Stokes Operator and Stability of Stationary Navier-Stokes Flows in Infinite Cylindrical Domains

Vom Fachbereich Mathematik
der Technischen Universität Darmstadt
zur Erlangung des akademischen Grades eines
Doktors der Naturwissenschaften
(Dr. rer. nat.)
genehmigte

Dissertation

von

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Tag der Einreichung:	20. März 2006
Tag der mündlichen Prüfung:	20. April 2006

Darmstadt 2006
D17

Abstract

Navier-Stokes equations in infinite cylindrical domains have been attracting great attention due to its theoretical and practical significance. However, in most cases, stationary Navier-Stokes problems were dealt with whereas instationary Navier-Stokes problems have been less studied. The L^q-approach to instationary Navier-Stokes problems is very important and convenient to analyze existence, uniqueness as well as strong energy inequality and partial regularity for solutions; to this end, the study of the Stokes operator is fundamental.

The aim of this dissertation is to get resolvent estimates, maximal regularity and boundedness of H^∞-calculus of Stokes operators in infinite cylindrical domains and to apply them to the stability of stationary Navier-Stokes flows in infinite cylindrical domains.

We start with a Stokes resolvent system in an infinite straight cylinder. The Stokes resolvent system on an infinite straight cylinder is reduced by the (one-dimensional) partial Fourier transform along the axis of the cylinder to a *parametrized Stokes system* with the Fourier variable as a parameter. Using the Fourier multiplier theory in weighted spaces we get estimates for the parametrized Stokes system with bound constants independent of parameters. Based on these estimates resolvent estimate and maximal regularity of the Stokes operator in weighted Lebesgue spaces on an infinite straight cylinder are shown using the techniques of operator-valued Fourier multiplier theory and Schauder decomposition in Banach spaces with UMD property.

Next we consider the Stokes operator in general infinite cylinders with several exits to infinity. A resolvent estimate of the Stokes operator in L^q-space is obtained using cut-off techniques based on the result of generalized Stokes resolvent system in an infinite straight cylinder. In particular, the Stokes operator is shown to generate a bounded and exponentially decaying analytic semigroup in any L^q-space on a general infinite cylinder. Moreover, it is proved that the Stokes operator admits a bounded H^∞-calculus in any L^q-space on an infinite cylinder with several exits to infinity.

As an application of the obtained properties of the Stokes operator we study stability of stationary Navier-Stokes flows in infinite cylindrical domains. First, existence and uniqueness for stationary Navier-Stokes systems in infinite cylinders are shown. Then the exponential stability of the stationary Navier-Stokes flow is proved based on $L^r - L^q$ estimates of the perturbed Stokes semigroup.

Zusammenfassung

Die Navier-Stokes Gleichungen in unendlichen zylindrischen Gebieten haben aufgrund ihrer theoretischen und praktischen Bedeutung großes Interesse geweckt. Jedoch wurden in den meisten Fällen stationäre Navier-Stokes Probleme betrachtet, während instationäre Navier-Stokes Probleme weniger behandelt wurden. Der L^q-Zugang zu instationären Navier-Stokes Problemen ist sehr wichtig und geeignet zur Analyse von Existenz, Eindeutigkeit sowie starken Energieabschätzungen und partieller Regularität von Lösungen. Hierfür ist die Untersuchung des Stokes Operators fundamental.

Das Ziel dieser Dissertation ist es, Resolventenabschätzungen, maximale Regularität und Beschränktheit des H^∞-Kalküls für den Stokes Operator zu beweisen und dies auf die Stabilität des stationären Navier-Stokes Flusses in unendlichen zylindrischen Gebieten anzuwenden.

Wir beginnen mit dem Stokes-Resolventen System in einem unendlichen geraden Zylinder. Das Stokes-Resolventen System in einem unendlichen geraden Zylinder wird durch (eindimensionale) partielle Fouriertransformation entlang der Achse des Zylinders zu einem *parametrisierten Stokes System* mit der Fouriervariablen als Parameter reduziert. Mit Hilfe von Fourier-Multiplikatoren Theorie in gewichteten Räumen erhalten wir Abschätzungen für das parametrisierte Stokes System mit Konstanten, die nicht von den Parametern abhängen. Auf der Basis dieser Abschätzungen werden Resolventenabschätzungen und maximale Regularität des Stokes Operators in gewichteten Lebesgue Räumen auf einem unendlichen zylindrischen Gebiet gezeigt. Hierfür werden Techniken aus der operatorwertigen Fourier-Multiplikatoren Theorie und Schauder Zerlegung in Banachräumen mit der UMD-Eigenschaft verwendet.

Als nächstes betrachten wir den Stokes Operator in allgemeinen Zylindern mit mehreren Ausgängen nach Unendlich. Man erhält eine Resolventenabschtzung für den Stokes Operator in L^q-Räumen unter Verwendung von Abschneidetechniken basierend auf dem Resultat für das verallgemeinerte Stokes Resolventensystem in einem unendlichen geraden Zylinder. Insbesondere wird gezeigt, dass der Stokes Operator eine beschränkte und exponentiell fallende analytische Halbgruppe in jedem L^q-Raum auf einem allgemeinen unendlichen Zylinder erzeugt. Auerdem wird bewiesen, dass der Stokes Operator einen beschränkten H^∞-Kalkül in jedem L^q-Raum auf einem unendlichen Zylinder mit mehreren Ausgängen nach Unendlich erlaubt.

Als Anwendung der bewiesenen Eigenschaften behandeln wir die Stabilität von stationären Navier-Stokes Flüssen in unendlichen zylindrischen Gebieten. Zuerst werden Existenz und Eindeutigkeit für stationäre Navier-Stokes Systeme gezeigt. Anschlieend werden exponentielle Stabilität des stationären Navier-Stokes Flusses mit Hilfe von L^r-L^q Abschätzungen der gestörten Stokes Halbgruppe bewiesen.

Acknowledgement

The present work was prepared during my stay at the Department of Mathematics, Darmstadt University of Technology by the support of Gottlieb Daimler- und Karl Benz-Stiftung with grant No. S 025/02-10/03. I would like to express my sincere thanks to Gottlieb Daimler- und Karl Benz-Stiftung.

I express my deep gratitude to Prof. Reinhard Farwig for his valuable advice, discussion and efforts during the preparation of this thesis.

I also dedicate my sincere thanks to Mrs. Christiane Herdler and Mrs. Magdalena Tabbert, the secretaries of the working group 6, for their consistent care and help.

Finally, I am grateful to Dr. Patrizio Neff and to all the other members of the working group 6 for a nice and stimulating atmosphere.

Contents

1 Introduction

The equations describing the motion of incompressible, Newtonian fluid are usually called Navier-Stokes equations. They were proposed by the French engineer C. L. M. H. Navier in 1822 and rederived by G. H. Stokes later in 1845. In the case where a fluid fills a domain Ω the *instationary Navier-Stokes equations* can be described by the following system of partial differential equations:

$$\begin{aligned} u_t - \Delta u + (u \cdot \nabla)u + \nabla p &= f \qquad \text{in } \ \Omega \times (0,T) \\ \operatorname{div} u &= 0 \qquad \text{in } \ \Omega \times (0,T) \\ u(x,t) &= 0 \qquad \text{on } \ \partial\Omega \times (0,T) \\ u(x,0) &= u_0 \qquad \text{in } \ \Omega, \end{aligned} \tag{1.1}$$

where the unknowns $u = u(x,t), p = p(x,t)$ are the velocity, pressure of the fluid field, respectively, $f = f(x,t)$ is the body force and $u_0 = u_0(x)$ is the initial velocity field; for simplicity we put the coefficient of kinematical viscosity and the density of the fluid equal to be 1.

Starting with the famous works of Leray [53] and Hopf [45], the existence and uniqueness of solutions to the problem (1.1) have been studied by many people with tremendous efforts. However, the uniqueness of *Leray-Hopf weak solutions* and the existence of *global strong solutions* to (1.1) still remain unsolved for the space dimension $n \geq 3$, which is a prominent open problem in the theory of Navier-Stokes equations. The Leray-Hopf solution is of significance since this solution is, up to now, the only solution to (1.1) for which global existence is proven without any restriction on the size of the data. The *Serrin's class* $L^{p,q} \equiv L^p(0,T; L^q(\Omega)), 2/p + n/q = 1, p > n$, is a very crucial functional class since Leray-Hopf weak solutions are unique in $L^{p,q}$ and any Leray-Hopf weak solution belonging to this class is regular. We refer to [35], [42], [74] for more details. Note that for a Leray-Hopf weak solution the initial value u_0 must be necessarily in $L^2(\Omega)$. Consideration of (1.1) by an L^q-space approach is known to be very convenient to study a suitable solution belonging to $L^{p,q}$ without imposing any smoothness of data. There are many papers dealing with (1.1) in L^q-spaces on domains with compact boundaries as well as with noncompact boundaries. However, in the case of unbounded cylindrical domains, there seems to be no result known yet for L^q-approach to the instationary problem (1.1). We would like to mention that there is only a few papers, so far as we know, dealing with instationary problems (1.1) in unbounded cylindrical domains in contrast to fairly many papers for stationary problems (e.g. [12], [51], [52], [60], [64], [65], [71]). In this respect we refer to [67] and [66] for recent results of instationary linear and nonlinear problems in L^2-space.

As is well known, the analytic semigroup approach to the instationary Stokes and Navier-Stokes equation is a very convenient tool; to this end, resolvent estimate of the Stokes operator must be obtained. The Stokes operator $A_q, 1 < q < \infty$, in Ω is defined by

$$A_q = -P_q \Delta, \quad D(A_q) = W^{2,q}(\Omega)^n \cap W_0^{1,q}(\Omega)^n \cap L^q_\sigma(\Omega),$$

where P_q is the Helmholtz projection in $L^q(\Omega)$ and $L^q_\sigma(\Omega) := P_q L^q(\Omega)$, see Section 2.1 for details of notations. Moreover, to analyze further properties of the Stokes operator such as maximal regularity, boundedness of imaginary powers is very important and useful for estimates of the nonlinear term of the Navier-Stokes equation which causes main difficulties in the study of stationary and instationary Navier-Stokes equations. Maximal L^p-regularity of the Stokes operator is a crucial property for the study of the *strong energy inequality* and *partial regularity* for Navier-Stokes equations. *Boundedness of imaginary powers* of sectorial operators is an important property which enables us to apply easily techniques of interpolation spaces to estimates for nonlinear problems and, moreover, this property yields, in a particular case, maximal regularity. We also mention that *boundedness of H^∞-calculus* of sectorial operators implies boundedness of imaginary powers, and moreover, the property of admitting a bounded H^∞-calculus is stable by small perturbation.

In the present contribution we consider an *infinite cylindrical domain*

$$\Omega = \bigcup_{i=0}^{m} \Omega_i \tag{1.2}$$

of $\mathbb{R}^n, n \geq 3$, of $C^{1,1}$-class, where Ω_0 is a bounded domain and $\Omega_i, i = 1, \ldots, m$, are disjoint semi-infinite straight cylinders, that is, in possibly different coordinates,

$$\Omega_i = \{x^i = (x^i_1, \ldots, x^i_n) \in \mathbb{R}^n : \ x^i_n > 0, \, x'^i = (x^i_1, \ldots, x^i_{n-1}) \in \Sigma^i\},$$

with $\Sigma^i \subset \mathbb{R}^{n-1}, i = 1, \ldots, m$, a bounded domain and $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$.

The main purpose of the present thesis is to study the resolvent estimate, maximal regularity and H^∞-calculus of the Stokes operator in the domain Ω. Moreover, we apply the obtained properties of the Stokes operator in Ω to prove the exponential stability of the *stationary Navier-Stokes equations* (SNS), see below.

First we study the *Stokes resolvent system*

$$(R_\lambda) \qquad \begin{array}{rcll} \lambda u - \Delta u + \nabla p &=& f & \text{in } \Omega \\ \operatorname{div} u &=& 0 & \text{in } \Omega \\ u &=& 0 & \text{on } \partial\Omega. \end{array}$$

We start with the consideration of the system (R_λ) with prescribed divergence

$$\operatorname{div} u = g$$

on an infinite straight cylinder $\Omega = \Sigma \times \mathbb{R}, \Sigma \subset \mathbb{R}^{n-1}$. Then, by the application of the partial Fourier transform $\mathcal{F} = \hat{}$ along the axis of the cylinder Ω the system (R_λ) is reduced to the *parametrized Stokes system* in the cross-section Σ

$$(R_{\lambda,\xi}) \qquad \begin{array}{rcll} (\lambda + \xi^2 - \Delta')\hat{u}' + \nabla'\hat{p} &=& \hat{f}' & \text{in } \Sigma \\ (\lambda + \xi^2 - \Delta')\hat{u}_n + i\xi\hat{p} &=& \hat{f}_n & \text{in } \Sigma \\ \operatorname{div}'\hat{u}' + i\xi\hat{u}_n &=& \hat{g} & \text{in } \Sigma \\ \hat{u}' = 0, \quad \hat{u}_n &=& 0 & \text{on } \partial\Sigma, \end{array}$$

which is elliptic in the sense of Agmon, Douglis and Nirenberg [9] and involves the Fourier phase variable $\xi \in \mathbb{R}$ as a parameter. We get parameter-independent estimates of solutions to $(R_{\lambda,\xi}), \xi \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$, in L^r-spaces with Muckenhoupt weights. To this end, we start with the case $\Sigma = \mathbb{R}^{n-1}$; using Fourier multiplier theory, we get weighted estimates of solutions to $(R_{\lambda,\xi})$ in $\mathbb{R}^{n-1}$ (Theorem 3.1). Next, for $(R_{\lambda,\xi})$ on the half space $\Sigma = \mathbb{R}^{n-1}_+$ (Theorem 3.4), we first consider an estimate for $\hat{p}$; for this a result on Fourier multipliers in trace spaces of Sobolev spaces with Muckenhoupt weights is crucial, see Lemma 3.2. Then the estimate for $\hat{u}$ is obtained using the Laplace resolvent equation. The result for the case of bent half spaces $\Sigma = H_\sigma$ (Theorem 3.5; see (3.2) for the definition of H_σ) is obtained by Kato's perturbation argument. For bounded domains Σ, we start with the Hilbert space setting of $(R_{\lambda,\xi})$ when $\omega \equiv 1$ (Lemma 3.7); for general $r \in (1,\infty)$, cut-off techniques are used based on results for the whole, half and bent half spaces to get a preliminary *a priori* estimate in weighted spaces (Lemma 3.9). Finally we are led to the main estimate of the solution to $(R_{\lambda,\xi})$ using a contradiction argument (Lemma 3.10, Theorem 3.11).

From the parameter-independent estimate for $(R_{\lambda,\xi})$ we get the resolvent estimate and maximal regularity of the Stokes operator, in particular, in *weighted spaces on an infinite straight cylinder*. Due to an extrapolation property of operators defined on L^r-spaces with Muckenhoupt weights, see Theorem 2.14, it follows that the family of solution operators $a_1(\xi)$ for $(R_{\lambda,\xi})$ with $\hat{g} = 0$ is $\mathcal{R}$-bounded. Since the solution u to (R_λ) in the whole cylinder Ω is represented by $u = \mathcal{F}^{-1}(a_1(\xi)\hat{f}(\xi))$, an operator-valued Fourier multiplier theorem (Theorem 2.12) implies the resolvent estimate of the Stokes operator in weighted spaces on an infinite straight cylinder, see Theorem 4.1. In order to prove maximal regularity in weighted spaces on an infinite straight cylinder, we use that maximal regularity of an operator A in a *UMD* space X is implied by the $\mathcal{R}$-boundedness of the operator family

$$\{\lambda(\lambda + A)^{-1} : \ \lambda \in \mathrm{i}\mathbb{R}\} \tag{1.3}$$

in $\mathcal{L}(X)$, see Theorem 2.18. We show the $\mathcal{R}$-boundedness of the family in (1.3) for the Stokes operator $A := A_{q,r;\omega}$ in $L^q(\mathbb{R} : L^r_\omega(\Sigma))$ by virtue of Schauder decomposition techniques; to be more precise, we use the dyadic Schauder decomposition $\{\Delta_j\}_{j\in\mathbb{Z}}$ where $\Delta_j = \mathcal{F}^{-1}\chi_{[2^j,2^{j+1})}\mathcal{F}$ and again the $\mathcal{R}$-boundedness of the family of solution operators for $(R_{\lambda,\xi})$.

Next we consider the general unbounded cylinders Ω, see (1.2). In order to get the L^q-resolvent estimate of the Stokes operator in Ω using the technique of cut-off functions we need to consider the *generalized Stokes resolvent system* (R_λ) with prescribed divergence $\operatorname{div} u = g \neq 0$ on an infinite straight cylinder. With the solution operator $a_2(\xi)$ for $(R_{\lambda,\xi})$ with $\hat{f} = 0$ the solution to (R_λ) with $f = 0$, $\operatorname{div} u = g \neq 0$ is represented by $u = \mathcal{F}^{-1}(a_2(\xi)\hat{g}(\xi))$. However, in this case, the application of Fourier multiplier theorems is not straightforward since the estimate for $(R_{\lambda,\xi})$ with $\hat{g} \neq 0$ involves a complicated parameter-dependent norm. We use techniques of unconditional Schauder decompositions of *UMD* spaces combined with a property of Muckenhoupt weights (see Lemma 5.5) to get estimates for the generalized Stokes system in an infinite straight cylinder (Theorem 5.7). The resolvent estimate of the Stokes operator in the general cylinder Ω is obtained by using standard techniques of cut-off functions as in [27] based on the result for the generalized Stokes system on infinite straight cylinders. In particular, we get that the Stokes operator in Ω generates a bounded and exponentially decaying analytic semigroup in $L^q_\sigma(\Omega)$ for $1 < q < \infty$ (Theorem 5.9).

An important application of our resolvent estimate concerns the H^∞-calculus of the Stokes operator in the general unbounded cylinder Ω in (1.2). A general theory for unbounded domains for which the shifted Stokes operator $c + A_q$ for some $c > 0$ admits a bounded H^∞-calculus was studied in [7], Theorem 1.3. We check that the unbounded cylindrical domain Ω satisfies the assumptions on the domain in that theory (see Assumption (A1) - (A3) in Section 5.4 for details). Then, since our resolvent estimate includes the case $\lambda = 0$, it follows that the Stokes operator admits a bounded H^∞-calculus in $L^q_\sigma(\Omega)$, see Theorem 5.13.

Up to now the Stokes resolvent system has been analyzed e.g. in [1] - [8], [15], [22], [25] - [29], [31], [33], [34] and [39]. Resolvent estimates for the Stokes operator in L^q-spaces in the case of $\operatorname{div} u = 0$ or $\operatorname{div} u \neq 0$ in (R_λ) were obtained for bounded and exterior domains as well as for bent, perturbed half spaces and aperture domains in [15], [26] - [28] and [39]; corresponding results in weighted L^q-spaces can be found in [29], [33], [34]. In [2], [3] and [8], L^q-resolvent estimates of the Stokes operator in an infinite layer $\mathbb{R}^{n-1} \times (0,1)$ were considered. Recently Stokes resolvent estimates in layer-like domains were obtained in [4] using the theory of pseudo-differential operators. General unbounded domains are considered in [31] by replacing the space L^q by $L^q \cap L^2$ or $L^q + L^2$. For infinite cylindrical domains one can find a result in the Bloch space of locally square integrable functions in [69].

We refer to e.g. [32], [34] for the maximal regularity of Stokes operators in bent half spaces, bounded and aperture domains.

Concerning the H^∞-calculus we mention that the Stokes operator admits a bounded H^∞-calculus for domains with compact boundaries [61], for half spaces [21], perturbed half spaces [61], aperture domains [7] and layer-like domains [5].

The next objective of this thesis is to apply the obtained properties of the Stokes operator in the unbounded cylindrical domain Ω to the study of stability of a strong solution to the stationary Navier-Stokes system (SNS) (see below) with prescribed flux $\Phi_i, i = 1, \ldots, m$, in the i-th exit of Ω. Let us consider the *stationary Navier-Stokes system*

$$
\text{(SNS)} \qquad \begin{array}{rcll}
-\Delta w + (w, \nabla) w + \nabla z & = & f & \text{in } \Omega \\
\operatorname{div} w & = & 0 & \text{in } \Omega \\
w & = & 0 & \text{on } \partial\Omega \\
w & = & u_\infty & \text{at infinity,}
\end{array}
$$

where u_∞ coincides with the *Poiseuille solution* corresponding to the given flux in each exit. Due to the solenoidalness of the fluid, a *flux condition*

$$\sum_{i=1}^{m} \Phi_i = 0$$

must necessarily be satisfied.

In order to prove existence and uniqueness to (SNS), first, a *carrier* $\mathbf{a}$ on Ω of the Poiseuille flows $\mathbf{v}_i$, corresponding to the given fluxes $\Phi_i, i = 1, \dots, m$, in each exit of the domain Ω is constructed. The original system (SNS) is reduced to a modified stationary Navier-Stokes system with respect to the new unknown $v = w - \mathbf{a}$, see the system (SNS′) in (6.21), with zero flux. Then a standard fixed point argument via a linearization of (SNS′) yields the existence and uniqueness of a solution to the system (SNS′) in L^r-spaces if the external force f and the total flux Φ are sufficiently small, see Theorem 6.4 for details.

If the stationary Navier-Stokes flow $\{w, \nabla q\}$ subject to (SNS) is perturbed by a velocity field u_0 at time $t = 0$, then the corresponding perturbed instationary flow $\{u(t) + w, \nabla(p(t) + q)\}$ is governed by the following system;

$$\begin{aligned} u_t - \Delta u + (u \cdot \nabla)w + (w \cdot \nabla)u + (u \cdot \nabla)u + \nabla p &= 0 \quad \text{in} \quad \Omega \times (0, \infty) \\ \operatorname{div} u &= 0 \quad \text{in} \quad \Omega \times (0, \infty) \\ u &= 0 \quad \text{on} \quad \partial\Omega \times (0, \infty) \\ u(0) &= u_0 \quad \text{in} \quad \Omega. \end{aligned} \tag{1.4}$$

The study of stability for (SNS) is reduced to the investigation of the behavior of solutions to (1.4) at infinity. We consider the abstract version of (1.4), i.e.,

$$u_t + S_r u + P_r(u \cdot \nabla)u = 0, \quad u(0) = u_0 \quad \text{in } L^r_\sigma(\Omega),$$

where $S_r u = A_r u + P_r((u \cdot \nabla)w + (w \cdot \nabla)u)$ with w the solution to (SNS); here P_r is the Helmholtz projection in $L^r(\Omega)$. We show using a perturbation technique that, if $\|f\|_r$ and *total flux* $\Phi := \sum_{i=1}^m |\Phi_i|$ is small enough, then the operator $-S_r$ generates a bounded analytic semigroup $\{e^{-tS_r}\}_{t\geq 0}$ and, moreover, admits a bounded H^∞-calculus in $L^r_\sigma(\Omega)$ for $r > \frac{n}{3}$ (Theorem 6.9). Then, based on $L^r - L^q$ estimates for $\{e^{-tS_r}\}_{t\geq 0}$ and $\{e^{-tS^*_{r'}}\}_{t\geq 0}$ where $S^*_{r'}$ is the adjoint operator of S_r (Lemma 6.13), we get by a fixed point argument the existence of a *global mild solution* in the sense of Definition 6.16 to (1.4) which decays exponentially as $t \to \infty$ (Theorem 6.19).

Moreover, the $L^r - L^q$ estimates of $\{e^{-tS_r}\}_{t\geq 0}$ and $\{e^{-tS^*_{r'}}\}_{t\geq 0}$ yield that the global mild solution has, at least, a certain regularity depending on $r \geq n$. Then, sharp estimates for the nonlinear term $(u \cdot \nabla)u$ (Lemma 6.1 (3)) combined with the theory of abstract parabolic equations yield that this global mild solution is actually a *strong solution* to (1.4) in the sense of Definition 6.14 (Theorem 6.21). Finally, we consider the uniqueness of strong solutions to (1.4) (Theorem 6.22). Summarizing these results we proved exponential stability of the stationary solution w. We note that, when $w = 0$, we get a result about existence and uniqueness of a global in time strong solution with zero flux to the instationary Navier-Stokes system in $L^r(\Omega)$ for $r \geq n$.

The existence of solutions to stationary Navier-Stokes systems in infinite cylindrical domains of $\mathbb{R}^n, n = 2, 3$, with ball cross-sections was considered in [52] for weak solution with arbitrary flux. It should be noted that the existence for large data were obtained without imposing *a priori* that the flow at infinity behaves like a Poiseuille flow and that it is not known whether the solutions obtained will tend to a Poiseuille flow as $|x| \to \infty$, see [65], §2.6 or [37], Ch. XI, Remark 3.1. In [64] the existence of a strong solution to the system (SNS) in cylindrical domains with ball cross-sections was obtained under a smallness condition on the flux; it is not clear if the technique in [64] is applicable to our domain with arbitrary cross-section. In the case of our domain, the existence of weak solutions was shown in [12] under the smallness condition for the flux, and the existence of so-called *quiet flows* to the stationary Navier-Stokes system was obtained in [60]. We refer to Introduction in [64] and [36], Ch. VI for more details of solvability of stationary Navier-Stokes systems in domains with noncompact boundaries.

There is a number of papers dealing with stability of stationary Navier-Stokes flows on various domains; we refer to [48], [57] for the whole space, [55] for half space, [50] for bounded domain and [47], [49], [70] for exterior domains and the references therein. In the case of our domain Ω the instationary Navier-Stokes system with time-dependent prescribed flux has been considered in Hilbert spaces using the Galerkin approximation method in [66].

The thesis is organized as follows:

In Chapter 2 we introduce preliminaries concerning notations, function spaces, definitions and theorems required for the forthcoming chapters.

Chapter 3 is devoted to the parametrized Stokes system on the cross-section of an infinite cylinder.

Chapter 4 concerns the resolvent estimate and maximal L^p-regularity of the Stokes operator in weighted spaces on an infinite straight cylinder.

In Chapter 5 we consider the resolvent problem and H^∞-calculus of the Stokes operator in general unbounded cylinders with several exits to infinity.

Finally, in Chapter 6 we study stability of solutions to a stationary Stokes system in a general unbounded cylinder Ω.

2 Preliminary

2.1 Notation, function spaces

In the following $\mathbb{N}$ denotes the set of all natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ the set of all integers, $\mathbb{R}$ the set of real numbers and $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$, and $\mathbb{C}$ denotes the set of all complex numbers.

If $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}_0^k$, is a multi-index, $|\alpha| = \alpha_1 + \ldots + \alpha_k$ and $\nabla^\alpha = D_{x_1}^{\alpha_k} \ldots D_{x_k}^{\alpha_k}$, where $\nabla = (D_{x_1}, \ldots, D_{x_k})$ and $D_{x_j} = \frac{\partial}{\partial x_j}$, $j = 1, \ldots, k$.

For $\varepsilon \in (0, \pi]$ the sector of the complex plane with angle 2ε around the positive real axis is denoted by

$$\Sigma_\varepsilon = \{\lambda \in \mathbb{C} :\ \lambda \neq 0,\ |\arg \lambda| < \varepsilon\}.$$

For a locally convex space X we denote usually by X^* the dual space of X and by $\langle \cdot, \cdot \rangle_{X^*,X}$ or $\langle \cdot, \cdot \rangle_{X,X^*}$ the duality pairing between X and X^*. The closure of a subset M of X is denoted by $\overline{M}$.

Let X, Y be linear normed spaces. For a linear operator A from X to Y its domain, range and kernel are denoted by $D(A), R(A)$ and $\operatorname{Ker} A$, respectively.

Let A be a linear operator whose domain and range both lie in the same complex linear normed space. Then $\rho(A)$, $\sigma(A)$ denote the resolvent set and the spectrum of A, respectively.

Let X, Y be Banach spaces. Then $X \hookrightarrow Y$ means that X is continuously embedded into Y. The Banach space of all linear bounded operators from X to Y endowed with the uniform convergence topology is denoted by $\mathcal{L}(X, Y)$, and $\mathcal{L}(X) := \mathcal{L}(X, X)$.

Let G be a domain of $\mathbb{R}^k, k \in \mathbb{N}$. Then, $C_0^\infty(G)$ is the set of all functions $f \in C^\infty(\mathbb{R}^k)$ such that $\operatorname{supp} f \subset G$ is compact, and

$$C_0^\infty(\overline{G}) = \{f\big|_{\overline{G}} :\ f \in C_0^\infty(\mathbb{R}^k)\},$$

where $f\big|_{\overline{G}}$ is the restriction of f onto $\overline{G}$. The space $C_0^\infty(G)$ topologized via an inductive limit argument ([77]) is denoted by $\mathcal{D}(G)$ and its dual, the space of distributions on G, by $\mathcal{D}'(G)$. The space $L^r(G; X)$ for $1 < r \leq \infty$ and a Banach space X denotes the vector space of all X-valued strongly measurable functions such that

$$\|u\|_{L^r(G;X)} := \left(\int_G \|u(x)\|_X^r \, dx \right)^{1/r} < \infty \qquad \text{for } 1 < r < \infty$$

$$\|u\|_{L^\infty(G;X)} := \text{ess sup}_{x \in G} \|u(x)\|_X < \infty \qquad \text{for } r = \infty,$$

and $L^r(G) := L^r(G; \mathbb{K})$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $\|\cdot\|_r := \|\cdot\|_{L^r(G;\mathbb{K})}$. Moreover, $W^{l,r}(G; X)$ for $1 < r < \infty, l \in \mathbb{N}$, denotes the vector-valued Sobolev space of all strongly measurable X-valued functions on G whose derivatives of order up to l exist in X and whose norm

$$\|u\|_{W^{l,r}(G;X)} := \left(\int_G \sum_{\alpha \in \mathbb{N}_0^m, |\alpha| \leq l} \|\nabla^\alpha u(x)\|_X^r \, dx \right)^{1/r}$$

is finite. As usual, $W^{l,r}(G) := W^{l,r}(G;\mathbb{K})$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $W_0^{l,r}(G)$ is the closure of the set $C_0^\infty(G)$ in $W^{l,r}(G)$ and $W^{-l,r'}(G) := (W_0^{l,r}(G))^*, r' = r/(r-1)$.

We define the vector-valued homogeneous Sobolev space $\widehat{W}^{1,q}(G;X)$ by

$$\widehat{W}^{1,q}(G;X) := \{u \in L^1_{\text{loc}}(G;X); \nabla u \in L^q(G;X)\}$$

endowed with the (semi–)norm

$$\|u\|_{\widehat{W}^{1,q}(G;X)} = \|\nabla u\|_{L^q(G;X)},$$

here we neglect the technicality that $\widehat{W}^{1,q}(G;X)$ should be defined as a quotient space (of functions modulo constants). Let $\widehat{W}^{1,r}(G) := \widehat{W}^{1,q}(G;\mathbb{K})$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $\widehat{W}^{-1,r}(G) := (\widehat{W}^{1,r'}(G))^*$.

Moreover, $\mathcal{S}(\mathbb{R}^k;X)$ is the Schwartz space of all rapidly decreasing X-valued functions, that is,

$$\mathcal{S}(\mathbb{R}^k;X) = \{f \in C^\infty(\mathbb{R}^k;X) : \sup_{x\in\mathbb{R}^k} |x|^\alpha |\nabla^\beta f(x)| < \infty \quad \text{for all } \alpha, \beta \in \mathbb{N}_0^k\},$$

and $\mathcal{S}'(\mathbb{R}^k;X)$ is the space of tempered distributions with values in X. In particular, $\mathcal{S}(\mathbb{R}^k) := \mathcal{S}(\mathbb{R}^k;\mathbb{C})$ and $\mathcal{S}'(\mathbb{R}^k) := \mathcal{S}'(\mathbb{R}^k;\mathbb{C})$.

The k-dimensional Fourier transform $\mathcal{F}f$ of $f \in \mathcal{S}(\mathbb{R}^k;X)$ is defined by

$$(\mathcal{F}f)(\xi) = \frac{1}{(2\pi)^{k/2}} \int_{\mathbb{R}^k} f(x) e^{-\mathrm{i}x\xi}\, dx,$$

and the inverse Fourier transform $\mathcal{F}^{-1}g$ of $g \in \mathcal{S}(\mathbb{R}^k;X)$ by

$$(\mathcal{F}^{-1}g)(x) = \frac{1}{(2\pi)^{k/2}} \int_{\mathbb{R}^k} g(\xi) e^{\mathrm{i}x\xi}\, dx.$$

For $f \in \mathcal{S}'(\mathbb{R}^k)$ the Fourier transform $\mathcal{F} : \mathcal{S}'(\mathbb{R}^k) \to \mathcal{S}'(\mathbb{R}^k)$ is defined by

$$\langle \mathcal{F}f, \varphi\rangle_{\mathcal{S}',\mathcal{S}} = \langle f, \mathcal{F}\varphi\rangle_{\mathcal{S}',\mathcal{S}},\ \varphi \in \mathcal{S}(\mathbb{R}^k).$$

For $s \geq 0$ and $1 < r < \infty$ we denote by $H^{s,r}(\mathbb{R}^k)$ the Bessel potential space

$$\begin{aligned} &H^{s,r}(\mathbb{R}^k) := \{f \in \mathcal{S}'(\mathbb{R}^k) : \mathcal{F}^{-1}(1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}f \in L^r(\mathbb{R}^k)\}, \\ &\|f\|_{H^{s,r}(\mathbb{R}^k)} = \|\mathcal{F}^{-1}(1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}f\|_{r;\mathbb{R}^k}, \end{aligned}$$

where $\mathcal{F}$ is the k-dimensional Fourier transform, and for a domain $G \subset \mathbb{R}^k$

$$H^{s,r}(G) := \{f = \tilde{f}\big|_G : \tilde{f} \in H^{s,r}(\mathbb{R}^k)\}, \quad \|f\|_{H^{s,r}(G)} = \inf_{\tilde{f}\in H^{s,r}(\mathbb{R}^k), \tilde{f}|_G = f} \|\tilde{f}\|_{H^{s,r}(\mathbb{R}^k)}.$$

For an interval $J \subset \mathbb{R}$ let $BC(J,X)$ denote the space of all bounded and continuous X-valued functions defined on J with norm

$$\|u\|_{BC(J,X)} = \sup_{s\in J} \|u(s)\|_X.$$

Throughout the thesis $\Omega \subset \mathbb{R}^n, n \geq 3$, is an infinite straight cylinder $\Sigma \times \mathbb{R}$ with $\Sigma \subset \mathbb{R}^{n-1}$ a bounded domain of $C^{1,1}$-class or a general unbounded cylinder given by (1.2). Let a generic point $x \in \Omega$ be written in the form $x = (x', x_n) \in \Omega$, where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. Similarly, differential operators in $\mathbb{R}^n$ are split, in particular, $\nabla = (\nabla', \partial_n)$, $\Delta = \Delta' + \partial_n^2$. Let

$$C_{0,\sigma}^\infty(\Omega) := \{u \in C_0^\infty(\Omega)^n : \operatorname{div} u = 0\}.$$

Then $L_\sigma^r(\Omega)$ denotes the closure of the set $C_{0,\sigma}^\infty(\Omega)$ in $L^r(\Omega)$.

Let Σ be a domain of $\mathbb{R}^{n-1}$. Given a Muckenhoupt weight $\omega \in L^1_{\mathrm{loc}}(\mathbb{R}^{n-1})$ (see Definition 2.1), the space $L^r_\omega(\Sigma)$, $1 < r < \infty$, denotes the Lebesgue space with weight ω endowed with norm

$$\|u\|_{r,\omega;\Sigma} := \left(\int_\Sigma |u(x')|^r \omega(x')\, dx'\right)^{1/r} < \infty;$$

we write shortly $\|u\|_{r,\omega}$ for $\|u\|_{r,\omega;\Sigma}$ as long as no confusion arises. We introduce, if Σ is bounded, the subspace of functions of mean value 0 on Σ, namely

$$L^r_{(m),\omega}(\Sigma) := \left\{u \in L^r_\omega(\Sigma) : \int_\Sigma u(x')\, dx' = 0\right\},$$

and $L^r_{(m)}(\Sigma) := L^r_{(m),\omega}(\Sigma)$ for $\omega \equiv 1$ on Σ. Moreover, $W^{k,r}_\omega(\Sigma)$ for $k \in \mathbb{N}$ denotes the Sobolev space with Muckenhoupt weight ω endowed with norm

$$\|u\|_{k,r,\omega} = \big(\sum_{|\alpha|\leq k} \|\nabla'^\alpha u\|^r_{r,\omega}\big)^{1/r};$$

moreover, $W^{k,r}_{0,\omega}(\Sigma) := \overline{C_0^\infty(\Sigma)}^{\|\cdot\|_{k,r,\omega}}$ and $W^{-k,r}_{0,\omega}(\Sigma) := (W^{k,r'}_{0,\omega'}(\Sigma))^*$, where $\omega' = \omega^{-1/(r-1)}$. We introduce the homogeneous Sobolev space with Muckenhoupt weight

$$\widehat{W}^{1,r}_\omega(\Sigma) = \big\{u \in L^1_{\mathrm{loc}}(\bar{\Sigma})/\mathbb{R}; \nabla' u \in L^r_\omega(\Sigma)\big\}$$

with norm $\|\nabla' u\|_{r,\omega}$ and its dual space $\widehat{W}^{-1,r'}_{\omega'} := (\widehat{W}^{1,r}_\omega)^*$ with norm $\|\cdot\|_{-1,r',\omega'} = \|\cdot\|_{-1,r',\omega';\Sigma}$.

Let $q, r \in (1,\infty)$, and let Ω be an infinite cylinder $\Sigma \times \mathbb{R}$, where Σ is a bounded $C^{1,1}$-domain of $\mathbb{R}^{n-1}$. We introduce the function space $L^q(L^r_\omega) := L^q(\mathbb{R}; L^r_\omega(\Sigma))$ with norm

$$\|u\|_{L^q(L^r_\omega)} = \left(\int_\mathbb{R} \Big(\int_\Sigma |u(x', x_n)|^r \omega(x')\, dx'\Big)^{q/r} dx_n\right)^{1/q}.$$

Furthermore, $W^{k;q,r}_\omega(\Omega), k \in \mathbb{N}$, denotes the Banach space of all functions in Ω whose derivatives of order up to k belong to $L^q(L^r_\omega)$ with norm $\|u\|_{W^{k;q,r}_\omega} = (\sum_{|\alpha|\leq k} \|D^\alpha u\|^2_{L^q(L^r_\omega)})^{1/2}$, where $\alpha \in \mathbb{N}_0^n$, and let $W^{1;q,r}_{0,\omega}(\Omega)$ be the completion of the set $C_0^\infty(\Omega)$ in $W^{1;q,r}_\omega(\Omega)$. The weighted homogeneous Sobolev space $\widehat{W}^{1;q,r}_\omega(\Omega)$ is defined by

$$\widehat{W}^{1;q,r}_\omega(\Omega) = \{u \in L^1_{\mathrm{loc}}(\Omega)/\mathbb{R} : \nabla u \in L^q(L^r_\omega)\}$$

with norm $\|\nabla u\|_{L^q(L^r_\omega)}$. Finally, $L^q(L^r_\omega)_\sigma$ is the completion in the space $L^q(L^r_\omega)$ of the set $C^\infty_{0,\sigma}(\Omega)$.

The duality pairing between $L^r(G)$ and $L^{r'}(G)$ is denoted by $\langle\cdot,\cdot\rangle$ or $(\cdot,\cdot)$. For $\theta \in (0,1), p \in [1,\infty]$ we denote by $[\cdot,\cdot]_\theta$ the complex interpolation functor and $(\cdot,\cdot)_{\theta,p}$ the real interpolation functor.

For notational convenience we do not distinguish spaces of vector functions from ones of scalar functions, for example, $L^r(\Omega)$ may mean a Lebesgue space of scalar functions or the one of vector functions, which will depend on the context. We use the short notation $\|u,v\|_X$ for $\|u\|_X + \|v\|_X$, even if u and v are tensors of different order. For notational convenience, as long as no confusion arises, we denote constants appearing in the proofs by the same symbol, say c or C, even though they may be different from line to line.

2.2 Muckenhoupt weights

Definition 2.1 (Muckenhoupt Weight) *Let $1 < r < \infty$. A function $0 \le \omega \in L^1_{loc}(\mathbb{R}^{n-1})$ is called A_r-weight (Muckenhoupt weight) on $\mathbb{R}^{n-1}$ iff*

$$\mathcal{A}_r(\omega) := \sup_Q \left(\frac{1}{|Q|}\int_Q \omega\, dx'\right)\cdot\left(\frac{1}{|Q|}\int_Q \omega^{-1/(r-1)}\, dx'\right)^{r-1} < \infty, \tag{2.1}$$

where the supremum is taken over all cubes of $\mathbb{R}^{n-1}$ and $|Q|$ denotes the $(n-1)$-dimensional Lebesgue measure of Q. We call $\mathcal{A}_r(\omega)$ the A_r-constant of ω and denote the set of all A_r-weights on $\mathbb{R}^{n-1}$ by $A_r = A_r(\mathbb{R}^{n-1})$.

Note that

$$\omega \in A_r \quad \text{iff} \quad \omega' := \omega^{-1/(r-1)} \in A_{r'},\ r' = r/(r-1) \tag{2.2}$$

and $\mathcal{A}_{r'}(\omega') = \mathcal{A}_r(\omega)^{r'/r}$. A constant $C = C(\omega)$ is called A_r-*consistent* if for every $d > 0$

$$\sup\{C(\omega) : \omega \in A_r, \mathcal{A}_r(\omega) < d\} < \infty. \tag{2.3}$$

We write $\omega(Q)$ for $\int_Q \omega\, dx'$.

It is well-known that $L^r_\omega(\Sigma)$ for any domain $\Sigma \subset \mathbb{R}^{n-1}$ is a separable reflexive Banach space with dense subspace $C^\infty_0(\Sigma)$. In particular $(L^r_\omega(\Sigma))^* = L^{r'}_{\omega'}(\Sigma)$.

Proposition 2.2 ([30], Lemma 2.4) *Let $1 < r < \infty$ and $\omega \in A_r(\mathbb{R}^{n-1})$.*

(1) Let $T : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be a bijective, bi-Lipschitz vector field. Then also $\omega \circ T \in A_r(\mathbb{R}^{n-1})$ and $\mathcal{A}_r(\omega \circ T) \le c\,\mathcal{A}_r(\omega)$ with a constant $c = c(T,r) > 0$ independent of ω.

(2) Define the weight $\tilde\omega(x') = \omega(|x_1|, x'')$ for $x' = (x_1, x'') \in \mathbb{R}^{n-1}$. Then $\tilde\omega \in A_r$ and $\mathcal{A}_r(\tilde\omega) \le 2^r\,\mathcal{A}_r(\omega)$.

(3) Let $\Sigma \subset \mathbb{R}^{n-1}$ be a bounded domain. Then there exist $\tilde s, s \in (1,\infty)$ satisfying

$$L^{\tilde s}(\Sigma) \hookrightarrow L^r_\omega(\Sigma) \hookrightarrow L^s(\Sigma).$$

Here $\tilde{s}$ *and* $\frac{1}{s}$ *are* A_r*-consistent. Moreover, the embedding constants can be chosen uniformly on a set* $W \subset A_r$ *provided that*

$$\sup_{\omega \in W} \mathcal{A}_r(\omega) < \infty, \qquad \int_Q \omega \, dx' = 1 \quad \textit{for all } \omega \in W, \tag{2.4}$$

for a cube $Q \subset \mathbb{R}^{n-1}$ *with* $\bar{\Sigma} \subset Q$.

Proposition 2.3 ([30], Proposition 2.5) *Let* $\Sigma \subset \mathbb{R}^{n-1}$ *be a bounded Lipschitz domain and let* $1 < r < \infty$.

(1) For every $\omega \in A_r$ *the continuous embedding* $W^{1,r}_\omega(\Sigma) \hookrightarrow L^r_\omega(\Sigma)$ *is compact.*

(2) Consider a sequence of weights $(\omega_j) \subset A_r$ *satisfying* (2.4) *for* $W = \{\omega_j : j \in \mathbb{N}\}$ *and a fixed cube* $Q \subset \mathbb{R}^{n-1}$ *with* $\bar{\Sigma} \subset Q$. *Further let* (u_j) *be a sequence of functions on* Σ *satisfying*

$$\sup_j \|u_j\|_{1,r,\omega_j} < \infty \quad \textit{and} \quad u_j \rightharpoonup 0 \quad \textit{in } W^{1,s}(\Sigma)$$

for $j \to \infty$ *where* s *is given by Proposition 2.2 (3). Then*

$$\|u_j\|_{r,\omega_j} \to 0 \quad \textit{as } j \to \infty.$$

(3) Under the same assumptions on $(\omega_j) \subset A_r$ *as in (2) consider a sequence of functions* (v_j) *on* Σ *satisfying*

$$\sup_j \|v_j\|_{r,\omega_j} < \infty \quad \textit{and} \quad v_j \rightharpoonup 0 \quad \textit{in } L^s(\Sigma)$$

for $j \to \infty$. *Then considering* v_j *as functionals on* $W^{1,r'}_{\omega'_j}(\Sigma)$

$$\|v_j\|_{(W^{1,r'}_{\omega'_j}(\Sigma))^*} \to 0 \quad \textit{as } j \to \infty.$$

Proposition 2.4 (Poincaré's inequality) *Let* $r \in (1, \infty)$, $\omega \in A_r$ *and* Σ *be a bounded Lipschitz domain. Then there exists an* A_r*-consistent constant* $c = c(r, \Sigma, \mathcal{A}_r(\omega)) > 0$ *such that*

$$\|u\|_{r,\omega} \le c \|\nabla' u\|_{r,\omega}$$

for all $u \in W^{1,r}_\omega(\Sigma)$ *with vanishing integral mean* $\int_\Sigma u \, dx' = 0$.

Proof: See the proof of [34], Corollary 2.1 and its conclusions; checking the proof, one can claim that the constant $c = c(r, \Sigma, \mathcal{A}_r(\omega))$ is A_r-consistent. ∎

Finally we recall the *Fourier multiplier theorem* in weighted spaces, cf. [38], Ch. IV, Theorem 3.9.

Theorem 2.5 (Fourier multiplier theorem) *Let* $m \in C^k(\mathbb{R}^k \setminus \{0\})$, $k \in \mathbb{N}$, *admit a constant* $M \in \mathbb{R}$ *such that the Hörmander-Michlin condition*

$$|\eta|^\gamma |D^\gamma m(\eta)| \le K \quad \forall \eta \in \mathbb{R}^k \setminus \{0\} \quad \forall \gamma \in \mathbb{N}_0^k \ \textit{with } |\gamma| \le k.$$

Then for all $1 < r < \infty$ *and* $\omega \in A_r(\mathbb{R}^k)$ *the multiplier operator* $Tf = \mathcal{F}^{-1} m(\cdot) \mathcal{F}$ *defined for all rapidly decreasing functions* $f \in \mathcal{S}(\mathbb{R}^k)$ *can uniquely be extended to a bounded linear operator from* $L^r_\omega(\mathbb{R}^k)$ *to* $L^r_\omega(\mathbb{R}^k)$. *Moreover, there exists an* A_r*-consistent constant* $C = C(r, \mathcal{A}_r(\omega))$ *such that*

$$\|Tf\|_{r,\omega} \le CK \|f\|_{r,\omega}, \quad f \in L^r_\omega(\mathbb{R}^k).$$

2.3 $\mathcal{R}$-boundedness, Schauder decomposition

Definition 2.6 (*UMD* space) *A Banach space X is called a* UMD *space if the Hilbert transform*

$$Hf(t) = -\frac{1}{\pi}\ \mathrm{PV} \int \frac{f(s)}{t-s}\,ds \quad \textit{for } f \in \mathcal{S}(\mathbb{R};X)$$

extends to a bounded linear operator in $L^q(\mathbb{R};X)$ for some $q \in (1,\infty)$.

Thus *UMD* spaces are those spaces such that the function $m(t) = \mathrm{sign}\,(t)I_X$ is a Fourier multiplier in $\mathcal{L}(X)$, in particular, the *Riesz projection* $R_0 := \mathcal{F}^{-1}\chi_{[0,\infty)}\mathcal{F}$, where $\mathcal{F}$ is the one-dimensional Fourier transform, is bounded in $\mathcal{L}(X)$. It is well known that, if X is a *UMD* space, the Hilbert transform is bounded in $L^q(\mathbb{R};X)$ for *all* $q \in (1,\infty)$ (see e.g. [68], Theorem 1.3). The dual space and closed subspaces of a *UMD* space are *UMD* spaces as well and for any open set Σ of $\mathbb{R}^{n-1}, 1 < r < \infty$, the weighted spaces $L^r_\omega(\Sigma), W^{1,r}_\omega(\Sigma)$ and $\widehat{W}^{1,r}_\omega(\Sigma)$ are *UMD* spaces.

Definition 2.7 ($\mathcal{R}$-boundedness of operator families) *Let X, Y be Banach spaces. An operator family $\mathcal{T} \subset \mathcal{L}(X;Y)$ is called $\mathcal{R}$-bounded if there is a constant $c > 0$ such that for all $T_1, \cdots, T_N \in \mathcal{T}$, $x_1, \cdots, x_N \in X$ and $N \in \mathbb{N}$*

$$\Big\| \sum_{j=1}^N \varepsilon_j(\cdot) T_j u_j \Big\|_{L^q(0,1;Y)} \le c \Big\| \sum_{j=1}^N \varepsilon_j(\cdot) u_j \Big\|_{L^q(0,1;X)} \tag{2.5}$$

for some $q \in [1,\infty)$, where $(\varepsilon_j(\cdot))$ is any sequence of independent, symmetric $\{-1,1\}$-valued random variables on $[0,1]$. The smallest constant c for which (2.5) holds is called $\mathcal{R}$-bound of $\mathcal{T}$ and denoted by $\mathcal{R}_q(\mathcal{T})$.

Remark 2.8 (1) Due to *Kahane's inequality* ([23])

$$\| \sum_{j=1}^N \varepsilon_j(s) x_j \|_{L^{q_1}(0,1;X)} \le c(q_1,q_2,X) \| \sum_{j=1}^N \varepsilon_j(s) x_j \|_{L^{q_2}(0,1;X)}, \ \ 1 \le q_1, q_2 < \infty, \tag{2.6}$$

the inequality (2.5) holds *for all* $q \in [1,\infty)$ if it holds for some $q \in [1,\infty)$.

(2) If an operator family $\mathcal{T} \subset \mathcal{L}(L^r_\omega(\Sigma))$, $1 < r < \infty$, $\omega \in A_r(\mathbb{R}^{n-1})$, is $\mathcal{R}$-bounded, then $\mathcal{R}_{q_1}(\mathcal{T}) \le C\mathcal{R}_{q_2}(\mathcal{T})$ for all $q_1, q_2 \in [1,\infty)$ with a constant $C = C(q_1,q_2,\Sigma) > 0$ independent of ω. In fact, introducing the isometric isomorphism

$$I_\omega : L^r_\omega(\Sigma) \to L^r(\Sigma), \quad I_\omega f = f\omega^{1/r},$$

for all $T \in \mathcal{L}(L^r_\omega(\Sigma))$ we have $\tilde{T}_\omega = I_\omega T I_\omega^{-1} \in \mathcal{L}(L^r(\Sigma))$ and $\|T\|_{\mathcal{L}(L^r_\omega(\Sigma))} = \|\tilde{T}_\omega\|_{\mathcal{L}(L^r(\Sigma))}$. Then it is easily seen that $\tilde{\mathcal{T}}_\omega := \{I_\omega T I_\omega^{-1} : T \in \mathcal{T}\} \subset \mathcal{L}(L^r(\Sigma))$ is $\mathcal{R}$-bounded and $\mathcal{R}_q(\tilde{\mathcal{T}}_\omega) = \mathcal{R}_q(\mathcal{T})$ for all $q \in (1,\infty)$. Thus the assertion follows.

Definition 2.9 *Let X be a Banach space and $(x_n)_{n=1}^\infty \subset X$. The series $\sum_{n=1}^\infty x_n$ is called unconditionally convergent if $\sum_{n=1}^\infty x_{\sigma(n)}$ is convergent in norm for every permutation $\sigma : \mathbb{N} \to \mathbb{N}$.*

Definition 2.10 (Schauder decomposition) *Let X be a Banach space. A sequence of projections $(\Delta_j)_{j\in\mathbb{N}} \subset \mathcal{L}(X)$ is called a Schauder decomposition of X if*

$$\Delta_i \Delta_j = 0 \qquad \textit{for all} \quad i \neq j$$

and

$$\sum_{j=1}^{\infty} \Delta_j x = x \qquad \textit{for each} \quad x \in X.$$

A Schauder decomposition $(\Delta_j)_{j\in\mathbb{N}}$ of X is called unconditional if the series $\sum_{j=1}^{\infty} \Delta_j x$ converges unconditionally for each $x \in X$.

Note that if $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent, then the sum $\sum_{n=1}^{\infty} x_{\sigma(n)}$ is independent of the permutation σ, see e.g. [19], §3.2.

Remark 2.11 (1) If $(\Delta_j)_{j\in\mathbb{N}}$ is an unconditional Schauder decomposition of a Banach space X, then for each $p \in [1,\infty)$ there is a constant $c_\Delta = c_\Delta(p, X) > 0$ such that for all x_j in the range $R(\Delta_j)$ of Δ_j the inequalities

$$c_\Delta^{-1} \Big\| \sum_{j=l}^{k} x_j \Big\|_X \quad \leq \quad \Big\| \sum_{j=l}^{k} \varepsilon_j(s) x_j \Big\|_{L^p(0,1;X)} \leq c_\Delta \Big\| \sum_{j=l}^{k} x_j \Big\|_X \tag{2.7}$$

are valid for any sequence $(\varepsilon_j(s))$ of independent, symmetric $\{-1,1\}$-valued random variables defined on $(0,1)$ and for all $l \leq k \in \mathbb{Z}$, see e.g. [19], (3.8).

