb{E}^{q,p}_{0,\mu}(0,T)} \leq C_{q,p,\mu,T}\bigg(&\|v\|_{\mathbb{E}^{q,p}_{1,\mu}(0,T)} + \|v\|^2_{\mathbb{E}^{q,p}_{1,\mu}(0,T)} \\ &+ \|f\|_{\mathbb{E}^{q,p}_{0,\mu}(0,T)} + \|t \cdot \partial_t \mathbb{P}f\|_{\mathbb{E}^{q,p}_{0,\mu}(0,T)}\bigg),\end{aligned}$$

where we treated $F(v)$ as in the proof of Lemma 5.1.13, we obtain

$$\begin{aligned}\|t\cdot\partial_t v\|_{\mathbb{E}^{q,p}_{1,\mu}(0,T')} &\le C_{q,p,\mu,T,v}\Big(\|v\|_{\mathbb{E}^{q,p}_{1,\mu}(0,T)}+\|v\|^2_{\mathbb{E}^{q,p}_{1,\mu}(0,T)}+\|f\|_{\mathbb{E}^{q,p}_{0,\mu}(0,T)}+\|t\cdot\partial_t\mathbb{P}f\|_{\mathbb{E}^{q,p}_{0,\mu}(0,T)}\Big)\\ &= B(q,p,\mu,T,v,f).\end{aligned}$$

Here we used that the family of constants $\{C_{q,p,\mu,T',v}>0 : T'\in(0,T)\}$ is uniformly bounded. This implies that the left-hand side is uniformly bounded for $T'\in(0,T)$ and so by the dominated convergence theorem the bound is also valid for $T'=T$ and we have $t\cdot\partial_t v\in\mathbb{E}^{q,p}_{1,\mu}(0,T)$. □

5.1.5 Global existence

A priori estimates for solutions are a key tool in establishing global well-posedness, compare Theorem 5.1.7. For the case $\Gamma_D=\Gamma_b$ and $f=0$, Hieber and Kashiwabara established in [49] an L^∞-H^2-estimate of the form

$$\|v(t)\|_{H^2(\Omega)}\le B(t,\|a\|_{H^2(\Omega)}),\quad t>0.$$

Analogous estimates for the more general case $\Gamma_D\neq\emptyset$ are obtained via similar arguments. They can also be adapted to cover the general case $f\neq 0$, see [31,48] for details. For the maximal-regularity approach, however, we only require *a priori* estimates of the H^1-L^2 and L^2-H^2 types. In the case $\Gamma_D=\emptyset$, the proof of such estimates is very similar to the ones given in [48,49], so we chose to omit it here and simply refer to [38, Theorem 6.9] for details. In either case, one obtains the following.

Theorem 5.1.15. *There exists a function $B\colon[0,\infty)^3\to(0,\infty)$ increasing in each variable such that for all $T>0$ and*

$$v\in H^1(0,T;L^2_{\overline{\sigma}}(\Omega))\cap L^2(0,T;D(A_2)),$$

satisfying the primitive equations (5.1.1) *for data*

$$a\in\{H^1_{per}(\Omega)^2 : v\big|_{\Gamma_D}=0\}\}\cap L^2_{\overline{\sigma}}(\Omega),\quad \mathbb{P}f\in L^2(0,T;L^2_{\overline{\sigma}}(\Omega)),$$

it holds that

$$\|v\|_{H^1(0,T;L^2_{\overline{\sigma}}(\Omega))}+\|v\|_{L^2(0,T;D(A_2))}\le B(\|a\|_{H^1(\Omega)},\|\mathbb{P}f\|_{L^2(0,T);L^2_{\overline{\sigma}}(\Omega)},T).$$

We now turn to the issue of global well-posedness. As in [49], we first consider the case $p=q=2$. We then utilize this result to obtain the general case.

Proof of Lemma 5.1.2. It suffices to show that the solution is global since the additional regularity for $t\cdot\partial_t v$ was already proven in Lemma 5.1.14. We thus set

$$T^*:=\sup\{T'\in(0,T] : \text{ there exists a solution } v\in\mathbb{E}^{2,2}_{1,1}(0,T')\}$$

and by Lemma 5.1.12 it holds that $T^* > 0$. First, we show the existence of a unique solution $v \in \mathbb{E}^{2,2}_{1,1}(0,T^*)$. For this purpose, assume that $v_1, v_2 \in \mathbb{E}^{2,2}_{1,1}(0,T')$ are solutions to the primitive equations (5.1.1) for some $T' \in (0,T^*]$ and set

$$t^* := \sup\{t > 0 : \|v_1(s) - v_2(s)\|_{X^{2,2}_{\gamma,1}} \text{ for all } s \in [0,t].\}$$

By the uniqueness of solutions for initial data a from Lemma 5.1.12 we have $t^* > 0$. Now assume that $t^* < T'$. Due to the embedding

$$\mathbb{E}^{2,2}_{1,1}(0,T) \hookrightarrow C([0,T]; X^{2,2}_{\gamma,1}), \quad T \in (0,\infty), \tag{5.1.11}$$

we then have $a^* := v_1(t^*) = v_2(t^*)$ and so applying the uniqueness of solutions for the new initial data a^* yields that $v_1 = v_2$ on $[0, t^* + t']$ for some $t' \in (T' - t^*)$ and this contradicts the definition of t^*. It follows that $t^* = T'$ and thus solutions are unique in $\mathbb{E}^{2,2}_{1,1}(0,T')$ for all $T' \in (0,T^*)$. In particular, we obtain the existence of a unique solution v on $(0,T^*)$.

We now show that $T^* = T$. For this purpose we take $T' \in (0,T^*)$ and apply Theorem 5.1.15 to obtain that $v \in \mathbb{E}^{2,2}_{1,1}(0,T')$ satisfies the estimate

$$\|v\|_{\mathbb{E}^{2,2}_{1,1}(0,T')} \le B(\|a\|_{H^1(\Omega)}, \|\mathbb{P}f\|_{\mathbb{E}^{2,2}_{0,1}(0,T)}, T^*).$$

and thus we have $v \in \mathbb{E}^{2,2}_{1,1}(0,T^*)$ by the dominated convergence theorem. By the embedding (5.1.11) we may further take $v(T^*) \in X^{2,2}_{\gamma,1}$ as new initial data and so by Lemma 5.1.12 we may extend the solution onto $(0, T^* + \varepsilon)$ for some $\varepsilon \in (0, T - T^*)$. Since this contradicts the definition of T^* we obtain that $T^* = T$ and thus the L^2-solution is global. □

We are now able to prove the main result of this section by combining the local L^q-L^p-theory with the global L^2-theory, yielding a suitable *a priori* bound via the additional L^q-L^p and L^2-regularity for $\mathbb{P}f$.

Proof of Theorem 5.1.1. In order to prove that the local solution v from Lemma 5.1.12 is global we show that the assumptions of Theorem 5.1.7 are satisfied. For this purpose, we apply Theorem 5.1.4 and the additional regularity of $\mathbb{P}f$, yielding

$$v \in H^{1,q}_\mu(0,T'; D(A_p)) \hookrightarrow H^{1,q}(\delta,T'; D(A_p)) \hookrightarrow C([\delta,T']; D(A_p))$$

for all $\delta \in (0,T')$. We now proceed as in [48, Section 3.2] to obtain that $v(t_0) \in X^{2,2}_{\gamma,1}$ for $t_0 > 0$. Since it holds that

$$D(A_p) \subset H^{2,p}(\Omega)^2 \hookrightarrow H^{2/r,r}(\Omega)^2$$

whenever $r \in (1,\infty)$ and $p \in (3r/(2r+1),\infty)$, we have $D(A_p) \hookrightarrow X^{2,2}_{\gamma,1}$ for $p \in [6/5,\infty)$, whereas for $p \in (1,6/5)$ we obtain an increasing sequence $p_0, p_1, \dots, p_N$ recursively given by

$$p_0 := p, \quad p_{n+1} := \frac{p_n}{3 - 2p_n}$$

and terminating for $p_N \geq 6/5$. Since solutions v with initial data $v(0) \in B^{2/p}_{p,q}(\Omega)$ become smooth with $v(t) \in D(A_p)$ for $t > 0$, we then obtain $0 < T_0 < T_1 < \ldots < T_{N-1} < t_0$ with

$$v(T_n) \in D(A_{p_n}) \hookrightarrow B^{2/p_{n+1}}_{p_{n+1},q}(\Omega)$$

for $0 \leq n \leq N-1$ and thus $v(T_N) \in D(A_{p_N}) \hookrightarrow X^{2,2}_{\gamma,1}$. Since we further have $\mathbb{P}f \in H^{1,2}(\delta, T; L^2_{\overline{\sigma}}(\Omega))$, where we take $\delta \in (0, T^*/2)$ with T^* as in Theorem 5.1.7, obtain that v is a global L^2-solution on (δ, T) by Lemma 5.1.2 and even satisfies

$$v \in H^1(\delta, T; D(A_2)) \hookrightarrow C([\delta, T]; D(A_2)).$$

We now set $\mu_0 := 1/p + 1/q$ and choose $\overline{\mu}_0 \in (\mu_0, 1]$ such that

$$2(\overline{\mu}_0 - \mu_0) < 1/2 + 1/p.$$

Then the embeddings $D(A_2) \hookrightarrow X^{q,p}_{\overline{\mu}_0,q} \hookrightarrow X^{q,p}_{\mu_0,q}$ yield that $v \in C([\delta, T]; X^{q,p}_{\overline{\mu}_0,q})$ and since they are compact by [89, Section 4.3.2, Remark 1], it follows from Theorem 5.1.7 that $v \in \mathbb{E}^{q,p}_{1,\mu_0}(\delta, T)$, which also yields

$$v \in \mathbb{E}^{q,p}_{1,1}(2\delta, T) \hookrightarrow \mathbb{E}^{q,p}_{1,\mu}(2\delta, T).$$

Since we have $2\delta < T^*$ as well as $v \in \mathbb{E}^{q,p}_{1,\mu}(0, T')$ for all $T' \in (0, T^*)$ we obtain $v \in \mathbb{E}^{q,p}_{1,\mu}(0, T)$ and thus the L^q-L^p-solution is global. This completes the proof. □

5.2 The case of Neumann boundary conditions

In this section, we consider the primitive equations in the form

$$\begin{array}{rcll} \partial_t v - \Delta v + (v \cdot \nabla_H) v + w \partial_z v + \nabla_H \pi &=& 0 & \text{in } \mathbb{L} \times (0,T), \\ \partial_z \pi &=& 0 & \text{in } \mathbb{L} \times (0,T), \\ \mathrm{div}_H \overline{v} &=& 0 & \text{in } \mathbb{R}^2 \times (0,T), \\ v(0) &=& a & \text{in } \mathbb{L}, \\ \partial_z v &=& 0 & \text{on } \partial\mathbb{L} \times (0,T), \end{array} \tag{5.2.1}$$

where the layer domain $\mathbb{L} = \mathbb{R}^2 \times (-h, 0)$ is as in Section 4.2 and $T \in (0, \infty]$. The results presented here have been previously published in [40, Sections 2 and 6].

Recall the definition of the spaces $L^{\infty,p}_{\overline{\sigma}}(\mathbb{L})^2$ and $X^{\infty,p}_{\overline{\sigma}}(\mathbb{L})$ from (4.2.2) and (4.2.8). Throughout this section, we will use the notations

$$\|v\|_{\infty,p} := \|v\|_{L^\infty_H L^p_z(\mathbb{L})}, \quad \|v\|_{1,\infty,p} := \|v\|_{L^\infty_H L^p_z(\mathbb{L})} + \|\nabla v\|_{L^\infty_H L^p_z(\mathbb{L})}.$$

The main result of this section consists of two parts: The first is that of the local well-posedness for initial data belonging to $X^{\infty,1}_{\overline{\sigma}}(\mathbb{L})$ with small, discontinuous perturbations.

