Chapter 5 Procrustean Statistical Inference of Deformations

5.1. General

Since the signal to noise ratio for the change in deformation is normally small, its assessment and interpretation requires sophisticated mathematical techniques. Mathematical statistics is a common mathematical tool for this purpose. The development and implementation of statistical methods have been systematically restricted to the analysis of displacement fields (for example: Caspary 1987; Caspary et al., 1999). The statistical inference of the deformation tensor itself can benefit from the invariance property of the deformation parameters at least in comparison with some of the traditional inference techniques.

Cai (2004) did the first comprehensive study on the statistical inference of the symmetric 2D- and 3D- deformation tensor. His work was based on the statistical inference of the eigenspace components of a random deformation tensor. For this purpose, using the eigenspace synthesis and eigenspace analysis of the symmetric deformation tensor in two- and three-dimensions, Cai formulated the deformation tensor elements as a set of nonlinear functions of the eigenspace components. Assuming that the strain tensor elements are normally distributed and that a time series of deformation tensor is available (either through their direct measurement or through their estimate from other observations) the eigenspace synthesis approach can give a least-squares estimate of the eigenspace elements and their variance-covariance information. He has then proposed the test statistics such as the Hotteling's T^2 and the likelihood ratio statistics (e.g. Papoulis and Pillai, 2002) as statistical apparatuses for the inference of estimated eigenspace elements (Cai, 2004).

The eigenspace elements of the deformation tensor are the standard parameters for the interpretation and representation of the accumulation of strain and stress in an area. Nevertheless, a method that can help us analyze deformation changes in a further detail is obviously more desirable. In case of significant change in deformations, such a method would enable us to assign the detected variations to the normal and/or shear strains. This goal is not attainable through the statistical inference of the eigenspace elements. This is because according to the theorem 2.4.3 of Chapter 4, a significant change in an eigenspace element of the deformation tensor is the cumulative result of changes in various parameters of deformation.

In this thesis, a new two-step method has been forwarded that can fulfill the abovementioned requirements. The method is capable of identifying significant changes in deformations between the stations of a network (change of deformations in space) and similar stations in a time series of deformation tensor (change of deformations in time). Instead of the standard multivariate test statistics such as the Hotteling's T^2 test, the method is based on 1) the Procrustes analysis (Mosier, 1939; Green, 1952; Cliff, 1966; Schönemann, 1966; Schönemann and Carroll, 1970; Gower, 1975; Lissitz and Schönemann, 1976; Ten Berg, 1977; Goodall, 1991; Dryden and Mardia, 2002) of the deformation tensors and 2) screening of the corresponding procrustean residual tensors, hence here is given the name: *Procrustean Statistical Inference of Deformation.* The basic assumptions in procrustean statistical inference are that the strain components are all normally distributed and no gross error is present in the deformation tensors to be analyzed.

To introduce the method, different solutions to the least-squares problem of Procrustes analysis are firstly introduced. It shall be shown in this section that available methods for inculcating the stochastic properties of observations in the solution of weighted Procrustes problem are not appropriate for this specific application of Procrustes analysis. For this reason, the problem will be formulated and solved using the standard least-squares algorithms for solving nonlinear constrained least-squares minimization problems. Later, the efficiency of the method will be verified for simulated deformations. The results of applying the method to the test area of this research will be given in Chapter 7 where the spatial variations of deformation will be analyzed.

5.2. Procrustes Analysis

Procrustes analysis is a mathematical technique for superimposing one or more configurations (shapes) onto another. This is done by the transformation of desired configuration(s) onto the target one under the choice of a rotation, a translation and a central dilation.

In statistical shape analysis, a configuration or shape is known as a set of *landmark* coordinates in an arbitrary coordinate system (Dryden and Mardia, 2002). Since the characteristic tensor of quadratic surfaces and the quadratic polynomials of the form:

$$f(x,y,z) = ax^2 + by^2 + cz^2 + dxy + exz + gyz \tag{5.1a}$$

$$f(x,y) = ax^2 + by^2 + cxy \tag{5.1b}$$

in three and two-dimensions respectively governs the geometrical shape of such a family of surfaces or curves, it is also one possible representation for the shapes or configurations of these forms.

Depending on the sign of the corresponding latent roots, Equation (5.1a) can geometrically represent an ellipsoid, a hyperboloid of one sheet or a hyperboloid of two sheets. Similarly, the characteristic tensor of the quadratic form (5.1b) represents an ellipse or a parabola depending on the sign of its latent roots. See Figure 1 and Figure 2 for further details. This shows the potentiality of applying Procrustes method for analyzing the change in deformations of a deformable body. Depending on the number of involved configurations, the number of involved parameters in the transformation and the incorporation of observational errors in the formulation of the problem, Procrustes problem is termed, formulated and solved

