Heron of Alexandria and the Dome of Hagia Sophia in Istanbul

Helge Svenshon
Technische Universität Darmstadt, Germany

ABSTRACT: Writings have been published under the name of Heron of Alexandria in a long period between the 1st century AD and the Byzantine middle ages. This extensive collection was issued for use by engineers, geodesists, architects as well as for other engineering related professionals and it belongs to the most important sources for the history of the ancient building trade, yet even until today it remains almost entirely overlooked.
These manuals, conceived for a wider specialised readership and broadly known, also contain the mathematical and technical prerequisites, that were essential to professionally realise the concept of a building. The groundwork for this, is found in a mathematical tradition, that has been almost consistently passed down from Old-Babylonian times to the geometrical treatises of the early modern times.
The comparison between Heron's texts with singular, well-preserved buildings and especially the Hagia Sophia in Istanbul, demonstrates the degree of influence this neglected source has had on the technical and built environment of antiquity.

INTRODUCTION

The broad reception of Roman architecture theorist Vitruvius, within architectural history, has notably contributed to scientific disregard for fields of knowledge, crucial for understanding the production of ancient architecture. While he presents a multitude of conceptual recipes for modular, ideal type architecture in his work, which had little impact in its time, further questions concerning the mathematical and technical groundwork essential for the transformation of the presented concepts into real structures were left unanswered, that is the practical realisation at the building site in all its various manifestations, such as building surveys, construction and logistics.
A reason for such a disparity might be found, among other things, in the predominant misconception, that architectural concepts in the antiquity were developed with compass and ruler along regular figures of *pure* geometry, that would grant the architectural product dignity and significance through their mathematical perfection and harmony. However, a sufficient amount of clues and contemporary evidence supports the notion that the engineers of antiquity worked principally along a system of rational numbers and approximation values. These often even conflicted with the exact geometry of Euclid's teachings.
Such calculating methods stand in a long tradition stretching from the Babylonian to the early modern times (Neugebauer 1969, p. 146; Friberg 2007). Essential for their transmission through history has been Heron the engineer, mechanic, mathematician and teacher at the Mouseion of Alexandria, especially as, from the first century AD up until the Byzantine Middle Ages, they were mainly published under his name. Although the practically oriented manuals of Heron, that were used as standard references during the antiquity and thus held in high regard, were translated in the beginning of the 20th century into German and English, their importance for the history of architecture has been completely overlooked (Schmidt 1903; Heiberg 1912; Heiberg 1914). The potential of this neglected source of mathematical history for the process of understanding ancient architectural production is revealed in the examples of singular, well preserved buildings, like the Theatre in Epidauros and the Tower of the Winds in Athens (Svenshon 2008; 2009). The reason for this is that the instruments from the field of applied mathematics and geometry revealed within those structures have left significantly more traces in the technical and built world of the antiquity than assumed, especially as the knowledge secured within these instruments originated from deeper layers of the past than Heron's and his works' dating would allow us to assume (Neugebauer 1969).