Theorem 5.2.1. *There exists a constant $C_0 > 0$ such that for any initial data of the form $a = a_1 + a_2$ with $a_1 \in X^{\infty,1}_{\overline{\sigma}}(\mathbb{L})$ and $a_2 \in L^{\infty,1}_{\overline{\sigma}}(\mathbb{L})$ such that*

$$\max\{\|a_2\|_{\infty,1}, \|a\|_{\infty,1} \cdot \|a_2\|_{\infty,1}\} \leq C_0,$$

there exists $T \in (0, \infty]$ such that the primitive equations (5.2.1) *with initial data a have a unique, local mild solution v satisfying*

$$v \in C_b((0,T); X^{\infty,1}_{\overline{\sigma}}(\mathbb{L})), \quad t^{1/2} \nabla v \in L^\infty(0,T; L^\infty_H L^1_z(\mathbb{L}))^2,$$

as well as

$$\limsup_{t \to 0} t^{1/2} \|\nabla v(t)\|_{\infty,1} \leq C \|a_2\|_{\infty,1}$$

for an absolute constant $C > 0$. In the case $a_2 = 0$ one even has $v \in C_b([0,T); X^{\infty,1}_{\overline{\sigma}}(\mathbb{L}))$.

The second part is that of strong, global well-posedness for horizontally periodic initial data. Following Hieber and Kashiwabara's definition (5.0.3), we say that v is strong if

$$v\big|_{\Omega \times (0,\infty)} \in C^1((0,\infty); L^p(\Omega))^2 \cap C((0,\infty); H^{2,p}(\Omega))^2,$$

where $\Omega = G \times (-h, 0)$, and $G = (0,1)^2$. Note that we may assume without loss of generality that the period is 1. Since global well-posedness results are known for the case of $\Omega = G \times (-h, 0)$ for $G = (0,1)^2$ with periodic boundary conditions on the lateral part of Ω, we further obtain the following.

Theorem 5.2.2. *Assume that a in Theorem 5.2.1 is horizontally periodic. Then v has an extension to a global, strong solution to the primitive equations* (5.2.1) *onto* $\mathbb{L}\times(0,\infty)$ *that is even real analytic.*

Remark 5.2.3. Note that since the perturbation term a_2 fails to be continuous with respect to the horizontal variables, it holds that the mapping $Sa_2 : t \mapsto S(t)a_2$ satisfies

$$Sa_2 \in C_b((0,\infty); X_{\overline{\sigma}}^{\infty,1}(\mathbb{L})),$$

but fails to be continuous at $t=0$. However, it holds that $v - Sa_2 \in C_b([0,T); X_{\overline{\sigma}}^{\infty,1}(\mathbb{L}))$.

Since the hydrostatic Stokes semigroup admits L^1-L^p-smoothing for the vertical regularity, see Theorem 4.2.1, we also show additional regularity for the local solution under the assumption that the initial data belongs to $L^\infty_H L^p_z(\mathbb{L})^2$ for some $p \in (1,\infty]$. This additional regularity also plays an important role in obtaining a global extension.

Lemma 5.2.4. *Let $p \in (1,\infty]$ and assume that a in Theorem 5.2.1 further satisfies*

$$\text{(i) } a \in L^\infty_H L^p_z(\mathbb{L})^2, \qquad \text{(ii) } a \in X_{\overline{\sigma}}^{\infty,p}(\mathbb{L}), \qquad \text{or} \qquad \text{(iii) } a \in BUC(\mathbb{R}^2 \times [-h,0])^2.$$

Then the solution v has the additional regularity

(i) $t^{(1-1/p)/2} v, t^{1-1/(2p)} \nabla v \in L^\infty(0,T; L^\infty_H L^p_z(\mathbb{L}))^2$,

(ii) $t^{(1-1/p)/2} v \in C([0,T); X_{\overline{\sigma}}^{\infty,p}(\mathbb{L}))$ *and* $t^{1-1/(2p)} \nabla v \in L^\infty(0,T; L^\infty_H L^p_z(\mathbb{L}))^2$,

(iii) $t^{1/2} v \in C([0,T); BUC(\mathbb{R}^2 \times [-h,0]))^2$ *and* $t\nabla v \in L^\infty(0,T; L^\infty(\mathbb{L}))^2$

for all $0 < T < \infty$.

Observe that, since $u = (v,w)$ satisfies $\operatorname{div} u = 0$, the nonlinear term of the primitive equations can be written as

$$(v \cdot \nabla_H) v + w\partial_z v = (u \cdot \nabla) v = \nabla \cdot (u \otimes v).$$

The key components to our proof of the existence of local mild solutions are the following estimates for the hydrostatic Stokes semigroup applied to this nonlinear term.

Lemma 5.2.5. *Let $\alpha \in [0,1)$. Then there exists a constant $C = C_\alpha > 0$ such that for all $v_1, v_2 \in L_{\overline{\sigma}}^{\infty,1}(\mathbb{L})$ with $\nabla v_1, \nabla v_2 \in L^\infty_H L^1_z(\mathbb{L})^2$ it holds that*

$$\left\| S(t)\mathbb{P}\nabla \cdot (u_1 \otimes v_2) \right\|_{\infty,1} \le C t^{-(1-\alpha)/2} G_\alpha(v_1, v_2),$$

where

$$G_\alpha(v_1,v_2) := \left(\|v_1\|_{1,\infty,1} \|v_2\|_{\infty,1} + \|v_2\|_{1,\infty,1} \|v_1\|_{\infty,1} \right)^{1-\alpha} \left(\|v_1\|_{1,\infty,1} \|v_2\|_{1,\infty,1} \right)^{\alpha}$$

and $u_i = (v_i, w_i)$ with $w_i = w_i(v_i)$ as in (5.0.2) *for* $i \in \{1,2\}$.

Proof. Using the notation $\|\cdot\|_{\infty,\infty} := \|\cdot\|_{L^\infty(\mathbb{L})}$, we apply the Poincaré inequality for average-free functions in the vertical direction to obtain, for $i \in \{1,2\}$, that

$$\|v_i - \overline{v_i}\|_{\infty,\infty} \le C\|\partial_z v_i\|_{\infty,1}, \quad \|\overline{v_i}\|_{\infty,\infty} \le \|v_i\|_{\infty,1},$$

which implies

$$\|v_i\|_{\infty,\infty} \le \|v_i - \overline{v_i}\|_{\infty,\infty} + \|\overline{v_i}\|_{\infty,\infty} \le C\|v_i\|_{1,\infty,1}.$$

The anisotropic Hölder inequality then yields

$$\begin{aligned} \|\nabla(v_1 \otimes v_2)\|_{\infty,1} &\le \|\nabla v_1\|_{\infty,1} \|v_2\|_{\infty,\infty} + \|v_1\|_{\infty,\infty}\|\nabla v_2\|_{\infty,1} \\ &\le C\|v_1\|_{1,\infty,1}\|v_2\|_{1,\infty,1}, \\ \|v_1 \otimes v_2\|_{\infty,1} &\le \|v_1\|_{\infty,1}\|v_2\|_{\infty,\infty} + \|v_1\|_{\infty,\infty}\|v_2\|_{\infty,1} \\ &\le C\left(\|v_1\|_{1,\infty,1}\|v_2\|_{\infty,1} + \|v_1\|_{\infty,1}\|v_2\|_{1,\infty,1}\right). \end{aligned} \tag{5.2.2}$$

We further have

$$\|w_i\|_{\infty,\infty} \le C\left\|\partial_z w_i\right\|_{\infty,1} \le C\left\|\nabla_H v_i\right\|_{\infty,1} \le C\|v_i\|_{1,\infty,1}$$

and therefore

$$\begin{aligned} \|\partial_z(w_1 v_2)\|_{\infty,1} &\le \|\partial_z w_1\|_{\infty,1}\|v_2\|_{\infty,\infty} + \|w_1\|_{\infty,\infty}\|\partial_z v_2\|_{\infty,1} \\ &\le C\|v_1\|_{1,\infty,1}\|v_2\|_{1,\infty,1}, \\ \|w_1 v_2\|_{\infty,1} &\le \|w_1\|_{\infty,\infty}\|v_2\|_{\infty,1} \\ &\le \|v_1\|_{1,\infty,1}\|v_2\|_{\infty,1}. \end{aligned} \tag{5.2.3}$$

Recall from (4.2.3) that $\mathbb{P}f = f + (R\otimes R)\overline{f}$ where $R \otimes R = (R_i R_j)_{1\le i,j\le 2}$ and $\overline{f}$ denotes the vertical average. By rewriting the bilinear term as

$$\nabla \cdot (u_1 \otimes v_2) = \nabla_H \cdot (v_1 \otimes v_2) + \partial_z(w_1 v_2)$$

and using the fact that w_i vanishes for $z = 0$ and $z = -h$, we obtain

$$\begin{aligned} \mathbb{P}\nabla \cdot (u_1 \otimes v_2) &= \mathbb{P}\nabla_H \cdot (v_1 \otimes v_2) + \partial_z(w_1 v_2) + (R_i R_j)_{1\le i,j\le 2}\overline{\partial_z(w_1 v_2)} \\ &= \mathbb{P}\nabla_H \cdot (v_1 \otimes v_2) + \partial_z(w_1 v_2). \end{aligned}$$

The case $\alpha = 0$ then follows from Theorem 4.2.1 (v) and (vi), as well as (5.2.2) and (5.2.3). For the case $\alpha \in (0,1)$ we observe that the horizontal operators $(-\Delta_H)^{(1-\alpha)/2}$, $(-\Delta_H)^{-(1-\alpha)/2}$ and ∇_H commute and therefore it holds that

$$S(t)\mathbb{P}\nabla_H \cdot (v_1 \otimes v_2) = S(t)\mathbb{P}(-\Delta_H)^{(1-\alpha)/2}\nabla_H \cdot (-\Delta_H)^{-(1-\alpha)/2}(v_1 \otimes v_2).$$

By Theorem 4.2.1.(iv), Lemma 3.2.3, and the estimates (5.2.2), we thus obtain

$$\begin{aligned} \|S(t)\mathbb{P}\nabla_H \cdot (v_1 \otimes v_2)\|_{\infty,1} &\le Ct^{-(1-\alpha)/2}\|\nabla_H \cdot (-\Delta_H)^{-(1-\alpha)/2}(v_1 \otimes v_2)\|_{\infty,1} \\ &\le Ct^{-(1-\alpha)/2}\|v_1 \otimes v_2\|_{\infty,1}^{1-\alpha}\|\nabla_H(v_1 \otimes v_2)\|_{\infty,1}^{\alpha} \\ &\le Ct^{-(1-\alpha)/2}G_\alpha(v_1, v_2). \end{aligned}$$

For the remaining term, we observe that the fact that w_1 vanishes for $z = -h, 0$ implies that

$$S(t)\partial_z(w_1v_2) = S(t)\partial_z I^1\partial_z(w_1v_2) = S(t)\partial_z I^\alpha I^{1-\alpha}\partial_z(w_1v_2) = S(t)\partial_z I^\alpha \partial_z^\alpha(w_1v_2),$$

compare Definition 3.3.5, and so, by Theorem 4.2.1.(vi) and the estimates (5.2.3), we thus have

$$\begin{aligned}\|S(t)\partial_z(w_1v_2)\|_{\infty,1} &\le Ct^{-(1-\alpha)/2}\|I^{1-\alpha}(w_1v_2)\|_{\infty,1}\\ &\le Ct^{-(1-\alpha)/2}\|w_1v_2\|_{\infty,1}^{1-\alpha}\|\partial_z(w_1v_2)\|_{\infty,1}^{\alpha}\\ &\le Ct^{-(1-\alpha)/2}G_\alpha(v_1,v_2),\end{aligned}$$

where in the last step we used that the interpolation inequality (ii) in Lemma 3.3.6 is also valid in $L^\infty_H L^p_z(\mathbb{L})$ by an obvious modification. This completes the proof. □

Before we proceed to apply these estimates to the primitive equations, we also require the following lemma concerning the boundedness of sequences subject to suitable recursive inequalities.

