Contributions to Passivity Theory and Dissipative Control Synthesis

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Genehmigte Dissertation von M.Sc. Diego de Sousa Madeira aus Fortaleza, Brasilien

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Ich versichere hiermit, dass ich die vorliegende Dissertation allein und nur unter Verwendung der angegebenen Literatur verfasst habe. Die Arbeit hat bisher noch nicht zu Prüfungszwecken gedient.

Darmstadt, 15. November 2017

Vorwort

Die vorliegende Arbeit enstand während meiner Tätigkeit als wissenschaftlicher Mitarbeiter am Fachgebiet Regelungsmethoden und Robotik der Technischen Universität Darmstadt. Mein besonderer Dank geht an Professor Jürgen Adamy, der mir die Bearbeitung dieses Promotionsthemas ermöglicht hat. Seine Unterstützung hat sehr zum Gelingen der Arbeit beigetragen.

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Darmstadt, November 2017 — Diego de Sousa Madeira

Contents

List of Symbols

The following tables are a selection of symbols that are important in more than one section of the thesis.

Abbreviations

LMI	linear matrix inequality	BMI	bilinear matrix inequality
QMI	quadratic matrix inequality	SP	strictly passive
SPR	strictly positive real	LTI	linear time-invariant
KYP	Kalman-Yakubovich-Popov	BIBO	bounded-input bounded output
SISO	single input single output	MIMO	multiple input multiple output
MISO	multiple input single output	LFT	linear fractional transformation
PID	proportional-integral-derivative	CbI	control by interconnection
SOF	static output feedback	ILMI	iterative LMI
SDP	semidefinite programming	ODE	ordinary differential equation
PDE	partial differential equation	ISP	input strictly passive
OSP	input strictly passive	VSP	input strictly passive
PDI	partial differential inequality	DOF	dynamic output feedback
ARE	algebraic Riccati equation	ARI	algebraic Riccati inequality

Notation

$\mathbb{R}$	the set of real numbers	$\mathbb{R}^+$	the set of nonnegative real numbers

$\lvert x \rvert$	absolute value of $x \in \mathbb{C}$	$\mathbb{C}$	the set of complex numbers
$\mathbb{N}$	the set of nonnegative integers	$A^\top$	transpose of a matrix $A \in \mathbb{C}^{n \times m}$
$\bar{A}$	conjugate of a matrix $A \in \mathbb{C}^{n \times m}$	A^*	conjugate transpose matrix of a matrix $A \in \mathbb{C}^{n \times m}$
$A > 0$	positive definite matrix	$A \geq 0$	positive semidefinite matrix
I_n	the $n \times n$ identity matrix	0_n	the $n \times n$ zero matrix
$\lvert\lvert x \rvert\rvert$	Euclidean norm in $\mathbb{R}^n$, $\lvert\lvert x \rvert\rvert = \sqrt{x^\top x})$	$tr(A)$	trace of a matrix A
$f(t)$	function in t	$A^\dagger$	the More-Penrose inverse of matrix A
$\frac{\partial f}{\partial t}$	partial derivative	$\dot{x}(t)$	derivative w.r.t. t
s	Laplace variable	$\det A$	determinant of matrix A
$Re[\,\cdot\,]$	real part	$Im[\,\cdot\,]$	imaginary part
$\mathcal{L}_{p,e}$	extended space associated with $\mathcal{L}_p$	$\mathcal{V}(\mathcal{X}_p)$	vertices of a polytope $\mathcal{X}_p$
$\mathcal{L}_p$	$\mathcal{L}_p$ norm of a signal	$\mathbb{I}_n$	a set of the integers $\{1,\ldots,n\}$
$dim(\,\cdot\,)$	dimension of a column vector		

Kurzfassung

Diese Doktorarbeit beinhaltet Beiträge zu einigen relevanten Problemen in den Feldern der Steuerungstheorie und Steuerungschnologie und zwar im Gebiet der Passivitätsanalyse und dissipativen Steuerungssynthese für lineare und nichtlineare dynamische Systeme. Der erste meiner Beiträge präsentiert eine Lösung zu einem Problem, dass seit vielen Jahren ungelöst ist: das Problem der Äquivalenz zwischen den Begriffe von streng positiver Reelheit und strenger Passivität von linearen Systemen. Beide Begriffe implizieren die asymptotische Stabilität eines linearen Systems, obwohl das eine auf einem Frequenzkeitsbereichkonzept und das andere auf einem Zeitbereichskonzept beruht.

Des Weiteren erörtern wir das gleichermaßen klassische Problem der Stabilisierung von linearen Systemen durch eine statische Ausgangsrückführung, einem Problem, dessen definitive Lösung noch offen ist. Präsentiert wird eine neue notwendige und hinreichende LMI-Bedingung für Stabilität basierend auf dem Begriff strenger Dissipativität und eine neue nicht-iterative Strategie für den Reglerentwurf, welche durch ein konvexes Optimierungsproblem gelöst wird.

Außerdem beinhaltet diese Arbeit eine neue konstruktive dissipativitätbasierte Strategie, die den Begriff linearer Linksannihilator und das Finsler's Lemma nutzt. Dieser Ansatz erlaubt die Analyse der dissipativen Eigenschaften von rational nichtlinearen Systemen unter dem Aspekt von einer polytopischen LMI-Bedingung. Präsentiert wird eine Bedingung für die Stabilisierbarkeit eines Systems, die anhand des Begriffs der Passivität nicht zulässig ist. Die Methode erlaubt auch die Bestimmung einer expliziten Formel für den Reglers Durchgriff, die den weiteren Entwurf wesentlicht vereinfacht, sowohl in dem Fall einer statischen als auch einer dynamischen Ausgangsrückführung.

Abstract

This thesis contains contributions to some relevant problems in the field of control theory and controller design technology, namely to the areas of passivity analysis and dissipative control synthesis for linear and nonlinear dynamical systems. The first of our contributions consists in presenting a solution to a problem which had been unsolved for many years: the problem of the equivalence between the notions of strict positive realness and strict passivity of linear systems. Both properties imply the asymptotic stability of a linear system, although the former is a frequency-domain concept and the latter is a time-domain concept.

Subsequently, we approach the equally classical topic of static output feedback stabilization of linear systems, a problem to which a definite solution remains to be given. We present a new necessary and sufficient LMI condition for stabilization based on the notion of strict dissipativity, and we propose a new noniterative strategy for controller design which consists in solving a single convex optimization problem.

In addition, we also introduce a new dissipativity-based strategy for feedback stabilization of nonlinear systems using the notion of linear annihilators and the celebrated Finsler's Lemma. This approach allows for analysing the dissipativity properties of rational nonlinear plants in terms of a polytopic LMI condition. A new stabilizability condition that would not be feasible in the case of a passive representation of the system is presented as well, making it possible to derive a closed-form expresion for the controller's feedthrough term as a direct consequence of the local dissipativity analysis of the plant. This feature simplifies the remaing steps of the controller design procedure considerably, both in the case of a static or a dynamic output feedback.

1 Introduction

The field of Control Systems Technology is a remarkably relevant and broad subject in the realm of the applied sciences. In fact, as the question of how to act upon a certain system in order to achieve a desired operating behaviour is a cornerstone of human activity, the idea of automatic control of dynamical systems came to play a key role in our society. In this regard, the notion of stability around a prescribed operation/equilibrium point is central and presents itself as matter of great interest among engineers [2]. Such a desired behaviour is usually achieved through the interconnection of a feedback controller with the plant, i.e. a control device which has at least partial access to the system dynamics and, from this information obtained via measurements, is able to act upon the system in such a manner that the closed-loop is stabilized around a desired operating regime.

For a long time already, it is the well-known Lyapunov Stability Theory that has been playing a major role in the domain of stability analysis of dynamical systems [7]. Whether one is dealing with open-loop systems or closed-loop stabilization via feedback control, the idea of determining a Lyapunov function for the system under consideration remains one of the cornerstones in the field of stability analysis and control [1]. In this context, a concept known as *dissipativity*, as well as one of its particular cases known as *passivity*, are of special interest. One of the main arguments in favor of a dissipativity-based approach for stabilization that it is closely related to the energy properties of a dynamical system, in the sense that it describes the phenomenon of energy dissipation in a number of physical plants [3].

Originally defined as an input-output property [16], the concept of dissipativity was unified with the classical state-space perspective through the introduction of the notion of energy function (storage function) [55]. This input-state-output framework has proved truly powerful for stabilization of linear and nonlinear systems. A key feature consists in its direct connection with the classical Lyapunov theory, as a storage function, under mild conditions, can be used as a Lyapunov function, which guarantees stability [3]. Another major feature of dissipative systems is that their interconnection via negative feedback preserves the dissipativity property [53], a fact that has contributed enormously for its widespread application.

Currently, the area of Dissipative Systems Analysis and Control presents itself as a vast research topic, where the notion of passivity also plays a particularly relevant role on its own. In the context of linear time-invariant systems, for instance, the time-domain notion of passivity has a frequency-domain counterpart known as *positive realness*, which essentially embodies the same features as passivity, though in a different domain. Numerous subdivisions of passivity and positive realness have been introduced in the literature and the study of the relationships between those classes is another field of intensive investigation, whereas many problems are still open [19]. Besides, various stabilization strategies for linear systems have been proposed using the concepts of both passivity and positive realness [30], [76], [90].

Similarly, a number of dissipativity and passivity-based control strategies have been introduced for feedback stabilization of nonlinear systems. A common approach consists in rendering the closed-loop passive subject to the existence of an appropriate storage function. If this function is positive definite and attains a strict local minimum at the desired equilibrium point, then it is equivalent to a Lyapunov function, so that stability is achieved [43]. A further approach regards the determination of a control law which assigns a port-Hamiltonian structure (a specific passive structure) [15] to the closed-loop, thus guaranteeing stability [12] under certain conditions. This port-Hamiltonian approach has been widely successful in applications, though the need for deriving an Hamiltonian state-space representation for the plant (or for the closed-loop) might be cumbersome in some cases.

Despite the number of achievements which have been reported in the field of dissipative control theory, there still exist many issues that need to be further investigated. Static output feedback for nonlinear systems, for example, has been mainly addressed in the context of open-loop stable systems whose stability region needs to be enlarged. An in-depth treatment of controller design for unstable plants via dissipativity-based strategies would be helpful from the perspective of future applications. In addition, with regards to applications of dissipativity theory for state feedback, it is generally assumed that the free system without disturbance is asymptotically stable, whereas the main contribution of a dissipativity analysis consists in rendering the closed-loop from disturbance to the controlled output strictly dissipative and, as a result, stable [102]. When compared with the definition of passivity, the notion of dissipativity offers extra degrees of freedom which are useful for controller design purposes.

1.1 Objectives

In this work, we are going to approach the following problems:

- The problem of the equivalence between strict positive realness and strict passivity of linear systems.
- The static output feedback control problem of linear plants.
- Linear static and dynamic output feedback control of rational nonlinear systems through a polytopic LMI condition.

1.2 Structure and Contributions

The thesis is structured as follows. Chapter 2 contains the mathematical background necessary for covering the topics approached in this work. It describes the classes of signals, systems, differential equations and the type of stability conditions we rely on throughout this thesis. In Chapter 3, the notions of passivity and dissipativity are described. A broad review with regard to the conditions under which those properties hold and the variety of manners they can be applied for controller design are presented. Existing passivity and dissipativity-based control methods are described, and well-known conditions for stability of interconnected nonlinear systems are discussed as well. At this point, the notion of *passivity indices*, as a particular case of dissipativity, is also presented.

Chapter 4 consists in our solution to the problem of the equivalence between two definitions in the field of linear systems theory which were not known to be fully equivalent yet. We prove that a controllable and observable linear time-invariant system with possibly nonzero feedthrough term is strictly positive real if and only if it is strictly passive. Chapter 5 contains a major literature review over the problem of static output feedback control of linear systems and our contributions to this problem, namely a new necessary and sufficient condition for stabilization and the respective strategy for controller design. Our strategy is noniterative and simpler than most of the methods available to date.

In Chapter 6, we address the question of feedback stabilization of nonlinear systems. We introduce a constructive dissipativity-based strategy for controller design using the notion of linear annihilators and the celebrated Finsler's Lemma. This approach allows for analysing the dissipativity properties of nonlinear plants in terms of a polytopic LMI condition, a framework to which very efficient semidefinite programming (SDP) tools can be applied. We present a procedure which allows to do without port-Hamiltonian representations and also apply for nonsquare models. In this chapter, we address both linear dynamic feedback and linear static output feedback design. Unlike most of the strategies reported in the literature, neither observer design nor feedback passivation are necessary for local stabilization. At this point, we restrict the class of systems in consideration to the class of the rational plants with possibly rational Lyapunov functions. Furthermore, the proposed strategy provides an alternative to overcome well-known problems in the field of passivity-based output feedback control. One could mention, for instance, the so-called dissipation obstacle, which is a severe impediment for asymptotic stabilization via interconnection of a dynamic output feedback (DOF) controller, in case of nonzero equilibrium point in closed-loop.

Chapter 7 contains our concluding remarks and suggestions for future research.

2 Mathematical Background

This chapter introduces much of the notation, nomenclature and definitions we will rely on throughout this thesis. Here, we specify the classes of signals and systems we are going to analyse, and the stability conditions we are going to apply for investigating open-loop and closed-loop behaviour.

2.1 Normed Spaces and $\mathcal{L}_p$ Norms

Let us consider a linear space E defined over a field $K \in \mathbb{R}$. A function $\varrho : E \mapsto R^+$ is said to be a *norm* on E if and only if the relations introduced below hold [3]

1. $x \in E$ and $x \neq 0 \Rightarrow \varrho(x) > 0,\ \varrho(0) = 0$,
2. $\varrho(\alpha x) = |\alpha|\varrho(x), \forall \alpha \in K, \forall x \in E$,
3. $\varrho(x+y) \leq \varrho(x) + \varrho(y), \forall x, y \in E$.

In this norm-based framework, a primary assumption throughout this chapter consists in restricting our analysis to the class of the so-called $\mathcal{L}_p$ signals. This is, indeed, a critical issue, as the notion of $\mathcal{L}_p$ norm plays a key role in establishing stability conditions for nonlinear systems described by input-output representations [5].

In the following, the standard definition of a *norm in a linear space* is applied. Consider a function $x : \mathbb{R} \mapsto \mathbb{R}$, whose absolute value is indicated by $|\cdot|$, whereas the signal x disposed in this section is not necessarily to be taken for the state variable considered in the remainder of this work. Then, according to [3], the following norms are of major interest

- $\mathcal{L}_1$ norm, defined as $||x||_1 \triangleq \int |x| dt$,
- $\mathcal{L}_2$ norm: $||x||_2 \triangleq (\int |x|^2 dt)^{\frac{1}{2}}$,
- $\mathcal{L}_p$ norm: $||x||_p \triangleq (\int |x|^p dt)^{\frac{1}{p}}$,

- $\mathcal{L}_\infty$ norm: $\inf\{a|\ |x(t)| < a, a.e.\} = \sup_{t>0}|x(t)|$,

where the integrals are defined on $\mathbb{R}$. We say that a function f belongs to $\mathcal{L}_p$ (or $f \in \mathcal{L}_p$) if and only if

- f is locally Lebesgue integrable, i.e. $\int_a^b |f(t)|dt < +\infty$ for any $a \leq b \in \mathbb{R}$, and
- $||f||_p < +\infty$.

Multivariable systems can be handled by this framework as well, if the following norm for vector functions $f : \mathbb{R} \mapsto \mathbb{R}^n$ is introduced [3]

$$||f||_p \triangleq \left[\sum_{i=1}^{n} ||f_i||_p^2\right]^{\frac{1}{2}},$$

where $f_i \in \mathcal{L}_p$, for each component $1 \leq i \leq n$.

With regard to the definition of an $\mathcal{L}_p$ norm described previously, the following results can be presented, whose proofs can be found in [3].

Proposition 2.1. *If $f \in \mathcal{L}_1 \cap \mathcal{L}_\infty$, then $f \in \mathcal{L}_p$ for all $1 \leq p \leq +\infty$.*

Proposition 2.2. *If $V : \mathbb{R} \mapsto \mathbb{R}$ is a non-decreasing function and if $V(t) \leq M$ for some $M \in \mathbb{R}$ and all $t \in \mathbb{R}$, then $V(\cdot)$ converges.*

Proposition 2.3. *$\dot{f} \in \mathcal{L}_1$ implies that f has a limit.*

Proposition 2.4. *If $f \in \mathcal{L}_2$ and $\dot{f} \in \mathcal{L}_2$, then $f(t) \to 0$ as $t \to +\infty$ and $f \in \mathcal{L}_\infty$.*

Proposition 2.5. *If $f \in \mathcal{L}_1$ and $\dot{f} \in \mathcal{L}_1 \Rightarrow f \to 0$ as $t \to +\infty$.*

From Propositions 2.1-2.5, we notice that the notion of $\mathcal{L}_p$ norm is closely connected with the idea of convergence of trajectories, which justifies the original $\mathcal{L}_p$-based approach of dissipativity as an input-output relation. In [16], for instance, stability conditions for dissipative system could be established relying firmly in $\mathcal{L}_p$ analysis .

Another concept mentioned in the coming sections is the notion of *extended spaces* [3], which is related to the previous norms. Consider then a function $f : \mathbb{R}^+ \mapsto \mathbb{R}$ and let $0 \leq T < +\infty$. Define the truncation

$$f_T(t) = \begin{cases} f(t) \text{ if } t \leq T \\ 0 \quad\ \text{if } t > T, \end{cases} \tag{2.1}$$

and let us introduce the following definitions

- $\mathcal{T}$: subset of $\mathbb{R}^+$,
- $\bar{\mathcal{V}}$: normed space with $||\cdot||$,
- $\mathcal{F} = \{f | f : \mathcal{T} \mapsto \bar{\mathcal{V}}\}$ the set of all functions mapping $\mathcal{T}$ into $\bar{\mathcal{V}}$.

The normed linear subspace $\mathcal{L}$ is given by

$$\mathcal{L} \triangleq \{f : \mathcal{T} \mapsto \bar{\mathcal{V}} \mid ||f|| < +\infty\}. \tag{2.2}$$

Associated with $\mathcal{L}$ is the extended space $\mathcal{L}_e$

$$\mathcal{L}_e \triangleq \{f : \mathcal{T} \mapsto \bar{\mathcal{V}} \mid \forall T \in \mathcal{T}, ||f_T|| < +\infty\}. \tag{2.3}$$

The set $\mathcal{L}_e$ (or $\mathcal{L}_{p,e}$) consists of all Lebesgue functions f such that every truncation of f belongs to the set $\mathcal{L}_p$ itself. Thus, for all $f \in \mathcal{L}_{p,e}$

- The map $t \mapsto ||f_t||$ is monotonically increasing.
- $||f_t|| \mapsto ||f||$ as $t \mapsto +\infty$.

A last alternative definition refers to the notion of $\mathcal{L}_{p,loc}$, which means that

$$(\int_{\mathcal{I}} |f(t)|^p dt)^{\frac{1}{p}} < +\infty, \tag{2.4}$$

for all compact intervals $\mathcal{I} \in \mathbb{R}$. Clearly, $\mathcal{L}_{p,loc} = \mathcal{L}_{p,e}$.

See [3] (Chapter 4) or [31] (Chapter 7) for an in-depth discussion on the application of $\mathcal{L}_p$ norms in the context of input-output stability conditions and dissipativity theory.

2.2 Lipschitz Continuity

Here, we report on a central issue in the context of existence of solutions of differential equations, namely the assumption of Lipschitz continuity. The relevance of this concept is going to be stressed in the next section. At this point, consider the subsequent set of definitions [3].

Definition 2.1. The function $(t,x) \mapsto f(t,x)$ is said to be globally Lipschitz (with respect to x) if there exists a bounded $k \in \mathbb{R}^+$ such that

$$|f(t,x) - f(t,x')| \leq k|x - x'|, \ \forall x, x' \in \mathbb{R}^n, t \in \mathbb{R}^+. \tag{2.5}$$

Definition 2.2. The function $(t,x) \mapsto f(t,x)$ is said to be locally Lipschitz (with respect to x) if (2.5) holds for all $x \in K$, where $K \subset \mathbb{R}^n$ is a compact set. Then k may depend on K.

Definition 2.3. The function $(t,x) \mapsto f(t,x)$ is said to be Lipschitz with respect to time if there exists a bounded k such that

$$|f(t,x) - f(t',x)| \leq k|t - t'|, \ \forall x \in \mathbb{R}^n, \, t, \, t' \in \mathbb{R}^+. \tag{2.6}$$

Definition 2.4. The function $f(\,\cdot\,)$ is uniformly continuous in a set $\mathcal{A}$ if for all $\epsilon > 0$, there exists $\delta(\epsilon) > 0$:

$$|t - t'| < \delta \Rightarrow |f(t) - f(t')| < \epsilon, \forall \, t, \, t' \in \mathcal{A}. \tag{2.7}$$

Any Lipschitz continuous function is uniformly continuous. The converse is not true. Lipschitz continuity is a standard assumption in the field of smooth feedback control and will also be considered so in this work.

2.3 Well-posedness of ODEs

The question of the well-posedness of an ordinary differential equation (ODE) of the form $\dot{x}(t) = f(x(t),t)$ is traditionally approached on the basis of the following results [3].

Theorem 2.1. *([37]) **Carathéodory Conditions.** Let $\dot{x}(t) = f(x(t),t)$ and*

$$\mathcal{I} = \left\{ (x,t) \mid ||x - x_0|| \leq b, \, |t - \tau| \leq a, \, a \in \mathbb{R}^+, \, b \in \mathbb{R}^+ \right\}.$$

In addition, assume that $f : \mathcal{I} \mapsto \mathbb{R}$ satisfies:

- *$f(x,\,\cdot\,)$ is measurable in t for each fixed x,*
- *$f(\,\cdot\,,t)$ is continuous in x for each fixed t,*
- *there exists a Lebesgue integrable function $m(\,\cdot\,)$ on the interval defined by $|t - \tau| \leq a$ such that $|f(x,t)| \leq m(t)$ for all $(x,t) \in \mathcal{I}$.*

Then, for some $\beta > 0$ there exists an absolutely continuous solution for $\dot{x}(t) = f(x(t),t)$ on the interval $|t - \tau| \leq \beta$, satisfying $x(\tau) = x_0$.

From the classic Carathéodory conditions, one deduces that the absolute continuity of a solution $x(t)$ implies the fulfillment of the ODE almost everywhere in the Lebesgue measure [3]. Uniqueness of a particular solution starting at x_0 is guaranteed if $f(\,\cdot\,,\cdot\,)$ satisfies $||f(x,t) - f(y,t)|| \leq \psi(||x - y||,|t - \tau|)$. The matters of existence and uniqueness of solutions of an ODE are key issues in the field of control systems. Consider, then, the following results.

Theorem 2.2. *([3]) **Local Existence and Uniqueness.** Let $f(x,t)$ be continuous in a neighborhood $\mathcal{N}$ of $(x_0,t_0) \in \mathbb{R}^n \times \mathbb{R}$, and be locally Lipschitz with a Lipschitz constant k. Then there exists $\alpha > 0$ such that the ODE $\dot{x}(t) = f(x(t),t)$ possesses in the interval $\mathcal{I} = [t_0 - \alpha, t_0 + \alpha]$ one and only one solution $x : \mathcal{I} \mapsto \mathbb{R}^n$ such that $x(t_0) = x_0$.*

Theorem 2.3. *([3]) **Global Uniqueness.** Let $f(x,t)$ be locally Lipschitz. Let $\mathcal{I} \subset \mathbb{R}$ be an interval ($\mathcal{I}$ may be open, closed, unbounded, compact, etc). If $x_1(t)$ and $x_2(t)$ are two solutions of $\dot{x}(t) = f(x(t),t)$ on $\mathcal{I}$ and if they are equal for some $t_0 \in \mathcal{I}$, then they are equal on the whole $\mathcal{I}$. If in addition $f(x,t)$ is continuous in some domain $\mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}$ and if $(x_0,t_0) \in \mathcal{U}$, then there exists a maximum interval $\mathcal{J}$ ($t_0 \in \mathcal{J}$) in which a solution exists, and this solution is unique.*

Theorem 2.4. *([3]) **Continuous Dependence on Initial Data.** Let $f : \mathcal{W} \mapsto \mathbb{R}^n$, $\mathcal{W} \subseteq \mathbb{R}^n$ an open set, be Lipschitz with constant k. Let $x_1(t)$ and $x_2(t)$ be solutions of $\dot{x}(t) = f(x(t))$ on the interval $[t_0,t_1]$. Then for all $t \in [t_0,t_1]$, one has $||x_1(t) - x_2(t)|| \leq ||x_1(t_0) - x_2(t_0)||exp(k(t-t_0))$.*

2.4 Dynamical System

Throughout this work, we consider causal nonlinear dynamical systems of the form $(\Gamma_{nl}) : u(t) \mapsto y(t)$, where $u(t) \in \mathcal{L}_{p,e}$ is the input and $y(t) \in \mathcal{L}_{p,e}$ is the output, described by an input-affine state-space representation as

$$(\Gamma_{nl}) \begin{cases} \dot{x}(t) = f(x(t)) + g(x(t))u(t) \\ y(t) = h(x(t)) + d(x(t))u(t) \\ x(0) = x_0, \end{cases} \tag{2.8}$$

where $x(t) \in \mathbb{R}^n$ is the state of the system. Besides, $f : \mathbb{R}^n \mapsto \mathbb{R}^n$, $g : \mathbb{R}^n \mapsto \mathbb{R}^{n \times m}$, $h : \mathbb{R}^n \mapsto \mathbb{R}^p$ and $d : \mathbb{R}^n \mapsto \mathbb{R}^{p \times m}$ possess sufficient regularity in such a manner that the system with input in $\mathcal{L}_{2,e}$ is well-posed. Evidently, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$.

In the following, what we call an admissible input u, simply means that the ODE (2.8) possesses a unique differentiable solution. Hence, it is sufficient that the vector field $f(x(t)) + g(x(t))u(t)$ satisfies the Carathèodory conditions of Theorem 2.1, where $u(t)$ may be a Lebesgue measurable function.

As we are also going to investigate linear systems, we specify the following (A,B,C,D) representation for this all-important special case of (Γ_{nl}),

$$(\Gamma_l) \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t), \end{cases} \tag{2.9}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$. Setting a specific representation for (Γ_l) is meaningful due to the fact that the class of the linear systems has a whole body of theoretical tools developed specifically for it, as its practical relevance is unquestionable.

2.5 Lyapunov Stability Theory

Let us consider a dynamical system described by

$$\dot{x} = f(x), \tag{2.10}$$

where $x(t) \in \mathbb{R}^n$, $x(0) = x_0$, is called the state variable. Notice that (2.10) represents a set of first order differential equations and that the existence and uniqueness of its solutions is guaranteed if $f(x)$ is Lipschitz continuous. This condition is firmly established in the literature and, throughout this dissertation, we assume that it is fulfilled. Trivial solutions are provided in case of $f(x^*) \equiv 0$, where x^* is said to be an *equilibrium point* of (2.10).

Equally relevant, though, is the question of analysing the *convergence* properties of a solution $x(t)$ of (2.10) as time evolves, specially in a neighborhood of an equilibrium point. In the area of state-space analysis, the celebrated Lyapunov stability conditions are the major paradigm we have at hand. It allows for investigating the convergence of a solution $x(t)$ without calculating it directly. Consider, without loss of generality, $x^* = 0$ and the following definitions [1].

Definition 2.5. ***Stability of an Equilibrium.*** The equilibrium point $x^* = 0$ of (2.10) is

(i) *stable (in the sense of Lyapunov)* if, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that

$$||x(0)|| < \delta \Rightarrow ||x(t)|| < \epsilon, \forall t \geq 0. \tag{2.11}$$

(ii) *unstable* if it is not stable.

(iii) *asymptotically stable* if it is stable and δ can be chosen such that

$$||x(0)|| < \delta \Rightarrow \lim_{t\to\infty} x(t) = 0. \tag{2.12}$$

In short, Definition 2.5 states that if all solutions starting in a neighborhood of an equilibrium point stay nearby, then this point is be stable, whereas it is otherwise said to be unstable. Moreover, an equilibrium x^* is asymptotically stable if all solutions starting at nearby points tend to it as $t \to \infty$. Notice that asymptotic stability assures not only that state trajectories stay nearby a certain equilibrium point, but it also guarantees convergence to it as time evolves [1]. In this work, stability is always referred to as *stability of an equilibrium point*, and the Lyapunov theory provides the most celebrated manner of approaching this question.

Once the concept of stability has been introduced, the question of establishing conditions for verifying it presents itself. In this regard, the problem of analysing the stability of a point can be formulated in terms of determining a continuously differentiable function whose derivative along the system's trajectory is nonpositive in a neighborhood of this point [1] (see theorem below).

Theorem 2.5. *([1])* ***Stability.*** *Let $x^* = 0$ be an equilibrium point for (2.10) in a domain $\mathcal{X} \subset \mathbb{R}^n$ containing $x^* = 0$. Let $V : \mathcal{X} \mapsto \mathbb{R}$ be a continuously differentiable function such that*

$$V(0) = 0 \text{ and } V(x) > 0 \text{ in } \mathcal{X} - \{0\}, \tag{2.13}$$

and

$$\dot{V}(x) \leq 0 \text{ in } \mathcal{X}. \tag{2.14}$$

Then, $x^* = 0$ is stable. Moreover, if

$$\dot{V}(x) < 0 \text{ in } \mathcal{X} - \{0\}, \tag{2.15}$$

then $x^* = 0$ is asymptotically stable.

Theorem (2.5) plays a central role in the field of control theory, not only because it allows to analyse the stability properties of open-loop systems, but also because the question of designing a stabilizing feedback controller is frequently approached as the task of specifying a Lyapunov function to the closed-loop [1].

Theorem 2.6. *([1])* ***Global Asymptotic Stability.*** *Let $x^* = 0$ be an equilibrium point of (2.10) and let $V : \mathbb{R}^n \mapsto \mathbb{R}$ be a continuously differentiable function such that*

$$V(0) = 0 \text{ and } V(x) > 0 \; \forall x \neq 0, \tag{2.16}$$

$$||x|| \to \infty \Rightarrow V(x) \to \infty, \tag{2.17}$$

$$\dot{V}(x) < 0, \forall x \neq 0, \tag{2.18}$$

then $x^* = 0$ is said to be globally asymptotically stable.

Global stability consists in a best case scenario, where $\mathcal{X} = \mathbb{R}^n$. Nonetheless, as it might be quite challenging to verify it in practice, one usually has to guarantee stability at least in a certain domain broad enough for practical purposes, a *domain of attraction*. In this context, the problem of estimating a domain of attraction constitutes a wide research topic itself and it will not be covered in details in this work. For more discussions on this issue please refer to [64]-[66], [113]-[117].

Finally, a remarkable result which guarantees asymptotic stability even if (2.18) is not fulfilled is presented below, namely the *LaSalle's invariance principle*.