(2) If $(\Delta_j)_{j\in\mathbb{N}}$ is a Schauder decomposition of a Banach space X, then the family $\{\sum_{j=l}^{k} \Delta_j\}_{l,k\in\mathbb{Z}}$ is uniform bounded in X due to the Banach-Steinhaus theorem. Moreover, if $(\Delta_j)_{j\in\mathbb{N}}$ is unconditional, then there is a constant $c > 0$ such that

$$\Big\| \sum_{j=1}^{N} \varepsilon_j \Delta_j x \Big\|_X \leq c \Big\| \sum_{j=1}^{N} \Delta_j x \Big\|_X \quad \text{for all } N \in \mathbb{N},\, x \in X,\ \varepsilon_j \in \{-1,1\},$$

see e.g. [19], Proposition 3.14.

(3) Let $X = L^q(\mathbb{R}; L^r_\omega(\Sigma))$ and let $(\Delta_j)_{j\in\mathbb{N}}$ be an unconditional Schauder decomposition of X such that each Δ_j commutes with the isomorphism introduced in Remark 2.8 (2). Then the constant c_Δ in (2.7) depends only on q, r and is independent of ω, Σ. In fact the constant c_Δ is easily seen to be independent of the weight ω. Moreover we can show that this constant is independent of Σ, by extending functions on Σ by 0 onto $\mathbb{R}^{n-1}$.

(4) In the previous definitions and results the set of indices $\mathbb{N}$ may be replaced by $\mathbb{Z}$ without any further changes.

(5) Given an interpolation couple $\mathcal{X}_1$, $\mathcal{X}_2$ of Banach spaces, it is easily seen that a Schauder decomposition of both $\mathcal{X}_1$ and $\mathcal{X}_2$ is a Schauder decomposition of $\mathcal{X}_1 \cap \mathcal{X}_2$ and $\mathcal{X}_1 + \mathcal{X}_2$ as well.

(6) Let X be a *UMD* space, and let $\chi_{[a,b)}$ denote the characteristic function for the interval $[a,b)$. Let

$$R_a := \mathcal{F}^{-1}\chi_{[a,\infty)}\mathcal{F} \text{ for } a \in \mathbb{R}, \quad \text{and } R_{a,b} := R_a - R_b \text{ for } a, b \in \mathbb{R}. \tag{2.8}$$

It is well known that the *Riesz projection* R_0 is bounded in $L^q(\mathbb{R};X)$, and moreover, $\{R_{a,b}:\ a,b\in\mathbb{R}\}$ is $\mathcal{R}$-bounded in $\mathcal{L}(L^q(\mathbb{R};X))$ for each $q\in(1,\infty)$. In particular, defining

$$\Delta_j = R_{2^j,2^{j+1}},\ j\in\mathbb{Z}, \tag{2.9}$$

the family $\{\Delta_j:\ j\in\mathbb{Z}\}$ is $\mathcal{R}$-bounded in $\mathcal{L}(L^q(\mathbb{R};X))$ and defines an unconditional Schauder decomposition of $R_0L^q(\mathbb{R};X)$, the image of $L^q(\mathbb{R};X)$ by R_0, see [19], proof of Theorem 3.19.

Now we recall an *operator-valued Fourier multiplier theorem* in Banach spaces, cf. [19], Theorem 3.19, [76], Theorem 3.4. Let $\mathcal{D}_0(\mathbb{R};X)$ denote the set of all C^∞-functions $f:\mathbb{R}\to X$ with compact support in $\mathbb{R}^*$.

Theorem 2.12 (Operator-valued Fourier multiplier theorem) *Let X and Y be* UMD *spaces and $1<q<\infty$. Let $M:\mathbb{R}^*\to\mathcal{L}(X,Y)$ be a differentiable function such that*

$$\mathcal{R}_q\big(\{M(t),\, tM'(t):\ t\in\mathbb{R}^*\}\big)\le A.$$

Then the operator

$$Tf=\big(M(\cdot)\hat{f}(\cdot)\big)^\vee,\quad f\in\mathcal{D}_0(\mathbb{R};X),$$

extends to a bounded operator $T:\ L^q(\mathbb{R};X)\to L^q(\mathbb{R};Y)$ with operator norm $\|T\|_{\mathcal{L}(L^q(\mathbb{R};X);L^q(\mathbb{R};Y))}\le CA$ where $C>0$ depends only on q,X and Y.

Remark 2.13 Checking the proof of [19], Theorem 3.19, one can see that the constant C in Theorem 2.12 satisfies

$$C\le\mathcal{R}(\mathcal{P})\cdot(c_\Delta)^2$$

where $\mathcal{R}(\mathcal{P})$ is the $\mathcal{R}$-bound of the operator family $\mathcal{P}=\{R_{a,b}:\ a,b\in\mathbb{R}\}$ in $\mathcal{L}(L^q(\mathbb{R};X))$ and c_Δ is the *unconditional constant* in (2.7) corresponding to the family $\{\Delta_j\}_{j\in\mathbb{Z}}$ in (2.9). In particular, for $X=L^r_\omega(\Sigma)$, $1<r<\infty$, $\omega\in A_r$, using the isometry I_ω of Remark 2.8 (2), we get that the constants $\mathcal{R}(\mathcal{P})$, see Remark 2.8 (2), and c_Δ do not depend on the weight ω; note that each Δ_j, $j\in\mathbb{Z}$, commutes with the isometry I_ω.

Theorem 2.14 (Extrapolation Theorem) *Let $1<r,s<\infty$, $\omega\in A_r(\mathbb{R}^{n-1})$ and $\Sigma\subset\mathbb{R}^{n-1}$ be an open set. Moreover let $\mathcal{T}\subset\mathcal{L}(L^r_\omega(\Sigma))$ be a family of linear operators with the property that there exists an A_s-consistent constant $C_\mathcal{T}=C_\mathcal{T}(\mathcal{A}_s(\nu))>0$ such that for all $\nu\in A_s$*

$$\|Tf\|_{s,\nu}\le C_\mathcal{T}\|f\|_{s,\nu}$$

for all $T\in\mathcal{T}$ and all $f\in L^r_\omega(\Sigma)\cap L^s_\nu(\Sigma)$. Then $\mathcal{T}$ is $\mathcal{R}$-bounded in $\mathcal{L}(L^r_\omega(\Sigma))$ with an A_r-consistent $\mathcal{R}$-bound $c_\mathcal{T}(q,r,\mathcal{A}_r(\omega))$, i.e.,

$$\mathcal{R}_q(\mathcal{T})\le c_\mathcal{T}(q,r,\mathcal{A}_r(\omega))\quad \textit{for all}\quad q\in(1,\infty). \tag{2.10}$$

Proof: From the proof of [34], Theorem 4.3, it can be deduced that $\mathcal{T}$ is $\mathcal{R}$-bounded in $\mathcal{L}(L^r_\omega(\Sigma))$ and that (2.10) is satisfied for $q=r$. Then Remark 2.8 yields (2.10) for every $1<q<\infty$. ∎

2.4 H^∞-calculus, maximal regularity

In this section we introduce the H^∞-calculus and maximal regularity for sectorial operators in a Banach space X.

Definition 2.15 *Let an operator A be closed, injective and densely defined in a Banach space X. The operator A is called sectorial if there is some $\omega \in (0, \pi)$ such that*

(1) $\sigma(A) \subset \overline{\Sigma}_\omega$

(2) For all $\omega' \in (\omega, \pi)$ there exists $M_{\omega'} > 0$ satisfying

$$\|\lambda(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \leq M_{\omega'} \quad \forall \lambda \in \mathbb{C} \setminus \Sigma_{\omega'}. \tag{2.11}$$

The spectral angle ω_A of A is defined by the infimum of the angles ω' for which the assertions (1), (2) hold.

Obviously $\omega_A \in [0, \pi)$. As is well known, if $\omega_A < \pi/2$, then $-A$ generates a bounded analytic semigroup e^{-tA} in X, see e.g. [63].

For a sectorial operator A in a Banach space X it is known that the set $D(A^k) \cap R(A^k), k \in \mathbb{N}$, is dense in X, see e.g. [19].

For $\theta \in (0, \pi)$ let $\mathcal{H}^\infty(\Sigma_\theta)$ be the algebra of all holomorphic and bounded functions on the sector Σ_θ and let

$$\mathcal{H}_0^\infty(\Sigma_\theta) := \left\{ h \in \mathcal{H}^\infty(\Sigma_\theta) : \exists\, k, s > 0 : \ |h(z)| < k \frac{|z|^s}{1 + |z|^{2s}} \quad \forall z \in \Sigma_\theta \right\}.$$

For a sectorial operator A and $h \in \mathcal{H}_0^\infty(\Sigma_\theta)$ with $\theta \in (\omega_A, \pi)$ we define the functional calculus $h(A)$ via the *Dunford integral*

$$h(A) = \frac{1}{2\pi i} \int_\Gamma h(\lambda)(\lambda - A)^{-1} \, d\lambda \in \mathcal{L}(X), \tag{2.12}$$

where the integral curve Γ is the oriented boundary of $\Sigma_{\theta'}$ for any $\theta' \in (\omega_A, \theta)$, i.e. $\Gamma = (\infty, 0)e^{i\theta'} \cup \{0\} \cup (0, \infty)e^{-i\theta'}$; note here that the integral in (2.12) is independent of the choice of θ'. Moreover, it is shown that $h(A) \in \mathcal{L}(X)$, cf. [19], Theorem 1.7.

Definition 2.16 (Boundedness of H^∞-calculus) *Let A be a sectorial operator on a Banach space X and let $\theta \in (\omega_A, \pi)$. The operator A is said to admit a bounded H^∞-calculus (or $H^\infty(\Sigma_\theta)$-calculus in X) if there is a constant $C_\theta > 0$ such that for all $h \in \mathcal{H}_0^\infty(\Sigma_\theta)$ the operator $h(A)$ satisfies the estimate*

$$\|h(A)\|_{\mathcal{L}(X)} \leq C_\theta \|h\|_\infty. \tag{2.13}$$

The $\mathcal{H}^\infty$-angle $\phi_A^\infty \in [\omega_A, \pi)$ of A is defined by

$$\phi_A^\infty := \inf\{\theta \in (\omega_A, \pi) : \ (2.13) \textit{ holds for all } h \in \mathcal{H}_0^\infty(\Sigma_\theta)\}.$$

Remark 2.17 We may define even for $h \in \mathcal{H}^\infty(\Sigma_\theta)$ the operator $h(A)$ with domain $D(A) \cap R(A)$ in X by

$$h(A) = \frac{1}{2\pi i} \int_\Gamma h(\lambda)\lambda(1+\lambda)^{-2}(\lambda - A)^{-1}\, d\lambda\, (1+A)^2 A^{-1}. \tag{2.14}$$

Note that $h \in \mathcal{H}^\infty(\Sigma_\theta)$ implies $h(\lambda)\lambda(1+\lambda)^{-2} \in \mathcal{H}_0^\infty(\Sigma_\theta)$. Then the definition of $h(A)$ by (2.14) is consistent with the definition by (2.12) in the sense that, if $h \in \mathcal{H}_0^\infty(\Sigma_\theta)$, then

$$\int_\Gamma h(\lambda)\lambda(1+\lambda)^{-2}(\lambda - A)^{-1}\, d\lambda = \int_\Gamma h(\lambda)(\lambda - A)^{-1}\, d\lambda\, A(I+A)^{-2},$$

and hence, $h(A)$ defined by (2.14) can be extended uniquely to the bounded operator $h(A)$ defined by (2.12), cf. [19], Theorem 2.1.

Moreover, if the operator A admits a bounded H^∞-calculus in X, then for $h \in \mathcal{H}^\infty(\Sigma_\theta)$ the operator $h(A)$ in (2.14) is bounded in X and (2.13) holds as well, cf. [19], p. 23.

We denote by $\mathcal{H}^\infty(X)$ the set of all sectorial operators admitting a bounded H^∞-calculus in Banach space X.

One of the most important properties of $A \in \mathcal{H}^\infty(X)$ is the boundedness of its imaginary powers. More precisely, if $A \in \mathcal{H}^\infty(X)$, then it has *bounded imaginary powers*, that is, $A^{it} \in \mathcal{L}(X)$ and

$$\|A^{it}\|_{\mathcal{L}(X)} \le Ce^{\mu|t|} \tag{2.15}$$

with some $C > 0, \mu > 0$ for all $t \in \mathbb{R}$; one can check (2.15) by putting $h(\lambda) = \lambda^{it}$ in (2.13) since $|h(\lambda)| \le e^{-t \arg \lambda}$. The infimum of the numbers μ for which (2.15) holds is called *power angle* of A and will be denoted by θ_A. Obviously,

$$\omega_A \le \theta_A \le \phi_A^\infty.$$

It is well known that if a sectorial operator A has bounded imaginary powers in X, then the domains of its fractional powers are represented by complex interpolation of the spaces $D(A)$ and X, i.e.,

$$D(A^\theta) = [X, D(A)]_\theta \quad \forall \theta \in (0,1), \tag{2.16}$$

([19] or [75], Theorem 1.15.3).

An important result for operators having bounded imaginary powers is *maximal L^p-regularity*. We say that the operator A has *maximal L^p-regularity* in X if the linear instationary problem

$$u_t + Au = f, \quad t \ge 0, \quad u(0) = 0 \tag{2.17}$$

for a given $f \in L^p(\mathbb{R}_+; X)$ has a unique solution u such that

$$\|u_t\|_{L^p(\mathbb{R}_+;X)} + \|Au\|_{L^p(\mathbb{R}_+;X)} \le C\|f\|_{L^p(\mathbb{R}_+;X)}. \tag{2.18}$$

It is known that if the power angle $\theta_A < \pi/2$ and X has the *UMD* property (see Definition 2.6), then A has *maximal L^p-regularity* for $1 < p < \infty$, ([19], Theorem 4.4, Theorem 4.5; see also [18] and [24], Theorem 3.2). Evidently, if $0 \in \rho(-A)$, then (2.18) is equivalent to the inequality obtained by replacing $\|u_t\|_{L^p(\mathbb{R}_+;X)}$ in (2.18) by $\|u\|_{W^{1,p}(\mathbb{R}_+;X)}$.

Theorem 2.18 ([76], [19]) *Let X be a* UMD *space, $1 < p < \infty$, and let A be a sectorial operator with spectral angle $\omega_A < \pi/2$ in X. Then the following statements (1) - (3) are equivalent.*

(1) A has maximal L^p-regularity.

(2) The operator family

$$\{\lambda(\lambda + A)^{-1};\ \lambda \in \Sigma_\theta\}$$

for some $\theta \in (\pi/2, \pi)$ is $\mathcal{R}$-bounded in $\mathcal{L}(X)$.

(3) The operator family

$$\{\lambda(\lambda + A)^{-1};\ \lambda \in \mathrm{i}\mathbb{R}\}$$

is $\mathcal{R}$-bounded in $\mathcal{L}(X)$.

Finally, we recall a perturbation result for operators in $\mathcal{H}^\infty(X)$.

Theorem 2.19 ([20], Theorem 3.2) *Let X be a* UMD *space and let $\mathcal{A}$ admit a bounded H^∞-calculus in X. Let $\mathcal{B}$ be a linear operator such that $D(\mathcal{B}) \supset D(\mathcal{A})$.*

(1) Assume that there exists $\kappa > 0$ such that

$$\|\mathcal{B}u\|_X \le \kappa \|\mathcal{A}u\|_X, \quad u \in D(\mathcal{A}).$$

(2) Suppose that there exist $\gamma \in (0,1)$ and $C > 0$ such that

$$\mathcal{B}(D(\mathcal{A}^{1+\gamma}) \subset D(\mathcal{A}^\gamma) \quad \text{and} \quad \|\mathcal{A}^\gamma \mathcal{B}u\|_X \le C\|\mathcal{A}^{1+\gamma}u\|_X \quad \forall u \in D(\mathcal{A}^{1+\gamma}).$$

Then $\mathcal{A}+\mathcal{B}$ admits a bounded H^∞-calculus provided κ is sufficiently small. Moreover, for each $\phi > \phi^\infty_{\mathcal{A}}$ there is $\kappa_0(\phi) > 0$ such that $\phi^\infty_{\mathcal{A}+\mathcal{B}} \le \phi$ if $\kappa < \kappa_0(\phi)$.

3 Parametrized Stokes System in Cross-sections

In this chapter we study the parametrized Stokes system

$$(R_{\lambda,\xi})\qquad \begin{array}{rcll} (\lambda+\xi^2-\Delta')u'+\nabla'p &=& f' & \text{in }\ \Sigma\\ (\lambda+\xi^2-\Delta')u_n+i\xi p &=& f_n & \text{in }\ \Sigma\\ \operatorname{div}'u'+i\xi u_n &=& g & \text{in }\ \Sigma\\ u'=0,\quad u_n &=& 0 & \text{on }\ \partial\Sigma, \end{array}$$

where Σ is a $C^{1,1}$-domain of $\mathbb{R}^{n-1}, n \geq 3$. We obtain parameter-independent estimates of solutions to $(R_{\lambda,\xi})$ for all $\xi \in \mathbb{R}^* = \mathbb{R}\setminus\{0\}$ and $\lambda \in \Sigma_\varepsilon, \frac{\pi}{2} < \varepsilon < \pi$ in L^r-spaces with Muckenhoupt weights. For notational convenience we omit the symbol $\hat{}$ for the one-dimensional Fourier transform; thus

$$u = (u', u_n), p, f, g \quad \text{ stand for } \quad \hat{u} = (\hat{u'}, \hat{u_n}), \hat{p}, \hat{f}, \hat{g}.$$

3.1 Whole and half Spaces

In this section Σ denotes either $\mathbb{R}^{n-1}$ or the half space

$$\Sigma = \mathbb{R}^{n-1}_+ = \{x' = (x_1, x''); x'' \in \mathbb{R}^{n-2}, x_1 > 0\}, \tag{3.1}$$

or a bent half space

$$H_\sigma = \{x' = (x_1, x''); x_1 > \sigma(x''), x'' \in \mathbb{R}^{n-2}\}, \tag{3.2}$$

where σ is a $C^{1,1}$-function.

Let $\omega \in A_r(\mathbb{R}^{n-1})$ be an arbitrary Muckenhoupt weight. For the divergence g ($\hat{=}\hat{g}$) we need for $r \in (1,\infty), \omega \in A_r(\mathbb{R}^{n-1})$ the definition of $\widehat{W}^{-1,r}_\omega(\Sigma) + L^r_{\omega,1/\xi}(\Sigma)$ parametrized by $\xi \in \mathbb{R}^*$. Consider the direct sum $L^r_\omega(\Sigma) \oplus \mathbb{R}$ and its quotient space

$$\hat{L}^r_\omega := L^r_\omega(\Sigma) \oplus \mathbb{R}/\mathbb{R}.$$

Since Σ has unbounded measure, the space $\hat{L}^r_\omega$ equipped with $\|\cdot\|_{r,\omega}$ is isometric to $L^r_\omega(\Sigma)$. This isomorphism allows to define the intersection of the Banach spaces $\widehat{W}^{1,r}_\omega(\Sigma)$ and $L^r_\omega(\Sigma)$, namely,

$$\widehat{W}^{1,r}_\omega(\Sigma) \cap L^r_{\omega,\xi}(\Sigma) \cong W^{1,r}_\omega(\Sigma) \quad \text{with norm} \quad \max\{\|\nabla' u, \xi u\|_{r,\omega}\}.$$

Moreover, since $C_0^\infty(\Sigma)$ is dense in $\widehat{W}^{1,r}_\omega(\Sigma)$ and in $L^r_{\omega,\xi}(\Sigma)$ ([33], Corollary 4.1), we may define the sum

$$\widehat{W}^{-1,r}_\omega + L^r_{\omega,1/\xi} := (\widehat{W}^{1,r'}_{\omega'} \cap L^{r'}_{\omega',\xi})^* \cong (W^{1,r'}_{\omega'})^*,\ r' = r/(r-1), \omega' = \omega^{-1/(r-1)}$$

with $\xi-$dependent norm

$$\|h; \widehat{W}^{-1,r}_\omega + L^r_{\omega,1/\xi}\| = \inf\{\|h_0\|_{-1,r,\omega} + \|h_1/\xi\|_{r,\omega}; h = h_0 + h_1, h_0 \in \widehat{W}^{-1,r}_\omega, h_1 \in L^r_\omega\}.$$

Assume that

$$f \in L^r_\omega(\Sigma), \quad g \in W^{1,r}_\omega(\Sigma).$$

Note that $W^{1,r}_\omega(\Sigma)$ is obviously contained in the sum $\widehat{W}^{-1,r}_\omega(\Sigma) + L^r_{\omega,1/\xi}(\Sigma)$.

Now we start with the case $\Sigma = \mathbb{R}^{n-1}$. Since $C_0^\infty(\mathbb{R}^{n-1})$ is dense in $\widehat{W}^{1,r'}_{\omega'}(\mathbb{R}^{n-1})$, if $g = g_0 + g_1$, $g_0 \in \widehat{W}^{-1,r}_\omega$ and $g_1 \in L^r_{\omega,1/\xi}$, is any splitting of g, Hahn-Banach's theorem implies the existence of a vector field $h \in L^r_\omega$ such that

$$g_0 = \operatorname{div}' h, \quad \|g_0\|_{-1,r,\omega} = \|h\|_{r,\omega}.$$

An elementary calculation shows that p in $(R_{\lambda,\xi})$ satisfies the equation

$$(\xi^2 - \Delta')p = (\lambda + \xi^2 - \Delta')g - (\operatorname{div}' f' + i\xi f_n). \tag{3.3}$$

Introducing the $(n-1)$-dimensional Fourier transform $\tilde{}$ with respect to x' and with phase variable $s \in \mathbb{R}^{n-1}$ we get

$$\begin{aligned} \tilde{p} &= \tilde{g} + \frac{\lambda}{\xi^2 + |s|^2}\tilde{g} - \frac{is}{\xi^2 + |s|^2} \cdot \tilde{f}' - \frac{i\xi}{\xi^2 + |s|^2}\tilde{f}_n \\ &= \tilde{g} + \frac{\lambda is}{\xi^2 + |s|^2} \cdot \tilde{h} + \frac{\lambda\xi}{\xi^2 + |s|^2}(\tilde{g}_1/\xi) - \frac{is}{\xi^2 + |s|^2} \cdot \tilde{f}' - \frac{i\xi}{\xi^2 + |s|^2}\tilde{f}_n. \end{aligned}$$

Obviously the functions

$$m_\xi(s) = \frac{s_j s_k}{\xi^2 + |s|^2}, \quad \frac{s_j \xi}{\xi^2 + |s|^2}, \quad \frac{\xi^2}{\xi^2 + |s|^2}, \quad 1 \le j, k \le n-1,$$

are classical multiplier functions satisfying the pointwise Hörmander-Michlin condition

$$|s|^\alpha |\nabla_s^\alpha m_\xi(s)| \le c_\alpha, \quad 0 \ne s \in \mathbb{R}^{n-1}, \alpha \in \mathbb{N}_0^{n-1}, |\alpha| \le n-1, \tag{3.4}$$

uniformly with respect to $\xi \in \mathbb{R}^*$. Then Theorem 2.5 applied to $\nabla' p$ and to ξp yields the estimate

$$\begin{aligned} \|\nabla' p, \xi p\|_{r,\omega} &\le c(\|f, \nabla' g, \xi g\|_{r,\omega} + \|\lambda h, \lambda g_1/\xi\|_{r,\omega}) \\ &\le c(\|f, \nabla' g, \xi g\|_{r,\omega} + \|\lambda g_0\|_{-1,r,\omega} + \|\lambda g_1/\xi\|_{r,\omega}). \end{aligned} \tag{3.5}$$

Next consider the Laplace resolvent equations for u' and u_n, i.e.,

$$\begin{aligned} (\lambda + \xi^2 - \Delta')u' &= F' \quad \text{in} \quad \mathbb{R}^{n-1}, \\ (\lambda + \xi^2 - \Delta')u_n &= F_n \quad \text{in} \quad \mathbb{R}^{n-1} \end{aligned} \tag{3.6}$$

with resolvent parameters $\lambda + \xi^2$, where $F' := f' - \nabla' p$, $F_n := f_n - i\xi p$ and p is the solution to (3.3) satisfying (3.5). Again applying the $(n-1)$-dimensional Fourier transform with respect to $x' \in \mathbb{R}^{n-1}$ to (3.6), we get

$$\tilde{u}' = \frac{\tilde{F}'}{\lambda + \xi^2 + |s|^2}, \quad \tilde{u}_n = \frac{\tilde{F}_n}{\lambda + \xi^2 + |s|^2}.$$

Therefore, using the fact that

$$\frac{\lambda+\xi^2}{\lambda+\xi^2+|s|^2}, \quad \frac{\sqrt{\lambda+\xi^2}s_j}{\lambda+\xi^2+|s|^2}, \quad \frac{s_j s_k}{\lambda+\xi^2+|s|^2}, \quad j,k=1,\cdots,n-1,$$

are Fourier multipliers satisfying (3.4), we get the existence of a solution $u=(u',u_n)$ to (3.6) satisfying

$$\begin{aligned} \|(\lambda+\xi^2)u, \sqrt{\lambda+\xi^2}\nabla' u, \nabla'^2 u\|_{r,\omega} &\le c\|f, \nabla' p, \xi p\|_{r,\omega} \\ &\le c(\|f,\nabla' g, \xi g\|_{r,\omega} + \|\lambda g_0\|_{-1,r,\omega} + \|\lambda g_1/\xi\|_{r,\omega}) \end{aligned} \tag{3.7}$$

with A_r-consistent constants $c=c(\varepsilon, r, \mathcal{A}_r(\omega))$.

Let $\mu = |\lambda+\xi^2|^{1/2}$. We can prove the following theorem.

Theorem 3.1 *Let $\Sigma=\mathbb{R}^{n-1}$, $1<r<\infty$ and $\omega\in A_r(\mathbb{R}^{n-1})$. If $f\in L^r_\omega(\Sigma)$ and $g\in W^{1,r}_\omega(\Sigma)$, then for every $\lambda\in\Sigma_\varepsilon$, $\frac{\pi}{2}<\varepsilon<\pi$, and $\xi\in\mathbb{R}^*$ $(R_{\lambda,\xi})$ has a unique solution $(u,p)\in W^{2,r}_\omega(\Sigma)\times W^{1,r}_\omega(\Sigma)$ satisfying*

$$\|\mu^2 u, \mu\nabla' u, \nabla'^2 u, \nabla' p, \xi p\|_{r,\omega} \le c\big(\|f,\nabla' g, \xi g\|_{r,\omega} + \|\lambda g; \widehat{W}^{-1,r}_\omega + L^r_{\omega,1/\xi}\|\big) \tag{3.8}$$

with an A_r-consistent constant $c=c(\varepsilon,r,\mathcal{A}_r(\omega))$.

Proof: Let u be a solution to (3.6) where p is a solution to (3.3). We have already seen that $(u,p)\in W^{2,r}_\omega(\Sigma)\times W^{1,r}_\omega(\Sigma)$ satisfies the estimate (3.8) since $g=g_0+g_1$ in the estimate (3.5), (3.7) is an arbitrary splitting of $g\in\widehat{W}^{-1,r}_\omega + L^r_{\omega,1/\xi}$. Therefore, for the proof of the existence of a solution, it is enough to show that (u,p) solves the divergence equation of $(R_{\lambda,\xi})$. A simple calculation with (3.3) and (3.6) yields

$$(\lambda+\xi^2-\Delta')(\operatorname{div}' u' + i\xi u_n - g) = 0 \quad \text{in} \quad \mathbb{R}^{n-1}.$$

Hence standard arguments from Fourier analysis show that $\operatorname{div}' u' + i\xi u_n = g$. The uniqueness of the solution is obvious from the above Fourier multiplier technique, i.e., if (u,p) is a solution to $(R_{\lambda,\xi})$ with $f=0, g=0$, then u satisfies (3.6) with $f=0$ and $(\xi^2-\Delta')p=0$ yielding $p=0$, and hence $u=0$. ■

In the next main step we consider the case $\Sigma=\mathbb{R}^{n-1}_+$, see (3.1). Just as for $x'=(x_1,x'')$ we write $u'=(u_1,u'')$, $f'=(f_1,f'')$. For a function $h:\Sigma\to\mathbb{R}$ define the even extension h_e by

$$h_e(x_1,x'') = \begin{cases} h(x_1,x'') & \text{for} \quad x_1>0 \\ h(-x_1,x'') & \text{for} \quad x_1<0, \end{cases}$$

while the odd extension h_o of h is defined by

$$h_o(x_1,x'') = -h(-x_1,x'') \quad \text{for } x_1<0.$$

Given $(R_{\lambda,\xi})$ in (Σ), take the even extension f''_e of f'', f_{ne} of f_n and g_e of g, but the odd extension f_{1o} of f_1. Then obviously

$$(f_{1o}, f''_e, f_{ne}) \in L^r_{\tilde\omega}(\mathbb{R}^{n-1}), \quad g_e\in W^{1,r}_{\tilde\omega}(\mathbb{R}^{n-1}),$$

where $\tilde\omega(x_1,x'') = \omega(|x_1|,x'')$. Note that $\mathcal{A}_r(\tilde\omega) \le 2^r\,\mathcal{A}_r(\omega)$, see Proposition 2.2 (2). It is clear that

$$\|h_o, h_e\|_{r,\tilde\omega;\mathbb{R}^{n-1}} \le c(r)\,\|h\|_{r,\omega;\Sigma}; \tag{3.9}$$

moreover, for a function $h \in L^r_\omega(\mathbb{R}^{n-1}_+) \cap \widehat{W}^{-1,r}_\omega(\mathbb{R}^{n-1}_+)$ we get

$$\begin{aligned} \|h_e\|_{\widehat{W}^{-1,r}_{\tilde\omega}(\mathbb{R}^{n-1})} &= \sup_\varphi \Big|\int_{\mathbb{R}^{n-1}} h_e\,\varphi\,dx'\Big| \\ &= \sup_\varphi \Big|\int_\Sigma h\,\varphi\,dx' + \int_\Sigma h\,\varphi(-x_1,x'')\,dx'\Big| \\ &\le 2\|h\|_{\widehat{W}^{-1,r}_\omega(\Sigma)}, \end{aligned} \tag{3.10}$$

where the supremum is taken over all $\varphi \in C_0^\infty(\mathbb{R}^{n-1})$ with $\|\nabla'\varphi\|_{r',\omega';\mathbb{R}^{n-1}} \le 1$.

Now we will solve $(R_{\lambda,\xi})$ in the whole space $\mathbb{R}^{n-1}$ with right-hand side $(f_{1o}, f''_e, f_{ne}), g_e$. By the uniqueness assertion it is easily seen that the solution (U,P) of this extended problem is even with respect to x_1 except for the component U_1 which is odd with respect to x_1. In particular $U_1 = 0$ for $x_1 = 0$ and, due to (3.8),

$$\begin{aligned} &\|\mu^2 U, \mu\nabla' U, \nabla'^2 U, \nabla' P, \xi P\|_{r,\omega;\Sigma} \\ &\le c\big(\|f_{1o}, f''_e, f_{ne}, \nabla' g_e, \xi g_e\|_{r,\tilde\omega;\mathbb{R}^{n-1}} + \|\lambda g_e; \widehat{W}^{-1,r}_{\tilde\omega}(\mathbb{R}^{n-1}) + L^r_{\tilde\omega,1/\xi}(\mathbb{R}^{n-1})\|\big) \end{aligned} \tag{3.11}$$

where $\mu = |\lambda + \xi^2|^{1/2}$ and the constant c is A_r-consistent due to Proposition 2.2. Thus, from (3.9)–(3.11) we get

$$\begin{aligned} &\|\mu^2 U, \mu\nabla' U, \nabla'^2 U, \nabla' P, \xi P\|_{r,\omega;\Sigma} \\ &\qquad\le c\big(\|f, \nabla' g, \xi g\|_{r,\omega;\Sigma} + \|\lambda g; \widehat{W}^{-1,r}_\omega + L^r_{\omega,1/\xi}\|\big) \end{aligned} \tag{3.12}$$

with an A_r-consistent constant $c = c(\varepsilon, r, \mathcal{A}_r(\omega))$.

Subtracting (U,P) in $(R_{\lambda,\xi})$, the parametrized resolvent problem $(R_{\lambda,\xi})$ is reduced to the homogeneous system

$$\begin{aligned} (\lambda + \xi^2 - \Delta')u' + \nabla' p &= 0 \quad \text{in} \quad \Sigma = \mathbb{R}^{n-1}_+ \\ (\lambda + \xi^2 - \Delta')u_n + i\xi p &= 0 \quad \text{in} \quad \Sigma \\ \operatorname{div}' u' + i\xi u_n &= 0 \quad \text{in} \quad \Sigma \end{aligned} \tag{3.13}$$

with inhomogeneous boundary values

$$u = \Phi := U|_{\partial\Sigma} \quad \text{on} \quad \partial\Sigma. \tag{3.14}$$

With the splittings $\Delta' = \partial_1^2 + \Delta''$, $\operatorname{div}' u' = \partial_1 u_1 + \operatorname{div}'' u''$ and $\nabla' = (\partial_1, \nabla'')$ elementary operations with (3.13), (3.14) yield the fourth order equation

$$\begin{aligned} (\lambda + \xi^2 - \Delta')(\xi^2 - \Delta')u_1 &= 0 && \text{in} \quad \Sigma \\ u_1 &= 0 && \text{on} \quad \partial\Sigma \\ \partial_1 u_1 &= -\operatorname{div}''\Phi'' - i\xi\Phi_n && \text{on} \quad \partial\Sigma. \end{aligned} \tag{3.15}$$

Let us introduce the additional partial Fourier transform $\mathcal{F}_\sigma = \tilde{}$ with respect to the variable $x'' \in \mathbb{R}^{n-2}$ and with phase variable $\sigma \in \mathbb{R}^{n-2}$. Applying $\tilde{}$ to (3.15), we get the fourth order ordinary differential equation ($s = |\sigma|$)

$$\begin{aligned}(\lambda + \xi^2 + s^2 - \partial_1^2)(\xi^2 + s^2 - \partial_1^2)\tilde{u}_1 &= 0 && \text{for} \quad x_1 > 0\\ \tilde{u}_1 &= 0 && \text{at} \quad x_1 = 0\\ \partial_1 \tilde{u}_1 &= -i\sigma \cdot \tilde{\Phi}'' - i\xi\tilde{\Phi}_n && \text{at} \quad x_1 = 0.\end{aligned} \tag{3.16}$$

For fixed $\lambda \in \Sigma_\varepsilon, \xi \in \mathbb{R}^*$ and $\sigma \in \mathbb{R}^{n-2}$ (3.16) has a unique bounded solution $\tilde{u}_1$ in $(0, \infty)$, namely

$$\tilde{u}_1(x_1, \sigma, \xi) = \frac{e^{-\sqrt{\lambda+\xi^2+s^2}x_1} - e^{-\sqrt{\xi^2+s^2}x_1}}{\sqrt{\lambda + \xi^2 + s^2} - \sqrt{\xi^2 + s^2}} (i\sigma \cdot \tilde{\Phi}'' + i\xi\tilde{\Phi}_n)|_{\partial\Sigma}. \tag{3.17}$$

Furthermore (3.13), (3.17) yield after some elementary calculations

$$\begin{aligned} p(x', \xi) &= -\mathcal{F}_\sigma^{-1}\big(\tfrac{1}{\xi^2+s^2}(\lambda + \xi^2 + s^2 - \partial_1^2)\partial_1\tilde{u}_1\big)\\ &= -\mathcal{F}_\sigma^{-1}\Big(\frac{\sqrt{\lambda+\xi^2+s^2}+\sqrt{\xi^2+s^2}}{\sqrt{\xi^2+s^2}} e^{-\sqrt{\xi^2+s^2}x_1}(i\sigma \cdot \tilde{\Phi}'' + i\xi\tilde{\Phi}_n)\Big)\\ &= \mathcal{F}_\sigma^{-1}\Big(\big(1 + \frac{\sqrt{\lambda+\xi^2+s^2}}{\sqrt{\xi^2+s^2}}\big)\tilde{v}\Big),\end{aligned} \tag{3.18}$$

where

$$v = \mathcal{F}_\sigma^{-1}\big(- e^{-\sqrt{\xi^2+s^2}x_1}(i\sigma \cdot \tilde{\Phi}'' + i\xi\tilde{\Phi}_n)\big). \tag{3.19}$$

For every nonzero complex number μ and $k = 1, 2$ let $W^{k,r}_{\omega,\mu}(\mathbb{R}^{n-1})$ denote the weighted Sobolev space $W^{k,r}_\omega(\mathbb{R}^{n-1})$ endowed with the norm

$$\|u\|_{W^{k,r}_{\omega,\mu}(\mathbb{R}^{n-1})} = \|\nabla'^k u, \mu u\|_{r,\omega;\mathbb{R}^{n-1}}, \quad k = 1, 2.$$

Similarly we define the space $W^{k,r}_{\omega,\mu}(\mathbb{R}^{n-1}_+), k = 1, 2$, on the half space $\mathbb{R}^{n-1}_+$. Using the trace operator γ, well-defined for functions from $W^{k,r}_{\text{loc}}(\mathbb{R}^{n-1}_+)$, we may define the trace space $T^{k,r}_{\omega,\mu}(\mathbb{R}^{n-2}), k = 1, 2$, by

$$T^{k,r}_{\omega,\mu}(\mathbb{R}^{n-2}) := \gamma W^{k,r}_{\omega,\mu}(\mathbb{R}^{n-1}_+), \quad \|\phi\|_{T^{k,r}_{\omega,\mu}(\mathbb{R}^{n-2})} = \inf_{\gamma u = \phi} \|u\|_{W^{k,r}_{\omega,\mu}(\mathbb{R}^{n-1}_+)}.$$

Obviously the set $C_0^\infty(\mathbb{R}^{n-1})$ is dense in the Banach space $T^{k,r}_{\omega,\mu}(\mathbb{R}^{n-2}), k = 1, 2$. We note that for $\phi \in T^{2,r}_{\omega,\mu}(\mathbb{R}^{n-2})$ and $\mu \in \Sigma_\varepsilon$ the function $R_\mu\phi := \mathcal{F}_\sigma^{-1}(e^{-\sqrt{\mu+s^2}x_1}\tilde{\phi}) \in W^{2,r}_\omega(\mathbb{R}^{n-1}_+)$ is the unique solution to the Laplace resolvent equation

$$(\mu - \Delta')q = 0 \quad \text{in } \mathbb{R}^{n-1}_+, \quad q|_{\mathbb{R}^{n-2}} = \phi \tag{3.20}$$

(see [33], Theorem 4.5). Furthermore, by standard techniques using Fourier multiplier theory one can easily see that $R_\mu\phi$ satisfies the estimates

$$\|R_\mu\phi\|_{W^{2,r}_{\omega,\mu}(\mathbb{R}^{n-1}_+)} \le c(r, \varepsilon, \mathcal{A}_r(\omega))\|\phi\|_{T^{2,r}_{\omega,\mu}(\mathbb{R}^{n-2})}, \tag{3.21}$$

$$\|R_\mu\phi\|_{W^{1,r}_{\omega,\sqrt{\mu}}(\mathbb{R}^{n-1}_+)} \le c(r, \varepsilon, \mathcal{A}_r(\omega))\|\phi\|_{T^{1,r}_{\omega,\sqrt{\mu}}(\mathbb{R}^{n-2})}. \tag{3.22}$$

Lemma 3.2 *Let $m(\cdot,\xi) \in C^{n-2}(\mathbb{R}^{n-2} \setminus \{0\})$ with a paramter $\xi \in \mathbb{R}^*$. If $m(\sigma,\xi)$ as well as $\frac{\sqrt{\xi^2+s^2}}{s} m(\sigma,\xi)$, $\xi \in \mathbb{R}^*$, are $(n-2)$-dimensional classical multiplier functions with respect to σ satisfying the pointwise Hörmander-Michlin condition, see Theorem 2.5, with a constant $K > 0$ independent of $\xi \in \mathbb{R}^*$, then the operator $M : \mathcal{S}(\mathbb{R}^{n-2}) \to \mathcal{S}'(\mathbb{R}^{n-2})$ defined by*

$$M\phi = \mathcal{F}_\sigma^{-1}(m(\sigma,\xi)\tilde{\phi})$$

is a bounded operator in $\mathcal{L}(T^{1,r}_{\omega,\xi}(\mathbb{R}^{n-2}))$ with $\|M\|_{\mathcal{L}(T^{1,r}_{\omega,\xi}(\mathbb{R}^{n-2}))} \le c(r,\varepsilon,\mathcal{A}_r(\omega))K$.

Proof: Let $\phi \in \mathcal{S}(\mathbb{R}^{n-2})$, let τ be the Fourier phase variable for the partial Fourier transform with respect to x_1, and let $\eta = (\tau,\sigma)$. Note that

$$\mathcal{F}_{x_1}\big(e^{-\sqrt{\xi^2+s^2}|x_1|}\big) = \frac{2\sqrt{\xi^2+s^2}}{\xi^2+s^2+\tau^2}$$

and

$$\mathcal{F}_\tau^{-1}\Big(\frac{\sqrt{\xi^2+s^2}+s}{s}\,\frac{s^2}{s^2+\tau^2}\mathcal{F}_{x_1} e^{-\sqrt{\xi^2+s^2}|x_1|}\Big)\Big|_{x_1=0} = 1.$$

Hence, by the definition of the space $T^{1,r}_{\omega,\xi}(\mathbb{R}^{n-2})$, we get

$$\begin{aligned} &\|M\phi\|_{T^{1,r}_{\omega,\xi}(\mathbb{R}^{n-2})} \\ &\quad\le \Big\|\mathcal{F}_\sigma^{-1}\big(m(\sigma,\xi)\mathcal{F}_\tau^{-1}\big(\tfrac{\sqrt{\xi^2+s^2}+s}{s}\tfrac{s^2}{s^2+\tau^2}\mathcal{F}_{x_1} e^{-\sqrt{\xi^2+s^2}|x_1|}\big)\widetilde{\phi}\big)\Big\|_{W^{1,r}_{\omega,\xi}(\mathbb{R}^{n-1}_+)} \\ &\quad\le \Big\|\mathcal{F}_\eta^{-1}\Big(m(\sigma,\xi)\big(\tfrac{\sqrt{\xi^2+s^2}+s}{s}\tfrac{s^2}{s^2+\tau^2}\mathcal{F}_{x_1} e^{-\sqrt{\xi^2+s^2}|x_1|}\big)\widetilde{\phi}\Big)\Big\|_{W^{1,r}_{\omega,\xi}(\mathbb{R}^{n-1})}. \end{aligned} \tag{3.23}$$

Since $m(\sigma,\xi)\frac{\sqrt{\xi^2+s^2}+s}{s}\frac{s^2}{s^2+\tau^2}$ is easily seen to be an $(n-1)$-dimensional Fourier multiplier by the assumptions on m, we get from (3.23), (3.22) that

$$\begin{aligned} \|M\phi\|_{T^{1,r}_{\omega,\xi}(\mathbb{R}^{n-2})} &\le c(\mathcal{A}_r(\omega))K\|\mathcal{F}_\sigma^{-1}(e^{-\sqrt{\xi^2+s^2}|x_1|}\tilde{\phi})\|_{W^{1,r}_{\omega,\xi}(\mathbb{R}^{n-1})} \\ &\le c(\mathcal{A}_r(\omega))K\|\mathcal{F}_\sigma^{-1}(e^{-\sqrt{\xi^2+s^2}x_1}\tilde{\phi})\|_{W^{1,r}_{\omega,\xi}(\mathbb{R}^{n-1}_+)} \\ &\le c(r,\varepsilon,\mathcal{A}_r(\omega))K\|\phi\|_{T^{1,r}_{\omega,\xi}(\mathbb{R}^{n-2})}. \end{aligned}$$

The proof of the lemma is complete. ■

Lemma 3.3 *For the function p defined by* (3.18) *we have*

$$\|\nabla' p, \xi p\|_{r,\omega;\Sigma} \le c\big(\|f,\nabla' g,\xi g\|_{r,\omega;\Sigma} + \|\lambda g; \widehat{W}^{-1,r}_\omega(\Sigma) + L^r_{\omega,1/\xi}(\Sigma)\|\big)$$

with an A_r-consistent constant $c = c(r,\varepsilon,\mathcal{A}_r(\omega))$.

Proof: First we shall show for the function v in (3.19) the estimate

$$\|\nabla' v, \xi v\|_{r,\omega;\Sigma} \le c\big(\|f,\nabla' g,\xi g\|_{r,\omega;\Sigma} + \|\lambda g; \widehat{W}^{-1,r}_\omega(\Sigma) + L^r_{\omega,1/\xi}(\Sigma)\|\big), \tag{3.24}$$

with an A_r-consistent constant $c = c(r,\varepsilon,\mathcal{A}_r(\omega))$. Since v solves the equation $(\xi^2-\Delta')v = 0$ in $\mathbb{R}^{n-1}_+$ with boundary condition $v|_{\partial\Sigma} = -\mathrm{div}''\Phi'' - i\xi\Phi_n$, standard techniques (see [33], Theorem 4.4) and a scaling argument yield a constant $c = c(r,\mathcal{A}_r(\omega)) > 0$ independent of $\xi\in\mathbb{R}^*$ such that

$$\|\nabla' v, \xi v\|_{r,\omega;\Sigma} \le c\|\nabla'(\mathrm{div}''U'' + i\xi U_n), \xi(\mathrm{div}''U'' + i\xi U_n)\|_{r,\omega;\Sigma}.$$

Hence (3.12) yields (3.24).

Now let $\mu = \lambda+\xi^2$. We shall show the auxiliary estimate

$$\begin{aligned}
&\|\mathcal{F}_\sigma^{-1}\big(\sqrt{\mu+s^2}e^{-\sqrt{\xi^2+s^2}x_1}(\sigma\cdot\tilde\Phi''+\xi\tilde\Phi_n)\big)\|_{r,\omega;\Sigma}\\
&\qquad\le c(r,\varepsilon,\mathcal{A}_r(\omega))\big(\|f,\nabla' g,\xi g\|_{r,\omega;\Sigma} + \|\lambda g; \widehat{W}_\omega^{-1,r}(\Sigma)+L^r_{\omega,1/\xi}(\Sigma)\|\big).
\end{aligned} \tag{3.25}$$

By (3.22) we get

$$\begin{aligned}
&\big\|\mathcal{F}_\sigma^{-1}\big(\sqrt{\mu+s^2}e^{-\sqrt{\xi^2+s^2}x_1}(\sigma\cdot\tilde\Phi''+\xi\tilde\Phi_n)\big)\big\|_{r,\omega;\Sigma}\\
&\quad= \big\|\partial_1\mathcal{F}_\sigma^{-1}\big(e^{-\sqrt{\xi^2+s^2}x_1}\sqrt{\mu+s^2}(\tfrac{\sigma}{\sqrt{\xi^2+s^2}}\cdot\tilde\Phi''+\tfrac{\xi}{\sqrt{\xi^2+s^2}}\tilde\Phi_n)\big)\big\|_{r,\omega;\Sigma}\\
&\quad\le c\big\|\mathcal{F}_\sigma^{-1}\big(\sqrt{\mu+s^2}(\tfrac{\sigma}{\sqrt{\xi^2+s^2}}\cdot\tilde\Phi''+\tfrac{\xi}{\sqrt{\xi^2+s^2}}\tilde\Phi_n)\big)\big\|_{T^{1,r}_{\omega,\xi}}
\end{aligned} \tag{3.26}$$

where $c = c(r,\varepsilon,\mathcal{A}_r(\omega)) > 0$. Note that $\frac{\sigma_k}{\sqrt{\xi^2+s^2}}, k = 2,\dots,n-1$, and $1-\frac{\xi}{\sqrt{\xi^2+s^2}}$ satisfy the assumption of Lemma 3.2 with a constant $K>0$ independent of $\xi\in\mathbb{R}^*$. Hence Lemma 3.2 and the fact that $\|\varphi\|_{T^{1,r}_{\omega,\xi}} \le c(\varepsilon)\|\varphi\|_{T^{1,r}_{\omega,\sqrt{\mu}}}$ for $\varphi\in T^{1,r}_{\omega,\xi}(\mathbb{R}^{n-2}_+)$ yield

$$\begin{aligned}
&\big\|\mathcal{F}_\sigma^{-1}\big(\sqrt{\mu+s^2}e^{-\sqrt{\xi^2+s^2}x_1}(\sigma\cdot\tilde\Phi''+\xi\tilde\Phi_n)\big)\big\|_{r,\omega;\Sigma}\\
&\quad\le c\big\|\mathcal{F}_\sigma^{-1}\big((\tfrac{\sigma}{\sqrt{\xi^2+s^2}}\cdot\sqrt{\mu+s^2}\,\tilde\Phi''+(1-\tfrac{\xi}{\sqrt{\xi^2+s^2}})\sqrt{\mu+s^2}\,\tilde\Phi_n)\big)\big\|_{T^{1,r}_{\omega,\xi}}\\
&\qquad+\|\mathcal{F}_\sigma^{-1}(\sqrt{\mu+s^2}\,\tilde\Phi_n)\|_{T^{1,r}_{\omega,\xi}}\\
&\quad\le c\|\mathcal{F}_\sigma^{-1}(\sqrt{\mu+s^2}\,\tilde\Phi)\|_{T^{1,r}_{\omega,\xi}} \qquad \le c\|\mathcal{F}_\sigma^{-1}(\sqrt{\mu+s^2}\,\tilde\Phi)\|_{T^{1,r}_{\omega,\sqrt{\mu}}}\\
&\quad\le c\|\mathcal{F}_\sigma^{-1}(\sqrt{\mu+s^2}e^{-\sqrt{\mu+s^2}x_1}\,\tilde\Phi)\|_{W^{1,r}_{\omega,\sqrt{\mu}}} = c\|\partial_1 R_\mu\Phi\|_{W^{1,r}_{\omega,\sqrt{\mu}}}
\end{aligned} \tag{3.27}$$

where $c = c(r,\varepsilon,\mathcal{A}_r(\omega)) > 0$. Then, by interpolation and (3.21), we get

$$\|\partial_1 R_\mu\Phi\|_{W^{1,r}_{\omega,\sqrt{\mu}}} \le c\|R_\mu\Phi\|_{W^{2,r}_{\omega,\mu}} \le c\|\Phi\|_{T^{2,r}_{\omega,\mu}} \le c\|\mu U, \nabla'^2 U\|_{r,\omega;\Sigma}$$

where $c = c(r,\varepsilon,\mathcal{A}_r(\omega)) > 0$. Hence, from (3.12), (3.27) we get (3.25).

To complete the proof, we must obtain an estimate for $h := \mathcal{F}_\sigma^{-1}\big(\frac{\sqrt{\mu+s^2}}{\sqrt{\xi^2+s^2}}\tilde v\big)$; see (3.18), (3.19). Note that $\partial_1 h$ is just the left-hand side of (3.25). Moreover, $\nabla'' h, \xi h$ are represented by the left-hand side of (3.25) with Φ replaced by $\mathcal{F}_\sigma^{-1}(\frac{\sigma\tilde\Phi}{\sqrt{\xi^2+s^2}}), \mathcal{F}_\sigma^{-1}(\frac{\xi\tilde\Phi}{\sqrt{\xi^2+s^2}})$, respectively. Therefore, using that $\frac{\sigma_j\sigma_k}{\xi^2+s^2}, j,k =$

$2,\dots,n-1$, $\frac{\sigma_k\xi}{\xi^2+s^2}$, and $1-\frac{\xi^2}{\xi^2+s^2}$ satisfy the assumptions of Lemma 3.2, we get by the same technique as before that

$$\|\nabla''h, \xi h\|_{r,\omega;\Sigma} \le c\big(\|f, \nabla' g, \xi g\|_{r,\omega;\Sigma} + \|\lambda g; \widehat{W}_\omega^{-1,r}(\Sigma) + L^r_{\omega,1/\xi}(\Sigma)\|\big)$$

with an A_r-consistent constant $c = c(r, \varepsilon, \mathcal{A}_r(\omega))$.

The proof of the lemma is complete. ■

Now we can prove the following theorem.

Theorem 3.4 *With $\Sigma = \mathbb{R}^{n-1}_+$ the assertions of Theorem 3.1 remain true. In particular the a priori estimate* (3.8) *holds.*

Proof: It is enough to show the existence of a unique solution to (3.13), (3.14) which satisfies (3.8). Consider the system

$$\begin{aligned} (\mu-\Delta')u' &= -\nabla' p && \text{in } \Sigma \\ (\mu-\Delta')u_n &= -i\xi p && \text{in } \Sigma \\ u &= \Phi && \text{on } \partial\Sigma \end{aligned} \tag{3.28}$$

for (u', u_n) where p is defined by (3.18). By standard techniques, cf. [33], §4.2, and the scaling argument $x' \to \mu^{-1/2}y'$ we get that (3.28) has a unique solution $u := (u', u_n) \in W^{2,r}_\omega(\Sigma) \cap W^{1,r}_{0,\omega}(\Sigma)$ satisfying

$$\|\mu u, \sqrt{\mu}\nabla' u, \nabla'^2 u\|_{r,\omega;\Sigma} \le c\|\nabla' p, \xi p, \mu U, \nabla'^2 U\|_{r,\omega;\Sigma}$$

with an A_r-consistent constant $c = c(r, \mathcal{A}_r(\omega))$. Thus, by Lemma 3.3 it follows that the functions u, p satisfy (3.8) with $\Sigma = \mathbb{R}^{n-1}_+$.

Now, for the proof of existence it remains to show that u satisfies the divergence equation. From the expression for p one can infer that

$$(-\Delta' + \xi^2)p = 0. \tag{3.29}$$

Hence, from (3.28) we get

$$(\mu-\Delta')(\operatorname{div}' u' + i\xi u_n) = 0 \quad \text{in} \quad \Sigma.$$

Furthermore (3.28), (3.29) imply (3.17), (3.18) with $(i\sigma\cdot\tilde{U}'' + i\xi\tilde{U}_n)|_{\partial\Sigma}$ replaced by $(-\partial_1\tilde{u}_1)|_{\partial\Sigma}$. Therefore we have $(i\sigma\cdot\tilde{U}''+i\xi\tilde{U}_n)|_{\partial\Sigma} = (-\partial_1\tilde{u}_1)|_{\partial\Sigma}$, i.e., $\operatorname{div}' u' + i\xi u_n = 0$ on $\partial\Sigma$. Thus $\operatorname{div}' u' + i\xi u_n = 0$ in Σ.

