The Stokes and Navier-Stokes equations in layer domains with and without a free surface

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Introduction

The Stokes and Navier-Stokes equations have first been formulated in the early 19^{th} century as a model for the behaviour of viscous incompressible fluid flow and have subsequently attracted the interest of mathematicians, physicists, and engineers alike. While much progress has been made in understanding the Stokes and Navier-Stokes equations during the 20^{th} century and in recent years many questions remain open, and many problems remain unsolved. Various attempts to further the understanding of the Stokes and Navier-Stokes equations have led to various concepts of solvability within various classes of functions.

Even though in general no explicit representations for solutions to the Stokes or Navier-Stokes equations are known, it turns out that in certain kinds of underlying geometries it is possible to derive explicit solutions for the linear Stokes equations e. g. by Fourier analytic means. In this way a fruitful study of various aspects of the Stokes and Navier-Stokes equations e. g. in the full space $\mathbb{R}^n$ or the half space $\mathbb{R}^n_+$ has been carried out.

Another kind of domain where the Stokes equations are accessible to investigations by Fourier analytic means are infinite layer domains $\Omega = \mathbb{R}^{n-1} \times (0,1)$ which, although clearly unbounded, share some common ground with bounded domains in that for instance Poincaré's inequality continues to hold.

Among other authors Abe [**Abe04**], Abe and Shibata [**AS03a**, **AS03b**], Abels [**Abe06**, **Abe05a**, **Abe05b**, **Abe05c**], Abels and Wiegner [**AW05**] studied the resolvent problem for the Stokes equation in an infinite layer domain in the Lebesgue spaces L_p for $1 < p < \infty$ by means of an explicit representation of solutions and established various properties of the associated solution operator. It was established that the Stokes operator in L_p does not only generate a holomorphic and strongly continuous semigroup, but even admits a bounded $\mathcal{H}_\infty$-calculus and in particular admits bounded imaginary powers and has maximal regularity. Abe and Yamazaki [**AY10**] made similar investigations by similar means in the setting of Besov spaces.

The endpoint cases L_1 and L_∞ though had not been treated thus far and it was unknown whether any of these results carry over. We attempt to close this gap in chapter III and chapter IV. In chapter III we show that the Stokes operator in a two-dimensional layer domain in the solenoidal L_∞-type space $C_{0,\sigma}$, once adequately defined, generates a holomorphic strongly continuous semigroup, and we show that this result does not extend to three- or higher-dimensional layer domains. In chapter IV we show a similar result in a solenoidal subspace $L_{1,\sigma}$ of L_1, namely a generation result in a two-dimensional layer and a rather strong non-generation result in higher-dimensional layer domains.

Our results stand in contrast to those in the reflexive range $1 < p < \infty$ in that the Stokes operator in L_p generates a semigroup regardless of the spacial dimension. Our results also stand in contrast to known results for the Stokes resolvent problem in a half space due to Desch et al. [**DHP01**] who showed that the Stokes operator in solenoidal subspaces of L_∞, including $C_{0,\sigma}$, generates a holomorphic semigroup regardless of the dimension n, while in $L_{1,\sigma}$ a non-generation result holds, also regardless of the dimension.

Even though the results of Desch et al. [**DHP01**] have been greatly generalised to a much larger class of domains by Abe and Giga [**AG13**, **AG14**], who employed a very different method of proof based on a blow-up argument, the Stokes resolvent problem in layer domains had remained unsolved.

Our treatment of the Stokes resolvent problem in layer domains in solenoidal subspaces of L_1 and L_∞ is based on an explicit representation of the solution which can be derived and estimated by Fourier analytic means. While this approach bears resemblance to the work by Abe and Shibata [**AS03a**, **AS03b**], which in fact was the starting point and main inspiration for our work, this resemblance is limited by the fact that Fourier analysis, and harmonic analysis in general, is a whole different matter in L_1 and L_∞ than in L_p for $1 < p < \infty$.

Where the chapters III and IV exclusively treat the linear Stokes equations, in the final chapter V we will direct our ambitions towards a nonlinear free boundary problem for the Navier-Stokes equations, albeit in spaces based on L_p with p from the reflexive range of parameters. The free boundary problem which is subject to our investigations models the motion of a viscous incompressible fluid in a layer domain with free upper surface and hence is often referred to as the water wave problem or the ocean problem.

When describing the motion of a fluid with a free surface one has a certain degree of freedom in that one can choose which physical effects and influences to include in the model and which to neglect. In particular one can study a model that includes the influence of surface tension, or one can study a model that willingly neglects the effect of surface tension on the motion of the fluid.

Both the water wave problem with and without consideration of the effect of surface tension have been studied for decades by a variety of authors. Among the contributions where the effect of surface tension was taken into account one should mention Beale [**Bea84**], Beale and Nishida [**BN85**], Solonnikov [**Sol86**, **Sol89**, **Sol91**], Mogilevskiĭ and Solonnikov [**MS91**], Tani [**Tan96**], Tani and Tanaka [**TT95**], who studied the water wave problem in various classed of domains in a Hilbert space setting, i. e. in function spaces based on L_2, except for Mogilevskiĭ and Solonnikov [**MS91**] who worked in the more classical Hölder space setting. The first results concerning the water wave problem with surface tension taken into account in an L_p-setting were obtained by Shibata and Shimizu [**SS11**].

The water wave problem with surface tension not taken into account has been studied among others by Beale [**Bea81**], Sylvester [**Syl90**], Tani and Tanaka [**TT95**] in an L_2-setting, and later by Abels [**Abe05a**] in an L_p-setting.

It is a natural question to which extent these two different models and their properties are related to one another, and we made this question the leitmotiv of chapter V. As the influence of surface tension is represented by a parameter $\sigma \geq 0$, with $\sigma > 0$ corresponding to the case where surface tension is taken into account and $\sigma = 0$ corresponding to the case where the effect of surface tension is neglected, the question how the two models are related can be rephrased as the question how the solutions of the free boundary problem corresponding to a surface tension parameter $\sigma > 0$ behave in the limit $\sigma \to 0$ of vanishing surface tension.

It turns out that this is a singular limit in that certain quantities will have a higher regularity in the presence of surface tension than in its absence. Investigating the singular limit of vanishing surface tension we thus have to deal with a loss of regularity. While it remains unclear how or whether one can obtain satisfactory results in a formulation in Eulerian coordinates, where one could base ones investigations on the approach and results by Denk et al. [**DGH$^+$11**], it turns out that an analysis based on Shibata and Shimizu [**SS07**, **SS11**], who used a formulation in Lagrangian coordinates, of the singular limit of vanishing surface tension for the water wave problem is feasible.

We will show that the solution to the water wave problem with surface tension exists in some time interval which can be chosen independently of the surface tension parameter $\sigma > 0$ and does indeed converge as $\sigma \to 0$ to the solution of the water wave problem without surface tension. Our analysis is based on a thorough understanding of a linearised problem which in turn relies heavily on an explicit solution formula for the Stokes resolvent problem in a layer domain with certain boundary conditions.

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Deutsche Zusammenfassung

Die Stokes- und Navier-Stokes-Gleichungen wurden im frühen 19. Jahrhundert als Modell für das physikalische Verhalten inkompressibler viskoser Fluide formuliert und wurden seither von Mathematikern, Physikern, und Ingenieuren gleichermaßen untersucht. Obwohl im 20. Jahrhundert und auch in den vergangenen Jahren im Verständnis der Stokes- und Navier-Stokes-Gleichungen große Fortschritte gemacht wurden, sind viele Fragen offen geblieben und viele Probleme ungelöst.

Während im allgemeinen keine explizite Darstellung von Lösungen der Stokes- oder Navier-Stokes-Gleichungen bekannt ist, stellt es sich doch heraus dass in bestimmten Geometrien explizite Lösungen der linearen Stokes-Gleichungen konstruiert werden können. Dies gilt in erster Linie für die Stokes-Gleichungen in $\mathbb{R}^n$, aber auch im Halbraum $\mathbb{R}^n_+$, wo man mit Hilfe Fourier-analytischer Methoden Lösungen konstruieren und untersuchen kann.

Eine weitere Klasse von Gebieten, die dies zulässt, sind die sogenannten Schichtgebiete $\mathbb{R}^{n-1} \times (0, 1)$. Obwohl diese offensichtlich unbeschränkt sind, teilen sie sich einige Eigenschaften mit beschränkten Gebieten insofern, dass beispielsweise die Poincaré'sche Ungleichung gilt.

Es ist bekannt, dass man für das Resolventenproblem für die Stokes-Gleichungen in einem Schichtgebiet mit Hilfe Fourier-analytischer Methoden zeigen kann, dass der Stokes-Operator in den Lebesgue-Räumen L_p für $1 < p < \infty$ nicht nur eine holomorphe stark-stetige Halbgruppe erzeugt, sondern darüber hinaus einen beschränkten $\mathcal{H}_\infty$-Kalkül gestattet, insbesondere also auch beschränkte imaginäre Potenzen besitzt und maximale Regularität hat.

Die Stokes-Gleichungen in Schichtgebieten in den Räumen L_1 und L_∞, also an den Endpunkten der Skala der Lebesgue-Räume, wurden bis dato nicht untersucht, und es war unbekannt ob und inwiefern die obigen Resulte sich übertragen lassen. In Kapitel III und Kapitel IV versuchen wir, diese Lücke zu schließen.

In Kapitel III zeigen wir, dass der Stokes-Operator in zweidimensionalen Schichtgebieten in dem Raum $C_{0,\sigma}$ divergenzfreier stetiger Funktionen eine holomorphe stark-stetige Halbgruppe erzeugt, und wir zeigen, dass sich dieses Resultat nicht auf drei- oder höherdimensionale Schichtgebiete übertragen lässt. In Kapitel IV zeigen wir im Raum $L_{1,\sigma}$ divergenzfreier integrierbarer Funktionen ein Erzeugerresultat in zweidimensionalen Schichtgebieten und geben ein Beispiel an, das zeigt, dass ein vergleichbares Erzeugerresultat in drei oder mehr Dimensionen nicht gelten kann.

Diese Resultate sollten den korrespondierenden Resultaten in L_p für $1 < p < \infty$ gegenüber gestellt werden, wo ein Erzeugerresultat unabhängig von der Raumdimension gilt, sie sollten allerdings auch mit bekannten Resultaten im Halbraum $\mathbb{R}^n_+$ kontrastiert werden, wo in $C_{0,\sigma}$ unabhängig von der Raumdimension ein Erzeugerresultat gilt, wohingegen in $L_{1,\sigma}$ kein Erzeugerresultat gilt, ebenfalls unabhängig von der Raumdimension.

In Kapitel V schließlich beschäftigen wir uns mit einem nichtlinearen freien Randwertproblem für die Navier-Stokes-Gleichungen in einem Schichtgebiet mit fixem unteren und freiem oberen Rand.

In der mathematischen Beschreibung der Bewegung eines Fluids mit freiem Rand hat man grundsätzlich einen gewissen Grad an Freiheit insofern, dass man wählen kann welche physikalischen Einflüsse man in dem Modell berücksichtigen, und welche man vernachlässigen möchte. Insbesondere kann man ein Modell studieren, dass den Einfluss der Oberflächenspannung miteinbezieht, oder man kann ein Modell studieren, das diesen Einfluss bewusst vernachlässigt.

Beide Modelle wurden bereits ausgiebig untersucht. Dennoch war nicht bekannt, inwiefern diese Modelle und ihre Eigenschaften zusammenhängen. Diese Fragestellung ist das Leitmotiv von Kapitel V.

Da man den Einfluss der Oberflächenspannung durch einen Parameter $\sigma \geq 0$ darstellen kann, wobei $\sigma > 0$ dem Fall dass man die Oberflächenspannung in das Modell miteinbezieht entspricht, und $\sigma = 0$ dem Fall dass man die Oberflächenspannung vernachlässigt, lässt sich die Frage ob und inwiefern diese beiden Modelle zusammenhängen auf die Frage zurückführen, wie sich Lösungen des freien Randwertproblems mit Oberflächenspannung im Grenzwert $\sigma \to 0$ verschwindender Oberflächenspannung verhalten.

Es stellt sich heraus, dass dieser Grenzwert ein singulärer Grenzwert ist, da bestimmte Größen in der Präsenz von Oberflächenspannung regulärer sind als in ihrer Abwesenheit. Wir werden zeigen, dass die Lösung des freien Randwertproblems mit Oberflächenspannung in einem Zeitintervall, das von dem Parameter $\sigma > 0$ unabhängig gewählt werden kann, existiert, und in der Tat für $\sigma \to 0$ gegen die Lösung des freien Randwertproblems ohne Oberflächenspannung konvergiert.

CHAPTER I

Preliminaries

1. Notation

We will for the most part adhere to standard notation. We use the symbols $\mathbb{R}$ and $\mathbb{C}$ to denote the sets of real and complex numbers, $\mathbb{N}$ denotes the positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We write $\mathbb{R}^d_+$ for the set of $x \in \mathbb{R}^d$ of the form (x', x_d) with $x' \in \mathbb{R}^{d-1}$ and $x_d > 0$. Given $0 < \rho < \pi$ we write

$$\Sigma_\rho = \{z \in \mathbb{C} \setminus \{0\} \colon \, |\arg z| < \rho\}$$

for the open sector in $\mathbb{C}$ of opening angle 2ρ.

Given $a, b \in \mathbb{C}^d$ we will write $a \cdot b = \sum_{j=1}^d a_j b_j$ and with $a \otimes b$ we denote the matrix with entries $(a_i b_j)_{ij}$. Given matrices $A, B \in \mathbb{C}^{d \times d}$ we write A^T for the transpose of A, as well as $A : B = \operatorname{tr} B^T A = \sum_{j,k=1}^d A_{jk} B_{jk}$. Given two Banach spaces X and Y we write $\mathcal{L}(X, Y)$ for the space of linear operators from X to Y, and if X and Y coincide then we will simply write $\mathcal{L}(X)$ instead of $\mathcal{L}(X, Y)$. We will write X' for the topological dual of X and given $x \in X$ and $x' \in X'$ we write $\langle x, x' \rangle$ for the dual pairing.

We will occasionally write C for a generic constant that may change from line to line but will be independent of the free variables unless otherwise stated. Similarly we will occasionally employ the notation $A \lesssim B$, by which we mean $A \leq CB$ with a generic constant C as above.

Additional notation will be introduced when needed.

2. Matrix identities

In this section let $\mathbf{A} \in \mathbb{C}^{n \times n}$, $\mathbf{B} \in \mathbb{C}^{n \times m}$, $\mathbf{C} \in \mathbb{C}^{m \times n}$ and $\mathbf{D} \in \mathbb{C}^{m \times m}$ for some $n, m \in \mathbb{N}$. We state some properties of matrices in general and the block matrix

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$$

in particular. The following results are taken from [**Ber09**].

LEMMA 2.1. *If* $\mathbf{A}$ *is invertible and* $\mathbf{u}, \mathbf{v}$ *are column vectors, then*

$$\det[\mathbf{A} + \mathbf{u}\mathbf{v}^T] = (1 + \mathbf{v}^T \mathbf{A}^{-1} \mathbf{u}) \det \mathbf{A}$$

and if additionally $1 + \mathbf{v}^T \mathbf{A}^{-1} \mathbf{u} \neq 0$ *then*

$$[\mathbf{A} + \mathbf{u}\mathbf{v}^T]^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1} \mathbf{u} \mathbf{v}^T \mathbf{A}^{-1}}{1 + \mathbf{v}^T \mathbf{A}^{-1} \mathbf{u}}.$$

LEMMA 2.2. *If the matrix* $\mathbf{A}$ *is invertible then*

$$\det \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \det \mathbf{A} \det(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}).$$

If $\mathbf{D}$ *is invertible then*

$$\det \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \det \mathbf{D} \det(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}).$$

LEMMA 2.3. *If the matrices* $\mathbf{D}$ *and* $\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C}$ *are invertible then*

$$\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}^{-1} = \begin{bmatrix}(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & -(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{B}\mathbf{D}^{-1}\\ -\mathbf{D}^{-1}\mathbf{D}(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & \mathbf{D}^{-1}+\mathbf{D}^{-1}\mathbf{C}(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{B}\mathbf{D}^{-1}\end{bmatrix}.$$

If $\mathbf{A}$ *and* $\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}$ *are invertible then*

$$\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}^{-1} = \begin{bmatrix}\mathbf{A}^{-1}+\mathbf{A}^{-1}\mathbf{B}(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\\ -(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1} & (\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\end{bmatrix}.$$

We will also use the following explicit representation for the inverse of a regular matrix.

LEMMA 2.4 (Cramer's Rule). *Let* $\mathbf{M}\in\mathbb{C}^{n\times n}$ *be a regular matrix. Then the inverse* $\mathbf{M}^{-1}$ *of* $\mathbf{M}$ *can be written in the form*

$$\mathbf{M}^{-1} = \frac{1}{\det\mathbf{M}}\mathbf{M}^{\sharp}$$

where $\mathbf{M}^{\sharp}$ *denotes the* adjugate matrix *of* $\mathbf{M}$ *with entries* $\mathbf{M}^{\sharp}_{ij}$ *given by* $(-1)^{i+j}$ *times the determinant of the submatrix of* $\mathbf{M}$ *formed by deleting the* i*-th column and the* j*-th row.*

In particular the solution $\mathbf{x}$ *to the linear equation* $\mathbf{M}\mathbf{x}=\mathbf{b}$ *has the representation*

$$\mathbf{x}_i = \frac{1}{\det\mathbf{M}}\sum_{j=1}^{n}\mathbf{M}^{\sharp}_{ij}\mathbf{b}_j$$

for $j=1,\dots,n$*.*

3. Function spaces and Fourier analysis

In this section we introduce certain classes of functions which will be used throughout this thesis, fix notations, and collect some results concerning these classes of functions for later reference. For the most part we will use the notation from the monograph by Triebel [**Tri83**], which we also use as a general reference for function spaces and their properties. It contains most of the results listed in this chapter. Given a set $M\subset\mathbb{R}^d$ we will write $C(M)$ for the Banach space continuous, bounded, complex-valued functions on M, provided with the supremum norm. We will write $C_0(M)$ for the functions $f\in C(M)$ such that for any $\varepsilon>0$ there is a compact set $K\subset M$ with $|f|\le\varepsilon$ outside of K. We provide $C_0(M)$ with the topology inherited from $C(M)$. For open sets $M\subset\mathbb{R}^d$ we write $C^m(M)$ with $m\in\mathbb{N}$ for the functions in $C(M)$ that have continuous classical derivatives up to order m, provided with the norm $\|\cdot\|_{C^m(M)} = \sum_{\alpha\le m}\|\partial^\alpha\cdot\|_{C(M)}$. Here we use the usual multiindex notation $\partial^\alpha=\partial_1^{\alpha_1}\cdots\partial_d^{\alpha_d}$ for $\alpha\in\mathbb{N}_0^d$. We write $C^\infty(M)$ for the functions which are contained in $C^m(M)$ for every $m\in\mathbb{N}$, and $C_c^\infty(M)$ for the functions in $f\in C^\infty(M)$ with compact support $\operatorname{supp} f\subset M$. We write $C_{c,\sigma}^\infty(M)$ for the functions $f\in C_c^\infty(M)^d$ with $\operatorname{div} f=0$. Following Schwartz [**Sch66**] we will occasionally write $\mathscr{D}(M)$ instead of $C_c^\infty(M)$, and we write $\mathscr{D}'(M)$ for the topological dual space of $\mathscr{D}(M)$ as defined in [**Sch66**]. The functions in $\mathscr{D}(M)$ will sometimes be referred to as *test functions*, and the elements of $\mathscr{D}'(M)$ will be referred to as *distributions*. In addition we write $\mathcal{S}(\mathbb{R}^d)$ for the Schwartz space of rapidly decaying functions, and $\mathcal{S}'(\mathbb{R}^d)$ for its dual space, the space of temperate distributions. We will use the monograph by Schwartz [**Sch66**] as a general reference for distributions and related topics.

We will write $\mathcal{F}$ for the Fourier transform

$$\mathcal{F}\colon\mathcal{S}(\mathbb{R}^d)\to\mathcal{S}(\mathbb{R}^d),\qquad \mathcal{F}f(\xi)=(2\pi)^{-d/2}\int_{\mathbb{R}^d}e^{-ix\cdot\xi}f(x)\,\mathrm{d}x \text{ for } \xi\in\mathbb{R}^d$$

and its extension to the space $\mathcal{S}'(\mathbb{R}^d)$ of temperate distributions. We write $\mathcal{F}^{-1}$ for its inverse. We will often use the shorthand notation $\hat{f}=\mathcal{F}f$ and $\check{f}=\mathcal{F}^{-1}f$. We refer to [**Sch66**,**Hör83**,**Gra08**] for an overview over basic properties of the Fourier transform.

The spaces we will encounter most often are the Lebesgue spaces $L_p(M,\mu)$ with $1 \le p \le \infty$ and μ a measure (see e. g. [**HS69**]) which are defined as usual as the (equivalence classes of) measurable functions $f\colon M \to \mathbb{C}$ such that, in the case $1 \le p < \infty$, the norm

$$\|f\|_{L_p(M,\mu)} = \left(\int_M |f|^p \, \mathrm{d}\mu\right)^{1/p}$$

is finite, and in the case $p = \infty$ the norm

$$\|f\|_{L_\infty(M,\mu)} = \operatorname*{esssup}_x |f(x)| = \inf\{t > 0\colon \mu(\{x \in M\colon \ |f(x)| > t\}) = 0\}$$

is finite. If μ is Lebesgue measure then we will simply write $L_p(M)$. We point to [**HS69**] as a general reference concerning Lebesgue spaces, and more generally the basics of real analysis. Given a Banach space X we write $L_p(M,\mu;X)$, and if μ is Lebesgue measure $L_p(M;X)$, for the Bochner-Lebesgue spaces. We will tacitly assume familiarity with the Bochner integral and its properties. As a general reference for the Bochner integral and spaces of vector-valued functions we point to the treatise by Diestel and Uhl [**DU77**].

Based on the Lebesgue spaces and Bochner-Lebesgue spaces one can introduce various function spaces that generalise the very classical spaces of differentiable functions. Following the notation in [**Tri83**] we will write W^s_p for the Sobolev-Slobodeckij spaces, H^s_p for the Bessel potential spaces, $B^s_{p,q}$ for the inhomogeneous Besov spaces, $\dot{B}^s_{p,q}$ for their homogeneous counterparts, $F^s_{p,q}$ and $\dot{F}^s_{p,q}$ for the inhomogeneous and homogeneous Triebel-Lizorkin spaces, and H_1 for the (homogeneous) Hardy space. We point to Triebel [**Tri83**] for definitions and properties of these spaces, but for the convenience of the reader let us recall here one possible definition of the homogeneous and inhomogeneous Besov spaces. Let $(\psi_j)_{j\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R}^d)$ with $\operatorname{supp}\psi_0 \subset \{x \in \mathbb{R}^d\colon \ |x| \le 2\}$ and $\operatorname{supp}\psi_j \subset \{x \in \mathbb{R}^d\colon 2^{j-1} \le |x| \le 2^{j+1}\}$ for $j = 1, 2, \ldots$ such that for every multi-index $\alpha \in \mathbb{N}_0^d$ there is $c_\alpha > 0$ such that

$$2^{j|\alpha|}\,|\partial^\alpha \psi_j(x)| \le c_\alpha$$

for all $j = 0, 1, 2, \ldots$ and $x \in \mathbb{R}^d$, and

$$\sum_{j\in\mathbb{N}_0} \psi_j(x) = 1$$

for every $x \in \mathbb{R}^d$. Given $0 < p, q \le \infty$ and $s \in \mathbb{R}$ the (inhomogeneous) Besov space $B^s_{p,q}(\mathbb{R}^d)$ can be defined as the subspace of $\mathcal{S}'(\mathbb{R}^d)$ consisting of all f such that the (quasi-) norm

$$\|f\|_{B^s_{p,q}(\mathbb{R}^d)} = \left\|\left(\left\|2^{sj}\mathcal{F}^{-1}\psi_j\mathcal{F}f\right\|_{L_p(\mathbb{R}^d)}\right)_{j\in\mathbb{N}_0}\right\|_{\ell_q(\mathbb{N}_0)}$$

is finite. It turns out that this definition does not depend on the specific choice of the functions $\psi_0, \psi_1, \psi_2, \ldots$ in the sense that the corresponding norms can be shown to be equivalent. The homogeneous Besov spaces can be defined in a similar manner. If we write

$$\mathscr{Z}(\mathbb{R}^d) = \left\{\psi \in \mathcal{S}(\mathbb{R}^d)\colon \partial^\alpha\mathcal{F}\psi(0) = 0 \text{ for every } \alpha \in \mathbb{N}_0^d\right\}$$

and provide $\mathscr{Z}(\mathbb{R}^d)$ with the topology inherited from $\mathcal{S}(\mathbb{R}^d)$ then $\mathscr{Z}(\mathbb{R}^d)$ becomes a locally convex topological vector space and we can form its dual space $\mathscr{Z}'(\mathbb{R}^d)$. Let $(\dot{\psi}_j)_{j\in\mathbb{Z}} \subset \mathcal{S}(\mathbb{R}^d)$ with $\operatorname{supp}\dot{\psi}_j \subset \{x \in \mathbb{R}^d\colon 2^{j-1} \le |x| \le 2^{j+1}\}$ for every $j \in \mathbb{Z}$ such that for every multi-index $\alpha \in \mathbb{N}_0^d$ there is $c_\alpha > 0$ such that

$$2^{j|\alpha|}\left|\partial^\alpha\dot{\psi}_j(x)\right| \le c_\alpha$$

for all $j \in \mathbb{Z}$ and $x \in \mathbb{R}^d$, and

$$\sum_{j\in\mathbb{Z}} \dot{\psi}_j(x) = 1$$

for every $x \in \mathbb{R}^d \setminus \{0\}$. Given $0 < p, q \leq \infty$ and $s \in \mathbb{R}$ the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^d)$ can be defined as the subspace of $\mathscr{Z}'(\mathbb{R}^d)$ consisting of all f such that the (quasi-) norm

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^d)} = \left\| \left(\left\| 2^{sj} \mathcal{F}^{-1} \dot{\psi}_j \mathcal{F} f \right\|_{L_p(\mathbb{R}^d)} \right)_{j \in \mathbb{Z}} \right\|_{\ell_q(\mathbb{N}_0)}$$

is finite. The homogeneous and inhomogeneous Triebel-Lizorkin spaces can be defined in a similar way.

We will also use the vector-valued counterparts of some of these spaces. We write $W^s_p(M;X)$, $H^s_p(M;X)$ for the vector-valued Sobolev-Slobodeckij and Bessel potential spaces of order $s \in \mathbb{R}$ as defined in e. g. [**Ama95**], as well as the vector-valued Besov spaces $B^s_{p,q}(M;X)$ as in [**Ama97**].

Throughout this thesis we will make use of interpolation theory for Banach spaces which we assume the reader to be familiar with. As a general reference we point again to Triebel [**Tri83**, **Tri78**], but we will also use other sources in some specific instances.

The inhomogeneous Besov spaces $B^s_{p,q}(\mathbb{R}^d)$ and their homogeneous counterparts $\dot{B}^s_{p,q}(\mathbb{R}^d)$ enjoy a variety of embedding properties, some of which we collect here. The relations in question are consequences of Theorem 2.2.2 of [**RS96**] and Propositon 2.39 of [**BCD11**].

PROPOSITION 3.1. *Let* $1 \leq p \leq \infty$. *Then*

$$\dot{B}^0_{p,1}(\mathbb{R}^d) \hookrightarrow B^0_{p,1} \hookrightarrow L_p(\mathbb{R}^d) \hookrightarrow B^0_{p,\infty}(\mathbb{R}^d) \hookrightarrow \dot{B}^0_{p,\infty}(\mathbb{R}^d).$$

In the endpoint cases $p = 1$ *and* $p = \infty$ *we have additionally*

$$\dot{B}^0_{1,1}(\mathbb{R}^d) \hookrightarrow H_1(\mathbb{R}^d) \hookrightarrow L_1(\mathbb{R}^d) \hookrightarrow \mathcal{M}(\mathbb{R}^d) \hookrightarrow B^0_{1,\infty}(\mathbb{R}^d) \hookrightarrow \dot{B}^0_{1,\infty}(\mathbb{R}^d)$$

and

$$\dot{B}^0_{\infty,1}(\mathbb{R}^d) \hookrightarrow B^0_{\infty,1} \hookrightarrow L_\infty(\mathbb{R}^d) \hookrightarrow \mathrm{BMO}(\mathbb{R}^d) \hookrightarrow \dot{B}^0_{\infty,\infty}(\mathbb{R}^d).$$

Let us write $(\cdot,\cdot)_{\theta,q}$ for the real interpolation functor as in e. g. [**Tri78**]. Then the following result holds.

PROPOSITION 3.2. *Let* $s_0, s_1 \in \mathbb{R}$ *with* $s_0 \neq s_1$ *and let* $1 \leq p, q, q_0, q_1 \leq \infty$. *Then*

$$\left(B^{s_0}_{p,q_0}(\mathbb{R}^d), B^{s_1}_{p,q_1}(\mathbb{R}^d)\right)_{\vartheta,q} = B^s_{p,q}(\mathbb{R}^d)$$

and

$$\left(\dot{B}^{s_0}_{p,q_0}(\mathbb{R}^d), \dot{B}^{s_1}_{p,q_1}(\mathbb{R}^d)\right)_{\vartheta,q} = \dot{B}^s_{p,q}(\mathbb{R}^d)$$

with

$$s = (1-\vartheta)s_0 + \vartheta s_1.$$

In particular the interpolation inequalities

$$\|f\|_{B^s_{p,q}(\mathbb{R}^d)} \lesssim \|f\|^{1-\vartheta}_{B^{s_0}_{p,q_0}(\mathbb{R}^d)} \|f\|^{\vartheta}_{B^{s_1}_{p,q_1}(\mathbb{R}^d)}$$

and

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^d)} \lesssim \|f\|^{1-\vartheta}_{\dot{B}^{s_0}_{p,q_0}(\mathbb{R}^d)} \|f\|^{\vartheta}_{\dot{B}^{s_1}_{p,q_1}(\mathbb{R}^d)}$$

hold uniformly in $f \in B^{s_0}_{p,q_0}(\mathbb{R}^d) \cap B^{s_1}_{p,q_1}(\mathbb{R}^d)$ *and* $f \in \dot{B}^{s_0}_{p,q_0}(\mathbb{R}^d) \cap \dot{B}^{s_1}_{p,q_1}(\mathbb{R}^d)$, *respectively.*

Here the assertion concerning inhomogeneous Besov spaces is Theorem 2.4.2 of [**Tri83**], and the assertion concerning homogeneous Besov spaces follows from the remarks in section 5.2.5 of [**Tri83**]. The interpolation inequalities follow from the interpolation property of the real interpolation method (Proposition 2.4.1 of [**Tri83**]).

An elementary but very useful property of the Fourier transform is known as the Lemma of Riemann-Lebesgue, see e. g. Theorem IX.7 of [**RS75**].

PROPOSITION 3.3. *Let $f \in L_1(\mathbb{R}^d)$. Then $\mathcal{F}f \in C_0(\mathbb{R}^d)$ with $\|\mathcal{F}f\|_{C_0(\mathbb{R}^d)} \le \|f\|_{L_1(\mathbb{R}^d)}$.*

We will frequently encounter operators on $L_p(\mathbb{R}^d)$ and related spaces of the form

$$T_m \colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d), \qquad f \mapsto \mathcal{F}^{-1} m \mathcal{F} f$$

for a given essentially bounded function $m \colon \mathbb{R}^d \to \mathbb{C}$. If T_m extends to a bounded operator on the spaces in question then we will refer to T_m as a Fourier integral operator or a Fourier multiplier operator, and we will refer to the function m as a Fourier multiplier, see e. g. [**Hör83**]. It is in general a difficult task to determine whether a given function $m \in L_\infty(\mathbb{R}^d)$ is a Fourier multiplier between two given spaces, and apart from certain special cases no exact characterisations are known. We refer to [**Tri83**] for specific properties of classes of Fourier multipliers. We will state some conditions on a function $m \in L_\infty(\mathbb{R}^d)$ that will ensure that the associated Fourier integral operator is bounded. In the parameter range $1 < p < \infty$ we have the following result.

PROPOSITION 3.4. *Let $1 < p < \infty$, $s \in \mathbb{R}$, $1 \le q \le \infty$, and $m \in L_\infty(\mathbb{R}^d)$. If m satisfies*

$$r^{\alpha_1 + \ldots + \alpha_d} \left(\frac{1}{r^d} \int_{\{r < |\xi| < 2r\}} |\partial^\alpha m(\xi)|^2 \, \mathrm{d}\xi \right)^{1/2} \le A$$

for every $r > 0$ and all $\alpha \in \{0,1\}^d$ with $\alpha_1 + \ldots + \alpha_d \le \lfloor d/2 \rfloor + 1$, then the associated operator $f \mapsto \mathcal{F} m \mathcal{F}^{-1} f$ extends to a bounded operator on $L_p(\mathbb{R}^d)$, $H_1(\mathbb{R}^d)$, $H^s_p(\mathbb{R}^d)$, $W^s_p(\mathbb{R}^d)$, $B^s_{p,q}(\mathbb{R}^d)$, and $\dot{B}^s_{p,q}(\mathbb{R}^d)$ with norm $\le CA$ with a constant $C > 0$ independent of m.

The assertion for $L_p(\mathbb{R}^d)$ and $H_1(\mathbb{R}^d)$ is due to Hytönen (Theorem 1.2 in [**Hyt04**]), and for L_p this result goes back to Hörmander [**Hör60**] and Mikhlin. The extension to the spaces $H^s_p(\mathbb{R}^d)$, $W^s_p(\mathbb{R}^d)$, $B^s_{p,q}(\mathbb{R}^d)$ is a consequence of the remarks in section 2.6.6 and Theorem 2.6.4 of [**Tri83**]. The result for $\dot{B}^s_{p,q}(\mathbb{R}^d)$ can be seen from the result for $L_p(\mathbb{R}^d)$ and the characterisation of homogeneous Besov spaces in Definition 5.1.3.2 of [**Tri83**], or alternatively from the lifting property of homogeneous Besov spaces (Theorem 5.2.3.1 of [**Tri83**]) and interpolation of homogeneous Besov spaces (Proposition 3.2).

The condition in Proposition 3.4 is not sharp in general. It is known that a function m is a Fourier multiplier on $L_2(\mathbb{R}^d)$ if and only if $m \in L_\infty(\mathbb{R}^d)$. It is known (Theorem 2.6.3 of [**Tri83**]) that a function m is a Fourier multiplier on $L_p(\mathbb{R}^d)$ if and only if m is a Fourier multiplier on $L_{p'}(\mathbb{R}^d)$, where p' is the Hölder conjugate exponent with $1/p + 1/p' = 1$. Furthermore, for $1 \le p \le q \le 2$, any function m that is a Fourier multiplier on L_p is also a Fourier multiplier on L_q. This shows that the endpoint cases $p = 1$ and $p = \infty$, which are not covered by Proposition 3.4, are the most restrictive cases. The following characterisation can be found in section 2.6.3 of [**Tri83**]).

PROPOSITION 3.5. *For a function $m \in L_\infty(\mathbb{R}^d)$ the following statements are equivalent:*

i) *The function m gives rise to a Fourier multiplier operator on $L_1(\mathbb{R}^d)$ of norm $\le A$.*
ii) *The function m gives rise to a Fourier multiplier operator on $L_\infty(\mathbb{R}^d)$ of norm $\le A$.*
iii) *There is a bounded regular Radon measure μ on $\mathbb{R}^d$ with $\mathcal{F}^{-1} m = \mu$ and $\|\mu\|_{Var} \le A$.*

First of all it follows from the Littlewood-Paley characterisation of homogeneous and inhomogeneous Besov spaces (Definition 2.3.1.2 and Definition 5.1.3.2 of [**Tri83**]) that a function m satisfying any of the conditions in 3.5 is a Fourier multiplier on the Besov spaces $B^s_{\infty,q}$ and $\dot{B}^s_{\infty,q}$ for $s \in \mathbb{R}$ and $1 \le q \le \infty$, and then also on $B^s_{p,q}$ and $\dot{B}^s_{p,q}$ for $s \in \mathbb{R}$ and $1 \le p, q \le \infty$. It turns out, however, that the classes of Fourier multipliers of the Besov spaces $B^s_{p,q}$ and $\dot{B}^s_{p,q}$ (which depend only on p) are larger than the class of Fourier multipliers on L_p. There are similar characterisations of the class of Fourier multipliers for certain endpoint Besov spaces.

Proposition 3.6. *Let $m \in L_\infty(\mathbb{R}^d)$. The following statements are equivalent:*

i) *m gives rise to a Fourier multiplier operator on $B^s_{1,q}(\mathbb{R}^d)$ of norm $\leq A$.*
ii) *m gives rise to a Fourier multiplier operator on $B^s_{\infty,q}(\mathbb{R}^d)$ of norm $\leq A$.*
iii) *$\mathcal{F}m \in B^0_{1,\infty}(\mathbb{R}^d)$ with norm $\leq A$.*

Also the following statements are equivalent:

i) *m gives rise to a Fourier multiplier operator on $\dot{B}^s_{1,q}(\mathbb{R}^d)$ of norm $\leq A$.*
ii) *m gives rise to a Fourier multiplier operator on $\dot{B}^s_{\infty,q}(\mathbb{R}^d)$ of norm $\leq A$.*
iii) *$\mathcal{F}m \in \dot{B}^0_{1,\infty}(\mathbb{R}^d)$ with norm $\leq A$.*

The assertion concerning the inhomogeneous Besov spaces is contained in Theorem 2.6.3 of [**Tri83**], and the corresponding result for the homogeneous Besov spaces was proved by Mizuhara [**Miz87**]. Recall that the class of Fourier multipliers on Besov spaces only depends on p, but not on s or q. For the inhomogeneous spaces this is contained in Proposition 2.6.2 of [**Tri83**], and for the homogeneous Besov spaces this was shown in [**Miz87**].

Proposition 3.7. *Let $m \in L_\infty(\mathbb{R}^d)$ and $1 \leq p \leq \infty$. Assume there are $s_0, s_1 > 0$ such that the functions*

$$\xi \mapsto |\xi|^{-s_0}\, m(\xi), \qquad \xi \mapsto |\xi|^{s_1}\, m(\xi)$$

give rise to Fourier multiplier operators on $\dot{B}^s_{p,q}(\mathbb{R}^d)$ of norm M_0 and M_1, respectively. Then m gives rise to a Fourier integral operator from $\dot{B}^s_{p,\infty}(\mathbb{R}^d)$ to $\dot{B}^s_{p,1}(\mathbb{R}^d)$ with

$$\left\|\mathcal{F}^{-1} m \mathcal{F} f\right\|_{\dot{B}^s_{p,1}(\mathbb{R}^d)} \lesssim M_0^{1-\vartheta} M_1^{\vartheta} \left\|f\right\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^d)}$$

uniformly in m and f, where $0 < \vartheta < 1$ with $\vartheta s_1 = (1-\vartheta) s_0$.

This result follows immediately from the lifting property of homogeneous Besov spaces (Theorem 5.2.3.1 of [**Tri83**]) and interpolation of homogeneous Besov spaces (Proposition 3.2). Observe that due to the embedding $\dot{B}^0_{p,1} \hookrightarrow L_p \hookrightarrow \dot{B}^0_{p,\infty}$ this result yields a sufficient condition for a function m to be a Fourier multiplier on L_p for $1 \leq p \leq \infty$.

The assertion of Proposition 3.4 concerning homogeneous Besov spaces admits an extension to the endpoint cases $p = 1$ and $p = \infty$.

Proposition 3.8. *Let $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. Let X denote a Banach space and assume $m \in L_\infty(\mathbb{R}^d; X')$. If m satisfies*

$$r^{\alpha_1 + \ldots + \alpha_d} \left(\frac{1}{r^d} \int_{\{r<|\xi|<2r\}} \left\|\partial^\alpha m(\xi)\right\|_{X'}^2 \, \mathrm{d}\xi \right)^{1/2} \leq A$$

for every $r > 0$ and all $\alpha \in \mathbb{N}_0^d$ with $\alpha_1 + \ldots + \alpha_d \leq \lfloor d/2 \rfloor + 1$, then the associated operator $f \mapsto \mathcal{F} m \mathcal{F}^{-1} f$ extends to a bounded operator from $\dot{B}^s_{p,q}(\mathbb{R}^d; X)$ to $\dot{B}^s_{p,q}(\mathbb{R}^d)$ with norm $\leq CA$, with a constant $C > 0$ independent of m.

This is a consequence of Theorem 7.1 of [**HW06**] combined with Proposition VI.4.4.2 of [**Ste93**].

The following result due to Trebels [**Tre73**] gives a sufficient condition for a radially symmetric function to be a Fourier multiplier on $L_\infty(\mathbb{R}^d)$. Given $k \in \mathbb{N}_0$ one can define a normed space $\mathrm{BV}_{k+1}(\mathbb{R}_+)$ as the collection of all functions $m \in C_0([0,\infty); \mathbb{C})$ with all derivatives up to order k being locally absolutely continuous on $(0,\infty)$ and vanishing at infinity such that

$$\|m\|_{\mathrm{BV}_{k+1}(\mathbb{R}_+)} = \frac{1}{k!} \int_0^\infty t^k \left| m^{(k+1)}(t) \right| \, \mathrm{d}t$$

is finite. In the case $k = 0$ we will write $\mathrm{BV}(\mathbb{R}_+)$ instead of $\mathrm{BV}_1(\mathbb{R}_+)$.