Theorem 2.7. *([1])* ***LaSalle's Invariance Principle.*** *Let $\Omega \subset \mathcal{X}$ be a compact set that is positively invariant with respect to (2.8). Let $V : \mathcal{X} \mapsto \mathbb{R}$ be a continuously differentiable function such that $\dot{V}(x) \leq 0$ in Ω. Let E be the set of all points in Ω where $\dot{V}(x) = 0$. Let M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$.*

This theorem provides a quite different perspective when compared with Theorem 2.5, as $\dot{V}$ is not required to be negative definite. Numerous control strategies apply this theory in order to guarantee asymptotic stability for the interconnection of a controller with a given plant. As a direct consequence of Theorem 2.5, the following conclusions can be established [1].

Corollary 2.1. *Let $x^* = 0$ be an equilibrium point for (2.10). Let $V : \mathcal{X} \mapsto \mathbb{R}$ be a continuously differentiable positive definite function on a domain $\mathcal{X}$ containing the origin, such that $\dot{V}(x) \leq 0$ in $\mathcal{X}$. Let $S = \{x \in \mathcal{X} | \dot{V}(x) = 0\}$ and suppose that no solution can stay identically in S, other than the trivial solution $x(t) \equiv 0$. Then, the origin is asymptotically stable.*

Corollary 2.2. *Let $x^* = 0$ be an equilibrium point for (2.10). Let $V : \mathcal{X} \mapsto \mathbb{R}$ be a continuously differentiable, radially unbounded, positive definite function such that $\dot{V}(x) \leq 0$ for all $x \in \mathbb{R}^n$. Let $S = \{x \in \mathcal{R}^n | \dot{V}(x) = 0\}$ and suppose that no solution can stay identically in S, other than the trivial solution $x(t) \equiv 0$. Then, the origin is globally asymptotically stable.*

3 Literature Review

This chapter contains a comprehensive overview of the work that has been done in the field of passivity/dissipativity analysis and control over the last decades. The definitions of passivity, dissipativity, passivity indices, for instance, are introduced, while the theoretical framework for testing whether a certain system possesses those properties is also presented. Closed-loop stabilization in a dissipativity-based framework is discussed in an in-depth manner, so that the major controller design strategies available in the literature are covered. The dynamical systems considered in the following are those introduced in Chapter 2, namely nonlinear models of the form (Γ_{nl}) as in (2.8), and linear plants (Γ_l) as in (2.9). The concepts and theorems introduced in the forthcoming sections will be referred to throughout this entire dissertation.

3.1 Passivity and Dissipativity Analysis

3.1.1 Dissipative Systems

The notion of dissipativity can be viewed as a phenomenon of loss of energy, which is widely recognized as a fundamental property of any physical system [3]. It was introduced in the field of control theory in [16] and has been the subject of intensive research ever since [20], [23], [26]. Typical examples of dissipative systems are the electrical circuits, in which the supplied energy is partially dissipated as heat in the resistors [3], [4]. From a theoretical perspective, dealing with the notion of dissipativity demands, as a first step, the introduction of two relevant functions:

- the *supply rate* $w(u,y)$, that is the rate at which energy flows into the system,
- the *storage function* $V(x)$, which measures the amount of energy stored inside the system.

Let us the supply rate be such that for all admissible $u(t)$ and $x(0)$, and for all $t \in \mathbb{R}^+$

$$\int_0^t |w(u(\tau),y(\tau))|d\tau < +\infty, \tag{3.1}$$

which means that w is assumed to be Lebesgue integrable independently of the input and the initial conditions. An *admissible input* means that the ODE (2.8) has a unique differentiable solution, for which a sufficient condition is that the vector field $f(x(t)) + g(x(t))u(t)$ satisfies the Carathéodory conditions (see Theorem 2.1), where $u(t)$ may be Lebesgue measurable.

The relation between $w(u,y)$ and $V(x)$ is established by the so-called *dissipation inequality*, which states that along the time trajectories of a dissipative system the increase in the stored energy is not greater than the supply rate. The system cannot store more energy than is supplied to it from the outside [3]. In addition, dissipativity is *invariant* under parallel and negative feedback interconnection and it is closely related to stability, as a dissipative system with a positive definite storage function $V(x)$ has, under mild assumptions, a Lyapunov stable equilibrium at the origin [6], [8], [9].

Definition 3.1. ([3]) ***Dissipative System.*** The system (Γ_{nl}) is said to be *dissipative* if there exists a so-called storage function $V(x) \geq 0$ such that the following dissipation inequality holds

$$V(x(t)) \leq V(x(0)) + \int_0^t w(u(s),y(s))ds, \tag{3.2}$$

along all possible trajectories of (Γ_{nl}) starting at $x(0)$, for all $x(0)$, $t \geq 0$.

Definition 3.1, frequently referred to as Willems's definition, is a very general one, which does not require any regularity assumptions. The existence of a storage function is sufficient for guaranteeing the dissipativity of a given system, whereas storage functions are defined up to an additive constant [22]. If the system is dissipative with respect to supply rates $w_i(u,y)$, $1 \leq i \leq m$, then it is also dissipative with respect to any supply rate of the form $\sum_{i=1}^{m} \alpha_i w_i(u,y)$, where $\alpha_i \geq 0$ for all $1 \leq i \leq m$ [3].

As it is going to be shown later in this section and in the remainder of this thesis, the notion of storage function no longer has to be attached to a physical interpretation of energy. Although this perspective was dominant at the time when this theory was introduced, the concept of storage function has acquired a much more general meaning with the latest developments in the field.

This approach which connects a supply rate defined in terms of an input-output relation with a storage function, which is a function of the state variable, has proven highly successful in the field of control theory. Certainly, it is a major achievement, as it has allowed for a number of applications and created a whole new field of investigation. Nevertheless, the first results reported in the realm of dissipativity theory were based rather on an input-output framework alone, relying critically on $\mathcal{L}_p$ analysis for e stablishing stability results. In this regard, dissipativity can also be defined as in the classical work of Hill and Moylan [52] reported below.

Definition 3.2. ([52]) ***Dissipative System.*** The system (Γ_{nl}) is said to be *dissipative* with respect to the supply rate $w(u,y)$ if for all admissible $u(t)$ and for all $t_1 \geq t_0$ one has

$$\int_{t_0}^{t_1} w(u(s),y(s))ds \geq 0, \tag{3.3}$$

with $x(t_0) = 0$ and along trajectories of (Γ_{nl}).

This corresponds to imposing that storage functions satisfy $V(0) = 0$. This is justified by the fact that storage functions will often, if not always, be used as Lyapunov functions for studying the stability of an equilibrium of (Γ_{nl}) with zero input $u(t)$. In a slightly more general setting, one may assume that the controlled system has an equilibrium x^*, correponding to some input u^*, and with

$$f(x^*) + g(x^*)u^* = 0, \quad y^* = h(x^*) + d(x^*)u^*$$

and

$$w(u^*,y^*) = 0, \quad V(x^*) < +\infty.$$

Then, changing $V(x)$ to $V(x) - V(x^*)$ one obtains $V(x^* = 0)$.

Another class of dissipative systems which is frequently referred to, specially in the realm of control by interconnection, is the notion of cyclo-dissipativity.

Definition 3.3. ([3]) ***Cyclo-dissipative System.*** The system (Γ_{nl}) is said to be *cyclo-dissipative* with respect to the supply rate $w(u,y)$ if

$$\int_{t_0}^{t_1} w(u(s),y(s))ds \geq 0, \tag{3.4}$$

for all $t_1 > t_0$, all admissible $u(t)$ whenever $x(t_0) = x(t_1) = 0$.

The difference with regard to Definition 3.2 is that the state boundary conditions are forced to be the equilibrium of the free system: trajectories that start and end at $x = 0$. A cyclo-dissipative system abosorbs energy for any cyclic motion passing through the origin. Cyclo-dissipativity and dissipativity are related as follows, where a proof of the subsequent theorem can be found in [55].

Theorem 3.1. *Suppose that the system* (Γ_{nl}) *defines a causal input-output operator* $H_{x(0)}$, *and that the supply rate is of the form*

$$w(u,y) = y^\top Qy + 2y^\top Su + u^\top Ru,$$

with Q and R symmetric. Suppose further that the system is zero state detectable. Then, dissipativity in the sense of Definition 3.2 and the cyclo-dissipativity of (Γ_{nl}) *are equivalent properties.*

3.1.2 Storage Functions

Having in mind this preliminary material, the next natural question is:

$$\textit{Given a system, how can we find V(x)?} \tag{3.5}$$

This question is closely related to the problem of finding a suitable Lyapunov function, in the Lyapunov second method. As it going to be stated next, a storage function can be found by computing the maximum amount of energy that can be extracted from the system.

Definition 3.4. ([3]) ***Available Storage.*** The *available storage* $V_a(\,\cdot\,)$ of the system (Γ_{nl}) is given by

$$0 \le V_a(x) = \sup_{x=x(0),\ u(t),\ t\ge 0} -\left\{ \int_0^t w(u(s),y(s))ds \right\}, \tag{3.6}$$

where $V_a(x)$ is the maximum amount of energy which can be extracted from the system with initial state $x = x(0)$.

The supremum taken over all admissible $u(t)$, all $t \ge 0$, all signals with initial value $x(0) = x$, and the terminal boundary conditions $x(t)$ is left free. It is clear that $V_a(x) \ge 0$ (take $t = 0$ to notice that the supremum cannot be negative). When the final state is not free but constrained to $x(t) = 0$ (the equilibrium of the uncontrolled system), then one speaks of the *virtual* available storage $V_a^*(\,\cdot\,)$ [55].

Another function that plays an important role in the framework of dissipative systems, called the *required supply*. We recall that the state of a system is said reachable from the state x^* if, given any x and t, there exist a time $t_0 \leq t$ and an admissible controller $u(t)$ such that the state can be driven from $x(t_0) = x^*$ to $x(t) = x$. The state-space $\mathcal{X}$ is connected provided every state is reachable from every other state.

Definition 3.5. ([3]) ***Required Supply.*** The *required supply* $V_r(\,\cdot\,)$ of the system (Γ_{nl}) is given by

$$V_r(x) = \inf_{u(\,\cdot\,),\ t\geq 0} \left\{ \int_{-t}^{0} w(u(s),y(s))ds \right\}, \tag{3.7}$$

with $x(-t) = x^*$, $x(0) = x$, and it is assumed that the system is reachable from x^*. The function $V_r(x)$ is the required amount of energy to be injected in the system to go from $x(-t)$ to $x(0)$.

The infimum is taken over all trajectories starting from x^* at $-t$ and ending at $x(0) = x$ at time 0, and all $t \geq 0$ (or said differently, over all admissible controllers $u(t)$ which drive the system from x^* to x on the interval $[-t,0]$). If the system is not reachable from x^*, one may define $V_r(x) = +\infty$. Contrary to the available storage, the required supply is not necessarily positive. When the system is reversible, the required supply and the available storage coincide [24]. It is interesting to define accurately what is meant by *reversibility* of a dynamical system. This is done thanks to the definition of a third energy function, the *cycle energy*.

Definition 3.6. ([3]) ***Cycle Energy.*** The *cycle energy* $V_c(\,\cdot\,)$ of the system (Γ_{nl}) is given by

$$V_c(x) = \inf_{u(t),\ t_0\leq t_1,\ x(t_0)=0} \int_{t_0}^{t_1} u(t)^\top y(t)dt, \tag{3.8}$$

where the infimum is taken over all admissible $u(t)$ which drive the system from $x(t_0) = 0$ to x.

The cycle energy is, thus, the minimum energy it takes to cycle a system between the equilibrium $x = 0$ and a given state. One has

$$V_a(x) + V_c(x) = V_r(x), \tag{3.9}$$

assuming that the system is reachable so that the required supply is not identically $+\infty$. Then, the following holds.

Definition 3.7. Let a dynamical system be passive, i.e. $w = u^\top y$, and let its state-space representation be reachable. The system is irreversible if $V_c(x) = 0$ only if $x = 0$. It is said uniformly irreversible if there exists a class $\mathcal{K}_\infty$ funcion $\alpha(x)$ such that for all $x \in \mathbb{R}^n : V_c(x) \geq \alpha(||x||)$. the system is said reversible if $V_c(x) = 0$ for all $x \in \mathbb{R}^n$, i.e. if $V_a(x) = V_r(x)$.

Theorem 3.2. *([22],[23]) The available storage $V_a(x)$ in (3.6) is finite for all $x \in \mathcal{X}$ if and only if (Γ_{nl}) in (2.8) is dissipative in the sense of Definition 3.1. Moreover, $0 \leq V_a(x) \leq V(x)$ for all $x \in \mathcal{X}$ for dissipative systems and V_a is itself a possible storage function.*

Therefore, dissipativity can be tested by attempting to compute $V_a(x)$: if it is locally bounded, it is a storage function and the system is dissipative with respect to the supply rate $w(u,y)$. The idea of determining a storage function for a certain system, particularly a positive definite function, is a central paradigm in this thesis and is going to be approached in a variety of manners.

Proposition 3.1. *([3]) Consider the system (Γ_{nl}) in (2.8). Assume that it is zero state observable ($u(t) = 0$ and $y(t) = 0$ for all $t \geq 0$ imply that $x(t) = 0$ for all $t \geq 0$), with a reachable state-space $\mathcal{X}$, and that it is dissipative with respect to $w(u,y) = u^\top y$. Let $d(x) + d^\top(x)$ has full rank for all $x \in \mathcal{X}$. Then, $V_a(x)$ and $V_r(x)$ are solutions of the PDE:*

$$\begin{aligned}&\nabla V^\top(x) f(x) + \\ &(h(x) - \frac{1}{2} g^\top(x) \nabla V(x))^\top (d(x) + d^\top(x))^{-1} (h(x) - \frac{1}{2} g^\top(x) \nabla V(x)) = 0.\end{aligned}$$

In the LTI case, and provided the system is observable and controllable, then $V_a(x) = x^\top P_a x$ and $V_r(x) = x \top P_r x$ satisfy the above PDE, which means that P_a and P_r are extremal solutions of the Riccati equation $A^\top P + PA + (PB - C^\top)(D + D^\top)^{-1}(B^\top P - C) = 0$. In the realm of the linear plants, the problem of determining a storage function which could be used as a Lyapunov function is efficiently solved via LMIs, in a variety of scenarios. In the case of the nonlinear plants, however, it is a much more complex challenge to succeed in it.

After having introduced the notion of dissipativity and storage function, we now focus on a particular though remarkably relevant special case of dissipative systems, namely the passive systems.

Definition 3.8. ([3]) ***Passive System.*** Suppose that the system (Γ_{nl}) in (2.8) is dissipative with supply rate $w(u,y) = u^\top y$ and storage function $V(\cdot)$ with $V(0) = 0$, i.e for all $t \geq 0$:

$$V(x(t)) \leq V(x(0)) + \int_0^t u^\top(s)y(s)ds. \tag{3.10}$$

Then the system is called *passive*.

Definition 3.9. ([3]) ***Strictly Passive System.*** The system (Γ_{nl}) in (2.8) is said to be *strictly passive* if it is dissipative with supply rate $w(u,y) = u^\top y$ and storage function $V(x)$ with $V(0) = 0$, and there exists a positive definite function $S(x)$ such that for all $t \geq 0$:

$$V(x(t)) \leq V(x(0)) + \int_0^t u^\top(s)y(s)ds - \int_0^t S(x(t))dt. \tag{3.11}$$

if the equality holds in the above and $S(x(t)) \equiv 0$, then the system is said to be lossless.

Some authors [7] also introduce a notion of *weak strict passivity* that is more general than the strict passivity: the function $S(x)$ is replaced by a dissipative function $D(x,u) \geq 0$, $D(0,0) = 0$. The notion of weak strict passivity is meant to generalize WSPR functions to nonlinear systems.

Theorem 3.3. *([22]) Suppose that the system (Γ_{nl}) in (2.8) is lossless with a minimum value at $x = x^*$ such that $V(x^*) = 0$. If the state-space is reachable from x^* and controllable to x^*, then $V_a(x) = V_r(x)$ and thus the storage function is unique and given by $V(x) = \int_{t_1}^0 w(u(t),y(t))dt$ with any $t_1 \leq 0$ and $u \in \mathcal{U}$ such that the state trajectory starting at x^* at t_1 is driven by $u(t)$ to $x = 0$ at $t = 0$. Equivalently $V(x) = -\int_0^{t_1} w(u(t),y(t))dt$ with any $t_1 \geq 0$ and $u \in \mathcal{U}$ such that the state trajectory starting at x at $t = 0$ is driven by $u(t)$ to x^* at t_1.*

Remark 3.1. If the system (Γ_{nl}) in (2.8) is dissipative with supply rate $w = u^\top y$, i.e. passive, and the storage function satisfies $V(0) = 0$ with $V(x)$ positive definite, then the system and its zero dynamics are Lyapunov stable. This can be seen from the dissipativity inequality (3.2) by taking u or y equal to zero.

This remark contains an outstanding result, as it connects the concept of passivity with the traditional Lyapunov stability. This statement is, indeed, a cornerstone of modern control technology and is particularly useful in the context of feedback stabilization of nonlinear plants, where a major strategy consists in determining a control law which renders the closed-loop passive with a positive definite storage, thus, stable.

In [52], a general supply rate identical to that of Theorem 3.1 was introduced, and it has proven very suitable to distinguish between different types of dissipative as well as strictly passive systems.

Definition 3.10. ([3]) ***General Supply Rate.*** Let us consider a dissipative system (Γ_{nl}) with supply rate

$$w(u,y) = y^\top Q y + 2y^\top S u + u^\top R u, \tag{3.12}$$

with $Q = Q^\top$, $R = R^\top$. If $Q = 0$, $R = -\epsilon I_m$, $\epsilon > 0$, $S = \frac{1}{2} I_m$, the system is said to be *input strictly passive* (ISP), i.e.

$$\int_0^t y^\top(s)u(s)ds \geq \beta + \epsilon \int_0^t u^\top(s)u(s)ds. \tag{3.13}$$

If $R = 0$, $Q = -\delta I_m$, $\delta > 0$, $S = \frac{1}{2} I_m$, the system is said to be *output strictly passive* (OSP), i.e.

$$\int_0^t y^\top(s)u(s)ds \geq \beta + \delta \int_0^t y^\top(s)y(s)ds. \tag{3.14}$$

If $Q = -\delta I_m$, $\delta > 0$, $R = -\epsilon I_m$, $\epsilon > 0$, $S = \frac{1}{2} I_m$, the system is said to be *very strictly passive* (VSP), i.e.

$$\int_0^t y^\top(s)u(s)ds \geq \beta + \delta \int_0^t y^\top(s)y(s)ds + \epsilon \int_0^t u^\top(s)u(s)ds. \tag{3.15}$$

Notice that Definitions 3.9 and 3.11 do not imply in general the *asymptotic stability* of the considered system. Then, at this point, it is important to discuss the circumstances under which dissipativity with (3.12) implies asymptotic convergence. A well-established approach relies on the relationship between the finite-gain property of an operator and dissipativity. Assume that (Γ_{nl}) is dissipative with respect to a general supply rate, i.e.

$$V(x(t)) - V(x(0)) \leq \int_0^t [y^\top(s)Qy(s)) + 2y^\top(s)Su(s) + u^\top(s)Ru(s)]ds,$$

for some storage function $V(x)$. Let $S = 0$. Then it follows that

$$-\int_0^t y^\top(s)Qy(s)ds \leq \int_0^t u^\top(s)Ru(s)ds + V(x(0)). \tag{3.16}$$

Let $Q = -\delta I_m$ and $R = \epsilon I_m$, $\delta > 0, \epsilon > 0$. Then we get

$$\int_0^t y^\top(s)y(s)ds \leq \frac{\epsilon}{\delta}\int_0^t u^\top(s)u(s)ds + V(x(0)), \tag{3.17}$$

so that the operator $u \mapsto y$ has a finite $\mathcal{L}_2$-gain with a bias equal to $V(x(0))$. Dissipativity with supply rates $w(u,y) = -\delta y^\top y + \epsilon u^\top u$ will be commonly met and it is sometimes called the H_∞-behaviour supply rate of the system. Therefore, dissipativity with $Q = -\delta I_m$ and $R = \epsilon I_m$ and $S = 0$ implies finite-gain stability. What about the converse? The following is true [54]:

Theorem 3.4. *The system is dissipative with respect to the general supply rate in (3.12) with zero bias ($\beta = 0$) and with $Q < 0$, if and only if it is finite-gain stable.*

A dynamical system may be dissipative with respect to several supply rates, and with different storage functions corresponding to those supply rates. Let us make an aside on LTI systems. We consider a general supply rate with $Q \leq 0$ and $\bar{R} = R + SD + D^\top S + D^\top QD > 0$. We denote $\bar{S} = S + D^\top Q$. Then

Theorem 3.5. *([102]) Consider the system (A,B,C,D) with A asymptotically stable. Suppose that*

$$-\int_0^t w(u(s),y(s))ds \leq -\frac{\epsilon}{2}\int_0^t u^\top(s)u(s)ds + \beta(x_0), \tag{3.18}$$

where $\beta(\cdot) \geq 0$ and $\beta(0)$. Then

- *There exists a solution $P \geq 0$ to the algebraic Riccati equation (ARE)*

$$A^\top P + PA + (PB - C^\top \bar{S}^\top)\bar{R}^\top(B^\top P - \bar{S}C) - C^\top QC = 0, \tag{3.19}$$

such that $A^* = A + B\bar{R}^{-1}(B^\top P - \bar{S}C)$ is asymptotically stable, and

- *There exists a solution $\bar{P} > 0$ to the algebraic Riccati inequality (ARI)*

$$A^\top \bar{P} + \bar{P}A + (\bar{P}B - C^\top \bar{S}^\top)\bar{R}^\top(B^\top \bar{P} - \bar{S}C) - C^\top QC < 0. \quad (3.20)$$

Conversely, suppose that there exists a solution $P \geq 0$ to the ARE (3.19) such that the matrix $A^* = A + B\bar{R}^{-1}(B^\top P - \bar{S}C)$ is asymptotically stable. Then the matrix A is asymptotically stable and the system (A,B,C,D) satisfies (3.18) with the above supply rate.

This theorem establishes a condition for asymptotic stability of free systems. Later in this work, we are going to approach the problem of how to use the dissipativity properties of a system, in terms of the matrices (Q,S,R), in order to guarantee closed-loop stabilizability by feedback control. In that case, the matrix Q, for example, cannot be assumed to be negative definite, and we have to investigate alternative conditions for stabilization.

With regards to nonlinear systems, there exist certain algebraic conditions for dissipativity as well. In order to approach those conditions, though, we have to consider the following assumptions firstly.

Assumption 3.1. The state-space of system (2.8) is reachable from the origin. More precisely, given any x_1 and t_1, there exists $t_0 \leq t_1$ and an admissible control $u(t)$ such that the state can be driven from $x(t_0) = 0$ to $x(t_1) = x_1$.

Assumption 3.2. The available storage $V_a(x)$, when it exists, is a differentiable function of x.

These two assumptions are supposed to hold when it comes to the following definition.

Definition 3.11. ([3]) ***QSR-dissipativity.*** A system is called QSR-dissipative if it is dissipative with the following supply rate

$$w(u,y) = y^\top Qy + 2y^\top Su + u^\top Ru = \begin{bmatrix} y^\top & u^\top \end{bmatrix} \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}, \quad (3.21)$$

where $Q = Q^\top$ and $R = R^\top$.

Theorem 3.6. *([3])* ***Nonlinear KYP Lemma.*** *The nonlinear system (2.8) is dissipative in the sense of Definition 3.2 with respect to the supply rate $w(u,y)$ in (3.20) if and only if there exists functions $V : \mathbb{R}^n \mapsto \mathbb{R}$, $L : \mathbb{R}^n \mapsto \mathbb{R}^q$,*

$W : \mathbb{R}^n \mapsto \mathbb{R}^{q\times m}$ (for some integer q), with $V(x)$ differentiable, such that:

$$\begin{aligned} V(x) &\geq 0, \\ V(0) &= 0, \\ \nabla V^\top(x) f(x) &= h^\top(x) Q h(x) - L^\top(x) L(x), \\ \frac{1}{2} g^\top(x) \nabla V(x) &= \hat{S}^\top(x) h(x) - W^\top(x) L(x), \\ \hat{R}(x) &= W^\top(x) W(x), \end{aligned} \tag{3.22}$$

where

$$\begin{cases} \hat{S}(x) \triangleq Qd(x) + S, \\ \hat{R}(x) \triangleq R + d^\top(x) S + S^\top d(x) + d^\top(x) Q d(x). \end{cases} \tag{3.23}$$

If cyclo-dissipativity is used instead of dissipativity, then the first two conditions on the storage function $V(x)$ can be replaced by the single condition $V(0) = 0$ [55]. A linear version of this lemma will be presented in Chapter 4 as a necessary and sufficient condition for passivity and positive realness of controllable and observable LTI plants.

Corollary 3.1. *([52]) If the system (2.8) is dissipative with respect to the supply rate in (3.21), then there exists $V(x) \geq 0$, $V(0) = 0$ and some $L : \mathcal{X} \mapsto \mathbb{R}^q$, $W : \mathcal{X} \mapsto \mathbb{R}^{q\times m}$ such that*

$$\frac{d(V \circ x)}{dt} = -[L(x) + W(x)u]^\top [L(x) + W(x)u] + w(u,y). \tag{3.24}$$

Notice that the notion of strict passivity can be introduced as a special case of Theorem 3.6 for

$$Q = 0,\ R = 0,\ S = \frac{1}{2},\ d = 0, \tag{3.25}$$

where (3.22) is reduced to

$$\begin{cases} \nabla V^\top(x) f(x) = -L^\top(x) L(x) = -S(x), \\ g^\top(x) \nabla V(x) = h(x). \end{cases} \tag{3.26}$$

3.1.3 Passivity Indices

An important special case of QSR-dissipativity is the notion of passivity indices [48], which can be interpreted as an estimate of the level of passivity of a system.

Definition 3.12. ([48]) ***Passivity Indices.*** The system (Γ_{nl}) in (2.8) has input feed-forward passivity index (IFP) $\nu \in \mathbb{R}$ and output feedback passivity index (OFP) $\rho \in \mathbb{R}$ if the following holds

$$\dot{V}(x(t)) \leq u^{\top}(t)y(t) - \nu u^{\top}(t)u(t) - \rho y^{\top}(t)y(t), \tag{3.27}$$

$\forall t \geq 0$, for some function $V : \mathbb{R}^n \mapsto \mathbb{R}$, $V \geq 0$.

If both $\rho > 0$ and $\nu > 0$, then the system has an excess of passivity. If either $\rho < 0$ or $\nu < 0$, then (Γ_{nl}) lacks passivity. If $\nu = \rho = 0$, the system is simply passive, which means that it is stable (if $V > 0$) and minimum-phase. The OFP index is a measure of the level of stability of a system and the IFP index is a measure of the extent that the minimum phase property is present [48].

Before generalizing our results to the case of QSR-dissipative systems, we have approached the problem of stabilization of nonlinear systems mainly in the framework of passivity indices [125]-[128]. We proved that, under certain circumstances, the indices can be used for determining a feedback through a closed-form expression which allows even for establishing an interval for such a stabilizing gain. In general, there is a rich literature in the field of controller design and systems analysis using the concept of passivity indices, embracing a variety of relevant applications [48], [109], [19], [49].

3.1.4 Stability of Interconnected Dissipative Systems

Throughout this chapter we have considered stability and dissipativity of individual systems. Another question of pivotal importance, though, concerns the stability analysis of interconnections of dissipative systems. An immediate application of such a framework lies in the domain of controller design technology, since a stabilizing controller can be interpreted as another dynamical system connected with the plant in an appropriate manner.

In [52], a general framework to compute storage functions for nonlinear input-affine systems was introduced. In [53], the stability properties of interconnected dissipative systems was investigated in terms of the (Q,S,R) matrices of both systems and their relationships. In the following, we present this important result which is crucial to our work, specially with regards to the contributions which we are going to provide in Chapter 6.

Consider, firstly, two subsystems H_1 and H_2 described by the following state-space representations based on (2.8)

$$H_i : \begin{cases} \dot{x}_i = f_i(x_i) + g_i(x_i)u_i \\ y_i = h_i(x_i) + d_i(x_i)u_i, \end{cases} \tag{3.28}$$

for $i = 1,2$. We also assume that the feedback system is well-defined, i.e.

$$I + d_2(x_2)d_1(x_1) \text{ is nonsingular } \forall\ (x_1,x_2). \tag{3.29}$$

This guarantees that an extended state $[x_1\ x_2]^\top$ appropriately describes the dynamics of the interconnected system, based on an augmented ODE whose existence and uniqueness of solutions is assured.

Theorem 3.7. *([53]) Suppose that the two subsystems H_1 and H_2 are dissipative with respect to the supply rates*

$$w_i(u_i,y_i) = y_i^\top Q_i y_i + 2y_i^\top S_i u_i + u_i^\top R_i u_i,\ \ i = 1,2.$$

Then the feedback interconnected system is stable (asymptotically stable) if the matrix

$$\hat{Q} = \begin{bmatrix} Q_1 + \alpha R_2 & -S_1 + \alpha S_2^\top \\ -S_1^\top + \alpha S_2 & R_1 + \alpha Q_2 \end{bmatrix}, \tag{3.30}$$

is negative semidefinite (negative definite) for some $0 < \alpha \in \mathbb{R}$.

Theorem 3.7 provides a powerful framework for stabilization purposes, as one of its direct applications implies that a dissipative controller with matrices (Q_2,S_2,R_2) can be designed for compensating for the dissipativity properties of a plant with (Q_1,S_1,R_1), in order to achieve a stable closed-loop interconnection. Furtermore, by strengthening the observability requirements, the conditions on $\hat{Q}$ can be weakened, as argued below.

Theorem 3.8. *([53]) With the same assumptions as in Theorem 3.7, suppose that $\hat{Q} \leq 0$ and $S_1 = \alpha S_2^\top$. Then the feedback system is asymptotically stable if either*

- *The matrix $(Q_1 + \alpha R_2)$ is nonsingular and the composite system H_1 $(-H_2)$ is zero-state detectable, or*
- *The matrix $(R_1 + \alpha Q_2)$ is nonsingular and the composite system $H_2 H_1$ is zero-state detectable.*

Theorems 3.7 and 3.8 provide stability criteria in terms of general quadratic supply rates. These theorems are of key importance for this work, as we rely decisively on them for controller design. Moreover, as previously discussed in Section 3.1.2, such a general supply rate allows for specifying some subclasses of dissipative systems, which includes the class of the passive systems, as well as ISP, OSP and VSP systems. The illustrative table below summarizes particular cases of QSR-dissipativity that are of major importance in the literature. Consider (ϵ,ϵ_1,ϵ_2) as positive constants defined in $\mathbb{R}^+$. See [53] for a detailed discussion.

Table 3.1: Special supply rates

Supply rate	Type of dissipativiity
$u^\top y$	passive
$u^\top y - \epsilon u^\top u$	input strictly passive (ISP)
$u^\top y - \epsilon y^\top y$	output strictly passive (OSP)
$u^\top y - \epsilon_1 u^\top u - \epsilon_2 y^\top y$	very strictly passive (VSP)

According to the type of dissipativity in consideration, a variety of assertions can be formulated with regard to the interconnection of H_1 and H_2.