For the proof of uniqueness let $(u,p) \in \big(W^{2,r}_\omega(\mathbb{R}^{n-1}_+) \cap W^{1,r}_{0,\omega}(\mathbb{R}^{n-1}_+)\big) \times W^{1,r}_\omega(\mathbb{R}^{n-1}_+)$ be a solution to (3.13), (3.14) with $\Phi \equiv 0$. Then, by (3.18) it follows that $p = 0$, and consequently, $u = 0$ due to the uniqueness result for Laplace resolvent equation in the half space, see e.g. [33], Theorem 4.3.

Now the proof of this theorem is complete. ■

3.2 Bent half spaces

In this section we consider $(R_{\lambda,\xi})$ in a bent half space $\Sigma = H_\sigma$, see (3.2).

Theorem 3.5 *Let $n \geq 3$, $1 < r < \infty$, $\omega \in A_r(\mathbb{R}^{n-1})$, $\pi/2 < \varepsilon < \pi$ and*

$$\Sigma = H_\sigma = \{x' = (x_1, x''); \, x_1 > \sigma(x''), x'' \in \mathbb{R}^{n-2}\}$$

for a given function $\sigma \in C^{1,1}(\mathbb{R}^{n-2})$. Then there are A_r-consistent constants $K_0 = K_0(r, \varepsilon, \mathcal{A}_r(\omega)) > 0$ and $\lambda_0 = \lambda_0(r, \varepsilon, \mathcal{A}_r(\omega)) > 0$ such that, provided $\|\nabla'\sigma\|_\infty \leq K_0$, for every $\lambda \in \Sigma_\varepsilon, |\lambda| \geq \lambda_0$, every $\xi \in \mathbb{R}^*$ and

$$f \in L^r_\omega(\Sigma), \quad g \in W^{1,r}_\omega(\Sigma), \tag{3.30}$$

the parametrized resolvent problem $(R_{\lambda,\xi})$ has a unique solution

$$(u, p) \in (W^{2,r}_\omega(\Sigma) \cap W^{1,r}_{0,\omega}(\Sigma)) \times W^{1,r}_\omega(\Sigma).$$

This solution satisfies the estimate ($\mu = |\lambda + \xi^2|^{1/2}$)

$$\begin{aligned} &\|\mu^2 u, \mu\nabla' u, \nabla'^2 u, \nabla' p, \xi p\|_{r,\omega} \\ &\qquad \leq c\big(\|f, \nabla' g, \xi g\|_{r,\omega} + \|\lambda g; \widehat{W}^{-1,r}_\omega(\Sigma) + L^r_{\omega,1/\xi}(\Sigma)\|\big) \end{aligned} \tag{3.31}$$

with an A_r-consistent constant $c = c(r, \varepsilon, \mathcal{A}_r(\omega))$. If (3.30) *is satisfied for an additional exponent $s \in (1, \infty)$ and weight $\nu \in A_r(\mathbb{R}^{n-1})$ and if $\|\nabla'\sigma\|_\infty \leq K_0$ for some constant $K_0 = K_0(r, s, \varepsilon, \mathcal{A}_r(\omega), \mathcal{A}_s(\nu)) > 0$, then the assertion* (3.31) *holds true with L^s_ν-norms for all $\lambda \in \Sigma_\varepsilon, |\lambda| \geq \lambda_0$, for some $\lambda_0 = \lambda_0(r, s, \varepsilon, \mathcal{A}_r(\omega), \mathcal{A}_s(\nu)) > 0$ as well.*

Proof: By the transformation

$$\Phi : H_\sigma \to \mathbb{R}^{n-1}_+, \quad x' \mapsto \tilde{x}' = (\tilde{x}_1, \tilde{x}'') = \Phi(x') = (x_1 - \sigma(x''), x''),$$

the problem $(R_{\lambda,\xi})$ in H_σ is reduced to a modified version of $(R_{\lambda,\xi})$ in the half space $H = \mathbb{R}^{n-1}_+$. Note that Φ is a bijection with Jacobian equal to 1. For a function u on H_σ define $\tilde{u}$ on H by

$$\tilde{u}(\tilde{x}') = u(\Phi^{-1}(\tilde{x}') = u(x').$$

Further let $\tilde{\partial}_i = \partial/\partial\tilde{x}_i, i = 1, \cdots, n-1, \tilde{\nabla}' = (\tilde{\partial}_1, \tilde{\nabla}'')$ etc. denote the standard differential operators acting on the variable $\tilde{x} \in H$.

Since $\partial_i u = (\tilde{\partial}_i - (\partial_i\sigma)\tilde{\partial}_1)\tilde{u}$ for $i = 1, \cdots, n-1$, we easily get

$$\begin{aligned} \Delta' u(x', \xi) &= \big(\tilde{\Delta}' + |\nabla'\sigma|^2\tilde{\partial}_1^2 - 2\nabla'\sigma \cdot (\tilde{\nabla}'\tilde{\partial}_1) - (\Delta''\sigma)\tilde{\partial}_1\big)\tilde{u}(\tilde{x}', \xi) \\ \nabla' p(x', \xi) &= \big(\tilde{\nabla}' - (\nabla'\sigma)\tilde{\partial}_1\big)\tilde{p}(\tilde{x}', \xi) \\ \operatorname{div}' u'(x', \xi) &= \big(\widetilde{\operatorname{div}'} - \nabla'\sigma \cdot \tilde{\partial}_1\big)\tilde{u}'(\tilde{x}', \xi) \end{aligned} \tag{3.32}$$

and a similar formula for $\nabla'^2 u(x', \xi)$. Note that by the change of variable $\tilde{x}' = \Phi(x'), x' \in \mathbb{R}^{n-1}$, the Muckenhoupt weight $\omega \in A_r(\mathbb{R}^{n-1})$ is mapped to $\tilde{\omega} \in A_r(\mathbb{R}^{n-1})$ satisfying

$$c^{-1}\mathcal{A}_r(\tilde{\omega}) \leq \mathcal{A}_r(\omega) \leq c\,\mathcal{A}_r(\tilde{\omega}) \tag{3.33}$$

with c independent of ω, cf. Proposition 2.2 (1). Therefore, it follows from (3.32) that for $u \in W^{2,r}(\Sigma)$

$$\begin{aligned}
\|u\|_{r,\omega;H_\sigma} &= \|\tilde u\|_{r,\tilde\omega;H} \\
\|\nabla' u\|_{r,\omega;H_\sigma} &\le c(1+K)\|\tilde\nabla'\tilde u\|_{r,\tilde\omega;H} \\
\|\nabla'^2 u\|_{r,\omega;H_\sigma} &\le c(1+K^2)\|\tilde\nabla'^2\tilde u\|_{r,\tilde\omega;H} + cL\|\tilde\partial_1\tilde u\|_{r,\tilde\omega;H},
\end{aligned} \tag{3.34}$$

where $K = \|\nabla'\sigma\|_\infty, L = \|\nabla'^2\sigma\|_\infty$ and c is independent of the weight ω. Furthermore, $\|f,\xi g\|_{r,\omega;H_\sigma} = \|\tilde f,\xi\tilde g\|_{r,\tilde\omega;H}$ and $\|\nabla' g\|_{r,\omega;H_\sigma} \le c(1+K)\|\tilde\nabla'\tilde g\|_{r,\tilde\omega;H}$ with $c>0$ independent of ω. Concerning the norm of g in $\widehat{W}^{-1,r}_\omega(H_\sigma) + L^r_{\omega,1/\xi}(H_\sigma)$ note that for a function $g_0 \in \widehat{W}^{-1,r}_\omega(H_\sigma) \cap L^r_\omega(H_\sigma)$ and all test functions $\varphi \in C_0^\infty(\bar H_\sigma)$

$$\begin{aligned}
\int_{H_\sigma} g_0\varphi dx' &= \int_H \tilde g_0\tilde\varphi\, d\tilde x' \\
&\le \|\tilde g_0\|_{-1,r,\tilde\omega;H}\|\tilde\nabla'\tilde\varphi\|_{r',(\tilde\omega)';H} \\
&\le c(1+\|\nabla'\sigma\|_\infty)\|\tilde g_0\|_{-1,r,\tilde\omega;H}\|\nabla'\varphi\|_{r',\omega';H_\sigma}
\end{aligned}$$

with a constant c independent of ω; here we used that $(\tilde\omega)' = \widetilde{(\omega')}$, $\omega' = \omega^{-\frac{1}{r-1}}$. Since $C_0^\infty(\bar H_\sigma)$ is dense in $\widehat{W}^{1,r'}_{\tilde\omega'}(H_\sigma)$ (see e.g. [33], Corollary 4.1), we get

$$\|g_0\|_{-1,r,\omega;H_\sigma} \le c(1+K)\|\tilde g_0\|_{-1,r,\tilde\omega;H}.$$

Then for every $\xi \in \mathbb{R}^*$ and every decomposition of g into $g = g_0 + g_1$ with $g_0 \in \widehat{W}^{-1,r}_\omega(H_\sigma), g_1 \in L^r_\omega(H_\sigma)$

$$\|g_0\|_{-1,r,\omega;H_\sigma} + \|g_1/\xi\|_{r,\omega;H_\sigma} \le c(1+K)(\|\tilde g_0\|_{-1,r,\tilde\omega;H} + \|\tilde g_1/\xi\|_{r,\tilde\omega;H}),$$

where $c>0$ is independent of ω; note that $\tilde g = \tilde g_0 + \tilde g_1$ gives all admissible decompositions of $\tilde g \in \widehat{W}^{-1,r}_{\tilde\omega}(H) + L^r_{\tilde\omega,1/\xi}(H)$. Consequently

$$\|g;\widehat{W}^{-1,r}_\omega(H_\sigma) + L^r_{\omega,1/\xi}(H_\sigma)\| \le c(1+K)\,\|\tilde g;\widehat{W}^{-1,r}_{\tilde\omega}(H) + L^r_{\tilde\omega,1/\xi}(H)\|. \tag{3.35}$$

To apply Kato's perturbation theorem we introduce for every $\xi \in \mathbb{R}^*$ on H_σ the ξ-dependent Banach spaces ($\mu = |\lambda+\xi^2|^{1/2}$)

$$\begin{aligned}
&\mathcal{X} = (W^{2,r}_\omega \cap W^{1,r}_{0,\omega})^n \times W^{1,r}_\omega, \quad \|u,p\|_{\mathcal{X}} = \|\mu^2 u, \mu\nabla' u, \nabla'^2 u, \nabla' p, \xi p\|_{r,\omega;H_\sigma}, \\
&\mathcal{Y} = (L^r_\omega)^n \times W^{1,r}_\omega, \quad \|f,g\|_{\mathcal{Y}} = \|f,\nabla' g,\xi g\|_{r,\omega;H_\sigma} + \|\lambda g;\widehat{W}^{-1,r}_\omega(H_\sigma) + L^r_{\omega,1/\xi}(H_\sigma)\|,
\end{aligned}$$

and on H similar spaces $(\tilde{\mathcal{X}},\|\cdot\|_{\tilde{\mathcal{X}}}), (\tilde{\mathcal{Y}},\|\cdot\|_{\tilde{\mathcal{Y}}})$ with the weight $\tilde\omega$ instead of ω. Then it follows from (3.34), (3.35) that

$$\|(u,p)\|_{\mathcal{X}} \le c(1+K+K^2+L/\mu)\|(\tilde u,\tilde p)\|_{\tilde{\mathcal{X}}}, \quad \|(f,g)\|_{\mathcal{Y}} \le c(1+K)\|(\tilde f,\tilde g)\|_{\tilde{\mathcal{Y}}}, \tag{3.36}$$

and exchanging the role of the variables x' and $\tilde x'$, we get

$$\|(\tilde u,\tilde p)\|_{\tilde{\mathcal{X}}} \le c(1+K+K^2+L/\mu)\|(u,p)\|_{\mathcal{X}}, \quad \|(\tilde f,\tilde g)\|_{\tilde{\mathcal{Y}}} \le c(1+K)\|(f,g)\|_{\mathcal{Y}}, \tag{3.37}$$

with constants $c > 0$ not depending on ω, λ and ξ. Further define the operators

$$\mathcal{S} : \mathcal{X} \to \mathcal{Y}, \quad \mathcal{S}(u,p) = \begin{pmatrix} (\lambda + \xi^2 - \Delta')u' + \nabla' p \\ (\lambda + \xi^2 - \Delta')u_n + i\xi p \\ \operatorname{div}' u' + i\xi u_n \end{pmatrix},$$

and analogously $\tilde{\mathcal{S}} : \tilde{\mathcal{X}} \to \tilde{\mathcal{Y}}$. By (3.32) we get the decomposition

$$\mathcal{S}(u,p) = \tilde{\mathcal{S}}(\tilde{u},\tilde{p}) + \mathcal{R}(\tilde{u},\tilde{p})$$

with a remainder term $\mathcal{R} : \tilde{\mathcal{X}} \to \tilde{\mathcal{Y}}$,

$$\begin{aligned} \mathcal{R}(\tilde{u},\tilde{p})(\tilde{x}',\xi) &= \begin{pmatrix} -(\nabla'\sigma)\tilde{\partial}_1 \tilde{p} \\ 0 \\ -(\nabla'\sigma)\cdot \tilde{\partial}_1 \tilde{u}' \end{pmatrix} \\ &\quad + \begin{pmatrix} -|\nabla'\sigma|^2 \tilde{\partial}_1^2 \tilde{u} + 2\nabla'\sigma \cdot \tilde{\nabla}' \tilde{\partial}_1 \tilde{u} + (\Delta''\sigma)\tilde{\partial}_1 \tilde{u} \\ 0 \end{pmatrix} \end{aligned}$$

not depending explicitly on λ and ξ. Since $\tilde{u}|_{\partial H} = 0$ and $\tilde{\partial}_1(\nabla'\sigma) = 0$, we have

$$\int_H -(\nabla'\sigma)\cdot \tilde{\partial}_1 \tilde{u}' \,\varphi \, d\tilde{x}' = \int_H (\nabla'\sigma)\cdot \tilde{u}' \,\tilde{\partial}_1 \varphi \, d\tilde{x}'$$

for all $\varphi \in C_0^\infty(\bar{H})$; consequently

$$\| -(\nabla'\sigma)\cdot \tilde{\partial}_1 \tilde{u}'; \widehat{W}_{\tilde{\omega}}^{-1,r}(H) + L^r_{\tilde{\omega},1/\xi}(H)\| \le \| -(\nabla'\sigma)\cdot \tilde{\partial}_1 \tilde{u}'\|_{-1,r,\tilde{\omega};H} \le K\|\tilde{u}\|_{r,\tilde{\omega};H}.$$

Hence

$$\begin{aligned} \|\mathcal{R}(\tilde{u},\tilde{p})\|_{\tilde{\mathcal{Y}}} &\le c(K+K^2)\|\lambda\tilde{u}, \xi\tilde{\nabla}'\tilde{u}, \tilde{\nabla}'^2\tilde{u}, \tilde{\nabla}'\tilde{p}\|_{r,\tilde{\omega};H} + L\|\tilde{\nabla}'\tilde{u}\|_{r,\tilde{\omega};H} \\ &\le c_\varepsilon(K+K^2+\tfrac{L}{\mu})\|(\tilde{u},\tilde{p})\|_{\tilde{\mathcal{X}}} \\ &\le c_\varepsilon(K+K^2+\tfrac{L}{\sqrt{|\lambda|}})\|(\tilde{u},\tilde{p})\|_{\tilde{\mathcal{X}}}, \end{aligned} \tag{3.38}$$

where $c, c_\varepsilon > 0$ are independent of $\omega, \tilde{\omega}$; note that $|\lambda| < \frac{\mu^2}{\cos\varepsilon}$ and $|\xi| < \mu(1+\frac{1}{\cos\varepsilon})^{1/2}$ for all $\lambda \in \Sigma_\varepsilon$.

Due to Theorem 3.4 and (3.33) $\tilde{\mathcal{S}} : \tilde{X} \to \tilde{Y}$ is an isomorphism such that $\|(\tilde{u},\tilde{p})\|_{\tilde{\mathcal{X}}} \le C_1\|\tilde{\mathcal{S}}(\tilde{u},\tilde{p})\|_{\tilde{\mathcal{Y}}}$ with an A_r-consistent constant $C_1 = C_1(r,\varepsilon,\mathcal{A}_r(\omega))$ independent of $\lambda \in \Sigma_\varepsilon, \xi \in \mathbb{R}^*$. Therefore, it follows from (3.38) that there exist A_r-consistent constants $\delta_0 = \delta(\varepsilon, r, \mathcal{A}_r(\omega)), \lambda_0 = \lambda(\varepsilon, r, \mathcal{A}_r(\omega))$ such that, if $K \le \delta_0$ and $\lambda \in \Sigma_\varepsilon, |\lambda| \ge \lambda_0$, then

$$\|\mathcal{R}(\tilde{u},\tilde{p})\|_{\tilde{\mathcal{Y}}} \le \frac{1}{2}\|\mathcal{S}(\tilde{u},\tilde{p})\|_{\tilde{\mathcal{Y}}} \quad \text{for all } (\tilde{u},\tilde{p}) \in \tilde{\mathcal{X}}.$$

Hence $\tilde{\mathcal{S}} + \mathcal{R}$ is an isomorphism from $\tilde{\mathcal{X}}$ to $\tilde{\mathcal{Y}}$ satisfying

$$\|(\tilde{u},\tilde{p})\|_{\tilde{\mathcal{X}}} \le 2C_1\|(\tilde{\mathcal{S}}+\mathcal{R})(\tilde{u},\tilde{p})\|_{\tilde{\mathcal{Y}}}.$$

Thus, considering (3.32), (3.36) and (3.37), if $\|\nabla''\sigma\|_\infty \le \delta_0$ and $\lambda \in \Sigma_\varepsilon, |\lambda| \ge \lambda_0$, we get

$$\begin{aligned}\|(u,p)\|_{\mathcal{X}} &\le C_2\|(\tilde u,\tilde p)\|_{\tilde{\mathcal{X}}}\\ &\le 2C_1C_2\|\tilde{\mathcal{S}}(\tilde u,\tilde p)\|_{\tilde{\mathcal{Y}}}\\ &\le C_3\|\mathcal{S}(u,p)\|_{\mathcal{Y}},\end{aligned}$$

where the constants $C_i = C_i(\varepsilon, r, \mathcal{A}_r(\omega)), i = 1,2,3$, are A_r-consistent and independent of $\lambda \in \Sigma_\varepsilon, |\lambda| \ge \lambda_0$ and $\xi \in \mathbb{R}^*$. Thus, existence of a unique solution to $(R_{\lambda,\xi})$ in H_σ has been proved.

Assume that (3.30) is satisfied for an additional exponent $s \ne r$ and weight $\nu \in A_s(\mathbb{R}^{n-1})$. Repeating the above argument for the index s, we see $\mathcal{S}$ to be an isomorphism from $\mathcal{X}_s \cap \mathcal{X}_r$ to $\mathcal{Y}_s \cap \mathcal{Y}_r$ for $|\lambda| \ge \lambda_0 = \lambda_0(r, s, \varepsilon, \mathcal{A}_r(\omega), \mathcal{A}_s(\nu))$ under the given smallness condition $\|\nabla''\sigma\|_\infty \le \delta_0(r, s, \varepsilon, \mathcal{A}_r(\omega), \mathcal{A}_s(\nu))$. Now the proof of Theorem 3.5 is complete. ■

3.3 Bounded domains

Let us consider the system $(R_{\lambda,\xi})$ in a bounded domain. For a bounded domain the definition of the space for the divergence g has to be modified since it is impossible to think of the sum of $\widehat{W}^{-1,r}_\omega(\Sigma)$ and $L^r_\omega(\Sigma)$ for any $\omega \in A_r$. On the bounded domain $\Sigma \subset \mathbb{R}^{n-1}$ of $C^{1,1}$-class let α_0 denote the smallest eigenvalue of the Laplacian, i.e.

$$0 < \alpha_0 = \inf\{\|\nabla u\|_2^2;\ u \in W^{1,2}_0(\Sigma), \|u\|_2 = 1\}.$$

For fixed $\lambda \in \mathbb{C} \setminus (-\infty, -\alpha_0], \xi \in \mathbb{R}^*$ and $\omega \in A_r$ we introduce the *parametrized Stokes operator* $S = S^\omega_{r,\lambda,\xi}$ by

$$S(u,p) = \begin{pmatrix} (\lambda + \xi^2 - \Delta')u' + \nabla' p \\ (\lambda + \xi^2 - \Delta')u_n + i\xi p \\ -\operatorname{div}_\xi u \end{pmatrix}$$

defined on $D(S) = D(\Delta'_{r,\omega}) \times W^{1,r}_\omega(\Sigma)$, where $D(\Delta'_{r,\omega}) = W^{2,r}_\omega(\Sigma) \cap W^{1,r}_{0,\omega}(\Sigma)$ and

$$\operatorname{div}_\xi u = \operatorname{div}' u' + i\xi u_n.$$

For $\omega \equiv 1$ the operator $S^\omega_{r,\lambda,\xi}$ will be denoted by $S_{r,\lambda,\xi}$. Note that the image of $D(S)$ by div_ξ is included in $W^{1,r}_\omega(\Sigma)$ and $W^{1,r}_\omega(\Sigma) \subset L^r_{(m),\omega}(\Sigma) + L^r_\omega(\Sigma)$, we recall that

$$L^r_{(m),\omega}(\Sigma) := \big\{u \in L^r_\omega(\Sigma); \int_\Sigma u\, dx' = 0\big\}.$$

Using Poincaré's inequality in weighted spaces, see Proposition 2.4, one can easily check the continuous embedding $L^r_{(m),\omega}(\Sigma) \hookrightarrow \widehat{W}^{-1,r}_\omega(\Sigma)$, more precisely

$$\|u\|_{-1,r,\omega} \le c\|u\|_{r,\omega}, \quad u \in L^r_{(m),\omega}(\Sigma),$$

with an A_r-consistent constant $c > 0$. For convenience we use the notation

$$\|g; L^r_{(m),\omega} + L^r_{\omega,1/\xi}\|_0 := \inf\{\|g_0\|_{-1,r,\omega} + \|g_1/\xi\|_{r,\omega}; g = g_0 + g_1, g_0 \in L^r_{(m),\omega}, g_1 \in L^r_\omega\};$$

note that this norm is equivalent to the norm $\|\cdot\|_{(W^{1,r'}_{\omega',\xi})^*}$ where $W^{1,r'}_{\omega',\xi}$ is the usual weighted Sobolev space on Σ with norm $\|\nabla' u, \xi u\|_{r',\omega'}$.

First, we deal with the Hilbert space setting of $(R_{\lambda,\xi})$. For $\xi \in \mathbb{R}^*$ define the closed subspace V_ξ of $W_0^{1,2}(\Sigma)$ by

$$V_\xi = \{u \in W_0^{1,2}(\Sigma); \mathrm{div}_\xi u = 0\}.$$

Lemma 3.6 *Suppose that* $\varphi = (\varphi', \varphi_n) \in W^{-1,2}(\Sigma) := (W_0^{1,2}(\Sigma))^*$ *satisfies* $(\varphi, v) = 0$ *for all* $v \in V_\xi$. *Then there is some* $p \in L^2(\Sigma)$ *such that*

$$\varphi = (\nabla' p, i\xi p).$$

Proof: It follows from the assumption that $< \varphi', v' >_{W^{-1,2}, W_0^{1,2}} = 0$ for all $v' \in W_0^{1,2}(\Sigma)$ satisfying $\mathrm{div}\,' v' = 0$. Therefore, by [36], Corollary III 5.1, we get

$$\varphi' = \nabla' p \quad \text{with some } p \in L^2(\Sigma). \tag{3.39}$$

Then, for all $v = (v', v_n) \in \mathcal{V}_\xi := \{u \in C_0^\infty(\Sigma)^n; \mathrm{div}_\xi u = 0\}$, by assumption

$$\begin{aligned} 0 &= \langle \nabla' p, v' \rangle_{W^{-1,2}, W_0^{1,2}} + \langle \varphi_n, v_n \rangle_{W^{-1,2}, W_0^{1,2}} \\ &= \langle \nabla' p, v' \rangle_{W^{-1,2}, W_0^{1,2}} + \langle \varphi_n, -\tfrac{\mathrm{div}\,' v'}{i\xi} \rangle_{W^{-1,2}, W_0^{1,2}} \\ &= \langle \nabla'(p - \tfrac{\varphi_n}{i\xi}), v' \rangle_{\mathcal{D}'(\Sigma), \mathcal{D}(\Sigma)}. \end{aligned} \tag{3.40}$$

Since $v' \in C_0^\infty(\Sigma)$ in (3.40) can be chosen arbitrarily due to the structure of $\mathcal{V}_\xi$, we get $\nabla'(p - \frac{\varphi_n}{i\xi}) = 0$ in the sense of distributions yielding $p - \frac{\varphi_n}{i\xi} = \text{const}$ and $\varphi_n \in L^2(\Sigma)$. Thus, choosing p in (3.39) such that $\int_\Sigma (p - \frac{\varphi_n}{i\xi})\, dx' = 0$, we get $p - \frac{\varphi_n}{i\xi} = 0$. The proof of this lemma is complete. ■

Lemma 3.7 *(1) For every* $g \in W^{1,2}(\Sigma)$ *and* $\xi \in \mathbb{R}^*$ *the divergence problem* $\mathrm{div}_\xi u = g$ *has at least one solution* $u \in W^{2,2}(\Sigma) \cap W_0^{1,2}(\Sigma)$ *such that*

$$\|u\|_{W^{2,2}} \le c \left(\|g\|_{W^{1,2}} + \frac{1}{|\xi|} \left| \int_\Sigma g\, dx' \right| \right). \tag{3.41}$$

Here $c > 0$ *is a constant independent of* ξ *and* g.

(2) Let $f \in L^2(\Sigma)$ *and* $g \in W^{1,2}(\Sigma)$. *For every* $\lambda \in -\alpha_0 + \Sigma_\varepsilon, \varepsilon \in (\pi/2, \pi)$, *and* $\xi \in \mathbb{R}^*$ *there exists a unique solution* (u, p) *of* $(R_{\lambda,\xi})$ *such that* $(u, p) \in (W^{2,2}(\Sigma) \cap W_0^{1,2}(\Sigma)) \times W^{1,2}(\Sigma)$.

Proof: (1) Choose an arbitrary, but fixed $w = (0, \cdots, 0, w_n) \in C_0^\infty(\Sigma)$ with $\int_\Sigma w_n\, dx' = 1$. Given $g \in W^{1,2}(\Sigma)$ with $\alpha = \int_\Sigma g\, dx'$ such that consequently $g - \alpha w_n \in W^{1,2}(\Sigma) \cap L^2_{(m)}(\Sigma)$, there exists by [27], Theorem 1.2, a velocity field $u = (u', 0) \in W^{2,2}(\Sigma) \cap W_0^{1,2}(\Sigma)$ satisfying $\mathrm{div}\, u = g - \alpha w_n$ and

$$\|u\|_{W^{2,2}} \le c \|\nabla'(g - \alpha w_n)\|_2 \le c \|g\|_{W^{1,2}}.$$

Then $v = u + \frac{\alpha}{i\xi} w$ solves the divergence problem and satisfies the estimate (3.41).

(2) In consideration of (1) we may assume without loss of generality that $g = 0$. Define, for $\lambda \in -\alpha_0 + \Sigma_\varepsilon$ and $\xi \in \mathbb{R}^*$, the bilinear form $a(\cdot, \cdot) : V_\xi \times V_\xi \to \mathbb{C}$ by

$$a(u, v) = \int_\Sigma ((\lambda + \xi^2) u \cdot \bar{v} + \nabla' u \cdot \nabla' \bar{v}) \, dx'.$$

Obviously a is continuous and elliptic in the sense that $|a(u, u)| \geq \alpha \|u\|_{1,2}^2$ for all $\lambda \in -\alpha_0 + S_\varepsilon, \xi \in \mathbb{R}^*$ and $u \in V_\xi$ with a constant $\alpha = \alpha(\lambda, \xi) > 0$. By the Lemma of Lax-Milgram the variational problem

$$a(u, v) = \int_\Sigma f \cdot \bar{v} \, dx' \quad \forall v \in V_\xi$$

has a unique solution $u \in V_\xi$, that is,

$$\langle (\lambda + \xi^2 - \Delta') u - f, v \rangle_{W^{-1,2}, W_0^{1,2}} = 0 \quad \forall v \in V_\xi.$$

Moreover, by Lemma 3.6 there is some $p \in L^2(\Sigma)$ such that

$$(\lambda + \xi^2 - \Delta') u' + \nabla' p = f', (\lambda + \xi^2 - \Delta') u_n + i\xi p = f_n.$$

Then standard regularity results for the Stokes and Poisson equation applied to the problems

$$-\Delta' u' + \nabla' p = f' - (\lambda + \xi^2) u', \quad \operatorname{div}' u' = -i\xi u_n \quad \text{in } \Sigma, \quad u'|_{\partial\Sigma} = 0,$$

and $-\Delta' u_n = f_n - (\lambda + \xi^2) u_n - i\xi p$ in Σ, $u_n|_{\partial\Sigma} = 0$, yield $(u, p) \in (W^{2,2}(\Sigma) \cap W_0^{1,2}(\Sigma)) \times W^{1,2}(\Sigma)$. Since the uniqueness of (u, p) is obvious, the proof of the lemma is complete. ∎

In the following we consider the resolvent problem $(R_{\lambda,\xi})$ for arbitrary $\lambda \in -\alpha_0 + \Sigma_\varepsilon$, $\pi/2 < \varepsilon < \pi$.

Lemma 3.8 *For every $\lambda \in -\alpha_0 + \Sigma_\varepsilon$, $\pi/2 < \varepsilon < \pi$, $\xi \in \mathbb{R}^*$ and $\omega \in A_r$ the operator $S = S^\omega_{r,\lambda,\xi}$ is injective and the range $R(S)$ of S is dense in $L^r_\omega(\Sigma) \times W^{1,r}_\omega(\Sigma)$.*

Proof: First we shall prove this lemma for $\omega \equiv 1$.

Let $S_{r,\lambda,\xi} := S^\omega_{r,\lambda,\xi}$ for $\omega \equiv 1$, $\lambda \in -\alpha_0 + \Sigma_\varepsilon$, $\pi/2 < \varepsilon < \pi$, $\xi \in \mathbb{R}^*$. In order to prove the injectivity of $S_{r,\lambda,\xi}$ let $S_{r,\lambda,\xi}(u, p) = 0$. By the regularity assertions in Theorem 3.1 and Theorem 3.5 for the case $\omega \equiv 1$ it can be proved in a finite number of steps using Sobolev's embedding theorem that $(u, p) \in D(S_{2,\lambda,\xi})$. We note that in order to apply Theorem 3.5 the partition of unity of Σ has to be refined, if necessary, such that all crucial smallness assumptions on $\|\nabla' \omega_j\|_\infty$ are fulfilled. Thus by Lemma 3.7 (2), $(u, p) = 0$.

Let us show that $R(S_{r,\lambda,\xi})$ is dense in $L^r \times W^{1,r}$. Note that $C_0^\infty(\Sigma) \times C^\infty(\bar{\Sigma})$ is dense in $L^r \times W^{1,r}$. By Lemma 3.7 (2), there is a unique solution (u, p) of $S_{2,\lambda,\xi}(u, p) = (f, -g)$ with $(f, g) \in C_0^\infty(\Sigma) \times C^\infty(\bar{\Sigma})$. Moreover, this solution can be shown to be in $D(S_{r,\lambda,\xi})$ for every $r \in (1,\infty)$ thus proving the denseness of $R(S)$ in $L^r \times W^{1,r}$.

Next we consider the general case of ω. Since, by Proposition 2.2 (3), there is an $s \in (1,r)$ such that $L^r_\omega(\Sigma) \subset L^s(\Sigma)$, one sees immediately that

$$D(S^\omega_{r,\lambda,\xi}) \subset D(S_{s,\lambda,\xi}).$$

Therefore, $S^\omega_{r,\lambda,\xi}(u,p) = 0$ for some $(u,p) \in D(S^\omega_{r,\lambda,\xi})$ yields $(u,p) \in D(S_{s,\lambda,\xi})$ and $S_{s,\lambda,\xi}(u,p) = 0$. Hence, the result already proved for $\omega \equiv 1$ implies that $u = 0, p = 0$.

On the other hand, by Proposition 2.2 (3), there is an $\tilde{s} \in (r,\infty)$ such that $S_{\tilde{s},\lambda,\xi} \subset S^\omega_{r,\lambda,\xi}$, and consequently

$$R(S_{\tilde{s},\lambda,\xi}) \subset R(S^\omega_{r,\lambda,\xi}).$$

Therefore, the denseness result for the case $\omega \equiv 1$ implies the denseness of $R(S^\omega_{r,\lambda,\xi})$ in the space $L^{\tilde{s}}(\Sigma) \times W^{1,\tilde{s}}(\Sigma)$ which is dense in $L^r_\omega(\Sigma) \times W^{1,r}_\omega(\Sigma)$. Thus the assertion on the denseness of $R(S)$ follows.

The proof of this lemma is complete. ■

The following lemma gives a preliminary *a priori* estimate for a solution (u,p) of $S(u,p) = (f,-g)$.

Lemma 3.9 *Let $1 < r < \infty$, $\omega \in A_r$ and $\varepsilon \in (\pi/2,\pi)$. Then there exists an A_r-consistent constant $c = c(\varepsilon, r, \Sigma, \mathcal{A}_r(\omega)) > 0$ such that for every $\lambda \in -\alpha_0 + \Sigma_\varepsilon, \xi \in \mathbb{R}^*$ and every $(u,p) \in D(S^\omega_{r,\lambda,\xi})$,*

$$\begin{aligned}\|\mu_+^2 u, \mu_+\nabla' u, \nabla'^2 u, \nabla' p, \xi p\|_{r,\omega} \le c\big(&\|f, \nabla' g, g, \xi g\|_{r,\omega} + |\lambda| \|g; L^r_{(m),\omega} + L^r_{\omega,1/\xi}\|_0 \\ &+ \|\nabla' u, \xi u, p\|_{r,\omega} + |\lambda| \|u\|_{(W^{1,r'}_{\omega'})^*}\big),\end{aligned} \tag{3.42}$$

where $\mu_+ = |\lambda + \alpha_0 + \xi^2|^{1/2}, (f,-g) = S(u,p)$ and $(W^{1,r'}_{\omega'})^*$ denotes the dual space of $W^{1,r'}_{\omega'}(\Sigma)$.

Proof: The proof is based on a partition of unity in Σ and on the localization procedure reducing the problem to a finite number of problems of type $(R_{\lambda,\xi})$ in bent half spaces and in the whole space $\mathbb{R}^{n-1}$. Since $\partial\Sigma \in C^{1,1}$, we can cover $\partial\Sigma$ by a finite number of balls $B_j, j \ge 1$, such that, after a translation and rotation of coordinates, $\Sigma \cap B_j$ locally coincides with a bent half space $H_j = H_{\sigma_j}$ where $\sigma_j \in C^{1,1}(\mathbb{R}^{n-1})$ has a compact support, $\sigma_j(0) = 0$ and $\nabla''\sigma_j(0) = 0$. Choosing the balls B_j small enough (and its number large enough) we may assume that $\|\nabla''\sigma_j\|_\infty \le K_0(\varepsilon, r, \Sigma, \mathcal{A}_r(\omega))$ for all $j \ge 1$ where K_0 was introduced in Theorem 3.5. According to the covering $\partial\Sigma \subset \bigcup_{j\ge1} B_j$ there are cut-off functions $0 \le \varphi_0, \varphi_j \in C^\infty(\mathbb{R}^{n-1})$ such that

$$\varphi_0 + \sum_{j\ge1} \varphi_j \equiv 1 \text{ in } \Sigma, \quad \operatorname{supp}\varphi_j \subset B_j \quad \text{and} \quad \operatorname{supp}\varphi_0 \subset \Sigma.$$

Given $(u,p) \in D(S)$ and $(f,-g) = S(u,p)$, we get for each φ_j, $j \ge 0$, the local $(R_{\lambda,\xi})$-problems

$$\begin{aligned}(\lambda + \xi^2 - \Delta')(\varphi_j u') + \nabla'(\varphi_j p) &= f'_j \\ (\lambda + \xi^2 - \Delta')(\varphi_j u_n) + i\xi(\varphi_j p) &= f_{jn} \\ \operatorname{div}_\xi(\varphi_j u) &= g_j\end{aligned} \tag{3.43}$$

for $(\varphi_j u, \varphi_j p), j \geq 0$, in $\mathbb{R}^{n-1}$ or H_j; here

$$\begin{aligned} f_j' &= \varphi_j f' - 2\nabla'\varphi_j \cdot \nabla' u' - (\Delta'\varphi_j)u' + (\nabla'\varphi_j)p \\ f_{jn} &= \varphi_j f_n - 2\nabla'\varphi_j \cdot \nabla' u_n - (\Delta'\varphi_j)u_n \\ g_j &= \varphi_j g + \nabla'\varphi_j \cdot u'. \end{aligned} \tag{3.44}$$

To control f_j and g_j note that $u = 0$ on $\partial\Sigma$; hence Poincaré's inequality for Muckenhoupt weighted space (Proposition 2.4) yields for all $j \geq 0$ the estimate

$$\|f_j, \nabla' g_j, \xi g_j\|_{r,\omega;H_j} \leq c(\|f, \nabla' g, g, \xi g\|_{r,\omega;\Sigma} + \|\nabla' u, \xi u, p\|_{r,\omega;\Sigma}), \tag{3.45}$$

where $H_0 \equiv \mathbb{R}^{n-1}$ and $c > 0$ is A_r-consistent. Moreover, let $g = g_0 + g_1$ denote any splitting of $g \in L^r_{(m),\omega} + L^r_{\omega,1/\xi}$. Defining the characteristic function χ_j of $\Sigma \cap H_j$ and the scalar

$$\begin{aligned} m_j &= \frac{1}{|\Sigma \cap H_j|} \int_{\Sigma\cap H_j} (\varphi_j g_0 + u' \cdot \nabla'\varphi_j) dx' \\ &= \frac{1}{|\Sigma \cap H_j|} \int_{\Sigma\cap H_j} (i\xi u_n - g_1)\varphi_j dx', \end{aligned}$$

we split g_j in the form

$$g_j = g_{j0} + g_{j1} := (\varphi_j g_0 + u' \cdot \nabla'\varphi_j - m_j\chi_j) + (\varphi_j g_1 + m_j \chi_j).$$

Concerning g_{j1} we get

$$\begin{aligned} \|g_{j1}\|^r_{r,\omega;H_j} &= \int_{\Sigma\cap H_j} |\varphi_j g_1 + m_j|^r \omega\, dx' \\ &\leq c(r)\big(\|g_1\|^r_{r,\omega;\Sigma} + |m_j|^r \omega(\Sigma \cap H_j)\big) \\ &\leq c(r)\Big(\|g_1\|^r_{r,\omega;\Sigma} + \frac{\omega(\Sigma\cap H_j)\cdot \omega'(\Sigma\cap H_j)^{r/r'}}{|\Sigma \cap H_j|^r}(\|\xi u_n\|^r_{(W^{1,r'}_{\omega'})^*} + \|g_1\|^r_{r,\omega;\Sigma})\Big) \end{aligned}$$

with $c(r) > 0$ independent of ω. Since we chose the balls B_j for $j \geq 1$ small enough, for each $j \geq 0$ there is a cube Q_j with $\Sigma \cap H_j \subset Q_j$ and $|Q_j| < c(n)|\Sigma \cap H_j|$ where the constant $c(n) > 0$ is independent of j. Therefore

$$\begin{aligned} \|g_{j1}\|_{r,\omega;H_j} &\leq c(r)\Big(\|g_1\|_{r,\omega} + \tfrac{c(n)\omega(Q_j)^{1/r}\cdot\omega'(Q_j)^{1/r'}}{|Q_j|}(\|\xi u_n\|_{(W^{1,r'}_{\omega'})^*} + \|g_1\|_{r,\omega})\Big) \\ &\leq c(r)(1 + \mathcal{A}_r(\omega)^{1/r})\big(\|\xi u_n\|_{(W^{1,r'}_{\omega'})^*} + \|g_1\|_{r,\omega;\Sigma}\big) \end{aligned} \tag{3.46}$$

for $j \geq 0$. Furthermore, for every test function $\Psi \in C_0^\infty(\bar{H}_j)$ let

$$\tilde{\Psi} = \Psi - \frac{1}{|\Sigma \cap H_j|}\int_{\Sigma\cap H_j} \Psi dx'.$$

By the definition of m_j, χ_j we have $\int_{H_j} g_{j0}\, dx' = 0$; hence by Poincaré's inequality (see Proposition 2.4)

$$\begin{aligned} \textstyle\int_{H_j} g_{j0}\Psi dx' &= \textstyle\int_{H_j} g_{j0}\tilde{\Psi} dx' \\ &= \textstyle\int_\Sigma g_0(\varphi_j\tilde{\Psi})dx' + \int_\Sigma u'\cdot(\nabla'\varphi_j)\tilde{\Psi}dx' \\ &\leq \|g_0\|_{-1,r,\omega}\|\nabla'(\varphi_j\tilde{\Psi})\|_{r',\omega'} + \|u'\|_{(W^{1,r'}_{\omega'})^*}\|(\nabla'\varphi_j)\tilde{\Psi}\|_{1,r',\omega'} \\ &\leq c(\|g_0\|_{-1,r,\omega} + \|u'\|_{(W^{1,r'}_{\omega'})^*})\|\nabla'\Psi\|_{r',\omega';H_j}, \end{aligned}$$

where $c > 0$ is A_r-consistent. Thus

$$\|g_{j0}\|_{-1,r,\omega;H_j} \le c\big(\|g_0\|_{-1,r,\omega} + \|u'\|_{(W^{1,r'}_{\omega'})^*}\big) \quad \text{for } j \ge 0. \tag{3.47}$$

Summarizing (3.46) and (3.47), we get for $j \ge 0$

$$\|g_j; \widehat{W}^{-1,r}_{\omega}(H_j) + L^r_{\omega,1/\xi}(H_j)\| \le c\big(\|u'\|_{(W^{1,r'}_{\omega'})^*} + \|g; L^r_{(m),\omega} + L^r_{\omega,1/\xi}\|_0\big) \tag{3.48}$$

with an A_r-consistent $c = c(r, \mathcal{A}_r(\omega)) > 0$.

To complete the proof, apply Theorem 3.1 to (3.43), (3.44) when $j = 0$. Further use Theorem 3.5 in (3.43), (3.44) for $j \ge 1$, but with λ replaced by $\lambda + M$ with $M = \lambda_0 + \alpha_0$, where $\lambda_0 = \lambda_0(\varepsilon, r, \mathcal{A}_r(\omega))$ is the A_r-consistent constant indicated in Theorem 3.5. This shift in λ implies that f_j has to be replaced by $f_j + M\varphi_j u$ and that (3.31) will be used with λ replaced by $\lambda + M$. Summarizing (3.8), (3.31) as well as (3.45), (3.48) and summing over all j we arrive at (3.42) with the additional terms

$$I = \|Mu\|_{r,\omega} + \|Mu'\|_{(W^{1,r'}_{\omega'})^*} + \|Mg; L^r_{(m),\omega} + L^r_{\omega,1/\xi}\|_0$$

on the right-hand side of the inequality. Note that $M = M(\varepsilon, r, \mathcal{A}_r(\omega))$ is A_r-consistent and that $g = \operatorname{div}' u' + i\xi u_n$ defines a natural splitting of $g \in L^r_{(m),\omega}(\Sigma) + L^r_{\omega}(\Sigma)$. Hence Poincaré's inequality yields

$$\begin{aligned} I &\le M\big(\|u\|_{r,\omega;\Sigma} + \|\operatorname{div}' u'\|_{-1,r,\omega} + \|u_n\|_{r,\omega;\Sigma}\big) \\ &\le c_1\|u\|_{r,\omega;\Sigma} \le c_2\|\nabla' u\|_{r,\omega;\Sigma} \end{aligned}$$

with A_r-consistent constants $c_i = c_i(\varepsilon, r, \Sigma, \mathcal{A}_r(\omega)) > 0$, $i = 1, 2$. Thus (3.42) is proved. ■

Lemma 3.10 *Let $1 < r < \infty$, $\omega \in A_r$ and $\lambda \in -\alpha + \Sigma_\varepsilon$, $\varepsilon \in (\frac{\pi}{2}, \pi)$ with $\alpha \in (0, \alpha_0)$. Then there is an A_r-consistent constant $c = c(\alpha, \varepsilon, r, \mathcal{A}_r(\omega))$ such that for every $(u, p) \in D(S)$ and $(f, -g) = S(u, p)$ the estimate*

$$\begin{aligned} &\|\mu_+^2 u, \mu_+\nabla' u, \nabla'^2 u, \nabla' p, \xi p\|_{r,\omega} \\ &\qquad \le c\big(\|f, \nabla' g, g, \xi g\|_{r,\omega} + (|\lambda| + 1)\|g; L^r_{(m),\omega} + L^r_{\omega,1/\xi}\|_0\big) \end{aligned} \tag{3.49}$$

holds; here $\mu_+ = |\lambda + \alpha + \xi^2|^{1/2}$.

Proof: Assume that this lemma is wrong. Then there is a constant $c_0 > 0$, a sequence $\{\omega_j\}_{j=1}^\infty \subset A_r$ with $\mathcal{A}_r(\omega_j) \le c_0$ for all j, sequences $\{\lambda_j\}_{j=1}^\infty \subset -\alpha + \Sigma_\varepsilon$, $\{\xi_j\}_{j=1}^\infty \subset \mathbb{R}^*$ and $(u_j, p_j) \in D(S^{\omega_j}_{r,\lambda_j,\xi_j})$ for all $j \in \mathbb{N}$ such that

$$\begin{aligned} &\|(\lambda_j + \alpha + \xi_j^2)u_j, (\lambda_j + \alpha + \xi_j^2)^{1/2}\nabla' u_j, \nabla'^2 u_j, \nabla' p_j, \xi_j p_j\|_{r,\omega_j} \\ &\qquad \ge j\big(\|f_j, \nabla' g_j, g_j, \xi_j g_j\|_{r,\omega_j} + (|\lambda_j| + 1)\|g_j; L^r_{(m),\omega_j} + L^r_{\omega_j,1/\xi_j}\|_0 \end{aligned} \tag{3.50}$$

where $(f_j, -g_j) = S^{\omega_j}_{r,\lambda_j,\xi_j}(u_j, p_j)$. Fix an arbitrary cube Q containing Σ. We may assume without loss of generality that

$$\mathcal{A}_r(\omega_j) \le c_0, \quad \omega_j(Q) = 1 \quad \forall j \in \mathbb{N}, \tag{3.51}$$

by using the A_r-weight $\tilde\omega_j := \omega_j(Q)^{-1}\omega_j$ instead of ω_j if necessary. Note that (3.51) also holds for $r', \{\omega_j'\}$ in the following form: $\mathcal{A}_r(\omega_j) \le c_0^{r'/r}$, $\omega_j'(Q) \le c_0^{r'/r}|Q|^{r'}$. Therefore, by a minor modification of Proposition 2.2 (3), there exist numbers s, s_1 such that

$$L^r_{\omega_j}(\Sigma) \hookrightarrow L^s(\Sigma), \quad L^{s_1}(\Sigma) \hookrightarrow L^{r'}_{\omega_j'}, \quad j \in \mathbb{N}, \tag{3.52}$$

with embedding constants independent of $j \in \mathbb{N}$. Furthermore, we may assume without loss of generality that

$$\|(\lambda_j + \alpha + \xi_j^2)u_j, (\lambda_j + \alpha + \xi_j^2)^{1/2}\nabla' u_j, \nabla'^2 u_j, \nabla' p_j, \xi_j p_j\|_{r,\omega_j} = 1 \tag{3.53}$$

and consequently that

$$\|f_j, \nabla' g_j, g_j, \xi_j g_j\|_{r,\omega_j} + (|\lambda_j| + 1)\|g_j; L^r_{(m),\omega_j} + L^r_{\omega_j,1/\xi_j}\|_0 \to 0 \quad \text{as} \quad j \to \infty. \tag{3.54}$$

From (3.52), (3.53) we have

$$\|(\lambda_j + \alpha + \xi_j^2)u_j, (\lambda_j + \alpha + \xi_j^2)^{1/2}\nabla' u_j, \nabla'^2 u_j, \nabla' p_j, \xi_j p_j\|_s \le K, \tag{3.55}$$

with some $K > 0$ for all $j \in \mathbb{N}$ and

$$\|f_j, \nabla' g_j, g_j, \xi_j g_j\|_s \to 0 \quad \text{as} \quad j \to \infty. \tag{3.56}$$

Without loss of generality let us suppose that as $j \to \infty$,

$$\begin{aligned} &\lambda_j \to \lambda \in -\alpha + \bar\Sigma_\varepsilon \quad \text{or} \quad |\lambda_j| \to \infty \\ &\xi_j \to 0 \quad \text{or} \quad \xi_j \to \xi \neq 0 \quad \text{or} \quad |\xi_j| \to \infty. \end{aligned}$$

Thus we have to consider six possibilities.

(i) The case $\lambda_j \to \lambda \in -\alpha + \bar\Sigma_\varepsilon, \quad \xi_j \to \xi \neq 0$.
Due to (3.55) $\{u_j\} \subset W^{2,s}$ and $\{p_j\} \subset W^{1,s}$ are bounded sequences. In virtue of the compactness of the embedding $W^{1,s}(\Sigma) \hookrightarrow L^s(\Sigma)$ for the bounded domain Σ, we may assume (suppressing indices for subsequences) that

$$\begin{array}{lll} u_j \to u, \nabla' u_j \to \nabla' u & \text{in } L^s & \text{(strong convergence)} \\ \nabla'^2 u_j \rightharpoonup \nabla'^2 u & \text{in } L^s & \text{(weak convergence)} \\ p_j \to p & \text{in } L^s & \text{(strong convergence)} \\ \nabla' p_j \rightharpoonup \nabla' p & \text{in } L^s & \text{(weak convergence)} \end{array} \tag{3.57}$$

for some $(u, p) \in D(S_{s,\lambda,\xi})$ as $j \to \infty$. Therefore, $S_{s,\lambda,\xi}(u, p) = 0$ and, consequently, $u = 0$, $p = 0$ by Lemma 3.8. On the other hand we get from (3.53) that $\sup_{j\in\mathbb{N}} \|u_j\|_{2,r,\omega_j} < \infty$ and $\sup_{j\in\mathbb{N}} \|p_j\|_{1,r,\omega_j} < \infty$ which, together with the weak convergences $u_j \rightharpoonup 0$ in $W^{2,s}(\Sigma)$, $p_j \rightharpoonup 0$ in $W^{1,s}(\Sigma)$, yields

$$\|u_j\|_{1,r,\omega_j} \to 0, \quad \|p_j\|_{r,\omega_j} \to 0$$

due to Proposition 2.3 (2). Moreover, since

$$\sup_{j\in\mathbb{N}} \|\lambda_j u_j\|_{r,\omega_j} < \infty \quad \text{and } \lambda_j u_j \rightharpoonup \lambda u = 0 \quad \text{in } L^s(\Sigma),$$

Proposition 2.3 (3) implies that

$$\|\lambda_j u_j\|_{(W^{1,r'}_{\omega'_j})^*} \to 0. \tag{3.58}$$

Thus (3.42), (3.53) and (3.54) yield the contradiction $1 \le 0$.

(ii) The case $\lambda_j \to \lambda \in -\alpha + \bar\Sigma_\varepsilon, \quad \xi_j \to 0$.
Since $u_j|_{\partial\Sigma} = 0, \|\nabla'^2 u_j\|_s \le K$, we have the convergence (3.57) for some $u \in W^{2,s}(\Sigma) \cap W^{1,s}_0(\Sigma)$, but concerning p we get the existence of $p \in \widehat{W}^{1,s}$ and $q \in L^s$ such that

$$\nabla' p_j \rightharpoonup \nabla' p, \quad \xi_j p_j \rightharpoonup q \quad \text{ in } L^s$$

as $j \to \infty$. Looking at (R_{λ_j,ξ_j}), the convergence of $\{u_j\}$, $\{p_j\}$ yields

$$\begin{aligned}(\lambda - \Delta')u' + \nabla' p &= 0\\ (\lambda - \Delta')u_n + iq &= 0\\ \operatorname{div}' u' &= 0\\ (u', u_n)\big|_\Sigma &= 0\end{aligned}$$

in Σ. Thus, the uniqueness result for the Stokes system on Σ yields $(u', \nabla' p) = (0,0)$, see [27]. Moreover, since $\xi_j \nabla' p_j \to 0$ in L^s and $\xi_j \nabla' p_j \rightharpoonup \nabla' q$ in $W^{-1,s}$ as $j \to \infty$, it is seen that q is a constant. Hence elliptic regularity theory implies $u_n \in W^{2,2}(\Sigma) \cap W^{1,2}_0(\Sigma)$.

By (3.54), for all $j \in \mathbb{N}$ there is a splitting $g_j = g_{j0} + g_{j1}$ such that

$$g_{j0} \in L^r_{(m),\omega_j},\ g_{j1} \in L^r_{\omega_j} \quad \text{and} \quad (|\lambda_j| + 1)\big(\|g_{j0}\|_{-1,r,\omega_j} + \|g_{j1}/\xi_j\|_{r,\omega_j}\big) \to 0. \tag{3.59}$$

Therefore, from the divergence equation $\operatorname{div}_{\xi_j} u_j = g_j$ we get

$$(|\lambda_j| + 1)\Big|\int_\Sigma u_{jn}\,dx'\Big| = \frac{|\lambda_j| + 1}{|\xi_j|}\Big|\int_\Sigma g_{j1}\,dx'\Big| \to 0 \quad \text{ as } j \to \infty,$$

and consequently $\int_\Sigma u_n\,dx' = 0$. Now, testing the equation $(\lambda - \Delta')u_n + iq = 0$ in Σ with u_n, we see that $\lambda \int_\Sigma |u_n|^2\,dx' + \int_\Sigma |\nabla' u_n|^2\,dx' = 0$ yielding $u_n = 0$ and also $q = 0$. Thus $u_j \rightharpoonup 0$ in $W^{2,s}(\Sigma)$ which, together with $\sup_{j\in\mathbb{N}} \|u_j\|_{2,r,\omega_j} < \infty$, yields

$$\|u_j\|_{1,r,\omega_j} \to 0 \quad \text{as} \quad j \to \infty \tag{3.60}$$

due to Proposition 2.3 (2).