PROPOSITION 3.9. *Let $k, d \in \mathbb{N}$ satisfy $k > d/2$ and let $m \in \mathrm{BV}_{k+1}(\mathbb{R}_+)$. Then the function $m(|\cdot|)\colon \mathbb{R}^d \to \mathbb{C}$, $\xi \mapsto m(|\xi|)$ satisfies*

$$\left\|\mathcal{F}^{-1}m(|\cdot|)\right\|_{L_1(\mathbb{R}^d)} \lesssim \|m\|_{\mathrm{BV}_{k+1}(\mathbb{R}_+)}.$$

In particular m is a Fourier multiplier on $L_p(\mathbb{R}^d)$ for $1 \le p \le \infty$ with

$$\left\|\mathcal{F}^{-1}m(|\cdot|)\mathcal{F}u\right\|_{L_p(\mathbb{R}^d)} \lesssim \|m\|_{\mathrm{BV}_{k+1}(\mathbb{R}_+)} \|u\|_{L_p(\mathbb{R}^d)}.$$

Now we will state some sufficient conditions for holomorphic functions m of one complex variable ensuring that $m(|\cdot|)$ gives rise to a Fourier multiplier. Given a domain $G \subset \mathbb{C}$ write $\mathcal{H}_\infty(G)$ for the Banach space of bounded holomorphic functions $f\colon G \to \mathbb{C}$ with the supremum norm. If X is a Banach space then we write $\mathcal{H}_\infty(G; X)$ for the holomorphic functions $f\colon G \to X$ with the norm $\|f\|_{\mathcal{H}_\infty(G;X)} = \sup_{z\in G} \|f(z)\|_X$. As a general reference for analysis of vector-valued functions of a complex variable we point to [**HP57**]. Given $0 < \rho < \pi$ we continue to write

$$\Sigma_\rho = \{z \in \mathbb{C} \setminus \{0\}\}\colon |\arg z| < \rho\}$$

for the open sector in $\mathbb{C}$ of opening angle 2ρ. The following criterion ensures that the assumptions of Proposition 3.9 are satisfied for holomorphic mappings m.

PROPOSITION 3.10. *Let $m \in \mathcal{H}_\infty(\Sigma_\varepsilon)$ for some $\varepsilon > 0$ and assume*

$$\left\|m(e^{i\alpha}\cdot)\right\|_{\mathrm{BV}(\mathbb{R}_+)} \le M$$

for all $|\alpha| < \varepsilon$. Then $m \in \mathrm{BV}_{k+1}(\mathbb{R}_+)$ for every $k \in \mathbb{N}_0$ with $\|m\|_{\mathrm{BV}_{k+1}(\mathbb{R}_+)} \le C_k M$.

PROPOSITION 3.11. *Let $\psi \in C^\infty(\mathbb{R})$ and let $S \subset \mathbb{R}$ open such that $\operatorname{supp}\psi \subset S$. Assume further that there is a compact set $K \subset S$ such that ψ is constant on $\mathbb{R} \setminus K$. Let $0 < \rho < \pi$, $G = \{z \in \Sigma_\rho\colon |z| \in S\}$, and assume $m \in \mathcal{H}_\infty(G)$. Then the function*

$$\tilde{m}\colon \mathbb{R}^d \to \mathbb{C}, \qquad \xi \mapsto \psi(|\xi|)m(|\xi|)$$

(with m extended by zero to $\mathbb{R}_+$) satisfies the assumptions of Proposition 3.4 and Proposition 3.8 with a constant $A \lesssim \|m\|_{\mathcal{H}_\infty(G)}$. In particular any function $m \in \mathcal{H}_\infty(\Sigma_\rho)$ satisfies the assumptions of Proposition 3.4 and Proposition 3.8 with constant $A \lesssim \|m\|_{\mathcal{H}_\infty(\Sigma_\rho)}$.

If m takes values in the dual space X' of some Banach space X instead of values in $\mathbb{C}$ then $\tilde{m} = \psi(|\cdot|)m(|\cdot|)$ satisfies the assumptions of Proposition 3.8 with a constant $A \lesssim \|m\|_{\mathcal{H}_\infty(G;X')}$.

PROPOSITION 3.12. *Let $\psi \in C^\infty(\mathbb{R})$ and let $S \subset \mathbb{R}$ open such that $\operatorname{supp}\psi \subset S$. Assume further that there is a compact set $K \subset S$ such that ψ is constant on $\mathbb{R} \setminus K$. Let $0 < \rho < \pi$, $G = \{z \in \Sigma_\rho\colon |z| \in S\}$, and assume $m \in \mathcal{H}_\infty(G)$ is such that*

$$z \mapsto z^\varepsilon m(z)$$

is holomorphic and bounded on G for some $\varepsilon > 0$ by a constant $M > 0$. Then the function

$$\tilde{m}\colon \mathbb{R}^d \to \mathbb{C}, \qquad \xi \mapsto \psi(|\xi|)m(|\xi|)$$

(with m extended by zero to $\mathbb{R}_+$) satisfies the assumptions of Proposition 3.5 with a constant $A \lesssim M$. In particular any function $m \in \mathcal{H}_\infty(\Sigma_\rho)$ such that $z \mapsto z^\varepsilon m(z)$ is bounded on Σ_ρ satisfies the assumptions of Proposition 3.5 with constant $A \lesssim \|z \mapsto z^\varepsilon m(z)\|_{\mathcal{H}_\infty(\Sigma_\rho)}$.

PROPOSITION 3.13. *Let $\psi \in C^\infty(\mathbb{R})$ and let $S \subset \mathbb{R}$ open such that $\operatorname{supp}\psi \subset S$. Assume further that there is a compact set $K \subset S$ such that ψ is constant on $\mathbb{R} \setminus K$. Let $0 < \rho < \pi$, $G = \{z \in \Sigma_\rho\colon |z| \in S\}$, and assume $m \in \mathcal{H}_\infty(G)$ is such that*

$$z \mapsto z^\varepsilon m(z) \qquad \text{and} \qquad z \mapsto z^{-\varepsilon} m(z)$$

are holomorphic and bounded on G for some $\varepsilon > 0$. Then the function

$$\tilde{m}\colon \mathbb{R}^d \to \mathbb{C}, \qquad \xi \mapsto \psi(|\xi|)m(|\xi|)$$

(with m extended by zero to $\mathbb{R}_+$) satisfies the assumptions of Proposition 3.7 with constants $M_0 \lesssim \|z \mapsto z^{-\varepsilon}m(z)\|_{\mathcal{H}_\infty(G)}$ and $M_1 \lesssim \|z \mapsto z^{\varepsilon}m(z)\|_{\mathcal{H}_\infty(G)}$. In particular any function $m \in \mathcal{H}_\infty(\Sigma_\rho)$ such that $z \mapsto z^{\varepsilon}m(z)$ and $z \mapsto z^{-\varepsilon}m(z)$ are bounded on Σ_ρ satisfies the assumptions of Proposition 3.7 with constants $M_0 \lesssim \|z \mapsto z^{-\varepsilon}m(z)\|_{\mathcal{H}_\infty(\Sigma_\rho)}$ and $M_1 \lesssim \|z \mapsto z^{\varepsilon}m(z)\|_{\mathcal{H}_\infty(\Sigma_\rho)}$.

Proposition 3.10, Proposition 3.11, Proposition 3.12, and Proposition 3.13 follow essentially from Cauchy's integral formula combined with the product rule of differentiation. We will only prove Proposition 3.10 and Proposition 3.11. The remaining results can be shown in a similar fashion.

Proof of Proposition 3.10. Given $\alpha \in (-\varepsilon, \varepsilon)$ we write $m_\alpha(z) = m(e^{i\alpha}z)$. Then $m_\alpha \in \mathcal{H}_\infty(\Sigma_{\varepsilon-\alpha})$. Given $t > 0$ there is a radius $r(t) > 0$ depending on the angle $\varepsilon > 0$ such that the ball of radius $r(t)$ around t is contained in Σ_ε. There is $\beta > 0$ such that we can choose $r(t) = 2\beta t$. Let $\gamma(t)$ denote the circle around t of radius βt. By Cauchy's integral formula we can write

$$m^{(k+1)}(t) = (2\pi i)^{-k-1}\int_{\gamma(t)} \frac{m'(z)}{(z-t)^{k+1}}\,\mathrm{d}z = (2\pi i)^{-k-1} i\int_0^{2\pi} \frac{m'(t(1+\beta e^{is}))}{(\beta t e^{is})^k}\,\mathrm{d}s.$$

We obtain the estimate

$$\left|m^{(k+1)}(t)\right| \lesssim \beta^{-k}t^{-k}\int_0^{2\pi}\left|m'(t(1+\beta e^{is}))\right|\,\mathrm{d}s$$

and thus

$$\|m\|_{\mathrm{BV}_{k+1}} = \int_0^\infty t^k\left|m^{(k+1)}(t)\right| \lesssim \int_0^\infty\int_0^{2\pi}\left|m'(t(1+\beta e^{is}))\right|\,\mathrm{d}s \lesssim M. \qquad \square$$

Proof of Proposition 3.11. Let us define functions

$$\Psi\colon \mathbb{R}_+ \to \mathbb{C}, \qquad \zeta \mapsto \psi(\sqrt{\zeta})$$
$$F\colon \Sigma_{2\rho} \to \mathbb{C}, \qquad \zeta \mapsto m(\sqrt{\zeta})$$

where we extend the functions ψ and m by zero to $\mathbb{R}_+$ and $\Sigma_{2\rho}$, respectively. Then $\tilde{m}(\xi) = \Psi(|\xi|^2)F(|\xi|^2)$ for $\xi \in \mathbb{R}^d$ and we have for $\alpha \in \{0,1\}^d$

$$\begin{aligned}
\partial_\xi^\alpha \tilde{m}(\xi) &= \sum_{\beta\le\alpha}\binom{\alpha}{\beta}\partial_\xi^{\alpha-\beta}\Psi(|\xi|^2)\partial_\xi^\beta F(|\xi|^2)\\
&= \sum_{\beta\le\alpha}\binom{\alpha}{\beta}2^{|\alpha-\beta|}\xi^{\alpha-\beta}\Psi^{(|\alpha-\beta|)}(|\xi|^2)2^{|\beta|}\xi^\beta F^{(|\beta|)}(|\xi|^2)\\
&= \sum_{\substack{\beta\le\alpha\\ \beta\ne\alpha}}\binom{\alpha}{\beta}2^{|\alpha-\beta|}\xi^{\alpha-\beta}\Psi^{(|\alpha-\beta|)}(|\xi|^2)2^{|\beta|}\xi^\beta F^{(|\beta|)}(|\xi|^2)\\
&\quad + 2^{|\alpha|}\Psi(|\xi|^2)\xi^\alpha F^{(|\alpha|)}(|\xi|^2).
\end{aligned}$$

Since the derivatives of Ψ are only supported on a compact set this shows the estimate

$$|\xi|^{|\alpha|}\left|\partial_\xi^\alpha\tilde{m}(\xi)\right| \lesssim \sum_{\substack{\beta\le\alpha\\ \beta\ne\alpha}}\chi_{|\xi|\in K}|\xi|^{2|\beta|}F^{(|\beta|)}(|\xi|^2) + \chi_{\operatorname{supp}\psi}|\xi|^{2|\alpha|}F^{(|\alpha|)}(|\xi|^2).$$

For general $\alpha \in \mathbb{N}_0^d$ an analogous estimate can be shown in the same fashion by using Faà di Bruno's formula. Given any $k \in \mathbb{N}_0$ and $z \in \operatorname{supp} \Psi$ we can use Cauchy's integral formula to obtain the representation

$$F^{(k)}(z) = \frac{1}{(2\pi i)^{k+1}} \int_{\Gamma_z} \frac{F(\zeta)}{(\zeta - z)^{k+1}} \, \mathrm{d}\zeta$$

with $\Gamma_z = \{\zeta \in \mathbb{C} \colon |z - \zeta| = \varepsilon |z|\}$ with $\varepsilon > 0$ small enough. We can choose $\varepsilon > 0$ such that this representation holds for every $z \in \operatorname{supp} \Psi$. In particular we have

$$F^{(k)}(|\xi|^2) = \frac{1}{(2\pi i)^{k+1}} \frac{1}{\varepsilon^k |\xi|^{2k}} \int_0^{2\pi} F(|\xi|^2 + \varepsilon |\xi|^2 e^{it}) e^{itk} \, \mathrm{d}t$$

and thus

$$|\xi|^{2k} \left| F^{(k)}(|\xi|^2) \right| \lesssim \sup_t \left| F(z + \varepsilon |\xi|^2 e^{it}) e^{itk} \right| \lesssim \sup_{z \in G} |m(z)|$$

for any $k \in \mathbb{N}_0$ and $\xi \in \mathbb{R}^d$ with $|\xi| \in \operatorname{supp} \psi$. In particular we obtain the estimate

$$|\xi|^{|\alpha|} \left| \partial_\xi^\alpha \tilde{m}(\xi) \right| \lesssim \|m\|_{\mathcal{H}_\infty(G)}$$

for every $\xi \in \mathbb{R}^d \setminus \{0\}$ and $\alpha \in \{0,1\}^d$. Now it follows from a direct calculation that the assumptions of Proposition 3.4 and Proposition 3.8 are indeed satisfied.

If m is X'-valued then one can replace absolute values with norms in the above proof to obtain the corresponding statement. □

4. Estimates for certain Fourier multiplier operators

In this section we collect estimates for certain Fourier multiplier operators which will show up at various occasions over the next chapters. Following the notation in [**Tri83**] we write M_∞ for the class of Fourier multipliers on $L_\infty(\mathbb{R}^d)$. Observe that by the results in section 2.6 of [**Tri83**] any $m \in M_\infty$ is also a Fourier multiplier on the full scale of Lebesgue spaces $L_p(\mathbb{R}^d)$, Besov spaces $B^s_{p,q}(\mathbb{R}^d)$, homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^d)$, Bessel-Potential spaces $H^s_p(\mathbb{R}^d)$, Sobolev-Slobodeckij spaces $W^s_p(\mathbb{R}^d)$ and in the spaces $H_1(\mathbb{R}^d)$ and $\mathrm{BMO}(\mathbb{R}^d)$.

We will use the notation $A \lesssim B$ again with the meaning $A \leq CB$ with a generic constant $C > 0$ independent of the free variables, in particular independent of the quantities λ and t below.

Lemma 4.1. *Let $\alpha, \beta \geq 0$, $t > 0$ and $\lambda \in \Sigma_\rho \cup \{0\}$ for some $0 < \rho < \pi$. Let $\omega = \omega(\xi, \lambda) = \sqrt{\lambda + |\xi|^2}$. Then*

i) $m_{\alpha,\beta,\lambda,t}(|\xi|) = |\xi|^\alpha \omega^\beta e^{-|\xi| t}$ *satisfies*

$$\|m_{\alpha,\beta,\lambda,t}\|_{M_\infty(\mathbb{R}^d)} \lesssim t^{-\alpha}(t^{-\beta} + |\lambda|^{\beta/2}).$$

ii) $m_{\alpha,\beta,\lambda,t}(|\xi|) = |\xi|^\alpha \omega^\beta e^{-\omega t}$ *satisfies*

$$\|m_{\alpha,\beta,\lambda,t}\|_{M_\infty(\mathbb{R}^d)} \lesssim t^{-\alpha} \left[t^{-\beta} + |\lambda|^{\beta/2} \right] e^{-c|\lambda|^{1/2} t}.$$

iii) $\varphi_{\alpha,\beta,\lambda,t}(|\xi|) = |\xi|^\alpha \omega^\beta \frac{e^{-\omega t} - e^{-|\xi| t}}{\omega - |\xi|}$ *satisfies*

$$\|\varphi_{\alpha,\beta,\lambda,t}\|_{M_\infty(\mathbb{R}^d)} \lesssim \frac{t^{1-\alpha}}{1 + t|\lambda|^{1/2}} (t^{-\beta} + |\lambda|^{\beta/2}).$$

iv) $m_{\lambda,t,\delta}(|\xi|) = \frac{e^{-\omega t} - e^{-\omega(\delta - t)}}{\omega}$ *satisfies*

$$\|m_{\lambda,t,\delta}\|_{M_\infty(\mathbb{R}^d)} \lesssim \frac{1}{1 + |\lambda|^{1/2}}.$$

Each of these estimates holds uniformly in $\lambda, t, \alpha, \beta$ from the specified range of values.

Proof. Since of the mappings in question are holomorphic and radially symmetric the proof of this Lemma essentially amounts to verifying the conditions of Proposition 3.10 in order to apply Proposition 3.9.

i) We begin with the case $\beta = 0$. We wish to estimate

$$m_{\alpha,0,\lambda,t}(r) = r^\alpha e^{-rt}.$$

This function obviously extends to a mapping in $\mathcal{H}_\infty(\Sigma_\varepsilon)$ for $0 < \varepsilon < \pi/2$. If $\alpha = 0$ as well then

$$\left\|m_{0,0,\lambda,t}(e^{i\phi}\cdot)\right\|_{\mathrm{BV}} \lesssim t \int_0^\infty e^{-crt} \,\mathrm{d}r \lesssim 1.$$

For $\alpha > 0$ we obtain

$$\left\|m_{\alpha,0,\lambda,t}(e^{i\phi}\cdot)\right\|_{\mathrm{BV}} \lesssim \int_0^\infty \alpha r^{\alpha-1} e^{-crt} \,\mathrm{d}r + t \int_0^\infty r^\alpha e^{-crt} \lesssim t^{-\alpha}.$$

Now for $\alpha = 0$ and $\beta > 0$ a similar calculation shows

$$\left\|m_{0,\beta,\lambda,t}(e^{i\phi}\cdot)\right\|_{\mathrm{BV}} \lesssim t^{-\beta} + |\lambda|^{-\beta/2}.$$

To obtain the general case we rely on M_∞ being an algebra and thus conclude

$$\begin{aligned}\|m_{\alpha,\beta,\lambda,t}\|_{M_\infty} &\lesssim \left\|m_{\alpha,0,\lambda,t/2}\right\|_{M_\infty} \left\|m_{0,\beta,\lambda,t/2}\right\|_{M_\infty} \\ &\lesssim t^{-\alpha}(t^{-\beta} + |\lambda|^{\beta/2}).\end{aligned}$$

ii) As above we begin with the case $\alpha = \beta = 0$. Then we have

$$\left\|m_{0,0,\lambda,t}(e^{i\phi}\cdot)\right\|_{\mathrm{BV}} \lesssim \int_0^\infty \frac{rt}{r + |\lambda|^{1/2}} e^{-crt} e^{-c|\lambda|^{1/2}t} \,\mathrm{d}r \lesssim e^{-c|\lambda|^{1/2}t}.$$

If $\beta = 0$ and $\alpha > 0$ then a similar calculation shows

$$\left\|m_{\alpha,0,\lambda,t}(e^{i\phi}\cdot)\right\|_{\mathrm{BV}} \lesssim t^{-\alpha} e^{-c|\lambda|^{1/2}t}.$$

Similarly for $\alpha = 0$ and $\beta > 0$ we obtain

$$\begin{aligned}\left\|m_{0,\beta,\lambda,t}(e^{i\phi}\cdot)\right\|_{\mathrm{BV}} &\lesssim e^{-c|\lambda|^{1/2}t} \int_0^\infty \left[\beta\,|\omega|^{\beta-1} e^{-crt} + t\,|\omega|^\beta e^{-crt}\right] \frac{r}{|\omega|} \,\mathrm{d}r \\ &\lesssim \left[t^{-\beta} + |\lambda|^{\beta/2}\right] e^{-c|\lambda|^{1/2}t}.\end{aligned}$$

Combining these results the assertion follows.

iii) We begin with the case $\alpha = \beta = 0$. We can write

$$\varphi_{0,0,\lambda,t}(|\xi|) = -t \int_0^1 e^{-\omega ts} e^{-|\xi|t(1-s)} \,\mathrm{d}s$$

and thus

$$\|\varphi_{0,0,\lambda,t}\|_{M_\infty} \lesssim t \int_0^1 e^{-c|\lambda|^{1/2}ts} \,\mathrm{d}s \lesssim \frac{1 - e^{-c|\lambda|^{1/2}t}}{|\lambda|^{1/2}} \lesssim \frac{t}{1 + t\,|\lambda|^{1/2}}.$$

We can write

$$\begin{aligned}\varphi_{\alpha,\beta,\lambda,t}(|\xi|) &= |\xi|^\alpha\,\omega^\beta\frac{e^{-\omega t}-e^{-|\xi|t}}{\omega-|\xi|}\\ &= \left[|\xi|^\alpha\,\omega^\beta\left(e^{-\omega t/2}+e^{-|\xi|t/2}\right)\right]\frac{e^{-\omega t/2}-e^{-|\xi|t/2}}{\omega-|\xi|}\\ &= \left[|\xi|^\alpha\,\omega^\beta\left(e^{-\omega t/2}+e^{-|\xi|t/2}\right)\right]\varphi_{0,0,\lambda,t/2}(|\xi|)\end{aligned}$$

and thus we obtain

$$\|\varphi_{\alpha,\beta,\lambda,t}\|_{M_\infty}\lesssim t^{-\alpha}(t^{-\beta}+|\lambda|^{\beta/2})\frac{t}{1+t\,|\lambda|^{1/2}}.$$

iv) We can write

$$m_{\lambda,t}(z)=\int_0^1 e^{-\omega[(\delta-t)(1-s)+ts]}\,\mathrm{d}s$$

and thus we obtain

$$\|m_{\lambda,t}\|_{M_\infty(\mathbb{R}^d)}\lesssim\int_0^1\left\|e^{-\omega[(\delta-t)(1-s)+ts]}\right\|_{M_\infty(\mathbb{R}^d)}\,\mathrm{d}s.$$

Now the assertion follows from the previous estimates. □

5. Estimates for certain operators on the boundary of a layer

In this section we will estimate operators of the form

$$T_m\colon f\mapsto\mathcal{F}_{\xi'}^{-1}m(\xi',x_n)\mathcal{F}_{x'}f(\xi').$$

These will occur in a natural way when constructing a function on e. g. a layer domain from some given boundary data.

LEMMA 5.1. *Let $\alpha,\beta\geq 0$ with $\alpha+\beta\leq 1$ and $\lambda\in\Sigma_\rho\cup\{0\}$ for some $0<\rho<\pi$. Let $s\in\mathbb{R}$, $1\leq q\leq\infty$, and $\omega=\omega(\xi,\lambda)=\sqrt{\lambda+|\xi|^2}$. Then the operator*

$$T_m\colon f\mapsto\mathcal{F}_{\xi'}^{-1}m(\xi',x_n)\mathcal{F}_{x'}f(\xi')$$

associated to

i) *$m(|\xi|,t)=e^{-|\xi|t}$ extends to a bounded operator from $L_\infty(\mathbb{R}^{n-1})$ to $L_\infty(\mathbb{R}^{n-1}\times(0,\delta))$ and from $\dot B^s_{\infty,q}(\mathbb{R}^{n-1})$ to $L_\infty(0,\delta;\dot B^s_{\infty,q}(\mathbb{R}^{n-1}))$ of norm $\lesssim 1$.*
ii) *$m_\lambda(|\xi|,t)=e^{-\omega t}$ extends to a bounded operator from $L_\infty(\mathbb{R}^{n-1})$ to $L_\infty(\mathbb{R}^{n-1}\times(0,\delta))$ and from $\dot B^s_{\infty,q}(\mathbb{R}^{n-1})$ to $L_\infty(0,\delta;\dot B^s_{\infty,q}(\mathbb{R}^{n-1}))$ of norm $\lesssim 1$.*
iii) *$\varphi_{\alpha,\beta,\lambda}(|\xi|,t)=|\xi|^\alpha\,\omega^\beta\frac{e^{-\omega t}-e^{-|\xi|t}}{\omega-|\xi|}$ extends to a bounded operator from $L_\infty(\mathbb{R}^{n-1})$ to $L_\infty(\mathbb{R}^{n-1}\times(0,\delta))$ and from $\dot B^s_{\infty,q}(\mathbb{R}^{n-1})$ to $L_\infty(0,\delta;\dot B^s_{\infty,q}(\mathbb{R}^{n-1}))$ of norm $\lesssim|\lambda|^{-\frac{1-\alpha-\beta}{2}}$.*
iv) *$m_{\lambda,t,\delta}(|\xi|)=\frac{e^{-\omega t}-e^{-\omega(\delta-t)}}{\omega}$ extends to a bounded operator from $L_\infty(\mathbb{R}^{n-1})$ to $L_\infty(\mathbb{R}^{n-1}\times(0,\delta))$ and from $\dot B^s_{\infty,q}(\mathbb{R}^{n-1})$ to $L_\infty(0,\delta;\dot B^s_{\infty,q}(\mathbb{R}^{n-1}))$ of norm $\lesssim\frac{1}{1+|\lambda|^{1/2}}$.*

These estimates hold uniformly in λ,α,β from the specified range of values, and are independent from the parameters s and q.

Proof. An estimate for the norm of T_m is given by

$$\begin{aligned}\|T_m\|_{L_\infty(\mathbb{R}^{n-1}\times(0,\delta)} &\leq \operatorname*{esssup}_{x_n} \left\|\mathcal{F}_{\xi'}^{-1} m(\xi',x_n)\mathcal{F}_{x'} f\right\|_{L_\infty(\mathbb{R}^{n-1})} \\ &\lesssim \operatorname*{esssup}_{x_n} \|m(\cdot,x_n)\|_{M_\infty} \|f\|_{L_\infty(\mathbb{R}^{n-1})}\end{aligned}$$

and analogously

$$\begin{aligned}\|T_m\|_{L_\infty(0,\delta;\dot B^s_{\infty,q}(\mathbb{R}^{n-1}))} &\leq \operatorname*{esssup}_{x_n} \left\|\mathcal{F}_{\xi'}^{-1} m(\xi',x_n)\mathcal{F}_{x'} f\right\|_{\dot B^s_{\infty,q}(\mathbb{R}^{n-1})} \\ &\lesssim \operatorname*{esssup}_{x_n} \|m(\cdot,x_n)\|_{M_\infty} \|f\|_{\dot B^s_{\infty,q}(\mathbb{R}^{n-1})}.\end{aligned}$$

Now the assertion follows from the estimates obtained in Lemma 4.1. □

Observe that the estimates in Lemma 5.1 are independent of $\delta > 0$.

6. The heat equation in a layer: Dirichlet boundary conditions

We collect some results concerning the equation

$$\begin{cases} \lambda w - \Delta w = f & \text{in } \Omega \\ \qquad\quad w = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

in a layer $\Omega = \mathbb{R}^{n-1}\times(0,\delta)$ in the spaces $L_\infty(\Omega)$ and $L_1(\Omega)$.

Proposition 6.1. *Let $0<\rho<\pi$ and $\lambda_0>0$. Then for every $\lambda\in\Sigma_\rho$ with $|\lambda|\geq\lambda_0$ and $f\in L_\infty(\Omega)$ there is a unique solution $w\in L_\infty(\Omega)$ of (1) satisfying the estimate*

$$|\lambda|\,\|w\|_{L_\infty(\Omega)} + |\lambda|^{1/2}\,\|\nabla w\|_{L_\infty(\Omega)} + \operatorname*{esssup}_{x_n}\left\|\nabla^2 w(\cdot,x_n)\right\|_{\dot B^0_{\infty,\infty}(\mathbb{R}^{n-1})} \lesssim \|f\|_{L_\infty(\Omega)}$$

uniformly in λ and f. If $f\in C^\infty_{c,\sigma}(\Omega)$ then the boundary values of $\partial_n w_n$ satisfy the estimates

$$\begin{aligned}|\lambda|\,\|\partial_n w_n(\cdot,\delta)\|_{\dot B^{-1}_{\infty,\infty}(\mathbb{R}^{n-1})} + |\lambda|^{1/2}\,\|\partial_n w_n(\cdot,x_n)\|_{\dot B^0_{\infty,1}(\mathbb{R}^{n-1})}& \\ + \|\partial_n w_n(\cdot,\delta)\|_{\dot B^1_{\infty,\infty}(\mathbb{R}^{n-1})} \lesssim \|f\|_{L_\infty(\Omega)}&\end{aligned}$$

for $x_n\in\{0,\delta\}$.

Proposition 6.2. *Let $0<\rho<\pi$ and $\lambda_0>0$. Then for every $\lambda\in\Sigma_\rho$ with $|\lambda|\geq\lambda_0$ and $f\in L_1(\Omega)$ there is a unique solution $w\in L_1(\Omega)$ of (1) satisfying the estimate*

$$|\lambda|\,\|w\|_{L_1(\Omega)} + |\lambda|^{1/2}\,\|\nabla w\|_{L_1(\Omega)} + \int_0^\delta \left\|\nabla^2 w(\cdot,x_n)\right\|_{\dot B^0_{1,\infty}(\mathbb{R}^{n-1})}\,\mathrm{d}x_n \lesssim \|f\|_{L_1(\Omega)}$$

uniformly in λ and f. If $f\in C^\infty_{c,\sigma}(\Omega)$ then the boundary values of $\partial_n w_n$ satisfy the estimates

$$|\lambda|^{1/2}\,\|\partial_n w_n(\cdot,x_n)\|_{\dot B^{-1}_{1,\infty}(\mathbb{R}^{n-1})} + \|\partial_n w_n(\cdot,x_n)\|_{L_1(\mathbb{R}^{n-1})} \lesssim \|f\|_{L_1(\Omega)}$$

for $x_n\in\{0,\delta\}$. If $f\in C^\infty_{c,\sigma}(\Omega)$ is such that the function $x_n\mapsto\|f(\cdot,x_n)\|_{H_1(\mathbb{R}^{n-1})}$ is integrable on $(0,\delta)$ then the boundary values satisfy

$$|\lambda|^{1/2}\,\|\partial_n w_n(\cdot,x_n)\|_{\dot F^{-1}_{1,2}(\mathbb{R}^{n-1})} + \|\partial_n w_n(\cdot,x_n)\|_{H_1(\mathbb{R}^{n-1})} \lesssim \int_0^\delta \|f(\cdot,x_n)\|_{H_1(\mathbb{R}^{n-1})}\,\mathrm{d}x_n$$

for $x_n\in\{0,\delta\}$.

The resolvent estimates are well-known and can be shown in a variety of ways. We will only show the estimates for the boundary values in Proposition 6.1 and Proposition 6.2. In order to show these assertions we will derive an explicit solution formula. Applying the Fourier transform in the tangential part of the spacial variable $x = (x', x_n)$ we obtain the boundary value problem

$$\begin{cases} \omega^2 \hat{w} - \partial_n^2 \hat{w} = \hat{f} & \text{in } (0, \delta) \\ \hat{w} = 0 & \text{in } \{0, \delta\}. \end{cases}$$

For fixed $\xi \in \mathbb{R}^{n-1}$ this is an ordinary differential equation, the solution of which is given by

$$\hat{w}(\xi', x_n) = \mathbf{a}_1 e^{-\omega(\delta - x_n)} + \mathbf{a}_2 e^{-\omega x_n} - \frac{1}{\omega} \int_0^{x_n} \sinh[\omega(x_n - t)] \hat{f}(\xi', t) \,\mathrm{d}t$$

with functions $\mathbf{a}_1, \mathbf{a}_2$ to be determined from the boundary conditions, i. e. these functions have to satisfy

$$\begin{pmatrix} 1 & e^{-\delta\omega} \\ e^{-\delta\omega} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\omega} \int_0^\delta \sinh[\omega(\delta - t)] \hat{f}(\xi', t) \,\mathrm{d}t \\ 0 \end{pmatrix}$$

and thus

$$\begin{aligned} \mathbf{a}_1 &= \frac{1}{1 - e^{-2\delta\omega}} \frac{1}{\omega} \int_0^\delta \sinh[\omega(\delta - t)] \hat{f}(\xi', t) \,\mathrm{d}t \\ \mathbf{a}_2 &= -\frac{e^{-\delta\omega}}{1 - e^{-2\delta\omega}} \frac{1}{\omega} \int_0^\delta \sinh[\omega(\delta - t)] \hat{f}(\xi', t) \,\mathrm{d}t. \end{aligned}$$

For the solution $\hat{w}$ we thus obtain

$$\hat{w}(\xi', x_n) = \int_0^\delta G(\xi', x_n, t) \hat{f}(\xi', t) \,\mathrm{d}t$$

with the Green's function

$$G(\xi', x_n, t) = \frac{1}{\omega} \frac{\sinh[\omega \min\{x_n, t\}] \sinh[\omega(\delta - \max\{x_n, t\})]}{\sinh(\omega\delta)}.$$

Representations for $\hat{w}$ are given by

$$\begin{aligned} \hat{w}(\xi', x_n) &= \int_0^\delta \frac{1}{\omega} \frac{\sinh[\omega \min\{x_n, t\}]}{\sinh(\omega\delta)} \sinh[\omega(\delta - \max\{x_n, t\})] \hat{f}(\xi', t) \,\mathrm{d}t \\ &= \int_0^{x_n} \frac{1}{\omega} \frac{\sinh(\omega t)}{\sinh(\omega\delta)} \sinh[\omega(\delta - x_n)] \hat{f}(\xi', t) \,\mathrm{d}t + \int_{x_n}^\delta \frac{1}{\omega} \frac{\sinh[\omega x_n] \sinh[\omega(\delta - t)]}{\sinh[\omega\delta]} \hat{f}(\xi', t) \,\mathrm{d}t \\ &= \int_0^{x_n} \frac{1}{2\omega} e^{-\omega(x_n - t)} \frac{(1 - e^{-2\omega t})(1 - e^{-2\omega(\delta - x_n)})}{1 - e^{-2\omega\delta}} \hat{f}(\xi', t) \,\mathrm{d}t \\ &\quad + \int_{x_n}^\delta \frac{1}{2\omega} e^{-\omega(t - x_n)} \frac{(1 - e^{-2\omega x_n})(1 - e^{-2\omega(\delta - t)})}{1 - e^{-2\omega\delta}} \hat{f}(\xi', t) \,\mathrm{d}t. \end{aligned}$$

In particular we obtain

$$\partial_n \hat{w}(\xi', x_n) = -\int_0^{x_n} \frac{\sinh(\omega t)}{\sinh(\omega\delta)} \cosh[\omega(\delta - x_n)] \hat{f}(\xi', t) \,\mathrm{d}t + \int_{x_n}^\delta \frac{\cosh[\omega x_n] \sinh[\omega(\delta - t)]}{\sinh[\omega\delta]} \hat{f}(\xi', t) \,\mathrm{d}t$$

and thus

$$\partial_n \hat{w}(\xi', 0) = \int_0^\delta \frac{\sinh[\omega(\delta - t)]}{\sinh[\omega\delta]} \hat{f}(\xi', t) \,\mathrm{d}t, \qquad \partial_n \hat{w}(\xi', \delta) = -\int_0^\delta \frac{\sinh(\omega t)}{\sinh(\omega\delta)} \hat{f}(\xi', t) \,\mathrm{d}t.$$

Assume $f \in C^\infty_{c,\sigma}(\Omega)$. Then we can use integration by parts to compute

$$\begin{aligned}\partial_n \hat{w}_n(\xi', 0) &= \int_0^\delta \frac{\sinh[\omega(\delta - t)]}{\sinh[\omega\delta]} \hat{f}_n(\xi', t)\,\mathrm{d}t \\ &= \frac{1}{\omega}\int_0^\delta \frac{\cosh[\omega(\delta - t)]}{\sinh[\omega\delta]} \partial_n \hat{f}_n(\xi', t)\,\mathrm{d}t \\ &= -\sum_{j=1}^{n-1} i \frac{\xi_j}{|\xi'|}\frac{z}{\omega}\int_0^\delta \frac{\cosh[\omega(\delta - t)]}{\sinh[\omega\delta]} \hat{f}_j(\xi', t)\,\mathrm{d}t,\end{aligned}$$

and similarly

$$\partial_n \hat{w}_n(\xi', \delta) = -\sum_{j=1}^{n-1} i \frac{\xi_j}{|\xi'|}\frac{z}{\omega}\int_0^\delta \frac{\cosh[\omega t]}{\sinh[\omega\delta]} \hat{f}_j(\xi', t)\,\mathrm{d}t.$$

We begin with the estimates for the boundary values of $\partial_n w_n$ in Proposition 6.1. Using Proposition 3.8 we immediately obtain

$$\begin{aligned}\|\partial_n w_n(\cdot, 0)\|_{\dot{B}^1_{\infty,\infty}(\mathbb{R}^{n-1})} &\lesssim \left[\sup_{z\in\Sigma_\varepsilon}\int_0^\delta \left| z\frac{\sinh(\omega t)}{\sinh(\omega\delta)}\right|\,\mathrm{d}t\right]\|f_n\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1};L_\infty(0,\delta))} \\ &\lesssim \sup_{x_n}\|f_n(\cdot, x_n)\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1})}.\end{aligned}$$

Using the previous calculations we can also show

$$\begin{aligned}\|\partial_n w_n(\cdot, 0)\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^{n-1})} &\lesssim \sum_{j=1}^{n-1}\left\| i\frac{\xi_j}{|\xi'|}\frac{1}{\omega}\int_0^\delta \frac{\cosh[\omega(\delta - t)]}{\sinh[\omega\delta]}\hat{f}_j(\xi', t)\,\mathrm{d}t\right\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1})} \\ &\lesssim \left[\sup_{z\in\Sigma_\varepsilon}\int_0^\delta \left|\frac{1}{\omega}\frac{\cosh(\omega t)}{\sinh(\omega\delta)}\right|\,\mathrm{d}t\right]\|f\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1};L_\infty(0,\delta))} \\ &\lesssim |\lambda|^{-1}\sup_{x_n}\|f(\cdot, x_n)\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1})}.\end{aligned}$$

In the same way one can estimate $\partial_n w_n(\cdot, \delta)$. Then the assertion of Proposition 6.1 follows from interpolation theory for homogeneous Besov spaces (Proposition 3.2).

The estimates for $\partial_n w_n(\cdot, 0)$ and $\partial_n w_n(\cdot, \delta)$ in $L_1(\mathbb{R}^{n-1})$ and $\dot{B}^{-1}_{1,\infty}(\mathbb{R}^{n-1})$ can be shown in much the same way as the corresponding estimates in $\dot{B}^s_{\infty,\infty}(\mathbb{R}^{n-1})$. The estimates in $H_1(\mathbb{R}^{n-1})$ follow in the same way if one uses Proposition 3.4 instead of Proposition 3.8.

CHAPTER II

Representation formulae for solutions to the Stokes equation in layer domains

This chapter is devoted to the derivation of explicit representation formulae for solutions to the Stokes equation in layer domains $\Omega = \mathbb{R}^{n-1} \times (0, \delta)$ with a constant $\delta > 0$. In which sense and to what extent these representations actually are solutions will be stated and proved in the forthcoming chapters. We will derive solution formulae to the following resolvent problems. We will begin with the Stokes resolvent equation in a layer with Dirichlet boundary condition on both the upper and lower boundary:

$$\begin{cases} \lambda u - \Delta u + \nabla\theta = f & \text{in } \Omega \\ \operatorname{div} u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Then we will proceed to the Stokes resolvent equation with free boundary condition at the upper boundary Γ^+ and a Dirichlet condition on the lower boundary Γ^- with and without surface tension σ and gravity γ:

$$\begin{cases} \lambda u - \Delta u + \nabla\theta = f & \text{in } \Omega \\ \operatorname{div} u = f_d & \text{in } \Omega \\ S(u,\theta)\nu + (\gamma - \sigma\eta)\nu = g^+ & \text{on } \Gamma^+ \\ \lambda\eta - u \cdot \nu = k^+ & \text{on } \Gamma^+ \\ u = 0 & \text{on } \Gamma^-. \end{cases} \tag{2}$$

Here $S(u,\theta) = -\theta \operatorname{Id} + \nabla u + (\nabla u)^T$ denotes the Cauchy stress tensor and ν the unit outer normal vector. We will employ a technique resembling that from e. g. [**AS03a**]. Since $\Omega = \mathbb{R}^{n-1} \times (0, \delta)$ we can apply the Fourier transform in the first $n-1$ variables to derive a system of ordinary differential equations. We will solve these ordinary differential equations explicitly and thus obtain a representation for the solution to the equation in question. Throughout this thesis we will write $x = (x', x_n)$ for a generic element of $\Omega = \mathbb{R}^{n-1} \times (0, \delta)$.

1. Dirichlet boundary conditions, no surface tension

Let $f \in C^\infty_{c,\sigma}(\Omega)$. Applying the Fourier transform in x', i. e. the tangential part of the space variable, we can derive the following system of equations:

$$\begin{cases} (\omega^2 - \partial_n^2)\hat{u} + (i\xi', \partial_n)\hat{\theta} = \hat{f} & \text{in } (0, \delta) \\ \displaystyle\sum_{j=1}^{n-1} i\xi_j \hat{u}_j + \partial_n \hat{u}_n = 0 & \text{in } (0, \delta) \\ \hat{u} = 0 & \text{for } x_n = \delta \\ \hat{u} = 0 & \text{for } x_n = 0. \end{cases}$$

Here we write $\hat{u}, \hat{\theta}, \hat{f}$ for the transformed functions, i. e.

$$\hat{u}(\xi', x_n) = (2\pi)^{-\frac{n-1}{2}} \int_{\mathbb{R}^{n-1}} e^{-ix'\cdot\xi'} f(x', x_n)\,\mathrm{d}x', \qquad (\xi', x_n) \in \mathbb{R}^{n-1} \times (0, \delta)$$

and analogously for $\hat{\theta}$ and $\hat{f}$. Throughout this chapter we will write $\xi' \in \mathbb{R}^{n-1}$ for the variable in the frequency domain corresponding to the spatial variable $x' \in \mathbb{R}^{n-1}$. As in the previous chapter we also write $\omega = \omega(\lambda, |\xi'|) = \sqrt{\lambda + |\xi'|^2}$ here and in the sequel. We will derive an explicit representation for u and θ. To this end apply $v \mapsto i\xi' \cdot v' + \partial_n v_n$ to the first equation. Using the second equation this gives the relation

$$(|\xi'|^2 - \partial_n^2)\hat{\theta} = 0$$

and we see that θ is harmonic in Ω.