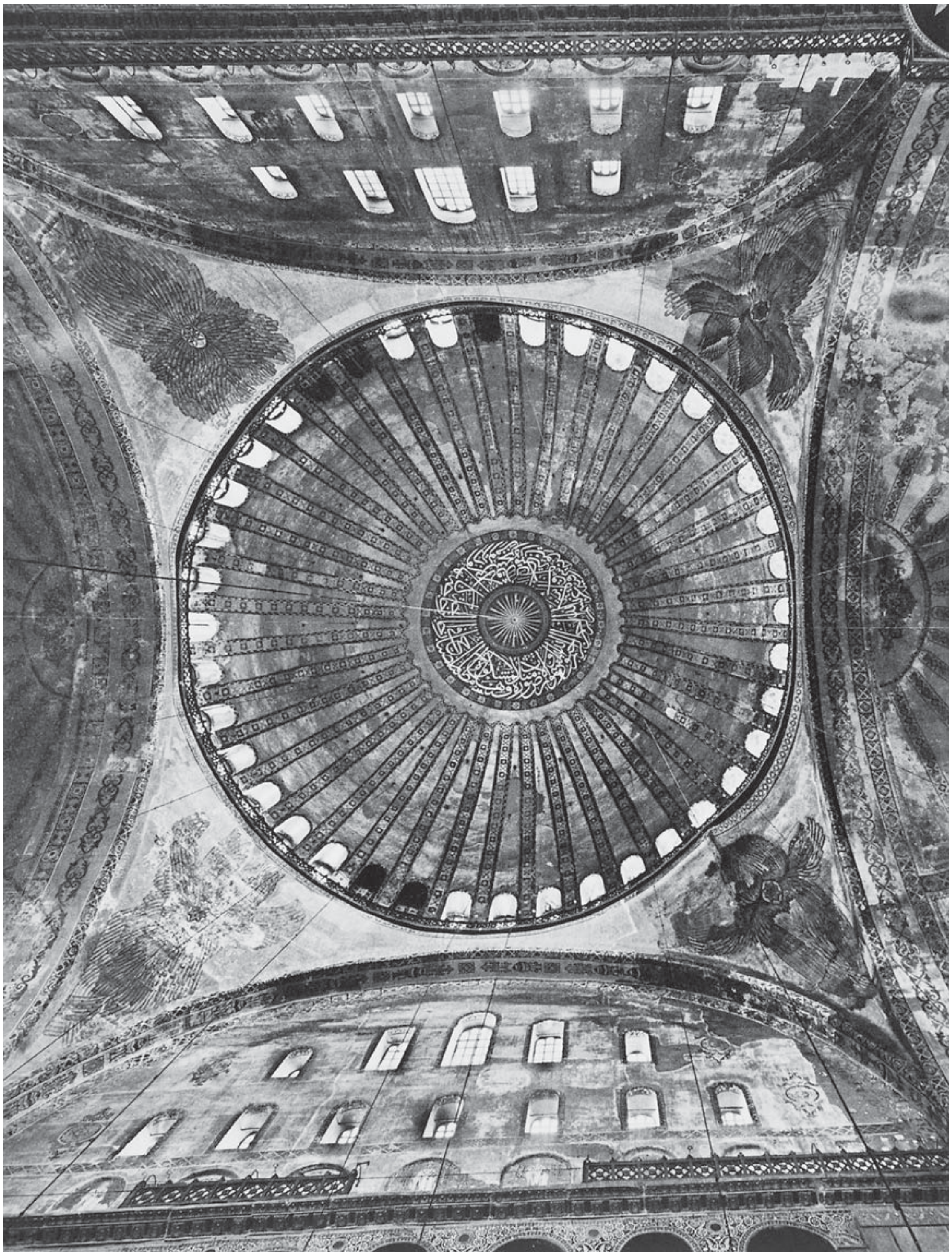

Figure 1: Hagia Sophia in Istanbul. View into the dome and the pendentives; (Kähler 1967, pl. 27)

THE HAGIA SOPHIA AND ITS ENGINEERS ANTHEMIUS OF TRALLES AND ISIDORUS OF MILETUS

Among the structures, of which concept and construction could only be fully understood by consulting these mathematical fundamentals, the Hagia Sophia in Constantinople, built on the instruction of Emperor Justinian, has to be particularly noted. This mighty church was erected in place of the Basilica of Theodosius II, ruined during the Nika riots, with the remarkably short construction time of only about six years, namely between the 23rd February 532 AD and the inauguration on the 27th December 537 AD. Placed on an extraordinary site, it is the result of the architectural idea of a fusion between the oblong floor plan of a basilica and the figure of a domed central-plan space. It is both its dimensions – like the 56 meters high free floating pendentive based dome – and bold conception – the asymmetrical nature of which produced constructive problems of a new quality concerning the load transfer of domes and vaults – that make this church a one of a kind example of the engineering arts (Mainstone 1988) Fig. 1. The uniqueness of the floor plan layout becomes even clearer

when compared to similar contemporary buildings. The complicated geometry is furthermore affected by the blending of these two rather conflicting floor plan types - central and oblong - and shows a curious over-expansion of typical details for a central-plan building, like the eccentrically placed conchs, that are usually found opposing each other, on crossed diagonals.
It seems that this extraordinary layout of the church, in its entirety, is the reason its geometry and therefore fundamentals for its measurements, so essential in the building process, have not undergone a rational analysis and explanation for such a long time. Up until now, none of the many studies dedicated to giving answers to these problems have been capable in providing plausible solutions in any way compatible with technical thinking of those times.

