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MATHEMATICS

# Foundations of E-Theory

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Differential geometry is a powerful tool in various branches of science, especially in theoretical physics. Ordinary differential geometry requires differentiable manifolds. This research paper shows how concepts of differential geometry can also be applied to pure topological spaces. Such a theory is based on concepts like cohomology theory. It allows to define a curvature operator also on pure topological spaces without connection. The main advantage of this theory is that the only required information about the topological spaces is the structure of these spaces. A formulation of quantum gravity is also possible with this theory.



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## 1. INTRODUCTION

Since differential geometry is applied very successful in sciences like theoretical physics (e.g. general relativity, gauge theory), computer graphics and many more, there are no significant doubts about the logical structure of this branch of mathematics. However, differential geometry is based mainly on the requirement of the smooth $C^{\infty}$-manifold. Also the theory of Lie groups which is used very frequently in theoretical physics bases on smooth manifold. Despite the great success of this concepts there is remaining a disadvantage which is the failure of derivatives at certain regions of the manifold. Such a problem is avoided in discrete differential geometry due to the fact that only finitely many objects (e.g. simplices) are used. However, many structures cannot be modeled sufficiently accurate with discrete differential geometry. Another possibility of introducing a differential even if the manifold is not differentiable in ordinary analysis is Nonstandard analysis (Schmieden 1958).

This research paper focuses also on the unification of quantum theory with general relativity. There is still not found a way how to quantize general relativity (Hamber 2009). The reason is because of the singularity at the Big Bang that is predicted by the classical general relativity theory which is based on the geometry of smooth manifolds. Heisenberg's incertainty principle $\Delta p \Delta x \geq \frac{\hbar}{2}$ with momentum uncertainty $\Delta p$ and length resolution $\Delta x$ states that for the continuum limit, i.e. $\Delta x \mapsto 0$, the incertainty of momentum becomes infinite which is clearly unphysical. To solve this problem, various theories of quantum gravity were proposed. An example of such a theory is String theory (Schwarz 2007) that assumes that elementary particles are not pointlike such that $\Delta x \neq 0$. Other approaches of quantum gravity are Loop Quantum Gravity (Ashtekar 1987) and Causal Dynamical Triangulation (Loll 1998). These theories provide a discretization of spacetime, where Causal Dynamical Triangulation relies on discrete differential geometry. There are still no experimental validations of these theories and therefore the plausibility of these theories is still an open question in theoretical physics.

Also an open question is how to define a proper directional derivative in general manifolds or even in topological spaces. Formal descriptions of manifolds are existing in mathematics literature; one of these are pseudogroups (Golab 1939). Another formal description of manifolds is synthetic differential geometry (Katz 1970). This research paper shows how topological spaces can be formalized with the use of semigroups and category theoretical elements. With the definition of the E-semigroups calculus on general topological spaces can be performed. The groups are called E groups because the letter

"E" is an acronym of the word "Equalizer"; one can also call this theory "Equalizer-Theory". After the basic definitions and theorems about such topological spaces the application to quantum gravity is shown. It will be shown the following

**Theorem 1.1:** A quantum gravity theory is possible without singularities.□

The proof of this theorem is given in section 2 of this research paper.

## 2. THEORETICAL CONCEPT

The E-Theory is based only on topological spaces $X$ that have finite cardinality. Here, the main ingredient of the E-Theory are the E-semigroups (groups without the inverse and identity property).

**Definition 2.1:** May be $S$ a semigroup, where operations between elements $s_{i_1 i_2 \cdots i_n} \in S, n \in \mathbb{N}, i_k \in \{1, \ldots, n\} \forall k \in \{1, \ldots, n\}$ are only multiplications. If the maximum number of indices that are attached on an element of $S$ is $n$, this semigroup has the characteristic $n$. The semigroup $S$ is called an E-semigroup if the following properties are satisfied:

(i) $s_{i_a} s_{i_b} = s_{i_a i_b}$ or more general $s_{i_a i_b \cdots i_c} = s_{i_a} s_{i_b} \cdots s_{i_c}$; here the elements $s_{i_a}, \ldots, s_{i_c}$ are called generators of $S$

(ii) The group contains the empty element $0 \in S$ with $0r = 0$ and arbitrary $r \in S$.

(iii) If there is an equality in indices, i.e. $\exists k, l \in \{1, \ldots, n\} \wedge k \neq l : i_k = i_l$ then $s_{i_1 i_2 \cdots i_n} = 0$

(iv) The semigroup is commutative. □

Property (iii) of the definition 2.1 contains the equalizer property: The set $Eq(s_{i_k}, s_{i_l})$ is clearly not empty for $k = l$ and if the element $s_{i_1 i_2 \cdots i_n} = s_{i_1} \cdots s_{i_n}$ has two or more factors $s_{i_k}$ that are equal, it holds $s_{i_1 i_2 \cdots i_n} = 0$. E-semigroups are strongly related to relations between objects.