Applying $(-|\xi'|^2 + \partial_n^2)$ to the first equation and $v \mapsto i\xi' \cdot v' + \partial_n v_n$ to the boundary conditions we see that the normal velocity $\hat{u}_n$ satisfies

$$(3)\qquad \begin{cases} (\omega^2 - \partial_n^2)(|\xi'|^2 - \partial_n^2)\hat{u}_n = (|\xi'|^2 - \partial_n^2)\hat{f}_n & \text{in } (0,\delta) \\ \hat{u}_n(\xi', \delta) = 0 \\ \partial_n \hat{u}_n(\xi', \delta) = 0 \\ \hat{u}_n(\xi', 0) = 0 \\ \partial_n \hat{u}_n(\xi', 0) = 0. \end{cases}$$

Then θ is given as the solution of the boundary value problem

$$(4)\qquad \begin{cases} (|\xi'|^2 - \partial_n^2)\hat{\theta} = 0 & \text{in } (0,\delta) \\ \partial_n \hat{\theta}(\xi', \delta) = \hat{f}_n(\xi', \delta) - (\omega^2 - \partial_n^2)\hat{u}_n(\xi', \delta) \\ \partial_n \hat{\theta}(\xi', 0) = \hat{f}_n(\xi', 0) - (\omega^2 - \partial_n^2)\hat{u}_n(\xi', 0) \end{cases}$$

and finally the tangential components $u_1, \dots, u_{n-1}$ of the velocity are determined by the boundary value problem

$$(5)\qquad \begin{cases} (\omega^2 - \partial_n^2)\hat{u}_j = \hat{f}_j - i\xi_j\hat{\theta} & \text{in } (0,\delta) \\ \hat{u}_j(\xi', \delta) = 0 \\ \hat{u}_j(\xi', 0) = 0 \end{cases}$$

for $j = 1, \dots, n-1$. Let w denote the solution of the Helmholtz equation

$$(6)\qquad \begin{cases} \lambda w - \Delta w = f & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Then we can write u in the form $u = v + w$ with w as above, and (u, θ) satisfies (3) to (5) if and only if (v, θ) satisfies the boundary value problems

$$(7)\qquad \begin{cases} (\omega^2 - \partial_n^2)(|\xi'|^2 - \partial_n^2)\hat{v}_n = 0 & \text{in } (0,\delta) \\ \hat{v}_n(\xi', \delta) = 0 \\ \hat{v}_n(\xi', 0) = 0 \\ \partial_n \hat{v}_n(\xi', \delta) = -\partial_n \hat{w}_n(\xi', \delta) \\ \partial_n \hat{v}_n(\xi', 0) = -\partial_n \hat{w}_n(\xi', 0) \end{cases}$$

as well as

$$
\begin{cases} (|\xi'|^2 - \partial_n^2)\hat{\theta} = 0 & \text{in } (0,\delta) \\ \quad \partial_n \hat{\theta}(\xi',\delta) = -(\omega^2 - \partial_n^2)\hat{v}_n(\xi',\delta) \\ \quad \partial_n \hat{\theta}(\xi',0) = -(\omega^2 - \partial_n^2)\hat{v}_n(\xi',0) \end{cases} \tag{8}
$$

and finally for $j = 1, \dots, n-1$

$$
\begin{cases} (\omega^2 - \partial_n^2)\hat{v}_j = -i\xi_j\hat{\theta} & \text{in } (0,\delta) \\ \quad \hat{v}_j(\xi',\delta) = 0 \\ \quad \hat{v}_j(\xi',0) = 0. \end{cases} \tag{9}
$$

Observe that for fixed $\xi' \in \mathbb{R}^{n-1}$ the equations (7) to (9) are boundary value problems for ordinary differential equations.

1.1. An explicit solution formula for the normal velocity u_n. In this section we will derive an explicit solution formula for (7) for a given solution w of (6). We will write $z = |\xi'|$ and, as above, $\omega = \omega(\lambda, z) = \sqrt{\lambda + z^2}$. Basic linear ODE theory suggests to look for a solution $\hat{v}_n$ that is a linear combination of the functions

$$
e^{-z(\delta - x_n)}, \quad e^{-zx_n}, \quad e^{-\omega(\delta - x_n)}, \quad e^{-\omega x_n}.
$$

We make the following ansatz for $\hat{v}_n$:

$$
\hat{v}_n(\xi', x_n) = \mathbf{a}_1^n \omega\varphi(x_n, z, \omega) + \mathbf{a}_2^n \omega\varphi(\delta - x_n, z, \omega) + \mathbf{a}_3^n e^{-zx_n} + \mathbf{a}_4^n e^{-z(\delta - x_n)} \tag{10}
$$

with

$$
\varphi(x_n) = \varphi(x_n, z, \omega) = \frac{e^{-\omega x_n} - e^{-zx_n}}{\omega - z}.
$$

Then

$$
\begin{aligned} \partial_n \hat{v}_n(\xi', x_n) = &-\omega\left[\omega\varphi(x_n) + e^{-zx_n}\right]\mathbf{a}_1^n + \omega\left[z\varphi(\delta - x_n) + e^{-\omega(\delta - x_n)}\right]\mathbf{a}_2^n \\ &- ze^{-zx_n}\mathbf{a}_3^n + ze^{-z(\delta - x_n)}\mathbf{a}_4^n \end{aligned}
$$

and the function (10) satisfies (7) if and only if $\mathbf{a}^n = (\mathbf{a}_1^n, \dots, \mathbf{a}_4^n)$ satisfy the linear equation

$$
\mathbf{M}(z,\omega)\mathbf{a}^n = \begin{bmatrix} \mathbf{0} \\ \mathbf{g} \end{bmatrix}
$$

with Lopatinskiĭ matrix $\mathbf{M}$ given by

$$
\mathbf{M}(z,\omega) = \begin{pmatrix} \omega\varphi(\delta) & 0 & e^{-z\delta} & 1 \\ 0 & \omega\varphi(\delta) & 1 & e^{-z\delta} \\ -\omega\left[\omega\varphi(\delta) + e^{-z\delta}\right] & \omega & -ze^{-z\delta} & z \\ -\omega & \omega\left[\omega\varphi(\delta) + e^{-z\delta}\right] & -z & ze^{-z\delta} \end{pmatrix}
$$

and right hand side $[\mathbf{0}, \mathbf{g}]$ with

$$
\mathbf{g} = \begin{pmatrix} -\partial_n \hat{w}_n(\xi',\delta) \\ -\partial_n \hat{w}_n(\xi',0) \end{pmatrix}.
$$

If we define

$$
\mathbf{A}(z) = \begin{pmatrix} e^{-z\delta} & 1 \\ 1 & e^{-z\delta} \end{pmatrix} \qquad \mathbf{D} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
$$

then we can write $\mathbf{M}(z,\omega)$ equivalently as block matrix

$$
\mathbf{M}(z,\omega) = \begin{bmatrix} \omega\varphi(\delta)\mathbf{I} & \mathbf{A}(z) \\ \omega[\omega\varphi(\delta)\mathbf{I} + \mathbf{A}(z)]\mathbf{D} & z\mathbf{A}(z)\mathbf{D} \end{bmatrix}.
$$

In order to find $\mathbf{a}^n$ we have to invert the matrix $\mathbf{M}$ at least for $z > 0$. That this is possible for every $\lambda \in \mathbb{C} \setminus (-\infty, 0)$ will be established in the following Lemma.

LEMMA 1.1. *Let* $\lambda \in \mathbb{C} \setminus (-\infty, 0)$ *and* $z > 0$. *Then* $\det \mathbf{M}(z, \omega) \neq 0$.

PROOF. Assume $\det \mathbf{M}(z, \omega) = 0$. Then there is a nonzero function v on $(0, \delta)$ of the form (10) satisfying the homogeneous equations

$$v(\delta) = 0 \qquad v(0) = 0 \qquad \partial_n v(\delta) = 0 \qquad \partial_n v(0) = 0.$$

Multiplying the equation $(\omega - \partial_n^2)(z^2 - \partial_n^2)v = 0$ by $\bar{v}$ and integrating the interval $(0, \delta)$, by integration by parts, we obtain

$$\begin{aligned}
0 &= \int_0^\delta \bar{v}(\omega - \partial_n^2)(z^2 - \partial_n^2)v \\
&= \omega^2 z^2 \|v\|^2_{L_2(0,\delta)} - (\omega^2 + z^2) \int_0^\delta \bar{v}\partial_n^2 v + \int_0^\delta \bar{v}\partial_n^4 v \\
&= \omega^2 z^2 \|v\|^2_{L_2(0,\delta)} + (\omega^2 + z^2) \|\partial_n v\|^2_{L_2(0,\delta)} + \left\|\partial_n^2 v\right\|^2_{L_2(0,\delta)} \\
&\quad - (\omega^2 + z^2)\bar{v}\partial_n v\big|_0^\delta + \bar{v}\partial_n^3 v\big|_0^\delta - \partial_n \bar{v}\partial_n^2 v\big|_0^\delta \\
&= \omega^2 z^2 \|v\|^2_{L_2(0,\delta)} + (\omega^2 + z^2) \|\partial_n v\|^2_{L_2(0,\delta)} + \left\|\partial_n^2 v\right\|^2_{L_2(0,\delta)}.
\end{aligned}$$

If z is real and nonnegative then, taking the imaginary part, we obtain

$$0 = (\Im\lambda) z^2 \|v\|^2_{L_2(0,\delta)} + (\Im\lambda) \|\partial_n v\|^2_{L_2(0,\delta)}.$$

If $\Im\lambda \neq 0$ then $v = 0$, in contradiction to our assumption. Thus λ must be real. Rewriting our equation we obtain

$$0 = (\lambda + z^2) z^2 \|v\|^2_{L_2(0,\delta)} + (\lambda + 2z^2) \|\partial_n v\|^2_{L_2(0,\delta)} + \left\|\partial_n^2 v\right\|^2_{L_2(0,\delta)}.$$

We see that if $\lambda \geq 0$ then again v must vanish. □

In fact one can use Poincaré's inequality in $(0, \delta)$ to show that the assertion of Lemma 1.1 holds for $\lambda \in \mathbb{C} \setminus (-\infty, -C\delta^{-1})$ for some $C > 0$. Given $0 < \varepsilon < \pi$ let us use the notation

$$\Sigma_\varepsilon = \{z \in \mathbb{C} \setminus \{0\} \colon\ |\arg z| < \varepsilon\}$$

for the open sector in the complex plane of opening angle 2ε. Then the mapping $(z, \lambda) \mapsto \mathbf{M}(z, \omega)$ obviously extends to a holomorphic function on $\Sigma_\varepsilon \times \Sigma_\rho$ for any $0 < \varepsilon, \rho < \pi$, and a simple compactness argument yields the following corollary.

COROLLARY 1.2. *Let* $0 < r < R$, $\lambda_1 > 0$ *and* $0 < \rho < \pi$. *Then there are* $\varepsilon > 0$ *and* $C > 0$ *such that*

$$|\det \mathbf{M}(z, \omega)| \geq C$$

for all $\lambda \in \Sigma_\rho \cup \{0\}$ *with* $|\lambda| \leq \lambda_1$ *and* $z \in \Sigma_\varepsilon$ *with* $r \leq |z| \leq R$.

Now first of all observe that $\mathbf{A}(z)$ is invertible for every $z \in \Sigma_\varepsilon$. Let

$$\begin{aligned}
\mathbf{X}(z, \omega) &= \varphi(\delta) z \mathbf{A}(z)\mathbf{D} - [\omega\varphi(\delta)\mathbf{I} + \mathbf{A}(z)]\mathbf{D}\mathbf{A}(z) \\
&= \varphi(\delta)[z\mathbf{A}(z)\mathbf{D} - \omega\mathbf{D}\mathbf{A}(z)] - \mathbf{A}(z)\mathbf{D}\mathbf{A}(z) \\
&= \begin{pmatrix} 1 - e^{-(\omega+z)\delta} & (\omega + z)\varphi(\delta) \\ (\omega + z)\varphi(\delta) & 1 - e^{-(\omega+z)\delta} \end{pmatrix} \mathbf{D}
\end{aligned}$$

with determinant

$$\det \mathbf{X}(z, \omega) = -(1 - e^{-(\omega+z)\delta})^2 + (\omega + z)^2 \varphi(\delta)^2$$

which is nonzero for any $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ and $z \in \Sigma_\varepsilon$ with $\varepsilon > 0$ sufficiently small by Corollary 1.2. Hence the inverse of $\mathbf{X}$ exists and is given by

$$\mathbf{X}(z,\omega)^{-1} = -\frac{1}{(1-e^{-(\omega+z)\delta})^2 - (\omega+z)^2\varphi(\delta)^2}\mathbf{D}\begin{pmatrix} 1-e^{-(\omega+z)\delta} & -(\omega+z)\varphi(\delta) \\ -(\omega+z)\varphi(\delta) & 1-e^{-(\omega+z)\delta} \end{pmatrix}.$$

Lemma I.2.3 allows us to compute the inverse of $\mathbf{M}$ to obtain the following explicit representation of $\mathbf{a}^n$:

$$\mathbf{a}^n = \mathbf{M}(z,\omega)^{-1}\begin{bmatrix}\mathbf{0}\\ \mathbf{g}\end{bmatrix} = \begin{bmatrix} -\frac{1}{\omega}\mathbf{A}(z)\mathbf{X}(z,\omega)^{-1}\mathbf{g} \\ \varphi(\delta)\mathbf{X}(z,\omega)^{-1}\mathbf{g}\end{bmatrix} = \begin{bmatrix} -\frac{1}{\omega}\mathbf{A}(z)\mathbf{h} \\ \varphi(\delta)\mathbf{h}\end{bmatrix}$$

with

$$\mathbf{h} = \mathbf{X}(z,\omega)^{-1}\mathbf{g} = \frac{1}{(1-e^{-(\omega+z)\delta})^2 - (\omega+z)^2\varphi(\delta)^2}\mathbf{D}\begin{pmatrix} 1-e^{-(\omega+z)\delta} & -(\omega+z)\varphi(\delta) \\ -(\omega+z)\varphi(\delta) & 1-e^{-(\omega+z)\delta} \end{pmatrix}\mathbf{g}.$$

Let us introduce the functions

$$\Phi_\pm(z,\omega) = 1 - e^{-(\omega+z)\delta} \pm (\omega+z)\varphi(\delta)$$

and

$$\begin{aligned}\mathbf{k} = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\mathbf{h} &= -\frac{1}{2\Phi_+(z,\omega)\Phi_-(z,\omega)}\begin{pmatrix}\Phi_+(z,\omega) & -\Phi_+(z,\omega)\\ \Phi_-(z,\omega) & \Phi_-(z,\omega)\end{pmatrix}\mathbf{g} \\ &= -\frac{1}{2\Phi_+(z,\omega)\Phi_-(z,\omega)}\begin{pmatrix}\Phi_+(z,\omega)(\mathbf{g}_1-\mathbf{g}_2)\\ \Phi_-(z,\omega)(\mathbf{g}_1+\mathbf{g}_2)\end{pmatrix}.\end{aligned}$$

Then

$$\mathbf{k}_1 = -\frac{1}{2}\frac{\mathbf{g}_1-\mathbf{g}_2}{\Phi_-(z,\omega)} \qquad \mathbf{k}_2 = -\frac{1}{2}\frac{\mathbf{g}_1+\mathbf{g}_2}{\Phi_+(z,\omega)}$$

and $\mathbf{a}^n$ admits the following representation in terms of $\mathbf{k}_1$ and $\mathbf{k}_2$:

$$\mathbf{a}^n = \begin{bmatrix} -\frac{1}{\omega}\mathbf{A}(z)\begin{pmatrix}\mathbf{k}_1+\mathbf{k}_2\\ \mathbf{k}_1-\mathbf{k}_2\end{pmatrix} \\ \varphi(\delta)\begin{pmatrix}\mathbf{k}_1+\mathbf{k}_2\\ \mathbf{k}_1-\mathbf{k}_2\end{pmatrix}\end{bmatrix}.$$

This gives rise to the following representation formula for v_n:

$$\begin{aligned}\hat{v}_n(\xi',x_n) = &\Big\{\varphi(\delta)(e^{-zx_n} + e^{-z(\delta-x_n)}) - (1+e^{-\delta z})(\varphi(x_n)+\varphi(\delta-x_n))\Big\}\mathbf{k}_1 \\ &+ \Big\{\varphi(\delta)(e^{-zx_n} - e^{-z(\delta-x_n)}) + (1-e^{-\delta x})(\varphi(x_n)-\varphi(\delta-x_n))\Big\}\mathbf{k}_2\end{aligned}$$

with

$$\mathbf{k}_1 = \frac{1}{2}\frac{\partial_n\hat{w}_n(\xi',\delta) - \partial_n\hat{w}_n(\xi',0)}{\Phi_-(z,\omega)} \qquad \mathbf{k}_2 = \frac{1}{2}\frac{\partial_n\hat{w}_n(\xi',\delta) + \partial_n\hat{w}_n(\xi',0)}{\Phi_+(z,\omega)}$$

as above and

$$\Phi_\pm(z,\omega) = 1 - e^{-(\omega+z)\delta} \pm (\omega+z)\varphi(\delta), \qquad \varphi(t) = \frac{e^{-\omega t} - e^{-zt}}{\omega - z}.$$

1.2. An explicit solution formula for the pressure θ. In this section we will derive an explicit formula for θ. The equation (8) suggests to look for a function θ of the form

$$\hat{\theta}(\xi', x_n) = \mathbf{b}_1 e^{-zx_n} + \mathbf{b}_2 e^{-z(\delta - x_n)} \tag{11}$$

with $\mathbf{b}_1, \mathbf{b}_2 \in \mathbb{C}$ depending only on z and λ to be determined from the boundary conditions. Now θ given by (11) satisfies (8) if and only if $\mathbf{b} = (\mathbf{b}_1, \mathbf{b}_2)$ satisfies the linear equation

$$\mathbf{N}(z,\omega)\mathbf{b} = \mathbf{m}$$

with matrix $\mathbf{N}$ given by

$$\mathbf{N}(z,\omega) = \begin{pmatrix} -ze^{-z\delta} & z \\ -z & ze^{-z\delta} \end{pmatrix} = z\mathbf{A}(z)\mathbf{D}$$

and right hand side

$$\mathbf{m} = \begin{pmatrix} -(\omega^2 - \partial_n^2)\hat{v}_n(\xi', \delta) \\ -(\omega^2 - \partial_n^2)\hat{v}_n(\xi', 0) \end{pmatrix} = \big[\omega(\omega + z)\mathbf{A}(z), -\lambda\mathbf{A}(z)\big]\, \mathbf{a}^n.$$

This yields

$$\mathbf{b} = \frac{1}{z}\mathbf{D}\mathbf{A}(z)^{-1}\mathbf{m} = \frac{1}{z}\big[\omega(\omega + z)\mathbf{D}, -\lambda\mathbf{D}\big]\, \mathbf{a}^n$$

and in particular, using the notation from the previous section, we obtain

$$z\mathbf{b} = -(\omega + z)\mathbf{D}\mathbf{A}(\omega)\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\mathbf{k}.$$

This in turn yields for θ the representation

$$\hat{\theta}(\xi', x_n) = \frac{z+\omega}{z}\Big\{(1 + e^{-\omega\delta})(e^{-zx_n} - e^{-z(\delta - x_n)})\mathbf{k}_1 - (1 - e^{-\omega\delta})(e^{-zx_n} + e^{-z(\delta - x_n)})\mathbf{k}_2\Big\}$$

with

$$\mathbf{k}_1 = \frac{1}{2}\frac{\partial_n\hat{w}_n(\xi', \delta) - \partial_n\hat{w}_n(\xi', 0)}{\Phi_-(z,\omega)} \qquad \mathbf{k}_2 = \frac{1}{2}\frac{\partial_n\hat{w}_n(\xi', \delta) + \partial_n\hat{w}_n(\xi', 0)}{\Phi_+(z,\omega)}$$

and

$$\Phi_\pm(z,\omega) = 1 - e^{-(\omega+z)\delta} \pm (\omega + z)\varphi(\delta)$$

as in the previous section.

1.3. An explicit solution formula for $u_1, \ldots, u_{n-1}$. The tangential part $v_1, \ldots, v_{n-1}$ of the velocity is the solution of the Helmholtz equation (9). It follows from Proposition I.6.1 and the considerations thereafter that v_j admits the integral representation

$$\hat{v}_j(\xi', x_n) = -\int_0^\delta \frac{\sinh[\omega \min\{x_n, t\}]\sinh[\omega(\delta - \max\{x_n, t\})]}{\omega\sinh[\omega\delta]} i\xi_j\hat{\theta}(\xi', t)\, \mathrm{d}t \tag{12}$$

or equivalently

$$\begin{aligned}\hat{v}_j(\xi', x_n) = &-\int_0^{x_n} \frac{1}{\omega}\frac{\sinh(\omega t)\sinh[\omega(\delta - x_n)]}{\sinh(\omega\delta)} i\xi_j\hat{\theta}(\xi', t)\, \mathrm{d}t \\ &-\int_{x_n}^{\delta} \frac{1}{\omega}\frac{\sinh[\omega x_n]\sinh[\omega(\delta - t)]}{\sinh[\omega\delta]} i\xi_j\hat{\theta}(\xi', t)\, \mathrm{d}t.\end{aligned}$$

Given that we already have a representation formula for θ we are able to evaluate these integrals explicitly to obtain

$$\hat{v}_j(\xi', x_n) = -(1 + e^{-\omega\delta})\Psi_-(x_n)\left[i\frac{\xi_j}{|\xi|}\mathbf{k}_1\right] + (1 - e^{-\omega\delta})\Psi_+(x_n)\left[i\frac{\xi_j}{|\xi|}\mathbf{k}_2\right]$$

with

$$\begin{aligned}\Psi_\pm(x_n,z,\omega) &= \frac{1}{\omega-z}\left[(e^{-zx_n}\pm e^{-z(\delta-x_n)}) - \frac{1\pm e^{-\delta z}}{1\pm e^{-\omega\delta}}\left(e^{-\omega x_n}\pm e^{-\omega(\delta-x_n)}\right)\right]\\ &= -\varphi(x_n)\mp\varphi(\delta-x_n)\pm\varphi(\delta)\frac{1}{1\pm e^{-\delta\omega}}\left(e^{-\omega x_n}\pm e^{-\omega(\delta-x_n)}\right)\end{aligned}$$

where $\Psi_\pm(x_n,z,\omega)$ are the solutions to the differential equations

$$(\omega^2-\partial_n^2)\Psi_\pm(x_n,z,\omega) = (\omega+z)\left[e^{-zx_n}\pm e^{-z(\delta-x_n)}\right]$$

with boundary values $\Psi_\pm(0,z,\omega)=\Psi_\pm(\delta,z,\omega)=0$.

1.4. The representation formula. All in all we obtain the following representation for the solution (u,θ).

FORMULA 1.3. *Given $f\in C^\infty_{c,\sigma}(\Omega)$ we can write the corresponding solution (u,θ) of (1) in the form $u=v+w$ where w is the solution to the Helmholtz equation*

$$\begin{cases}\lambda w-\Delta w = f & \text{in } \Omega\\ w=0 & \text{on } \partial\Omega\end{cases}$$

and $v=(v_1,\dots,v_n)$ is given by

$$\begin{aligned}\hat v_n(\xi',x_n) = &\left\{\varphi(\delta)e^{-zx_n}+\varphi(\delta)e^{-z(\delta-x_n)} - (1+e^{-\delta z})(\varphi(x_n)+\varphi(\delta-x_n))\right\}\mathbf{k}_1\\ &+\left\{\varphi(\delta)(e^{-zx_n}-e^{-z(\delta-x_n)}) + (1-e^{-\delta z})(\varphi(x_n)-\varphi(\delta-x_n))\right\}\mathbf{k}_2\end{aligned}$$

and

$$\hat v_j(\xi',x_n) = -(1+e^{-\omega\delta})\Psi_-(x_n)\left[i\frac{\xi_j}{|\xi|}\mathbf{k}_1\right] + (1-e^{-\omega\delta})\Psi_+(x_n)\left[i\frac{\xi_j}{|\xi|}\mathbf{k}_2\right]$$

for $j=1,\dots,n-1$. The pressure θ admits the representation

$$\hat\theta(\xi',x_n) = \frac{z+\omega}{z}\left\{(1+e^{-\omega\delta})(e^{-zx_n}-e^{-z(\delta-x_n)})\mathbf{k}_1 - (1-e^{-\omega\delta})(e^{-zx_n}+e^{-z(\delta-x_n)})\mathbf{k}_2\right\}.$$

Here

$$\Psi_\pm(x_n) = -\varphi(x_n)\mp\varphi(\delta-x_n)\pm\varphi(\delta)\frac{1}{1\pm e^{-\delta\omega}}\left(e^{-\omega x_n}\pm e^{-\omega(\delta-x_n)}\right)$$

as well as

$$\mathbf{k}_1 = \frac{1}{2}\frac{\partial_n\hat w_n(\xi',\delta)-\partial_n\hat w_n(\xi',0)}{\Phi_-(z,\omega)} \qquad \mathbf{k}_2 = \frac{1}{2}\frac{\partial_n\hat w_n(\xi',\delta)+\partial_n\hat w_n(\xi',0)}{\Phi_+(z,\omega)}$$

and

$$\Phi_\pm(z,\omega) = 1-e^{-(\omega+z)\delta}\pm(\omega+z)\varphi(\delta)$$

with $\varphi(t)=\varphi(t,z,\lambda)$ given by

$$\varphi(t) = \frac{e^{-\omega t}-e^{-zt}}{\omega-z}.$$

That the functions (u,θ) are, in fact, a solution to (1) will be established in chapter III and chapter IV. In particular we will show that for $f\in C^\infty_{c,\sigma}(\Omega)$ the solution (u,θ) is a classical solution which is unique within certain classes of functions.

We have already established that the functions $(z,\lambda)\mapsto\Phi_\pm(z,\omega)$ do not vanish for z in a neighbourhood of $z\in\mathbb{R}_+$. We will, however, need considerably stronger estimates in chapter III and chapter IV. These will be established in the following lemma.

LEMMA 1.4. *Let $0 < \rho < \pi$. Given any $r > 0$ and $\lambda_0 > 0$ there is $0 < \varepsilon < \pi - \rho$ such that the following estimates hold for all $z \in \Sigma_\varepsilon$ and $\lambda \in \Sigma_\rho$ with $|\lambda| \geq \lambda_0$.*

i) *If $|z| \leq r$ then*

$$|\Phi_+(z,\omega)| \gtrsim |z|, \qquad |\Phi_-(z,\omega)| \gtrsim 1.$$

ii) *If $|z| \geq r$ then*

$$|\Phi_+(z,\omega)| \gtrsim 1, \qquad |\Phi_-(z,\omega)| \gtrsim 1.$$

PROOF. The assertion holds for some $r > 0$ if and only if it holds for any $r > 0$, albeit with different constants. We begin with the estimates for Φ_+. Replacing z and λ by z/δ and λ/δ^2 we can reduce to the case where $\delta = 1$. Treating z and ω as independent of each other the leading term of the Taylor expansion of Φ_+ is

$$\Phi_+(z,\omega) = \left[1 + e^{-\omega} - \frac{2}{\omega}(1 - e^{-\omega})\right] z + O(z^2).$$

We wish to show that the coefficient of the leading term is bounded from below, with a bound that can be chosen independently of λ. It can be written in the form

$$(1 + e^{-\omega}) \left[1 - \frac{2}{\omega}\frac{1 - e^{-\omega}}{1 + e^{-\omega}}\right] = (1 + e^{-\omega}) \left[1 - \frac{\tanh(\omega/2)}{\omega/2}\right].$$

With $\omega = x + iy$, where $x \geq x_0 > 0$ and $y \in \mathbb{R}$, we have

$$\frac{\tanh(\omega/2)}{\omega/2} = 2\frac{x\sinh(x) + y\sin(y)}{(x^2 + y^2)(\cosh(x) + \cos(y))} + 2i\frac{x\sin(y) - y\sinh(x)}{(x^2 + y^2)(\cosh(x) + \cos(y))}.$$

We can write the imaginary part as

$$2\frac{x\sin(y) - y\sinh(x)}{(x^2 + y^2)(\cosh(x) + \cos(y))} = -\frac{2xy}{x^2 + y^2}\frac{\sinh(x)/x - \sin(y)/y}{\cosh(x) + \cos(y)}$$

and hence

$$|\Im\{\tanh(\omega/2)/(\omega/2)\}| \geq \frac{2\,|y|}{x^2 + y^2}\frac{\sinh(x) - x}{\cosh(x) + 1}.$$

The mapping

$$x \mapsto \frac{\sinh(x) - x}{\cosh(x) + 1}$$

is strictly increasing and positive in the interval $(0, \infty)$ and thus we obtain

$$\frac{\sinh(x) - x}{\cosh(x) + 1} \geq \frac{\sinh(x_0) - x_0}{\cosh(x_0) + 1} > 0$$

for $x \in [x_0, \infty)$. In particular we have

$$|\Im\{\tanh(\omega/2)/(\omega/2)\}| \gtrsim \frac{|y|}{x^2 + y^2}$$

uniformly in $x \geq x_0 > 0$ and $y \in \mathbb{R}$.

Writing $x = r\cos(\psi)$ and $y = r\sin(\psi)$ with $r \geq c\sqrt{\lambda_0} > 0$ and $|\psi| \leq \pi/2 - \varepsilon$ we obtain

$$|\Im\{\tanh(\omega/2)/(\omega/2)\}| \gtrsim |\sin(\psi)|/r.$$

This shows in particular

$$\left|1 - \frac{\tanh(\omega/2)}{\omega/2}\right| \gtrsim \frac{|\sin(\psi)|}{|\omega|}$$

whenever $\arg\omega = \psi$. Take any $0 < \varepsilon < \pi - \rho$. Let us assume that the function $f(\omega) = 1 - \tanh(\omega/2)/(\omega/2)$ takes values arbitrarily close to zero. Then there is a sequence $(\omega_n) \subset \Sigma_{(\rho+\varepsilon)/2}$ with $|\omega_n| \geq c\sqrt{\lambda_0} > 0$ such that $f(\omega_n)$ converges to zero. We can write $\omega_n = r_n e^{i\psi_n}$ with

$|\psi_n| \leq (\rho+\varepsilon)/2$. Then there is a subsequence of (ψ_n), which we denote again by (ψ_n), such that $\psi_n \to \psi$. Now either $\psi = 0$ or $\psi \neq 0$. If $\psi \neq 0$ then for $f(\omega_n)$ to converge to zero it is necessary that $r_n \to \infty$. But then $f(\omega_n) \to 1$, hence (r_n) has to be bounded and then necessarily $\psi = 0$. Hence there is a subsequence of (r_n) that converges to a finite $r \gtrsim \sqrt{\lambda_0}$, and then also $\omega_n \to r$. Then $f(\omega_n) \to f(r)$. But the function $r \mapsto f(r)$ is strictly monotonically increasing on $[0, \infty)$, and thus $f(r) \geq f(x_0) > f(0) = 0$, contradicting the assumption $f(\omega_n) \to 0$. Hence the function f cannot take values arbitrarily close to zero, i. e. there is a constant $C > 0$ such that $|f(\omega)| \geq C$ for every relevant ω.

The function Φ_- is considerably easier to handle since $\Phi_-(0,\omega) = 2 - 2e^{-\omega}$ is evidently nonzero for $|\lambda| \geq \lambda_0 > 0$. Thus the first assertion follows for $r > 0$ sufficiently small.

We turn to the second assertion. It is an immediate consequence of Corollary II.1.2 that the second assertion holds for $r \leq |z| \leq R$ and $\lambda_0 \leq |\lambda| \leq \lambda_1$ for any choice of $R > r$ and $\lambda_1 > \lambda_0$ if we adjust $\varepsilon > 0$ accordingly. It remains to treat the case where either $|\lambda| \geq \lambda_0$ and $|z| \geq R$, or $|\lambda| \geq \lambda_1$ and $|z| \geq r$.

We can write

$$\Phi_\pm(z,\omega) = 1 \mp e^{-\delta z} \mp \frac{2z}{\omega - z} e^{-\delta z} \pm \frac{\omega + z}{\omega - z} e^{-\delta\omega} - e^{-\delta(\omega+z)}.$$

Thus, for $|z| \geq r$ and $|\lambda| \geq \lambda_1$, we can use the relation $(\omega + z)(\omega - z) = \lambda$ to estimate

$$\begin{aligned}
|\Phi_\pm(z,\omega)| &\geq \left|1 \mp e^{-\delta z}\right| - \left|\frac{2z}{\omega - z}\right| e^{-\delta\Re z} - \left|\frac{\omega + z}{\omega - z}\right| e^{-\delta\Re\omega} - e^{-\delta\Re\omega - \delta\Re z} \\
&\geq 1 - e^{-c\delta r} - 2C\,|z|\,|\lambda|^{-1}\,(|z| + |\lambda|^{1/2}) e^{-c\delta|z|} - C\,|\lambda|^{-1}\,(|\lambda| + |z|^2) e^{-c\delta|z|} e^{-c\delta|\lambda|^{1/2}} \\
&\geq 1 - e^{-c\delta r} - C\,|\lambda_1|^{-1/2}
\end{aligned}$$

Choosing λ_1 large enough we obtain $|\Phi_\pm(z,\omega)| \gtrsim 1$.

Similarly, if $|\lambda| \geq \lambda_0$ and $|z| \geq R$ then we obtain the estimate

$$|\Phi_\pm(z,\omega)| \geq 1 - e^{-c\delta R/2} > 0. \qquad \square$$

1.5. The Stokes resolvent problem in a half space. The case $\delta \to \infty$. One may interpret the case where Ω is a half space $\mathbb{R}^n_+$ as the limiting case of a layer $\mathbb{R}^{n-1} \times (0, \delta)$ with $\delta \to \infty$. The corresponding resolvent problem reads

$$\left\{\begin{aligned} \lambda u - \Delta u + \nabla\theta &= f && \text{in } \Omega \\ \operatorname{div} u &= 0 && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}\right. \tag{13}$$

and it turns out that the (formal) limit $\delta \to \infty$ applied to the representation formula 1.3 derived above actually is a representation formula for the solution in the half space case.

FORMULA 1.5. *Given $f \in C^\infty_{c,\sigma}(\mathbb{R}^n_+)$ the solution (u,θ) of (13) can be written in the form $u = v + w$ where w is the solution to the corresponding Helmholtz equation*

$$\left\{\begin{aligned} \lambda w - \Delta w &= f && \text{in } \mathbb{R}^n_+ \\ w &= 0 && \text{on } \partial\mathbb{R}^n_+ \end{aligned}\right.$$

and $v = (v_1, \ldots, v_n)$ is given by

$$\hat{v}_n(\xi', x_n) = \varphi(x_n)\partial_n \hat{w}_n(\xi, 0)$$

and the tangential components v_j for $j = 1, \ldots, n-1$ are given by

$$\hat{v}_j(\xi', x_n) = \varphi(x_n) i \frac{\xi_j}{|\xi|} \partial_n \hat{w}_n(\xi, 0).$$

The pressure θ admits the representation

$$\hat{\theta}(\xi', x_n) = -\frac{z+\omega}{z} e^{-zx_n} \partial_n \hat{w}_n(\xi, 0)$$

where we write as above

$$\varphi(t) = \frac{e^{-\omega t} - e^{-zt}}{\omega - z}.$$

The function $\partial_n w_n(\cdot, 0)$ is given by

$$\partial_n \hat{w}_n(\xi', 0) = \int_0^\infty e^{-\omega t} \hat{f}_n(\xi, t)\,\mathrm{d}t.$$

This is precisely the representation formula of [**Saa07**] in the case of pure Dirichlet boundary conditions. We will later on use our techniques to give a short proof of a result of Saal [**Saa07**] and Desch et al. [**DHP01**].

2. Free boundary/Dirichlet condition, gravity, surface tension

In this section we will reduce the Stokes resolvent problem (2) to a system of ordinary differential equations in Fourier space. For our purposes it will be sufficient to study reduced versions of (2), thus we will derive an ODE representation for u, θ, η and derive explicit solution formulae only for the special cases of interest. In addition we will only treat the case where λ has large modulus. Assume $f \in C^\infty_{c,\sigma}(\Omega)$, $f_d \in C^\infty_c(\Omega)$ and $g^+, k^+ \in C^\infty_c(\mathbb{R}^{n-1})$.

Applying the Fourier transform in the tangential part x' of $x = (x', x_n)$ we obtain the following system of equations:

$$\begin{cases} (\omega^2 - \partial_n^2)\hat{u} + (i\xi', \partial_n)\hat{\theta} = \hat{f} & \text{in } (0,\delta) \\ \sum_{j=1}^{n-1} i\xi_j \hat{u}_j + \partial_n \hat{u}_n = \hat{f}_d & \text{in } (0,\delta) \\ -\hat{\theta} e_n + (i\xi', \partial_n)\hat{u}_n + \partial_n \hat{u} + (\gamma + \sigma\,|\xi'|)\hat{\eta} e_n = \hat{g}^+ & \text{for } x_n = \delta \\ \lambda\hat{\eta} - \hat{u}_n = \hat{k}^+ & \text{for } x_n = \delta \\ \hat{u} = 0 & \text{for } x_n = 0. \end{cases}$$

Applying the divergence operator to the first equation we obtain, together with the second equation, the relation

$$(|\xi'|^2 - \partial_n^2)\hat{\theta} = (\omega^2 - \partial_n^2)\hat{f}_d - \sum_{j=1}^{n-1} i\xi_j \hat{f}_j - \partial_n \hat{f}_n = (\omega^2 - \partial_n^2)\hat{f}_d.$$

Applying the Laplacian to the first equation we thus obtain for u_n the equation

$$(\omega^2 - \partial_n^2)(z^2 - \partial_n^2)\hat{u}_n = (z^2 - \partial_n^2)\hat{f}_n - (\omega^2 - \partial_n^2)\partial_n \hat{f}_d.$$

Similarly as in the previous section we obtain the following boundary conditions for u_n, θ, and η. On the upper boundary Γ^+, i. e. for $x_n = \delta$, we obtain

$$-\hat{\theta} + 2\partial_n \hat{u}_n + (\gamma + z^2\sigma)\hat{\eta} = \hat{g}^+_n$$

$$(z^2 + \partial_n^2)\hat{u}_n = \partial_n \hat{f}_d - \sum_{j=1}^{n-1} i\xi_j \hat{g}^+_j$$

$$\lambda\hat{\eta} - \hat{u}_n = \hat{k}^+,$$

whereas on the lower boundary Γ^-, i. e. for $x_n = 0$, we obtain

$$\begin{aligned} \partial_n \hat{u}_n &= \hat{f}_d \\ \hat{u}_n &= 0. \end{aligned}$$

If $\lambda \neq 0$ then we can solve the equation $\lambda\hat{\eta} - \hat{u}_n = \hat{k}^+$ for η. We require additionally that on both the upper and lower boundary the equation

$$(\omega^2 - \partial_n^2)\hat{u}_n + \partial_n\hat{\theta} = \hat{f}$$

is satisfied. Since we assume $f \in C^\infty_{c,\sigma}(\Omega)$ we obtain the equation

$$(\omega^2 - \partial_n^2)\hat{u}_n + \partial_n\hat{\theta} = 0$$

on $\Gamma^\pm$, i. e. for $x_n = 0$ and $x_n = \delta$. This gives rise to the following system of ordinary differential equations for $\hat{u}_n$, $\hat{\theta}$, and $\hat{\eta}$:

$$\begin{cases} (\omega^2 - \partial_n^2)(z^2 - \partial_n^2)\hat{u}_n = (z^2 - \partial_n^2)\hat{f}_n - (\omega^2 - \partial_n^2)\partial_n\hat{f}_d & \text{in } (0,\delta) \\ (|\xi'|^2 - \partial_n^2)\hat{\theta} = (\omega^2 - \partial_n^2)\hat{f}_d & \text{in } (0,\delta) \end{cases} \tag{14}$$

with boundary conditions

$$\begin{cases} -\hat{\theta} + 2\partial_n\hat{u}_n + (\gamma + z^2\sigma)\hat{\eta} = \hat{g}_n^+ & \text{for } x_n = \delta \\ (z^2 + \partial_n^2)\hat{u}_n = \partial_n\hat{f}_d - \sum_{j=1}^{n-1} i\xi_j\hat{g}_j^+ & \text{for } x_n = \delta \\ (\omega^2 - \partial_n^2)\hat{u}_n + \partial_n\hat{\theta} = 0 & \text{for } x_n = \delta \\ \lambda\hat{\eta} - \hat{u}_n = \hat{k}^+ & \text{for } x_n = \delta \\ \partial_n\hat{u}_n = \hat{f}_d & \text{for } x_n = 0 \\ \hat{u}_n = 0 & \text{for } x_n = 0 \\ (\omega^2 - \partial_n^2)\hat{u}_n + \partial_n\hat{\theta} = 0 & \text{for } x_n = 0. \end{cases} \tag{15}$$

And then, once u_n, θ, η are known, we obtain the tangential velocity components $u_1, \dots, u_{n-1}$ from the equations

$$\begin{cases} (\omega^2 - \partial_n^2)\hat{u}_j = \hat{f}_j - i\xi_j\hat{\theta} & \text{in } (0,\delta) \\ \partial_n\hat{u}_j = \hat{g}_j^+ - i\xi_j\hat{u}_n & \text{for } x_n = \delta \\ \hat{u}_j = 0 & \text{for } x_n = 0. \end{cases} \tag{16}$$

We begin with the case where the data are zero except for k^+. This gives rise to the following system of ordinary differential equations for $\hat{u}_n$, $\hat{\theta}$, and $\hat{\eta}$:

$$\begin{cases} (\omega^2 - \partial_n^2)(z^2 - \partial_n^2)\hat{u}_n = 0 & \text{in } (0,\delta) \\ (|\xi'|^2 - \partial_n^2)\hat{\theta} = 0 & \text{in } (0,\delta) \end{cases}$$

with boundary conditions

$$\begin{cases} -\hat{\theta} + 2\partial_n \hat{u}_n + (\gamma + z^2\sigma)\hat{\eta} = 0 & \text{for } x_n = \delta \\ (z^2 + \partial_n^2)\hat{u}_n = 0 & \text{for } x_n = \delta \\ (\omega^2 - \partial_n^2)\hat{u}_n + \partial_n \hat{\theta} = 0 & \text{for } x_n = \delta \\ \lambda\hat{\eta} - \hat{u}_n = \hat{k}^+ & \text{for } x_n = \delta \\ \partial_n \hat{u}_n = 0 & \text{for } x_n = 0 \\ \hat{u}_n = 0 & \text{for } x_n = 0 \\ (\omega^2 - \partial_n^2)\hat{u}_n + \partial_n \hat{\theta} = 0 & \text{for } x_n = 0. \end{cases}$$

And then, once u_n, θ, η are known, we obtain the tangential velocity components $u_1, \dots, u_{n-1}$ as the solutions to the equations

$$\begin{cases} (\omega^2 - \partial_n^2)\hat{u}_j = -i\xi_j \hat{\theta} & \text{in } (0, \delta) \\ \partial_n \hat{u}_j = -i\xi_j \hat{u}_n & \text{for } x_n = \delta \\ \hat{u}_j = 0 & \text{for } x_n = 0. \end{cases}$$