Corollary 3.2. *([53]) If both H_1 and H_2 are passive, then the feedback system is stable. Asymptotic stability follows if, in addition, any one of the following (nonequivalent) conditions is satisfied:*

1. *One of H_1 and H_2 is VSP.*
2. *Both H_1 and H_2 are ISP.*
3. *Both H_1 and H_2 are OSP.*
4. *$H_1(-H_2)$ is zero-state detectable, and either*
   - *H_2 is ISP or,*
   - *H_1 is OSP.*
5. *H_2 H_1 is zero-state detectable, and either*
   - *H_2 is OSP,*
   - *H_1 is ISP.*

This corollary contains valuable information as to how different types of feedback controllers can be selected in order to guarantee asymptotic stability when connected with a plant whose dissipativity properties are known or have been, at least, estimated.

3.1.5 Locally Dissipative Interconnected Systems

In the previous sections, a variety of conditions for a system to be dissipative have been introduced. These conditions were considered to be globally valid, i.e. in the whole domain composed of the state and input variables (x,u). In practical applications, however, it may not be possible to ensure global QSR-dissipativity with regards to a certain quadratic supply rate, while local dissipativity, though, may be achievable. If this property is fulfilled in a domain broad enough for the specific application, then certain stabilization problems can be eventually solved through such a local analysis. In [38], the property of local dissipativity was introduced and conditions for the stability of the interconnection of locally dissipative nonlinear systems was presented. In our work, we rely heavily on this results, also presented below.

Firstly, let U be defined as an *inner product* space whose elements are functions $(\,\cdot\,) : \mathbb{R} \mapsto \mathbb{R}$. Furthermore, let U^n be the space of n-tuples over U with inner product given by

$$\langle u,v\rangle = \sum_{i=1}^{n} \langle u_i,v_i\rangle.$$

Then, as introduced in Chapter 2, for any $u \in U^n$ and any $T \in \mathbb{R}$, a truncation u_T can be defined via

$$u_T(t) = \begin{cases} u(t) \text{ for } t < T \\ 0 \quad \text{otherwise.} \end{cases}$$

In addition, *truncated inner product* is defined by

$$\langle u,v\rangle_T = \langle u_T,v_T\rangle.$$

Finally, let us specify an extended space $U_e^n = \{u \mid u_T \in U^n \text{ for all } T \in \mathbb{R}\}$, as discussed in Section 2.1. A system with m inputs and p outputs may now be formally defined as a relation on $U_e^m \times U_e^p$, that is, a pair $(u \in U_e^m,\, y \in U_e^p)$. We assume that there exists a state-space $\mathcal{X}$ for the dynamical system.

Definition 3.13. ([38]) ***Locally Dissipative System.*** A dynamical system is called locally QSR-dissipative in a region $\Omega \subset \mathcal{X}$ if

$$\int_0^T w(u,y)dt = \langle y,Qy\rangle_T + 2\langle y,Su\rangle_T + \langle u,Ru\rangle_T \geq 0,$$

$\forall u \in U_e$ and $T \in \mathbb{R}^+$, such that $x(0) = 0$ and $x(t) \in \Omega$ for $0 \leq t \leq T$.

Next, consider the following conditions on a system and its local properties such that it is internally stable. Such properties will be applied below for establishing stability conditions for interconnections of dissipative systems.

Theorem 3.9. *([38]) Let the dynamical system*

(i) *Be locally QSR-dissipative in a region* $\Omega \subseteq \mathcal{X}$.

(ii) *Be locally connected in a region* Ω_c *with respect to* Ω.

(iii) *Be locally uniformly controllable in a region* Ω_{uc} *with respect to* Ω.

(iv) *Be locally uniformly zero state detectable in a region* Ω_z *with respect to* Ω.

(v) *Be locally Lipschitz continuous in the region* Ω.

(vi) *There exists a well-defined feedback law* $u^*(\,\cdot\,)$ *such that* $w(u^*(\,\cdot\,),y) < 0$, $\forall y \neq 0$, $u^*(0) = 0$.

Furthermore, suppose that the region $\Omega_s \triangleq \Omega_c \cap \Omega_{uc} \cap \Omega_z$ *is nonempty, in the sense that it contains an open neighborhood of the origin. Then, if* $Q < 0$, *the origin is asymptotically stable.*

The Lipschitz continuity arguments employed in Theorem 3.9 are necessary in order to ensure that the system's trajectory, when started in a given subset of Ω, remains inside the domain Ω for a finite time, irrespective of the system stability [38]. The next theorem extends the content of Theorem 3.9 to the case of a linear interconnection of N locally dissipative subsystems, where this interconnection is described by

$$u_i = u_{ei} - \sum_{j=1}^{N} H_{ij} y_j, \quad i = 1, \cdots, n, \tag{3.31}$$

u_i is input to subsystem i, y_i is the output, u_{ei} is an external input, and H_{ij} are constant matrices. A compact matrix notation is, with obvious definitions

$$U = U_e - Hy. \tag{3.32}$$

Theorem 3.10. *([38]) Let the dynamical system by formed by interconnecting N subsystems via the interconnection (3.31) and suppose that:*

(i) *The* i^{th} *subsystem is locally QSR-dissipative with* (Q_i, S_i, R_i) *in a region* Ω_i *satisfying conditions* (i)-(vi) *of Theorem 3.9.*

(ii) The interconnection (3.31) is such that the dynamic state-space $\hat{\mathscr{X}}$ equal the Cartesian product of the state-space of individual subsystems and $\Omega = \Omega_1 \times \cdots \times \Omega_N$ is a nonempty region containing a neighborhood of the origin ($\Omega \subseteq \hat{\mathscr{X}}$).

(iii) The overall system is uniformly zero-state detectable in a region Ω_z with respect to Ω, where Ω_z is a nonempty region containing a neighborhood of the origin ($\Omega \subseteq \hat{\mathscr{X}}$).

(iv) The overall system is locally Lipschitz continuous in Ω.

If $\hat{Q} > 0$, then the origin is asymptotically stable, where

$$\hat{Q} = SH + H^\top S^\top - H^\top RH - Q, \tag{3.33}$$

$$Q = \{Q_1, \cdots, Q_N\}, \; S = \{S_1, \cdots, S_N\}, \; R = \{R_1, \cdots, R_N\}. \tag{3.34}$$

Theorem 3.10 provides a powerful framework for local dissipativity and stability analysis of interconnected nonlinear systems. The idea of estimating the parameters (Q_i,S_i,R_i) for each individual subsystem becomes, then, a key problem to which a number of strategies can be applied. For LTI systems, the problem can be frequently solved via LMIs. In the domain of nonlinear models, the question of determining those matrices and, at the same time the respective storage functions, is more involved. Polynomial systems, for example, can be investigated through SOS techniques [109], whereas rational models can be dealt with via polytopic LMI strategies [128], as we are going to show later in this work.

3.2 Dissipative Control Synthesis

Among the classical references in the area of dissipative control systems synthesis one includes the contribution [102], which summarizes the results of [103] and [104], and introduces new material as well. In those references, the problem of expressing strict dissipativity as a necessary and sufficient condition for (asymptotic) stabilizability of a system is addressed. In [102], for instance, the following class of systems is considered

$$(\Gamma)\begin{cases} \dot{x}(t) = f(x(t),u(t),\bar{d}(t)) \\ z(t) = h(x(t),u(t),\bar{d}(t)) \\ x(0) = x_0, \end{cases} \tag{3.35}$$

where $z \in \mathbb{R}^q$ is the output to be controlled and $\bar{d} \in \mathbb{R}^d$ is a disturbance. In addition, a general supply rate of the form

$$r_q(z,\bar{d}) = \frac{1}{2}\left(z^\top Q z + 2z^\top S\bar{d} + \bar{d}^\top R\bar{d}\right) \tag{3.36}$$

is used, where the additional assumption that $Q \leq 0$ implies

$$r_q(z,0) \leq 0. \tag{3.37}$$

As a result, the open-loop system is stable in the sense of Lyapunov, if it is dissipative with a positive definite storage function. In this context, the *dissipative control problem* can be formulated as follows

Find $u = K(x)$ such that the closed-loop system is dissipative with respect to the supply rate $r_q(z,\bar{d})$.

In [102]-[104], the analysis is restricted to the class of the open-loop stable models. Moreover, the notion of strict dissipativity employed in order to achieve asymptotic stability was the one introduced in relation (3.18) of Theorem 3.5, where a function of the input signal is added to the right-hand side of that relation. The main result with regards to that topic is presented below in terms of solutions of a PDI.

Theorem 3.11. *([102]) Consider the system (Γ) in (3.35) and the supply rate $r_q(z,\bar{d})$ satisfying (3.37). Assume that there exists $u(x) = k(x)$ such that the closed-loop system is dissipative with respect to the supply rate $r_q(z,\bar{d})$. Then there exists a solution V to the following PDI.*

$$\inf_{u\in\mathbb{R}^m} \sup_{\bar{d}\in\mathbb{R}^d} \left\{\nabla_x V(x)\cdot f(x,u,\bar{d}) - r_q(z,\bar{d})\right\} \leq 0, \tag{3.38}$$

such that $V \geq 0$ and $V(0) = 0$. Conversely, assume that there exists a smooth function V solving the PDI (3.38) in the classical sense such that $V \geq 0$ and $V(0) = 0$. Suppose that the control law $u(x)$ attains the infimum on the left-hand side of (3.38). Then the closed-loop is dissipative with respect to the supply rate $r_q(z,\bar{d})$.

The closed-loop is guaranteed to be stable if V is positive definite, so that it qualifies as a Lyapunov function. Zero-state detectability is the property usually associated with positive definiteness of storage functions. In [102], another theorem regarding asymptotic stability is presented, based in a notion of strict dissipativity similar to (3.18). In the present work, though, we adopt the following definition of strict QSR-dissipativity

$$\dot{V} + N(x) \leq w(u,y), \tag{3.39}$$

where N is a positive definite function in the domain $\mathcal{X}$. This definition is similar to that of a strict passive system, approached in Definition 3.9. Furthermore, we do not consider the presence of a disturbance $\bar{d}$ or an extra output z, as we address the problem of stabilization of systems of the form (Γ_{nl}) introduced in Chapter 2. Our main results on this topic are provided in Chapter 6, where we adapt this dissipative control synthesis framework also for the general problem of linear static and dynamic output feedback stabilization.

3.3 Passivity-based Control

3.3.1 Standard PBC

In this section, we discuss an idea that has been central in the domain of nonlinear control theory, namely the strategy of rendering a certain system passive in closed-loop by the application of a suitable control law, combined with the specification of a positive definite storage function. This approach is denominated *Standard PBC* and, as classical references in this field, we emphasize [17] and [51], which addressed the problem of smooth state feedback for general nonlinear *square* systems.

In [51], a sufficient condition for state feedback asymptotic stabilization of nonaffine systems was presented, based on the assumption that the open-loop is locally stable in the sense of Lyapunov and that a Lyapunov function exists. In [17], it was proven that a system can be rendered passive if and only if it is weakly minimum-phase and has relative degree 1. The authors also established the fact that a passive system can be globally asymptotically stabilized by pure gain feedback if it is detectable. Subsequently, a number of passivity-based strategies have been proposed, allowing for numerous applications of passivity, which is a special case of dissipativity, for feedback stabilization [13].

A control action $u = \hat{u}(x) + v$ solves the *Standard PBC Problem* if the closed-loop system satisfies the desired power-balance equation

$$\dot{V}_d(x) = v^\top z - d_d(x), \tag{3.40}$$

where $V_d : \mathbb{R}^n \mapsto \mathbb{R}^+$ is the desired energy function, $d_d : \mathbb{R}^n \mapsto \mathbb{R}^+$ is the desired damping, and $z \in \mathbb{R}^m$ is a new passive output. The problem above has too many degrees of freedom, i.e. V_d, d_d, z, $\hat{u}$. Selecting various desired dissipation functions, d_d, generates different versions of Standard PBC [10].

Consider the nonlinear system (2.8) in closed-loop with $u = \hat{u}(x) + v$. Then, (3.40) holds if and only if

$$\nabla V_d^\top (f + g\hat{u}) = -d_d, \tag{3.41}$$

$$z = g^\top \nabla V_d, \tag{3.42}$$

for some function $d_d : \mathbb{R}^n \mapsto \mathbb{R}^+$. By specifying the new passive output z via (3.42), our problem amounts to find $(V_d, d_d, \hat{u})$ that solve (3.41) for a given triple (f, g, V_d). If we fix the desired damping d_d, then we must be able to define a control signal $\hat{u}$ - function of V_d - so that (3.41) becomes a *linear PDE* in the unknown assignable energy function V_d.

3.3.2 IDA-PBC

Over the last decades, a particular form of state-space structure known as a port-Hamiltonian (pH) representation assumed a prominent role in the field of PBC, as they are natural candidates to describe many physical systems. Unfortunately, in some engineering applications, physical pH models are too complex for control design and a reduction stage, which commonly destroys the pH structure, is usually needed. On the other hand, they yield well-established models that are widely accepted in practice.

In principle, PBC does not demand the open-loop system to be port-Hamiltonian, although if this is the case, then certain particular strategies can be applied in order to design a stabilizing controller [45], [46]. A pH model is given by [10]

$$\Sigma_{nl} \begin{cases} \dot{x} = [\mathcal{J}(x) - \mathcal{R}(x)]\nabla V(x) + g(x)u \\ y = g^\top(x)\nabla V(x), \end{cases} \tag{3.43}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $m \leq n$, $V : \mathbb{R}^n \mapsto \mathbb{R}$ is the total storage energy, $\mathcal{J}, \mathcal{R} : \mathbb{R}^n \mapsto \mathbb{R}^{n\times n}$, with $\mathcal{J} = -\mathcal{J}^\top$ and $\mathcal{R} = \mathcal{R}^\top \geq 0$ are the natural interconnection and damping matrices, respectively, $u, y \in \mathbb{R}^m$, are conjugated variables whose product has units of power and $g : \mathbb{R}^n \mapsto \mathbb{R}^{n\times m}$ is assumed full rank, a condition that is important for deriving an explicit state feedback control law. Besides, we define the matrix function $F : \mathbb{R}^n \mapsto \mathbb{R}^{n\times n}$ as follows

$$F(x) \triangleq \mathcal{J}(x) - \mathcal{R}(x), \tag{3.44}$$

which fulfills the inequality

$$F + F^\top = -2\mathcal{R} \le 0. \tag{3.45}$$

The open-loop function V is not assumed to be positive semi-definite nor bounded from below. Finally, the power conservation property of pH systems is captured by the power-balance equation

$$\dot{V} = -(\nabla V)^\top \mathcal{R} \nabla V + u^\top y. \tag{3.46}$$

Using the fact that $\mathcal{R} \ge 0$, we obtain the relation

$$\dot{V} \le u^\top y, \tag{3.47}$$

which we refer to as *cyclo-passivity inequality*.

Systems satisfying such an inequality are called cyclo-passive, which should be distinguished from passive systems where V is positive semi-definite. As stated previously in this chapter, a system is cyclo-passive if it cannot create energy over closed paths in the state-space. It might, however, produce energy along some initial portion of such a trajectory. Clearly, every passive system is cyclo-passive, whereas the converse is not necessarily true [55].

Based on Hamiltonian representations of a model, a strategy known as IDA-PBC (Interconnection and Damping Assignment Passivity-based Control) was introduced in [11] and [12]. The method can be summarized in the following proposition.

Proposition 3.2. ***IDA-PBC Strategy.*** *Consider the system*

$$\dot{x} = f(x) + g(x)u. \tag{3.48}$$

Assume there are matrices $g^\perp(x)$, $\mathcal{J}_d(x) = -\mathcal{J}_d^\top(x)$, $\mathcal{R}_d(x) = \mathcal{R}_d^\top(x) \ge 0$ and a function $H_d : \mathbb{R}^n \mapsto \mathbb{R}$ that verifies the PDE

$$g^\perp(x)f(x) = g^\perp(x)[\mathcal{J}_d(x) - \mathcal{R}_d(x)]\nabla V_d, \tag{3.49}$$

where $g^\perp(x)$ is a full-rank left annihilator of $g(x)$, that is, $g^\perp(x)g(x) = 0$, and $V_d(x)$ is such that

$$x^* = arg\ min\ V_d(x), \tag{3.50}$$

with $x^* \in \mathbb{R}^n$ the equilibrium to be stabilized. Then, the closed-loop system (3.48) with $u = \beta(x)$, where

$$\beta(x) = [g^\top(x)g(x)]^{-1}g^\top(x) \times \{[\mathcal{J}_d(x) - \mathcal{R}_d(x)]\nabla V_d - f(x)\}, \tag{3.51}$$

takes the pH form

$$\dot{x} = [\mathcal{J}_d(x) - \mathcal{R}_d(x)]\nabla V_d, \tag{3.52}$$

with x^* a (locally) stable equilibrium. It will be asymptotically stable if, in addition, x^* is an isolated minimum of $H_d(x)$ and the largest invariant set under the closed-loop dynamics (3.52) contained in

$$\left\{x \in \mathbb{R}^n \mid (\nabla V_d)^\top \mathcal{R}_d \nabla V_d = 0\right\}, \tag{3.53}$$

equals $\{x_*\}$. An estimate of its domain of attraction is given by the largest bounded level set $\{x \in \mathbb{R}^n \mid V_d(x) \leq c\}$.

The key step in IDA-PBC is the solution of linear PDE (3.49). In this regard, the following assertions hold.

- Matrices $\mathcal{J}_d(x)$ and $\mathcal{R}_d(x)$ are free, up to the constraint of skew-symmetry and positive semidefiniteness, respectively,
- $V_d(x)$ may be totally, or partially, fixed provided we can ensure (3.50), and probably a properness condition. In this case, the problem would be posed rather as a set of algebraic equations than as a PDE,
- There is an additional degree-of-freedom in $g^\perp(x)$ which is not uniquely defined by $g(x)$.

3.3.3 Control by Interconnection

In the well-known Control by Interconnection (CbI) strategy [44], [47], the controller is another pH system with its own state variables and energy function, i.e. a dynamic feedback controller. The overall system is still cyclo-passive with a new energy function given by the sum of the energy functions of the plant and the controller [44]. As the closed-loop energy function is not guaranteed to have a minimum at the desired equilibrium point, it is necessary to relate the states of the plant and the controller via generation of invariant sets defined by the so-called *Casimir functions*.

In its basic formulation, CbI assumes that only the plant output is measurable and considers the classical output feedback interconnection. Generally formulated, CbI is a dynamic output feedback strategy based on the idea of assigning a Hamiltonian structure to the closed-loop interconnection, with an appropriate closed-loop storage function that attains a strict minimum at the desired equilibrium point.

First, let us define the assignable equilibria of the nonlinear system (3.43) as the elements of the set

$$\mathcal{E}_x \triangleq \left\{ g^{\perp} \mathcal{R} \nabla V = 0 \right\}, \tag{3.54}$$

with $g^{\perp} : \mathbb{R}^n \mapsto \mathbb{R}^{(n-m)\times n}$ a full rank left-annihilator of g, that is, $g^{\perp} g = 0$ and rank $g = n - m$. Associated to each $x_* \in \mathcal{E}_x$ there is a uniquely defined constant control

$$u_* =\triangleq -g^{\dagger}(x_*) F(x_*) \nabla V(x_*), \tag{3.55}$$

where $g^{\dagger}$ is the More-Penrose pseudo-inverse of g, which is well-defined since g is assumed full rank.

In CbI, a pH controller of the form

$$\Sigma_c : \begin{cases} \dot{\xi} = u_c \\ y_c = \nabla V_c(\xi), \end{cases} \tag{3.56}$$

is usually considered, where $\xi \in \mathbb{R}^m$ is the state of the controller, u_c, y_c are its input and output of the controller, respectively, and $V_c : \mathbb{R}^m \mapsto \mathbb{R}$ is a controller storage function to be designed. Alternative forms of dynamic are also possible.

In the literature, CbI comes in two basic variants. In the *standard* version, Σ_{nl} and Σ_c are coupled using the classical unitary feedback power-preserving interconnection [44]

$$\Sigma_I : \left\{ \begin{bmatrix} u \\ u_c \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ y_c \end{bmatrix} + \begin{bmatrix} v \\ 0 \end{bmatrix} \right. , \tag{3.57}$$

where v is a new virtual input. As the pH structure is invariant under power-preserving interconnection, we have an interconnected pH system described by

$$\Sigma_{Ts} : \begin{cases} \begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} F & -g \\ g^{\top} & 0 \end{bmatrix} \nabla V_T + \begin{bmatrix} g \\ 0 \end{bmatrix} v, \\ y_{Ts} = \begin{bmatrix} g^{\top} & 0 \end{bmatrix} \nabla V_T, \end{cases} \tag{3.58}$$

with

$$V_T(x,\xi) \triangleq V(x) + V_c(\xi), \tag{3.59}$$

the new total energy.

A new version of CbI has been recently introduced in [12] that, being related to the power shaping procedure of [14], is called *power shaping CbI*. In this case,

F is assumed to be non-singular and a modified port-Hamiltonian system with a new passive output is generated as

$$\Sigma_{ps} : \begin{cases} \dot{x} = F\nabla V + gu, \\ y_{ps} = -g^\top F^{-T}(F\nabla V + gu). \end{cases} \tag{3.60}$$

Noticing that $y_{ps} = -g^\top F^{-T}\dot{x}$ it is easy to show that (3.60) satisfies $\dot{V} \leq u^\top y_{ps}$. The interconnection is then given by [44]

$$\Sigma_{Ips} : \left\{ \begin{bmatrix} u \\ u_c \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_{ps} \\ y_c \end{bmatrix} + \begin{bmatrix} v \\ 0 \end{bmatrix} \right. , \tag{3.61}$$

so that the closed-loop system is given by

$$\Sigma_{Tps} : \begin{cases} \begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} F & -g \\ -g^\top F^{-T} F & g^\top F^{-T} g \end{bmatrix} \nabla V_T + \begin{bmatrix} g \\ -g^\top F^{-T} g \end{bmatrix} v, \\ y_{Ts} = \begin{bmatrix} g^\top & -g^\top F^{-T} g \end{bmatrix} \nabla V_T, \end{cases} \tag{3.62}$$

Since V_c can be modified at will, it seems reasonable to use it to shape the total storage function V_T. We are interested in shaping V_T along the x coordinates but, unfortunately, V_c is a function of ξ, so this idea cannot be applied directly. One way to get around this, is to relate x and ξ in the following manner.

Assumption 3.3. There exist a differentiable mapping $C : \mathbb{R}^n \mapsto \mathbb{R}^m$, the Jacobian of which has rank m and at least one of the following condition is satisfied.

- (Standard CbI)

$$\begin{bmatrix} g^\top \\ F \end{bmatrix} \nabla C = \begin{bmatrix} 0 \\ g \end{bmatrix}. \tag{3.63}$$

- (Power shaping CbI) $detF(x) \neq 0$ and

$$F\nabla C = -g. \tag{3.64}$$

It is assumed that, for the given F and g, a solution of the partial differential equations (3.63) and (3.64) is known. Also, to simplify the presentation, it is assumed that F is full rank. In [44], it is shown that condition (3.63) (resp., (3.64)) of Assumption 3.3 ensures that, for any $\kappa \in \mathbb{R}^m$, the manifolds $\mathcal{M}_k = \{(x,\xi) \mid C(x) - \xi = k\}$ are *invariant* under the flow of the system (3.58) (resp., (3.62)). This means that the condition $C(x(t)) - \xi(t) = C(x_0) - \xi_0 \; \forall t$,

$(x_0,\xi_0) \triangleq (x(0),\xi(0))$, needs to be fulfilled, which is a severe restriction to the application of the method.

The construction of this Casimir function $C(x) - \xi$ is the key step of CbI that allows to shape the storage function in the state coordinates x. In order to reveal this property and, at the same time, provide a unified framework to study both versions of CbI, we find it convenient to define the pH system

$$\Sigma_T : \begin{cases} \begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = F_T \nabla V_T + g_T v, \\ y_T = g_T^\top \nabla V_T, \end{cases} \tag{3.65}$$

where

$$F_T \triangleq \begin{bmatrix} I \\ \nabla C^\top \end{bmatrix} \begin{bmatrix} F & -g \end{bmatrix}, \quad g_T \triangleq \begin{bmatrix} I \\ \nabla C^\top \end{bmatrix} g. \tag{3.66}$$

The proposition below opens the possibility of creating appropriate storage functions that can shape along x.

Proposition 3.3. *([12]) The pH system (3.65) is cyclo-passive with storage function*

$$W(x,\xi) \triangleq V_T(x,\xi) + \phi(C(x) - \xi), \tag{3.67}$$

for any differentiable $\phi : \mathbb{R}^m \mapsto \mathbb{R}$.

Proposition 3.3 can be used for stabilization of an arbitrary element of the assignable equilibrium set $\mathcal{E}_x$ defined in (3.54). In [47], the authors propose functions V_c and ϕ and give conditions on C that ensure the stabilization requirement. As a first step, define the set of equilibria $\mathcal{E}$ for the system (3.65) in open-loop ($v = 0$). According to (3.65), (3.66)

$$\mathcal{E} = \{(x,\xi) \mid F\nabla V - g\nabla V_c = 0\}. \tag{3.68}$$

previously, we have shown that

$$\dot{W} = y_T^\top v - d_T, \tag{3.69}$$

with $d_T \geq 0$. It follows from standard Lyapunov theory that if W has a strict minimum at a point $(x_*,\xi_*) \in \mathcal{E}$ and we set $v = 0$, then (x_*,ξ_*) is stable. Our goal is thus, to find appropriate ϕ and H_c, and impose conditions on C, such that

$$(x_*,\xi_*) = arg\ min\ W(x,\xi). \tag{3.70}$$

Clearly, negativity of $\dot{W}$ can be reinforced by setting

$$v = -K_v y_T, \quad K_v = K_v^\top > 0. \tag{3.71}$$

This damping injection is usually adopted in PBC to try to make the equilibrium *asymptotically stable*, which is the case if y_T is a detectable output. Unfortunately, the latter condition is not satisfied for CbI and we must adopt another strategies. Recent results in this field can be found in [47], which contain a novel adaptive approach that can be applied for stabilization of an arbitrary element of $\mathcal{E}_x$.

In Chapter 6, we are going to approach dynamic output feedback through a framework similar to that of standard CbI, although we will rely on a dissipativity-based line of investigation. Neither Hamiltonian representations nor Casimir function will no longer be a part of the controller design procedure.

4 On the Equivalence Between Strict Positive Realness and Strict Passivity

In this chapter, we restrict our analysis to the case of linear systems, where specific necessary and sufficient conditions for passivity can be established in terms of solving algebraic equations of constant matrices. In addition, an equivalent frequency-domain concept to the notion of passivity known as positive realness is presented, as well as necessary and sufficient conditions for it to hold. The definitions of strict passivity and strict positive realness are also introduced. Indeed, this chapter contains a solution to an important problem that had been open for many years in the field of linear systems theory. Although it was well known that the strict passivity of LTI systems with $D = 0$ is equivalent to their strict positive realness [3], [6], a relevant open problem consisted of proving equivalence in the case of $D \neq 0$, and that is precisely the question we deal with in this chapter. We present a proof of this equivalence for the case of controllable and observable LTI models, see also [124].

The text is organized as follows. We present in Section 4.1 an introduction to the research topic, and in Section 4.2 a brief review of some relevant known results. Here, we mathematically define (strict) passivity and (strict) positive realness and provide an overview of the work that has been done concerning the relationships among these concepts. In Section 4.3 we prove our main results, demonstrating that in the case of LTI minimal systems strict passivity is equivalent to strict positive realness. Section 4.4 presents a numerical example of our contribution. Finally, the concluding remarks are provided in Section 4.5.

4.1 Introduction

As stressed in Chapter 3, passivity is a special case of the property of dissipativity where the supply rate is given by the simplified expression $w = y^\top u$. Passivity applies, then, only to square systems [6]. In this chapter, our focus relies on *linear time-invariant* (LTI) passive systems with a state-space representation as introduced in Chapter 2 and repeated here for ease of reference [3]

$$(\Gamma_l)\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t), \end{cases} \tag{4.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$. We assume that (Γ_l) is controllable and observable, i.e. a realization (A,B,C,D) is minimal. The respective transfer function matrix is given by

$$H(s) = C(sI - A)^{-1}B + D. \tag{4.2}$$

$H(s) \in \mathbb{C}^{m\times m}$, $s \in \mathbb{C}$, is considered as rational and proper. A rational transfer function matrix is proper if all its entries are finite at $s = \infty$ [30].

In the LTI case, passive systems with positive storage functions are minimum-phase and have relative degree not greater than one, i.e. the real part of their transfer function matrices satisfy $Re[H(s)] \geq 0$ for all $Re[s] > 0$ [9]. In short, the passivity of (Γ_l) is equivalent to the *positive realness* of $H(s)$ [3], [6]. In a Nyquist diagram, it ensures that the poles of $H(s)$ are located in the closed left-half plane. Thus, the system is *stable*. Moreover, the feedback interconnection of positive real systems is also guaranteed to be stable. Furthermore, positive realness and passivity have been extensively applied in the context of robust stability and robust stabilizability of control systems [30].

While passivity applies both to linear and nonlinear systems, positive realness is a notion related to linear dynamical systems only. The relationship between the many subcategories of passivity, a time-domain concept, and positive realness, a frequency-domain notion, is a major field of research in linear control theory. There still exist many open problems which are worth a deeper analysis. Beyond passivity and positive realness, we discuss the notions of *strict passivity* and *strict positive realness*. Both properties guarantee *asymptotic stability* to an LTI system. We address the question of determining the conditions under which an equivalence between these concepts arises. Although it is clear that passivity and positive realness are equivalent, it is not so in the *strict* sense [3].

Numerous papers have been published concerning this topic, e.g. [19], [32]. A variety of necessary and sufficient conditions for strict positive realness have been derived [21], [25], [18]. Some of them are frequency-domain conditions, others rely on state-space representations.

4.2 Preliminaries

4.2.1 Passivity

We adopt the perspective and notation introduced in [21], [23]. Consider then the LTI system (Γ_l) introduced in (4.1). According to the theory introduced in Chapter 3, the system (Γ_l) is said to be *passive* (P) if there exists a storage function $V(x) \geq 0$, $V(0) = 0$, such that the following dissipation inequality holds [3]

$$V(x(t)) \leq V(x(0)) + \int_0^t u^\top(s)y(s)ds. \tag{4.3}$$

Assuming that the storage function is differentiable, we have the following relation for a passive system [1]

$$\dot{V}(x(t)) \leq u^\top(t)y(t). \tag{4.4}$$

If $V(x) > 0$, the connection with the stability of the free system is a well-known fact. The necessary and sufficient conditions for the passivity of (Γ_l) are given by the Kalman-Yakubovich-Popov (KYP) equations.

Theorem 4.1. *([3]) An LTI system with a minimal realization (Γ_l) is passive (P) if and only if the set of equations*

$$\begin{aligned} A^\top R_1 + R_1 A &= -L_1^\top L_1 \\ C^\top - R_1 B &= L_1^\top W_1 \\ D + D^\top &= W_1^\top W_1 \end{aligned} \tag{4.5}$$

is satisfied for some $R_1 = R_1^\top > 0$, $R_1 \in \mathbb{R}^{n\times n}$, $L_1 \in \mathbb{R}^{n\times m}$ and $W_1 \in \mathbb{R}^{m\times m}$.