It is therefore imperative that this structure be recognised as a serious source of its contemporary knowledge and be placed in the context of the available ancient literature. Especially in the case of the Haghia Sophia we are in the fortunate position that the conveyed knowledge concerning the two of its architects, Anthemius of Tralles and Isidorus of Miletus, is indicating directly to the texts, that are urgently needed to understand this exceptional building. These two "brilliant engineers", as described by Procopius of Ceasarea, cast a unique light on the qualifications of the building engineers of the time (Veh 1977, p. 23). Of both it has been reported that they were not only real specialists of their practical field, but also knowledgeable in theoretical aspects. About their building activities sadly only very little is known. Procopius mentions only the intended planing of a retaining dam in the east-roman fortified town of Dara in Mesopotamia, for which both engineers were active only in an advisory role (Veh 1977, pp. 95-97).
On the other hand, the sources present quite a differentiated picture about their knowledge, especially in the field of theoretical fundamentals and methods of the disciplines of ancient engineering. Both seem to have studied in Alexandria as both names appear as dedications in writings of Eutokios of Askalon, a mathematician and philosopher, that taught in the Mouseion of Alexandria. It is known that Anthemius occupied himself, among others, with concave mirrors and that he authored a scholarly piece under the title "Peri Paradoxon Mechanematon" (Huxley 1959). As for Isidor, there are also different clues to his theoretical writings. It is being reported that he studied the works of Euclid and Archimedes and that he authored a number of texts on those subjects. His activities in the field of applied mathematics seem to include the design of a compass-like object used for drawing parabolas, that was probably very useful when drawing conic sections (Downey 1948, pp. 112-118; Meek 1952; Cameron 1990, p. 116, pp. 119-127).

HERON'S TEACHINGS ON VAULTING

In connection to the planning and erection of the Hagia Sophia, one piece of information is of special importance; Isidor had written a commentary on the vault-teachings of Heron of Alexandria, an author so important for engineering of antiquity (Downey 1948, p. 113; Cameron 1990, p. 120). Unfortunately nothing has survived of this commentary, yet Heron's own writings are to be found, even if only in fragment, in a manuscript from Constantinople. This seems to be a collection of mathematical problems, that along with other examples are presented under the combined title of "Stereometrica" in the fifth volume of the German edition of Heron and has also been published as a commented reprint of the Codex Constantinopolitanus (Heiberg 1914, pp. 59-121; Bruins 1964).
All the calculations collected within "Stereometrica" are characterised by a strong practical orientation, collecting measuring methods principally for volumes and surfaces of a variety of architectural components. That way, for example, it is possible to calculate adequate figures for the necessary quantities of material for a building project. Since the problems presented are primarily used for practical evaluations of construction operations, the solutions that are produced consist solely of rational numbers and fractions, even if that means utilising *inaccurate* approximations.
These practically oriented processes become especially sensitive in examples concerning vault calculations, because the Archimedean formulas for the measuring of circles, spheres and cylinders are used without exception. These are of course defined by the irrational constant π and therefore produce solution values that are *unpronounceable* for Greek mathematics and therefore unusable. Considering that the approximation for π was set by Archimedes between 3 1/7 and 3 10/71, it wouldn't have been surprising if these calculations would have become so complicated that they would have been of no use to the construction trade. It is because these "numbers are uncomfortable for measuring" that Heron in his Metrica suggests the use of the less accurate value of 3 1/7 (= 22/7), that seems considerably more practicable for the calculation of circle related problems (Schmidt 1903, p. 67). In most of Heron's examples concerning circles, cylinders and spheres – or their practical manifestations as vaults and domes – the values for radius or diameter are a multiple of 7, equal to the denominator of the approximation for π. This choice makes it easy to cancel the denominator in calculations of either surface, circumference or volume and produce integral and thereby effectively standardised solutions.
Therefore, when ascertaining the circle circumference using the formula $U = D \cdot 22/7$, the following elegant solutions are produced, which can also be easily achieved with mental arithmetic, because of the simplicity of this procedure. When using a diameter of $D = 7$ and $D = 14$, by simply cancelling the denominator, the circumference values of $U = 22$ and $U = 44$ respectively are produced. The situation is similar for the calculations of circle surfaces, for which Heron offers a variety of approaches. One of these variants, that is closely related to an Euclidean approach, is expressed in the formula $A = D2 – (1/7 +1/14)$ that also produces

rational results with these diameter values; using D = 7 and D = 14 the results are A = 38 ½ and A = 154 respectively. These type of values also dominate the examples concerning spheres and cylinders, so that it becomes apparent what the advantages of such a systematised approach for the calculation of such complicated construction elements could have been. It is even imaginable that charts collecting such calculation-results, comparable to the extensive conversion tables in Heron's "Geometrica", existed and were used for the constructive processes (Heiberg 1912, pp. 183-203).