To come to a contradiction consider the equivalent equation $S^{\omega_j}_{r,\lambda_j,\xi_j}(u_j, p_j - p_{jm}) = (f_j - i\xi_j p_{jm} e_n, -g_j)$ with $p_{jm} = \frac{1}{|\Sigma|}\int_\Sigma p_j\,dx'$. Due to Lemma 3.9

$$\begin{aligned}&\|(\lambda_j + \alpha + \xi_j^2)u_j, (\lambda_j + \alpha + \xi_j^2)^{1/2}\nabla' u_j, \nabla'^2 u_j, \nabla' p_j, \xi_j(p_j - p_{jm})\|_{r,\omega_j}\\ &\qquad \le c\big(\|f_j, \nabla' g_j, g_j, \xi_j g_j\|_{r,\omega_j} + (|\lambda_j| + 1)\|g_j; L^r_{(m),\omega_j} + L^r_{\omega_j,1/\xi}\|_0\\ &\qquad\quad + \|\xi_j p_{jm}\|_{r,\omega_j} + \|\nabla' u_j, \xi_j u_j, p_j - p_{jm}\|_{r,\omega_j} + \|\lambda_j u_j\|_{(W^{1,r'}_{\omega'_j})^*}\big)\end{aligned} \tag{3.61}$$

where $c > 0$ is independent of $j \in \mathbb{N}$ due to $\mathcal{A}_r(\omega_j) \le c_0$, $j \in \mathbb{N}$. Since $\xi_j p_j \rightharpoonup q = 0$ in L^s, we have $\xi_j p_{jm} \to 0$ and, considering (3.51),

$$\|\xi_j p_{jm}\|_{r,\omega_j} = |\xi_j p_{jm}|\omega_j(\Sigma)^{1/r} \le |\xi_j p_{jm}| \to 0. \tag{3.62}$$

From Poincaré's inequality (Proposition 2.4) and (3.53), we conclude that

$$\sup_j \|p_j - p_{jm}\|_{1,r,\omega_j} < \infty,$$

which, together with $p_j - p_{jm} \rightharpoonup 0$ in $W^{1,s}(\Sigma)$, yields

$$\|p_j - p_{jm}\|_{r,\omega_j} \to 0 \quad \text{as} \quad j \to \infty, \tag{3.63}$$

cf. Proposition 2.4 (2). Now, (3.53), (3.54), (3.58), (3.60), (3.62) and (3.63) lead in (3.61) to the contradiction $1 \le 0$.

(iii) The case $\lambda_j \to \lambda \in -\alpha + \bar{\Sigma}_\varepsilon$, $\quad |\xi_j| \to \infty$.
From (3.53) we get $\|\nabla' u_j, \xi_j u_j, p_j\|_{r,\omega_j} \to 0$. On the other hand, since

$$\|u_j\|_{r,\omega_j} \to 0 \quad \text{and} \quad u_j \to 0 \text{ in } L^s \text{ as } j \to \infty,$$

Proposition 2.3 (3) implies (3.58). Thus, from (3.42), (3.53) and (3.54) we get the contradiction $1 \le 0$.

(iv) The case $|\lambda_j| \to \infty$, $\quad \xi_j \to \xi \neq 0$.
By (3.53)

$$\|\nabla' u_j, \xi_j u_j\|_{r,\omega_j} \to 0 \quad \text{as} \quad j \to \infty. \tag{3.64}$$

Further, (3.55) yields the convergence

$$\begin{array}{lll} u_j \to 0, \nabla' u_j \to 0 & \text{and} & \nabla'^2 u_j \rightharpoonup 0, \lambda_j u_j \rightharpoonup v, \\ p_j \to p & \text{and} & \nabla' p_j \rightharpoonup \nabla' p, \end{array}$$

in L^s, which, together with (3.56), leads to

$$v' + \nabla' p = 0, \quad v_n + i\xi p = 0. \tag{3.65}$$

From (3.52), (3.59) we see that

$$\begin{aligned} |\langle \lambda_j g_j, \varphi \rangle| &= |\langle \lambda_j g_{j0}, \varphi \rangle + \langle \lambda_j g_{j1}, \varphi \rangle| \\ &\le \|\lambda_j g_{j0}\|_{-1,r,\omega_j} \|\nabla' \varphi\|_{r',\omega_j'} + \|\lambda_j g_{j1}\|_{r,\omega_j} \|\varphi\|_{r',\omega_j'} \\ &\le c\big(\|\lambda_j g_{j0}\|_{-1,r,\omega_j}\| + \|\lambda_j g_{j1}\|_{r,\omega_j}\big)\|\varphi\|_{W^{1,s_1}(\Sigma)}. \end{aligned}$$

Consequently,

$$\lambda_j g_j \in (W^{1,s_1}(\Sigma))^* \quad \text{and} \quad \|\lambda_j g_j\|_{(W^{1,s_1}(\Sigma))^*} \to 0 \quad \text{as } j \to \infty. \tag{3.66}$$

Therefore, it follows from the divergence equation $\operatorname{div}'_{\xi_j} u_j = g_j$ that for all $\varphi \in C^\infty(\bar{\Sigma})$

$$\begin{aligned} \langle v', -\nabla' \varphi \rangle + \langle i\xi v_n, \varphi \rangle &= \lim_{j\to\infty} \langle \operatorname{div}' \lambda_j u_j' + i\lambda_j \xi_j u_{jn}, \varphi \rangle \\ &= \lim_{j\to\infty} \langle \lambda_j g_j, \varphi \rangle = 0, \end{aligned}$$

yielding $\operatorname{div}'v' = -i\xi v_n,\ v'\cdot N'|_{\partial\Sigma} = 0$, where N' is the outward normal vector on $\partial\Sigma$. Therefore (3.65) implies

$$-\Delta' p + \xi^2 p = 0 \text{ in } \Sigma, \quad \frac{\partial p}{\partial N'} = 0 \text{ on } \partial\Sigma;$$

hence $p \equiv 0$ and also $v \equiv 0$. Now, due to Proposition 2.3 (2), (3), we get (3.58) and the convergence $\|p_j\|_{r,\omega_j} \to 0$, since $\lambda_j u_j \rightharpoonup 0$ in L^s, $p_j \rightharpoonup 0$ in $W^{1,s}$ and $\sup_{j\in\mathbb{N}} \|\lambda_j u_j\|_{r,\omega_j} < \infty$, $\sup_{j\in\mathbb{N}} \|p_j\|_{1,r,\omega_j} < \infty$. Thus (3.42), (3.53), (3.54) and (3.64) lead to the contradiction $1 \le 0$.

(v) The case $|\lambda_j| \to \infty, \quad \xi_j \to 0$.
It follows from (3.53) that in L^s

$$\begin{aligned} &u_j \to 0, \nabla' u_j \to 0 \quad \text{and} \quad \nabla'^2 u_j \rightharpoonup 0, \lambda_j u_j \rightharpoonup v,\\ &\nabla' p_j \rightharpoonup \nabla' p, \quad \xi_j p_j \rightharpoonup q, \end{aligned}$$

which, looking at $(R_{\lambda,\xi})$, yields in the weak limit

$$v' + \nabla' p = 0, \quad v_n + iq = 0;$$

moreover, q is a constant. Note that (3.66) holds true in this case as well. Therefore, using (3.66), for any function φ in $C^\infty(\bar\Sigma)$

$$0 = -\lim_{j\to\infty} \langle \lambda_j g_j, \varphi\rangle = \lim_{j\to\infty} \big(\langle \lambda_j u_j', \nabla'\varphi\rangle - \langle i\lambda_j\xi_j u_{jn}, \varphi\rangle\big) = \int_\Sigma v'\cdot \overline{\nabla'\varphi}\, dx'$$

yielding $\operatorname{div}'v' = 0, v'\cdot N'|_{\partial\Sigma} = 0$. Thus the equation $v' + \nabla' p = 0$ is just the Helmholtz decomposition of the null vector field; therefore, $v' \equiv 0, \nabla' p \equiv 0$.

On the other hand, looking at (3.59) we get from the divergence equation and (3.52) that

$$\int_\Sigma \lambda_j u_{jn}\, dx' = \int_\Sigma \frac{\lambda_j}{\xi_j}(g_{j0} + g_{j1} - \operatorname{div}' u_j')\, dx' = \int_\Sigma \frac{\lambda_j g_{j1}}{\xi_j}\, dx' \to 0.$$

Consequently, the weak convergence $\lambda_j u_{jn} \rightharpoonup v_n$ in L^s yields $\int_\Sigma v_n\, dx' = 0$; since q is a constant, we get $v_n = 0$, $q = 0$. Then Proposition 2.3 (3) implies (3.58).

Now we repeat the argument as in the case (ii) to get (3.61), (3.62) and (3.63), and are finally led to the contradiction $1 \le 0$.

(vi) The case $|\lambda_j| \to \infty, \quad |\xi_j| \to \infty$.
To come to a contradiction, it is enough to prove (3.58) since $\|\nabla' u_j, \xi_j u_j, p_j\|_{r,\omega_j} \to 0$ as $j \to \infty$. From (3.53) we get the convergence

$$\begin{aligned} &u_j \to 0, \nabla' u_j \to 0 \quad &&\text{and} \quad \nabla'^2 u_j \rightharpoonup 0, (\lambda_j + \xi_j^2) u_j \rightharpoonup v,\\ &p_j \to 0 &&\text{and} \quad \nabla' p_j \rightharpoonup 0, \quad \xi_j p_j \rightharpoonup q \end{aligned}$$

in L^s with some $v, q \in L^s$. Therefore, (3.56) and (R_{λ_j,ξ_j}) yield

$$v' = 0, \quad v_n + iq = 0.$$

Since $\|\lambda_j u_j\|_s \le c_\varepsilon \|(\lambda_j + \xi_j^2)u_j\|_s$, there exists $w = (w', w_n) \in L^s$ such that, for a suitable subsequence, $\lambda_j u_j \rightharpoonup w$. Let $g_j = g_{j0} + g_{j1}$, $j \in \mathbb{N}$, be a sequence of splittings satisfying (3.59). By (3.52) we get for all $\varphi \in C^\infty(\bar{\Sigma})$

$$|\langle \lambda_j g_{j0}, \varphi\rangle| + \Big|\langle \frac{\lambda_j g_{j1}}{\xi_j}, \varphi\rangle\Big| \to 0 \quad \text{as} \quad j \to \infty,$$

cf. (3.66) and (3.66). Hence, the divergence equation implies that for $j \to \infty$

$$\langle \lambda_j u_{jn}, \varphi\rangle = \frac{1}{i\xi_j}\langle \lambda_j g_{j0}, \varphi\rangle + \langle \frac{\lambda_j g_{j1}}{i\xi_j}, \varphi\rangle + \frac{1}{i\xi_j}\langle \lambda_j u_j', \nabla'\varphi\rangle \to 0$$

for all $\varphi \in C^\infty(\bar{\Sigma})$ yielding $\langle w_n, \varphi\rangle = 0$ and consequently $w_n = 0$.

Obviously, $\xi_j u_j \to 0$ in L^s as $j \to \infty$. Therefore, by (3.56) and the boundedness of the sequence $\big\{\|\xi_j \nabla u_j\|_{r,\omega_j}\big\}$, we get from the identity $\operatorname{div}'(\xi_j u_j') + i\xi_j^2 u_{jn} = \xi_j g_j$ that

$$\xi_j^2 u_{jn} \rightharpoonup 0 \quad \text{in } L^s \text{ as } j \to \infty.$$

Thus we proved $v_n = 0$. Now $v = 0$ together with the estimate $\|(\lambda_j + \xi_j^2)u_j\|_{r,\omega_j} \le 1$ imply due to Proposition 2.3 (3) that $\|(\lambda_j + \xi_j^2)u_j\| \to 0$ in $(W^{1,r'}_{\omega_j'})^*$ as $j \to \infty$. Hence also (3.58) is proved.

Now the proof of this lemma is complete. ■

Theorem 3.11 *Let $\Sigma \subset \mathbb{R}^{n-1}$ be a bounded domain of $C^{1,1}$-class, $1 < r < \infty$, $\omega \in A_r(\mathbb{R}^{n-1})$ and $\alpha \in (0, \alpha_0)$, $\frac{\pi}{2} < \varepsilon < \pi$. Then for every $\lambda \in -\alpha + \Sigma_\varepsilon$, $\xi \in \mathbb{R}^*$ and $f \in L^r_\omega(\Sigma)$, $g \in W^{1,r}_\omega(\Sigma)$ the parametrized resolvent problem $(R_{\lambda,\xi})$ has a unique solution $(u,p) \in \big(W^{2,r}_\omega(\Sigma) \cap W^{1,r}_{0,\omega}(\Sigma)\big) \times W^{1,r}_\omega(\Sigma)$. Moreover, this solution satisfies the estimate* (3.49) *with an A_r-consistent constant $c = c(\alpha, \varepsilon, r, \Sigma, \mathcal{A}_r(\omega)) > 0$.*

Proof: The existence is obvious since, for every $\lambda \in -\alpha + \Sigma_\varepsilon, \xi \in \mathbb{R}^*$ and $\omega \in A_r(\mathbb{R}^{n-1})$, the range $R(S^\omega_{r,\lambda,\xi})$ is closed and dense in $L^r_\omega(\Sigma) \times W^{1,r}_\omega(\Sigma)$ by Lemma 3.10 and by Lemma 3.8, respectively. Here note that for fixed $\lambda \in \mathbb{C}$, $\xi \in \mathbb{R}^*$ the norm $\|\nabla' g, g, \xi g\|_{r,\omega} + (1 + |\lambda|)\|g; L^r_{(m),\omega} + L^r_{\omega,1/\xi}\|_0$ is equivalent to the norm of $W^{1,r}_\omega(\Sigma)$. The uniqueness of solutions is obvious from Lemma 3.8. ■

Now, for fixed $\omega \in A_r, 1 < r < \infty$, define the operator-valued functions

$$a_1 : \mathbb{R}^* \to \mathcal{L}(L^r_\omega(\Sigma); W^{2,r}_{0,\omega}(\Sigma) \cap W^{1,r}_\omega(\Sigma)),$$
$$b_1 : \mathbb{R}^* \to \mathcal{L}(L^r_\omega(\Sigma); W^{1,r}_\omega(\Sigma))$$

by

$$a_1(\xi) f := u_1(\xi), \quad b_1(\xi) f := p_1(\xi), \tag{3.67}$$

where $(u_1(\xi), p_1(\xi))$ is the solution to $(R_{\lambda,\xi})$ corresponding to $f \in L^r_\omega(\Sigma)$ and $g = 0$. Further, define

$$a_2 : \mathbb{R}^* \to \mathcal{L}(W^{1,r}_\omega(\Sigma); W^{2,r}_{0,\omega}(\Sigma) \cap W^{1,r}_\omega(\Sigma)),$$
$$b_2 : \mathbb{R}^* \to \mathcal{L}(W^{1,r}_\omega(\Sigma); W^{1,r}_\omega(\Sigma))$$

by

$$a_2(\xi)g := u_2(\xi), \quad b_2(\xi)g := p_2(\xi). \tag{3.68}$$

with $(u_2(\xi), p_2(\xi))$ the solution to $(R_{\lambda,\xi})$ corresponding to $f = 0$ and $g \in W^{1,r}_\omega(\Sigma)$.

Corollary 3.12 *For every $\alpha \in (0, \alpha_0)$ and $\lambda \in -\alpha + \Sigma_\varepsilon$ the operator-valued functions a_1, b_1 and a_2, b_2 defined by (3.67), (3.68) are Fréchet differentiable in $\xi \in \mathbb{R}^*$. Furthermore, their derivatives $w_1 = \frac{d}{d\xi}a_1(\xi)f,\ q_1 = \frac{d}{d\xi}b_1(\xi)f$ for fixed $f \in L^r_\omega(\Sigma)$ and $w_2 = \frac{d}{d\xi}a_2(\xi)g,\ q_2 = \frac{d}{d\xi}b_2(\xi)g$ for fixed $g \in W^{1,r}_\omega(\Sigma)$ satisfy the estimates*

$$\|(\lambda+\alpha)\xi w_1, \xi\nabla'^2 w_1, \xi^3 w_1, \xi\nabla' q_1, \xi^2 q_1\|_{r,\omega} \le c\|f\|_{r,\omega} \tag{3.69}$$

and

$$\begin{aligned} &\|(\lambda+\alpha)\xi w_2, \xi\nabla'^2 w_2, \xi^3 w_2, \xi\nabla' q_2, \xi^2 q_2\|_{r,\omega} \\ &\qquad \le c\big(\|\nabla' g, g, \xi g\|_{r,\omega} + (|\lambda|+1)\|g; L^r_{(m),\omega} + L^r_{\omega,1/\xi}\|_0\big), \end{aligned} \tag{3.70}$$

with an A_r-consistent constant $c = c(\alpha, r, \varepsilon, \Sigma, \mathcal{A}_r(\omega))$ independent of $\lambda \in -\alpha + \Sigma_\varepsilon$ and $\xi \in \mathbb{R}^*$.

Proof: Since ξ enters in $(R_{\lambda,\xi})$ in a polynomial way, it is easy to prove that $a_j(\xi), b_j(\xi), j = 1, 2$, are Fréchet differentiable and their derivatives w_j, q_j solve the system

$$\begin{aligned} (\lambda + \xi^2 - \Delta')w'_j + \nabla' q_j &= -2\xi u'_j \\ (\lambda + \xi^2 - \Delta')w_{jn} + i\xi q_j &= -2\xi u_{jn} - ip_j \\ \operatorname{div}' w'_j + i\xi w_{jn} &= -iu_{jn}, \end{aligned} \tag{3.71}$$

where $(u_1, p_1), (u_2, p_2)$ are the solutions to $(R_{\lambda,\xi})$ for $f \in L^r_\omega(\Sigma)$, $g = 0$ and $f = 0$, $g \in W^{1,r}_\omega(\Sigma)$, respectively.

We get from (3.71) and Theorem 3.11 for $j = 1, 2$,

$$\begin{aligned} &\|(\lambda+\alpha)\xi w_j, \xi\nabla'^2 w_j, \xi^3 w_j, \xi\nabla' q_j, \xi^2 q_j\|_{r,\omega} \\ &\quad \le c\big(\|\xi^2 u'_j, \xi p_j, \nabla'\xi u_{jn}, \xi^2 u_{jn}\|_{r,\omega} + (|\lambda|+1)\|i\xi u_{jn}; L^r_{(m),\omega} + L^r_{\omega,1/\xi}\|_0\big) \\ &\quad \le c\big(\|\xi^2 u_j, \xi p_j, \nabla'\xi u_j\|_{r,\omega} + (|\lambda|+1)\|u_j\|_{r,\omega}\big) \\ &\quad \le c\|u_j, (\lambda+\alpha+\xi^2)u_j, \sqrt{\lambda+\alpha+\xi^2}\nabla' u_j, \xi p_j\|_{r,\omega} \\ &\quad \le c\|(\lambda+\alpha+\xi^2)u_j, \sqrt{\lambda+\alpha+\xi^2}\nabla' u_j, \nabla'^2 u_j, \xi p_j\|_{r,\omega}, \end{aligned} \tag{3.72}$$

with an A_r-consistent constant $c = c(\alpha, r, \varepsilon, \Sigma, \mathcal{A}_r(\omega))$; here we used the fact that $\xi^2 + |\lambda+\alpha| \le c(\varepsilon, \alpha)|\lambda + \alpha + \xi^2|$ for all $\lambda \in -\alpha + \Sigma_\varepsilon, \xi \in \mathbb{R}$ and $\|u_j\|_{r,\omega} \le c(\mathcal{A}_r(\omega))\|\nabla'^2 u_j\|_{r,\omega}$ (see [34], Corollary 2.2). Thus Theorem 3.11 and (3.72) prove (3.69), (3.70). ∎

4 Resolvent Estimate and Maximal Regularity in Weighted Spaces; Infinite Straight Cylinders

In this chapter Ω is an infinite straight cylinder $\Sigma \times \mathbb{R}$ with cross-section $\Sigma \subset \mathbb{R}^{n-1}, n \geq 3$, a bounded domain of $C^{1,1}$-class. We study the resolvent estimate and maximal regularity of the Stokes operator in weighted L^q-spaces on Ω. The proofs use the operator-valued Fourier multiplier theorem, Schauder decomposition techniques and $\mathcal{R}$-boundedness of operator families based on results of Chapter 3.

4.1 Resolvent estimate

Let us consider the Stokes resolvent system

$$(R_\lambda) \qquad \begin{aligned} \lambda u - \Delta u + \nabla p &= f \quad \text{in } \Omega \\ \operatorname{div} u &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Based on the estimate for the parametrized Stokes system $(R_{\lambda,\xi})$ in Chapter 3 we can prove the following theorem.

Theorem 4.1 (Weighted Resolvent Estimates) *Let Σ be a bounded domain of $C^{1,1}$-class with $\alpha_0 > 0$ being the least eigenvalue of the Dirichlet Laplacian in Σ, and let $\pi/2 < \varepsilon < \pi$, $1 < q, r < \infty$ and $\omega \in A_r$. Then for every $f \in L^q(\mathbb{R}; L^r_\omega(\Sigma))$, every $\alpha \in (0, \alpha_0)$ and $\lambda \in -\alpha + \Sigma_\varepsilon$ there exists a unique solution*

$$(u,p) \in \big(W^{2;q,r}_\omega(\Omega) \cap W^{1;q,r}_{0,\omega}(\Omega)\big) \times \widehat{W}^{1;q,r}_\omega(\Omega)$$

to (R_λ) satisfying the estimate

$$\|(\lambda+\alpha)u, \nabla^2 u, \nabla p\|_{L^q(L^r_\omega)} \leq C\|f\|_{L^q(L^r_\omega)} \tag{4.1}$$

with an A_r-consistent constant $C = C(q, r, \alpha, \varepsilon, \Sigma, \mathcal{A}_r(\omega))$ independent of λ.

Proof: Let a_1, b_1 be the operator-valued functions defined in (3.67) and let us define u, p in the cylinder $\Omega = \Sigma \times \mathbb{R}$ by

$$u(x) = \mathcal{F}^{-1}(a_1\hat{f})(x), \quad p(x) = \mathcal{F}^{-1}(b_1\hat{f})(x). \tag{4.2}$$

We will show that (u,p) is the unique solution to (R_λ) satisfying

$$(u,p) \in \big(W^{2;q,r}_\omega(\Omega) \cap W^{1;q,r}_{0,\omega}(\Omega)\big) \times \widehat{W}^{1;q,r}_\omega(\Omega) \tag{4.3}$$

and the estimate (4.1). Obviously, (u,p) solves the resolvent problem (R_λ). For $\xi \in \mathbb{R}^*$ define $m_\lambda(\xi) : L^r_\omega(\Sigma) \to L^r_\omega(\Sigma)$ by

$$m_\lambda(\xi)f := \big((\lambda+\alpha)a_1(\xi)\hat{f}, \xi\nabla' a_1(\xi)\hat{f}, {\nabla'}^2 a_1(\xi)\hat{f}, \xi^2 a_1(\xi)\hat{f}, \nabla' b_1(\xi)\hat{f}, \xi b_1(\xi)\hat{f}\big). \tag{4.4}$$

Theorem 3.11 and Corollary 3.12 show that the operator family $\{m_\lambda(\xi), \xi m'_\lambda(\xi) : \xi \in \mathbb{R}^*\}$ satisfies the assumptions of Theorem 2.14, e.g., with $s = r$. Therefore, this operator family is $\mathcal{R}$-bounded in $\mathcal{L}(L^r_\omega(\Sigma))$; to be more precise,

$$\mathcal{R}_q\big(\{m_\lambda(\xi), \xi m'_\lambda(\xi) : \xi \in \mathbb{R}^*\}\big) \leq c(q, r, \alpha, \varepsilon, \Sigma, \mathcal{A}_r(\omega)) < \infty. \tag{4.5}$$

Hence Theorem 2.12 and Remark 2.13 imply that

$$\|(m_\lambda \hat{f})^\vee\|_{L^q(L^r_\omega)} \leq C\|f\|_{L^q(L^r_\omega)}$$

with an A_r-consistent constant $C = C(q, r, \alpha, \varepsilon, \Sigma, \mathcal{A}_r(\omega)) > 0$ independent of the resolvent parameter $\lambda \in -\alpha + \Sigma_\varepsilon$. Note that, due to the definition of the multiplier $m_\lambda(\xi)$, we have $(\lambda + \alpha)u, \nabla^2 u, \nabla p \in L^q(L^r_\omega)$ and

$$\|(\lambda + \alpha)u, \nabla^2 u, \nabla p\|_{L^q(L^r_\omega)} \leq \|(m\hat{f})^\vee\|_{L^q(L^r_\omega)}.$$

Thus the existence of a solution satisfying (4.1) is proved.

For the uniqueness of solutions let $(u, p) \in \big(W^{2;q,r}_\omega(\Omega) \cap W^{1;q,r}_{0,\omega}(\Omega)\big) \times \widehat{W}^{1;q,r}_\omega(\Omega)$ satisfy (R_λ) with $f = 0$. Fix $h \in L^{q'}(L^{r'}_{\omega'})$ arbitrarily and let $(v, z) \in \big(W^{2;q',r'}_{\omega'}(\Omega) \cap W^{1;q',r'}_{0,\omega'}(\Omega) \cap L^{q'}(L^{r'}_{\omega'})_\sigma\big) \times \widehat{W}^{1;q',r'}_{\omega'}(\Omega)$ be a solution to $(R_{\bar\lambda})$ with right-hand side h. Then using the denseness of $C^\infty_{0,\sigma}(\Omega)$ in $W^{1;q',r'}_{0,\omega}(\Omega) \cap L^{q'}(L^{r'}_{\omega'})_\sigma$ we get

$$0 = \langle \lambda u - \Delta u + \nabla p, v\rangle = \langle u, \bar\lambda v - \Delta v + \nabla z\rangle = \langle u, h\rangle_{L^q(L^r_\omega), L^{q'}(L^{r'}_{\omega'})}$$

yielding $u = 0$, and consequently, $\nabla p = 0$. Now the proof of Theorem 4.1 is complete. ■

Corollary 4.2 (Stokes' Operator and Stokes' Semigroup) *Let $1 < q, r < \infty$, $\omega \in A_r(\mathbb{R}^{n-1})$ and define the Stokes operator $A = A_{q,r;\omega}$ on Ω by*

$$D(A) = W^{2;q,r}_\omega(\Omega) \cap W^{1;q,r}_{0,\omega}(\Omega) \cap L^q(L^r_\omega)_\sigma \subset L^q(L^r_\omega)_\sigma,\ Au = -P_{q,r;\omega}\Delta u, \tag{4.6}$$

where $P_{q,r;\omega}$ is the Helmholtz projection in $L^q(\mathbb{R}; L^r_\omega(\Sigma))$ (see [30]). Then, for every $\varepsilon \in (\frac{\pi}{2}, \pi)$ and $\alpha \in (0, \alpha_0)$, $-\alpha + \Sigma_\varepsilon$ is contained in the resolvent set of $-A$, and the estimate

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(L^q(L^r_\omega)_\sigma)} \leq \frac{C}{|\lambda + \alpha|} \quad \forall \lambda \in -\alpha + \Sigma_\varepsilon \tag{4.7}$$

holds with an A_r-consistent constant $C = C(\Sigma, q, r, \alpha, \varepsilon, \mathcal{A}_r(\omega))$.

As a consequence, the Stokes operator generates a bounded analytic semigroup $\{e^{-tA_{q,r;\omega}}; t \geq 0\}$ on $L^q(L^r_\omega)_\sigma$ satisfying the estimate

$$\|e^{-tA_{q,r;\omega}}\|_{\mathcal{L}(L^q(L^r_\omega)_\sigma)} \leq C\, e^{-\alpha t} \quad \forall \alpha \in (0, \alpha_0), \forall t > 0 \tag{4.8}$$

with a constant $C = C(q, r, \alpha, \varepsilon, \Sigma, \mathcal{A}_r(\omega))$.

Proof: Defining the Stokes operator $A = A_{q,r;\omega}$ by (4.6), due to the Helmholtz decomposition of the space $L^q(L^r_\omega)$ on the cylinder Ω (see [30]), we see that for $F \in L^q(L^r_\omega)_\sigma$ the solvability of the equation

$$(\lambda + A)u = F \quad \text{in} \quad L^q(L^r_\omega)_\sigma \tag{4.9}$$

is equivalent to the solvability of (R_λ). By virtue of Theorem 4.1 for every $\lambda \in -\alpha + \Sigma_\varepsilon$ there exists a unique solution $u = (\lambda + A)^{-1}F \in D(A)$ to (4.9) satisfying the estimate

$$\|(\lambda + \alpha)u\|_{L^q(L^r_\omega)_\sigma} \leq C\|F\|_{L^q(L^r_\omega)_\sigma}$$

with $C = C(q, r, \alpha, \varepsilon, \Sigma, \mathcal{A}_r(\omega))$ independent of λ; hence (4.7) is proved. Then (4.8) is a direct consequence of (4.7) using semigroup theory. ∎

4.2 Maximal regularity

In this section we prove maximal L^p-regularity of the Stokes operator in weighted spaces $L^q(L^r_\omega)$.

Theorem 4.3 (Maximal Regularity) *Let $1 < p, q, r < \infty$ and $\omega \in A_r(\mathbb{R}^{n-1})$. Then the Stokes operator $A = A_{q,r;\omega}$ has maximal L^p-regularity in $L^q(L^r_\omega)_\sigma$. To be more precise, for each $f \in L^p(\mathbb{R}_+; L^q(L^r_\omega)_\sigma)$ the instationary system*

$$u_t + Au = f, \quad u(0) = 0 \tag{4.10}$$

has a unique solution $u \in W^{1,p}(\mathbb{R}_+; L^q(L^r_\omega)_\sigma) \cap L^p(\mathbb{R}_+; D(A))$ such that

$$\|u, u_t, Au\|_{L^p(\mathbb{R}_+;L^q(L^r_\omega)_\sigma)} \leq C\|f\|_{L^p(\mathbb{R}_+;L^q(L^r_\omega)_\sigma)}. \tag{4.11}$$

Analogously, for every $f \in L^p(\mathbb{R}_+; L^q(L^r_\omega))$, the instationary system

$$u_t - \Delta u + \nabla p = f, \quad \operatorname{div} u = 0, \quad u(0) = 0$$

has a unique solution $(u, \nabla p) \in \big(W^{1,p}(\mathbb{R}_+; L^q(L^r_\omega)_\sigma) \cap L^p(\mathbb{R}_+; D(A))\big) \times L^p(\mathbb{R}_+; L^q(L^r_\omega))$ satisfying the a priori estimate

$$\|u_t, u, \nabla u, \nabla^2 u, \nabla p\|_{L^p(\mathbb{R}_+;L^q(L^r_\omega))} \leq C\|f\|_{L^p(\mathbb{R}_+;L^q(L^r_\omega))}. \tag{4.12}$$

Moreover, if $e^{\alpha t} f \in L^p(\mathbb{R}_+; L^q(L^r_\omega)_\sigma)$ for some $\alpha \in (0, \alpha_0)$, then the solution u satisfies the estimate

$$\|e^{\alpha t}u, e^{\alpha t}u_t, e^{\alpha t}Au\|_{L^p(\mathbb{R}_+;L^q(L^r_\omega)_\sigma)} \leq C\|e^{\alpha t}f\|_{L^p(\mathbb{R}_+;L^q(L^r_\omega)_\sigma)}. \tag{4.13}$$

In each estimate $C = C(\Sigma, q, r, \mathcal{A}_r(\omega))$ and $C = C(\Sigma, q, r, \mathcal{A}_r(\omega), \alpha)$, respectively.

Remark 4.4 (1) We note that in (4.10) we may take nonzero initial values $u(0) = u_0$ in the real interpolation space $(L^q(L^r_\omega)_\sigma, D(A_{q,r;\omega}))_{1-1/p,p}$.

(2) By [7], Theorem 1.3, maximal regularity in $L^q(\Omega)$ of $cI + A_q$ with some $c > 0$, where A_q is the Stokes operator in $L^q(\Omega)$, will follow; this result is weaker than the particular case $q = r$ and $\omega \equiv 1$ in Theorem 4.3.

Proof of Theorem 4.3: Based on Theorem 2.18 we shall show that the operator family

$$\mathcal{T} = \{\lambda(\lambda + A_{q,r;\omega})^{-1} :\ \lambda \in \mathrm{i}\mathbb{R}\}$$

is $\mathcal{R}$-bounded in $\mathcal{L}(L^q(L^r_\omega))$. To this end, for $\xi \in \mathbb{R}^*$ and $\lambda \in \Sigma_\varepsilon, \varepsilon \in (\pi/2, \pi)$, let $m_\lambda(\xi) := \lambda a_1(\xi)$ where $a_1(\xi)$ is the solution operator for $(R_{\lambda,\xi})$ with $g = 0$ defined by (3.67). Then $\lambda(\lambda + A_{q,r;\omega})^{-1} f = (m_\lambda(\xi)\hat{f})^\vee$ for $f \in \mathcal{S}(\mathbb{R} : L^r_\omega(\Sigma)_\sigma)$. In view of Definition 2.7 and the denseness of $\mathcal{S}(\mathbb{R}; L^r_\omega(\Sigma)_\sigma)$ in $L^q(\mathbb{R}; L^r_\omega(\Sigma)_\sigma)$ we will prove that there is a constant $C > 0$ such that

$$\Big\| \sum_{i=1}^N \varepsilon_i (m_{\lambda_i} \hat{f}_i)^\vee \Big\|_{L^q(0,1;L^q(\mathbb{R}:L^r_\omega(\Sigma)))} \le C \Big\| \sum_{i=1}^N \varepsilon_i f_i \Big\|_{L^q(0,1;L^q(\mathbb{R}:L^r_\omega(\Sigma)))} \tag{4.14}$$

for any independent, symmetric and $\{-1,1\}$-valued random variables $(\varepsilon_i(s))$ defined on $(0,1)$, for all $(\lambda_i) \subset \mathrm{i}\mathbb{R}$ and $(f_i) \subset \mathcal{S}(\mathbb{R}; L^r_\omega(\Sigma)_\sigma)$. Without loss of generality we may assume that $(f_i) \subset Y := R_0 L^q(\mathbb{R} : L^r_\omega(\Sigma)_\sigma)$, since the Riesz projection R_0 is continuous in $L^q(\mathbb{R}; L^r_\omega(\Sigma)_\sigma)$, see Section 2.3, and

$$f_i(x', x_n) = (\chi_{[0,\infty)} \hat{f}_i(\xi))^\vee(x', x_n) + (\chi_{[0,\infty)} \hat{f}_i(-\xi))^\vee(x', -x_n).$$

Therefore, we shall show that $\mathcal{T}$ is $\mathcal{R}$-bounded in $\mathcal{L}(Y)$; note that, if $\operatorname{supp} \hat{f} \subset [0,\infty)$, then $\operatorname{supp}(m_\lambda \hat{f}) \subset [0,\infty)$ as well.

Obviously $m_\lambda(\xi) = m_\lambda(2^j) + \int_{2^j}^\xi m'_\lambda(\tau)\, d\tau$ for $\xi \in [2^j, 2^{j+1})$, $j \in \mathbb{Z}$, and $\big(m_\lambda(2^j)\widehat{\Delta_j f}\big)^\vee = m_\lambda(2^j)\Delta_j f$ for $f \in \mathcal{S}(\mathbb{R}; L^r_\omega(\Sigma)_\sigma)$. Furthermore,

$$\begin{aligned}
\Big(\int_{2^j}^\xi m'_\lambda(\tau)\, d\tau\, \widehat{\Delta_j f}(\xi)\Big)^\vee &= \Big(\int_{2^j}^{2^{j+1}} m'_\lambda(\tau)\chi_{[2^j,\xi)}(\tau)\widehat{\Delta_j f}(\xi)\, d\tau\Big)^\vee \\
&= \Big(\int_0^1 2^j m'_\lambda(2^j(1+t))\chi_{[2^j,\xi)}(2^j(1+t))\chi_{[2^j,2^{j+1})}(\xi)\hat{f}(\xi)\, dt\Big)^\vee \\
&= \int_0^1 2^j m'_\lambda(2^j(1+t)) B_{j,t}\Delta_j f\, dt,
\end{aligned}$$

where

$$B_{j,t} = R_{2^j(1+t),2^{j+1}}, \quad t \in (0,1); \tag{4.15}$$

we recall the notation $R_{a,b} = R_a - R_b$ and $R_a = \mathcal{F}^{-1}\chi_{[a,\infty)}\mathcal{F}$ for $a, b \in \mathbb{R}$, see (2.8). Thus we get

$$\begin{aligned}
\big(m_\lambda(\xi)\hat{f}(\xi)\big)^\vee &= \sum_{j\in\mathbb{Z}} \Big(\big(m_\lambda(2^j) + \int_{2^j}^\xi m'_\lambda(\tau)\, d\tau\big)\, \widehat{\Delta_j f}\Big)^\vee \\
&= \sum_{j\in\mathbb{Z}} \big(m_\lambda(2^j)\widehat{\Delta_j f}\big)^\vee + \sum_{j\in\mathbb{Z}} \Big(\int_{2^j}^\xi m'_\lambda(\tau)\, d\tau\, \widehat{\Delta_j f}\Big)^\vee \qquad (4.16) \\
&= \sum_{j\in\mathbb{Z}} m_\lambda(2^j)\Delta_j f + \sum_{j\in\mathbb{Z}} \int_0^1 2^j m'_\lambda(2^j(1+t)) B_{j,t}\Delta_j f\, dt.
\end{aligned}$$

First let us prove

$$\Big\|\sum_{i=1}^{N}\varepsilon_i(s)\sum_{j\in\mathbb{Z}}m_{\lambda_i}(2^j)\Delta_j f_i\Big\|_{L^q(0,1;Y)} \le C\Big\|\sum_{i=1}^{N}\varepsilon_i(s)f_i\Big\|_{L^q(0,1;Y)}. \tag{4.17}$$

Note that the operator $m_{\lambda_i}(2^j)$ commutes with Δ_j, $j\in\mathbb{Z}$; hence, for almost all $s\in(0,1)$, the sum $\sum_{i=1}^{N}\varepsilon_i(s)m_{\lambda_i}(2^j)\Delta_j f_i$ belongs to the range of Δ_j. Therefore, for any $l,k\in\mathbb{Z}$ we get by (2.7) that

$$\begin{aligned}
&\Big\|\sum_{i=1}^{N}\varepsilon_i\sum_{j=l}^{k}m_{\lambda_i}(2^j)\Delta_j f_i\Big\|_{L^q(0,1;Y)}\\
&\qquad=\Big(\int_0^1\Big\|\sum_{j=l}^{k}\sum_{i=1}^{N}\varepsilon_i(s)m_{\lambda_i}(2^j)\Delta_j f_i\Big\|_Y^q\,ds\Big)^{1/q}\\
&\qquad\le c_\Delta\Big(\int_0^1\int_0^1\Big\|\sum_{j=l}^{k}\varepsilon_j(\tau)\sum_{i=1}^{N}\varepsilon_i(s)m_{\lambda_i}(2^j)\Delta_j f_i\Big\|_Y^q\,d\tau\,ds\Big)^{1/q}\\
&\qquad=c_\Delta\Big\|\sum_{i=1}^{N}\sum_{j=l}^{k}\varepsilon_{ij}(s,\tau)m_{\lambda_i}(2^j)\Delta_j f_i\Big\|_{L^q((0,1)^2;Y)}
\end{aligned}\tag{4.18}$$

where $\varepsilon_{ij}(s,\tau)=\varepsilon_i(s)\varepsilon_j(\tau)$; note that $(\varepsilon_{ij})_{i,j\in\mathbb{Z}}$ is a sequence of independent, symmetric and $\{-1,1\}$-valued random variables defined on $(0,1)\times(0,1)$. Furthermore, due to Theorem 3.11, the operator family $\{m_\lambda(\xi)\,:\,\lambda\in\mathrm{i}\mathbb{R},\ \xi\in\mathbb{R}^*\}\subset\mathcal{L}(L^r_\omega(\Sigma))$ is uniformly bounded by an A_r-consistent constant, and hence it is $\mathcal{R}$-bounded by Theorem 2.14. Therefore, using Fubini's theorem and (2.7), we proceed in (4.18) as follows:

$$\begin{aligned}
&=c_\Delta\Big\|\sum_{i=1}^{N}\sum_{j=l}^{k}\varepsilon_{ij}(s,\tau)m_{\lambda_i}(2^j)\Delta_j f_i\Big\|_{L^q(\mathbb{R};L^q((0,1)^2;L^r_\omega(\Sigma)))}\\
&\le Cc_\Delta\Big\|\sum_{i=1}^{N}\sum_{j=l}^{k}\varepsilon_{ij}(s,\tau)\Delta_j f_i\Big\|_{L^q(\mathbb{R};L^q((0,1)^2;L^r_\omega(\Sigma)))}\\
&=Cc_\Delta\Big\|\sum_{i=1}^{N}\sum_{j=l}^{k}\varepsilon_{ij}(s,\tau)\Delta_j f_i\Big\|_{L^q((0,1)^2;Y)}\le Cc_\Delta^2\Big\|\sum_{i=1}^{N}\varepsilon_i\sum_{j=l}^{k}\Delta_j f_i\Big\|_{L^q(0,1;Y)}.
\end{aligned}\tag{4.19}$$

Since $\{\sum_{j=l}^{k}\Delta_j\,:\,l,k\in\mathbb{Z}\}$ is $\mathcal{R}$-bounded in $\mathcal{L}(Y)$ and (Δ_j) is a Schauder decomposition of Y, we see by Lebesgue's theorem that the right-hand side of (4.19) converges to 0 as either $l,k\to\infty$ or $l,k\to-\infty$. Thus, by (4.18), (4.19), the series $\sum_{i=1}^{N}\varepsilon_i(s)\sum_{j\in\mathbb{Z}}m_{\lambda_i}(2^j)\Delta_j f_i$ converges in $L^q(0,1;Y)$, and (4.17) holds.

Next let us show that

$$\Big\|\sum_{i=1}^{N}\varepsilon_i(s)\sum_{j\in\mathbb{Z}}\int_0^1 2^j m'_{\lambda_i}(2^j(1+t))B_{j,t}\Delta_j f_i\,dt\Big\|_{L^q(0,1;Y)}\le C\Big\|\sum_{i=1}^{N}\varepsilon_i(s)f_i\Big\|_{L^q(0,1;Y)}. \tag{4.20}$$

Using the same argument as in the proof of (4.17) and the $\mathcal{R}$-boundedness of the operator families $\{B_{j,t}: j\in\mathbb{Z}, t\in(0,1)\}\subset\mathcal{L}(Y)$ and $\{2^j(1+t)m'_\lambda(2^j(1+t)): \lambda\in \mathrm{i}\mathbb{R}, j\in\mathbb{Z}, t\in(0,1)\}\subset\mathcal{L}(L^r_\omega(\Sigma))$, see Corollary 3.12, we have

$$\begin{aligned}
&\Big\|\sum_{i=1}^N \varepsilon_i(s)\sum_{j=l}^k\int_0^1 2^j m'_{\lambda_i}(2^j(1+t))B_{j,t}\Delta_j f_i\,dt\Big\|_{L^q(0,1;Y)}\\
&\qquad\le \int_0^1\Big\|\sum_{i=1}^N\varepsilon_i(s)\sum_{j=l}^k 2^j m'_{\lambda_i}(2^j(1+t))B_{j,t}\Delta_j f_i\Big\|_{L^q(0,1;Y)}\,dt\\
&\qquad\le c_\Delta\int_0^1\Big\|\sum_{i=1}^N\sum_{j=l}^k\varepsilon_{ij}(s,\tau)2^j m'_{\lambda_i}(2^j(1+t))B_{j,t}\Delta_j f_i\Big\|_{L^q((0,1)^2;Y)}\,dt\\
&\qquad\le c_\Delta\int_0^1\Big\|\sum_{i=1}^N\sum_{j=l}^k\varepsilon_{ij}(s,\tau)2^j(1+t) m'_{\lambda_i}(2^j(1+t))\Delta_j f_i\Big\|_{L^q((0,1)^2;Y)}\,dt\\
&\qquad\le Cc_\Delta^2\Big\|\sum_{i=1}^N\varepsilon_i(s)\sum_{j=l}^k\Delta_j f_i\Big\|_{L^q((0,1);Y)}
\end{aligned}$$

for all $l,k\in\mathbb{Z}$. Thus (4.20) is proved.

By (4.17), (4.20) we conclude that the operator family $\mathcal{T}=\{\lambda(\lambda+A_{q,r;\omega})^{-1}: \lambda\in\mathrm{i}\mathbb{R}\}$ is $\mathcal{R}$-bounded in $\mathcal{L}(L^q(L^r_\omega))$. Then, by Theorem 2.18, for each $f\in L^p(\mathbb{R}_+;L^q(L^r_\omega)_\sigma), 1<p<\infty$, the mild solution u to the system

$$u_t + A_{q,r;\omega}u = f,\quad u(0)=0 \tag{4.21}$$

belongs to $L^p(\mathbb{R}_+;L^q(L^r_\omega)_\sigma)\cap L^p(\mathbb{R}_+;D(A_{q,r;\omega}))$ and satisfies the estimate

$$\|u_t, A_{q,r;\omega}u\|_{L^p(\mathbb{R}_+;L^q(L^r_\omega)_\sigma)}\le C\|f\|_{L^p(\mathbb{R}_+;L^q(L^r_\omega)_\sigma)}.$$

Furthermore, (4.7) with $\lambda=0$ implies that even u satisfies this inequality. If $f\in L^p(\mathbb{R}_+;L^q(L^r_\omega))$, let u be the solution of (4.21) with f replaced by Pf, where $P=P_{q,r;\omega}$ denotes the Helmholtz projection in $L^p(\mathbb{R}_+;L^q(L^r_\omega))$, and define p by $\nabla p=(I-P)(f-u_t+\Delta u)$. By (4.1) with $\lambda=0$ and the boundedness of P we get (4.12). Finally, assume $e^{\alpha t}f\in L^p(\mathbb{R}_+;L^q(L^r_\omega)_\sigma)$ for some $\alpha\in(0,\alpha_0)$ and let v be the solution of the system $v_t+(A-\alpha)v=e^{\alpha t}f$, $v(0)=0$. Obviously, replacing A by $A-\alpha$ in the previous arguments, v is easily seen to satisfy estimate (4.11). Then $u(t)=e^{-\alpha t}v(t)$ solves (4.21) and satisfies (4.13). In each case the constant C depends only on $\mathcal{A}_r(\omega)$ due to Remark 2.13.

The proof of Theorem 4.3 is complete. ■

5 Resolvent Estimate and H^∞-calculus; General Cylinders

In this chapter we consider the resolvent problem and H^∞-calculus of the Stokes operator A_q in the general unbounded cylinder Ω, see (1.2). Section 5.1 includes the investigation of dyadic Schauder decompositions of vector-valued homogeneous Sobolev spaces, in particular, in $L^q(\mathbb{R}; L^r_\omega(\Sigma))$ for $1 < q, r < \infty$ and Muckenhoupt weights ω. In Section 5.2 the generalized Stokes resolvent system with a prescribed divergence in an infinite straight cylinder with bounded cross-section is studied. Stokes resolvent estimates for general unbounded cylinders are considered in Section 5.3. In Section 5.4 we prove that the the Stokes operator A_q has a bounded H^∞-calculus.

5.1 Dyadic Schauder decompositions

Let X be a reflexive Banach space and $1 < q < \infty$. First let us consider *vector-valued homogeneous Sobolev spaces* $\widehat{W}^{1,q}(\mathbb{R}; X)$. Using the one-dimensional Fourier transform $\mathcal{F} \equiv \hat{}$ the space $\widehat{W}^{1,q}(\mathbb{R}; X)$ may be rewritten as

$$\widehat{W}^{1,q}(\mathbb{R}; X) = \{u \in L^1_{\text{loc}}(\mathbb{R}; X); \mathcal{F}^{-1}(\xi \hat{u}) \in L^q(\mathbb{R}; X)\}$$

with norm

$$\|u\|_{\widehat{W}^{1,q}(\mathbb{R};X)} = \|\mathcal{F}^{-1}(\xi \hat{u})\|_{L^q(\mathbb{R};X)},$$

where ξ is the phase variable of the Fourier transform $\mathcal{F}$. It is easy to see that $\widehat{W}^{1,q}(\mathbb{R}; X), 1 < q < \infty$, is a reflexive Banach space.

Let $\mathcal{D}(\mathbb{R}; X)$ be the space of all compactly supported and infinitely differentiable X-valued functions and $\mathcal{D}'(\mathbb{R}; X^*)$ the space of X^*-valued distributions.

Lemma 5.1 *$\mathcal{D}(\mathbb{R}; X)$ is dense in $\widehat{W}^{1,q}(\mathbb{R}; X)$ for each $q \in (1, \infty)$.*

Proof: Let $f \in (\widehat{W}^{1,q}(\mathbb{R}; X))^*$ vanish on $\mathcal{D}(\mathbb{R}; X)$. Then, due to the Hahn-Banach theorem, there exists $h \in L^{q'}(\mathbb{R}; X^*)$, $q' = q/(q-1)$, such that

$$0 = \langle f, \phi \rangle = \langle h, D\phi \rangle \quad \forall \phi \in \mathcal{D}(\mathbb{R}; X).$$

In particular, for all $\varphi \in \mathcal{D}(\mathbb{R})$ and $x \in X$, we have

$$0 = \langle h, D\varphi \cdot x \rangle = \langle\, \langle h(\cdot), x \rangle_{X^*,X}, D\varphi \rangle_{\mathcal{D}'(\mathbb{R}),\mathcal{D}(\mathbb{R})}$$

which together with $\langle h(\cdot), x \rangle_{X^*,X} \in L^{q'}(\mathbb{R})$ yields

$$\langle h(\cdot), x \rangle_{X^*,X} = \text{const} = 0 \quad \text{for all } x \in X.$$

Hence $h = 0$, and $f = 0$. ■

By the Hahn-Banach theorem, for every $f \in (\widehat{W}^{1,q}(\mathbb{R}; X))^*$ there is some $h \in L^{q'}(\mathbb{R}; X^*)$ such that

$$f = Dh \quad \text{and} \quad \|f\|_{(\widehat{W}^{1,q}(\mathbb{R};X))^*} = \|h\|_{L^{q'}(\mathbb{R};X^*)},$$

cf. Lemma 5.1. Conversely, it is obvious from Lemma 5.1 that, if $h \in L^{q'}(\mathbb{R}; X^*)$, then $Dh \in (\widehat{W}^{1,q}(\mathbb{R}; X))^*$. Thus we conclude that

$$\begin{aligned} (\widehat{W}^{1,q}(\mathbb{R}; X))^* &= \{f \in \mathcal{S}'(\mathbb{R}; X^*);\ \mathcal{F}^{-1}(\tfrac{1}{\xi}\hat{f}) \in L^{q'}(\mathbb{R}; X^*)\}, \\ \|f\|_{(\widehat{W}^{1,q}(\mathbb{R};X))^*} &= \|\mathcal{F}^{-1}(\tfrac{1}{\xi}\hat{f})\|_{L^{q'}(\mathbb{R};X^*)}. \end{aligned} \tag{5.1}$$

In consideration of (5.1) we shall denote the space $(\widehat{W}^{1,q}(\mathbb{R}; X))^*$ by $\widehat{W}^{-1,q'}(\mathbb{R}; X^*)$ for $1 < q < \infty$.

