Assertion Games
to Justify Classical Reasoning
Simplified and expanded version of [20]

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Abstract. To establish and justify general methods of linguistic reasoning, we introduce a use of sentences by means of 'assertion rules' which partially have a narrow relationship to the proceeding in [7: 1., 2.] and help us to demonstrate that one can rely on certain logical inference rules (§1, §2). As assertion rules we choose only such rules which can be formulated with words which are as unambiguous as possible. Therefore, we at first introduce the particles $\wedge, \vee, \exists$, and $\neg$ only. Let the resulting use of sentences be said to be the 'primary' use. However, we cannot define a subjunction (material implication) which indicates that, according to this use, one may conclude the succedent from the antecedent.

Therefore, we subsequently liberalize the primary use of sentences in §3. This liberalization establishes a 'classical' use which permits to apply classical logic, and can be justified by the following (and some other) facts: An elementary sentence may be asserted classically iff (i.e. if and only if) it may be asserted primarily. A sentence of the form $\forall x\,[A(x) \to B(x)]$ (which is defined suitably) expresses that, for any value r of the variable x, if $A(r)$ may be asserted classically, then $B(r)$ may *at once* (and generally later on) be asserted so. For this 'inferential purpose', the classical use of sentences of that form is also not unnecessarily restricted. If we replace the primary use by the classical use, only dispensable means of speech get lost. (Details will be discussed in §6.)

In §4 we especially deal with the concept of infinity on the example of the set $\mathbb{N}$ of natural numbers. The infinity of $\mathbb{N}$ is considered as a 'deontic' one. This means that we shall never be *obliged* to terminate the construction of natural numbers. So we avoid the ontological assumption concerning the infinity of $\mathbb{N}$.

In §5 we investigate a use of sentences which include indicators (as "this ant", e.g.) or objectual variables. This use depends on situations. To eliminate this dependency we introduce objectual quantification.

In §7 - §10 we deal with a ramified type theory in a cumulative version: In §7 and §8 we introduce an extension of a union of higher order languages by means of variables $x, y, \ldots$ for constants of arbitrary order, and variables for tuples $(c_1, \ldots, c_j)$ of arbitrary order and arbitrary length $j \in \mathbb{N}^+$. So we may simply identify types with orders. In that language, 'type-free' equations $x = y$ are definable. We extend the primary and the classical use to that language, show that their sentences are non-circular, and that their formulas are invariant under $(=)$.

In §9 we deal with singular description terms.

In §10 we even introduce higher order languages which also contain formulas with indicators and objectual variables, and enable a quantification which combines both substitutional and objectual quantification. We show that we may commute any consecutive existential quantifiers that occur in formulas of those expanded languages.

In §11 we extend the language introduced before by proper names for parts of that language such that we can speak in that language about itself.

By this means, we progressively introduce a (not completely) universal frame of language.

Contents. §0. Introduction

§0. Introduction

By an **assertion** we understand an act: One asserts a proposition in general simply by pronouncing it as a complete sentence to a listener or writing it for a reader. Statements we also include among assertions. However, there exist linguistic acts in the shape of assertions which are not to be understood as assertions or are not meant seriously. (Examples are fictitious, fictional or jocular expressions.) This can be said additionally or can result from the context. Nevertheless, we can in general decide whether one has - or ourselves have - asserted a particular sentence because assertions which are unjustified (as lies, e.g.) or have not been accepted or believed are assertions as well (or are said to be assertions here).

Let us raise the question how it is possible to understand assertions. To this we consider the sentence: "Paul had a temperature of 39.2° C yesterday evening." A listener can understand this in so far as he is used to the rule by which this sentence should be asserted only after measuring temperature and getting the corresponding result. Such examples lead us to the

Thesis: Sentences become understandable because it is usual or agreed to restrict assertions in a regular way, and so to omit asserting particular sentences finally or temporarily. (This explanation requires, of course, some completion.)

The underlying standards of assertion are generally only tacitly valid. To obtain means of linguistic reasoning (especially inference rules) which can be shown to be serviceable and reliable we shall contrive a use of sentences by fixing explicit 'assertion rules' in addition to certain conventional standards. However, we do *not* intend to describe or explain how fluent speakers argue *factually*. Instead, we intend to establish and justify an argumentation technique which is efficient, uncomplicated, clearly arranged, and easy to use.

As assertion rules we shall choose only such rules that can be formulated with words which are as unambiguous as possible - so that we can check in as many cases as possible whether we have broken such a rule. However, it would not be sufficient for the mentioned purpose to take only 'formal' rules like such of a calculus since it follows from a result of GÖDEL that there does not exist a calculus, K, such that any first order arithmetic sentence is assertible iff it is deducible by the rules of K.

Recourse to prohibition rules: Which kind of rules are particularly appropriate to stipulate a use of assertions? General positive commands of the form "Whenever a happens, then do b !" have the disadvantage that one mostly has no opportunity to act upon them. An example for this fact is given by the following rule: "Whenever two sentences A, B have been asserted, then assert $A \wedge B$ too". Since we would have to speak without end by this rule, it suggests itself to replace it by an analogous permission. That an act is permitted means that it is not forbidden or that it must not be forbidden. To forbid an act means to ask or demand, not to do it. Note that the meaning of prohibitions can be demonstrated by means of punishment or blame, for instance. Therefore and with regard to the above mentioned thesis we shall take prohibition rules as assertion rules.

A calculus, e.g., is a system K of rules that *allow* to successively perform certain schematic operations on strings of symbols. This means, however, that in the context of K such an operation is *forbidden* unless it is explicitly permitted by the rules of K.

Material inference rules and means for their formulations: Requisite for reasoning are inference rules by which, for particular formulas $A_1(x)$, $A_2(x)$, and $B(x)$, e.g., we may conclude $B(r)$ from $A_1(r)$ and $A_2(r)$, for any value r of the variable x. To indicate that one may conclude so we intend to write

$$\forall x \, [A_1(x) \wedge A_2(x) \rightarrow B(x)]$$

(cf. [12, p.104]). To this end the assertion of this sentence should be restricted to the condition that, for all values r of x, if $A_1(r)$ and $A_2(r)$ may be asserted, then $B(r)$ may be asserted at once. But our understanding of this condition is particularly problematic, because it contains the words "for all" and "if - then". So we shall define sentences of the forms $A \rightarrow B$ and $\forall x \, A(x)$ by means of $\wedge$ (and), $\neg$ (not), and $\exists$ (for some). However, those sentences will serve their 'inferential purposes' only after a subsequent liberalization (§3) of the initially introduced use of that language.

Remarks on some well-known approaches to logic.

(1) An intuitionistic approach starts from a concept of proof which can inductively be defined (cf. [1], [4]). We formulate that for subjunction only:

> A proof of $A \rightarrow B$ is a 'construction' c that converts each proof of A into a proof of B.

That c is a proof of $A \rightarrow B$ means in detail that, for all pertinent p, if p is a proof of A, then $c(p)$ is a proof of B. Especially for $0 = 1$ in place of B, this means that there does not exist a proof of A. In these explanations, however, we have already used the words "for all", "if - then", and "not" of everyday speech. Besides, (for compound A) it is in general not decidable whether a pertinent construction p is a proof of A. For this reason, in [5] c as above has been replaced by a pair (c, d) where d is a 'demonstration' which shows that c converts each proof of A into a proof of B. However, this concept of proof - which is critically discussed in [1, p. 232] and [14] - is rather intricate and yet somewhat problematic. - The following approach avoids definitions which are circular or seriously involved in an infinite regressus:

(2) An approach to logic by dialogues (see [9, pp. 60ff.]). Let us consider material dialogue games as in [9, pp. 75 - 83]. To show their suitability we need especially the following theorem by which the 'modus ponens' may be applied: If there exist strategies to win dialogues for A as well as for $A \to B$, then there also exists a strategy to win dialogues for B. This theorem is a composite proposition of a metalanguage. Even the attempt to dialogically interprete it meets with difficulties. Moreover, to formulate a proof of that theorem - or of a more general so called *Cut Theorem* - we need sentences of the metalanguage in which the connective "if - then" and the quantifier "for all" occur iteratively. (Note also that the mentioned strategies are meant to be strategies to win dialogues against *any* opponents.) In the proof of the Cut Theorem, the comprehension of those particles succeed by linguistic habits and on the context. However, some ways of reasoning applied in that proof are just to be justified for the object language by means of the Cut Theorem.

In [6], K. LORENZ has taken other frame rules for dialogues as a basis, and has used constructive ordinal numbers. For the pertinent investigations one must have seen before that those alleged ordinals can in fact be used as ordinals, i.e. enable transfinite induction. - In **Proof Theory** (cf. [13] or [16], e.g.) occur analogous problems.

The problem of beginning

As we have partially seen, the considered approaches to logic are involved in different kinds of circularity or regressus. However, there is a more general problem: How can one be initially entitled to any reasoning at all (cf. [2], [15])? For a justification of rules for arguing, or for a proof that they are reliable we already need reliable argumentations.

To reduce this problem, we do not only stipulate certain rules to restrict assertions; we also agree that those assertions must not be restricted by further rules. Then we can see that the stipulated rules for compound sentences may be 'inverted' (cf. the 'Inversionsprinzip' in [7, §4]). So the latter rules and their inverses may also be used as inference rules. - Since our colloquial language can partially be understood to be ruled in that way, we may also perform particular colloquial reasonings by applying the mentioned rules. (Later we shall see that also certain other rules may be applied.)

If we start any metatheoretical reasoning with sentences which are said to be *assumed*, *presupposed*, or the like, we shall treat them like asserted sentences - so that the conclusions which we draw from them may be asserted as soon as those assumptions have rightly been asserted. In this way we understand colloquial conditionals.

We shall not presuppose that there ('actually' or 'potentially') exist infinitely many objects like (mental or other) constructions, proofs, or numbers, for instance.

§1. An assertion game

Elementary sentences and their use

We presuppose that we already dispose of concepts of certain '**elementary sentences**', of '**internal rules**' (of assertion concerning elementary sentences), and of '**external rules**'. (Here we include usage among rules.) For our purposes it suffices to formulate some claims on these concepts and some explanations.

Let elementary (or 'atomic') sentences be not as usual composed of other formulas (i.e. sentences or sentence forms) by means of connectives, quantifiers, or set theoretical particles. - In the following we write $E, E_1, \ldots$ for elementary sentences.

For numerous elementary sentences, E, we have learned in practice that E may be asserted only after we have made a particular perception or observation or have got a special result of an act as a measuring, for instance. (Details are beyond the scope of our topic). Let at least such rules of assertion (which are valid in our community) be said to be **external**. We say that E has been **anchored** to mean that the present assertion of E would not violate an external rule.

Several elementary sentences can also depend on each other by rules concerning assertions of those sentences. Examples are the following rules by which one may assert "(All) beetles are insects" and "Beetles are no flies": If the sentence "This is a beetle" may be asserted, then also the sentence "This is an insect" may be asserted, but "This is a fly" must not be asserted. (See also the 'Prädikatorenregeln' in [9, p. 182].) Thus sentences as "Beetles are insects" and "Beetles are no flies" are based on our common linguistic usage (see [3]). [Here we need not deal with the question whether they can also be justified by (fictitious) definitions as "beetle $\rightleftharpoons$ insect and winged and with wing-cases and ...".]

Rules which are valid in our community and by which elementary sentences depend on each other as in the above examples can be considered as particular **internal rules**. As an internal rule we also take the prohibition to assert a particular elementary sentence, $\perp$. Let us say that E has been **rejected** to mean that the present or later assertion of E would (generally together with already accomplished assertions of other elementary sentences, $D_1, \ldots, D_n$) violate an internal rule. However, if D_i has been rejected at the moment t, let any assertion of D_i at t or later pass for failed, i.e. not performed. Therefore, if E has been asserted (without failing), E cannot become rejected thereafter.

As internal rules for elementary sentences we at first admit - apart from the mentioned prohibition to assert $\perp$ - only rules with instances (substitution instances) by which, for certain elementary sentences $E_1, \ldots, E_n$, and E, it is forbidden to do both, to assert all of the sentences $E_1, \ldots, E_n$ and to reject E. For such an instance of an internal rule we simply write

(Int) $$E_1, \dots, E_n \;\Rightarrow\; E.$$

Commentary: From (Int) the following results:
If $E_1, \dots, E_n$ have been asserted, it is forbidden to reject E.
But if E has been rejected, it is forbidden to assert all of the sentences $E_1, \dots, E_n$.
If, moreover, $E_2, \dots, E_n$ have been asserted, then E_1 is rejected.

In §5 we shall also admit other similar internal rules for elementary sentences.

Example 1: Let a and b range over signs for length (as '25 cm', e.g.). Let $L(s,a)$ mean that a stick s has the length a, and suppose that we have the internal rules

$$\begin{aligned} a = b,\, a \neq b \;&\Rightarrow\; \bot \\ L(s,a),\, L(s,b) \;&\Rightarrow\; a = b. \end{aligned}$$

$\bot$ is agreed to be rejected. So if we assert $a \neq b$, then $a = b$ becomes rejected. So if we also assert $L(s,a)$, then $L(s,b)$ becomes rejected.

By the latter rule, we may assert only one result of a measurement of s. This is not suitable for a branch s which can grow or for a stick which can change its length otherwise. In this case, a formulation as "s has the length a at t" (where t denotes a moment) may be more adequate. Nevertheless, in certain cases only *experience* can show (in general without giving final certainty, however) whether or how far a particular linguistic rule can be useful.

Now we consider the internal rules of the form (Int) as a calculus which operates on elementary sentences where '⇒' indicates the permitted deduction steps. Let us agree that if E is deducible by those rules from other elementary sentences which have already been anchored and asserted, then also E passes for anchored. (Let this agreement be external.)

Notes: 1. An instance $\Rightarrow E$ of an internal rule without premises can be inconsistent with other internal rules. In this case it should not be accepted.
2. It is not totally impossible that an elementary sentence becomes both, anchored and rejected. (Therefore we do not use the word "verified" for "anchored".)
3. If $E_1, \dots, E_n$ have been asserted from separate persons at the same moment and if the internal rules have been violated by those assertions, then let them pass for failed.

A material first order language $\mathcal{L}_0$

In the following we only deal with sentences that do not depend by rules of assertion (i.e. exclusive of rules of politeness or regard, e.g) on particular linguistic contexts or situations (or on the involved persons, e.g.). In §5, however, we shall also deal with other sentences as "*This* is a beetle" that may only be asserted in particular situations (as while showing a particular animal).

At first we only consider formulas that are elementary formulas (of a certain class) or are composed of them by means of ∧ (and), ∨ (or), ¬ (not), and ∃ (for some) as usual.

Let the class of those formulas be denoted by $\mathcal{L}_0$. We assume here that the concepts of variables (of $\mathcal{L}_0$) and of values of variables have already been introduced. Those values are supposed to be constants, i.e. certain strings of symbols in which no variables occur. For short we say

"formula" for "formula of $\mathcal{L}_0$",
"sentence" for "sentence of $\mathcal{L}_0$".

(Sentences are formulas in which all occurrences of variables are bound.)

In more detail, we presuppose the following: For any string x of symbols we know how to decide whether x is a variable. For any variable x and any string c of symbols we know how to decide whether c is a value of x. Any two occurrences of variables in any string of symbols do not overlap. - Elementary formulas are not as usual composed of other formulas by means of logical or set theoretical particles. Elementary formulas include outer brackets (which, however, we generally omit). Further brackets occur only pairwise as usual in elementary formulas. From an elementary formula there results an elementary sentence if we replace the occurring variables by arbitrary values of them.

If F is a formula and $x_1, \ldots, x_n$ $(n \geq 1)$ are distinct variables, then let also

$$\exists\, x_1, \ldots, x_n\, F$$

be a formula (in which $x_1, \ldots, x_n$ occur bound). - As 'metavariables' we use

F, G, H	for	formulas
A, B, C	for	sentences
x, y, z	for	variables
$\underline{x}$	for	lists $x_1, \ldots, x_n$ of distinct variables
$\underline{r}$	for	lists $r_1, \ldots, r_m$ of constants
$A\underline{x}$ or $A(\underline{x})$	for	formulas in which at most the variables $\underline{x}$ occur free.

Definition: $\underline{r}$ is a **value** of $\underline{x}$ iff $\underline{r}$ has as many members as $\underline{x}$ (i.e. $m = n$) and r_i is a value of x_i for $i = 1, \ldots, n$. In this case, by $A\underline{r}$ we denote the sentence which is obtained from a given formula $A\underline{x}$ by substituting $\underline{r}$ for $\underline{x}$, i.e. r_i for each free occurrence of x_i $(i = 1, \ldots, n)$. - We write

$$\natural\, A \text{ for: to assert } A.$$

Now, (Int) can also more detailed be written as: $\natural\, E_1, \ldots, \natural\, E_n \;\Rightarrow\; \natural\, E$.

Though we have partially formalized the considered language $\mathcal{L}_0$ by using symbols as logical particles, $\mathcal{L}_0$ is a material (assertoric) language, not merely a formal language. Since we shall introduce a use of sentences such that some but not all sentences may be asserted, we need not additionally give interpretations which assign meanings to sentences (in a realistic, mentalistic, or other manner). Nevertheless, some relation to 'reality' is given by external rules. - In §4, §5, and §7 - §11 we shall also consider some expansions of $\mathcal{L}_0$.

The primary game

Leading idea: We should generally assert a sentence only if we know arguments for it, which we could, therefore, also assert to ourselves in our mind before asserting that sentence. Accordingly, we choose rules of assertion by which an assertion is forbidden until a certain condition is satisfied. Thus, that a sentence *may* be asserted means that its assertion would no longer violate an appertaining rule.

Given certain external rules as well as internal rules for elementary sentences (as above), the **primary game** is defined to be the 'assertion game' with those rules and the below quoted assertion rules for compound sentences. Also these rules, which are conditional, possibly temporary prohibitions of assertions, are said to be **internal**. The first of them is to be read as: Assert $(A \wedge B)$ only after A has been asserted and B has been asserted. (Accordingly, let ": $\Rightarrow$" be short for "only after".)

P($\wedge$)	$\natural\,(A \wedge B)$	$:\Rightarrow$	$\natural\,A$ and $\natural\,B$
P($\vee$)	$\natural\,(A \vee B)$	$:\Rightarrow$	$\natural\,A$ or $\natural\,B$ (or both)
P($\exists$)	$\natural\,\exists\underline{x}\,A\underline{x}$	$:\Rightarrow$	for some value $\underline{r}$ of $\underline{x}$: $\natural\,A\underline{r}$
P($\neg$)	$\natural\,\neg A$	$:\Rightarrow$	A has been rejected,

where the latter condition means that, by the internal rules, A must not (or no longer) be asserted. (This does in general require that certain elementary sentences have been asserted. - A more detailed explanation of "rejected" will be given below.)

Let this game not contain other rules of assertion. - The rule P($\wedge$) can also be substituted by two rules. The internal rules are formulated by means of our colloquial language, which we have learned exemplarily. This is not problematic for the rules P($\wedge$), P($\vee$), and P($\exists$) since we know how to decide at any time, and for any sentence of the form $A \wedge B$, $A \vee B$, or $\exists\underline{x}\,A\underline{x}$, whether (we know that) the present assertion of it would not violate the pertinent assertion rule. (This is a reason for which we have chosen those rules.) - To explain P($\neg$) we at first give some examples:

Example 2: For sentences A and B which have been asserted according to the assertion rules, and for any sentence C we may successively also assert $A \wedge B$, and $(A \wedge B) \vee C$ since this would not violate the corresponding assertion rules. Therefore, $\neg\,[(A \wedge B) \vee C]$ must not be asserted.

Example 3: $\exists x\,(Ax \wedge \neg Ax)$ is rejected, since before asserting this sentence we should have asserted $Ar \wedge \neg Ar$ for some value r of x, and hence also Ar as well as $\neg Ar$, for which, however, Ar should also have been rejected. Accordingly, $\neg\exists x\,(Ax \wedge \neg Ax)$ may be asserted.

Example 4: Let s denote a stick with a length of *approximately* 25 cm, and let E now be short for "s is 24.8 cm in length", which can only be anchored by a measurement of s with the result 24.8 cm. Suppose that E can be rejected only by asserting another result of a measurement of s. Let us assume, however, that s has been burnt before measuring. Then E can neither be anchored nor rejected. So we must not assert E, neither $\neg E$.

This example shows that if E is an elementary sentence, the assertion of $\neg E$ does in general not become permitted by a mere hindrance to anchor E.

By the rule P($\neg$), the assertability of $\neg A$ is restricted to the condition that A is rejected, which means that performing any series of assertions which ends with that of A and satisfies P($\wedge$), P($\vee$), and P($\exists$) would (generally together with already accomplished assertions of elementary sentences) violate another internal rule. If we speak so about *any* assertion series of such a kind we do not only mean assertion series which will *really* individually be performed, imagined, or considered. We ignore want of time and opportunity to perform, imagine, or consider assertions.

However, the condition that a sentence A is rejected is not in any case decidable. This fact corresponds to a theorem of GÖDEL by which there does not exist an effective procedure by which one can, for any first order arithmetical sentence A, decide whether A may be asserted. (This theorem is also relevant to other approaches to logic.)

As in the above Examples 2 and 3, the assertion rules for compound sentences can be inverted for the following reasons: The primary game does not contain other rules to restrict assertions of those sentences, and the use of every compound sentence is determined non-circularly since it only depends on the use of its predecessors in the following sense.

Definition: C is said to be a **predecessor** of D iff C can be deduced *from* D by at least one application of the following rules (where '$\Rightarrow$' indicates the deduction steps):

$$\begin{array}{lll} A \wedge B \;\Rightarrow\; A; & A \wedge B \;\Rightarrow\; B; & \\ A \vee B \;\Rightarrow\; A; & A \vee B \;\Rightarrow\; B; & \\ \neg A \;\Rightarrow\; A; & \exists \underline{x}\, A\underline{x} \;\Rightarrow\; A\underline{r} & \text{(for values } \underline{r} \text{ of } \underline{x}\text{).} \end{array}$$

(Thus, A and B are the 'immediate predecessors' of $A \wedge B$ and of $A \vee B$, etc.)

The mentioned non-circularity means that no sentence is a predecessor of itself. Though this proposition belongs to a metalanguage, it can be understood as in the primary game. Sentences of $\mathcal{L}_0$ or of a similar metalanguage which have a sufficiently small complexity are obviously non-circular. Moreover, as generally known, the noncircularity of all sentences of $\mathcal{L}_0$ can be proved by induction on their complexity. However, we have not yet established that method of proof. So we stipulate for the present that by a sentence is to be understood a non-circular sentence only. - Now we consider the following '**inverse rules**' (of internal rules):

$$\begin{array}{llcll} \text{I}(\wedge) & A, B & \Rightarrow & A \wedge B & \\ \text{I}(\vee) & A & \Rightarrow & A \vee B & \\ & B & \Rightarrow & A \vee B & \\ \text{I}(\exists) & A\underline{r} & \Rightarrow & \exists \underline{x}\, A\underline{x} & \text{(for values } \underline{r} \text{ of } \underline{x}\text{).} \end{array}$$

These rules have the following property: After asserting the premises (on the left) of an instance of an inverse rule the assertion of its conclusion (on the right) would not violate the pertinent rule. According to this property, in certain cases we may successively assert several sentences in a proper order.

This shows a narrow relationship of our introduction of compound sentences to that given in [7] (see specially [7, §4, §7]) which starts with rules as I(∧), I(∨), and I(∃) as permission rules. The latter approach seems to have the advantage that colloquial phrases as "or" and "for some" do not occur in its rules. However, it must be supplied by the analogous agreement that one may assert a sentence (which is neither elementary nor a negation) only if *there exists* a deduction of it from asserted elementary sentences or negations by the indicated rules.

Also the rule $P(\neg)$ can be inverted: If A has been rejected, then $\neg A$ may be asserted (cf. Example 3).

However, the following example shows that a rule to restrict assertions can generally be inverted for non-circular sentences only: Suppose, e.g., that an extension of the language $\mathcal{L}_0$ contains a particular sentence A_0 (as $\{x : x \notin x\} \in \{x : x \notin x\}$, e.g.) whose assertion is restricted by the rule: $\natural\, A_0 :\Rightarrow\ \natural\, \neg A_0$. Then this rule cannot be inverted. Note that A_0 is a predecessor of itself.

A corresponding assertion rule for universal sentences would be the following:

$$\natural\, \forall \underline{x}\, A\underline{x} \quad :\Rightarrow \quad \text{for all values } \underline{r} \text{ of } \underline{x} : \natural\, A\underline{r}.$$

This would not be useful if $\underline{x}$ has infinitely many values. So we define instead

$$\forall \underline{x}\, F \;\rightleftharpoons\; \neg\exists \underline{x}\, \neg F.$$

Moreover, we define 'subjunction' by

$$F \to G \;\rightleftharpoons\; \neg\,(F \wedge \neg G).$$

However, a sentence $\forall x\,(Ax \to Bx)$ defined so does in general not yet express that, for any value r of x, one may conclude Br from Ar (cf. §0). To this end we shall liberalize the primary game in §3. We shall also return to this point in §6.

§2. Admissibility of inference rules

Now we deal with further rules which may be applied. For the present, we restrict our investigations to inference rules of the form

$$\mathcal{R}: \qquad \mathcal{A}_1, \ldots, \mathcal{A}_n \;\Rightarrow\; \neg\,\mathcal{B}$$

with sentence schemes $\mathcal{A}_i$ and $\mathcal{B}$ in which metavariables (for sentences, formulas, variables, constants, or proper terms) may occur - and from which sentences are obtained by replacing those metavariables by arbitrary values of them. We say that an instance (substitution instance) of $\mathcal{R}$ results from $\mathcal{R}$ by such a substitution.

We define a condition on which $\mathcal{R}$ may be 'applied' (as we shall show):

Definition: $\mathcal{R}$ is said to be **admissible** iff, by the internal rules, it is forbidden for any instance $A_1, \ldots, A_n \Rightarrow \neg B$ of $\mathcal{R}$ to assert all of the sentences $A_1, \ldots, A_n$, and B. - This condition belongs to a metalanguage and can be formalized by

$$\neg\exists \,..\, (\mathcal{A}_1 \wedge \ldots \wedge \mathcal{A}_n \wedge \mathcal{B})$$

where '..' indicates a list of all metavariables occurring behind.

The just mentioned metalanguage is an expansion of the object language $\mathcal{L}$. We have just used abbreviations as

$$\mathcal{A}_1 \wedge \mathcal{A}_2 \wedge \mathcal{B} \rightleftharpoons (\mathcal{A}_1 \wedge \mathcal{A}_2) \wedge \mathcal{B}.$$

Only for rules of the special form $\mathcal{R}$ with a negation on the right we could define the concept of admissibility in terms of the introduced particles $\wedge, \neg$, and $\exists$. But in §3 we shall define a concept of 'classical admissibility' for rules of the general form $\mathcal{A}_1, \ldots, \mathcal{A}_n \Rightarrow \mathcal{B}$.

By the following Lemma admissible rules of the form $\mathcal{R}$ may be applied. To formulate it, we use the explication: That A has been asserted without violating any internal rule means that there has been performed a series of assertions which ends with that of A, begins only with assertion of elementary sentences that have not been rejected or with assertions of negations of sentences that have been rejected (or with assertions of both sorts), and then only applies the inverse rules (see Example 2).

2.1 Basic Lemma: If $A_1, \ldots, A_n \Rightarrow \neg B$ is an instance of an admissible rule and $A_1, \ldots, A_n$ have been asserted without violating an internal rule, then B has been rejected (so that $\neg B$ may be asserted at once).