Since we are only interested in the case of large λ we will look for a solution (u_n, θ, η) with $\hat{u}_n$ and $\hat{\theta}$ of the form

$$\begin{cases} \hat{u}_n = \mathbf{a}_1 e^{-z(\delta - x_n)} + \mathbf{a}_2 e^{-zx_n} + \mathbf{a}_3 e^{-\omega(\delta - x_n)} + \mathbf{a}_4 e^{-\omega x_n} \\ \hat{\theta} = \mathbf{b}_1 e^{-z(\delta - x_n)} + \mathbf{b}_2 e^{-zx_n}. \end{cases}$$

These functions satisfy the above equations and $\mathbf{a}_1, \dots, \mathbf{a}_4$ and $\mathbf{b}_1, \mathbf{b}_2$ are to be determined from the boundary conditions. The functions $\hat{u}_n$ and $\hat{\theta}$ satisfy

$$\begin{aligned} \partial_n \hat{u}_n(\xi', x_n) &= a_1 z e^{-z(\delta - x_n)} - a_2 z e^{-zx_n} + a_3 \omega e^{-\omega(\delta - x_n)} - a_4 \omega e^{-\omega x_n} \\ \partial_n^2 \hat{u}_n(\xi', x_n) &= a_1 z^2 e^{-z(\delta - x_n)} + a_2 z^2 e^{-zx_n} + a_3 \omega^2 e^{-\omega(\delta - x_n)} + a_4 \omega^2 e^{-\omega x_n} \\ \partial_n \hat{\theta}(\xi', x_n) &= b_1 z e^{-z(\delta - x_n)} - b_2 z e^{-zx_n}. \end{aligned}$$

The functions (u_n, θ, η) as above satisfy the boundary conditions if and only if $\mathbf{a}, \mathbf{b}$ satisfy the linear equation

$$\begin{cases} a_1 \lambda e^{-z\delta} + a_2 \lambda + b_1 z e^{-z\delta} - b_2 z = 0 \\ a_1 \lambda + a_2 \lambda e^{-z\delta} + b_1 z - b_2 z e^{-z\delta} = 0 \\ a_1 2z - a_2 2z e^{-z\delta} + a_3 2\omega - a_4 2\omega e^{-\omega\delta} - b_1 - b_2 e^{-z\delta} + (\gamma + \sigma z^2)\hat{\eta} = 0 \\ a_1 2z^2 + a_2 2z^2 e^{-z\delta} + a_3(2z^2 + \lambda) + a_4(2z^2 + \lambda)e^{-\omega\delta} = 0 \\ -a_1 - a_2 e^{-z\delta} - a_3 - a_4 e^{-\omega\delta} + \lambda\hat{\eta} = \hat{k}^+ \\ a_1 z e^{-z\delta} - a_2 z + a_3 \omega e^{-\omega\delta} - a_4 \omega = 0 \\ a_1 e^{-z\delta} + a_2 + a_3 e^{-\omega\delta} + a_4 = 0. \end{cases}$$

Rearrangeing the order of equations we obtain the linear system of equations

$$\mathbf{M}(\lambda, |\xi'|, \sigma) \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \hat{\eta} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \hat{k}^+ \end{bmatrix} \tag{17}$$

with Lopatinskiĭ matrix $\mathbf{M}$ given by

$$\mathbf{M}(\lambda,z,\gamma,\sigma)=\begin{pmatrix} e^{-z\delta} & 1 & e^{-\omega\delta} & 1 & 0 & 0 & 0\\ ze^{-z\delta} & -z & \omega e^{-\omega\delta} & -\omega & 0 & 0 & 0\\ 2z^2 & 2z^2e^{-z\delta} & z^2+\omega^2 & \left(z^2+\omega^2\right)e^{-\omega\delta} & 0 & 0 & 0\\ 2z & -2ze^{-z\delta} & 2\omega & -2\omega e^{-\omega\delta} & -1 & -e^{-z\delta} & \gamma+\sigma z^2\\ \lambda e^{-z\delta} & \lambda & 0 & 0 & ze^{-z\delta} & -z & 0\\ \lambda & \lambda e^{-z\delta} & 0 & 0 & z & -ze^{-z\delta} & 0\\ -1 & -e^{-z\delta} & -1 & -e^{-\omega\delta} & 0 & 0 & \lambda \end{pmatrix}.$$

The mapping $\mathbf{M}$ obviously admits a unique extension to a function

$$\mathbf{M}\colon \Sigma_\varepsilon\times\Sigma_\delta\times[0,\gamma^*]\times[0,\sigma^*]\to\mathbb{C}^{7\times7}$$

which is holomorphic in the first two and real analytic in the last two entries, i. e. we can insert $z\in\Sigma_\varepsilon$ instead of $|\xi'|$. Then the determinant of $\mathbf{M}$ is again holomorphic in (λ,z) and given by

$$\begin{aligned}\det\mathbf{M}(z,\lambda,\gamma,\sigma)&=8z^6\lambda+8z^6e^{-4z\delta}\lambda-16z^6e^{-2z\delta}\lambda-8z^6e^{-2\delta\omega}\lambda-8z^6e^{-4z\delta-2\delta\omega}\lambda\\ &+16z^6e^{-2z\delta-2\delta\omega}\lambda+8z^4\lambda^2+8z^4e^{-4z\delta}\lambda^2-16z^4e^{-2z\delta}\lambda^2-8z^4e^{-2\delta\omega}\lambda^2\\ &-8z^4e^{-4z\delta-2\delta\omega}\lambda^2+16z^4e^{-2z\delta-2\delta\omega}\lambda^2+z^2\lambda^3+z^2e^{-4z\delta}\lambda^3-2z^2e^{-2z\delta}\lambda^3\\ &-z^2e^{-2\delta\omega}\lambda^3-z^2e^{-4z\delta-2\delta\omega}\lambda^3+2z^2e^{-2z\delta-2\delta\omega}\lambda^3-8z^5\lambda\omega+8z^5e^{-4z\delta}\lambda\omega\\ &-8z^5e^{-2\delta\omega}\lambda\omega+8z^5e^{-4z\delta-2\delta\omega}\lambda\omega-32z^5e^{-3z\delta-\delta\omega}\lambda\omega+32z^5e^{-z\delta-\delta\omega}\lambda\omega\\ &-4z^3\lambda^2\omega+4z^3e^{-4z\delta}\lambda^2\omega-4z^3e^{-2\delta\omega}\lambda^2\omega+4z^3e^{-4z\delta-2\delta\omega}\lambda^2\omega\\ &+16z^3e^{-z\delta-\delta\omega}\lambda^2\omega-z\lambda^3\omega+ze^{-4z\delta}\lambda^3\omega-ze^{-2\delta\omega}\lambda^3\omega+ze^{-4z\delta-2\delta\omega}\lambda^3\omega\\ &-16z^3e^{-3z\delta-\delta\omega}\lambda^2\omega\\ &+(\gamma+\sigma z^2)\left\{z^3\lambda-z^3e^{-4z\delta}\lambda-z^3e^{-2\delta\omega}\lambda+z^3e^{-4z\delta-2\delta\omega}\lambda-z^2e^{-4z\delta}\lambda\omega\right.\\ &\left.-z^2\lambda\omega+2z^2e^{-2z\delta}\lambda\omega-z^2e^{-2\delta\omega}\lambda\omega-z^2e^{-4z\delta-2\delta\omega}\lambda\omega+2z^2e^{-2z\delta-2\delta\omega}\lambda\omega\right\}.\end{aligned}$$

The Lopatinskiĭ determinant satisfies the following estimates.

LEMMA 2.1. *Let $\sigma^*>0$ and $\gamma^*>0$. There are $\lambda_0>0$, $\rho\in(0,\pi/2)$, $\varepsilon\in(0,(\pi-\delta)/2)$ and a constant $C>0$ such that the following estimates hold for all $\lambda\in\Sigma_{\pi/2+\rho}$ with $|\lambda|\ge\lambda_0$, $z\in\Sigma_\varepsilon$, $\gamma\in[0,\gamma^*]$ and $\sigma\in[0,\sigma^*]$:*

i) *If $|z|\le1$ then*

$$|\det\mathbf{M}(z,\lambda,\gamma,\sigma)|\ge C\,|z|^2\,|\lambda|^{7/2}\,.$$

ii) *If $|z|\ge1$ then*

$$|\det\mathbf{M}(z,\lambda,\gamma,\sigma)|\ge C\frac{|z|\,|\lambda|^2}{|z|^2+|\lambda|}\left\{|\lambda|^{5/2}+|z|^3\,|\lambda|+\sigma\,|z|^4\right\}.$$

In particular the matrix $\mathbf{M}$ is invertible in the specified range of parameters. We will postpone the proof of Lemma 2.1 to the end of this chapter. From Cramer's Rule (Lemma 2.4) we can infer that an explicit representation of η is given by

$$\hat\eta(\xi')=\frac{\mathbf{M}^\sharp_{7,7}}{\det\mathbf{M}}\hat k^+(\xi') \tag{18}$$

where $\mathbf{M}^\sharp_{ij}$ denote the entries of the adjugate matrix $\mathbf{M}^\sharp$ associated to $\mathbf{M}$. The entry $\mathbf{M}^\sharp_{7,7}$ is given by

$$\begin{aligned}\mathbf{M}^\sharp_{7,7}(z,\omega) = {}& -\lambda\omega^3 z e^{-2\delta\omega} + \lambda\omega^3 z e^{-4\delta z} + \lambda\omega^3 z e^{-2\delta\omega-4\delta z} - \lambda\omega^2 z^2 e^{-2\delta\omega} + \lambda\omega^2 z^2 e^{-4\delta z}\\ & - 2\lambda\omega^2 z^2 e^{-2\delta z} + 2\lambda\omega^2 z^2 e^{-2\delta\omega-2\delta z} - \lambda\omega^2 z^2 e^{-2\delta\omega-4\delta z} + 2\lambda z^4 e^{-2\delta\omega-2\delta z}\\ & - \lambda z^4 e^{-2\delta\omega-4\delta z} - \lambda\omega z^3 e^{-2\delta\omega} + \lambda\omega z^3 e^{-4\delta z} + 8\lambda\omega z^3 e^{-\delta\omega-\delta z} - 8\lambda\omega z^3 e^{-\delta\omega-3\delta z}\\ & + \lambda\omega z^3 e^{-2\delta\omega-4\delta z} + \lambda z^4 e^{-4\delta z} - 2\lambda z^4 e^{-2\delta z} - 2\omega^3 z^3 e^{-2\delta\omega} + 2\omega^3 z^3 e^{-4\delta z}\\ & + 8\omega^3 z^3 e^{-\delta\omega-\delta z} - 8\omega^3 z^3 e^{-\delta\omega-3\delta z} + 2\omega^3 z^3 e^{-2\delta\omega-4\delta z} - 6\omega^2 z^4 e^{-2\delta\omega}\\ & + 6\omega^2 z^4 e^{-4\delta z} - 12\omega^2 z^4 e^{-2\delta z} + 12\omega^2 z^4 e^{-2\delta\omega-2\delta z} - 6\omega^2 z^4 e^{-2\delta\omega-4\delta z}\\ & - 2z^6 e^{-2\delta\omega} + 4z^6 e^{-2\delta\omega-2\delta z} - 2z^6 e^{-2\delta\omega-4\delta z} - 6\omega z^5 e^{-2\delta\omega} + 6\omega z^5 e^{-4\delta z}\\ & + 24\omega z^5 e^{-\delta\omega-\delta z} - 24\omega z^5 e^{-\delta\omega-3\delta z} + 6\omega z^5 e^{-2\delta\omega-4\delta z} + 2z^6 e^{-4\delta z} - \lambda z^4 e^{-2\delta\omega}\\ & - 4z^6 e^{-2\delta z} - \lambda\omega^3 z + \lambda\omega^2 z^2 - \lambda\omega z^3 + \lambda z^4 - 2\omega^3 z^3 + 6\omega^2 z^4 - 6\omega z^5 + 2z^6\end{aligned}$$

and satisfies the following estimates.

LEMMA 2.2. *There are $\lambda_0 > 0$, $\rho \in (0, \pi/2)$, $\varepsilon \in (0, (\pi-\delta)/2)$ and a constant $C > 0$ such that the following estimates hold for all $\lambda \in \Sigma_{\pi/2+\rho}$ with $|\lambda| \geq \lambda_0$ and $z \in \Sigma_\varepsilon$:*

i) *If $|z| \leq 1$ then*

$$\left|\mathbf{M}^\sharp_{7,7}(z,\lambda)\right| \leq C\,|z|^2\,|\lambda|^{5/2}.$$

ii) *If $|z| \geq 1$ then*

$$\left|\mathbf{M}^\sharp_{7,7}(z,\lambda)\right| \leq C\,|z|\,|\lambda|^2\,(|z| + |\lambda|^{1/2}).$$

Once η is known we can obtain u and θ from the equation

$$\begin{cases} \lambda u - \Delta u + \nabla\theta = 0 & \text{in } \Omega\\ \operatorname{div} u = 0 & \text{in } \Omega\\ S(u,\theta)\nu = -(\gamma - \sigma\Delta'\eta)\nu & \text{on } \Gamma^+\\ u = 0 & \text{on } \Gamma^- \end{cases}$$

which has been treated by e. g. Abels [**Abe05a, Abe06**]. Let us summarise this representation of the solution (u, θ, η).

FORMULA 2.3. *Given $\lambda_0 > 0$ sufficiently large and $\lambda \in \Sigma_\rho$ for some $0 < \rho < \pi$ with $|\lambda| \geq \lambda_0$ the solution (u, θ, η) of (2) in the case where $f = 0$, $f_d = 0$, $g^+ = 0$ and $k^+ \in C_c^\infty(\mathbb{R}^{n-1})$ admits the following representation: The function η is given by*

$$\hat{\eta} = \frac{\mathbf{M}^\sharp_{7,7}}{\det \mathbf{M}}\hat{k}^+$$

with the Lopatinskiĭ matrix $\mathbf{M}$ as above, and (u, θ) are the unique solution to the resolvent problem

$$\begin{cases} \lambda u - \Delta u + \nabla\theta = 0 & \text{in } \Omega\\ \operatorname{div} u = 0 & \text{in } \Omega\\ S(u,\theta)\nu = -(\gamma - \sigma\Delta'\eta)\nu & \text{on } \Gamma^+\\ u = 0 & \text{on } \Gamma^- \end{cases}$$

*which is known to have a unique solution by [**Abe05a, Abe06**].*

One can use Abels' results [**Abe05a**, **Abe06**] to generalise this representation to the linear problem (2) with nonzero data f, f_d, g^+, k^+.

It remains to prove Lemma 2.1 and Lemma 2.2. While the proof of Lemma 2.2 is simple and elementary, it turns out that proving Lemma 2.1 is much more involved.

Proof of Lemma 2.2. As we are only interested in λ of large modulus it suffices to study the dominant part of $\mathbf{M}^\sharp_{7,7}$ as $\lambda \to \infty$, which is given by

$$\begin{aligned}\mathbf{M}^\sharp_{7,7}(z,\omega) \approx{}& \lambda\omega^3 z e^{-4\delta z} + \lambda\omega^2 z^2 e^{-4\delta z} - 2\lambda\omega^2 z^2 e^{-2\delta z} + \lambda\omega z^3 e^{-4\delta z} \\ &+ \lambda z^4 e^{-4\delta z} - 2\lambda z^4 e^{-2\delta z} + 2\omega^3 z^3 e^{-4\delta z} + 6\omega^2 z^4 e^{-4\delta z} - 12\omega^2 z^4 e^{-2\delta z} \\ &+ 6\omega z^5 e^{-4\delta z} + 2z^6 e^{-4\delta z} - 4z^6 e^{-2\delta z} - \lambda\omega^3 z + \lambda\omega^2 z^2 - \lambda\omega z^3 + \lambda z^4 \\ &- 2\omega^3 z^3 + 6\omega^2 z^4 - 6\omega z^5 + 2z^6.\end{aligned}$$

For large z, i. e. for $|z| \geq R$ with sufficiently large $R > 0$, the dominant part of this expression is

$$\begin{aligned}\mathbf{M}^\sharp_{7,7}(z,\omega) &\approx -\lambda\omega^3 z + \lambda\omega^2 z^2 - \lambda\omega z^3 + \lambda z^4 - 2\omega^3 z^3 + 6\omega^2 z^4 - 6\omega z^5 + 2z^6 \\ &= -\frac{z\lambda^2}{(\omega+z)^2}\left(4z^2\omega + z\lambda + \lambda\omega\right)\end{aligned}$$

while for $|z| \leq R$ the dominant part as $\lambda \to \infty$ is simply $-\lambda\omega^3 z(1 - e^{-4\delta z})$. This shows for $|z| \leq R$ and $|\lambda| \geq \lambda_0$, for $R > 0$ and $\lambda_0 > 0$ sufficiently large, the estimate

$$\left|\mathbf{M}^\sharp_{7,7}(z,\omega)\right| \lesssim |\lambda|^{7/2}|z|^2$$

and for $|z| \geq R$ and $|\lambda| \geq \lambda_0$ the estimate

$$\left|\mathbf{M}^\sharp_{7,7}(z,\omega)\right| \lesssim \frac{|z|\,|\lambda|^2}{|z|^2 + |\lambda|}\left(|z|^3 + |\lambda|^{3/2}\right) \lesssim |z|\,|\lambda|^2\left(|z| + |\lambda|^{1/2}\right). \qquad \square$$

We will show Lemma 2.1 in a series of steps. Observe that $\det \mathbf{M}$ is of the form

$$P(z,\lambda) = \sum_{m\in\mathcal{I}} a_m z^{m_1}\lambda^{m_2}\omega^{m_3} e^{-\delta z m_4} e^{-\delta\omega m_5}$$

with $(a_m) \subset \mathbb{C}$ and $\mathcal{I} \subset \mathbb{N}_0^5$ a finite set. We will treat high frequencies, i. e. z of large modulus, low frequencies and the remaining range of frequencies separately. The main difficulty is imposed by the high frequencies. For λ and z of large modulus we can absorb the terms containing $e^{-\omega\delta}$ and $e^{-z\delta}$ into the remaining part. Thus we are led to study the function

$$P_{\infty,\infty}(z,\lambda,\gamma,\sigma) = \sum_{\substack{m\in\mathcal{I}\\ m_4=m_5=0}} a_m z^{m_1}\lambda^{m_2}\omega^{m_3}$$

which in our case is given by

$$P_{\infty,\infty}(z,\lambda) = -\frac{z\lambda^2}{(z+\omega)^2}\left\{\left[4z^2\lambda\omega + z\lambda^2 + \lambda^2\omega\right] + (\gamma + \sigma z^2)z(z+\omega)\right\}.$$

A careful application of the Newton polygon method [**DSS08**, **GV92**] to the function $P_{\infty,\infty}$ yields the following lemma.

Lemma 2.4. *Let $\gamma^*, \sigma^* > 0$. There are $\lambda_0 > 0$, $\rho \in (\pi/2, \pi)$, $\varepsilon \in (0, (\pi-\rho)/2)$ and a constant $C > 0$ such that the estimate*

$$\left|\left[4z^2\lambda\omega + z\lambda^2 + \lambda^2\omega\right] + \sigma z^3(z+\omega)\right| \geq C\left\{|\lambda|^{5/2} + |z|^3|\lambda| + \sigma|z|^4\right\}$$

holds for all $\lambda \in \Sigma_\rho$ with $|\lambda| \geq \lambda_0$ and $z \in \Sigma_\varepsilon$ as well as $\gamma \in [0,\gamma^*]$, $\sigma \in [0,\sigma^*]$.

We postpone the proof of Lemma 2.4 to the end of this chapter. As an immediate consequence of Lemma 2.4 we obtain the estimate

$$|P_{\infty,\infty}(z,\lambda,\gamma,\sigma)| \geq C\frac{|z|\,|\lambda|^2}{|z|^2+|\lambda|}\left\{|\lambda|^{5/2}+|z|^3\,|\lambda|+\sigma\,|z|^4\right\},$$

and via an absorption argument

$$|P(z,\lambda,\gamma,\sigma)| \geq C\frac{|z|\,|\lambda|^2}{|z|^2+|\lambda|}\left\{|\lambda|^{5/2}+|z|^3\,|\lambda|+\sigma\,|z|^4\right\}$$

for z and λ of sufficiently large modulus.

In order to treat the case where λ is of large modulus, but z is close to zero we employ the approximations $e^{-m_5\delta\omega}\approx 0$ and $e^{-m_4\delta z}\approx 1-m_4\delta z$ if $m_4,m_5\geq 1$. Thus we are led to study the function

$$P_{\infty,0}(z,\lambda,\gamma,\sigma) = -4\delta z^2\lambda\omega(\lambda^2+4\lambda z^2+8z^4)+4\delta z^4\lambda(\gamma+\sigma z^2)$$

An absorption argument shows that for $0\leq\sigma\leq\sigma^*$ and $0\leq\gamma\leq\gamma^*$ it is sufficient to consider the function

$$\tilde{P}_{\infty,0}(z,\lambda) = -4\delta z^2\lambda^{7/2}.$$

Thus we obtain, again with an absorption argument,

$$|P(z,\lambda)| \geq C\,|z|^2\,|\lambda|^{7/2}$$

whenever λ is of sufficiently large modulus and z is sufficiently close to zero. The remaining range of frequencies can be dealt with rather easily in that it is sufficient to consider the part of P corresponding to the highest appearing order of λ. In this case it is thus sufficient to consider $P_{\infty,c}=C\lambda^{7/2}$, which obviously does not do any harm.

In order to establish Lemma 2.1 it remains to show Lemma 2.4. This can be done along the lines of the proof of Theorem 3.1 of [**DSS08**]. In order to do so let us begin by recalling the following result, which is contained in the proof of Theorem 3.1 in [**DSS08**], and a proof of which can be found in chapter 4 of [**GV92**]. We adopt the methods, notation, and nomenclature from [**DSS08**].

LEMMA 2.5. *Let $\rho\in(0,\pi)$, $\varepsilon\in(0,(\pi-\rho)/2)$ and $\mu>0$. Let $\tilde{I}=\{v_1,\ldots,v_{J+1}\}$ denote the vertices of a regular Newton polygon in the sense of [**DSS08**], with $v_j=(r_j,s_j)$ for $j=1,\ldots,J+1$ and $\gamma_j=(r_j-r_{j+1})/(s_{j+1}-s_j)$ for $j=1,\ldots,J$. Then there are constants $\lambda_0>0$ and $M>0$, and a partition of the form*

$$\{(z,\lambda)\in\overline{\Sigma}_\varepsilon\times\overline{\Sigma}_\rho\colon\ |\lambda|\geq\lambda_0\}\subset\bigcup_{j=1}^{J}G_j\cup\bigcup_{j=1}^{J+1}\tilde{G}_j$$

with the following properties:

i) *Let $j\in\{1,\ldots,J\}$. Then*

$$G_j=\{(z,\lambda)\in\overline{\Sigma}_\varepsilon\times\overline{\Sigma}_\rho\colon M^{-1}\,|z|^{\gamma_j}\leq|\lambda|\leq M\,|z|^{\gamma_j}\}$$

and for each $n=(n_1,n_2)\in\tilde{I}\setminus[v_jv_{j+1}]$ we have

$$|z|^{n_1}\,|\lambda|^{n_2}\leq\mu\sum_{(n_1',n_2')\in[v_jv_{j+1}]\cap\tilde{I}}|z|^{n_1'}\,|\lambda|^{n_2'}$$

for all $(z,\lambda)\in G_j$.

ii) *Let $j\in\{1,\ldots,J+1\}$. Then for every $(n_1,n_2)\in\tilde{I}\setminus\{v_j\}$ we have*

$$|z|^{n_1}\,|\lambda|^{n_2}\leq\mu\,|z|^{r_j}\,|\lambda|^{s_j}$$

for all $(z,\lambda)\in\tilde{G}_j$.

This puts us in a position to show Lemma 2.4.

Proof of Lemma 2.4. Write

$$P(z,\lambda) = \left[4z^2\lambda\omega + z\lambda^2 + \lambda^2\omega\right] + (\alpha + \sigma z^2)z(z+\omega).$$

In the course of this proof we write α for the gravity parameter so as to avoid confusion with the values $\gamma, \gamma_1, \ldots, \gamma_J$ appearing in the nomenclature around the Newton polygon method. The associated sets I and $\tilde{I}$ are given by

$$I = \{(2,1,1),(1,2,0),(0,2,1),(2,0,0),(1,0,1),(4,0,0),(3,0,1)\}$$

and

$$\begin{aligned}\tilde{I} &= \{(0,0)\} \cup \{(3,1),(1,2),(2,0),(4,0)\} \cup \{(1,2),(2,3/2),(0,5/2),(2,0),(1,1/2),(4,0),(3,1/2)\}\\ &= \{(0,0),(3,1),(1,2),(4,0),(2,3/2),(2,0),(1,1/2),(0,5/2),(3,1/2)\}.\end{aligned}$$

Let $N(P) = \operatorname{conv}(\tilde{I})$. Then the vertices of $N(P)$ are given by

$$v_0 = (0,0), \qquad v_1 = (4,0), \qquad v_2 = (3,1), \qquad v_3 = (0,5/2)$$

and we have

$$\gamma_1 = 1, \qquad \gamma_2 = 2.$$

Thus, for $\gamma > 0$, the γ-degree

$$d_\gamma(P) = \max\{m_1 + \gamma m_2 + m_3 \max\{1, \gamma/2\} \colon m \in I\}$$

is given by

$$d_\gamma(P) = \begin{cases} 4 & 0 < \gamma \le 1 \\ 3 + \gamma & 1 < \gamma \le 2 \\ \frac{5}{2}\gamma & 2 < \gamma. \end{cases}$$

Then the leading exponents for the weight $\gamma > 0$ are given by

$$I_\gamma = \begin{cases} \{(4,0,0),(3,0,1)\} & 0 < \gamma < 1 \\ \{(2,1,1),(4,0,0),(3,0,1)\} & \gamma = 1 \\ \{(2,1,1)\} & 1 < \gamma < 2 \\ \{(1,2,0),(2,1,1),(0,2,1)\} & \gamma = 2 \\ \{(0,2,1)\} & \gamma > 2 \end{cases}$$

and the corresponding γ-principal parts of P are given by

$$P_\gamma(z,\lambda) = \begin{cases} 2\sigma z^4 & 0 < \gamma < 1 \\ 4z^3\lambda + 2\sigma z^4 & \gamma = 1 \\ 4z^3\lambda & 1 < \gamma < 2 \\ z\lambda^2 + 4z^2\lambda\omega + \lambda^2\omega & \gamma = 2 \\ \lambda^{5/2} & \gamma > 2. \end{cases}$$

It is not too difficult to show (and contained in the proof of Proposition 5.3 of [**DGH+11**]) that none of these functions have any zeros in $\overline{\Sigma}_\varepsilon \setminus \{0\} \times \overline{\Sigma}_\rho \setminus \{0\}$. At this point Theorem 3.1 of [**DSS08**] immediately yields an estimate from below for the function in question, which, however, is too rough for our purposes. Hence we will have to apply Lemma 2.5 and carefully estimate the appearing terms. We will treat the regions G_j, $\tilde{G}_j$ separately.

But first of all let us remark that either by an application of the Newton Polygon method or a compactness and homogeneity argument one can show the following estimates:

$$\left|z\lambda^2 + 4z^2\lambda\omega + \lambda^2\omega\right| \ge C(|\lambda|^{5/2} + |z|^3|\lambda|)$$

and

$$\left|4z^3\lambda + 2\sigma z^4\right| \geq C\,|z|^3\,(|\lambda| + \sigma\,|z|)$$

for $|\lambda| \geq \lambda_0$ sufficiently large.

Let us start by stating the assertion of Lemma 2.5 in the present situation. More precisely, the assertion of Lemma 2.5 reads that there is $\lambda_0 > 0$, $M > 0$ and a partition of the form

$$\{(z,\lambda) \in \overline{\Sigma}_\varepsilon \times \overline{\Sigma}_\rho \colon\ |\lambda| \geq \lambda_0\} \subset \bigcup_{j=1}^{2} G_j \cup \bigcup_{j=1}^{3} \tilde{G}_j$$

with the following properties:

i) For $(z,\lambda) \in G_1$ we have

$$M^{-1}\,|z| \leq |\lambda| \leq M\,|z|$$

and

$$|\lambda|^{5/2} \leq \mu(|z|^4 + |z|^3\,|\lambda|).$$

ii) For $(z,\lambda) \in G_2$ we have

$$M^{-1}\,|z|^2 \leq |\lambda| \leq M\,|z|^2$$

and

$$|z|^4 \leq \mu(|\lambda|^{5/2} + |z|^3\,|\lambda|).$$

iii) For $(z,\lambda) \in \tilde{G}_1$ we have

$$|z|^3\,|\lambda| \leq \mu\,|z|^4 \quad \text{and} \quad |\lambda|^{5/2} \leq \mu\,|z|^4\,.$$

iv) For $(z,\lambda) \in \tilde{G}_2$ we have

$$|z|^4 \leq \mu\,|z|^3\,|\lambda| \quad \text{and} \quad |\lambda|^{5/2} \leq \mu\,|z|^3\,|\lambda|\,.$$

v) For $(z,\lambda) \in \tilde{G}_3$ we have

$$|z|^4 \leq \mu\,|\lambda|^{5/2} \quad \text{and} \quad |z|^3\,|\lambda| \leq \mu\,|\lambda|^{5/2}\,.$$

Now let us estimate the function P separately on the regions G_j, $\tilde{G}_j$.

For $(z,\lambda) \in G_1$ we have

$$|P(z,\lambda)| \geq |P_1(z,\lambda)| - |P(z,\lambda) - P_1(z,\lambda)| \geq C(\sigma\,|z|^4 + |z|^3\,|\lambda|) - |P(z,\lambda) - P_1(z,\lambda)|$$

and

$$\begin{aligned}|P(z,\lambda) - P_1(z,\lambda)| &= \left|z\lambda^2 + \lambda^2\omega + 4z^2\lambda(\omega - z) + \alpha z(z+\omega) + \sigma z^3(\omega - z)\right| \\ &\lesssim |z|\,|\lambda|^2 + |\lambda|^{5/2} + |z|\,|\lambda|^2 + \alpha^*\,|z|^2 + \alpha^*\,|z|\,|\lambda|^{1/2} + \sigma^*\,|z|^2\,|\lambda| \\ &\lesssim \lambda_0^{-1}\,|z|^3\,|\lambda|\end{aligned}$$

and thus we have, choosing λ_0 sufficiently large,

$$|P(z,\lambda)| \gtrsim \sigma\,|z|^4 + |z|^3\,|\lambda|\,.$$

uniformly in $z, \lambda, \alpha, \sigma$. Since $|\lambda|^{5/2} \lesssim \lambda_0^{-3/2}\,|z|^3\,|\lambda|$ we can conclude that, again for λ_0 sufficiently large,

$$|P(z,\lambda)| \gtrsim \sigma\,|z|^4 + |z|^3\,|\lambda| \gtrsim |\lambda|^{5/2}$$

holds. This yields the desired estimate on G_1. For $(z,\lambda) \in G_2$ we have

$$|P(z,\lambda)| \geq |P_2(z,\lambda)| - |P(z,\lambda) - P_2(z,\lambda)| \geq C(|\lambda|^{5/2} + |z|^3\,|\lambda|) - |P(z,\lambda) - P_2(z,\lambda)|$$

and

$$\begin{aligned}|P(z,\lambda)-P_2(z,\lambda)| &= \left|\alpha z(z+\omega)+\sigma z^3(z+\omega)\right| \\ &\lesssim \alpha^* |z|^2+\alpha^* |z| |\lambda|^{1/2}+\sigma^* |z|^4+\sigma^* |z|^3 |\lambda|^{1/2} \\ &\lesssim \lambda_0^{-1/2} |z|^3 |\lambda|\end{aligned}$$

and thus we have for λ_0 sufficiently large the estimate

$$|P(z,\lambda)| \gtrsim \sigma |\lambda|^{5/2} + |z|^3 |\lambda| .$$

uniformly in $z, \lambda, \alpha, \sigma$. Since, as just seen, $|z|^4 \lesssim \lambda_0^{-1/2} |\lambda|^{5/2}$ we can conclude that, again for λ_0 sufficiently large,

$$|P(z,\lambda)| \gtrsim |\lambda|^{5/2} + |z|^3 |\lambda| \gtrsim \sigma |z|^4$$

holds. This yields the desired estimate on G_2. For $(z,\lambda) \in \tilde{G}_1$ we have

$$|P(z,\lambda)| \geq |P_1(z,\lambda)| - |P(z,\lambda)-P_1(z,\lambda)| \geq C(\sigma |z|^4 + |z|^3 |\lambda|) - |P(z,\lambda)-P_1(z,\lambda)|$$

and

$$\begin{aligned}|P(z,\lambda)-P_1(z,\lambda)| &= \left|z\lambda^2+\lambda^2\omega+4z^2\lambda(\omega-z)+\alpha z(z+\omega)+\sigma z^3(\omega-z)\right| \\ &\lesssim |z| |\lambda|^2 + |\lambda|^{5/2} + |z| |\lambda|^2 + \alpha^* |z|^2 + \alpha^* |z| |\lambda|^{1/2} + \sigma^* |z|^2 |\lambda| \\ &\lesssim \left(\mu+\mu^3\lambda_0^{-3/2}+\mu^2\lambda_0^{-3/2}+\mu\lambda_0^{-1}\right) |z|^3 |\lambda|\end{aligned}$$

and thus we have, for λ_0 sufficiently large and $\mu > 0$ sufficiently small,

$$|P(z,\lambda)| \gtrsim \sigma |z|^4 + |z|^3 |\lambda| .$$

Since $|\lambda|^{5/2} \lesssim \mu^3 \lambda_0^{-3/2} |z|^3 |\lambda|$ we can conclude that, again for λ_0 sufficiently large,

$$|P(z,\lambda)| \gtrsim \sigma |z|^4 + |z|^3 |\lambda| \gtrsim |\lambda|^{5/2}$$

holds. This yields the desired estimate on $\tilde{G}_1$. Now for $(z,\lambda) \in \tilde{G}_2$ we have

$$|P(z,\lambda)| \geq |P_2(z,\lambda)| - |P(z,\lambda)-P_2(z,\lambda)| \geq C(|\lambda|^{5/2} + |z|^3 |\lambda|) - |P(z,\lambda)-P_2(z,\lambda)|$$

and

$$\begin{aligned}|P(z,\lambda)-P_2(z,\lambda)| &= \left|\alpha z(z+\omega)+\sigma z^3(z+\omega)\right| \\ &\lesssim \alpha^* |z|^2+\alpha^* |z| |\lambda|^{1/2}+\sigma^* |z|^4+\sigma^* |z|^3 |\lambda|^{1/2} \\ &\lesssim \lambda_0^{-1/2} |z|^3 |\lambda| + \mu |\lambda|^{5/2}\end{aligned}$$

and thus we have, for λ_0 sufficiently large and $\mu > 0$ sufficiently small,

$$|P(z,\lambda)| \gtrsim \sigma |\lambda|^{5/2} + |z|^3 |\lambda| .$$

Since $|z|^4 \lesssim \lambda_0^{-1/2} |\lambda|^{5/2}$ we can conclude that for λ_0 sufficiently large we have

$$|P(z,\lambda)| \gtrsim |\lambda|^{5/2} + |z|^3 |\lambda|) \gtrsim \sigma |z|^4 .$$

This yields the desired estimate on $\tilde{G}_2$. Now for $(z,\lambda) \in \tilde{G}_3$ we have

$$|P(z,\lambda)| \geq |P_2(z,\lambda)| - |P(z,\lambda)-P_2(z,\lambda)| \geq C(|\lambda|^{5/2} + |z|^3 |\lambda|) - |P(z,\lambda)-P_2(z,\lambda)|$$

and

$$\begin{aligned}|P(z,\lambda) - P_2(z,\lambda)| &= \left|\alpha z(z+\omega) + \sigma z^3(z+\omega)\right| \\ &\lesssim \alpha^* |z|^2 + \alpha^* |z| |\lambda|^{1/2} + \sigma^* |z|^4 + \sigma^* |z|^3 |\lambda|^{1/2} \\ &\lesssim \mu |\lambda|^{5/2} + \mu\lambda_0^{-1/2} |\lambda|^{5/2}\end{aligned}$$

and thus we have, for λ_0 sufficiently large and $\mu > 0$ sufficiently small,

$$|P(z,\lambda)| \gtrsim \sigma |\lambda|^{5/2} + |z|^3 |\lambda| .$$

Since $|z|^4 \lesssim \mu |\lambda|^{5/2}$ we can conclude that again for $\lambda_0 > 0$ sufficiently large and $\mu > 0$ sufficiently small

$$|P(z,\lambda)| \gtrsim |\lambda|^{5/2} + |z|^3 |\lambda| \gtrsim \sigma |z|^4$$

holds. This yields the desired estimate on $\tilde{G}_3$ and concludes the proof. □

CHAPTER III

Analysis of the Stokes equation in a layer in spaces of bounded functions

1. Introduction and main results

Let $\Omega = \mathbb{R}^{n-1} \times (0, \delta)$ denote a layer of infinite extent. In this chapter we study the linear evolution equation

$$
(1) \qquad \begin{cases} \partial_t u - \Delta u + \nabla\theta = 0 & \text{in } J \times \Omega \\ \operatorname{div} u = 0 & \text{in } J \times \Omega \\ u = 0 & \text{on } J \times \partial\Omega \\ u(0) = u_0 & \text{in } \Omega \end{cases}
$$

via the associated resolvent problem

$$
(2) \qquad \begin{cases} \lambda u - \Delta u + \nabla\theta = f & \text{in } \Omega \\ \operatorname{div} u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}
$$

in solenoidal subspaces of $L_\infty(\Omega)$ and related spaces.

Similar problems in different classes of domains have been studied by a variety of authors. The resolvent problem (2) for $\Omega = \mathbb{R}^n_+$ has been studied in solenoidal subspaces of $L_\infty(\Omega)$ by, among others, Desch et al. [**DHP01**] and under more general boundary conditions by Saal [**Saa07**]. These results were later improved upon by Abe and Giga [**AG13, AG14**] who proved generation results for the Stokes operator in solenoidal subspaces of $L_\infty(\Omega)$ for a large class of domains Ω, which they refer to as *admissible* domains. It turns out, however, that layer domains are not admissible in the sense of Abe and Giga.

In this chapter we attempt to close this gap left by the work of Abe and Giga. Let

$$
C^\infty_{c,\sigma}(\Omega) = \{f \in C^\infty_c(\Omega) \colon \operatorname{div} f = 0\}
$$

denote the space of divergence-free test functions. We denote with $C_{0,\sigma}(\Omega)$ the closure of $C^\infty_{c,\sigma}(\Omega)$ in $L_\infty(\Omega)$. We will show the following results.

Theorem 1.1. *Let $n \geq 2$, $\lambda_0 > 0$ and $0 < \rho < \pi$. Then there is $C > 0$ such that for all $\lambda \in \Sigma_\rho$ with $|\lambda| \geq \lambda_0$ and $f \in C^\infty_{c,\sigma}(\Omega)$ there is a solution (u, θ) with $u \in C_{0,\sigma}(\Omega)$ of (2) satisfying the estimates*

$$
\begin{aligned} |\lambda| \sup_{x_n} \|u(\cdot, x_n)\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1})} + |\lambda|^{1/2} \sup_{x_n} \|\nabla u(\cdot, x_n)\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1})} \\ + \sup_{x_n} \left\|\nabla^2 u(\cdot, x_n)\right\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1})} + \sup_{x_n} \|\nabla\theta(\cdot, x_n)\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1})} \leq C \|f\|_{L_\infty(\Omega)} . \end{aligned}
$$

By continuity this result extends to all functions $f \in C_{0,\sigma}(\Omega)$, but then the solution u will, in general, be no longer in $C_{0,\sigma}(\Omega)$. In fact, the closure of $C_{0,\sigma}(\Omega)$ in the space of solutions considered in Theorem 1.1 contains unbounded functions. In contrary to what one might expect this result will turn out to be rather sharp. If we restrict ourselves to the two-dimensional setting however, then we can improve this result considerably insofar as we can estimate u in $L_\infty(\Omega)$.

THEOREM 1.2. *Let $n = 2$, $\lambda_0 > 0$ and $0 < \rho < \pi$. Then there is $C > 0$ such that for all $\lambda \in \Sigma_\rho$ with $|\lambda| \geq \lambda_0$ and $f \in C^\infty_{c,\sigma}(\Omega)$ there is a unique solution (u, θ) with $u \in C_{0,\sigma}(\Omega)$ of (2) satisfying the estimates*

$$|\lambda| \, \|u\|_{L_\infty(\Omega)} + |\lambda|^{1/2} \, \|\nabla u\|_{L_\infty(\Omega)} + \sup_{x_n} \left\|\nabla^2 u(\cdot, x_n)\right\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1})} + \sup_{x_n} \|\nabla\theta(\cdot, x_n)\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1})} \leq C \, \|f\|_{L_\infty(\Omega)} \, .$$

Here uniqueness of solutions is to be understood as uniqueness of u in the class $C_{0,\sigma}(\Omega)$ and uniqueness of θ modulo an additive constant, i. e. we identify two solutions (u, θ) and (v, π) if $u = v$ and $\nabla\theta = \nabla\pi$. By continuity we can extend the assertion of Theorem 1.2 to all functions $f \in C_{0,\sigma}(\Omega)$. As a corollary we obtain that the Stokes operator on $C_{0,\sigma}(\Omega)$ generates a holomorphic semigroup.

COROLLARY 1.3. *Let $n = 2$. Then the Stokes operator generates a strongly continuous holomorphic semigroup of angle $\pi/2$ on $C_{0,\sigma}(\Omega)$.*

However, this result does not extend to $n \geq 3$ dimensions. We can show the following result.