In this framework, passive LTI systems are stable with a quadratic Lyapunov function $V(x) = \frac{1}{2}\, x^\top R_1 x$. In addition, it is straightforward to ensure that only systems with a positive semidefinite feedthrough, i.e. $D \geq 0$, can be passive, since $D + D^\top = W_1^\top W_1 \geq 0$ and $D + D^\top \geq 0 \Leftrightarrow D \geq 0$ (see Appendix A). Moreover, if $\dot{V}(x) < 0$ then (Γ_l) is asymptotically stable. This is the case of the *strictly passive* (SP) systems [1], [17], to which there exists a positive definite function $S(x)$ such that for all $t \geq 0$

$$V(x(t)) \leq V(x(0)) + \int_0^t u^\top(s)y(s)ds - \int_0^t S(x(t))dt. \tag{4.6}$$

This definition is referred to as *state strict passivity* in [3] and [4], and can also be stated as

$$\dot{V}(x(t)) + S(x(t)) \leq u^\top(t)y(t). \tag{4.7}$$

Following the definition of [1] and [17] we call this kind of passivity as strict passivity here. The conditions for an LTI system to be SP are given in the following theorem, in terms of linear KYP Equations.

Theorem 4.2. *([3]) An LTI system with a minimal realization* (Γ_l) *is strictly passive (SP) if and only if the set of equations*

$$\begin{array}{rcl} A^\top R_1 + R_1 A & = & -L_1^\top L_1 - \mu_1 P_1 \\ C^\top - R_1 B & = & L_1^\top W_1 \\ D + D^\top & = & W_1^\top W_1 \end{array} \tag{4.8}$$

is satisfied for some $R_1 = R_1^\top > 0$, $R_1 \in \mathbb{R}^{n\times n}$, $P_1 = P_1^\top > 0$, $P_1 \in \mathbb{R}^{n\times n}$, $L_1 \in \mathbb{R}^{n\times m}$, $W_1 \in \mathbb{R}^{m\times m}$ *and a real parameter* $\mu_1 > 0$.

This result can be derived from (4.7) and Theorem 4.1, with a function $S(x) = \frac{1}{2}\,\mu_1 x^\top P_1 x$. SP systems are asymptotically stable. Any SP system is P, but the converse is not true.

4.2.2 Positive Realness

As reported in [21], the concept of positive realness appears frequently in a variety of aspects of engineering. Stability theory, in particular, has benefited greatly from the introduction and further development of this notion. Consider then the transfer function (4.2) of (Γ_l) and the following definition.

Definition 4.1. ([3]) A rational and proper transfer function matrix $H(s) \in \mathbb{C}^{m\times m}$ is said to be *positive real (PR)* if

- all elements of $H(s)$ are analytic in $Re[s] > 0$,
- $H(s)$ is real for all positive real s,
- $H(s) + H^*(s) \geq 0$ for all $Re[s] > 0$.

The poles of $H(s)$ are located in the closed lef-half plane, i.e a minimal realization of the system is Lyapunov stable. PR systems are dissipative and phase bounded [25], [30]. A minimal transfer function matrix that is PR is BIBO stable, minimum-phase, and has relative degree zero or one [19]. The necessary and sufficient conditions for positive realness are also provided by the linear KYP equations.

Theorem 4.3. *([25],[3]) The rational and proper transfer function matrix (4.2) is positive real (PR) if and only if there exists matrices $R_2 = R_2^\top > 0$, $R_2 \in \mathbb{R}^{n\times n}$, $L_2 \in \mathbb{R}^{n\times m}$ and $W_2 \in \mathbb{R}^{m\times m}$ that fulfill (4.5).*

In Theorem 4.3, the matrices R_2, L_2 and W_2 replace, respectively, R_1, L_1 and W_1 of Theorem 4.1. Equations (4.5) connect time-domain conditions for passivity with frequency-domain conditions for positive realness. Thus, passivity and positive realness are equivalent features. Strict positive realness is defined as follows from [3], [4].

Definition 4.2. A rational and proper transfer function matrix $H(s) \in \mathbb{C}^{m\times m}(s)$ that is not identically zero for all s, is said to be *strictly positive real (SPR)* if $H(s-\epsilon)$ is *PR* for some $\epsilon > 0$.

The definition of SPR implies that the poles of $H(s)$ are in the open left-half plane, i.e. the system is asymptotically stable [25]. The theorem below provides the necessary and sufficient conditions for strict positive realness.

Theorem 4.4. *([3]) The rational and proper transfer function matrix (4.2) is strictly positive real (SPR) if and only if there exists matrices $R_2 = R_2^\top > 0$, $R_2 \in \mathbb{R}^{n\times n}$, $L_2 \in \mathbb{R}^{n\times m}$ and $W_2 \in \mathbb{R}^{m\times m}$ and a real parameter $\mu_2 > 0$ such that*

$$\begin{aligned} A^\top R_2 + R_2 A &= -L_2^\top L_2 - \mu_2 R_2 \\ C^\top - R_2 B &= L_2^\top W_2 \\ D + D^\top &= W_2^\top W_2. \end{aligned} \tag{4.9}$$

Any SPR system is also PR. The converse is not true. Furthermore, a remarkable difference between (4.8) and (4.9) is that the latter must be satisfied for the same matrix R_2 which appears both on the left-hand and on the right-hand side of the first KYP equation. That feature seems to turn strict positive realness into a more *restrictive* condition than strict passivity.

Based on Theorems 4.1 and 4.3, an LTI system is P if and only if it is PR. Moreover, if (Γ_l) is SPR then it is SP. But it is not known whether the converse is true. Nevertheless, it is a firmly established result that strict passivity implies strict positive realness if $D = 0$ [3]. The case in which $D \neq 0$ remained unsolved until the present work.

There are in the literature many different and sometimes conflicting definitions of strict passivity and strict positive realness. The definitions of some special cases of strict positive realness such as *strong* and *weak strict positive realness*, as well as the nomenclature used, may differ depending on the author. Important to emphasize is that it is not only the kind of definitions of SPR and SP systems that makes for the necessity and sufficiency of Theorems 4.2 and 4.4, but mainly the fact that the systems are controllable and observable, i.e. minimal. Our definitions are consistent with [19] , [25], [3] and [6].

In [3], the authors refer to strict passivity as *state strict passivity*. In [32], what is termed in [3] and [19] as *input strict passivity* and *strong strict positive realness* is referred to as strict passivity and strict positive realness, respectively. The authors proved equivalence between those concepts, and the nomenclature employed may induce the reader to argue that our problem has already been solved. We need then to stress that the authors have not really dealt with the same problem as ours. What has been proved in [32] is that input strict passivity is equivalent to strong strict positive realness, if we intend to be consonant with [19] and [3]. To the best of our knowledge, we present the first proof of equivalence between strict passivity and strict positive realness in the general case of nonzero feedthrough matrices. The following lemma summarizes much of knowledge available in the literature regarding the equivalences between classes of positive real and passive systems.

Lemma 4.1. *([18]) Consider an LTI, minimal (controllable and observable) system (2.9) whose transfer function matrix is given by (4.2), where the minimum singular value $\sigma_{min}(B) > 0$. Assume that the system is exponentially stable. Consider the following statements:*

1) There exist $P > 0$, $P, L \in \mathbb{R}^{n\times n}$, $\mu_{min}(L) \triangleq \epsilon > 0$, $Q \in \mathbb{R}^{m\times n}$, $W \in \mathbb{R}^{m\times m}$ that satisfy the Lur'e equations

$$\begin{aligned} A^\top P + PA &= -Q^\top Q - L && (4.10)\\ B^\top P - C &= W^\top Q && (4.11)\\ W^\top W &= D^\top + D. && (4.12)\end{aligned}$$

1') Same as 1) except L is related to P by

$$L = 2\mu P, \tag{4.13}$$

for some $\mu > 0$.
2) There exists $\eta > 0$ such that for all $\omega \in \mathbb{R}$

$$H(j\omega) + H^*(j\omega) \geq \eta I. \tag{4.14}$$

3) For all $\omega \in \mathbb{R}$

$$H(j\omega) + H^*(j\omega) > 0. \tag{4.15}$$

4) For all $\omega \in \mathbb{R}$

$$H(j\omega) + H^*(j\omega) > 0. \tag{4.16}$$

and

$$\lim_{\omega\to\infty} \omega^2 (H(j\omega) + H^*(j\omega)) > 0. \tag{4.17}$$

5) The system can be realized as the driving point impedance of a port dissipative network.
6) The Lur'e equations with $L = 0$ are satisfied by the internal parameter set $(A + \mu I, B, C, D)$ corresponding to $T(j\omega - \mu)$ for some $\mu > 0$.
7) For all $\omega \in \mathbb{R}$, there exists $\mu > 0$ such that

$$H(j\omega - \mu) + H^*(j\omega - \mu) \geq 0. \tag{4.18}$$

8) There exists a positive constant ρ and a constant $\xi(x_0) \in \mathbb{R}$, $\xi(0) = 0$, such that for all $t \geq 0$

$$\int_0^t u^\top(s) y(s) ds \geq \xi(x_0) + \rho \int_0^t ||u(s)||^2 ds. \tag{4.19}$$

9) There exists a positive constant γ and a constant $\xi(x_0) \in \mathbb{R}$, $\xi(0) = 0$, such that for all $t \geq 0$

$$\int_0^t e^{\gamma s} u^\top(s) y(s) ds \geq \xi(x_0). \tag{4.20}$$

10) There exists a positive constant α such that the following kernel is positive in $\mathcal{L}_2$ ($\mathbb{R}_+; \mathbb{R}^{m\times m}$):

$$K(t-s) = D\delta(t-s) + Ce^{(A+I)(t-s)} BI(t-s), \tag{4.21}$$

where δ and I denote the Dirac measure and the step function, respectively.
11) There following kernel is coercive in $\mathcal{L}_2$ ($[0,T]; \mathbb{R}^{m\times m}$), for all T:

$$K(t-s) = D\delta(t-s) + Ce^{A(t-s)} BI(t-s). \tag{4.22}$$

These statements are related as follows:

$$(1)\left\{\begin{array}{ccccl} \overset{\Leftarrow}{\underset{(if\ D>0)}{\Rightarrow}} & (2) & \Leftrightarrow & (8) & \Leftrightarrow (11) \\ & & & \Downarrow & \\ \overset{\Leftarrow}{\underset{(if\ D=0)}{\Rightarrow}} & (1') & \Leftrightarrow & (4) & \Leftrightarrow (5) \Leftrightarrow (6) \Leftrightarrow (7) \Leftrightarrow (9) \Leftrightarrow (10) \\ & & & \Downarrow & \\ & & & (3) & \end{array}\right.$$

The following picture summarizes the relationships between (strict) passivity and (strict) positive realness as they are currently known.

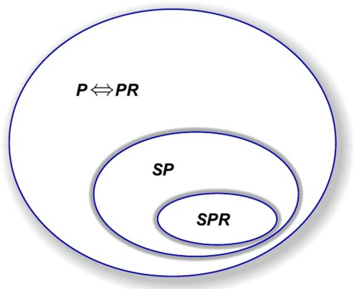

Figure 4.1: Relations between P, PR, SP and SPR systems.

4.3 Main Results

The existence of a solution to problem (4.8) is equivalent to the feasibility of the following LMI [23], [6]

$$\begin{bmatrix} A^\top R_1 + R_1 A + \mu_1 P_1 & R_1 B - C^\top \\ B^\top R_1 - C & -(D + D^\top) \end{bmatrix} \leq 0. \tag{4.23}$$

An LTI minimal system is SP if and only if symmetric matrices $R_1 > 0$, $P_1 > 0$ and a real parameter $\mu_1 > 0$ ensures the negative semidefiniteness of

(4.23). By the same token, an LTI minimal system is SPR if and only if the following holds [6]

$$\begin{bmatrix} A^\top R_2 + R_2 A + \mu_2 R_2 & R_2 B - C^\top \\ B^\top R_2 - C & -(D + D^\top) \end{bmatrix} \leq 0, \tag{4.24}$$

for some symmetric matrix $R_2 > 0$ and some $\mu_2 > 0$. (4.9) is equivalent to (4.24).

In order to solve the LMIs (4.23) and (4.24) we must first of all consider the conditions for semidefiniteness of a partitioned Hermitian matrix. We accomplish this task by presenting the following result due to [27] and [36]. The lemma presented below states the necessary and sufficient conditions for a partitioned Hermitian matrix to be positive semidefinite, which can be easily arranged to prove negative semidefiniteness.

Lemma 4.2. *([27]) Let S be symmetric*

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^\top & S_{22} \end{bmatrix}.$$

$S \geq 0$ if and only if

$$\begin{aligned} &S_{22} \geq 0 \\ &S_{12}^\top = S_{22} S_{22}^\dagger S_{12}^\top \\ &S_{11} - S_{12} S_{22}^\dagger S_{12}^\top \geq 0, \end{aligned} \tag{4.25}$$

where $S_{22}^\dagger$ is any generalized inverse of S_{22}.

One may state Lemma 4.2 regarding $S_{22}^\dagger$ as the Moore-Penrose pseudoinverse. Nevertheless, the choice of the generalized inverse is not a key issue, as the relevant terms are invariant with respect to it [27].

We use Lemma 4.2 and the fact that $S \geq 0$ if and only if $-S \leq 0$ to analyse the feasibility of the LMIs (4.23) and (4.24). We intend to prove that (4.23) is true if and only if (4.24) is also satisfied. In other words, the sets of solutions R_1 and R_2 to problems (4.8) and (4.9), respectively, are the same. We are now in conditions to state our main result.

Theorem 4.5. *The controllable and observable LTI system (Γ_l) is strictly passive (SP) if and only if the rational and proper transfer function matrix (4.2) is strictly positive real (SPR).*

Proof. Sufficiency: This part of the proof is widely known [3], [6]. If the system is SPR then it is SP. For it is evident that we can arrange

$$R_1 = R_2,\ P_1 = R_2,\ \mu_1 = \mu_2,$$

in (4.23) regardless of any condition on D.

Necessity: We consider the general case $D \geq 0$, as the system cannot be passive otherwise (see Appendix A). The case $D > 0$, i.e. D is positive definite, is a particular case of the former. A proof for $D = 0$ can be found in [3].

As a consequence of Lemma 4.2 and (4.23), (Γ_l) given by (2.9) is SP if and only if the following holds for some $R_1 > 0$, $P_1 > 0$ and some $\mu_1 > 0$

(i) $(D + D^\top) \geq 0$,
(ii) $(B^\top R_1 - C) = (D + D^\top)(D + D^\top)^\dagger(B^\top R_1 - C)$,
(iii) $(A^\top R_1 + R_1 A + \mu_1 P_1) + (R_1 B - C^\top)(D + D^\top)^\dagger(B^\top R_1 - C) \leq 0$.

Similarly, (Γ_l) is SPR if and only if the conditions below are fulfilled

(iv) $(D + D^\top) \geq 0$,
(v) $(B^\top R_2 - C) = (D + D^\top)(D + D^\top)^\dagger(B^\top R_2 - C)$,
(vi) $(A^\top R_2 + R_2 A + \mu_2 R_2) + (R_2 B - C^\top)(D + D^\top)^\dagger(B^\top R_2 - C) \leq 0$.

We set $R_2 = R_1$. Then, due to (i) and (ii), conditions (iv) and (v) hold. We rearrange (vi) as:

$$(A^\top R_1 + R_1 A) + \mu_2 R_1 + (R_1 B - C^\top)(D + D^\top)^\dagger(B^\top R_1 - C) \leq 0. \quad (4.26)$$

What still remains to prove is that (4.26) is fulfilled for some $\mu_2 > 0$. We know from (iii) that

$$(A^\top R_1 + R_1 A) + (R_1 B - C^\top)(D + D^\top)^\dagger(B^\top R_1 - C) \leq -\mu_1 P_1 < 0,$$

so that the matrix on the left-hand side of (4.26) can be presented as a symmetric matrix M which is the sum of a negative definite matrix

$$N_1 = (A^\top R_1 + R_1 A) + (R_1 B - C^\top)(D + D^\top)^\dagger(B^\top R_1 - C),$$

and a positive definite matrix $\mu_2 R_1$

$$M = N_1 + \mu_2 R_1. \quad (4.27)$$

Both N_1 and R_1 are symmetric. If the largest eigenvalue of the resulting matrix M is zero for some $\mu_2 > 0$, then $M \leq 0$ and μ_2 and $R_2 = R_1$ are a solution to (4.24) and (4.9), which ensures necessity to our result.

We deal in this problem with symmetric matrices, i.e. matrices whose eigenvalues are real numbers. In this context, let the eigenvalues of the symmetric matrix $M \in \mathbb{R}^{n\times n}$ be

$$\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_n(M), \tag{4.28}$$

where $\lambda_1(M)$ is the largest eigenvalue of the matrix.

We prove at first that $\lambda_1(M)$ is negative for some $\mu_2 > 0$. For that, we apply the property that for two symmetric matrices N_1,$R_1 \in \mathbb{R}^{n\times n}$ ([28], p. 396)

$$\lambda_1(N_1 + R_1) \leq \lambda_1(N_1) + \lambda_1(R_1), \tag{4.29}$$

and that

$$\lambda_1(\alpha R_1) = \alpha\lambda_1(R_1),\ \alpha \in \mathbb{R}, \tag{4.30}$$

thus

$$\lambda_1(M) \leq \lambda_1(N_1) + \mu_2\lambda_1(R_1). \tag{4.31}$$

Since $\lambda_1(N_1) < 0$ and $\lambda_1(R_1) > 0$, we obtain

$$\lambda_1(M) < 0, \tag{4.32}$$

if

$$0 < \mu_2 < \frac{-\lambda_1(N_1)}{\lambda_1(R_1)}. \tag{4.33}$$

For any μ_2 in this interval a matrix $M < 0$ is obtained, as its largest eigenvalue is negative.

Second, we prove that there also exists $\mu_2 > 0$ such that $\lambda_1(M) > 0$. With this aim, we evoke the so-called Cauchy Interlacing Law ([28], p. 396)

$$\lambda_{i+1}(M_k) \leq \lambda_i(M_{k-1}) \leq \lambda_i(M_k),\ 1 \leq i \leq k, \tag{4.34}$$

for all $k = 2,\cdots,n$, where M_{k-1} is the $(k-1)\times(k-1)$ principal minor of M. By refering to the elements of M as $m_{i,j}$, for i,$j = 1,2,\cdots,n$, we have from (4.34) that if $m_{1,1} > 0$ then $\lambda_1(M) > 0$. Equivalently, if there exists some $\mu_2 > 0$ such that $m_{1,1} > 0$, then $\lambda_1(M) > 0$. The following relation holds

$$m_{1,1} = n_{1,1} + \mu_2 r_{1,1}, \tag{4.35}$$

where $n_{i,j}$,$r_{i,j}$, i,$j = 1,2,\cdot,n$, are the entries of N_1 and R_1, respectively. Moreover, since $N_1 < 0$ and $R_1 > 0$, then $n_{1,1} < 0$ and $r_{1,1} > 0$. Thus, $m_{1,1} > 0$ in the interval

$$\mu_2 > \frac{-n_{1,1}}{r_{1,1}} > 0, \tag{4.36}$$

and this is a sufficient condition to obtain

$$\lambda_1(M) > 0. \tag{4.37}$$

As $\lambda_1(M)$ is a continuous function of μ_2 (see Figure 4.2 below), we deduce from the Intermediate Value Theorem of ordinary calculus [29] and from (4.36) and (4.37) that there exists a value of $\mu_2 > 0$ such that $\lambda_1(M) = 0$ and M is negative semidefinite. This value of μ_2 can be determined by the well known bisection method for finding roots of continuous functions that assume values of opposite signs on a given closed interval [33].

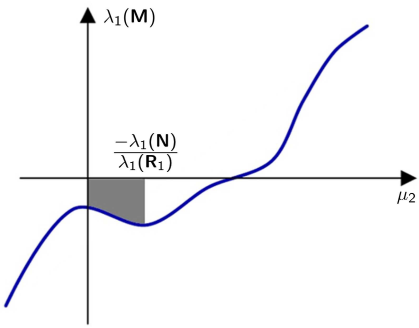

Figure 4.2: Curve of the highest eigenvalue of a matrix M.

Finally, it is very important to stress that there is no conflict between (4.33) and (4.36), as these conditions are simply existence results. Because the system is SP we can always determine a set of values of μ_2 in the interval (4.33) such that (4.32) holds and another set of values (4.36) such that (4.37) is valid. We are then allowed to apply the Intermediate Value Theorem to solve our problem. □

Our framework covers all scenarios of the proposed problem. The existence result we provide is also constructive, in the sense that we show how a SPR solution can be determined from its SP counterpart. Values of μ_2 that ensure opposite signs to $\lambda_1(M)$ can be easily obtained and used as entries to a bisection-like method. As long as the system is SP, a SPR solution is guaranteed to exist and the least conservative value for μ_2 can also be determined. Thus, we conclude that the relationships of Figure 4.1 are actually not the case, whereas Figure 4.3 below contains the actual state-of-the-art in this regard.

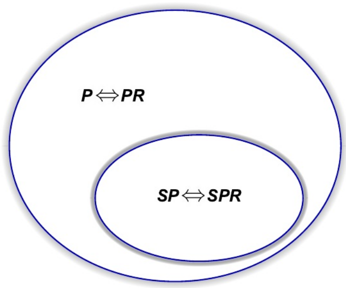

Figure 4.3: SP/SPR equivalence.

It is also evident that the method can provide alternative proofs for some equivalence results found in the literature, such as the equivalence between input strict passivity and strong strict positive realness. This question is addressed in [3], which presents a solution based on Riccati-like arguments.

4.4 Example

We consider the following linear system

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

This system is SP with

$$R_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; \mu_1 = 1, \; P_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

The respective matrix (4.23) is given by

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which is negative semidefinite. The KYP matrices are

$$W_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \ L_1 = \begin{bmatrix} 0 & 0 \\ 0 & 2.2361 \end{bmatrix}.$$

Condition (4.33) is in this case given by $0 < \mu_2 < 2$ and if, for example, $\mu_2 = 1.9$ then $\lambda_1(M) = -0.1$. (4.36) is equivalent to $\mu_2 > 2$ and, in addition, $\lambda_1(M) = 0.1$ if $\mu_2 = 2.1$. By applying the bisection method we obtain $\lambda_1(M) = 0$ for $\mu_2 = 2$, i.e. M is negative semidefinite. We have, thus, a solution to (4.9)

$$R_2 = R_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \mu_2 = 2.$$

The system is SPR and (4.24) assumes the structure below

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which is negative semidefinite. The KYP matrices are

$$W_2 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \ L_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

4.5 Discussion

In this chapter, the equivalence of strict passivity and strict positive realness of a controllable and observable LTI system was proved. Thus, we obtained the equivalence between a time-domain criteria and a frequency-domain concept. We regard this result, to the best of our knowledge, as a new contribution in the field of linear control theory. A simple algorithm for determining the least conservative value of μ_2 such that strict positive realness can be guaranteed, for the same matrix R_1 which proves strict passivity, was provided as well.

5 On the Application of Strict QSR-Dissipativity for Static Output Feedback

In the following, we address another relevant open problem in the field of linear control theory, namely the static output feedback (SOF) control problem. We provide a new necessary and sufficient condition for stabilizability of continuous-time LTI systems by SOF. It consists in solving an ordinary LMI subject to a quadratic equality constraint, whereas its feasibility also guarantees that the system is strictly QSR-dissipative. In addition, a stabilizing gain can be determined by a simple closed-form expression. As we are going to show, the dissipativity-based framework introduced in this chapter allows for analysing this all-important problem from a very general perspective.

Based on a new stabilizability condition, we propose a new convex optimization approach for controller design, which is noniterative and consists in fulfilling the necessary condition for stabilization in such a manner that it comes as close as possible to a necessary and sufficient one. Extensions to the H_∞ control problem are provided as well. No restrictions with regard to symmetry of the state-space matrices or the number of inputs/outputs of the system are involved, i.e. we deal with the general MIMO case. Numerical examples are also presented in order to demonstrate the usefulness of our strategy. Pole placement by SOF is not covered in this work. For a comprehensive discussion on this topic refer to [93], [94].

This chapter is organized as follows. In Section 5.1 and 5.2, the SOF problem is introduced and mathematically formulated. Relevant feasibility issues and a few existence conditions are also addressed. In Section 5.3, our main results are presented, namely: a new necessary and sufficient condition for stabilizability by SOF, a new noniterative LMI-based strategy for SOF design, and an extension to the related H_∞ control problem. In Section 5.4, a few applications of our method are provided, which prove that our approach is able to produce better results than existing strategies, being at the same time considerably simple to implement. Our concluding remarks are presented in Section 5.5.

5.1 Introduction

The static output feedback control problem is one of the major open problems in the field of controller design technology [74]. For many reasons, it has been attracting considerable attention over the last decades, as its theoretical and practical relevance remains an indisputable matter. On the one hand, it deals with the case where a full state feedback cannot be applied, as some components of the state vector might not be available for measurement. On the other hand, it consists in the simplest feedback strategy one could possibly implement in practice, where the application of an ordinary static gain is able to stabilize the closed-loop. The problem can be shortly stated as follows [65], [70]

Given an LTI system, find a static output feedback such that the closed-loop is asymptotically stable, or prove that such a feedback does not exist.

A first underlying question in this regard comprises the existence conditions for such a stabilizing gain. At the same time, the issue of how it could be numerically and efficiently determined is also posed. Let us consider a continuous-time LTI system given by [65]

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \tag{5.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$. Next, apply the following SOF to this plant

$$u(t) = Ky(t), \quad K \in \mathbb{R}^{m\times p}, \tag{5.2}$$

such that the closed-loop is described by

$$\dot{x}(t) = (A + BKC)x(t). \tag{5.3}$$

The fundamental question of the stabilizability of (5.1) by a static gain (5.2) is still open, although its mathematical formulation is remarkably simple [68]. Numerous necessary and/or sufficient conditions for stabilizability have been proposed, as well as many (computational) strategies for determining a stabilizing gain [69]. Unfortunately, most of these conditions are either not testable or rely on numerical methods that have proved quite inefficient [65]. A simple testable criterion that fails *if and only if* the system is not stabilizable, finding otherwise a solution regardless of the particular state-space representation, is not known.

Much of the work available in the literature insists on iterative algorithms in which a set of LMIs are solved until a certain convergence criterion is reached

[68]. These algorithms, however, cannot be proven to be convergent in general [71]-[73]. A major difficulty involved is that the set of solutions of the SOF problem is not convex. Among the first iterative algorithms ever published, we include [59]-[62], whose authors addressed the question of the stabilizability of (5.1) by solving a min/max optimization problem. In [80], a novel necessary and sufficient condition was established, as well as an iterative LMI (ILMI) method for controller design. The introduction of additional variables allows for expressing nonconvex conditions for stabilizability as convex quadratic conditions. Nevertheless, feasibility remains dependent on the state-space representation, which is a major obstacle to the application of the method. In [85], an iterative algorithm without additional variables was proposed and was also applied to PID control. The method introduced in [96] is also iterative and demands the intialization of a number of variables. In [92], an algorithmic procedure based on polynomial matrix inequalities was presented as well. Another recent advancement in the field of iterative strategies was presented in [91], where the idea of decomposing a bilinear matrix inequality (BMI) [112] as the difference between two positive semidefinite convex mappings was applied for SOF stabilization.

Another major contribution to the field was provided by [83], where a non-iterative LMI approach was applied for solving the coupled matrix inequalities that characterize the problem of static gain design. In order to satisfy a sufficient condition (not necessary) for stabilizability, a change of coordinates is also demanded, though. Alternatively, using the fact that the task of designing a stabilizing gain can be expressed as an LMI subject to a set of quadratic inequality constraints, [71] showed that the SOF control problem can be formulated as the minimization of a concave function on a convex set. The strategy applies to the class of the multiple-input single-output (MISO) systems alone, and the algorithm is not guaranteed to converge to the global solution in general. A further necessary and sufficient condition computable in terms of a concave programming problem was provided in [84], where an algorithm for achieving sufficiency was also presented. A closed-form expression for the feedback gain is given, whereas its validity is also restricted to a certain class of plants. In [65] and [66], sufficient convex (LMI) conditions were stated in terms of the so-called W-Problem and P-Problem, whereas feasibility is dependent on the system's state-space representation. While stabilizability was proven to be equivalent to the existence of a transformation that leads those problems to be feasible, no procedure for determining such a coordinate transformation was proposed.

In [95], a novel two-step LMI strategy for controller design was presented. First, a related state feedback problem is solved via LMIs. Later, the output feedback gain is determined via transformation of the LMI variables and the specification of certain intermediary variables whose choice is a key issue for a successful application of the method. In [63], a noniterative necessary and sufficient LMI condition was proposed, though limited to the class of the minimum-phase plants with CB full row rank. The two-step LMI-based strategy introduced in [79] is quite interesting too, although it is a sufficient (not necessary) condition which was not extended to H_∞ design.

In [68], [86]-[88], new necessary and sufficient conditions for the stabilization of state-space symmetric systems were provided through the application of the concept of strict dissipativity. If a certain BMI condition is fulfilled, then a closed-form expression for a stabilizing gain exists. Only symmetric systems were considered and the iterative LMI involved can be efficiently solved only if certain decision variables are fixed. This is also the case of the necessary and sufficient conditions proposed in [81], which cannot be solved by LMI techniques directly.

Finally, nonsmooth optimization strategies as the ones introduced in [118] and [122] changed the way under which SOF was handled, as a considerable improvement in performance was achieved and many difficult problems proved feasible via the aforementioned approaches.

In the forthcoming sections, we present a new necessary and sufficient condition for SOF stabilizability of LTI systems as well as for controller design. Unlike most of the strategies known in the field, our LMI-based strategy is noniterative, does not involve any state-space transformations, and can be very efficiently implemented via semidefinite programming (SDP) tools. This feature simplifies the determination of the stabilizing static gain considerably. We also do not demand the LTI system to be state-space symmetric. In addition, unlike the systems considered in Chapter 4, the LTI models analysed here are not demanded to be square.

5.2 Preliminaries

As in [83] and [85], we proceed on the following assumptions with regard to the linear system (5.1).

Assumption 5.1. (A,B) is stabilizable and (C,A) is detectable.

Having said that, we are now able to present the first fundamental result related to static output feedback stabilization.

Lemma 5.1. *([80]) A static gain K stabilizes (5.3) if and only if there exists a symmetric matrix $0 < P \in \mathbb{R}^{n\times n}$ such that*

$$(A+BKC)^\top P + P(A+BKC) < 0. \tag{5.4}$$

Unlike the equivalent state feedback problem, (5.4) is not convex and cannot be easily solved. It is, indeed, a highly cumbersome problem for which a complete solution is not known. In addition to (5.4), another interesting necessary and sufficient condition for stabilization if provided in the following lemma.