Lemma 5.2.6. *Let $c_0, c_1, c_2 > 0$ be constant coefficients and let $\gamma, \delta > 0$ be such that*

$$18c_0c_2^2\gamma \le 1 \quad \text{and} \quad 6c_0\delta \le 1.$$

Then there exists $\varepsilon > 0$ such that any two sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ of positive real numbers satisfying $\alpha_0 \le c_0$, $\beta_0 \le \varepsilon$, as well as the recursive growth bounds

$$\alpha_{n+1} \le c_0 + c_1\alpha_n\beta_n, \quad \beta_{n+1} \le \varepsilon + c_2\alpha_n^{1/2}\beta_n^{3/2} + \delta\alpha_n\beta_n,$$

are bounded with $\alpha_n \le 2c_0$ and $\beta_n \le \gamma$ for all $n \in \mathbb{N}$.

Proof. We prove the estimates in two steps via induction. Let $\varepsilon > 0$ be so small that $p_\varepsilon(x) := \varepsilon + c_2(2c_0)^{1/2}x^{3/2} + x/3$ has fixed points $x_0 = x_0(\varepsilon)$ and $x_1 = x_1(\varepsilon)$ with $0 < x_0 < x_1$. Since $\lim_{\varepsilon\to 0} x_0(\varepsilon) = 0$, we further take $\varepsilon > 0$ to be so small that $x_0 < 1/2c_1$. Then we have

$$b_0 \le \varepsilon \le p_\varepsilon(x_0) = x_0 < 1/2c_1$$

and if it holds that $\alpha_n \le 2c_0$ and $\beta_n \le x_0$ for some $n \in \mathbb{N}$, then the recursive growth bounds imply that

$$\alpha_{n+1} \le c_0 + c_1 2c_0\frac{1}{2c_1} \le 2c_0, \quad \beta_{n+1} \le p_\varepsilon(\beta_n) \le p_\varepsilon(x_0) = x_0 < 1/2c_1,$$

so by induction it follows that $\alpha_n \le 2c_0$ and $\beta_n \le x_0 < 1/2c_1$ for all $n \in \mathbb{N}$. Now take $\varepsilon > 0$ to be so small that $3\varepsilon \le \gamma$. Then we have $\beta_0 \le \gamma$ and the recursive growth bound for $(\beta_n)_{n\in\mathbb{N}}$ yields that if it holds that $\beta_n \le \gamma$ for some $n \in \mathbb{N}$, then we also have

$$\begin{aligned}\beta_{n+1} &\le \varepsilon + c_2\sqrt{2c_0}\gamma^{3/2} + \delta 2c_0\gamma\\ &\le \left(\frac{1}{3} + c_2\sqrt{2c_0\gamma} + \frac{1}{3}\right)\gamma\\ &\le \gamma,\end{aligned}$$

which implies the estimate for $(\beta_n)_{n\in\mathbb{N}}$. □

Furthermore, we utilize the following Lemma concerning the continuity of convolution integrals.

Lemma 5.2.7. *Let* $v \in C_b((0,\infty); X^{\infty,1}_{\overline{\sigma}}(\mathbb{L}))$ *be a function such that*

$$t^{1/2}\nabla v \in C_b((0,T); L^\infty_H L^1_z(\mathbb{L}))^2$$

and $a \in L^{\infty,p}_{\overline{\sigma}}(\mathbb{L})$. *Then the function given by*

$$V(t) := S(t)a - \int_0^t S(t-s)\mathbb{P}\nabla \cdot (u \otimes v)(s)\, ds$$

satisfies $V, \nabla V \in C((0,T); L^\infty_H L^1_z(\mathbb{L}))^2$.

Proof. Let $t, \delta > 0$. Then we have for the first term $V_1(t) := S(t)a$

$$\begin{aligned} S(t+\delta)a &= S(\delta)S(t)a, \\ \nabla_H S(t+\delta)a &= S(\delta)\nabla_H S(t)a, \\ \partial_z S(t+\delta)a &= S_H(\delta)S_D(\delta)\partial_z S(t)a. \end{aligned}$$

Here we used the fact that ∇_H commutes with $S(t) = S_H(t)S_N(t)$, that ∂_z commutes with $S_H(t)$, as well as the elementary relation $\partial_z S_N(t) = S_D(t)\partial_z$ with S_D from Lemma 3.3.2. Since S acts on the horizontal component via convolution with the Gaussian kernel we have $S(t)a, \nabla S(t)a \in BUC(\mathbb{R}^2; L^1(-h,0))^2$ for $t > 0$ and thus the continuity V_1 and ∇V_1 follows from Theorem 4.2.1 and Lemma 3.3.2.

For the convolution term $V_2(t) := \int_0^t S(t-s)\mathbb{P}\nabla \cdot (u(s) \otimes v(s))\, ds$ we have

$$V_2(t+\delta) - V_2(t) = I_1(t,\delta) + I_2(t,\delta),$$

with auxiliary terms

$$I_1(t,\delta) := \int_t^{t+\delta} S(t+\delta-s)\mathbb{P}\nabla \cdot (u \otimes v)(s)\, ds,$$

and

$$I_2(t,\delta) := \int_0^t S(s)\mathbb{P}\nabla \cdot \left[(u \otimes v)(t+\delta-s) - (u \otimes v)(t-s)\right]\, ds.$$

For I_1 we have by Lemma 5.2.5 with $\alpha = 0$ that

$$\|I_1(t,\delta)\|_{\infty,1} \le C \int_t^{t+\delta} (t+\delta-s)^{-1/2} \|v(s)\|_{1,\infty,1} \|v(s)\|_{\infty,1}\, ds.$$

Setting

$$K(t) := \sup_{0<s<t} s^{1/2}\|v(s)\|_{1,\infty,1}, \quad H(t) := \sup_{0<s<t} \|v(s)\|_{\infty,1},$$

we then have

$$\|I_1(t,\delta)\|_{\infty,1} \le CK(t+\delta)H(t+\delta) \int_t^{t+\delta} (t+\delta-s)^{-1/2} s^{-1/2}\, ds.$$

Replacing $\delta > 0$ with $t\delta > 0$, the latter integral can be further estimated via

$$\begin{aligned}\int_t^{t+t\cdot\delta} (t+\delta-s)^{-1/2}s^{-1/2}\,ds &= \int_1^{1+\delta} (1+\delta-u)^{-1/2}u^{-1/2}\,du\\ &\le \int_1^{1+\delta} (1+\delta-u)^{-1/2}\,du\\ &= \int_0^{\delta} x^{-1/2}\,dx\\ &= 2\delta^{1/2},\end{aligned}$$

where we used the substitutions $u = t\cdot s$ and $x = 1+\delta-u$. This yields $I_1(t,t\delta) \to 0$ for $\delta \to 0$.

For ∇I_1 we write $\nabla S(t) = \nabla S(t/2)S(t/2)$ and apply Theorem 4.2.1.(i) and Lemma 5.2.5 with $\alpha = 1/2$ to obtain

$$\|I_2(t,t\delta)\|_{\infty,1} \le CK(t+t\delta)^{3/2}H(t+t\delta)^{1/2}\int_t^{t+t\delta} (t+t\delta-s)^{-3/4}s^{-3/4}\,ds.$$

Using the same substitutions yields

$$\begin{aligned}\int_t^{t+t\delta} (t+t\delta-s)^{-3/4}s^{-3/4}\,ds &= t^{-1/2}\int_1^{1+\delta} (1+\delta-u)^{-3/4}u^{-3/4}\,du\\ &\le t^{-1/2}\int_1^{1+\delta} (1+\delta-u)^{-3/4}\,du\\ &= t^{-1/2}\int_0^{\delta} x^{-3/4}\,dx\\ &= t^{-1/2}4\delta^{1/4},\end{aligned}$$

and therefore $\nabla I_1(t,t\delta) \to 0$ for $\delta \to 0$ and $t > 0$.

For the term $I_2(t,\delta)$ we likewise apply Lemma 5.2.5 with $\alpha = 0$, yielding

$$\|I_2(t,\delta)\|_{\infty,1} \le C\int_0^t s^{-1/2}\left\|(u\otimes v)(t+\delta-s)-(u\otimes v)(t-s)\right\|_{\infty,1}\,ds.$$

Here we further have

$$\begin{aligned}(u\otimes v)(t+\delta-s)-(u\otimes v)(t-s) =&[u(t+\delta-s)-u(t-s)]\otimes v(t+\delta-s)\\ &+u(t-s)\otimes[v(t+\delta-s)-v(t-s)],\end{aligned}$$

yielding

$$\begin{aligned}&\|(u\otimes v)(t+\delta-s)-(u\otimes v)(t-s)\|_{\infty,1}\\ &\le C\|v(t+\delta-s)-v(t-s)\|_{1,\infty,1}\|v(t+\delta-s)\|_{\infty,1},\\ &+C\|v(t-s)\|_{1,\infty,1}\|v(t+\delta-s)-v(t-s)\|_{\infty,1},\end{aligned}$$

compare the proof of Lemma 5.2.5. We now utilize the estimate

$$\int_0^t s^{-1/2}\|v(t+\delta-s)-v(t-s)\|_{1,\infty,1}\|v(t+\delta-s)\|_{\infty,1}\,ds$$
$$\leq K(t+\delta)\int_0^t s^{-1/2}(t-s)^{-1/2}(t-s)^{1/2}\|v(t+\delta-s)-v(t-s)\|_{1,\infty,1}\,ds.$$

By our assumption on v, the term $(t-s)^{1/2}\|v(t+\delta-s)-v(t-s)\|_{1,\infty,1}$ vanishes for $\delta\to 0$ for almost all $s\in(0,t)$ and is uniformly bounded for $s\in(0,t)$, $\delta\in(0,1)$. Since the value of the integral

$$\int_0^t s^{-1/2}(t-s)^{-1/2}=\int_0^1 u^{-1/2}(1-u)^{-1/2}\,du$$

is finite, it follows that the right-hand side vanishes for $\delta\to 0$ by the dominated convergence theorem. The remaining term

$$\int_0^t s^{-1/2}\|v(t-s)\|_{1,\infty,1}\|v(t+\delta-s)-v(t-s)\|_{\infty,1}\,ds$$

is treated analogously. This yields $\nabla I_2(t,\delta)\to 0$ for $\delta\to 0$. The remaining term $\nabla I_2(t,\delta)$ is treated similarly via the same arguments we used for $\nabla I_1(t,\delta)$. This concludes the proof. □

We are now able to prove our first main result.

Proof of Theorem 5.2.1. We construct the solution v as the limit of the sequence $(v_n)_{n\in\mathbb{N}}$, recursively defined by $v_0(t):=S(t)a$ and

$$v_{n+1}(t):=S(t)a-\int_0^t S(t-s)\mathbb{P}\nabla\cdot(u_n\otimes v_n)(s)\,ds,\quad n\in\mathbb{N}.$$

For this purpose we consider the space

$$\mathcal{S}(T):=\{v\in C_b((0,T);X_{\overline{\sigma}}^{\infty,1}(\mathbb{L})):t^{1/2}\nabla v\in L^\infty(0,T;L_H^\infty L_z^1(\mathbb{L}))^2\}$$

endowed with the norm

$$\|v\|_{\mathcal{S}(T)}:=\max\left\{\sup_{0<t<T}\|v(t)\|_{\infty,1},\ \sup_{0<t<T}t^{1/2}\|\nabla v(t)\|_{\infty,1}\right\}.$$

The fact that this sequence belongs to $\mathcal{S}(T)$ for suitable $T>0$ follows from Lemma 5.2.7 together with the uniform estimates established in the next step.

