Chapter 6 The Effect of the GPS Systematic Errors on Deformation Parameters

6.1. General

Beutler et al., (1988) did the first comprehensive study on the GPS systematic errors. Based on a geometric approach and assuming a uniform satellite sky distribution, they analytically analyzed the effect of relative troposphere, neglected troposphere, ionosphere, wrong GM-value (gravitational constant times the mass of the Earth), wrong fixed coordinates and along track orbital errors on the GPS results. Later, Santerre (1991) studied the impact of the GPS non-uniform satellite sky distribution on these results. Kaniuth (1997) showed, in his study on the reference frame realization errors on relative positioning in regional GPS networks, that: firstly; biases of the reference frame elements (satellites' orbit, earth orientation parameters and fiducial point coordinates) will not only affect the absolute but also the relative positions of the network stations. Secondly, even improper selection of fiducial points has a similar impact on the relative positions as the biases of the fiducial point coordinates. He also showed that, the fiducial point coordinate biases and their improper selection have the largest impact on the relative point positions among the other studied reference frame systematic errors. Dermanis and Grafarend (1993) also analytically analyzed the impact of reference frame variations on deformation parameters.

In the pioneering work of Beutler et al. (1988), systematic errors in GPS measurements have been classified to two groups:

1) The first group includes systematic errors that produce height biases in the network positions. These systematic errors are termed as Class 1 biases.

2) The second group includes errors that result in scale biases in the network's solution. This group of errors is introduced as Class 2 systematic errors.

In this chapter, the effect of these systematic errors on the parameters of deformation will be analytically worked out. This will be done for both infinitesimal and finite deformations. Since by definition strain is the change in length per unit of length, it is having a scale nature.

Therefore, Class 2 systematic errors are expected to have a greater impact on the strain tensor components compared to the Class 1 biases. This encourages one to firstly analyze the effect of Class 2 biases.

Isoparametric representation of deformation provides an easy and efficient way for this study. Therefore, the corresponding observation equations, that is:

$$q = \frac{s^2 - s^1}{s^1} = e_{xx} l^2 + e_{yy} m^2 + e_{zz} n^2 + e_{yz} mn + e_{zx} nl + e_{xy} lm \tag{6.1a}$$

$$Q = \left(\frac{s^2}{s^1}\right)^2 = (1+2E_{xx})l^2 + (1+2E_{yy})m^2 + (1+E_{zz})n^2 + 2E_{yz} mn + 2E_{zx} nl + 2E_{xy} lm \tag{6.1b}$$

are basic equations in this study. Here, similar to the notation of Chapter 3, s^1 and s^2 are the baseline length before and after deformation, l, m and n are its direction cosines, e_{xx}, etc are the infinitesimal strain components and finally E_{xx}, etc are the finite ones.

If $\bar{\mathbf{s}}^i$ is the vector of the unbiased estimate of baseline lengths at epoch i and $\mathbf{s}^i$ is its corresponding vector of biased estimate due to some systematic errors $\boldsymbol{\delta}^i$, theoretically; the biased and unbiased estimates of the baseline lengths give the biased (e_{xx}, e_{yy} etc) versus unbiased estimate ($\bar{e}_{xx}$, $\bar{e}_{yy}$, etc) for strain parameters respectively. Clearly, for Class 1 and Class 2 systematic errors we can write:

Class 1:

$$(s^i)^2 = (\bar{s}^i)^2 + (\delta^i)^2 \tag{6.2a}$$

$$(\delta^i)^2 = (\delta_z^i)^2 + 2\delta_z^i \overline{dz}^i \tag{6.2b}$$

Class 2:

$$s^i = \bar{s}^i \left(1 + \delta^i\right) \tag{6.2c}$$

where $\overline{s}^i$, s^i, δ^i are the elements of corresponding vectors, δ_z^i is the systematic height bias and $\overline{dz}^i$ is the unbiased height difference between the end points of a baseline.