THEOREM 1.4. *Let $n \geq 3$ and $\lambda > 0$. Then there is $f \in C_{0,\sigma}(\Omega) \cap C^\infty(\Omega)$ such that the solution (u, θ) from Theorem 1.1 satisfies $u \notin L_\infty(\Omega)$, $\nabla u \notin L_\infty(\Omega)$ and $\nabla\theta \notin L_\infty(\Omega)$.*

For the case $\lambda = 0$ Abe and Yamazaki [**AY10**] obtained similar results in homogeneous Besov spaces. A particularly interesting result of theirs is that in the case $\lambda = 0$ the only solution to the homogeneous problem is Poiseuille flow. A corresponding result for general λ is the following.

THEOREM 1.5. *Assume (u, θ) is a solution to (2) with zero data such that the functions*

$$x_n \mapsto \|u(\cdot, x_n)\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1})} , \qquad x_n \mapsto \|\nabla u(\cdot, x_n)\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1})}$$
$$x_n \mapsto \left\|\nabla^2 u(\cdot, x_n)\right\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1})} , \qquad x_n \mapsto \|\nabla\theta(\cdot, x_n)\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1})}$$

are essentially bounded. Then there are $d \in \mathbb{R}^{n-1}$ and $c \in \mathbb{R}$ such that

$$u_j(x', x_n) = \frac{d_j}{\lambda} \left[1 - \frac{e^{-\sqrt{\lambda}x_n} + e^{-\sqrt{\lambda}(\delta - x_n)}}{1 + e^{-\sqrt{\lambda}\delta}}\right]$$

for $j = 1, \ldots, n-1$, $u_n = 0$, and $\theta(x', x_n) = -d \cdot x' + c$.

This shows in particular that the solution constructed in Theorem 1.1 is unique 'up to Poiseuille flow', and that any solution (u, θ) with $u \in C_{0,\sigma}(\Omega)$ to the homogeneous Stokes resolvent problem must be zero. Furthermore, Theorem 1.1 shows that *any* solution to the Stokes resolvent problem for the function f constructed in Theorem 1.4 is unbounded.

This closes a gap that was left by the work of Abe and Giga [**AG13**,**AG14**] and complements their results. While Corollary 1.3 shows that in a two-dimensional layer domain Ω the Stokes operator generates a holomorphic and strongly continuous semigroup of angle $\pi/2$ in $C_{0,\sigma}(\Omega)$, Theorem 1.4 shows that for layer domains in $n \geq 3$ dimensions any attempt at a generation result for the Stokes operator in $C_{0,\sigma}$ is futile. An extension of Theorem 1.2 and Corollary 1.3 to the larger spaces $\mathrm{BUC}_\sigma(\Omega)$ and $L_{\infty,\sigma}(\Omega)$ as in e. g. [**AG13**] will be considered in a subsequent project.

The proofs for Theorem 1.1 and Theorem 1.2 essentially boil down to estimating the explicit solution formula II.1.3 that we have constructed in Chapter II. Once these results are established the counterexample Theorem 1.4 refers to will be constructed. This construction again relies heavily on the explicit solution formula II.1.3. Finally we will show Theorem 1.5 the proof of which is essentially an adaptation of the proof of the corresponding result in [**AY10**].

This work is mainly inspired by Desch et al. [**DHP01**] and Saal [**Saa07**], who studied the Stokes resolvent problem in solenoidal spaces of bounded functions in a half space $\mathbb{R}^n_+$, and by Abe and Shibata [**AS03a**, **AS03b**] who studied the Stokes resolvent problem in L_p for $1 < p < \infty$ in layer domains.

As an application of the techniques we will establish in this chapter we will give a short proof of the resolvent estimates for the Stokes equation in a half space that were established in [**DHP01**, **Saa07**].

2. A characterisation of $C_{0,\sigma}$

Aim of this section is to characterise the space $C_{0,\sigma}(M)$ for a class of domains $M \subset \mathbb{R}^n$, where $n \geq 2$. In addition to the spaces $C^\infty_{c,\sigma}(M)$ and $C_{0,\sigma}(M)$ introduced above we define the space $L_{\infty,\sigma}(M)$ as the space of all functions $f \in L_\infty(M)$ such that

$$\int_M f \cdot \nabla\phi = 0$$

for all $\phi \in L_{1,\mathrm{loc}}(M)$ with $\nabla\phi \in L_1(M)$. We provide $L_{\infty,\sigma}(M)$ with the norm of $L_\infty(M)$. Then $L_{\infty,\sigma}(M)$ is a closed subspace of $L_\infty(M)$ and in particular a Banach space. We will use de Rham's Theorem [**dR84**] in the version presented in Theorem 4.1 of [**Mar08**].

Theorem 2.1 (de Rham). *Let $M \subset \mathbb{R}^n$ be a domain. If $\phi \in \mathscr{D}'(M)$ satisfies $\langle f, \phi\rangle = 0$ for every $f \in C^\infty_{c,\sigma}(M)$ then there is $\pi \in \mathscr{D}'(M)$ such that $\phi = \nabla\pi$.*

Here we write $\mathscr{D}(M)$ for the space of smooth functions compactly supported in M, and $\mathscr{D}'(M)$ for its dual space, i. e. the space of distributions as in [**Sch66**]. We will show the following simple characterisation of $C_{0,\sigma}(M)$.

Lemma 2.2. *Let $M \subset \mathbb{R}^n$ be a domain. Then $f \in C_{0,\sigma}(M)$ if and only if $f \in C_0(M) \cap L_{\infty,\sigma}(M)$.*

Proof. Since $C_{0,\sigma}(M)$ is obviously a closed subspace of $C_0(M) \cap L_{\infty,\sigma}(M)$ for any domain M, it is sufficient to show that $C_{0,\sigma}(M)$ is in fact dense in $C_0(M) \cap L_{\infty,\sigma}(M)$. Let φ denote a functional on the space $C_0(M) \cap L_{\infty,\sigma}(M)$ such that φ vanishes on $C_{0,\sigma}(M)$. We wish to show that φ is the zero functional.

By Hahn-Banach we can extend φ to a functional on $C_0(M)$, which we denote by φ again, and it follows from the Theorem of Riesz-Markov (Theorem V.20.48 in [**HS69**]) that φ is given by a bounded Radon measure μ, i. e.

$$\langle f, \varphi\rangle = \int_M f \,\mathrm{d}\mu$$

for $f \in C_0(M)$, and μ has finite total variation, i. e. $\|\mu\|_{\mathrm{Var}} < \infty$. We can interpret μ as a distribution in $\mathscr{D}'(M)$, and since μ vanishes on $C_{0,\sigma}(M)$ it does so in particular on $C^\infty_{c,\sigma}(M)$. Then it is a consequence of de Rham's Theorem that there is a distribution $\pi \in \mathscr{D}'(M)$ with $\mu = \nabla\pi$.

Since φ is given by a measure $\mu \in C_0(M)'$ it follows from Théorème XV of §6 in Chapter VI of [**Sch66**] and the remarks thereafter that π is a regular distribution, i. e. $\pi \in L_{1,\mathrm{loc}}(M)$, and hence $\pi \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ by definition, see e. g. Chapter 9 of [**Maz85**].

By Theorem 9.1.2 and Lemma 9.1.2.2 of [**Maz85**] we can find a sequence $(\pi_k)_k$ of smooth functions with $\nabla\pi_k \in L_1(M)$ and $\lim_{k\to\infty} \|\nabla\pi_k\|_{L_1(M)} = \|\mu\|_{\mathrm{Var}}$ converging to π in $L_{1,\mathrm{loc}}(M)$

such that for any function $f \in C_0(M)$ we have

$$\lim_{k\to\infty} \int_M f \nabla \pi_k = \int_M f \, \mathrm{d}\mu. \tag{3}$$

Indeed, if $f \in C_0(M)$ then for any given $\varepsilon > 0$ we find $f^\varepsilon \in \mathscr{D}(M)$ with $\|f - f^\varepsilon\|_{C_0(M)} < \varepsilon$, and with $\mu_k = \nabla \pi_k$ we can compute

$$\begin{aligned}\int_M f \, \mathrm{d}\mu - \int_M f \nabla \pi_k \, \mathrm{d}\lambda &= \int_M f - f^\varepsilon \, \mathrm{d}\mu - \int_M (f - f^\varepsilon) \nabla \pi_k \, \mathrm{d}\lambda + \int_M f^\varepsilon \, \mathrm{d}(\mu - \mu_k) \\ &= \int_M f - f^\varepsilon \, \mathrm{d}\mu - \int_M (f - f^\varepsilon) \nabla \pi_k \, \mathrm{d}\lambda - \int_M (\operatorname{div} f^\varepsilon)(\pi - \pi_k) \, \mathrm{d}\lambda\end{aligned}$$

where λ denotes Lebesgue measure. Hence we can estimate

$$\left| \int_M f \, \mathrm{d}\mu - \int_M f \nabla \pi_k \, \mathrm{d}\lambda \right| \le \varepsilon \|\mu\|_{\mathrm{Var}} + \varepsilon \|\nabla \pi_k\|_{L_1(M)} + \|\operatorname{div} f^\varepsilon\|_{C_0(M)} \|\pi - \pi_k\|_{L_1(\operatorname{supp} f^\varepsilon)}.$$

The last term converges to zero as $k \to \infty$ since π_k converges to π in $L_{1,\mathrm{loc}}(M)$, and we obtain

$$\limsup_{k\to\infty} \left| \int_M f \, \mathrm{d}\mu - \int_M f \nabla \pi_k \, \mathrm{d}\lambda \right| \le 2\varepsilon \|\mu\|_{\mathrm{Var}}.$$

Since $\varepsilon > 0$ was chosen arbitrarily we obtain (3).

If $f \in C_0(M) \cap L_{\infty,\sigma}(M)$ then for any such π_k we have

$$\int_M f \nabla \pi_k \, \mathrm{d}\lambda = 0$$

and due to (3) we obtain

$$\int_M f \, \mathrm{d}\mu = 0.$$

This shows that the functional φ given by the measure μ vanishes on $C_0(M) \cap L_{\infty,\sigma}(M)$. □

We will need the following Lemma concerning approximation of functions with integrable gradient in order to improve upon Lemma 2.2

LEMMA 2.3. *Let $M \subset \mathbb{R}^n$ a domain and $\phi \in L_{1,loc}(M)$ such that $\nabla \phi \in L_1(M)$. Let $N \subset M$ a domain such that* $\operatorname{dist}(N, \partial M) > 0$. *Then there is a sequence $(\phi_k)_k \subset \mathscr{D}(M)$ such that $\nabla \phi_k \to \nabla \phi$ in $L_1(N)$.*

PROOF. Let $\eta \in \mathscr{D}(\mathbb{R}^n)$ such that $\eta(x) = 1$ for $|x| \le 1$ and $\eta(x) = 0$ for $|x| \ge 2$. For any integer $k \in \mathbb{N}$ let $A_k = \{x \in M \colon k < |x| < 2k\}$, $B_{2k} = \{x \in M \colon |x| < 2k\}$, and $\eta_k = \eta(\cdot/k)$. Then $\eta_k(x) = 1$ for $|x| \le k$ and $\eta_k(x) = 0$ for $|x| \ge 2k$. The derivative of η_k satisfies

$$|\nabla \eta_k(x)| \le C k^{-1} \chi_{A_k}(x).$$

We define a sequence $(c_k)_k \subset \mathbb{C}$ via

$$c_k = \frac{1}{\lambda(A_k)} \int_{A_k} \phi.$$

Define $f_k = \eta_k(\phi - c_k)$. Then $f_k \in L_{1,\mathrm{loc}}(M)$ with $\operatorname{supp} f_k \subset M \cap B_{2k}$ and

$$\nabla f_k = \nabla \eta_k (\phi - c_k) + \eta_k \nabla \phi.$$

We use Poincaré's inequality to estimate

$$\|\nabla f_k\|_{L_1(N)} \le \|\nabla \phi\|_{L_1(N)} + C k^{-1} \|\phi - c_k\|_{L_1(A_k)} \le C \|\nabla \phi\|_{L_1(M)}$$

since the diameter of A_k is at most $2k$. Furthermore, we can use Poincaré's inequality again to obtain

$$\|\nabla f_k - \nabla \phi\|_{L_1(N)} \le \|(1-\eta_k)\nabla\phi\|_{L_1(N)} + C\,\|\nabla\phi\|_{L_1(A_k)}\,.$$

It follows from Lebesgue's Dominated Convergence Theorem that the right hand side converges to zero for $k \to \infty$.

Now take $\psi_k \in \mathscr{D}(M)$ such that $\eta \equiv 1$ on $N \cap B_{2k}$ for every $k \in \mathbb{N}$, and let $(\rho_\varepsilon)_{\varepsilon>0}$ denote a standard mollifier. For $\varepsilon > 0$ sufficiently small the function

$$\varphi_{k,\varepsilon} = (\psi_k f_k) * \rho_\varepsilon$$

is containded in $\mathscr{D}(M)$. For any $k \in \mathbb{N}$ there is $\varepsilon_k > 0$ sufficiently small such that $\varphi_{k,\varepsilon_k} \in \mathscr{D}(M)$ and

$$\|\nabla\varphi_{k,\varepsilon_k} - \nabla f_k\|_{L_1(N)} \le k^{-1}.$$

Define $\phi_k = \varphi_{k,\varepsilon_k}$. Then $\phi_k \in \mathscr{D}(M)$ and

$$\|\nabla\phi_k - \nabla\phi\|_{L_1(N)} \le k^{-1} + \|\nabla f_k - \nabla\phi\|_{L_1(N)}$$

and since the right hand side converges to zero for $k \to \infty$ the assertion follows. □

Finally we are able to prove the following characterisation of $C_{0,\sigma}(M)$ for layer domains. Our proof is essentially an elaboration of the arguments in Lemma 6.1 of [**AG13**].

PROPOSITION 2.4. *For any layer domain $M = \mathbb{R}^{n-1} \times (0,\delta)$ a function f is contained in $C_{0,\sigma}(M)$ if and only if $f \in C_0(M)$ and $\operatorname{div} f = 0$ in $\mathscr{D}'(M)$.*

PROOF. For the sake of notational simplicity we assume $M = \mathbb{R}^{n-1} \times (-1,1)$. Let E denote the vector space consisting of all $f \in C_0(M)$ with $\operatorname{div} f = 0$ in the sense of distributions, i. e.

$$\int_M f\nabla\phi = 0$$

for all $\phi \in \mathscr{D}(M)$. In light of Lemma 2.2 it remains to show that any function in $f \in E$ satisfies

$$\int_M f\nabla\phi = 0$$

for every $\phi \in L_{1,\mathrm{loc}}(M)$ with $\nabla\phi \in L_1(M)$.

First let us assume that $f \in C_0(M)$, extended by zero to a function in $C_0(\mathbb{R}^n)$, satisfies $\operatorname{div} f = 0$ in $\mathscr{D}'(\mathbb{R}^n)$. Take $\phi \in L_{1,\mathrm{loc}}(M)$ with $\nabla\phi \in L_1(M)$. Given $\lambda \ge 1$ let $f_\lambda(x) = f(\lambda x)$. Then $f_1 = f$ and for $\lambda > 1$ we have $\operatorname{supp} f_\lambda \subset \mathbb{R}^{n-1} \times [-\lambda^{-1}, \lambda^{-1}]$ and $\operatorname{div} f_\lambda = 0$ in $\mathscr{D}'(\mathbb{R}^n)$.

Let $\lambda > 1$. By Lemma 2.3 we can find a sequence $(\phi_k)_k \subset \mathscr{D}(M)$ such that $\nabla\phi_k \to \nabla\phi$ in $L_1(\operatorname{supp} f_\lambda)$. Then we have for $k \in \mathbb{N}$

$$\int_M f_\lambda \nabla\phi_k = 0$$

and thus we can estimate

$$\begin{aligned}\left|\int_M f_\lambda\nabla\phi\right| &\le \|f_\lambda\|_{L_\infty(M)}\,\|\nabla\phi - \nabla\phi_k\|_{L_1(\operatorname{supp} f_\lambda)}\\ &\le \|f\|_{L_\infty(M)}\,\|\nabla\phi - \nabla\phi_k\|_{L_1(\operatorname{supp} f_\lambda)}\,.\end{aligned}$$

The right hand side converges to zero for $k \to \infty$ and thus

$$\int_M f_\lambda\nabla\phi = 0.$$

Now it follows from Lebesgue's Dominated Convergence Theorem that

$$\int_M f\nabla\phi = \lim_{\lambda\to 1^+}\int_M f_\lambda \nabla\phi = 0$$

and thus $f \in C_0(M)\cap L_{\infty,\sigma}(M)$. Now the assertion follows from Lemma 2.2.

It remains to show that any $f\in C_0(M)$ with $\operatorname{div} f = 0$ in $\mathscr{D}'(M)$ also satisfies $\operatorname{div} f = 0$ in $\mathscr{D}'(\mathbb{R}^n)$, but this follows at once from the Gauss-Green formula. Indeed, for any $\phi\in\mathscr{D}(\mathbb{R}^d)$ we have

$$\int_M f\nabla\phi = \int_{\partial M}\phi f\cdot\nu$$

and since f vanishes on ∂M we see that $\langle f,\nabla\phi\rangle = 0$ for any $\phi\in\mathscr{D}(\mathbb{R}^n)$. □

The assertion of Proposition 2.4 holds for a considerably larger class of domains. In fact, the proof of Proposition 2.4 can be copied verbatim whenever M is star-shaped (without loss of generality with respect to $0\in M$) and $\lambda^{-1}M = \{m/\lambda\colon m\in M\}$ satisfies $\operatorname{dist}(\lambda^{-1}M,\partial M) > 0$ whenever $\lambda > 1$. This holds e. g. for bounded star-shaped domains M with sufficiently smooth boundary.

One can employ an approximation procedure to show that Proposition 2.4 remains true for general bounded domains M with Lipschitz boundary. This has been done in [**AG13**], and the same result has been obtained with a different method in [**Mar09**]. For exterior domains M with sufficiently smooth boundary an analogous characterisation of $C_{0,\sigma}(M)$ has been proved in [**AG14**].

3. Estimates for the velocity u and the pressure θ

In this section we will show Theorem 1.1 and Theorem 1.2. In Chapter II we derived for (u,θ) the representation formula II.1.3 which states that we can write u as $u = v + w$ where w is the solution to the Helmholtz equation

$$\begin{cases} \lambda w - \Delta w = f & \text{in } \Omega\\ \hphantom{\lambda w - \Delta} w = 0 & \text{on } \partial\Omega\end{cases}$$

and $v = (v_1,\dots,v_n)$ is given by

$$\begin{aligned}\hat v_n(\xi',x_n) = &\Big\{\varphi(\delta)e^{-zx_n} + \varphi(\delta)e^{-z(\delta-x_n)} - (1+e^{-\delta z})(\varphi(x_n)+\varphi(\delta-x_n))\Big\}\,\mathbf{k}_1\\ &+\Big\{\varphi(\delta)(e^{-zx_n}-e^{-z(\delta-x_n)}) + (1-e^{-\delta z})(\varphi(x_n)-\varphi(\delta-x_n))\Big\}\,\mathbf{k}_2\end{aligned}$$

and for $j = 1,\dots,n-1$

$$\hat v_j(\xi',x_n) = -(1+e^{-\omega\delta})\Psi_-(x_n)\left[i\frac{\xi_j}{|\xi|}\mathbf{k}_1\right] + (1-e^{-\omega\delta})\Psi_+(x_n)\left[i\frac{\xi_j}{|\xi|}\mathbf{k}_2\right].$$

The pressure θ admits the representation

$$\hat\theta(\xi',x_n) = \frac{z+\omega}{z}\Big\{(1+e^{-\omega\delta})(e^{-zx_n}-e^{-z(\delta-x_n)})\mathbf{k}_1 - (1-e^{-\omega\delta})(e^{-zx_n}+e^{-z(\delta-x_n)})\mathbf{k}_2\Big\}.$$

Here

$$\Psi_\pm(x_n) = -\varphi(x_n)\mp\varphi(\delta-x_n)\pm\varphi(\delta)\frac{1}{1\pm e^{-\delta\omega}}\left(e^{-\omega x_n}\pm e^{-\omega(\delta-x_n)}\right)$$

as well as

$$\mathbf{k}_1 = \frac12\frac{\partial_n\hat w_n(\xi',\delta)-\partial_n\hat w_n(\xi',0)}{\Phi_-(z,\omega)}\qquad \mathbf{k}_2 = \frac12\frac{\partial_n\hat w_n(\xi',\delta)+\partial_n\hat w_n(\xi',0)}{\Phi_+(z,\omega)}$$

and

$$\Phi_\pm(z,\omega) = 1 - e^{-(\omega+z)\delta} \pm (\omega+z)\varphi(\delta)$$

with $\varphi(t) = \varphi(t,z,\lambda)$ given by

$$\varphi(t) = \frac{e^{-\omega t} - e^{-zt}}{\omega - z}.$$

In order to estimate the functions v and θ we need to be able to estimate the absolute value of $\Phi_\pm$ from below. The relevant results have been shown in Lemma II.1.4.

Combining the estimates for $\partial_n w_n$ from Proposition I.6.1 with those for $\Phi_\pm$ in Lemma II.1.4 we are able to show the following estimates for $\mathbf{k}_1$ and $\mathbf{k}_2$. To this end let ψ_0 denote a smooth cut-off function with $0 \le \psi_0 \le 1$ on $[0,\infty)$ such that $\psi_0 = 1$ on $[0,1]$ and $\psi_0 = 0$ on $[2,\infty)$. In addition let $\psi_\infty = 1 - \psi_0$.

LEMMA 3.1. *Under the assumptions of Theorem 1.1 we have the following estimates for* $\mathbf{k}_1$ *and* $\mathbf{k}_2$. *We have*

$$\left\|\mathcal{F}^{-1}\mathbf{k}_1\right\|_{\dot{B}^s_{\infty,\infty}(\mathbb{R}^{n-1})} \lesssim |\lambda|^{-\frac{1-s}{2}} \|f\|_{L_\infty(\Omega)}$$

for $-1 \le s \le 1$ *and*

$$\left\|\mathcal{F}^{-1}\mathbf{k}_2\right\|_{\dot{B}^s_{\infty,\infty}(\mathbb{R}^{n-1})} \lesssim |\lambda|^{-\frac{1-s}{2}} \|f\|_{L_\infty(\Omega)}$$

for $0 \le s \le 1$, *both uniformly in* λ *and* $f \in L_\infty(\Omega)$.

PROOF. We begin with the estimates for $\mathbf{k}_1$. It follows from Proposition I.3.11 and the estimates in Lemma II.1.4 that $\Phi_-(z,\omega)^{-1}$ is a Fourier multiplier on homogeneous Besov spaces of norm $\lesssim 1$. Combining this with the estimates for $\partial_n w_n$ from Proposition I.6.1 and interpolation of homogeneous Besov spaces (Proposition I.3.2) the first assertion follows.

In order to show the second assertion we will decompose $\mathbf{k}_2$ into a low frequency part and a part with Fourier transform vanishing in a neighbourhood of zero. We write

$$\mathbf{k}_2 = \mathbf{k}_2^0 + \mathbf{k}_2^\infty = \psi_0\mathbf{k}_2 + \psi_\infty\mathbf{k}_2.$$

It follows as above that $\mathbf{k}_2^\infty$ satisfies the estimate

$$\left\|\mathcal{F}^{-1}\mathbf{k}_2^\infty\right\|_{\dot{B}^s_{\infty,\infty}(\mathbb{R}^{n-1})} \lesssim |\lambda|^{-\frac{1-s}{2}} \|f\|_{L_\infty(\Omega)}$$

for $-1 \le s \le 1$. In order to estimate the low frequency part we write

$$\mathbf{k}_2^0 = \frac{1}{2}\frac{z\psi_0}{\Phi_+(z,\omega)}\frac{\partial_n\hat{w}_n(\xi',\delta) + \partial_n\hat{w}_n(\xi',0)}{z}$$

It follows from Proposition I.3.11 and the estimates in Lemma II.1.4 that $z\psi_0\Phi_+(z,\omega)^{-1}$ is a Fourier multiplier on the homogeneous Besov spaces $\dot{B}^s_{\infty,\infty}(\mathbb{R}^{n-1})$. Now it follows from the lifting property of homogeneous Besov spaces as stated in Theorem 5.2.3.1 of [**Tri83**] and Proposition I.6.1 that $\mathbf{k}_2^0$ satisfies the estimate

$$\left\|\mathcal{F}^{-1}\mathbf{k}_2^0\right\|_{\dot{B}^s_{\infty,\infty}(\mathbb{R}^{n-1})} \lesssim |\lambda|^{-\frac{1-s}{2}} \|f\|_{L_\infty(\Omega)}$$

for $s = 0, 1$ and then, by Proposition I.3.2, also for $0 \le s \le 1$. Combining the estimates for $\mathbf{k}_2^0$ and $\mathbf{k}_2^\infty$ the second assertion follows. □

This Lemma enables us to show the following results.

PROPOSITION 3.2. *Under the assumptions of Theorem 1.1, for $\lambda \in \Sigma_\rho$ with $|\lambda| \geq \lambda_0$ and $f \in C^\infty_{c,\sigma}(\Omega)$ the normal component v_n given by the representation formula II.1.3 is contained in $C_0(\Omega)$ and satisfies the estimates*

$$|\lambda| \|v_n\|_{L_\infty(\Omega)} + |\lambda|^{1/2} \|\nabla v_n\|_{L_\infty(\Omega)} + \sup_{x_n} \left\|\nabla^2 v_n(\cdot, x_n)\right\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1})} \lesssim \|f\|_{L_\infty(\Omega)}.$$

PROOF. We use the representation formula II.1.3, and decompose v_n into a low frequency part and a part with Fourier transform vanishing in a neighbourhood of zero via

$$\hat{v}_n = \hat{v}_n^0 + \hat{v}_n^\infty = \psi_0 \hat{v}_n + \psi_\infty \hat{v}_n.$$

We can write the low frequency part v_n^0 as

$$\begin{aligned}\hat{v}_n^0(\xi', x_n) = \psi_0 \Big\{\varphi(\delta)(e^{-zx_n} + e^{-z(\delta - x_n)}) - (1 + e^{-\delta z})(\varphi(x_n) + \varphi(\delta - x_n))\Big\} \mathbf{k}_1 \\ + \left\{\varphi(\delta)\frac{e^{-zx_n} - e^{-z(\delta-x_n)}}{z} + \frac{1 - e^{-\delta z}}{z}(\varphi(x_n) - \varphi(\delta - x_n))\right\} z\mathbf{k}_2^0\end{aligned}$$

and now Lemma I.4.1 and Lemma I.5.1 enable us estimate $v_n^0(\cdot, x_n)$ in $\dot{B}^{-1}_{\infty,\infty(\mathbb{R}^{n-1})}$. First of all we have

$$\begin{aligned}\left\|v_n^0(\cdot, x_n)\right\|_{\dot{B}^{-1}_{\infty,\infty(\mathbb{R}^{n-1})}} \lesssim& \left\|\mathcal{F}^{-1}\varphi(\delta)(e^{-zx_n} + e^{-z(\delta-x_n)})\mathbf{k}_1\right\|_{\dot{B}^{-1}_{\infty,\infty(\mathbb{R}^{n-1})}} \\ &+ \left\|\mathcal{F}^{-1}(1 + e^{-\delta z})(\varphi(x_n) + \varphi(\delta - x_n))\mathbf{k}_1\right\|_{\dot{B}^{-1}_{\infty,\infty(\mathbb{R}^{n-1})}} \\ &+ \left\|\mathcal{F}^{-1}\varphi(\delta)\frac{e^{-zx_n} - e^{-z(\delta-x_n)}}{z} z\mathbf{k}_2^0\right\|_{\dot{B}^{-1}_{\infty,\infty(\mathbb{R}^{n-1})}} \\ &+ \left\|\mathcal{F}^{-1}\frac{1 - e^{-\delta z}}{z}(\varphi(x_n) - \varphi(\delta - x_n))z\mathbf{k}_2^0\right\|_{\dot{B}^{-1}_{\infty,\infty(\mathbb{R}^{n-1})}}.\end{aligned}$$

Immediately from Lemma I.4.1 and Lemma I.5.1 we obtain the estimates

$$\begin{aligned}|\lambda|^{1/2} \left\|v_n^0(\cdot, x_n)\right\|_{\dot{B}^{-1}_{\infty,\infty(\mathbb{R}^{n-1})}} \lesssim& \left\|\mathcal{F}^{-1}\mathbf{k}_1\right\|_{\dot{B}^{-1}_{\infty,\infty(\mathbb{R}^{n-1})}} + \left\|\mathcal{F}^{-1}\frac{e^{-zx_n} - e^{-z(\delta-x_n)}}{z} z\mathbf{k}_2^0\right\|_{\dot{B}^{-1}_{\infty,\infty(\mathbb{R}^{n-1})}} \\ &+ \left\|\mathcal{F}^{-1}\frac{1 - e^{-\delta z}}{z} z\mathbf{k}_2^0\right\|_{\dot{B}^{-1}_{\infty,\infty(\mathbb{R}^{n-1})}}\end{aligned}$$

uniformly in x_n and λ. Now we can use the fourth assertion of Lemma I.4.1 in the case $\lambda = 0$, i. e. $\omega = z$, with $t = 0$ and $t = x_n$, respectively, to obtain

$$(1 + |\lambda|^{1/2}) \left\|v_n^0(\cdot, x_n)\right\|_{\dot{B}^{-1}_{\infty,\infty(\mathbb{R}^{n-1})}} \lesssim \left\|\mathcal{F}^{-1}\mathbf{k}_1\right\|_{\dot{B}^{-1}_{\infty,\infty(\mathbb{R}^{n-1})}} + \left\|\mathcal{F}^{-1} z\mathbf{k}_2^0\right\|_{\dot{B}^{-1}_{\infty,\infty(\mathbb{R}^{n-1})}}.$$

At this point we can use the lifting property of homogeneous Besov spaces (Theorem 5.2.3.1 of [**Tri83**]) and the estimates from Lemma 3.1 to obtain

$$\begin{aligned}(1 + |\lambda|^{1/2}) \left\|v_n^0(\cdot, x_n)\right\|_{\dot{B}^{-1}_{\infty,\infty(\mathbb{R}^{n-1})}} &\lesssim |\lambda|^{-1} \|f\|_{L_\infty(\Omega)} + \left\|\mathcal{F}^{-1}\mathbf{k}_2^0\right\|_{\dot{B}^0_{\infty,\infty(\mathbb{R}^{n-1})}} \\ &\lesssim |\lambda|^{-1} \|f\|_{L_\infty(\Omega)} + |\lambda|^{-1/2} \|f\|_{L_\infty(\Omega)}.\end{aligned}$$

In a similar fashion we can obtain the estimate

$$\begin{aligned}\left\|\partial_n v_n^0(\cdot, x_n)\right\|_{\dot{B}^{-1}_{\infty,\infty(\mathbb{R}^{n-1})}} &\lesssim \left\|\mathcal{F}^{-1}\mathbf{k}_1\right\|_{\dot{B}^{-1}_{\infty,\infty(\mathbb{R}^{n-1})}} + \left\|\mathcal{F}^{-1} z\mathbf{k}_2^0\right\|_{\dot{B}^{-1}_{\infty,\infty(\mathbb{R}^{n-1})}} \\ &\lesssim |\lambda|^{-1} \|f\|_{L_\infty(\Omega)} + |\lambda|^{-1/2} \|f\|_{L_\infty(\Omega)}.\end{aligned}$$

For the high frequency part the estimates

$$(1+|\lambda|^{1/2}) \|v_n^\infty(\cdot,x_n)\|_{\dot B^{-1}_{\infty,\infty}(\mathbb{R}^{n-1})} \lesssim \left\|\mathcal{F}^{-1}\mathbf{k}_1\right\|_{\dot B^{-1}_{\infty,\infty}(\mathbb{R}^{n-1})} + \left\|\mathcal{F}^{-1}\mathbf{k}_2^\infty\right\|_{\dot B^{-1}_{\infty,\infty}(\mathbb{R}^{n-1})}$$
$$\lesssim |\lambda|^{-1}\|f\|_{L_\infty(\Omega)}$$

follow from Lemma I.4.1 and Lemma I.5.1 and the last inequality follows from Lemma 3.1. Similarly we obtain

$$\|\partial_n v_n^\infty(\cdot,x_n)\|_{\dot B^{-1}_{\infty,\infty}(\mathbb{R}^{n-1})} \lesssim |\lambda|^{-1}\|f\|_{L_\infty(\Omega)}.$$

This shows

$$|\lambda|^{3/2}\|v_n(\cdot,x_n)\|_{\dot B^{-1}_{\infty,\infty}(\mathbb{R}^{n-1})} + |\lambda|\,\|\partial_n v_n(\cdot,x_n)\|_{\dot B^{-1}_{\infty,\infty}(\mathbb{R}^{n-1})} \lesssim \|f\|_{L_\infty(\Omega)}$$

for $0<x_n<\delta$. Using Lemma I.4.1, Lemma I.5.1 and Lemma 3.1 the following estimates can be read off directly from the representation formula for v_n:

$$\|v_n(\cdot,x_n)\|_{\dot B^{2}_{\infty,\infty}(\mathbb{R}^{n-1})} \lesssim \|f\|_{L_\infty(\Omega)}$$
$$\left\|\partial_n^2 v_n(\cdot,x_n)\right\|_{\dot B^{0}_{\infty,\infty}(\mathbb{R}^{n-1})} \lesssim \|f\|_{L_\infty(\Omega)}$$
$$\|\partial_n v_n(\cdot,x_n)\|_{\dot B^{1}_{\infty,\infty}(\mathbb{R}^{n-1})} \lesssim \|f\|_{L_\infty(\Omega)}.$$

Now the assertion follows from interpolation theory for Besov spaces, i. e. Proposition I.3.2. □

The proof above shows actually more than we stated in Proposition 3.2 since the interpolation results in Proposition I.3.2 in fact yields estimates in the smaller space $\dot B^0_{\infty,1}$. What we actually just proved is

$$|\lambda|\,\|v_n(\cdot,x_n)\|_{\dot B^0_{\infty,1}(\mathbb{R}^{n-1})} + |\lambda|^{1/2}\|\nabla v_n(\cdot,x_n)\|_{\dot B^0_{\infty,1}(\mathbb{R}^{n-1})} + \left\|\nabla^2 v_n(\cdot,x_n)\right\|_{\dot B^0_{\infty,\infty}(\mathbb{R}^{n-1})} \lesssim \|f\|_{L_\infty(\Omega)}$$

uniformly in $0<x_n<\delta$ and λ.

PROPOSITION 3.3. *Under the assumptions of Theorem 1.1, for $\lambda\in\Sigma_\rho$ with $|\lambda|\geq\lambda_0$ and $f\in C^\infty_{c,\sigma}(\Omega)$ the pressure θ given by the representation formula II.1.3 satisfies the estimate*

$$\sup_{x_n}\|\nabla\theta(\cdot,x_n)\|_{\dot B^0_{\infty,\infty}(\mathbb{R}^{n-1})} \lesssim \|f\|_{L_\infty(\Omega)}.$$

PROOF. It follows immediately from the representation formula II.1.3, boundedness of the Riesz transforms on homogeneous Besov spaces, the lifting property, and the estimates in Lemma 4.1 and Lemma 5.1 of Chapter I that we have for $j=1,\dots,n-1$ an estimate

$$\|\partial_j\theta(\cdot,x_n)\|_{\dot B^0_{\infty,\infty}(\mathbb{R}^{n-1})} \lesssim \left\|\mathcal{F}^{-1}(\omega+z)\mathbf{k}_1\right\|_{\dot B^0_{\infty,\infty}(\mathbb{R}^{n-1})} + \left\|\mathcal{F}^{-1}(\omega+z)\mathbf{k}_2\right\|_{\dot B^0_{\infty,\infty}(\mathbb{R}^{n-1})}$$
$$\lesssim |\lambda|^{1/2}\left\|\mathcal{F}^{-1}\mathbf{k}_1\right\|_{\dot B^0_{\infty,\infty}(\mathbb{R}^{n-1})} + \left\|\mathcal{F}^{-1}\mathbf{k}_1\right\|_{\dot B^1_{\infty,\infty}(\mathbb{R}^{n-1})}$$
$$+ |\lambda|^{1/2}\left\|\mathcal{F}^{-1}\mathbf{k}_2\right\|_{\dot B^0_{\infty,\infty}(\mathbb{R}^{n-1})} + \left\|\mathcal{F}^{-1}\mathbf{k}_2\right\|_{\dot B^1_{\infty,\infty}(\mathbb{R}^{n-1})}$$
$$\lesssim \|f\|_{L_\infty(\Omega)}$$

where the last inequality is a consequence of Lemma 3.1. It remains to estimate

$$\partial_n\hat\theta(\xi',x_n) = -(z+\omega)\left\{(1+e^{-\omega\delta})(e^{-zx_n}+e^{-z(\delta-x_n)})\mathbf{k}_1 - (1-e^{-\omega\delta})(e^{-zx_n}-e^{-z(\delta-x_n)})\mathbf{k}_2\right\}$$

and in the same way as above we obtain

$$\|\partial_n\theta(\cdot,x_n)\|_{\dot B^0_{\infty,\infty}(\mathbb{R}^{n-1})} \lesssim \|f\|_{L_\infty(\Omega)}.$$

□

In a similar way we can estimate the tangential components $v_1, \dots, v_{n-1}$.

PROPOSITION 3.4. *Under the assumptions of Theorem 1.1, for $\lambda \in \Sigma_\rho$ with $|\lambda| \geq \lambda_0$ and $f \in C^\infty_{c,\sigma}(\Omega)$ the components $v_1, \dots, v_{n-1}$ given by the representation formula II.1.3 satisfy the estimate*

$$|\lambda| \sup_{x_n} \|v_j(\cdot, x_n)\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1})} + |\lambda|^{1/2} \sup_{x_n} \|\nabla v_j(\cdot, x_n)\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1})} + \sup_{x_n} \left\|\nabla^2 v_j(\cdot, x_n)\right\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1})} \lesssim \|f\|_{L_\infty(\Omega)} .$$

If $n = 2$ then the improved estimate

$$|\lambda| \|v_j\|_{L_\infty(\Omega)} + |\lambda|^{1/2} \|\nabla v_j\|_{L_\infty(\Omega)} + \sup_{x_n} \left\|\nabla^2 v_j(\cdot, x_n)\right\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1})} \lesssim \|f\|_{L_\infty(\Omega)}$$

holds.

PROOF. We begin with the first assertion. As in the proof of Proposition 3.2 we can use the representation formula II.1.3 and the estimates proved in Lemma I.4.1, Lemma I.5.1, and Lemma 3.1 to show the estimates

$$\begin{aligned}|\lambda| \|v_j(\cdot, x_n)\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1})} &\lesssim |\lambda|^{1/2} \left\|\mathcal{F}^{-1}\mathbf{k}_1\right\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1})} + |\lambda|^{1/2} \left\|\mathcal{F}^{-1}\mathbf{k}_2\right\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1})} \\ &\lesssim \|f\|_{L_\infty(\Omega)}\end{aligned}$$

as well as

$$\begin{aligned}\|v_j(\cdot, x_n)\|_{\dot{B}^2_{\infty,\infty}(\mathbb{R}^{n-1})} &\lesssim \left\|\mathcal{F}^{-1}\mathbf{k}_1\right\|_{\dot{B}^1_{\infty,\infty}(\mathbb{R}^{n-1})} + \left\|\mathcal{F}^{-1}\mathbf{k}_2\right\|_{\dot{B}^1_{\infty,\infty}(\mathbb{R}^{n-1})} \lesssim \|f\|_{L_\infty(\Omega)} \\ |\lambda|^{1/2} \|\partial_n v_j(\cdot, x_n)\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1})} &\lesssim |\lambda|^{1/2} \left\|\mathcal{F}^{-1}\mathbf{k}_1\right\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1})} + |\lambda|^{1/2} \left\|\mathcal{F}^{-1}\mathbf{k}_2\right\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1})} \\ &\lesssim \|f\|_{L_\infty(\Omega)}\end{aligned}$$

and

$$\|\partial_n v_j(\cdot, x_n)\|_{\dot{B}^1_{\infty,\infty}(\mathbb{R}^{n-1})} \lesssim \left\|\mathcal{F}^{-1}\mathbf{k}_1\right\|_{\dot{B}^1_{\infty,\infty}(\mathbb{R}^{n-1})} + \left\|\mathcal{F}^{-1}\mathbf{k}_2\right\|_{\dot{B}^1_{\infty,\infty}(\mathbb{R}^{n-1})} \lesssim \|f\|_{L_\infty(\Omega)} .$$

For $\partial_n^2 \hat{v}_j = \omega^2 \hat{v}_j - i\xi_j \hat{\theta}$ this shows the estimates

$$\begin{aligned}\left\|\partial_n^2 v_j(\cdot, x_n)\right\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1})} &\lesssim |\lambda| \|v_j(\cdot, x_n)\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1})} + \|v_j(\cdot, x_n)\|_{\dot{B}^2_{\infty,\infty}(\mathbb{R}^{n-1})} \\ &\quad + \|\partial_j \theta(\cdot, x_n)\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1})} \\ &\lesssim \|f\|_{L_\infty(\Omega)}\end{aligned}$$

where the last inequality is a consequence of the estimates for v_j we showed above and Proposition 3.3. Estimates for $\partial_1 v_j, \dots, \partial_{n-1} v_j$ can be derived via interpolation from those for v_j by Proposition I.3.2:

$$\|\nabla' v_j(\cdot, x_n)\|_{L_\infty(\mathbb{R}^{n-1})} \lesssim \|v_j(\cdot, x_n)\|^{1/2}_{\dot{B}^2_{\infty,\infty}(\mathbb{R}^{n-1})} \|v_j(\cdot, x_n)\|^{1/2}_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1})} \lesssim |\lambda|^{-1/2} \|f\|_{L_\infty(\Omega)} .$$

All in all we obtain the first assertion.