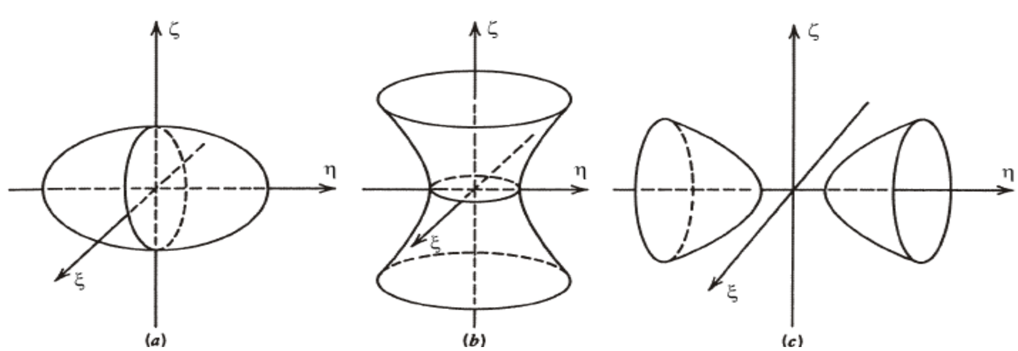

Figure 5.1: Geometrical representation of the deformation quadratic in three-dimensions: (a) Ellip soid: $\lambda_1 > 0,\ \lambda_2 > 0$ and $\lambda_3 > 0$, (b) Hyperboloid of one sheet: $\lambda_1 > 0,\ \lambda_2 > 0$ and and $\lambda_3 < 0$ (c) Hyperboloid of two sheets: $\lambda_1 < 0,\ \lambda_2 > 0$ and $\lambda_3 < 0$.

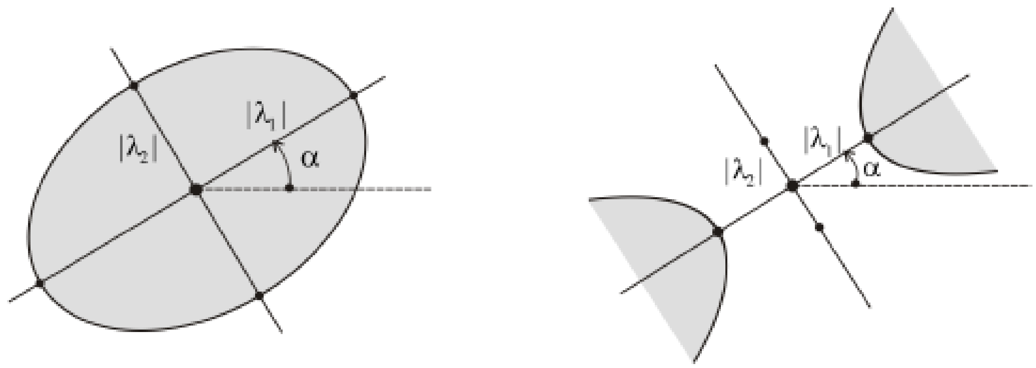

Figure 5.2: Geometrical representation of the deformation quadratic in two-dimensions: (a) Ellipse: $\mathrm{sign}\lambda_1 = \mathrm{sign}\lambda_2$ **: and (b) Hyperbola:** $\mathrm{sign}\lambda_1 \neq \mathrm{sign}\lambda_2$

differently. When one configuration is transformed onto another by an orthogonal transformation in such a way that the sums of the squares of residuals is minimum, the Procrustes problem is termed as Orthogonal Procrustes Analysis (OPA) (Mosier, 1939; Green, 1952; Cliff, 1966; Schönemann, 1966). Schönemann proposed the general solution to the least-squares problem of orthogonal Procrustes analysis in 1966. The problem and its solution can be mathematically formulated in the following theorem:

Theorem 1.5.2: If $\mathbf{A} \in R^{n\times m}$ and $\mathbf{B} \in R^{n\times m}$ $(m \geq n)$ are two arbitrary real matrices of the same dimension, the necessary and sufficient condition to have a unique orthogonal transformation $\mathbf{T}$ to satisfy the least-squares problem:

$$\mathbf{B} + \mathbf{E} = \mathbf{AT} \tag{5.2a}$$

$$\mathbf{TT}^T = \mathbf{T}^T\mathbf{T} = \mathbf{I} \tag{5.2b}$$

$$\mathrm{trace}(\mathbf{E}^T\mathbf{E}) = \min \tag{5.2c}$$

is that the matrix $\mathbf{SS}^T$, where $\mathbf{S} = \mathbf{A}^T\mathbf{B}$, has not multiple zero latent roots and all the singular values are nonnegative. The unique solution $\mathbf{T}$ is then given by the equation:

$$\mathbf{T} = \mathbf{WV}^T \tag{5.3a}$$

where **W**, **V** and $\mathbf{D}_s$ are latent vectors and diagonal matrix of latent roots in orthogonal decomposition of matrices $\mathbf{SS}^T$ and $\mathbf{S}^T\mathbf{S}$ respectively, that is:

$$\mathbf{SS}^T = \mathbf{WD}_s\mathbf{W}^T \tag{5.3b}$$

$$\mathbf{S}^T\mathbf{S} = \mathbf{VD}_s\mathbf{V}^T \tag{5.3c}$$

Proof. Please refer to (Schönemann, 1966).