Proof: Let $A_1, A_2 \Rightarrow \neg B$ (e.g.) be an instance of an admissible rule $\mathcal{A}_1, \mathcal{A}_2 \Rightarrow \neg\mathcal{B}$. So $\exists \,..\, (\mathcal{A}_1 \wedge \mathcal{A}_2 \wedge \mathcal{B})$ has been rejected. Assume, that A_1 and A_2 have been asserted without violating an internal rule. Then any assertion series ending with the assertion $\natural\, B$ may - by the inverse rules - be proceeded with the assertions of $A_1 \wedge A_2$, $A_1 \wedge A_2 \wedge B$, and $\exists \,..\, (\mathcal{A}_1 \wedge \mathcal{A}_2 \wedge \mathcal{B})$ which, however, has been rejected. So any assertion series ending with $\natural\, B$ violates an internal rule, i.e. B is rejected. □

Definition: For formulas F in which the distinct variables $x_1, \ldots, x_n$ but no others occur free,

$$\exists.F \rightleftharpoons \exists x_1, \ldots, x_n F,$$
$$\text{specially} \quad \exists.A \rightleftharpoons A, \quad \text{for sentences } A \ \ (\text{i.e. } n = 0).$$

Proposition: The following inference rules are admissible:

R01	$A, \neg(A \wedge B) \Rightarrow \neg B$	
	$B, \neg(A \wedge B) \Rightarrow \neg A$	
R02	$\neg(A \vee B) \Rightarrow \neg A$	
	$\neg(A \vee B) \Rightarrow \neg B$	
R03	$\neg \exists \underline{x}\, A\underline{x} \Rightarrow \neg A\underline{r}$	(for values $\underline{r}$ of $\underline{x}$)

$$
\begin{array}{ll}
\text{R1} & \Rightarrow\ \neg\,\exists.(F \wedge \neg\,F) \\
\text{R2} & \Rightarrow\ \neg\,\exists.[(F \wedge G) \wedge \neg\,(G \wedge F)] \\
\text{R3}a & \Rightarrow\ \neg\,\exists.\{[(F \wedge G) \wedge H] \wedge \neg\,[F \wedge (G \wedge H)]\} \\
\text{R3}b & \Rightarrow\ \neg\,\exists.\{[F \wedge (G \wedge H)] \wedge \neg\,[(F \wedge G) \wedge H]\} \\
\text{R4} & \neg\,\exists.G\ \ \Rightarrow\ \ \neg\exists.(F \wedge G) \\
\text{R5} & \neg\,\exists.(F \wedge G),\ \neg\,\exists.(F \wedge \neg\,G)\ \ \Rightarrow\ \ \neg\,\exists.F \\
\text{R6}a & \Rightarrow\ \neg\,\exists.[F \wedge \neg\,(F \vee G)] \\
\text{R6}b & \Rightarrow\ \neg\,\exists.[G \wedge \neg\,(F \vee G)] \\
\text{R7} & \Rightarrow\ \neg\,\exists.[\neg F \wedge \neg\,G \wedge (F \vee G)].
\end{array}
$$

These rules have the form $\mathcal{R}\colon\ \mathcal{A}_1, \ldots, \mathcal{A}_n \Rightarrow \neg\mathcal{B}$ from above. To show that such a rule is admissible, we at first note the sentence schemes

$$\mathcal{B}, \mathcal{A}_1, \ldots, \mathcal{A}_n$$

with „a.“ for „assumptions” behind. Thereafter we tread them like sentences and show that their assertions would together violate an internal rule: To this end we infer from those assumptions further sentences by applying P($\wedge$), P($\vee$), P($\exists$), their inverses or rules that have already been shown to be admissible (cf. 2.1). We proceed so until we obtain a contradiction $\mathcal{C}, \neg\,\mathcal{C}$. According to the internal rules, we ought to assert those assumptions *only if* all sentences inferred from them - inclusive of $\mathcal{C}, \neg\,\mathcal{C}$ - may also be asserted, which, however, is not the case. - Substitutions of all variables occurring free in the considered formulas by values of them will sometimes be indicated by $*$.

Ad R01:

$$
\begin{array}{ll}
B, A, \neg\,(A \wedge B) & \text{a.} \\
A \wedge B & \text{I}(\wedge).
\end{array}
$$

Ad R03:

$$
\begin{array}{ll}
A\underline{r},\ \neg\exists\underline{x}A\underline{x} & \text{a.} \\
\exists\underline{x}A\underline{x} & \text{I}(\exists).
\end{array}
$$

Ad R02: Analogously. - Ad R1: See §1, Example 3.

Ad R2:

$$
\begin{array}{rll}
 & \exists.[(F \wedge G) \wedge \neg\,(G \wedge F)] & \text{a.} \\
\text{for some } * : & (F^* \wedge G^*) \wedge \neg\,(G^* \wedge F^*) & \text{P}(\exists) \\
 & F^* \wedge G^*,\ \ \neg\,(G^* \wedge F^*) & \text{P}(\wedge) \\
 & F^*,\ \ G^* & \text{P}(\wedge) \\
 & G^* \wedge F^* & \text{I}(\wedge).
\end{array}
$$

Ad R3: Analogously by means of the partial scheme:

$$
\begin{array}{c}
(F^* \wedge G^*) \wedge H^* \\
F^* \wedge G^*,\ \ H^* \\
F^*,\ \ G^* \\
G^* \wedge H^* \\
F^* \wedge (G^* \wedge H^*).
\end{array}
$$

Ad R4:

$$\begin{array}{rll} & \exists.(F \wedge G), \neg\, \exists.G & \text{a.} \\ \text{for some } * : & F^* \wedge G^*, G^*, \exists.G & \mathrm{P}(\exists), \mathrm{P}(\wedge), \mathrm{I}(\exists). \end{array}$$

Ad R5:

$$\begin{array}{rllll} & \exists.F, & \neg\, \exists.(F \wedge G), & \neg\, \exists.(F \wedge \neg\, G) & \text{a.} \\ \text{for some } * : & F^*, & \neg\,(F^* \wedge G^*), & \neg\,(F^* \wedge \neg\, G^*) & \mathrm{P}(\exists), \mathrm{R03} \\ & & \neg G^* & \neg\neg\, G^* & \mathrm{R01}. \end{array}$$

R6 can immediately be checked.
Ad R7:

$$\begin{array}{rll} & \exists.[(\neg\, F \wedge \neg\, G) \wedge (F \vee G)] & \text{a.} \\ \text{for some } * : & (\neg\, F^* \wedge \neg\, G^*) \wedge (F^* \vee G^*) & \mathrm{P}(\exists) \\ & \neg\, F^* \wedge \neg\, G^*,\ \ F^* \vee G^* & \mathrm{P}(\wedge) \\ & \neg\, F^*,\ \ \neg\, G^*,\ \ \ F^* \mid G^* & \mathrm{P}(\wedge),\ \mathrm{P}(\vee). \end{array}$$

At last $F^* \vee G^*$ has to be 'defended' by one of the sentences F^*, G^*; in both cases we obtain a contradiction. □

By a **term** t we understand a string of symbols in which variables may occur and from which a constant results by any substitution of all free occurring variables by values of them. - The literal equality of any two strings of symbols ϱ and σ will be indicated by '$\varrho \equiv \sigma$'.

Definition: For lists $\underline{x} \equiv x_1, \ldots, x_n$ $(n \geq 0)$ of distinct variables and lists $\underline{t} \equiv t_1, \ldots, t_n$ of terms, let

$$F^{\underline{x}}_{\underline{t}} \quad \text{result from} \quad F$$

by substituting t_i for each free occurrence of x_i $(i = 1, \ldots, n)$. - Moreover, we use the following sentences of a metalanguage:

$$\begin{array}{rcl} \mathrm{N}(x, F) & \rightleftharpoons & x \text{ does not occur free in } F. \\ \mathrm{Fr}(t, x, F) & \rightleftharpoons & t \text{ is free for } x \text{ in } F, \text{ this means} \end{array}$$

1) every substitution instance of t is a value of x,
2) F^x_t is a formula of $\mathcal{L}_0$, and
3) each free occurrence of a variable in t is also free in F^x_t wherever t is substituted for x in F.

Example: y is *not* free for x in $\exists y\,(x < y)$ since y occurs free in y but the occurrence of y substituted for x is bound in $\exists y\,(y < y)$.

Remark: $\mathrm{Fr}(t, x, F)$ is an abbreviation of a composite formula of a metalanguage. We use it in the same way as a composite formula of our object language $\mathcal{L}_0$ in the primary game. [In that metalanguage we may also apply 3.2 (below).]

2.2 Lemma: If $\mathrm{Fr}(t, x, F)$, if $x, \underline{y}$ is a list of all distinct variables occurring free in F or t, and if $r, \underline{s}$ is a value of $x, \underline{y}$, then

$$(*) \qquad (F^x_t)^{x,\underline{y}}_{r,\underline{s}} \equiv (F^{\underline{y}}_{\underline{s}})^x_{t^*} \quad \text{for} \quad t^* \equiv t^{x,\underline{y}}_{r,\underline{s}}.$$

Proof: Consider the following diagrams of partially simultaneous and partially successive substitutions of the free occurrences of $x, \underline{y}$:

$$
\begin{array}{cc}
x, \underline{y} & x, \underline{y} \\
t, \underline{y} & x, \underline{s} \\
t^*, \underline{s} & t^*, \underline{s}.
\end{array}
$$

Proposition: The following rules are admissible:

$$
\begin{array}{lrcl}
\text{R8} & \mathrm{Fr}(t,x,F) & \Rightarrow & \neg\exists.(F^x_t \wedge \neg\exists x\, F). \\
\text{R9} & \mathrm{N}(x,F),\ \neg\exists.(F \wedge G) & \Rightarrow & \neg\exists.(F \wedge \exists x\, G).
\end{array}
$$

Proofs: Ad R8: Let $\mathrm{Fr}(t,x,F)$. By using the denotations from above we may argue thus:

$$
\begin{array}{rll}
 & \exists.(F^x_t \wedge \neg\,\exists x\, F) & \text{a.} \\
\text{for some } r, \underline{s}: & (F^x_t \wedge \neg\,\exists x\, F)^{x,\underline{y}}_{r,\underline{s}} & \mathrm{P}(\exists) \\
 & (F^{\underline{y}}_{\underline{s}})^x_{t*},\ \ \neg\,\exists x\, F^{\underline{y}}_{\underline{s}} & \mathrm{P}(\wedge), (*) \\
 & \exists x\, F^{\underline{y}}_{\underline{s}} & \mathrm{I}(\exists).
\end{array}
$$

Ad R9: Let $\mathrm{N}(x,F)$, and let $\underline{y}$ be a list of all distinct variables occurring free in $F \wedge \exists x\, G$. Then wen may argue thus:

$$
\begin{array}{rll}
 & \exists\underline{y}\,(F \wedge \exists x\, G),\ \ \neg\,\exists x, \underline{y}\,(F \wedge G) & \text{a.} \\
\text{for some } r, \underline{s}: & F^{\underline{y}}_{\underline{s}},\ G^{x,\underline{y}}_{r,\underline{s}} & \mathrm{P}(\exists), \mathrm{P}(\wedge) \\
 & (F \wedge G)^{x,\underline{y}}_{r,\underline{s}} & \mathrm{I}(\wedge) \\
 & \exists x, \underline{y}(F \wedge G). & \mathrm{I}(\exists).
\end{array}
$$

On occasion we write '$\Leftrightarrow$' to combine two inference rules, and we use the definition

$$
\forall.F \ \rightleftharpoons\ \neg\exists.\neg F, \quad \text{specially} \quad \forall.A \ \rightleftharpoons \neg\neg A.
$$

Proposition: Admissible are the following rules:

$$
\begin{array}{lrcl}
\text{R10}a & \neg\exists.G,\ \neg\exists.(F \wedge \neg G) & \Rightarrow & \neg\exists.F \\
\text{R10}b & \neg\exists.\neg G,\ \neg\exists.(F \wedge G) & \Rightarrow & \neg\exists.F \\
\text{R11} & \neg\exists.(G \wedge F) & \Rightarrow & \neg\exists.(F \wedge G) \\
\text{R12} & \neg\exists.[F \wedge (G \wedge H)] & \Leftrightarrow & \neg\exists.[(F \wedge G) \wedge H] \\
\text{R13}a & \forall.\neg F & \Leftrightarrow & \neg\exists.F \\
\text{R13}b & \neg\neg\neg A & \Leftrightarrow & \neg A \\
\text{R13}c & \forall.(F \rightarrow G) & \Leftrightarrow & \neg\exists.(F \wedge \neg G) \\
\text{R13}d & \forall.(F \rightarrow \neg G) & \Leftrightarrow & \neg\exists.(F \wedge G)
\end{array}
$$

For the proof that a rule $\mathcal{A}_1, \ldots, \mathcal{A}_n \Rightarrow \neg\mathcal{B}$ is admissible, it also suffices to deduce $\neg\mathcal{B}$ from $\mathcal{A}_1, \ldots, \mathcal{A}_n$ by repeated application of admissible rules (since $\neg\mathcal{B}$ contradicts the omited assumption $\mathcal{B}$). We can proceed so in the following.

Ad R10a:

$\neg\,\exists.G,\ \ \neg\,\exists.(F \wedge \neg\, G)$	premises
$\neg\,\exists.(F \wedge G)$	by R4
$\neg\,\exists.F$	by R5.

Ad R11:

$\neg\,\exists.(G \wedge F)$	prem.
$\neg\,\exists.[(F \wedge G) \wedge \neg\,(G \wedge F)]$	R2
$\neg\,\exists.(F \wedge G)$	R10a.

The proof of R12 by means of R3 and R10 is left to the reader.
Ad R13a:

$\neg\,\exists.\neg\,\neg\,F$	prem.
$\neg\,\exists.(F \wedge \neg\,F)$	R1
$\neg\,\exists.F$	R10b

R13b and R13c are special cases of R13a.

Ad R13d(⇒):

$\forall.(F \to \neg G)$	prem.
$\neg\exists.(F \wedge \neg\neg G)$	R13c
$\neg\exists.(\neg\neg G \wedge F)$	R11
$\neg\exists.(G \wedge \neg\neg G \wedge F)$	R4, 12
$\neg\exists.(F \wedge G \wedge \neg\neg G)$	R11, 12
$\neg\exists.(F \wedge G \wedge \neg G)$	R1, 4, 12
$\neg\exists.(F \wedge G)$	R5.

Also for further inference rules we can prove their admissibility in this way 'deductively'. To this end we need not yet justify the corresponding *general* method of deduction. This will later be possible by means of induction on the number of deduction steps (cf. §4).

§3. An approach to classical logic

To justify classical logic in §6 we need the following result concerning the use of double negations of elementary sentences in the primary game:

3.1 Theorem: For elementary sentences E as considered so far, if $\neg\neg\,E$ may be asserted, then E may be asserted at the same time. So we ought to assert $\neg\neg E$ only after E has been anchored (see §1).

To obtain this theorem (whose proof follows below) we assign to every elementary sentence E the new 'auxiliary sentence', $-E$, for which we lay down these rules:

Int(−): $$E, -E \;\Rightarrow\; \bot.$$

Aux(−) : Assert $-E$ only after E has been rejected without regard to Int(−).

Let Int($-$) be an internal rule and Aux($-$) an 'auxiliary' (not 'internal') rule, and let the primary game not contain other rules concerning $-E$.

Note that E may be rejected due to Int($-$) (i.e. by $\natural -E$) only if E has been rejected without regard to this rule. For the proof of 3.3 we need some further preliminaries:

Definitions: A tuple $(E_1, \ldots, E_n)$ of elementary sentences is said to be *absolutely rejected* iff it is forbidden by the internal rules to assert all of its members $E_1, \ldots, E_n$.
(Here "absolutely" means: "independent of which other elementary sentences have been asserted".)

3.2 Lemma: If $(E_1, \ldots, E_n)$ is absolutely rejected, then $\perp$ is deducible from $E_1, \ldots, E_n$ by the internal rules.

Proof. An internal rule $E_1, \ldots, E_n \Rightarrow E$ demands that if $D_1, \ldots, D_m$ are elementary sentences such that $D_1, \ldots, D_m, E$ is absolutely rejected, then not to assert all of the sentences $D_1, \ldots, D_m, E_1, \ldots, E_n$. Therefore, all absolutely rejected tuples can be derived by the following rules.
1. $\Rightarrow (\perp)$;
2. $(D_1, \ldots, D_m, E) \Rightarrow (D_1, \ldots, D_m, E_1, \ldots, E_n)$
   if $E_1, \ldots, E_n \Rightarrow E$ an internal rule $(m, n \geq 0)$;
3. $(D_1, \ldots, D_m) \Rightarrow (E_1, \ldots, E_n)$ if $\{D_1, \ldots, D_m\} \subseteq \{E_1, \ldots, E_n\}$.

So we easily obtain 3.2 by induction on the number of applications of these rules (cf. §4, arithmetical induction). □

Proof of 3.1: We assume that $\neg\neg E$ may be asserted at present. Then $\neg E$ has been rejected, i.e. it has been caused that E can no more be rejected. Hence, $-E$ has been rejected (cf. Int($-$)). Now let Γ be the list of all elementary sentences asserted to the present. So $\Gamma, -E$ is absolutely rejected. Therefore, by 3.4, $\perp$ is deducible from $\Gamma, -E$ by the internal rules. But $E, -E \Rightarrow \perp$ is the only internal rule with $-E$ as premise. So E must belong to or be deducible from Γ by those rules. All members of Γ ought to be anchored. Then also E passes for anchored. □

Unsolved problems as the (arithmetical) conjecture of GOLDBACH yield examples of sentences A for which neither A nor $\neg A$ may be asserted up to now so that $A \vee \neg A$ must also not yet be asserted in the primary game. Correspondingly, the *'tertium non datur'* was said to be 'onbetrouwbaar' by L.E.J. BROUWER (1908). Nevertheless, $\neg(A \vee \neg A)$ must not be asserted for any sentence A. This follows from the admissibility of the rules

$$\begin{array}{rcl} \neg(A \vee \neg A) & \Rightarrow & \neg A \\ \neg(A \vee \neg A) & \Rightarrow & \neg\neg A \end{array}$$

(see R02). Hence, for arbitrary sentences A, we have

$$\neg\neg(A \vee \neg A).$$

Accordingly, there exist sentences B such that $\neg\neg B$ may indeed be asserted but B must not yet be asserted. Hence, the rule

$$\neg\neg\, B \Rightarrow B$$

should not be applied merely thoughtlessly. Note, however, that the inverse rule

$$B \;\; \Rightarrow \;\; \neg\neg\, B$$

is admissible due to R1 and R01. - To make the previous rule admissible, too, and so to obtain the *tertium non datur* and even the whole classical logic we liberalize the primary game by using the complete *asserted* sentences as abbreviations of their double negations.

In §6 we shall justify this '**classical use**' of assertions by showing that it satisfies what has been stated in the second section of the abstract.

Definition: An inference rule $\mathcal{A}_1, \ldots, \mathcal{A}_n \Rightarrow \mathcal{B}$ is said to be **classically admissible** iff

$$\neg\neg\, \mathcal{A}_1, \ldots, \neg\neg\, \mathcal{A}_n \Rightarrow \neg\neg\, \mathcal{B}$$

is admissible.

3.3. Proposition: If an inference rule of the form

$$\mathcal{A}_1, \ldots, \mathcal{A}_n \Rightarrow \neg\, \mathcal{B}$$

$(n \geq 0)$ is admissible, then it is also classically admissible.

Proof for $n = 2$: Let $\mathcal{A}_1, \mathcal{A}_2 \Rightarrow \neg\mathcal{B}$ be admissible, i.e., let

$$\neg\, \exists..(\mathcal{A}_1 \wedge \mathcal{A}_2 \wedge \mathcal{B}).$$

Then the following rules are also admissible:

$$\mathcal{A}_2, \mathcal{B} \Rightarrow \neg\, \mathcal{A}_1 \Rightarrow \neg\neg\neg\, \mathcal{A}_1$$

and hence likewise

$$\neg\neg\, \mathcal{A}_1, \mathcal{A}_2 \Rightarrow \neg\, \mathcal{B}$$
$$\neg\neg\mathcal{A}_1, \neg\neg\, \mathcal{A}_2 \Rightarrow \neg\, \mathcal{B} \Rightarrow \neg\neg\neg\, \mathcal{B}. \;\Box$$

Due to our definition $\forall \underline{x}\, F \rightleftharpoons \neg\, \exists \underline{x}\, \neg F$, R13b, and R03,

$$\forall \underline{x}\, A\underline{x} \Rightarrow A\underline{r} \quad \text{(for values } \underline{r} \text{ of } \underline{x}\text{)}$$

is classically admissible. Therefore, $\forall.F$ may be read as "F is universally true". Accordingly, a rule of the form

$$\forall.\mathcal{F}_1, \ldots, \forall.\mathcal{F}_n \Rightarrow \forall.\mathcal{G}$$

will be abbreviated by

$$\mathcal{F}_1, \ldots, \mathcal{F}_n \Rightarrow \mathcal{G}.$$

Moreover, in the below rules R24 - R27 we shall (as usual) remove the 'syntactical' premises $\mathrm{Fr}(t, x, F)$ and $\mathrm{N}(x, H)$ to the rear.

3.4. Proposition: The following rules are admissible (by the latter abbreviation):

R14 $\Rightarrow \; F \to F$

R15 $F,\ F \to G \;\Rightarrow\; G$ (modus ponens)

R16 $F \to G,\ G \to H \;\Rightarrow\; F \to H$

R17a $\Rightarrow \; F \wedge G \to F$

R17b $\Rightarrow \; F \wedge G \to G$

R18 $H \to F,\ H \to G \;\Rightarrow\; H \to F \wedge G$

R19 $F \wedge G \to H \;\Leftrightarrow\; F \to (G \to H)$

R20 $F \;\Rightarrow\; H \to F$

R21 $F,\ G \;\Leftrightarrow\; F \wedge G$ (three rules)

Definition: $F \leftrightarrow G \;\rightleftharpoons\; (F \to G) \wedge (G \to F)$.

R22 $\Rightarrow \; F \leftrightarrow \neg\neg F$

R23 $F \to G \;\Leftrightarrow\; \neg G \to \neg F$

R24 $\Rightarrow \; F^x_t \to \exists x\, F,$ if $\mathrm{Fr}(t, x, F)$

R25 $F \to H \;\Rightarrow\; \exists x\, F \to H,$ if $\mathrm{N}(x, H)$

R26 $\Rightarrow \; \forall x\, F \to F^x_t,$ if $\mathrm{Fr}(t, x, F)$

R27 $H \to F \;\Rightarrow\; H \to \forall x\, F,$ if $\mathrm{N}(x, H)$.

R28a $\Rightarrow \; F \to F \vee G$

R28b $\Rightarrow \; G \to F \vee G$

R29 $F \to H,\ G \to H \;\Rightarrow\; F \vee G \to H.$

Here we only prove the admissibility of R18. To this end we give a deduction of the following rule at first:

R18* $H \to F \;\Rightarrow\; H \to H \wedge F.$

$\forall.\,(H \to F)$	premise
$\neg\,\exists.\,(H \wedge \neg F)$	R13
$\neg\,\exists.\,(\neg\,(H \wedge F) \wedge H \wedge \neg\, F)$	R4, 12
$\neg\,\exists.\,(\neg\,(H \wedge F) \wedge H \wedge F)$	R1, 11, 12
$\neg\,\exists.\,(\neg\,(H \wedge F) \wedge H)$	R5
$\forall.\,(H \to H \wedge F)$	R11, 13.

Ad R18 (sketch):

$$\begin{array}{rl} H \to F, \quad H \to G & \text{premises} \\ H \wedge F \to G & \text{R17, 16} \\ H \to H \wedge F \to H \wedge F \wedge G \to F \wedge G & \text{R18* etc.} \ \square \end{array}$$

It is well-known that also other inference rules and other methods of classical reasoning are due to the rules R1 - R29. We shall make use of this fact within the framework of $\mathcal{L}$ and other languages.

§4. An approach to arithmetic

The following is due to [7, §13] or [9, II.1]. As natural numbers we can simply use the **numerals** which are defined to be the figures $0, 0', 0'', 0''', \ldots$ constructible in the calculus K(IN) with the two rules

$$\begin{array}{rcll} & \Rightarrow & 0 & \text{(start with 0)} \\ k & \Rightarrow & k' & \text{(from } k \text{ infer } k'). \end{array}$$

Here, k may be replaced by any figure already constructed in K(IN). For 'original' constants r we read $r \ \varepsilon$ IN as "r is constructible in K(IN)". (In another context we shall write '$\in$' for 'ε'.) - However, also certain '*new*' signs may be used as abbreviations or singular descriptions for elements of IN (e.g. 3 for $0'''$, and $357 \rightleftharpoons 3 \times 10^2 + 5 \times 10 + 7$ after introducing addition etc.). - In the following, k, m, n stand for arbitrary numerals.

Let the **equality** on IN be the literal equality. Accordingly, $k_0 = m_0$ is to mean that this equation is deducible in the calculus K($=$) with the two rules

$$\Rightarrow 0 = 0; \quad k = m \ \Rightarrow \ k' = m'.$$

The 'deontic infinity' of IN: Because of lack of time and material we can *really* construct only finitely many numerals. However, we shall never be *obliged* to terminate the constructions in K(IN). - Moreover, by the rules of K(IN) we successively obtain only *diverse* numerals $0, 0', 0'', \ldots$. These facts suggest to say that IN is *infinite*. (We shall return to this point.)

For 'external reasons' it is impossible to apply the rules of K(IN) infinitely many times. So we cannot construct infinitely long 'numerals' $0'' \ldots '$. However, with respect to arithmetical induction, such figures should also be excluded from IN *by internal rules*. Therefore, we replace the rules of K(IN) by the following Γ-rules and, similarly, the rules of K($=$) by the following Δ-rules. (An infinitely long 'numeral' $\Omega \equiv 0'' \ldots '$ cannot be distinguished from Ω'. So $\Gamma|\Omega|$ is not rejected by the rules fixed below . We shall, however, not make use of these informal remarks.)

Given any first order language $\mathcal{L}_0$ as considered so far, we extend it by introducing new sentences of the forms $\Gamma|\varrho|$, $r \ \varepsilon$ IN, $\Delta|\varrho = \sigma|$, and $r = s$ where $\Gamma, |, \varepsilon,$ IN, Δ, and $=$ (which is short for $=_{\mathrm{IN}}$) are particular *new* symbols, r and s range over arbitrary values of variables of $\mathcal{L}$, and ϱ and σ range over strings of (atomic) symbols occurring in those values. To introduce the use of sentences of these forms we include just the following 'Γ-' and 'Δ-rules' among the **internal rules** of the language expanded so:

$$\begin{array}{rcl} \Gamma|0| & \Rightarrow & \bot \\ \Gamma|\varrho'| & \Rightarrow & \Gamma|\varrho| \\ r \in \mathbb{N} & :\Rightarrow & \neg\,\Gamma|r| \end{array} \qquad \begin{array}{rcl} \Delta|0=0| & \Rightarrow & \bot \\ \Delta|\varrho'=\sigma'| & \Rightarrow & \Delta|\varrho=\sigma| \\ r=s & :\Rightarrow & \neg\,\Delta|r=s|. \end{array}$$

(But we do not fix any external rule to restrict assertions of sentences considered here.) For any constant r, $\Delta|r'=0|$ does not occur as premise of an instance of an internal rule. Therefore $\Delta|r'=0|$ cannot be rejected, $\neg\,\Delta|r'=0|$ must not be asserted, $r'=0$ must also not be asserted, and hence $\neg\, r'=0$ may be asserted.

By the Γ-rules it is just forbidden to assert $\Gamma|0|,\ \Gamma|0'|,\ \Gamma|0''|,\ \ldots$. So we may assert $0 \in \mathbb{N},\ 0' \in \mathbb{N},\ 0'' \in \mathbb{N},\ \ldots$ but no other sentences of the form $r \in \mathbb{N}$. - Similarly, by the Δ-rules it is just forbidden to assert $\Delta|0=0|,\ \Delta|0'=0'|,\ \Delta|0''=0''|,\ \ldots$. So we may assert $0=0,\ 0'=0',\ 0''=0'',\ \ldots$ but no other equations between numerals.

In this §4 we write x, y for variables ranging over numerals, and z for variables ranging over numerals *at least.*

Propositions:
(a) $\forall z\,(z \in \mathbb{N} \leftrightarrow z' \in \mathbb{N})$ (cf. the infinity of $\mathbb{N}$)
(b) $\forall x, y\,(x=y \leftrightarrow x'=y')$
(c) $\forall x\,\neg\,(x'=0),\quad \forall y\,\neg\,(0=y')$.

Proofs: (a)($\leftarrow$) We ought to assert $\exists z\,\big(\neg\,(z \in \mathbb{N}) \wedge z' \in \mathbb{N}\big)$ only if, for some r, $r \in \mathbb{N}$ is rejected and $r' \in \mathbb{N}$ has been asserted. To this, $\Gamma|r'|$ should be rejected by the rule $\Gamma|r'| :\Rightarrow \Gamma|r|$. So also $\Gamma|r|$ must be rejected (inversion). But then $r \in \mathbb{N}$ may be asserted ('contradiction'). By these arguments, $\exists z\,\big(\neg\,(z \in \mathbb{N}) \wedge z' \in \mathbb{N}\big)$ is rejected. So we may assert its negation and so, by R13, $\forall z\,(z \in \mathbb{N} \leftarrow z' \in \mathbb{N})$. - (a)($\rightarrow$), (b), and (c) can be proved similarly. □

Principle of arithmetical induction: Admissible is the rule

$$A(0),\ \forall x\,[A(x) \rightarrow A(x')] \ \Rightarrow\ \forall x\, A(x).$$

Proof: Since *not all* figures denoted by ϱ are constants, we shall use new sentences $\Lambda|\varrho|$ (where the symbol '|' does not occur in ϱ) instead of $\varrho \in \mathbb{N} \wedge A(\varrho)$. As additional internal rules for assertions of the form $\natural\,\Lambda|\varrho|$ we fix just the rules $\Lambda|\varrho| :\Rightarrow \neg\,\Gamma|\varrho|$ and $\Lambda|k| :\Rightarrow A(k)$, for numerals k. (So we have $\Lambda|k| \leftrightarrow A(k)$.) - Assume now that we have $A(0),\ \forall x\ [A(x) \rightarrow A(x')]$, and $n \in \mathbb{N}$. Then the rules $\neg\,\Lambda|0| \Rightarrow \bot$ and $\neg\,\Lambda|\varrho'| \Rightarrow \neg\,\Lambda|\varrho|$ are admissible, and (because of $n \in \mathbb{N}$) it is forbidden by the rules $\Gamma|0| \Rightarrow \bot$ and $\Gamma|\varrho'| \Rightarrow \Gamma|\varrho|$ to assert $\Gamma|n|$. Here we may replace Γ by $\neg\,\Lambda$. This means, it is in the same way forbidden by the rules $\neg\,\Lambda|0| \Rightarrow \bot$ and $\neg\,\Lambda|\varrho'| \Rightarrow \neg\,\Lambda|\varrho|$ to assert $\neg\,\Lambda|n|$. So we have $\neg\neg\,A(n)$ for all $n \in \mathbb{N}$, and so $\neg\,\exists x\,\neg A(x)$. □

‘Induction principle for equations’: Admissible is the rule

$$A(0,0),\ \forall x,y\,[A(x,y) \to A(x',y')] \ \ \Rightarrow\ \ \forall x,y\,[x = y \to A(x,y)].$$

For the proof of it we have to consider the Δ-rules instead of the Γ-rules. By this principle we can conclude: $r = s \to r \varepsilon \mathbb{N} \wedge s \varepsilon \mathbb{N}$. - By (b) and arithmetical induction, we easily obtain $n = n$. Moreover, we also have

(d) $\qquad k = m \ \wedge\ A(k) \ \to\ A(m).$

Proof: Let e be a variable for the ‘empty figure’ as well as for figures which can be constructed from it by applying the calculus rule: $q \Rightarrow {}'q$. Then we have

$$\forall e\,[A(0e) \to A(0e)] \ \text{and}\ \forall x,y\,\{\forall e\,[A(xe) \to A(ye)] \to \forall e\,[A(x'e) \to A(y'e)]\}.$$

Now we obtain (d) by the induction principle for equations. □

From (d) follows the comparativity, $k = m \wedge k = n \to m = n$, and hence the symmetry and transitivity of the equality on $\mathbb{N}$. This relation is, therefore, an equivalence relation under which all formulas considered are invariant.