We turn to the case of a two-dimensional layer domain, i. e. the case $n = 2$. In order to show the second assertion it remains to estimate v_j and $\partial_n v_j$ in $L_\infty(\Omega)$. First observe that if $\mathbf{k}_2$ were zero, then we could use Lemma 3.1 to show estimates for $v_j(\cdot, x_n)$ and $\partial_n v_j(\cdot, x_n)$ in $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^{n-1})$ and then the second assertion would be a consequence of the interpolation results in Proposition I.3.2 as we would have the estimates

$$\begin{aligned}\|v_j(\cdot, x_n)\|_{L_\infty(\mathbb{R}^{n-1})} &\lesssim \|v_j(\cdot, x_n)\|^{1/2}_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^{n-1})} \|v_j(\cdot, x_n)\|^{1/2}_{\dot{B}^1_{\infty,\infty}(\mathbb{R}^{n-1})} \lesssim |\lambda|^{-1} \|f\|_{L_\infty(\Omega)} \\ \|\partial_n v_j(\cdot, x_n)\|_{L_\infty(\mathbb{R}^{n-1})} &\lesssim \|\partial_n v_j(\cdot, x_n)\|^{1/2}_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^{n-1})} \|\partial_n v_j(\cdot, x_n)\|^{1/2}_{\dot{B}^1_{\infty,\infty}(\mathbb{R}^{n-1})} \lesssim |\lambda|^{-1/2} \|f\|_{L_\infty(\Omega)} .\end{aligned}$$

Thus, in order to show the assertion concerning a two-dimensional layer it suffices to show that the parts of v_j and $\partial_n v_j$ involving $\mathbf{k}_2$ are in $L_\infty(\Omega)$. We will show

$$\mathcal{F}^{-1} i \frac{\xi_1}{|\xi'|} \mathbf{k}_2 \in L_\infty(\mathbb{R}^{n-1}).$$

This can be seen as follows. We can use integration by parts and the divergence-free condition to calculate

$$\begin{aligned} i\frac{\xi_1}{z}\partial_n \hat{w}_n(\xi',\delta) &= i\frac{\xi_1}{z}\int_0^\delta \frac{\sinh(\omega t)}{\sinh(\omega\delta)} \hat{f}_n(\xi',t)\,\mathrm{d}t \\ &= -\frac{\xi_1^2}{z\omega}\int_0^\delta \frac{\cosh(\omega t)}{\sinh(\omega\delta)} \hat{f}_1(\xi',t)\,\mathrm{d}t \\ &= -\frac{z}{\omega}\int_0^\delta \frac{\cosh(\omega t)}{\sinh(\omega\delta)} \hat{f}_1(\xi',t)\,\mathrm{d}t \end{aligned}$$

where the last equality is due to $z^2 = |\xi_1|^2 = \xi_1^2$. Now we can use Proposition I.3.12 to show

$$|\lambda| \left\| \mathcal{F}^{-1} i\frac{\xi_1}{z^2}\partial_n \hat{w}_n(\cdot,\delta) \right\|_{L_\infty(\mathbb{R}^{n-1})} + |\lambda|^{1/2} \left\| \mathcal{F}^{-1} i\frac{\xi_1}{z}\partial_n \hat{w}_n(\cdot,\delta) \right\|_{L_\infty(\mathbb{R}^{n-1})} \lesssim \|f\|_{L_\infty(\Omega)}.$$

Analogously one obtains the estimates

$$|\lambda| \left\| \mathcal{F}^{-1} i\frac{\xi_1}{z^2}\partial_n \hat{w}_n(\cdot,0) \right\|_{L_\infty(\mathbb{R}^{n-1})} + |\lambda|^{1/2} \left\| \mathcal{F}^{-1} i\frac{\xi_1}{z}\partial_n \hat{w}_n(\cdot,0) \right\|_{L_\infty(\mathbb{R}^{n-1})} \lesssim \|f\|_{L_\infty(\Omega)}.$$

We can write the low-frequency part $\mathbf{k}_2^0$ of $\mathbf{k}_2$ in the form

$$i\frac{\xi_1}{z}\mathbf{k}_2^0 = -\frac{1}{2}\frac{z\psi_0}{\Phi_+(z,\omega)}\left[i\frac{\xi_1}{z^2}\partial_n \hat{w}_n(\cdot,\delta) + i\frac{\xi_1}{z^2}\partial_n \hat{w}_n(\cdot,0) \right].$$

It follows from Proposition I.3.11 and Lemma II.1.4 that

$$\frac{z\psi_0}{\Phi_+(z,\omega)}, \qquad \frac{\psi_\infty}{\Phi_+(z,\omega)}$$

are Fourier multipliers on $L_\infty(\mathbb{R}^{n-1})$ of norm $\lesssim 1$ and thus we obtain the estimate

$$|\lambda| \left\| \mathcal{F}^{-1}\frac{\xi_1}{z}\mathbf{k}_2^0 \right\|_{L_\infty(\mathbb{R}^{n-1})} \lesssim \|f\|_{L_\infty(\Omega)}.$$

For the high-frequency part $\mathbf{k}_2^\infty$ of $\mathbf{k}_2$ we can write

$$i\frac{\xi_1}{z}\mathbf{k}_2^\infty = -\frac{1}{2}\frac{\psi_\infty}{\Phi_+(z,\omega)}\left[i\frac{\xi_1}{z}\partial_n \hat{w}_n(\cdot,\delta) + i\frac{\xi_1}{z}\partial_n \hat{w}_n(\cdot,0) \right]$$

and now the estimates above combined with Proposition I.3.12 and Lemma II.1.4 show the estimate

$$|\lambda|^{1/2} \left\| \mathcal{F}^{-1}\frac{\xi_1}{z}\mathbf{k}_2^\infty \right\|_{L_\infty(\mathbb{R}^{n-1})} \lesssim \|f\|_{L_\infty(\Omega)}.$$

Combining these estimates we obtain

$$|\lambda|^{1/2} \left\| \mathcal{F}^{-1}\frac{\xi_1}{z}\mathbf{k}_2 \right\|_{L_\infty(\mathbb{R}^{n-1})} \lesssim \|f\|_{L_\infty(\Omega)}.$$

It remains to show that $\Psi_+(x_n)$ and $\partial_n \Psi_+(x_n)$ are Fourier multipliers on L_∞, but this follows easily from Lemma I.4.1. □

Finally we are able to show our first two main results. This essentially boils down to collecting the estimates we have obtained in this section.

Proof of Theorem 1.1 and Theorem 1.2. Setting $u = v + w$ with v the solution constructed in Proposition 3.2 and Proposition 3.4 and w the solution of the corresponding Helmholtz equation we see that u satisfies, by construction, the Stokes resolvent problem (2). The divergence free condition is satisfied as well since $\operatorname{div} u$ satisfies the Helmholtz equation with zero data. Combining the estimates obtained for v and θ in Proposition 3.2, Proposition 3.4 and Proposition 3.3 with the estimates for w shown in Proposition I.6.1 the desired estimates for u and θ follow.

We show that u is not only in $L_\infty(\Omega)$ but actually in $C_{0,\sigma}(\Omega)$. Due to Proposition 2.4 it suffices to show that $u(x)$ vanishes for $x \to \infty$. An inspection of the proofs of Proposition 3.2 and Proposition 3.4 shows that for fixed $x_n \in (0, \delta)$ the solution operator $f \mapsto u(\cdot, x_n)$ is given by a bounded Fourier multiplier operator and thus by the Lemma of Riemann-Lebesgue we have $u(\cdot, x_n) \in C_0(\mathbb{R}^{n-1})$, and then also $u \in C_0(\Omega)$. Since $\operatorname{div} u$ vanishes by construction it follows from Proposition 2.4 that u is indeed contained in $C_{0,\sigma}(\Omega)$.

It remains to show that solutions are unique in the class $C_{0,\sigma}(\Omega)$. This follows from Theorem 1.5 which will be proved later on. □

4. Generation of a semigroup in the two-dimensional case

The estimates in Theorem 1.2 will allow us to show that, at least in the two-dimensional case, the Stokes operator generates a holomorphic semigroup of angle $\pi/2$ and thus prove Corollary 1.3. Given $\lambda \in \Sigma_\rho$ for some $0 < \rho < \pi$ let

$$R_\lambda \colon C^\infty_{c,\sigma}(\Omega) \to C_{0,\sigma}(\Omega), \qquad f \mapsto u$$

denote the solution operator constructed in Theorem 1.2. This operator extends to a bounded operator on $C_{0,\sigma}(\Omega)$ which we again denote by R_λ. A direct calculation shows that R_λ satisfies the resolvent identity

$$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu$$

whenever $\lambda, \mu \in \Sigma_\rho$ and thus R_λ is a pseudo-resolvent on $C_{0,\sigma}(\Omega)$. It follows from Theorem VIII.4.1 in [**Yos74**] that the null space of R_λ is independent of λ and that R_λ is the resolvent of a linear operator A precisely if the null space of R_λ is trivial. In this case the domain $D(A)$ of A coincides with the range of R_λ, which is independent of λ as well.

We show that the pseudo-resolvent $R_\lambda \colon C_{0,\sigma}(\Omega) \to C_{0,\sigma}(\Omega)$ has trivial kernel. To this end we will show in the following computations that $\lim_{\lambda\to\infty} \lambda R_\lambda f = f$ for $f \in C_{0,\sigma}(\Omega)$, with the limit to be understood in a sufficiently weak sense.

We start with the case $f \in C^\infty_{c,\sigma}(\Omega)$. From the representation formula II.1.3 we can see

$$\lambda R_\lambda f = \lambda R(\lambda, \Delta_D) f + \lambda v_\lambda.$$

Given that the Dirichlet-Laplacian Δ_D is sectorial on $C_0(\Omega)$ we know that $\lambda R(\lambda, \Delta_D) f$ converges in $C_0(\Omega)$ to f as $\lambda \to \infty$, and it remains to show that λv_λ vanishes for $\lambda \to \infty$.

We begin by taking a look at the behaviour of the boundary values of the normal derivative of $w_n = R(\lambda, \Delta_D) f_n$. We have

$$\begin{aligned}\partial_n \hat{w}_n(\xi, \delta) &= \int_0^\delta \frac{\sinh(\omega t)}{\sinh(\omega\delta)} \hat{f}_n(\xi, t)\, \mathrm{d}t \\ &= \int_r^{\delta - r} \frac{\sinh(\omega t)}{\sinh(\omega\delta)} \hat{f}_n(\xi, t)\, \mathrm{d}t\end{aligned}$$

for some $r > 0$ since f is compactly supported, and we obtain the estimate

$$|\partial_n \hat{w}_n(\xi,\delta)| \lesssim \left(\int_r^{\delta-r} \left|\frac{\sinh(\omega t)}{\sinh(\omega\delta)}\right| \,\mathrm{d}t\right) \sup_t \|f_n(\cdot,t)\|_{L_1(\mathbb{R}^{n-1})}$$
$$\lesssim e^{-cr|\lambda|^{1/2}} \sup_t \|f_n(\cdot,t)\|_{L_1(\mathbb{R}^{n-1})},$$

and similarly

$$|\partial_n \hat{w}_n(\xi,0)| \lesssim e^{-cr|\lambda|^{1/2}} \sup_t \|f_n(\cdot,t)\|_{L_1(\mathbb{R}^{n-1})}.$$

It follows that $\mathbf{k}_1, \mathbf{k}_2$ satisfy the same pointwise estimates, and thus also

$$|\hat{v}_\lambda(\xi,x_n)| \lesssim e^{-cr|\lambda|^{1/2}} \sup_t \|f_n(\cdot,t)\|_{L_1(\mathbb{R}^{n-1})}$$

for λ sufficiently large. Take a function $\psi \in C_c^\infty(\Omega)$. Then we have

$$\langle \psi, v_\lambda\rangle = \int_\Omega \psi(x) v_\lambda(x)\,\mathrm{d}x = \int_0^\delta \int_{\mathbb{R}^{n-1}} \hat{\psi}(\xi,x_n)\hat{v}_\lambda(\xi,x_n)\,\mathrm{d}\xi\,\mathrm{d}x_n$$

and thus

$$|\langle \psi, \lambda v_\lambda\rangle| \lesssim |\lambda|\,\|\hat{\psi}\|_{L_1(\Omega)}\,\|\hat{v}_\lambda\|_{L_\infty(\Omega)} \lesssim |\lambda|\, e^{-cr|\lambda|^{1/2}} \|\hat{\psi}\|_{L_1(\Omega)} \sup_t \|f_n(\cdot,t)\|_{L_1(\mathbb{R}^{n-1})}$$

and we see that $\lambda\langle\psi,v_\lambda\rangle$ converges to zero as $\lambda\to\infty$, i. e. λv_λ converges to zero in a distributional sense. Hence we see that $\lambda R_\lambda f$ converges to f in the sense of distributions for any $f \in C_{c,\sigma}^\infty(\Omega)$.

Now let $f \in C_{0,\sigma}(\Omega)$. For any given $\varepsilon > 0$ there is $f^\varepsilon \in C_{c,\sigma}^\infty(\Omega)$ with $\|f - f^\varepsilon\|_{L_\infty(\Omega)} < \varepsilon$, and thus for any $\psi \in C_c^\infty(\Omega)$ we have

$$|\langle \psi, f - \lambda R_\lambda f\rangle| \le |\langle\psi, f^\varepsilon - f\rangle| + |\langle\psi, f^\varepsilon - \lambda R_\lambda f^\varepsilon\rangle| + |\langle\psi, \lambda R_\lambda(f - f^\varepsilon)\rangle|$$
$$\lesssim \varepsilon\,\|\psi\|_{L_1(\Omega)} + |\langle\psi, f^\varepsilon - \lambda R_\lambda f^\varepsilon\rangle|.$$

We obtain by the above calculations

$$\limsup_{\lambda\to\infty} |\langle\psi, f - \lambda R_\lambda f\rangle| \lesssim \varepsilon\,\|\psi\|_{L_1(\Omega)}$$

and for $\varepsilon \to 0$ we see that $\lambda R_\lambda f$ converges to f for any $f \in C_{0,\sigma}(\Omega)$ at least in a distributional sense. Now assume f were in the null space of R_λ, i. e. $R_\lambda f = 0$. Then in particular

$$0 = \lim_{\lambda\to\infty} \langle\psi, \lambda R_\lambda f\rangle = \langle\psi, f\rangle$$

for any $\psi \in C_c^\infty(\Omega)$ and we see that f must vanish. In particular R_λ has trivial null space and thus there is an operator A such that $R_\lambda = R(\lambda, A)$ is the resolvent of A. We call this operator A the Stokes operator on $C_{0,\sigma}(\Omega)$. The domain $D(A)$ of the Stokes operator A coincides with the range of $R(\lambda, A)$, and since any function $u \in C_{c,\sigma}^\infty(\Omega)$ arises as the solution to (2) for some $f \in C_{c,\sigma}^\infty(\Omega)$ we can infer that $C_{c,\sigma}^\infty(\Omega) \subset D(A)$. This shows that the closure of $D(A)$ coincides with $C_{0,\sigma}(\Omega)$, which in turn shows that the Stokes operator in $C_{0,\sigma}(\Omega)$ is densely defined.

It follows from Theorem 1.2 that the Stokes operator on $C_{0,\sigma}(\Omega)$ is sectorial of angle π and thus generates a holomorphic and strongly continuous semigroup of angle $\pi/2$.

5. Construction of a counterexample in $n \geq 3$ dimensions

This section is devoted to a proof of Theorem 1.4, which is to say that we will construct a function $f \in C_{0,\sigma}(\Omega)$ such that the solution (u, θ) of (2) given by Theorem 1.1 is *not* contained in $L_\infty(\Omega)$. Since $u = w + v$ with $w = R(\lambda, \Delta_D)f \in L_\infty(\Omega)$ it is sufficient to construct a function f such that some component of v is unbounded. For simplicity we will take a function $f \in C_{0,\sigma}(\Omega)$ such that $f_n(x', \delta - x_n) = -f_n(x', x_n)$ for all $x \in \Omega$, i. e. we assume f_n to be antisymmetric with respect to the plane $\mathbb{R}^{n-1} \times \{\delta/2\}$. Then, in the notation of the previous sections, $\mathbf{k}_1$ vanishes. In particular we obtain as a representation for the tangential components $v_1, \ldots, v_{n-1}$

$$\hat{v}_j(\xi', x_n) = -(1 - e^{-\omega\delta})\Psi_+(x_n)\left[i\frac{\xi_j}{z}\mathbf{k}_2\right]$$

with

$$\mathbf{k}_2 = \frac{\partial_n \hat{w}_n(\xi', 0)}{\Phi_+(z, \omega)}$$

and the functions Φ_+, Ψ_+ given by

$$\begin{aligned}\Phi_+(z, \omega) &= 1 - e^{-(\omega+z)\delta} + (\omega + z)\varphi(\delta)\\ \Psi_+(x_n) &= -\varphi(x_n) - \varphi(\delta - x_n) + \varphi(\delta)\frac{1}{1 + e^{-\delta\omega}}\left(e^{-\omega x_n} + e^{-\omega(\delta - x_n)}\right).\end{aligned}$$

Assume v were in $L_\infty(\Omega)$. It follows from Theorem 1.1 that in that case $v \in \mathrm{BUC}^{1,s}(\Omega)$ for every $s \in (0, 1)$ and in particular that the restriction of v to the hyperplane $\mathbb{R}^{n-1} \times \{\delta/2\}$ would have to be in $L_\infty(\mathbb{R}^{n-1})$ as well. Similarly $\nabla v(\cdot, 0)$ would have to be in $L_\infty(\mathbb{R}^{n-1})$. If $\nabla\theta$ were in $L_\infty(\Omega)$ then the restriction to some hyperplane $\mathbb{R}^{n-1} \times \{x_n\}$ would have to be bounded as well. Thus we take for $j = 1, \ldots, n-1$ a closer look at

$$\begin{aligned}\hat{v}_j(\xi', \delta/2) &= -(1 - e^{-\omega\delta})\Psi_+(\delta/2)\left[i\frac{\xi_j}{z}\mathbf{k}_2\right]\\ \partial_n\hat{v}_j(\xi', 0) &= -(1 - e^{-\omega\delta})\partial_n\Psi_+(0)\left[i\frac{\xi_j}{z}\mathbf{k}_2\right]\\ i\xi_j\hat{\theta}(\xi', x_n) &= -2(\omega + z)(1 - e^{-\omega\delta})(e^{-zx_n} + e^{-z(\delta - x_n)})\left[i\frac{\xi_j}{z}\mathbf{k}_2\right]\end{aligned}$$

with

$$\begin{aligned}\Psi_+(\delta/2) &= -2\varphi(\delta/2) + 2\frac{e^{-\omega\delta/2}}{1 + e^{-\omega\delta}}\varphi(\delta)\\ \partial_n\Psi_+(0) &= 1 + \frac{(\omega + z)^2}{\lambda}e^{-\delta z}.\end{aligned}$$

It is sufficient to construct a function f such that the low frequency parts of $v(\cdot, \delta/2)$, $\partial_n v_j(\cdot, 0)$ and $\partial_j\theta(\cdot, x_n)$ are unbounded because we have in general, for $\psi_0 \in C_c^\infty(\mathbb{R}^{n-1})$ with $0 \in \operatorname{supp}\psi_0$,

$$\|g\|_{L_\infty(\mathbb{R}^{n-1})} \gtrsim \|g\|_{B^0_{\infty,\infty}(\mathbb{R}^{n-1})} \gtrsim \left\|\check{\psi}_0 * g\right\|_{L_\infty(\mathbb{R}^{n-1})}.$$

Assume f is of the special form

$$f\colon \Omega \to \mathbb{C}^n, \qquad (x', x_n) \mapsto (-g(x')\partial_n h(x_n), \operatorname{div}' g(x')h(x_n))$$

for some $g\colon \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ and $h\colon (0, \delta) \to \mathbb{R}$ such that g and $\operatorname{div}' g$ are contained in $C_0(\mathbb{R}^{n-1})$ and $h \in C_c^\infty(0, \delta)$ with $h(x_n) = -h(\delta - x_n)$. Since $\operatorname{div} f = 0$ it follows from Proposition 2.4 that any such f is contained in $C_{0,\sigma}(\Omega)$. The solution w_n of the Helmholtz equation with data f_n satisfies

$$\begin{aligned}\partial_n \hat{w}_n(\xi', 0) &= \int_0^\delta \frac{\sinh(\omega t)}{\sinh(\omega\delta)} \hat{f}_n(\xi', \delta - t)\, \mathrm{d}t \\ &= (i\xi' \cdot \hat{g}(\xi')) \int_0^\delta \frac{\sinh(\omega t)}{\sinh(\omega\delta)} h(\delta - t)\, \mathrm{d}t\end{aligned}$$

and

$$\partial_n \hat{w}_n(\xi', \delta) = \partial_n \hat{w}_n(\xi', 0).$$

In particular we have

$$i\frac{\xi_j}{|\xi|}\mathbf{k}_2 = -i\frac{\xi_j}{|\xi|}\frac{\partial_n \hat{w}_n(\xi', 0)}{\Phi_+(z,\omega)} = \frac{\xi_j \xi' \cdot \hat{g}}{|\xi|^2}\frac{z}{\Phi_+(z,\omega)}\int_0^\delta \frac{\sinh(\omega t)}{\sinh(\omega\delta)} h(\delta - t)\, \mathrm{d}t.$$

In the following we will construct suitable functions g and h. We will use the following result, the proof of which will be presented later on.

Lemma 5.1. *Let $d \geq 2$ and let $\psi_0 \in \mathscr{D}(\mathbb{R}^d)$ denote a cut-off function with $\psi_0(0) = 1$, and define $\psi_\infty = 1 - \psi_0$. There is a function $g\colon \mathbb{R}^d \to \mathbb{R}^d$ such that*

i) $g \in C_0^\infty(\mathbb{R}^d)$.
ii) *The function g satisfies*

$$\mathcal{F}^{-1}\psi_0 \frac{\xi\xi^T}{|\xi|^2}\mathcal{F}g \in \dot{B}^0_{\infty,\infty}(\mathbb{R}^d) \setminus L_\infty(\mathbb{R}^d), \qquad \mathcal{F}^{-1}\psi_\infty \frac{\xi\xi^T}{|\xi|^2}\mathcal{F}g \in B^1_{\infty,\infty}(\mathbb{R}^d)$$

and in particular

$$\mathcal{F}^{-1}\frac{\xi\xi^T}{|\xi|^2}\mathcal{F}g \notin L_\infty(\mathbb{R}^d).$$

Here we write $C_0^\infty(\mathbb{R}^d)$ for the space of all smooth functions f on $\mathbb{R}^d$ with $\partial^\alpha f \in C_0(\mathbb{R}^d)$ for every multiindex $\alpha \in \mathbb{N}_0^d$. We show that Lemma 5.1 implies that for some $j = 1, \ldots, n-1$ we have

$$\mathcal{F}^{-1} i \frac{\xi_j}{|\xi|}\mathbf{k}_2 \notin L_\infty(\mathbb{R}^{n-1}). \tag{4}$$

We can write

$$i\frac{\xi}{|\xi|}\mathbf{k}_2 = \mathcal{M}_\lambda(z)\frac{\xi\xi^T}{|\xi|^2}\hat{g} = \mathcal{M}_\lambda(0)\frac{\xi\xi^T}{|\xi|^2}\hat{g} + \tilde{\mathcal{M}}_\lambda(z)\frac{\xi\xi^T}{|\xi|^2}\hat{g}$$

with

$$\mathcal{M}_\lambda(z) = \frac{z}{\Phi_+(z,\omega)}\int_0^\delta \frac{\sinh(\omega t)}{\sinh(\omega\delta)} h(\delta - t)\, \mathrm{d}t, \qquad \tilde{\mathcal{M}}_\lambda(z) = \mathcal{M}_\lambda(z) - \mathcal{M}_\lambda(0).$$

In particular

$$\mathcal{M}_\lambda(0) = \Big[\partial_1 \Phi_+(0, \lambda^{1/2})\Big]^{-1}\int_0^\delta \frac{\sinh(\lambda^{1/2} t)}{\sinh(\lambda^{1/2}\delta)} h(\delta - t)\, \mathrm{d}t$$

and we can choose h such that $\mathcal{M}_\lambda(0) \neq 0$. We show

$$\mathcal{F}^{-1}\mathcal{M}_\lambda(0)\frac{\xi\xi^T}{|\xi|^2}\hat{g} \notin L_\infty(\mathbb{R}^{n-1}), \qquad \mathcal{F}^{-1}\tilde{\mathcal{M}}_\lambda(z)\frac{\xi\xi^T}{|\xi|^2}\hat{g} \in L_\infty(\mathbb{R}^{n-1}). \tag{5}$$

This would imply (4). First of all, since $\mathcal{M}_\lambda(0)$ is constant with respect to z and nonzero we have

$$\mathcal{F}^{-1}\mathcal{M}_\lambda(0)\frac{\xi\xi^T}{|\xi|^2}\hat{g} = \mathcal{M}_\lambda(0)\mathcal{F}^{-1}\frac{\xi\xi^T}{|\xi|^2}\hat{g} \notin L_\infty(\mathbb{R}^{n-1})$$

by Lemma 5.1. To show the second assertion in (5) we decompose the function in question into its low-frequency part and a remainder part. Let ψ_0 denote a cut-off function with $\psi_0 = 1$ in a neighbourhood of zero as in the previous sections, and let $\psi_\infty = 1 - \psi_0$. We will show that

$$\mathcal{F}^{-1}\tilde{\mathcal{M}}_\lambda(z)\frac{\xi\xi^T}{|\xi|^2}\hat{g} = \mathcal{F}^{-1}\psi_0\tilde{\mathcal{M}}_\lambda(z)\frac{\xi\xi^T}{|\xi|^2}\hat{g} + \mathcal{F}^{-1}\psi_\infty\tilde{\mathcal{M}}_\lambda(z)\frac{\xi\xi^T}{|\xi|^2}\hat{g}$$

is in $L_\infty(\mathbb{R}^{n-1})$. We start with the remainder part. We can write

$$\mathcal{F}^{-1}\psi_\infty\tilde{\mathcal{M}}_\lambda(z)\frac{\xi\xi^T}{|\xi|^2}\hat{g} = -\mathcal{M}_\lambda(0)\mathcal{F}^{-1}\psi_\infty\frac{\xi\xi^T}{|\xi|^2}\hat{g} + \mathcal{F}^{-1}\psi_\infty\mathcal{M}_\lambda(z)\frac{\xi\xi^T}{|\xi|^2}\hat{g}.$$

Since $0 \notin \operatorname{supp}\psi_\infty$ we can estimate

$$\begin{aligned}
\left\|\mathcal{M}_\lambda(0)\mathcal{F}^{-1}\psi_\infty\frac{\xi\xi^T}{|\xi|^2}\hat{g}\right\|_{L_\infty(\mathbb{R}^{n-1})} &\lesssim |\mathcal{M}_\lambda(0)|\left\|\mathcal{F}^{-1}\psi_\infty\frac{\xi\xi^T}{|\xi|^2}\hat{g}\right\|_{B^1_{\infty,\infty}(\mathbb{R}^{n-1})}\\
&\lesssim |\mathcal{M}_\lambda(0)|\left\|\mathcal{F}^{-1}\psi_\infty\frac{\xi\xi^T}{|\xi|^2}\hat{g}\right\|_{\dot{B}^1_{\infty,\infty}(\mathbb{R}^{n-1})}\\
&\lesssim |\mathcal{M}_\lambda(0)|\left\|\mathcal{F}^{-1}\psi_\infty\frac{\xi}{|\xi|}\mathcal{F}\operatorname{div}' g\right\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1})}\\
&\lesssim |\mathcal{M}_\lambda(0)|\left\|\operatorname{div}' g\right\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1})}
\end{aligned}$$

which is finite since $\operatorname{div}' g \in C_0(\mathbb{R}^{n-1})$ and thus in particular $\operatorname{div}' g \in \dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1})$. We turn to

$$\mathcal{F}^{-1}\psi_\infty\mathcal{M}_\lambda(z)\frac{\xi\xi^T}{|\xi|^2}\hat{g}.$$

Since $h \in C_c(0,\delta)$ there is $r > 0$ such that $\operatorname{supp} h \subset [r, \delta - r]$, and thus

$$\mathcal{M}_\lambda(z) = \frac{z}{\Phi_+(z,\omega)}\int_r^{\delta-r}\frac{\sinh(\omega t)}{\sinh(\omega\delta)}h(\delta - t)\,\mathrm{d}t.$$

Thus we can show the following estimate for $\mathcal{M}_\lambda$ for $|z| \geq \rho > 0$.

$$\begin{aligned}
|\mathcal{M}_\lambda(z)| &\lesssim \frac{|z|}{|\Phi_+(z,\omega)|}\int_r^{\delta-r}\left|\frac{\sinh(\omega t)}{\sinh(\omega\delta)}\right||h(\delta - t)|\,\mathrm{d}t\\
&\lesssim |z|\int_r^{\delta-r} e^{-(\delta-t)\Re\omega}\,\mathrm{d}t\\
&\lesssim \frac{|z|}{|\omega|}\left[e^{-cr|\omega|} - e^{-c(\delta-r)|\omega|}\right]\\
&\lesssim |z|\,|\lambda|^{-1/2}\,e^{-cr|z|}.
\end{aligned}$$

At this point it follows from Proposition I.3.13 that $\psi_\infty\mathcal{M}_\lambda$ gives rise to a Fourier integral operator from $\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1})$ into $\dot{B}^0_{\infty,1}(\mathbb{R}^{n-1})$. In particular we obtain

$$\left\|\mathcal{F}^{-1}\psi_\infty\mathcal{M}_\lambda(z)\frac{\xi\xi^T}{|\xi|^2}\hat{g}\right\|_{L_\infty(\mathbb{R}^{n-1})} \lesssim \|g\|_{L_\infty(\mathbb{R}^{n-1})}.$$

This concludes the estimates for the remainder part and we can move on to the low-frequency part. For fixed $\lambda \in \Sigma_\rho$ the function $\mathcal{M}_\lambda$ is holomorphic on a sector Σ_ε and in a pointed neighbourhood around zero. Since $\mathcal{M}_\lambda$ is bounded in a neighbourhood of zero, it has a holomorphic extension to $\Sigma_\varepsilon \cup B(0,r)$ for some $r > 0$. Since $\tilde{\mathcal{M}}_\lambda$ vanishes in $z = 0$, we have $\tilde{\mathcal{M}}_\lambda(z) = O(z)$ in a neighbourhood of zero. Again this shows that $\psi_0 \tilde{\mathcal{M}}_\lambda$ gives rise to a Fourier integral operator from $\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1})$ into $\dot{B}^0_{\infty,1}(\mathbb{R}^{n-1})$. In particular we obtain

$$\begin{aligned}\left\| \mathcal{F}^{-1}\psi_0 \tilde{\mathcal{M}}_\lambda(z) \frac{\xi\xi^T}{|\xi|^2}\hat{g} \right\|_{L_\infty(\mathbb{R}^{n-1})} &\lesssim \left\| \mathcal{F}^{-1}\frac{\xi\xi^T}{|\xi|^2}\hat{g}\right\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1})} \\ &\lesssim \|g\|_{L_\infty(\mathbb{R}^{n-1})}.\end{aligned}$$

All in all we see that (5) is satisfied and hence also (4). In particular the low-frequency part

$$\mathcal{F}^{-1}\psi_0 i \frac{\xi_j}{|\xi|}\mathbf{k}_2$$

is unbounded.

Now we are ready to show that the low-frequency part of v_j is unbounded for some j. To this end we can write

$$\begin{aligned}\mathcal{F}^{-1}\psi_0 \hat{v}_j(\xi,\delta/2) &= -\mathcal{F}^{-1}\psi_0(1-e^{-\omega\delta})\Psi_+(\delta/2,z)\left[i\frac{\xi_j}{z}\mathbf{k}_2\right] \\ &= -(1-e^{-\sqrt{\lambda}\delta})\Psi_+(\delta/2,0)\left[\mathcal{F}^{-1}\psi_0 i\frac{\xi_j}{z}\mathbf{k}_2\right] \\ &\quad - \mathcal{F}^{-1}\psi_0\left\{(1-e^{-\omega\delta})\Psi_+(\delta/2,z)-(1-e^{-\sqrt{\lambda}\delta})\Psi_+(\delta/2,0)\right\}\left[i\frac{\xi_j}{z}\mathbf{k}_2\right]\end{aligned}$$

and the same reasoning as above shows that the second summand is bounded while the first is, by the above calculations, unbounded. In particular the low-frequency part of $v_j(\cdot,\delta/2)$ is unbounded for some j and hence $v_j \notin L_\infty(\Omega)$. The assertion for $\partial_n v_j$ and $\partial_j\theta$ follows analogously. This shows Theorem 1.4, and it remains to show Lemma 5.1.

Proof of Lemma 5.1. Fix $j,k \in \{1,\dots,d\}$. It follows from Proposition I.3.6 that $m(\xi) = \psi_0(\xi)\xi_j\xi_k/|\xi|^2$ is a Fourier multiplier on $B^0_{1,1}(\mathbb{R}^d)$ if and only if m is a Fourier multiplier on $B^0_{\infty,\infty}(\mathbb{R}^d)$ if and only if $\mathcal{F}^{-1}m \in B^0_{1,\infty}(\mathbb{R}^d)$. Since m is only supported in a neighbourhood of zero this is the case precisely if $\mathcal{F}^{-1}m \in L_1(\mathbb{R}^d)$. Assume this were the case, i. e. $\mathcal{F}^{-1}m \in L_1(\mathbb{R}^d)$. It follows from the Lemma of Riemann-Lebesgue that then $m \in C_0(\mathbb{R}^d)$. But m is obviously not continuous in $\xi = 0$, and hence m cannot be a Fourier multiplier on $B^0_{1,1}(\mathbb{R}^d)$ or $B^0_{\infty,\infty}(\mathbb{R}^d)$. This in turn implies that there is a function $\phi \in B^0_{1,1}(\mathbb{R}^d)$ such that $\mathcal{F}^{-1}m\mathcal{F}\phi \notin B^0_{1,1}(\mathbb{R}^d)$. Since m is compactly supported we obtain the stronger statement $\mathcal{F}^{-1}m\mathcal{F}\phi \notin L_1(\mathbb{R}^d)$.

Now assume that the operator

$$f \mapsto \mathcal{F}^{-1}m\mathcal{F}f$$

were bounded from $C_0(\mathbb{R}^d)$ to $B^0_{\infty,\infty}(\mathbb{R}^d)$. Then, since the dual space of $B^0_{1,1}$ is $B^0_{\infty,\infty}$, the mapping

$$f \mapsto \langle \phi, \mathcal{F}^{-1}m\mathcal{F}f\rangle = \langle \mathcal{F}^{-1}m\mathcal{F}\phi, f\rangle$$

is a continuous functional on $C_0(\mathbb{R}^d)$. It follows from the Theorem of Riesz-Markov (Theorem V.20.48 in [**HS69**]) that this functional coincides with a finite regular Radon measure μ such that

$$\int_{\mathbb{R}^d} f \,\mathrm{d}\mu = \langle \mathcal{F}^{-1}m\mathcal{F}\phi, f\rangle.$$

It follows from Remark 2.11.2.2 in [**Tri83**] and the embedding in Theorem 2.2.2.iv of [**RS96**] that any finite regular Radon measure is contained in the Besov space $B^0_{1,\infty}(\mathbb{R}^d)$. Since $\hat{\mu} = m\hat{\phi}$ is compactly supported this is equivalent to $\mu \in L_1(\mathbb{R}^d)$. But ϕ was constructed such that $\mu = \mathcal{F}^{-1}m\mathcal{F}\phi \notin L_1(\mathbb{R}^d)$, and thus we arrive at a contradiction and we see that the operator $f \mapsto \mathcal{F}^{-1}m\mathcal{F}f$ cannot map $C_0(\mathbb{R}^d)$ boundedly into $B^0_{\infty,\infty}(\mathbb{R}^d)$.

This implies that there is a function $w \in C_0(\mathbb{R}^d)$ such that $\mathcal{F}^{-1}m\mathcal{F}w \notin B^0_{\infty,\infty}(\mathbb{R}^d)$. Let $\eta \in C_c^\infty(\mathbb{R}^d)$ denote a cut-off function with $\eta \equiv 1$ on $\operatorname{supp}\psi_0$, and define $g = (g_1, \dots, g_d)$ via $g_k = \mathcal{F}^{-1}\eta\hat{w}$ and $g_l = 0$ for $k \neq l$. Then $g \in C_0^\infty(\mathbb{R}^d)^d$ and

$$\mathcal{F}^{-1}\psi_0\frac{\xi\xi^T}{|\xi|^2}\mathcal{F}g = \mathcal{F}^{-1}\psi_0\frac{\xi\xi_k}{|\xi|^2}\eta\mathcal{F}w = \mathcal{F}^{-1}\psi_0\frac{\xi\xi_k}{|\xi|^2}\mathcal{F}w \notin B^0_{\infty,\infty}(\mathbb{R}^d).$$

In particular

$$\mathcal{F}^{-1}\psi_0\frac{\xi\xi^T}{|\xi|^2}\mathcal{F}g \notin L_\infty(\mathbb{R}^d).$$

Concerning the high-frequency part, we have

$$\mathcal{F}^{-1}\psi_\infty\frac{\xi\xi^T}{|\xi|^2}\mathcal{F}g = \mathcal{F}^{-1}\psi_\infty\eta\frac{\xi\xi_k}{|\xi|^2}\mathcal{F}w \in B^1_{\infty,\infty}(\mathbb{R}^d)$$

since the support of $\psi_\infty\eta$ is contained in some annulus. □

6. Uniqueness of solutions to the resolvent problem

The proof of Theorem 1.5 is basically an adaptation of the corresponding part of the proof of Theorem 1.2 of [**AY10**]. Their proof relies essentially on solvability of the Stokes resolvent problem (2) in spaces related to $L_1(\Omega)$. We will show a corresponding result in the following chapter.

Assume (u, θ) satisfies the Stokes resolvent problem (2) and satisfies the assumptions of Theorem 1.5. Let $f \in C^\infty_{c,\sigma}(\Omega)$, and given $j = 1, \dots, n-1$ let (v, π) denote the solution to the Stokes resolvent problem with data $\partial_j f$. Assume for the moment that v is such that the functions

$$\|v(\cdot, x_n)\|_{\dot{B}^0_{1,1}(\mathbb{R}^{n-1})}, \quad \|\nabla v(\cdot, x_n)\|_{\dot{B}^0_{1,1}(\mathbb{R}^{n-1})}, \quad \left\|\nabla^2 v(\cdot, x_n)\right\|_{\dot{B}^0_{1,1}(\mathbb{R}^{n-1})}, \quad \|\nabla\pi(\cdot, x_n)\|_{\dot{B}^0_{1,1}(\mathbb{R}^{n-1})}$$

are integrable on $(0, \delta)$. Then we can calculate

$$\begin{aligned}\langle\partial_j u, f\rangle &= \langle u, \partial_j f\rangle \\ &= \langle u, \lambda v - \Delta v + \nabla\pi\rangle \\ &= \langle\lambda u - \Delta u + \nabla\theta, v\rangle \\ &= 0.\end{aligned}$$

By de Rham's Theorem (Theorem 2.1) this implies $\partial_j u = \nabla\pi$ for some distribution $\pi \in \mathscr{D}'(\Omega)$, and since $\partial_j u$ is divergence-free it follows from Weyl's Lemma that the distribution π must, in fact, be harmonic. Then also $\partial_j u$ is harmonic. Since $\partial_j u$ is bounded and vanishes on the boundary $\partial\Omega$ it follows from the maximum principle that $\partial_j u$ vanishes in all of Ω. This implies that u depends only on x_n and then the conclusion of Theorem 1.5 follows by a direct calculation.

It remains to show that v has the desired regularity properties. This will be established in Chapter IV.

7. Symmetric data

We have shown that in $n \geq 3$ dimensions solutions to the Stokes resolvent equation (2) are not necessarily bounded, i. e. a result analogous to Theorem 1.2 fails in $n \geq 3$ dimension. It is not clear, however, to which extent it fails, and which additional conditions one would have to impose on the data to recover Theorem 1.2 in three dimensions or more. An inspection of the proof of Theorem 1.1 shows that if $\mathcal{F}^{-1}\mathbf{k}_2$ satisfied

$$\mathcal{F}^{-1} i \frac{\xi_j}{|\xi|} \mathbf{k}_2 \in L_\infty(\mathbb{R}^{n-1})$$

for $j = 1, \ldots, n-1$ then the solution u would be, in fact, bounded. This is the case if e. g. the functions

$$x_n \mapsto \|f_j(\cdot, x_n) + f_j(\cdot, \delta - x_n)\|_{\dot{B}^0_{\infty,1}(\mathbb{R}^{n-1})}$$

are bounded on $(0, \delta)$ for $j = 1, \ldots, n-1$. In particular any function $f \in C_{0,\sigma}(\Omega)$ with

$$f_n(x', x_n) = f_n(x', \delta - x_n)$$

for every $x \in \Omega$ gives rise to a bounded solution u, even in $n \geq 3$ dimension. It would certainly be interesting to investigate necessary conditions on the data f to give rise to a bounded solution.

8. The Stokes equation in a half space. The limiting case $\delta \to \infty$.

In this section we use our representation formula II.1.5 and the tools and techniques we used to tackle the Stokes equation in a layer to sketch a short proof of a result by Saal [**Saa07**] and Desch et al. [**DHP01**]. The essential difference between the case of a layer of finite width and a half space is that in the case of a layer the function Φ_+ vanishes at $z = 0$, whereas in the half space case $\Phi_\pm \equiv 1$. This enables one to extend the estimates for $\mathbf{k}_2$ in Lemma 3.1 to $-1 \leq s \leq 1$. This in turn makes it possible to estimate $v_j(\cdot, x_n)$ and $\partial_n v_j(\cdot, x_n)$ in $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^{n-1})$ and then it follows via interpolation that v and ∇v are, in fact, bounded.

While the counterexample we constructed for the proof of Theorem 1.4 does not seem to take the layer structure into account, an inspection of the proof of Theorem 1.4 shows that it was crucial that the function $\mathcal{M}_\lambda$ does not vanish at $z = 0$. This, however, is not the case in $\mathbb{R}^n_+$. The function $f \in C_{0,\sigma}(\Omega)$ that we constructed in the proof of Theorem 1.4 can be extended by zero to a function $f \in C_{0,\sigma}(\mathbb{R}^n_+)$, and this function is mapped to a bounded $u \in C_{0,\sigma}(\mathbb{R}^n_+)$. This illustrates the very nonlocal character of the Stokes equation.