Lemma 5.2. *([80]) The system (5.1) is stabilizable by static output feedback if and only if there exists matrices $P > 0$ and K satisfying the following relation*

$$A^\top P + PA - PBB^\top P + (B^\top P + KC)^\top (B^\top P + KC) < 0. \tag{5.5}$$

The nonconvex quadratic matrix inequality (QMI) (5.5) cannot be solved directly using LMIs. However, by introducing an additional variable $0 < X \in \mathbb{R}^{n\times n}$, and using Schur complement, a sufficient convex condition for stabilizability (a convex QMI) can be formulated [80]. Namely,

$$\begin{bmatrix} A^\top P + PA - XBB^\top P - PBB^\top X + XBB^\top X & (B^\top P + KC)^\top \\ (B^\top P + KC) & -I \end{bmatrix} < 0.$$

For a fixed X, this is an LMI in the decision variables (P,K), a problem that can be solved very efficiently using semidefinite programming, though adding some conservativeness. A further remarkable feature of the SOF control problem appears in the form of the alternative necessary and sufficient condition presented below.

Lemma 5.3. *([83]) The LTI system (5.1) is stabilizable via static output feedback if and only if there exist a symmetric matrix $P > 0$ and positive scalars (σ_1,σ_2) such that*

$$A^\top P + PA - \sigma_1 PBB^\top P < 0, \tag{5.6}$$

$$A^\top P + PA - \sigma_2 C^\top C < 0. \tag{5.7}$$

By multiplication of both sides of (5.6) by P^{-1} we obtain

$$P^{-1}A^\top + AP^{-1} - \sigma_1 BB^\top < 0, \tag{5.8}$$

which is an LMI, as (5.7) is. The task of fulfilling (5.7) and (5.8) simultaneously, however, cannot be handled by conventional LMI tools, as the solution of the former is the inverse of the solution of the latter. The feasibility of these coupled matrix inequalities constitutes a challenging problem which can be solved rather by iterative strategies as in [83], than directly by a simple LMI condition.

As argued before, many different approaches have been taken for addressing the question of stabilizability by SOF. In this work, we present a dissipativity-based line of investigation, relying on previous results on this topic [127]. The notion of dissipativity generalizes the concept of passivity indices applied in [127], allowing for less conservative results and for a deeper understanding of key features of the problem. According to the theory presented in Chapter 3, a system is said to be QSR-dissipative if the following holds

$$\dot{V}(x(t)) \leq y(t)^\top Q y(t) + 2y(t)^\top S u(t) + u(t)^\top R u(t), \tag{5.9}$$

for a differentiable storage function $V(x) > 0, \forall x \neq 0\,(V(0) = 0)$, and matrices $Q = Q^\top \in \mathbb{R}^{p\times p}$, $S \in \mathbb{R}^{p\times m}$, $R = R^\top \in \mathbb{R}^{m\times m}$.

In [75], a necessary and sufficient LMI condition for a *stable* LTI model (5.1) to be QSR-dissipative was presented, which requires $Q \leq 0$. From [52], in addition, the free system is said to be asymptotically stable if $Q < 0$ and $y(t)$ is detectable. In this chapter, though, we suppose an *unstable* open-loop (5.1) to which a static gain has to be applied in order to achieve the asymptotic stability of (5.3). In this context, a slightly different notion of dissipativity is more suitable for our purposes, namely the concept of *strict* QSR-dissipativity. To wit, a system is said to be strictly QSR-dissipative if [77]

$$\dot{V}(x(t)) + T(x(t)) \leq y(t)^\top Q y(t) + 2y(t)^\top S u(t) + u(t)^\top R u(t), \tag{5.10}$$

where $T(x) > 0$, $\forall x \neq 0$ ($T(0) = 0$), is a continuous function. A simple sufficient LMI condition for (5.10) is given below.

Lemma 5.4. *The LTI plant (5.1) is strictly QSR-dissipative if the following LMI is fulfilled*

$$\begin{bmatrix} (A^\top P + PA + N - C^\top QC) & (PB - C^\top S) \\ (PB - C^\top S)^\top & -R \end{bmatrix} \leq 0, \tag{5.11}$$

for some S and for symmetric matrices (Q,R), $(P,N) > 0$.

Proof. Define

$$V(x) = x^\top P x,\, P = P^\top \in \mathbb{R}^{n\times n}, \tag{5.12}$$

$$T(x) = x^\top N x,\, N = N^\top \in \mathbb{R}^{n\times n}, \tag{5.13}$$

where $(P,N) > 0$. Condition (5.10) is equivalent to

$$x^\top P[Ax + Bu] + [Ax + Bu]^\top P x + x^\top N x$$
$$\leq x^\top C^\top Q C x + 2x^\top C^\top S u + u^\top R u, \tag{5.14}$$

which leads to

$$\begin{bmatrix} x \\ u \end{bmatrix}^\top \begin{bmatrix} (A^\top P + PA + N - C^\top QC) & (PB - C^\top S) \\ (PB - C^\top S)^\top & -R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq 0. \tag{5.15}$$

Clearly, if (5.11) holds, then (5.15) and (5.10) are also verified. Furthermore, if $R > 0$, then (5.11) holds if and only if [27]

$$(A^\top P + PA + N - C^\top QC) + (PB - C^\top S)R^{-1}(PB - C^\top S)^\top \leq 0. \quad (5.16)$$

□

A necessary and sufficient condition for dissipativity similar to (5.11) was introduced in [77], subject, though, to the constraint $Q \leq 0$. An LMI formulation was used to investigate open-loop stability, whereas the free system cannot be guaranteed to be stable if Q is assumed otherwise. The problems of state feedback and dynamic output feedback were considered, while SOF stabilization was not discussed. Unlike the case of open-loop stability, which was analysed in terms of an LMI (noniterative) condition, the whole theory proposed in that paper for controller design consisted in fulfilling a BMI, which is reduced to an LMI only if certain decision variables are fixed in advance. A slightly different definition of strict QSR-dissipativity was considered as well, as a positive definite function of the input is used instead of a function $T(x)$ of the state vector. Closed-loop stability is achieved by rendering the plant/controller interconnection strictly QSR-dissipative. In [78], the results of [77] are extended to the case of LTI systems with time-varying uncertainty.

In general, our approach differs considerably from the ideas employed in the aforementioned references and in [76] as well. For example, we suppose that the open-loop is unstable, which means that the restriction $Q \leq 0$ is not necessary. In fact, $Q \leq 0$ is not only not necessary, it hinders the feasibility of (5.11) for open-loop unstable plants. Among the strategies which aim at rendering the closed loop dissipative by SOF one might also include [89] and [90].

5.3 Main Results

5.3.1 A New Necessary and Sufficient Condition for SOF Stabilizability

Based on Lemma 5.4 and condition (5.16), we establish the first one of our main results.

Theorem 5.1. *The LTI system*

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \quad (5.17)$$

is asymptotically stabilizable by static output feedback if and only if it is strictly QSR-dissipative with

$$SR^{-1}S^\top - Q = 0, \tag{5.18}$$

where $R > 0$. A stabilizing gain is given by the following analytical closed-form

$$K = -R^{-1}S^\top. \tag{5.19}$$

Proof. Necessity: If (5.17) is stabilizable, then there exist $P > 0$ and K such that (5.4) is feasible. As a consequence, there exist $N > 0$ and $0 < \beta \in \mathbb{R}$ such that

$$(A + BKC)^\top P + P(A + BKC) + N + \beta PBB^\top P \le 0, \tag{5.20}$$

which is equivalent to

$$A^\top P + PA + N \le -C^\top K^\top B^\top P - PBKC - \beta PBB^\top P. \tag{5.21}$$

Here, without loss of generality, we let it be that $R = \nu I$, with $0 < \nu \in \mathbb{R}$. Then, recover that if (5.11) holds with $\nu = \frac{1}{\beta}$, we have

$$(A^\top P + PA + N - C^\top QC) + \beta(PB - C^\top S)(PB - C^\top S)^\top \le 0, \tag{5.22}$$

and after reorganization of this equation

$$\begin{aligned} A^\top P + PA + N \le C^\top QC &- \beta C^\top SS^\top C \\ &+ \beta PBS^\top C + \beta C^\top SB^\top P - \beta PBB^\top P. \end{aligned} \tag{5.23}$$

Stabilizability by SOF implies the feasibility of (5.11) if the right-hand side of (5.21) is not greater than the expression on the right-hand side of (5.23), i.e.

$$\begin{aligned} -C^\top K^\top B^\top P - PBKC - \beta PBB^\top P \le C^\top QC& \\ -\beta C^\top SS^\top C + \beta PBS^\top C + \beta C^\top SB^\top P - \beta PBB^\top P&, \end{aligned} \tag{5.24}$$

which is satisfied if

$$PB(K + \beta S^\top)C + C^\top (K + \beta S^\top)^\top B^\top P + C^\top (Q - \beta SS^\top)C \ge 0. \tag{5.25}$$

Now, let us set

$$K = -\beta S^\top = -R^{-1}S^\top, \text{which means } S = -\frac{1}{\beta}K^\top = -\nu K^\top. \tag{5.26}$$

Then, if we define

$$\Delta = SR^{-1}S^\top - Q, \tag{5.27}$$

relation (5.25) is reduced to

$$C^\top(Q - \nu K^\top K)C = -C^\top \Delta C \geq 0, \tag{5.28}$$

and the inequality holds if $Q - \nu K^\top K \geq 0$, i.e. $\Delta \leq 0$. In short, (5.17) is stabilizable only if (5.11) is feasible with $\Delta \leq 0$. Notice that the assumption of $R = \nu I$ is not an impediment to establish necessity.

Sufficiency: Suppose that (5.11) is feasible for (Q,S,R,P,N), $R > 0$, i.e. (5.17) is strictly QSR-dissipative. From (5.16), we obtain

$$\begin{aligned}(A^\top P + PA + N - C^\top QC) + PBR^{-1}B^\top P \\ -PBR^{-1}S^\top C - C^\top SR^{-1}B^\top P + C^\top SR^{-1}S^\top C \leq 0,\end{aligned} \tag{5.29}$$

Applying (5.19), we get

$$(A + BKC)^\top P + P(A + BKC) \leq -PBR^{-1}B^\top P - N - C^\top \Delta C, \tag{5.30}$$

If $\Delta \geq 0$, then (5.4) certainly holds and the system is stabilizable. Thus, the feasibility of (5.11) with $\Delta = 0$ is a necessary and sufficient condition for SOF stabilization. The results are also valid if we consider R as a general square matrix not necessarily given by νI. □

Notice that if (5.11) is feasible subject to $\Delta \geq 0$, then there exists another solution with $\Delta \leq 0$, as the system is guaranteed to be stabilizable. The converse is not true. Besides, it is also evident that $\Delta \geq 0$ can be a quite conservative sufficient condition for stabilizability, as one only has to fulfill the following inequality

$$PBR^{-1}B^\top P + N + C^\top \Delta C > 0, \tag{5.31}$$

which might be solvable even if $\Delta < 0$, supposing that the other matrices involved compensate for it. Moreover, observe that in order to fulfill (5.22) it is necessary to satisfy $(A^\top P + PA + N - C^\top QC) < 0$, which is very similar to (5.7), as the latter relation is a special case of the former.

Furthermore, the application of dissipativity-based arguments to the problem of SOF stabilization of (5.17) allows to approach some interesting features not covered yet in the literature. For instance, from the definition of strict QSR-dissipativity given in (5.10), we have

$$\begin{aligned}x^\top(A^\top P + PA)x + 2x^\top PBu + x^\top Nx \\ -x^\top C^\top QCx - 2x^\top C^\top Su - u^\top Ru \leq 0.\end{aligned} \tag{5.32}$$

For any $R > 0$, the left-hand side of (5.32) is a concave function of the control signal $u(t)$ and its *global maximum* is attained for

$$u^* = R^{-1}(PB - C^\top S)^\top x, \tag{5.33}$$

which is an explicit function of the state. By substitution of (5.33) into (5.32), we obtain

$$(A - BR^{-1}S^\top C)^\top P + P(A - BR^{-1}S^\top C) + PBR^{-1}B^\top P + N + C^\top \Delta C \leq 0.$$

Thus, if the conditions of Theorem 5.1 hold, (5.4) is fulfilled for $K = -R^{-1}S^\top$. The fact to be stressed here is that the full state feedback law (5.33) which maximizes (5.32) and stabilizes the closed-loop contains the solution to the SOF stabilization problem explicitly.

5.3.2 A Convex Optimization Approach for Controller Design

In order to apply Theorem 5.1 for solving the proposed control problem, we ought to analyse constraint (5.18) more carefully. The *necessary* condition of having strict QSR-dissipativity with $\Delta \leq 0$ is convex and equivalent to a simple LMI, as follows

$$\begin{cases} R > 0 \\ \Delta = SR^{-1}S^\top - Q \leq 0 \end{cases} \Leftrightarrow X_d = \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \geq 0. \tag{5.34}$$

As a result, we can formulate the following convex optimization problem for controller design, which can be solved very efficiently using semidefinite programming

$$\begin{aligned} &\textit{minimize} \;\; tr(X_d), \\ &\textit{subject to} \;\; (5.11),\, X_d \geq 0, \end{aligned} \tag{5.35}$$

whereas $tr(X_d) = 0 \Leftrightarrow X_d = 0$ [99]. By solving the optimization problem (5.35), we guarantee that the necessary condition for stabilizability is fulfilled and, at the same time, we try to come as close as possible to the necessary and sufficient constraint $\Delta = 0$. A stabilizing gain is, of course, given by (5.19).

Remark 5.1. An alternative formulation which may result in the feasibility of (5.35) with $\Delta > 0$ consists in the following strategy. Specify some $0 < \alpha \in \mathbb{R}$ and define

$$X_d = \begin{bmatrix} Q + \alpha I & S \\ S^\top & R \end{bmatrix} \geq 0, \tag{5.36}$$

which is equivalent to

$$\begin{cases} R > 0, \\ \Delta = SR^{-1}S^\top - Q \leq \alpha I. \end{cases} \tag{5.37}$$

Then, apply condition (5.35). By minimizing $tr(X_d)$ we may approach the positive upper bound of $\Delta = \alpha I$.

Here, it also important to stress that the matrix $SR^{-1}S^\top - Q$ used in our work is not related to the one introduced in [81] as a part of another necessary and sufficient condition for stabilizability. In that work, the authors did not apply any dissipativity-based arguments, making use of an expression identical to Δ rather for setting an upper bound for a cost function of the form

$$\int_0^\infty (x^\top Qx + 2u^\top Sx + u^\top Ru)dt,$$

which means a completely diverse interpretation of Δ. Finally, in [81], no LMI approach for SOF design was provided and the authors suggested, indeed, the application of iterative strategies for solving the problem. On the contrary, we provided a noniterative strategy which consists in solving the simple convex optimization problem (5.35) subject to ordinary LMI constraints.

Furthermore, the reader may also argue that our results are similar to those presented in [102]. Again, we emphasize that our perspective is quite different. Firstly, [102] deals with open-loop stable systems and state feedback stabilization, not covering the SOF control problem. As in [81], an expression similar to Δ is employed, whose interpretation is analogous to the one we provide in our work. The differences appear, though, as [102] assumes that $Q \leq 0$, due to the fact that A is supposed asymptotically stable, which clearly results in the sufficient condition $\Delta \geq 0$.

5.3.3 The H_∞ Control Problem

An extension of the previous results to the well-known problem of H_∞ control seems natural, as it can be formulated in terms of solving an ordinary SOF design problem. In this regard, consider the following continuous-time LTI system given by [80]

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t), \\ z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t), \\ y(t) &= C_2 x(t) + D_{21} w(t), \end{aligned} \tag{5.38}$$

where $w \in \mathbb{R}^r$ is the external signal, $u \in \mathbb{R}^m$ is the controlled input, $z \in \mathbb{R}^q$ is defined as the controlled output and $y \in \mathbb{R}^p$ is the measured output. $A \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times r}$, $B_2 \in \mathbb{R}^{n\times m}$, $C_1 \in \mathbb{R}^{q\times n}$, $D_{11} \in \mathbb{R}^{q\times r}$, $D_{12} \in \mathbb{R}^{q\times m}$, $C_2 \in \mathbb{R}^{p\times n}$ and $D_{21} \in \mathbb{R}^{p\times r}$. In addition, assumption 5.1 holds for (A,B_2) and (C_2,A).

The H_∞ control problem consists in finding a static output feedback K such that the (closed-loop) transfer function matrix T_{zw} from w to z is stable and the following norm constraint is satisfied [85]

$$||T_{zw}||_\infty < \gamma, \quad 0 < \gamma \in \mathbb{R}, \tag{5.39}$$

where γ is a maximum gain in any direction and at any frequency. This holds true if and only if

$$\begin{bmatrix} PA_{cl} + A_{cl}^\top P & PB_{cl} & C_{cl}^\top \\ B_{cl}^\top P & -\gamma I & D_{cl}^\top \\ C_{cl} & D_{cl} & -\gamma I \end{bmatrix} < 0, \tag{5.40}$$

where

$$\begin{aligned} A_{cl} &= A + B_2KC_2, \\ B_{cl} &= B_1 + B_2KD_{21}, \\ C_{cl} &= C_1 + D_{12}KC_2, \\ D_{cl} &= D_{11} + D_{12}KD_{21}. \end{aligned}$$

As pointed out in [80], condition (5.40) is equivalent to

$$\bar{P}\bar{B}K\bar{C} + (\bar{P}\bar{B}K\bar{C})^\top + \bar{A}^\top \bar{P} + \bar{P}\bar{A} < 0, \tag{5.41}$$

where

$$\bar{P} = \begin{bmatrix} P & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \; \bar{A} = \begin{bmatrix} A & B_1 & 0 \\ 0 & -\frac{\gamma}{2}I & 0 \\ C_1 & D_{11} & -\frac{\gamma}{2}I \end{bmatrix},$$
$$\bar{B} = \begin{bmatrix} B_2 \\ 0 \\ D_{12} \end{bmatrix}, \; \bar{C} = \begin{bmatrix} C_2 & D_{21} & 0 \end{bmatrix}. \tag{5.42}$$

One easily deduces that solving (5.41) amounts to solve the SOF control problem for the state-space representation $(\bar{A},\bar{B},\bar{C})$. Clearly, a solution K to the H_∞ control problem can be determined through the application of Theorem 5.1 and of (5.35).

Remark 5.2. Notice that even if γ is not fixed, due to the particular structure o $\bar{P}$, algorithm (5.35) can still be applied directly. If A is Hurwitz, the following adaptation of (5.35) can be used for designing an SOF that minimizes the H_∞ norm of w to z.

$$\begin{aligned} \textit{minimize} \;\; & tr(X_d) + \gamma, \\ \textit{subject to} \;\; & (5.11),\, X_d \geq 0,\, \gamma > 0. \end{aligned} \tag{5.43}$$

5.4 Examples

In this section, we provide a couple of applications of our method which will make it clear how straightforward its implementation is and how competitive the technique is when compared with the existing ones.

5.4.1 Example 5.1

As a first application of our strategy, we consider the following system also found in [79]

$$A = \begin{bmatrix} -2.5 & 1 & -0.5 \\ 8.125 & 8.25 & 6.375 \\ -11.25 & -16.5 & -10.75 \end{bmatrix}, B = \begin{bmatrix} -0.5 & 1 \\ -0.125 & 0.5 \\ 2.25 & 0 \end{bmatrix}, C = \begin{bmatrix} 7 & 6 & 5 \\ 2 & 4 & 2 \end{bmatrix}.$$

The stabilization problem is solved in a straightforward manner using condition (5.35) and semidefinite programming[1)]. As a set of initializations necessary for employing the solver, we assumed $P \geq \varepsilon_P I$, $N \geq \varepsilon_N I$, $X_d \geq \varepsilon_X I$, for $\varepsilon_P = \varepsilon_N = 10^{-2}$ and $\varepsilon_X = 10^{-10}$. Then, we obtained the following solution

$$P = \begin{bmatrix} 0.0515 & 0.0371 & 0.0217 \\ 0.0371 & 0.0475 & 0.0204 \\ 0.0217 & 0.0204 & 0.0216 \end{bmatrix}, \; N = 10^{-2} I, \; Q = \begin{bmatrix} 0.0039 & 0.0172 \\ 0.0172 & 0.0987 \end{bmatrix},$$

$$S = \begin{bmatrix} -0.0063 & 0.0086 \\ -0.0476 & 0.0465 \end{bmatrix}, \; R = \begin{bmatrix} 0.0272 & -0.0213 \\ -0.0213 & 0.0223 \end{bmatrix}.$$

Consequently,

$$\Delta = 10^{-9} \begin{bmatrix} -0.1544 & -0.0936 \\ -0.0936 & -0.3969 \end{bmatrix},$$

[1)]The well-known solver YALMIP [97] can be applied to solve the problem.

which is considerably small when compared with P and N. Finally, from (5.19)

$$K = \begin{bmatrix} -0.2782 & 0.4510 \\ -0.6531 & -1.6586 \end{bmatrix}.$$

The eigenvalues of the closed-loop (5.3) are given by: $(-15.5511, -2.2085, -1)$. Moreover, the sufficient matrix condition (5.31) is also satisfied, as its eigenvalues are: (3.0270, 0.0100, 0.0225). In [79], a stabilizing gain to this system was designed in terms of a sufficient condition, whereas our method allowed for fulfilling both necessary and sufficient conditions for stabilization.

For this system, the optimization problem is also feasible for $\alpha = 0.1$ in (5.36), for example, and applying (5.37) in the algorithm. In this case, we obtain

$$\Delta = 10^{-2} I > 0, \;\; K = \begin{bmatrix} -0.5162 & -0.0553 \\ -0.8980 & -2.1823 \end{bmatrix},$$

which means that our sufficient condition is also attainable.

5.4.2 Example 5.2

In this second example of SOF design using the framework of Theorem 5.1, we deal with the following state-space model borrowed from [85] and also investigated in [80].

$$A = \begin{bmatrix} -0.0266 & -36.6170 & -18.8970 & -32.0900 & 3.2509 & -0.7626 \\ 0.0001 & -1.8997 & 0.9831 & -0.0007 & -0.1708 & -0.0050 \\ 0.0123 & 11.7200 & -2.6316 & 0.0009 & -31.6040 & 22.3960 \\ 0 & 0 & 1.0000 & 0 & 0 & \\ 0 & 0 & 0 & 0 & -30.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 & -30.0000 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 30 & 0 \\ 0 & 0 & 0 & 0 & 0 & 30 \end{bmatrix}^{\top}, \; C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

We assume $P \geq \varepsilon_P I$, $N \geq \varepsilon_N I$, $X_d \geq \varepsilon_X I$, where $\varepsilon_P = 10^{-5}$ and $\varepsilon_N = \varepsilon_X = 10^{-10}$, for solving (5.35) using semidefinite programming[2)]. Again, the LMI is feasible and we have

$$\Delta = \begin{bmatrix} -0.0093 & -0.0392 \\ -0.0391 & -0.1650 \end{bmatrix}.$$

[2)] Again, YAMLMIP can be applied, as in remaining examples later on in this section.

In the following table, we provide a comparison between the results obtained through the application of our strategy and the data available in [85] and [80] regarding to the iterative approaches proposed in these works. Considering that the system is not trivial in any respects, in fact a six dimensional model, the usefulness of our method could be attested, as it provides a noniterative and simple manner for solving the problem of SOF design with a performance similar to the results reported in the literature.

Table 5.1: SOF Results - Example 5.2.

Method	Gain	Poles	Iterations
[80]	$K = \begin{bmatrix} 7.0158 & -4.3414 \\ 2.1396 & -4.4660 \end{bmatrix}$	$\begin{bmatrix} -0.0475 \pm j0.0853 \\ -0.7576 \pm j0.7543 \\ -29.2613,\ -33.6825 \end{bmatrix}$	20
[85]	$K = \begin{bmatrix} 0.6828 & 0.2729 \\ -0.1024 & -0.0348 \end{bmatrix}$	$\begin{bmatrix} -1.3274 \pm j4.6317 \\ -0.7735,\ -0.0665 \\ -31.0626,\ -30.0006 \end{bmatrix}$	5
Proposed	$K = \begin{bmatrix} 0.1819 & 1.9074 \\ -0.1470 & -0.0346 \end{bmatrix}$	$\begin{bmatrix} -0.1485 \pm j7.4380 \\ -0.0322,\ -2.0120 \\ -32.1999,\ -30.0169 \end{bmatrix}$	–

5.4.3 Example 5.3

Our first application of Theorem 5.1 to H_∞ control addresses the following model found in [80], the model of the longitudinal motion of a VTOL helicopter. See also [82] for more details with regard to this plant.

$$A = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & 0.3681 & -0.7070 & 1.4200 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ C_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}^\top,$$

$$B_2 = \begin{bmatrix} 0.4422 & 3.5446 & -5.5200 & 0 \\ 0.1761 & -7.5922 & 4.4900 & 0 \end{bmatrix}^\top,\ C_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix},$$

$$D_{11} = 0,\ D_{12} = \begin{bmatrix} 1 & 0 \end{bmatrix},\ D_{21} = 1.$$

We applied $\varepsilon_P = 10^{-2}$ and $\varepsilon_N = \varepsilon_X = 10^{-10}$ to solve the SOF control problem for the state-space representation $(\bar{A},\bar{B},\bar{C})$ considered in (5.42). It amounts for applying Theorem 5.1 without any changes rather than this transformation. In the following table, the best results reported in [80] and [63] are compared with our solution.

Table 5.2: SOF Results - Example 5.3.

Method	Gain	γ	Iterations
[80]	$K = \begin{bmatrix} 0.0074 \\ 0.7832 \end{bmatrix}$	0.62	257
[63]	$K = \begin{bmatrix} -2.139 \\ 4.347 \end{bmatrix}$	14.65	–
Proposed	$K = \begin{bmatrix} -0.0025 \\ 0.3934 \end{bmatrix}$	0.05	–

Although it is iterative, the strategy proposed in [80] does not provide competitive performance. On the other hand, the noniterative method of [63] is clearly conservative when compared with our condition, where the H_∞ control problem is proven to be feasible for a fixed γ as small as 0.05.

5.4.4 Example 5.4

Let us take the unstable LTI system borrowed from [100] and [101]

$$A = \begin{bmatrix} 0.1 & 0 & 2 \\ 0 & -0.2 & 1 \\ 0 & 0.3 & -1 \end{bmatrix}, \; B_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \; C_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}^\top, \; B_2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \; D_{11} = 1, \; D_{12} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \; D_{21} = 0.$$

Consider the state-space representation $(\bar{A},\bar{B},\bar{C})$ as in (5.42). We set $\varepsilon_P = 1$ and $\varepsilon_N = \varepsilon_X = 10^{-10}$, and obtained the results presented below.

Table 5.3: SOF Results - Example 5.4.

Method	Gain	γ
[101]	$K = \begin{bmatrix} -0.0029 \\ -0.1664 \end{bmatrix}$	6.04
[100]	$K = \begin{bmatrix} 0.1244 \\ -0.2151 \end{bmatrix}$	4.17
Proposed	$K = \begin{bmatrix} 0.3710 \\ -0.2480 \end{bmatrix}$	4

In [100] and [101], sufficient dilated LMI conditions were applied for controller design, a procedure which involved defining extra variables to be used as additional degrees of freedom for solving the SOF BMI. Our strategy is, again, simpler and provides better results as well.

5.5 Discussion

In this chapter, a new necessary and sufficient condition for the stabilizability of continuous-time LTI systems by SOF was proposed. We introduced a noniterative LMI-based procedure for controller design which can also be extended to the H_∞ control problem. A static gain can be determined in a straightforward manner through a closed-form expression. Numerical applications were also provided in order to attest the usefulness of our strategy.

We have presented new features of the SOF problem, proving that it is fundamentally related to the property of strict QSR-dissipativity. We also discussed the similarities between our strategy and other results available in the literature, stressing the novelty of our approach and specifying precisely in which sense it can be advantageous. We derived a necessary condition for stabilization which is a simple convex optimization problem subject to LMI constraints and appears to be less conservative than some existing methods. Then, focusing on a dissipativity-based approach for SOF could be fruitful from the perspective of further contributions to the field. An interesting research direction, for instance, would be an investigation regarding the extent to which the feasibility of our condition is dependent of the particular state-space representation.

The results presented in this chapter still have to be applied to a collection of benchmark models as those available in [121]. A comparison between our strategy and well-established tools as *HIFOO* [119], [120] and *Hinfstruct* [123] is also necessary in order to investigate the full potential of a dissipativity-based method.

6 Dissipative Control Synthesis for Rational Nonlinear Systems

The main subject of this chapter is the application of strict QSR-dissipativity as a general framework for feedback stabilization of nonlinear systems. This notion embraces the definition of passivity and the idea of passivity indices as particular cases. New dissipativity-based conditions for linear output feedback stabilization are presented, extending our previous results on this topic, where the concept of passivity indices was applied instead [125]-[128]. One of the drawbacks of using passivity indices or even the notion of passivity is that they are restricted to the class of square systems and offer fewer degrees of freedom than a QSR-dissipativity approach, as we are going to show. State feedback is addressed superficially here, as it is not the main subject of this work.

6.1 Introduction

A fundamental question in the area of control theory concerns the design of a controller that asymptotically stabilizes a given nonlinear system at a prescribed equilibrium point. If any state component is not available for measurement or observation, then a full state feedback control law can not be specified and, as a result, an output feedback compensator must be implemented. Stability has to be guaranteed at least locally and the aforementioned controller can be either a static or a dynamic one.

In order to solve this import problem, a number of control strategies have been developed over the last decades [1]. In this context, the so-called passivity-based control (PBC) plays a major role, as passivity is an inherent property of numerous physical systems [3], [43], [51]. Simultaneously, alternative methods not based on passivity or dissipativity theory have been presented as well, see for instance [56], [50], [40] and [39]. In this regard, observer construction is usually a part of the controller design procedure even in the case of local stabilization, let alone for global stability.

In the realm of PBC, one of the most celebrated approaches for output feedback stabilization is denominated Control by Interconnection [44], as presented in Chapter 3. The method applies for systems with a port-Hamiltonian representation (not necessarily a passive one) centered at the origin and consists in designing a dynamic output feedback controller which is connected with the plant in a power preserving way and asymptotically stabilizes the whole interconnection. It is necessary to assign an energy function for the closed-loop that assumes its minimum at the desired equilibrium point, playing the role of a Lyapunov function. As the equilibrium may not be origin, the derivation of Casimir functions to relate the states of the plant to the states of the controller might be necessary. This feature of CbI may lead to the so-called dissipation obstacle, a restriction that can severely hamper the applicability of the method [47].

Here, we present constructive semidefinite procedures for output feedback design which allow to do without port-Hamiltonian representations and also apply for nonsquare models. In addition, systems that in the framework of control by interconnection (CbI) would suffer from the dissipation obstacle may be asymptotically stabilizable through our strategy even without a redefinition of their possibly undetectable output variables. In this chapter, we address both linear dynamic feedback and linear static output feedback design. Unlike most of the strategies reported in the literature, neither observer design nor feedback passivation are necessary for local stabilization. The usefulness of our results is illustrated through a few applications.

6.2 Problem Formulation

Let us consider multivariable input-affine systems $(\mathcal{P})$ of the form (Γ_{nl}), whose state-space equations were given in (2.8) and are repeated below for ease of reference:

$$\dot{x} = f(x) + g(x)u, \tag{6.1}$$

and

$$y = h(x) + d(x)u, \tag{6.2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$, with $m \leq n$. Functions $f : \mathbb{R}^n \mapsto \mathbb{R}^n$, $g : \mathbb{R}^n \mapsto \mathbb{R}^{n\times m}$, $h : \mathbb{R}^n \mapsto \mathbb{R}^p$ and $d : \mathbb{R}^n \mapsto \mathbb{R}^{p\times m}$ are smooth functions of the state, whose initial value is given by $x(0) = x_0$.