We shall also apply other induction principles that can be explained by arithmetical induction.

Recursion as a way of generating relations: Addition in $\mathbb{N}$, e.g., can be introduced by fixing the following assertion rules for new sentences:

$$\begin{aligned} \mathrm{Add}(k,0,n) &\ :\Rightarrow\ \ n = k \\ \mathrm{Add}(k,m',n) &\ :\Rightarrow\ \ \exists x[\mathrm{Add}(k,m,x) \wedge n = x']. \end{aligned}$$

These rules can be considered as special cases of

$$\begin{aligned} \underline{r}\ \varepsilon\ S_0 &\ :\Rightarrow\ \ A(\underline{r}) \\ \underline{r}\ \varepsilon\ S_{m'} &\ :\Rightarrow\ \ B(\underline{r},m,S_m) \end{aligned}$$

where $S \rightleftharpoons \mathrm{I}\underline{x}yZ(A(\underline{x}),B(\underline{x},y,Z))$. $A(\ldots)$ and $B(\ldots)$ are permitted to be formulas of an *extended* object language $\mathcal{L}^+$ which is the least language containing certain elementary formulas as well as formulas of the shape $\underline{t}\ \varepsilon\ Z$, and is closed under $\wedge, \vee, \neg, \exists$, and $\varepsilon\ \mathrm{I}$ (with the ‘*induction operator*’ ‘I’). Z is assumed to be a variable for sets or relations (as S_m) that are definable in $\mathcal{L}^+$. Such variables may be bound by ‘I’ but must not be bound by ‘∃’ in formulas of $\mathcal{L}^+$.

The latter rules can also be inverted since the assertions on their left are not to be subjected to additional restrictions, and even the language $\mathcal{L}^+$ is non-circular as can be shown by induction on the outlined construction of formulas of $\mathcal{L}^+$ (see [19, pp. 426 ff., 452] or 7.3). The predicator ‘Add’ represents a function, ‘+’. An introduction of definite description terms (as terms of the form $s+t$, e.g.) will be sketched in §9. - As easily seen, all recursive functions are definable in $\mathcal{L}^+$.

For constructive or predicative **analysis** in the sense of [8] inclusive of measure theory and functional analysis (as in [18], e.g) there suffice real numbers which are given by rational Cauchy sequences definable in $\mathcal{L}^+$. (See also the end of §7.)

We have unproblematically obtained the above result (a), by which $\mathbb{N}$ is infinite. This result, however, has substantial consequences. To give an example, we consider the power (Pow) of natural numbers. The proposition

$$9^{9^9} \varepsilon \mathbb{N}$$

(in which the iterated definite description '9^{9^9}' occurs) can be considered as an abbreviation of the composite sentence

$$\exists x, y \, [\mathrm{Pow}(9, 9, x) \ \wedge \ \mathrm{Pow}(9, x, y) \ \wedge \ y \, \varepsilon \, \mathbb{N}],$$

a generalization of which can inductively be proved by a well known procedure. It is, however, not possible *really* to construct a figure n by the rules of K($\mathbb{N}$) which satisfies $9^{9^9} = n$. The existential sentence $9^{9^9} \varepsilon \mathbb{N}$ must, therefore, not actually be asserted in the primary game for the whole history of mankind. Nevertheless, it would *not* violate a rule successively to perform proper assertions and ultimately to assert $9^{9^9} \varepsilon \mathbb{N}$. Accordingly, we may assert the double negation of this sentence, which, therefore, can be understood classically.

A more general problem concerns sentences of the form $\forall x \, \varepsilon \, K. \, \exists y \, A(x, y)$, i.e. $\forall x \, (x \, \varepsilon \, K \to \exists y \, A(x, y))$. At best we can proof such a sentence *directly* by describing an effective procedure, p, and showing that

(*) $$\forall x \, \varepsilon \, K. \left\{ \exists y \, (p : x \mapsto y) \ \wedge \ \forall y \, [(p : x \mapsto y) \to A(x, y)] \right\}.$$

Here, $p : x \mapsto y$ is to mean that p with the input x prescribes to produce the output y finally.

The assertion of (*) shows for any $k \, \varepsilon \, K$ how one can 'on principle' find an m satisfying $\neg\neg A(k, m)$. In many cases, however, p with a 'large' input k will not really yield an output m in available time. How can we understand the existence of such an m? To this, we consider p as a system of rules by which certain successions of action steps are permitted (or even required) and the others are forbidden. Every permitted step is assumed to be uniquely determined by the input and the preceding steps.

If an input, k, is given, the 'classical existence' of a corresponding output (i.e. $\neg\neg\exists y \, (p : k \to y)$) means that it is permitted (i.e. not forbidden) by the rules of p to perform certain steps which finally yield an output.

§5. Substitutional combined with objectual quantification

Sentences as "All ants are mortal" or "Some apples are red" have the form "All P are Q" or "Some P are Q", respectively, or - in a 'modern' manner of writing - $\forall x\,(Px \rightarrow Qx)$ or $\exists x\,(Px \wedge Qx)$, respectively. However, the use of such sentences cannot adequately be reconstructed as in §1 since we have not enough proper names for ants or apples, e.g., as values of the variable x at our disposal. So we also consider sentences as "This is an ant" or "This ant has only five legs" with '**indicators**' as "this" or "this ant", which can *temporarily* be used like proper names for objects (as solids or events, e.g.). We include indicators among constants (though we use them instead of 'objectual variables'). They may also occur in other constants and in sentences of the language consideed here.

Under a **denotation** of an object by an indicator or of an indicator by an object we understand a naming which, however, is in general only valid in a special situation (or context). Such a denotation can result, for instance, from pointing at that object and pronouncing that indicator at the same time.

In many cases, an object in question cannot be shown to a listener so that the corresponding denotation is restricted to the speaker. However, if he has said to *himself* "This ant has only five legs", e.g., then he may say to any listener that there *exists* an ant with five legs only (cf. P($\exists$ den) below). Accordingly, we need *not presuppose* here that the denoted objects do not depend on the concerned persons. (Constellations of stars, e.g., do so). - We shall not deal with several problems concerning denotations as, for instance, the danger of misunderstandings, which can result from not sufficiently clear bounds of situations.

A denotation of a single indicator is said to be **simple**. We distinguish any two simple denotations that are created by different acts of naming. If $u_1, \ldots, u_k$ are different indicators, and α_i is a simple denotation of u_i $(i = 1, \ldots, k)$, then we say that $\alpha_1, \ldots, \alpha_k$ is a denotation of $u_1, \ldots, u_k$. Let all denotations considered in the following be composed in this way from simple denotations of different indicators.

If α is a denotation of $\underline{u}$, and $\beta \rightleftharpoons \beta_1, \ldots, \beta_m$ is a further denotation, then let $\alpha\,\beta$ result from α by adding only those members β_i of β that are not denotations of members of $\underline{u}$. (That is, let $\alpha\,\beta$ coincide with α on $\underline{u}$ and otherwise with β.)

In the following α, β range over denotations (which may also be empty).

If Φ is a formula or a list of constants, let $\alpha\,\Delta\Phi$ mean that α is a denotation of at least all indicators occurring in Φ. We compactly write $\natural\, A|\,\alpha$ for the assertion of a sentence A 'in α', i.e. in a situation in which α is valid. Then α is assumed to satisfy $\alpha\,\Delta A$. We identify $\natural\, A|\,\alpha$ with $\natural\, A|\,\alpha_A$ where α_A is the restriction of α to all indicators occurring in A.

Now we extend the primary game by transferring its rules (see §1) to such assertions as follows:

P(∧)	$\natural\,(A \wedge B)\vert\,\alpha$	$:\Rightarrow$	$\natural\,A\vert\,\alpha$ and $\natural\,B\vert\,\alpha$
P(∨)	$\natural\,(A \vee B)\vert\,\alpha$	$:\Rightarrow$	$\natural\,A\vert\,\alpha$ or $\natural\,B\vert\,\alpha$
P(∃ den)	$\natural\,\exists \underline{x}\,A(\underline{x})\vert\,\alpha$	$:\Rightarrow$	for some value $\underline{r}$ of $\underline{x}$: for some $\beta\Delta\underline{r}$: $\natural\,A(\underline{r})\vert\,\alpha\,\beta$
P(¬)	$\natural\,\neg A\vert\,\alpha$	$:\Rightarrow$	A has been rejected in α.

Let the primary game also contain certain corresponding external rules and internal rules for elementary sentences (cf. §1), but let it not contain further rules for compound sentences. Accordingly, the above primary rules for compound sentences may be inverted.

We assume that all instances of internal rules for elementary sentences containing indicators have the form $E_1\vert\,\alpha, \ldots, E_n\vert\,\alpha \Rightarrow E\vert\,\alpha$ with $\alpha\,\Delta(E_1, \ldots, E_n, E)$. Then, by a corresponding external rule, we let E pass for anchored in α, if $E_1, \ldots, E_n$ pass for anchored in α.

5.1 Remark: The existential quantifiers introduced by P(∃ den) combine both substitutional and (in general) objectual quantification. In 10.2 (where we consider a more extensive language) we shall show that we may commute consecutive existential quantifiers introduced there. This result does not contradict the well-known fact that in the context of Quines discussion (see [10, §28]) it is in general not allowed to commute consecutive existential quantifiers when one is objectual and the other substitutional.

Example: In the context of Quines discussion it is not true that every non-empty set has a subset containing a unique element. Nevertheless, we may argue as follows. Let a set be given by a constant $a \rightleftharpoons \{x\colon A(x)\}$ together with a denotation $\alpha\,\Delta\{a\}$. Suppose that this set contains at least one element:

$$\exists x\,(x \,\varepsilon\, a)\vert\,\alpha.$$

Then there exists a constant r and a denotation β with $\beta\Delta r$ such that

$$(r \,\varepsilon\, a)\vert\,\alpha\beta.$$

From this we can conclude the following (where $\{r\}$ is short for $\{y\colon y = r\}$):

$$\begin{gathered}(r \,\varepsilon\, a \wedge \{r\} = \{r\})\vert\,\alpha\beta\\ \exists x\,(x \,\varepsilon\, a \wedge \{r\} = \{x\})\vert\,\alpha\beta\\ \exists Z\,\exists x\,(x \,\varepsilon\, a \wedge Z = \{x\})\vert\,\alpha.\end{gathered}$$

which means that a has a subset, Z, containing a unique element.
(Here, the variable Z ranges over certain sets of values of x.) So in α we may assert $\exists x\,(x \,\varepsilon\, a) \rightarrow \exists Z\,\exists x\,(x \,\varepsilon\, a \wedge Z = \{x\})$ and, therefore,

$$\exists x\,\exists Z\,(x \,\varepsilon\, a \wedge Z = \{x\}) \rightarrow \exists Z\,\exists x\,(x \,\varepsilon\, a \wedge Z = \{x\}),$$

which, however, becomes untrue for a merely substitutional variable instead of Z. Notice that the above unit set is given by the ‘pair’ $\{r\}, \beta$.

3.1 and the following proposition show how far the means of speech of the classical game are sufficient to inform about empirical datas (cf. §6). To this end we consider sentences of the form $A \equiv \exists \underline{x}\,(E_1(\underline{x}) \wedge \ldots \wedge E_n(\underline{x}))$ $(n > 0)$ in which no indicators occur. For such sentences, A, we introduce new 'auxiliary sentences' $-A$, and fix these rules: For values $\underline{r}$ of $\underline{x}$ and denotations $\beta\Delta\underline{r}$,

Int($-\exists$) $\quad\quad \natural\, E_1(\underline{r})|\,\beta, \ldots, \natural\, E_n(\underline{r})|\,\beta,\ \natural -A \;\; :\Rightarrow \;\; \natural \perp,$

Aux($-\exists$) $\quad\quad \natural -A \;\; :\Rightarrow$ $\;A$ has been rejected without regard to Int($-\exists$).

Let Int($-\exists$) be an internal rule and Aux($-\exists$) an 'auxiliary' (not 'internal') rule.

5.2 Proposition: Let $E_1(\underline{x}), \ldots, E(\underline{x})$ $(n \geq 1)$ be elementary formulas in which no indicators occur, and let $A \equiv \exists \underline{x}\,(E_1(\underline{x}) \wedge \ldots \wedge E_n(\underline{x}))$. Then, in the primary game, we ought to assert $\neg\neg\, A$ only if, for some tuple $\underline{r}$ of values of $\underline{x}$ and some denotation $\beta\Delta\underline{r}$, the sentences $E_1(\underline{r}), \ldots, E_n(\underline{r})$ have been anchored in β. If $\neg\neg\, A$ may be asserted in the primary game, then A may be asserted in this game at the same time.

Proof: If we assert $-A$ without violating an internal rule, then, by Int($-\exists$), $E_1(\underline{r}) \wedge \ldots \wedge E_n(\underline{r})$ becomes rejected in β for every value $\underline{r}$ of $\underline{x}$ and every β with $\beta\Delta\underline{r}$, hence A becomes rejected, and hence $\neg\, A$ is not rejected. Now we assume, that $\neg\neg\, A$ may already be asserted. Then $\neg\, A$ has been rejected, and then, by the latter argument, $\natural\, -A$ would violate an internal rule, i.e. $-A$ has been rejected. By 3.4 and since Int($-\exists$) is the only internal rule concerning $-A$, it follows that, for some $\underline{r}$ and some $\beta\Delta\underline{r}$, $E_1(\underline{r}), \ldots, E_n(\underline{r})$ have been asserted in β or are deducible by the internal rules from some elementary sentences $D_1, \ldots, D_m$ that have been asserted in α for some $\alpha\Delta(D_1, \ldots, D_m)$. Then $\natural\, A$ would not violate an internal rule, and $D_1, \ldots, D_m$ should have been anchored in α, and then also $E_1(\underline{r}), \ldots, E_n(\underline{r})$ pass for anchored in β (by a rule mentioned above). □

Further investigations on assertions under denotations will be performed in §10.

Literal equality of strings of symbols is a relation of individual *occurrences* of such strings at arbitrary places, also outside of corresponding equations. Therefore, literal equality can be introduced and investigated by means of *indicators* for such occurrences. (Here we do not go into details and do also not discuss problems that arise from the fact that (particularly in §1 and §2) we have tacitly used 'syntactical' properties of formulas and their components which concern literal equality.)

§6. Purposes of assertions in the classical game

Let the **classical game** be that assertion game in which a sentence A of $\mathcal{L}$ may be asserted iff $\neg\neg A$ may be asserted in the primary game.

The rules R1 - R29 (see §2 and §3) inclusive of $\neg\neg A \Rightarrow A$ and analogous rules for sentences with indicators are admissible in the classical game. This means that in this game we may apply classical logic. - In the following we show which purposes assertions of different kinds of sentences can serve in the classical game, and that this game preserves all means of speech which are indispensable for those purposes.

Due to 3.1 (which also holds for elementary sentences containing indicators), we should assert an elementary sentence, E, in the classical game only if E has been anchored. Accordingly, for the listener or reader the 'classical' assertion of E can substitute a first hand knowledge of an anchoring of E, in particular a perception or observation, or the result of an investigation of objects.

Due to R01 and R03, in the classical game the rules

$$\begin{array}{rcl} A,\ A \to B & \Rightarrow & B \\ \forall x\, Ax & \Rightarrow & Ar \quad \text{(for values } r \text{ of } x\text{)} \end{array}$$

have the property that their premises may be asserted only if the pertinent conclusion may *already* be asserted. So we have in the classical game:

$\natural\,(A \to B)$ can serve the listener or reader as the advice to assert B (perhaps to himself only) as soon as A may be asserted.

$\natural\,\forall x\, Ax$ can serve as a substitute for $\natural\, Ar$, for any value r of x.

Similarly, in the context of §5,
$\natural\,\forall x\, A(x)|\alpha$ can serve as a substitute for $\natural\, A(b)|\alpha\beta$,
for any constant b and any denotation β of the indicators occurring in b.

By means of conjunction we can - more clearly arranged - write $A_1 \wedge A_2 \wedge A_3 \to B$ for $A_1 \to [A_2 \to (A_3 \to B)]$.

The following holds in the primary game. An assertion of the form $\natural\, A \vee B$ can without loss of information be replaced by the shorter assertion $\natural\, A$ or $\natural\, B$. In the same way, the assertion of an existential sentence, $\exists x\, Ax$, is dispensable since it can be replaced by the assertion of Ar, for some value r of x.

However, what we have just stated does not hold for existential sentences $\exists x\, A(x)$ with a variable x whose values may contain indicators. Note that sentences A containing indicators may in general be asserted only in particular situations. On the other hand, many empirically obtained facts can - in any situations - be summarized to sentences of the form $\exists \underline{x}\, (E_1(\underline{x}) \wedge \ldots \wedge E_n(\underline{x}))$ with elementary components $E_i(\underline{x})$. Accordingly, we have adduced 5.3 by which even from the classical assertion of $\exists \underline{x}\, (E_1(\underline{x}) \wedge \ldots \wedge E_n(\underline{x}))$ we can conclude that $E_1(\underline{r}), \ldots, E_n(\underline{r})$ should have been *anchored* for some value $\underline{r}$ of $\underline{x}$ and some denotation of $\underline{r}$. This shows how far the means of speech of the classical game are sufficient to inform about empirical datas.

Sentences of the form $\forall x \,\varepsilon\, K.\, \exists y\, A(x, y)$ have been investigated at the end of §4. A generalization of those investigations should still be worked out.

Since we dispose of certain admissible inference rules, composite formulas can be used as marks for something of data processing. As is well known, all inference rules of 'constructive' or 'intuitionistic' logic are also admissible in classical logic. Hence, (especially mathematical) composite formulas are in the classical game at least as useful as processing marks as in a language in which only intuitionistic logic is available. - For purposes, however, which have not been regarded here, a more restrictive use of assertions may be more suitable than the classical use.

Hypothetical assertions

In everyday speech and in empirical sciences one necessarily proceeds more liberally than in our classical game. So one does not only assert established facts but also uses universal hypotheses or conjectures, which often do not even get cited. If H is the conjunction of all current hypotheses, we could use (assert) certain sentences B as short for $H \to B$. Indeed, for any admissible inference rule $\mathcal{A}_1, \dots, \mathcal{A}_n \Rightarrow \mathcal{B}$ the rule $H \to \mathcal{A}_1, \dots, H \to \mathcal{A}_n \Rightarrow H \to \mathcal{B}$ is also admissible. But as soon as H becomes rejected, it becomes obviously unserviceable to assert sentences of the form $H \to B$ (or abbreviations of them).

Accordingly, if H contains (probably) untrue hypotheses (such as simplifications of conjectures) we can instead of $H \to B$ better use the statement that B has been *deduced* from H and certain already verified sentences by the rules of classical logic (e.g.). This statement reminds of *necessity* (cf. [9, p.111], e.g.). - Our investigations of §11 will show how we can include sentences of of the form "B is deducible from H by given rules" in the object language.

Sometimes we are convinced that if we perform a certain action a, then - after an additional time δ - we shall obviously have attained a purpose e_1 *or* another purpose e_2, for instance. We explain the intended effect of the advice then to act as if the according hypothesis $\forall\tau\,(A^{\tau-\delta} \to E_1^\tau \vee E_2^\tau)$ (with τ for moments) holds in the classical game: By this advice, we should act as if the following holds: If $A^{\tau-\delta}$ may be asserted in the classical game, then $E_1^\tau \vee E_2^\tau$ may also be asserted in that game - and hence E_1^τ or E_2^τ will have been anchored (due to an analogue to 5.3). (This anchoring will be anticipated, if we assert $A^{\tau-\delta}$ before τ.

§7. Preliminaries on higher order languages

In the following we speak 'about' so-called abstract objects like sets and relations. But we do not presuppose that they exist independently of the signs by which they are given ('designated', 'denoted', or the like). What we shall say of sets can be regarded as a mere manner of speaking. If we say that a sign S is (or designates) a set we only mean this:
1. For all relevant constants (or, especially, proper names) c, we dispose of sentences with the meaning "c is an element of S". 2. S is to be used 'abstractively' (within a given language) so that any occurrence of S in any asserted sentence (of that language) may be replaced by any other set sign that is said to be *equal* to S. (However, we do not simply identify sets that are extensionally equal. So one may prefer the word

"attribute" or "property" in place of "set".) For "is an element of" we shall write '$\in$' in the metalanguage (for which we use our colloquial language), and 'ε' in the object language to be introduced. In the metalanguage we also use other familiar set theoretical signs.

We shall introduce a ramified type theory (cf. [7], [11], [13], [17], e.g.) in a cumulative version.

Given a set $\mathcal{E}$ of elementary formulas in which certain constants may occur. Those constants are said to be of order 0. We shall introduce sets of order 1, whose elements are constants of order 0 (or objects denoted by them), sets of order 2, whose elements are constants of order 0 or sets of order 1, etc. So a set of order n contains only elements that have orders $< n$. However, a set of order n will also be said to have any order larger than n.

To this end, we shall construct a set $\mathcal{A}$ of (first or) higher order sentences and introduce an assertion game, which contains certain '*primary rules*' to restrict assertions of sentences belonging to $\mathcal{A}$. Since this '*primary game*' does not contain further rules of assertion, and since all sentences of $\mathcal{A}$ can be shown to be non-circular, the primary rules for sentences of $\mathcal{A} \setminus \mathcal{E}$ can be *inverted* so that both, those rules and their inverses, can also be used as inference rules. By this means, even all usual inference rules of classical logic can (as in §2, §3) be shown to be admissible in the '*classical game*' which is given by the agreement that a sentence may be asserted in this game iff its double negation may be asserted in the primary game (cf. §3).

So our **first main task** will be to show that all sentences of $\mathcal{A}$ are non-circular.

Now we incompletely sketch the higher order languages that will be introduced in §8. Assume that we already dispose of certain **elementary formulas** and terms, which are said to be **original terms**. All variables that occur in those formulas or terms are said to be of order 0. Let

$\mathcal{V}_0$ = set of all variables of order 0

$\mathcal{T}_{\text{or}}$ = set of all original terms, $\mathcal{V}_0 \subset \mathcal{T}_{\text{or}}$

$\mathcal{E}$ = set of all elementary formulas (to be considered).

$\mathcal{V}_0$ is permitted to contain variables of several sorts. (Of course, $\mathcal{V}_0$ is supposed to contain denumerably many variables of every of those sorts.) Let **constants / sentences** be terms / formulas, respectively, without free occurring variables.

We shall introduce the following sets of higher order terms and formulas:

$\mathcal{T}_n$ = set of all (simple) terms of order n,

$\mathcal{F}_n$ = set of all formulas of order n.

Here and in the following, m, n range over (signs of) ordinal numbers belonging to a given set Ω with $\mathbb{N} \subseteq \Omega \subseteq \mathcal{C}_0$. We define

$\mathcal{C}_n \rightleftharpoons$ set of all constants belonging to $\mathcal{T}_n$,

$\overline{\mathcal{C}}_n \rightleftharpoons \bigcup_{j \in \mathbb{N}^+} C_n^j$,

which is the set of all j-tuples $(c_1, \ldots, c_j)$ of constants $c_i \in \mathcal{C}_n$ with arbitrary length $j \in \mathbb{N}^+ \rightleftharpoons \mathbb{N} \setminus \{0\}$. Let also be given two disjunct denumerable sets $\mathcal{V}$ and $\overline{\mathcal{V}}$ of 'new' variables, which do not occur in elements of $\mathcal{T}_{\text{or}} \cup \mathcal{E}$. We shall use the elements of $\mathcal{V}$ as

variables for elements of $\bigcup_{n\in\Omega}\mathcal{C}_n$, i.e. for constants of arbitrary order, and the elements of $\overline{\mathcal{V}}$ as variables for elements of $\bigcup_{n\in\Omega}\overline{\mathcal{C}}_n$, i.e. for arbitrary tuples of constants. - Moreover, let
$\overline{\mathcal{T}}_n \rightleftharpoons \bigcup_{j\in\mathbb{N}^+}\mathcal{T}_n^j \cup \overline{\mathcal{V}}$.
So $\overline{\mathcal{C}}_n$ is the set of all constants belonging to $\overline{\mathcal{T}}_n$.

As signs of the object language for $\mathcal{C}_n, \overline{\mathcal{C}}_n$, and $\in$ we shall use $C_n, \overline{C}_n$, and ε, respectively. In this introduction, $x, x_1, x_2, \ldots$ range over variables of $\mathcal{V}_0 \cup \mathcal{V}$, and $\overline{x}, \overline{y}$ over variables of $\overline{\mathcal{V}}$.

All elements of $\mathcal{C}_n \setminus \mathcal{C}_0$ will be introduced as subsets of $\bigcup_{m<n}\overline{\mathcal{C}}_m$. A constant of the form $\{\overline{x}\ \varepsilon\ \overline{C}_m : A(\overline{x})\}$ will denote the set of all elements $c \in \overline{\mathcal{C}}_m$ satisfying $A(c)$. A sentence of the form $\exists x\ \varepsilon\ C_m.\, A(x)$ is to mean that there exists a constant c of order m satisfying $A(c)$. By this means, j**-ary relations** ($j \in \mathbb{N}^+$) can be described in the form

$$\begin{array}{l}\{(x_1,\ldots,x_j)\ \varepsilon\ C_m^j : A(x_1,\ldots,x_j)\} \rightleftharpoons \\ \{\overline{y}\ \varepsilon\ \overline{C}_m : \exists x_1\ \varepsilon\ C_m.\ \ldots \exists x_j\ \varepsilon\ C_m.\ (\overline{y} =_m (x_1,\ldots,x_j) \wedge A(x_1,\ldots,x_j))\},\end{array}$$

(where $\overline{y}$ does not occur in $A(x_1,\ldots,x_j)$). To this end, the sign '$=_m$' must previously be introduced suitably. - So we at first demand that

$$\begin{array}{rcl} s \in \mathcal{T}_n & \text{if} & s \in \mathcal{T}_{\text{Or}} \cup \mathcal{V}, \\ \{\overline{x}\ \varepsilon\ \overline{C}_m : F\} \in \mathcal{T}_n & \text{if} & F \in \mathcal{F}_n,\ m < n, \\ E \in \mathcal{F}_n & \text{if} & E \in \mathcal{E}, \\ (F \wedge G), (F \vee G) \in \mathcal{F}_n & \text{if} & F, G \in \mathcal{F}_n, \\ (\neg F) \in \mathcal{F}_n & \text{if} & F \in \mathcal{F}_n, \\ (\exists x\ \varepsilon\ C_m.\ F) \in \mathcal{F}_n & \text{if} & F \in \mathcal{F}_n,\ m < n, \\ (s\ \varepsilon\ t) \in \mathcal{F}_n & \text{if} & s \in \overline{\mathcal{T}}_n,\ t \in \mathcal{T}_n. \end{array}$$

We shall replace these and certain further demands by corresponding rules of constructi- on. - Note that we need not deal with complicated types that include information about 'arities' of relations. So we may simply identify types with orders.