6.2. Class 2 Systematic Errors

6.2.1. Infinitesimal Strains

Let us assume that the elements of the vector of baseline lengths $\mathbf{s}^i$ are contaminated by different scaling errors, which construct the coordinates of the error vector $\boldsymbol{\delta}^i$ at two different epochs $i = 1,2$. Theoretically, at each point of the network and in each epoch one can estimate the *virtual strains* e_{xx}^i, e_{xy}^i, e_{yy}^i, etc. by forming the following system of linear equations:

$$\delta_j^i = e_{xx}^i l_j^2 + e_{yy}^i m_j^2 + e_{zz}^i n_j^2 + e_{yz}^i m_j n_j + e_{zx}^i n_j l_j + e_{xy}^i l_j m_j, \; j = 1,...,N \tag{6.3a}$$

$$q_j^{(i)} = \delta_j^i = \frac{s_j^i - \overline{s}_j^i}{\overline{s}_j^i} \tag{6.3b}$$

Where N is the number of contributing points to be used for estimating the strain parameters at the position of the network stations, s_j^i and $\overline{s}_j^i$ are the biased and unbiased estimates of the jth baseline length in the ith epoch respectively and finally, l_j, m_j and n_j are the direction cosines of the jth baseline in both epochs 1 and 2. Similarly, the unbiased estimate of the baseline lengths theoretically provide us the *unbiased strains* $\overline{e}_{xx}$, $\overline{e}_{yy}$ etc. of the strain tensors through:

$$\overline{q}_j = \overline{e}_{xx} l_j^2 + \overline{e}_{yy} m_j^2 + \overline{e}_{zz} n_j^2 + \overline{e}_{yz} m_j n_j + \overline{e}_{zx} n_j l_j + \overline{e}_{xy} l_j m_j, \; j = 1,...,N \tag{6.4a}$$

$$\overline{q}_j = \frac{\overline{s}_j^2 - \overline{s}_j^1}{\overline{s}_j^1} \tag{6.4b}$$

To analyze the effect of the scaling errors (6.2c) on infinitesimal deformations, we should try to establish a functional relation between the unbiased deformations of Equations (6.4) and the biased ones. The biased deformation parameters e_{xx}, e_{yy}, e_{zz}, etc are theoretically derived from the biased estimates of the baseline lengths, i.e. $\mathbf{s}^i$, by solving the over determined simultaneous system of equations:

$$q_j = e_{xx} l_j^2 + e_{yy} m_j^2 + e_{zz} n_j^2 + e_{yz} m_j n_j + e_{zx} n_j l_j + e_{xy} l_j m_j, \; j = 1, \ldots, N \tag{6.5a}$$

$$q_j = \frac{s_j^2 - s_j^1}{s_j^1} \tag{6.5b}$$

If $\delta_j^i \left(i = 1, 2\right)$ is an scaling error on a single baseline of length s_j^i, we can write:

$$\frac{s_j^2}{s_j^1} = \frac{\bar{s}_j^2 \left(1 + \delta_j^2\right)}{\bar{s}_j^1 \left(1 + \delta_j^1\right)} \tag{6.6}$$

Applying Equation (6.6) to Equation (6.5b) results:

$$q_j = \frac{s_j^2 - s_j^1}{s_j^1} = \frac{\bar{s}_j^2 \left(1 + \delta_j^2\right) - \bar{s}_j^1 \left(1 + \delta_j^1\right)}{\bar{s}_j^1 \left(1 + \delta_j^1\right)} = \frac{1}{\left(1 + \delta_j^1\right)} \frac{\bar{s}_j^2 - \bar{s}_j^1}{\bar{s}_j^1} + \frac{\delta_j^2 \bar{s}_j^2 - \delta_j^1 \bar{s}_j^1}{\left(1 + \delta_j^1\right) \bar{s}_j^1} \tag{6.7a}$$

Using Equations (6.4b) and (6.2c) gives

$$q_j = \frac{1}{\left(1 + \delta_j^1\right)} \left[\bar{q}_j + \frac{\delta_j^2 \bar{s}_j^2 - \delta_j^1 \bar{s}_j^1}{\bar{s}_j^1} \right] = \frac{1}{\left(1 + \delta_j^1\right)} \left[\bar{q}_j + \frac{\left(s_j^2 - \bar{s}_j^2\right) - \left(s_j^1 - \bar{s}_j^1\right)}{\bar{s}_j^1} \right] \tag{6.7b}$$

$$q_j = \frac{1}{\left(1 + \delta_j^1\right)} \left[\bar{q}_j + \frac{\left(s_j^2 - \bar{s}_j^2\right) - \left(s_j^1 - \bar{s}_j^1\right)}{\bar{s}_j^1} \right] = \frac{1}{\left(1 + \delta_j^1\right)} \left[\bar{q}_j + \frac{s_j^2 - \bar{s}_j^2}{\bar{s}_j^1} - \frac{s_j^1 - \bar{s}_j^1}{\bar{s}_j^1} \right] \tag{6.7c}$$