In functional relations (5.2a) and (5.2b), $\mathbf{E}$ and $\mathbf{I}_{n\times m}$ are the residual and identity matrices respectively. In the particular application of Procrustes analysis for the statistical inference of deformation parameters matrices **A** and **B** are the full column rank strain tensors, that are directly observed or obtained from the 2D- or 3D-analysis of deformation. Matrices $\mathbf{SS}^t$ and $\mathbf{S}^t\mathbf{S}$ are therefore, necessarily positive definite. In other words, the uniqueness of the Procrustes solution for the orthogonal transformation **T** is already assured and needs not to be verified.

Extended Orthogonal Procrustes analysis (EOP) is an extension of OPA in which the transformation involves a rotation **T**, translation γ and a central dilation *c* for matching two configurations (Schönemann and Carroll, 1970). The problem and its solution is mathematically formulated in the following theorem:

Theorem 2.5.2: If $\mathbf{A} \in R^{n\times m}$ and $\mathbf{B} \in R^{n\times m}$ $(m \geq n)$ are two arbitrary real matrices of the same dimension, the necessary and sufficient condition to have a unique orthogonal transformation $\mathbf{T}$ to satisfy the least-squares problem:

$$\mathbf{B} + \mathbf{E} = c\mathbf{AT} + \mathbf{J}\gamma^T \tag{5.4a}$$

$$\mathbf{TT}^T = \mathbf{T}^T\mathbf{T} = \mathbf{I} \tag{5.4b}$$

$$\text{tarce}(\mathbf{E}^T\mathbf{E}) = \min \tag{5.4c}$$

is that the matrix $\mathbf{SS}^T$, where $\mathbf{S} = \mathbf{A}^T\left(\mathbf{I} - \frac{1}{m}\mathbf{JJ}^T\right)\mathbf{B}$, has non-negative latent roots. The unique solution for the transformation parameters is then given by:

$$\mathbf{T} = \mathbf{VW}^T \tag{5.5a}$$

$$c = \text{tarce}\left[\mathbf{T}^T\mathbf{A}^T(\mathbf{I} - \frac{1}{m}\mathbf{JJ}^T)\mathbf{B}\right] \Big/ \text{tarce}\left[\mathbf{A}^T(\mathbf{I} - \frac{1}{m}\mathbf{JJ}^T)\mathbf{A}\right] \tag{5.5b}$$

$$\gamma = \frac{1}{m}\left(\mathbf{B} - c\mathbf{AT}\right)^T\mathbf{J} \tag{5.5c}$$

$$\mathbf{E} = \left(\mathbf{I} - \frac{1}{m} \mathbf{J}\mathbf{J}^T \right) \left(\mathbf{B} - c\mathbf{A}\mathbf{T} \right) \tag{5.5d}$$

Scalar c is the scale factor of transformation (dilation parameter), $\gamma_{n\times 1}$ is its translation vector, $\mathbf{T}$ is the corresponding rotation matrix of the transformation, $\mathbf{J} = \begin{pmatrix} 1 & 1 & \dots & 1 \end{pmatrix}^T_{1\times n}$ and $m = \mathbf{J}^T\mathbf{J}$. $\mathbf{W}$ and $\mathbf{V}$ are the latent vectors in orthogonal decomposition of matrices $\mathbf{S}\mathbf{S}^T$ and $\mathbf{S}^T\mathbf{S}$ respectively, that is:

$$\mathbf{S}\mathbf{S}^T = \mathbf{W}\mathbf{D}_s\mathbf{W}^T \tag{5.5e}$$

$$\mathbf{S}^T\mathbf{S} = \mathbf{V}\mathbf{D}_s\mathbf{V}^T \tag{5.5f}$$

Proof. Please refer to (Schönemann and Carroll, 1970).

Corollary 1.5.2*:* The residual tensor in the least-squares problem of Procrustes analysis is independent of the translation between the involved configurations.

Proof: Since the translation vector γ does not contribute in the matrix of residuals (5.5d), the residual tensor in the problem of EOP is independent of the translation between the involved configurations. Since OPA is an special case of EOP, the residual tensor in orthogonal Procrustes analysis also is not sensitive to the translation between the involved configurations. This can also be verified by putting $c = 1$ in theorem 2.5.2 and following similar derivation steps.

Corollary 1.5.2 ensures that in general, Procrustes analysis can also be applied for the analysis of the shape change in space. This is because Procrustean residuals are not sensitive to the location of configurations.

Further generalization to the Procrustes problem involved the development of mathematical models that were necessary for transforming more than one configuration to the target shape by a set of transformations. This problem is known in the literature as Generalized Procrustes Analysis (GPA). Kristof and Wingersky (1971) solved this problem for a set of transformations that include different orthogonal rotations. They also proved that the solution of this problem is the geometrical centroid of involved configurations. They couldn't prove the uniqueness of proposed solution. Later, generalized Procrustes problem was set up and solved for transformations that include different scaling, translations and rotations (Gower, 1975 and Ten Berg, 1977; Goodall, 1991).