For mathematical purposes we want also to dispose of sequences R of relations $R(0), R(1), R(2), \ldots \in \mathcal{C}_n$ satisfying

$$(\underline{c}, k)\ \varepsilon\ R(l) \leftrightarrow (\underline{c})\ \varepsilon\ \overline{C}_m\ \wedge\ k < l\ \wedge\ A((\underline{c}), k, R(k))$$

for all tuples $(\underline{c}) \equiv (c_1,\ldots,c_j)$ of constants and all $k, l \in \Omega$, if any formula $A(\overline{x}, \mu, z) \in \mathcal{F}_n$ and any ordinal $m < n$ are given. By this 'recursive characterization', $R(l)$ depends upon the relations $R(k)$ with ordinals $k < l$ only. - We designate R by $(\mathrm{J}\overline{x}\ \varepsilon\ \overline{C}_m, \mu, z : A(\overline{x}, \mu, z))$. Accordingly, we demand:

$$(\mathrm{J}\overline{x}\ \varepsilon\ \overline{C}_m, \mu, z : F)(q) \in \mathcal{T}_n \quad \text{if} \quad F \in \mathcal{F}_n,\ q \in \mathcal{T}(\Omega),\ m < n,\ \mu \in \mathcal{V}(\Omega), z \in \mathcal{V}$$

where $\mathcal{T}(\Omega)$ ($\subseteq \mathcal{T}_{\text{Or}}$) is a given set of terms whose substitution instances are elements of Ω, and $\mathcal{V}(\Omega) = \mathcal{V}_0 \cap \mathcal{T}(\Omega)$ is a set of variables for elements of Ω. ('J' is an 'induction operator'; cf. §4) - Then it can be shown that there also exists a sequence S of relations $S(0), S(1), S(2), \ldots \in \mathcal{C}_n$ satisfying

$$\begin{array}{rcl} c \,\varepsilon\, S(0) & \leftrightarrow & c \,\varepsilon\, \overline{C}_m \ \wedge\ A(c) \\ c \,\varepsilon\, S(k+1) & \leftrightarrow & c \,\varepsilon\, \overline{C}_m \ \wedge\ B(c, k, S(k)) \end{array}$$

for all $c \in \bigcup_{n \in \Omega} \overline{\mathcal{C}}_n$ and all $k \in \mathbb{N}$, if the formulas $A(\overline{x}), B(\overline{x}, \mu, z) \in \mathcal{F}_n$ and the order $m < n$ are given. (For purposes of classical reasoning, the particles $\rightarrow$, $\leftrightarrow$, and $\forall$ can be defined as in §1 and §3.)

We want to introduce equations $x = y$ such that all formulas considered are invariant under $(=)$, i.e. satisfy $c = d \wedge A(c) \rightarrow A(d)$ for all constants c, d and all formulas $A(x)$ of arbitrary orders. To this end, equal constants must especially have the same order, and equal sets must contain the same elements:

$$\begin{array}{rcl} c = d & \rightarrow & \forall \mu \,\varepsilon\, C_0 .\, (c \,\varepsilon\, C_\mu \ \leftrightarrow\ d \,\varepsilon\, C_\mu) \\ c = d \ \wedge\ \neg\, (c \,\varepsilon\, C_0) & \rightarrow & c \subseteq d \ \wedge\ d \subseteq c \end{array}$$

where $\mu \in \mathcal{V}(\Omega)$ (again), and $c \subseteq d$ means that c is a subset of d (see below). Since the formulas $c \,\varepsilon\, C_\mu$ and $c \subseteq d$ should belong to the object language to be introduced, we demand and define the following (where $\exists \overline{x} \,\varepsilon\, t .\, F$ is to be read as "For some $\overline{x}$, $\overline{x} \,\varepsilon\, t$ and F"):

$$\begin{array}{rcl} (t \,\varepsilon\, C_q) \in \mathcal{F}_n & \text{if} & t \in \mathcal{T}_n,\ q \in \mathcal{T}(\Omega) \\ (\exists \overline{x} \,\varepsilon\, t .\, F) \in \mathcal{F}_n & \text{if} & t \in \mathcal{T}_n,\ F \in \mathcal{F}_n \\ \forall \overline{x} \,\varepsilon\, s .\, F & \rightleftharpoons & \neg\, \exists \overline{x} \,\varepsilon\, s .\, \neg F \\ s \subseteq t & \rightleftharpoons & \forall \overline{x} \,\varepsilon\, s .\, \overline{x} \,\varepsilon\, t \ \wedge\ \neg\, (s \,\varepsilon\, C_0) \ \wedge\ \neg\, (t \,\varepsilon\, C_0). \end{array}$$

However, if q (is or) contains a variable, we do not rank C_q with the terms of $\bigcup_{n \in \Omega} \mathcal{T}_n$.

Now we presuppose that $(=_0)$ represents an equivalence relation on $\mathcal{C}_0$ (which is suitable for certain purposes), and that $s =_0 t$ is a formula of $\mathcal{F}$ for terms s, t. Assume that all terms of $\mathcal{T}_{\mathrm{OR}}$ and all formulas of $\mathcal{E}$ are invariant under $(=_0)$. For terms s, t of any order we define

$$\begin{array}{rcl} s \sim t & \rightleftharpoons & \forall \mu \,\varepsilon\, C_0 .\, (s \,\varepsilon\, C_\mu \ \leftrightarrow\ t \,\varepsilon\, C_\mu) \\ s = t & \rightleftharpoons & s =_0 t \ \vee\ (s \subseteq t \ \wedge\ t \subseteq s \ \wedge\ s \sim t). \end{array}$$

Of course, we demand that

$$(s =_0 t) \in \mathcal{F}_n \quad \text{if} \quad s, t \in \mathcal{T}_n.$$

Then it can be shown that all formulas of $\bigcup_{n \in \Omega} \mathcal{F}_n$ are invariant under $(=)$. This is our **second main task.**

The 'type-free' relations $(\subseteq)$, $(\sim)$, and $(=)$ are definable in our object language but they are neither elements of $\mathcal{C}$ nor elements of elements of $\mathcal{C}$.

Given a formula $A(x)$, a tuple $c \equiv (c_1, \ldots, c_j) \in \overline{\mathcal{C}}_m$ of constants, and some $i = 1, \ldots, j$. Then $A(c_i)$ means that the i^{th} component of c satisfies $A(x)$. Since our object language also contains variables $\overline{y}$ for such tuples c of constants, we postulate, in addition, that the object language contains a formula expressing that the i^{th} component of any given value of $\overline{y}$ belongs to $\mathcal{C}_m$ and satisfies $A(x)$. For that formula we take $\exists x \, \varepsilon \, \pi_m(\overline{y}, i).\, A(x)$ (with π for "projection"). Generalizing we demand

$$(\exists x \, \varepsilon \, \pi_m(s,p).\, F) \in \mathcal{F}_n \;\text{ if }\; m < n,\; s \,\varepsilon\, \overline{\mathcal{T}}_n,\; p \in \mathcal{T}(\mathbb{N}^+),\; F \in \mathcal{F}_n$$

where $\mathcal{T}(\mathbb{N}^+)$ ($\subseteq \mathcal{T}_{\text{or}}$) is a given set of terms (inclusive of variables) whose substitution instances are elements of $\mathbb{N}^+$. Of course, we want to obtain that

$$\exists x \, \varepsilon \, \pi_m((c_1, \ldots, c_j), i).\, A(x) \;\leftrightarrow\; c_i \,\varepsilon\, C_m \wedge A(c_i)$$

$(i = 1, \ldots, j)$ holds in the object language.

For constructive or predicative *analysis* in the sense of [8] inclusive of measure theory and functional analysis (as in [18], e.g.) there suffice real numbers that are given by first order Cauchy sequences of rational numbers. In the domain of those real numbers there converges every real Cauchy sequence that is given by a corresponding *first order double sequence of rational numbers.* Suitable for predicative analysis are functions $f : A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^j$, such that if $\alpha_1, \ldots, \alpha_j$ are sequences *of the mentioned sort* which satisfy $(\alpha_1, \ldots, \alpha_j) : \mathbb{N} \to A$ then $f \circ (\alpha_1, \ldots, \alpha_j)$ is a sequence *of that sort.* So we can pursue predicative analysis in languages of low orders (as stressed in [17], see also [8, p.3]). - Nevertheless, to designate orders or types we also admit transfinite ordinal numbers.

§8. Higher order languages

In this §8 we introduce only sentences in which no indicators (cf. §5) occur. But in §10 we shall also consider sentences containing indicators, whose meaning depends on situations or contexts.

The following sets (which we have already mentioned in §7) are supposed to be decidable: $\mathcal{E}, \mathcal{T}_{\text{or}}, \mathcal{V}_0, \mathcal{V}, \overline{\mathcal{V}}$, and the set $\mathcal{C}_0$ of all constants of order 0, i.e belonging to $\mathcal{T}_{\text{or}}$. For all $w \in \mathcal{V}_0$ let $\mathcal{C}(w) \subseteq \mathcal{C}_0$ be the set of all values of w, which is also supposed to be decidable. Two variables $w, x \in \mathcal{V}_0$ are said to be of the **same sort** iff $\mathcal{C}(w) = \mathcal{C}(x)$. In the metalanguage we sometimes use the particles $\underline{\wedge}$, $\underline{\rightarrow}$, $\underline{\leftrightarrow}$, and $\underline{\forall}$. In place of $\underline{\wedge}$ we sometimes write the comma.

For any element Φ of $\mathcal{T}_{\text{or}} \cup \mathcal{E}$ we define:

$$\begin{array}{rcl} \mathcal{V}_0(\Phi) & \rightleftharpoons & \text{set of all variables occurring (free) in } \Phi, \\ * \in \mathcal{S}_0(\Phi) & \rightleftharpoons & * \text{ is a substitution of all variables } w \in \mathcal{V}_0(\Phi) \text{ by} \\ & & \text{values } w^* \text{ of them (so that } \underline{\forall}\, w \in \mathcal{V}_0(\Phi).\, w^* \in \mathcal{C}(w)), \\ \mathcal{T}(w) & \rightleftharpoons & \{r \in \mathcal{T}_{\text{or}} : \underline{\forall} * \in \mathcal{S}_0(r).\, r^* \in \mathcal{C}(w)\} \quad \text{for} \quad w \in \mathcal{V}_0. \end{array}$$

$\mathcal{T}(w)$ is the set of all original terms whose substitution instances are elements of $\mathcal{C}(w)$. For all $w \in \mathcal{V}_0$ we have $\mathcal{T}(w) \cap \mathcal{C}_0 = \mathcal{C}(w)$.

Let an ordered set $(\Omega, <)$, $\Omega \subseteq \mathcal{C}_0$, be introduced, which includes $\mathbb{N}$ (in the usual succession) as an initial segment and permits all applications of (transfinite) induction that will be performed in the following. The equality $(=)$ in Ω is assumed to be the literal equality $(\equiv)$. The elements of Ω are said to be **ordinals** (ordinal numbers). We suppose the following: If $k \in \Omega$ then $k' \in \Omega$ where k' is the (immediate) successor of k; for any two ordinals k, l it is decidable whether $k < l$. - Examples for Ω are $\mathbb{N}$ and the set of all ordinals of the form $\omega^k \cdot n_k + \omega^{k-1} \cdot n_{k-1} + \ldots + \omega \cdot n_1 + n_0$ where $k, n_0, \ldots, n_k \in \mathbb{N}$, and $n_k > 0$ if $k > 0$.

As variables for elements of $\Omega \,/\, \mathbb{N}^+$, respectively, we use elements of a denumerable and decidable set $\mathcal{V}(\Omega) \,/\, \mathcal{V}(\mathbb{N}^+) \subseteq \mathcal{V}_0$. As metavariables we take: i, j for elements of $\mathbb{N}^+$; k, l, m, n for elements of Ω; and λ, μ, ν for elements of $\mathcal{V}(\Omega)$. Let $\mathcal{T}(\Omega) = \mathcal{T}(\lambda) = \Omega \cup \bigcup_{x \in \mathcal{V}(\Omega) \cup \mathcal{V}(\mathbb{N}^+)} \{x, x', x'', \ldots\}$. The equations $(q = r)$ and inequations $(q < r)$ with $q, r \in \mathcal{T}(\Omega)$ are assumed to be elements of $\mathcal{E}$. - Let $\mathcal{T}(\mathbb{N}^+) = \mathcal{T}(\kappa) = \mathbb{N}^+ \cup \bigcup_{x \in \mathcal{V}(\mathbb{N}^+)} \{x, x', x'', \ldots\}$ where $\kappa \in \mathcal{V}(\mathbb{N}^+)$.

We admit that $\mathcal{E}$ contains formulas of the shape $((s_1, \ldots, s_j) \;\varepsilon\; P)$ with $s_1, \ldots, s_j \in \mathcal{T}_{\mathrm{or}}$ but $P \notin \mathcal{T}_{\mathrm{or}}$. Here, P may especially be $\mathbb{N}$ or Ω (if $j = 1$). So let $\mathbb{N}, \Omega \notin \mathcal{C}_0$.

For original terms and elementary formulas we presuppose (where s_r^w is defined as in §2):

P1: $s \in \mathcal{T}_{\mathrm{or}},\ w \in \mathcal{V}_0,\ r \in \mathcal{T}(w) \;\underline{\rightarrow}\; s_r^w \in \mathcal{T}_{\mathrm{or}}$.
P2: $E \in \mathcal{E},\ w \in \mathcal{V}_0,\ r \in \mathcal{T}(w) \;\underline{\rightarrow}\; E_r^w \in \mathcal{E}$.

$$\text{Let} \qquad \mathcal{W} \rightleftharpoons \mathcal{V}_0 \cup \mathcal{V},\ \ \overline{\mathcal{W}} \rightleftharpoons \mathcal{W} \cup \overline{\mathcal{V}} \qquad \text{(see §7).}$$

In the following $w, x, y, z, x_1, x_2, \ldots$ range over arbitrary variables (belongig to $\overline{\mathcal{W}}$), and $\overline{x}, \overline{y}$ over elements of $\overline{\mathcal{V}}$. Distinctly denoted variables are assumed to be distinct. Accordingly, a list $x_1, \ldots, x_n$ of variables is assumed to be a list of distinct variables.

Induktive definitions of $\mathcal{T}_n$, $\overline{\mathcal{T}}_n$ and $\mathcal{F}_n$ $(n \in \Omega)$:
Sentences (belonging to a metalanguage) of the forms $(s \in \mathcal{T}_n)$, $(s \in \overline{\mathcal{T}}_n)$ and $(F \in \mathcal{F}_n)$ are to be verified by their deductions by the following '$\mathcal{T}, \mathcal{F}$-**rules**'. (In these rules, $\Rightarrow$ indicates the permitted deduction steps; the pertinent conditions for applications of these rules are quoted behind the word "if".)

$$
\begin{array}{rcll}
 & \Rightarrow & s \in \mathcal{T}_n & (\text{if } s \in \mathcal{T}_{\mathrm{or}}) \\
 & \Rightarrow & z \in \mathcal{T}_n & (\text{if } z \in \mathcal{V}) \\
F \in \mathcal{F}_n & \Rightarrow & \{\overline{x}\,\varepsilon\,\overline{C}_m : F\} \in \mathcal{T}_n & (\text{if } m < n) \\
F \in \mathcal{F}_n & \Rightarrow & (\mathrm{J}\overline{x}\,\varepsilon\,\overline{C}_m, \mu, z : F)(q) \in \mathcal{T}_n & (\text{if } m < n,\ z \in \mathcal{V},\ q \in \mathcal{T}(\Omega)\,) \\
 & \Rightarrow & \overline{x} \in \overline{\mathcal{T}}_n & \\
s_1, \ldots, s_j \in \mathcal{T}_n & \Rightarrow & (s_1, \ldots, s_j) \in \overline{\mathcal{T}}_n & \\
 & \Rightarrow & E \in \mathcal{F}_n & (\text{if } E \in \mathcal{E}) \\
F, G \in \mathcal{F}_n & \Rightarrow & (F \wedge G), (F \vee G) \in \mathcal{F}_n & (\text{two rules}) \\
F \in \mathcal{F}_n & \Rightarrow & (\neg F) \in \mathcal{F}_n & \\
F \in \mathcal{F}_n & \Rightarrow & (\exists x\,\varepsilon\,C_m.\,F) \in \mathcal{F}_n & (\text{if } m < n,\ x \in \mathcal{W}) \\
s \in \overline{\mathcal{T}}_n,\ t \in \mathcal{T}_n & \Rightarrow & (s\,\varepsilon\,t) \in \mathcal{F}_n & \\
t \in \mathcal{T}_n,\ F \in \mathcal{F}_n & \Rightarrow & (\exists \overline{x}\,\varepsilon\,t.\,F) \in \mathcal{F}_n & \\
s \in \overline{\mathcal{T}}_n,\ F \in \mathcal{F}_n & \Rightarrow & (\exists x\,\varepsilon\,\pi_m(s,p).\,F) \in \mathcal{F}_n & (\text{if } m < n,\ x \in \mathcal{W},\ p \in \mathcal{T}(\mathbb{N}^+)) \\
s, t \in \mathcal{T}_n & \Rightarrow & (s =_0 t) \in \mathcal{F}_n & \\
s \in \overline{\mathcal{T}}_n & \Rightarrow & (s\,\varepsilon\,C^p_q) \in \mathcal{F}_n & (\text{if } p \in \mathcal{T}(\mathbb{N}^+),\ q \in \mathcal{T}(\Omega)).
\end{array}
$$

Thus, $\mathcal{T}_0 = \mathcal{T}_{\mathrm{or}} \cup \mathcal{V}$, and $\mathcal{E} \subset \mathcal{F}_0$. - The following occurrences of variables in terms or formulas are said to be *bound*: x in $(\exists x\ \varepsilon\ C_m.\,F)$ and in $(\exists x\ \varepsilon\ \pi_m(s,p).\,F)$, $\overline{x}$ in $\{\overline{x}\,\varepsilon\,\overline{C}_m : F\}$ and in $(\exists \overline{x}\,\varepsilon\,t.\,F)$, and $\overline{x}, \mu, z$ in $(J\overline{x}\,\varepsilon\,\mathcal{C}_m, \mu, z : F)$. All other occurrences of variables in terms or formulas are said to be *free*, i.e. not bound. - We presuppose, of course, that if $(\Phi \in \mathcal{T}_n), (\Phi \in \overline{\mathcal{T}}_n)$, or $(\Phi \in \mathcal{F}_n)$ is a conclusion of one of the latter rules except the first or seventh, then Φ does not belong to $\mathcal{T}_{\mathrm{or}} \cup \mathcal{E}$. - Sometimes we shall as usual omit brackets from formulas. - Definitions:

$$
\begin{array}{l}
\mathcal{A}_n \rightleftharpoons \text{ set of all sentences of order } n\ (\subset \mathcal{F}_n); \\
\mathcal{T} \rightleftharpoons \bigcup_{n\in\Omega} \mathcal{T}_n;\ \overline{\mathcal{T}} \rightleftharpoons \bigcup_{n\in\Omega} \overline{\mathcal{T}}_n;\ \mathcal{F} \rightleftharpoons \bigcup_{n\in\Omega} \mathcal{F}_n; \\
\mathcal{C} \rightleftharpoons \bigcup_{n\in\Omega} \mathcal{C}_n;\ \overline{\mathcal{C}} \rightleftharpoons \bigcup_{n\in\Omega} \overline{\mathcal{C}}_n;\quad \text{and } \mathcal{A} \rightleftharpoons \bigcup_{n\in\Omega} \mathcal{A}_n.
\end{array}
$$

Notice that $C^j_m, C_m, \overline{C}_m \notin \mathcal{C}$. Similarly, $C^p_q, \pi_m(s,p) \notin \mathcal{T}$. - As metavariables we shall use: p for elements of $\mathcal{T}(\mathbb{N}^+)$; q for elements of $\mathcal{T}(\Omega)$; r, s, t for terms (i.e. elements of $\mathcal{T} \cup \overline{\mathcal{T}}$); F, G, H for formulas (i.e. elements of $\mathcal{F}$); a, b, c, d for constants (i.e. elements of $\mathcal{C} \cup \overline{\mathcal{C}}$); A, B for sentences (i.e. elements of $\mathcal{A}$); and, for instance, $A(x_1, \ldots, x_j)$ for formulas, in which at most the variables $x_1, \ldots, x_j$ occur free.

We shall write $\underline{s}$ for lists $s_1, \ldots, s_j$; $(\underline{s})$ for tuples $(s_1, \ldots, s_j)$ of terms $s_i \in \mathcal{T}$; and '$s_1, \ldots, s_j \in \mathcal{T}_n$' for '$s_1 \in \mathcal{T}_n, \ldots, s_j \in \mathcal{T}_n$', e.g. $\underline{x}$ is to be distinguished from $\overline{x}$.
To formulate assertion rules for sentences of $\mathcal{A}$ we use some **definitions**:

$$
\mathcal{C}_m(x) \ \rightleftharpoons\ \begin{cases} \mathcal{C}(x) & \text{for } x \in \mathcal{V}_0 \\ \mathcal{C}_m & \text{for } x \in \mathcal{V} \\ \overline{\mathcal{C}}_m & \text{for } x \in \overline{\mathcal{V}} \end{cases}
$$

$$
(c_1, \ldots, c_j) \in \mathcal{C}_m(x_1, \ldots, x_k) \ \rightleftharpoons\ j = k \ \underline{\wedge}\ \underline{\forall} i \leq j.\ c_i \in \mathcal{C}_m(x_i).
$$

$\mathcal{C}_m(x)$ is the set of all elements of $\mathcal{C}_m$ that are values of x.
For $\Phi \in \mathcal{T} \cup \overline{\mathcal{T}} \cup \mathcal{F}$ and $\mathcal{U} \subseteq \overline{\mathcal{W}}$ let

$$
\mathcal{U}(\Phi) \ \rightleftharpoons\ \text{set of all variables} \in \mathcal{U} \text{ that occur free in } \Phi.
$$

A term s / a formula F is said to be *invariant* under an equivalence relation $(\approx)$ on $\mathcal{C}_n$ iff for all $(\underline{c}), (\underline{d}) \in \mathcal{C}_n(\underline{w})$ with $\{\underline{w}\} = \{w_1, \ldots, w_j\} = \overline{\mathcal{W}}(\underline{s})$ / $\overline{\mathcal{W}}(F)$, respectively, the following holds:

$$\underline{c} \approx \underline{d} \quad \to \quad s^{\underline{w}}_{\underline{c}} \approx s^{\underline{w}}_{\underline{d}},$$
$$\underline{c} \approx \underline{d} \quad \to \quad (F^{\underline{w}}_{\underline{c}} \leftrightarrow F^{\underline{w}}_{\underline{d}}), \text{ respectively,}$$

where $$(\underline{c}) \approx (\underline{d}) \quad \rightleftharpoons \quad \underline{c} \approx \underline{d} \quad \rightleftharpoons \quad c_1 \approx d_1 \wedge \ldots \wedge c_j \approx d_j$$

and the substitutions $\frac{\underline{w}}{\underline{c}}$ and $\frac{\underline{w}}{\underline{d}}$ are defined as in §2. (This has been formulated somewhat beforehand.)

Assumptions: 1. We have agreed upon certain primary rules (or usage) for sentences of $\mathcal{E}$ which do not refer to assertions of sentences of $\mathcal{A} \setminus \mathcal{E}$.
2. $\mathcal{E}$ contains all formulas $s =_0 t$ with $s, t \in \mathcal{T}_{\text{or}}$.
3. $(=_0)$ represents a given equivalence relation on $\mathcal{C}_0$ under which all terms of $\mathcal{T}_{\text{or}}$ and all formulas of $\mathcal{E}$ are invariant. For all $w \in \mathcal{V}_0$, the scope $\mathcal{C}(w)$ of values of w is invariant, i.e., for all $c, d \in \mathcal{C}_0$, if $c =_0 d$ and $c \in \mathcal{C}(w)$, then $d \in \mathcal{C}(w)$. $(=_0) \notin \mathcal{C}$.

Let the '**primary game**' contain the following '**primary rules**' of assertion for sentences of $\mathcal{A} \setminus \mathcal{E}$; we include these rules among the internal rules:

$$\begin{array}{rcl}
\natural\,(A \wedge B) & :\Rightarrow & \natural\,A \text{ and } \natural\,B \\
\natural\,(A \vee B) & :\Rightarrow & \natural\,A \text{ or } \natural\,B \\
\natural\,\neg A & :\Rightarrow & A \text{ rejected (see §1)} \\
\natural\,\exists x \,\varepsilon\, C_m.\, A(x) & :\Rightarrow & \text{for some } c: \ \natural\,c \in \mathcal{C}_m(x), \ \natural\,A(c) \\
\natural\,c \,\varepsilon\, d & :\Rightarrow & \natural\,d \notin \mathcal{C}_0 \\
\natural\,c \,\varepsilon\, \{\overline{x} \,\varepsilon\, \overline{C}_m \colon A(\overline{x})\} & :\Rightarrow & \natural\,c \in \overline{\mathcal{C}}_m, \ \natural\,A(c),
\end{array}$$

for $R(\nu) \rightleftharpoons (\mathrm{J}\overline{x} \,\varepsilon\, \overline{C}_m, \mu, z \colon A(\overline{x}, \mu, z))(\nu) \in \mathcal{T}$:

$$\begin{array}{rcl}
\natural\,(\underline{c}, k) \,\varepsilon\, R(l) & :\Rightarrow & \natural\,(\underline{c}) \in \overline{\mathcal{C}}_m, \ \natural\,k < l, \ \natural\,A((\underline{c}), k, R(k)) \\
\natural\,a \,\varepsilon\, R(l) & :\Rightarrow & \natural\,a \in \overline{\mathcal{C}}_m \times \Omega \\
\natural\,\exists \overline{x} \,\varepsilon\, b(\overline{x}).\, A(\overline{x}) & :\Rightarrow & \text{for some } c \in \overline{\mathcal{C}}: \ \natural\,c \,\varepsilon\, b(c), \ \natural\,A(c) \\
\natural\,\exists x \,\varepsilon\, \pi_m((c_1, \ldots, c_j), i).\, A(x) & :\Rightarrow & \natural\,c_i \in \mathcal{C}_m(x), \ \natural\,A(c_i) \quad (\text{if } i \le j) \\
\natural\,\exists x \,\varepsilon\, \pi_m((c_1, \ldots, c_j), i).\, A(x) & :\Rightarrow & \natural\,\bot \quad (\text{if } i > j) \\
\natural\,\exists x \,\varepsilon\, \pi_m(s, p).\, A(x) & :\Rightarrow & \natural\,\bot \quad (\text{if } \overline{\mathcal{W}}(s, p) = \{x\}) \\
\natural\,c =_0 d & :\Rightarrow & \natural\,c, d \in \mathcal{C}_0 \\
\natural\,c \,\varepsilon\, C^j_m & :\Rightarrow & \natural\,c \in \mathcal{C}^j_m.
\end{array}$$

Assertions of 'auxiliary sentences' of the forms $c \in \mathcal{C}_m(x)$, $d \notin \mathcal{C}_0$, $c \in \overline{\mathcal{C}}_m$, $a \in \overline{\mathcal{C}}_m \times \Omega$, and $c \in \mathcal{C}^j_m$ ought, of course, to be justified additionally. - *Assertions of sentences of* $\mathcal{A}$ *are not to be restricted besides.*

To see that the primary rules for sentences of $\mathcal{A} \setminus \mathcal{E}$ can be inverted we shall prove that all sentences of $\mathcal{A}$ are non-circular in the following sense.

Definitions: A sentence $C\,(\in \mathcal{A})$ is said to be a **predecessor** of D iff C is deducible from D by at least one application of the following rules (where $\Rightarrow$ again indicates the deduction steps):

$$\begin{array}{rcll} A\wedge B & \Rightarrow & A,\ B & \text{(two rules)}\\ A\vee B & \Rightarrow & A,\ B & \text{(two rules)}\\ \neg A & \Rightarrow & A & \\ \exists x\,\varepsilon\, C_m.\, A(x) & \Rightarrow & A(c), & \text{if } c\in \mathcal{C}_m(x)\\ c\,\varepsilon\,\{\overline{x}\,\varepsilon\,\overline{C}_m : A(\overline{x})\} & \Rightarrow & A(c), & \text{if } c\in \overline{\mathcal{C}}_m, \end{array}$$

for $R(\nu) \rightleftharpoons (\mathrm{J}\overline{x}\,\varepsilon\,\overline{C}_m, \mu, z\colon A(\overline{x},\mu,z))(\nu)\in\mathcal{T}$:

$$(\underline{c},k)\,\varepsilon\, R(l) \quad\Rightarrow\quad A((\underline{c}),k,R(k)), \qquad \text{if } (\underline{c})\in\overline{\mathcal{C}}_m \text{ and } k<l,$$

for terms $b(\overline{x})$ of the form $\{\overline{y}\,\varepsilon\,\overline{C}_m\colon G\}$ or $(\mathrm{J}\overline{y}\,\varepsilon\,\overline{C}_m,\mu,z\colon G)(l)$:

$$\begin{array}{rcll} \exists\overline{x}\,\varepsilon\, b(\overline{x}).\, A(\overline{x}) & \Rightarrow & c\,\varepsilon\, b(c),\ A(c), & \text{if } c\in\overline{\mathcal{C}}_m\\ \exists x\,\varepsilon\,\pi_m((c_1,\ldots,c_j),i).\, A(x) & \Rightarrow & A(c_i), & \text{if } i\le j. \end{array}$$

In every instance of any of these rules the conclusion is said to be an **immediate predecessor** of the premise, iff the conditions quoted behind the word "if" are satisfied. Sentences that do not occur as premises of the just mentioned rules have no predecessors. So sentences that belong $\mathcal{E}$ or have the form $c\,\varepsilon\, C^j_m$ have no immedate predecessors.

A sentence of $\mathcal{A}$ is said to be **non-circular** iff it is not a predecessor of itself.

For the announced proof that all sentences are non-circular we need some preliminaries. At first we define

$$\mathcal{T}_n(w) \quad\rightleftharpoons\quad \begin{cases} \mathcal{T}(w) & \text{for} \quad w\in\mathcal{V}_0\\ \mathcal{T}_n & \text{for} \quad w\in\mathcal{V}\\ \overline{\mathcal{T}}_n & \text{for} \quad w\in\overline{\mathcal{V}}. \end{cases}$$

8.1. Lemma: For all $w,y\in\overline{\mathcal{W}}$ we have

$$\begin{array}{rcl} s\in\mathcal{T}_n(y),\ r\in\mathcal{T}_n(w) & \underset{-}{\rightarrow} & s^w_r\in\mathcal{T}_n(y);\\ F\in\mathcal{F}_n,\ r\in\mathcal{T}_n(w) & \underset{-}{\rightarrow} & F^w_r\in\mathcal{F}_n. \end{array}$$

Regard that $\mathcal{T}(\Omega),\mathcal{T}(\mathbb{N}^+)\in\{\mathcal{T}(y)\colon y\in\mathcal{V}_0\}$.