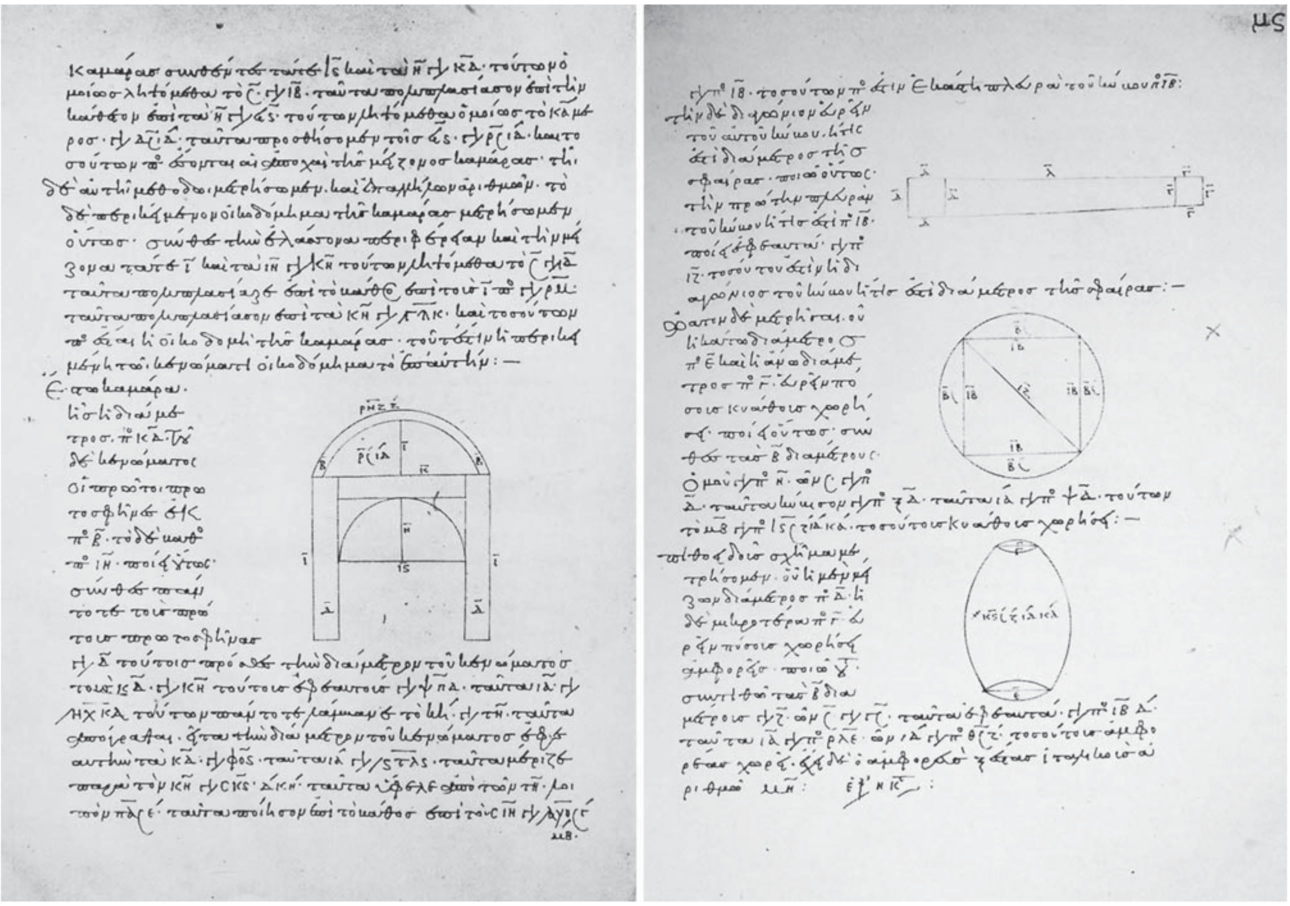

Figure 2: Calculation of overlaying vaults (left) and measurement of the pendentive, middle sketch drawing (right); (Bruins 1964, Vol. 1, pls. 87, 92)

This system is then used in the few extant examples of a variety of vault calculations, from the simplest barrel vaults to the complicated forms combining several vault elements. All presented examples have a strong practical orientation and, without exception, mirror the reality of the construction trade of their time. This becomes apparent when observing a sketch depicting two overlaying barrel vaults with arched lintels and round arches that rest on retaining walls, which strongly resemble the Hagia Sophia narthex (Bruins 1964, Vol. 1, pl. 92) Fig. 2.