Step 1: *Uniform boundedness.* We begin by establishing recursive inequalities for the quantities

$$H_n(t):=\sup_{0<s<t}\|v_n(s)\|_{\infty,1},\quad K_n(t):=\sup_{0<s<t}s^{1/2}\|v_n(s)\|_{1,\infty,1}.$$

By Theorem 4.2.1 it holds for all $t > 0$ that

$$\sup_{0<s<t} \|v_0(s)\|_{\infty,1} \le \|a\|_{\infty,1}, \quad \sup_{0<s<t} s^{1/2}\|\nabla v_0(s)\|_{\infty,1} \le C\|a\|_{\infty,1}.$$

Let $n \in \mathbb{N}$ and $t > 0$ be arbitrary. We apply Lemma 5.2.5 for $\alpha = 0$, yielding

$$\begin{aligned} \|v_{n+1}(t)\|_{\infty,1} &\le \|S(t)a\|_{\infty,1} + C\int_0^t (t-s)^{-1/2}\|v_n(s)\|_{1,\infty,1}\|v_n(s)\|_{\infty,1}\, ds \\ &\le \|a\|_{\infty,1} + C_1 H_n(t)K_n(t) \end{aligned} \tag{5.2.4}$$

for a constant $C_1 > 0$, where we used that S is contractive and that the value of the integral

$$\int_0^t (t-s)^{-1/2}s^{-1/2}\, ds = \int_0^1 (1-s)^{-1/2}s^{-1/2}\, ds < \infty$$

does not depend on $t > 0$. This yields the estimate

$$H_{n+1}(t) \le \|a\|_{\infty,1} + C_1 H_n(t)K_n(t), \tag{5.2.5}$$

and by multiplying both sides in the first line of (5.2.4) we obtain

$$\sup_{0<s<t} s^{1/2}\|v_{n+1}(s)\|_{\infty,1} \le \sup_{0<s<t} s^{1/2}\|S(s)a\|_{\infty,1} + C_1 t^{1/2} H_n(t)K_n(t). \tag{5.2.6}$$

For the gradient estimate, we split the derivative of the semigroup into

$$\nabla S(t-s) = \nabla S\left(\frac{t-s}{2}\right) S\left(\frac{t-s}{2}\right),$$

and so using the estimates (i) in Theorem 4.2.1 as well as Lemma 5.2.5 for $\alpha = 1/2$ yields

$$\|\nabla v_{n+1}(t)\|_{\infty,1} \le \|\nabla S(t)a\|_{\infty,1} + C\int_0^t (t-s)^{-3/4}\|v_n(s)\|_{1,\infty,1}^{3/2}\|v_n(s)\|_{\infty,1}^{1/2}\, ds.$$

By multiplying both sides of this inequality by $t^{1/2}$, it follows that

$$\sup_{0<s<t} s^{1/2}\|\nabla v_{n+1}(s)\|_{\infty,1} \le \sup_{0<s<t} s^{1/2}\|\nabla S(s)a\|_{\infty,1} + C_2 H_n(t)^{1/2}K_n(t)^{3/2} \tag{5.2.7}$$

since the value of the integral

$$t^{1/2}\int_0^t (t-s)^{-3/4}s^{-3/4}\, ds = \int_0^1 (1-s)^{-3/4}s^{-3/4}\, ds < \infty$$

likewise does not depend on $t > 0$. We now add estimates (5.2.6) and (5.2.7) together to obtain

$$K_{n+1}(t) \le K_0(t) + C_2 H_n(t)^{1/2}K_n(t)^{3/2} + C_3 t^{1/2} H_n(t)K_n(t). \tag{5.2.8}$$

We now apply Lemma 5.2.6 to the sequences $(H_n(t))_{n\in\mathbb{N}}$ and $(K_n(t))_{n\in\mathbb{N}}$. For this purpose, we note that

$$H_0(t) = \sup_{0<s<t} \|S(s)a\|_{\infty,1} \le \|a\|_{\infty,1}$$

since S is contractive, as well as

$$\begin{aligned} K_0(t) &= \sup_{0<s<t} s^{1/2}\|S(t)a\|_{1,\infty,1} \\ &\le t^{1/2}\|a\|_{\infty,1} + \sup_{0<s<t} s^{1/2}\|\nabla S(t)a_1\|_{\infty,1} + \sup_{0<s<t} s^{1/2}\|\nabla S(t)a_2\|_{\infty,1} \\ &\le t^{1/2}\|a\|_{\infty,1} + \sup_{0<s<t} s^{1/2}\|\nabla S(t)a_1\|_{\infty,1} + C_4\|a_2\|_{\infty,1}, \end{aligned}$$

where $C_4 > 0$ is from estimate (i) in Theorem 4.2.1. By Theorem 4.2.1.4 and the continuity of a_1, the first and second right-hand side terms converge to 0 for $t \to 0$. Therefore, we may take $T_0 = T_0(\|a\|_{\infty,1}, a_1) > 0$ to be so small that

$$T_0^{1/2}\|a\|_{\infty,1} + \sup_{0<s<T_0} s^{1/2}\|\nabla S(t)a_1\|_{\infty,1} \le C_4\|a_2\|_{\infty,1}$$

and this implies that $K_0(t) \le 2C_4\|a_2\|_{\infty,1}$ for all $t \in (0, T_0)$. We now choose the parameters

$$c_0 := \|a\|_{\infty,1}, \quad c_1 := C_1, \quad c_2 := C_2.$$

Due to the assumption

$$\|a\|_{\infty,1} \cdot \|a_2\|_{\infty,1} \le C_0,$$

we may set $\gamma := 3C_4\|a_2\|_{\infty,1}$ and take $C_0, \delta = \delta(\|a\|_{\infty,1}) > 0$ to be so small that

$$18c_0c_2^2\gamma = 54C_2^2\|a\|_{\infty,1} \cdot \|a_2\|_{\infty,1} \le 54C_0C_2^2 \le 1, \quad 6c_0\delta = 6\delta\|a\|_{\infty,1} \le 1.$$

We further take $C_0, T_0 > 0$ to be so small that

$$K_0(t) \le 2C_4\|a_2\|_{\infty,1} \le 2C_4C_0 \le \varepsilon = \varepsilon(c_0, c_1, c_2, \gamma, \delta),$$

as well as $C_3T_0^{1/2} \le \delta$. Then, by the recursive estimates (5.2.5) and (5.2.8), the assumptions of Lemma 5.2.6 are satisfied and we obtain

$$H_n(t) \le 2\|a\|_{\infty,1}, \quad K_n(t) \le 3C_4\|a_2\|_{\infty,1} \tag{5.2.9}$$

for all $t \in (0, T_0)$ and $n \in \mathbb{N}$. In particular, we have that $(v_n)_{n\in\mathbb{N}}$ is a bounded sequence in $\mathcal{S}(T)$.

Step 2: *Convergence.* Consider the auxiliary sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$ defined by

$$\mathcal{V}_n := v_{n+1} - v_n, \quad n \in \mathbb{N},$$

as well as

$$\mathcal{H}_n(t) := \sup_{0<s<t} \|\mathcal{V}_n(s)\|_{\infty,1}, \quad \mathcal{K}_n(t) := \sup_{0<s<t} s^{1/2}\|\mathcal{V}_n(s)\|_{1,\infty,1}.$$

We use the representation

$$\mathcal{V}_{n+1}(t) = \int_0^t S(t-s)\mathbb{P}\nabla \cdot (\mathcal{U}_n(s) \otimes v_n(s) + u_n(s) \otimes \mathcal{V}_n(s))\, ds,$$

where $\mathcal{U}_n = (\mathcal{V}_n, \mathcal{W}_n)$ and $\mathcal{W}_n$ is determined by $\mathcal{V}_n$ via the relation (5.0.2), and apply Lemma 5.2.5 for $\alpha = 1/2$ to obtain

$$\|\mathcal{V}_{n+1}(t)\|_{\infty,1} \le 2C \int_0^t (t-s)^{-1/4} G_{1/2}(v_n(s), \mathcal{V}_n(s))\, ds.$$

Since we have

$$G_{1/2}(v_n(s), \mathcal{V}_n(s)) \le s^{-3/4} \left(K_n(t)\mathcal{H}_n(t) + H_n(t)\mathcal{K}_n(t)\right)^{1/2} K_n(t)^{1/2}\mathcal{K}_n(t)^{1/2}$$

for all $0 < s < t < T$ and the value of the integral

$$\int_0^t (t-s)^{-1/4} s^{-3/4}\, ds = \int_0^1 (1-s)^{-1/4} s^{-3/4}\, ds < \infty$$

does not depend on $t > 0$, it follows that

$$\mathcal{H}_{n+1}(t) \le C_5 \left(K_n(t)\mathcal{H}_n(t) + H_n(t)\mathcal{K}_n(t)\right)^{1/2} K_n(t)^{1/2}\mathcal{K}_n(t)^{1/2}.$$

Proceeding analogously for $t^{1/2}\|\mathcal{V}_n(t)\|_{\infty,1}$ and $t^{1/2}\|\nabla\mathcal{V}_n(t)\|_{\infty,1}$ then yields

$$\mathcal{K}_{n+1}(t) \le C_5(1+t^{1/2}) \left(K_n(t)\mathcal{H}_n(t) + H_n(t)\mathcal{K}_n(t)\right)^{1/2} K_n(t)^{1/2}\mathcal{K}_n(t)^{1/2}.$$

Setting $\mathcal{N}_n(t) := \max\{\mathcal{H}_n(t), \mathcal{K}_n(t)\}$, we obtain the recursive estimate

$$\mathcal{N}_{n+1}(t) \le C_5(1+T_0^{1/2}) \left(H_n(t) + K_n(t)\right)^{1/2} K_n(t)^{1/2}\mathcal{N}_n(t)$$

for all $t \in (0, T_0)$. Due to (5.2.9), we may take $C_0, T_0 > 0$ to be so small that

$$\mathcal{N}_{n+1}(t) \le \frac{1}{2}\mathcal{N}_n(t)$$

for all $t \in (0, T_0)$ and $n \in \mathbb{N}$. This implies that $(v_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{S}(T_0)$. A similar approach shows that $(v_n - Sa_2)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T_0]; X_{\overline{\sigma}}^{\infty,1}(\mathbb{L}))$. Denoting the limit of $(v_n)_{n\in\mathbb{N}}$ by v, it is clear that v is a mild solution to the primitive equations, i.e., that it satisfies the integral equation (5.0.6). The gradient estimate

$$\limsup_{t\to 0} t^{1/2}\|\nabla v(t)\|_{\infty,1} \le C\|a_2\|_{\infty,1}$$

follows from (5.2.9).