Proof: At first let $y\in\mathcal{V}_0$, and $s\in\mathcal{T}(y)$. This means that if $\underline{x}$ is a list of all distinct elements of $\mathcal{V}(s)$, and $\underline{a}\in\mathcal{C}(\underline{x})$, then $s^{\underline{x}}_{\underline{a}}\in\mathcal{C}(y)$. Let $w\in\mathcal{V}_0(s)$ (otherwise $s^w_r\equiv s$) and $r\in\mathcal{T}(w)$. Let $\underline{z}$ be a list of all distinct elements of $\mathcal{V}_0(s^w_r)$, and let $\underline{u}$ be $\underline{z}$ without w. Then for all $\underline{c}\in\mathcal{C}(\underline{z})$, by setting $r^\circ\rightleftharpoons r^{\underline{z}}_{\underline{c}}$ and $\underline{b}\rightleftharpoons\underline{u}^{\underline{z}}_{\underline{c}}$, we have $r^\circ\in\mathcal{C}(w)$ and hence $(s^w_r)^{\underline{z}}_{\underline{c}}\equiv s^{w,\,\underline{u}}_{r^\circ,\underline{b}}\in\mathcal{C}(y)$. So $s^w_r\in\mathcal{T}(y)$. -

Now let $y\in\mathcal{V}\cup\overline{\mathcal{V}}$. We prove 8.1 in this case by **induction on the $\mathcal{T},\mathcal{F}$-rules** (i.e. on the number of steps of construction by the $\mathcal{T},\mathcal{F}$-rules): Let $r\in\mathcal{T}_n(w)$, and let $*$ denote the substitution of r for w. We write "I.H." for "induction hypothesis". - For arbitrary

elements Φ of $\mathcal{T}_n \cup \overline{\mathcal{T}}_n \cup \mathcal{F}_n$ we conclude $\Phi^* \in \mathcal{T}_n \cup \overline{\mathcal{T}}_n \cup \mathcal{F}_n$ from the
I.H.: $s^* \in \mathcal{T}_n / \; s^* \in \overline{\mathcal{T}}_n / \; F^* \in \mathcal{F}_n$, respectively, holds for all terms s and formulas F for which a previous deduction of $(s \in \mathcal{T}_n) \; / \; (s \in \overline{\mathcal{T}}_n) \; / \; (F \in \mathcal{F}_n)$ by the $\mathcal{T}, \mathcal{F}$-rules is required for a deduction of $(\Phi \in \mathcal{T}_n)$, $(\Phi \in \overline{\mathcal{T}}_n)$ or $(\Phi \in \mathcal{F}_n)$.

- ▷ Let $\Phi \in \mathcal{T}_{\text{Or}} \subset \mathcal{T}_n$. If $w \in \mathcal{V}_0$ then $r \in \mathcal{T}(w)$, so that (by **P1**) $\Phi^* \in \mathcal{T}_{\text{Or}}$. If $w \notin \mathcal{V}_0$ then w does not occur in Φ; therefore, $\Phi^* \equiv \Phi \in \mathcal{T}_n$.
- ▷ Let $\Phi \equiv z \in \mathcal{V} \subset \mathcal{T}_n$. If $w \equiv z$ then $z^* \equiv r \in \mathcal{T}_n(w) = \mathcal{T}_n$. If $w \not\equiv z$ then $z^* \equiv z \in \mathcal{T}_n$.
- ▷ Let $\Phi \equiv (\mathrm{J}\overline{x} \; \varepsilon \; \overline{C}_m, \mu, z \colon F)(q) \in \mathcal{T}_n$. So $F \in \mathcal{F}_n, m < n$, and so (by I.H.) $F^* \in \mathcal{F}_n$. If $w \notin \{\overline{x}, \mu, z\}$ then $\Phi^* \equiv (\mathrm{J}\overline{x} \; \varepsilon \; \overline{C}_m, \mu, z \colon F^*)(q^*) \in \mathcal{T}_n$ (since $q^* \in \mathcal{T}(\Omega)$). If $w \in \{\overline{x}, \mu, z\}$ then $\Phi^* \equiv (\mathrm{J}\overline{x} \; \varepsilon \; \overline{C}_m, \mu, z \colon F)(q^*)$, thus again $\Phi^* \in \mathcal{T}_n$.
- ▷ Let $\Phi \equiv (s_1, \ldots, s_j) \in \overline{\mathcal{T}}_n$. Then $s_1, \ldots, s_j \in \mathcal{T}_n$. So (by I.H.) $s_1^*, \ldots, s_j^* \in \mathcal{T}_n$, and so $\Phi^* \equiv (s_1^*, \ldots, s_j^*) \in \overline{\mathcal{T}}_n$.

The remaining steps of induction can be performed analogously. □

Definitions: For $\Phi \in \mathcal{T} \cup \overline{\mathcal{T}} \cup \mathcal{F}$and $n > 0$ we define:

$$\begin{array}{rcl} \mathcal{A}^{\text{nc}} & \rightleftharpoons & \text{set of all non-circular sentences.} \\ * \in \mathcal{S}_n(\Phi) & \rightleftharpoons & * \text{ is a substitution of all variables } w \in \overline{\mathcal{W}}(\Phi) \\ & & \text{by constants } w^* \in \mathcal{C}_n(w) \text{ and satisfies} \\ & & \underline{\forall}\, w \in \mathcal{V}(\Phi).\ \underline{\forall}\ a \in \overline{\mathcal{C}}_n.\ (a \; \varepsilon \; w^*) \in \mathcal{A}^{\text{nc}} \\ F \in \mathcal{F}_n^{\text{nc}} & \rightleftharpoons & \underline{\forall} * \in \mathcal{S}_n(F).\ F^* \in \mathcal{A}^{\text{nc}}. \end{array}$$

Remark: '$* \in \mathcal{S}_0(\Phi)$' has been defined on p.32. - We have:

$$\begin{array}{rcl} s \in \mathcal{T}_n,\ * \in \mathcal{S}_n(s) & \underline{\rightarrow} & s^* \in \mathcal{C}_n. \\ s \in \overline{\mathcal{T}}_n,\ * \in \mathcal{S}_n(s) & \underline{\rightarrow} & s^* \in \overline{\mathcal{C}}_n. \\ F \in \mathcal{F}_n,\ * \in \mathcal{S}_n(F) & \underline{\rightarrow} & F^* \in \mathcal{A}_n \quad \text{(by 8.1).} \end{array}$$

Definition: Let $t\binom{x}{c}^*$ be the term that results from t by replacing all free occurrences of x by c, and applying the substitution $*$ thereafter. Let $\binom{x}{c}^*$ be the corresponding compound substitution.

8.2. Lemma: (a) $* \in \mathcal{S}_n(\exists x \; \varepsilon \; C_m.\, F),\ c \in \mathcal{C}_m(x),\ \mathcal{F}_m \subseteq \mathcal{F}_m^{\text{nc}},\ m < n \ \underline{\rightarrow}\ \binom{x}{c}^* \in \mathcal{S}_n(F)$.
(b) $* \in \mathcal{S}_n(\exists \overline{x} \; \varepsilon \; t.\, F),\ a \in \mathcal{C}_m(\overline{x}) \ \underline{\rightarrow}\ \binom{\overline{x}}{a}^* \in \mathcal{S}_n(\overline{x} \; \varepsilon \; t) \cap \mathcal{S}_n(F)$.

Proof: (a) Let $* \in \mathcal{S}_n(\exists x \; \varepsilon \; C_m.\, F)$, $c \in \mathcal{C}_m(x)$, $\mathcal{F}_m \subseteq \mathcal{F}_m^{\text{nc}}$, $m < n$, as well as $w \in \mathcal{V}(F)$ with $w^* \in \mathcal{C}_n$, and $a \in \overline{\mathcal{C}}_n$. We have to show that $(a \; \varepsilon \; w\binom{x}{c}^*) \in \mathcal{A}^{\text{nc}}$. If $w \not\equiv x$ then $w \in \mathcal{V}(\exists x \; \varepsilon \; C_m.\, F)$ and hence $(a \; \varepsilon \; w\binom{x}{c}^*) \equiv (a \; \varepsilon \; w^*) \in \mathcal{A}^{\text{nc}}$. Now let $w \equiv x$. So $(a \; \varepsilon \; w\binom{x}{c}^*) \equiv (a \; \varepsilon \; c)$. If $a \in \overline{\mathcal{C}}_n \backslash \overline{\mathcal{C}}_m$ then $(a \; \varepsilon \; c)$ has no predecessors (since $c \in \mathcal{C}_m$), and hence $(a \; \varepsilon \; c)$ is non-circular. If $a \in \overline{\mathcal{C}}_m$ then $(a \; \varepsilon \; c) \in \mathcal{A}_m \subseteq \mathcal{A}^{\text{nc}}$.

(b) Let $* \in \mathcal{S}_n(\exists \overline{x} \; \varepsilon \; t.\, F)$, $a \in \mathcal{C}_m(\overline{x})$, $w \in \mathcal{V}(\overline{x} \; \varepsilon \; t) \cup \mathcal{V}(F)$ with $w^* \in \mathcal{C}_n$, and $a \in \overline{\mathcal{C}}_n$. We have to show that $(a \; \varepsilon \; w\binom{\overline{x}}{a}^*) \in \mathcal{A}^{\text{nc}}$. Because of $w \in \mathcal{V}$ we have $w \not\equiv \overline{x}$, hence $w \in \mathcal{V}(\exists \overline{x} \; \varepsilon \; t.\, F)$, and hence $(a \; \varepsilon \; w\binom{\overline{x}}{a}^*) \equiv (a \; \varepsilon \; w^*) \in \mathcal{A}^{\text{nc}}$.□

8.3. Theorem: All sentences of $\mathcal{A}$ are non-circular.

Proof: By 'composite induction' we show that $\mathcal{F}_n \subseteq \mathcal{F}_n^{\text{nc}}$ for all $n \in \Omega$: We start from the induction hypothesis

I.H.1: $\mathcal{F}_m \subseteq \mathcal{F}_m^{\text{nc}}$ for all $m < n$.

From this we conclude $\mathcal{F}_n \subseteq \mathcal{F}_n^{\text{nc}}$ by induction on the $\mathcal{T}, \mathcal{F}$-rules. To this end, for any formula $H \in \mathcal{F}_n$ we infer $H \in \mathcal{F}_n^{\text{nc}}$ from I.H.1 and the further hypothesis

I.H.2: $F \in \mathcal{F}_n^{\text{nc}}$ for all formulas F such that the deduction of $(H \in \mathcal{F}_n)$ by the $\mathcal{T}, \mathcal{F}$-rules requires a previous deduction of $(F \in \mathcal{F}_n)$ by those rules.

Let $H \in \mathcal{F}_n$ and $* \in \mathcal{S}_n(H)$. We have to show that H^* is non-circular. To this end it suffices to show that all immediate predecessors of H^* are non-circular.

▷ If $H \in \mathcal{E}$ or $H \equiv (s \; \varepsilon \; C_q^p)$, then H^* has no immedate predecessor ($\in \mathcal{A}$). So $H^* \in \mathcal{A}^{\text{nc}}$.

▷ Let $H \in \{(F \wedge G), (F \vee G)\}$ with $F, G \in \mathcal{F}_n$. By I.H.2 we have $F, G \in \mathcal{F}_n^{\text{nc}}$, and so $F^*, G^* \in \mathcal{A}^{\text{nc}}$.

▷ For $H \equiv (\neg F)$ we can conclude similarly.

▷ Let $H \equiv (\exists x \; \varepsilon \; C_m . \, F)$ with $m < n$ and $F \in \mathcal{F}_n$. By I.H.2, $F \in \mathcal{F}_n^{\text{nc}}$. Every immediate predecessor of H^* has the form $F\binom{x}{c}^*$ with $c \in \mathcal{C}_m(x)$ and is, therefore, non-circular (since $\binom{x}{c}^* \in \mathcal{S}_n(F)$ holds by 8.2 and I.H.1).

▷ Let $H \equiv (s \; \varepsilon \; t)$ with $s \in \overline{\mathcal{T}}_n$, $t \in \mathcal{T}_n$. By 8.1, $s^* \in \overline{\mathcal{C}}_n$.
Case 1: Let $t \in \mathcal{T}_{\text{or}}$. Then $t^* \in \mathcal{C}_0$. So $H^* \equiv (s^* \; \varepsilon \; t^*)$ has no predecessors.
Case 2: Let $t \in \mathcal{V}$. Because of $* \in \mathcal{S}_n(H)$ and $s^* \in \overline{\mathcal{C}}_n$ we have $H^* \equiv (s^* \; \varepsilon \; t^*) \in \mathcal{A}^{\text{nc}}$.
Case 3: $t \equiv R(q)$ with $R \equiv (\mathrm{J}\overline{x} \; \varepsilon \; \overline{C}_m, \mu, z : F)$, $F \in \mathcal{F}_n$ and $m < n$. By I.H.2, $F \in \mathcal{F}_n^{\text{nc}}$. Let $F \equiv A(\overline{x}, \mu, z)$. $H^* \equiv s^* \; \varepsilon \; t^*$ has the form $(\underline{a}, k) \; \varepsilon \; R(l)$, which has the immediate predecessor $A((\underline{a}), k, R(k))$ iff $k < l$. Now we suppose that, for all $k < l$ and all $(\underline{a}, h) \in \overline{\mathcal{C}}_m \times \Omega$, we have $((\underline{a}, h) \; \varepsilon \; R(k)) \in \mathcal{A}^{\text{nc}}$, which implies $\binom{\overline{x},\, \mu,\, z}{(\underline{a}), k, R(k)}^* \in \mathcal{S}_n(F)$ (since $\overline{x}, \mu \notin \mathcal{V}$), $A((\underline{a}), k, R(k)) \equiv F\binom{\overline{x},\, \mu,\, z}{(\underline{a}), k, R(k)}^* \in \mathcal{A}^{\text{nc}}$ (since $F \in \mathcal{F}_n^{\text{nc}}$), and hence $((\underline{a}, k) \; \varepsilon \; R(l)) \in \mathcal{A}^{\text{nc}}$. By induction on Ω we obtain especially $H^* \equiv (s^* \; \varepsilon \; R(q^*)) \in \mathcal{A}^{\text{nc}}$.
The residual case 4: $t \equiv \{\overline{x} \; \varepsilon \; \overline{C}_m : F\}$ can even be treated simpler.

▷ Let $H \equiv (\exists \overline{x} \; \varepsilon \; t. \, F)$ with $t \in \mathcal{T}_n$ and $F \in \mathcal{F}_n$. By I.H.2, $F \in \mathcal{F}_n^{\text{nc}}$. As in the case "$H \equiv (s \; \varepsilon \; t)$" we also obtain $(\overline{x} \; \varepsilon \; t) \in \mathcal{F}_n^{\text{nc}}$. Every immediate predecessor of H^* has the form $(\overline{x} \; \varepsilon \; t)\binom{\overline{x}}{a}^* \equiv (a \; \varepsilon \; t\binom{\overline{x}}{a}^*)$ or $F\binom{\overline{x}}{a}^*$ with $a \in \overline{\mathcal{C}}_m$ where $t\binom{\overline{x}}{a}^*$ has the form $\{\overline{y} \; \varepsilon \; \overline{C}_m \ldots\}$ or $(\mathrm{J}\overline{y} \; \varepsilon \; \overline{C}_m \ldots)(l)$. If $\overline{x} \notin \overline{\mathcal{V}}(t)$ then $t\binom{\overline{x}}{a}^* \equiv t^* \in \mathcal{C}_n$, so that $m < n$. If $\overline{x} \in \overline{\mathcal{V}}(t)$, then $t \notin \mathcal{T}_0$, hence $t \, (\in \mathcal{T}_n)$ has the indicated form, and hence $m < n$, again. Therefore, in every case, $a \in \overline{\mathcal{C}}_n$. So $(a \in t\binom{\overline{x}}{a}^*)$ is non-circular (since $(\overline{x} \; \varepsilon \; t) \in \mathcal{F}_n^{\text{nc}}$ and $\binom{\overline{x}}{a}^* \in \mathcal{S}_n(\overline{x} \in t)$ by 8.2). Also $F\binom{\overline{x}}{a}^*$ is non-circular (since $F \in \mathcal{F}_n^{\text{nc}}$ and $\binom{\overline{x}}{a}^* \in \mathcal{S}_n(F)$ by 8.2). Hence all immediate predecessors of H^* are non-circular.

▷ For $H \equiv (\exists x \; \varepsilon \; \pi_m(s, p). \, F)$ we can argue as in the above case "$H \equiv (\exists x \; \varepsilon \; C_m . \, F)$".

We have shown that $\mathcal{F}_n \subseteq \mathcal{F}_n^{\text{nc}}$ for all $n \in \Omega$. It follows especially that all sentences are non-circular. □ - By 8.3 and the results of §2 and §3 we obtain:

8.3*. Corollary: All primary rules for sentences of $\mathcal{A} \setminus \mathcal{E}$ can be inverted. So we may argue classically with sentences of $\mathcal{A}$ in the classical game.

Now we repeat some former definitions, define an equivalence relation $(=)$ on $\mathcal{C}$, and show that all terms and formulas are invariant under that relation.

Definitions: For $x \in \mathcal{W}$ and $s, t \in \mathcal{T}$ we define:

$$\begin{array}{rcll}
\forall x \,\varepsilon\, C_n.\, F & \rightleftharpoons & \neg\,\exists x \,\varepsilon\, C_n.\, \neg F & \\
\exists\, w\, F & \rightleftharpoons & \exists\, w \,\varepsilon\, C_0.\, F & (\text{if } w \in \mathcal{V}_0) \\
\forall\, w\, F & \rightleftharpoons & \forall\, w \,\varepsilon\, C_0.\, F & (\text{if } w \in \mathcal{V}_0) \\
\forall \overline{x} \,\varepsilon\, s.\, F & \rightleftharpoons & \neg\,\exists \overline{x} \,\varepsilon\, s.\, \neg F & \\
s \,\varepsilon\, C_q & \rightleftharpoons & (s) \,\varepsilon\, C_q^1 & \\
s \subseteq t & \rightleftharpoons & \forall \overline{x} \,\varepsilon\, s.\, \overline{x} \,\varepsilon\, t \;\wedge\; \neg\,(s \,\varepsilon\, C_0) \;\wedge\; \neg\,(t \,\varepsilon\, C_0) & \\
s \sim t & \rightleftharpoons & \forall \mu\, (s \,\varepsilon\, C_\mu \;\leftrightarrow\; t \,\varepsilon\, C_\mu) & \\
s = t & \rightleftharpoons & s =_0 t \;\vee\; (s \subseteq t \;\wedge\; t \subseteq s \;\wedge\; s \sim t) & \\
s =_n t & \rightleftharpoons & s = t \;\wedge\; s \,\varepsilon\, C_n \;\wedge\; t \,\varepsilon\, C_n & (\text{for } n > 0) \\
s \,\varepsilon\, t & \rightleftharpoons & (s) \,\varepsilon\, t. &
\end{array}$$

In the definitions of $(s \subseteq t)$ and $(s \sim t)$ let the variables $\overline{x}$ and μ not occur in s or t. - Because of $c = d \to c \sim d$ we obtain:

8.4. Lemma: $\quad c \,\varepsilon\, C_m \;\wedge\; c = d \;\to\; c =_m d.$

Definition: For $s, t \in \overline{\mathcal{T}}$ we define by means of variables $x, y, z \in \mathcal{V} \setminus \mathcal{V}(s,t)$ and κ for elements of $\mathbb{N}^+$:

$$\begin{array}{rcl}
s \,\varepsilon\, \overline{C}_m & \rightleftharpoons & s \,\varepsilon\, \{\overline{x} \,\varepsilon\, \overline{C}_m : 0 = 0\} \\
s =_m t & \rightleftharpoons & s \,\varepsilon\, \overline{C}_m \;\wedge\; t \,\varepsilon\, \overline{C}_m \;\wedge \\
& & \wedge\; \forall \kappa\, \forall x \,\varepsilon\, C_m.\, [\,\exists y \,\varepsilon\, \pi_m(s,\kappa).\, x = y \;\leftrightarrow\; \exists z \,\varepsilon\, \pi_m(t,\kappa).\, x = z\,].
\end{array}$$

8.5. Lemma: If $a \equiv (a_1, \ldots, a_j),\; b \equiv (b_1, \ldots, b_k)$ then

$$a =_m b \;\underline{\leftrightarrow}\; j = k \;\underline{\wedge}\; a_1 =_m b_1 \;\underline{\wedge}\; \ldots \;\underline{\wedge}\; a_j =_m b_j.$$

Proof: We now write A as short for $(a \,\varepsilon\, \overline{C}_m \;\wedge\; b \,\varepsilon\, \overline{C}_m)$ und use i as a metavariable for elements of $\mathbb{N}^+$. Then we have

$$\begin{array}{rcl}
a =_m b & \underline{\leftrightarrow} & A \;\underline{\wedge}\; \underline{\forall} i\, \forall x \,\varepsilon\, C_m.\, [\,\exists y \,\varepsilon\, \pi_m(a,i).\, x = y \;\leftrightarrow\; \exists z \,\varepsilon\, \pi_m(b,i).\, x = z\,] \\
& \leftrightarrow & A \;\underline{\wedge}\; \underline{\forall} i\, \forall x \,\varepsilon\, C_m.\, [\, i \le j \;\wedge\; x = a_i \;\leftrightarrow\; i \le k \;\wedge\; x = b_i\,] \\
& \underline{\leftrightarrow} & j = k \;\underline{\wedge}\; \underline{\forall} i \le j.\, a_i =_m b_i. \quad \Box
\end{array}$$

Definition: For $s \in \overline{\mathcal{T}}$, $\quad s \,\varepsilon\, \overline{C}_m \rightleftharpoons s \,\varepsilon\, \{\overline{x} \,\varepsilon\, \overline{C}_m : 0 = 0\}$.

Remark: For all $c \in \overline{\mathcal{C}}$: $\quad c \,\varepsilon\, \overline{C}_m \;\underline{\leftrightarrow}\; c \in \overline{\mathcal{C}}_m$.

8.6. Lemma: $c = d \ \rightarrow \ (c \,\varepsilon\, \overline{C}_m \leftrightarrow d \,\varepsilon\, \overline{C}_m)$. (Proof: 8.4, 8.5. $\Box$)

Definitions: For $\Phi \in \mathcal{T} \cup \overline{\mathcal{T}} \cup \mathcal{F}$, let $|\Phi|$ be the least $n \in \Omega$ with $\Phi \in \mathcal{T}_n \cup \overline{\mathcal{T}}_n \cup \mathcal{F}_n$. Moreover, for $\underline{w} \in \overline{\mathcal{W}}$ we define: $\mathcal{C}(\underline{w}) \rightleftharpoons \bigcup_{n\in\Omega} \mathcal{C}_n(\underline{w})$.

8.7. Lemma: If $\Phi \in \mathcal{T} \cup \overline{\mathcal{T}} \cup \mathcal{F}$, $w \in \overline{\mathcal{W}}(\Phi)$, and $c \in \mathcal{C}(w)$, then $|\Phi^w_c| = \max\{|\Phi|, |c|\}$.

Proof by induction on the construction of Φ by the $\mathcal{T}, \mathcal{F}$-rules: To this we give only some induction steps as examples. Let $\Phi \in \mathcal{T} \cup \overline{\mathcal{T}} \cup \mathcal{F}$, $w \in \overline{\mathcal{W}}(\Phi)$, and $c \in \mathcal{C}(w)$. We write $m+1$ for m'; $*$ for the substitution ^w_c, and "I.H." for "induction hypothesis".

▷ Let $\Phi \equiv (s \,\varepsilon\, C^p_q)$. Then $|\Phi| = |s|$. By 8.1, $p^* \in \mathcal{T}(\mathrm{IN}^+)$, $q^* \in \mathcal{T}(\Omega)$, and so $\Phi^* \in \mathcal{F}$. If $w \in \overline{\mathcal{W}}(s)$ then (by I.H.): $|s^*| = \max\{|s|, |c|\}$, and so $|\Phi^*| = |(s^* \,\varepsilon\, C^{p^*}_{q^*})| = |s^*| = \max\{|s|, |c|\} = \max\{|\Phi|, |c|\}$. If $w \notin \overline{\mathcal{W}}(s)$ then $w \in \mathcal{V}_0(p) \cup \mathcal{V}_0(q)$, so $c \in \mathcal{C}(w) \subset \mathcal{C}_0$, $|c| = 0$, and so $|\Phi^*| = |(s \,\varepsilon\, C^{p^*}_{q^*})| = |s| = |\Phi| = \max\{|\Phi|, |c|\}$.

▷ Let $\Phi \equiv (\mathrm{J}\overline{x} \,\varepsilon\, \overline{C}_m, \mu, z : F)(q) \in \mathcal{T}$. Then $|\Phi| = \max\{m+1, |F|\}$. If $w \in \{\overline{x}, \mu, z\}$ then $w \in \mathcal{V}_0(q)$, so again $|c| = 0$, moreover $\Phi^* \equiv (\mathrm{J}\overline{x} \,\varepsilon\, \overline{C}_m, \mu, z : F)(q^*)$, and so $|\Phi^*| = \max\{m+1, |F|\} = |\Phi| = \max\{|\Phi|, |c|\}$. In case $w \notin \overline{\mathcal{W}}(F)$ we may conclude in the same way. Now let $w \notin \{\overline{x}, \mu, z\}$ and $w \in \overline{\mathcal{W}}(F)$. Then $\Phi^* \equiv (\mathrm{J}\overline{x} \,\varepsilon\, \overline{C}_m, \mu, z : F^*)(q^*)$, and by I.H.: $|F^*| = \max\{|F|, |c|\}$. So we also obtain
$|\Phi^*| = \max\{m+1, |F^*|\} = \max\{m+1, |F|, |c|\} = \max\{|\Phi|, |c|\}$.

▷ Let $\Phi \equiv (\exists x \,\varepsilon\, \pi_m(s,p).\, F)$ with $s \in \overline{\mathcal{T}}$, $p \in \mathcal{T}(\mathrm{IN}^+)$ und $F \in \mathcal{F}$. Then $|\Phi| = \max\{m+1, |s|, |F|\}$. Because of $w \in \overline{\mathcal{W}}(\Phi)$ we have $w \not\equiv x$. Again we have $p^* \in \mathcal{T}(\mathrm{IN}^+)$ and so $\Phi^* \equiv (\exists x \,\varepsilon\, \pi_m(s^*, p^*).\, F^*) \in \mathcal{F}$. If $w \in \overline{\mathcal{W}}(s) \cup \overline{\mathcal{W}}(F)$ then
$|\Phi^*| = \max\{m+1, |s^*|, |F^*|\} =_{\text{I.H.}} \max\{m+1, |s|, |F|, |c|\} = \max\{|\Phi|, |c|\}$.
If $w \notin \overline{\mathcal{W}}(s) \cup \overline{\mathcal{W}}(F)$ then $w \in \mathcal{V}_0(p)$, so again $|c| = 0$ and thus
$|\Phi^*| = |\exists x \,\varepsilon\, \pi_m(s, p^*).\, F| = \max\{m+1, |s|, |F|\} = |\Phi| = \max\{|\Phi|, |c|\}$. -
The remaining steps of induction can be performed analogously. $\Box$ -
From 8.7 we obtain:

8.8. Corollary: If $s \in \mathcal{T} \cup \overline{\mathcal{T}}$, $\overline{\mathcal{W}}(s) = \{\underline{w}\}$, and $\underline{c}, \underline{d} \in \mathcal{C}(\underline{w})$ then: $\underline{c} \sim \underline{d} \ \rightarrow \ s^{\underline{w}}_{\underline{c}} \sim s^{\underline{w}}_{\underline{d}}$.

Definitions:

$$\begin{aligned} \exists \overline{x} \,\varepsilon\, \overline{C}_m.\, F \ &\rightleftharpoons\ \exists \overline{x} \,\varepsilon\, \{\overline{x} \,\varepsilon\, \overline{C}_m : 0 = 0\}.\, F \\ \forall \overline{x} \,\varepsilon\, \overline{C}_m.\, F \ &\rightleftharpoons\ \neg \exists \overline{x} \,\varepsilon\, \overline{C}_m.\, \neg F. \end{aligned}$$

Let I_n denote the set of all elements a of $\mathcal{C}_n$ for which the formula $(\overline{x} \,\varepsilon\, a)$ is invariant under $(=_n)$. Accordingly, for $a, b \in \mathcal{C}$ we define:

$$\begin{aligned} a \,\varepsilon\, I_n \ &\rightleftharpoons\ a \,\varepsilon\, C_n \ \wedge\ \forall \overline{x} \,\varepsilon\, \overline{C}_n.\, \forall \overline{y} \,\varepsilon\, \overline{C}_n.\, [\, \overline{x} =_n \overline{y} \ \rightarrow\ (\overline{x} \,\varepsilon\, a \ \leftrightarrow\ \overline{y} \,\varepsilon\, a)\,] \\ a \doteqdot_n b \ &\rightleftharpoons\ a =_n b \ \wedge\ a \,\varepsilon\, I_n \ \wedge\ b \,\varepsilon\, I_n. \end{aligned}$$

So we especially have: $a \,\varepsilon\, I_0 \ \leftrightarrow\ a \,\varepsilon\, C_0$, and $a \doteqdot_0 b \ \leftrightarrow\ a =_0 b$. - For $a \equiv (a_1, \ldots, a_j)$ and $b \equiv (b_1, \ldots, b_j)$ define

$$a \doteqdot_n b \ \rightleftharpoons\ a_1 \doteqdot_n b_1 \ \wedge\ \ldots\ \wedge\ a_j \doteqdot_n b_j.$$

8.9. Lemma: If all elements of $\mathcal{F}_m$ are invariant under $(=_m)$, then for all $a, b \in \mathcal{C}_m \cup \overline{\mathcal{C}}_m: \quad a =_n b \;\rightarrow\; a \doteqdot_n b$.