The first attempt to include the stochastic model into the solution of Procrustes problem is due to Lissitz and Schönemann (Lissitz and Schönemann, 1976). They proved that the inclusion of the stochastic model through the least-squares minimization of weighted errors of the form: $\operatorname{trace}\left[\mathbf{E}^T \mathbf{D}_1^2 \mathbf{E}\right] = \min$ and of the form: $\operatorname{trace}\left[\mathbf{E}\mathbf{D}_2^2 \mathbf{E}^T\right] = \min$ are equivalent to weighting rows and columns of the residual matrix respectively and minimizing the sum of the weighted residuals. In addition to the corresponding solutions to these weighting approaches, they also proposed the solution of the weighted Procrustes problem in which different matrices are used for weighting rows and columns of the residual matrix simultaneously.

5.3. Procrustean Statistical Inference of Deformations

In the problem of the analysis of the change in the Earth's surface crustal deformations, change in shape and change in size are equally of interest. Therefore, among possible formulations of the Procrustes problem, OPA is the most appropriate one. More specifically, the functional model in orthogonal Procrustes analysis does not involve the size and shape information of deformation tensors. Therefore, any possible changes between deformation tensors in terms of size and shape will be reflected as the misfit of the functional model to reality. Similar to any least-squares problem, the adequacy of the functional model can be assessed by screening the residuals. For this purpose, the null hypothesis "The model is correct and complete" is firstly analyzed. This hypothesis test is usually known in literature as Global Test of the Model (Caspary, 1987). The inadequacy of functional model in orthogonal Procrustes analysis of deformations, that is its rejection by the global model test, indicates significant change(s) in deformations in the form of size and/or shape. Further inspection of residuals (see Equation 5.2a) can help us in localizing the variation(s) that was statistically asserted in previous step. For this purpose, deformation changes are treated as outliers. By definition an outlier is a residual which, according to some testing rule, is in contradiction to the assumption (Caspary 1987). Therefore, using a test strategy and a clear statistical concept, outliers can be theoretically localized. Although according to the classical theory of errors an outlier can refer to a systematic or a gross error, in the theory of least-squares it is normally taken as an indication for the presence of gross errors. This is due to the implicit assumption that the functional model is normally taken to be completely in accord with reality. Since in Procrustean statistical inference of deformations outliers are expected to represent the misfit of the functional model to reality, it has been implicitly assumed that no gross errors are present in the deformation tensors under study.

This method for analyzing the change in deformations is naturally a relative method in the sense that it depends on the selected level of risk, the assumed distribution and the testing procedure. To reduce the sensitivity of the method to possible deviations from statistical concepts, robust estimation has been preferred to traditional outlier detection techniques. For this purpose, BIBER robust estimation has been used (Wicki, 2001). With regard to the application of robust estimation for data snooping, it is not possible to assign any probability to detected outliers. BIBER robust estimator has been modified in such a way that the modified estimator can assign a certain probability to the detected outliers.

Since the method of Lissitz and Schönemann doesn't minimize the sum of standardized residuals, it is not an appropriate method for inculcating the stochastic properties of the configurations in Procrustean statistical inference of deformations. The most straightforward solution for the abovementioned problem is using the standard least-squares algorithm for solving the non-linear mathematical model of the orthogonal Procrustes problem. That is, linearizing the model and minimizing the sum of the squares of weighted residuals. Mathematically the problem is formulated as follows:

$$\mathbf{f}(\mathbf{x},\mathbf{l}) = \mathbf{0} \tag{5.6a}$$
$$\mathbf{f}_c(\mathbf{x}) = \mathbf{0} \tag{5.6b}$$
$$\mathbf{r}^t\mathbf{P}\mathbf{r} \rightarrow \min \tag{5.6c}$$

where $\mathbf{l}$ is the vector of observations, $\mathbf{P}$ is the weighting matrix, $\mathbf{r}$ is the residual vector and $\mathbf{x}$ is the vector of unknown parameters. Equations (5.6a) and (5.6b) are the implicit representation of Equation (5.2a) and (5.2b) above. In this problem, the non-linearity resides in the constraints. The constraints ensure the orthogonality of the rotation tensor. The initial values for the rotation tensor components are computed form the direct solution to the orthogonal

Procrustes problem, which was given in the theorem 1.5.2 above. Linearizing the nonlinear model (5.6) above leads to the following system of simultaneous equations:

$$\begin{aligned} \mathbf{A}\delta + \mathbf{B}\mathbf{r} + \mathbf{w} &= \mathbf{0} \\ \mathbf{D}\delta + \mathbf{w}_c &= \mathbf{0} \end{aligned} \tag{5.7a}$$

$$\mathbf{A} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}^0} \tag{5.7b}$$

$$\mathbf{B} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{l}}\right|_{\mathbf{l}=\mathbf{l}^0} \tag{5.7c}$$

$$\mathbf{D} = \left.\frac{\partial \mathbf{f}_c}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}^0} \tag{5.7d}$$

$$\mathbf{w} = \mathbf{f}(\mathbf{x}^0, \mathbf{l}^0) \tag{5.7e}$$

$$\mathbf{w}_c = \mathbf{f}_c(\mathbf{x}^0) \tag{5.7f}$$

In these equations $\mathbf{w}$ and $\mathbf{w}_c$ are the misclosure vectors for observation equations and constraints. The vector of observations $\mathbf{l}$ is given by:

$$\mathbf{l} = vec\left(\mathbf{a}; \mathbf{b}\right) \tag{5.7g}$$

where $\mathbf{a}$ and $\mathbf{b}$ are assumed to represent the deformation tensors to be transformed one onto the other respectively. By definition, the operator $vec(.)$ in Equation (5.7g) changes an m-by-n matrix $\mathbf{a}$ to a column vector of length $m \times n$ by stacking it columns to each other.
The solution of the linear constrained implicit model (5.7a) and (5.7b) is given by (Vanicek and Krakiwsky, 1986):

$$\hat{\delta} = -\mathbf{N}^{-1}\mathbf{u} - \mathbf{N}^{-1}\mathbf{D}^T\left(\mathbf{D}\mathbf{N}^{-1}\mathbf{D}^T\right)^{-1}\left(\mathbf{w}_c + \mathbf{D}\delta^{(1)}\right) \tag{5.8a}$$

$$\mathbf{N} = \mathbf{A}^T\left(\mathbf{B}\mathbf{Q}_\mathbf{l}\mathbf{B}^T\right)^{-1}\mathbf{A} \tag{5.8b}$$

$$\mathbf{u} = \mathbf{A}^T\left(\mathbf{B}\mathbf{Q}_\mathbf{l}\mathbf{B}^T\right)^{-1}\mathbf{w} \tag{5.8c}$$

$$\delta^{(1)} = -\mathbf{N}^{-1}\mathbf{u} \tag{5.8d}$$

$$\mathbf{C}_{\hat{\delta}} = \mathbf{N}^{-1} - \mathbf{N}^{-1}\mathbf{D}^T(\mathbf{D}\mathbf{N}^{-1}\mathbf{D}^T)^{-1}\mathbf{D}\mathbf{N}^{-1} \tag{5.8e}$$

$$\hat{\mathbf{r}} = -\mathbf{Q}_\mathbf{l}\mathbf{B}^T\left(\mathbf{B}\mathbf{Q}_\mathbf{l}\mathbf{B}^T\right)^{-1}(\mathbf{A}\hat{\delta} + \mathbf{w}) \tag{5.8f}$$

$$\mathbf{Q}_{\hat{\mathbf{r}}} = \mathbf{Q}_\mathbf{l}\mathbf{B}^T(\mathbf{B}\mathbf{Q}_\mathbf{l}\mathbf{B}^T)^{-1}\mathbf{B}\mathbf{Q}_\mathbf{l} \tag{5.8g}$$

In the following sub-sections, first the global model test is reviewed. The stochastic concepts for the Procrustean statistical inference of deformations are also established in this section. BIBER robust estimation (Wicki, 2001) together with implemented modifications to this estimator are then introduced. Finally, the method is applied to synthetic deformations for three different case studies:

1. Two identical deformation tensors,
2. Two deformation tensors that are characteristic tensors for different geometric shapes are then examined and

3.1. Two deformation tensors with significantly different dilatational parameters

3.2. Two deformation tensors with significantly different shear parameters

3.3. For a variation of all strain parameters

5.3.1. Global Model Test

Global model test is based on the following theorem from mathematical statistics:

Theorem 1.5.3: The sum of the squares of *n* independent random variables z_i that are normally distributed with distribution parameters $\mu = 0$ and $\sigma = 1$, i.e. $x = z_1^2 + z_2^2 + \ldots + z_n^2$, has the probability density function:

$$f_x(x) = \begin{cases} \dfrac{x^{n/2-1}}{2^n \Gamma(n/2)} e^{-x/2} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{5.9a}$$

where $\Gamma(x)$ represents the gamma function defined as:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx \tag{5.9b}$$

Proof. Please refer to Papoulis and Pillai, (2002).

A random variable with the probability density function (5.9) is said to have chi-square distribution with *n* degrees of freedom and is normally denoted by $\chi^2(n)$. Assuming that observations are normally distributed, that is $\mathbf{l} \sim N(\bar{\mathbf{l}}, \mathbf{\Sigma})$ where $\bar{\mathbf{l}} = E(\mathbf{l})$ and $\mathbf{\Sigma}$ is the variance-covariance matrix of observations, it can be easily seen that residuals $\mathbf{r} = \mathbf{l} - \bar{\mathbf{l}}$ are also normally distributed with distribution parameters: $\mathbf{r} \sim N\left(\mathbf{0}, \sigma_0^2 \mathbf{Q_r}\right)$ where σ_0^2 and $\mathbf{Q_r}$ are the a-priori variance of unit weight and the cofactor matrix of residuals. Therefore, according to theorem 1.5.3, when no outliers are present, the loss function in the least-squares estimation characterizes a random variable with stochastic properties: $\mathbf{r}^T \mathbf{Pr} \sim \sigma_0^2 \chi^2\left(df\right)$. The global model test questions these assumptions by comparing the a-posteriori variance factor s_0^2 with σ_0^2 under the null hypothesis that "The model is correct; the distributional assumptions meet the reality". Following theorem 1.5.3, the test statistics to be used is given by:

$$T = \mathbf{r}^T \mathbf{Pr} / \sigma_0^2 = df \ \frac{s_0^2}{\sigma_0^2} \sim \chi^2\left(df\right) \tag{5.10}$$