Step 3: *Uniqueness.* Let $\tilde{v}$ be another such solution on $[0, T_0]$ and consider the auxiliary quantities

$$\begin{aligned}
\mathcal{H}(t) &:= \sup_{0<s<t} \|v(s) - \tilde{v}(s)\|_{\infty,1}, & \mathcal{K}(t) &:= \sup_{0<s<t} s^{1/2} \|\nabla v(s) - \nabla \tilde{v}(s)\|_{\infty,1}, \\
H(t) &:= \sup_{0<s<t} \|v(s)\|_{\infty,1}, & K(t) &:= \sup_{0<s<t} s^{1/2} \|\nabla v(s)\|_{\infty,1}, \\
\tilde{H}(t) &:= \sup_{0<s<t} \|\tilde{v}(s)\|_{\infty,1}, & \tilde{K}(t) &:= \sup_{0<s<t} s^{1/2} \|\nabla \tilde{v}(s)\|_{\infty,1}.
\end{aligned}$$

Then the same arguments as in the previous step yield the estimates

$$\begin{aligned}
\mathcal{H}(t) \le{}& C_5 (H(t)\mathcal{K}(t) + K(t)\mathcal{H}(t))^{1/2} K(t)^{1/2} \mathcal{K}(t)^{1/2} \\
&+ C_5 (\tilde{H}(t)\mathcal{K}(t) + \tilde{K}(t)\mathcal{H}(t))^{1/2} \tilde{K}(t)^{1/2} \mathcal{K}(t)^{1/2}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{K}(t) \le{}& C_5 (1 + t^{1/2}) (H(t)\mathcal{K}(t) + K(t)\mathcal{H}(t))^{1/2} K(t)^{1/2} \mathcal{K}(t)^{1/2} \\
&+ C_5 (1 + t^{1/2}) (\tilde{H}(t)\mathcal{K}(t) + \tilde{K}(t)\mathcal{H}(t))^{1/2} \tilde{K}(t)^{1/2} \mathcal{K}(t)^{1/2}.
\end{aligned}$$

Setting $\mathcal{N}(t) := \max\{\mathcal{H}(t), \mathcal{K}(t)\}$, this yields

$$\mathcal{N}(t) \le C_5 (1 + T_0^{1/2}) \Big((H(t) + K(t))^{1/2} K(t)^{1/2} + (\tilde{H}(t) + \tilde{K}(t))^{1/2} \tilde{K}(t)^{1/2} \Big) \mathcal{N}(t). \tag{5.2.10}$$

By our assumption on the solutions v and $\tilde{v}$ we have

$$\lim_{t \to 0} K(t) \le C_6 \|a_2\|_{\infty,1}, \quad \lim_{t \to 0} \tilde{K}(t) \le C_6 \|a_2\|_{\infty,1}.$$

The same argument used to derive (5.2.5) then yields

$$\lim_{t \to 0} \tilde{H}(t) \le \|a\|_{\infty,1} + C_1 \tilde{H}(t) \tilde{K}(t)$$

and thus

$$H(t) \le 2\|a\|_{\infty,1}, \quad \tilde{H}(t) \le 2\|a\|_{\infty,1}$$

for all $t \in (0, T_1)$ if $T_1 > 0$ is sufficiently small, where we also used (5.2.9). Applying these estimates to (5.2.10) yields

$$\mathcal{N}(t) \le C_6 (1 + T_1^{1/2}) \left(\|a\|_{\infty,1} + \|a_2\|_{\infty,1} \right) \|a_2\|_{\infty,1}^{1/2} \mathcal{N}(t) \tag{5.2.11}$$

for $t \in (0, T_1)$, so by taking $T_0 > 0$ and the upper bound $C_0 > 0$ to be sufficiently small we obtain $\mathcal{N}(t) \le \frac{1}{2}\mathcal{N}(t)$ for all $t \in (0, T_0]$ and this yields $v(t) = \tilde{v}(t)$ for all $t \in [0, T_0]$. Now let

$$t^* := \sup\{t \in (0, T_0) : v(s) = \tilde{v}(s) \text{ for all } 0 \le s \le t\}$$

and assume that $t^* < T_0$. By the argument above we have $t^* \ge T_1 > 0$ and thus

$$a^* := v(t^*) = \tilde{v}(t^*)$$

by the continuity of solutions on $(0,T_0)$. We now repeat the argument for the new initial value a^*. Since $t^{1/2}\nabla v, t^{1/2}\nabla\tilde v \in L^\infty((0,T_0);L^\infty_H L^1_z(\mathbb{L}))^2$ yields

$$\nabla v, \nabla\tilde v \in L^\infty((t^*,T_0);L^\infty_H L^1_z(\mathbb{L}))^2,$$

we obtain

$$\limsup_{t\to0} t^{1/2}\|\nabla v(t^*+t)\|_{\infty,1} = \limsup_{t\to0} t^{1/2}\|\nabla\tilde v(t^*+t)\|_{\infty,1} = 0$$

as well as

$$\limsup_{t\to0}\|v(t^*+t)\|_{\infty,1} \le 2\|a^*\|_{\infty,1}, \quad \limsup_{t\to0}\|\tilde v(t^*+t)\|_{\infty,1} \le 2\|a^*\|_{\infty,1}.$$

It follows that estimate (5.2.11) also holds for all $t\in(t^*,T_2)$ with $t^*<T_2<T_0$. Taking $C_0>0$ to be sufficiently small we thus obtain $v=\tilde v$ on $(0,T_2)$ which contradicts the definition of t^* and thus we have $v=\tilde v$ on $(0,T_0)$. This completes the proof. □

We now prove the additional regularity of the solution for initial data belonging to $L^\infty_H L^p_z(\mathbb{L})^2$.

Proof of Proposition 5.2.4. We write $S(t)=S(t/2)S(t/2)$ and use the vertical L^1-L^p-smoothing estimate for S from Theorem 4.2.1.(b), the fact that S is contractive as well as Lemma 5.2.5 for $\alpha=0$ to obtain

$$\|v(t)\|_{\infty,p} \le \|a\| + \left(\int_0^t \left[1+(t-s)^{-(1-1/p)/2}\right](t-s)^{-1/2}s^{-1/2}\,ds\right)K(t)H(t)$$

with $K(t):=\sup_{0<s<t}\|v(s)\|_{1,\infty,1}$ and $H(t):=\sup_{0<s<t}\|v(s)\|_{\infty,1}$. Since the values of the integrals

$$\int_0^t (t-s)^{-1/2}s^{-1/2}\,ds, \quad t^{(1-1/p)/2}\int_0^t (t-s)^{-(1-1/p)/2}(t-s)^{-1/2}s^{-1/2}\,ds$$

do not depend on $t>0$, the estimate for $t^{(1-1/p)/2}v$ follows. For the gradient estimate we write $\nabla S(t)=S(t/3)S(t/3)S(t/3)$ and use Theorem 4.2.1.(i), the $L^\infty_H L^1_z(\mathbb{L})$-$L^\infty_H L^p_z(\mathbb{L})$-smoothing estimate and Lemma 5.2.5 for $\alpha\in(1-1/p,1)$ to obtain

$$\begin{aligned}\|\nabla v(t)\|_{\infty,p} &\le \|\nabla S(t)a\|_{\infty,p}\\ &\quad + C\int_0^t \left[1+(t-s)^{-(1-1/p)/2}\right](t-s)^{-(1-\alpha/2)}s^{-(1+\alpha)/2}\,ds K(t)^{1+\alpha}H(t)^{1-\alpha}.\end{aligned}$$

Since the values of the integrals

$$t^{1/2}\int_0^t (t-s)^{-(1-\alpha/2)}s^{-(1+\alpha)/2}\,ds, \quad t^{1-1/2p}\int_0^t (t-s)^{-(1-1/p)/2-(1-\alpha/2)}s^{-(1+\alpha)/2}\,ds$$

likewise do not depend on $t>0$ and are finite for $\alpha\in(1-1/p,1)$, the estimate for $t^{1-1/2p}\nabla v$ follows as well. In the cases (ii) and (iii) the continuity at $t=0$ follows from the strong continuity of S, and in the case (iii) the continuity in the variables $(x,y,z)\in\mathbb{L}$ follows from the fact that both S_H and S_N preserve continuity. □

With Proposition 5.2.4 and our result for local existence, we are now able to derive our second main result concerning global, strong well-posedness.

Proof of Theorem 5.2.2. Let a be horizontally periodic. Since S_H and $\mathbb{P}$ and the nonlinear term $(u \cdot \nabla)v$ preserve horizontal periodicity, the way we constructed the solution v in the proof of Theorem 5.2.1 yields that v is also horizontally periodic. We may assume without loss of generality that the period is 1. Since we have $v(t), \nabla v(t) \in L^\infty_H L^1_z(\mathbb{L})^2$ for all $t > 0$, the embeddings

$$W^{1,1}(-h,0) \subset L^p(-h,0) \quad \text{and} \quad L^\infty(G) \hookrightarrow L^p(G)$$

for $G = (0,1)^2$ and all $p \in (1,\infty)$ imply that

$$v(t)\big|_\Omega \in H^{1,p}(\Omega)^2$$

for all $t > 0$ and $\Omega = G \times (-h,0)$. Due to the global, strong well-posedness of the primitive equations with Neumann boundary conditions on $\Gamma_u \cup \Gamma_b$ as in (3.4.2), we take $v(t_0)$ for $t_0 > 0$ as the initial data and obtain a strong extension onto $\Omega \times (t_0,\infty)$ that is even real analytic by Theorem 5.1.4. Due to the uniqueness of mild solutions and horizontal periodicity, we obtain global, strong well-posedness on $\mathbb{L} \times (0,\infty)$. $\square$

5.3 The case of Dirichlet boundary conditions

In this section, we consider the primitive equations (5.0.1) for the case $f = 0$ and the boundary conditions (5.0.4) as in [49]. The results presented here have been previously published in [39, Sections 3 and 8]. For $p \in (1, \infty)$, recall the spaces

$$\begin{aligned} L^{\infty,p}_{\overline{\sigma}}(\Omega) &= L^\infty_H L^p_z(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega), \\ X^{\infty,p}_{\overline{\sigma}}(\Omega) &= C_{\mathrm{per}}([0,1]^2; L^p(-h,0))^2 \cap L^p_{\overline{\sigma}}(\Omega), \end{aligned}$$

from (4.3.3) and (4.3.5), respectively. Then, using the notation $\|\cdot\|_{\infty,p} := \|\cdot\|_{L^\infty_H L^p_z(\Omega)}$, the main result of this section is the following.

Theorem 5.3.1. *Let $f = 0$ and $p \in (3, \infty)$. Then there exists a constant $C_0 = C_0(\Omega, p) > 0$ such that for any initial data of the form $a = a_1 + a_2$ with*

$$a_1 \in X^{\infty,p}_{\overline{\sigma}}(\Omega) \quad \text{and} \quad a_2 \in L^{\infty,p}_{\overline{\sigma}}(\Omega)$$

such that

$$\|a_2\|_{\infty,p} \leq C_0,$$

there exists a unique, global strong solution (v, π) to the primitive equations (5.0.1) with boundary conditions (5.0.4) satisfying

$$v \in C_b((0,\infty); X^{\infty,p}_{\overline{\sigma}}(\Omega)) \cap C_b([0,\infty); L^p_{\overline{\sigma}}(\Omega)), \quad t^{1/2}\nabla v \in L^\infty(0,\infty; L^\infty_H L^p_z(\Omega))^2,$$

as well as

$$\limsup_{t\to 0+} t^{1/2}\|\nabla v(t)\|_{\infty,p} \leq C_2 \|a_2\|_{\infty,p}$$

for a constant $C_2 = C_2(\Omega, p) > 0$. The solution decays exponentially as in (5.0.5) and is real analytic on $\Omega \times (0,\infty)$. If $a_2 = 0$, then it even holds that $v \in C([0,\infty); X^{\infty,p}_{\overline{\sigma}}(\Omega))$.

As in Section 5.2, we utilize a number of $L^\infty_H L^p_z$-estimates for the hydrostatic Stokes semigroup S applied to the nonlinear term $(u \cdot \nabla)v$.