Proof: Let $a \in \mathcal{C}_m$. Then the formula $(\overline{x} \;\varepsilon\; a)$ is a member of $\mathcal{F}_m$ and therefore, by hypothesis, invariant under $(=_m)$. So $a \;\varepsilon\; I_m$. Moreover, we have $\forall \overline{x} \;\varepsilon\; \overline{C}_n\, (\overline{x} \;\varepsilon\; a \;\rightarrow\; \overline{x} \;\varepsilon\; \overline{C}_m)$. By 8.4 (with $\overline{x}, \overline{y}$ in place of c, d) it follows that

$$\begin{aligned} \forall \overline{x} \;\varepsilon\; \overline{C}_n . \forall \overline{y} \;\varepsilon\; \overline{C}_n . \, [\, \overline{x} =_n \overline{y} \;\; \rightarrow \;\; (\overline{x} \;\varepsilon\; a \;\leftrightarrow\; \overline{x} \;\varepsilon\; a \;\wedge\; \overline{x} =_m \overline{y} \;\leftrightarrow \\ \leftrightarrow\; \overline{y} \;\varepsilon\; a \;\wedge\; \overline{x} =_m \overline{y} \;\leftrightarrow\; \overline{y} \;\varepsilon\; a) \,]. \end{aligned}$$

So $a \;\varepsilon\; I_n$. From this we obtain 8.9 for $a, b \in \mathcal{C}_m$ and so, by 8.5, also for $a, b \in \overline{\mathcal{C}}_m$. $\square$

8.10. Theorem: All elements of $\mathcal{T}_n \cup \overline{\mathcal{T}}_n \cup \mathcal{F}_n$ are invariant under $(=_n)$.

Proof by compound induction: We start from the hypothesis

I.H.1: For all $m < n$, all elements of $\mathcal{T}_m \cup \overline{\mathcal{T}}_m \cup \mathcal{F}_m$ are invariant under $(=_m)$.

From this we conclude *at first*: All elements of $\mathcal{T}_n \cup \overline{\mathcal{T}}_n \cup \mathcal{F}_n$ are invariant under $(\doteqdot_n)$. To this end, we consider any element $\Phi \in \mathcal{T}_n \cup \overline{\mathcal{T}}_n \cup \mathcal{F}_n$ and infer from I.H.1 and the following hypothesis I.H.2 that Φ is invariant under $(\doteqdot_n)$:

I.H.2: Invariant under $(\doteqdot_n)$ are all terms and formulas that must have been shown to be elements of $\mathcal{T}_n \cup \overline{\mathcal{T}}_n \cup \mathcal{F}_n$ in order to show (by the $\mathcal{T}, \mathcal{F}$-rules) that Φ is an element of $\mathcal{T}_n \cup \overline{\mathcal{T}}_n \cup \mathcal{F}_n$.

So we suppose that $\Phi \in \mathcal{T}_n \cup \overline{\mathcal{T}}_n \cup \mathcal{F}_n$ and $\overline{\mathcal{W}}(\Phi) = \{\underline{w}\}$ with $\underline{w} \rightleftharpoons w_1, \ldots, w_j$. Moreover, let $\underline{c}, \underline{d} \in \mathcal{C}(\underline{w})$ and $\underline{c} \doteqdot_n \underline{d}$. We consider the substitutions $* \rightleftharpoons \frac{\underline{w}}{\underline{c}}$ und $\dagger \rightleftharpoons \frac{\underline{w}}{\underline{d}}$, and write "invariant" as short for "invariant under $(\doteqdot_n)$".

▷ Let $\Phi \equiv s \in \mathcal{T}_{\mathrm{Or}}$. In s occur only variables w_i of $\mathcal{V}_0$. So we have $c_i, d_i \in \mathcal{C}_0$. Because of $c_i =_n d_i$ it follows (by 8.4) that $c_i =_0 d_i$, so (by the hypothesis on $(=_0)$): $s^* =_0 s^\dagger$, $s^*, s^\dagger \in \mathcal{C}_0$, and so $s^*, s^\dagger \;\varepsilon\; I_n$, too.

▷ Let $\Phi \equiv w_1 \in \mathcal{V} \cup \overline{\mathcal{V}}$. Then $\Phi^* \equiv c_1$ and $\Phi^\dagger \equiv d_1$. Since $c_1 \doteqdot_n d_1$ we have $\Phi^* \doteqdot_n \Phi^\dagger$.

▷ Let $\Phi \equiv R(q) \equiv (\mathrm{J}\overline{x} \;\varepsilon\; \overline{C}_m, \mu, z \colon F)(q) \in \mathcal{T}_n$ with $m < n$ and $F \in \mathcal{F}_n$. F is invariant by I.H.2. Let $F \equiv A(\overline{x}, \mu, z, \underline{w})$. We want to prove $R^*(l) \doteqdot_n R^\dagger(l)$ for all $l \in \Omega$ by induction on Ω. To this end we may use the (third) induction hypothesis that for all $k < l$ we have $R^*(k) \doteqdot_n R^\dagger(k)$. Then for all $\underline{a}, \underline{b}$ with $\underline{a} =_n \underline{b}$ we obtain by 8.6, 8.9 and I.H.1: $(\underline{a}) \;\varepsilon\; \overline{C}_m \;\rightarrow\; (\underline{b}) \;\varepsilon\; \overline{C}_m \;\rightarrow\; \underline{a} \doteqdot_n \underline{b}$. So (by 8.6) for all $k \in \Omega$:

$$\begin{aligned} (\underline{a}, k) \;\varepsilon\; R^*(l) \;\; &\leftrightarrow \;\; (\underline{a}) \;\varepsilon\; \overline{C}_m \;\wedge\; k < l \;\wedge\; A((\underline{a}), k, R^*(k), \underline{c}) \\ &\leftrightarrow \;\; (\underline{b}) \;\varepsilon\; \overline{C}_m \;\wedge\; k < l \;\wedge\; A((\underline{b}), k, R^\dagger(k), \underline{d}) \;\;\leftrightarrow\;\; (\underline{b}, k) \;\varepsilon\; R^\dagger(l). \end{aligned}$$

From this it follows, by 8.8, that $R^*(l), R^\dagger(l) \;\varepsilon\; I_n$ and hence $R^*(l) \doteqdot_n R^\dagger(l)$. This result holds for all $l \in \Omega$. So we obtain especially $R^*(q^*) \doteqdot_n R^\dagger(q^\dagger)$, i.e. $\Phi^* \doteqdot_n \Phi^\dagger$.

▷ The case $\Phi \equiv \{\overline{x} \;\varepsilon\; \overline{C}_m \colon F\}$ can even be treated simpler.

$\triangleright$ Let $\Phi \equiv (s_1, \ldots, s_k) \in \overline{\mathcal{T}}_n$. By I.H.2 we have $s_i^* \doteqdot_n s_i^\dagger$. So $\Phi^* \doteqdot_n \Phi^\dagger$.

$\triangleright$ Let $\Phi \equiv E \in \mathcal{E}$. As in the case "$\Phi \equiv s \in \mathcal{T}_{\mathrm{or}}$" we obtain $c_i =_0 d_i$, and so: $E^* \;\leftrightarrow\; E^\dagger$.

$\triangleright$ Let $\Phi \in \{(F \wedge G), (F \vee G), (\neg F), (\exists x \,\varepsilon\, C_m. F)\}$ with invariant formulas $F, G \in \mathcal{F}_n$, and $m < n$. Then it easily follows that also Φ is invariant. Concerning the formula $(\exists x \,\varepsilon\, C_m. F)$ regard that, by I.H.1 and 8.9, we have: $a \,\varepsilon\, C_m \;\rightarrow\; a \doteqdot_n a$.

$\triangleright$ Let $\Phi \equiv (s \,\varepsilon\, t)$ where $s \in \overline{\mathcal{T}}_n$ and $t \in \mathcal{T}_n$ are invariant. So we have $s^* =_n s^\dagger$, $t^* \,\varepsilon\, I_n$ and $t^* =_n t^\dagger$. It follows that: $s^* \,\varepsilon\, t^* \;\leftrightarrow\; s^\dagger \,\varepsilon\, t^* \;\leftrightarrow\; s^\dagger \,\varepsilon\, t^\dagger$.

$\triangleright$ Let $\Phi \equiv (\exists \overline{x} \,\varepsilon\, t. F)$ where $t \in \mathcal{T}_n$, $F \in \mathcal{F}_n$ are invariant. First let $\overline{x} \in \overline{\mathcal{V}}(t^*)$. For all $a \in \overline{\mathcal{C}}$ with $a \,\varepsilon\, t\binom{\overline{x}}{a}^*$ we have $|a| < |t\binom{\overline{x}}{a}^*| = \max\{|t^*|, |a|\}$ (by 8.7), so $|a| < |t^*| \leq n$. This also holds for $\overline{x} \notin \overline{\mathcal{V}}(t^*)$. So in any case, by I.H.1 and 8.9: $a \,\varepsilon\, t\binom{\overline{x}}{a}^* \;\rightarrow\; a \doteqdot_n a \;\rightarrow\; t\binom{\overline{x}}{a}^* \doteqdot_n t\binom{\overline{x}}{a}^\dagger$. So: $a \,\varepsilon\, t\binom{\overline{x}}{a}^* \;\wedge\; F\binom{\overline{x}}{a}^* \;\rightarrow\; a \,\varepsilon\, t\binom{\overline{x}}{a}^\dagger \;\wedge\; F\binom{\overline{x}}{a}^\dagger$. So we obtain: $\Phi^* \;\rightarrow\; \Phi^\dagger$ and, in the same way: $\Phi^\dagger \;\rightarrow\; \Phi^*$.

$\triangleright$ Let $\Phi \equiv (\exists x \,\varepsilon\, \pi_m(s,p). F)$ where $s \in \overline{\mathcal{T}}_n$, $p \in \mathcal{T}(\mathbb{N}^+)$, and $F \in \mathcal{F}_n$, which are invariant. At first let $x \in \mathcal{W} \setminus \mathcal{W}(s,p)$. Then $s^*, s^\dagger$ are constants, $p^*, p^\dagger \in \mathbb{N}^+$, $s^* \doteqdot_n s^\dagger$ and $p^* \equiv p^\dagger$. Let $s^* \equiv (a_1, \ldots, a_j)$, $s^\dagger \equiv (b_1, \ldots, b_j)$, and $i \rightleftharpoons p^*$. In case $i \leq j$ we have $a_i \doteqdot_n b_i$, and hence $\Phi^* \;\leftrightarrow\; F\binom{x}{a_i}^* \leftrightarrow F\binom{x}{b_i}^\dagger \;\leftrightarrow\; \Phi^\dagger$. In case $i > j$ we have: $\Phi^* \leftrightarrow \bot \leftrightarrow \Phi^\dagger$. Finally, let $x \in \mathcal{W}(s,p)$. Then $x \in \mathcal{W}(s^*, p^*) \cap \mathcal{W}(s^\dagger, p^\dagger)$, so that again: $\Phi^* \leftrightarrow \bot \leftrightarrow \Phi^\dagger$.

$\triangleright$ Let $\Phi \equiv (s \,\varepsilon\, C_q^p)$ with an invariant term $s \in \overline{\mathcal{T}}_n$. Since $s^* \sim s^\dagger$, $p^* \equiv p^\dagger$ and $q^* \equiv q^\dagger$ we have: $s^* \,\varepsilon\, C_{q^*}^{p^*} \;\leftrightarrow\; s^\dagger \,\varepsilon\, C_{q^*}^{p^*} \;\leftrightarrow\; s^\dagger \,\varepsilon\, C_{q^\dagger}^{p^\dagger}$.

$\triangleright$ Let $\Phi \equiv (s =_0 t)$ with invariant $s, t \in \mathcal{T}_n$. Then $\Phi^* \;\rightarrow\; s^* \,\varepsilon\, C_0 \;\wedge\; t^* \,\varepsilon\, C_0$, so $\Phi^* \;\rightarrow\; s^\dagger =_0 s^* =_0 t^* =_0 t^\dagger \;\rightarrow\; \Phi^\dagger$, and likewise: $\Phi^\dagger \;\rightarrow\; \Phi^*$.

On the assumption I.H.1 we have shown that all elements of $\mathcal{T}_n \cup \mathcal{F}_n$ are invariant under $(\doteqdot_n)$. Now we consider any element $c \in \mathcal{C}_n$. All 'elements' a, b of c (if there are any) have a lower order than n, so that (by I.H.1 and 8.9): $a =_n b \;\rightarrow\; a \doteqdot_n b$. Since the formula $(\overline{x} \,\varepsilon\, c)$ belongs to $\mathcal{F}_n$, it is invariant under $(\doteqdot_n)$ and so under $(=_n)$. Therefore, $c \,\varepsilon\, I_n$. For all $c, d \in \mathcal{C}_n$ we obtain: $c =_n d \;\leftrightarrow\; c \doteqdot_n d$. So all elements of $\mathcal{T}_n \cup \mathcal{F}_n$ are invariant under $(=_n)$. $\square$

Definition:

$$\begin{array}{lcl} (s_1, \ldots, s_j) = (t_1, \ldots, t_j) & \rightleftharpoons & s_1 = t_1 \wedge \ldots \wedge s_j = t_j \\ (s_1, \ldots, s_j) = (t_1, \ldots, t_k) & \rightleftharpoons & \bot \qquad \text{if } j \neq k. \end{array}$$

It seems not to be possible to adequately define $(\overline{x} = \overline{y})$, $(\overline{x} = (t_1, \ldots, t_j))$, and $((s_1, \ldots, s_j) = \overline{y})$ as formulas of $\mathcal{F}$. Nevertheless, we obtain:

8.11. Corollary: All elements of $\mathcal{T} \cup \overline{\mathcal{T}} \cup \mathcal{F}$ are invariant under $(=)$.

Proof, for formulas, e.g.: Let $F \in \mathcal{F}$, $\{\underline{w}\} = \overline{\mathcal{W}}(F)$, $(\underline{c}), (\underline{d}) \in \mathcal{C}(\underline{w})$ and $\underline{c} = \underline{d}$. Then there exist $k, m \in \Omega$ such that $F \in \mathcal{F}_k$ and $\underline{c} =_m \underline{d}$. Let $n \rightleftharpoons \max\{k, m\}$. Then $F \in \mathcal{F}_n$, $\underline{c} =_n \underline{d}$, and so $(F\frac{\underline{w}}{\underline{c}} \;\leftrightarrow\; F\frac{\underline{w}}{\underline{d}})$ (by 8.10). $\square$

By 8.3 and 8.11 we have solved our main tasks mentioned in §7.

Definitions of j-ary relations: For $\underline{x} \equiv x_1, \dots, x_j$ and $\overline{y} \notin \mathcal{V}(F)$,

$$\begin{array}{rcl} \exists \underline{x} \,\varepsilon\, C_m.\, F & \rightleftharpoons & \exists x_1 \,\varepsilon\, C_m \dots \exists x_j \,\varepsilon\, C_m.\, F \\ \{\underline{x} \,\varepsilon\, C_m \colon F\} & \rightleftharpoons & \{\overline{y} \,\varepsilon\, \overline{C}_m \colon \exists \underline{x} \,\varepsilon\, C_m.\, (\overline{y} =_m (\underline{x}) \;\wedge\; F)\} \\ (\mathrm{J}\underline{x} \,\varepsilon\, C_m, \mu, z \colon F) & \rightleftharpoons & (\mathrm{J}\overline{y} \,\varepsilon\, \overline{C}_m, \mu, z \colon \exists \underline{x} \,\varepsilon\, C_m.\, (\overline{y} =_m (\underline{x}) \;\wedge\; F)\}. \end{array}$$

So we have

$$(\underline{c}) \,\varepsilon\, \{\underline{x} \,\varepsilon\, C_m \colon A(\underline{x})\} \;\leftrightarrow\; (\underline{c}) \,\varepsilon\, \overline{C}_m \;\wedge\; A(\underline{c}),$$

Similarly, for $R \rightleftharpoons (\mathrm{J}\underline{x} \,\varepsilon\, C_m, \mu, z : A(\underline{x}, \mu, z))$ we have

$$(\underline{c}, k) \,\varepsilon\, R(l) \;\leftrightarrow\; (\underline{c}) \,\varepsilon\, \overline{C}_m \;\wedge\; k < l \;\wedge\; A(\underline{c}, k, R(k)).$$

Now we can prove the following Corollary by which we can 'define' sequences of relations $S(k),\ k \in \mathbb{N}$, by ordinary *recursion* on $\mathbb{N}$ (cf. §4: Addition in $\mathbb{N}$).

8.12. Corollary: For any two formulas $A(\underline{x}), B(\underline{x}, \mu, z) \in \mathcal{F}_n$ with $\underline{x} \in \mathcal{W},\ z \in \mathcal{V}$, and $m < n$ there exists a term $S(\nu) \in \mathcal{T}_n$ such that for all $(\underline{c}) \in \overline{\mathcal{C}}$ and all $k \in \mathbb{N}$,

$$\begin{array}{rcl} (\underline{c}) \,\varepsilon\, S(0) & \leftrightarrow & (\underline{c}) \,\varepsilon\, \overline{C}_m \;\wedge\; A(\underline{c}) \\ (\underline{c}) \,\varepsilon\, S(k') & \leftrightarrow & (\underline{c}) \,\varepsilon\, \overline{C}_m \;\wedge\; B(\underline{c}, k, S(k)). \end{array}$$

Proof: Let

$$\begin{array}{rcl} D(\underline{x}, \mu, z) & \rightleftharpoons & \big[\, \mu = 0 \;\wedge\; A(\underline{x}) \,\big] \vee \\ & & \vee \exists \lambda \big[\, \mu = \lambda' \;\wedge\; B(\underline{x}, \lambda, \{\underline{x} \,\varepsilon\, C_m \colon (\underline{x}, \lambda) \,\varepsilon\, z\}) \,\big] \\ R(k) & \rightleftharpoons & (\mathrm{J}\underline{x} \,\varepsilon\, C_m, \mu, z \colon D(\underline{x}, \mu, z))(k) \\ S(k) & \rightleftharpoons & \{\underline{x} \,\varepsilon\, C_m \colon (\underline{x}, k) \,\varepsilon\, R(k')\}. \end{array}$$

Then we have

$$\begin{array}{rcccl} (\underline{c}) \,\varepsilon\, S(0) & \leftrightarrow & (\underline{c}, 0) \,\varepsilon\, R(0') & \leftrightarrow & (\underline{c}) \,\varepsilon\, \overline{C}_m \;\wedge\; A(\underline{c}) \\ (\underline{c}) \,\varepsilon\, S(k') & \leftrightarrow & (\underline{c}, k') \,\varepsilon\, R(k'') & \leftrightarrow & (\underline{c}) \,\varepsilon\, \overline{C}_m \;\wedge\; B(\underline{c}, k, S(k)). \quad \square \end{array}$$

§9. ι-terms (definite description terms)

In this section we introduce 'ι-terms', i.e. terms of the form $(\iota\, y \,\varepsilon\, C_k.\, \Phi y)$ ["the (unique) element y of C_k that satisfies Φy"] (where Φy is a 'proper' formula). Then, for instance, the notation of function application can be defined by $f(x) \rightleftharpoons (\iota y \,\varepsilon\, C_k.\, (x, y) \,\varepsilon\, f)$.

Definitions: We shall use these abbreviations (for formulas Φy and variables u not occurring in them and satisfying $\mathcal{C}(u) = \mathcal{C}(y)$):

$$\begin{array}{rcl} \Phi(= u) & \rightleftharpoons & \forall y \,\varepsilon\, C_k\, (\Phi y \leftrightarrow y = u) \\ \varphi & \rightleftharpoons & \iota y \,\varepsilon\, C_k.\, \Phi y \\ \psi & \rightleftharpoons & \iota z \,\varepsilon\, C_\ell.\, \Psi z. \end{array}$$

φ is said to be a **proper** ι-term iff every instance of $\exists u \,\varepsilon\, C_k.\, \Phi(= u)$ holds (which means that there is a unique $y \,\varepsilon\, C_k$ satisfying Φy).
φ is said to be **prime** iff no other ι-term occurs in φ.
All occurrences of the variable y in φ are said to be bound.
A ι-term without free occurring variables is said to be a ι-*constant* (or a definite description).

In the following we adapt well-known introductions of ι-terms (as in [7, §9] or [9, pp. 170f.]) to higher order languages. To this end, terms and formulas are permitted to contain proper ι-terms. We spezify this thus: In the $\mathcal{T}, \mathcal{F}$-rules adduced in §8 (by which terms and formulas are to be constructed) we now write $\mathcal{T}_n^\iota, \mathcal{F}_n^\iota$, etc. instead of $\mathcal{T}_n, \mathcal{F}_n$, etc., and we complete those rules by this rule:

$$\Phi u \in \mathcal{F}_n^\iota \;\;\Rightarrow\;\; \varphi \in \mathcal{T}_n^\iota, \quad \text{if } k < n \text{ and } \varphi \text{ is proper.}$$

The words "term" / "formula" are to be understood as elements of $\mathcal{T}^\iota$ / $\mathcal{F}^\iota$, respectively. However, all constants c referred to in the following rules are supposed to be ι-free, i.e. not to contain ι-terms:

$$\begin{aligned} \natural\, \exists x \,\varepsilon\, C_m.\, A(x) \quad &:\Rightarrow \quad \text{for some } c : \natural\, c \in \mathcal{C}_m,\ \natural\, A(c) \\ \natural\, \exists \overline{x} \,\varepsilon\, b(\overline{x}).\, A(\overline{x}) \quad &:\Rightarrow \quad \text{for some } c : \natural\, c \,\varepsilon\, b(c),\ \natural\, A(c). \end{aligned}$$

We say that φ **occurs free** in a formula F iff φ occurs in F and every free occurrence of a variable in φ is also free in F.

For formulas $F \in \mathcal{F}^\iota$ let the **reduction** $*F$ of F be recursively defined thus:

$$*F \;\;\rightleftharpoons\;\; F, \quad \text{if no } \iota\text{-term occurs free in } F.$$

$$*F\varphi \;\;\rightleftharpoons\;\; \exists y \,\varepsilon\, C_k.\, (\Phi y \wedge *Fy), \quad \text{if (1) is satisfied:}$$

(1) $F\varphi \rightleftharpoons (Fy)^y_\varphi$, φ is a prime and proper ι-term that is free for y in Fy and does not occur in Fy, and some occurrence of φ in $F\varphi$ begins on the left of all other free occurrences of a prime and proper ι-terms in $F\varphi$.

Presupposition: In the following, φ and ψ are **proper** ι**-constants** as above.
A, $A\varphi$, $B\psi$ etc. are sentences of $\mathcal{A}^\iota$ and contain proper ι-terms only.

Now we accept all primary rules fixed in §8 for sentences of $\mathcal{A}$ in which no ι-constant occurs, and we fix this rule, in addition:

R(ι) : $\quad A \;:\Rightarrow\; *A, \quad$ if some ι-constant occurs.

Let assertions of sentences containing ι-constants not be restricted by other rules.
So the inverse of R(ι) is classically admissible.

Notes: 1. The use of $A\varphi \wedge B\varphi$, e.g., is ruled by R(ι) and its inverse. However, due to 9.2 (below) we may also apply the rules $A\varphi \wedge B\varphi \Leftrightarrow A\varphi, B\varphi$. So $A\varphi \wedge B\varphi$ may be used like a common conjunction.

2. Assume that $Ey \in \mathcal{E}$, φ is prime, and y is the only variable occurring free in $\Phi y \wedge Ey$. Then we have $y \in \mathcal{V}_0$, hence $\mathcal{C}(y) \subseteq \mathcal{C}_0$, and hence

$$\exists y \,\varepsilon\, C_k.\,(\Phi y \wedge Ey) \;\leftrightarrow\; \exists y \,\varepsilon\, C_0.\,(\Phi y \wedge Ey).$$

In the following we write $A \Rightarrow B$ / $A \Leftrightarrow B$ to express that these rules are classically admissible.

9.1 Lemma: If $A\psi \rightleftharpoons (Az)^z_\psi$ and ψ is prime, then

$$A\psi \;\Leftrightarrow\; \exists z \,\varepsilon\, C_\ell.\,(\Psi z \wedge *Az).$$

So $\exists z \,\varepsilon\, C_\ell.\,\Psi z$ implies $\psi \,\varepsilon\, C_\ell$ and $\Psi\psi$.

Proof: If φ satisfies (1), we obtain

$$A\varphi \;\Leftrightarrow\; *A\varphi \;\Leftrightarrow\; \exists y \in C_k.\,(\Phi y \wedge *Ay)$$

by R(ι) and its inverse. Now let φ and ψ be prime and occur in $A(\varphi,\psi)$ where φ satisfies (1). We prove

$$A(\varphi,\psi) \Leftrightarrow \exists z \,\varepsilon\, C_\ell.\,(\Psi z \wedge *A(\varphi,z))$$

by induction on the number of ι-constants occurring in $A(.,\psi)$ (with 'I.H.' for 'induction hypothesis').

$$\begin{array}{rll} A(\varphi,\psi) & \Leftrightarrow & \exists y \,\varepsilon\, C_k.\,(\Phi y \wedge *A(y,\psi)) \quad \text{(as above)} \\ & \Leftrightarrow & \exists y \,\varepsilon\, C_k.\,(\Phi y \wedge \exists z \,\varepsilon\, C_\ell.\,(\Psi z \wedge *A(y,z))) \;\; \text{(by I.H.)} \\ & \Leftrightarrow & \exists z \,\varepsilon\, C_\ell.\,(\Psi z \wedge \exists y \,\varepsilon\, C_k.\,(\Phi y \wedge *A(y,z))) \\ & \Leftrightarrow & \exists z \,\varepsilon\, C_\ell.\,(\Psi z \wedge *A(\varphi,z)). \quad \square \end{array}$$

9.2 Proposition (by which we may argue in $\mathcal{A}^\iota$ in the same way as in $\mathcal{A}$ and may apply classical logic; cf. Note 1 above): For sentences of $\mathcal{A}^\iota$ we have:

$$A \circ B \;\Leftrightarrow\; *A \circ *B \quad \text{if } \circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$$

$$\neg A \;\Leftrightarrow\; \neg *A$$

$$\exists x \,\varepsilon\, C_m.\,A(x) \;\Leftrightarrow\; \exists x \,\varepsilon\, C_m.\, *A(x))$$

$$\exists z \,\varepsilon\, C_\ell.\,(\Psi z \wedge Az) \;\Leftrightarrow\; \exists z \,\varepsilon\, C_\ell.\,(*\Psi z \wedge *Az)$$

For $b(\overline{x}) \equiv \{\overline{y} \,\varepsilon\, \overline{C}_m.\, B(\overline{x},\overline{y})\}$ or $b(\overline{x}) \equiv (J\overline{y} \,\varepsilon\, \overline{C}_m, \mu, z : \ldots)(l)$:

$$\exists \overline{x} \,\varepsilon\, b(\overline{x}).\,A(\overline{x}) \;\Leftrightarrow\; \exists \overline{x} \,\varepsilon\, \overline{C}_m.*(\overline{x} \,\varepsilon\, b(\overline{x}) \wedge A(\overline{x}))$$

$$c \,\varepsilon\, \{\overline{x} \,\varepsilon\, \overline{C}_m :\, A(\overline{x})\} \;\Leftrightarrow\; c \,\varepsilon\, \overline{C}_m \wedge A(c)$$

$$\exists x \,\varepsilon\, \pi_m((c_1,\ldots,c_j),i).A(x) \;\Leftrightarrow\; c_i \,\varepsilon\, C_m \wedge A(c_i) \quad (\text{for } i = 1,\ldots,j).$$

For $R \rightleftharpoons (J\overline{x}\, \varepsilon\, \overline{C}_m, \mu, z : A(\overline{x}, \mu, z))$, $i, j \in \Omega^\iota$:

$$(\underline{c}, i)\ \varepsilon\ R(j) \ \Leftrightarrow\ (\underline{c})\ \varepsilon\ \overline{C}_m \wedge i < j \wedge A((\underline{c}), i, R(i)).$$

Proof: Let φ be prime, and let u be a variable that is free for y in $\Phi y \wedge Dy$ and does not occur in $\Phi y \wedge Dy$.

ASSUMPTION: $u \in C_k \wedge \Phi u$.