However, solutions to the Stokes resolvent problem in a half space are not unique either. The nontrivial solutions in $L_{\infty,\sigma}(\mathbb{R}^n_+)$ to the homogeneous Stokes resolvent problem are given by

$$u_j(x', x_n) = \frac{d_j}{\lambda}\left[1 - e^{-\sqrt{\lambda} x_n}\right]$$

for $j = 1, \ldots, n-1$, $u_n = 0$, and $\theta(x', x_n) = -d \cdot x' + c$, where $d \in \mathbb{R}^{n-1}$ and $c \in \mathbb{R}$ are some arbitrary constants.

Since the Stokes resolvent problem in L_∞ should be in some sense dual to the Stokes resolvent problem in L_1 it does not come as much of a surprise that in L_1 we need additional conditions on the data f for the Stokes resolvent problem to possess an L_1-solution. This is also hinted at by a result of Kozono [**Koz98**] who showed that in exterior domains a necessary condition for a solution of the evolution equation (1) to be an L_1-solution is that the net force exerted by the fluid on $\partial\Omega$ is equal to zero.

9. Addendum

After the preparation of this thesis we learned that a recent result by Abe, Giga, Schade, Suzuki [**AGSS14a**] implies the assertion of Corollary 1.3.

In addition to the class of admissible domains in the sense of Abe and Giga, which does not include layer domains, they introduced a class of domains which they refer to as 'Neumann admissible domains', which includes cylindrical domains and in particular also two-dimensional layer domains. They were able to show generation results in solenoidal subspaces of L_∞ for domains within the class of Neumann admissible domains.

CHAPTER IV

Analysis of the Stokes equation in a layer in spaces of integrable functions

1. Introduction and main results

In this chapter we study the linear evolution equation

$$\begin{cases} \partial_t u - \Delta u + \nabla\theta = 0 & \text{in } J \times \Omega \\ \operatorname{div} u = 0 & \text{in } J \times \Omega \\ u = 0 & \text{on } J \times \partial\Omega \\ u(0) = u_0 & \text{in } \Omega_0. \end{cases} \tag{1}$$

via the associated resolvent problem

$$\begin{cases} \lambda u - \Delta u + \nabla\theta = f & \text{in } \Omega_0 \\ \operatorname{div} u = 0 & \text{in } \Omega_0 \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

in $L_1(\Omega)$ and related spaces. It was shown by Desch et al. [**DHP01**] and Saal [**Saa07**] that there are divergence-free functions $f \in L_1(\mathbb{R}^n_+)$ with $n \geq 2$ such that the corresponding solution u given by the solution formula in Chapter II satisfies $u \notin L_1(\mathbb{R}^n_+)$. It is known however, that the gradient of the solution ∇u will still be in $L_1(\mathbb{R}^n_+)$. In this chapter we show how this result extends to the case of a layer of finite width.

We write $H_1 = \dot{F}^0_{1,2}$ for the usual Hardy space as defined e. g. in section 5.4.2 of [**Tri83**], and $L_{1,\infty}$ for the weak L_1-space, which is a specific case of a Lorentz space as defined e. g. in [**Gra08**]. We also write

$$\|f\|_{L_{1,\infty}(0,\delta;L_1(\mathbb{R}^{n-1}))} := \left\| x_n \mapsto \|\nabla\theta(\cdot, x_n)\|_{L_1(\mathbb{R}^{n-1})} \right\|_{L_{1,\infty}(0,\delta)}.$$

Let $\Omega = \mathbb{R}^{n-1} \times (0,\delta)$ with some constant $\delta > 0$ as in the previous chapters. Let $L_{1,\sigma}(\Omega)$ denote the closure of $C^\infty_{c,\sigma}(\Omega)$ in $L_1(\Omega)$. In general we can show the following positive result.

THEOREM 1.1. *Let $n \geq 2$ and $0 < \rho < \pi$. Let $\lambda_0 > 0$. Then for all $\lambda \in \Sigma_\rho$ with $|\lambda| \geq \lambda_0$ and $f \in C^\infty_{c,\sigma}(\Omega)$ such that $x_n \to \|f(\cdot, x_n)\|_{H_1(\mathbb{R}^{n-1})}$ is integrable there is a unique solution (u, θ) of (2) such that u is in $L_{1,\sigma}(\Omega)$ satisfying the estimate*

$$|\lambda| \, \|u\|_{L_1(\Omega)} + |\lambda|^{1/2} \|\nabla u\|_{L_1(\Omega)} + |\lambda|^{-1/2} \|\nabla\theta\|_{L_{1,\infty}(0,\delta;L_1(\mathbb{R}^{n-1}))} \lesssim \int_0^\delta \|f(\cdot, x_n)\|_{H_1(\mathbb{R}^{n-1})} \, \mathrm{d}x_n.$$

Here uniqueness of solutions is to be understood in the sense of uniqueness in the class of all solutions (u, θ) with $u, \nabla u \in L_1(\Omega)$ and $\theta \in L_{1,\mathrm{loc}}(\Omega)$ with $\nabla\theta \in L_{1,\infty}(0,\delta; L_1(\mathbb{R}^{n-1}))$ and $\nabla\theta \in L_1(\mathbb{R}^{n-1} \times (\varepsilon, \delta - \varepsilon))$ for every $0 < \varepsilon < \delta/2$. We identify two solutions (u, θ) and (v, ϑ) if $u = v$ and $\nabla\theta = \nabla\vartheta$.

As in Chapter III we can improve upon Theorem 1.1 in the case of a two-dimensional layer. More precisely, we can weaken the assumptions on the data without losing control over the L_1-norm of u and ∇u.

THEOREM 1.2. *Let $n = 2$ and $0 < \rho < \pi$. Let $\lambda_0 > 0$. Then there is $C > 0$ such that for all $\lambda \in \Sigma_\rho$ with $|\lambda| \geq \lambda_0$ and $f \in C^\infty_{c,\sigma}(\Omega)$ there is a unique solution (u, θ) of (2) such that u is in $L_{1,\sigma}(\Omega)$ satisfying the estimate*

$$|\lambda| \, \|u\|_{L_1(\Omega)} + |\lambda|^{1/2} \, \|\nabla u\|_{L_1(\Omega)} + |\lambda|^{-1/2} \, \|\nabla\theta\|_{L_{1,\infty}(0,\delta;L_1(\mathbb{R}^{n-1})} \lesssim \|f\|_{L_1(\Omega)} \, .$$

We obtain as a corollary of Theorem 1.2 that the Stokes operator in $L_{1,\sigma}(\Omega)$ generates a holomorphic semigroup.

COROLLARY 1.3. *Let $n = 2$. Then the Stokes operator generates a strongly continuous holomorphic semigroup of angle $\pi/2$ on $L_{1,\sigma}(\Omega)$.*

However, in $n \geq 3$ dimensions we can show the following rather strong non-generation result.

THEOREM 1.4. *Let $n \geq 3$ and $\lambda > 0$. Then there is $f \in C^\infty_{c,\sigma}(\Omega)$ such that the solution (u, θ) from Theorem 1.1 satisfies $u \notin L_1(\Omega)$, $\nabla u \notin L_1(\Omega)$, and $\nabla\theta \notin L_1(\Omega)$.*

As in the previous chapter, the proof of Theorem 1.1 and Theorem 1.2 essentially amounts to estimating the functions (u, θ) given by the solution formula I.1.3. Then Corollary 1.3 follows from standard arguments. The main tool in the proof of Theorem 1.4 is the Lemma of Riemann-Lebesgue, which also provides us with the means to show a slightly stronger version of Theorem 5.1 of [**DHP01**].

2. A characterisation of the space $L_{1,\sigma}$

Given a domain M we let $L_{1,\sigma}(M)$ denote the closure of $C^\infty_{c,\sigma}(M)$ in $L_1(M)$. Provided with the norm of $L_1(M)$ the space $L_{1,\sigma}(M)$ becomes a Banach space. Given a function $f \in L_1(M)$ it can be difficult to verify whether f can be approximated by divergence-free test functions. We will show the following criterion.

PROPOSITION 2.1. *Let $M \subset \mathbb{R}^n$ denote a Lipschitz domain. Then $f \in L_{1,\sigma}(M)$ if and only if $f \in L_1(M)$ and $\operatorname{div} f = 0$ in $\mathscr{D}'(\mathbb{R}^n)$, i. e.*

$$\int_M f \nabla\phi = 0$$

for every $\phi \in \mathscr{D}(\mathbb{R}^n)$.

PROOF. This condition is obviously necessary. We show that it is also sufficient. Let E denote the space of all $f \in L_1(M)$ satisfying $\operatorname{div} f = 0$ in $\mathscr{D}'(\mathbb{R}^n)$, provided with the norm of $L_1(M)$. Then $L_{1,\sigma}(M) \subset E \subset L_1(M)$. We wish to show that $L_{1,\sigma}(M)$ and E coincide. To this end let $\varphi \in E'$ denote a functional on E that vanishes on $L_{1,\sigma}(M)$. By Hahn-Banach we can extend φ to a functional on $L_1(M)$, and we see that there must be $\phi \in L_\infty(M)$ with

$$\langle f, \varphi \rangle = \int_M f\phi$$

for $f \in E$. Since φ vanishes on $L_{1,\sigma}(M)$ it does so in particular on $C^\infty_{c,\sigma}(M)$ and it follows from de Rham's Theorem (Theorem III.2.1) that there is $\pi \in \mathscr{D}'(M)$ with $\phi = \nabla\pi$. Since $\phi \in L_\infty(M)$ we can infer that π is not only a regular distribution but in fact uniformly Lipschitz continuous on M, though not necessarily bounded. Now φ satisfies

$$\langle f, \varphi \rangle = \int_M f \nabla\pi$$

for $f \in E$. It is a consequence of Theorem 1.31 of [**Sch69**], which in turn is a consequence of a classical result by Kirszbraun [**Kir34**], that π can be extended to a Lipschitz function on $\mathbb{R}^n$ with the same Lipschitz constant. We denote this extension with $\tilde{\pi}$.

Take $f \in L_1(\mathbb{R}^n)$. Applying the approximation procedure employed in the proof of Lemma III.2.3 to the function $\tilde{\pi}$ we can find a sequence $(\tilde{\pi}_k)_k \subset \mathscr{D}(\mathbb{R}^n)$ with $\sup_k \|\nabla\tilde{\pi}_k\|_{L_\infty(\mathbb{R}^n)} \lesssim \|\nabla\pi\|_{L_\infty(M)}$ such that

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} f\nabla\tilde{\pi}_k = \int_{\mathbb{R}^n} f\nabla\tilde{\pi}.$$

In particular if f is not only in $L_1(\mathbb{R}^n)$ but actually $f \in E$, extended by zero to a function on $\mathbb{R}^n$, then by assumption

$$\int_M f\nabla\tilde{\pi}_k = 0$$

for every k, and thus

$$\langle f, \varphi\rangle = \int_M f\nabla\pi = \lim_{k\to\infty}\int_M f\nabla\tilde{\pi}_k = 0.$$

This shows that $L_{1,\sigma}(M)$ is dense in E and since both are closed subspaces of $L_1(M)$ they must coincide. □

Proposition 2.1 is a generalisation of a classical result by Miyakawa [**Miy96**] who proved an analogous characterisation of $L_{1,\sigma}(M)$ in the case $M = \mathbb{R}^n$.

Proposition 2.1 should be contrasted with Proposition III.2.4, which gives a characterisation of the space $C_{0,\sigma}(M)$ in the case that M is a layer domain. There are two major differences. First of all Proposition 2.1 holds in every Lipschitz domain, whereas it is known that a result analogous to Proposition III.2.4 is not valid for general domains. This appears to be essentially due to the possibility to extend any Lipschitz function to the full space, which is in general not possible for locally integrable functions with integrable gradient.

The second major difference is that in Proposition III.2.4 we require $\operatorname{div} f$ to vanish in $\mathscr{D}'(M)$, whereas in Proposition 2.1 we need $\operatorname{div} f$ to vanish in $\mathscr{D}'(\mathbb{R}^n)$, which is a considerably stronger assumption. It seems reasonable to expect that one could weaken the assumptions in Proposition 2.1 and require $f \in L_1(M)$ to satisfy $\operatorname{div} f = 0$ in $\mathscr{D}'(M)$ if one imposes additional conditions on f at the boundary of M. However, we will not pursue this further as Proposition 2.1 is more than sufficient for our purposes.

3. Estimates for the velocity u and the pressure θ

We can use the representation formula II.1.3 again. Then the proof of Theorem 1.1 boils down to estimating the solution as in Chapter III. This can be done in complete analogy to the proof of Theorem III.1.1 and Theorem III.1.2. The reason for this analogy stems from two facts. First, while L_1 and L_∞ both provide arguably inconvenient settings for Fourier analysis, they do so in a rather similar way. This is largely due to L_∞ being dual to L_1. And second, the homogeneous Besov and Triebel-Lizorkin spaces, of which $\dot{B}^0_{\infty,\infty}$ and $H_1 = \dot{F}^0_{1,2}$ are two examples, provide an excellent environment for harmonic analysis.

Given that we already have estimates for $\Phi_\pm$ at our disposal due to Lemma II.1.4 we can show the following counterpart of Lemma III.3.1.

LEMMA 3.1. *Under the assumptions of Theorem 1.1 we have the following estimates for* $\mathbf{k}_1$ *and* $\mathbf{k}_2$*.*

$$\left\|\mathcal{F}^{-1}\mathbf{k}_1\right\|_{\dot{F}^s_{1,2}(\mathbb{R}^{n-1})} \lesssim |\lambda|^{\frac{s}{2}} \int_0^\delta \|f(\cdot, x_n)\|_{H_1(\mathbb{R}^{n-1})}\, \mathrm{d}x_n$$

for $-1 \leq s \leq 0$ *and*

$$\left\|\mathcal{F}^{-1}\mathbf{k}_2\right\|_{H_1(\mathbb{R}^{n-1})} \lesssim \int_0^\delta \|f(\cdot,x_n)\|_{H_1(\mathbb{R}^{n-1})}.$$

In addition,

$$\left\|\mathcal{F}^{-1}\mathbf{k}_1\right\|_{\dot{B}^s_{1,\infty}(\mathbb{R}^{n-1})} \lesssim |\lambda|^{\frac{s}{2}} \|f\|_{L_1(\Omega)}$$

for $-1 \leq s \leq 0$ *and*

$$\left\|\mathcal{F}^{-1}\mathbf{k}_2\right\|_{\dot{B}^0_{1,\infty}(\mathbb{R}^{n-1})} \lesssim \|f\|_{L_1(\Omega)}\ \mathrm{d}x_n.$$

PROOF. We begin with the estimates for $\mathbf{k}_1$ in $\dot{F}^s_{1,2}$. It follows from Proposition I.3.11 and the estimates in Lemma II.1.4 that $\Phi_-(z,\omega)^{-1}$ is a Fourier multiplier on H_1 of norm $\lesssim 1$. Combining this with the estimates for $\partial_n w_n$ from Proposition I.6.2 and the lifting property stated in Theorem 5.2.3.1 of [**Tri83**] the first assertion follows.

In order to show the second assertion we will decompose $\mathbf{k}_2$ into a low frequency part and a part with Fourier transform vanishing in a neighbourhood of zero as in the proof of Lemma III.3.1. We write

$$\mathbf{k}_2 = \mathbf{k}_2^0 + \mathbf{k}_2^\infty = \psi_0\mathbf{k}_2 + \psi_\infty\mathbf{k}_2.$$

It follows as above that $\mathbf{k}_2^\infty$ satisfies the estimate

$$\left\|\mathcal{F}^{-1}\mathbf{k}_2^\infty\right\|_{\dot{F}^s_{1,2}(\mathbb{R}^{n-1})} \lesssim |\lambda|^{\frac{s}{2}} \int_0^\delta \|f(\cdot,x_n)\|_{H_1(\mathbb{R}^{n-1})}\ \mathrm{d}x_n$$

for $-1 \leq s \leq 0$. In order to estimate the low frequency part we write

$$\mathbf{k}_2^0 = -\frac{1}{2}\frac{z\psi_0}{\Phi_+(z,\omega)}\frac{\mathbf{g}_1+\mathbf{g}_2}{z}.$$

It follows from Proposition I.3.11 and the estimates in Lemma II.1.4 that $z\psi_0\Phi_+(z,\omega)^{-1}$ is a Fourier multiplier on H_1. Now it follows from Proposition I.6.2 that $\mathbf{k}_2^0$ satisfies the estimate

$$\left\|\mathcal{F}^{-1}\mathbf{k}_2^0\right\|_{H_1(\mathbb{R}^{n-1})} \lesssim \int_0^\delta \|f(\cdot,x_n)\|_{H_1(\mathbb{R}^{n-1})}\ \mathrm{d}x_n.$$

Combining the estimates for $\mathbf{k}_2^0$ and $\mathbf{k}_2^\infty$ the second assertion follows.

The remaining estimates can be shown in a similar fashion with Proposition I.3.11 and the embedding $L_1 \hookrightarrow \dot{B}^0_{1,\infty}$. □

Observe that the estimates for $\mathbf{k}_1$ in the spaces $\dot{B}^s_{1,\infty}$ for $-1 \leq s \leq 0$ imply corresponding estimates in the smaller spaces $\dot{B}^s_{1,1}$ for $-1 < s < 0$ due to interpolation of homogeneous Besov spaces. Since $\dot{B}^s_{1,1} \hookrightarrow \dot{F}^s_{1,2}$ we see that the relevance of the first assertion lies in the endpoint cases $s \in \{0,1\}$.

PROPOSITION 3.2. *Under the assumptions of Theorem 1.1, for* $\lambda \in \Sigma_\rho$ *with* $|\lambda| \geq \lambda_0$ *and* $f \in C^\infty_{c,\sigma}(\Omega)$ *the function* v_n *given by the representation formula II.1.3 is contained in* $L_1(\Omega)$ *and satisfies the estimates*

$$|\lambda|\,\|v_n\|_{L_1(\Omega)} + |\lambda|^{1/2}\,\|\nabla v_n\|_{L_1(\Omega)} \lesssim \|f\|_{L_1(\Omega)}.$$

PROOF. We use the representation formula II.1.3, and decompose v_n into a low frequency part and a part with Fourier transform vanishing in a neighbourhood of zero via

$$\hat{v}_n = \hat{v}_n^0 + \hat{v}_n^\infty = \psi_0\hat{v}_n + \psi_\infty\hat{v}_n.$$

We can write the low frequency part v_n^0 as

$$\hat{v}_n^0(\xi', x_n) = \psi_0 \left\{ \varphi(\delta) e^{-zx_n} + \varphi(\delta) e^{-z(\delta - x_n)} - (1 + e^{-\delta z})(\varphi(x_n) + \varphi(\delta - x_n)) \right\} \mathbf{k}_1 + \left\{ \varphi(\delta) \frac{e^{-zx_n} - e^{-z(\delta-x_n)}}{z} + \frac{1 - e^{-\delta z}}{z} (\varphi(x_n) - \varphi(\delta - x_n)) \right\} z\mathbf{k}_2^0$$

and now as a consequence of Lemma 3.1 and Lemma I.4.1 we obtain the estimate

$$\begin{aligned} \left\| v_n^0(\cdot, x_n) \right\|_{\dot{B}^{-1}_{1,\infty}(\mathbb{R}^{n-1})} &\lesssim \left\{ |\lambda|^{-1/2} + \frac{x_n}{1 + x_n |\lambda|^{1/2}} + \frac{\delta - x_n}{1 + (\delta - x_n) |\lambda|^{1/2}} \right\} \\ &\quad \cdot \left(\left\| \mathcal{F}^{-1} \mathbf{k}_1 \right\|_{\dot{B}^{-1}_{1,\infty}(\mathbb{R}^{n-1})} + \left\| \mathcal{F}^{-1} z \mathbf{k}_2^0 \right\|_{\dot{B}^{-1}_{1,\infty}(\mathbb{R}^{n-1})} \right) \\ &\lesssim \left\{ |\lambda|^{-1/2} + \frac{x_n}{1 + x_n |\lambda|^{1/2}} + \frac{\delta - x_n}{1 + (\delta - x_n) |\lambda|^{1/2}} \right\} |\lambda|^{-1/2} \|f\|_{L_1(\Omega)} \end{aligned}$$

where the last inequality follows from Lemma 3.1. Since this is the low frequency part we immediately get a corresponding estimate in $\dot{B}^s_{1,\infty}$ for any $s > -1$. We can use interpolation theory for homogeneous Besov spaces (Proposition I.3.2) to obtain

$$\left\| v_n^0(\cdot, x_n) \right\|_{\dot{B}^s_{1,1}(\mathbb{R}^{n-1})} \lesssim \left\{ |\lambda|^{-1/2} + \frac{x_n}{1 + x_n |\lambda|^{1/2}} + \frac{\delta - x_n}{1 + (\delta - x_n) |\lambda|^{1/2}} \right\} |\lambda|^{-1/2} \|f\|_{L_1(\Omega)}$$

for any $s > -1$. Integrating over $(0, \delta)$ with respect to x_n we obtain

$$\int_0^\delta \left\| v_n^0(\cdot, x_n) \right\|_{\dot{B}^s_{1,1}(\mathbb{R}^{n-1})} \, \mathrm{d}x_n \lesssim |\lambda|^{-1} \|f\|_{L_1(\Omega)}$$

for any $s > -1$.

For the high frequency part v_n^∞ we can use Lemma 3.1 and Lemma I.4.1 again to show for $0 < s < 1$ the estimate

$$\begin{aligned} \|v_n^\infty(\cdot, x_n)\|_{\dot{B}^{s+1}_{1,\infty}(\mathbb{R}^{n-1})} &\lesssim \left\{ |\lambda|^{-1/2} x_n^{-s} + |\lambda|^{-1/2} (\delta - x_n)^{-s} + \frac{x_n^{-s}}{1 + x_n |\lambda|^{1/2}} + \frac{(\delta - x_n)^{-s}}{1 + (\delta - x_n) |\lambda|^{1/2}} \right\} \\ &\quad \cdot \left(\left\| \mathcal{F}^{-1} \mathbf{k}_1 \right\|_{\dot{B}^0_{1,\infty}(\mathbb{R}^{n-1})} + \left\| \mathcal{F}^{-1} \mathbf{k}_2^\infty \right\|_{\dot{B}^0_{1,\infty}(\mathbb{R}^{n-1})} \right). \end{aligned}$$

We can use Lemma 3.1 to estimate $\mathbf{k}_1, \mathbf{k}_2$. Integrating over $(0, \delta)$ we obtain the estimate

$$\int_0^\delta \|v_n^\infty(\cdot, x_n)\|_{\dot{B}^{s+1}_{1,\infty}(\mathbb{R}^{n-1})} \, \mathrm{d}x_n \lesssim |\lambda|^{-\frac{1-s}{2}} \|f\|_{L_1(\Omega)}$$

for any $0 < s < 1$. Similarly we can estimate

$$\begin{aligned} \|v_n^\infty(\cdot, x_n)\|_{L_1(\mathbb{R}^{n-1})} &\lesssim \left\{ |\lambda|^{-1/2} + \frac{1}{1 + x_n |\lambda|^{1/2}} + \frac{1}{1 + (\delta - x_n) |\lambda|^{1/2}} \right\} \\ &\quad \cdot \left(\left\| \mathcal{F}^{-1} z^{-1} \mathbf{k}_1^\infty \right\|_{L_1(\mathbb{R}^{n-1})} + \left\| \mathcal{F}^{-1} z^{-1} \mathbf{k}_2^\infty \right\|_{L_1(\mathbb{R}^{n-1})} \right). \end{aligned}$$

Integrating over $(0, \delta)$ we obtain, together with the results from Proposition I.6.2 and Proposition I.3.7 the estimate

$$\|v_n^\infty\|_{L_1(\Omega)} \lesssim |\lambda|^{-1} \|f\|_{L_1(\Omega)}.$$

Combining results concerning interpolation of Bochner spaces [**LP64**] with interpolation of homogeneous Besov spaces this also shows

$$\int_0^\delta \|\partial_j v_n^\infty(\cdot,x_n)\|_{\dot{B}^s_{1,1}(\mathbb{R}^{n-1})}\ \mathrm{d}x_n \lesssim |\lambda|^{-1/2}\|f\|_{L_1(\Omega)}$$

for $j=1,\dots,n-1$. Combining this with the estimates for the low frequency part we obtain

$$|\lambda|\,\|v_n\|_{L_1(\Omega)} + |\lambda|^{1/2}\|\nabla' v_n\|_{L_1(\Omega)} \lesssim \|f\|_{L_1(\Omega)}.$$

The normal derivative $\partial_n v_n$ can be treated in much the same way. □

For the pressure θ we can show the following result.

PROPOSITION 3.3. *Under the assumptions of Theorem 1.1, for $\lambda\in\Sigma_\rho$ with $|\lambda|\geq\lambda_0$ and $f\in C^\infty_{c,\sigma}(\Omega)$ the pressure θ given by the representation formula II.1.3 satisfies the estimates*

$$\Big\|\|\nabla\theta(\cdot,x_n)\|_{L_1(\mathbb{R}^{n-1})}\Big\|_{L_{1,\infty}(0,\delta)} \lesssim |\lambda|^{1/2}\int_0^\delta \|f(\cdot,x_n)\|_{H_1(\mathbb{R}^{n-1})}\ \mathrm{d}x_n$$

whenever the right hand side is finite. In the case $n=2$ we have the better estimate

$$\Big\|\|\nabla\theta(\cdot,x_n)\|_{L_1(\mathbb{R}^{n-1})}\Big\|_{L_{1,\infty}(0,\delta)} \lesssim |\lambda|^{1/2}\|f\|_{L_1(\Omega)}.$$

PROOF. It follows immediately from the representation formula II.1.3, boundedness of the Riesz transforms on Hardy spaces and the estimates in Lemma I.4.1 that we have for $j=1,\dots,n-1$ an estimate

$$\begin{aligned}\|\partial_j\theta(\cdot,x_n)\|_{L_1(\mathbb{R}^{n-1})} &\lesssim \left(|\lambda|^{1/2}+\frac{1}{x_n}+\frac{1}{\delta-x_n}\right)\left(\left\|\mathcal{F}^{-1}\mathbf{k}_1\right\|_{H_1(\mathbb{R}^{n-1})}+\left\|\mathcal{F}^{-1}\mathbf{k}_2\right\|_{H_1(\mathbb{R}^{n-1})}\right)\\ &\lesssim \left(|\lambda|^{1/2}+\frac{1}{x_n}+\frac{1}{\delta-x_n}\right)\int_0^\delta \|f(\cdot,x_n)\|_{H_1(\mathbb{R}^{n-1})}\ \mathrm{d}x_n\end{aligned}$$

where the last inequality is a consequence of Lemma 3.1. This shows the estimate

$$\Big\|\|\partial_j\theta(\cdot,x_n)\|_{L_1(\mathbb{R}^{n-1})}\Big\|_{L_{1,\infty}(0,\delta)} \lesssim |\lambda|^{1/2}\int_0^\delta \|f(\cdot,x_n)\|_{H_1(\mathbb{R}^{n-1})}\ \mathrm{d}x_n$$

for $j=1,\dots,n-1$. The function

$$\partial_n\hat{\theta}(\xi',x_n) = -(z+\omega)\left\{(1+e^{-\omega\delta})(e^{-zx_n}+e^{-z(\delta-x_n)})\mathbf{k}_1-(1-e^{-\omega\delta})(e^{-zx_n}-e^{-z(\delta-x_n)})\mathbf{k}_2\right\}$$

can be estimated in much the same way, and since there are no Riesz transforms involved we obtain the better estimate

$$\Big\|\|\partial_n\theta(\cdot,x_n)\|_{L_1(\mathbb{R}^{n-1})}\Big\|_{L_{1,\infty}(0,\delta)} \lesssim |\lambda|^{1/2}\|f\|_{L_1(\Omega)}.$$

This shows the first assertion. In the case $n=2$ we have

$$\left\|\mathcal{F}^{-1}i\frac{\xi_1}{|\xi'|}\mathbf{k}_1\right\|_{L_1(\mathbb{R}^{n-1})}+\left\|\mathcal{F}^{-1}i\frac{\xi_1}{|\xi'|}\mathbf{k}_2\right\|_{L_1(\mathbb{R}^{n-1})}\lesssim\|f\|_{L_1(\Omega)}$$

and in the same way as above we obtain

$$\Big\|\|\partial_1\theta(\cdot,x_n)\|_{L_1(\mathbb{R}^{n-1})}\Big\|_{L_{1,\infty}(0,\delta)} \lesssim |\lambda|^{1/2}\|f\|_{L_1(\Omega)}.$$

This shows the second assertion. □

In particular we see that $\nabla\theta$ is integrable on every set of the form $\mathbb{R}^{n-1}\times(\varepsilon,\delta-\varepsilon)$ with $0<\varepsilon<\delta/2$.

PROPOSITION 3.4. *Under the assumptions of Theorem 1.1, for* $\lambda \in \Sigma_\rho$ *with* $|\lambda| \geq \lambda_0$ *and* $f \in C^\infty_{c,\sigma}(\Omega)$ *the components* $v_1, \dots, v_{n-1}$ *given by the representation formula II.1.3 satisfy the estimate*

$$|\lambda| \|v_j\|_{L_1(\Omega)} + |\lambda|^{1/2} \|\nabla v_j\|_{L_1(\Omega)} \lesssim \int_0^\delta \|f(\cdot, x_n)\|_{H_1(\mathbb{R}^{n-1})} \, \mathrm{d}x_n$$

whenever the right hand side is finite. If $n = 2$ *then the improved estimate*

$$|\lambda| \|v_j\|_{L_1(\Omega)} + |\lambda|^{1/2} \|\nabla v_j\|_{L_1(\Omega)} \lesssim \|f\|_{L_1(\Omega)}$$

holds.

PROOF. The first assertion is an immediate consequence of the estimates in Lemma 3.1 and in Lemma I.4.1. For the second assertion, i. e. the case $n = 2$, it suffices to note that

$$\left\| \mathcal{F}^{-1} i \frac{\xi_1}{|\xi'|} \mathbf{k}_1 \right\|_{L_1(\mathbb{R}^{n-1})} + \left\| \mathcal{F}^{-1} i \frac{\xi_1}{|\xi'|} \mathbf{k}_2 \right\|_{L_1(\mathbb{R}^{n-1})} \lesssim \|f\|_{L_1(\Omega)}$$

which can be seen to hold as in the proof of Proposition III.3.4, and apply Lemma I.4.1. □

Proving Theorem 1.1 and Theorem 1.2 now essentially amounts to collecting the results we obtained in this chapter so far.

PROOF OF THEOREM 1.1 AND THEOREM 1.2. The estimates in Theorem 1.1 and 1.2 follow immediately from the estimates in Proposition I.6.2 as well as Proposition 3.2, Proposition 3.3, and Proposition 3.4. Since the divergence of u vanishes by construction it is a consequence of Proposition 2.1 that the solution u is contained in $L_{1,\sigma}(\Omega)$.

It remains to address the issue of uniqueness of solutions. Assume (u, θ) is a solution to (2) with data $f \equiv 0$ such that u and θ satisfy $u \in L_{1,\sigma}(\Omega)$, $\nabla u \in L_1(\Omega)$, $\Delta u, \nabla\theta \in L_{1,\infty}(0, \delta; L_1(\mathbb{R}^{n-1}))$. Let us also assume that $\Delta u, \nabla\theta \in L_1(\mathbb{R}^{n-1} \times (\varepsilon, \delta - \varepsilon))$ for any $0 < \varepsilon < \delta/2$. Observe that this is satisfied by the solution constructed in Proposition 3.2, Proposition 3.3, and Proposition 3.4.

Take a function $g \in C^\infty_{c,\sigma}(\Omega)$ and an integer $j \in \{1, \dots, n-1\}$. Then there is $\varepsilon > 0$ such that $\operatorname{supp} g \subset \mathbb{R}^{n-1} \times (\varepsilon, \delta - \varepsilon) =: \Omega_\varepsilon$ and hence $g \in C^\infty_{c,\sigma}(\Omega_\varepsilon)$. Let (v, π) denote the solution of the Stokes resolvent equation in Ω_ε corresponding to $\partial_j g$. It is a consequence of Theorem III.1.1 and interpolation of homogenous Besov spaces that we have $v \in C_{0,\sigma}(\Omega_\varepsilon) \cap W^2_\infty(\Omega_\varepsilon)$. We can compute

$$\begin{aligned} \langle \partial_j u, g \rangle &= \langle u, \partial_j g \rangle \\ &= \langle u, \lambda v - \Delta v + \nabla \pi \rangle \\ &= \langle \lambda u - \Delta u + \nabla\theta, v \rangle \\ &= 0. \end{aligned}$$

It follows from de Rham's Theorem (Theorem III.2.1) that $\partial_j u = \nabla w$ for some distribution $w \in \mathscr{D}'(\Omega)$. Since $\partial_j u$ is divergence-free w must be a harmonic distribution and it follows from Weyl's Lemma that w is a harmonic function. But then $\partial_j u$ must be harmonic as well.

Since $\partial_j u \in L_1(\Omega)$ vanishes on the boundary $\partial\Omega$ it follows that $\partial_j u$ must vanish almost everywhere. This can be seen as follows. One can extend $\partial_j u$ from Ω antisymmetrically through the boundary of Ω to a larger layer and repeat this procedure to obtain a harmonic extension $F\colon \mathbb{R}^n \to \mathbb{C}^n$ of $\partial_j u$ that coincides (modulo sign) with $\partial_j u$ on each layer $\mathbb{R}^{n-1} \times (k\delta, k\delta + \delta)$ for $k \in \mathbb{Z}$. Then the mean value property of harmonic functions yields for $R > 0$ and $x \in \mathbb{R}^n$

$$F(x) = \frac{1}{c_n R^n} \int_{|y| \leq R} F(x + y) \, \mathrm{d}y$$

with c_n denoting the volume of the unit ball in $\mathbb{R}^n$. Observe that the ball of radius R in $\mathbb{R}^n$ intersects with at most $\lceil R/\delta \rceil + 1$ layers $\mathbb{R}^{n-1} \times (k\delta, k\delta + \delta)$. Thus we can estimate for $x \in \Omega$ and $R > 0$

$$\begin{aligned} |\partial_j u(x)| &= |F(x)| \\ &\leq \frac{1}{c_n R^n} \int_{|y| \leq R} |F(x+y)| \, \mathrm{d}y \\ &\leq \frac{\lceil R/\delta \rceil + 1}{c_n R^n} \int_\Omega |F(y)| \, \mathrm{d}y \\ &\leq \frac{\lceil R/\delta \rceil + 1}{c_n R^n} \left\| \partial_j u \right\|_{L_1(\Omega)} . \end{aligned}$$

Taking the limit $R \to \infty$ shows that $\partial_j u(x)$ is zero, and since $x \in \Omega$ was chosen arbitrarily we obtain that $\partial_j u$ vanishes for $j = 1, \ldots, n-1$. In particular u only depends on x_n.

In order for u to be integrable it is necessary that u vanishes. But then $\nabla\theta$ must vanish as well. In particular solutions to (2) are unique. □

Now we are able to show that in the case $n = 2$ the Stokes operator, if appropriately defined, generates a holomorphic and strongly continuous semigroup on $L_{1,\sigma}(\Omega)$.

Proof of Corollary 1.3. Let $n = 2$. Given $0 < \rho < \pi$ and $\lambda_0 > 0$ we write for any $\lambda \in \Sigma_\rho$ with $|\lambda| \geq \lambda_0$

$$R_\lambda \colon C_{c,\sigma} \to L_{1,\sigma}(\Omega), \qquad f \mapsto u$$

for the solution operator from Theorem 1.2. This operator extends to a bounded operator on $L_{1,\sigma}(\Omega)$, which we denote again with R_λ. One can show that the operators R_λ satisfy the resolvent identity, and, as in the proof of Corollary III.1.3 one can show that the kernel of R_λ is trivial. To this end it suffices to show that λv_λ converges to zero in $\mathscr{D}'(\Omega)$. This can be shown much like as in the proof of Corollary III.1.3. Then it follows from Theorem VIII.4.1 in [**Yos74**] that there is an operator A in $L_{1,\sigma}(\Omega)$ with $D(A) = R_\lambda L_{1,\sigma}(\Omega)$ such that $R_\lambda = R(\lambda, A)$. We will refer to this operator A as the Stokes operator in $L_{1,\sigma}(\Omega)$. It remains to show that A is densely defined, but this is trivially the case since $D(A)$ contains $C^\infty_{c,\sigma}(\Omega)$. It follows from Theorem 1.2 that A is sectorial of angle π, and hence is the generator of a strongly continuous holomorphic semigroup on $L_{1,\sigma}(\Omega)$ of angle $\pi/2$. □

4. Cancellation properties. Necessity and sufficiency for $n \geq 3$.

Throughout this section we assume $n \geq 3$. We will begin with a proof of Theorem 1.4. We will discuss several conditions on the data f such that the resulting solution u is integrable.

The considerations in this section rely chiefly on the classical Lemma of Riemann-Lebesgue (Proposition I.3.3) that states that the Fourier transform of an integrable function is continuous and vanishes at infinity, i. e. $\mathcal{F}L_1(\mathbb{R}^d) \subset C_0(\mathbb{R}^d)$.

If $u \in L_1(\Omega)$ then for almost every $x_n \in (0, \delta)$ the function $u(\cdot, x_n)$ must be in $L_1(\mathbb{R}^{n-1})$. The Lemma of Riemann-Lebesgue implies that for almost every $x_n \in (0, \delta)$ we have $\hat{u}(\cdot, x_n) \in C_0(\mathbb{R}^{n-1})$, and in particular that $\hat{u}(\cdot, x_n)$ is continuous at $\xi' = 0$. For $j = 1, \ldots, n-1$ this implies that the limit

$$\lim_{\xi' \to 0} \hat{v}_j(\xi', x_n) = -\left(1 - e^{-\delta\sqrt{\lambda}}\right) \left[\Psi_+(x_n)\big|_{z=0}\right] \lim_{\xi' \to 0} i \frac{\xi_j}{|\xi'|} \mathbf{k}_2$$

exists. For this limit to exist for every $j = 1, \ldots, n-1$ it is both necessary and sufficient that

$$\lim_{\xi' \to 0} \mathbf{k}_2 = 0.$$

Since we can write $\mathbf{k}_2$ as

$$\mathbf{k}_2 = -\frac{z}{2\Phi_+(z,\omega)} \int_0^\delta \frac{\sinh[\omega t]}{\sinh[\omega\delta]} \frac{\hat{f}_n(\xi', \delta - t) - \hat{f}_n(\xi', t)}{|\xi'|} \, \mathrm{d}t$$

we can infer that for functions f of the form

$$f \colon \Omega \to \mathbb{C}^n, \qquad x \mapsto (-g(x')\partial_n h(x_n), \operatorname{div}' g(x')h(x_n))$$

with $g \in \mathscr{D}(\mathbb{R}^{n-1})$ and $h \in \mathscr{D}(0,\delta)$ it is necessary and sufficient for $\lim_{\xi' \to 0} \mathbf{k}_2 = 0$ to hold that we have

$$\lim_{\xi' \to 0} \frac{\xi'}{|\xi'|} \cdot \hat{g} \int_0^\delta \frac{\sinh[\sqrt{\lambda} t]}{\sinh[\sqrt{\lambda}\delta]} \left[h(\delta - t) - h(t)\right] \, \mathrm{d}t = 0.$$

If

$$\int_0^\delta \frac{\sinh[\sqrt{\lambda} t]}{\sinh[\sqrt{\lambda}\delta]} \left[h(\delta - t) - h(t)\right] \, \mathrm{d}t \neq 0$$

then this is equivalent to

$$\hat{g}_j(0) = 0$$

for $j = 1, \ldots, n-1$, which in turn is equivalent to

$$\int_{\mathbb{R}^{n-1}} g_j(x') \, \mathrm{d}x' = 0$$

for $j = 1, \ldots, n-1$.

This shows that any function $f = (-g\partial_n h, \operatorname{div}' gh)$ as above satisfying

$$\int_0^\delta \frac{\sinh[\sqrt{\lambda} t]}{\sinh[\sqrt{\lambda}\delta]} \left[h(\delta - t) - h(t)\right] \, \mathrm{d}t \neq 0$$

and

$$\int_{\mathbb{R}^{n-1}} g_j(x') \, \mathrm{d}x' \neq 0$$

for some $j \in \{1, \ldots, n-1\}$ provides a counterexample to Theorem 1.2 in $n \geq 3$ dimensions. In particular we find counterexamples in the class $C^\infty_{c,\sigma}(\Omega)$. Since any compactly supported bounded function with vanishing mean value is contained in the Hardy space H_1, it follows from Theorem 1.1 that the relations

$$\int_{\mathbb{R}^{n-1}} g_j(x') \, \mathrm{d}x' = 0 \quad \text{for all } j = 1, \ldots, n-1, \text{ or} \qquad \int_0^\delta \frac{\sinh[\sqrt{\lambda} t]}{\sinh[\sqrt{\lambda}\delta]} \left[h(\delta - t) - h(t)\right] \, \mathrm{d}t = 0$$

give a characterisation of admissible functions f of the form above. Let us elaborate on that point. Assume $h \in \mathscr{D}(0,\delta)$ satisfies

$$\int_0^\delta \frac{\sinh[\sqrt{\lambda} t]}{\sinh[\sqrt{\lambda}\delta]} \left[h(\delta - t) - h(t)\right] \, \mathrm{d}t = 0$$

for every $\lambda \geq \lambda_0$ for some $\lambda_0 > 0$. From the power series expansion of the hyperbolic sine we can infer that this is equivalent to

$$\sum_{k=0}^\infty \frac{\sqrt{\lambda}^{2k+1}}{(2k+1)!} \int_0^\delta t^{2k+1} \left[h(\delta - t) - h(t)\right] \, \mathrm{d}t = 0.$$

Uniqueness of power series implies that we have

$$\int_0^\delta t^{2k+1} \left[h(\delta - t) - h(t)\right] \, \mathrm{d}t = 0$$

for every $k \in \mathbb{N}_0$. Uniqueness of solutions to the classical Hausdorff moment problem, or Weierstrass' Approximation Theorem, yields that h necessarily satisfies $h(t-\delta) = h(t)$ for every t. This shows that a function f of the above structure gives rise to an integrable solution u for every $\lambda \geq \lambda_0$ if and only if

$$h \equiv h(\delta - \cdot) \qquad \text{or} \qquad \int_{\mathbb{R}^{n-1}} g_j(x')\,\mathrm{d}x' = 0 \text{ for } j = 1, \dots, n-1.$$

This is equivalent to f satisfying either $f_n(x', x_n) = f_n(x', \delta - x_n)$ for almost all $x \in \Omega$ or

$$\int_{\mathbb{R}^{n-1}} f(x', x_n)\,\mathrm{d}x' = 0$$

for almost all $x_n \in (0, \delta)$. The same reasoning can be applied to the functions $\partial_n v_j$ and $\partial_j \theta$ for $j = 1, \dots, n-1$, with the same outcome. Choosing g and h appropriately we arrive at the conclusion of Theorem 1.4.