Lemma 5.3.2. *Let $p \in (3, \infty)$. Then there exists a constant $C > 0$ such that for all $t > 0$ and $v_1, v_2 \in L^{\infty,p}_{\overline{\sigma}}(\Omega)$ with $\nabla v_1, \nabla v_2 \in L^\infty_H L^p_z(\Omega)^2$, as well as $v_1\big|_{\Gamma_b} = v_1\big|_{\Gamma_b} = 0$ it holds that*

$$\|e^{tA}\mathbb{P}(u_1 \cdot \nabla)v_2\|_{\infty,p} \leq Ct^{-1/2}\|\nabla v_1\|_{\infty,p}\|v_2\|_{\infty,p}, \tag{i}$$
$$\|\nabla e^{tA}\mathbb{P}(u_1 \cdot \nabla)v_2\|_{\infty,p} \leq Ct^{-1/2}\|\nabla v_1\|_{\infty,p}\|\nabla v_2\|_{\infty,p}, \tag{ii}$$
$$\|\nabla e^{tA}\mathbb{P}(u_1 \cdot \nabla)v_2\|_{\infty,p} \leq Ct^{-1}\|\nabla v_1\|_{\infty,p}\|v_2\|_{\infty,p}, \tag{iii}$$

and for $\{i,j\} = \{1,2\}$ it holds that

$$\|e^{tA}\mathbb{P}(u_1 \cdot \nabla)v_2\|_{\infty,p} \leq C\left(t^{-1/2}\|\nabla v_i\|_{\infty,p}\|v_j\|_{\infty,p} + \|\nabla v_1\|_{\infty,p}\|\nabla v_2\|_{\infty,p}\right), \tag{iv}$$

where $u_k = (v_k, w_k)$ with $w_k = w_k(v_k)$ as in (5.0.2) for $k \in \{1,2\}$.

Proof. Let $t > 0$. The embeddings

$$W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega) \hookrightarrow L^\infty_H L^p_z(\Omega) \hookrightarrow L^p(\Omega), \quad p \in (3, \infty),$$

as well as the vertical Poincaré inequality for $k \in \{1, 2\}$ imply the estimates

$$\|v_k\|_{L^\infty(\Omega)} \le C\|v_k\|_{W^{1,p}(\Omega)} \le C\|\nabla v_k\|_{L^p(\Omega)} \le C\|\nabla v_k\|_{\infty,p}$$

and

$$\|w_k\|_\infty \le C\|\partial_z w_k\|_{\infty,p} \le C\|\mathrm{div}_H v_k\|_{\infty,p} \le C\|\nabla v\|_{\infty,p}.$$

Using the anisotropic Hölder inequality then yields

$$\begin{aligned} \|v_1 \otimes v_2\|_{\infty,p} &\le \|\nabla v_i\|_{L^\infty}\|v_j\|_{\infty,p} \\ &\le C\|v_i\|_{\infty,p}\|v_j\|_{\infty,p}, \quad \{i,j\} = \{1,2\}, \\ \|w_1 v_2\|_{\infty,p} &\le C\|w_1\|_{L^\infty(\Omega)}\|v_2\|_{\infty,p} \\ &\le C\|\nabla v_1\|_{\infty,p}\|v_2\|_{\infty,p}. \end{aligned} \tag{5.3.1}$$

We now obtain estimates (i) and (iii) by using $\mathrm{div}\, u_k = 0$ for $k \in \{1, 2\}$ to rewrite the bilinear term as

$$(u_1 \cdot \nabla)v_2 = \nabla \cdot (u_1 \otimes v_2) = \nabla_H \cdot (v_1 \otimes v_2) + \partial_z(w_1 v_2) \tag{5.3.2}$$

and applying Theorem 4.3.2.(ii) and (iv), yielding

$$\begin{aligned} \|S(t)\mathbb{P}(u_1 \cdot \nabla)v_2\|_{\infty,p} &\le \|S(t)\mathbb{P}\nabla_H \cdot (v_1 \otimes v_2)\|_{\infty,p} + \|S(t)\mathbb{P}\partial_z(w_1 v_2)\|_{\infty,p} \\ &\le Ct^{-1}\|v_1 \otimes v_2\|_{\infty,p} + Ct^{-1/2}\|w_1 v_2\|_{\infty,p} \\ &\le Ct^{-1/2}\|\nabla v_1\|_{\infty,p}\|v_2\|_{\infty,p}, \\ \|\nabla S(t)\mathbb{P}(u_1 \cdot \nabla)v_2\|_{\infty,p} &\le Ct^{-1/2}\|S(t/2)\mathbb{P}(u_1 \cdot \nabla)v_2\|_{\infty,p} \\ &\le Ct^{-1}\|\nabla v_1\|_{\infty,p}\|v_2\|_{\infty,p}. \end{aligned}$$

For the estimate (ii), we use that

$$\begin{aligned} \|(u_1 \cdot \nabla)v_2\|_{\infty,p} &\le \|(v_1 \cdot \nabla_H)v_2\|_{\infty,p} + \|w_1 \partial_z v_2\|_{\infty,p} \\ &\le \|v_1\|_{L^\infty(\Omega)}\|\nabla_H v_2\|_{\infty,p} + \|w_1\|_{L^\infty(\Omega)}\|\partial_z v_2\|_{\infty,p} \\ &\le C\|\nabla v_1\|_{\infty,p}\|\nabla v_2\|_{\infty,p}, \end{aligned}$$

so the claim follows from Theorem 4.3.2.(iii). For estimate (iv), we apply the hydrostatic Stokes projection $\mathbb{P}$ to (5.3.2) to obtain

$$\begin{aligned} \mathbb{P}(u_1 \cdot \nabla)v_2 &= \mathbb{P}\nabla_H \cdot (v_1 \otimes v_2) + \mathbb{P}\partial_z(w_1 v_2) \\ &= \mathbb{P}\nabla_H \cdot (v_1 \otimes v_2) + \partial_z(w_1 v_2), \end{aligned}$$

where we used that $w_1 = 0$ on $\Gamma_u \cup \Gamma_b$ as well as $\mathbb{P}f = f + (1-Q)\overline{f}$ where $\overline{f}$ is the vertical average. Since it holds that $\partial_z(w_1 v_2) = -(\mathrm{div}_H v_1)\, v_2 + w_1 \partial_z v_2$ with

$$\begin{aligned}\|-(\mathrm{div}_H v_1)\, v_2\|_{\infty,p} &\leq \|\mathrm{div}_H v_1\|_{\infty,p}\|v_2\|_{L^\infty(\Omega)} \\ &\leq C\|\nabla v_1\|_{\infty,p}\|\nabla v_2\|_{\infty,p}, \\ \|w_1\partial_z v_2\|_{\infty,p} &\leq \|w_1\|_{L^\infty(\Omega)}\|\partial_z v_2\|_{\infty,p} \\ &\leq C\|\nabla v_1\|_{\infty,p}\|\nabla v_2\|_{\infty,p},\end{aligned}$$

we thus obtain the estimate

$$\begin{aligned}\|S(t)\mathbb{P}(u_1\cdot\nabla)v_2\|_{\infty,p} &\leq \|S(t)\mathbb{P}\nabla_H\cdot(v_1\otimes v_2)\|_{\infty,p} + \|S(t)\partial_z(w_1v_2)\|_{\infty,p} \\ &\leq Ct^{-1/2}\|v_1\otimes v_2\|_{\infty,p} + C\|\partial_z(w_1v_2)\|_{\infty,p} \\ &\leq C\left(t^{-1/2}\|\nabla v_i\|_{\infty,p}\|v_j\|_{\infty,p} + \|\nabla v_1\|_{\infty,p}\|\nabla v_2\|_{\infty,p}\right)\end{aligned}$$

for $\{i,j\} = \{1,2\}$, where we used (5.3.1) again. This completes the proof. □

We further make use of the following result concerning the existence of smooth solutions for smooth data, obtained from the maximal regularity approach to the primitive equations in Section 5.1. Recall the trace space

$$X_{\gamma,p,q} = (L^p_{\overline{\sigma}}(\Omega), D(A_p))_{1-1/q,q} \subset B^{2-2/q,q}_{p,q}(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega), \quad p,q\in(1,\infty).$$

Then, by Theorem 5.1.1 and Remark 2.7.4.3, the following holds.

Lemma 5.3.3. *Let $f = 0$, $p,q \in (1,\infty)$ with $1/p + 1/q \leq 1$, and $a \in X_{\gamma,p,q}$. Then there exists a unique, global strong solution v to the primitive equations* (5.0.1) *with boundary conditions* (5.0.4)*, satisfying*

$$v \in C([0,\infty); X_{\gamma,p,q}).$$

Due to this result, we deviate from the approach in Section 5.2 in the following way. After decomposing the initial data $a = a_{\mathrm{ref}} + a_0$ into a smooth part a_{ref} and a small part a_0, we consider the corresponding reference solution v_{ref} to the primitive equations with initial data $v_{\mathrm{ref}}(0) = a_{\mathrm{ref}}$ and then construct $V := v - v_{\mathrm{ref}}$ by an iteration scheme for the small initial data $V(0) = a_0$. For this purpose we also utilize the following result.

Lemma 5.3.4. *Let $c_0, c_1 > 0$ and $c_2 \in (0,1)$ be coefficients satisfying*

$$4c_0c_1 < (1-c_2)^2.$$

Further let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers satisfying $\alpha_0 \leq c_0$ as well as the recursive growth bound

$$\alpha_{n+1} \leq c_0 + c_1\alpha_n^2 + c_2\alpha_n \quad \textit{for all } n\in\mathbb{N}.$$

Then $(\alpha_n)_{n\in\mathbb{N}}$ is uniformly bounded with

$$\alpha_n < \frac{2}{1-c_2}c_0 \quad \textit{for all } n\in\mathbb{N}.$$

Proof. The assumptions on the coefficients imply that the polynomial

$$p(x) := c_0 + c_1 x^2 + c_2 x$$

has fixed points $0 < x_0 < x_1$ with

$$\begin{aligned} 0 < x_0 &= \frac{(1-c_2) - \sqrt{(1-c_2)^2 - 4c_1c_0}}{2c_1} \\ &= \frac{1}{2c_1} \frac{4c_1c_0}{(1-c_2) + \sqrt{(1-c_2)^2 - 4c_1c_0}} \\ &< \frac{2}{1-c_2} c_0. \end{aligned}$$

We further have

$$(1-c_2) + \sqrt{(1-c_2)^2 - 4c_1c_0} < 2(1-c_2) < 2,$$

which yields

$$\alpha_0 \leq c_0 = 2c_1 \frac{(1-c_2) + \sqrt{(1-c_2)^2 - 4c_1c_0}}{4c_1} x_0 < x_0.$$

Since p is an increasing function on $[0, \infty)$, we therefore obtain that

$$p(\alpha_0) \leq p(x_0) = x_0 < \frac{2}{1-c_2} c_0$$

and since we also have $\alpha_{n+1} \leq p(\alpha_n)$, the claim follows by induction. □

We are now able to prove our main result. The following proof uses many of the same arguments we have previously used to establish Theorem 5.2.1, with some modifications arising due to our different approach to the initial data.

Proof of Theorem 5.3.1.