In the following, we discuss the problem of asymptotically stabilizing the nonlinear plant $(\mathcal{P})$ around an equilibrium point x^* through the interconnection of a feedback controller $(\mathcal{C})$, as depicted in Figure 6.1. We approach the classical *output feedback* control problem, that amounts to design a *static* or a *dynamic* output

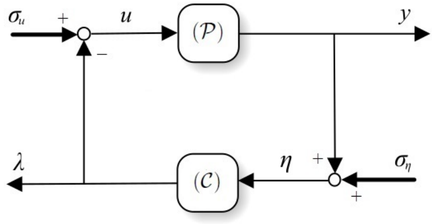

Figure 6.1: Feedback interconnection plant/controller.

feedback controller to be connected with the plant. The former is represented by a static control law $u(y)$, whereas the latter is another dynamical system which is connected with the plant in such a manner that the closed-loop is stable. A dynamic controller has its own state-space representation and equilibrium point.

6.3 A Local Stability Condition for Interconnected Nonlinear Systems

The celebrated references [52] and [53] contain the first results relating dissipativity theory to the stabilization problem of interconnected nonlinear systems. In the sequel, we present a few adaptations of these well-established results in the form of a condition for dynamic output feedback design which resembles the CbI framework, without setting a port-Hamiltonian structure to the closed-loop dynamics. In the forthcoming sections, variations of these results are going to be presented for the case of static output feedback, as well as for state feedback stabilization.

Let us consider a plant $(\mathcal{P})$ described by (6.1)-(6.2) and a dynamic controller $(\mathcal{C})$ represented by

$$\dot{\kappa} = z(\kappa) + q(\kappa)\eta, \tag{6.3}$$

and

$$\lambda = r(\kappa) + s(\kappa)\eta. \tag{6.4}$$

Here, $\kappa \in \mathbb{R}^{n_\kappa}$ is the state, $\eta \in \mathbb{R}^p$ is the input and $\lambda \in \mathbb{R}^m$ is the output of the controller's state-space model. Functions $z : \mathbb{R}^{n_\kappa} \mapsto \mathbb{R}^{n_\kappa}$, $q : \mathbb{R}^{n_\kappa} \mapsto \mathbb{R}^{n_\kappa \times p}$, $r : \mathbb{R}^{n_\kappa} \mapsto \mathbb{R}^m$ and $s : \mathbb{R}^{n_\kappa} \mapsto \mathbb{R}^{m\times p}$ are smooth functions of the state κ. Then, we present the following result based on Theorems 3.7 and 3.10. This result is not new, it is simply a formulation of the aforementioned theorems in such a manner that it beomes more explicit for the reader how these framework could be directly applied for controller design.

Theorem 6.1. *Consider the feedback interconnection of Figure 6.1. Then, suppose that a plant ($\mathcal{P}$) described by (6.1)-(6.2) fulfills*

$$\dot{V} + T \leq y^\top Q y + 2y^\top S u + u^\top R u + \phi, \ \forall (x,u) \in (\mathcal{X} \times \mathcal{U}), \tag{6.5}$$

for matrices (Q,S,R) and functions $(V,T) > 0$, $\forall x \neq x^*$, $x \in \mathcal{X}$, with $V(x^*) = T(x^*) = 0$. Notice that due to (6.2), (6.5) can be interpreted as a function of (x,u) in the domain $(\mathcal{X} \times \mathcal{U}) \subset \mathbb{R}^{n+m}$ containing the steady-state values (x^*,u^*). The additional variable $\phi \in \mathbb{R}$ is a constant.

Furthermore, suppose that a controller ($\mathcal{C}$) and matrices (Q_c,S_c,R_c) satisfy the inequality

$$\dot{V}_c + T_c \leq \lambda^\top Q_c \lambda + 2\lambda^\top S_c \eta + \eta^\top R_c \eta - \phi, \ \forall (\kappa,\eta) \in (\mathcal{K} \times \mathcal{N}), \tag{6.6}$$

where $(\mathcal{K} \times \mathcal{N}) \subset \mathbb{R}^{n_\kappa + p}$ is a domain about the steady-state values (κ^*,η^*). In addition, $V_c(\kappa^*) = T_c(\kappa^*) = 0$ and $(V_c,T_c) > 0$, $\forall \kappa \neq \kappa^*$, $\kappa \in \mathcal{K}$.

If

$$Q_c = -R, \tag{6.7}$$

$$S_c = S^\top, \tag{6.8}$$

$$R_c = -Q, \tag{6.9}$$

and

$$\lambda^* = -u^*, \tag{6.10}$$

with

$$\sigma_u = 0 \text{ and } \sigma_\eta = 0, \tag{6.11}$$

then $\xi^{*\top} = [x^{*\top} \ \ \kappa^{*\top}]$ is an asymptotically stable equilibrium of the closed-loop in the following domain of attraction

$$\mathcal{E} = (\mathcal{X} \times \mathcal{K}) \subset \mathbb{R}^{n+n_\kappa}, \tag{6.12}$$

of the extended state $\xi^\top = [x^\top \ \ \kappa^\top]$. If $\mathcal{X} = \mathbb{R}^n$, $\mathcal{K} = \mathbb{R}^{n_\kappa}$, $\mathcal{U} = \mathbb{R}^m$, $\mathcal{N} = \mathbb{R}^p$ and the functions (V,T,V_c,T_c) are radially unbounded, then the equilibrium point is globally asymptotically stable.

Proof. Define the following functions

$$V_T(\xi) \triangleq V(x) + V_c(\kappa), \tag{6.13}$$

$$T_T(\xi) \triangleq T(x) + T_c(\kappa), \tag{6.14}$$

and consider the loop relationships of Figure 6.1

$$u = \sigma_u - \lambda, \tag{6.15}$$

$$\eta = \sigma_\eta + y. \tag{6.16}$$

By substitution of (6.13)-(6.16) into (6.5) and (6.6), we derive

$$\dot{V}_T(\xi) + T_T(\xi) \leq \omega^\top Q_T \omega + 2\omega^\top S_T \sigma + \sigma^\top R_T \sigma, \tag{6.17}$$

where $\sigma^\top = \begin{bmatrix}\sigma_u^\top & \sigma_\eta^\top\end{bmatrix}$, $\omega^\top = \begin{bmatrix}y^\top & \lambda^\top\end{bmatrix}$ and

$$Q_T = \begin{bmatrix} (Q+R_c) & (-S+S_c^\top) \\ (-S+S_c^\top)^\top & (R+Q_c) \end{bmatrix},$$

$$S_T = \begin{bmatrix} S & R_c \\ -R & S_c \end{bmatrix}, \text{ and } R_T = \begin{bmatrix} R & 0 \\ 0 & R_c \end{bmatrix}.$$

We take σ as the new input, ω as the new output and ξ as the new state of the interconnected system. From (6.7)-(6.9), we obtain $Q_T = 0$. Moreover, it is evident that $V_T(\xi^*) = T_T(\xi^*) = 0$ and $(V_T,T_T) > 0$, $\forall \xi \neq \xi^*$.

Clearly, $V_T > 0$ ($\dot{V}_T < 0$) is a Lyapunov function to the free system ($\sigma = 0$) [1], as

$$\dot{V}_T(\xi) \leq -T_T(\xi) < 0, \ \forall \xi \neq \xi^*, \xi \in \mathcal{E}. \tag{6.18}$$

From (6.11) and (6.15), $u = -\lambda$. Suppose that $(\mathcal{C})$ is designed in such a way that (6.10) holds, then matching of the desired steady-state values is achieved, which ensures equilibrium assignment: $\kappa \to \kappa^*$, $x \to x^*$ and $\lambda \to \lambda^*$, as $t \to \infty$. As a consequence, $\xi^{*\top} = \begin{bmatrix}x^{*\top} & \kappa^{*\top}\end{bmatrix}$ is an asymptotically stable equilibrium of the closed-loop. For any initial state $\xi(0) \in \mathcal{E}$, the dynamic controller $(\mathcal{C})$ asymptotically stabilizes the plant $(\mathcal{P})$ at x^*. If $\mathcal{E} = \mathbb{R}^{n+n_\kappa}$ and (V_T,T_T) are radially unbounded functions, then the stability condition holds globally [1].

Notice that the domain $(\mathcal{K} \times \mathcal{N})$ must be compatible with the region $(\mathcal{X} \times \mathcal{U})$ established for the plant. As $u = -\lambda$, the state of the controller is limited to the region around κ^* where $u \in \mathcal{U}$. Even if (6.6) holds, in principle, globally for the controller, the set of actually admissible initial values $(\mathcal{K} \times \mathcal{N})$ is still bounded if the dissipativity condition (6.5) is fulfilled only locally for the plant. □

Theorem 6.1 applies to the stabilization problem of input-affine systems. After having determined matrices (Q,S,R), and functions (V,T) that fulfill (6.5) for the plant ($\mathcal{P}$), a stabilizing controller ($\mathcal{C}$) that verifies (6.6) has to be designed. At the same time, conditions (6.7)-(6.11) must hold.

In [125], [126], [128], we have applied a similar framework for output feedback stabilization of nonlinear systems, with the difference that the notion of passivity indices was employed instead of QSR-dissipativity. That concept can be regarded as a special case of dissipativity, in the sense that the systems are considered to be square ($m = p$) and $S = \frac{1}{2}I$, $Q = -\rho$, $R = -\nu$. As a result, (6.5) can be rewritten as (with $\phi = 0$)

$$\dot{V} + T \leq u^\top y - \nu u^\top u - \rho y^\top y, \tag{6.19}$$

in a domain $(\mathcal{X} \times \mathcal{U}) \subset \mathbb{R}^{n+m}$ of (x,u). Similarly, a controller ($\mathcal{C}$) and indices (ν_c,ρ_c) must satisfy the inequality

$$\dot{V}_c + T_c \leq \eta^\top \lambda - \nu_c \eta^\top \eta - \rho_c \lambda^\top \lambda, \tag{6.20}$$

in $(\mathcal{K} \times \mathcal{N}) \subset \mathbb{R}^{n_\kappa+m}$, which is similar to relation (6.6).

6.4 Strict QSR-dissipativity in a Domain $\mathcal{X} \times \mathcal{U}$

Although it is possible to specify a combination (Q,S,R,V,T,ϕ) so that (6.5) holds globally for a certain system, that might not be the case in general. Thus, one usually has to search for a local solution, i.e. a solution that guarantees strict dissipativity in a neighborhood ($\mathcal{X} \times \mathcal{U}$) of ($x^*$,$u^*$). In this regard, note that (6.5) is satisfied if the function $l_p(x,u)$ introduced below attains a (possibly local) *maximum* ϕ at (x^*,u^*), which means

$$l_p(x,u) = \dot{V} + T - y^\top Q y - 2y^\top S u - u^\top R u \leq \phi, \tag{6.21}$$

$\forall (x,u) \in (\mathcal{X} \times \mathcal{U})$. Provided that $\nabla l_p(x^*,u^*) = 0$, this holds true, for example, if the Hessian matrix with respect to (x,u) is negative definite at the equilibrium, i.e. if

$$\nabla^2 l_p < 0 \text{ at } (x^*,u^*), \tag{6.22}$$

which is a sufficient condition for an isolated local maximum $l_p(x^*,u^*) = \phi$ to exist, if all second-order partial derivatives are continuous [35]. The same rationale can be applied to design a stabilizing controller, where a function $l_c(\kappa,\eta)$ must admit a *maximum* $-\phi$ at (κ^*,η^*):

$$l_c(\kappa,\eta) = \dot{V}_c + T_c - \lambda^\top Q_c \lambda - 2\lambda^\top S_c \eta - \eta^\top R_c \eta \leq -\phi, \tag{6.23}$$

$\forall(\kappa,\eta) \in (\mathcal{K} \times \mathcal{N})$, whereas if $\nabla l_c(\kappa^*,\eta^*) = 0$, then

$$\nabla^2 l_c < 0 \text{ at } (\kappa^*,\eta^*) \tag{6.24}$$

is a sufficient condition for the existence of a local maximum.

In Section 6.10, a constructive strategy for determining a local solution of (6.5) or (6.21) will be introduced, where the notion of a polytopic LMI and the Finsler's lemma are applied. This strategy is neither based on linearizations nor on Hessians, consisting in a completely diverse approach. Though restricted to the class of the possibly rational systems, the method provides an interesting and fairly useful way of determining a solution (Q,S,R,V,T) and is able to handle a number of practically relevant plants.

6.5 Strict QSR-dissipativity in a Domain $\mathcal{X} \times \mathbb{R}^m$

The feasibility of conditions (6.5) and (6.6) can be investigated through a variety of strategies. A possible approach is provided by the following matrix inequality, which enables us to establish a sufficient condition for dissipativity in terms of the state x alone.

Lemma 6.1. *A plant $(\mathcal{P})$ described by (6.1)-(6.2) is strictly QSR-dissipative in a domain $\mathcal{X} \subset \mathbb{R}^n$ if*

$$\begin{bmatrix} \nabla V^\top f + T - \phi & \frac{1}{2}\nabla V^\top g - h^\top S \\ [\frac{1}{2}\nabla V^\top g - h^\top S]^\top & -[S^\top d + d^\top S] \end{bmatrix} - \begin{bmatrix} h^\top \\ d^\top \end{bmatrix} Q \begin{bmatrix} h & d \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix} = \mathcal{H}(x) \le 0, \ \forall x \in \mathcal{X}, \tag{6.25}$$

for functions $(V,T) > 0$, $\forall x \in \mathcal{X}$, matrices (Q,S,R), and $\phi \in \mathbb{R}$ as defined in Theorem 6.1.

Proof. According to (6.5), $(\mathcal{P})$ is strictly QSR-dissipative if

$$\begin{aligned} \nabla V^\top (f + gu) + T \le {} & (h + du)^\top Q(h + du) \\ & 2(h + du)^\top Su + u^\top Ru + \phi, \end{aligned} \tag{6.26}$$

which is equivalent to

$$\begin{bmatrix}1\\u\end{bmatrix}^\top \left\{ \begin{bmatrix} \nabla V^\top f + T - \phi & \frac{1}{2}\nabla V^\top g - h^\top S \\ [\frac{1}{2}\nabla V^\top g - h^\top S]^\top & -[S^\top d + d^\top S] \end{bmatrix} \right.$$
$$\left. - \begin{bmatrix} h^\top Qh & h^\top Qd \\ [h^\top Qd]^\top & d^\top Qd \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix} \right\} \begin{bmatrix}1\\u\end{bmatrix} \leq 0.$$

One easily concludes that (6.25) is a sufficient condition to the validity of the relation above in a domain $\mathcal{X}$, for all $u \in \mathbb{R}^m$. □

An interesting feature of condition (6.25) is that it does not dependent on the control vector u, which means that the existence of a domain of validity for strict dissipativity can be guaranteed by verifying the state variables alone. In fact, estimating the size of such a domain $\mathcal{X}$ is closely related to the task of estimating a domain of attraction for asymptotic stability.

Notice that $\mathcal{H}(x) \leq 0$ constitutes an infinite dimensional LMI, whose negative semidefiniteness may be hard to test [34]. Nevertheless, for each value of x, $\mathcal{H}(x)$ is a partitioned symmetric matrix of the form

$$\mathcal{H}(x) = \begin{bmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} \\ \mathcal{H}_{12}^\top & \mathcal{H}_{22} \end{bmatrix}, \tag{6.27}$$

so that $\mathcal{H} \leq 0$ in $\mathcal{X}$ if and only if [27]

$$\mathcal{H}_{22} \leq 0, \tag{6.28}$$
$$\mathcal{H}_{12}^\top = \mathcal{H}_{22}\mathcal{H}_{22}^\dagger\mathcal{H}_{12}^\top, \tag{6.29}$$
$$\mathcal{H}_{11} - \mathcal{H}_{12}\mathcal{H}_{22}^\dagger\mathcal{H}_{12}^\top \leq 0, \tag{6.30}$$

$\forall x \in \mathcal{X}$, where $\mathcal{H}_{22}^\dagger$ is the generalized inverse of $\mathcal{H}_{22}$. Besides,

$$\mathcal{H}_{11} = \nabla V^\top f + T - h^\top Qh - \phi, \tag{6.31}$$

and

$$\mathcal{H}_{12} = \frac{1}{2}\nabla V^\top g - h^\top S - h^\top Qd. \tag{6.32}$$

Finally, suppose that for a given system ($\mathcal{P}$), the following holds $\forall x \in \mathcal{X}$

$$\mathcal{H}_{22} = -d^\top Qd - [S^\top d + d^\top S] - R < 0. \tag{6.33}$$

This guarantees that (6.28) and (6.29) hold, as the inverse matrix $\mathcal{H}_{22}^{\dagger} = \mathcal{H}_{22}^{-1}$ exists. Thus, according to (6.30), (6.25) is fulfilled in a region $\mathcal{X} \subset \mathbb{R}^n$ of the state-space if

$$c_p(x) = \mathcal{H}_{11} - \mathcal{H}_{12}\mathcal{H}_{22}^{-1}\mathcal{H}_{12}^{\top} \leq 0, \tag{6.34}$$

$\forall x \in \mathcal{X}$. For fixed ($Q$,$S$,$R$,$T$,$\phi$), relation (6.34) is a *nonlinear first-order* PDI and can also be expressed as

$$c_p(x) = \nabla V^{\top} M_1 \nabla V + \nabla V^{\top} M_2 + M_3 \leq 0, \tag{6.35}$$

with

$$M_1 = \frac{1}{4} g \Omega g^{\top} \geq 0, \quad \Omega = -H_{22}^{-1}, \tag{6.36}$$

$$M_2 = f - g\Omega(d^{\top} Q + S^{\top})h, \tag{6.37}$$

$$\begin{aligned} M_3 = T - \phi + h^{\top}(S\Omega S^{\top} - Q)h \\ + h^{\top}\left[Qd\Omega d^{\top} Q + Qd\Omega S^{\top} + \left(Qd\Omega S^{\top} \right)^{\top} \right] h, \end{aligned} \tag{6.38}$$

whereas (Q,S,R,T,ϕ) must be specified in such manner that the feasibility of the PDI with respect to V is achieved.

An alternative formulation for condition (6.35) consists in presenting it as the following PDE

$$\nabla V^{\top} M_1 \nabla V + \nabla V^{\top} M_2 + M_3 + \tau(x) = 0, \tag{6.39}$$

where $\tau : R^n \to \mathbb{R}$, $\tau(x) \geq 0$, is a suitable continuous function. In fact, for an appropriate combination (Q,S,R,T,τ,ϕ), (6.39) is a first order nonlinear PDE, which can be solved for V around an equilibrium point x^* via the well-known *method of the characteristics* [57]. If a solution can be determined, then (6.35) is also valid.

Again, given that $\nabla c_p(x^*) = 0$, a sufficient condition for the local validity of (6.35) is that the Hessian of c_p with regard to x is negative definite at the desired equilibrium, i.e.

$$\nabla^2 c_p < 0 \text{ at } x = x^*, \tag{6.40}$$

which means that x^* is an isolated maximum of c_p [35]. In this case, there is a neighborhood around the equilibrium where this function is strictly negative.

In the particular case of the LTI systems, where $\phi = 0$ and $(x^*, u^*) = (0,0)$, a sufficient condition for (6.5) can be formulated in terms of a simple LMI. Consider the following linear model (Γ_l) introduced in Chapter 3 and repeated below

$$(\Gamma_l)\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du, \end{cases} \tag{6.41}$$

Assume positive definite functions

$$V(x) = x^\top P x, \quad P > 0, \quad P = \mathbb{R}^{n\times n}, \tag{6.42}$$

$$T(x) = x^\top N x, \quad N > 0, \quad N = \mathbb{R}^{n\times n}. \tag{6.43}$$

Lemma 6.2. *([49]) The linear plant (6.41) is strictly QSR-dissipative if*

$$\begin{bmatrix} (A^\top P + P^\top A + N) & (PB - C^\top S) \\ (PB - C^\top S)^\top & -(S^\top D + D^\top S) \end{bmatrix} - \begin{bmatrix} C^\top \\ D^\top \end{bmatrix} Q \begin{bmatrix} C & D \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix} \leq 0, \tag{6.44}$$

for (Q,S,R), and $(P,N) > 0$ as introduced in (6.42) and (6.43).

6.6 Dynamic Output Feedback Design

Once we have solved (6.21) or (6.34) for the $(\mathcal{P})$, we can use this information to design a stabilizing controller $(\mathcal{C})$. First of all, we fix (Q_c, S_c, R_c) according to (6.7)-(6.9). In the next step, we have to fulfill (6.6), i.e. a local maximum of $l_c(\kappa, \eta)$ has to be found. The linear or nonlinear dynamics (z, q, r, s) in (6.3)-(6.4) and the functions $(V_c, T_c) > 0$ are parameters to be specified for the controller. Moreover, the following set of steady-state relations must be verified

$$\begin{aligned} 0 &= z(\kappa^*) + q(\kappa^*)\eta^* \\ \lambda^* &= r(\kappa^*) + s(\kappa^*)\eta^*. \end{aligned} \tag{6.45}$$

Alternatively, a sufficient condition similar to that of Lemma 6.1 can be employed as well, which means

$$\begin{bmatrix} \nabla V_c^\top f + T_c + \phi & \frac{1}{2}\nabla V_c^\top g_c - h_c^\top S_c \\ [\frac{1}{2}\nabla V_c^\top g_c - h_c^\top S_c]^\top & -[S_c^\top d_c + d_c^\top S_c] \end{bmatrix} - \begin{bmatrix} h_c^\top \\ d_c^\top \end{bmatrix} Q_c \begin{bmatrix} h_c & d_c \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & R_c \end{bmatrix} = \mathcal{H}_c(\kappa) \leq 0, \tag{6.46}$$

which is analogous to (6.34) and involves the definition of a function $c_c(\kappa)$ similar to c_p for the controller. Finally, given (6.11), (6.15) and (6.16), $(\mathcal{C})$ must be designed in such a manner that (6.45) and (6.10) hold. Then, the conditions of Theorem 6.1 are satisfied and the interconnected system of Figure 6.1 has an asymptotically stable equilibrium at $\xi^{*\top} = [x^{*\top} \;\; \kappa^{*\top}]$.

If we restrict our analysis to the case of linear dynamic output feedback, then we have the following state-space representation for the controller

$$(\mathcal{C}) \begin{cases} \dot{\kappa} = A_c\kappa + B_c\eta \\ \lambda = C_c\kappa + D_c\eta, \end{cases} \tag{6.47}$$

where (A_c,B_c,C_c,D_c) are real matrices with compatible dimensions. According to Lemma 6.2, a sufficient condition for such a controller to be strictly QSR-dissipative is

$$\begin{bmatrix} (A_c^\top P_c + P_c^\top A_c + N_c) & (P_cB_c - C_c^\top S_c) \\ (P_cB_c - C_c^\top S_c)^\top & -(S_c^\top D_c + D_c^\top S_c) \end{bmatrix} \\ - \begin{bmatrix} C_c^\top \\ D_c^\top \end{bmatrix} Q_c \begin{bmatrix} C_c & D_c \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & R_c \end{bmatrix} \leq 0, \tag{6.48}$$

where

$$V_c(x) = x_c^\top P_c x_c, \quad P_c > 0, \quad P_c = \mathbb{R}^{n\times n}, \tag{6.49}$$

$$T_c(x) = x_c^\top N_c x_c, \quad N_c > 0, \quad N_c = \mathbb{R}^{n\times n}. \tag{6.50}$$

6.7 Static Output Feedback Design

In the following theorem, a simple adaptation of the previous results for the problem of static output feedback is presented. The proof will be omitted, as it is almost identical to that of the dynamic output feedback case. The difference is that, here, we have $V_c = N_c = 0$, as a static controller does not have a state.

Theorem 6.2. *Consider the feedback interconnection of Figure 6.1. Then, suppose that a plant $(\mathcal{P})$ described by (6.1)-(6.2) fulfills*

$$\dot{V} + T \leq y^\top Q y + 2y^\top S u + u^\top R u + \phi, \;\; \forall (x,u) \in (\mathcal{X} \times \mathcal{U}), \tag{6.51}$$

for matrices (Q,S,R) and functions $(V,T) > 0$, $\forall x \neq x^*$, $x \in \mathcal{X}$, with $V(x^*) = T(x^*) = 0$. The additional variable $\phi \in \mathbb{R}$ is a constant.

Furthermore, suppose that a static controller ($\mathcal{C}$) and matrices (Q_c,S_c,R_c) satisfy the inequality

$$0 \leq \lambda^\top Q_c \lambda + 2\lambda^\top S_c \eta + \eta^\top R_c \eta - \phi, \ \forall \eta \in \mathcal{N}, \tag{6.52}$$

where $\mathcal{N} \subset \mathbb{R}^p$ is a domain of η around the steady-state values η^*. As λ is considered as a function of η alone, inequality (6.52) can be interpreted as a function of η in a domain $\mathcal{N} \subset \mathbb{R}^p$.

If

$$Q_c = -R, \tag{6.53}$$

$$S_c = S^\top, \tag{6.54}$$

$$R_c = -Q, \tag{6.55}$$

and

$$\lambda^* = -u^*, \tag{6.56}$$

with

$$\sigma_u = 0 \text{ and } \sigma_\eta = 0, \tag{6.57}$$

then x^* is an asymptotically stable equilibrium of the closed-loop in $\mathcal{X}$. If $\mathcal{X} = \mathbb{R}^n$, $\mathcal{U} = \mathbb{R}^m$, $\mathcal{N} = \mathbb{R}^p$ and (V,T) are radially unbounded, then the equilibrium point is globally asymptotically stable.

This theorem provides a framework for solving the linear static output feedback control problem, as a controller of the form $\lambda = D_c \eta$ ($d_c = D_c$) can be applied, where D_c is a constant gain. In this regard, suppose that (6.51) is fulfilled for the plant, with $\phi \geq 0$. As a consequence, a necessary condition for solving (6.52) is given by

$$\mathcal{H}_c = -D_c^\top Q_c D_c^\top - S_c^\top D_c - D_c^\top S_c - R_c \leq 0, \tag{6.58}$$

which is also sufficient if $\phi < 0$. Consider then the following statement regarding the feasibility of (6.58) in this scenario.

Proposition 6.1. *Let us suppose that a plant ($\mathcal{P}$) described by (6.1)-(6.2) fulfills (6.51) with $\phi = 0$ and matrices (Q,S,R) such that*

$$\Delta = S R^{-1} S^\top - Q \geq 0, \tag{6.59}$$

where $R = \nu I$, $\nu \in \mathbb{R}$. Then, by applying relations (6.53)-(6.57), there exist real gains

$$D_{c1} = R^{-1}(S^\top - \bar{\Delta}^{\frac{1}{2}}), \quad D_{c2} = R^{-1}(S^\top + \bar{\Delta}^{\frac{1}{2}}), \tag{6.60}$$

where $\bar{\Delta} = SS^\top - RQ$, to which (6.58) is fulfilled with equality. In addition, if $R > 0$ ($Q_c < 0$), then any gain in the interval $D_{c1} \leq D_c \leq D_{c2}$ is also a solution to $\mathcal{H}_c \leq 0$.

Proof. We prove that D_{c2} satisfies (6.58). The proof for D_{c1} is similar and will be omitted. As (6.59) holds, there exists a symmetric square root matrix $\bar{\Delta}^{\frac{1}{2}} \geq 0$ [34] and a gain D_{c2} as proposed in (6.60). By substitution of D_{c2} into (6.58) and from (6.53)-(6.55), we obtain

$$\begin{aligned}
\mathcal{H}_c &= D_c^\top R D_c - S D_c - S^\top D_c^\top + Q \\
&= (S^\top - \bar{\Delta}^{\frac{1}{2}})^\top R^{-1}(S^\top - \bar{\Delta}^{\frac{1}{2}}) - SR^{-1}(S^\top - \bar{\Delta}^{\frac{1}{2}}) \\
&\qquad\qquad\qquad - (S^\top - \bar{\Delta}^{\frac{1}{2}})^\top R^{-1} S^\top + Q \\
&= (\bar{\Delta}^{\frac{1}{2}})^\top R^{-1} \bar{\Delta}^{\frac{1}{2}} - SR^{-1}S^\top + Q.
\end{aligned} \tag{6.61}$$

As $R = \nu I$, we have

$$\begin{aligned}
\mathcal{H}_c &= R^{-1}\Big((\bar{\Delta}^{\frac{1}{2}})^\top \bar{\Delta}^{\frac{1}{2}} - SS^\top\Big) + Q, \\
&= R^{-1}(\bar{\Delta} - SS^\top) + Q = 0,
\end{aligned} \tag{6.62}$$

which proves that D_{c2} is a solution. The same holds true for D_{c1}.

Then, consider $\nu > 0$ and a convex combination of D_{c1} and D_{c2} given by

$$D_\alpha = \alpha D_{c1} + (1-\alpha) D_{c2} = R^{-1} S^\top + (1 - 2\alpha) R^{-1} \bar{\Delta}^{\frac{1}{2}},$$

where $0 \leq \alpha \leq 1$. By substitution of D_α in (6.58) and setting $\beta = 1 - 2\alpha$, we have

$$\mathcal{H}_c = R^{-1}(\beta^2 \bar{\Delta} - SS^\top) + Q = \beta^2 \Delta - \Delta.$$

Since $-1 \leq \beta \leq 1$ and $\Delta \geq 0$, we obtain $\mathcal{H}_c \leq 0$ for any gain D_α. □

If $(x^*, u^*) = (0,0)$, then any matrix D_α is a stabilizing gain. If this is not the case, then we have to determine which value of D_α satisfies (6.56).

Proposition 6.1 is important because it determines whether a system is static output feedback stabilizable or a dynamic controller is necessary. The latter is the case if the gain D_c necessary for verifying (6.56) is not inside the interval defined by $D_{c1} \leq D_c \leq D_{c2}$. It means that $\Delta \geq 0$ is a necessary condition for output feedback stabilization through this framework, though it may not be sufficient.

6.8 State Feedback Stabilization

The problem on the existence of a Lyapunov function which guarantees asymptotic stability for a dynamical system around a certain equilibrium x^* is a key issue in the field of control theory. In the case of linear systems, for instance, it is a well-known fact that a plant is asymptotically stable if and only if there exists a Lyapunov function for the closed-loop [80], an issue that we have already addressed previously in this thesis. In the nonlinear case, though, such an equivalence cannot be as widely established as in the framework of the linear plants, as Lipschitz continuity of the vector field $\dot{x} = f(x)$ is required [41], [42]. In this section, exceptionally, we restrict our analysis to the class of systems to which this equivalence holds, i.e. we suppose that the following result applies.