**Example 2.2:** The set of all possible (generalized) relations between $n$ objects which also includes the objects is an E-semigroup of characteristic $n$. □

A very important fact is that E-semigroups can formalize topological spaces.

**Lemma 2.3:** Let $X$ be a topological space which can be covered by minimal closed subsets $U_i$, i.e. $X = \cup_{i=1}^{\#X} U_i$ and $U_i$ cannot be subdivided into smaller subsets. Then it exists a functor $F: \mathrm{Top} \to \mathrm{ES}$ between the category of topological spaces $\mathrm{Top}$ and the category of E-semigroups $\mathrm{ES}$.

Proof: A closed subset $U_i$ has a boundary $\partial U_i$ that can be determined by computing the following map: $U_i \xrightarrow{\partial} \partial U_i$. Since $U_i$ is a minimal subset, it can be regarded as an element of $X$. An E-semigroup of characteristic n has only one element $s_{i_1\cdots i_n}$ with $i_k \in \{1,\dots,n\}\forall k \in \{1,\dots,n\}$ due to property (iii) of definition 2.1. Therefore, one can define an isomorphism $F: U_i \to S_i$ where $S_i$ is the E-semigroup associated with subset $i$. The set of all elements $Q := \{s_{i_1\cdots i_{n-1}} | i_k \in \{1,\dots,n\}\forall k \in \{1,\dots,n-1\}\} \subset S$ can be obtained by removing one generator from the factorization of $s_{i_1\cdots i_n}$; this map is denoted by $\Delta$. Finally, one can construct a commutative diagram:

$$\begin{array}{ccc} U_i & \overset{\partial}{\rightarrow} & \partial U_i \\ \downarrow F & & \downarrow F \\ \{s_{i_1\cdots i_n}\} & \overset{\Delta}{\rightarrow} & Q_i \end{array}$$

and therefore the functor $F$ exists. □

From Lemma 2.3 a calculus on topological spaces can be defined that is similar to the calculus on manifolds (exterior calculus).

**Lemma 2.4:** For every closed subset $U_i$ that covers a topological space $X$ a long exact sequence $R_{i,n} \overset{\delta}{\rightarrow} R_{i,n-1} \overset{\delta}{\rightarrow} R_{i,n-2} \overset{\delta}{\rightarrow} \cdots$ with $R_{i,n-q} := \left\{s_{i_1\cdots i_{n-q}} | i_k \in \{1,\dots,n\}\forall k \in \{1,\dots,n-q\}\right\} \subset S, R_{i,0} = \{0\}$ can be constructed if $Eq(\alpha,\beta) = 0$ for arbitrary $\alpha,\beta \in S$.

Proof: If $Eq(\alpha,\beta) = 0$ for arbitrary $\alpha,\beta \in S$, there is no empty element contained in the sets $R_{i,n-q}$ for $q \neq n$ due to the equalizer property. Hence, every element of $R_{i,n-q}$ is well-defined. Defining the map $\delta$ as $\delta s_{i_1\cdots i_{n-q}} := \sum_{p=1}^{n-q}(-1)^{p+1} s_{i_1\cdots \widehat{i_p}\cdots i_{n-q}}$ where the superscript $\widehat{\cdots}$ denotes that this index is omitted. Then it is easy to show that it holds the exactness condition $\delta^2 = 0$. □

Clearly one can apply a functor $\mathrm{Hom}(\cdot,G)$ with a group $G$ to the exact sequence constructed in Lemma 2.4. This leads to an exact sequence in functions on a topological space (by respecting Lemma 2.3). Such an operation is very similar to the conversion of the simplicial complex to the deRham complex; a Hodge dual can be defined analogously.

From ordinary differential geometry it is known that for a scalar $\phi$ it follows $D^2\phi = T_a \partial^a \phi$ with the 2-form torsion tensor $T_a$ and for a vector $\theta^a$ it follows $D^2\theta^a = R^a_b \theta^b$ with the 2-form curvature tensor $R^a_b$ if $D$ is the exterior covariant derivative. Both quantities are based on the loss of exactness in a chain complex. Since the E-Theory related to topological spaces is based on a chain complex, one can define a curvature in topological spaces.

**Theorem 2.5:** A curvature value $\Omega_i$ (the analogous quantity is the curvature 2-form in differential geometry) can be assigned to every closed subset $U_i \subset X$.