Step 1: *Decomposition of data and solution.* Since S is strongly continuous on $X_{\overline{\sigma}}^{\infty,p}(\Omega)$ by Theorem 4.3.2, $D(A_{\infty,p})$ is dense in $X_{\overline{\sigma}}^{\infty,p}(\Omega)$. Given $a = a_1 + a_2$ with $a_1 \in X_{\overline{\sigma}}^{\infty,p}(\Omega)$, we may thus take $a_{\text{ref}} \in D(A_{\infty,p})$ and assume the remainder

$$a_0 := a - a_{\text{ref}} = (a_1 - a_{\text{ref}}) + a_2$$

to be sufficiently small. Due to $p \in (3, \infty)$, we can take $q \in (1, \infty)$ such that $2/q + 3/p < 1$ and apply Lemma 5.3.3 to obtain a solution v_{ref} to the primitive equations with initial data $v_{\text{ref}}(0) = a_{\text{ref}}$. The condition on p and q further yields the embedding

$$X_{\gamma,p,q} \subset B_{p,q}^{2-2/q}(\Omega)^2 \hookrightarrow C^1(\overline{\Omega})^2$$

by [89, Section 3.3.1], and therefore the auxiliary quantity

$$R(T) := \sup_{t \in [0,T]} \|\nabla v_{\text{ref}}(t)\|_{\infty,p} < \infty$$

is finite for all $T \in (0,\infty)$. For $T \in (0,\infty]$ we now consider the space

$$\mathcal{S}(T) := \{V \in C_b((0,\infty); X^{\infty,p}_{\overline{\sigma}}(\Omega)) : t^{1/2}\nabla V \in L^\infty(0,T; L^\infty_H L^p_z(\Omega))^2\}$$

endowed with the norm

$$\|V\|_{\mathcal{S}(T)} := \max\left\{\sup_{0<t<T}\|V(t)\|_{\infty,p}, \sup_{0<t<T} t^{1/2}\|\nabla V(t)\|_{\infty,p}\right\}$$

and further set $F(v_1,v_2) := -\mathbb{P}(u_1\cdot\nabla)v_2$ as in Lemma 5.3.2. We will construct a time $T \in (0,\infty]$ and a function $V \in \mathcal{S}(T)$ satisfying

$$V(t) = S(t)a_0 + \int_0^t S(t-s)\left(F(V,V)(s) + F(V,v_{\mathrm{ref}})(s) + F(v_{\mathrm{ref}},V)(s)\right)\, ds$$

for all $t \in (0,T)$. Then $v := V + v_{\mathrm{ref}}$ is a local mild solution to the primitive equations. For this purpose we consider the recursively defined sequence $(V_n)_{n\in\mathbb{N}}$ given by

$$V_0(t) = S(t)a_0, \quad V_{n+1}(t) = S(t)a_0 + \int_0^t S(t-s)F_n(s)\, ds,$$

where we further set $F_n := F(V_n,V_n) + F(V_n,v_{\mathrm{ref}}) + F(v_{\mathrm{ref}},V_n)$.

The fact that this sequence belongs to $\mathcal{S}(T)$ follows from the uniform boundedness we establish in the next step, which also implies their continuity for $t > 0$ by adapting the following arguments to the setting of Lemma 5.2.7.

Step 2: *Uniform boundedness.* It follows from Theorem 4.3.2 that

$$\|V_0\|_{\mathcal{S}(T)} = \max\left\{\sup_{0<t<T}\|S(t)a_0\|_{\infty,p}, \sup_{0<t<T} t^{1/2}\|\nabla S(t)a_0\|_{\infty,p}\right\} \le C\|a_0\|_{\infty,p} \tag{5.3.3}$$

for all $T \in (0,\infty]$. In order to estimate V_{n+1} with $n \in \mathbb{N}$, we set

$$H_n(t) := \sup_{0<s<t}\|V_n(s)\|_{\infty,p}, \quad K_n(t) := \sup_{0<s<t} s^{1/2}\|\nabla V_n(s)\|_{\infty,p},$$

and begin by establishing estimates for the gradient term. For this purpose, we take $t \in (0,T)$ for arbitrary $T \in (0,\infty)$ and use Lemma 5.3.2.(iii) and (ii) to obtain

$$\begin{aligned}\left\|\int_0^{t/2} \nabla S(t-s)F(V_n,V_n)(s)\, ds\right\|_{\infty,p} &\le C\int_0^{t/2}(t-s)^{-1}\|\nabla V_n(s)\|_{\infty,p}\|V_n(s)\|_{\infty,p}\, ds\\ &\le Ct^{-1/2}K_n(t)H_n(t),\end{aligned}$$

as well as

$$\begin{aligned}\left\|\int_{t/2}^{t} \nabla S(t-s)F(V_n,V_n)(s)\, ds\right\|_{\infty,p} &\le C\int_{t/2}^{t}(t-s)^{-1/2}\|\nabla V_n(s)\|^2_{\infty,p}\, ds\\ &\le Ct^{-1/2}K_n(t)^2.\end{aligned}$$

Here we used that the values of the integrals

$$t^{1/2} \int_0^{t/2} (t-s)^{-1} s^{-1/2}\, ds = \int_0^{1/2} (1-s)^{-1} s^{-1/2}\, ds < \infty,$$
$$t^{1/2} \int_{t/2}^{t} (t-s)^{-1/2} s^{-1}\, ds = \int_{1/2}^{1} (1-s)^{-1/2} s^{-1}\, ds < \infty$$

do not depend on $t > 0$. Applying Lemma 5.3.2.(ii) to the remaining terms yields

$$\begin{aligned}
\left\| \int_0^t \nabla S(t-s) F(V_n, V_{\mathrm{ref}})(s)\, ds \right\|_{\infty,p} &\leq C \int_0^t (t-s)^{-1/2} \|\nabla V_{\mathrm{ref}}(s)\|_{\infty,p} \|\nabla V_n(s)\|_{\infty,p}\, ds \\
&\leq C R(T) K_n(t), \\
\left\| \int_0^t \nabla S(t-s) F(V_{\mathrm{ref}}, V_n)(s)\, ds \right\|_{\infty,p} &\leq C R(T) K_n(t),
\end{aligned}$$

where we used that the value of the integral

$$\int_0^t (t-s)^{-1/2} s^{-1/2}\, ds = \int_0^1 (1-s)^{-1/2} s^{-1/2}\, ds < \infty \tag{5.3.4}$$

likewise does not depend on $t > 0$. By combining these estimates we obtain

$$K_{n+1}(t) \leq C_1 \left(\|a_0\|_{\infty,p} + K_n(t) H_n(t) + K_n(t)^2 + t^{1/2} R(T) K_n(t) \right), \tag{5.3.5}$$

where $C_1 = C_1(\Omega, p) > 0$ is a constant. We now turn to estimates without the gradient. For the first term, Lemma 5.3.2.(i) yields

$$\begin{aligned}
\left\| \int_0^t S(t-s) F(V_n, V_n)(s)\, ds \right\|_{\infty,p} &\leq C \int_0^t (t-s)^{-1/2} \|\nabla V_n(s)\|_{\infty,p} \|V_n(s)\|_{\infty,p}\, ds \\
&= C K_n(t) H_n(t),
\end{aligned}$$

where we used (5.3.4) again. By applying Lemma 5.3.2.(iv), it further follows that

$$\begin{aligned}
\left\| \int_0^t S(t-s) F(V_n, V_{\mathrm{ref}})(s)\, ds \right\|_{\infty,p} &\leq C \int_0^t (t-s)^{-1/2} \|\nabla v_{\mathrm{ref}}(s)\|_{\infty,p} \|V_n(s)\|_{\infty,p}\, ds \\
&\quad + C \int_0^t \|\nabla v_{\mathrm{ref}}(s)\|_{\infty,p} \|\nabla v_n(s)\|_{\infty,p}\, ds \\
&\leq C t^{1/2} R(T) \left(H_n(t) + K_n(t) \right), \\
\left\| \int_0^t S(t-s) F(V_{\mathrm{ref}}, V_n)(s)\, ds \right\|_{\infty,p} &\leq C t^{1/2} R(T) \left(H_n(t) + K_n(t) \right).
\end{aligned}$$

Here we used that

$$\int_0^t (t-s)^{-1/2}\, ds = t^{1/2} \int_0^1 (1-s)^{-1/2}\, ds = t^{1/2} \int_0^1 s^{-1/2}\, ds = \int_0^t s^{-1/2}\, ds < \infty$$

for $t > 0$. Combining these estimates yields

$$H_{n+1}(t) \le C_1 \bigg(\|a_0\|_{\infty,p} + K_n(t)H_n(t) + t^{1/2}R(T)H_n(t) + t^{1/2}R(T)K_n(t) \bigg). \tag{5.3.6}$$

We now chose $\|a_0\|_{\infty,p}$ and $T_0 > 0$ to be so small that

$$8C_1^2\|a_0\|_{\infty,p} < (1-1/4)^2, \quad 8C_1T_0^{1/2}R_0 < 1, \quad R_0 := R(T_0),$$

and combine (5.3.5) and (5.3.6) to obtain

$$\|V_{n+1}\|_{\mathbb{S}(t)} \le C_1\|a_0\|_{\infty,p} + 2C_1\|V_m\|_{\mathbb{S}(t)}^2 + \frac{1}{4}\|V_m\|_{\mathbb{S}(t)},$$

for all $t \in (0,T_0)$. Lemma 5.3.4 then yields the estimate

$$\|V_n\|_{\mathbb{S}(t)} \le \frac{8}{3}C_1\|a_0\|_{\infty,p} \tag{5.3.7}$$

for all $t \in (0,T_0)$. We now set $L_n := \limsup_{t\to 0+} K_n(t)$. Then we have

$$L_0 = \limsup_{t\to 0} t^{1/2}\|\nabla V_0\|_{\infty,p} \le \limsup_{t\to 0} t^{1/2}\|\nabla S(t)a_2\|_{\infty,p} \le C_1\|a_2\|_{\infty,p},$$

where we used Theorem 4.3.2.(d). By using this estimate instead of (5.3.3) we obtain

$$\begin{aligned} L_{n+1} &\le C_1\|a_2\|_{\infty,p} + C_1\left(\limsup_{t\to 0} H_n(t)\right)L_n + C_1L_n^2 + C_1\left(\limsup_{t\to 0} t^{1/2}\right)R_0L_n \\ &\le C_1\|a_2\|_{\infty,p} + \frac{8}{3}C_1^2\|a_0\|_{\infty,p}L_n + C_1L_n^2 \end{aligned}$$

via the same arguments used to derive estimate (5.3.5). We also estimated H_n via (5.3.7). We now take $\|a_0\|_{\infty,p}$ and $\|a_2\|_{\infty,p}$ to be so small that

$$\frac{8}{3}C_1\|a_0\|_{\infty,p} < \frac{1}{2}, \quad 4C_1^2\|a_2\|_{\infty,p} < \frac{1}{2}.$$

It follows that if $L_n \le 2C_1\|a_2\|_{\infty,p}$, then it also holds that

$$L_{n+1} \le \left(1 + \frac{8}{3}C_1\|a_0\|_{\infty,p} + 4C_1^2\|a_2\|_{\infty,p}\right)C_1\|a_2\|_{\infty,p} \le 2C_1\|a_2\|_{\infty,p}$$

and so by induction we obtain the estimate

$$L_n \le 2C_1\|a_2\|_{\infty,p} \tag{5.3.8}$$

for all $n \in \mathbb{N}$.