HERON'S CALCULATION OF A PENDENTIVE VAULT AND THE DESIGN OF THE HAGIA SOPHIA

One of the further problems presented directly concerns the building of the Justinian church and in fact its most challenging area; the central dome. Immediately after the completion of this building, the dome was damaged by a strong earthquake and collapsed in 558 A.D. According to contemporary sources it was then rebuilt about 6.5 meters higher to increase its bearing capacity. It was probably due to this change that it obtained the still visible form of a pendentive; a two-part vault, the crowning hemisphere of which rises over a larger vault that has been reduced to triangular segments. Heron's writings explain the exact calculation method for such spherical triangles by subtracting four sphere sections from a hemisphere after inscribing within it the half of a cube. The resulting surface, as well as the volume, can thereby easily be ascertained. This calculating method is in turn based on Archimedean laws and corresponds to the still valid method that can be found in many formularies (Bruins 1964, Vol 1, p. 81, p. 87; Vol 3, pp. 121-123) Figs 2, 3.

$$d^2 = 2\,a^2,\ k = (d - a)/2,$$
$$V = {}^1/_{12}\,\pi\, d^3 - 2\,.\,{}^1/_6\,\pi\,[3(a/2)^2 + k^2]\,k.$$

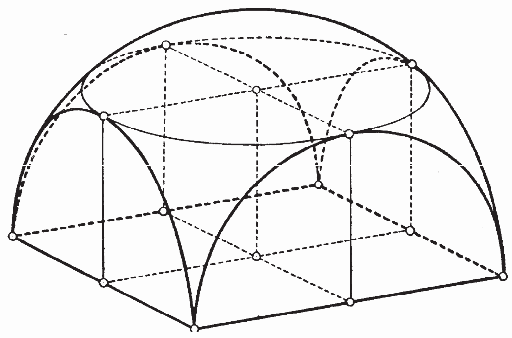

Figure 3: Schematic drawing with reconstructed calculation method; (Bruins 1964, Vol. 3, p. 122)

Besides its importance for the history of mathematics, this mathematical problem can lead to a deeper understanding of the planning and construction processes of Hagia Sophia. It is also essential in clarifying the conceptual geometry along with the measuring system, which forms the base for the realisation of the entire design. That is because on the contrary to all the other vault calculations, no diameter values compatible to the π approximation (22/7) were used (i.e. 7 or 14), but rather measurements derived from the so-called side-and-diagonal number progression.

This discovery of this numeric system has been attributed to the Pythagoreans by sources of late antiquity, but it was probably already known substantially earlier; in Babylonian times. With its help, approximations for √2 can be produced, which is essential for measuring all types of square or square-related objects and surfaces. Starting with number one and with the help of the formula:

s1, s2 = s1 + d1, s3 = s2 + d2, ... *and* d1, d2 = 2s1 + d1, d3 = 2s2 + d2, … (s = side, d = diagonal) (1)

a progression of squares is created (s/d = 1/1, 2/3, 5/7, 12/17, 29/41, 70/99, etc.), the side to diagonal ratio of which converges progressively to √2, since the difference between the square of the diagonal value and the double square of the side value alternates between either +1 or -1 (Heller 1965) Fig. 4.