Proof: The exact sequence of Lemma 2.4 requires that the E-semigroup has no nonempty equalizers. If there are nonempty equalizers, the chain complex $R_{i,n} \overset{\delta}{\rightarrow} R_{i,n-1} \overset{\delta}{\rightarrow} R_{i,n-2} \overset{\delta}{\rightarrow} \cdots$ loses the exactness. Now one can define an indicator function $I$ that is applied on a E-semigroup element. This indicator function has always the value 1 with exception if it is applied on an empty element; this indicator function has the value 0 when applied on an empty element. therefore, this indicator function measures the presence of equalizers. The curvature value can be obtained by computing the inexactness function $\Xi := \delta^2 I(s_{i_1 \dots i_n})(*)$. Here, the indicator function is evaluated on the element $s_{i_1 \cdots i_n}$, because it represents (by applying the functor $F^{-1}$ used in Lemma 2.3) the whole closed subset $U_i$. Equation (*) can also be written as $\Omega_i I(s_{i_1 \dots i_n}) = \delta^2 I(s_{i_1 \dots i_n})$ since the inexactness function lies also on $F^{-1}(s_{i_1 \dots i_n})$. Comparing this equation with the equation $D^2\theta^a = \check{R}\theta^a$ with curvature operator $\check{R}$ of differential geometry leads to the choice that it can be set $\Omega_i = \delta^2 I(s_{i_1 \dots i_n})$; it holds $I(s_{i_1 \dots i_n}) = 1$, because the element $s_{i_1 \cdots i_n}$ has to be well-defined (in other words: this element is not the empty element). □

Theoretical frameworks given in this section of this research papers can be used to rewrite General relativity in a form such that it can be quantized without UV-divergences. It is clear that the action functional of General relativity has the form $S_{GR} = \kappa \int \epsilon_{abcd} R^{ab} \wedge E^c \wedge E^d$ with a coupling constant $\kappa$ and tetrads $E^a$. Original General relativity has well-defined distance and angle values, whereas general topological spaces have not such values. To assign distance measures to a general topological space governed by E-semigroups (as described in Lemma 2.3) it is assumed that the two neighboring closed subsets $U_i, U'_i$ are separated exactly one Planck length (or one Planck time). With this assumption one can get rid of pure geometrical quantities like the tetrads.

Proof of Theorem 1.1: The integration over spacetime is replaced by summation over all closed subsets of the topological space. Also the tetrads and connections are deformed in a manner such that the physical spacetime coincides with a topological space governed by E-semigroups. Hence, the Lagrangian density can be rewritten as $L_{GR} = \kappa' \sum_{a,b=1}^{4} \widetilde{R^{ab}}$, where $\widetilde{R^{ab}}$ is the curvature tensor after the deformation process and $\kappa'$ is a modified coupling constant. The sum $\sum_{a,b=1}^{4} \widetilde{R^{ab}}$ can be interpreted as the average curvature value times 16 and hence it makes sense to redefine the action of General relativity in the following way: $S_{GR} = \nu \sum_{i=1}^{\#X} \Omega_i$. Here, $\nu$ is a new coupling constant. Finally, the Feynman path integral for quantum gravity has the form:

$$Z = \sum_{Eq(\alpha,\beta) \wedge \alpha,\beta \in S} \sum_{Q} \exp(iS_{GR}).$$

The set $Q$ denotes the set of all possible E-semigroup generators and has the cardinality $n$. For compact topological spaces, $Z$ is finite if $Q$ is also finite. For the standard E-Theory it is assumed that the sum over $Q$ can be omitted, because $\#Q = 4$ (since the Minkowski spacetime has 4 topological dimensions). □

The E-Theory applied to gravity is a quantum field theory that is based on the non-homogeny of spacetime; by Noether's theorem it is a theory that does not conserve energy and momentum.

## 3. CONCLUSIONS

With the use of commutative semigroup theory, a quantization of gravity is possible due to the introduction of a curvature measure. Such a theory has only equalizers as a degree of freedom that are simply Boolean variables (is there an equalizer in semigroup element or not?). This makes the theory easy to implement in computer simulations. A disadvantage of this theory is that the new action which is linked to the loss of exactness in a chain complex has a small deviation from the original General relativity.

**REFERENCES**

Schmieden, C. et al. "Eine Erweiterung der Infinitesimalrechnung." Mathematische Zeitschrift, 1958, 69: 1-39

Hamber, H. W. "Quantum Gravitation - The Feynman Path Integral Approach." Springer Publishing, 2009. ISBN 978-3-540-85292-6

Schwarz, J. H. "String Theory: Progress and Problems." Progress of Theoretical Physics Supplement, 2007, 170: 214–226. arXiv:hep-th/0702219. Bibcode:2007PThPS.170..214S. doi:10.1143/PTPS.170.214.

Ashtekar, A. "New Hamiltonian formulation of general relativity." Physical Review D, 1987, 36 (6): 1587–1602. Bibcode:1987PhRvD..36.1587A. doi:10.1103/PhysRevD.36.1587

Loll, R ."Discrete Approaches to Quantum Gravity in Four Dimensions". Living Reviews in Relativity, 1998, 1: 13. arXiv:gr-qc/9805049. Bibcode:1998LRR.....1...13L. doi:10.12942/lrr-1998-13

St. Golab. "Über den Begriff der "Pseudogruppe von Transformationen"." Mathematische Annalen, 1939, 116: 768–780. doi: 10.1007/BF01597390

Katz, N. "Nilpotent connections and the monodromy theorem." IHES Publ. Math., 1970, 39: 175-232.