Applying Equation (6.3b) to the relative length changes of the first epoch in Equation (6.7c) gives rise to the equation:

$$q_j = \frac{1}{\left(1+\delta_j^1\right)}\left[\bar{q}_j + \frac{s_j^2-\bar{s}_j^2}{\bar{s}_j^1} - \frac{s_j^1-\bar{s}_j^1}{\bar{s}_j^1}\right] = \frac{1}{\left(1+\delta_j^1\right)}\left[\bar{q}_j + \frac{\bar{s}_j^2}{\bar{s}_j^1}\frac{s_j^2-\bar{s}_j^2}{\bar{s}_j^2} - q_j^{(1)}\right] \tag{6.7d}$$

Again, by applying Equation (6.3b) to the relative length changes of the second epoch in Equation (6.7d), this equation can be further reduced to:

$$q_j = \frac{1}{\left(1+\delta_j^1\right)}\left[\bar{q}_j + \frac{\bar{s}_j^2}{\bar{s}_j^1}\frac{s_j^2-\bar{s}_j^2}{\bar{s}_j^2} - q_j^{(1)}\right] = \frac{1}{\left(1+\delta_j^1\right)}\left[\bar{q}_j + \frac{\bar{s}_j^2}{\bar{s}_j^1}q_j^{(2)} - q_j^{(1)}\right] \tag{6.7e}$$

Finally, the application of equation (6.6) to equation (6.7e) results:

$$q_j = \frac{1}{\left(1+\delta_j^1\right)}\left[\bar{q}_j + \frac{\bar{s}_j^2}{\bar{s}_j^1}q_j^2 - q_j^1\right] = \frac{1}{\left(1+\delta_j^1\right)}\left[\bar{q}_j + \frac{s_j^2}{s_j^1}\frac{1+\delta_j^1}{1+\delta_j^2}q_j^2 - q_j^1\right] \tag{6.7f}$$

Expressing q_j, $\bar{q}_j$, $q_j^{(2)}$ and $q_j^{(1)}$ in terms of the corresponding strain parameters that are given in Equations (6.5), (6.4) and (6.3), one will come up with the following set of simultaneous equations for each baseline:

$$\begin{aligned}
e_{xx} &= \bar{e}_{xx}\left(1-\delta_j^1\right) + \frac{s_j^2}{s_j^1}\frac{1+\delta_j^1}{1+\delta_j^2}e_{xx}^2 - e_{xx}^1 \\
e_{yy} &= \bar{e}_{yy}\left(1-\delta_j^1\right) + \frac{s_j^2}{s_j^1}\frac{1+\delta_j^1}{1+\delta_j^2}e_{yy}^2 - e_{yy}^1 \\
e_{zz} &= \bar{e}_{zz}\left(1-\delta_j^1\right) + \frac{s_j^2}{s_j^1}\frac{1+\delta_j^1}{1+\delta_j^2}e_{zz}^2 - e_{zz}^1 \\
e_{yz} &= \bar{e}_{yz}\left(1-\delta_j^1\right) + \frac{s_j^2}{s_j^1}\frac{1+\delta_j^1}{1+\delta_j^2}e_{yz}^2 - e_{yz}^1
\end{aligned} \tag{6.8}$$

$$e_{xz} = \bar{e}_{xz}\left(1-\delta_j^1\right) + \frac{s_j^2}{s_j^1}\frac{1+\delta_j^1}{1+\delta_j^2}e_{xz}^2 - e_{xz}^1$$

$$e_{xy} = \bar{e}_{xy}\left(1-\delta_j^1\right) + \frac{s_j^2}{s_j^1}\frac{1+\delta_j^1}{1+\delta_j^2}e_{xy}^2 - e_{xy}^1$$