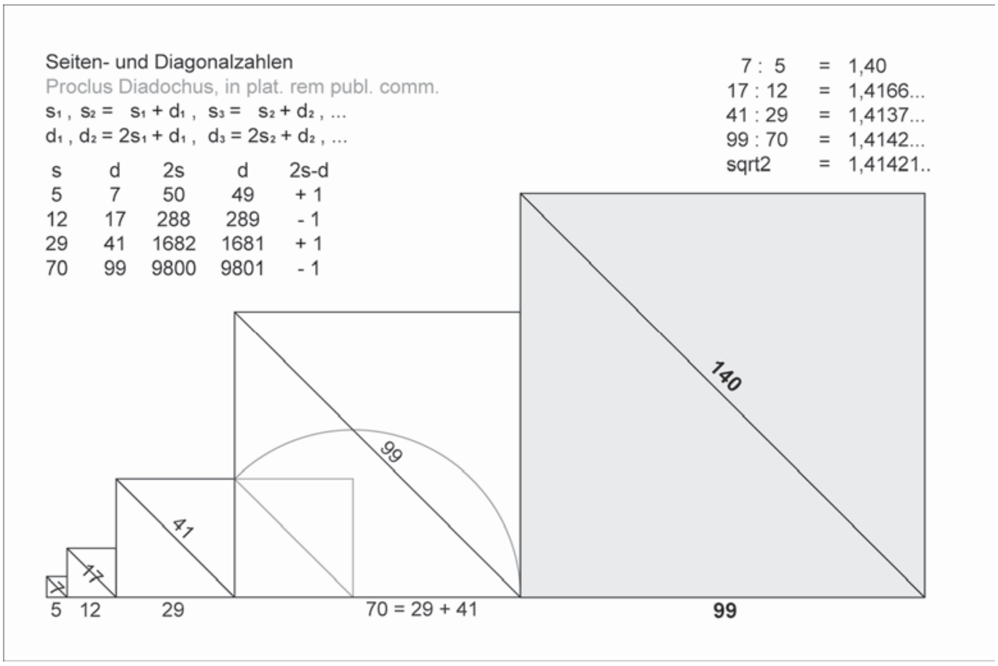

Figure 4: System of side and diagonal numbers; (Svenshon after Heller 1965, p.334)

The square defined by the numbers 12 and 17, whereas 12 defines the side of the square and 17 its diagonal, has been used as a standard value – similar to the use of 7 and 14 for the circle calculations – as early as in cuneiform Babylonian texts (Bruins; Rutten 1961, pp. 25-26, pl. 4-5). It is therefore often found in Heron's examples for square and octagon calculations and is also used to calculate the spherical triangles of the pendentive (Smily 1944). This utilisation of 12/17 is perfectly logical, since the specific form of the pendentive dome is defined by the square of its floor plan, which, according to ancient mathematical praxis, can only be described with "pronounceable" values with the help of the rational approximations produced by the side-and-diagonal number (Plato, Rep. VIII. 546b).

The afore mentioned square corresponds exactly to the one underneath the mighty pendentive vault in the centre of the Hagia Sophia, which has been rendered nearly invisible because of the inner sides of its pillars and which supplies the fundamental geometrical figure for the design of this building. Considering this architectural disposition, it appears almost imperative to visualise the necessary geodetic calculation methods within the measurements and ratios between segments of the building, especially since the design of nearly the entire floor plan has been developed in consequent dependency to that square.

The sides of the square with their highly accurate length of 31m reveal a very clear picture. Up until now it was thought that this length corresponded to 100 Byzantine feet, an assumption that was quickly adopted by the scientific world if nothing else for its persuasive elegance, even if there are no historical sources defining the size of one Byzantine foot (Schilbach 1970, p. 13). Yet such a *round* figure for the square side would lead to a diagonal with the irrational length of 141,421... feet, because of the factor √2. This would mean that the square and all dimensions related to it would not be manageable by the instruments of the surveyors of those times. If one realises that in this context, the diagonal is nothing else but the diameter of the circle defined by the vault's circumference, while at the same time remembering Heron's circle calculations, in which the practical diameter-values 7 and 14 were used, then it becomes obvious that the diagonal of the square, or diameter of the circle, has been calculated with the tenfold of the exemplary value of 14, or else 140 feet.