Step 3: *Convergence.* We now consider the sequence

$$\mathcal{V}_n := V_{n+1} - V_n, \quad n \in \mathbb{N}.$$

In order to estimate it, we make use of the representations

$$\mathcal{V}_n(t) = \int_0^t S(t-s)\left(F_n - F_{n-1}\right)(s)\,ds$$

and

$$F_n - F_{n-1} = F(\mathcal{V}_{n-1}, V_n) + F(V_{n-1}, \mathcal{V}_{n-1}) + F(\mathcal{V}_{n-1}, v_{\text{ref}}) + F(v_{\text{ref}}, \mathcal{V}_{n-1}).$$

Setting

$$\mathcal{H}_n(t) := \sup_{0<s<t} \|\mathcal{V}_n(t)\|_{\infty,p}, \quad \mathcal{K}_n(t) := \sup_{0<s<t} s^{1/2}\|\nabla\mathcal{V}_n(t)\|_{\infty,p},$$

we use the same arguments as in Step 2 to obtain the recursive inequalities

$$\mathcal{K}_n(t) \le C_1\left(H_n(t) + K_n(t) + K_{n-1}(t) + 2t^{1/2}R_0\right)\mathcal{K}_{n-1}(t) + C_1 K_{n-1}(t)\mathcal{H}_{n-1}(t)$$

and

$$\mathcal{H}_n(t) \le C_1\left(H_n + 2R_0 t^{1/2}\right)\mathcal{K}_{n-1}(t) + C_1\left(K_{n-1} + 2R_0 t^{1/2}\right)\mathcal{H}_{n-1}(t).$$

By combining them with estimate (5.3.7) and taking $\|a_0\|_{\infty,p}$ and $T_0 > 0$ to be sufficiently small, we thus obtain

$$\begin{aligned}
\|\mathcal{V}_n(t)\|_{\mathcal{S}(t)} &\le 2C_1\left(\|V_n\|_{\mathcal{S}(t)} + \|V_{n-1}\|_{\mathcal{S}(t)} + 2R_0 t^{1/2}\right)\|\mathcal{V}_{n-1}\|_{\mathcal{S}(t)} \\
&\le 2C_1\left(2\frac{8}{3}C_1\|a_0\|_{\infty,p} + 2R_0 T^{1/2}\right)\|\mathcal{V}_{n-1}\|_{\mathcal{S}(t)} \\
&\le \frac{1}{2}\|\mathcal{V}_{n-1}\|_{\mathcal{S}(t)},
\end{aligned}$$

for all $t \in (0, T_0)$. Since we have $V_{n+1} = V_0 + \sum_{k=0}^n \mathcal{V}_n$, it follows that $(V_n)_{n\in\mathbb{N}}$ converges in $\mathcal{S}(T_0)$. We denote its limit by V and set $v := V + v_{\text{ref}}$. Then v is a local mild solution to the primitive equations. The smoothness of v_{ref} then yields that $v \in \mathcal{S}(T_0)$ and the estimate (5.3.8) and

$$\limsup_{t\to 0} t^{1/2}\|\nabla v_{\text{ref}}(t)\|_{\infty,p} \le R \limsup_{t\to 0} t^{1/2} = 0, \tag{5.3.9}$$

yield that

$$\limsup_{t\to 0} t^{1/2}\|\nabla v(t)\|_{\infty,p} \le \limsup_{t\to 0} t^{1/2}\|\nabla V(t)\|_{\infty,p} \le 2C_1\|a_2\|_{\infty,p}.$$

Step 4: *Uniqueness of mild solutions.* We now show that v is unique among such local mild solutions. For this purpose we assume that $\tilde{v}$ is another such solution on $(0, T_0)$ and set $\tilde{V} := \tilde{v} - v_{\text{ref}}$. We then have the representation

$$v - \tilde{v} = V - \tilde{V} = \int_0^t S(t-s)\left(F(V-\tilde{V}, V) + F(\tilde{V}, V-\tilde{V})\right)(s)\, ds.$$

We set

$$\begin{aligned} &\mathcal{H}(t) := \sup\nolimits_{0<s<t}\|V(t)-\tilde{V}(t)\|_{\infty,p}, && \mathcal{K}(t) := \sup\nolimits_{0<s<t}\|\nabla V(t)-\nabla\tilde{V}(t)\|_{\infty,p},\\ &H(t) := \sup\nolimits_{0<s<t}\|V(t)\|_{\infty,p}, && K(t) := \sup\nolimits_{0<s<t}\|\nabla V(t)\|_{\infty,p},\\ &\tilde{H}(t) := \sup\nolimits_{0<s<t}\|\tilde{V}(t)\|_{\infty,p}, && \tilde{K}(t) := \sup\nolimits_{0<s<t}\|\nabla\tilde{V}(t)\|_{\infty,p}, \end{aligned} \tag{5.3.10}$$

and proceed as in the previous steps to obtain the estimates

$$\begin{aligned} \mathcal{K}(t) &\le C_1\left(H(t)+K(t)+\tilde{K}(t)\right)\mathcal{K}(t) + C_1\tilde{K}(t)\mathcal{H}(t),\\ \mathcal{H}(t) &\le C_1 H(t)\mathcal{K}(t) + C_1\tilde{K}(t)\mathcal{H}(t), \end{aligned}$$

which combined yield

$$\|v-\tilde{v}\|_{\mathcal{S}(t)} \le C_1\left(H(t)+K(t)+2\tilde{K}(t)\right)\|v-\tilde{v}\|_{\mathcal{S}(t)}. \tag{5.3.11}$$

By our assumption on the regularity of v and $\tilde{v}$ as well as (5.3.9), it follows that

$$\lim_{t\to 0} K(t) + 2\lim_{t\to 0}\tilde{K}(t) \le 3C_2\|a_2\|_{\infty,p},$$

whereas same argument used to derive (5.3.6) yields

$$\|V(t)\|_{\infty,p} \le C_1\left(\|a_0\|_{\infty,p} + K(t)H(t) + t^{1/2}R_0H(t) + t^{1/2}R_0K(t)\right)$$

and thus we obtain

$$\begin{aligned} \lim_{t\to 0} H(t) = \limsup_{t\to 0+}\|V(t)\|_{\infty,p} &\le C_1\|a_0\|_{\infty,p} + C_1\left(\lim_{t\to 0}K(t)\right)\lim_{t\to 0}H(t)\\ &\le C_1\|a_0\|_{\infty,p} + C_1C_2\|a_2\|_{\infty,p}\lim_{t\to 0}H(t). \end{aligned}$$

We now take $\|a_2\|_{\infty,p}$ to be so small that $C_1C_2\|a_2\|_{\infty,p} < 1/2$, yielding

$$\lim_{t\to 0} H(t) \le 2C_1\|a_0\|_{\infty,p}. \tag{5.3.12}$$

By applying this to (5.3.11) and taking $\|a_0\|_{\infty,p}$ and $\|a_2\|_{\infty,p}$ to be sufficiently small, we obtain

$$\begin{aligned} \|v-\tilde{v}\|_{\mathcal{S}(t)} &\le 2C_1\left(2C_1\|a_0\|_{\infty,p} + 3C_3\|a_2\|_{\infty,p}\right)\|v-\tilde{v}\|_{\mathcal{S}(t)}\\ &\le \frac{1}{2}\|v-\tilde{v}\|_{\mathcal{S}(t)}, \quad t\in(0,T_1), \end{aligned}$$

and thus $v = \tilde{v}$ on $(0, T_1)$ whenever $T_1 > 0$ is sufficiently small. We now set

$$t^* := \sup\{t \in [0, T_0) : v(s) = \tilde{v}(s) \text{ for all } 0 \le s \le t\}.$$

By the argument above we have $t^* > 0$. Now assume that $t^* < T_0$. Then the continuity of v and $\tilde{v}$ on $(0, T_0)$ yield that

$$a^* := v(t^*) = \tilde{v}(t^*).$$

Decomposing this new data as in Step 1, we write $a^* = a^*_{\mathrm{ref}} + a^*_0$ and consider a new reference solution v^*_{ref} to the primitive equations with $v^*_{\mathrm{ref}}(0) = a^*_{\mathrm{ref}}$. Replacing V and $\tilde{V}$ by

$$V^*(t) := v(t^* + t) - v^*_{\mathrm{ref}}(t), \quad \tilde{V}^*(t) := \tilde{v}(t^* + t) - v^*_{\mathrm{ref}}(t), \quad t \in [0, T_0 - t^*)$$

and repeating the argument yields

$$\|V^* - \tilde{V}^*\|_{\mathbb{S}(t)} \le C_1 \left(H^*(t) + K^*(t) + 2\tilde{K}^*(t) \right) \|V^* - \tilde{V}^*\|_{\mathbb{S}(t)} \tag{5.3.13}$$

for all $0 < t < \min\{T_1, T_0 - t^*\}$ and

$$\begin{aligned} H^*(t) &:= \sup_{0<s<t} \|V^*(s)\|_{\infty,p}, \\ K^*(t) &:= \sup_{0<s<t} s^{1/2} \|\nabla V^*(s)\|_{\infty,p}, \\ \tilde{K}^*(t) &:= \sup_{0<s<t} s^{1/2} \|\nabla \tilde{V}^*(s)\|_{\infty,p}. \end{aligned}$$

Due to the fact that

$$t^{1/2} \nabla v, t^{1/2} \nabla \tilde{v} \in L^\infty((0, T_0); L^\infty_H L^p_z(\Omega))^2,$$

we have $\nabla v, \nabla \tilde{v} \in L^\infty((t^*, T_0); L^\infty_H L^p_z(\Omega))^2$. By combining this with (5.3.9) we obtain

$$\lim_{t \to 0} K^*(t) = \lim_{t \to 0} \tilde{K}^*(t) = 0.$$

The same estimate used to derive (5.3.12) then yields $\lim_{t\to 0} H^*(t) \le C_1 \|a^*_0\|_{\infty,p}$. By taking $\|a^*_0\|_{\infty,p}$ and $T_2 > 0$ to be sufficiently small, it follows from (5.3.13) that

$$\|V^* - \tilde{V}^*\|_{\mathbb{S}(t)} \le \frac{1}{2} \|V^* - \tilde{V}^*\|_{\mathbb{S}(t)}$$

for all $t \in (0, T_2)$. We therefore have $v - \tilde{v} = V - \tilde{V} = 0$ on $(0, t^* + T_2)$, which contradicts the definition of t^*. This implies that $t^* = T_0$ and $v = \tilde{v}$.

Step 5: *Global extension and smoothing.* Due to the embedding $L^\infty_H L^p_z(\Omega) \hookrightarrow L^p(\Omega)$, we may employ the semigroup smoothing estimates

$$t^\vartheta \|S(t)\mathbb{P}f\|_{D((-A)^{\vartheta_0+\vartheta})} \le C\|f\|_{D((-A)^{\vartheta_0})}, \quad t^{1/2}\|S(t)\mathbb{P}\partial_i f\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)}$$

for $t > 0$, $\vartheta_0, \vartheta \in [0,1]$ with $\vartheta_0 + \vartheta \leq 1$ and $\partial_i \in \{\partial_x, \partial_y, \partial_z\}$ by [49, Lemma 4.6] and Theorem 4.1.7, respectively. This and $D((-A_p)^0) = L^p_{\overline{\sigma}}(\Omega)$ yield

$$\begin{aligned} t^{1/p}\|v(t)\|_{D((-A_p)^{1/p})} &\leq C\|a\|_{L^p(\Omega)} + Ct^{1/p}\int_0^t (t-s)^{-(1/p+1/2)}\|v(s)\otimes v(s)\|_{L^p(\Omega)}\,ds \\ &\leq C\|a\|_{L^p(\Omega)} + Ct^{1/p}\int_0^t (t-s)^{-(1/p+1/2)}\|v(s)\otimes v(s)\|_{\infty,p}\,ds \\ &\leq C\|a\|_{L^p(\Omega)} + C\left(\sup_{0<s<t} s^{1/2}\|\nabla v(s)\|_{\infty,p}\right)\left(\sup_{0<s<t}\|v(s)\|_{\infty,p}\right) \\ &< \infty \end{aligned}$$

for all $t \in (0, T_0)$. Here we used (5.3.1) and the fact that the value of the integral

$$t^{1/p}\int_0^t (t-s)^{-(1/p+1/2)}s^{-1/2}\,ds = \int_0^1 (1-s)^{-(1/p+1/2)}s^{-1/2}\,ds < \infty, \quad p \in (2,\infty),$$

does not depend on $t > 0$. By Theorem 5.0.1 we may take $v(t_0) \in D((-A_p)^{1/p})$ for arbitrary $t_0 > 0$ as new initial data to obtain a global extension for v that is strong on (t_0, ∞). By the uniqueness of mild solutions it follows that v is strong on $(0, \infty)$. In particular, v is real analytic by Theorem 5.1.4 and decays exponentially by Theorem 5.0.1. This completes the proof. □

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