Theorem 6.3. *Consider a plant ($\mathcal{P}$) described by (6.1)-(6.2). There exists a feedback control $u(x)$ that asymptotically stabilizes the closed-loop at x^* in a neighborhood $\mathcal{X} \subset \mathbb{R}^n$ in such a way that there exists a function $V : \mathbb{R}^n \mapsto \mathbb{R}$, with $V(x^*) = 0$, which is positive definite in $\mathcal{X}$, and the following relation holds*

$$\nabla V^\top [f(x) + g(x)u(x)] < 0, \;\; \forall x \in \mathcal{X}, \;\; x \neq x^*. \tag{6.63}$$

Due to the multiplication between $V(x)$ and $u(x)$, condition (6.63) constitutes a nonlinear design problem. For the plants which fulfill Theorem 6.3, it is possible to establish dissipativity-based stabilizability conditions similar to those introduced in Chapter 5 for LTI models. Firstly, consider the following sufficient matrix condition for local strict QSR-dissipativity obtained from Lemma 6.1 with feedthrough $d = 0$:

$$\begin{bmatrix} \nabla V^\top f + T - h^\top Q h & \frac{1}{2}\nabla V^\top g - h^\top S \\ (\frac{1}{2}\nabla V^\top g - h^\top S)^\top & -R \end{bmatrix} \leq 0, \tag{6.64}$$

$\forall x \in \mathcal{X}$, where the function $T : \mathbb{R}^n \mapsto \mathbb{R}$, with $T(x^*) = 0$ is positive definite in $\mathcal{X}$. Then, we present the subsequent result based on Theorem 5.1, though valid for nonlinear systems and involving sets of linear and possibly nonlinear PDEs rather than LMIs and SDP conditions.

Theorem 6.4. *Suppose that the nonlinear plant (6.1) belongs to the class of systems covered by Theorem 6.3, i.e. condition (6.63) is feasible. This plant is asymptotically stabilizable by a certain control law $u(x)$ if it is feedback equivalent to a (locally) strictly QSR-dissipative system whose output is given by (6.2) (with $d = 0$), and the following condition holds*

$$\Delta = SR^{-1}S^\top - Q \geq 0, \tag{6.65}$$

where $R > 0$. A stabilizing state feedback is given by the closed-form expression

$$u(x) = -R^{-1}S^\top h(x). \tag{6.66}$$

Proof. Without loss of generality, we assume $R = \nu I$ in (6.64), with $0 < \nu \in \mathbb{R}$. And if the inequality holds with $\nu = \frac{1}{\beta}$, then we have that $\forall x \in \mathcal{X}$

$$\nabla V^\top f + T - h^\top Qh + \beta\left[\frac{1}{2}\nabla V^\top g - h^\top S\right]\left[\frac{1}{2}\nabla V^\top g - h^\top S\right]^\top \leq 0. \tag{6.67}$$

Suppose that (6.64) is feasible in $\mathcal{X}$ for (Q,S,R), (V,T)> 0, $R > 0$, i.e. (6.1) is strictly QSR-dissipative. From (6.67), we obtain

$$\begin{aligned}\nabla V^\top f + T - h^\top Qh + \frac{1}{4}\nabla V^\top gR^{-1}g^\top\nabla - \frac{1}{2}\nabla V^\top gR^{-1}S^\top h\\ -\frac{1}{2}h^\top SR^{-1}g^\top\nabla V + h^\top SR^{-1}S^\top h \leq 0,\end{aligned} \tag{6.68}$$

Equivalently,

$$\nabla V^\top[f - gR^{-1}S^\top h] \leq -\frac{1}{4}\nabla V^\top gR^{-1}g^\top\nabla V - T - h^\top\Delta h, \tag{6.69}$$

with $\Delta = SR^{-1}S^\top - Q$. If $\Delta \geq 0$, then (6.63) certainly holds with the control $u(x) = -R^{-1}S^\top h(x)$, and the system is asymptotically stabilizable. □

This framework includes systems that are state feedback or linear static output feedback stabilizable. In the case of state feedback, the stabilization problem amounts for determining a fictitious output $h(x)$ which makes it possible to render the closed-loop strictly dissipativity with $\Delta \geq 0$. Notice that, as $R > 0$, it is necessary to solve the partial differential inequality (PDI) below in order to fulfill (6.64)

$$\nabla V^\top f + T - h^\top Qh < 0. \tag{6.70}$$

The solvability of (6.70) is equivalent to the existence of a positive definite function $\tau(x)$ such that the following linear PDE has a local solution $V > 0$, with (Q,h,τ) fixed.

$$\nabla V^\top f + T - h^\top Qh + \tau(x) = 0. \tag{6.71}$$

This expression can be used in order to reduce the number of free parameters we encounter when solving (6.68). A papameterized solution of (6.70) can be substituted into (6.68), which has to be solved for (Q,S,R). In the context of SOF, the output $h(x)$ is given and we have to determine matrices (Q,S,R) and a function $V(x) > 0$ by solving the respective PDEs.

6.9 Polynomial Systems - An SOS Approach

If we restrict the systems in consideration to be polynomial, then sum of squares (SOS) strategies can be employed for dissipativity analysis. A brief introduction to this topic is provided in Appendix B, and a detailed discussion is available in [111]. In this context, notice that (6.5) depends affinely on the parameters (Q,S,R,ϕ) and on the coefficients of the polynomial functions (V,T). Hence, there exists no multiplication involving these terms and the question of the feasibility of (6.5) can be approached by a standard SOS optimization problem [106].

Consider, then, polynomial functions $V_s(x) > 0$ and $T_s(x) > 0$. A sufficient condition for (6.5) is given by the following SOS program

$$\begin{aligned} \textit{find} \quad & (Q,S,R,V,T,\phi), \\ \textit{such that} \quad & V - V_s \textit{ is SOS}, \\ & T - T_s \textit{ is SOS}, \\ & -[\dot{V} + T - y^\top Q y - 2y^\top S u - u^\top R u - \phi] \textit{ is SOS}, \end{aligned} \tag{6.72}$$

where the degrees and general structures of (V,T) must be specified in advance. An implementation of this program with semidefinite programming tools[1)] is straightforward. If the optimization problem terminates successfully, then the coefficients of (V,T), the dissipativity matrices (Q,S,R) and a real number ϕ are determined.

Analogously, a sufficient condition for (6.25) can be presented using SOS, as $\mathcal{H}(x) \leq 0$ if the following program is feasible

$$\begin{aligned} \textit{find} \quad & (Q,S,R,V,T,\phi), \\ \textit{such that} \quad & V - V_s \textit{ is SOS}, \\ & T - T_s \textit{ is SOS}, \\ & -b^\top \mathcal{H}(x) b \text{ is } \textit{SOS}, \end{aligned} \tag{6.73}$$

where $b \in \mathbb{R}^{(m+1)}$ is a vector composed of additional degrees of freedom, i.e. additional variables, which may depend on the state x. The introduction of b allows us to preserve the linearity of condition $-b^\top \mathcal{H}(x) b \geq 0$ with respect to (Q,S,R,V,T,ϕ), which does not apply to (6.34), for instance.

Evidently, it is quite a challenging task to solve (6.72) or (6.73), as they represent a global optimization problem in (x,u). Thus, a more realistic approach consists, again, in investigating the possibility of having local dissipativity. In the case of condition (6.34), for example, c_p is a quadratic function in the decision

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1) For instance, using MATLAB® and SOSTOOLS [110].

variables, which means that conventional SOS strategies cannot be applied in order to determine all parameters (Q,S,R,V,T,ϕ) simultaneously. Nevertheless, for fixed values of (Q,S,R,V,T,ϕ), an estimate of the domain $\mathcal{X}$ where (6.34) holds can be obtained by the program below

$$\begin{array}{ll} \textit{max} & c \\ \textit{subject to} & (V(x)-c)+p_s(x)(-c_p(x)) \textit{ is SOS,} \end{array} \tag{6.74}$$

where a constant $0 < c \in \mathbb{R}$ and a polynomial $p_s(x) \geq 0$ have to be determined. If (6.74) is feasible, then a domain $\mathcal{X}$ is given by the compact set

$$\mathcal{X} = \{x : V(x) \leq c\}.$$

If $c = \infty$, then the QSR-dissipativity holds globally.

In [126], a similar local SOS strategy based on the notion of passivity indices was presented in order to estimate a domain where a given pair of indices hold for a polynomial nonlinear plant. In [128], an extension of these results for possibly rational systems was proposed in terms of an LMI condition based on Finsler's lemma and linear annihilators, which allows for less conservative estimates of the domain. Both approaches require, though, the assumption of a fixed Lyapunov function V, in such a manner that the local validity of the indices is guaranteed by the existence of a nonempty set $\mathcal{X}$. However, a more realistic scenario comprises precisely the necessity of determining V and (Q,S,R,T,ϕ) in a constructive way, as it might not be a simple task to guess a Lyapunov function in general. This is the subject of the next section, which contains the main contributions of this chapter and provides relevant improvements when compared to the framework introduced in [128], which is not constructive.

6.10 Rational Systems - A Polytopic LMI Condition

Though simple and useful, an SOS approach for local dissipativity analysis usually provides quite conservative results. Then, in this section, we apply the so-called Finsler's Lemma and the notion of linear annihilators for enlarging the estimates of the domains involved by reformulating the estimation problem as a polytopic LMI condition. We also extend previous results published in [128], as the concept of QSR-dissipativity is employed instead of the idea of passivity indices and a constructive method, which allows for determining all parameters (Q,S,R,V,T,ϕ) simultaneously, is introduced. We consider the class of the rational nonlinear systems with possibly rational Lyapunov functions, which includes polynomial systems as a particular case. Without loss of generality, we assume the equilibrium point of (6.1)-(6.2) to be the origin, i.e. $(x^*,u^*) = (0,0)$. Nonetheless, our results apply for any possible point (x^*,u^*).

Lemma 6.3. *([64]) **Finsler's Lemma.** Consider $\mathcal{X}_p \subseteq \mathbb{R}^{n_p}$ a given polytopic set. $\hat{S} : \mathcal{X}_p \mapsto \mathbb{R}^{n_q \times n_q}$ and $\hat{K} : \mathcal{X}_p \mapsto \mathbb{R}^{n_r \times n_q}$ are given matrix functions, where $\hat{S}$ is symmetric. The following conditions are equivalent:*

(i) $\forall x_p \in \mathcal{X}_p$ the condition that $z^\top \hat{S}(x_p) z > 0$ is satisfied $\forall z \in \mathbb{R}^{n_q}$: $\hat{K}(x_p) z = 0$.

(ii) $\forall x_p \in \mathcal{X}_p$ there exists a matrix function $L : \mathcal{X}_p \mapsto \mathbb{R}^{n_q \times n_r}$ such that $\hat{S}(x_p) + L(x_p)\hat{K}(x_p) + \hat{K}(x_p)^\top L^\top(x_p) > 0$.

If $\hat{S}$ and $\hat{K}$ are affine functions of x_p and L is a constant matrix, then *(ii)* becomes a polytopic LMI condition. Although *(i)* and *(ii)* are no longer equivalent, *(ii)* is still a sufficient condition for *(i)*. Moreover, there are numerically efficient tools to test condition *(ii)*, while *(i)* is very hard to test [64].

Definition 6.1. ([64]) **Annihilator.** Given a function $\hat{l} : \mathbb{R}^{n_q} \mapsto \mathbb{R}^{n_s}$ and a positive integer n_r, a matrix function $\mathcal{N}_{\hat{l}}$: $\mathbb{R}^{n_q} \mapsto \mathbb{R}^{n_r \times n_s}$ is called an *annihilator* of $\hat{l}$ if

$$\mathcal{N}_{\hat{l}}(z)\, \hat{l}(z) = 0, \tag{6.75}$$

$\forall z \in \mathbb{R}^{n_q}$ of interest. If in addition $\mathcal{N}_{\hat{l}}$ is a linear function, then it is said to be a *linear annihilator*.

As in [64], which is the main reference of this section, suppose that $\hat{l}(z) = z = [z_1 \ldots z_{n_q}]^\top \in \mathbb{R}^{n_q}$. A linear annihilator $\mathcal{N}_{\hat{l}} \in \mathbb{R}^{(n_q-1)\times n_q}$ can be expressed by

$$\mathcal{N}_{\hat{l}}(z) = \begin{bmatrix} z_2 & -z_1 & 0 & 0 & \ldots & 0 \\ 0 & z_3 & -z_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & z_{n_q} & -z_{n_q-1} \end{bmatrix}, \tag{6.76}$$

which is one of the simplest solutions possible for (6.75). Then, considering all possible pairs (z_i,z_j) for $i \neq j$ without repetition, i.e. $\forall (i,j) \in \mathbb{I}_{n_q}$ $(j > i)$, a general closed-form expression for a linear annihilator can be deduced [64]

$$\mathcal{N}_{\hat{l}}(z) = \begin{bmatrix} \Phi_1(z) & \Upsilon_1(z) \\ \vdots & \vdots \\ \Phi_{(n_q-1)}(z) & \Upsilon_{(n_q-1)}(z) \end{bmatrix}, \tag{6.77}$$

where

$$\begin{aligned} \Upsilon_i(z) &= -z_i I_{(n_q-i)},\ i \in \mathbb{I}_{(n_q-1)}, \\ \Phi_1(z) &= [z_2\ \ldots\ z_{n_q}]^\top, \\ \Phi_i(z) &= \begin{bmatrix} & z_{(i+1)} \\ 0_{(n_q-i)\times(i-1)} & \vdots \\ & z_{n_q} \end{bmatrix},\ i \in \{2,\ldots,n_q-1\}, \end{aligned} \tag{6.78}$$

$\mathcal{N}_{\hat{l}}(z) \in \mathbb{R}^{n_r \times n_q}$ with $n_r = \sum_{j=1}^{n_q-1} j$.

Our next step consists in connecting Lemma 6.3 and Definition 6.1 to the stabilization framework introduced in the previous sections and based on the notion of local QSR-dissipativity. Consider, in the following, a possibly rational function $t(x_p)$ whose non-negativeness in a polytopic set $\mathcal{X}_p$ is to be investigated. In Lemma 6.3, x_p represents the extended vector $[x^\top\ u^\top]^\top$, if we employ condition (6.21) with $\phi = 0$, which means $t(x_p) = -l_p(x,u)$. In addition, suppose that

$$0 = G(x_p)x_p + F(x_p)\pi, \tag{6.79}$$

and the following:

1. $x_p \in \mathbb{R}^{n_p}$ denotes the variable in consideration and $\mathcal{X}_p$ is a given polytope containing $x_p^* = 0$.

2. $\pi : \mathcal{X}_p \mapsto \mathbb{R}^{n_\pi}$ is a vector of nonlinear functions. It is a basis from which we can represent the set of nonlinear rational functions of interest. π is a function of x_p.

3. $G : \mathcal{X}_p \mapsto \mathbb{R}^{n_\pi \times n_p}$ and $F : \mathcal{X}_p \mapsto \mathbb{R}^{n_\pi \times n_\pi}$ are affine matrix functions of x_p.

4. The matrix $F(x_p)$ is invertible for all values of $x_p \in \mathcal{X}_p$. Under this regularity condition, the decomposition (6.79) of $t(x_p)$ in terms of a basis π is well-posed as $\pi(x_p) = -(F^{-1}(x_p)G(x_p))x_p$ is well-defined $\forall x_p \in \mathcal{X}_p$.

In the sequel, we establish our main result of this chapter, which connects well-established concepts in the field of LMI feasibility tests to the output feedback control framework introduced previously. The theorem below provides a constructive approach for investigating the local dissipativity of a possibly rational dynamical system.

Theorem 6.5. *Let $x_p = [x^\top \ u^\top]^\top$ and $t(x_p) = -l_p$ be a rational function representing dissipativity condition (6.21) with $(x^*,u^*) = (0,0)$ and $\phi = 0$. Furthermore, let $x_p \in \mathcal{X}_p$ be a given polytope around (x^*,u^*) and suppose that $t(x_p)$ can be decomposed in the following manner*

$$t(x_p) = \pi_p^\top Q_p \pi_p, \tag{6.80}$$

where Q_p is symmetric, independent of x_p, and depends affinely on all the unkown parameters of (Q,S,R,V,T). Vector π_p contains all basis functions (including polynomial and rational terms) necessary for expressing $t(x_p)$ as in (6.80). Consider

$$\pi_p = \begin{bmatrix} x_p \\ \pi \end{bmatrix} \text{ and } C_p(x_p) = \begin{bmatrix} G(x_p) & F(x_p) \end{bmatrix}, \tag{6.81}$$

whereas equation (6.79) defines a basis function π and C_p is a linear annihilator of π_p. Assume F invertible in $\mathcal{X}_p$. Then, testing whether $t(x_p) \geq 0$ in $\mathcal{X}_p$ can be rewritten as the following LMI condition, which has to be satisfied $\forall x_p \in \mathcal{V}(\mathcal{X}_p)$ (at the vertices of $\mathcal{X}_p$)

$$Q_p + L_p C_p + C_p^\top L_p^\top \geq 0, \tag{6.82}$$

where L_p is a scaling and constant matrix to be determined, as well as (Q,S,R) and the coefficients of (V,T). If (6.82) holds at the vertices of $\mathcal{X}_p$, then the system in consideration is said to be locally strict QSR-dissipative in this domain defined by this polytope.

Proof. This result is a direct application of the theory presented in [64] and Lemma 6.3. Function $t(x_p)$ can always be decomposed as in (6.80), where Q_p is a symmetric matrix. Besides, C_p is a linear annihilator of π_p. Then, according to [64], $t(x_p) \geq 0$, $\forall x_p \in \mathcal{X}_p$, if condition (6.82) if fulfilled at the vertices of the polytope $\mathcal{X}_p$, by convexity. If $\phi \neq 0$, the theorem is still valid if we consider $x_p = [1 \ x^\top \ u^\top]^\top$, which allows for handling constant terms of $t(x_p)$ in this case. If $(x^*,u^*) \neq (0,0)$, a polytopic condition around this point can be tested as well. □

If condition (6.35) is considered instead of (6.21), we have $t(x_p) = -c_p$,where $x_p = x$. In order to estimate a domain $\mathcal{X}_p$, one has to fix (Q,S,R,V,T,ϕ), as (6.35) is not an affine function of those variables. A polytopic LMI condition for estimating $\mathcal{X}_p$ for a fixed Lyapunov function was presented in [128], for instance, where the notion of passivity indices was employed. In that publication, we compared the SOS approach with polytopic LMI estimates and could verify that the latter, as expected, provides much less conservative results than the former.

The core of the proposed approach consists in specifying suitable annihilators and, in fact, they can be easily obtained and there exist systematic ways to do this. The following examples borrowed from [64] illustrate the procedure. Consider, for example, a two-dimensional polynomial system of degree three, with. $x_p = [x_1\ x_2]^\top$. Suppose that

$$\pi_2 = \begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_2^2 \end{bmatrix}, \ \pi_3 = \begin{bmatrix} x_1^3 \\ x_1^2 x_2 \\ x_1 x_2^2 \\ x_2^3 \end{bmatrix}, \text{ and } \pi = \begin{bmatrix} \pi_2 \\ \pi_3 \end{bmatrix}. \tag{6.83}$$

Besides,

$$\pi_2 = \Phi_1 x_p, \ \pi_3 = \Phi_2 \pi_2, \tag{6.84}$$

$$\Phi_1 = \begin{bmatrix} x_1 row_1(I_2) \\ x_2 I_2 \end{bmatrix} \text{ and } \Phi_2 = \begin{bmatrix} x_1 row_1(I_3) \\ x_2 I_3 \end{bmatrix}, \tag{6.85}$$

where $row_i(\cdot)$ represents the i^{th} row of the matrix in the argument. Furthermore, G and F are given by

$$G(x_p) = \begin{bmatrix} \Phi_1 \\ 0_{dim(\pi_3)\times dim(x_p)} \end{bmatrix}, \tag{6.86}$$

$$F(x_p) = \begin{bmatrix} -I_{dim(\pi_2)} & 0_{dim(\pi_2)\times dim(\pi_3)} \\ \Phi_2 & -I_{dim(\pi_3)} \end{bmatrix}. \tag{6.87}$$

This is a basis for any polynomial of order not greater than 3. $G(x_p)$ and $F(x_p)$ can be generalized to deal with polynomials of any finite order in x_p.

Moreover, as the main topic of this paper relies on rational nonlinear systems, consider a rational function as follows

$$\bar{r}(x_p) = \frac{\bar{p}(x_p)}{\bar{q}(x_p)}, \tag{6.88}$$

where $\bar{p}(0) = 0$ and $\bar{q}(x_p) \neq 0$, $\forall x_p \in \mathcal{X}_p$ as in [64]. Assume that the polynomials $\bar{p}(x_p)$, $\bar{q}(x_p)$ can be decomposed as $\bar{p}(x_p) = \alpha_1 x_p + \alpha_2 \pi$ and $\bar{q}(x_p) = \beta_0 + \beta_1 x_p + \beta_2 \pi$, where α_i, β_i are given coefficients and π is given in (6.83). Define

$$\bar{\xi}_1 = \bar{r}(x_p)x_p, \ \bar{\xi}_2 = \bar{r}(x_p)\pi_2, \ \bar{\xi}_3 = \bar{r}(x_p)\pi_3 : \ \bar{\xi} = \begin{bmatrix} \bar{\xi}_1 \\ \bar{\xi}_2 \\ \bar{\xi}_3 \end{bmatrix}.$$

A decomposition of this rational system is as follows.

$$G_r(x_p)x_p + F_r(x_p)\pi_r = 0,\ \pi_r = \begin{bmatrix} \pi(x_p) \\ \bar{r}(x_p) \\ \bar{\xi}(x_p) \end{bmatrix}, \tag{6.89}$$

$$G_r(x_p) = \begin{bmatrix} G(x_p) \\ -\alpha_1 \\ 0_{dim(\bar{\xi})\times dim(x_p)} \end{bmatrix}, \tag{6.90}$$

$$F_r(x_p) = \begin{bmatrix} F(x_p) & 0 & 0 \\ -\alpha_2 & \beta_0 & \beta \\ 0 & F_a(x_p) & F_b(x_p) \end{bmatrix}, \tag{6.91}$$

$$F_a(x_p) = \begin{bmatrix} x_p \\ 0_{dim(\pi_2)\times 1} \\ 0_{dim(\pi_3)\times 1} \end{bmatrix}, \tag{6.92}$$

$$F_b(x_p) = \begin{bmatrix} -I_{dim(x_p)} & 0 & 0 \\ \Phi_1(x_p) & -I_{dim(\pi_2)} & 0 \\ 0 & \Phi_2(x_p) & -I_{dim(\pi_3)} \end{bmatrix}, \tag{6.93}$$

where

$$\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix},\ \pi_p = \begin{bmatrix} x_p \\ \pi_r \end{bmatrix}, \tag{6.94}$$

and G, F, $\Phi_1(x_p)$ and Φ_2 are defined in (6.85), (6.86) and (6.87). As demonstrated in [64], it is possible to extend the previous expressions to represent any set of rational functions of x_p. The condition $det(F_r(x_p)) \neq 0,\ \forall x_p \in \mathcal{X}_p$ is equivalent to the aforementioned regularity assumption on the rational function $\bar{r}(x_p) = \bar{p}(x_p)/\bar{q}(x_p)$ for the values of x_p of interest.

As an example of how condition (6.21) can be analysed in terms of the framework of Theorem 6.5, consider the following rational nonlinear model ($\mathcal{P}$) given by

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 2x_1^2 + 0.1\frac{x_1}{x_2^2+1} \\ -x_1 + 2x_2 - 0.1x_1^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \\ y &= x_2, \end{aligned}$$

where $(x^*,u^*) = (0,0)$ and $\phi = 0$. Suppose that V and T are polynomial functions such as

$$V(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^\top \begin{bmatrix} v_1 & v_2 \\ v_2 & v_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\ T(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^\top \begin{bmatrix} n_1 & n_2 \\ n_2 & n_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{6.95}$$

where (v_1,v_2,v_3) and (n_1,n_2,n_3) are coefficients to be determined, as well as the real parameters (Q,S,R). From condition (6.21) and applying $t(x_p) = -l_p$, we obtain the following function whose nonnegativity in a polytopic domain $\mathcal{X}_p$ has to be investigated, whereas $x_p = [x_1^\top \; x_2^\top \; u^\top]^\top$ in Theorem 6.5.

$$\begin{aligned} t(x_p) &= t_1x_1^2 + t_2x_1x_2 + t_3x_2^2 - 2v_2x_1u + t_4x_2u + Ru^2 \\ &\quad + t_5x_1^2x_2 - 3.8v_1x_1^3 - 0.2v_1\frac{x_1^2}{x_2^2+1} - 0.2v_2\frac{x_1x_2}{x_2^2+1}, \end{aligned}$$

with

$$\begin{aligned} t_1 = 2v_2 - n_1,\; t_2 = 2v_3 - 4v_2,\; t_3 = Q - 4v_3 - n_2, \\ t_4 = 2S - 2v_3,\; t_5 = 0.2v_3 - 4v_2. \end{aligned}$$

Considering $\bar{p}_x = x_2^2 + 1$, we have

$$G = \begin{bmatrix} u & 0 & 0 \\ 0 & 0 & u \\ 0 & u & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & x_1 \\ 0 & 0 & x_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} -I_3 & 0_3 & 0_{3\times2} & 0_3 \\ 0_3 & -I_3 & 0_{3\times2} & 0_3 \\ 0_{2\times3} & 0_{2\times3} & -I_2 & F_1 \\ 0_3 & 0_{3\times2} & F_2 & -I_3 \end{bmatrix}, \tag{6.96}$$

in (6.79), with

$$F_1 = \begin{bmatrix} 0 & -x_2 & 0 \\ 0 & 0 & -x_2 \end{bmatrix}, \quad F_2 = \begin{bmatrix} x_1 & 0 \\ 0 & x_1 \\ 0 & -x_2 \end{bmatrix},$$

and

$$\pi_p = \begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix}^\top, \tag{6.97}$$

$$\pi_1 = \begin{bmatrix} u & x_1 & x_2 & u^2 & ux_2 & ux_1 & x_1^2 & x_1x_2 & x_2^2 \end{bmatrix}, \tag{6.98}$$

$$\pi_2 = \begin{bmatrix} \frac{x_1}{\bar{p}_x} & \frac{x_2}{\bar{p}_x} & \frac{x_1^2}{\bar{p}_x} & \frac{x_1x_2}{\bar{p}_x} & \frac{x_2^2}{\bar{p}_x} \end{bmatrix}. \tag{6.99}$$

A matrix Q_p is given by

$$Q_p = \begin{bmatrix} R & -v_2 & \frac{t_4}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -v_2 & t_1 & \frac{t_2}{2} & 0 & 0 & 0 & -1.9v_1 & 0 & 0 & -0.1v_1 & 0 & 0 & 0 & 0 \\ \frac{t_4}{2} & \frac{t_2}{2} & t_3 & 0 & 0 & 0 & \frac{t_5}{2} & 0 & 0 & -0.1v_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.9v_1 & \frac{t_5}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.1v_1 & -0.1v_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{6.100}$$

6.10.1 An SDP Strategy for Linear Output Feedback Design

In contrast to the results of [128], Theorem 6.5 enable us to formulate an algorithm for constructive dissipativity analysis which is analogous to the one presented in Chapter 5 and applied to the class of the LTI plants. As fully detailed in that chapter and in Section 6.8, an asymptotically stabilizable system may be locally strictly dissipative with $\Delta \geq 0$, which is a condition that can be relaxed into a simple LMI. In the following, an algorithm for analysing the dissipativity properties of a rational nonlinear system around the origin is presented.

Algorithm 6.1

1. Consider a nonlinear plant described by (6.1)-(6.2). Define a *structure* for the possibly rational positive definite functions (V,T), i.e specify their degrees and assign a set of coefficients to be determined, subject to $V(0) = T(0) = 0$.

2. Then, consider condition (6.21) after substitution of (V,T). Set $t(x_p) = -l_p$, $x_p = [1 \; x^\top \; u^\top]^\top$ and determine a decomposition (6.80), a basis vector π_p and a linear annihilator C_b according to (6.81) and (6.79) .

3. Initialize values $0 < \gamma_i \in \mathbb{R}$, $i = \{1, \cdots, n_p\}$, for all components of x_p. This describes a polytope $\mathcal{X}_p : \{x_p \in \mathbb{R}^{n_p} \mid |x_{p_i}| \leq \gamma_i\}$.

4. Specify some $0 < \alpha \in \mathbb{R}$ and set up the condition

$$M_d = \begin{bmatrix} Q + \alpha I & S \\ S^\top & R \end{bmatrix} \geq 0, \tag{6.101}$$

which is equivalent to

$$\begin{cases} R > 0, \\ \Delta = SR^{-1}S^\top - Q \leq \alpha I. \end{cases} \tag{6.102}$$

5. Solve the following linear semidefinite program $\forall x_p \in \mathcal{V}(\mathcal{X}_p)$

$$\begin{aligned} &\textit{minimize} \;\; tr(M_d), \\ &\textit{subject to} \;\; (6.82),\, M_d \geq 0, \end{aligned} \tag{6.103}$$

whereas $tr(M_d) = 0 \Leftrightarrow M_d = 0$. If feasible, this program provides a solution (Q,S,R,V,T,ϕ,L_p).

6. Larger domains $\mathcal{X}_p$ can be obtained by returning to Step 3 and setting larger values for γ_i, $i = \{1,\cdots,n_p\}$, until Step 5 is no longer feasible.

By applying Algorithm 6.1, we guarantee local QSR-dissipativity in a domain $(\mathcal{X} \times \mathcal{U}) = \mathcal{X}_p$ and, at the same time, try to ensure stabilizability by fufilling $\Delta > 0$[2)]. If the LMI condition is not feasible with $\phi = 0$, we have to consider $\phi > 0$ as a new coefficient to be determined. If $(x^*,u^*) \neq (0{,}0)$, a polytopic condition around this point can be tested in the same way. If the algorithm is not feasible even if $\phi > 0$ is added as a new LMI variable, then one may set functions (V,T) of higher degrees and return to Step 2.

After guaranteeing local dissipativity with (Q,S,R,V,T,ϕ) and $\Delta > 0$, we can use this information to design a stabilizing controller. In this regard, we analyse the following scenarios, where theorems 6.1 and 6.2 can be applied for stabilization. In both situations, we suppose that Algorithm 6.1 is feasible with $\Delta > 0$, $(x^*,u^*) = (0{,}0)$.

Scenario 1: $\phi = 0$

If $\phi = 0$, then the origin is stabilizable by static output feedback and any gain inside the interval described by (6.60) is a stabilizing controller.

Scenario 2: $\phi > 0$

If $\phi > 0$, the equilibrium is not stabilizable by SOF under this framework, and a dynamic controller has to be designed. In this case, a linear controller can be determined through the application of relations (6.47)-(6.50).

[2)] The whole procedure can be implemented in MATLAB® with the SDP tool YALMIP [97] and the solver SeDuMi [98].