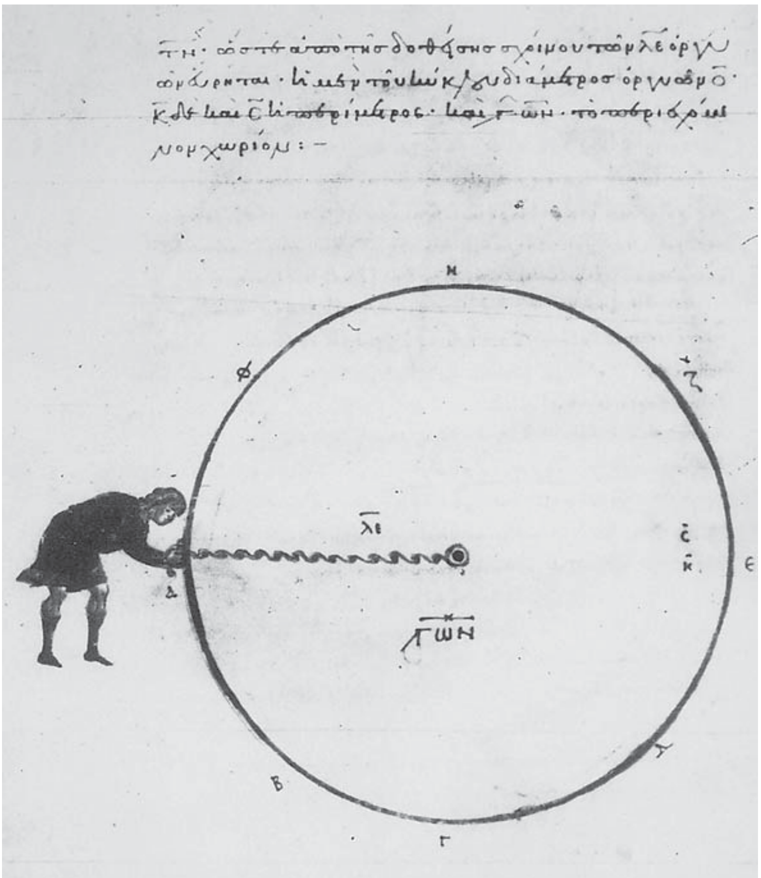

Figure 5: Measurement of a circle by Heron of Byzantium; (Sullivan 2000, pl. 36)

Yet in Heron's mathematical problems only small values up to a max. of 14 feet or cubits are used, with which calculations can easily and coherently be attempted, but that do not correspond to dimensions found in actual buildings. However, a late compilation of Heronian knowledge, that was published under the name of Heron of Byzantium, probably in the 10th century, reveals that written information concerning construction-fitting measurements existed as well. In the Geodaesia compendium, among other things, mathematical problems concerning circle measuring are presented using diameter-values of between 70 and 210 cubits (Sullivan 2000, pp. 129-133, figs. 35, 36) Fig. 5. Such dimensions are better suited for monumental circular and central-plan buildings and especially for the dome diameter of the Hagia Sophia and still also remain compatible to the approximation of π

$$70 \cdot 22/7 = 220;\ 105 \cdot 22/7 = 330;\ 140 \cdot 22/7 = 440;\ ...\ 210 \cdot 22/7 = 660 \qquad (2)$$

Additionally only after such large dimensions are utilised, the relation becomes apparent between the

approximation for π of 22/7, used in circle and square calculation, and specific numbers of the side-and-diagonal number progression, exactly the values 70 and 140, the ones relevant for the construction of cupolas. Using value of 140 feet for the diameter of the pendentive dome, and therefore also the diagonal of the floor plan square, then, after a modification of the assumed length by exactly one percent, the size of its side is described by the highly accurate approximation of 99 feet. This measurement is not only rational, but it is also embedded in the system of the side-and-diagonal number progression and therefore a usable value by the applied mathematics of antiquity. It has already been elaborately stated elsewhere, how this conventional and verifiable system of approximations, which was used throughout antiquity from the Babylonian time to the Byzantine middle ages, can be helpful in revealing the – up until now considered as complicated – geometrical structure of the Hagia Sophia floor plan and in describing it with rational numbers
(Svenshon; Stichel 2002; Svenshon 2003; Svenshon; Stichel 2006) Fig. 6.