Let us recall the notation $R_a := \mathcal{F}^{-1}\chi_{[a,\infty)}\mathcal{F}, R_{a,b} = R_a - R_b$ for $a, b \in \mathbb{R}$ in (2.8) and the family of dyadic decompositions $\{\Delta_j : \ j \in \mathbb{Z}\}$ in (2.9). We note that $\{\Delta_j : \ j \in \mathbb{Z}\}$ is $\mathcal{R}$-bounded in $\mathcal{L}(L^q(\mathbb{R}; X))$ and an unconditional Schauder decomposition of $R_0 L^q(\mathbb{R}; X)$, the image of $L^q(\mathbb{R}; X)$ by the Riesz projection R_0, see Remark 2.11 (6). Furthermore, $\{\Delta_j : j \in \mathbb{Z}\}$ is an unconditional Schauder decomposition of both $R_0\widehat{W}^{1,q}(\mathbb{R}; X)$ and $R_0\widehat{W}^{-1,q}(\mathbb{R}; X)$ for each $q \in (1, \infty)$ since for every permutation σ of $\mathbb{N}$, every $l < k \in \mathbb{Z}$ and any $u \in R_0\widehat{W}^{1,q}(\mathbb{R}; X)$

$$\Big\|u - \sum_{j=l}^{k} \Delta_{\sigma(j)} u\Big\|_{\widehat{W}^{1,q}(\mathbb{R};X)} = \Big\|Du - \sum_{j=l}^{k} \Delta_{\sigma(j)} Du\Big\|_{L^q(\mathbb{R};X)},$$

as well as for any $v \in R_0\widehat{W}^{-1,q}(\mathbb{R}; X)$

$$\Big\|v - \sum_{j=l}^{k} \Delta_{\sigma(j)} v\Big\|_{\widehat{W}^{-1,q}(\mathbb{R};X)} = \Big\|\mathcal{F}^{-1}(\xi^{-1}\hat{v}) - \sum_{j=l}^{k} \Delta_{\sigma(j)}\mathcal{F}^{-1}(\xi^{-1}\hat{v})\Big\|_{L^q(\mathbb{R};X)}.$$

Lemma 5.2 *Let X be a* UMD *space, $1 < q < \infty$, and let $-\infty < a < b < \infty$.*

(1) If $g \in \widehat{W}^{-1,q}(\mathbb{R}; X)$, then $R_{a,b}g \in L^q(\mathbb{R}; X)$ and there exists a constant $c(q, X) > 0$ such that

$$\|R_{a,b}g\|_{L^q(\mathbb{R};X)} \le c(q, X) \max\{|a|, |b|\} \|R_{a,b}g\|_{\widehat{W}^{-1,q}(\mathbb{R};X)}.$$

In particular, if $a > 0$, then

$$\frac{1}{b\, c(q, X)} \|R_{a,b}g\|_{L^q(\mathbb{R};X)} \le \|R_{a,b}g\|_{\widehat{W}^{-1,q}(\mathbb{R};X)} \le \frac{c(q, X)}{a} \|R_{a,b}g\|_{L^q(\mathbb{R};X)}.$$

(2) There is a constant $c > 0$ such that for all $g \in L^q(\mathbb{R}; X)$ and for any $l \le k \in \mathbb{Z}$ the following two formulae hold:

$$c^{-1}\Big\| \sum_{j=l}^{k} 2^j \Delta_j g\Big\|_{L^q(\mathbb{R};X)} \le \Big\| \sum_{j=l}^{k} \Delta_j g\Big\|_{\widehat{W}^{1,q}(\mathbb{R};X)} \le c\Big\| \sum_{j=l}^{k} 2^j \Delta_j g\Big\|_{L^q(\mathbb{R};X)} \tag{5.2}$$

$$c^{-1}\Big\| \sum_{j=l}^{k} 2^{-j} \Delta_j g\Big\|_{L^q(\mathbb{R};X)} \le \Big\| \sum_{j=l}^{k} \Delta_j g\Big\|_{\widehat{W}^{-1,q}(\mathbb{R};X)} \le c\Big\| \sum_{j=l}^{k} 2^{-j} \Delta_j g\Big\|_{L^q(\mathbb{R};X)}. \tag{5.3}$$

Proof: (1) Let $m_1(\xi)$ be a continuously differentiable function on $\mathbb{R}$ such that $m_1(\xi)=\xi$ in (a,b) and

$$\sup_{\xi\in\mathbb{R}}\{|m_1(\xi)|,|\xi m_1'(\xi)|\}\le 2\max\{|a|,|b|\}.$$

Then, by [78], Proposition 3, m_1 is a Fourier multiplier in $L^q(\mathbb{R};X)$, and we get

$$\begin{aligned}\|R_{a,b}g\|_{L^q(\mathbb{R};X)} &= \|\mathcal{F}^{-1}(m_1(\xi)\xi^{-1}\chi_{[a,b)}\hat{g})\|_{L^q(\mathbb{R};X)}\\ &\le c(q,X)\max\{|a|,|b|\}\|R_{a,b}g\|_{\widehat{W}^{-1,q}(\mathbb{R};X)}.\end{aligned}$$

If $a>0$, we define a C^1-function $m_2(\xi)$ on $\mathbb{R}$ such that $m_2(\xi)=\frac{1}{\xi}$ in (a,b) and

$$\sup_{\xi\in\mathbb{R}}\{|m_2(\xi)|,|\xi m_2'(\xi)|\}\le\frac{2}{a}.$$

Then we get for $g\in L^q(\mathbb{R};X)$

$$\begin{aligned}\|R_{a,b}g\|_{\widehat{W}^{-1,q}(\mathbb{R};X)} &= \|\mathcal{F}^{-1}(\xi^{-1}\chi_{[a,b)}\hat{g})\|_{L^q(\mathbb{R};X)}\\ &= \|\mathcal{F}^{-1}(m_2(\xi)\chi_{[a,b)}\hat{g})\|_{L^q(\mathbb{R};X)}\\ &\le \tfrac{c(q,X)}{a}\|R_{a,b}g\|_{L^q(\mathbb{R};X)}.\end{aligned}$$

Thus (1) is proved.

(2) Define the functions m_1,m_2 by

$$m_1(\xi)=\sum_{j\in\mathbb{Z}}\frac{2^j}{\xi}\chi_{[2^j,2^{j+1})}(\xi),\quad m_2(\xi)=\sum_{j\in\mathbb{Z}}\frac{\xi}{2^j}\chi_{[2^j,2^{j+1})}(\xi).$$

Obviously $\sup_{j\in\mathbb{Z}}\mathrm{Var}(\chi_{[2^j,2^{j+1})}m_i)<\infty$ for $i=1,2$, where 'Var' means the total variation on $\mathbb{R}$. Note that for $i=1,2$,

$$m_i(\xi)=\sum_{j\in\mathbb{Z}}\chi_{[2^j,2^{j+1})}(\xi)m_i(\xi)\quad\forall\xi\in\mathbb{R}\quad\text{and}\quad m_i(\xi)=0\quad\text{for }\xi<0.$$

Then by [76], Theorem 3.2, $m_i, i=1,2$, is a Marcinkiewicz type multiplier in $L^q(\mathbb{R};X)$, that is, there is a constant $c>0$ satisfying

$$\|\mathcal{F}^{-1}(m_i\hat{f})\|_{L^q(\mathbb{R};X)}\le c\|f\|_{L^q(\mathbb{R};X)}\quad\text{for all } f\in L^q(\mathbb{R};X).$$

Consequently, we get for each $g\in L^q(\mathbb{R};X)$

$$\begin{aligned}\|\textstyle\sum_{j=l}^k 2^j\Delta_j g\|_{L^q(\mathbb{R};X)} &= \|\mathcal{F}^{-1}\big(\sum_{j=l}^k\frac{2^j}{\xi}\chi_{[2^j,2^{j+1})}(\xi)\widehat{Dg}(\xi)\big)\|_{L^q(\mathbb{R};X)}\\ &= \|\mathcal{F}^{-1}\big(m_1\mathcal{F}(D(\textstyle\sum_{j=l}^k\Delta_j g))\big)\|_{L^q(\mathbb{R};X)}\\ &\le c\|\textstyle\sum_{j=l}^k\Delta_j g\|_{\widehat{W}^{1,q}(\mathbb{R};X)}.\end{aligned}$$

The second inequality of (5.8) is proved using the multiplier m_2, that is, we have

$$\begin{aligned}\|\sum_{j=l}^k \Delta_j g\|_{\widehat{W}^{1,q}(\mathbb{R};X)} &= \|\sum_{j=l}^k \mathcal{F}^{-1}\big(\xi\,\chi_{[2^j,2^{j+1})}(\xi)\hat{g}(\xi)\big)\|_{L^q(\mathbb{R};X)}\\ &= \|\mathcal{F}^{-1}\big(m_2\mathcal{F}(\sum_{j=l}^k 2^j\Delta_j g)\big)\|_{L^q(\mathbb{R};X)}\\ &\leq c\|\sum_{j=l}^k 2^j\Delta_j g\|_{L^q(\mathbb{R};X)}.\end{aligned}$$

The formula (5.9) is proved similarly. ∎

Lemma 5.3 *Let $(H, (\cdot,\cdot), \|\cdot\|_H)$ be a Hilbert space and let $1 < q < \infty$. Then there is a constant $c > 0$ such that for all $u_j = \Delta_j u_j \in L^q(\mathbb{R}; H)$ the inequalities*

$$\frac{1}{c}\Big\|\Big(\sum_{j=l}^k \|u_j\|_H^2\Big)^{1/2}\Big\|_{q;\mathbb{R}} \leq \Big\|\sum_{j=l}^k u_j\Big\|_{L^q(\mathbb{R};H)} \leq c\Big\|\Big(\sum_{j=l}^k \|u_j\|_H^2\Big)^{1/2}\Big\|_{q;\mathbb{R}} \tag{5.4}$$

hold for all $l < k \in \mathbb{Z}$.

Proof: Choose a sequence $(\varepsilon_j(s))$ of $\{-1,1\}-$valued symmetric, independent random variables on $[0,1]$. Then by (2.7), Fubini's Theorem and Kahane's inequality (5.3)

$$\begin{aligned}\|\sum_{j=l}^k x_j\|_{L^q(\mathbb{R};H)} &\leq c_\Delta\|\sum_{j=l}^k \varepsilon_j(s)x_j\|_{L^q(0,1;L^q(\mathbb{R};H))}\\ &= c_\Delta\|\sum_{j=l}^k \varepsilon_j(s)x_j\|_{L^q(\mathbb{R};L^q(0,1;H))}\\ &\leq c_\Delta\cdot c\|\sum_{j=l}^k \varepsilon_j(s)x_j\|_{L^q(\mathbb{R};L^2(0,1;H))}.\end{aligned} \tag{5.5}$$

Since $\int_0^1 \varepsilon_j(s)\varepsilon_i(s)\,ds = \delta_{ji}$ by the assumption on $(\varepsilon_j(s))$, we get due to the Hilbert space structure of H

$$\|\sum_{j=l}^k \varepsilon_j(s)x_j\|_{L^2(0,1;H)} = \big(\sum_{j=l}^k \|x_j\|_H^2\big)^{1/2}.$$

Therefore (5.5) leads to the estimate

$$\|\sum_{j=l}^k x_j\|_{L^q(\mathbb{R};H)} \leq c\|\big(\sum_{j=l}^k \|x_j\|_H^2\big)^{1/2}\|_{q;\mathbb{R}}. \tag{5.6}$$

Since in (5.5) the reversed inequality holds as well, (5.4) is proved. ∎

To generalize Lemma 5.3 to L^r-spaces, $r \neq 2$, we recall a crucial technical lemma from harmonic analysis ([38]).

Lemma 5.4 *Let $1 < p < r < \infty, \dfrac{1}{s} = 1 - \dfrac{p}{r}$ and $\omega \in A_r$. Then for every nonnegative function $u \in L^s_\omega(\Sigma)$ there is a nonnegative function $v \in L^s_\omega(\mathbb{R}^{n-1})$ such that*

(1) $u(x') \le v(x')$ for a.a. $x' \in \Sigma$.

(2) $\|v\|_{s,\omega;\mathbb{R}^{n-1}} \le 2\|u\|_{s,\omega;\Sigma}$.

(3) $\omega v \in A_p$ and $\mathcal{A}_p(\omega v) \le c$ with $c = c(\mathcal{A}_r(\omega)) > 0$ depending only on the A_r-constant of ω and independent of u, v.

If the function u above has a parameter τ running in a Lebesgue measurable set E of $\mathbb{R}^k, k \in \mathbb{N}$, and is Lebesgue measurable with respect to $(x', \tau) \in \Sigma \times E$, then the function v is also Lebesgue measurable with respect to $(x', \tau) \in \mathbb{R}^{n-1} \times E$.

Proof: We extend u onto $\mathbb{R}^{n-1}$ by 0 and again denote it by u. Then the assertion is a particular case of [38], Ch. IV, Lemma 5.18. Checking details of its proof, one can see that the constant in (2) may be taken as 2, cf. (5.7) below.

Let u have a parameter $\tau \in E$. By the proof of [38], Ch. IV, Lemma 5.18 the function v may be taken as

$$v(\cdot, \tau) = \sum_{j=0}^{\infty} (2\|S\|)^{-j} S^j u(\cdot, \tau), \tag{5.7}$$

where $Su = M(|u|\omega) \cdot \omega^{-1}$ with $M(|u|\omega)$ the Hardy-Littlewood maximal function of $|u|\omega$ on $\mathbb{R}^{n-1}$ and $\|S\|$ is the norm of the sublinear operator S in $L^s_\omega(\mathbb{R}^{n-1})$. Looking into the structure of the Hardy-Littlewood maximal function, it is seen that $Su(\cdot, \tau)$ is Lebesgue measurable with respect to $(x', \tau) \in \mathbb{R}^{n-1} \times E$, and hence each summand of the series in (5.7) is Lebesgue measurable with respect to (x', τ) as well. Then the function v as a limit of an increasing sequence of nonnegative measurable functions on $\mathbb{R}^{n-1} \times E$ is Lebesgue measurable on $\mathbb{R}^{n-1} \times E$.

The proof of the lemma is complete. ∎

Lemma 5.5 *Let $1 < q < \infty$, $2 < r < \infty$, $\frac{1}{s} = 1 - \frac{2}{r}$ and $\omega \in A_r$. Then there exist constants $C_1 = C_1(\mathcal{A}_r(\omega)) > 0$ and $C_2 = C_2(q, r) > 0$ independent of ω such that for $l, k \in \mathbb{Z}$, $l \le k$, and for each finite sequence $u_j = \Delta_j u_j \in L^q(\mathbb{R}; L^r_\omega(\Sigma)), j = l, \dots, k$, there is some measurable function v on $\mathbb{R}^n$ satisfying $v(\cdot, x_n) \in L^s_\omega(\mathbb{R}^{n-1})$ for a.a. $x_n \in \mathbb{R}$ and*

$$\begin{aligned} &\|v(\cdot, x_n)\|_{s,\omega} \le 2, \;\; \omega v(\cdot, x_n) \in A_2(\mathbb{R}^{n-1}) \;\; \text{and } \mathcal{A}_2(\omega v(\cdot, x_n)) \le C_1, \\ &\Big\| \sum_{j=l}^{k} u_j \Big\|_{L^q(\mathbb{R}; L^r_\omega(\Sigma))} \le C_2 c_\Delta \Big\| \Big(\sum_{j=l}^{k} \|u_j(\cdot, x_n)\|^2_{2,\omega v(x_n)} \Big)^{1/2} \Big\|_{L^q(\mathbb{R})}. \end{aligned} \tag{5.8}$$

Moreover, for all sequences $v_j = \Delta_j v_j \in L^q(\mathbb{R}; L^r_\omega(\Sigma)), j = l, \dots, k$,

$$\Big\| \Big(\sum_{j=l}^{k} \|v_j(\cdot, x_n)\|^2_{2,\omega v(x_n)} \Big)^{1/2} \Big\|_{L^q(\mathbb{R})} \le C_2 c_\Delta \Big\| \sum_{j=l}^{k} v_j \Big\|_{L^q(\mathbb{R}; L^r_\omega(\Sigma))}, \tag{5.9}$$

where $c_\Delta = c_\Delta(q,r) > 0$ *is the constant in* (2.7) *for* $X = L^q(\mathbb{R}; L^r_\omega(\Sigma))$. *In particular,* (5.9) *holds for* $(u_j)_{j=l}^k$ *as well.*

Proof: Choose a sequence $(\varepsilon_j(s))$ of $\{-1,1\}-$valued symmetric, independent random variables on $[0,1]$. By (2.7), Fubini's theorem and Kahane's inequality (2.6)

$$\begin{aligned} \Big\| \sum_{j=l}^k u_j \Big\|_{L^q(\mathbb{R};L^r_\omega(\Sigma))} &\le c_\Delta \Big\| \sum_{j=l}^k \varepsilon_j u_j \Big\|_{L^q(0,1;L^q(\mathbb{R};L^r_\omega(\Sigma)))} \\ = c_\Delta \Big\| \sum_{j=l}^k \varepsilon_j u_j \Big\|_{L^q(\mathbb{R};L^q(0,1;L^r_\omega(\Sigma)))} &\le c \Big\| \sum_{j=l}^k \varepsilon_j u_j \Big\|_{L^q(\mathbb{R};L^r(0,1;L^r_\omega(\Sigma)))}, \end{aligned} \tag{5.10}$$

where $c_\Delta = c_\Delta(q,r), c = c(q,r) > 0$; note that for $X = L^q(\mathbb{R}; L^r_\omega(\Sigma)$ the constants c_Δ in (2.7) and c in (2.6) are independent of the weight ω, see Remark 2.11 (3), Remark 2.8 (2). Let us recall *Khintchine's inequality* for complex numbers a_j, i.e.,

$$K^{-1} \Big\| \sum_{j=1}^N \varepsilon_j a_j \Big\|_{L^p(0,1)} \le \Big(\sum_{j=1}^N |a_j|^2 \Big)^{1/2} \le K \Big\| \sum_{j=1}^N \varepsilon_j a_j \Big\|_{L^p(0,1)}, \ p \in [1,\infty), \tag{5.11}$$

where the constant $K = K(p)$ does not depend on the choice of the sequence of independent, symmetric and $\{-1,1\}$-valued random variables $(\varepsilon_j(\cdot))$ on $[0,1]$ and on (a_j). By Fubini's theorem and (5.11), for a.a. $x_n \in \mathbb{R}$ we get

$$\begin{aligned} \Big\| \sum_{j=l}^k \varepsilon_j u_j(\cdot, x_n) \Big\|_{L^r(0,1;L^r_\omega(\Sigma))} &= \Big(\int_\Sigma \int_0^1 \Big| \sum_{j=l}^k \varepsilon_j(s) u_j(x', x_n) \Big|^r \, ds \, \omega \, dx' \Big)^{1/r} \\ &\le K(r) \Big(\int_\Sigma \Big(\sum_{j=l}^k |u_j(\cdot, x_n)|^2 \Big)^{r/2} \omega(x') \, dx' \Big)^{1/r} \\ &= K(r) \Big\| \Big(\sum_{j=l}^k |u_j(\cdot, x_n)|^2 \Big)^{1/2} \Big\|_{r,\omega} = K(r) \Big\| \sum_{j=l}^k |u_j(\cdot, x_n)|^2 \omega^{1/s'} \Big\|_{s'}^{1/2}. \end{aligned} \tag{5.12}$$

For a.a. $x_n \in \mathbb{R}$ we have

$$\begin{aligned} \Big\| \sum_{j=l}^k |u_j(\cdot, x_n)|^2 \omega^{1/s'} \Big\|_{s'}^{1/2} &= \Big(\int_\Sigma \sum_{j=l}^k |u_j(\cdot, x_n)|^2 \omega^{1/s'} \tilde{u}(\cdot, x_n) \, dx' \Big)^{1/2} \\ &= \Big(\int_\Sigma \sum_{j=l}^k |u_j(\cdot, x_n)|^2 \omega u(\cdot, x_n) \, dx' \Big)^{1/2}, \end{aligned} \tag{5.13}$$

where $u(x_n) := \tilde{u}(\cdot, x_n)\omega^{-1/s}$ and, if $\sum_{j=l}^k |u_j(\cdot, x_n)|^2 \neq 0$,

$$\tilde{u}(\cdot, x_n) := \frac{\big(\sum_{j=l}^k |u_j(\cdot, x_n)|^2 \omega^{1/s'} \big)^{s'-1}}{\big\| \sum_{j=l}^k |u_j(\cdot, x_n)|^2 \omega^{1/s'} \big\|_{s'}^{s'-1}},$$

or if $\sum_{j=l}^k |u_j(\cdot,x_n)|^2 = 0$, then

$$\tilde u(\cdot,x_n) := |\Sigma|^{-1/s}.$$

Note that $\tilde u(x',x_n)\ge 0$ and $\tilde u(\cdot,x_n)\in L^s(\Sigma)$ with $\|\tilde u(\cdot,x_n)\|_{s;\Sigma}=1$, and hence, for a.a. $x_n\in\mathbb{R}$ we get that $u(x_n)\in L^s_\omega(\Sigma), \|u(x_n)\|_{s,\omega}=1$. Moreover the function u is Lebesgue measurable with respect to $(x',x_n)\in\Sigma\times\mathbb{R}$. Therefore, by Lemma 5.4 there is a Lebesgue measurable function v on $\mathbb{R}^n$ such that $v(x_n)=v(\cdot,x_n)\in L^s_\omega(\mathbb{R}^{n-1})$ and

$$\begin{aligned} &u(x',x_n)\le v(x',x_n) \quad \text{for a.a } x'\in\Sigma,\ \|v(x_n)\|_{s,\omega}\le 2,\\ &\omega v(x_n)\in A_2(\mathbb{R}^{n-1}) \quad \text{and}\ \ \mathcal{A}_2(\omega v(x_n))\le C, \end{aligned} \tag{5.14}$$

where the constant C in (5.14) depends only on the A_r-constant of ω and is independent of u,v; see Lemma 5.4. Therefore, by (5.10), (5.12) and (5.13) it follows that (5.8) holds with the function v chosen above and some constant $C=C(q,r)>0$.

Let $v_j=\Delta_j v_j\in L^q(\mathbb{R};L^r_\omega(\Sigma)), j=l,\dots,k$, be an arbitrary sequence. Then, by Hölder's inequality, (5.14), (5.11) and (2.6) we get for almost all $x_n\in\mathbb{R}$ that

$$\begin{aligned} \Big(\sum_{j=l}^k \|v_j(\cdot,x_n)\|^2_{2,\omega v(x_n)}\Big)^{1/2} &= \Big(\int_\Sigma \sum_{j=l}^k |v_j(x',x_n)|^2\omega(x')^{1/s'}\cdot v(x',x_n)\omega(x')^{1/s}\,dx'\Big)^{1/2}\\ &\le \Big\|\sum_{j=l}^k |v_j(\cdot,x_n)|^2\Big\|^{1/2}_{s',\omega}\|v(x_n)\|^{1/2}_{s,\omega} \qquad \le \sqrt{2}\Big\|\Big(\sum_{j=l}^k |v_j(\cdot,x_n)|^2\Big)^{1/2}\Big\|_{r,\omega}\\ &\le K(r)\sqrt{2}\Big\|\sum_{j=l}^k \varepsilon_j v_j(\cdot,x_n)\Big\|_{L^r(0,1;L^r_\omega(\Sigma))} \le c(q,r)\Big\|\sum_{j=l}^k \varepsilon_j v_j(\cdot,x_n)\Big\|_{L^q(0,1;L^r_\omega(\Sigma))}. \end{aligned}$$

Therefore, using a similar technique as in (5.10), and by Fubini's theorem we get (5.9). ■

5.2 Generalized Stokes resolvent system in a straight cylinder

In this section Ω is an infinite cylinder $\Sigma\times\mathbb{R}\subset\mathbb{R}^n$, $n\ge 3$, with a bounded cross-section $\Sigma\subset\mathbb{R}^{n-1}$ of $C^{1,1}$-class. We consider the *generalized Stokes resolvent system* with prescribed divergence in Ω:

$$\begin{aligned} \lambda u-\Delta u+\nabla p &= f \quad &&\text{in } \Omega\\ \operatorname{div} u &= g &&\text{in } \Omega\\ u &= 0 &&\text{on } \partial\Omega. \end{aligned} \tag{5.15}$$

Let $1<q,r<\infty$ and $\omega\in A_r(\mathbb{R}^{n-1})$. In Section 4.1 the estimate for the system (5.15) for $g=0$ was obtained in $L^q(\mathbb{R};L^r_\omega(\Sigma))$. In this section we analyze (5.15) in $L^q(\mathbb{R};L^r_\omega(\Sigma))$ for general $g\ne 0$. For an infinite cylinder $\Omega=\Sigma\times\mathbb{R}$ let us recall the notation for the weighted spaces $W^{l;q,r}_\omega(\Omega), l\in\mathbb{N}$, $W^{1;q,r}_{0,\omega}(\Omega), \widehat{W}^{1;q,r}_\omega(\Omega)$, $\widehat{W}^{-1;q,r}_\omega(\Omega)$, see Section 2.1. By the Hahn-Banach theorem it is seen that

$$\widehat{W}^{-1;q,r}_\omega(\Omega)=\widehat{W}^{-1,q}(\mathbb{R};L^r_\omega(\Sigma))+L^q(\mathbb{R};\widehat{W}^{-1,q}_\omega(\Sigma)). \tag{5.16}$$

Lemma 5.6 *Let $1 < q, r < \infty$ and $\omega \in A_r(\mathbb{R}^{n-1})$.*

(1) For $d > 1$ let

$$\Omega_d = \{(x', x_n) \in \Omega :\ |x_n| < d\}.$$

Then Poincaré's inequality

$$\|\varphi\|_{L^q(-d,d;\, L^r_\omega(\Sigma))} \le C\, d \|\nabla\varphi\|_{L^q(-d,d;\, L^r_\omega(\Sigma))} \tag{5.17}$$

holds with an A_r-consistent constant $C = C(\mathcal{A}_r(\omega), \Sigma) > 0$ for all $\varphi \in C^\infty(\bar\Omega_d)$ with $\int_{\Omega_d} \varphi\, dx = 0$.

(2) The set $C_0^\infty(\bar\Omega)$ is dense in $\widehat{W}^{1;q,r}_\omega(\Omega)$.

(3) The set $C_0^\infty(\mathbb{R}; W^{1,r}_\omega(\Sigma)) \cap \widehat{W}^{-1;q,r}_\omega(\Omega)$ is dense in the space $W^{1;q,r}_\omega(\Omega) \cap \widehat{W}^{-1;q,r}_\omega(\Omega)$.

Proof: (1) Let

$$\zeta(x_n) = \frac{1}{|\Sigma|} \int_\Sigma \varphi(x', x_n)\, dx', x_n \in (-d, d),$$

and define

$$\psi(x', x_n) = \varphi(x', x_n) - \zeta(x_n).$$

Obviously, $\int_{-d}^d \zeta(x_n)\, dx_n = 0$ and $\int_\Sigma \psi(x', x_n)\, dx' = 0$ for all $x_n \in (-d, d)$. Therefore, by Poincare's inequalities on Σ and on $(-d, d)$ we get

$$\begin{aligned}
\|\varphi\|_{L^q(-d,d;\, L^r_\omega(\Sigma))} &\le \|\psi\|_{L^q(-d,d;\, L^r_\omega(\Sigma))} + \|\zeta\|_{L^q(-d,d;\, L^r_\omega(\Sigma))} \\
&\le \left(\int_{-d}^d \|\psi(\cdot, x_n)\|^q_{r,\omega;\Sigma}\, dx_n \right)^{1/q} + \omega(\Sigma)^{1/r} \|\zeta\|_{L^q(-d,d)} \\
&\le C(\mathcal{A}_r(\omega), \Sigma) \left(\int_{-d}^d \|\nabla'\psi(\cdot, x_n)\|^q_{r,\omega;\Sigma}\, dx_n \right)^{1/q} + d c_1 \omega(\Sigma)^{1/r} \|\partial_n \zeta\|_{L^q(-d,d)}.
\end{aligned}$$

Note that $\nabla'\psi = \nabla'\varphi$ and, due to Hölder's inequality and $\omega(x)^{1/r}\omega'(x)^{1/r'} = 1$ for $x' \in \Sigma$,

$$\begin{aligned}
\omega(\Sigma)^{1/r} \|\partial_n \zeta\|_{L^q(-d,d)} &= \frac{\omega(\Sigma)^{1/r}}{|\Sigma|} \Big\| \int_\Sigma \partial_n \varphi(x', x_n)\, dx' \Big\|_{L^q(-d,d)} \\
&\le \frac{\omega(\Sigma)^{1/r} \omega'(\Sigma)^{1/r'}}{|\Sigma|} \|\partial_n \varphi\|_{L^q(-d,d;\, L^r_\omega(\Sigma))} \\
&\le c(\Sigma) \mathcal{A}_r(\omega) \|\partial_n \varphi\|_{L^q(-d,d;\, L^r_\omega(\Sigma))}.
\end{aligned}$$

Thus (5.17) is proved.

(2) Given $u \in \widehat{W}^{1;q,r}_\omega(\Omega)$ define $u_0(x_n) = \frac{1}{|\Sigma|} \int_\Sigma u(x', x_n) dx'$ where $|\Sigma|$ denotes the $(n-1)$-dimensional Lebesgue measure of Σ. Since $u_0 \in \widehat{W}^{1,q}(\mathbb{R}; \mathbb{R})$, we may apply Lemma 5.1 and assume that $u \in \widehat{W}^{1;q,r}_\omega(\Omega)$ has vanishing means on Σ for almost all $x_n \in \mathbb{R}$. Then by Poincaré's inequality applied to $u(\cdot, x_n)$ on Σ it is seen that u belongs to $W^{1;q,r}_\omega(\Omega)$. Hence u may be approximated in $\widehat{W}^{1;q,r}_\omega(\Omega)$ by elements of the space

$$\{v \in \widehat{W}^{1;q,r}(\Omega);\ \text{supp } v \text{ is compact in } \overline{\Omega}\};$$

for example, we may choose an approximate sequence $u_j(x',x_n)=h(\frac{x_n}{j})u(x',x_n)$, $j\in\mathbb{N}$, where $h\in C_0^\infty(\mathbb{R})$ satisfies $\operatorname{supp} h\subset[-2,2]$ and $h(x_n)=1$ for $|x_n|\le 1$. Then by a standard argument using mollifiers each u_j, $j\in\mathbb{N}$, can be approximated by elements of $C_0^\infty(\overline{\Omega})$ with respect to the norm $\|\cdot\|_{\widehat{W}^{1;q,r}(\Omega)}$.

(3) Let $\{\rho_\varepsilon\}_{\varepsilon>0}$ be a *one-dimensional* mollifier defined by $\rho_\varepsilon(x_n)=\frac{1}{\varepsilon}\rho(\frac{x_n}{\varepsilon})$, $\varepsilon>0$, with $\rho\in C_0^\infty(\mathbb{R})$ satisfying $\operatorname{supp}\rho\subset[-1,1]$ and $\int_{\mathbb{R}}\rho(x_n)\,dx_n=1$. In the subsequent proof, for a function f defined on Ω let $\rho_\varepsilon * f$ denote the convolution with respect to x_n, that is,

$$\rho_\varepsilon * f(x',x_n):=\int_{\mathbb{R}} f(x',x_n-y_n)\rho_\varepsilon(y_n)\,dy_n.$$

Further choose $\eta\in C_0^\infty(\mathbb{R})$ such that

$$\eta(x_n):=\begin{cases}1 & \text{for } |x_n|<1\\ 0 & \text{for } |x_n|\ge 2,\end{cases}$$

and let $\eta_j(x_n):=\eta(\frac{x_n}{j})$ for $j\in\mathbb{N}$.

Now, for $g\in W_\omega^{1;q,r}(\Omega)\cap\widehat{W}_\omega^{-1;q,r}(\Omega)$, define the functions $g_j,\bar{g}_j$, $j\in\mathbb{N}$, by $g_j(x):=\eta_j(x_n)g(x)$, $x\in\Omega$, and

$$\bar{g}_j(x):=\begin{cases} g_j(x)-\dfrac{1}{|\Omega_{2j}|}\displaystyle\int_{\Omega_{2j}} g_j\,dx & \text{for } x\in\Omega_{2j}\\[2ex] 0 & \text{otherwise,}\end{cases}$$

respectively. Further let $g_{j\varepsilon}:=\bar{g}_j*\rho_\varepsilon$ for $\varepsilon>0$.

Evidently, $g_{j\varepsilon}\in C_0^\infty(\mathbb{R};W_\omega^{1,r}(\Sigma))\subset W_\omega^{1;q,r}(\Omega)$. To prove $g_{j\varepsilon}\in\widehat{W}_\omega^{-1;q,r}(\Omega)$ note that $\operatorname{supp} g_{j\varepsilon}\subset\Omega_{2j+\varepsilon}$ and that $\int_\Omega g_{j\varepsilon}\,dx=0$ since $\int_\Omega \overline{g}_j\,dx=0$. Therefore, by (5.17), for $\varphi\in C_0^\infty(\overline{\Omega})$

$$\begin{aligned}\int_\Omega g_{j\varepsilon}\varphi\,dx &= \int_{\Omega_{2j+\varepsilon}} g_{j\varepsilon}\varphi\,dx = \int_{\Omega_{2j+\varepsilon}} g_{j\varepsilon}\bar{\varphi}\,dx\\ &\le \|g_{j\varepsilon}\|_{L^q(L^r_\omega)}\|\bar{\varphi}\|_{L^{q'}(-2j-\varepsilon,2j+\varepsilon;L^{r'}_{\omega'}(\Sigma))}\\ &\le c(2j+\varepsilon)\|g_{j\varepsilon}\|_{L^q(L^r_\omega)}\|\nabla\varphi\|_{L^{q'}(L^{r'}_{\omega'})},\end{aligned}$$

where $\bar{\varphi}=\varphi-\frac{1}{|\Omega_{2j+\varepsilon}|}\int_{\Omega_{2j+\varepsilon}}\varphi\,dx$ and $c=c(\mathcal{A}_r(\omega),\Sigma)>0$. Thus $g_{j\varepsilon}\in\widehat{W}_\omega^{-1;q,r}(\Omega)$.

Now we will show that the sequence $\{g_{j\varepsilon}\}$ with carefully chosen $\varepsilon=\varepsilon(j)$ converges to g in $W_\omega^{1;q,r}(\Omega)\cap\widehat{W}_\omega^{-1;q,r}(\Omega)$ as $j\to\infty$. First let us prove the convergence in $W_\omega^{1;q,r}(\Omega)$. Since $\operatorname{supp} g_j\subset\Omega_{2j}$, we obtain

$$g_{j\varepsilon}-g=(g*\rho_\varepsilon-g)+(g_j-g)*\rho_\varepsilon-\Big(\frac{1}{|\Omega_{2j}|}\int_{\Omega_{2j}} g_j\,dx\Big)\int_{-2j}^{2j}\rho_\varepsilon(x_n-y_n)\,dy_n\,. \quad (5.18)$$

Since $g\in\widehat{W}_\omega^{-1;q,r}(\Omega)$, by Hahn-Banach's theorem there is some $u\in L^q(L^r_\omega)$ such that

$$g=\operatorname{div} u,\quad u\cdot N|_{\partial\Omega}=0\quad\text{and}\quad \|u\|_{L^q(L^r_\omega)}=\|g\|_{\widehat{W}_\omega^{-1;q,r}(\Omega)},$$

where N is the outward normal vector to $\partial\Omega$. By elementary calculations we have

$$\begin{aligned}\Big|\int_{\Omega_{2j}} g_j\,dx\Big| &= \Big|\int_{\Omega_{2j}} \eta_j \mathrm{div}\, u\,dx\Big| = \Big|\int_{\Omega_{2j}} \nabla\eta_j\cdot u\,dx\Big| \\ &\le \frac{1}{j}\Big\|(\partial_n\eta)(\frac{x_n}{j})\Big\|_{L^{q'}(L^{r'}_{\omega'})}\|\chi_{j,2j}u\|_{L^q(L^r_\omega)} \\ &= c_1(q)j^{-1/q}\omega'(\Sigma)^{1/r'}\|\chi_{j,2j}u\|_{L^q(L^r_\omega)},\end{aligned}\tag{5.19}$$

where $\chi_{j,2j}$ is the characteristic function of the set $[-2j,-j]\cup[j,2j]$ and $c_1(q) = \big(\int_{-2}^{2}|\partial_n\eta(y_n)|^{q'}\,dy_n\big)^{1/q'}$. Further we get

$$\Big\|\int_{-2j}^{2j}\rho_\varepsilon(x_n-y_n)\,dy_n\Big\|_{W^{1;q,r}_\omega(\Omega)} = \omega(\Sigma)^{1/r}\Big\|\int_{-2j}^{2j}\rho_\varepsilon(x_n-y_n)\,dy_n\Big\|_{W^{1,q}(\mathbb{R})}.\tag{5.20}$$

Note that, if $0<\varepsilon<2j$,

$$\Big\|\int_{-2j}^{2j}\rho_\varepsilon(x_n-y_n)\,dy_n\Big\|_{L^q(-2j-\varepsilon,2j+\varepsilon)} \le (4j+2\varepsilon)^{1/q}\le 8^{1/q}j^{1/q}$$

and that

$$\begin{aligned}&\Big\|\frac{\partial}{\partial x_n}\int_{-2j}^{2j}\rho_\varepsilon(x_n-y_n)\,dy_n\Big\|_{L^q(-2j-\varepsilon,2j+\varepsilon)} \\ &\qquad = \|\rho_\varepsilon(x_n+2j)-\rho_\varepsilon(x_n-2j)\|_{L^q(-2j-\varepsilon,2j+\varepsilon)} \\ &\qquad \le 2\|\rho_\varepsilon\|_{L^q(\mathbb{R})} = c_2(q)\varepsilon^{-1/q'},\end{aligned}$$

where $c_2(q) = 2\|\rho\|_{L^q(-1,1)}$. Therefore, taking $\varepsilon=\varepsilon(j) := j^{-q'/q}$, it follows from (5.19), (5.20) that the $W^{1;q,r}_\omega(\Omega)$-norm of the third term of (5.18) is estimated by

$$\frac{c(q)\omega(\Sigma)^{1/r}\omega'(\Sigma)^{1/r'}}{|\Omega_{2j}|}\|\chi_{j,2j}\,u\|_{L^q(L^r_\omega)} \le \frac{c(q,\Sigma)\mathcal{A}_r(\omega)}{j}\|\chi_{j,2j}\,u\|_{L^q(L^r_\omega)}$$

which tends to 0 as $j\to\infty$.

Obviously $\|g*\rho_{\varepsilon(j)}-g\|_{W^{1;q,r}_\omega(\Omega)}\to 0$ and $\|(g_j-g)*\rho_{\varepsilon(j)}\|_{W^{1;q,r}_\omega(\Omega)}\to 0$ as $j\to\infty$. Summarizing the previous results we get that $\|g_{j\varepsilon(j)}-g\|_{W^{1;q,r}_\omega(\Omega)}\to 0$ as $j\to\infty$.

Next we will prove $\|g_{j\varepsilon(j)}-g\|_{\widehat{W}^{-1;q,r}_\omega(\Omega)}\to 0$ as $j\to\infty$. For $j\in\mathbb{N}$ define f_j on Ω by

$$f_j(x',x_n) = \begin{cases} u_n\,\partial_n\eta_j + \frac{1}{|\Omega_{2j}|}\int_{\Omega_{2j}} g_j\,dx, & |x_n|<2j \\ 0, & |x_n|\ge 2j.\end{cases}$$

Then $\bar g_j = \mathrm{div}\,(\eta_j u) - f_j$ and, using (5.19), we have

$$\begin{aligned}\|f_j\|_{L^q(L^r_\omega)} &\le \|u_n\,\partial_n\eta_j\|_{L^q(-2j,2j;\,L^r_\omega)} + \Big\|\frac{1}{|\Omega_{2j}|}\int_{\Omega_{2j}} g_j\,dx\Big\|_{L^q(-2j,2j;\,L^r_\omega)} \\ &\le \|\partial_n\eta_j\|_\infty\|\chi_{j,2j}\,u\|_{L^q(L^r_\omega)} + \frac{(4j)^{1/q}\omega(\Sigma)^{1/r}}{|\Omega_{2j}|}\Big|\int_{\Omega_{2j}} g_j\,dx\Big| \\ &\le \Big(\frac{c}{j}+\frac{c(q)\omega(\Sigma)^{1/r}\omega'(\Sigma)^{1/r'}}{j|\Sigma|}\Big)\|\chi_{j,2j}u\|_{L^q(L^r_\omega)} \\ &\le \frac{c(q)}{j}(1+\mathcal{A}_r(\omega))\|\chi_{j,2j}\,u\|_{L^q(L^r_\omega)}.\end{aligned}\tag{5.21}$$

Note that $\int_{\Omega_{2j}} f_j\,dx = 0$. Therefore, defining $\langle f_j, \varphi\rangle := \int_\Omega f_j \varphi\,dx$ for $\varphi \in C^\infty(\bar\Omega)$, we get by (5.17), (5.21) that

$$\begin{aligned} |\langle f_j,\varphi\rangle| &= \Big|\int_{\Omega_{2j}} f_j\varphi\,dx\Big| = \Big|\int_{\Omega_{2j}} f_j\bar\varphi\,dx\Big| \\ &\le \|f_j\|_{L^q(L^r_\omega)}\,\|\bar\varphi\|_{L^{q'}(-2j,2j;L^{r'}_{\omega'})} \\ &\le C(\mathcal{A}_r(\omega),\Sigma)\|\chi_{j,2j}\,u\|_{L^q(L^r_\omega)}\,\|\nabla\varphi\|_{L^{q'}(L^{r'}_{\omega'})}, \end{aligned}$$

where $\bar\varphi = \varphi - \frac{1}{|\Omega_{2j}|}\int_{\Omega_{2j}}\varphi\,dx$. Hence $\langle f_j,\cdot\rangle \in \widehat{W}^{-1;q,r}_\omega(\Omega)$ and

$$\|\langle f_j,\cdot\rangle\|_{\widehat{W}^{-1;q,r}_\omega(\Omega)} \le C(\mathcal{A}_r(\omega),\Sigma)\|\chi_{j,2j}\,u\|_{L^q(L^r_\omega)}.$$

By Hahn-Banach's theorem there exists some $w_j \in L^q(L^r_\omega)$ such that

$$\operatorname{div} w_j = f_j,\quad w_j\cdot N|_{\partial\Omega} = 0,\quad \text{and}\quad \|w_j\|_{L^q(L^r_\omega)} = \|\langle f_j,\cdot\rangle\|_{\widehat{W}^{-1;q,r}_\omega(\Omega)}.$$

Therefore, with $u_j := \eta_j u - w_j$, we get $\bar g_j = \operatorname{div} u_j$ and

$$\begin{aligned} &\|g_{j\varepsilon(j)} - g\|_{\widehat{W}^{-1;q,r}_\omega(\Omega)} = \|\operatorname{div}(u - u_j * \rho_{\varepsilon(j)})\|_{\widehat{W}^{-1;q,r}_\omega(\Omega)} \\ &\qquad\le \|u - u_j*\rho_{\varepsilon(j)}\|_{L^q(L^r_\omega)} \\ &\qquad\le \|u - u*\rho_{\varepsilon(j)}\|_{L^q(L^r_\omega)} + \|u-\eta_j u\|_{L^q(L^r_\omega)} + \|w_j\|_{L^q(L^r_\omega)} \to 0 \end{aligned}$$

as $j\to\infty$.

The proof of this lemma is complete. ■

Now we are in a position to prove the main theorem of this section.

Theorem 5.7 *Let $\Sigma \subset \mathbb{R}^{n-1}$ be a bounded domain of $C^{1,1}$-class and let $\alpha_0 > 0$ be the smallest eigenvalue of the Dirichlet Laplacian in Σ. Moreover, let $1<q<\infty$, $2\le r<\infty$, $\omega \in A_r(\mathbb{R}^{n-1})$, $\alpha\in(0,\alpha_0)$ and let $\lambda \in -\alpha+\Sigma_\varepsilon$, $\varepsilon\in(\pi/2,\pi)$. Then, for every $f\in L^q(\mathbb{R};L^r_\omega(\Sigma))^n$, $g\in W^{1;q,r}_\omega(\Omega)\cap \widehat{W}^{-1;q,r}_\omega(\Omega)$ there exists a unique solution*

$$(u,p) \in \big(W^{2;q,r}_\omega(\Omega)^n \cap W^{1;q,r}_{0,\omega}(\Omega)^n\big)\times \widehat{W}^{1;q,r}_\omega(\Omega)$$

to (5.15) *satisfying the estimate*

$$\begin{aligned} &\|(\lambda+\alpha)u,\nabla^2 u,\nabla p\|_{L^q(L^r_\omega)} \\ &\qquad\le C\big(\|f\|_{L^q(L^r_\omega)} + \|g\|_{W^{1;q,r}_\omega(\Omega)} + (|\lambda|+1)\|g\|_{\widehat{W}^{-1;q,r}_\omega(\Omega)}\big) \end{aligned} \tag{5.22}$$

with an A_r-consistent constant $C = C(q,r,\alpha,\varepsilon,\Sigma,\mathcal{A}_r(\omega))$ independent of λ.

Proof: For the special case $g=0$ this theorem was treated in Theorem 4.1. Therefore, we shall consider only the case $f=0$ and assume, due to Lemma 5.6 (3), that $g\in C_0^\infty(\mathbb{R};W^{1,r}_\omega(\Sigma))\cap\widehat{W}^{-1;q,r}_\omega(\Omega)$.

Let a_2, b_2 be the operator-valued functions defined by (3.68). Then, by Theorem 3.11 and Corollary 3.12, the operator $M(\xi): W^{1,r}_\omega(\Sigma)\to L^r_\omega(\Sigma)$, $\xi\in\mathbb{R}^*$, defined by

$$M(\xi)g := \big((\lambda+\alpha)a_2(\xi)g, \xi^2 a_2(\xi)g, \xi\nabla' a_2(\xi)g, \nabla'^2 a_2(\xi)g, \xi b_2(\xi)g, \nabla' b_2(\xi)g\big)$$

is Frechét differentiable and, for all $\lambda \in -\alpha + \Sigma_\varepsilon, \varepsilon \in (\pi/2, \pi)$ and $\xi \in \mathbb{R}^*$,

$$\|M(\xi)g, \xi M'(\xi)g\|_{r,\omega,\Sigma} \le c\big(\|\nabla' g, g, \xi g\|_{r,\omega;\Sigma} + (|\lambda|+1)\|g; L^r_{(m),\omega} + L^r_{\omega,1/\xi}\|_0\big), \quad (5.23)$$

where $c = c(r, \alpha, \varepsilon, \Sigma, \mathcal{A}_r(\omega)) > 0$.

Obviously $(u,p) = \big((a_2(\xi)\hat{g}(\xi))^\vee, (b_2(\xi)\hat{g}(\xi))^\vee\big)$ solves (5.15) with right-hand side $(0,g)$ in the sense of distributions. Therefore, to prove (5.22) it is enough to show that

$$\|\big(M(\xi)\hat{g}(\xi)\big)^\vee\|_{L^q(\mathbb{R};L^r_\omega(\Sigma))} \le C\big(\|g\|_{W^{1;q,r}_\omega(\Omega)} + (|\lambda|+1)\|g\|_{\widehat{W}^{-1;q,r}_\omega(\Omega)}\big) \qquad (5.24)$$

with an A_r-consistent constant $C = C(q, r, \alpha, \varepsilon, \Sigma, \mathcal{A}_r(\omega))$ independent of λ. We may assume without loss of generality that $\operatorname{supp} \hat{g} \subset [0,\infty)$ due to the relation

$$\begin{aligned} g(x', x_n) &= (\chi_{[0,\infty)}\hat{g}(\xi))^\vee(x', x_n) + (\chi_{(-\infty,0]}\hat{g}(\xi))^\vee(x', x_n) \\ &= (\chi_{[0,\infty)}\hat{g}(\xi))^\vee(x', x_n) + (\chi_{[0,\infty)}\hat{g}(-\xi))^\vee(x', -x_n) \end{aligned}$$

and due to the linearity of the problem (5.15). For notational convenience, we introduce the space

$$\begin{aligned} \mathcal{X} &= W^{1;q,r}_\omega(\Omega) \cap \widehat{W}^{-1;q,r}_\omega(\Omega) \\ &= \big(W^{1,q}(\mathbb{R}; L^r_\omega(\Sigma)) \cap L^q(\mathbb{R}; W^{1,r}_\omega(\Sigma))\big) \cap \big(\widehat{W}^{-1,q}(\mathbb{R}; L^r_\omega(\Sigma)) + L^q(\mathbb{R}; \widehat{W}^{-1,r}_\omega(\Sigma))\big). \end{aligned}$$

As mentioned in Section 5.1 the operator family $\{\Delta_j : j \in \mathbb{Z}\}$ defined by (2.9) is an unconditional Schauder decomposition of $R_0\mathcal{X}$, the image of $\mathcal{X}$ by the Riesz projection R_0; hence $g = \sum_{j\in\mathbb{Z}} \Delta_j g$ in $\mathcal{X}$.

Note that $M(\xi) = M(2^j) + \int_{2^j}^{\xi} M'(\tau)\,d\tau$ for $\xi \in [2^j, 2^{j+1}), j \in \mathbb{Z}$, and that obviously $(M(2^j)\widehat{\Delta_j g})^\vee = M(2^j)\Delta_j g$; furthermore,

$$\begin{aligned} \Big(\int_{2^j}^{\xi} M'(\tau)\,d\tau\, \widehat{\Delta_j g}(\xi)\Big)^\vee &= \Big(\int_{2^j}^{2^{j+1}} M'(\tau)\chi_{[2^j,\xi)}(\tau)\widehat{\Delta_j g}(\xi)\,d\tau\Big)^\vee \\ &= \Big(\int_0^1 2^j M'(2^j(1+t))\chi_{[2^j,\xi)}(2^j(1+t))\chi_{[2^j,2^{j+1})}(\xi)\hat{g}(\xi)\,dt\Big)^\vee \\ &= \int_0^1 2^j M'(2^j(1+t)) B_{j,t}\Delta_j g\,dt, \end{aligned}$$

where $B_{j,t} := R_{2^j(1+t),2^{j+1}}$, see (4.15). Thus we get

$$\begin{aligned} \big(M(\xi)\hat{g}(\xi)\big)^\vee &= \Big(\sum_{j\in\mathbb{Z}} \chi_{[2^j,2^{j+1})}(\xi) M(\xi)\widehat{\Delta_j g}\Big)^\vee \\ &= \sum_{j\in\mathbb{Z}} \Big((M(2^j) + \int_{2^j}^{\xi} M'(\tau)\,d\tau)\,\widehat{\Delta_j g}\Big)^\vee \\ &= \sum_{j\in\mathbb{Z}} M(2^j)\Delta_j g + \sum_{j\in\mathbb{Z}} \int_0^1 2^j M'(2^j(1+t)) B_{j,t}\Delta_j g\,dt. \end{aligned} \qquad (5.25)$$

First let $2 < r < \infty$. To estimate the first term on the right-hand side of (5.25) in the norm of $L^q(\mathbb{R}; L^r_\omega(\Sigma))$, note that for each $j \in \mathbb{Z}$ the operator $M(2^j)$ commutes with Δ_j and that $\{\Delta_j; j \in \mathbb{Z}\}$ is a Schauder decomposition of $R_0 L^q(\mathbb{R}; L^r_\omega(\Sigma))$. Then, by Lemma 5.5, for a.a. $x_n \in \mathbb{R}$ and for any $l, k \in \mathbb{Z}$ there is some $v(x_n) \in L^s_\omega(\mathbb{R}^{n-1})$ depending on $u_j = M(2^j)\Delta_j g$, $j = l, \ldots, k$, such that (5.8), (5.9) are satisfied with $(u_j)_{j=l}^k$. Therefore, in view of (5.23), we get

$$\begin{aligned}
\Big\| \sum_{j=l}^{k} M(2^j)\Delta_j g \Big\|_{L^q(\mathbb{R}; L^r_\omega(\Sigma))} &\le c \Big\| \Big(\sum_{j=l}^{k} \|M(2^j)\Delta_j g\|^2_{2;\omega v(x_n)} \Big)^{1/2} \Big\|_{q;\mathbb{R}} \\
&\le c \Big\{ \Big\| \Big(\sum_{j=l}^{k} \|\Delta_j g\|^2_{W^{1,2}_{\omega v(x_n)}(\Sigma)} \Big)^{1/2} \Big\|_{q,\mathbb{R}} + \Big\| \Big(\sum_{j=l}^{k} 2^{2j} \|\Delta_j g\|^2_{2;\omega v(x_n)} \Big)^{1/2} \Big\|_{q,\mathbb{R}} \\
&\quad + (|\lambda| + 1) \Big\| \Big(\sum_{j=l}^{k} \|\Delta_j g; L^2_{(m),\omega v(x_n)} + L^2_{\omega v(x_n),1/2^j}\|^2_0 \Big)^{1/2} \Big\|_{q,\mathbb{R}} \Big\}
\end{aligned} \tag{5.26}$$

with $c = c(q, r, \alpha, \varepsilon, \Sigma, \mathcal{A}_r(\omega)) > 0$ independent of $l, k \in \mathbb{Z}$.