This assumption and $D\varphi \Leftrightarrow \exists y\ \varepsilon\ C_k. (\Phi y \wedge *Dy)$ (cf. 9.1) imply

$$\begin{aligned} &D\varphi \Rightarrow \exists y \in C_k. (u = y \wedge *Dy) \Rightarrow *Du \Rightarrow \exists y\ \varepsilon\ C_k. (\Phi y \wedge *Dy) \Rightarrow D\varphi, \\ &\text{and hence} \qquad *\, D\varphi \ \Leftrightarrow\ D\varphi \ \Leftrightarrow\ *Du \ \Leftrightarrow\ Du. \end{aligned}$$

In the following we apply induction on the number of ι-constants occurring in the formulas considered. For instance, we use the induction hypothesis (I.H.) $Au \circ Bu \ \Leftrightarrow\ *Au \circ *Bu$ to show that $A\varphi \circ B\varphi \ \Leftrightarrow\ *A\varphi \circ *B\varphi$. Here we assume that φ occurs in $A\varphi \circ B\varphi$.

$$A\varphi \circ B\varphi \ \Leftrightarrow\ Au \circ Bu \Leftrightarrow_{I.H.} *Au \circ *Bu \ \Leftrightarrow\ *A\varphi \circ *B\varphi$$

$$\neg\, A\varphi \ \Leftrightarrow\ \neg\, Au \Leftrightarrow_{I.H.} \neg *Au \ \Leftrightarrow\ \neg *A\varphi$$

$$\begin{aligned} \exists x\ \varepsilon\ C_m. A(x, \varphi) \ &\Leftrightarrow\ \exists x\ \varepsilon\ C_m. A(x, u) \Leftrightarrow_{I.H.} \exists x\ \varepsilon\ C_m. *A(x, u) \\ &\Leftrightarrow\ \exists x\ \varepsilon\ C_m. *A(x, \varphi) \end{aligned}$$

$$\begin{aligned} c(\varphi)\ \varepsilon\ \{\overline{x}\ \varepsilon\ \overline{C}_m. A(\varphi, \overline{x})\} \ &\Leftrightarrow\ c(u)\ \varepsilon\ \{\overline{x}\ \varepsilon\ \overline{C}_m. A(u, \overline{x})\} \\ &\Leftrightarrow\ c(u)\ \varepsilon\ \overline{C}_m \wedge A(u, c(u)) \quad (I.H.) \\ &\Leftrightarrow\ c(\varphi)\ \varepsilon\ \overline{C}_m \wedge A(\varphi, c(\varphi)). \end{aligned}$$

The remainder can be proved similarly. ☐

9.3 Corollary: If $A \Rightarrow B$, then $A \to B$.

Proof: Assume that $A \Rightarrow B$. Then, by R(ι) and its inverse, we obtain $*A \Rightarrow *B$, hence $*A \to *B$, and hence (by 9.2) $A \to B$. ☐

Now we omit the assumption of 9.1 that ψ is prime:

9.4 Proposition: We have

$$\begin{aligned} A\psi \ &\leftrightarrow\ \exists z\ \varepsilon\ C_\ell. (\Psi z \wedge Az) \\ &\leftrightarrow\ \forall z\ \varepsilon\ C_\ell. (\Psi z \to Az), && \text{hence} \\ \forall z\ \varepsilon\ C_\ell. Az \ &\to\ A\varphi \ \to\ \exists z\ \varepsilon\ C_\ell. Az, && \text{moreover} \\ A(\psi, \psi)) \ &\leftrightarrow\ \exists z\ \varepsilon\ C_\ell. (\Psi z \wedge A(z, \psi)). \end{aligned}$$

So proper ι-terms may be used like ordinary terms in the scope of nice formulas.

Proof: If ψ is prime, we have $*A\psi \Leftrightarrow \exists z \,\varepsilon\, C_\ell.\, (*\Psi z \wedge *Az)$ by 9.1. Now let φ be prime, and let $\psi(\varphi)$ be a proper ι-constant containing φ. Since in $\psi(y)$ occur fewer ι-constants than in $\psi(\varphi)$, we may assume that

$$\forall y \in C_k.\, (*A(\psi(y)) \Leftrightarrow \exists z \in C_\ell.\, (*\Psi(y,z) \wedge *A(z))).$$

This implies:

$$\begin{array}{rcl} *A\psi(\varphi) & \Leftrightarrow & \exists y \in C_k.\, (\Phi y \wedge *A\psi(y)) \\ & \Leftrightarrow & \exists y \in C_k.\, (\Phi y \wedge \exists z \in C_\ell.\, (*\Psi(y,z) \wedge *Az)) \\ & \Leftrightarrow & \exists z \in C_\ell.\, \exists y \in C_k.\, (\Phi y \wedge *\Psi(y,z) \wedge *Az) \\ & \Leftrightarrow & \exists z \in C_\ell.\, (*\Psi(\varphi,z) \wedge *Az). \end{array}$$

So, for all proper ψ,

$$\begin{array}{rcl} *A\psi & \Leftrightarrow & \exists z \,\varepsilon\, C_\ell.\, (*\Psi z \wedge *Az) \\ & \Rightarrow & \exists z \,\varepsilon\, C_\ell.\, \forall y \,\varepsilon\, C_\ell.\, (*\Psi y \to y = z \to *Ay) \\ & \Rightarrow & \forall y \,\varepsilon\, C_\ell.\, (*\Psi y \to *Ay) \\ & \Rightarrow & \exists z \,\varepsilon\, C_\ell.\, (*\Psi z \wedge *Az) \quad \text{(since } \exists z \,\varepsilon\, C_\ell.\, *\Psi z\text{)}. \end{array}$$

$$\begin{array}{rcl} *A(\psi,\psi) & \Leftrightarrow & \exists z, y \,\varepsilon\, C_\ell.\, (*\Psi z \wedge y = z \wedge *A(z,y)) \\ & \Leftrightarrow & \exists z, y \,\varepsilon\, C_\ell.\, (*\Psi z \wedge *\Psi y \wedge *A(z,y)) \\ & \Leftrightarrow & \exists z \,\varepsilon\, C_\ell.\, (*\Psi z \wedge \exists y \,\varepsilon\, C_\ell.\, (*\Psi y \wedge *A(z,y)) \\ & \Leftrightarrow & \exists z \,\varepsilon\, C_\ell.\, (*\Psi z \wedge *A(z,\psi)). \end{array}$$

By R(ι), its inverse, and 9.2 we may omit '$*$' in these results. By 9.3 we may replace '$\Leftrightarrow$' by '$\leftrightarrow$'. □

Note: Proper ι-terms with free variables can be eliminated as in the following examples (in which $\varphi(\overline{x})$ / $\varphi(x)$, resp., is not free for y on the left):

$$\begin{array}{rcl} c \,\varepsilon\, \{\overline{x} \,\varepsilon\, \overline{C}_m : A(\overline{x}, \varphi(\overline{x}))\} & \leftrightarrow & c \,\varepsilon\, \overline{C}_m \wedge A(c, \varphi(c)) \\ & \leftrightarrow & c \,\varepsilon\, \overline{C}_m \wedge \exists y \,\varepsilon\, C_k.\, (\Phi(c,y) \wedge *A(c,y)). \end{array}$$

$$\begin{array}{rcl} \exists x \,\varepsilon\, C_m.\, A(x, \varphi(x)) & \leftrightarrow & \exists x \,\varepsilon\, C_m.\, *A(x, \varphi(x)) \\ & \leftrightarrow & \exists x \,\varepsilon\, C_m.\, \exists y \in C_k.\, (\Phi(x,y) \wedge *A(x,y)). \end{array}$$

§10. Substitutional combined with objectual quantification in higher order languages

Here we want to extend our investigations of §5 to higher order languages. To this end we include certain **indicators** ("this flower", e.g.) in $\mathcal{T}_{\text{or}}$ and admit that they occur in other terms of $\mathcal{T}_{\text{or}}$ and in formulas of $\mathcal{E}$ (cf. p.28), and thus also in higher order terms and formulas. However, we do not include indicators in $\mathcal{V}_0 \cup \mathcal{V} \cup \overline{\mathcal{V}}$ (pp. 28, 29). **Denotations** of indicators (i.e. of objects by indicators) are to be understood as in §5. A denotation

$\alpha_1, \ldots, \alpha_k$ $(k \geq 0)$ is empty or composed of *simple* denotations α_i (i.e. denotations of *single* indicators) which are valid in particular situations only. To imitate objectual quantification by substitutional quantification, we shall also use certain '**d-names**' of simple denotations, which can also be considered as proper names of the objects denoted by those denotations. Such a d-name can be given, e.g., by an indicator, u, the name of an actor, and the date at which he denotes an object by u. However, we shall sometimes not explicitely distinguish simple denotations from their d-names.

Compound denotations generally result from acts of naming objects at different places at different times. However, their d-names can be used anywhere at any time and thus independent of situations.

Definition: If Φ is a term or a formula or a list of d-names, let $I(\Phi)$ be the set of all distinct indicators occurring in Φ. For single d-names α let $I\alpha$ be the (unique) indicator occurring in α.

As metavariables we use here: x, y for variables of $\mathcal{V}_0 \cup \mathcal{V}$, $\overline{x}$ for elements of $\overline{\mathcal{V}}$, and $\alpha, \beta, \gamma, \gamma'$ for lists $\alpha_1, \ldots, \alpha_k$ of d-names with $I\alpha_i \not\equiv I\alpha_j$ for all i, j with $1 \leq i < j \leq k$ (or corresponding denotations).

If Φ is a formula or a term and $\alpha = \alpha_1, \ldots, \alpha_k$, then let Φ^α result from Φ by substituting the *d-name* of α_i for the indicator occurring in it $(i = 1, \ldots, k)$. However, occurrences of indicators in d-names are to be treated as 'bound', i.e. not 'free', i.e. they are not to be replaced by the mentioned substitutions.

For $\mathcal{U} \in \bigcup_{x \in \mathcal{V}_0} \{\mathcal{T}_{\mathrm{or}}, \mathcal{T}(x), \mathcal{E}\}$, we demand that $t^\alpha \in \mathcal{U}$ iff $t \in \mathcal{U}$. So by induction on the $\mathcal{T}, \mathcal{F}$-rules of §8 we obtain: $t^\alpha \in \mathcal{T}_n$ iff $t \in \mathcal{T}_n$, and $F^\alpha \in \mathcal{F}_n$ iff $F \in \mathcal{F}_n$.

If A is a formula in which no variable occurs free and no d-name occurs, we compactly write $\natural A|\,\alpha$ for the assertion of A 'under/in α', i.e. in a situation in which α is valid. Then we assume that $I(A) \subseteq I(\alpha)$. Now we again accept the rules of assertion quoted in §8 and fix, in addition, this rule for assertions under denotations:

$$\natural A|\,\alpha \;:\Rightarrow\; \natural A^\alpha.$$

By this rule we want to imitate objectual by substitutional quantification and so to simplify the theory.

Sentences of the form E^α with $E \in \mathcal{E}$ remind on *protocol sentences.* However, in many cases it suffices for practice to ignore the latter rule and to act upon suitable given rules or usage for elementary assertions $\natural E|\,\alpha$ and to apply the rules quoted in 10.1 (below).

Since we do not admit other rules than $\natural A|\,\alpha \;:\Rightarrow\; \natural A^\alpha$ with $\natural A|\,\alpha$ as premise, this rule may be inverted.

Definitions: Let $\mathcal{I}$ be the set of all indicators. ($\mathcal{I}$ is assumed to be infinite.)
Let $\mathcal{C}_m$ be the set of all elements of $\mathcal{T}_m$ in which no elements of $\mathcal{V}_0 \cup \mathcal{V} \cup \overline{\mathcal{V}} \cup \mathcal{I}$ occur free.
Let $\mathcal{C}_m(x)$ be the set of are values of x belonging to $\mathcal{C}_m$ (cf. pp. 31, 33).
Let $\overline{\mathcal{C}}_m$ and $\overline{\mathcal{C}}$ be defined analogously.

Let $\alpha\,\beta$ result from the list α, β by canceling all those members β_j of β with $I\beta_j \in I(\alpha)$. We write '$(b,\beta)\in\mathcal{R}_m(x,\alpha)$' for '$I(b) \subseteq I(\alpha\,\beta) \;\underline{\wedge}\; b^{\alpha\,\beta} \in \mathcal{C}_m(x)$',
and '$(b,\beta)\in\overline{\mathcal{R}}(\alpha)$' for '$I(b) \subseteq I(\alpha\,\beta) \;\underline{\wedge}\; b^{\alpha\,\beta} \in \overline{\mathcal{C}}$'.

We complete the rules fixed in §8 (p.34) by this rule for sentences in which d-names occur but indicators do not occur free:

$$\natural\, \exists x \;\varepsilon\; C_m .\, A(x) \quad :\Rightarrow \quad \text{for some } (b,\beta)\in\mathcal{R}_m(x,\alpha).\ \natural\, A(b^{\,\alpha\,\beta}),$$

if α ist the list of all d-names occurring in $A(x)$.
(For empty α, this rules coincides with a rule fixed in §8.)

10.1 Proposition: In the primary game we may apply the following rules to formulas in which no elements of $\mathcal{V}_0 \cup \mathcal{V} \cup \overline{\mathcal{V}}$ occur free and no d-names occur:

$$\begin{array}{rcl}
\natural\,(A \wedge B)|\,\alpha & \Leftrightarrow & \natural\, A|\,\alpha \text{ and } \natural\, B|\,\alpha \\
\natural\,(A \vee B)|\,\alpha & \Leftrightarrow & \natural\, A|\,\alpha \text{ or } \natural\, B|\,\alpha \\
\natural\,\neg A|\,\alpha & \Leftrightarrow & A \text{ rejected in } \alpha \\
\natural\,\exists x \;\varepsilon\; C_m .\, A(x)|\,\alpha & \Leftrightarrow & \text{for some } (b,\beta)\in\mathcal{R}_m(x,\alpha).\ \natural\, A(b)|\,\alpha\,\beta \\
\natural\, c \;\varepsilon\; \{\overline{x} \;\varepsilon\; \overline{C}_m \colon A(\overline{x})\}|\,\alpha & \Leftrightarrow & \natural\, c^{\alpha} \in \overline{\mathcal{C}}_m,\ \natural\, A(c)|\,\alpha,
\end{array}$$

for $R \rightleftharpoons (\mathrm{J}\overline{x} \;\varepsilon\; \overline{C}_m, \mu, z \colon A(\overline{x}, \mu, z))$ and $k, l \in \Omega$:

$$\begin{array}{rcl}
\natural\,(\underline{c}, k) \;\varepsilon\; R(l)|\,\alpha & \Leftrightarrow & \natural\,(\underline{c})^{\alpha} \in \overline{\mathcal{C}}_m,\ \natural\, k < l,\ \natural\, A((\underline{c}), k, R(k))|\,\alpha \\
\natural\,\exists \overline{x} \;\varepsilon\; c(\overline{x}).\, A(\overline{x})|\,\alpha & \Leftrightarrow & \text{for some } (b,\beta)\in\overline{\mathcal{R}}(\alpha).\ \natural\,(b \;\varepsilon\; c(b) \wedge A(b))|\,\alpha\,\beta \\
\natural\,\exists x \;\varepsilon\; \pi_m((c_1,\dots,c_j),i).\, A(x)|\,\alpha & \Leftrightarrow & \natural\, c_i^{\alpha} \in \mathcal{C}_m(x),\ \natural\, A(c_i)|\,\alpha \qquad (\text{if } 1 \le i \le j) \\
\natural\, c \;\varepsilon\; C_m^{j}|\,\alpha & \Leftrightarrow & \natural\, c^{\alpha} \in \mathcal{C}_m^{j} \\
\natural\, A(\varphi)|\,\alpha & \Leftrightarrow & \exists y \;\varepsilon\; C_k .\,(\Phi(y) \wedge A(y))|\,\alpha,
\end{array}$$

if $\varphi \rightleftharpoons \iota y \;\varepsilon\; C_k.\Phi(y)$ is a proper ι-constant, in which indicators may occur.

Note: The above rule for $\exists x \;\varepsilon\; C_m .\, A(x)|\,\alpha$ combines both substitutional and (in general) objectual quantification.

Proof: We may apply the following rules (with '$\underline{\exists}$' for 'for some'):

$$\natural\,(A \wedge B)|\,\alpha \;\Leftrightarrow\; \natural\,(A \wedge B)^{\,\alpha} \;\Leftrightarrow_{\S 8}\; (\natural\, A^{\,\alpha} \text{ and } \natural\, B^{\alpha}) \;\Leftrightarrow\; (\natural\, A|\,\alpha \text{ and } \natural\, B|\,\alpha)$$

$$\natural\,\neg\, A|\,\alpha \;\Leftrightarrow\; \natural\,\neg\, A\,\alpha \;\Leftrightarrow_{\S 8}\; A^{\,\alpha} \text{ rejected } \;\Leftrightarrow\; A \text{ rejected in } \alpha.$$

$$\begin{array}{rcl}
\natural\,\exists x \;\varepsilon\; C_m .\, A(x)|\,\alpha & \Leftrightarrow & \natural\,\exists x \;\varepsilon\; C_m .\, A(x)^{\,\alpha} \\
& \Leftrightarrow & \underline{\exists}\,(b,\beta)\in\mathcal{R}_m(x,\alpha).\ \natural\, A(b^{\alpha\,\beta})^{\,\alpha} \\
& \Leftrightarrow & \underline{\exists}\,(b,\beta)\in\mathcal{R}_m(x,\alpha).\ \natural\, A(b)^{\,\alpha\,\beta} \\
& \Leftrightarrow & \underline{\exists}\,(b,\beta)\in\mathcal{R}_m(x,\alpha).\ \natural\, A(b)|\,\alpha\,\beta.
\end{array}$$

The remainder can be proved similarly. □

10.2 Theorem: We may commute consecutive existential quantifiers (cf. the remark 5.1).

Proof: Let $\gamma_{-\beta}$ be the list of those members of γ that contain only indicators belonging to $I(\gamma)\setminus I(\beta)$, and let $\gamma' \rightleftharpoons \gamma_{-\beta}\,\beta$. Then we have $I(\beta)\cap I(\gamma_{-\beta}) = \emptyset$ and hence $I(\alpha\,\beta\,\gamma) \subseteq I(\alpha\,\beta\,\gamma_{-\beta}) = I(\alpha\,\gamma')$. Therefore and due to the latter proof we may apply these rules:

$$\begin{array}{rl} & \exists x\ \varepsilon\ C_m.\,\exists y\ \varepsilon\ C_n.\,A(x,y)|\,\alpha \\ \Leftrightarrow & \underline{\exists}\,(b,\beta)\in\mathcal{R}_m(x,\alpha).\ \exists y\ \varepsilon\ C_n.\,A(b,y)^{\,\alpha\,\beta} \\ \Leftrightarrow & \underline{\exists}\,(b,\beta)\in\mathcal{R}_m(x,\alpha).\ \underline{\exists}\,(c,\gamma)\in\mathcal{R}_n(y,\alpha\,\beta).\ A(b,c)^{\,\alpha\,\beta\,\gamma_{-\beta}} \\ \Rightarrow & \underline{\exists}\,(c,\gamma')\in\mathcal{R}_n(y,\alpha).\ \underline{\exists}\,(b,\beta)\in\mathcal{R}_m(x,\alpha\,\gamma').\ A(b,c)^{\,\alpha\,\gamma'\,\beta} \\ \ldots\ \Rightarrow & \exists y\ \varepsilon\ C_n.\,\exists x\ \varepsilon\ C_m.\,A(x,y)|\,\alpha.\quad\Box \end{array}$$

§11 On the Use of Hypotheses in Cumulative Type Theory

1. Deducibility of sentences from hypotheses considered modal-logically

This section contains only the main considerations of my TU-Darmstadt Preprint 2428. A motivation for them is given above in the section "Hypothetical assertions" of §6.

In the previous sections we have constructed a comprehensive language of a ramified cumulative type theory. We shall construct and investigate an extension, $\mathcal{L}$, of it, which is a language of the same sort, but also contains sentences which express that certain sentences of $\mathcal{L}$ are deducible from others by given rules. To this we shall introduce 'names' of sentences of $\mathcal{L}$ and include them in $\mathcal{L}$. So in $\mathcal{L}$ we can not only use sentences of $\mathcal{L}$ but also 'speak about' them. (We do so though A. Tarski and others have shown that ramified type theory can, in a certain sense, be used as a metalanguage of itself.)

We especially deal with the deducibility of sentences from 'hypotheses' by given axioms and rules, and we shall show how we can formulate that deducibility in $\mathcal{L}$. Assume that $A \equiv A_1 \wedge \ldots \wedge A_j$ is the conjunction of all 'current hypotheses'. We shall introduce sentences of the form $A \rhd B$ which are to mean that B is *deducible* from A and certain additional axioms by certain rules. The system of those axioms and rules will be denoted by $\mathcal{S}$.

Let be given a language of type theory as described in §8. We have defined: $\mathcal{W} \rightleftharpoons \mathcal{V}_0\cup\mathcal{V}$ and $\overline{\mathcal{W}} \rightleftharpoons \mathcal{W}\cup\overline{\mathcal{V}}$. We extend the set $\mathcal{F} \rightleftharpoons \bigcup_{n\in\Omega}\mathcal{F}_n$ of formulas as follows: Let $\mathcal{F}^{\rhd}$ $(\supset \mathcal{F})$ be the set of all formulas constructible by the following six rules (where '$\Rightarrow$' indicates the steps of construction):

$$\begin{array}{rll} & \Rightarrow\ F, & \text{if } F\in\mathcal{F} \\ F & \Rightarrow\ (\neg F),\ (\exists x\,F), & \text{if } x\in\overline{\mathcal{W}} \\ F,G & \Rightarrow\ (F\wedge G),\ (F\vee G),\ (F\rhd G). & \end{array}$$

In the following, x, y, z range over $\overline{\mathcal{W}}$, $\overline{x}$ over $\overline{\mathcal{V}}$ only, and $\underline{y}, \underline{z}$ over all lists $z_1, \ldots, z_k$ of variables $z_i \in \overline{\mathcal{W}}$ with arbitrary length $k \in \mathbb{N}$; t ranges over $\mathcal{T} \cup \overline{\mathcal{T}}$; m over Ω; F, G, H over $\mathcal{F}^{\triangleright}$; and A, B, C over $\mathcal{A}^{\triangleright}$, i.e. the set of all sentences belonging to $\mathcal{F}^{\triangleright}$. - We let $\forall \underline{y}$ and $\forall \underline{z}$ range over all prefices of the form $\forall z_1 \ldots \forall z_k$ with $k \geq 0$. For $k = 0$ let $\forall z_1, \ldots, z_k\, F$ stand for F. Note that the quantifications in $\exists x\, F$ and in $\forall \underline{z}\, F$ are not restricted to any order, and that formulas of $\mathcal{F}^{\triangleright} \setminus \mathcal{F}$ do not occur in terms of $\mathcal{T}$.

Now we specify the **axioms** of $\mathcal{S}$ in A1. - A4.:

A1. Let PL be the 'propositional language' whose formulas are as usual composed of 'propositional variables' and $\perp$ ($\rightleftharpoons\ 0 = 1$) by means of $\wedge, \vee, \neg$, and $(,)$. Let TAU be a particular finite set of tautologies that are formulated in PL. TAU with the rule of *modus ponens* is assumed to be 'complete'. As axioms of $\mathcal{S}$ we take all formulas $\forall \underline{z}\, F$ where F results from an element of TAU by replacing all occurrences of propositional variables with formulas of $\mathcal{F}^{\triangleright}$. Here and in the following, formulas represented by $\forall \underline{z}\, F$ are permitted to contain further free variables.

By an *instance* of a formula, F, of $\mathcal{F}^{\triangleright}$ we understand a *sentence* that results from F by replacing all free occurrences of variables with values of them, which are constants. Also the following axioms of $\mathcal{S}$ have the form $\forall \underline{z}\, F$. $\mathrm{Fr}(t, x, F)$ is to mean that t is free for x in F, and $\mathrm{N}(y, G)$ is to mean that y does not occur free in G.

A2. Let all formulas of the following forms be axioms of $\mathcal{S}$:

$$
\begin{array}{rcll}
 & & \forall \underline{z}\, (t = t) & \text{with } t \in \mathcal{T}; \\
\forall \underline{z}\, (x = t & \rightarrow & (F \leftrightarrow F^x_t)) & \text{with } \mathrm{Fr}(t, x, F),\ t \in \mathcal{T},\ x \in \mathcal{W}; \\
\forall \underline{z}\, (F^x_t & \rightarrow & \exists x\, F) & \text{with } \mathrm{Fr}(t, x, F); \\
\forall \underline{z}\, (\forall y\, (F^x_y \rightarrow H) & \rightarrow & (\exists x\, F \rightarrow H)) & \text{with } \mathrm{Fr}(y, x, F),\ \mathrm{N}(y, (\exists x\, F \rightarrow H)); \\
\forall \underline{z}\, (\exists x \,\varepsilon\, C_m.\, F & \leftrightarrow & \exists x\, (x \,\varepsilon\, C_m \wedge F)) & \text{with } x \in \mathcal{W},\ F \in \mathcal{F}; \\
\forall \underline{z}\, (\exists \overline{x} \,\varepsilon\, t.\, F & \leftrightarrow & \exists \overline{x}\, (\overline{x} \,\varepsilon\, t\ \wedge\ F)) & \text{with } t \in \mathcal{T},\ F \in \mathcal{F}; \\
\forall \underline{z}\, (s \,\varepsilon\, \{\overline{x} \,\varepsilon\, \overline{C}_m \colon F\} & \leftrightarrow & s \,\varepsilon\, \overline{C}_m\ \wedge\ F^{\overline{x}}_s) & \text{with } \mathrm{Fr}(s, \overline{x}, F),\ s \in \overline{\mathcal{T}},\ F \in \mathcal{F};
\end{array}
$$

$$\forall \underline{z}\, \big((\underline{s}, p) \,\varepsilon\, T(q)\ \leftrightarrow\ (\underline{s}) \,\varepsilon\, \overline{C}_m\ \wedge\ p < q\ \wedge\ F((\underline{s}), p, T(p)) \big)$$

with $(\underline{s}) \in \overline{\mathcal{T}}$, $p, q \in \mathcal{T}(\Omega)$, $T \equiv (\mathrm{J}\overline{x} \,\varepsilon\, \overline{C}_m, \mu, z : F(\overline{x}, \mu, z))$, $\mu \in \mathcal{V}(\Omega)$, $z \in \mathcal{V}$, $\mathrm{Fr}((\underline{s}), \overline{x}, F(\ldots))$, and $\mathrm{Fr}(p, \mu, F(\ldots))$;

$$
\begin{array}{l}
\forall \underline{z}\, (\exists x \,\varepsilon\, \pi_m((t_1, \ldots, t_j), i).\, F\ \leftrightarrow\ t_i \,\varepsilon\, C_m\ \wedge\ F^x_{t_i}), \\
\forall \underline{z}\, (\exists x \,\varepsilon\, \pi_m((t_1, \ldots, t_j), p).\, F\ \rightarrow\ p = 1\ \vee\ \ldots\ \vee\ p = j)
\end{array}
$$

with $x \in \mathcal{W}$; $t_1, \ldots, t_j \in \mathcal{T}$; $i = 1, \ldots, j$; $p \in \mathcal{T}(\mathbb{N}^+)$, and $F \in \mathcal{F}$.

A3. The following axiom schemes, which we include in $\mathcal{S}$, concern the connective $\triangleright$:

$$
\begin{array}{cr}
\forall \underline{z}\, (F \triangleright F) & [1] \\
\forall \underline{z}\, (F \triangleright G\ \wedge\ G \triangleright H\ \rightarrow\ F \triangleright H) & [2] \\
\forall \underline{z}\, (F \triangleright \forall \underline{y}\, (G \rightarrow H)\ \rightarrow\ (F \triangleright \forall \underline{y}\, G\ \rightarrow\ F \triangleright \forall \underline{y}\, H)) & [3] \\
\forall \underline{z}\, (F \triangleright \forall \underline{y}\, H\ \rightarrow\ F \triangleright \forall \underline{y}\, (G \triangleright H)) & [4] \\
\forall \underline{z}\, ((F \wedge G) \triangleright H\ \rightarrow\ F \triangleright (G \triangleright H)) & [5].
\end{array}
$$

Note. [3] and [4] remind of the following axiom schemes of labelled modal logic: $\boxed{\mathrm{i}}(A \to B) \to (\boxed{\mathrm{i}}A \to \boxed{\mathrm{i}}B)$ and $\boxed{\mathrm{i}}A \to \boxed{\mathrm{i}}\,\boxed{\mathrm{j}}A$, respectively, which are in case $i = j$ (or without labelles i, j) usually designated by (K) and (4) (cf. (Popkorn 1994), chap. 2).

A4. As axioms of $\mathcal{S}$ we can (for certain purposes) also take other formulas whose instances may be asserted due to certain rules of assertion, especially formulas of the shape $\forall \underline{z}\,(E_1 \wedge \ldots \wedge E_n \to E)$ with $E_1, \ldots, E_n, E \in \mathcal{E}$, where $E_1, \ldots, E_n \Rightarrow E$ (with metavariables for certain elements of $\mathcal{C}_0$ in place of variables) is an agreed rule of assertion (cf. §1).

As **rules** of $\mathcal{S}$ we take

$$\begin{array}{rcll} \forall \underline{z}\, F,\ \forall \underline{z}\,(F \to G) & \Rightarrow & \forall \underline{z}\, G & (\textit{modus ponens}) \\ \forall \underline{z}\, H & \Rightarrow & \forall \underline{z}\,(G \rhd H) & (\textit{necessitation}). \end{array}$$

(Instances of these rules are $F, F \to G \Rightarrow G$ and $H \Rightarrow G \rhd H$.)