In statistical inference of the change in deformations, the rejection of global model test is an indication for the inadequacy of functional model in the orthogonal Procrustes analysis of deformations. This may be due to significant change(s) in the form of size and/or shape in deformations as well as inadequacy of stochastic components in the functional model (distributional assumptions). In the later case, some of strain components may not be perfectly normally distributed. If $0 \le \varepsilon < 1$ is a known parameter and H is an unknown contaminating distribution, for such a type of deformation parameters the probability density function can be written as (Huber, 1964):

$$F = \left(1 - \varepsilon\right) N + \varepsilon H \tag{5.11}$$

5.3.2. BIBER-Estimator

Robust statistics is the appropriate mathematical tool when the stochastic assumptions are only approximations to reality (Huber, 1964; Hampel et al., 1981). Among different kinds of robust estimators, maximum likelihood estimators (also known as M-estimators) are more in commensurate with the least-squares estimator which in principle is also a maximum likelihood estimator. For example: Huber (1964) proposed the following M-estimator in which, in contrary to the least-squares estimator, residuals (r_i) are bounded by a constant parameter ($\bar{c}$):

$$\Psi_{\bar{c}}\left(r_i\right)=\begin{cases} r_i & \text{for } \left|r_i\right|<\bar{c} \\ sign\left(r_i\right)\bar{c} & \text{for } \left|r_i\right|\geq\bar{c} \end{cases} \tag{5.12}$$

When $\bar{c}\rightarrow\infty$, the Huber estimator gives identical results to the least-squares estimator. Based on this estimator, Wicki (2001) has proposed a maximum likelihood estimator for the adjustment of geodetic networks. In contrary to Huber's estimator in which residuals are treated independently of their quality, the new estimator takes the quality of residuals into account. This was done by including the standard deviations of residuals (σ_{r_i}) in the formulation of this M-estimator:

$$\Psi_c\left(w_i\right)=\begin{cases} w_i=\dfrac{r_i}{\sigma_{r_i}} & \text{for } \left|w_i\right|<c \\ sign\left(w_i\right)c & \text{for } \left|w_i\right|\geq c \end{cases} \tag{5.13a}$$

From experience, a value in the range $2.5\leq c\leq 4$ has been proposed for the constant parameter c in Equation (5.13a) (Wicki 2001). The M-estimator (5.13a) ***b**ounds the **i**nfluence of residuals **b**y standardiz**e**d **r**esiduals*, hence is given the name BIBER-estimator. When no outliers are present (all w_i exceed the threshold c), BIBER-estimator gives the same results as the least-squares method. The M-estimator (5.13a) can be equally represented by:

$$\Psi_{k_i}\left(r_i\right)=\begin{cases} r_i & \text{for } \left|r_i\right|<k_i \\ sign\left(r_i\right)k_i & \text{for } \left|r_i\right|\geq k_i \end{cases} \tag{5.13b}$$

$$k_i=c\sigma_{r_i} \tag{5.13c}$$

Figure 5.3 illustrates how the M-estimator (5.13b) bounds the effect of large residuals by selecting different values for the bounding parameter k_i.

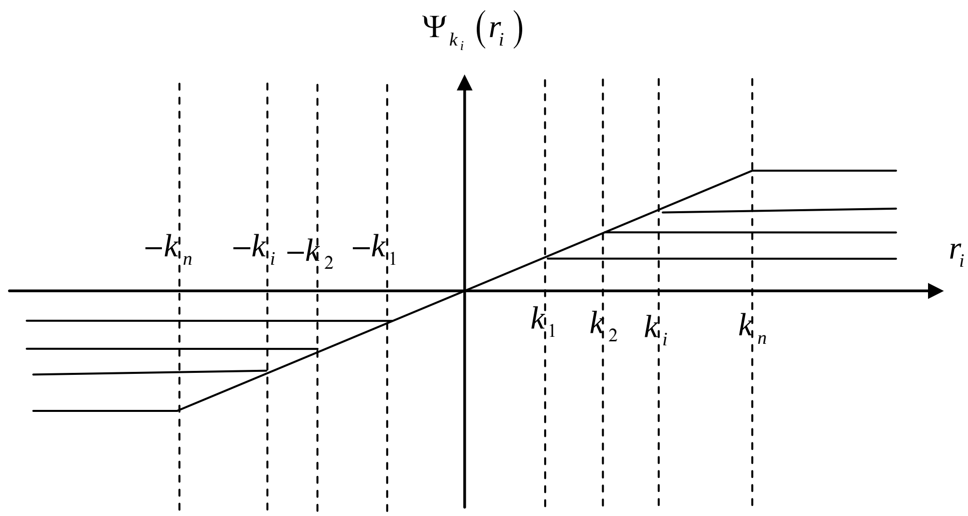

Figure 5.3: The function Ψ for BIBER-estimator, after Wicki, 2001.

The application of robust estimation techniques for estimating the unknown parameters of a functional model with stochastic components can not only provide a solution which is less sensitive to possible deviations in stochastic assumptions, but also can inform us on the outliers that may exist in the acquired measurements. This is because outliers are one of the causes for the deviation of stochastic assumptions from reality. In this respect, one should be cautious that it is not possible to assign any probability to the outliers to be identified by robust estimation techniques.