6.2.2. Finite strains

Similar to infinitesimal deformations, to analyze the effect of scaling errors on finite deformations we should try to establish a functional relation between the unbiased strains derived from:

$$\bar{Q}_j = (1+2\bar{E}_{xx})l_j^2 + (1+2\bar{E}_{yy})m_j^2 + (1+2\bar{E}_{zz})n_j^2 + 2\bar{E}_{yz}m_j n_j + 2\bar{E}_{zx}n_j l_j + 2\bar{E}_{xy}l_j m_j,\ j = 1,\ldots,N \qquad (6.9a)$$

$$\bar{Q}_j = \left(\frac{\bar{s}_j^2}{\bar{s}_j^1}\right)^2 \qquad (6.9b)$$

and the biased ones that are given by:

$$Q_j = (1+2E_{xx})l^2 + (1+2E_{yy})m^2 + (1+2E_{zz})n^2 + 2E_{yz}mn + 2E_{zx}nl + 2E_{xy}lm,\ j = 1,\ldots,N \qquad (6.10a)$$

$$Q_j = \left(\frac{s_j^2}{s_j^1}\right)^2 \qquad (6.10b)$$

Using Equations (6.6) and accepting the approximation $O\left(\delta^2\right)$, i.e. ignoring products and squares of systematic errors, we can write:

$$Q_j = \left(\frac{s_j^2}{s_j^1}\right)^2 = \left(\frac{\bar{s}_j^2}{\bar{s}_j^1}\frac{1+\delta_j^2}{1+\delta_j^1}\right)^2 \qquad (6.11a)$$

$$Q_j = \left(\frac{\bar{s}_j^2}{\bar{s}_j^1} \frac{1+\delta_j^2}{1+\delta_j^1} \right)^2 = \overline{Q}_j \frac{1+2\delta_j^2 + O\left[\left(\delta_j^2\right)^2\right]}{1+2\delta_j^1 + O\left[\left(\delta_j^1\right)^2\right]} \tag{6.11b}$$

$$Q_j = \overline{Q}_j \frac{1+2\delta_j^2 + O\left[\left(\delta_j^2\right)^2\right]}{1+2\delta_j^1 + O\left[\left(\delta_j^1\right)^2\right]} = \overline{Q}_j \left(1+2\delta_j^2\right)\left(1-2\delta_j^1 + O\left[\left(2\delta_j^1\right)^2\right]\right) \tag{6.11c}$$

And finally:

$$Q_j = \overline{Q}_j \left[1+2\left(\delta_j^2 - \delta_j^1\right)\right] \tag{6.12a}$$

Substituting Equations (6.9a) and (6.10a) in Equation (6.12a) gives:

$$\begin{aligned}
E_{xx} &= \overline{E}_{xx} \left[1+2\left(\delta_j^2 - \delta_j^1\right)\right] \\
E_{yy} &= \overline{E}_{yy} \left[1+2\left(\delta_j^2 - \delta_j^1\right)\right] \\
E_{zz} &= \overline{E}_{zz} \left[1+2\left(\delta_j^2 - \delta_j^1\right)\right] \\
E_{yz} &= \overline{E}_{yz} \left[1+2\left(\delta_j^2 - \delta_j^1\right)\right] \\
E_{xz} &= \overline{E}_{xz} \left[1+2\left(\delta_j^2 - \delta_j^1\right)\right] \\
E_{xy} &= \overline{E}_{xy} \left[1+2\left(\delta_j^2 - \delta_j^1\right)\right]
\end{aligned} \tag{6.12b}$$

6.3. Class 1 Systematic Errors

6.3.1. Infinitesimal strains

Using Equations (6.5b) and (6.2a) we can write:

$$q_j = \frac{s_j^2 - s_j^1}{s_j^1} = \frac{\left[\left(\bar{s}_j^2\right)^2 + \left(\delta_j^2\right)^2\right]^{1/2} - \left[\left(\bar{s}_j^1\right)^2 + \left(\delta_j^1\right)^2\right]^{1/2}}{\left[\left(\bar{s}_j^1\right)^2 + \left(\delta_j^1\right)^2\right]^{1/2}} \tag{6.13a}$$

Where δ_j^1 and δ_j^2 are defined in Equation (6.2b). Binomial expansion of contributing terms in Equation (6.13a) gives:

$$q_j = \frac{\left[\bar{s}_j^2 + \frac{1}{2}\frac{\left(\delta_j^2\right)^2}{\bar{s}_j^2} + O\left[\frac{\left(\delta_j^2\right)^4}{\left(\bar{s}_j^2\right)^3}\right]\right] - \left[\bar{s}_j^1 + \frac{1}{2}\frac{\left(\delta_j^1\right)^2}{\bar{s}_j^1} + O\left[\frac{\left(\delta_j^1\right)^4}{\left(\bar{s}_j^1\right)^3}\right]\right]}{\bar{s}_j^1 + \frac{1}{2}\frac{\left(\delta_j^1\right)^2}{\bar{s}_j^1} + O\left[\frac{\left(\delta_j^1\right)^4}{\left(\bar{s}_j^1\right)^3}\right]} \tag{6.13b}$$