Let $\mathcal{S} \vdash B$ be short for "B is deducible in $\mathcal{S}$ (i.e. from the axioms of $\mathcal{S}$ by the rules of $\mathcal{S}$)", and $\mathcal{S}(A) \vdash B$ for "B is deducible in $\mathcal{S}(A)$ (i.e. from A and the axioms of $\mathcal{S}$ by the rules of $\mathcal{S}$)." We now interprete $A \rhd B$ as $\mathcal{S}(A) \vdash B$, i.e. we fix the following rule in addition to the 'primary rules' quoted in §8:

$$\natural\, A \rhd B \ :\Rightarrow\ \natural\, \mathcal{S}(A) \vdash B.$$

This rule is invertible, since we do not restrict the assertion of $A \rhd B$ by other rules. But all sentences of $\mathcal{A}^{\rhd}$ are to be understood classically, i.e. with respect to the classical game of assertion (cf. §3).

Note. We have: $\mathcal{S} \vdash \forall \underline{z}\, F$ if and only if $\mathcal{S} \vdash F$. This can be shown by induction on $\mathcal{S}$ (i.e. on the number of corresponding deduction steps). The same also holds for $\mathcal{S}(A)$ instead of $\mathcal{S}$. Moreover, we have $\mathcal{S} \vdash \forall \underline{z}\,\big(F \rhd \forall x\, G \to \forall x\,(F \rhd G)\big)$; this reminds of the inverse Barcan formula, $\boxed{\mathrm{i}}\forall x\, G \to \forall x \boxed{\mathrm{i}} G$.

11.1 Lemma: For all $A \in \mathcal{A}^{\rhd}$ and all $F \in \mathcal{F}^{\rhd}$, if $\mathcal{S}(A) \vdash F$ then $\mathcal{S} \vdash A \rhd F$.

Proof, by induction on $\mathcal{S}(A)$: Let $\mathcal{S}(A) \vdash F$. If F is an axiom of $\mathcal{S}$, then $\mathcal{S} \vdash A \rhd F$ by *necessitation.* If $F \equiv A$, then $\mathcal{S} \vdash A \rhd F$ by axiom [1]. - If $F \equiv \forall \underline{y}\, H$ has been deduced in $\mathcal{S}(A)$ by applying *modus ponens* from the premises $\forall \underline{y}\, G$ and $\forall \underline{y}\,(G \to H)$, say, then we may use the induction hypotheses $\mathcal{S} \vdash A \rhd \forall \underline{y}\, G$ and $\mathcal{S} \vdash A \rhd \forall \underline{y}\,(G \to H)$. Then, by axiom [3] and *modus ponens*, $\mathcal{S} \vdash A \rhd \forall \underline{y}\, H$. - If $F \equiv \forall \underline{y}\,(G \rhd H)$ has been deduced in $\mathcal{S}(A)$ by *necessitation* from the premise $\forall \underline{y}\, H$, then, by induction hypothesis, we have $\mathcal{S} \vdash A \rhd \forall \underline{y}\, H$ and so, by [4] and *modus ponens,* $\mathcal{S} \vdash A \rhd \forall \underline{y}\,(G \rhd H)$. □

Substituting B for A and $\mathcal{S}(A)$ for $\mathcal{S}$ in the latter proof, we also obtain:

11.2 Lemma: For all $A, B \in \mathcal{A}^{\rhd}$ and all $F \in \mathcal{F}^{\rhd}$, if $\mathcal{S}(A \wedge B) \vdash F$ then $\mathcal{S}(A) \vdash B \rhd F$.

11.3 Proposition: If $\mathcal{S} \vdash F$, then all instances of F are true (assertible).

Proof (cf. (Smullyan 1987), chap. 26, proof of Theorem 1, e.g.): At first we show that all instances of the axioms [1] - [4] are true. To this, we consider any instances A, B, C of F, G, H, respectively.
Ad [1]: Let A be deducible from itself, i.e. $\mathcal{S}(A) \vdash A$.
Ad [2]: If $\mathcal{S}(A) \vdash B$ and $\mathcal{S}(B) \vdash C$, then $\mathcal{S}(A) \vdash C$.
Ad [3]: Let $A \triangleright \forall \underline{y}\,(B\underline{y} \to C\underline{y})$ be a instance of $F \triangleright \forall \underline{y}\,(G \to H)$.
If $\mathcal{S}(A) \vdash \forall \underline{y}\,(B\underline{y} \to C\underline{y})$ and $\mathcal{S}(A) \vdash \forall \underline{y}\, B\underline{y}$, then, by *modus ponens*, $\mathcal{S}(A) \vdash \forall \underline{y}\, C\underline{y}$.
Ad [4]: If $\mathcal{S}(A) \vdash \forall \underline{y}\, C\underline{y}$, then, by *necessitation*, $\mathcal{S}(A) \vdash \forall \underline{y}\,(B\underline{y} \triangleright C\underline{y})$.
Ad [5]: By 11.2, every sentence of the from [5] is true.

Also all instances of the residual axioms of $\mathcal{S}$ are true. Now we easily obtain 11.3 by induction on $\mathcal{S}$. To this note the following: To obtain $\mathcal{S} \vdash \forall \underline{z}\,(G \triangleright H)$ by *necessitation*, we must previously have $\mathcal{S} \vdash \forall \underline{z}\, H$. But then, for any instance $B \triangleright C$ of $G \triangleright H$, C is deducible in $\mathcal{S}$ and so in $\mathcal{S}(B)$ so that $B \triangleright C$ is true. Thus every instance of $\forall \underline{z}\,(G \triangleright H)$ is true. □

From 11.1 and 11.3 we obtain:

11.4 Corollary: For all $A, B \in \mathcal{A}^{\triangleright}$, $\mathcal{S}(A) \vdash B$ if and only if $\mathcal{S} \vdash A \triangleright B$.

2. A language of cumulative type theory with quotation marks

In the following we construct a language of cumulative type theory that contains 'names' of sentences. By means of first order sentences of that language we can also speak about higher order sentences of it. Despite this 'reduction' of order, all sentences of that language are non-circular.

In §8 we have defined the sets $\mathcal{T}, \overline{\mathcal{T}}$ and $\mathcal{F}$. We say that the language $\mathcal{L}$ consisting of them results from $\mathcal{T}_{\mathrm{or}}, \mathcal{E}, \mathcal{V}_0, \mathcal{V}, \overline{\mathcal{V}}$ by the $\mathcal{T}, \mathcal{F}$-rules. Now we presuppose that a given language $\mathcal{L}^\circ$ results from $\mathcal{T}^\circ_{\mathrm{or}}, \mathcal{E}^\circ, \mathcal{V}^\circ_0, \mathcal{V}, \overline{\mathcal{V}}$ by those rules. We shall construct extensions $\mathcal{T}_{\mathrm{or}} \supset \mathcal{T}^\circ_{\mathrm{or}}$, $\mathcal{E} \supseteq \mathcal{E}^\circ$, and $\mathcal{V}_0 \supset \mathcal{V}^\circ_0$ such that the language $\mathcal{L}$ that results from $\mathcal{T}_{\mathrm{or}}, \mathcal{E}, \mathcal{V}_0, \mathcal{V}, \overline{\mathcal{V}}$ contains sentences expressing that $\mathcal{S}(A) \vdash B$, for $\mathcal{S}$ as above and any sentences A, B of $\mathcal{A}^{\triangleright}$. *Note that $\mathcal{L}$ is a language of cumulative type theory.*

Let $\mathcal{V}_{\mathcal{N}}$ be a denumerable set of 'new variables' that do not occur in the elements of $\mathcal{T}^\circ \cup \overline{\mathcal{T}}^\circ \cup \mathcal{F}^\circ$. ($\mathcal{N}$ will be defined below.) Let the set Σ° contain all atomic symbols occurring in elements of $\mathcal{T}^\circ \cup \overline{\mathcal{T}}^\circ \cup \mathcal{F}^\circ \cup \mathcal{V}_{\mathcal{N}}$, and the additional symbols $\triangleright$, N, Fr, and Sub. Let the symbol set Σ result from Σ° by adding the new symbols $\lceil\alpha\rceil, \lceil\lceil\alpha\rceil\rceil, \lceil\lceil\lceil\alpha\rceil\rceil\rceil, \ldots$, for every $\alpha \in \Sigma^\circ$. The symbols $\lceil, \rceil$ are supposed not to belong to Σ°. We do also *not* include them in Σ. So we may consider all elements of Σ as *atomic* symbols.

Let Σ^* be the set of all strings $\alpha_1 \ldots \alpha_j$ ($j \geq 0$) of symbols $\alpha_i \in \Sigma$. So, especially, the 'empty word' belongs to Σ^* (case $j = 0$). For $\alpha_1, \ldots, \alpha_j \in \Sigma$ ($j \geq 0$) we define

$$\lceil\alpha_1\alpha_2 \ldots \alpha_j\rceil \rightleftharpoons \lceil\alpha_1\rceil\lceil\alpha_2\rceil \ldots \lceil\alpha_j\rceil.$$

Let this figure be said to be the **name** of $\alpha_1\alpha_2\ldots\alpha_j$. Especially, $\ulcorner\,\urcorner$ stands for the empty word, which is its own name. Let $\mathcal{N}$ be the set of all such names of elements of Σ^*. We shall use the elements of $\mathcal{V}_{\mathcal{N}}$ as variables for (all or particular) elements of $\mathcal{N}$. All variables occurring in an element of $\mathcal{N}$ are considered to be *bound* (by the 'quotation marks' $\ulcorner, \urcorner$).

Let $\mathcal{T}(\mathcal{N})$ be the set of all figures of the form

$$\ulcorner S_0\urcorner X_{11}\ldots X_{1k_1}\ulcorner S_1\urcorner X_{21}\ldots X_{2k_2}\ulcorner S_2\urcorner\ldots X_{j1}\ldots X_{jk_j}\ulcorner S_j\urcorner$$

with $S_i\in\Sigma^*$ for $0\le i\le j$, $S_i\not\equiv\ulcorner\,\urcorner$ for $0<i<j$, variables $X_{ik}\in\mathcal{V}_{\mathcal{N}}$, $j\ge 0$, and $k_1,k_2,\ldots,k_j\ge 1$.

Note. We have $\mathcal{N}\cup\mathcal{V}_{\mathcal{N}}\subseteq\mathcal{T}(\mathcal{N})$. If we replace all free occurrences of a variable in an element of $\mathcal{T}(\mathcal{N})$ by an element of $\mathcal{T}(\mathcal{N})$ - or, especially, by the empty word -, then we again obtain an element of $\mathcal{T}(\mathcal{N})$.

$\mathcal{V}_0^\circ$ ($\subseteq\mathcal{T}_{\mathrm{or}}^\circ$, see above) is assumed to be a set of variables (of several given sorts) for certain constants belonging to $\mathcal{T}_{\mathrm{or}}^\circ$. Let $\mathcal{V}_0 \rightleftharpoons \mathcal{V}_0^\circ\cup\mathcal{V}_{\mathcal{N}}$, $\mathcal{T}_{\mathrm{or}}\rightleftharpoons\mathcal{T}_{\mathrm{or}}^\circ\cup\mathcal{T}(\mathcal{N})$, and let $\mathcal{E}$ contain all elements of $\mathcal{E}^\circ$, all equations $(\sigma =_0 \tau)$ with $\sigma,\tau\in\mathcal{T}(\mathcal{N})$, and certain further formulas, which we shall specify below. Let, as announced, $\mathcal{T},\overline{\mathcal{T}},\mathcal{F}$ result from $\mathcal{T}_{\mathrm{or}},\mathcal{E},\mathcal{V}_0,\mathcal{V},\overline{\mathcal{V}}$ by the $\mathcal{T},\mathcal{F}$-rules.

Examples of tuples containing the empty word are: ( ), $(t,\,)$, (,), $(\,,\,,t)$, for any $t\in\mathcal{T}$. Such tuples are particular elements of $\overline{\mathcal{T}}$.

Given a system $\mathcal{S}$ of axioms and rules as indicated above. We want to define sets $R_0,R_1,R_2,\ldots\in\mathcal{C}_1$ such that, for all $n\in\mathbb{N}$, R_n is the set of all names of sentences that are deducible in $\mathcal{S}$ by $\le n$ steps of deduction.

For any $\mathcal{U}\subseteq\Sigma^*$ let $\mathcal{U}^{\ulcorner,\urcorner}$ be the set of all names $\ulcorner u\urcorner$ of elements u of $\mathcal{U}$. (So we have $\mathcal{U}^{\ulcorner,\urcorner}\subseteq\Sigma^{*\ulcorner,\urcorner}=\mathcal{N}$.) The sign '$=_0$' between elements of $\mathcal{N}$ is to mean their literal equality.

So long we have used the letters $x,y,\overline{x},s,t,m,F,G,H,\ldots$ as metavariables.
However, to make the following definitions easier to read, we now use these and some other letters to indicate particular variables of $\mathcal{V}_{\mathcal{N}}$ that range over certain subsets of $\mathcal{N}$. That is, we provisionally let w ($\in\mathcal{V}_{\mathcal{N}}$) range over $\mathcal{W}^{\ulcorner,\urcorner}$, x,y over $\overline{\mathcal{W}}^{\ulcorner,\urcorner}$, $\overline{x}$ over $\overline{\mathcal{V}}^{\ulcorner,\urcorner}$, r over $\mathcal{T}^{\ulcorner,\urcorner}$, s over $\overline{\mathcal{T}}^{\ulcorner,\urcorner}$, t over $\mathcal{T}^{\ulcorner,\urcorner}\cup\overline{\mathcal{T}}^{\ulcorner,\urcorner}$, m over $\Omega^{\ulcorner,\urcorner}$, F,G,H,X over $\mathcal{F}^{\triangleright\ulcorner,\urcorner}$, P,Q over $\mathcal{F}^{\ulcorner,\urcorner}$ only, and η,ζ over names $\ulcorner\forall z_1\ldots\forall z_k\urcorner$ of prefices with variables $z_i\in\overline{\mathcal{W}}$ and length $k\ge 0$. (Of course, we presuppose that $\mathcal{V}_{\mathcal{N}}$ contains denumerably many variables of each of those sorts.) We write $\ulcorner\ldots\check{x}\text{- - -}\urcorner$ for $\ulcorner\ldots\urcorner x\ulcorner\text{- - -}\urcorner$ (wherein x occurs free), $\ulcorner\ldots\check{x}\check{F}\text{- - -}\urcorner$ for $\ulcorner\ldots\urcorner xF\ulcorner\text{- - -}\urcorner$, e.g., $\ulcorner\ldots\forall\underline{\check{y}}\text{ - - -}\urcorner$ for $\ulcorner\ldots\urcorner\eta\ulcorner\text{- - -}\urcorner$, and $\ulcorner\forall\underline{\check{z}}\text{ - - -}\urcorner$ for $\zeta\ulcorner\text{- - -}\urcorner$.

The following formula 'Axiom(X)' of $\mathcal{F}_1$ can be read as "X is the name of an axiom of $\mathcal{S}$":

$$
\begin{array}{rcl}
\mathrm{Axiom}(X) & \rightleftharpoons & \exists\, F,G,H,P,Q,w,x,y,\overline{x},\eta,\zeta,r,s,t,m\ \varepsilon\ C_0.\ \big(X =_0 \ulcorner \forall\underline{\check{z}}\,(\check{F} \triangleright \check{F})\urcorner \\
& & \vee\ X =_0 \ulcorner \forall\underline{\check{z}}\,(\check{F} \triangleright \check{G}\ \wedge\ \check{G} \triangleright \check{H}\ \rightarrow\ \check{F} \triangleright \check{H})\urcorner \\
& & \vee\ X =_0 \ulcorner \forall\underline{\check{z}}\,(\check{F} \triangleright \forall\underline{\check{y}}\,(\check{G} \rightarrow \check{H}) \rightarrow (\check{F} \triangleright \forall\underline{\check{y}}\,\check{G} \rightarrow \check{F} \triangleright \forall\underline{\check{y}}\,\check{H}))\urcorner \\
& & \vee\ X =_0 \ulcorner \forall\underline{\check{z}}\,(\check{F} \triangleright \forall\underline{\check{y}}\,\check{H}\ \rightarrow\ \check{F} \triangleright \forall\underline{\check{y}}\,(\check{G} \triangleright \check{H}))\urcorner \\
& & \vee\ X =_0 \ulcorner \forall\underline{\check{z}}\,((\check{F} \wedge \check{G}) \triangleright \check{H} \rightarrow \check{F} \triangleright (\check{G} \triangleright \check{H}))\urcorner \\
& & \vee\ \ldots \\
& & \vee\ \big(X =_0 \ulcorner \forall\underline{\check{z}}\,(\check{G} \rightarrow \exists\check{x}\,\check{F})\urcorner\ \wedge\ \mathrm{Sub}(G,F,t,x)\ \wedge\ \mathrm{Fr}(t,x,F)\big) \\
& & \vee\ \big(X =_0 \ulcorner \forall\underline{\check{z}}\,(\forall\check{y}\,(\check{G} \rightarrow \check{H}) \rightarrow (\exists\check{x}\,\check{F} \rightarrow \check{H}))\urcorner \\
& & \qquad \wedge\ \mathrm{Sub}(G,F,y,x)\ \wedge\ \mathrm{Fr}(y,x,F)\ \wedge\ \mathrm{N}(y,\ulcorner \exists\check{x}\,\check{F} \rightarrow \check{H}\urcorner)\big) \\
& & \vee\ X =_0 \ulcorner \forall\underline{\check{z}}\,(\exists\check{w}\ \varepsilon\ C_{\check{m}}.\,\check{P}\ \leftrightarrow\ \exists\check{w}\,(\check{w}\ \varepsilon\ C_{\check{m}} \wedge \check{P}))\urcorner \\
& & \vee\ X =_0 \ulcorner \forall\underline{\check{z}}\,(\exists\check{\overline{x}}\ \varepsilon\ \check{r}.\,\check{P}\ \leftrightarrow\ \exists\check{\overline{x}}\,(\check{\overline{x}}\ \varepsilon\ \check{r} \wedge \check{P}))\urcorner \\
& & \vee\ \big(X =_0 \ulcorner \forall\underline{\check{z}}\,(\check{s}\ \varepsilon\ \{\check{\overline{x}}\ \varepsilon\ \overline{C}_{\check{m}} \colon \check{P}\}\ \leftrightarrow\ \check{s}\ \varepsilon\ \overline{C}_{\check{m}}\ \wedge\ \check{Q})\urcorner \\
& & \qquad \wedge\ \mathrm{Sub}(Q,P,s,\overline{x})\ \wedge\ \mathrm{Fr}(s,\overline{x},P)\big) \\
& & \vee\ \ldots\big).
\end{array}
$$

(We have ommited several brackets in the definiens of Axiom(X), which is to be completed by means of names for further axioms alleged under A1, A2 and A4 (above).) Of course, a *sentence* of the form $\mathrm{Fr}(t,x,F)$ with *names* t,x,F *(in place of variables)* is to mean that t' is free for x' in F' where t' is the term denoted by t, x' is the variable denoted by x, and F' is the formula denoted by F. Similarly, $\mathrm{N}(y,G)$ is to mean that y' does not occur free in G', and $\mathrm{Sub}(G,F,t,x)$ is to mean that G' results from F' by substituting t' for x'. We include all *formulas* of those shapes in $\mathcal{E}$. To formulate this in more detail, we at first define: For $\mathcal{U} \subseteq \Sigma^*$ let $\mathcal{T}(\mathcal{U}^{\ulcorner,\urcorner})$ be the set of all elements of $\mathcal{T}(\mathcal{N})$ whose instances are elements of $\mathcal{U}^{\ulcorner,\urcorner}$. Now let $\mathcal{E}$ contain all elements of $\mathcal{E}^\circ$ and all formulas $\mathrm{Fr}(t,x,F)$, $\mathrm{N}(y,G)$, and $\mathrm{Sub}(G,F,t,x)$ with $x,y \in \mathcal{T}(\overline{\mathcal{W}}^{\ulcorner,\urcorner})$; $F,G \in \mathcal{T}(\mathcal{F}^{\triangleright\,\ulcorner,\urcorner})$, and $t \in \mathcal{T}(\mathcal{T}^{\ulcorner,\urcorner} \cup \overline{\mathcal{T}}^{\ulcorner,\urcorner})$.

We now recursively define $R_0, R_1, R_2, \ldots$:

$$
\begin{array}{rcl}
R_0 & = & \big\{X\ \varepsilon\ C_0 \colon \mathrm{Axiom}(X)\big\} \\
R_{n+1} & = & \big\{X\ \varepsilon\ C_0 \colon\ X\ \varepsilon\ R_n \\
& & \quad \vee\ \exists\, F,G,\zeta\ \varepsilon\ C_0.\,(\ulcorner \forall\underline{\check{z}}\,\check{F}\urcorner\ \varepsilon\ R_n\ \wedge\ \ulcorner \forall\underline{\check{z}}\,(\check{F} \rightarrow \check{G})\urcorner\ \varepsilon\ R_n\ \wedge\ X =_0 \ulcorner \forall\underline{\check{z}}\,\check{G}\urcorner) \\
& & \quad \vee\ \exists\, G,H,\zeta\ \varepsilon\ C_0.\,(\ulcorner \forall\underline{\check{z}}\,\check{H}\urcorner\ \varepsilon\ R_n\ \wedge\ X =_0 \ulcorner \forall\underline{\check{z}}\,(\check{G} \triangleright \check{H})\urcorner)\big\}.
\end{array}
$$

It is easy to see that R_n is the set of all names of sentences that are deducible in $\mathcal{S}$ by $\leq n$ steps (cf. (Zahn 1993, p. 425f.)). For any $\nu \in \mathcal{V}(\Omega)$, R_ν can also be defined as an element of $\mathcal{T}_1$ (see the proof of 8.12). Let $R \rightleftharpoons \bigcup_{\nu\ \varepsilon\ \mathbb{N}} R_\nu$. So $R \in \mathcal{C}_1$, and for any $B \in \mathcal{A}^\triangleright$, the sentence $\ulcorner B\urcorner\ \varepsilon\ R$ belongs to $\mathcal{A}_1$ and means that B is deducible in $\mathcal{S}$. So, by 11.4, $\ulcorner A \triangleright B\urcorner\ \varepsilon\ R$ means that B is deducible in $\mathcal{S}(A)$ (i.e. 'from A in $\mathcal{S}$').

Remarks: In the above definition of $R_0, R_1, R_2, \ldots$ it has been convenient to use variables of several sorts, namely for every set $\mathcal{U} \in \{\mathcal{W}, \overline{\mathcal{V}}, \overline{\mathcal{W}}, \mathcal{T}, \overline{\mathcal{T}}, \mathcal{T} \cup \overline{\mathcal{T}}, \Omega, \mathcal{F}, \mathcal{F}^\triangleright\}$ variables ranging over $\mathcal{U}^{\ulcorner,\urcorner}$, and variables ranging over names $\ulcorner \forall z_1 \ldots \forall z_k\urcorner$ of prefices with $z_i \in \overline{\mathcal{W}}$ and $k \geq 0$.

But instead of those variables (which belong to $\mathcal{V}_{\mathcal{N}}$) we need only *one* sort of variables, namely variables ranging over $\mathcal{N}$. Then we have to reformulate Axiom(X) thus:

$$\exists F, \ldots, \zeta, \ldots \;\varepsilon\; C_0.\, \big(\, F \;\varepsilon\; \mathcal{F}^{\rhd\ulcorner,\urcorner} \;\wedge\; \ldots \;\wedge\; \zeta \;\varepsilon\; \Pi^{\ulcorner,\urcorner} \;\wedge\; \ldots \;\wedge\; (X =_0 \zeta \ulcorner(\check{F} \rhd \check{F})\urcorner \;\vee\; \ldots)\big)$$

where $X, F, \zeta, \ldots$ are elements of $\mathcal{V}(\mathcal{N})$ ranging over $\mathcal{N}$. Here we have added the clauses $F \;\varepsilon\; \mathcal{F}^{\rhd\ulcorner,\urcorner}$, $\zeta \;\varepsilon\; \Pi^{\ulcorner,\urcorner}, \ldots$, where Π denotes the set of the above mentioned prefices. (To avoid misunderstandings we can replace '$\mathcal{F}^{\rhd}$' by a new sign in this context.) We can effect that the latter clauses are in $\mathcal{F}_1$ - provided that $\mathcal{T}^\circ_{\mathrm{Or}}, \mathcal{E}^\circ, \mathcal{V}^\circ_0, \mathcal{V}$, and $\overline{\mathcal{V}}$, are recursively enumerable (i.e. constructible by formal rules). Indeed, in this case also the sets $\mathcal{T}, \overline{\mathcal{T}}, \mathcal{F}, \mathcal{F}^{\rhd}, \Pi, \ldots$ are recursively enumerable so that the corresponding sets of names for elements of those sets can be introduced as elements of $\mathcal{C}_1$ (namely on the model of the above introduction of $R \rightleftharpoons \bigcup_{\nu \,\varepsilon\, \mathbb{N}} R_\nu$). - However, we dispense with complete reformulations of 'Axiom(X)' and the definition of R_{n+1}.

The predicates $\mathrm{N}(\cdot,\cdot)$, $\mathrm{Fr}(\cdot,\cdot,\cdot)$, and $\mathrm{Sub}(.,.,.)$ have recursively enumerable extents and can, therefore, be defined to be elements of $\mathcal{C}_1$. So it suffices to take $\mathcal{E}$ to be $\mathcal{E}^\circ$. (Recall that, by a demand given in §8, we have $(s =_0 t) \in \mathcal{F}_0$ for all $s, t \in \mathcal{T}(\mathcal{N})$.) We may also omit the signs N, Fr, and Sub from Σ°.

When we say that a sentence B is deducible in $\mathcal{S}(A)$, we do not *use* the sentences A and B, we only *refer* to them. To indicate this fact we can put them in quotation marks. Accordingly, it would be adequate to understand $A \rhd B$ as a shorthand of $\ulcorner A \urcorner \rhd \ulcorner B \urcorner$. But then the definiens of 'Axiom(X)' turns in

$$\exists F, \ldots, \zeta, \ldots \;\varepsilon\; C_0.\, \big(\, X =_0 \ulcorner \forall \check{\underline{z}}\, (F \rhd F) \urcorner \;\vee\; \ldots \,\big),$$

where several occurrences of F are bound by $\ulcorner, \urcorner$, which misses the intended meaning.
We do no further discuss that matter.

3. A version of the Theorem of Löb

Modifying an idea of Craig (see (Smullyan 1987), chap. 26, e.g.) we now extend $\mathcal{F}^{\rhd}$ thus: For all $F, G \in \mathcal{F}^{\rhd}$ let $\Delta(F, G)$ be a formula of $\mathcal{E}$ $(\subset \mathcal{F} \subset \mathcal{F}^{\rhd})$. For all $A, B \in \mathcal{A}^{\rhd}$ let

$$\Delta(A, B) \text{ mean that } \mathcal{S}(A) \vdash (\Delta(A, B) \to B).$$

(Note that the latter deducibility relation does not depend on the meaning of $\Delta(A, B)$.)
So for all $F, G \in \mathcal{F}^{\rhd}$, all instances of

$$\forall \underline{z}\, \big(\, \Delta(F, G) \;\leftrightarrow\; F \rhd (\Delta(F, G) \to G)\,\big)$$

are true. We now take all formulas of this form as additional axioms of $\mathcal{S}$. (These axioms can easily be enclosed in 'Axiom(X)'.) So all formulas of the following form are deducible in $\mathcal{S}$:

$$\forall \underline{z}\, \big\{\, (\Delta(F, G) \to G) \;\leftrightarrow\; [\, F \rhd (\Delta(F, G) \to G) \to G]\,\big\}.$$

Writing H for $(\Delta(F, G) \to G)$ we obtain this version of the

Diagonal Lemma: For all $F, G \in \mathcal{F}^{\triangleright}$ there is an $H \in \mathcal{F}^{\triangleright}$ satisfying $\mathcal{S} \vdash \forall \underline{z} \left\{ H \;\leftrightarrow\; (F \triangleright H \rightarrow G) \right\}$.

This implies especially the following version of the

Theorem of Löb: For all $B, C \in \mathcal{A}^{\triangleright}$, if $\mathcal{S} \vdash (B \triangleright C \rightarrow C)$, then $\mathcal{S} \vdash C$.

The proof given in (Boolos 1989), p.187, can easily be transformed into a proof of this version of Löb's theorem. By another well known theorem of modal logic, this version yields

$$\mathcal{S} \vdash (B \triangleright (B \triangleright C \rightarrow C) \rightarrow B \triangleright C),$$

which reminds of the modal scheme $\boxed{\mathrm{i}}(\boxed{\mathrm{i}}C \rightarrow C) \rightarrow \boxed{\mathrm{i}}C)$. Obviously, all results of this section also hold for $\mathcal{S}(A)$ instead of $\mathcal{S}$.

Notes: 1. Modal systems satisfying the latter scheme together with (K) and (4) are said to be of type G.
2. Let $\top \rightleftharpoons \neg\bot$, e.g. Because of $\mathcal{S} \nvdash \bot$, Löb's theorem and 11.4 especially imply $\mathcal{S} \nvdash \neg(\top \triangleright \bot)$ (cf. Gödel's second incompleteness theorem).

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