According to Baarda's method of data snooping (Baarda, 1968), for a certain error level of type I ($\alpha = \alpha_0$) and a certain error level of type II ($\beta = \beta_0$) the outlier on an observation l_i can be identified if it reaches the value:

$$\Delta_i = \sqrt{\lambda_0}\,\frac{\sigma_{l_i}}{\sqrt{z_i}} \qquad (5.14)$$

In this equation $\lambda_0 = \lambda(\alpha_0, \beta_0, df)$ is non-centrality parameter in non-central chi-square distribution with df degree of freedom and $z_i = [\mathbf{Q_r P}]_{ii}$ is the local redundancy number (Baarda, 1968). For the conventional complementary hypothesis H_a: "One of the measurements is an outlier": $df = 1$ and therefore, the non-centrality parameter is only a function of α_0 and β_0. Figure 5.4 illustrates the ratio $\sqrt{\lambda_0}/\sqrt{z_i}$ for $\alpha_0 = 0.001$, 0.005, 0.01, 0.05, 0.1, 1.0, 2.5 and 5 percent against $\beta_0 = 10$, 20 and 30 percent when the conventional alternative hypothesis is taken into account. It is seen in this figure that for a risk level of $\alpha_0 \geq 1$ and for $\beta_0 = 10$, 20 and 30, when local redundancy numbers are larger or as large as 0.9 the range of this ratio coincides with the proposed range for the constant parameter c in Equation (5.13a). For a significance level of 95% $(\alpha_0 = 0.05)$ the range of redundancy numbers for which a similar property is seen enlarges to $z_i \geq 0.6$. Implementing this ratio instead of the empirical bounding parameter c in Equation (5.13a) not only strengthens the theoretical basis of the BIBER-estimator but also assigns a certain probability to the outliers that are to be detected by using this estimator. Therefore, the following modification is proposed here to the BIBER-estimator:

$$\Psi_{\sqrt{\lambda_0}\frac{\sigma_{l_i}}{\sqrt{z_i}}}(w_i) = \begin{cases} w_i = \dfrac{r_i}{\sigma_{r_i}} & \text{for } |w_i| < \sqrt{\lambda_0}\dfrac{\sigma_{l_i}}{\sqrt{z_i}} \\ sign(w_i)\sqrt{\lambda_0}\dfrac{\sigma_{l_i}}{\sqrt{z_i}} & \text{for } |w_i| \geq \sqrt{\lambda_0}\dfrac{\sigma_{l_i}}{\sqrt{z_i}} \end{cases} \tag{5.15}$$

The M-estimator above is called here the modified BIBER-estimator and is proposed for a detailed analysis of the change in deformations.