Now let us estimate each term on the right-hand side of (5.26). By (5.9) we get

$$\Big\| \Big(\sum_{j=l}^{k} \|\Delta_j g\|^2_{W^{1,2}_{\omega v(x_n)}(\Sigma)} \Big)^{1/2} \Big\|_{q,\mathbb{R}} \le c(q, r) \Big\| \sum_{j=l}^{k} \Delta_j g \Big\|_{L^q(\mathbb{R}; W^{1,r}_\omega(\Sigma))}; \tag{5.27}$$

note that Δ_j is an operator with respect to the variable x_n. By analogy, exploiting Lemma 5.2 (2),

$$\begin{aligned}
\big\| \big(\textstyle\sum_{j=l}^k 2^{2j} \|\Delta_j g\|^2_{2,\omega v(x_n)} \big)^{1/2} \big\|_{q,\mathbb{R}} &\le c(q, r) \big\| \textstyle\sum_{j=l}^k 2^j \Delta_j g \big\|_{L^q(\mathbb{R}; L^r_\omega(\Sigma))} \\
&\le c(q, r) \big\| \textstyle\sum_{j=l}^k \Delta_j g \big\|_{\widehat{W}^{1,q}(\mathbb{R}; L^r_\omega(\Sigma))}.
\end{aligned} \tag{5.28}$$

In order to get an estimate of the last term on the right-hand side of (5.26), let

$$\sum_{j=l}^{k} \Delta_j g = g_0 + g_1, \quad g_0 \in L^q(\mathbb{R}; \widehat{W}^{-1,r}_\omega(\Sigma)),\ g_1 \in \widehat{W}^{-1,q}(\mathbb{R}; L^r_\omega(\Sigma)),$$

be any splitting of $\sum_{j=l}^k \Delta_j g$. Due to the properties of Δ_j we see that $\Delta_j g = \Delta_j g_0 + \Delta_j g_1$ for all $j = l, \ldots, k$, and that, by Lemma 5.2 (1), $\Delta_j g_1 \in L^q(\mathbb{R}; L^r_\omega(\Sigma))$ and consequently even $\Delta_j g_0 \in L^q(\mathbb{R}; \widehat{W}^{-1,r}_\omega(\Sigma) \cap L^r_\omega(\Sigma)) = L^q(\mathbb{R}; L^r_{(m),\omega}(\Sigma))$. Furthermore, by (5.8) and Hölder's inequality it is easily proved that for a.a $x_n \in \mathbb{R}$

$$L^r_\omega(\Sigma) \hookrightarrow L^2_{\omega v(x_n)}(\Sigma), \quad \|\varphi\|_{2,\omega v(x_n)} \le \|\varphi\|_{r,\omega} \|v(x_n)\|^{1/2}_{s,\omega} \le \sqrt{2} \|\varphi\|_{r,\omega} \tag{5.29}$$

for all $\varphi \in L^r_\omega(\Sigma)$, and hence

$$\widehat{W}^{-1,r}_\omega(\Sigma) \hookrightarrow \widehat{W}^{-1,2}_{\omega v(x_n)}(\Sigma), \quad \|h\|_{-1,2,\omega v(x_n)} \le \sqrt{2} \|h\|_{-1,r,\omega} \tag{5.30}$$

for all $h \in \widehat{W}^{-1,r}_{\omega}(\Sigma)$. By the triangle inequality,

$$\Big\|\Big(\sum_{j=l}^{k}\|\Delta_j g; L^2_{(m),\omega v(x_n)}+L^2_{\omega v(x_n),1/2^j}\|_0^2\Big)^{1/2}\Big\|_{q,\mathbb{R}}$$
$$\leq \Big\|\Big(\sum_{j=l}^{k}\|\Delta_j g_0\|^2_{-1,2,\omega v(x_n)}\Big)^{1/2}\Big\|_{q,\mathbb{R}} + \Big\|\Big(\sum_{j=l}^{k}2^{-2j}\|\Delta_j g_1\|^2_{2,\omega v(x_n)}\Big)^{1/2}\Big\|_{q,\mathbb{R}}$$

Then using the Hilbert space structure of $\widehat{W}^{-1,2}_{\omega v(x_n)}(\Sigma)$ and the properties of any independent symmetric $\{-1,1\}$-valued random variables $(\varepsilon_j(\cdot))$ on $(0,1)$ as well as (5.30), Kahane's inequality (2.6), Fubini's theorem and (2.7) we get that

$$\begin{aligned}
\Big\|\Big(\sum_{j=l}^{k}\|\Delta_j g_0\|^2_{-1,2,\omega v(x_n)}\Big)^{1/2}\Big\|_{q,\mathbb{R}} &= \Big\|\Big\|\sum_{j=l}^{k}\varepsilon_j(s)\Delta_j g_0\Big\|_{L^2(0,1;\widehat{W}^{-1,2}_{\omega v(x_n)}(\Sigma))}\Big\|_{q,\mathbb{R}}\\
&\leq \sqrt{2}\Big\|\Big\|\sum_{j=l}^{k}\varepsilon_j(s)\Delta_j g_0\Big\|_{L^2(0,1;\widehat{W}^{-1,r}_{\omega}(\Sigma))}\Big\|_{q,\mathbb{R}}\\
&\leq c(q,r)\Big\|\Big\|\sum_{j=l}^{k}\varepsilon_j(s)\Delta_j g_0\Big\|_{L^q(0,1;\widehat{W}^{-1,r}_{\omega}(\Sigma))}\Big\|_{q,\mathbb{R}}\\
&\leq c(q,r)\Big\|\sum_{j=l}^{k}\Delta_j g_0\Big\|_{L^q(\mathbb{R};\widehat{W}^{-1,r}_{\omega}(\Sigma))}.
\end{aligned}$$

Similarly, using (5.29) and (5.3), we get that

$$\begin{aligned}
\big\|\big(\textstyle\sum_{j=l}^{k}2^{-2j}\|\Delta_j g_1\|^2_{2,\omega v(x_n)}\big)^{1/2}\big\|_{q,\mathbb{R}} &\leq c(q,r)\big\|\textstyle\sum_{j=l}^{k}2^{-j}\Delta_j g_1\big\|_{L^q(\mathbb{R};L^r_\omega(\Sigma))}\\
&\leq c(q,r)\big\|\textstyle\sum_{j=l}^{k}\Delta_j g_1\big\|_{\widehat{W}^{-1,q}(\mathbb{R};L^r_\omega(\Sigma))}.
\end{aligned}$$

Then the uniform boundedness of $\{\sum_{j=l}^{k}\Delta_j\}_{l,k\in\mathbb{Z}}$ in $\mathcal{L}(L^q(\mathbb{R};\widehat{W}^{-1,r}_{\omega}(\Sigma)))$ and in $\mathcal{L}(\widehat{W}^{-1,q}(\mathbb{R};L^r_\omega(\Sigma)))$ implies the estimate

$$\begin{aligned}
&\Big\|\Big(\sum_{j=l}^{k}\|\Delta_j g; L^2_{(m),\omega v(x_n)}+L^2_{\omega v(x_n),1/2^j}\|_0^2\Big)^{1/2}\Big\|_{q,\mathbb{R}}\\
&\leq c\Big(\Big\|\sum_{j=l}^{k}\Delta_j g_0\Big\|_{L^q(\mathbb{R};\widehat{W}^{-1,r}_{\omega}(\Sigma))} + \Big\|\sum_{j=l}^{k}\Delta_j g_1\Big\|_{\widehat{W}^{-1,q}(\mathbb{R};L^r_\omega(\Sigma))}\Big)\\
&\leq c\big(\|g_0\|_{L^q(\mathbb{R};\widehat{W}^{-1,r}_{\omega}(\Sigma))} + \|g_1\|_{\widehat{W}^{-1,q}(\mathbb{R};L^r_\omega(\Sigma))}\big)
\end{aligned}$$

with $c = c(q,r) > 0$ independent of $l,k \in \mathbb{Z}$. Now (5.16) implies the estimate

$$\Big\|\Big(\sum_{j=l}^{k}\|\Delta_j g; L^2_{(m),\omega v(x_n)}+L^2_{\omega v(x_n),1/2^j}\|_0^2\Big)^{1/2}\Big\|_{q,\mathbb{R}} \leq c(q,r)\Big\|\sum_{j=l}^{k}\Delta_j g\Big\|_{\widehat{W}^{-1;q,r}_{\omega}(\Omega)}. \quad (5.31)$$

Summarizing (5.26)-(5.28) and (5.31) we get that

$$\begin{aligned} &\big\| \textstyle\sum_{j=l}^k M(2^j)\Delta_j g\big\|_{L^q(\mathbb{R};L^r_\omega(\Sigma))} \\ &\qquad \le c\big(\big\| \textstyle\sum_{j=l}^k \Delta_j g\big\|_{W^{1;q,r}_\omega(\Omega)} + (|\lambda|+1)\big\| \textstyle\sum_{j=l}^k \Delta_j g\big\|_{\widehat{W}^{-1;q,r}_\omega(\Omega)}\big) \end{aligned} \tag{5.32}$$

with $c = c(q,r,\alpha,\varepsilon,\Sigma,\mathcal{A}_r(\omega)) > 0$ for all $l,k \in \mathbb{Z}$ and for all $\lambda \in -\alpha + \Sigma_\varepsilon$. Since $(\Delta_j)_{j\in\mathbb{Z}}$ defines unconditional Schauder decompositions of $R_0W^{1;q,r}_\omega(\Omega)$ and of $R_0\widehat{W}^{-1;q,r}_\omega(\Omega)$, (5.32) implies that the series $\sum_{j\in\mathbb{Z}} M(2^j)\Delta_j g$ converges in $L^q(\mathbb{R};L^r_\omega(\Sigma))$ and

$$\Big\| \sum_{j\in\mathbb{Z}} M(2^j)\Delta_j g\Big\|_{L^q(\mathbb{R};L^r_\omega(\Sigma))} \le c\big(\|g\|_{W^{1;q,r}_\omega(\Omega)} + (|\lambda|+1)\|g\|_{\widehat{W}^{-1;q,r}_\omega(\Omega)}\big)$$

with $c = c(q,r,\alpha,\varepsilon,\Sigma,\mathcal{A}_r(\omega)) > 0$. This is the desired estimate of the first term on the right-hand side of (5.25).

Next let us estimate the second term on the right-hand side of (5.25). Note that the operator family

$$\{B_{j,t} : j\in\mathbb{N}, t\in(0,1)\} \subset \mathcal{L}(L^q(\mathbb{R};L^r_\omega(\Sigma)))$$

is $\mathcal{R}$-bounded, see Remark 2.11 (6). Moreover, for $t\in(0,1)$, the operator $M'(2^j(1+t))$ commutes with the operator $B_{j,t}$ and the range of $B_{j,t}$ is contained in the range of Δ_j. Hence it follows from (2.7), (2.5) that for any independent symmetric $\{-1,1\}$-valued random variables $\{\varepsilon_j(\cdot)\}$ on $(0,1)$

$$\begin{aligned} &\Big\| \sum_{j=l}^k \int_0^1 2^j M'(2^j(1+t))B_{j,t}\Delta_j g\,dt\Big\|_{L^q(\mathbb{R};L^r_\omega(\Sigma))} \\ &\quad\le \int_0^1 \Big\| \sum_{j=l}^k 2^j(1+t)B_{j,t}M'(2^j(1+t))\Delta_j g\Big\|_{L^q(\mathbb{R};L^r_\omega(\Sigma))} dt \\ &\quad\le c_\Delta \int_0^1 \Big\| \sum_{j=l}^k \varepsilon_j B_{j,t}2^j(1+t)M'(2^j(1+t))\Delta_j g\Big\|_{L^q(0,1;L^q(\mathbb{R};L^r_\omega(\Sigma)))} dt \\ &\quad\le c\int_0^1 \Big\| \sum_{j=l}^k \varepsilon_j 2^j(1+t)M'(2^j(1+t))\Delta_j g\Big\|_{L^q(0,1;L^q(\mathbb{R};L^r_\omega(\Sigma)))} dt \\ &\quad\le c\int_0^1 \Big\| \sum_{j=l}^k 2^j(1+t)M'(2^j(1+t))\Delta_j g\Big\|_{L^q(\mathbb{R};L^r_\omega(\Sigma))} dt. \end{aligned} \tag{5.33}$$

By Lemma 5.5 (5.8) holds with $u_j = u_j(t) := 2^j(1+t)M'(2^j(1+t))\Delta_j g$ and with corresponding functions $v = v(\cdot,x_n,t) \in L^s_\omega(\mathbb{R}^{n-1})$ for $(x_n,t)\in\mathbb{R}\times(0,1)$, where v is Lebesgue measurable with respect to $(x',x_n,t)\in\mathbb{R}^n\times(0,1)$ by Lemma 5.4, see the proof of Lemma 5.5. Therefore, using (5.23) we get that

$$
\begin{aligned}
&\text{the r.h.s. of (5.33)}\\
&\le c\int_0^1 \Big\|\Big(\sum_{j=l}^k \|2^j(1+t)M'(2^j(1+t))\Delta_j g(\cdot,x_n)\|^2_{2,\omega v(\cdot,x_n,t)}\Big)^{1/2}\Big\|_{q,\mathbb{R}}\,dt\\
&\le c\Big(\int_0^1 \Big\|\Big\{\sum_{j=l}^k \big[\|\Delta_j g\|^2_{W^{1,2}_{\omega v(\cdot,x_n,t)}(\Sigma)} + 2^{2j}(1+t)^2\|\Delta_j g\|^2_{2,\omega v(\cdot,x_n,t)}\\
&\qquad\qquad +|\lambda+1|^2\,\|\Delta_j g; L^2_{(m),\omega v(\cdot,x_n,t)} + L^2_{\omega v(\cdot,x_n,t),2^{-j}(1+t)^{-1}}\|_0^2\big]\Big\}^{1/2}\Big\|_{q,\mathbb{R}}\,dt\Big)\\
&\le c\Big(\int_0^1 \Big\|\Big(\sum_{j=l}^k \|\Delta_j g\|^2_{W^{1,2}_{\omega v(\cdot,x_n,t)}(\Sigma)}\Big)^{1/2}\Big\|_{q,\mathbb{R}} + \Big\|\Big(\sum_{j=l}^k 2^{2j}\|\Delta_j g\|^2_{2,\omega v(\cdot,x_n,t)}\Big)^{1/2}\Big\|_{q,\mathbb{R}}\\
&\qquad\qquad +|\lambda+1|\,\Big\|\Big(\sum_{j=l}^k \|\Delta_j g; L^2_{(m),\omega v(\cdot,x_n,t)} + L^2_{\omega v(\cdot,x_n,t),2^{-j}}\|_0^2\Big)^{1/2}\Big\|_{q,\mathbb{R}}\,dt\Big),
\end{aligned}
$$

where $c = c(q, r, \alpha, \varepsilon, \Sigma, \mathcal{A}_r(\omega))$. Thus, by the same argument leading from (5.26) to (5.32) we get the estimate

$$
\begin{aligned}
&\Big\|\sum_{j=l}^k \int_0^1 2^j M'(2^j(1+t))B_{j,t}\Delta_j g\,dt\Big\|_{L^q(\mathbb{R};L^r_\omega(\Sigma))}\\
&\qquad \le c\Big(\Big\|\sum_{j=l}^k \Delta_j g\Big\|_{W^{1;q,r}_\omega(\Omega)} + (|\lambda|+1)\Big\|\sum_{j=l}^k \Delta_j g\Big\|_{\widehat{W}^{-1;q,r}_\omega(\Omega)}\Big)
\end{aligned}
$$

with $c = c(q, r, \alpha, \varepsilon, \Sigma, \mathcal{A}_r(\omega)) > 0$. Summarizing, we proved in the case $r > 2$ the existence of a solution to (R_λ) satisfying the estimate (5.22).

In the case $r = 2$ the same proof as before, but with $v \equiv 1$, may be used.

The uniqueness of solution is obvious from the uniqueness result for $f \neq 0, g = 0$, see Theorem 4.1. Now the proof of the theorem is complete. ■

5.3 Stokes resolvent system for general cylinders

In this section $\Omega \subset \mathbb{R}^n, n \geq 3$, is the cylindrical domain $\Omega = \bigcup_{i=0}^m \Omega_i$ in (1.2). We consider the Stokes resolvent system

$$
\begin{aligned}
\lambda u - \Delta u + \nabla p &= f \quad \text{in } \Omega\\
\operatorname{div} u &= 0 \quad \text{in } \Omega \qquad (5.34)\\
u &= 0 \quad \text{on } \partial\Omega.
\end{aligned}
$$

Let $\bar\alpha = \min\{\alpha^{(i)} :\ i = 0, \ldots, m\}$ where $\alpha^{(0)} > 0$ and $\alpha^{(i)} > 0$, $i = 1, \ldots, m$, are the smallest eigenvalues of the Dirichlet Laplacians in Ω_0 and in Σ^i, $i = 1, \ldots, m$, respectively.

For fixed $\lambda \in \mathbb{C} \setminus (-\infty, -\bar\alpha]$ let us define the operator $S_{q,\lambda}$ by

$$
\begin{aligned}
&D(S_{q,\lambda}) = \big(W^{2,q}(\Omega)^n \cap W^{1,q}_0(\Omega)^n \cap L^q_\sigma(\Omega)\big) \times \widehat{W}^{1,q}(\Omega),\\
&S_{q,\lambda}(u,p) = \lambda u - \Delta u + \nabla p.
\end{aligned}
$$

Obviously the range $R(S_{q,\lambda})$ of $S_{q,\lambda}$ is contained in $L^q(\Omega)^n$.

Lemma 5.8 *Let $2 \le q < \infty$, $\varepsilon \in (\pi/2, \pi)$ and $\lambda \in -\alpha + \Sigma_\varepsilon$, where $\alpha \in (0, \bar{\alpha})$.*

(1) If $(u,p) \in \big(W^{2,q}(\Omega)^n \cap W_0^{1,q}(\Omega)^n\big) \times \widehat{W}^{1,q}(\Omega)$ is a solution to the resolvent problem (5.34) *with $f \in L^q(\Omega)^n$, then (u,p) satisfies the estimate*

$$\begin{aligned}&\|(\lambda+\alpha)u, \nabla^2 u, \nabla p\|_{L^q(\Omega)}\\ &\quad \le C\big(\|f\|_{L^q(\Omega)} + \|\nabla u, u, p\|_{L^q(\Omega_0)} + (|\lambda|+1)\|u\|_{(W^{1,q'}(\Omega_0))^*}\big),\end{aligned} \tag{5.35}$$

with a constant $C = C(q, \alpha, \varepsilon, \Omega_0, \Sigma^1, \dots, \Sigma^m) > 0$ independent of $\lambda \in -\alpha + \Sigma_\varepsilon$; here $q' = q/(q-1)$.

(2) The operator $S_{q,\lambda}$ is injective.

(3) The range $R(S_{q,\lambda})$ of $S_{q,\lambda}$ is dense in $L^q(\Omega)^n$.

Proof: The proof uses a cut-off technique and, in principle, follows the same argument as in the proof of Lemma 4.1 of [27]. Without loss of generality we may assume that there exist cut-off functions $\{\varphi_i\}_{i=0}^m$ such that

$$\begin{aligned}&\textstyle\sum_{i=0}^m \varphi_i(x) = 1, \quad 0 \le \varphi_i(x) \le 1 \quad \text{for } x \in \Omega,\\ &\varphi_i \in C^\infty(\bar{\Omega}_i), \quad \operatorname{dist}(\operatorname{supp}\varphi_i,\, \partial\Omega_i \cap \Omega) \ge \delta > 0,\ i = 0, \dots, m,\end{aligned} \tag{5.36}$$

where 'dist' means the distance. For $i = 1, \dots, m$ let $\widetilde{\Omega}_i$ be the infinite straight cylinder extending the semi-infinite cylinder Ω_i, and denote the zero extension of v to $\widetilde{\Omega}_i$ by $\tilde{v}$. Then $\{\varphi_0 u, \varphi_0 p\}$ on Ω_0 satisfies

$$(R_\lambda)_0 \qquad \begin{aligned} \lambda(\varphi_0 u) - \Delta(\varphi_0 u) + \nabla(\varphi_0 p) &= f^0 && \text{in } \Omega_0\\ \operatorname{div}(\varphi_0 u) &= g^0 && \text{in } \Omega_0\\ \varphi_0 u &= 0 && \text{on } \partial\Omega_0,\end{aligned}$$

and $\{\widetilde{\varphi_i u}, \widetilde{\varphi_i p}\}$ on $\widetilde{\Omega}_i$, $i = 1, \dots, m$, satisfy

$$(R_\lambda)_i \qquad \begin{aligned} \lambda(\widetilde{\varphi_i u}) - \Delta(\widetilde{\varphi_i u}) + \nabla(\widetilde{\varphi_i p}) &= \tilde{f}^i && \text{in } \widetilde{\Omega}_i\\ \operatorname{div}(\widetilde{\varphi_i u}) &= \tilde{g}^i && \text{in } \widetilde{\Omega}_i\\ \widetilde{\varphi_i u} &= 0 && \text{on } \partial\widetilde{\Omega}_i,\end{aligned}$$

where

$$f^i := \varphi_i f + (\nabla\varphi_i)p - (\Delta\varphi_i)u - 2\nabla\varphi_i \cdot \nabla u, \quad g^i := \nabla\varphi_i \cdot u, \quad i = 0, \dots, m.$$

Note that $\operatorname{supp} g^i \subset \Omega_0$ and $\int_{\Omega_0} g^i\, dx = 0$ for $i = 0, \dots, m$. Therefore,

$$\int_{\Omega_0} g^0 \psi\, dx = \int_{\Omega_0} u \cdot (\overline{\psi}\, \nabla\varphi_0)\, dx \qquad \text{for all } \psi \in C^\infty(\overline{\Omega}_0)$$

where $\overline{\psi} = \psi - \frac{1}{|\Omega_0|}\int_{\Omega_0} \psi\, dx$. Hence, using Poincaré's inequality, we get that $g^0 \in \widehat{W}^{-1,q}(\Omega_0)$ and

$$\|g^0\|_{\widehat{W}^{-1,q}(\Omega_0)} \le c(\Omega_0)\|\nabla^2\varphi_0, \nabla\varphi_0\|_{L^\infty(\Omega_0)}\|u\|_{(W^{1,q'}(\Omega_0))^*}.$$

In the same way it follows that $\tilde{g}^i \in \widehat{W}^{-1,q}(\widetilde{\Omega}_i)$ and

$$\|\tilde{g}^i\|_{\widehat{W}^{-1,q}(\widetilde{\Omega}_i)} \leq c\|u\|_{(W^{1,q'}(\Omega_0))^*} \quad \text{for } i = 1, \dots, m.$$

Therefore, by [27], Theorem 1.2, for all $\lambda \in -\alpha + \Sigma_\varepsilon$

$$\begin{aligned} &\|(\lambda+\alpha)(\varphi_0 u), \nabla^2(\varphi_0 u), \nabla(\varphi_0 p)\|_{L^q(\Omega_0)} \\ &\qquad \leq c\big(\|f^0, \nabla g^0, g^0\|_{L^q(\Omega_0)} + |\lambda|\|g^0\|_{\widehat{W}^{-1,q}(\Omega_0)}\big) \\ &\qquad \leq c\big(\|f\|_{L^q(\Omega)} + \|\nabla u, u, p\|_{L^q(\Omega_0)} + |\lambda|\|u\|_{(W^{1,q'}(\Omega_0))^*}\big) \end{aligned} \tag{5.37}$$

with $c = c(q, \alpha, \varepsilon, \Omega_0) > 0$. Furthermore, by Theorem 5.7, for $i = 1, \dots, m$

$$\begin{aligned} &\|(\lambda+\alpha)(\varphi_i u), \nabla^2(\varphi_i u), \nabla(\varphi_i p)\|_{L^q(\Omega_i)} \\ &\qquad = \|(\lambda+\alpha)(\widetilde{\varphi_i u}), \nabla^2(\widetilde{\varphi_i u}), \nabla(\widetilde{\varphi_i p})\|_{L^q(\widetilde{\Omega}_i)} \\ &\qquad \leq c\big(\|\tilde{f}^i, \nabla \tilde{g}^i, \tilde{g}^i\|_{L^q(\widetilde{\Omega}_i)} + (|\lambda|+1)\|\tilde{g}^i\|_{\widehat{W}^{-1,q}(\widetilde{\Omega}_i)}\big) \\ &\qquad \leq c\big(\|f\|_{L^q(\Omega)} + \|\nabla u, u, p\|_{L^q(\Omega_0)} + (|\lambda|+1)\|u\|_{(W^{1,q'}(\Omega_0))^*}\big), \end{aligned} \tag{5.38}$$

with $c = c(q, \alpha, \varepsilon, \Sigma^i) > 0$. Finally, summing (5.37) and (5.38) for $i = 1, \dots, m$, we get the estimate (5.35) for $u = \sum_{i=0}^m \varphi_i u$ and $p = \sum_{i=0}^m \varphi_i p$. Thus (1) is proved.

To prove the injectivity of $S_{q,\lambda}$ let $S_{q,\lambda}(u,p) = 0$ with $(u,p) \in D(S_{q,\lambda})$. If $q = 2$, one directly gets $(u, \nabla p) = 0$ by testing with u.

Let $2 < q < \infty$. Looking at $(R_\lambda)_0$ and $(R_\lambda)_i$, $i = 1, \dots, m$, it is obvious that $f^0 \in L^2(\Omega_0)$, $g^0 \in W^{1,2}(\Omega_0) \cap \widehat{W}^{-1,2}(\Omega_0)$ and $\tilde{f}^i \in L^2(\widetilde{\Omega}_i)$, $\tilde{g}^i \in W^{1,2}(\widetilde{\Omega}_i) \cap \widehat{W}^{-1,2}(\widetilde{\Omega}_i)$; note that $f = 0$ and that f^i, g^i, $i = 0, \dots, m$ are compactly supported in Ω_0. Therefore, by [27], Theorem 1.2 and Theorem 5.7 with $q = r = 2$ and $\omega \equiv 1$, we get that

$$(\varphi_i u, \varphi_i p) \in \big(W^{2,2}(\Omega_i)^n \cap W_0^{1,2}(\Omega_i)^n\big) \times \widehat{W}^{1,2}(\Omega_i), \quad i = 0, \dots, m.$$

Thus $(u,p) \in D(S_{2,\lambda})$ yielding $(u,p) = 0$, and (2) is proved .

Next let us show that $R(S_{q,\lambda})$ is dense in $L^q(\Omega)^n$. By the lemma of Lax-Milgram and regularity theory of the Stokes system we conclude that $R(S_{2,\lambda}) = L^2(\Omega)^n$. For $q > 2$ and $f \in C_0^\infty(\Omega)^n$ which is dense in $L^q(\Omega)^n$, there is $(u,p) \in D(S_{2,\lambda})$ such that $S_{2,\lambda}(u,p) = f$. Looking at $(R_\lambda)_0$ and $(R_\lambda)_i$ and using regularity results for Stokes resolvent systems on bounded domains and on infinite cylinders (Theorem 5.7), one can see that

$$(\varphi_i u, \varphi_i p) \in \big(W_\omega^{2;\tilde{q},r}(\tilde{\Omega}_i)^n \cap W_{0,\omega}^{1;\tilde{q},r}(\tilde{\Omega}_i)^n\big) \times \widehat{W}_\omega^{1;\tilde{q},r}(\Omega_i), \quad i = 1, \dots, m,$$

with $\omega \equiv 1$ for all $\tilde{q} \in (1, \infty)$, $r \in [2, \infty)$, in particular,

$$(\varphi_i u, \varphi_i p) \in \big(W^{2,q}(\Omega_i)^n \cap W_0^{1,q}(\Omega_i)^n\big) \times \widehat{W}^{1,q}(\Omega_i), \quad i = 0, \dots, m,$$

yielding the denseness of $R(S_{q,\lambda})$ in $L^q(\Omega)^n$.

The proof of this lemma is complete. ∎

Now we can prove the main theorem of this section.

Theorem 5.9 *Let $1 < q < \infty$ and $\lambda \in -\alpha + \Sigma_\varepsilon$, where $\alpha \in (0, \bar{\alpha})$, and let $\varepsilon \in (\pi/2, \pi)$. If $f \in L^q(\Omega)^n$, then the resolvent problem (5.34) has a unique solution*

$$(u, p) \in \big(W^{2,q}(\Omega)^n \cap W_0^{1,q}(\Omega)^n \cap L^q_\sigma(\Omega)\big) \times \widehat{W}^{1,q}(\Omega)$$

satisfying the estimate

$$\|(\lambda + \alpha)u, \nabla^2 u, \nabla p\|_{L^q(\Omega)} \leq C\|f\|_{L^q(\Omega)} \tag{5.39}$$

with a constant $C = C(q, \alpha, \varepsilon, \Omega_0, \Sigma^1, \ldots, \Sigma^m)$ independent of $\lambda \in -\alpha + \Sigma_\varepsilon$.

As a consequence, for every $\varepsilon \in (\pi/2, \pi)$ and $\alpha \in (0, \bar{\alpha})$ the set $-\alpha + \Sigma_\varepsilon$ is contained in $\rho(-A_q)$ and the resolvent estimate

$$\|(\lambda + A_q)^{-1}\|_{\mathcal{L}(L^q_\sigma(\Omega))} \leq \frac{C}{|\lambda + \alpha|} \quad \forall \lambda \in -\alpha + \Sigma_\varepsilon. \tag{5.40}$$

with $C = C(q, \alpha, \varepsilon, \Omega_0, \Sigma^1, \ldots, \Sigma^m)$ holds. In particular, $-A_q$ generates a bounded analytic semigroup e^{-tA_q} in $L^q_\sigma(\Omega)$ satisfying

$$\|e^{-tA_q}\|_{\mathcal{L}(L^q_\sigma(\Omega))} \leq Ce^{-\alpha t} \quad \text{for all } \ t \geq 0 \tag{5.41}$$

with a constant $C = C(q, \alpha, \varepsilon, \Omega_0, \Sigma^1, \ldots, \Sigma^m)$ independent of $t \geq 0$.

Proof: First let $2 \leq q < \infty$. Let us prove the *a priori* estimate (5.39) which will imply by Lemma 5.8 that the operator $S_{q,\lambda}$ is an isomorphism from $D(S_{q,\lambda})$ to $L^q(\Omega)^n$. Instead of proving (5.39) we shall show a slightly stronger estimate

$$\|(\lambda + \beta)u, \nabla^2 u, \nabla p\|_{L^q(\Omega)} \leq C\|f\|_{L^q(\Omega)} \quad \forall \lambda \in -\alpha + \Sigma_\varepsilon \tag{5.42}$$

with a constant $C = C(q, \alpha, \varepsilon, \Omega)$ independent of λ where $\beta = \frac{1}{2}(\alpha + \bar{\alpha})$; note that $|\lambda + \alpha| \leq c(\varepsilon, \alpha)|\lambda + \beta|$ for all $\lambda \in -\alpha + \Sigma_\varepsilon$.

Assume that (5.42) does not hold. Then there are sequences $\{\lambda_j\} \subset -\alpha + \Sigma_\varepsilon$, $\{(u_j, p_j)\} \subset D(S_{q,\lambda_j})$ such that

$$\|(\lambda_j + \beta)u_j, \nabla^2 u_j, \nabla p_j\|_{L^q(\Omega)} = 1, \quad \|f_j\|_{L^q(\Omega)} \to 0 \quad \text{as } j \to \infty, \tag{5.43}$$

where $f_j = S_{q,\lambda_j}(u_j, p_j)$. Without loss of generality we may assume that

$$(\lambda_j + \beta)u_j \rightharpoonup v, \quad \nabla^2 u_j \rightharpoonup \nabla^2 u, \quad \nabla p_j \rightharpoonup \nabla p \quad \text{as } j \to \infty \tag{5.44}$$

with some $v \in L^q(\Omega)$, $u \in \widehat{W}^{2,q}(\Omega)$ and $p \in \widehat{W}^{1,q}(\Omega)$. Moreover, we may assume $\int_{\Omega_0} p_j \, dx = 0$, $\int_{\Omega_0} p \, dx = 0$ and that $\lambda_j \to \lambda \in \{-\alpha + \bar{S}_\varepsilon\} \cup \{\infty\}$.

(i) Let $\lambda_j \to \lambda \in -\alpha + \bar{S}_\varepsilon$.

Note that $\lambda + \beta \neq 0$. Then by (5.44) $v = (\lambda + \beta)u$, $u_j \rightharpoonup u$ in $W^{2,q}(\Omega)$ and $u \in D(S_{q,\lambda})$. It follows from (5.34), (5.43) that $S_{q,\lambda}(u, p) = 0$ yielding $(u, p) = 0$ by Lemma 5.8 (2). On the other hand, we have the strong convergences

$$u_j \to 0 \ \text{ in } W^{1,q}(\Omega_0), \ \ p_j \to 0 \ \text{ in } L^q(\Omega_0), \ \ (|\lambda_j| + 1)u_j \to 0 \ \text{ in } (W^{1,q'}(\Omega_0))^* \tag{5.45}$$

due to the compact embeddings $W^{2,q}(\Omega_0) \subset\subset W^{1,q}(\Omega_0) \subset\subset L^q(\Omega_0) \subset\subset (W^{1,q'}(\Omega_0))^*$, Poincaré's inequality on Ω_0 and (5.44). Thus Lemma 5.8 (1) together with (5.43) yields the contradiction $1 \le 0$.

(ii) Let $|\lambda_j| \to \infty$. Then, besides (5.44), we conclude that $\nabla^2 u = 0$, and consequently $v + \nabla p = 0$ where $v \in L^q_\sigma(\Omega)$. Note that this is the L^q-Helmholtz decomposition of the null vector field on Ω. Therefore, $v = 0$, $\nabla p = 0$. Again we get (5.45) and finally the contradiction $1 \le 0$.

Thus (5.42) holds true proving existence of a unique solution to (R_λ) in the case $2 \le q < \infty$.

The case $1 < q < 2$ can be proved by a duality argument. As is well known, (5.34) is equivalent to

$$(\lambda + A_q)u = P_q f$$

with the Stokes operator A_q and the Helmholtz decomposition P_q of $L^q(\Omega)$. Moreover, if $0 \in \rho(A_q)$, then the resolvent estimate of type (5.40) implies by the open mapping theorem the estimate (5.39) as well as the uniqueness and existence of a solution to (R_λ). If we show

$$A_q^* = A_{q'}, \tag{5.46}$$

where A_q^* is the dual of A_q in $L^q_\sigma(\Omega)$, then $\lambda \in -\alpha + \Sigma_\varepsilon \subset \rho(-A_{q'})$ and the estimate (5.40) for $2 < q' < \infty$ implies, by [77], Ch. 8, Theorem 1, that $\bar{\lambda} \in -\alpha + \Sigma_\varepsilon \subset \rho(-A_q)$ and the estimate (5.40) for $1 < q < 2$.

Since $P_q^* = P_{q'}$, it is easily seen that $A_{q'} \subset A_q^*$. Let $v \in D(A_q^*)$ and let $w \in D(A_{q'})$ satisfy $A_{q'}w = A_q^* v$; note that $0 \in \rho(A_{q'})$ due to the result already proved for $q' > 2$. Then for all $u \in D(A_q)$

$$\langle A_q u, v\rangle = \langle u, A_q^* v\rangle = \langle u, A_{q'} w\rangle = \langle A_q u, w\rangle.$$

Since $R(A_q)$ is dense in $L^q_\sigma(\Omega)$, see the last paragraph of the proof of Lemma 5.8, we conclude that $v = w \in D(A_{q'})$, and (5.46) is proved.

Finally, (5.41) follows from (5.40) by the well-known theory of analytic semigroups. ■

5.4 H^∞-calculus of the Stokes operator

Let the domain Ω be given as in Section 5.3. In this section we consider the H^∞-calculus of the Stokes operator A_q in Ω. An approach to the H^∞-calculus of the Stokes operator in general unbounded domains has been considered in [7] and some partial results have been obtained. Based on the result of [7] and combined with our result on resolvent estimates we prove that the Stokes operator A_q in Ω admits a bounded H^∞-calculus with H^∞-angle $\phi^\infty_{A_q} = 0$.

In [7] it is proved that for any $\theta \in (0, \pi)$ there is a constant $c_\theta > 0$ such that the shifted Stokes operator $c_\theta + A_q$ on $L^q_\sigma(G)$ admits a bounded $H^\infty(\Sigma_\theta)$-calculus provided the domain $G \subset R^n$, $n \ge 2$, satisfies the following assumptions (A1)-(A3):

(A1) There is a finite covering of $\bar{G}$ with relatively open sets $U_j, j = 1, \dots, l$, such that U_j coincides (after rotation) with a relatively open set of $\overline{\mathbb{R}^n_{\gamma_j}}$, where

$\overline{\mathbb{R}^n_{\gamma_j}} := \{(x_1, \tilde{x}) \in \mathbb{R}^n : \; x_1 > \gamma_j(\tilde{x})\}$, $\gamma_j \in C^{1,1}$, $j = 1, \ldots, l$. Moreover, suppose that there are cut-off functions $\varphi_j, \psi_j \in C_b^\infty(\bar{G})$, $j = 1, \ldots, l$, such that $\{\varphi_j\}$ is a partition of unity subordinated to $\{U_j\}_{j=1}^l$, $\psi_j \equiv 1$ on $\operatorname{supp} \varphi_j$ and $\operatorname{supp} \psi_j \subset U_j$, $j = 1, \ldots, l$; here $C_b^\infty(\bar{G})$ means the space of all infinitely differentiable and bounded functions on $\bar{G}$.

(A2) The Helmholtz decomposition is valid for $L^r(G)^n$ with $r = q$ and $r = q'$, i.e., for every $f \in L^r(G)^n$ there is a unique decomposition $f = f_0 + \nabla p$ with $f_0 \in L^r_\sigma(G)$ and $p \in \widehat{W}^{1,r}(G)$. Moreover,

$$L^q_\sigma(G) = \{f \in L^q(G)^n : \; \operatorname{div} f = 0, \; f \cdot N|_{\partial G} = 0\}. \tag{5.47}$$

(A3) For every $p \in \widehat{W}^{1,r}(G), r = q, q'$, there is a decomposition $p = p_1 + p_2$ such that $p_1 \in W^{1,r}(G), p_2 \in L^r_{\text{loc}}(G)$ with $\nabla p_2 \in W^{1,r}(G)$ and

$$\|p_1, \nabla p_2\|_{W^{1,r}(G)} \leq C \|\nabla p\|_r.$$

It is easily seen that the domain Ω satisfies the assumption (A1). Furthermore the Helmholtz decomposition of $L^q(\Omega)^n$ was proved in [14], Theorem 4(c). Through the following lemmata we shall see that the remaining assumptions are satisfied as well.

Lemma 5.10 *The set $C_0^\infty(\bar{\Omega})$ is dense in $\widehat{W}^{1,q}(\Omega)$ for $1 < q < \infty$.*

Proof: Fix $u \in \widehat{W}^{1,q}(\Omega)$. Using the same notation as in the proof of Lemma 5.8 and the cut-off functions φ_j, $j = 0, \ldots, m$, see (5.36), we have $u = \sum_{j=0}^m \varphi_j u$. Without loss of generality assume that $\int_{\Omega_0} u \, dx = 0$. Thus, by Poincaré's inequality on the bounded domain Ω_0,

$$\varphi_0 u \in W^{1,q}(\Omega_0) \quad \text{and} \quad \varphi_j u \in \widehat{W}^{1,q}(\Omega_j), \quad \widetilde{\varphi_j u} \in \widehat{W}^{1,q}(\tilde{\Omega}_j), \; j = 1, \ldots, m.$$

Then there are sequences $\{v_k^{(0)}\} \subset C_0^\infty(\bar{\Omega}_0)$, $\{v_k^{(j)}\} \subset C_0^\infty(\bar{\tilde{\Omega}}_j)$, $j = 1, \ldots, m$, such that

$$\|v_k^{(0)} - \varphi_0 u\|_{W^{1,q}(\Omega_0)} \to 0, \quad \|v_k^{(j)} - \widetilde{\varphi_j u}\|_{\widehat{W}^{1,q}(\tilde{\Omega}_j)} \to 0 \tag{5.48}$$

as $k \to \infty$ due to the denseness of $C_0^\infty(\bar{\Omega}_0)$ in $W^{1,q}(\Omega_0)$ and Lemma 5.6 (2). Let

$$\Omega_j^\delta := \{x \in \Omega_j : \operatorname{dist}(x, \Omega \cap \partial\Omega_j) \geq \delta\} \quad \text{for } j = 0, \ldots, m.$$

Note that

$$\operatorname{supp} \varphi_j u \subset \Omega_j^\delta, \quad j = 0, \ldots, m, \tag{5.49}$$

due to the construction of $\{\varphi_j\}_{j=0}^m$. Without loss of generality we may assume that

$$\int_{\Omega_j \setminus \Omega_j^\delta} v_k^{(j)} \, dx = 0 \quad \text{for } j = 1, \ldots, m. \tag{5.50}$$

Let us choose functions $\eta_0 \in C_0^\infty(\bar{\Omega})$ and $\eta_j \in C_0^\infty(\bar{\tilde{\Omega}}_j), j = 1, \ldots, m$ such that

$$\begin{aligned} &\eta_0(x) \equiv 1 \; \text{ for } x \in \Omega_0^\delta \quad \text{and } \eta_0(x) \equiv 0 \; \text{ for } x \in \Omega \setminus \Omega_0^{\delta/2}, \\ &\eta_j(x) \equiv 1, \, x \in \Omega_j^\delta, \quad \text{and} \quad \eta_j(x) \equiv 0, \, x \in \tilde{\Omega}_j \setminus \Omega_j, \; j = 1, \ldots, m. \end{aligned} \tag{5.51}$$

For $k \in \mathbb{N}$ let $w_k^{(0)} = \eta_0 v_k^{(0)}$ and let $w_k^{(j)}$ be the zero extension of $\eta_j v_k^{(j)}$ onto Ω.

Now let $w_k := \sum_{j=0}^m w_k^{(j)}$. Obviously $w_k \in C_0^\infty(\bar{\Omega})$, $k \in \mathbb{N}$, and

$$\|\nabla(u - w_k)\|_{L^q(\Omega)} \leq \sum_{j=0}^m \|\nabla(\varphi_j u - w_k^{(j)})\|_{L^q(\Omega)}. \tag{5.52}$$

Due to (5.49) and (5.51) we get for each $j = 0, \dots, m$ that

$$\begin{aligned} \|\nabla(\varphi_j u - w_k^{(j)})\|_{L^q(\Omega)} &\leq \|\nabla(\varphi_j u - v_k^{(j)})\|_{L^q(\Omega_j^\delta)} + \|\nabla(\eta_j v_k^{(j)})\|_{L^q(\Omega_j \setminus \Omega_j^\delta)} \\ &\leq \|\nabla(\varphi_j u - v_k^{(j)})\|_{L^q(\Omega_j^\delta)} + c_j \|v_k^{(j)}, \nabla v_k^{(j)}\|_{L^q(\Omega_j \setminus \Omega_j^\delta)}. \end{aligned} \tag{5.53}$$

Note that for $j = 1, \dots, m$, using (5.50) and Poincaré's inequality, $\|v_k^{(j)}\|_{L^q(\Omega_j \setminus \Omega_j^\delta)} \leq c(q, \Omega_0)\|\nabla v_k^{(j)}\|_{L^q(\Omega_j \setminus \Omega_j^\delta)}$. Therefore, by (5.48), (5.49) the right-hand side of (5.53) for $j = 0, \dots, m$ tends to 0 as $k \to \infty$, and so does the right-hand side of (5.52).

The proof of the lemma is complete. ■

Corollary 5.11 *For the domain Ω the assertion* (5.47) *holds.*

Proof: Obviously,

$$L_\sigma^q(\Omega) \subset \{f \in L^q(\Omega)^n : \text{ div} f = 0, \; f \cdot N|_{\partial\Omega} = 0\}.$$

Since the right-hand side of (5.47) is 'orthogonal' to $\{\nabla h : h \in C_0^\infty(\bar{\Omega})\}$, the same result holds for $\{\nabla h : h \in \widehat{W}^{1,q'}(\Omega)\}$ by Lemma 5.10. Therefore, [36], Ch. III, Lemma 2.1, accomplishes the proof. ■

Lemma 5.12 *The assumption (A3) is satisfied for the domain Ω.*

Proof: First consider the case of Ω being an infinite straight cylinder $\Sigma \times \mathbb{R}$ with $\Sigma \subset \mathbb{R}^{n-1}$, a bounded domain of $C^{1,1}$-class. For $p \in \widehat{W}^{1,q}(\Omega)$ let $p_0(x', x_n) \equiv p_0(x_n) := \frac{1}{|\Sigma|}\int_\Sigma p(x', x_n)\, dx'$ and $\tilde{p} := p - p_0$. Then it follows that

$$\begin{aligned} &p_0 \in \widehat{W}^{1,q}(\Sigma \times \mathbb{R}), \quad \|p_0\|_{\widehat{W}^{1,q}(\Sigma\times\mathbb{R})} \leq c(\Sigma, q)\|p\|_{\widehat{W}^{1,q}(\Sigma\times\mathbb{R})}, \\ &\tilde{p} \in W^{1,q}(\Sigma \times \mathbb{R}), \quad \|\tilde{p}\|_{W^{1,q}(\Sigma\times\mathbb{R})} \leq c(\Sigma, q)\|p\|_{\widehat{W}^{1,q}(\Sigma\times\mathbb{R})}; \end{aligned} \tag{5.54}$$

here we used Poincaré's inequality for $\tilde{p}(\cdot, x_n)$ on Σ. On the other hand the whole space $\mathbb{R}^k$, $k \in \mathbb{N}$, was proved to satisfy assumption (A3), see [6], Remark 2.7. Therefore, as a function on $\mathbb{R}$, p_0 is decomposed by

$$p_0 = p_{01} + p_{02}, \quad \|p_{01}, \partial_1 p_{02}\|_{W^{1,q}(\mathbb{R})} \leq c\|p_0\|_{\widehat{W}^{1,q}(\mathbb{R})}.$$

Then $p_1 := \tilde{p} + p_{01}, p_2 := p_{02}$ satisfy assumption (A3) due to (5.54).

Next let Ω be the general unbounded cylinder introduced in the beginning of this section. We use the same notation for $\{\varphi_j\}_{j=0}^m, \Omega_j, \tilde{\Omega}_j$ and Ω_j^δ as in the proof of

Lemma 5.10. Fix $p \in \widehat{W}^{1,q}(\Omega)$ and write it in the form $p = \sum_{j=0}^m \varphi_j p$. Without loss of generality we assume that $\int_{\Omega_0} p\,dx = 0$; therefore, by Poincaré's inequality

$$\|p\|_{W^{1,q}(\Omega_0)} \le c\|p\|_{\widehat{W}^{1,q}(\Omega)}. \tag{5.55}$$

By the fact already proved for infinite straight cylinders, we have for $j = 1, \dots, m$, a decomposition $\widetilde{\varphi_j p} = p_{j1} + p_{j2}$ such that $p_{j1}, \nabla p_{j2} \in W^{1,q}(\tilde{\Omega}_j)$ and

$$\|p_{j1}, \nabla p_{j2}\|_{W^{1,q}(\Omega_j)} \le \|p_{j1}, \nabla p_{j2}\|_{W^{1,q}(\tilde{\Omega}_j)} \le c\|\widetilde{\varphi_j p}\|_{\widehat{W}^{1,q}(\tilde{\Omega}_j)} \le c\|p\|_{\widehat{W}^{1,q}(\Omega)}; \tag{5.56}$$

here we used $\int_{\Omega_0} p\,dx = 0$. Now define the functions $\eta \in C^\infty(\Omega)$ by

$$\eta(x) = \begin{cases} 1, & x \in \Omega_j^{2\delta},\ j = 1, \dots, m \\ 0, & x \in \Omega \setminus \bigcup_{j=1}^m \Omega_j^\delta, \end{cases}$$

with $\delta > 0$ as in (5.36), and w_i, $i = 1, 2$, on Ω by

$$w_i(x) = \begin{cases} p_{ji}(x), & x \in \Omega_j,\ j = 1, \dots, m \\ 0, & \text{otherwise.} \end{cases}$$

Then we get the decomposition

$$p = p_1 + p_2 \quad \text{with} \quad p_1 = \psi p + \eta w_1,\ p_2 = \eta w_2, \tag{5.57}$$

where $\psi = (1 - \eta)\sum_{j=1}^m \varphi_j + \varphi_0$; note that $\psi \in C^\infty(\Omega)$ and $\operatorname{supp}\psi \in \bar{\Omega}_0$. Hence, in view of (5.55), $\psi p \in W^{1,q}(\Omega)$ and $\|\psi p\|_{W^{1,q}(\Omega)} \le c\|p\|_{\widehat{W}^{1,q}(\Omega)}$. Moreover, $\eta w_1 \in W^{1,q}(\Omega)$ and, due to (5.56), $\|\eta w_1\|_{W^{1,q}(\Omega)} \le c\|p\|_{\widehat{W}^{1,q}(\Omega)}$. Thus we conclude that

$$p_1 \in W^{1,q}(\Omega), \quad \|p_1\|_{W^{1,q}(\Omega)} \le c\|p\|_{\widehat{W}^{1,q}(\Omega)}. \tag{5.58}$$

On the other hand, we have $\nabla p_2 = \nabla(\eta w_2) = \eta \nabla w_2 + w_2 \nabla \eta$ and, due to (5.56),

$$\|\eta \nabla w_2\|_{W^{1,q}(\Omega)} = \|\eta \nabla w_2\|_{W^{1,q}(\cup_{j=1}^m \Omega_j^\delta)} \le c \sum_{j=1}^m \|\nabla p_{j2}\|_{W^{1,q}(\Omega_j^\delta)} \le c\|p\|_{\widehat{W}^{1,q}(\Omega)};$$

moreover, $\operatorname{supp}\nabla\eta \subset \bigcup_{j=1}^m \left(\Omega_j^\delta \setminus \Omega_j^{2\delta}\right) \subset \Omega_0$ and obviously $w_2 = p - \varphi_0 p - w_1 \in W^{1,q}\big(\bigcup_{j=1}^m \left(\Omega_j^\delta \setminus \Omega_j^{2\delta}\right)\big)$ implying that

$$\|w_2 \nabla \eta\|_{W^{1,q}(\Omega)} \le c \sum_{j=1}^m \|p, p_{j1}\|_{W^{1,q}(\Omega_j^\delta \setminus \Omega_j^{2\delta})} \le c\|p\|_{\widehat{W}^{1,q}(\Omega)},$$

due to (5.56). Therefore we get that

$$\nabla p_2 \in W^{1,q}(\Omega), \quad \|\nabla p_2\|_{W^{1,q}(\Omega)} \le c\|p\|_{\widehat{W}^{1,q}(\Omega)},$$

which together with (5.57), (5.58) completes the proof of this lemma. ∎

Now we are in a position to formulate the main theorem of this section.

Theorem 5.13 *For $1 < q < \infty$ the Stokes operator A_q admits a bounded $H^\infty(\Sigma_\theta)$-calculus in $L^q_\sigma(\Omega)$ for any $\theta \in (0,\pi)$, i.e., the H^∞-angle $\phi^\infty_{A_q} = 0$. In particular, the Stokes operator A_q has maximal regularity in $L^q_\sigma(\Omega)$.*

Proof: By Theorem 5.9 the spectral angle ω_{A_q} of A_q is 0. Fix $\theta \in (0,\pi)$ arbitrarily. We must show that there is a constant $C > 0$ depending on θ such that for all $h \in \mathcal{H}^\infty(\Sigma_\theta)$ the operator

$$h(A_q) = \int_\Gamma h(\lambda)(\lambda - A_q)^{-1}\,d\lambda \in \mathcal{L}(L^q_\sigma(\Omega))$$

satisfies the estimate

$$\|h(A_q)\|_{\mathcal{L}(L^q_\sigma(\Omega))} \le C_\theta \|h\|_\infty, \tag{5.59}$$

where Γ is the oriented boundary of the sector $\Sigma_{\theta'}$ for any fixed $\theta' \in (0,\theta)$.

Since the domain Ω has been shown to satisfy the assumptions (A1)-(A3), by [1], Theorem 1.3, there are constant $R = R(q,\theta) > 0$ and $C = C(q,\theta) > 0$ such that

$$\left\| \int_{\Gamma_{R,\infty}} h(\lambda)(\lambda - A_q)^{-1}\,d\lambda \right\|_{\mathcal{L}(L^q_\sigma(\Omega))} \le C\|h\|_\infty,$$

where $\Gamma_{R,\infty} = \{\lambda \in \Gamma :\ |\lambda| > R\}$. On the other hand, due to Theorem 5.9, we get

$$\left\| \int_{\Gamma\setminus\Gamma_{R,\infty}} h(\lambda)(\lambda - A_q)^{-1}\,d\lambda \right\|_{\mathcal{L}(L^q_\sigma(\Omega))} \le C_{q,\theta}\|h\|_\infty.$$

Thus we proved (5.59).

Maximal regularity of A_q in $L^q_\sigma(\Omega)$ follows directly, since A_q admits a bounded $H^\infty(\Sigma_\theta)$-calculus for $\theta \in (0,\pi/2)$ and $L^q_\sigma(\Omega)$ is a *UMD* space, see Section 2.4.

Now the proof is complete. ■

6 Stability of Stationary Navier-Stokes Flows

Let $\Omega = \bigcup_{i=0}^{m} \Omega_i$ be the general unbounded cylinder given in (1.2) and

$$\bar{\alpha} = \min\{\alpha^{(i)} : \ i = 0, \dots, m\}, \tag{6.1}$$

where $\alpha^{(0)} > 0$ and $\alpha^{(i)} > 0$, $i = 1, \dots, m$, are the smallest eigenvalues of Dirichlet Laplacians in Ω_0 and in Σ^i, the bounded cross-section of the semi-infinite cylinder Ω_i, $i = 1, \dots, m$, respectively. In this chapter we consider the existence, uniqueness and stability of a stationary Navier-Stokes flow with prescribed flux in Ω. In Section 6.1 we deal with the existence and uniqueness of a stationary Navier-Stokes flow in Ω. Section 6.2 is devoted to the analysis of a perturbed Stokes operator related with stationary Navier-Stokes flows in Ω. Finally, in Section 6.3 we study the exponential stability of stationary Navier-Stokes flows.

6.1 Existence of stationary Navier-Stokes flows

Let us consider the stationary Navier-Stokes system

$$\text{(SNS)} \qquad \begin{aligned} -\Delta w + (w \cdot \nabla) w + \nabla q &= f && \text{in } \Omega \\ \operatorname{div} w &= 0 && \text{in } \Omega \\ w &= 0 && \text{on } \partial\Omega \\ w &= u_\infty && \text{at infinity,} \end{aligned} \tag{6.2}$$

where u_∞ is a function depending on the variables $x'^i = (x_1^i, \dots, x_{n-1}^i)$ in the cross section Σ^i of $\Omega_i, i = 1, \dots, m$.

It is well known that for the existence of a unique solution to (6.2) some additional conditions, e.g. a *flux condition* in each exit, must be given, i.e.,

$$\Phi_i = \int_{\Sigma_i} u \cdot \mathbf{n}^i \, ds, \quad i = 1, \dots, m, \tag{6.3}$$

where $\mathbf{n}^i$ is the unit vector along the positive axial direction of Ω_i, is prescribed. Note that, due to the solenoidalness of the fluid, $\Phi_i \equiv$ const, $i = 1 \dots, m$, and

$$\sum_{i=1}^{m} \Phi_i = 0. \tag{6.4}$$

Let us assume that the velocity at infinity u_∞ in each $\Omega_i, i = 1, \dots, m$, equals the *Poiseuille flow* $\mathbf{v}_i$ corresponding to the flux Φ_i.

The Poiseuille flow $(\mathbf{v}_0, p_0)$ corresponding to a given flux Φ_0 in an infinite straight cylinder $\Sigma \times \mathbb{R}$ with $\Sigma \subset \mathbb{R}^{n-1}$ a bounded domain is the solution to the stationary Stokes system in $\Sigma \times \mathbb{R}$ such that $\mathbf{v}_0 = v_0(x')\mathbf{n}$, $\nabla p_0 = -k\mathbf{n}$ with constant $k = k(\Phi_0)$ and

$$\int_{\Sigma} \mathbf{v}_0 \cdot \mathbf{n} \, ds = \Phi_0,$$

where $\mathbf{n}$ is the unit vector along the positive direction of the cylinder $\Sigma \times \mathbb{R}$. Then it is easily seen that

$$-\Delta v_0 = k, \quad v_0|_{\partial\Sigma} = 0;$$

in particular, if Σ is a Lipschitz domain, one gets the explicit representation $v_0 = \frac{\Phi_0}{k_0} w_0(x')$, $k = \frac{\Phi_0}{k_0}$, where w_0 is the (unique) solution to the Dirichlet problem $-\Delta' w_0 = 1$, $w_0|_{\partial\Sigma} = 0$ and $k_0 = \int_\Sigma |\nabla' w_0|^2 \, dx'$. Moreover, if Σ is of $C^{1,1}$-class, then

$$v_0 \in H^{2,s}(\Sigma) \cap H_0^{1,s}(\Sigma), \quad \|v_0\|_{H^{2,s}(\Sigma)} \le c(s, \Sigma)|\Phi_0| \quad \forall s \in (1, \infty), \tag{6.5}$$

in particular,

$$v_0 \in L^\infty(\Sigma), \quad \|v_0\|_{L^\infty(\Sigma)} \le c(n, \Sigma)|\Phi_0|$$

due to the Sobolev embedding theorem. Note that the Poiseuille solution $(\mathbf{v}_0, p_0)$ also solves the stationary Navier-Stokes system in $\Sigma \times \mathbb{R}$.