From the LMI condition, we have $l_p \leq \phi$ in $\mathcal{X}_p$, where

$$l_p = \dot{V} + T - (y^\top Q y + 2y^\top S u + u^\top R u). \tag{6.104}$$

As $\Delta > 0$, the matrix M_d in (6.101) is indefinite, which means that the function $w(y,u) = y^\top Q y + 2y^\top S u +^\top Ru$ assumes both positive and negative values in $\mathcal{X}_p$. Nevertheless, if we define a new matrix $\bar{Q} = \rho Q$, with $\rho > 1$, such that $\Delta = SR^{-1}S\top - \bar{Q} = 0$, then we have a new suppy rate given by

$$y^\top \bar{Q} y + 2y^\top S u + u^\top R u \geq 0. \tag{6.105}$$

Notice that this function still fulfills $l_p \leq \phi$, as $\bar{Q} > Q$. Furthermore, the necessary conditions for a local minimum of the (6.105) are the following

$$\nabla w = \begin{bmatrix} 2\bar{Q}y + 2Su \\ 2Ru + 2S^\top y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{6.106}$$

which is equivalent to

$$u = -R^{-1}S^\top y,$$
$$(\bar{Q} - SR^{-1}S^\top)y = 0.$$

As $\Delta = 0$ with $\bar{Q}$, the second equality holds true, and the first one represents the expression of an SOF, as we have already discussed previously. As a result, it is the static gain which minimizes the function $w(y,u)$, whereas $w > 0$ otherwise. This means that in case of a dynamic feedback, the suppy rate is zero only at the equilibrium, since in this case the controller output

$$y_c = C_c x_c + D_c u_c = -u, \tag{6.107}$$

with $D_c = R^{-1}S^\top$, equals the expression of an SOF only at $x_c = 0$. Notice that the necessary condition $u = -R^{-1}S^\top y$ defines a set of points (not only a single point), where w is zero. As a consequence, a point (x,u) in this set is not able to compensate for a possible term $\dot{V} + T > 0$ (unstable). This is the reason why a dynamic controller is necessary in this case, as it guarantees that the suppy rate is zero only at the equilibrium (x^*,x_c^*) to which the interconnected system asymptotically converges.

For any point $(x,u) \in \mathcal{X}_p$, there is combination $(\mu\bar{Q},\mu S,\mu R)$, $\mu > 1$, such that $\dot{V} + T - \mu(y^\top \bar{Q} y + 2y^\top S u + u^\top R u) \leq 0$. A stabilizing controller fulfills (6.48), where $D_c = R^{-1}S^\top$, C_c and P_c have to be specified in advance, while (A_c,B_c,N_c) are obtained by solving the resulting LMI. For a certain combination (A_c,B_c,C_c,D_c), expression (6.48) is linear in (Q_c,S_c,R_c,P_c). As a result, for any $\mu > 1$, the inequality is also fulfilled for $(\mu\bar{Q},\mu S,\mu R,\mu P_c)$. The same linear controller (A_c,B_c,C_c,D_c) is strictly dissipative for any combination $(\mu\bar{Q},\mu S,\mu R,\mu P_c)$, $\mu > 1$, which in turn guarantees that for the plant a domain around the origin is asymptotically stabilizable. This feature of our dissipativity-based framework is a quite interesting one: in the case of $\phi > 0$, it is in general not possible to use a single combination $(\mu\bar{Q},\mu S,\mu R)$ in order to guarantee dissipativity and at the same time stabilizability in a certain domain about the equilibrium.

Notice that if $(x^*,u^*) \neq (0,0)$, the necessary conditions (6.106) imply that the following linear constraint must be added to Algorithm 6.1

$$Ru^* + S^\top y^* = 0, \tag{6.108}$$

which is automatically fulfilled if the desired equilibrium is the origin. By fufilling this condition, we guarantee equilibrium assignment through $D_c = R^{-1}S^\top$, as $\Delta = 0$ can be achieved by setting a suitable $\bar{Q} > Q$.

In [107] an SOS-based constructive procedure for dynamic output feedback control of polynomial system was presented. The approach is aimed to achieve global stabilization, as it relies on an explicit SOS solution. A disadvantage is that observer design is necessary and the strategy is restricted to square systems. In this regard, we believe that our method contains relevant improvements when compared to this framework, as not only polynomial systems can be investigated, nonsquare models can be handled as well, and local stability can be obtained in a constructive manner, which also allows for achieving global stabilization, in principle.

A potential drawback of this polytopic LMI approach is that the number of monomials in the basis π increases considerably as the dimension of x_p or the degree of the dynamics (6.1)-(6.2) increases. Nevertheless, in the same sense that there is a systematic procedure for specifying linear annihilators, there exist procedures to decrease the number of monomials in π. In this regard, the so-called *Linear Fractional Transformation* (LFT) [67], can be applied in order to eliminate repetitive terms of π. It is out of the scope of this work to discuss LFT in details. At this point, we simply would like to emphasize that constructing linear annihilators and simplifying basis vectors, as well as solving the respective polytopic LMIs, are procedures which can be automatically handled and systematically carried out. An investigation with regards to reducing the size of the basis needed will be left for future research.

6.11 Examples

6.11.1 Example 6.1

Let us consider the model of an isothermal nonlinear continuous fermenter with constant volume and constant physico-chemical properties as described in [58]. A standard input-affine representation in the form (6.1)-(6.2) for this SISO system is:

$$f(x) = \begin{bmatrix} \mu(x_2 + x_{2_0})(x_1 + x_{1_0}) - \frac{(x_1+x_{1_0})F_0}{V} \\ -\frac{\mu(x_2+x_{2_0})(x_1+x_{1_0})}{Y} + \frac{(S_F-(x_2+x_{2_0}))F_0}{V} \end{bmatrix},$$

$$g(x) = \begin{bmatrix} -\frac{(x_1+x_{1_0})}{V} \\ \frac{(S_F-(x_2+x_{2_0}))}{V} \end{bmatrix},$$

and

$$y(x) = x_2,$$

where,

$$\mu(x_2) = \mu_{max}\frac{x_2}{K_2 x_2^2 + x_2 + K_1}.$$

This rational nonlinear process is based on the *centered* state $x = [x_1 \ \ x_2]^\top = [\bar{x}_1 - x_{1_0} \ \ \bar{x}_2 - x_{2_0}]$, where $x_{1_0} = 4.8907$ [g/l] and $x_{2_0} = 0.2187$ [g/l] are the *centered* biomass and substrate concentrations. The component x_1 is the biomass concentration [g/l] and x_2 is the substrate concentration [g/l], defined as deviations from their respective reference values x_{1_0} and x_{2_0}. The control variable is the centered input flow rate given by $u = \bar{F} = F - F_0$, with $F_0 = 3.2089$ [l/h]. The remaining parameters are presented below.

Table 6.1: Parameters of the fermentation process.

V	Volume	4	[l]
F	Feed flow rate	-	[l/h]
S_F	Substrate feed concentration	10	[g/l]
Y	Yield coefficient	0.5	-
μ_{max}	Maximal growth rate	1	[l/h]
K_1	Saturation parameter	0.03	[g/l]
K_2	Inhibition parameter	0.5	[l/g]

Our control objective is to apply Algorithm 6.1 to design a static gain that asymptotically stabilizes the closed-loop around the equilibrium point $(x_1^*, x_2^*) =$

(0,0), $u^* = 0$. This fermentation process is highly nonlinear and the domain of stability in open loop is very small, as there is an unstable equilibrium close to the origin. Thus, the task of broadening the region of stability is a quite challenging and relevant one.

As a first step towards the application of our control method, we suppose that V and T are polynomial functions given by (6.95), where $n_2 = 0$. Then, with $x_p = [x_1\ x_2\ u]^\top$, the rational function $t(x_p)$ in (6.80) is described by

$$\begin{aligned} t(x_p) &= t_1 x_2^2 + t_2 x_2 u + R u^2 + t_3 x_1 u + t_4 x_1^2 + t_5 x_1 \\ &\quad + t_6 x_2 + t_7 x_1 x_2 + t_8 x_1^2 u + t_9 x_1 x_2 u + t_{10} x_2^2 u \\ &\quad + \frac{1}{\bar{p}_x}\left(t_{11} x_2^2 + t_{12} x_1 x_2 + t_{13} x_1^2 + t_{14} x_2 + t_{15} x_1 t_{16} x_1^2 x_2 + t_{17} x_1 x_2^2\right), \end{aligned}$$

whereas

$$\begin{aligned} \bar{p}_x &= a x_2^2 + b x_2 + c, \\ a = K_2,\ b &= (2K_2 x_{2_0} + 1),\ c = K_2 x_{2_0}^2 + x_{2_0} + K_1, \end{aligned}$$

and

$$\begin{aligned} t_1 &= Q - n_3 + \frac{2v_3 F_0}{V}, \quad t_2 = 2S + \frac{2v_2 x_{1_0}}{V} + \frac{2v_3(x_{2_0} - S_F)}{V}, \\ t_3 &= \frac{2x_{1_0} v_1}{V} + \frac{2v_2(x_{2_0} - S_F)}{V}, \quad t_4 = \frac{2F_0 v_1}{V} - n_1, \\ t_5 &= \frac{2v_2(x_{2_0} - S_F)F_0}{V} + \frac{2F_0 x_{1_0} v_1}{V}, \quad t_6 = \frac{2F_0 x_{1_0} v_2}{V} + \frac{2v_3(x_{2_0} - S_F)F_0}{V}, \\ t_7 &= \frac{4F_0 v_2}{V}, \quad t_8 = \frac{2v_1}{V}, \quad t_9 = \frac{4v_2}{V}, \quad t_{10} = \frac{2v_3}{V}, \\ t_{11} &= \frac{2v_3 x_{1_0}}{V} - 2v_2 x_{1_0}, \quad t_{12} = -2v_1 x_{1_0} + \frac{2v_2 x_{1_0}}{Y} - 2v_2 x_{2_0} + \frac{2v_3 x_{2_0}}{Y}, \\ t_{13} &= \frac{2v_2 x_{2_0}}{Y} - 2v_1 x_{2_0}, \quad t_{14} = \frac{2v_3 x_{1_0} x_{2_0}}{Y} - 2v_2 x_{1_0} x_{2_0}, \\ t_{15} &= \frac{2v_2 x_{1_0} x_{2_0}}{Y} - 2v_1 x_{1_0} x_{2_0}, \quad t_{16} = \frac{2v_2}{Y} - 2v_1, \quad t_{17} = \frac{2v_3}{Y} - 2v_2. \end{aligned}$$

Notice that t_1 and t_2 contain the dissipativity parameters Q and S, respectively, which we need to determine. All the unknown parameters appear linearly in t_1-t_{17}. In addition, for a vector basis given by

$$\pi_p = \begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix}^\top,$$

$$\pi_1 = \begin{bmatrix} u & x_2 & x_1 & 1 & u^2 & ux_2 & ux_1 & x_1^2 & x_1x_2 & x_2^2 \end{bmatrix},$$

$$\pi_2 = \begin{bmatrix} \frac{x_1}{\bar{p}_x} & \frac{x_2}{\bar{p}_x} & \frac{x_1^2}{\bar{p}_x} & \frac{x_1x_2}{\bar{p}_x} & \frac{x_2^2}{\bar{p}_x} \end{bmatrix},$$

a possible combination (G,F) in (6.81) is as follows

$$G = \begin{bmatrix} -u & 0 & 0 & 0 \\ 0 & -u & 0 & 0 \\ 0 & 0 & -u & 0 \\ 0 & 0 & -x_1 & 0 \\ 0 & -x_1 & 0 & 0 \\ 0 & -x_2 & 0 & 0 \\ 0 & 0 & 0 & -x_1 \\ 0 & 0 & 0 & -x_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & b & ax_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c+b & 0 & 0 & ax_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -x_1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_2 & 0 & 0 & 1 \end{bmatrix}$$

where $det(F) = 0.1363x_2^2 + 0.4066$, i.e. $det(F) \neq 0$ everywhere in (x_1,x_2,u). Finally, a matrix Q_p in (6.80) is given by

$$Q_p = \begin{bmatrix} R & \frac{t_2}{2} & \frac{t_3}{2} & 0 & 0 & 0 & 0 & \frac{t_8}{2} & \frac{t_9}{2} & \frac{t_{10}}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{t_2}{2} & t_1 & \frac{t_7}{2} & \frac{t_6}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{t_3}{2} & \frac{t_7}{2} & t_4 & \frac{t_5}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{t_6}{2} & \frac{t_5}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{t_{15}}{2} & \frac{t_{14}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{t_8}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{t_{16}}{2} & 0 & 0 & 0 & 0 \\ \frac{t_9}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{t_{10}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{t_{17}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{t_{15}}{2} & 0 & 0 & 0 & 0 & 0 & \frac{t_{17}}{2} & t_{13} & \frac{t_{12}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{t_{14}}{2} & 0 & 0 & 0 & \frac{t_{16}}{2} & 0 & 0 & \frac{t_{12}}{2} & t_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

By proceeding so, we are able to apply Algorithm 6.1 for determining a solution (Q,S,R,V,T). A domain $\mathcal{X}_p$ can be enlarged by iteratively increasing the components γ_i, $i = 1,2,3$. Indeed, we have in this case

$$|x_1| \leq \gamma_1, \ |x_2| \leq \gamma_2, \ |u| \leq \gamma_3.$$

As initial values for the algorithm we used $\gamma_1 = \gamma_2 = \gamma_3 = 0.1$. Then, for the nonlinear continuous fermenter, we obtained the following results with $\phi = 0$

$$R = 1.5577 \times 10^{-4}, \ Q = 5.3254 \times 10^{-4}, \ S = 3.0024 \times 10^{-4},$$

$$V = 10^{-2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^\top \begin{bmatrix} 0.2878 & 0.1423 \\ 0.1423 & 0.0831 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

$$T = 10^{-3} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^\top \begin{bmatrix} 0.2267 & 0 \\ 0 & 0.1319 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

From the values of (Q,S,R), we obtain $\Delta = SR^{-1}S^\top - Q = 4.6170 \times 10^{-5} > 0$. As $\phi = 0$ and $\Delta > 0$, the origin is stabilizable by SOF and a stabilizing gain is given by Proposition 6.1, as interval for the controller $D_c = -K$. As a result,

$$D_{c1} = 1.3831, \; D_{c2} = 2.4719$$

in (6.60), where we adopt for example $D_c = 2$. Notice that as the free system is atable $D_c = 0$ is also a admissible gain. Nevertheless, stability cannot be established in such a broad domain as with $D_c = 2$.

We were able to stabilize this system in a quite broad domain of attraction by applying a very simple control law, $u = -2x_2$ (see Figure 6.2, where the equilibrium is marked in red). The polytopic LMI condition was proven feasible in the region defined by

$$\mathcal{X}_p = \{(x_1,x_2,u) \mid |x_1| \leq 2, |x_2| \leq 0.2, |u| \leq F_0\},$$

which means that stability is guaranteed in the following domain of attraction

$$\mathcal{X} = \{(x_1,x_2) \mid |x_1| \leq 2, |x_2| \leq 0.2\},$$

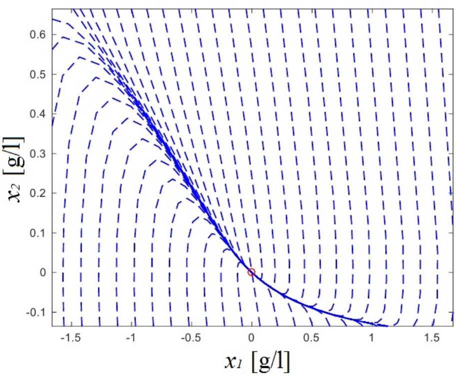

Figure 6.2: Phase diagram of the controlled system - Example 6.1

We were able to guarantee a broad region of stability for this complex fermentation process using a simple quadratic Lyapunov function. Rational Lyapunov functions can also be employed, as well as polynomial functions of degree higher than two. As a result, the domain of attraction could be further enlarged, although the complexity of implementing our algorithm would increase.

Notice that the following dynamic controller, for example, also solves the stabilization problem

$$(\mathcal{C})\begin{cases} \dot{\kappa} = -3\kappa + 0.1\eta \\ \lambda = 0.1\kappa + 2\eta, \end{cases}$$

as condition (6.48) is fulfilled for these parameters.

6.11.2 Example 6.2

In this example we consider the following nonlinear model which was also investigated in [47]:

$$f(x) = \begin{bmatrix} -\frac{1}{2}x_1 + x_2 \\ -x_2^2 \end{bmatrix}, \quad g(x) = \begin{bmatrix} \frac{1}{2} - x_2^2 \\ x_2^3 \end{bmatrix},$$

$$y(x) = x_1\left(\frac{1}{2} - x_2^2\right) + x_2^3.$$

This system has a port-Hamiltonian representation with

$$F(x) = \begin{bmatrix} -\frac{1}{2} & x_2 \\ 0 & -x_2^2 \end{bmatrix}, \quad H(x) = \frac{1}{2}x_1^2 + x_2,$$

in (3.43)-(3.44). As the function H is not bounded from below, one cannot affirm that the open-loop equilibrium $(x_1^*,x_2^*) = (0,0)$ is stable. Indeed, as reported in [47], the origin is an unstable equilibrium of the uncontrolled system ($u^* = 0$). Furthermore, this plant suffers from the dissipation obstacle described in Chapter 3, and CbI is able to stabilize it at an arbitrary equilibrium $(x_1^*,x_2^*) \neq (0,0)$ only if its output y is redefined according to (3.60), i.e through power shaping control by interconnection.

In the following, we apply the dissipativity-based control method introduced earlier in this chapter to stabilize this plant, though doing without port-Hamiltonian representations or output shaping. As in [47], we consider $(x_1^*,x_2^*) = (-1,-1)$, whereas $(u^*,y^*) = (-1,-\frac{1}{2})$. In order apply to apply Algorithm 6.1, we consider polynomial functions V and T given by

$$V(x) = \begin{bmatrix} x_1 - x_1^* \\ x_2 - x_2^* \end{bmatrix}^\top \begin{bmatrix} v_1 & v_2 \\ v_2 & v_3 \end{bmatrix} \begin{bmatrix} x_1 - x_1^* \\ x_2 - x_2^* \end{bmatrix},$$

and

$$T(x) = n \begin{bmatrix} x_1 - x_1^* \\ x_2 - x_2^* \end{bmatrix}^\top \begin{bmatrix} x_1 - x_1^* \\ x_2 - x_2^* \end{bmatrix}.$$

Then, with $x_p^\top = [1\ x_1^\top\ x_2^\top\ u^\top]$, we have a polynomial function $t(x_p)$ in (6.80) described by

$$\begin{aligned} t(x_p) = {} & t_1 + t_2 x_1 + t_3 x_2 + t_4 u + t_5 x_1^2 + t_6 x_1 x_2 + t_7 x_1 u + t_8 x_2^2 \\ & -v_2 x_2 u + R u^2 + 2v_2 x_1 x_2^2 + t_9 x_2^2 u + Q x_1 x_2^3 - Q x_1^2 x_2^2 + t_{10} x_2^3 u \\ & t_{11} x_1 x_2^2 u - 2v_2 x_1 x_2^3 u - 2v_3 x_2^4 u - 2Q x_1 x_2^5 + Q x_1^2 x_2^4 + Q x_2^6, \end{aligned}$$

where

$$\begin{aligned} & t_1 = \phi - 2n, \quad t_2 = -2n + v_1 + v_2, \quad t_3 = -2n - 2v_1 - 2v_2, \\ & t_4 = -v_1 - v_2, \quad t_5 = \frac{Q}{4} + v_1 - n, \quad t_6 = v_2 - 2v_1, \quad t_7 = v_1 - S, \\ & t_8 = 2v_3 - n, \quad t_9 = 2v_1 + 2v_2, \quad t_{10} = 2S - 2v_3, \quad t_{11} = 2v_1 - 2S. \end{aligned}$$

Again, all the unknown parameters appear linearly in $t(x_p)$. A vector basis is given by

$$\pi_p^\top = \begin{bmatrix} x_p^\top & \pi_1^\top \end{bmatrix},$$

$$\pi_1^\top = \begin{bmatrix} x_1^2 & x_1 x_2 & x_2^2 & x_2 u & x_2^2 u & x_2^3 & x_1 x_2^3 \end{bmatrix},$$

and a possible combination (G,F) is as follows

$$G = \begin{bmatrix} u & -x_1 & 0 & -1 \\ -u & 0 & -x_1 & 1 \\ 0 & 0 & -x_2 & 0 \\ 0 & 0 & 0 & -x_2 \end{bmatrix},$$

$$F = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x_2 & 1 & 0 & 0 \\ 0 & 0 & -x_2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x_1 & 1 \end{bmatrix},$$

where $det(F) = 1$ everywhere in (x_1,x_2,u). Finally, a matrix Q_p in (6.80) is given by

$$Q_p = \begin{bmatrix} t_1 & \frac{t_2}{2} & \frac{t_3}{2} & \frac{t_4}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{t_2}{2} & t_5 & \frac{t_6}{2} & \frac{t_7}{2} & 0 & 0 & v_2 & 0 & 0 & 0 & 0 \\ \frac{t_3}{2} & \frac{t_6}{2} & t_8 & -\frac{v_2}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{t_4}{2} & \frac{t_7}{2} & -\frac{v_2}{2} & R & 0 & 0 & \frac{t_9}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -Q & \frac{Q}{2} & \frac{t_{11}}{2} & -v_2 & 0 & \frac{Q}{2} \\ 0 & v_2 & 0 & \frac{t_9}{2} & 0 & \frac{Q}{2} & 0 & \frac{t_{10}}{2} & 0 & 0 & Q \\ 0 & 0 & 0 & 0 & 0 & \frac{t_{11}}{2} & \frac{t_{10}}{2} & 0 & 0 & -v_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -v_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -v_3 & 0 & Q & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{Q}{2} & Q & 0 & 0 & 0 & 0 \end{bmatrix}.$$

By proceeding so, we are able to apply Algorithm 6.1 for determining a solution (Q,S,R,V,T,ϕ). A domain $\mathcal{X}_p$ can be enlarged by iteratively increasing the components γ_i, $i = 1,2,3$. Indeed, we have in this case

$$|x_1| \leq \gamma_1, \ |x_2| \leq \gamma_2, \ |u| \leq \gamma_3,$$

where the polytopic LMI is feasible for values as high as $\gamma_1 = \gamma_2 = \gamma_3 = 0.4$. Then, we obtained the following results

$$Q = 0.8928, \quad S = -0.7144, \quad R = 0.3572, \quad \phi = 3.8211,$$
$$\begin{bmatrix} v_1 & v_2 \\ v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 0.9521 & 0.4126 \\ 0.4126 & 1.8185 \end{bmatrix} > 0, \quad n = 0.1237.$$

The parameter L_p which solves the LMI condition at all vertices of the polytope is given by

$$L_b = \begin{bmatrix} 0.2473 & -0.6871 & 1.6113 & 0.9467 & -0.3965 & -0.9397 & 0.4495 \\ 0.1378 & -0.2914 & 0.7927 & -0.0634 & 0.0467 & -0.8360 & 0.4803 \\ -0.1296 & 0.0531 & -1.0355 & 0.6922 & -1.0010 & 0.1843 & 0.0846 \\ -0.2418 & 0.4169 & -1.0702 & -0.1786 & 0.0645 & -0.9162 & 0.2999 \\ 0.3200 & 0.0096 & 0.1191 & 0.4082 & -0.1871 & -0.1165 & 0.1222 \\ 0.2019 & 0.9395 & -0.7807 & -1.0997 & 0.0093 & 0.4513 & -0.4013 \\ -0.0065 & -0.3209 & 1.0023 & 0.9358 & -0.5574 & -0.2626 & -0.0743 \\ -0.0810 & -0.6282 & 0.9475 & 0.6981 & -0.0594 & 1.2017 & 0.1201 \\ 0.0494 & 0.4506 & 0.5381 & -0.5166 & 0.6299 & 0.0238 & -0.0598 \\ 0.0001 & 0.1460 & -0.2746 & 0.4645 & 0.0351 & 0.4679 & -0.1798 \\ -0.1291 & -0.2394 & 0.0114 & -0.1835 & 0.0599 & -0.6874 & 0.7277 \end{bmatrix}.$$

From the values of (Q,S,R), we obtain $\Delta = SR^{-1}S^{\top} - Q = 0.5360 > 0$, and for $\bar{Q} = SR^{-1}S^{\top} = 1.4288$ we obtain another solution though with $\Delta = 0$. According to (6.58) and Proposition 6.1, the following closed-form expression gives the value of D_c for the case $(\bar{Q},S,R)$

$$D_c = R^{-1}S^{\top} = -2.$$

The last step of the controller design procedure consits in solving (6.48) for the controller, subject to (6.49)-(6.50). Notice that D_c has already been determined from the local dissipativity analysis of the plant. This gain is the one that fulfills (6.56), which means that if the controller is asymptotically stable and has an equilibrium at $\kappa^* = 0$, then the gain D_c gurarantees equilibrium assignment.

Next, fix $C_c = 1$ and a matrix $P_c = 1$, for example. By setting fxed values for this parameters, conditions (6.48) becomes a simple LMI on the (A_c, B_c, N_c). The following dynamic controller solves the stabilization problem

$$(\mathcal{C}) \begin{cases} \dot{\kappa} = -6.9255\kappa \\ \lambda = \kappa - 2\eta, \end{cases}$$

where $(A_c, B_c, N_c) = (-6.9255, 0, 0.01)$. Figure 6.3 contains the phase diagram for the closed-loop system.

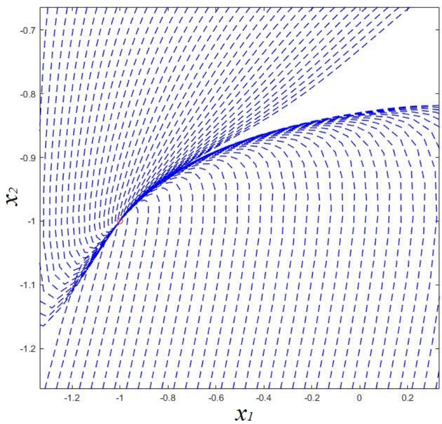

Figure 6.3: Phase diagram of the controlled system - Example 6.2

The polytopic LMI condition was proven feasible in the region defined by

$$\mathcal{X}_p = \{(x_1, x_2, u) \mid |x_1| \leq 0.4, |x_2| \leq 0.4, |u| \leq 0.4\},$$

which means that stability is guaranteed in the following domain of attraction

$$\mathcal{X} = \{(x_1, x_2) \mid |x_1| \leq 0.4, |x_2| \leq 0.4\},$$

where the controller's state must be initialized in a certain domain aroud $\kappa = 0$, in order to be compatible with the domain $\mathcal{X}$ specified for the plant.

6.12 Discussion

In this chapter, we introduced new methods for asymptotic stabilization of nonlinear systems using the notion of strict QSR-dissipativity. We applied the notions of Finsler's Lemma and linear annihilators to enlarge the estimates of the domains of attraction of rational systems and for controller design. The constructive polytopic LMI condition which we have presented provides extra degrees of freedom when compared to a passivity-based approach.

The strategy presents itself in an alternative for control by interconnection, as it allows to stabilize systems that suffer from the dissipation obstacle. In No port-Hamiltonian representation is needed, and the derivation of Casimir functions is obliterated too. In short, we can affirm that investigating the dissipativity properties of a plant around a desired equilibrium provides, indeed, valuable information for controller design purposes. An efficient and simple SDP strategy is able to cope with complex stabilization problems, involving even undetectable outputs. Both linear SOF and dynamic output feedback (DOF) can be handled by the same stabilization framework.

7 Conclusion

In this work, we proposed new controller design strategies for both linear and nonlinear dynamical systems. Indeed, a wide spectrum of the field of passivity and dissipativity analysis and control was covered.

7.1 Summary

- In Chapter 4, as a first contribution of this thesis, we presented a solution for a problem that had been open for many years in the field of linear time-invariant systems: the problem of the equivalence between strict positive realness and strict passivity. A simple proof of this equivalence was provided and published in an early stage of this doctorate [124].
- In Chapter 5, a new strategy for the classical static output feedback stabilization problem of linear systems was presented. Our dissipativity-based strategy is remarkably simple and based on a novel and straightforward LMI condition. No initializations rather the ones necessary for computationally implementing the semidefinite program are necessary.
- In Chapter 6, novel constructive methods for controller design based on local strict QSR-dissipativity were provided. We applied the notion linear annihilators combined with the Finsler's Lemma, which allows for formulating the stabilization problem as a simple polytopic LMI condition. This strategy can be interpreted as a very general approach for handling rational systems, as the computational power available allows for solving large-scale LMIs quite efficiently and, in addition, algorithms can also be applied for reducing the dimension of the monomial bases involved.

7.2 Future Work

- Although we have introduced a new perspective for handling the SOF problem of LTI systems, a definite solution remains to be found. It is worth investigating how useful a dissipativity-based approach could be in the search for a testable necessary and sufficient condition which is feasible regardless of the system's state-space representation. A broader comparison of our method with existing strategies (for instance [119], [123]) and on a much more comprehensive group of benchmark models (such as in [121]) is still necessary. The technique works quite well for small and middle-sized plants, but it is not clear whether it is capable of solving large-scale problems.

- The question of the analysing local dissipativity by directly solving the PDEs involved may constitute a fruitful research line as well. In this case, the problem of the ever increasing number of LMI parameters to be determined as the dimension of the plant gets larger may be avoided.

- I would also mention extensions to state feedback as a potential application of the ideas introduced in this work. Either by solving the PDEs or by solving polytopic LMI conditions.

A On The Equivalence: $(D + D^\top) \geq 0 \Leftrightarrow D \geq 0$

A matrix $D \in \mathbb{R}^{m \times m}$ can be expressed as

$$D = \frac{1}{2}(D + D^\top) + \frac{1}{2}(D - D^\top), \tag{A.1}$$

where $(D + D^\top)$ is symmetric and $(D - D^\top)$ is skew-symmetric. From [34], $D \geq 0$ if

$$x^\top D x \geq 0, \ \forall x \in \mathbb{R}^m.$$

And from (A.1)

$$x^\top D x = \frac{1}{2}\, x^\top (D + D^\top) x,$$

since

$$x^\top (D - D^\top) x = 0.$$

Thus it is evident that

$$D \geq 0 \Leftrightarrow (D + D^\top) \geq 0.$$

B Sum of Squares

SOS techniques are relaxation methods for testing if a polynomial function is positive semidefinite. Although the existence of a SOS decomposition is only sufficient for positivity, testing it is a convex optimization problem to which very efficient semidefinite programming techniques can be applied [105]. An SOS optimization problem is formulated as follows [110]

$$\begin{aligned} &\textit{min} && \sum_{i=1}^{J} w_i n_i \\ &\textit{subject to} && a_0(x) + \sum_{j=1}^{J} n_i a_i(x) \textit{ is SOS}, \end{aligned} \tag{B.1}$$

where n_i's are the scalar and real decision variables, w_i's are some given real numbers, and $a_i(x)$'s are some given polynomials of the variable x (possibly a state variable) with fixed coefficients [111].

Another very important feature of SOS methods is the possibility of specifying optimization problems with semidefinite constraints. For instance, the problem of determining the largest value of a constant $c > 0$ such that $q(x) \leq c$ ($q(x) \geq 0$) implies $s(x) \geq 0$, where $q(x)$ and $s(x)$ are given polynomials. This question assumes the following SOS formulation [108]

$$\begin{aligned} &\textit{max} && c \\ &\textit{subject to} && (q(x) - c) + p(x)s(x) \textit{ is SOS}, \end{aligned} \tag{B.2}$$

where $p(x) \geq 0$ is an unknown polynomial function. The set $\mathcal{D} = \{x : q(x) \leq c\}$ is said to be an invariant set.

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Index

Curriculum Vitae

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