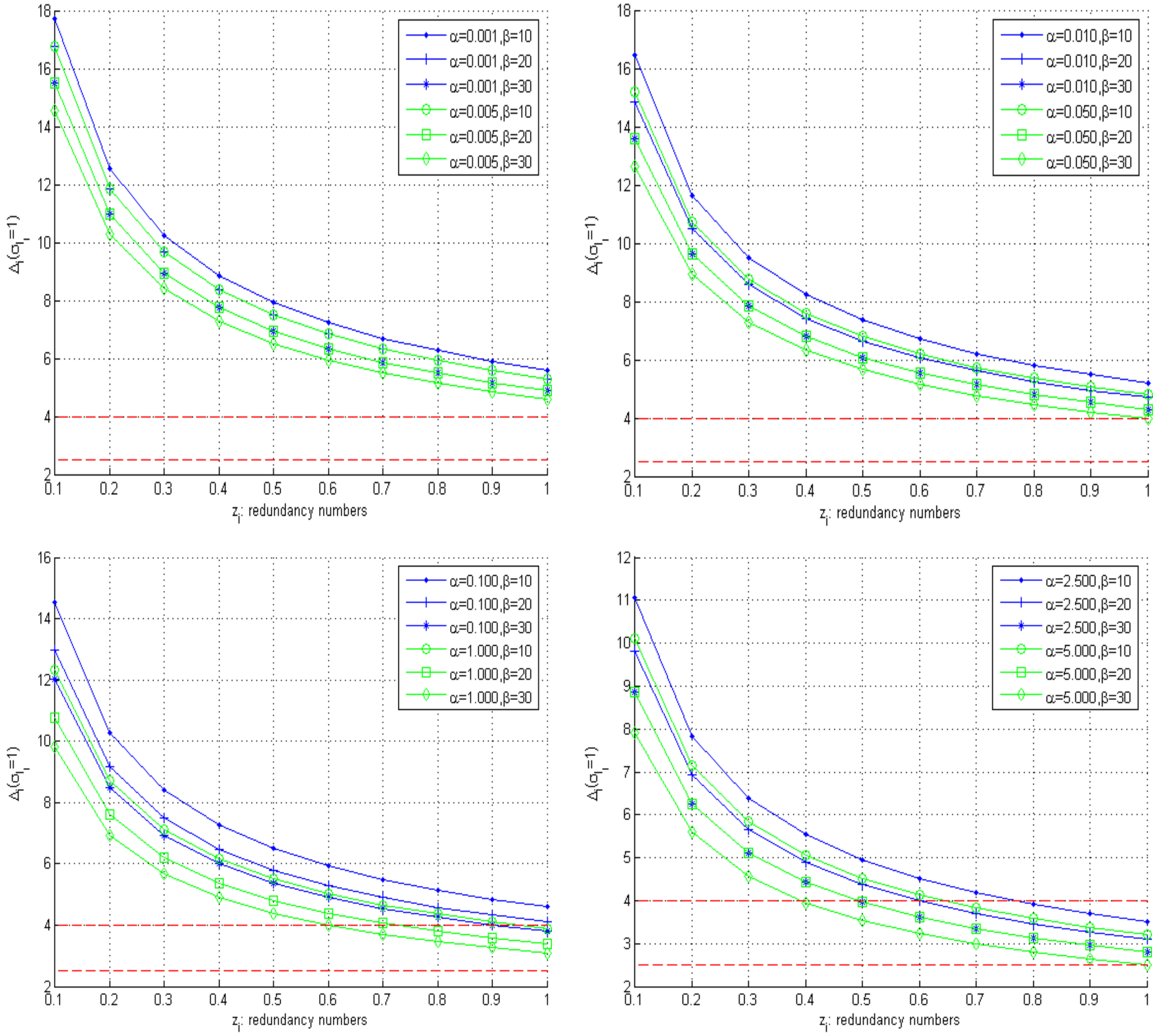

Figure 5.4: the ratio $\sqrt{\lambda_0}\Big/\sqrt{z_i}$ for $\alpha_0 = 0.001$, 0.005, 0.01, 0.05, 0.1, 1.0, 2.5 and 5 percent against $\beta_0 = 10$, 20 and 30 when the conventional null hypothesis is taken into account.

5.4. Application to Simulated Deformations

To check the efficiency of the forwarded method for the statistical inference of deformations, three different simulated case studies have been analyzed. Procrustes analysis is firstly done on two identical synthetic deformations (Case I in Table 5.1). This special case is a good measure for checking the computer codes that has been developed for practical applications. Deformation tensors that have been taken into account in the second case (Case II in Table 5.1) are the characteristic tensors of the deformation quadratics that represent completely different geometric shapes (ellipsoid and hyperboloid). In the last case, once significant changes are made on dilatational strains of a synthetic deformation tensor (Case III-a in Table 5.1). Then, only the shear components are subjected to significant changes (Case III-b in Table 5.1). Finally, deformation changes have been implemented on both shear and normal strains (Case III-c in Table 5.1). Simulated deformation changes fulfill the criterion of Equation (5.14). The statistical significance of simulated deformation changes have been analyzed using the Procrustes method of this thesis. For this purpose a risk level of $\alpha = 1\%$ and a power of $\beta = 10\%$ has been selected. Table 5.1 provides the obtained numerical results.

It can be seen in Table 5.1 that for the case I, the method can efficiently realize that there is no change in the involved deformation tensors. For the case study II, significant changes on all elements of the deformation tensors has been statistically asserted by the method. This is completely in accord with the fact that the deformation tensors of this case study are geometrically the characteristic tensors of two different shapes: an ellipsoid and a hyperboloid of two sheets (see Figure 5.1). Similarly, when all elements of a deformation tensor are subjected to detectable changes and the deformation tensors are still geometrically representing a similar shape, like an ellipsoid in the case study III-c, the method can efficiently identify the variations. The efficiency of the method is also clear when only the dilatational parameters are subjected to detectable variations (see Table 5.1 for the results of the case study III-a). Nevertheless, similar to any other statistical inference technique, it is always possible that the method mistakably assigns significant changes to the deformation parameters that have not experienced any variation. In statistical inference techniques this type of error is known as error type II. This type of error is tried to be avoided by increasing the power of test (reducing the probability β of this type of error). Table 5.1 shows that when only the shear parameters are subjected to detectable variations in the synthetic deformations of III-b, the method has committed a type II error in assigning significant changes to the deformation parameter ezz.

Case	Kind of deformation	The Procrustean Statistical Inference Results
I	Two Identical Deformation Tensors	The null hypothesis cannot be rejected, no significant change in deformations can be statistically asserted at the probability of 99%
II	Variation in the geometric shapes	The null hypothesis is rejected, significant change in deformations can be statistically asserted at the probability of 99%
		Significant change has occurred on the parameters: exx, eyx, ezx, exy, eyy, ezy, exz, eyz, ezz
III-a	Variation in dilatational parameters	The null hypothesis is rejected, Significant change in deformations can be statistically asserted at the probability of 99%
		Significant change has occurred on the parameters: exx, eyy, ezz
III-b	Variation in shear parameters	The null hypothesis is rejected, significant change in deformations can be statistically asserted at the probability of 99%
		Significant change has occurred on the parameters: eyx, ezx, exy, ezy, exz, eyz, ezz
III-c	Variation in all parameters of strain	The null hypothesis is rejected, significant change in deformations can be statistically asserted at the probability of 99%
		Significant change has occurred on the parameters: exx, eyx, ezx, exy, eyy, ezy, exz, eyz, ezz

Table 5.1: Results of Procrustean statistical inference of simulated deformations