We consider the system (SNS), see (6.2), with

$$u_\infty = \mathbf{v}_i,$$

in each exit $\Omega_i, i = 1, \dots, m$, where $\mathbf{v}_i$ is the Poiseuille flow corresponding to the flux Φ_i through the cross section Σ^i of Ω_i and (6.4) is assumed. Note that $\mathbf{v}_i, i = 1, \dots, m$, depends only on the variable $x'^i \in \Sigma^i$.

First of all, we construct a *carrier* $\mathbf{a}$ of the Poiseuille flows $\mathbf{v}_i, i = 1, \dots, m$. Let $1 < r < \infty$. A carrier $\mathbf{a}$ is defined as a function on Ω such that

$$\mathbf{a} \in H^{2,r}_{\mathrm{loc}}(\overline{\Omega}), \quad \operatorname{div} \mathbf{a} = 0 \text{ in } \Omega, \quad \mathbf{a} = \mathbf{0} \text{ on } \partial\Omega, \quad \mathbf{a} = \mathbf{v}_i \text{ in } \Omega_i \setminus \Omega_0, i = 1, \dots, m.$$

In [36], Ch. 6, §1, a carrier $\mathbf{a}$ for the case $r = 2$ is constructed. The idea used there can be applied to the general case $r \in (1, \infty)$. Without loss of generality we may assume that there exist cut-off functions $\{\varphi_i\}_{i=0}^m$ such that

$$\begin{aligned} &\textstyle\sum_{i=0}^m \varphi_i(x) = 1, \quad 0 \le \varphi_i(x) \le 1 \quad \text{for } x \in \Omega, \\ &\varphi_i \in C^\infty(\bar{\Omega}_i), \quad \operatorname{dist}(\operatorname{supp}\varphi_i,\, \partial\Omega_i \cap \Omega) \ge d > 0,\ i = 0, \dots, m. \end{aligned}$$

For $i = 1, \dots, m$ let $\tilde{\mathbf{v}}_i = \chi_i \mathbf{v}_i$, where the function χ_i on Ω is the characteristic function of Ω_i, and set

$$\mathbf{v}(x) := \sum_{i=1}^m \varphi_i(x) \tilde{\mathbf{v}}_i(x) \quad \text{for } x \in \Omega. \tag{6.6}$$

Then, from the construction of $\{\varphi_i\}$ and (6.5) we get

$$\mathbf{v}|_{\Omega_0} \in H^{2,r}(\Omega_0), \quad \|\mathbf{v}|_{\Omega_0}\|_{H^{2,r}(\Omega_0)} \le c(r, \Omega)\Phi \quad \forall r \in (1, \infty), \tag{6.7}$$

where and in what follows we use the notation

$$\Phi := \sum_{i=1}^m |\Phi_i|$$

for the total flux. Note that $\operatorname{div} \mathbf{v}|_{\Omega_0} \in H_0^{1,r}(\Omega_0)$ for all $r \in (1,\infty)$ and by (6.4)

$$\int_{\Omega_0} \operatorname{div} \mathbf{v}\, dx = \sum_{i=1}^m \int_{\Sigma_i} \mathbf{v}_i(x'^i) \cdot \mathbf{n}^i\, dx'^i = \sum_{i=1}^m \Phi_i = 0.$$

Then, by [36], Ch. III, Theorem 3.2, Remark 3.6, (cf. [16], Theorem 2.4) there is a vector field $\mathbf{z}$ such that

$$\mathbf{z} \in H_0^{2,r}(\Omega_0) \quad \text{and} \quad \operatorname{div} \mathbf{z} = -\operatorname{div} \mathbf{v}|_{\Omega_0} \quad \text{for all } r \in (1,\infty)$$

and

$$\|\mathbf{z}\|_{H_0^{2,r}(\Omega_0)} \le c(r,\Omega_0)\|\operatorname{div} \mathbf{v}|_{\Omega_0}\|_{H_0^{1,r}(\Omega_0)} \le c(r,\Omega)\Phi \quad \forall r \in (1,\infty), \tag{6.8}$$

where we used (6.7). Hence we get

$$\mathbf{z} \in L^\infty(\Omega_0), \quad \|\mathbf{z}\|_{L^\infty(\Omega_0)} \le c(n,\Omega_0)\Phi$$

due to the Sobolev embedding theorem. Now extend the function $\mathbf{z}$ from Ω_0 to Ω by 0 and denote it again by $\mathbf{z}$. Then

$$\mathbf{a} := \mathbf{z} + \mathbf{v} \tag{6.9}$$

is a carrier of the Poiseuille flows $\mathbf{v}_i, i = 1,\ldots,m$, and, by (6.7), (6.8) satisfies the estimate

$$\|\mathbf{a}\|_{H^{2,r}(\Omega_0)} \le C_0(r,\Omega)\Phi \quad \forall r \in (1,\infty). \tag{6.10}$$

In particular,

$$\mathbf{a} \in L^\infty(\Omega), \quad \|\mathbf{a}\|_{L^\infty(\Omega)} \le c(n,\Omega)\Phi. \tag{6.11}$$

Lemma 6.1 *Let $n \ge 3, 1 < r < \infty$ and let*

$$\delta = \begin{cases} \frac{n}{r} - 2 & \text{for } 1 < r < \frac{n}{2} \\ \delta' & \text{for } r = \frac{n}{2} \\ 0 & \text{for } r > \frac{n}{2}, \end{cases} \tag{6.12}$$

with $\delta' > 0$ arbitrarily small.

(1) For all $u \in H^{1+\delta,r}(\Omega)$ and $v \in H^{2,r}(\Omega)$ we have $(u\cdot\nabla)v, (v\cdot\nabla)u \in L^r(\Omega)$ and

$$\|(u\cdot\nabla)v,\, (v\cdot\nabla)u\|_r \le c\|u\|_{H^{1+\delta,r}(\Omega)}\|v\|_{H^{2,r}(\Omega)} \tag{6.13}$$

where $c = c(r,\Omega) > 0$ is independent of δ unless $r = \frac{n}{2}$.

(2) Let $r \in (1,\infty)$ and $r \ge \frac{n}{3}$. Then for all $u, v \in H^{2,r}(\Omega)$ we have $(u\cdot\nabla)v \in H^{1-\delta,r}(\Omega)$ and

$$\|(v\cdot\nabla)u\|_{H^{1-\delta,r}(\Omega)} \le c\|u\|_{H^{2,r}(\Omega)}\|v\|_{H^{2,r}(\Omega)}. \tag{6.14}$$

(3) Let

$$\eta = \begin{cases} \frac{n+r}{2r}, & r < n \\ 1+\delta', & r = n \\ 1, & r > n \end{cases} \tag{6.15}$$

with $\delta' > 0$ arbitrarily small. Then for $r \in (1,\infty), r \geq \frac{n}{3}$, and $\xi \in [\eta, 2]$

$$\|(v\cdot\nabla)u\|_{H^{(1-\delta)\frac{\xi-\eta}{2-\eta},r}(\Omega)} \leq c\|u\|_{H^{\xi,r}(\Omega)}\|v\|_{H^{\xi,r}(\Omega)}, \tag{6.16}$$

where $c = c(r,\xi,\Omega) > 0$ is independent of δ (δ') unless $r = \frac{n}{2}$ ($r = n$).

Proof: First of all, we note that for the unbounded cylinder Ω the usual Sobolev embedding theorems hold since Ω has a *minimally smooth boundary* and hence, extension theorems for Sobolev spaces hold for Ω, cf. [10], Ch. V, Theorem 2.4.5 (cf. [73], Theorem 3.21). In the proof we shall write shortly $H^{s,r}$, L^q in place of $H^{s,r}(\Omega)$, $L^q(\Omega)$, respectively.

(1) First let $1 < r < \frac{n}{2}$. Observe that for $\delta = \frac{n}{r} - 2$ the Sobolev embeddings $H^{1+\delta,r} \hookrightarrow L^n$ and $H^{1,r} \hookrightarrow L^{nr/(n-r)}$ hold. Hence we get for all $u \in H^{1+\delta,r}, v \in H^{2,r}$ that

$$\|(u\cdot\nabla)v\|_r \leq \|u\|_n\|\nabla v\|_{\frac{nr}{n-r}} \leq c\|u\|_{H^{1+\delta,r}}\|v\|_{H^{2,r}}$$

with $c = c(r,\Omega) > 0$. Moreover, by the embeddings $H^{2,r} \hookrightarrow L^{nr/(n-2r)}$, $H^{\delta,r} = H^{\frac{n}{r}-2,r} \hookrightarrow L^{n/2}$, we get

$$\|(v\cdot\nabla)u\|_r \leq \|v\|_{\frac{nr}{n-2r}}\|\nabla u\|_{n/2} \leq c(r,\Omega)\|v\|_{H^{2,r}}\|u\|_{H^{1+\delta,r}}.$$

Now let $\frac{n}{2} < r < \infty$. Then

$$\|(u\cdot\nabla)v\|_r \leq \|u\|_{2r}\|\nabla v\|_{2r} \leq c\|u\|_{H^{1+\delta,r}}\|v\|_{H^{2,r}}$$

with $\delta = 0$. Note that the embedding $H^{2,r} \hookrightarrow L^\infty$ holds for $r > \frac{n}{2}$. Hence,

$$\|(v\cdot\nabla)u\|_r \leq c\|v\|_\infty\|\nabla u\|_r \leq c\|v\|_{H^{2,r}}\|u\|_{H^{1+\delta,r}}$$

with $\delta = 0$.

In the limit case $r = \frac{n}{2}$ note that $H^{2,r} \hookrightarrow L^p$ for all $p \in [r,\infty)$ and that for all $\delta \in (0,1)$ there exists an $\varepsilon = \varepsilon(r,\delta,\Omega) > 0$ such that $H^{\delta,r} \hookrightarrow L^{r+\varepsilon}$. Hence for $u \in H^{1+\delta,r}, v \in H^{2,r}$

$$\|(u\cdot\nabla)v\|_r \leq c\|u\|_{2r}\|v\|_{2r} \leq c\|u\|_{H^{1,r}}\|v\|_{H^{2,r}},$$

and there exists $p_\varepsilon > r$ such that

$$\|(v\cdot\nabla)u\|_r \leq \|\nabla u\|_{r+\varepsilon}\|v\|_{p_\varepsilon} \leq c_\delta\|u\|_{H^{1+\delta,r}}\|v\|_{H^{2,r}}.$$

(2) Observe that for all $u \in H^{2+\delta,r}, v \in H^{2,r}$

$$\|(v\cdot\nabla)u\|_{H^{1,r}} \leq c\|u\|_{H^{2+\delta,r}}\|v\|_{H^{2,r}}. \tag{6.17}$$

Actually $D(v\cdot\nabla)u = (Dv\cdot\nabla)u + (v\cdot\nabla)Du$, where D is any first order derivative. By (6.13) we get that

$$\|(Dv\cdot\nabla)u\|_r \leq c\|\nabla u\|_{H^{1+\delta,r}}\|v\|_{H^{2,r}} \leq c\|u\|_{H^{2+\delta,r}}\|v\|_{H^{2,r}},$$
$$\|(v\cdot\nabla)Du\|_r \leq c\|\nabla u\|_{H^{1+\delta,r}}\|v\|_{H^{2,r}} \leq c\|u\|_{H^{2+\delta,r}}\|v\|_{H^{2,r}},$$

proving (6.17). Note that $1-\delta \in (0,1]$ for $r \geq \frac{n}{3}$ and that by complex interpolation $[H^{1+\delta,r}, H^{2+\delta,r}]_{1-\delta} = H^{2,r}$ and $[L^r, H^{1,r}]_{1-\delta} = H^{1-\delta,r}$, cf. [13], [75]. Therefore, by complex interpolation of (6.13), (6.17) with the index $1-\delta$, we get for all $u, v \in H^{2,r}$ that $(v\cdot\nabla)u \in H^{1-\delta,r}$ and

$$\|(v\cdot\nabla)Du\|_{H^{1-\delta,r}} \leq c\|u\|_{H^{2,r}}\|v\|_{H^{2,r}},$$

where $c = c(r,\delta,\Omega)$ for $r = \frac{n}{2}$ and arbitrarily small δ. Thus (6.14) is proved.

(3) First let us prove for η given by (6.15) and for $u, v \in H^{\eta,r}$ that

$$\|(v\cdot\nabla)u\|_r \leq c\|u\|_{H^{\eta,r}}\|v\|_{H^{\eta,r}}, \tag{6.18}$$

with $c = c(r,\Omega) > 0$ ($c = c(r,\delta',\Omega) > 0$ for $r = n$). Actually, for $1 < r < n$ we get with $\alpha = \frac{1}{2}(1-\frac{r}{n}) \in (0,1)$ that

$$\|(v\cdot\nabla)u\|_r \leq \|v\|_{\frac{r}{\alpha}}\|\nabla u\|_{\frac{r}{1-\alpha}} \leq c\|v\|_{H^{\eta,r}}\|u\|_{H^{\eta,r}},$$

where we used that $H^{\eta,r} \hookrightarrow L^{r/\alpha}$ and $H^{\eta-1,r} \hookrightarrow L^{r/(1-\alpha)}$. For $r = n$

$$\|(v\cdot\nabla)u\|_r \leq \|v\|_\infty\|\nabla u\|_r \leq c\|v\|_{H^{1+\delta',r}}\|u\|_{H^{1,r}},$$

and finally, for $r > \frac{n}{2}$ we get

$$\|(v\cdot\nabla)u\|_r \leq \|v\|_\infty\|\nabla u\|_r \leq c\|v\|_{H^{1,r}}\|u\|_{H^{1,r}},$$

thus proving (6.18). Now bilinear complex interpolation of (6.14) and (6.18) (see [75], 1.19.5) yields

$$\|(v\cdot\nabla)u\|_{H^{(1-\delta)\theta,r}} \leq c\|u\|_{H^{\eta(1-\theta)+2\theta,r}}\|v\|_{H^{\eta(1-\theta)+2\theta,r}}, \quad \theta \in [0,1],$$

which coincides with (6.16) for $\theta = \frac{\xi-\eta}{2-\eta}$.

The proof of the lemma is complete. ■

Lemma 6.2 *Let $1 < r < \infty$, let the constant δ be given as in Lemma 6.1, and let $\mathbf{a}$ be defined by (6.9).*

(1) For all $u \in H^{1+\delta,r}(\Omega)$ we have $(\mathbf{a}\cdot\nabla)u, (u\cdot\nabla)\mathbf{a} \in L^r(\Omega)$ and

$$\|(\mathbf{a}\cdot\nabla)u, (u\cdot\nabla)\mathbf{a}\|_r \leq c(r,\Omega)\Phi\|u\|_{H^{1+\delta,r}}.$$

(2) Let $1 < r < \infty, r \geq \frac{n}{3}$. For all $u \in H^{2,r}(\Omega)$ we have $(\mathbf{a}\cdot\nabla)u, (u\cdot\nabla)\mathbf{a} \in H^{1-\delta,r}(\Omega)$ and

$$\|(\mathbf{a}\cdot\nabla)u, (u\cdot\nabla)\mathbf{a}\|_{H^{1-\delta,r}} \leq c(r,\Omega)\Phi\|u\|_{H^{2,r}}.$$

Proof: Since Lemma 6.1 (1) holds for Ω_0 as well in place of Ω, we get by (6.10) that

$$(u\cdot\nabla)\mathbf{a} \in L^r(\Omega_0), (\mathbf{a}\cdot\nabla)u \in L^r(\Omega_0)$$

and

$$\begin{aligned}\|(u\cdot\nabla)\mathbf{a},(\mathbf{a}\cdot\nabla)u\|_{L^r(\Omega_0)} &\le c(r,\Omega_0)\|u\|_{H^{1+\delta,r}(\Omega)}\|\mathbf{a}\|_{H^{2,r}(\Omega_0)}\\ &\le c(r,\Omega)\Phi\|u\|_{H^{1+\delta,r}(\Omega)}.\end{aligned} \tag{6.19}$$

Now it remains to show that $(\mathbf{a}\cdot\nabla)u,(u\cdot\nabla)\mathbf{a}\in L^r(\Omega\setminus\Omega_0)$ and

$$\|(\mathbf{a}\cdot\nabla)u,(u\cdot\nabla)\mathbf{a}\|_{L^r(\Omega\setminus\Omega_0)}\le c(r,\Omega)\Phi\|u\|_{H^{1+\delta,r}(\Omega)}, \tag{6.20}$$

which is obvious since $\mathbf{a}|_{\Omega_i\setminus\Omega_0}=\mathbf{v}_i, i=1,\dots,m$, due to the construction of $\mathbf{a}$ and $\mathbf{v}_i|_{\Omega_i\setminus\Omega_0},\nabla\mathbf{v}_i|_{\Omega_i\setminus\Omega_0}\in L^\infty(\Omega_i\setminus\Omega_0), i=1,\dots,m$ (see (6.5)). Hence (1) is proved.

The proof of (2) can be done similarly to the proof of Lemma 6.1 (2) using complex interpolation and will be omitted. ■

Let $1<r<\infty$. By the transform $v:=w-\mathbf{a}$ the system (SNS) is reduced to

$$(\text{SNS})'\qquad \begin{aligned}-\Delta v+(v\cdot\nabla)\mathbf{a}+(\mathbf{a}\cdot\nabla)v+(v\cdot\nabla)v+\nabla q &= F &&\text{in }\ \Omega\\ \operatorname{div} v &= 0 &&\text{in }\ \Omega\\ v &= 0 &&\text{on }\partial\Omega\\ v(x) &= 0 &&\text{at infinity,}\end{aligned} \tag{6.21}$$

where $F=f-(\mathbf{a}\cdot\nabla)\mathbf{a}$.

It is easily seen that the reduced system (SNS)′ is equivalent to

$$G_r v+P_r(v\cdot\nabla)v=P_rF; \tag{6.22}$$

for P_r see the Introduction, the operator G_r is defined by

$$\begin{aligned}&D(G_r)=D(A_r)=H^{2,r}(\Omega)\cap H_0^{1,r}(\Omega)\cap L^r_\sigma(\Omega),\\ &G_rv:=A_rv+P_r\big((v\cdot\nabla)\mathbf{a}+(\mathbf{a}\cdot\nabla)v\big)\end{aligned}$$

with the Stokes operator $A_r=-P_r\Delta$ in $L^r_\sigma(\Omega)$.

First we consider the linearization of (6.22):

$$G_rv+P_r(y\cdot\nabla)v=P_rF, \tag{6.23}$$

for a fixed $y\in D(A_r)$.

Lemma 6.3 *Let $1<r<\infty, r\ge\frac{n}{3}$. There exists a constant $K_0=K_0(r,\Omega)>0$ such that, if $\Phi\le K_0$ and $\|y\|_{H^{2,r}(\Omega)}\le K_0$, then problem* (6.23) *has a unique solution $v_y\in H^{2,r}(\Omega)$ satisfying the estimate*

$$\|v_y\|_{H^{2,r}(\Omega)}\le M\big(\|f\|_r+\Phi^2\big) \tag{6.24}$$

with a constant $M=M(r,\Omega)>0$.

Proof: For $v \in H^{2,r}(\Omega)$ let

$$E_y v := P_r\big((v\cdot\nabla)\mathbf{a} + (\mathbf{a}\cdot\nabla)v + (y\cdot\nabla)v\big).$$

Then (6.23) is represented by

$$(A_r + E_y)v = P_r F.$$

By Lemma 6.1 (1) and Lemma 6.2 (1)

$$\begin{aligned}
\|E_y v\|_{L^r_\sigma(\Omega)} &\le C_1(r,\Omega)(\Phi + \|y\|_{H^{2,r}(\Omega)})\|v\|_{H^{2,r}(\Omega)} \\
&\le C_1(r,\Omega)\|A_r^{-1}\|_{\mathcal{L}(L^r_\sigma, H^{2,r})}(\Phi + \|y\|_{H^{2,r}(\Omega)})\|A_r v\|_r \\
&\le C_2(r,\Omega)(\Phi + \|y\|_{H^{2,r}(\Omega)})\|A_r^{-1}\|_{\mathcal{L}(L^r_\sigma, H^{2,r})}\|v\|_{H^{2,r}(\Omega)};
\end{aligned}$$

note that, by Theorem 5.9, $A_r^{-1} \in \mathcal{L}(L^r_\sigma(\Omega), H^{2,r}(\Omega))$ and

$$\|v\|_{H^{2,r}(\Omega)} \le c(r,\Omega)\|A_r v\|_{L^r_\sigma(\Omega)}$$

for all $v \in D(A_r)$. Therefore, if

$$\Phi \le K_0 := \frac{1}{4C_2}, \quad \|y\|_{H^{2,r}(\Omega)} \le K_0, \tag{6.25}$$

then $\|E_y\|_{\mathcal{L}(H^{2,r}, L^r_\sigma)} \le \frac{1}{2}\|A_r^{-1}\|_{\mathcal{L}(L^r_\sigma, H^{2,r})}$, and consequently, we get

$$\|(A_r + E_y)v\|_r = \|(I + E_y A_r^{-1})A_r v\|_r \ge \frac{1}{2}\|A_r v\|_r \ge c(r,\Omega)\|v\|_{H^{2,r}}$$

for all $v \in D(A_r)$. Note that $R(A_r + E_y) = L^r_\sigma(\Omega)$ due to $R(I + E_y A_r^{-1}) = R(A_r) = L^r_\sigma(\Omega)$. Hence

$$(A_r + E_y)^{-1} \in \mathcal{L}(L^r_\sigma(\Omega), H^{2,r}(\Omega)) \quad \text{and} \quad \|(A_r + E_y)^{-1}\|_{\mathcal{L}(L^r_\sigma, H^{2,r})} \le M_0 \tag{6.26}$$

with some $M_0 = M_0(r,\Omega)$. Thus the equation (6.23) has a unique solution $v_y = (A_r + E_y)^{-1}F \in H^{2,r}(\Omega)$ satisfying

$$\begin{aligned}
\|v_y\|_{H^{2,r}(\Omega))} &\le c\|F\|_r \le c(\|f\|_r + \|(\mathbf{a}\cdot\nabla)\mathbf{a}\|_{L^r(\Omega_0)}) \\
&\le c(\|f\|_r + \|\mathbf{a}\|^2_{H^{2,r}(\Omega_0)}) \le c(\|f\|_r + \Phi^2)
\end{aligned}$$

with $c = c(r,\Omega) > 0$, where we used $(\mathbf{a}\cdot\nabla)\mathbf{a} = 0$ in $\Omega \setminus \Omega_0$, Lemma 6.2 (1) for Ω_0 and (6.10). ■

Now we state the theorem on the existence of solutions for (SNS).

Theorem 6.4 *Let $1 < r < \infty, r \ge \frac{n}{3}$ and let $f \in L^r(\Omega)$. Furthermore, let the velocity u_∞ at infinity for each exit $\Omega_i, i = 1,\dots,m$, be the Poiseuille flow corresponding to the given flux $\Phi_i, i = 1,\dots,m$, satisfying* (6.4). *Then there is a constant*

$K_1 = K_1(r,\Omega) > 0$ such that, if $\|f\|_r + \Phi^2 < K_1$, then (SNS) has a solution $w = \mathbf{a} + v$ satisfying $v \in H^{2,r}(\Omega)$ and

$$\|v\|_{H^{2,r}(\Omega)} \le c(r,\Omega)(\|f\|_r + \Phi^2).$$

This solution w is the only solution to (SNS) in the class

$$\left\{ w \in H^{2,r}_{\mathrm{loc}}(\overline{\Omega}) : \sum_{i=1}^m \|w - \mathbf{v}_i\|_{H^{2,r}(\Omega_i \setminus \Omega_0)} + \|w\|_{H^{2,r}(\Omega_0)} \le \bar{K}_0 \right\} \tag{6.27}$$

with some $\bar{K}_0 = \bar{K}_0(r,\Omega) > 0$.

Proof: It is enough to show the unique solvability of (SNS)$'$ in a ball of $H^{2,r}(\Omega)$. Let K_0 be the number given by Lemma 6.3 and let

$$U_{K_0} = \{v \in H^{2,r}(\Omega);\ \|v\|_{H^{2,r}} \le K_0\}.$$

Assuming $\Phi < K_0$, let us define the mapping

$$\Psi : U_{K_0} \to H^{2,r}(\Omega), \quad \Psi y = v_y,$$

where v_y is the unique solution to the linearized problem (6.23). Then, for $y_1, y_2 \in U_{K_0}$

$$G_r v_{y_j} + P_r(y_j \cdot \nabla) v_{y_j} = P_r(f - (\mathbf{a} \cdot \nabla)\mathbf{a}),\ j = 1, 2,$$

which, by subtraction, yields

$$G_r(v_{y_1} - v_{y_2}) + P_r(y_1 \cdot \nabla)(v_{y_1} - v_{y_2}) = -P_r((y_1 - y_2) \cdot \nabla) v_{y_2},$$

i.e.,

$$(A_r + E_{y_1})(v_{y_1} - v_{y_2}) = -P_r((y_1 - y_2) \cdot \nabla) v_{y_2}.$$

Hence, (6.26), Lemma 6.1 (1) and (6.24) yield

$$\begin{aligned} \|v_{y_1} - v_{y_2}\|_{H^{2,r}} &\le M_0 \|P_r((y_1 - y_2) \cdot \nabla) v_{y_2}\|_r \\ &\le M_0 \tilde{C} \|v_{y_2}\|_{H^{2,r}} \|y_1 - y_2\|_{H^{2,r}} \\ &\le M_0 M \tilde{C} (\|f\|_r + \Phi^2) \|y_1 - y_2\|_{H^{2,r}} \end{aligned}$$

where $\tilde{C} = \tilde{C}(r,\Omega) > 0$. Therefore, if

$$\|f\|_r + \Phi^2 < K_1 := \min\left\{ \frac{1}{M_0 M \tilde{C}}, \frac{K_0}{4M}, \frac{K_0^2}{16 C_0^2 + 1} \right\} \tag{6.28}$$

where C_0 is the constant in (6.10), then $\Psi(U_{K_0}) \subset U_{K_0}$ due to Lemma 6.3 and $\Psi : U_{K_0} \to U_{K_0}$ is a contraction mapping. Thus, by Banach fixed point theorem, there is a unique fixed point $\tilde{y} \in U_{K_0}$ of Ψ, which implies that, if (6.28) is satisfied, (SNS)$'$ has a solution $v = v_{\tilde{y}} \in H^{2,r}(\Omega)$ which is unique in U_{K_0}. Moreover, this solution satisfies

$$\|v\|_{H^{2,r}(\Omega)} \le M(\|f\|_r + \Phi^2)\ \ (< \frac{K_0}{4})$$

by Lemma 6.3. In consideration of (6.28), one can easily check that $w = v + \mathbf{a}$ belongs to the class given in (6.27) with $\bar{K}_0 := \frac{K_0}{2}$.

In order to consider the uniqueness of solutions, let $\tilde{w}$ be a solution to (SNS) in the class given in (6.27) with $\bar{K}_0 = \frac{K_0}{2}$. Obviously, $\tilde{v} := \tilde{w} - \mathbf{a} \in H^{2,r}(\Omega)$ solves (SNS′) and, due to (6.10),

$$\begin{aligned} \|\tilde{v}\|_{H^{2,r}(\Omega)} &\leq \textstyle\sum_{i=1}^m \|\tilde{w} - \mathbf{v}_i\|_{H^{2,r}(\Omega_i \setminus \Omega_0)} + \|\tilde{w}\|_{H^{2,r}(\Omega_0)} + \|\mathbf{a}\|_{H^{2,r}(\Omega_0)} \\ &\leq \bar{K}_0 + C_0|\Phi| \ \leq \ \tfrac{3}{2}\bar{K}_0 \ < \ K_0 \end{aligned}$$

Therefore, $\tilde{v}$ is in U_{K_0} and, hence, is the (unique) fixed point of the mapping Ψ, i.e. $v = \tilde{v}$ yielding $\tilde{w} = w$. ■

6.2 Perturbed Stokes operator

Let us introduce the operator

$$S_r := A_r + B_r \tag{6.29}$$

with

$$B_r u := P_r((u \cdot \nabla)w + (w \cdot \nabla)u), \tag{6.30}$$

where δ is given by (6.12) and w is the unique solution to (SNS) given by Theorem 6.4. It is easily seen that B_r with domain

$$D(B_r) = \{u \in L^r_\sigma(\Omega) : \ (u \cdot \nabla)w + (w \cdot \nabla)u \in L^r(\Omega)\}$$

is closed. We will call the operator S_r *perturbed Stokes operator*.

First we need a result on the domain of fractional power of the Stokes operator.

Lemma 6.5 *Let* $1 < r < \infty$ *and*

$$D(\Delta_r) = H^{2,r}(\Omega) \cap H^{1,r}_0(\Omega), \quad \Delta_r u = \Delta u.$$

Then there is a continuous projection Q_r *such that*

$$Q_r \in \mathcal{L}(D(\Delta_r), D(A_r)) \cap \mathcal{L}(L^r(\Omega), L^r_\sigma(\Omega)).$$

Proof: This lemma can be proved in the same way as [40], Lemma 6 using that $P_r^* = P_{r'}$, $A_r^* = A_{r'}$ (see e.g. the proof of Theorem 5.9) and $\Delta_r^* = \Delta_{r'}, r' = r/(r-1)$, for all $r \in (1, \infty)$. ■

Corollary 6.6 *Let* $1 < r < \infty$, $0 < \theta < 1$. *Then*

$$D(A_r^\theta) = [L^r_\sigma(\Omega), D(A_r)]_\theta = [L^r(\Omega), H^{2,r}(\Omega) \cap H^{1,r}_0(\Omega)]_\theta \cap L^r_\sigma(\Omega).$$

In particular, if $\theta < \frac{1}{2r}$, *then*

$$D(A_r^\theta) = H^{2\theta,r}(\Omega) \cap L^r_\sigma(\Omega). \tag{6.31}$$

Proof: Since A_r admits a bounded H^∞-calculus in $L^r_\sigma(\Omega)$ for $r \in (1,\infty)$, see Theorem 5.13, we get that $D(A_r^\theta) = [L^r_\sigma(\Omega), D(A_r)]_\theta$ for $\theta \in (0,1)$, see Section 2.4, (2.16). On the other hand, due to Lemma 6.5 we can apply [75], Theorem 1.17.1/1, that is,

$$\begin{aligned}[L^r_\sigma(\Omega), D(A_r)]_\theta &= [L^r(\Omega) \cap L^r_\sigma(\Omega), H^{2,r}(\Omega) \cap H^{1,r}_0(\Omega) \cap L^r_\sigma(\Omega)]_\theta \\ &= [L^r(\Omega), H^{2,r}(\Omega) \cap H^{1,r}_0(\Omega)]_\theta \cap L^r_\sigma(\Omega).\end{aligned}$$

It is well known that, if $\theta < \frac{1}{2r}$, then

$$H^{2\theta,r}(\Omega) = [L^r(\Omega), H^{2,r}_0(\Omega)]_\theta = [L^r(\Omega), H^{2,r}(\Omega)]_\theta$$

yielding $H^{2\theta,r}(\Omega) = [L^r(\Omega), H^{2,r}(\Omega) \cap H^{1,r}_0(\Omega)]_\theta$, cf. [75], 4.3.2. ∎

Lemma 6.7 *Let $1 < r < \infty$ and let the assumption of Theorem 6.4 be satisfied. Then, for all $u \in H^{1+\delta,r}(\Omega) \cap L^r_\sigma(\Omega))$ and*

$$\|B_r u\|_r \le c(r,\Omega)(\|f\|_r + \Phi + \Phi^2)\|u\|_{H^{1+\delta,r}(\Omega)}.$$

Proof: Since $w = v + \mathbf{a}$, Lemma 6.1 (1), Lemma 6.2 (1) and Theorem 6.4 yields that

$$\begin{aligned}\|B_r u\|_{L^r_\sigma} &\le \|(v\cdot\nabla)u\|_r + \|(u\cdot\nabla)v\|_r + \|(\mathbf{a}\cdot\nabla)u\|_r + \|(u\cdot\nabla)\mathbf{a}\|_r \\ &\le c(r,\Omega)(\|v\|_{H^{2,r}(\Omega)} + \Phi)\|u\|_{H^{1+\delta,r}(\Omega)} \\ &\le c(r,\Omega)(\|f\|_r + \Phi + \Phi^2)\|u\|_{H^{1+\delta,r}(\Omega)}\end{aligned}$$

for all $u \in H^{1+\delta,r}(\Omega) \cap L^r_\sigma(\Omega)$. ∎

Remark 6.8 By Theorem 5.9, for any $r \in (1,\infty)$, $\alpha \in (0,\bar\alpha)$ (see (6.1) for $\bar\alpha$) and $\varepsilon \in (\pi/2,\pi)$

$$\|u\|_{H^{2,r}(\Omega)} \le c(r,\Omega,\alpha,\varepsilon)\|(\lambda + A_r)u\|_{L^r_\sigma(\Omega)} \quad \forall u \in D(A_r)\ \forall \lambda \in -\alpha + \Sigma_\varepsilon.$$

Note that $r \ge \frac{n}{3}$ implies $\delta \le 1$. Hence, from Lemma 6.7 we get that if $r \ge \frac{n}{3}$ and $\|f\|_r + \Phi + \Phi^2$ is small enough, then $-\alpha + \Sigma_\varepsilon \subset \rho(-S_r)$ and

$$\|(\lambda + S_r)^{-1}\|_{\mathcal{L}(L^r_\sigma(\Omega))} \le \frac{C}{|\lambda + \alpha|} \tag{6.32}$$

with some constant $C = C(r,\Omega,\alpha,\varepsilon) > 0$. In particular, for $r \in (1,\infty), r \ge \frac{n}{3}$ the analytic semigroup $\{e^{-tS_r}\}_{t\ge 0}$ generated by $-S_r$ satisfies the estimate

$$\|e^{-tS_r}\|_{\mathcal{L}(L^r_\sigma(\Omega))} \le Ce^{-\alpha t} \quad \forall t > 0\ \forall \alpha \in (0,\bar\alpha) \tag{6.33}$$

with some constant $C = C(r,\Omega,\alpha) > 0$. Moreover, for $r \ge \frac{n}{3}$, under the same smallness condition for f and Φ as above, the adjoint operator $S^*_{r'} = A_{r'} + B_{r'}, r' = r/(r-1)$, generates a bounded analytic semigroup $e^{-tS^*_{r'}}$ in $L^{r'}_\sigma(\Omega)$ with the same estimate as in (6.33) due to [63], Ch. 1, Corollary 10.6 and $0 \in \rho(S^*_{r'})$.

In the next theorem we shall show that the operator $S_r, r \in (\frac{n}{3}, \infty), n \geq 3$, admits a bounded H^∞-calculus in $L^r_\sigma(\Omega)$ under smallness conditions on f and Φ. Note that $L^r(\Omega), L^r_\sigma(\Omega)$ are *UMD* spaces, see e.g. [10].

Theorem 6.9 *Let $r \in (\frac{n}{3}, \infty)$, and let $w = v + \mathbf{a}$ be the solution to (SNS) given by Theorem 6.4. Then there is a constant $K_2 = K_2(r, \Omega) > 0$ such that, if $\|f\|_r + \Phi + \Phi^2 < K_2$, then the operator S_r defined by* (6.29) *admits a bounded H^∞-calculus with H^∞-angle less than $\pi/2$ in $L^r_\sigma(\Omega)$. Moreover, the adjoint operator $S^*_{r'}$ of S_r in $L^{r'}_\sigma(\Omega)$ has a bounded H^∞-calculus with H^∞-angle less than $\pi/2$ as well.*

Proof: Based on the fact that the Stokes operator A_r admits a bounded H^∞-calculus with H^∞-angle 0 in $L^r_\sigma(\Omega)$, see Theorem 5.9, we shall use the perturbation theorem 2.19 for H^∞-calculus. Hence, let us show that the operator B_r given by (6.30) satisfies the assumptions (1), (2) of Theorem 2.19 with $\mathcal{A} = A, \mathcal{B} = B_r$. By Lemma 6.7, for all $u \in D(A_r), r \in (1, \infty)$,

$$\begin{aligned} \|B_r u\|_{L^r_\sigma(\Omega)} &\leq c(r,\Omega)(\|f\|_r + \Phi + \Phi^2)\|u\|_{H^{1+\delta,r}(\Omega)} \\ &\leq c(r,\Omega)(\|f\|_r + \Phi + \Phi^2)\|u\|_{H^{2,r}(\Omega)} \end{aligned} \tag{6.34}$$

proving (1) of Theorem 2.19.

In view of $w = v + \mathbf{a}$, Lemma 6.1 (2), Lemma 6.2 (2) and Theorem 6.4 yield

$$\begin{aligned} \|B_r u\|_{H^{1-\delta,r}(\Omega)} &= \|(u\cdot\nabla)w + (w\cdot\nabla)u\|_{H^{1-\delta,r}(\Omega)} \\ &\leq c(r,\Omega)(\|v\|_{H^{2,r}(\Omega)} + \Phi)\|u\|_{H^{2,r}(\Omega)} \\ &\leq c(r,\Omega)(\|f\|_r + \Phi + \Phi^2)\|u\|_{H^{2,r}(\Omega)} \end{aligned} \tag{6.35}$$

for all $u \in D(A_r)$. Note that for $\gamma \in (0,1)$ the complex interpolation space $[L^r_\sigma(\Omega), D(A_r)]_\gamma$ coincides with the domain $D(A^\gamma_r)$ of A^γ_r since A_r has bounded imaginary powers, cf. [75], Theorem 1.15.3. Therefore, by (6.31), (6.35) we get that if $0 < \gamma < \min\{\frac{1-\delta}{2}, \frac{1}{2r}\}$, then $B_r u \in D(A^\gamma_r)$ for all $u \in D(A_r)$ and

$$\|A^\gamma_r B_r u\|_{L^r_\sigma(\Omega)} \leq c(\delta,\gamma,r,\Omega)\|B_r u\|_{H^{1-\delta,r}(\Omega)} \leq c(\delta,\gamma,r,\Omega)(\|f\|_r + \Phi + \Phi^2)\|A_r u\|_{L^r_\sigma(\Omega)}$$

which is a stronger estimate than the one in Theorem 2.19 (2). Now fix δ, γ suitably depending on r, n. Thus Theorem 2.19 implies that, there is a sufficiently small number K_2 depending only on r, Ω such that, if $\|f\|_r + \Phi + \Phi^2 < K_2$, then $S_r = A_r + B_r$ admits a bounded H^∞-calculus in $L^r_\sigma(\Omega)$ with H^∞-angle less than $\pi/2$.

Finally [19], Proposition 2.11, proves the assertion on the adjoint operator $S^*_{r'}$.

■

As an important corollary of Theorem 6.9 we have the following maximal regularity result for the linearization of (6.43), cf. [19].

Proposition 6.10 *Let $1 < p < \infty$, $\frac{n}{3} < r < \infty, n \geq 3$. Furthermore, let $h \in L^p(0,\infty; L^r(\Omega))$ and $u_0 \in D(A_r)$. Then the linear system*

$$\begin{aligned} u_t - \Delta u + (u\cdot\nabla)w + (w\cdot\nabla)u + \nabla p &= h \quad \text{in } \Omega\times(0,\infty) \\ \operatorname{div} u &= 0 \quad \text{in } \Omega\times(0,\infty) \\ u &= 0 \quad \text{on } \partial\Omega\times(0,\infty) \\ u(0) &= u_0 \quad \text{in } \Omega, \end{aligned}$$

where $w \in H^{2,r}(\Omega)$ is the solution to (SNS) given by Theorem 6.4, has a unique solution

$$u \in L^p(0,\infty; H^{2,r}(\Omega)), u_t \in L^p(0,\infty; L^r_\sigma(\Omega))$$

satisfying

$$\|u\|_{L^p(0,\infty;H^{2,r}(\Omega))} + \|u_t\|_{L^p(0,\infty;L^r_\sigma(\Omega))} \le c(\|h\|_{L^p(0,\infty;L^r_\sigma(\Omega))} + \|u_0\|_{D(A_r)}).$$

Proposition 6.11 *Let $\frac{n}{3} < r < \infty, n \ge 3$. If $\|f\|_r + \Phi + \Phi^2$ is small enough depending on r, δ, Ω, for $\theta \in (0,1)$ we have $D(S_r^\theta) = D(A_r^\theta)$. In particular,*

$$D(S_r^\theta) = [L^r(\Omega), H^{2,r}(\Omega) \cap H_0^{1,r}(\Omega)]_\theta \cap L^r_\sigma(\Omega) \tag{6.36}$$

with equivalent norms, and

$$\|u\|_{H^{2\theta,r}(\Omega)} \le C\|S_r^\theta u\|_{L^r_\sigma(\Omega)} \quad \forall u \in D(S_r^\theta) \tag{6.37}$$

with $C = C(r,\theta,\Omega) > 0$. Moreover, for $\theta < \frac{1}{2r}$, the norms $\|\cdot\|_{D(S_r^\theta)} = \|\cdot\|_{H^{2\theta,r}(\Omega)}$ are equivalent.

Proof: Due to Theorem 6.9, we get that $D(S_r^\theta) = [L^r_\sigma(\Omega), D(S_r)]_\theta$ for all $\theta \in (0,1)$, see (2.16). Note that $D(S_r) = D(A_r)$. Then Corollary 6.6 implies the assertions. ∎

Let us have a closer look at the adjoint operator $S^*_{r'}$ of S_r in $L^r_\sigma(\Omega)$ and characterize the domains of its fractional powers. Note that for all $\varphi \in C^\infty_{0,\sigma}(\Omega)$

$$\begin{aligned}(B_r u, \varphi)_{L^r,L^{r'}} &= \int_\Omega \big((w\cdot\nabla)u + (u\cdot\nabla)w\big)\cdot\varphi\,dx \\ &= -\int_\Omega \Big((w\cdot\nabla)\varphi + \sum_{j=1}^n w_j\nabla\varphi_j\Big)\cdot u\,dx,\end{aligned}$$

where we used that $\operatorname{div} w = \operatorname{div} v + \operatorname{div} \mathbf{a} = 0$. Let us prove that, if $r > \max\{\frac{n}{3}, \frac{2n}{n+2}\}$, then $(w\cdot\nabla)\varphi + \sum_{j=1}^n w_j\nabla\varphi_j \in L^{r'}(\Omega)$ and

$$\|(w\cdot\nabla)\varphi + \textstyle\sum_{j=1}^n w_j\nabla\varphi_j\|_{r'} \le c\big(\|v\|_{H^{2,r}(\Omega)} + \|\mathbf{a}\|_{L^\infty(\Omega)}\big)\|\varphi\|_{H^{1+\delta,r'}(\Omega)}, \tag{6.38}$$

where $\delta \in [0,1)$ is given by (6.12). In fact, if $\max\{\frac{n}{3}, \frac{2n}{n+2}\} < r < \frac{n}{2}$, then $H^{2,r}(\Omega) \hookrightarrow L^{\frac{nr}{n-2r}}(\Omega)$, $\frac{nr}{n-2r} > r'$ and $\frac{n-2r}{nr} + \frac{1}{s} = \frac{1}{r'}$, $H^{\delta,r'}\Omega) = H^{\frac{n}{r}-2,r'}(\Omega) \hookrightarrow L^s(\Omega)$. Hence

$$\|(v\cdot\nabla)\varphi\|_{r'} \le \|v\|_{\frac{nr}{n-2r}}\|\nabla\varphi\|_s \le c\|v\|_{H^{2,r}}\|\nabla\varphi\|_{H^{\delta,r'}};$$

in the case $r \ge \frac{n}{2}$ the inequality $\|(v\cdot\nabla)\varphi\|_{r'} \le c\|v\|_{H^{2,r}}\|\nabla\varphi\|_{H^{\delta,r'}}$ can be proved in a similar way as in the proof Lemma 6.1 (1). The remaining estimate for $(\mathbf{a}\cdot\nabla)\varphi$ is trivial since $\mathbf{a} \in L^\infty(\Omega)$ (see (6.11)).

Let $B^*_{r'}$ denote the adjoint of the operator B_r in $L^r_\sigma(\Omega)$. Then (6.38) and the embedding $D(A_{r'}^{(1+\delta)/2}) \hookrightarrow H^{1+\delta,r'}(\Omega)$ imply that

$$D(A_{r'}^{\frac{1+\delta}{2}}) \hookrightarrow D(B^*_{r'}) \tag{6.39}$$

with $B^*_{r'}\varphi = -(w\cdot\nabla)\varphi - \sum_{j=1}^n w_j \nabla\varphi_j$ for $\varphi \in D(A_{r'}^{\frac{1+\delta}{2}})$ and

$$\begin{aligned} \|B^*_{r'}\varphi\|_{L^{r'}_\sigma(\Omega)} &\le c(r,\delta,\Omega)\big(\|v\|_{H^{2,r}(\Omega)} + \|\mathbf{a}\|_{L^\infty(\Omega)}\big)\|\varphi\|_{H^{1+\delta,r'}(\Omega)} \\ &\le c(r,\delta,\Omega)\big(\|f\|_r + \Phi + \Phi^2\big)\|\varphi\|_{D(A_{r'}^{\frac{1+\delta}{2}})}. \end{aligned} \tag{6.40}$$

Since $L^r_\sigma(\Omega)$ is reflexive, also $S^*_{r'} = A_{r'} + B^*_{r'}$ generates a bounded analytic semigroup in $L^{r'}_\sigma(\Omega)$, see [63], Ch. 1, Corollary 10.6. Note that (6.40) and an interpolation inequality ([63], Ch. 2, Theorem 6.10) imply the $A_{r'}$-boundedness of $B^*_{r'}$ with $A_{r'}$-bound less than 1. Hence $A_{r'}+B^*_{r'}$ is closed and $D(A_{r'}+B^*_{r'}) = D(A_{r'})$, see [46], Ch. IV, Theorem 1.1. Moreover, (6.40) shows that $A_{r'} + B^*_{r'}$ is invertible if $\|f\|_r$ and Φ are sufficiently small. Since it is easily seen that $A_{r'}+B^*_{r'} \subset S^*_{r'}$ and since both operators $A_{r'} + B^*_{r'}$ and $S^*_{r'}$ are invertible, we conclude that $D(A_{r'}) = D(S^*_{r'})$. Now Theorem 6.9 and (2.16) imply for all $\theta \in [0,1]$ that

$$D((S^*_{r'})^\theta) = [L^{r'}_\sigma(\Omega), D(S^*_{r'})]_\theta = [L^{r'}_\sigma(\Omega), D(A_{r'})]_\theta = D(A^\theta_{r'}). \tag{6.41}$$

In particular, for all $r > \max\left\{\frac{n}{3}, \frac{2n}{n+2}\right\}$ and $\theta \in (0,1)$

$$\|u\|_{H^{2\theta,r'}(\Omega)} \le c(r,\theta,\Omega)\|(S^*_{r'})^\theta u\|_{L^{r'}_\sigma(\Omega)} \quad \forall u \in D((S^*_{r'})^\theta) = D(A^\theta_{r'}). \tag{6.42}$$

In the remainder of this paper we shall assume that the constant K_2 in Theorem 6.9 is so small that (6.41), (6.42) hold as well.

Remark 6.12 If $\max\{\frac{n}{3}, \frac{2n}{n+2}\} < r < q < \infty$, then obviously

$$e^{-tS^*_{r'}}\varphi = e^{-tS^*_{q'}}\varphi \quad \forall\varphi \in C^\infty_{0,\sigma}(\Omega)\ \forall t > 0.$$

Therefore, we shall write $e^{-tS^*}\varphi$ for $e^{-tS^*_{r'}}\varphi$ in the following.

The following lemma is crucial for the study of the stability of a solution to (SNS).

Lemma 6.13 *(L^r-L^q estimates) Let $\frac{n}{2} < r < q < \infty, n \ge 3$ and let $\alpha \in (0,\bar\alpha)$ be fixed, where $\bar\alpha$ is given by (6.1). Then the following estimates hold for all $u \in L^r_\sigma(\Omega)$ and $t > 0$:*

(1) $\|e^{-tS_r}u\|_q \le c(r,q,\alpha,n,\Omega)\, t^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\, e^{-\alpha t}\|u\|_r.$

(2) $\|\nabla S^\beta_r e^{-tS_r}u\|_q \le c(r,q,\alpha,\beta,\Omega)\, t^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{2}-\beta}\, e^{-\alpha t}\|u\|_r \quad \forall\beta \in (0,\frac{1}{2}).$

Moreover, for all $\varphi \in L^{r'}_\sigma(\Omega)$ and $\xi > r'$:

(1′) $\|e^{-tS^*_{r'}}\varphi\|_\xi \le c(r,\xi,\alpha,n,\Omega)\, t^{-\frac{n}{2}(\frac{1}{r'}-\frac{1}{\xi})}\, e^{-\alpha t}\|\varphi\|_{r'}.$

(2′) $\|\nabla(S^*_{r'})^\beta e^{-tS^*_{r'}}\varphi\|_\xi \le c(r,\xi,\alpha,\beta,\Omega)\, t^{-\frac{n}{2}(\frac{1}{r'}-\frac{1}{\xi})-\frac{1}{2}-\beta}\, e^{-\alpha t}\|\varphi\|_{r'} \quad \forall\beta \in (0,\frac{1}{2}).$

Proof: First let us prove (1). Let $\gamma = \frac{n}{2}(\frac{1}{r} - \frac{1}{q})$. Obviously, $\gamma \in (0,1)$. By the embedding $H^{2\gamma,r}(\Omega) \hookrightarrow L^q(\Omega)$ and (6.37) we get for all $u \in L^r_\sigma(\Omega)$ that

$$\begin{aligned}
\|e^{-tS_r}u\|_q &\le c_1(r,q,\Omega)\|e^{-tS_r}u\|_{H^{2\gamma,r}} \\
&\le c_2(r,q,\Omega)\|S_r^\gamma e^{-tS_r}u\|_{L^r_\sigma} \\
&= c_2(r,q,\Omega)\|S_r^\gamma e^{-\frac{\bar\alpha-\alpha}{\bar\alpha+\alpha}tS_r}\, e^{-\frac{2\alpha}{\bar\alpha+\alpha}tS_r}u\|_{L^r} \\
&\le c_3(r,\alpha,q,\Omega)\, t^{-\gamma}\, \|e^{-\frac{2\alpha}{\bar\alpha+\alpha}tS_r}u\|_{L^r},
\end{aligned}$$

where we used the well-known estimate $\|S_r^\theta e^{-tS_r}\|_{\mathcal{L}(L^r_\sigma(\Omega))} \le c(r,\theta,\Omega)t^{-\theta}$ for $\theta \in (0,1)$, $t>0$, for analytic semigroups. Thus by (6.33) with α replaced by $\frac{\bar\alpha+\alpha}{2}$ we get (1).

The assertion (2) can be proved in a similar way as (1) using additionally that

$$\|\nabla u\|_q \le c\|u\|_{H^{1,q}} \le c\|S_r^{1/2}u\|_q \quad \text{for all} \quad u \in C^\infty_{0,\sigma}(\Omega).$$

The proofs of (1′) and (2′) are similar and are omitted. ■

6.3 Exponential stability of stationary Navier Stokes flows

In this section we consider the exponential stability of stationary Navier Stokes flows in Ω. If the stationary solution $\{w, \nabla q\}$ is perturbed by a velocity field u_0 at time $t=0$, then the corresponding perturbed instationary flow $\{u(t)+w, \nabla(p(t)+q)\}$ is governed by the system

$$\begin{aligned}
u_t - \Delta u + (u\cdot\nabla)w + (w\cdot\nabla)u + (u\cdot\nabla)u + \nabla p &= 0 \quad \text{in} \quad \Omega\times(0,T) \\
\operatorname{div} u &= 0 \quad \text{in} \quad \Omega\times(0,T) \\
u &= 0 \quad \text{on} \quad \partial\Omega\times(0,T) \\
u(0) &= u_0 \quad \text{in} \quad \Omega.
\end{aligned} \tag{6.43}$$

Note that (6.43) is equivalent to the abstract problem

$$\begin{aligned}
u_t + S_r u + P_r(u\cdot\nabla)u &= 0, \\
u(0) &= u_0,
\end{aligned} \tag{6.44}$$

where S_r is the perturbed Stokes operator defined by (6.29) with w the solution to (SNS) given by Theorem 6.4. Hence the study of stability for (SNS) is reduced to the investigation of the behavior of solutions to (6.43) for $t\to\infty$.

In the following we fix $r \in [n,\infty)$ and an initial value $u_0 \in L^r_\sigma(\Omega)$.

Definition 6.14 *A function u is called a strong solution to* (6.43) *on* $[0,T)$, $0 < T \le \infty$, *if*

$$u \in BC([0,T), L^r_\sigma(\Omega)) \cap C^1((0,T), L^r_\sigma(\Omega)) \cap C((0,T), D(A_r)) \tag{6.45}$$

and u satisfies (6.43) *pointwise in* $t \in (0,T)$.

Remark 6.15 *Due to the Sobolev embedding $H^{2,r}(\Omega) \hookrightarrow L^q(\Omega)$ for $q \geq r$, any strong solution to* (6.43) *on* $(0,T)$ *belongs to* $C((0,T), L^q(\Omega))$ *for any* $q \geq r$.