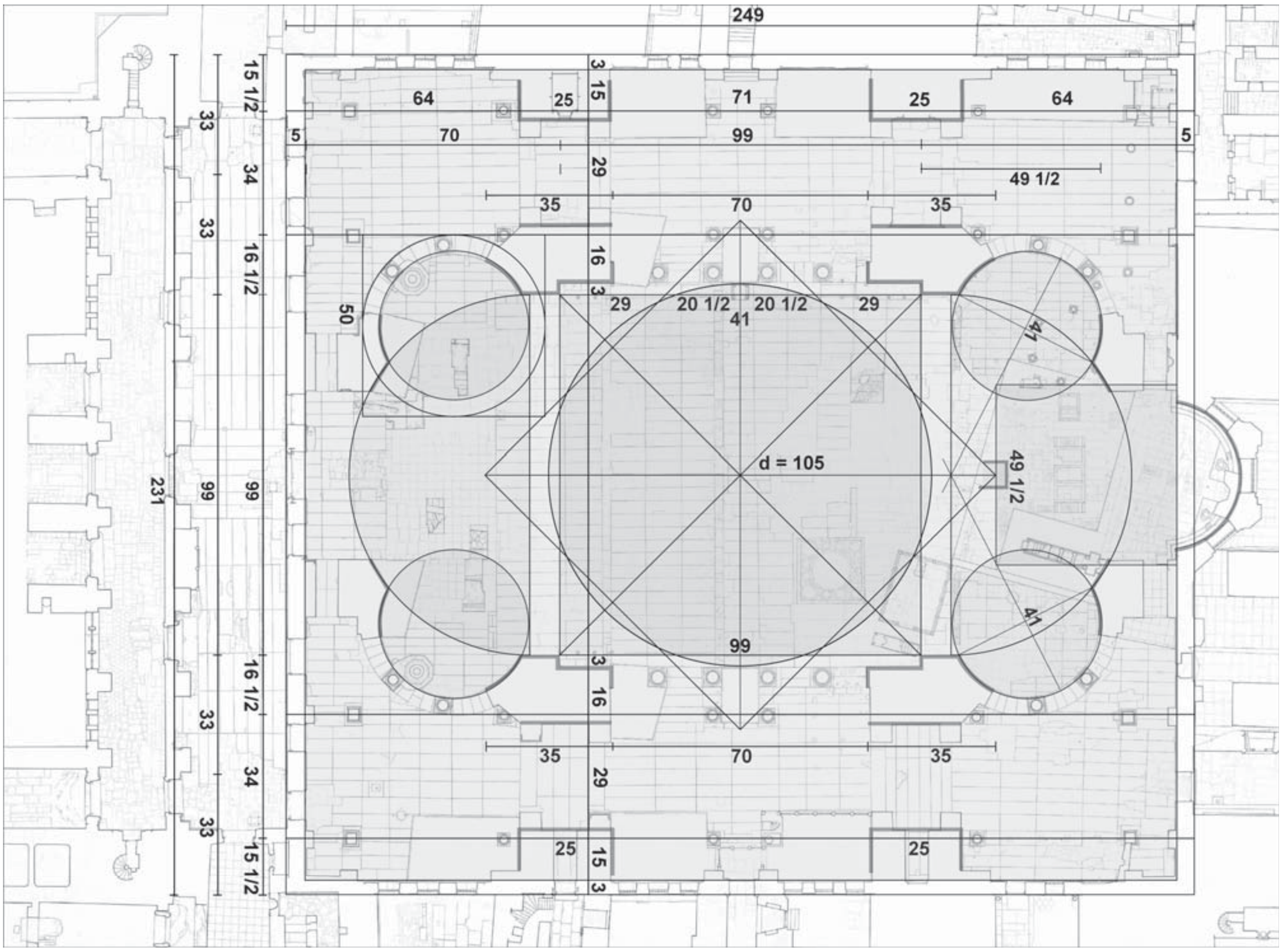

Figure 6: Hagia Sophia, floor plan with the most important measurements; (Svenshon)

CONCLUSION

With these design methods as a backdrop it is not surprising that further domed and central-plan buildings have been conceived using this system. At nearly the same time as the Hagia Sophia, the central octagon of the Sergius-and-Bacchus church was built. In this case the central square is only half as large as the one in the Hagia Sophia and the dome has a diameter of 70 feet, a value that is also found in both the approximation systems. The Justinian church of San Vitale in Ravenna must be placed in the same context. Its octagonal core is only slightly different in dimension from the one of the Sergius-and-Bacchus church and therefore it must be assumed that the same methods were used.
Therefore it is apparent that the handbooks of Heron constitute a rediscovered source, that can offer an extended perspective concerning ancient design processes and therefore at the same time a new field for interpretation for the entirety of ancient architecture. Furthermore the extant calculation examples document the theoretically funded yet pragmatically simple instruments ancient architects and engineers used to realise their intent. Only under those circumstances was it possible for them to precisely pre-calculate large projects with all their specific requirements, from the complicated construction to the transnational transfer of material.

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