Therefore:

$$q_j = \frac{\bar{s}_j^2 - \bar{s}_j^1 + \frac{1}{2}\left(\frac{\left(\delta_j^2\right)^2}{\bar{s}_j^2} - \frac{\left(\delta_j^1\right)^2}{\bar{s}_j^1}\right)}{\bar{s}_j^1 + \frac{1}{2}\frac{\left(\delta_j^1\right)^2}{\bar{s}_j^1}} \tag{6.13c}$$

Substituting Equation (6.2b) in this equation and using the approximation $O\left[\frac{\left(\delta_{zj}^i\right)^2}{\bar{s}_j^i}\right]$ gives:

$$q_j = \frac{\bar{s}_j^2 - \bar{s}_j^1 + \left(\frac{\delta_{zj}^2 \overline{dz}_j^2}{\bar{s}_j^2} - \frac{\delta_{zj}^1 \overline{dz}_j^1}{\bar{s}_j^1}\right)}{\bar{s}_j^1 + \frac{\delta_{zj}^1 \overline{dz}_j^1}{\bar{s}_j^1}} \tag{6.14a}$$

or approximately:

$$q_j = \bar{q}_j\left(1 - \frac{1}{2}\frac{\delta_{zj}^1 \overline{dz}_j^1}{\left(\bar{s}_j^1\right)^2}\right) + \frac{1}{2}\left(1 - \frac{1}{2}\frac{\delta_{zj}^1 \overline{dz}_j^1}{\left(\bar{s}_j^1\right)^2}\right)\left(\frac{\delta_{zj}^2 \overline{dz}_j^2}{\bar{s}_j^1 \bar{s}_j^2} - \frac{\delta_{zj}^1 \overline{dz}_j^1}{\left(\bar{s}_j^1\right)^2}\right) + \ldots \tag{6.14b}$$

6.3.2. Finite strains

Using Equations (6.10b) and (6.2a) one can write:

$$Q_j = \left(\frac{s_j^2}{s_j^1}\right)^2 = \frac{(\bar{s}_j^2)^2 + \left(\delta_j^2\right)^2}{(\bar{s}_j^1)^2 + \left(\delta_j^1\right)^2} \tag{6.15a}$$

Again, substituting Equation (6.2b) in this equation and using the approximation $O\left[\left(\frac{\delta_{zj}^i}{\bar{s}_j^i}\right)^2\right]$ gives:

$$Q_j = \bar{Q}_j \frac{1 + 2\delta_{zj}^2 \frac{\overline{dz}_j^2}{\left(\bar{s}_j^2\right)^2}}{1 + 2\delta_{zj}^1 \frac{\overline{dz}_j^1}{\left(\bar{s}_j^1\right)^2}} \tag{6.15b}$$

$$Q_j = \bar{Q}_j \left(1 + 2\delta_{zj}^2 \frac{\overline{dz}_j^2}{\left(\bar{s}_j^2\right)^2}\right)\left(1 - 2\delta_{zj}^1 \frac{\overline{dz}_j^1}{\left(\bar{s}_j^1\right)^2} + \ldots\right) \tag{6.15c}$$

And finally:

$$Q_j = \bar{Q}_j \left[1 + 2\left(\delta_{zj}^2 \frac{\overline{dz}_j^2}{\left(\bar{s}_j^2\right)^2} - \delta_{zj}^1 \frac{\overline{dz}_j^1}{\left(\bar{s}_j^1\right)^2}\right)\right] \tag{6.16}$$

Since to a good approximation:

$$\overline{dz}_j^2 \big/ \bar{s}_j^2 = \overline{dz}_j^1 \big/ \bar{s}_j^1 \tag{6.17a}$$

$$\overline{dz}_j^2 \Big/ \left(\overline{s}_j^2\right)^2 = \overline{dz}_j^1 \Big/ \left(\overline{s}_j^1\right)^2 \tag{6.17b}$$

Equation (6.16) can be further simplified to

$$Q_j = \overline{Q}_j \left[1 + 2\left(\delta_{zj}^2 - \delta_{zj}^1\right) \frac{\overline{dz}_j^p}{\left(\overline{s}_j^p\right)^2} \right] \tag{6.18}$$

where $p = 1$ or 2.