If u is a strong solution to (6.43), then u satisfies the integral equation

$$u(t) = e^{-tS_r}u_0 - \int_0^t e^{-(t-s)S_r} P_r(u\cdot\nabla)u(s)\,ds,\ t\in(0,T), \tag{6.46}$$

hence, in consideration of Remark 6.12, for all $\varphi \in C^\infty_{0,\sigma}(\Omega)$ and $t \in (0,T)$

$$(u(t),\varphi) = (e^{-tS_r}u_0,\varphi) + \int_0^t \big((u\cdot\nabla)e^{-(t-s)S^*}\varphi, u(s)\big)\,ds. \tag{6.47}$$

For $r < q < \infty$ and $\alpha \in [\frac{\bar\alpha}{2},\bar\alpha)$ let

$$\begin{aligned} X_q(\alpha) := \{u : \ & e^{\alpha t}u \in BC([0,\infty), L^r_\sigma(\Omega)),\\ & t^{\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}e^{\alpha t}u \in BC((0,\infty), L^q_\sigma(\Omega)), \lim_{t\to+0} t^{\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\|u(t)\|_q = 0\},\\ \|u\|_{X_q(\alpha)} = \ & \|e^{\alpha t}u\|_{BC([0,\infty),L^r_\sigma(\Omega))} + \|t^{\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}e^{\alpha t}u\|_{BC((0,\infty),L^q_\sigma(\Omega))}. \end{aligned}$$

Obviously, $X_q(\alpha)$ is a Banach space. Moreover, $X_{q_1}(\alpha) \hookrightarrow X_{q_2}(\alpha)$ for $q_1 \geq q_2$ and

$$X_q(\alpha_1) \hookrightarrow X_q(\alpha_2) \quad \text{for } \alpha_1 > \alpha_2. \tag{6.48}$$

Definition 6.16 *A function u belonging to X_q for any $q > r, \alpha \in [\frac{\bar\alpha}{2},\bar\alpha)$ and satisfying* (6.47) *for all $t \in (0,\infty)$ is called a global mild solution to* (6.43).

For each $u, z \in X_q$ define the functional $F(u,z)(t), t \geq 0$, on $C^\infty_{0,\sigma}(\Omega)$ by

$$\langle F(u,z)(t),\varphi\rangle = \int_0^t \big((u(s)\cdot\nabla)e^{-(t-s)S^*}\varphi, z(s)\big)\,ds. \tag{6.49}$$

Then (6.47) can be rewritten formally as

$$u(t) = e^{-tS_r}u_0 + F(u,u)(t), \quad t > 0. \tag{6.50}$$

Lemma 6.17 *Let $n \leq r < q < \infty$ and $\alpha \in [\bar\alpha/2,\bar\alpha)$.*

(1) The operator $F(\cdot,\cdot)$ is a bilinear continuous mapping from $X_q(\alpha)\times X_q(\alpha)$ to $X_q(\alpha)$, i.e.,

$$\|F(u,z)\|_{X_q(\alpha)} \leq c\|u\|_{X_q(\alpha)}\|z\|_{X_q(\alpha)} \quad \forall u,z \in X_q(\alpha)$$

with $c = c(r,q,\alpha,\Omega) > 0$.

(2) For all $q \in (r,\infty)$ the operator $F(\cdot,\cdot)$ is a bilinear continuous mapping from $X_{2r}(\alpha)\times X_{2r}(\alpha)$ to $X_q(\alpha)$, i.e.,

$$\|F(u,z)\|_{X_q(\alpha)} \leq c\|u\|_{X_{2r}(\alpha)}\|z\|_{X_{2r}(\alpha)}$$

with $c = c(r,q,\alpha,\Omega) > 0$ for all $u,z \in X_{2r}(\alpha)$.

Proof: (1) For simplicity we write $X_q = X_q(\alpha)$ and $\gamma = \frac{n}{2}\big(\frac{1}{r} - \frac{1}{q}\big) \in \big(0, \frac{1}{2}\big)$. For $u, z \in X_q$ and $\varphi \in C_{0,\sigma}^\infty(\Omega)$

$$\begin{aligned}
|\langle F(u,z)(t), \varphi\rangle| &\leq \int_0^t \|u(s)\|_r \|\nabla e^{-(t-s)S^*}\varphi\|_\xi \|z(s)\|_q \, ds \\
&\leq \sup_{0<s<t} \big\{e^{\alpha s}\|u(s)\|_r\big\} \cdot \sup_{0<s<t} \big\{s^\gamma e^{\alpha s}\|z(s)\|_q\big\} \\
&\quad \times \int_0^t s^{-\gamma} e^{-2\alpha s} \|\nabla e^{-(t-s)S^*}\varphi\|_\xi \, ds,
\end{aligned} \tag{6.51}$$

for all $t > 0$, where $\frac{1}{\xi} = 1 - \frac{1}{r} - \frac{1}{q}$. By Lemma 6.13 (2′) with α replaced by $\frac{\bar\alpha + \alpha}{2}$

$$\|\nabla e^{-(t-s)S^*}\varphi\|_\xi \leq c(t-s)^{-\frac{n}{2}(\frac{1}{r'} - \frac{1}{\xi}) - \frac{1}{2}} e^{-\frac{\bar\alpha+\alpha}{2}(t-s)} \|\varphi\|_{r'} = c(t-s)^{-\frac{n}{2q} - \frac{1}{2}} e^{-\frac{\bar\alpha+\alpha}{2}(t-s)} \|\varphi\|_{r'}$$

with $c = c(r, q, \alpha, \Omega) > 0$. Hence (6.51) yields for all $t > 0$ that

$$|\langle F(u,z)(t), \varphi\rangle| \leq c \sup_{0<s<t} \big\{e^{\alpha s}\|u(s)\|_r\big\} \cdot \sup_{0<s<t} \big\{s^\gamma e^{\alpha s}\|z(s)\|_q\big\} e^{-\alpha t} \cdot I_1(t) \|\varphi\|_{r'},$$

where

$$\begin{aligned}
I_1(t) &= e^{-\frac{\bar\alpha-\alpha}{2}t} \int_0^t s^{-\gamma}(t-s)^{-\frac{n}{2q} - \frac{1}{2}} e^{-2\alpha + \frac{\bar\alpha+\alpha}{2}s} \, ds \\
&= e^{-\frac{\bar\alpha-\alpha}{2}t} t^{\frac{1}{2} - \frac{n}{2r}} \int_0^1 \tau^{-\gamma}(1-\tau)^{-\frac{n}{2q} - \frac{1}{2}} \, d\tau \\
&\leq cB(1-\gamma, \tfrac{1}{2} - \tfrac{n}{2q})
\end{aligned}$$

and $B(\cdot,\cdot)$ denotes the Beta function; note here that $-2\alpha + \frac{\bar\alpha+\alpha}{2} < 0$ for all $\alpha \in [\bar\alpha/2, \bar\alpha)$. Therefore, for $u, z \in X_q$ we have $F(u,z)(t) \in L^r_\sigma(\Omega)$ for all $t > 0$ and

$$e^{\alpha t}\|F(u,z)(t)\|_r \leq c \sup_{0\leq s\leq t} \big\{e^{\alpha s}\|u(s)\|_r\big\} \cdot \sup_{0<s\leq t} \big\{s^\gamma e^{\alpha s}\|z(s)\|_q\big\}, \tag{6.52}$$

where $c = c(r, q, \alpha, \Omega) > 0$.

Furthermore, for $u, z \in X_q$ we have

$$F(u,z) \in BC([0,\infty), L^r_\sigma(\Omega)), \tag{6.53}$$

since $t \to F(u,z)(t)$ is continuous from $[0,\infty)$ to $L^r_\sigma(\Omega)$. In fact, $t \to F(u,z)(t)$ is continuous at $t = 0$ in $L^r_\sigma(\Omega)$ due to (6.52). Moreover, for $t_1, t_2 \in (0,\infty), t_1 > t_2$,

$$\begin{aligned}
|\langle F(u,z)(t_1) - F(u,z)(t_2), \varphi\rangle| = \Big| &\int_{t_2}^{t_1} \big((u(s)\cdot\nabla)e^{-(t_1-s)S^*}\varphi,\ z(s)\big)\, ds \\
&+ \int_0^{t_2} \big((u(s)\cdot\nabla)(e^{-(t_1-t_2)S^*} - I)e^{-(t_2-s)S^*})\varphi,\ z(s)\big)\, ds\Big|
\end{aligned} \tag{6.54}$$

Then, by the same technique as in the proof of (6.52)

$$\begin{aligned}
&\Big|\int_{t_2}^{t_1} \big((u(s)\cdot\nabla)e^{-(t_1-s)S^*}\varphi,\ z(s)\big)\, ds\Big| \\
&\qquad \leq c\|u\|_{X_q}\|z\|_{X_q} \int_{t_2}^{t_1} s^{-\gamma}(t_1-s)^{-\frac{n}{2q} - \frac{1}{2}} \, ds\, \|\varphi\|_{r'}
\end{aligned} \tag{6.55}$$

where

$$\int_{t_2}^{t_1} s^{-\gamma}(t_1-s)^{-\frac{n}{2q}-\frac{1}{2}}\,ds \le c\,t_2^{-\gamma}(t_1-t_2)^{\frac{1}{2}-\frac{n}{2q}} \to 0$$

as $t_1 \to t_2$ or $t_2 \to t_1$. Moreover, we have

$$\begin{aligned}\Big|\int_0^{t_2} \big((u(s)\cdot\nabla)(e^{-(t_1-t_2)S^*}-I)e^{-(t_2-s)S^*}\varphi,\, z(s)\big)\,ds\Big| \\ \le \|u\|_{X_q}\|z\|_{X_q}\int_0^{t_2} s^{-\gamma}\|\nabla(e^{-(t_1-t_2)S^*}-I)e^{-(t_2-s)S^*}\varphi\|_\xi\,ds\end{aligned} \tag{6.56}$$

where $\frac{1}{\xi}=1-\frac{1}{r}-\frac{1}{q}$. Note that $0\in\rho(S^*_{r'})$, and by [63], Ch. 2, Theorem 6.13 (d), and Lemma 6.13 ($2'$)

$$\begin{aligned}\|\nabla(e^{-(t_1-t_2)S^*}-I)e^{-(t_2-s)S^*}\varphi\|_\xi &= \|\nabla e^{-\frac{t_2-s}{2}S^*}(e^{-(t_1-t_2)S^*}-I)e^{-\frac{t_2-s}{2}S^*}\varphi\|_\xi \\ &\le c(t_2-s)^{-\frac{n}{2q}-\frac{1}{2}}\|(e^{-(t_1-t_2)S^*}-I)e^{-\frac{t_2-s}{2}S^*}\varphi\|_{r'} \\ &\le c_\zeta(t_2-s)^{-\frac{n}{2q}-\frac{1}{2}}(t_1-t_2)^\zeta\|(S^*)^\zeta e^{-\frac{t_2-s}{2}S^*}\varphi\|_{r'} \\ &\le c_\zeta(t_2-s)^{-\frac{n}{2q}-\frac{1}{2}-\zeta}(t_1-t_2)^\zeta\|\varphi\|_{r'},\end{aligned}$$

where ζ is arbitrarily fixed in $(0,\frac{1}{2}-\frac{n}{2q})$. Thus, from (6.56) we get

$$\begin{aligned}\Big|\int_0^{t_2} \big((u(s)\cdot\nabla)(e^{-(t_1-t_2)S^*}-I)e^{-(t_2-s)S^*}\varphi,\, z(s)\big)\,ds\Big| \\ \le c_\zeta(t_1-t_2)^\zeta\|u\|_{X_q}\|z\|_{X_q}\int_0^{t_2} s^{-\gamma}(t_2-s)^{-\frac{n}{2q}-\frac{1}{2}-\zeta}\,ds\,\|\varphi\|_{r'} \\ \le \tilde{c}_\zeta(t_2)(t_1-t_2)^\zeta\|u\|_{X_q}\|z\|_{X_q}\|\varphi\|_{r'},\end{aligned}$$

which together with (6.54), (6.55) and (6.56) implies that the function $t\to F(u,z)(t)$ is continuous from $(0,\infty)$ to $L^r_\sigma(\Omega)$.

By a similar technique as in the proof of (6.52) we get for all $t>0$ that

$$\begin{aligned}|\langle F(u,z)(t),\varphi\rangle| &\le \int_0^t \|u(s)\|_q\|\nabla e^{-(t-s)S^*}\varphi\|_{q/(q-2)}\|z(s)\|_q\,ds \\ &\le c\sup_{0<s\le t}\big\{s^\gamma e^{\alpha s}\|u(s)\|_q\big\}\cdot\sup_{0<s\le t}\big\{s^\gamma e^{\alpha s}\|z(s)\|_q\big\} \\ &\qquad\cdot e^{-\frac{\bar\alpha+\alpha}{2}t}\int_0^t s^{-2\gamma}(t-s)^{-\frac{n}{2q}-\frac{1}{2}}\,ds\|\varphi\|_{q'}.\end{aligned}$$

Hence for all $t>0$ we have $F(u,z)(t)\in L^q_\sigma(\Omega)$ and

$$\begin{aligned}\|F(u,z)(t)\|_q \le c\sup_{0<s\le t}\big\{s^\gamma e^{\alpha s}\|u(s)\|_q\big\}\cdot\sup_{0<s\le t}\big\{s^\gamma e^{\alpha s}\|z(s)\|_q\big\} \\ \cdot t^{-\frac{n}{r}+\frac{n}{2q}+\frac{1}{2}}e^{-\frac{\bar\alpha+\alpha}{2}t}\int_0^1 \tau^{-2\gamma}(1-\tau)^{-\frac{n}{2q}-\frac{1}{2}}\,d\tau,\end{aligned}$$

yielding

$$\begin{aligned}t^\gamma e^{\alpha t}\|F(u,z)(t)\|_q &\le c_1\sup_{0<s\le t}\big\{s^\gamma e^{\alpha s}\|u(s)\|_q\big\}\cdot\sup_{0<s\le t}\big\{s^\gamma e^{\alpha s}\|z(s)\|_q\big\} \\ &\qquad\cdot t^{\frac{1}{2}-\frac{n}{2r}}e^{-\frac{\bar\alpha-\alpha}{2}t}\int_0^1 \tau^{-2\gamma}(1-\tau)^{-\frac{n}{2q}-\frac{1}{2}}\,d\tau \\ &\le c_2\sup_{0<s\le t}\big\{s^\gamma e^{\alpha s}\|u(s)\|_q\big\}\cdot\sup_{0<s\le t}\big\{s^\gamma e^{\alpha s}\|z(s)\|_q\big\},\end{aligned} \tag{6.57}$$

where $c_i = c_i(r, q, \alpha, \Omega) > 0, i = 1, 2$. In particular, (6.57) implies that

$$\lim_{t \to 0} t^{\gamma} \|F(u, z)(t)\|_q = 0.$$

Moreover, by a similar argument as in the proof of (6.53) we get that

$$t^{\gamma} e^{\alpha t} F(u, z)(t) \in BC((0, \infty), L^q(\Omega)).$$

Thus we proved (1).

(2) For $u, z \in X_{2r}$ and $\varphi \in C_{0,\sigma}^{\infty}(\Omega)$ we get

$$|\langle F(u, z)(t), \varphi \rangle| \leq \int_0^t \|u(s)\|_{2r} \|z(s)\|_{2r} \|\nabla e^{-(t-s)S^*} \varphi\|_{\frac{r}{r-1}} \, ds \quad \forall t > 0. \tag{6.58}$$

By Lemma 6.13 ($2'$)

$$\|\nabla e^{-(t-s)S^*} \varphi\|_{\frac{r}{r-1}} \leq c(r, q, \alpha, \Omega) \, (t-s)^{-\gamma - \frac{1}{2}} \, e^{-\frac{\bar{\alpha}+\alpha}{2}(t-s)} \|\varphi\|_{q'}.$$

Hence (6.58) yields for all $t > 0$ that

$$\begin{aligned} |\langle F(u, z)(t), \varphi \rangle| &\leq c \sup_{0 < s \leq t} \left\{ s^{\frac{n}{4r}} e^{\alpha s} \|u(s)\|_{2r} \right\} \cdot \sup_{0 < s \leq t} \left\{ s^{\frac{n}{4r}} e^{\alpha s} \|z(s)\|_{2r} \right\} \\ &\qquad \cdot e^{-\frac{\bar{\alpha}+\alpha}{2} t} \int_0^t s^{-\frac{n}{2r}} (t-s)^{-\gamma - \frac{1}{2}} \, ds \, \|\varphi\|_{q'} \\ &\leq c \sup_{0 < s \leq t} \left\{ s^{\frac{n}{4r}} e^{\alpha s} \|u(s)\|_{2r} \right\} \cdot \sup_{0 < s \leq t} \left\{ s^{\frac{n}{4r}} e^{\alpha s} \|z(s)\|_{2r} \right\} t^{-\gamma} e^{-\alpha t} I_2(t) \|\varphi\|_{q'}, \end{aligned}$$

where

$$I_2(t) = t^{\frac{1}{2} - \frac{n}{2r}} e^{-\frac{\bar{\alpha}-\alpha}{2} t} \int_0^1 \tau^{-\frac{n}{2r}} (1-\tau)^{-\gamma - \frac{1}{2}} \, d\tau \leq c B\big(1 - \frac{n}{2r}, \frac{1}{2} - \gamma\big)$$

for all $t > 0$. Therefore, for all $t > 0$ we have $F(u, z)(t) \in L_{\sigma}^q(\Omega)$ and

$$t^{\frac{n}{2}(\frac{1}{r} - \frac{1}{q})} e^{\alpha t} \|F(u, z)(t)\|_q \leq c \sup_{0 < s \leq t} \left\{ s^{\frac{n}{4r}} e^{\alpha s} \|u(s)\|_{2r} \right\} \cdot \sup_{0 < s \leq t} \left\{ s^{\frac{n}{4r}} e^{\alpha s} \|z(s)\|_{2r} \right\}, \tag{6.59}$$

where $c = c(r, q, \alpha, \Omega) > 0$. It follows directly from (6.59) that

$$\lim_{t \to +0} t^{\frac{n}{2}(\frac{1}{r} - \frac{1}{q})} \|F(u, z)(t)\|_q = 0.$$

Moreover, as in the proof of (6.53), it is easily seen that the mapping $t \mapsto t^{\frac{n}{2}(\frac{1}{r} - \frac{1}{q})} F(u, z)(t)$ is continuous from $(0, \infty)$ to $L_{\sigma}^q(\Omega)$. Therefore, from (6.52) with $q = 2r$ and (6.59) we get $F(u, z) \in X_{2r}$ and the inequality in (2).

The proof of this lemma is complete. ■

Remark 6.18 Due to Lemma 6.17 (2) and Lemma 6.13 (1), it follows that, if a function u satisfying $u \in X_{2r}(\alpha)$ for all $\alpha \in [\bar{\alpha}/2, \bar{\alpha})$ solves the equation (6.50), then it is a global mild solution to the system (6.43).

Theorem 6.19 *(Existence of Global Mild Solutions) Let $n \le r < \infty$, $f \in L_r(\Omega)$ and let the fluxes $\Phi_1, \dots, \Phi_m \in \mathbb{R}$ satisfy*

$$\|f\|_r + \Phi + \Phi^2 < \min\{K_1, K_2\},$$

where $\Phi = \sum_{i=1}^m |\Phi_i|$ and $K_i = K_i(r, \Omega), i = 1, 2$, are the constants in Theorem 6.4, Theorem 6.9, respectively. Then there exists a constant $\delta_0 = \delta_0(r, \Omega) > 0$ such that for all $u_0 \in L^r_\sigma(\Omega)$ satisfying $\|u_0\|_r < \delta_0$ the system (6.43) – with the unique solution w to (SNS) corresponding to $f, \Phi_1, \dots, \Phi_m$ given by Theorem 6.4 – has a global mild solution u which is unique in a small ball of $X_{2r}(\bar{\alpha}/2)$. This solution u has the following properties for all $\alpha \in (0, \bar{\alpha})$ and $\theta \in (0, \frac{1}{2} + \frac{n}{2r})$:

$$\lim_{t\to\infty} e^{\alpha t}\|u(t)\|_q = 0 \quad \textit{for all} \quad q \ge r, \tag{6.60}$$

$$\lim_{t\to+0} t^{\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\|u(t)\|_q = 0 \quad \textit{for all} \quad q > r, \tag{6.61}$$

$$t^\theta e^{\alpha t} u \in BC((0,\infty), D(S_r^\theta)), \tag{6.62}$$

$$\lim_{t\to\infty} e^{\alpha t}\|u(t)\|_{D(S_r^\theta)} = 0, \tag{6.63}$$

$$\lim_{t\to+0} t^\theta \|u(t)\|_{D(S_r^\theta)} = 0. \tag{6.64}$$

Remark 6.20 It follows from (6.62) that the global mild solution given by Theorem 6.19 solves the integral equation (6.46).

Proof of Theorem 6.19: First we note that

$$\lim_{t\to+0} t^{\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\|e^{-tS_r}u_0\|_q = 0 \quad \text{for all } q > r. \tag{6.65}$$

In fact, for $\gamma = \frac{n}{2}(\frac{1}{r} - \frac{1}{q}) \in \left(0, \frac{1}{2}\right)$ and with the embedding $H^{2\gamma,r}(\Omega) \hookrightarrow L^q(\Omega)$, Proposition 6.11 yields

$$\begin{aligned} t^\gamma\|e^{-tS_r}u_0\|_q &\le ct^\gamma\|e^{-tS_r}u_0\|_{H^{2\gamma,r}(\Omega)} \\ &\le ct^\gamma\|S_r^\gamma e^{-tS_r}u_0\|_r \\ &\le c\|e^{-tS_r}u_0\|_r^{1-\gamma}\|tS_r\, e^{-tS_r}u_0\|_r^\gamma, \end{aligned} \tag{6.66}$$

where $c = c(r, q, \Omega) > 0$. Since $\|tS_r e^{-tS_r}u_0\|_r \to 0$ as $t \to 0$, the denseness of $D(S_r)$ in $L^r_\sigma(\Omega)$ and the boundedness of the operator family $\{tS_r\, e^{-tS_r}\}_{t\ge0}$ in $\mathcal{L}(L^r_\sigma(\Omega))$ imply (6.65).

By Theorem 5.9, Lemma 3.10 (1) and (6.65) we get for all $\alpha \in [\bar{\alpha}/2, \bar{\alpha})$ that

$$e^{-tS_r}u_0 \in X_q(\alpha), \quad \forall q > r \tag{6.67}$$

and, in particular,

$$\|e^{-tS_r}u_0\|_{X_{2r}(\alpha)} \le \sup_{t>0} e^{\alpha t}\|e^{-tS_r}u_0\|_r + \sup_{t>0} t^{\frac{n}{4r}} e^{\alpha t}\|e^{-tS_r}u_0\|_{2r} < C_*\|u_0\|_r \tag{6.68}$$

with some constant $C_* = C_*(r, \alpha, \Omega) > 0$.

Now let us define the mapping $\Psi_{\alpha,u_0} : X_{2r}(\alpha) \to X_{2r}(\alpha)$ by

$$\Psi_{\alpha,u_0} u = e^{-tS_r} u_0 + F(u,u)$$

for a fixed $\alpha \in [\bar\alpha/2, \bar\alpha)$. Let $C_{**} = C_{**}(r, \alpha, \Omega)$ denote the constant in the inequality of Lemma 6.17 (1) with $q = 2r$. Then

$$\|\Psi_{\alpha,u_0} u\|_{X_{2r}(\alpha)} \le \|e^{-tS_r} u_0\|_{X_{2r}(\alpha)} + \|F(u,u)\|_{X_{2r}(\alpha)} \le C_* \|u_0\|_r + C_{**} \|u\|^2_{X_{2r}(\alpha)}. \tag{6.69}$$

Note that, if

$$\|u_0\|_r < C_0(r, \alpha, \Omega) := \frac{1}{8 C_* C_{**}}, \tag{6.70}$$

then

$$K = K(\alpha, \|u_0\|_r) := \frac{1 - \sqrt{1 - 4 C_* C_{**} \|u_0\|_r}}{2 C_{**}} < \frac{1}{2 C_{**}} \tag{6.71}$$

and the inequality $C_* \|u_0\|_r + C_{**} K^2 \le K$ holds. Therefore, we get from (6.69) that

$$\Psi_{\alpha,u_0}(U_{K,\alpha}) \subset U_{K,\alpha} := \{u \in X_{2r}(\alpha) : \|u\|_{X_{2r}(\alpha)} \le K\}.$$

For any $u, z \in U_{K,\alpha}$

$$\begin{aligned}
\|\Psi_{\alpha,u_0} u - \Psi_{\alpha,u_0} z\|_{X_{2r}(\alpha)} &= \|F(u, u - z) - F(u - z, z)\|_{X_{2r}(\alpha)} \\
&\le C_{**}(\|u\|_{X_{2r}(\alpha)} + \|z\|_{X_{2r}(\alpha)}) \|u - z\|_{X_{2r}(\alpha)} \\
&\le 2 C_{**} K \|u - z\|_{X_{2r}(\alpha)}.
\end{aligned}$$

Hence, in view of $2C_{**}K < 1$, see (6.71), $\Psi_\alpha : U_{K,\alpha} \to U_{K,\alpha}$ is a contraction mapping, and by the Banach fixed point theorem it has a unique fixed point u in $U_{K,\alpha}$.

Now let $u \in X_{2r}(\bar\alpha/2)$ be the unique fixed point of $\Psi_{\bar\alpha/2}$ in $U_{K(\bar\alpha/2, \|u_0\|_r)}$. We shall show that $u \in X_{2r}(\alpha)$ for all $\alpha \in [\bar\alpha/2, \bar\alpha)$. Since $\|u(t)\|_r$ decays as time tends to infinity, for any $\alpha \in (\bar\alpha/2, \bar\alpha)$ there is a (sufficiently large) $t_1(\alpha) > 0$ such that

$$\|u(t_1)\|_r \le \min\{C_0(r, \bar\alpha/2, \Omega), C_0(r, \alpha, \Omega)\}, \tag{6.72}$$

see (6.70), and

$$U_{K(\alpha, \|u(t_1)\|_r), \alpha} \subset U_{K(\bar\alpha/2, \|u_0\|_r), \bar\alpha/2} \tag{6.73}$$

due to (6.48) and the fact that $K(\alpha, \|u_0\|_r) \to 0$ as $\|u_0\|_r \to 0$, see (6.71). Then, due to (6.72), there is a fixed point $\tilde u \in U_{K(\alpha, \|u(t_1)\|_r), \alpha} \subset X_{2r}(\alpha)$ of $\Psi_{\alpha, u(t_1)}$. Note that $\tilde u$ is also a fixed point of $\Psi_{\bar\alpha/2, u(t_1)}$ in $U_{K(\bar\alpha/2, \|u_0\|_r), \bar\alpha/2}$ due to (6.73). We shall show that $\tilde u(t), t \ge 0$, coincides with $u(t + t_1), t \ge 0$. Obviously, $u(\cdot + t_1) \in X_{2r}(\bar\alpha/2)$ and $\|u(\cdot + t_1)\|_{X_{2r}(\bar\alpha/2)} \le K(\bar\alpha/2, \|u_0\|_r)$. Moreover, we can check that $u(\cdot + t_1)$ solves (6.47), hence (6.50), since for all $t > t_1$ and $\varphi \in C^\infty_{0,\sigma}(\Omega)$

$$\begin{aligned}
(u(t), \varphi) &= (e^{-tH_r} u_0, \varphi) + \int_0^t \big((u(s) \cdot \nabla) e^{-(t-s)H^*} \varphi, u(s)\big)\, ds \\
&= (e^{-(t-t_1)H_r} u(t_1), \varphi) + \int_{t_1}^t \big((u(s) \cdot \nabla) e^{-(t-s)H^*} \varphi, u(s)\big)\, ds.
\end{aligned}$$

and $\lim_{t\to+t_1}(t-t_1)^{\frac{n}{4r}}\|u(t)\|_{2r}=0$. Therefore, in view of (6.72), $u(\cdot+t_1)$ is the unique fixed point of $\Psi_{\bar\alpha/2,u(t_1)}$ in $U_{K(\bar\alpha/2,\|u_0\|_r),\bar\alpha/2}$. Consequently, we get $\tilde u(\cdot)=u(\cdot+t_1)$ yielding $u\in X_{2r}(\alpha)$.

Formulae (6.60) and (6.61) are direct consequences of $u\in X_q(\alpha)$ for all $q\in(r,\infty)$ and $\alpha\in[\bar\alpha/2,\bar\alpha)$.

Now let $\theta\in(0,\frac12+\frac{n}{2r})$ and fix $p\in(r,\infty)$ such that

$$\frac{n}{p}<\frac{n}{2r}+\frac12-\theta.$$

It is enough to prove (6.62)-(6.64) for $\alpha\in[\bar\alpha/2,\bar\alpha)$. By Lemma 6.13 (2′) with α replaced by $\frac{\bar\alpha+\alpha}{2}$ we get for all $\varphi\in D(S_{r'}^{*\theta})$ that

$$\begin{aligned}
\big|\langle F(u,u)(t),(S_{r'}^*)^\theta\varphi\rangle_{L^r,L^{r'}}\big| &= \Big|\int_0^t \big((u(s)\cdot\nabla)(S^*)^\theta e^{-(t-s)S_{r'}^*}\varphi,u(s)\big)\,ds\Big|\\
&\le \int_0^t \|u(s)\|_p^2\|\nabla(S_{r'}^*)^\theta e^{-(t-s)S_{r'}^*}\varphi\|_{p/(p-2)}\,ds\\
&\le c\sup_{0<s\le t}\big\{s^\gamma e^{\alpha s}\|u(s)\|_p\big\}^2\int_0^t s^{-2\gamma}(t-s)^{-\frac{n}{p}+\frac{n}{2r}-\frac12-\theta}e^{-2\alpha s}e^{-\frac{\alpha+\bar\alpha}{2}(t-s)}\,ds\,\|\varphi\|_{r'}\\
&\le c\sup_{0<s\le t}\big\{s^\gamma e^{\alpha s}\|u(s)\|_p\big\}^2 t^{-\theta}e^{-\alpha t}I_3(t)\|\varphi\|_{r'}
\end{aligned}$$

for all $t>0$, where $\gamma=\frac{n}{2}(\frac1r-\frac1p)$ and

$$I_3(t)\equiv t^{\frac12-\frac{n}{2r}}e^{-\frac{\bar\alpha-\alpha}{2}t}\int_0^1 s^{-2\gamma}(1-s)^{-\frac{n}{p}+\frac{n}{2r}-\frac12-\theta}\,ds\le c\quad\forall t>0.$$

Therefore, in view of $(S_{r'}^*)^\theta=(S_r^\theta)^*$, we get $F(u,u)(t)\in D(S_r^\theta)$ for all $t>0$ and

$$t^\theta e^{\alpha t}\|S_r^\theta F(u,u)(t)\|_r\le c\big\{\sup_{0<s\le t}s^{\frac n2(\frac1r-\frac1p)}e^{\alpha s}\|u(s)\|_p\big\}^2. \tag{6.74}$$

On the other hand, by the same technique to prove (6.53) it can be seen that the function $t\to S_r^\theta F(u,u)(t)$ is continuous from $(0,\infty)$ to $L_\sigma^r(\Omega)$, which together with (6.74) yields (6.62), (6.63). Moreover, (6.74) implies (6.64) due to $u\in X_p(\alpha)$.

Finally let us prove that this fixed point is unique in the whole space $X_{2r}(\alpha)$ rather than only in $U_{K(\alpha,\|u_0\|_r),\alpha}$. Given fixed points $u_1,u_2\in X_{2r}(\alpha)$ of Ψ_α we get from (6.59) with $q=2r$ that for all $t>0$

$$\begin{aligned}
&\sup_{0<s\le t}\big\{s^{\frac{n}{4r}}e^{\alpha s}\|u_1(s)-u_2(s)\|_{2r}\big\}\\
&\quad\le c\sup_{0<s\le t}\big\{s^{\frac{n}{4r}}e^{\alpha s}\big(\|u_1(s)\|_{2r}+\|u_2(s)\|_{2r}\big)\big\}\cdot\sup_{0<s\le t}\big\{s^{\frac{n}{4r}}e^{\alpha s}\|u_1(s)-u_2(s)\|_{2r}\big\}.
\end{aligned}$$

Since $s^{\frac{n}{4r}}\big(\|u_1(s)\|_{2r}+\|u_2(s)\|_{2r}\big)\to0$ as $s\to0$, there exists $t_1=t_1(u_1,u_2)>0$ such that $u_1\equiv u_2$ in $[0,t_1]$. Defining $T=\sup\{t_1>0: u_1\equiv u_2 \text{ on } [0,t_1]\}$, a continuity argument yields $u_1\equiv u_2$ on $[0,T]$. If $T<\infty$, we repeat the above argument by starting at T and conclude that $u_1\equiv u_2$ on $[0,T+t_2]$ for some $t_2=t_2(u_1,u_2)>0$ in contradiction with the definition of T.

The proof of this theorem is complete. ■

We shall see in the next theorem that the global mild solution given by Theorem 6.19 is actually a strong solution to system (6.43). More precisely we have the following theorem.

Theorem 6.21 *(Existence of Global Strong Solution) The global weak solution given by Theorem 6.19 is a strong solution to* (6.43)*.*

Proof: Let u be the global mild solution to (6.43) given by Theorem 6.19. We shall prove that for all $\varepsilon > 0$ and $T > \varepsilon$

$$P_r(u \cdot \nabla)u \in C([\varepsilon, T], D(S_r^\zeta)) \tag{6.75}$$

with some $\zeta \in (0,1)$. Then by well-known results on analytic semigroups (see e.g. [63], Ch. 4, Theorem 3.6 or [10], Ch. II, Theorem 1.2.2)

$$u(t) = e^{-tS_r}u_0 - \int_0^t e^{-(t-s)S_r} P_r(u \cdot \nabla)u(s)\, ds$$

is a strong solution on $(\varepsilon, T]$ to (6.43) for any $0 < \varepsilon < T < \infty$, i.e.,

$$u \in C([\varepsilon, T], L^r_\sigma(\Omega)) \cap C^1((\varepsilon, T], L^r_\sigma(\Omega)) \cap C((\varepsilon, T], D(S_r)) \quad \forall t \in (\varepsilon, T].$$

Note $D(S_r) = D(A_r)$, see (6.29). Therefore $u \in C^1((0,\infty), L^r_\sigma(\Omega)) \cap C((0,T]; D(A_r))$ and consequently, u is a global strong solution to (6.43) since u belongs to $BC([0,\infty), L^r_\sigma(\Omega))$ as a global mild solution.

Fix $\theta \in (\frac{1}{2}, \frac{1}{2} + \frac{1}{2r})$, and let $\xi = 2\theta$ and $\zeta = \frac{1}{2}\frac{\xi - \eta}{2 - \eta}$ where $\eta \geq 1$ is defined by (6.15) (with $\delta' > 0$ arbitrarily small when $r = n$) so that $2\zeta \leq \xi - 1 < \frac{1}{r}$. Then by Lemma 6.1 (3) (with $\delta = 0$) and Proposition 6.11

$$\|P_r(u \cdot \nabla)v\|_{H^{2\zeta,r}(\Omega)} \leq c\|u\|_{H^{\xi,r}(\Omega)}\|v\|_{H^{\xi,r}(\Omega)} \leq c\|u\|_{D(S_r^\theta)}\|v\|_{D(S_r^\theta)}.$$

Since $\|\cdot\|_{D(S_r^\zeta)}$ is equivalent to $\|\cdot\|_{H^{2\zeta,r}(\Omega)}$, see Proposition 6.11, we conclude that $P_r(u(t) \cdot \nabla)u(t) \in D(S_r^\zeta)$ for all $t \in [\varepsilon, T]$ and

$$\begin{aligned}
&\|P_r((u(t_1) \cdot \nabla)u(t_1) - (u(t_2) \cdot \nabla)u(t_2))\|_{D(S_r^\zeta)} \\
&\qquad \leq c\big(\|u(t_1)\|_{D(S_r^\theta)} + \|u(t_2)\|_{D(S_r^\theta)}\big)\|u(t_1) - u(t_2)\|_{D(S_r^\theta)}.
\end{aligned}$$

Hence (6.62) yields (6.75). ■

Theorem 6.22 *(Uniqueness of Strong Solution)*

(1) Let $r \in (n, \infty)$. If $u_0 \in L^r_\sigma(\Omega)$, then the strong solution to (6.43) *is unique.*

(2) If $u_0 \in H^{s,n}(\Omega) \cap L^n_\sigma(\Omega)$ for some $s > 0$, then the strong solution to (6.43) *is unique.*

(3) Let $u_0 \in L^n_\sigma(\Omega)$ and let u_1, u_2 be strong solutions to (6.43) *on $[0,T)$ satisfying*

$$\lim_{t \to +0} t^{\frac{1}{2} - \frac{n}{2q}} u_i(t) = 0 \quad in \quad L^q(\Omega),\ i = 1, 2, \tag{6.76}$$

for some $q > n$. Then $u_1 \equiv u_2$.

Proof: (1) Let $r > n$. If u_1, u_2 are strong solutions on $[0,T)$, $0 < T < \infty$, to (6.43), we have by Lemma 6.13 (2′) for all $\varphi \in C^\infty_{0,\sigma}(\Omega)$ and $t \in (0,T)$ that

$$\begin{aligned}
\big|\langle u_1(t) - u_2(t), \varphi\rangle\big| &= \Big| \int_0^t \big(e^{-(t-s)S_r} P_r((u_1(s)\cdot\nabla)u_1(s) - (u_2(s)\cdot\nabla)u_2(s)), \varphi\big)\, ds\Big| \\
&\le \Big| \int_0^t \big((u_1(s) - u_2(s)\cdot\nabla)e^{-(t-s)S^*}\varphi, u_1(s)\big)\, ds\Big| \\
&\quad + \Big| \int_0^t \big((u_2(s)\cdot\nabla)e^{-(t-s)S^*}\varphi, u_1(s) - u_2(s)\big)\, ds\Big| \\
&\le \|u_1, u_2\|_{BC([0,T),L^r_\sigma(\Omega))} \int_0^t \|\nabla e^{-(t-s)S^*}\varphi\|_{\frac{r}{r-2}} \|u_1(s) - u_2(s)\|_r\, ds \\
&\le c\|u_1, u_2\|_{BC([0,T),L^r_\sigma(\Omega))} \int_0^t (t-s)^{-\frac{n}{2r}-\frac{1}{2}} \|u_1(s) - u_2(s)\|_r\, ds \|\varphi\|_{r'}.
\end{aligned}$$

Hence we get

$$\|u_1(t) - u_2(t)\|_r \le c\int_0^t (t-s)^{-\frac{n}{2r}-\frac{1}{2}} \|u_1(s) - u_2(s)\|_r\, ds \quad \forall t \in (0,T),$$

which implies after a finite number of integrations and due to Gronwall's lemma that $u_1(t) = u_2(t)$ on $(0,T)$.

(2) Due to the Sobolev embedding $H^{s,n}(\Omega) \hookrightarrow L^r(\Omega)$ for some $r > n$, the assertion (2) follows from the assertion (1).

(3) Next let $u_0 \in L^n_\sigma(\Omega)$. Let u_1, u_2 be strong solutions to (6.43) satisfying (6.76). In view of the above proved uniqueness result (1), (2) and the fact that any strong solution belongs to $C([\varepsilon,T], L^r(\Omega))$ for any $0 < \varepsilon < T$ and some $r > n$, it is enough to show $u_1(t) = u_2(t)$ for some $\delta > 0$. This can be done in the same way as in [49], Lemma 3.2. By the same technique as above, see also (6.51), (6.52), we get for all $t > 0$ and $\varphi \in C^\infty_{0,\sigma}(\Omega)$ that

$$\begin{aligned}
|\langle u_1(t) - u_2(t), \varphi\rangle| &\le \int_0^t \|\nabla e^{-(t-s)S^*}\varphi\|_\xi \|u_1(s) - u_2(s)\|_n (\|u_1(s)\|_q + \|u_2(s)\|_q)\, ds \\
&\le D(t)K(t) \int_0^t s^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})} \|\nabla e^{-(t-s)H^*}\varphi\|_\xi\, ds \\
&\le cD(t)K(t) \textstyle\int_0^t s^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})} (t-s)^{-\frac{n}{2q}-\frac{1}{2}} \|\varphi\|_{n'},
\end{aligned}$$

where

$$D(t) = \sup_{0<s\le t} \|u_1(s) - u_2(s)\|_n, \quad K(t) = \sum_{i=1}^2 \sup_{0<s\le t} s^{\frac{n}{2}(\frac{1}{n}-\frac{1}{q})} \|u_i(s)\|_q$$

and $\frac{1}{\xi} = 1 - \frac{1}{n} - \frac{1}{q}$. Therefore we get $\|u_1(t) - u_2(t)\|_n \le C_0 K(t)D(t)$, with some $C_0 > 0$ and even

$$D(t) \le C_0 K(t) D(t)$$

for all $t > 0$. By assumption, we have $\lim_{t\to 0} K(t) = 0$; hence, there is some $\delta > 0$ such that $C_0 K(t) < 1$ for all $t \in (0,\delta)$. Thus $D(t) = 0$, i.e., $u_1(t) = u_2(t)$ for $t \in (0,\delta)$.

The proof of the theorem is complete. ■

References

[1] T. Abe, On a resolvent estimate of the Stokes equation with Neumann-Dirichlet-type boundary condition on an infinite layer, Math. Methods Appl. Sci. 27 (2004), 1007-1048

[2] T. Abe and Y. Shibata, On a resolvent estimate of the Stokes equation on an infinite layer, J. Math. Soc. Japan 55 (2003), 469-497

[3] T. Abe and Y. Shibata, On a resolvent estimate of the Stokes equation on an infinite layer, II: $\lambda = 0$ case, J. Math. Fluid Mech. 5 (2003), 245-274

[4] H. Abels, Reduced and generalized Stokes resolvent estimates in asymptotically flat layers, Part I: unique solvability, J. Math. Fluid Mech. 7 (2005), 201-222

[5] H. Abels, Reduced and generalized Stokes resolvent estimates in asymptotically flat layers, Part II: H^∞-calculus. J. Math. Fluid Mech. 7 (2005), 223-260

[6] H. Abels, Stokes Equations in Asymptotically Flat Domains and the Motion of a Free Surface, Dissertation, Shaker, 164 p., 2003

[7] H. Abels, Bounded imaginary powers and H_∞-calculus of the Stokes operator in unbounded domains, Preprint No. 2362, FB Mathematik, TU Darmstadt, 2004

[8] H. Abels and M. Wiegner, Resolvent estimates for the Stokes operator on an infinite layer. Preprint No. 2363, FB Mathematik, TU Darmstadt, 2004

[9] S. Agmon, A. Douglis and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I, Comm. Pure Appl. Math. 12 (1959), 623-727

[10] H. Amann, *Linear and Quasilinear Parabolic Problems, Vol. I.: Abstract Linear Theory*, Birkhäuser, 1995

[11] H. Amann, Operator-valued Fourier multipliers, vector-valued Besov spaces, and applications, Math. Nachr. 186 (1997), 5-56

[12] C. J. Amick, Steady solutions of the Navier-Stokes equations in unbounded channels and pipes, Ann. Scuola Norm. Sup. Pisa 4 (1977), 473-513

[13] J. Bergh and J. Löfström, *Interpolation Spaces,* Springer, 1976

[14] M. E. Bogovskii, Decomposition of $L_p(\Omega; \mathbb{R}^n)$ into the direct sum of subspaces of solenoidal and potential vector fields, Sov. Math., Dokl. 33 (1986), 161-165; translation from Dokl. Akad. Nauk SSSR 286 (1986), 781-786

[15] W. Borchers and H. Sohr, On the semigroup of the Stokes operator for exterior domains in L^q-spaces, Math. Z. 196 (1987), 415-425

[16] W. Borchers and H. Sohr, On the equations rot $\mathbf{v} = g$ and div $\mathbf{u} = f$ with zero boundary conditions, Hokkaido Math. J. 19, 67-87 (1990).

[17] J. Bourgain, Some remarks on Banach spaces in which martigale difference sequences are unconditional, Arkiv Math. 21 (1983), 163-168

[18] M. Cowling, I. Doust, A. McIntosh, and A. Yagi, Banach space operators with a bounded H^∞ functional calculus, J. Aust. Math. Soc., Ser. A 60 (1996), 51-89

[19] R. Denk, M. Hieber and J. Prüss, $\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type, Mem. Am. Math. Soc. 788, 114 p., 2003

[20] R. Denk, G. Dore, M. Hieber, J. Prüss and A. Venni, New thoughts on old results of R. T. Seeley, Math. Ann. 328 (2004), 545-583

[21] W. Desch, M. Hieber and J. Prüss, L^p-theory of the Stokes equation in a half-space, J. Evol. Equ., 1 (2001), 115-142

[22] P. Deuring, The resolvent problem for the Stokes system in exterior domains: An elementary approach, Math. Methods Appl. Sci. 13 (1990), 335-349

[23] J. Diestel, H. Jarchow and A. Tonge, *Absolutely Summing Operators,* Cambridge University Press, 1995

[24] G. Dore and A. Venni, On the closedness of the sum of two closed operators, Math. Z. 196 (1987), 189-201

[25] R. Farwig, Note on the flux condition and pressure drop in the resolvent Problem of the Stokes System, Manuscripta Math. 89 (1996), 139–158

[26] R. Farwig and H. Sohr, An approach to resolvent estimates for the Stokes equations in L^q- spaces, Lecture Notes in Math. 1530, 97-110 (1992).

[27] R. Farwig and H. Sohr, Generalized Resolvent estimates for the Stokes system in bounded and unbounded domains, J. Math. Soc. Japan 46 (1994), 607-643

[28] R. Farwig and H. Sohr, Helmholtz decomposition and Stokes resolvent system for aperture domains in L^q–spaces, Analysis 16 (1996), 1–26

[29] R. Farwig and H. Sohr, Weighted L^q-theory for the Stokes resolvent in exterior domains, J. Math. Soc. Japan 49 (1997), 251-288

[30] R. Farwig, Weighted L^q-Helmholtz decompositions in infinite cylinders and in infinite layers, Adv. Differ. Equ. 8 (2003), 357-384

[31] R. Farwig, H. Kozono and H. Sohr, An L^q–approach to Stokes and Navier-Stokes equations in general domains, Acta Math. 195 (2005), 21-53

[32] A. Fröhlich, Maximal regularity for the non-stationary Stokes system in an aperture domain, J. Evol. Eq. 2 (2002), 471-493

[33] A. Fröhlich, The Stokes operator in weighted L^q-spaces I: Weighted estimates for the Stokes resolvent problem in a half space, J. Math. Fluid Mech. 5 (2003), 166-199

[34] A. Fröhlich, The Stokes operator in weighted L^q-spaces II: Weighted resolvent estimates and maximal regularity, Preprint No. 2173, FB Mathematik, TU Darmstadt, 2001

[35] G. P. Galdi, An introduction to the Navier-Stokes initial-boundary value problem, G. P. Galdi (ed.) et al., Fundamental directions in mathematical fluid mechanics, Birkhäuser, 1-70 (2000)

[36] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations, Vol. 1: Linearized Steady Problems,* Springer Tracts in Natural Philosophy, 38, Springer, 1994

[37] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations, Vol. II: Nonlinear Steady Problems,* Springer Tracts in Natural Philosophy, 39, Springer, 1994

[38] J. Garcia-Cuerva, J.L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics,* North Holland, Amsterdam, 1985

[39] Y. Giga, Analyticity of the semigroup generated by the Stokes operator in L_r spaces, Math. Z. 178 (1981), 297-329

[40] Y. Giga, Domains of fractional powers of the Stokes operator in L_r spaces, Arch. Ration. Mech. Anal. 89 (1985), 251-265

[41] R. Haller, H. Heck and A. Noll, Mikhlin's theorem for operator-valued Fourier multipliers in n variables, Math. Nachr. 244 (2002), 110-130

[42] J. G. Heywood, The Navier-Stokes equations: On the existence, regularity and decay of solutions, Indiana Univ. Math. J. 29 (1980), 639-681

[43] J. G. Heywood, Open problems in the theory of the Navier-Stokes equations for viscous incompressible flow, Lect. Notes Math. 1431, 1-22 (1990)

[44] M. Hieber and J. Prüss, Heat kernels and maximal $L^p - L^q$ estimate for parabolic evolution equations, Comm. Partial Diff. Equat. 22 (1997), 1647-1669

[45] E. Hopf, Über die Anfangswertaufabe fr die hydrodynamischen Grundgleichungen, Math. Nachr. 4 (1951), 213-231

[46] T. Kato, *Perturbation theory for linear operators*, Springer, 1966

[47] H. Kozono and M. Yamazaki, Exterior problem for the Navier-Stokes equations, existence, uniqueness and stability of stationary solutions, Heywood, J. G. (ed.) et al., Theory of the Navier-Stokes equations, World Scientific. Ser. Adv. Math. Appl. Sci. 47 (1998), 86-98

[48] H. Kozono and M. Yamazaki, The stability of small stationary solutions in Morrey spaces of the Navier-Stokes equation, Indiana Univ. Math. J. 44, (1995), 1307-1336

[49] H. Kozono and T. Ogawa, On stability of Navier-Stokes flows in exterior domains, Arch. Ration. Mech. Anal. 128, (1994), 1-31

[50] H. Kozono and T. Ozawa, Stability in L^r for the Navier-Stokes flow in an n-dimensional bounded domain, J. Math. Anal. Appl. 152, (1990), 35-45

[51] O. A. Ladyzhenskaya and V. A. Solonnikov, On the solvability of boundary and initial boundary value problems for Navier-Stokes equations with noncompact boundaries, Vestnik Leningrad Univ. 13 (1977), 39-47

[52] O. A. Ladyzhenskaya and V. A. Solonnikov, Determination of solutions of boundary value problems for stationary Stokes and Navier-Stokes equations having an unbounded Dirichlet integral, J. Sov. Math. 21 (1983), 728 -761

[53] J. Leray, Sur le mouvement d'un liquide visqueux emplissant l'espace, Acta Math. 63 (1934), 193-248

[54] J. L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications,* Springer, 1972

[55] P. Maremonti, Some results of pointwise stability for solutions to the Navier-Stokes system, [CA] Fabrizio, Mauro (ed.) et al., Mathematical models and methods for smart materials. Proceedings of the conference, Cortona, 2001. River Edge, NJ: World Scientific. Ser. Adv. Math. Appl. Sci. 62, 225-230 (2002)

[56] T. Miyakawa, On nonstationary solutions of the Navier-Stokes equations in an exterior domain, Hiroshima Math. J. 12 (1982), 115-140

[57] T. Miyakawa, On L^1-stability of stationary Navier-Stokes flows in $\mathbb{R}^n$, J. Math. Sci. Tokyo 4 (1997), 67-119

[58] S. A. Nazarov and K. I. Pileckas, Asymptotic conditions at infinity for the Stokes and Navier-Stokes problems in domains with cylindrical outlets to infinity, Maremonti, P. (ed.), Advances in fluid dynamics, Rome: Aracne, Quad. Mat. 4, 141-243 (1999)

[59] S. A. Nazarov, M. Specovius-Neugebauer and G. Thäter, Full steady Stokes system in domains with cylindrical outlets, Math. Ann. 314 (1999), 729-762

[60] S. A. Nazarov, M. Specovius-Neugebauer and G. Thäter, Quiet flows for Stokes and Navier-Stokes problems in domains with cylindrical outlets to infinity, Kyushu J. Math. 53 (1999), 369-394

[61] A. Noll and J. Saal, H^∞-calculus for the Stokes operator on L_q-spaces, Math. Z. 244 (2003), 651-688

[62] A. Passerini and G. Thäter, The Stokes system in domains with outlets of bounded connected cross-sections, Z. Anal. Anwend. 17 (1998), 615-639

[63] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, 1983

[64] K. Pileckas, Strong solutions of the steady nonlinear Navier-Stokes system in domains with exits to infinity, Rend. Semin. Mat. Univ. Padova 97 (1997), 235-267

[65] K. Pileckas, Recent advances in the theory of Stokes and Navier-Stokes equations in domains with non-compact boundaries, Galdi, G. P. (ed.) et al., Mathematical theory in fluid mechanics, Lectures of the 4th winter school, Paseky, 1995, Pitman Res. Notes Math. Ser. 354, 30-85 (1996)

[66] K. Pileckas, Existence of solutions with the prescribed flux of the Navier-Stokes system in an infinite cylinder, manuscript, 2005, to appear in J. Math. Fluid Mech.

[67] K. Pileckas, On the nonstationary linearized Navier-Stokes problem in domains with cylindrical outlets to infinity, Math. Ann. 332 (2005), 395-419

[68] J. L. Rubio de Francia, F. J. Ruiz and J. L. Torrea, Calderón-Zygmund theory for operator-valued kernels, Adv. in Math. 62 (1986), 7-48

[69] G. Schneider, Nonlinear stability of Taylor vortices in infinite cylinders, Arch. Rational Mech. Anal. 144 (1998), 121-200

[70] Y. Shibata and M. Yamazaki, On some stability theorem of the Navier-Stokes equation in the three dimensional exterior domain, RIMS Kokyuroku 1146 (2000), 73-91

[71] V. A. Solonnikov and K. I. Pileckas, On some spaces of divergence-free vector fields and on the solvability of a boundary-value problem for Navier-Stokes equations in domains with non- compact boundaries, Zap. Nauchn. Semin. Leningr. Otd. Mat. Inst. Steklova 73 (1977), 136-151

[72] M. Specovius-Neugebauer, Approximation of the Stokes Dirichlet problem in domains with cylindrical outlets, SIAM J. Math. Anal. 30 (1999), 645-677

[73] H. Tanabe, *Functional Analytic Methods for Partial Differential Equations*, Marcel Dekker, Inc., 1997

[74] R. Temam, *Navier-Stokes Equations*, North-Holland, 1979

[75] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators,* North Holland, 1978

[76] L. Weis, Operator-valued Fourier multiplier theorems and maximal L_p-regularity, Math. Ann. 319 (2001), 735-758

[77] K. Yosida, *Functional Analysis*, Springer, 1974

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Erklärung

Ich erkläre hiermit, dass ich die Dissertation selbstständig und nur unter Zuhilfenahme der angegebenen Hilfsmittel und Quellen angefertigt habe.

Darmstadt, den 20. März 2006

Ri Myong-Hwan