5. The Stokes equation in a half space. The limiting case $\delta \to \infty$.

Saal [**Saa07**] and Desch et al. [**DHP01**] showed that there is a divergence-free function $f \in L_1(\mathbb{R}^n_+)$ such that the corresponding solution (u, θ) given by the representation formula II.1.3 satisfies $u \notin L_1(\Omega)$.

We will give a short proof of a slight improvement of their result and discuss necessary and sufficient conditions on the data f for the solution to be integrable. We begin with the following simple observation.

LEMMA 5.1. *Let $f \in L_1(\mathbb{R}^n_+)$ with $\operatorname{div} f = 0$ in $\mathscr{D}'(\mathbb{R}^n_+)$, and for $\lambda > 0$ let $(u_\lambda, \theta_\lambda)$ denote the solution given by representation formula II.1.3. Then $u_\lambda \in L_1(\Omega)$ if and only if*

$$\int_0^\infty \|v_n(\cdot, x_n)\|_{H_1(\mathbb{R}^{n-1})}\,\mathrm{d}x_n < \infty.$$

In this case we have the estimate

$$\int_0^\infty \|v_n(\cdot, x_n)\|_{H_1(\mathbb{R}^{n-1})}\,\mathrm{d}x_n \leq \|v\|_{L_1(\Omega)}.$$

PROOF. The function u_λ is in $L_1(\Omega)$ if and only if $v_\lambda \in L_1(\Omega)$ with

$$\hat{v}_{\lambda,n}(\xi', x_n) = -\varphi(x_n)\partial_n \hat{w}_n(\xi, 0)$$
$$\hat{v}_{\lambda,j}(\xi', x_n) = -\varphi(x_n) i \frac{\xi_j}{|\xi|} \partial_n \hat{w}_n(\xi, 0) = i \frac{\xi_j}{|\xi'|} \hat{v}_{\lambda,n}(\xi', x_n).$$

It is a consequence of the characterisation of the Hardy space H_1 in Theorem 6.7.5 of [**Gra09**] that this is the case precisely if

$$\int_0^\infty \|v_n(\cdot, x_n)\|_{H_1(\mathbb{R}^{n-1})}\,\mathrm{d}x_n < \infty,$$

and then we obtain

$$\int_0^\infty \|v_n(\cdot, x_n)\|_{H_1(\mathbb{R}^{n-1})}\,\mathrm{d}x_n \leq \|v\|_{L_1(\Omega)}.$$

□

We specialise to the case where the function f is of the form

$$f \colon \Omega \to \mathbb{C}^n, \qquad f(x) = \big(g(x')\partial_n h(x_n), -\operatorname{div}' g(x') h(x_n)\big)$$

for $g \in \mathscr{D}(\mathbb{R}^{n-1})^{n-1}$ and $h \in \mathscr{D}(0, \delta)$. Then $f \in C^\infty_{c,\sigma}(\Omega)$.

Given such a function f we have

$$\hat{v}_n(\xi', x_n) = \varphi(x_n)\left[\int_0^\infty e^{-\omega t} h(t)\,\mathrm{d}t\right] i\xi' \cdot \hat{g}(\xi').$$

Applying the Laplace transform with respect to x_n we obtain

$$\mathcal{F}_{\xi'}^{-1}\int_0^\infty e^{-sx_n}\hat{v}_n(\xi', x_n)\,\mathrm{d}x_n = -\mathcal{F}_{\xi'}^{-1}\frac{1}{s+\omega}\frac{|\xi'|}{s+|\xi'|}\left[\int_0^\infty e^{-\omega t}h(t)\,\mathrm{d}t\right] i\frac{\xi'}{|\xi'|}\cdot\hat{g}(\xi') \in H_1(\mathbb{R}^{n-1}).$$

Let $s = 0$. Since functions in H_1 generally have vanishing mean value we obtain for $\xi' \to 0$ the relation

$$0 = \left[\int_0^\infty e^{-\sqrt{\lambda}t}h(t)\,\mathrm{d}t\right]\lim_{\xi'\to 0} i\frac{\xi'}{|\xi'|}\cdot\hat{g}(\xi').$$

If

$$\int_0^\infty e^{-\sqrt{\lambda}t}h(t)\,\mathrm{d}t \neq 0$$

then necessarily $\hat{g}_j(0) = 0$ for every $j = 1, \ldots, n-1$, which is equivalent to

$$\int_{\mathbb{R}^{n-1}} g_j(x')\,\mathrm{d}x' = 0$$

for $j = 1, \ldots, n-1$. There are functions f of this structure that do not satisfy these compatibility conditions, and by our calculations these functions provide examples for data f such that the corresponding solution u is not integrable. The pressure θ can be treated in the same way, with the same outcome. This shows the following Theorem, which is a slightly stronger version of Theorem 5.1 of [**DHP01**].

THEOREM 5.2. *Let $\lambda > 0$ and $n \geq 2$. There is $f \in C^\infty_{c,\sigma}(\mathbb{R}^n_+)$ such that the corresponding solution (u, θ) of (2) satisfies $u \notin L_1(\mathbb{R}^n_+)$ and $\nabla\theta \notin L_1(\mathbb{R}^n_+)$.*

It was shown by Giga et al. [**GMS99**] that $\nabla u \in H_1(\mathbb{R}^n_+)$ for any given $f \in L_{1,\sigma}(\mathbb{R}^n_+)$. This stands in contrast to the case of a layer Ω of finite width, where we could show that for $n \geq 3$ even the inclusion $\nabla u \in L_1(\Omega)$ may fail.

6. The Stokes resolvent problem in L_p with $1 < p < \infty$

Even though we produced examples showing that the Stokes resolvent problem in L_1 and L_∞ is not in general well-posed in layer domains of dimension $n \geq 3$ the weaker estimates we obtained in Theorem III.1.1 and Theorem 1.1 are still strong enough to show resolvent estimates for the Stokes resolvent problem in subspaces of L_p for $1 < p < \infty$.

In complete analogy to the proof of Theorem III.1.1 one can show a slight improvement in that one can show that the solution operator is actually continuous from $C^\infty_{c,\sigma}(\Omega)$ to the space of functions $u\colon \Omega \to \mathbb{C}^n$ such that $x_n \to \|u(\cdot, x_n)\|_{\mathrm{BMO}(\mathbb{R}^{n-1})}$ is essentially bounded.

Let us introduce for the moment the following short-hand notation. We write

$$A_1 = L_1(0,\delta; H_1(\mathbb{R}^{n-1})),\quad A_\infty = L_\infty(\Omega),\quad B_1 = L_1(\Omega),\quad B_\infty = L_\infty(0,\delta;\mathrm{BMO}(\mathbb{R}^{n-1})),$$

and write $A_{\cdot,\sigma}, B_{\cdot,\sigma}$ for the closure of $C^\infty_{c,\sigma}(\Omega)$ in $A_\cdot$ and $B_\cdot$, respectively. If $R_\lambda\colon f \mapsto u$ denotes the solution operator constructed in Theorem III.1.1 and Theorem 1.1 then general interpolation theory shows that

$$R_\lambda\colon\ (A_{1,\sigma}, A_{\infty,\sigma})_{\theta,p} \to (B_{1,\sigma}, B_{\infty,\sigma})_{\theta,p}$$

is bounded for $0 < \theta < 1$ and $1 \leq p \leq \infty$. For $\theta = 1 - 1/p$ we can use interpolation results for Bochner spaces [**LP64**] and for the interpolation couple $L_1(\mathbb{R}^{n-1})$ and $\mathrm{BMO}(\mathbb{R}^{n-1})$ due to Hanks [**Han77**] to show

$$\begin{aligned}(B_1, B_\infty)_{\theta,p} &= \big(L_1(0,\delta; L_1(\mathbb{R}^{n-1})), L_\infty(0,\delta; \mathrm{BMO}(\mathbb{R}^{n-1}))\big)_{\theta,p} \\ &= L_p(0,\delta; (L_1(\mathbb{R}^{n-1}), \mathrm{BMO}(\mathbb{R}^{n-1}))_{\theta,p} \\ &= L_p(\Omega).\end{aligned}$$

Similarly, interpolation results due to Rivière and Sagher [**RS73**] for the interpolation couple $H_1(\mathbb{R}^{n-1})$ and $L_\infty(\mathbb{R}^{n-1})$ yield

$$\begin{aligned}(A_1, A_\infty)_{\theta,p} &= \big(L_1(0,\delta; H_1(\mathbb{R}^{n-1})), L_\infty(0,\delta; L_\infty(\mathbb{R}^{n-1}))\big)_{\theta,p} \\ &= L_p(0,\delta; (H_1(\mathbb{R}^{n-1}), L_\infty(\mathbb{R}^{n-1}))_{\theta,p} \\ &= L_p(0,\delta; L_p(\mathbb{R}^{n-1})) \\ &= L_p(\Omega).\end{aligned}$$

It follows immediately that for $\theta = 1 - 1/p$ we have

$$(A_{1,\sigma}, A_{\infty,\sigma})_{\theta,p} \hookrightarrow L_{p,\sigma}(\Omega), \qquad (B_{1,\sigma}, B_{\infty,\sigma})_{\theta,p} \hookrightarrow L_{p,\sigma}(\Omega).$$

If one can show that, in fact, $(A_{1,\sigma}, A_{\infty,\sigma})_{\theta,p}$ coincides with $L_{p,\sigma}(\Omega)$ then this would imply resolvent estimates in $L_{p,\sigma}(\Omega)$.

Similarly in $n = 2$ dimensions the resolvent operator $R(\lambda, A)$ from Theorem III.1.2 and Theorem 1.2 extends to a bounded operator on the real interpolation space $(L_{1,\sigma}(\Omega), C_{0,\sigma}(\Omega))_{1-1/p,p}$ as well as on the complex interpolation space $[L_{1,\sigma}(\Omega), C_{0,\sigma}(\Omega)]_{1-1/p}$. It is easily seen that these interpolation spaces are continuously embedded into $L_{p,\sigma}(\Omega)$, but whether these spaces coincide with $L_{p,\sigma}(\Omega)$ seems to be open. This question is closely connected to the problem of interpolation of intersections of spaces [**KMP99**]. In spaces where the Helmholtz projection exists and is bounded, e. g. in $L_p(\Omega)$ for $1 < p < \infty$, this difficulty can be overcome by means of the method of retraction and coretraction [**Tri78**], the essential ingredient being that $L_{p,\sigma}(\Omega)$ is a complemented subspace of $L_p(\Omega)$.

If one succeeded to show the desired interpolation results then this would yield a rather novel approach to the Stokes resolvent problem in L_p. This idea is also present in recent work by Abe et al. [**AGSS14b**] where they apply interpolation theory to derive estimates in subspaces of L_p from estimates in solenoidal subspaces of L_2 and L_∞ in domains where the Helmholtz decomposition does not hold.

CHAPTER V

The water wave problem in the singular limit of vanishing surface tension

1. Introduction

In this chapter we study the free boundary problem

$$(1)\qquad \begin{cases} \partial_t v - \Delta v + v\nabla v + \nabla\pi = 0 & \text{in } \Omega(t) \\ \operatorname{div} v = 0 & \text{in } \Omega(t) \\ -S(v,\pi)\nu = \sigma\kappa\nu & \text{on } \Gamma^+(t) \\ V = v\nu & \text{on } \Gamma^+(t) \\ v = 0 & \text{on } \Gamma^- \\ v(0) = v_0 & \text{in } \Omega(0) \end{cases}$$

for $t > 0$ where $\Omega(t)$ is an unknown layer-like domain with Γ^- and $\Gamma^+(t)$ denoting the fixed lower and free upper boundary of $\Omega(t)$, respectively. The initial domain $\Omega(0) = \Omega_0$ is assumed to be parametrised by a given function h via

$$(2)\qquad \Omega_0 = \{x \in \mathbb{R}^n_+ : 0 < x_n < \delta + h(x')\}$$

with some constant $\delta > 0$ such that $\delta + h$ is bounded away from zero. We denote the upper boundary of Ω_0 with Γ_0^+. Then $\Gamma_0^+ = \{x \in \mathbb{R}^n_+ : x_n = \delta + h(x')\}$.

Here v and π denote the velocity field and pressure of the fluid in question. We write $S(v,\pi) = -\pi I + (\nabla v + (\nabla v)^T)$ for the Cauchy stress tensor, $\sigma \geq 0$ denotes the surface tension parameter, κ the mean curvature on the upper surface $\Gamma^+(t)$, and V denotes the velocity of the upper surface in normal direction.

For $\sigma > 0$ a system closely resembling (1) has been studied by Denk et al. [**DGH⁺11**]. They studied (1) with partial slip conditions on the lower boundary Γ^- and included the effect of rotation in their model. For $\sigma > 0$ their approach carries over to the system (1) without any difficulties. For $\sigma = 0$, i. e. not taking the effect of surface tension into account, we arrive at the equations studied in e. g. [**Abe05a**].

These two results, the first one covering the case $\sigma > 0$, and the second for $\sigma = 0$, have been obtained by different methods and a priori it is not clear at all how solutions to (1) behave for $\sigma \to 0^+$. Furthermore, since the mean curvature κ essentially involves the Laplace-Beltrami operator on the upper surface we expect to have in general higher regularity of the free upper surface Γ^+ if $\sigma > 0$ than in the case $\sigma = 0$. In particular the limit $\sigma \to 0^+$ is a singular limit. Our aim is to investigate the singular limit of vanishing surface tension and to show that solutions to (1) corresponding to $\sigma > 0$ do in fact converge to the solution corresponding to $\sigma = 0$ under suitable hypotheses.

Our methods resemble those of [**PSS12**] where similar investigations aimed at the two-phase Stefan problem were carried out.

The case $\sigma > 0$ in [**DGH**$^+$**11**] was treated in Eulerian coordinates by means of the Hanzawa transform. This is not possible in the case $\sigma = 0$ since the upper surface Γ^+ is, in general, not regular enough. The estimates shown in [**DGH**$^+$**11**] break down as $\sigma \to 0^+$. The case $\sigma = 0$, however, was treated in [**Abe05a**] in Lagrangian coordinates.

Thus we will transform the system (1) to Lagrangian coordinates and solve the transformed system. This will allows us to show uniform estimates in $\sigma \geq 0$ and ultimately also convergence of the corresponding solutions. The drawback of this approach, however, is that it is not clear how to recapture the higher boundary regularity for $\sigma > 0$ that was shown to hold in [**DGH**$^+$**11**].

The system (1) with and without surface tension has been studied by many authors from various different points of view.

The water wave problem without taking the influence of surface tension into account has been studied extensively by, among others, Beale [**Bea81**], Sylvester [**Syl90**], Tani and Tanaka [**TT95**] in an L_2-setting, and later by Abels [**Abe05a**] in an L_p-setting for $1 < p < \infty$.

A considerable amount of work has also been dedicated by various authors to the water wave problem in the presence of surface tension, i. e. in the case $\sigma > 0$. Solonnikov [**Sol86, Sol89, Sol91**] studied the free boundary problem (1) in bounded domains and proved local-in-time existence of solutions in a Hilbert space setting. Similar results in Hölder spaces have been obtained by Mogilevskiĭ and Solonnikov [**MS91**]. Tani [**Tan96**] studied the water-wave problem in an infinite layer domain in a Hilbert space setting proving local-in-time existence of solutions to the free boundary problem (1). Beale [**Bea84**] and Beale and Nishida [**BN85**] studied the long-term behaviour of solutions to (1) in a Hilbert space setting and proved global-in-time solvability, Tani and Tanaka [**TT95**] obtained similar results with and without inclusion of the effect of surface tension. An extensive survey of results concerning the water wave problem (1) and related free boundary problems can be found in [**Zad04**].

The first results concerning free boundary problems related to (1) in an L_p-setting were obtained by Prüss and Simonett [**PS10**] for the two-phase Navier-Stokes equations in a half space, and by Shibata and Shimizu [**SS11**] who proved local-in-time existence of solutions for (1) in a rather general class of domains.

Up to our knowledge the behaviour of solutions to (1) in the singular limit of vanishing surface tension has not been investigated thus far. Our aim is to close this gap.

Our analysis of (1) will be carried out to some extent along the lines of Shibata and Shimizu [**SS11**], but we will track the dependence of solutions on the surface tension parameter $\sigma \geq 0$ in order to obtain information about convergence of solutions in the singular limit $\sigma \to 0^+$. This is possible in a Lagrangian formulation, and in fact this is the reason why we chose to investigate (1) in a Lagrangian formulation.

As in [**SS11**] we pass to Lagrangian coordinates in (1) by means of the mapping

$$X_u \colon J \times \Omega_0 \to \bigcup_{t \in J} \{t\} \times \Omega(t), \qquad (t, \xi) \mapsto \xi + \int_0^t u(\tau, \xi) \, \mathrm{d}\tau$$

where u and v are related via $u(t,\xi) = v(t, X_u(t,\xi))$ for $(t,\xi) \in J \times \Omega_0$. If we write $\theta(t,\xi) = \pi(t, X_u(t,\xi))$ then we obtain the system

$$\tag{3} \begin{cases} \partial_t u - \Delta u + \nabla\theta = F_1(u,\theta) & \text{in } J \times \Omega_0 \\ \operatorname{div} u = F_d(u) & \text{in } J \times \Omega_0 \\ \Pi_0 E u \nu_0 = G_\parallel^+(u) & \text{on } J \times \Gamma_0^+ \\ \nu_0 \cdot S(u,\theta)\nu_0 - \sigma(m - \Delta_{\Gamma_0^+})\eta = G_\perp^+(u,\pi) & \text{on } J \times \Gamma_0^+ \\ \partial_t \eta - u \cdot \nu_0 = K^+(u) & \text{on } J \times \Gamma_0^+ \\ u = 0 & \text{on } J \times \Gamma^- \\ u(0) = u_0 & \text{in } \Omega_0 \\ \eta(0) = 0 & \text{in } \Gamma_0^+ \end{cases}$$

with some sufficiently large but fixed $m > 0$ and mappings $\eta, F_1, F_d, G_\parallel^+, G_\perp^+, K^+$ which we will specify later on, together with a more detailed derivation of (3). Here $\Delta_{\Gamma_0^+}$ denotes the Laplace-Beltrami operator on Γ_0^+, ν_0 denotes the outer normal vector on $\partial\Omega_0$ and Π_0 denotes the projection

$$\Pi_0 \colon \mathbb{R}^n \to \mathbb{R}^n, \qquad d \mapsto d - (d \cdot \nu_0)\nu_0.$$

In the sequel we will construct a solution to (3) and show that this solution converges as $\sigma \to 0^+$ to the solution of the corresponding equation with $\sigma = 0$ in a suitable topology. Our method of proof relies essentially on a careful study of the associated resolvent problem by means of the Newton Polygon method [**DSS08**].

2. Main results

Let us introduce the following notation for the spaces for the solution and data of (3). The initial value u_0 of the velocity u will be an element of the Sobolev-Slobodeckij space

$$\mathbb{I}_u = W_p^{2-2/p}(\Omega_0).$$

For data from this space satisfying certain compatibility conditions we look for solutions

$$u \in \mathbb{E}_u(J) = H_p^1(J; L_p(\Omega_0)) \cap L_p(J; H_p^2(\Omega_0))$$
$$\theta \in \mathbb{E}_\theta(J) = \{\theta \in L_p(J; \hat{H}_p^1(\Omega_0)) \colon \theta\big|_{\Gamma^+} \in W_p^{1/2-1/2p}(J; L_p(\Gamma_0^+)) \cap L_p(J; W_p^{1-1/p}(\Gamma_0^+))\}$$

and

$$\eta \in \mathbb{E}_\eta^\sigma(J) = H_p^1(J; W_p^{2-1/p}(\Gamma_0^+)) \cap L_p(J; W_p^{3-1/p}(\Gamma_0^+))$$

with norm

$$\|\eta\|_{\mathbb{E}_\eta^\sigma(J)} = \|\eta\|_{H_p^1(J; W_p^{2-1/p}(\Gamma_0^+))} + \sigma \|\Delta_{\Gamma_0^+}\eta\|_{L_p(J; W_p^{1-1/p}(\Gamma_0^+))}$$

if $\sigma > 0$, and

$$\eta \in \mathbb{E}_\eta^0(J) = H_p^1(J; W_p^{2-1/p}(\Gamma_0^+))$$

with norm

$$\|\eta\|_{\mathbb{E}_\eta^0(J)} = \|\eta\|_{H_p^1(J; W_p^{2-1/p}(\Gamma_0^+))}$$

if $\sigma = 0$. Here $J = (0,T)$ denotes a finite time interval. We will occasionally write

$$\mathbb{E}^\sigma(J) = \mathbb{E}_u(J) \times \mathbb{E}_\theta(J) \times \mathbb{E}_\eta^\sigma(J)$$

for the solution space. Our main results read as follows.

THEOREM 2.1. *Let $n \geq 2$, $n < p < \infty$ and $\sigma^* > 0$. Let $R > 0$, and assume that Ω_0 is parametrised as in* (2) *with $h \in W_p^{3-1/p}(\mathbb{R}^{n-1})$ of sufficiently small norm. Then there is a time interval $J = (0, T)$ such that for every $0 \leq \sigma \leq \sigma^*$ and initial velocity $u_0 \in \mathbb{I}_u$ with $\|u_0\|_{\mathbb{I}_u} \leq R$ satisfying the compatibility conditions*

$$\begin{cases} \operatorname{div} u_0 = 0 & \text{in } \Omega_0 \\ \Pi_0 E u_0 \cdot \nu_0 = 0 & \text{on } \Gamma_0^+ \\ u_0 = 0 & \text{on } \Gamma^- \end{cases} \tag{4}$$

there is a unique solution $(u, \theta, \eta) \in \mathbb{E}^\sigma(J)$ of (3). *Moreover,*

$$\|u\|_{\mathbb{E}_u(J)} + \|\theta\|_{\mathbb{E}_\theta(J)} + \|\eta\|_{\mathbb{E}_\eta^\sigma(J)} \leq C \|u_0\|_{\mathbb{I}_u} + C\sigma \|h\|_{W_p^{3-1/p}(\mathbb{R}^{n-1})}$$

with a constant $C > 0$ independent of σ and u_0.

While the constant $C > 0$ in Theorem 4.1 does not depend on σ and u_0, it may very well depend on σ^* and the initial height function h.

Theorem 4.1 stands in contrast to known results insofar as it provides us with intervals of existence and bounds for a solution (u, θ, η) which do not depend on the surface tension parameter σ. Once this result is established it is an obvious question whether the solutions $(u^\sigma, \theta^\sigma, \eta^\sigma)$ corresponding to a surface tension parameter converge to the solution of the equations without surface tension as $\sigma \to 0$ and if so in which topology. This is adressed in the following Theorem.

THEOREM 2.2. *Let $n \geq 2$, $n < p < \infty$ and $\sigma^* > 0$. Assume that Ω_0 is parametrised as in* (2) *with $h \in W_p^{3-1/p}(\mathbb{R}^{n-1})$ of sufficiently small norm. For $0 \leq \sigma \leq \sigma^*$ let u_0^σ belong to $\mathbb{I}_u$ and satisfy the compatibility conditions* (4). *Assume further $\lim_{\sigma \to 0} u_0^\sigma = u_0^0$ in $\mathbb{I}_u$.*

Given $0 \leq \sigma \leq \sigma^*$ let $(u^\sigma, \theta^\sigma, \eta^\sigma) \in \mathbb{E}^\sigma(J)$ denote the unique solution of (3) *with surface tension parameter σ and initial value u_0^σ. Then*

$$\lim_{\sigma \to 0^+} \left\|(u^\sigma, \theta^\sigma, \eta^\sigma) - (u^0, \theta^0, \eta^0)\right\|_{\mathbb{E}^0(J)} = 0.$$

As an immediate consequence of Theorem 2.1 we can see that the free upper surface $\Gamma^+(t)$, which is parametrised as

$$\Gamma^+(t) = X_u(t, \Gamma_0^+) = \left\{\xi + \int_0^t u(\tau, \xi)\,\mathrm{d}\tau \colon \xi \in \Gamma_0^+\right\},$$

can be represented as the image of the function $X_u(t, \cdot) = \operatorname{Id} + S(t, \cdot)$ with

$$S \in H_p^1(J; W_p^{2-1/p}(\Gamma_0^+)) \cap W_p^{2-1/2p}(J; L_p(\Gamma_0^+)).$$

However, for $\sigma > 0$ we would expect from the results in [**DGH+11**] that there should be a parametrisation of $\Gamma^+(t)$ via a height function of regularity

$$H_p^1(J; W_p^{2-1/p}(\Gamma_0^+)) \cap W_p^{2-1/2p}(J; L_p(\Gamma_0^+)) \cap L_p(J; W_p^{3-1/p}(\Gamma_0^+)).$$

It is not clear how one could show this higher regularity of the free upper surface in a Lagrangian formulation. We will give some partial results concerning boundary regularity later on.

In order to investigate the system (3) and prove these results we will start with an analysis of the linearised problem. We will start with the case where Ω_0 is a layer $\mathbb{R}^{n-1} \times (0, \delta)$ and then proceed to the case where the upper boundary of Ω_0 is the graph of a sufficiently regular function. Once we have sufficient information about the linear problem we proceed to the nonlinear system (3) and show Theorem 2.1 by means of a fixed point iteration. But first we will give an account of the derivation of (3).

3. Transformation to Lagrangian coordinates

This section is devoted the derivation of (3) from (1). We follow the presentation in [**SS07**, **SS11**, **Abe05a**]. If a velocity field $u(t,\xi)$ as a function of Lagrangian coordinates (t,ξ) is known then the corresponding Eulerian coordinates (t,x) are given by $(t,x)=(t,X_u(t,\xi))$ where $X_u(t,\cdot)$ is defined for $t\geq 0$ as the mapping

$$X_u(t,\cdot)\colon \Omega_0\to\Omega(t), \qquad \xi\mapsto x=\xi+\int_0^t u(\tau,\xi)\,\mathrm{d}\tau. \tag{5}$$

If v defined on the moving domain $\bigcup_{t\in J}\{t\}\times\Omega(t)$ denotes the same velocity field in Eulerian coordinates then this relation can be written as

$$u(t,\xi)=v(t,x)=v(t,X_u(t,\xi)), \qquad (t,\xi)\in J\times\Omega_0.$$

It follows that

$$\partial_t u(t,\xi)=\partial_t v+v\cdot\nabla v.$$

In the same fashion we can transfom any other function f defined on the moving domain $\bigcup_{t\in J}\{t\}\times\Omega(t)$ to a function g on the fixed domain $J\times\Omega_0$ by setting

$$g(t,\xi)=f(t,X_u(t,\xi)).$$

We define $\theta(t,\xi)=\pi(t,x)$, $u_0(\xi)=v_0(x)$, and $\mathcal{H}(t,\xi)=\kappa(t,x)$. Set

$$A(u)=\nabla_\xi X_u(t,\xi)=\left(\frac{\partial x_i}{\partial\xi_j}\right)_{i,j=1,\dots,n}=\left(\delta_{ij}+\int_0^t\frac{\partial u_i}{\partial\xi_j}(\tau,\xi)\,\mathrm{d}\tau\right)_{i,j=1,\dots,n}.$$

It follows from the chain rule that, whenever the mapping (5) has a differentiable inverse then the inverse of $A(u)$ is given by

$$A(u)^{-1}=\left(\frac{\partial\xi_i}{\partial x_j}\right)_{ij}\circ X_u.$$

If $v(t,x)$ is divergence-free and $u(t,\xi)=v(t,x)$ then it was shown in [**SS07**] that $X_u(t,\cdot)$ is volume-preserving, i. e.

$$\det(\nabla_\xi X_u)=1$$

for $t>0$, $\xi\in\Omega_0$. We define operators ∇_u, div_u, Δ_u, E_u, and S_u via

$$\begin{aligned} \nabla_u&=A(u)^{-T}\nabla_\xi & \qquad \mathrm{div}_u\, v&=\mathrm{tr}[A(u)^{-T}\nabla_\xi v]\\ \Delta_u&=\mathrm{div}_u\nabla_u & E_u v&=\nabla_u v+(\nabla_u v)^T\\ S_u(v,\pi)&=-\pi\,\mathrm{Id}+E_u v \end{aligned}$$

and we set

$$\nu_u(t,\xi)=\frac{A(u)^{-T}\nu_0(\xi)}{|A(u)^{-T}\nu_0(\xi)|}$$

with $\nu_0(\xi)$ denoting the outer normal vector at $\xi\in\Gamma_0^+$. For small times t the matrix $A(u)$ will be close to the identity matrix, and then we can consider the operators ∇_u, div_u, Δ_u as small pertubations of the usual differential operators ∇, div, Δ.

Now assume f is a function in Eulerian coordinates and g is a function in Lagrangian coordinates with $g(t,\xi)=f(t,X_u(t,\xi))=f(t,x)$. Then we write $f\rightsquigarrow g$, and we can use the chain rule to show the correspondences

$$\begin{aligned} \nabla_x f&\rightsquigarrow\nabla_u g & \qquad \mathrm{div}_x f&\rightsquigarrow\mathrm{div}_u g\\ \Delta_x f&\rightsquigarrow\Delta_u g & E_x f&\rightsquigarrow E_u g. \end{aligned}$$

The outer normal vector ν satisfies

$$\nu(t,x)=\nu_u(t,\xi).$$

This can be seen as follows. The surface Γ_0^+ is parametrised via $\Gamma_0^+ = \{\xi \in \mathbb{R}^n_+ : \xi_n = h(\xi')\}$. With $F(\xi) = \xi_n - h(\xi')$ we can write Γ_0^+ as the set of zeros of F. Then $\Gamma^+(t)$ must be the set of zeros of $F \circ X_u(t,\cdot)^{-1}$. Thus the direction of $\nu(t,x)$ must be given by

$$\nabla_x F \circ X_u(t,\cdot)^{-1} = [\nabla_x X_u(t,\cdot)^{-1}]^T (\nabla_\xi F) \circ X_u(t,\cdot)^{-1}.$$

Thus $\nu_u(t,\xi) = \nu(t, X_u(t,\xi))$ must be a multiple of

$$\nabla_x F \circ X_u(t,\cdot)^{-1} \circ X_u(t,\cdot) = A(u)^{-T} \nabla_\xi F = A(u)^{-T} \nu_0.$$

And since $|\nu_u| = 1$ we obtain the desired result for ν_u.

This shows that by introducing Lagrangian coordinates (1) is transformed to

$$\begin{cases} \partial_t u - \Delta_u u + \nabla_u \theta = 0 & \text{in } J \times \Omega_0 \\ \operatorname{div}_u u = 0 & \text{in } J \times \Omega_0 \\ S_u(u,\theta)\nu_u - \sigma \mathcal{H} \nu_u = 0 & \text{on } J \times \Gamma_0^+ \\ u = 0 & \text{on } J \times \Gamma^- \\ u(0) = u_0 & \text{in } \Omega_0. \end{cases}$$

The kinematic condition $V = v \cdot \nu$ is automatically satisfied. This is implicitly contained in the assumption that a formulation in Lagrangian coordinates exists. Introducing the mappings

$$F_1(u,\theta) = (\Delta_u - \Delta)u + (\nabla - \nabla_u)\theta \qquad\qquad F_d(u) = (\operatorname{div} - \operatorname{div}_u)u$$

we can write this equation equivalently as

$$\begin{cases} \partial_t u - \Delta u + \nabla \theta = F_1(u,\theta) & \text{in } J \times \Omega_0 \\ \operatorname{div} u = F_d(u) & \text{in } J \times \Omega_0 \\ S_u(u,\theta)\nu_u - \sigma \mathcal{H} \nu_u = 0 & \text{on } J \times \Gamma_0^+ \\ u = 0 & \text{on } J \times \Gamma^- \\ u(0) = u_0 & \text{in } \Omega_0. \end{cases}$$

It is known that the mean curvature κ satisfies the relation $\kappa\nu = \Delta_{\Gamma^+(t)} x$ at any point $x \in \Gamma^+(t)$, where $\Delta_{\Gamma^+(t)}$ denotes the Laplace-Beltrami operator on the manifold $\Gamma^+(t)$, see e. g. [**Tay06**] and [**Tri92**] for an account of analysis on manifolds and the Laplace-Beltrami operator acting in various function spaces.

Recall that the Laplace-Beltrami operator Δ_M on a manifold M parametrised (locally) over $\mathbb{R}^m$ by a function f has the representation

$$[\Delta_M u] \circ f = |\det g|^{-1/2} \operatorname{div} \left\{ |\det g|^{1/2} g^{-1} \nabla \tilde{u} \right\}$$

for $u \colon M \to \mathbb{C}$ sufficiently smooth, $g = (\nabla f)^T (\nabla f)$, and $\tilde{u} = u \circ f$. This shows that $u \mapsto [\Delta_M u] \circ f$ is an elliptic operator that arises as a perturbation of the usual Laplacian Δu whenever g is close to the identity matrix.

By slight abuse of notation we will not distinguish between $\Delta_M u$ and $[\Delta_M u] \circ f$, i. e. we will not distinguish between a function on a manifold and its representation in local coordinates. In our case $M = \Gamma^+(t)$ and $f = X_u(t,\cdot) \circ \Phi$.

Similar to Δ_M we can also define operators div_M and ∇_M via

$$[\operatorname{div}_M u] \circ f = \frac{1}{\sqrt{\det g}} \sum_{i=1}^m \partial_i \left\{ \sqrt{\det g}\, \tilde{u}_i \right\}$$

and

$$[\nabla_M u]_i \circ f = \sum_{k=1}^{m} g_{ik}^{-1} \partial_k \tilde{u}.$$

Then we have the usual relation $\operatorname{div}_M \circ \nabla_M = \Delta_M$. One can also derive product identities such as e. g.

$$\Delta_M(fg) = f\Delta_M g + \nabla_M f \cdot \nabla g + \nabla f \cdot \nabla_M g + g\Delta_M f = f\Delta_M g + 2\nabla_M f \cdot \nabla g + g\Delta_M f.$$

Transforming the relation $\kappa\nu = \Delta_{\Gamma^+(t)} x$ into a formulation in Lagrangian coordinates we obtain for $\kappa\nu$ the expression $\mathcal{H}\nu_u$, or equivalently

$$\Delta_{\Gamma^+(t)} X_u(t, \cdot).$$

In analogy to the projection $\Pi_0 \colon d \mapsto d - (d \cdot \nu_0)\nu_0$ we define a second projection

$$\Pi_u \colon \mathbb{R}^n \to \mathbb{R}^n, \qquad d \mapsto d - (d \cdot \nu_u)\nu_u.$$

It was shown in [**SS11**] that if $\nu_0 \cdot \nu_u \neq 0$ then a vector $d \in \mathbb{R}^n$ is zero if and only if $\Pi_0\Pi_u d = 0$ and $\nu_0 \cdot d = 0$. Note that for small times t we will always have $\nu_0 \cdot \nu_u \neq 0$.

Then the boundary condition

$$S_u(u, \theta)\nu_u - \sigma\mathcal{H}\nu_u = 0 \quad \text{ on } J \times \Gamma_0^+$$

can be written equivalently as

$$\begin{cases} \Pi_0\Pi_u \left[S_u(u, \theta)\nu_u - \sigma\mathcal{H}\nu_u\right] = 0 & \text{on } J \times \Gamma_0^+ \\ \quad \nu_0 \cdot \left[S_u(u, \theta)\nu_u - \sigma\mathcal{H}\nu_u\right] = 0 & \text{on } J \times \Gamma_0^+. \end{cases}$$

The first equation can be simplified to

$$\Pi_0\Pi_u E_u u\nu_u = 0 \qquad \text{on } J \times \Gamma_0^+,$$

which in turn can be written as

$$\begin{aligned} \Pi_0 E u\nu_0 &= \Pi_0 E u\nu_0 - \Pi_0\Pi_u E_u u\nu_u \\ &= \Pi_0 E u(\nu_0 - \nu_u) + \Pi_0(Eu - E_u u)\nu_u + \Pi_0(\Pi_0 - \Pi_u)E_u u\nu_u \end{aligned}$$

on $J \times \Gamma_0^+$.

We turn to the second boundary condition on the upper surface, which we can write as

$$\begin{aligned} 0 &= \nu_0 \cdot \left[S_u(u, \theta)\nu_u - \sigma\mathcal{H}\nu_u\right] \\ &= \nu_0 \cdot S_u(u, \theta)\nu_u - \sigma\nu_0 \cdot \Delta_{\Gamma^+(t)} X_u(t, \xi). \end{aligned}$$

We have

$$\Delta_{\Gamma^+(t)} X_u(t, \xi) = \Delta_{\Gamma_0^+} X_u(t, \cdot) + [\Delta_{\Gamma^+(t)} - \Delta_{\Gamma_0^+}] X_u(t, \cdot)$$

and, using the product rule for the Laplace-Beltrami operator, we can write

$$\begin{aligned} \nu_0 \cdot \Delta_{\Gamma_0^+} \int_0^t u(\tau, \cdot)\, \mathrm{d}\tau = \Delta_{\Gamma_0^+} \int_0^t \nu_0 \cdot u(\tau, \cdot)\, \mathrm{d}\tau - (\Delta_{\Gamma_0^+}\nu_0) \cdot \int_0^t u(\tau, \cdot)\, \mathrm{d}\tau \\ - 2\left(\nabla \int_0^t u\, \mathrm{d}\tau\right) : \left(\nabla_{\Gamma_0^+}\nu_0\right). \end{aligned}$$

This shows

$$\begin{aligned}\nu_0 \cdot \Delta_{\Gamma^+(t)} X_u(t,\xi) &= \nu_0 \cdot \Delta_{\Gamma_0^+} X_u(t,\xi) + \nu_0 \cdot [\Delta_{\Gamma^+(t)} - \Delta_{\Gamma_0^+}] X_u(t,\xi) \\ &= \nu_0 \cdot \Delta_{\Gamma_0^+} \xi + \nu_0 \cdot \Delta_{\Gamma_0^+} \int_0^t u \,\mathrm{d}\tau + \nu_0 \cdot [\Delta_{\Gamma^+(t)} - \Delta_{\Gamma_0^+}]\xi \\ &\quad + \nu_0 \cdot [\Delta_{\Gamma^+(t)} - \Delta_{\Gamma_0^+}] \int_0^t u \,\mathrm{d}\tau \\ &= \kappa_0 + \Delta_{\Gamma_0^+} \int_0^t \nu_0 \cdot u(\tau,\cdot)\,\mathrm{d}\tau - (\Delta_{\Gamma_0^+}\nu_0) \cdot \int_0^t u(\tau,\cdot)\,\mathrm{d}\tau \\ &\quad - 2\left(\nabla \int_0^t u \,\mathrm{d}\tau\right) : \left(\nabla_{\Gamma_0^+}\nu_0\right) + \nu_0 \cdot [\Delta_{\Gamma^+(t)} - \Delta_{\Gamma_0^+}]\xi \\ &\quad + \nu_0 \cdot [\Delta_{\Gamma^+(t)} - \Delta_{\Gamma_0^+}] \int_0^t u \,\mathrm{d}\tau\end{aligned}$$

where we write $\kappa_0 = \nu_0 \cdot \Delta_{\Gamma_0^+}\xi$ for twice the mean curvature of Γ_0^+. All in all this shows that we can write the second boundary condition on Γ_0^+ as

$$\begin{aligned}&\nu_0 \cdot S(u,\theta)\nu_0 - \sigma \Delta_{\Gamma_0^+} \int_0^t \nu_0 \cdot u \,\mathrm{d}\tau \\ &\quad + \sigma \left\{ (\Delta_{\Gamma_0^+}\nu_0) \cdot \int_0^t u(\tau,\cdot)\,\mathrm{d}\tau + \nu_0 \cdot [\Delta_{\Gamma^+(t)} - \Delta_{\Gamma_0^+}] \int_0^t u \,\mathrm{d}\tau + \nu_0 \cdot [\Delta_{\Gamma^+(t)} - \Delta_{\Gamma_0^+}]\xi \right\} \\ &\qquad = \nu_0 \cdot [Eu - E_u u]\nu_0 + \nu_0 \cdot S_u(u,\theta)[\nu_0 - \nu_u] - 2\sigma \left(\nabla \int_0^t u \,\mathrm{d}\tau\right) : \left(\nabla_{\Gamma_0^+}\nu_0\right) + \sigma\kappa_0.\end{aligned}$$

Let us write

$$F(u) = (\Delta_{\Gamma_0^+}\nu_0) \cdot \int_0^t u(\tau,\cdot)\,\mathrm{d}\tau + \nu_0 \cdot [\Delta_{\Gamma^+(t)} - \Delta_{\Gamma_0^+}] \int_0^t u \,\mathrm{d}\tau + \nu_0 \cdot [\Delta_{\Gamma^+(t)} - \Delta_{\Gamma_0^+}]\xi.$$

Choose $m > 0$ large enough such that $m - \Delta_{\Gamma_0^+}$ is invertible. Then we can write this as

$$\begin{aligned}&\nu_0 \cdot S(u,\theta)\nu_0 - \sigma \Delta_{\Gamma_0^+} \int_0^t \nu_0 \cdot u \,\mathrm{d}\tau + \sigma(m - \Delta_{\Gamma_0^+}) \left\{ \nu_0 \cdot \int_0^t u \,\mathrm{d}\tau + (m - \Delta_{\Gamma_0^+})^{-1} F(u) \right\} \\ &\quad - \sigma m \nu_0 \cdot \int_0^t u \,\mathrm{d}\tau = \nu_0 \cdot [Eu - E_u u]\nu_0 + \nu_0 \cdot S_u(u,\theta)[\nu_0 - \nu_u] \\ &\qquad\qquad - 2\sigma \left(\nabla \int_0^t u \,\mathrm{d}\tau\right) : \left(\nabla_{\Gamma_0^+}\nu_0\right) + \sigma\kappa_0.\end{aligned}$$

If we define

$$\eta = \nu_0 \cdot \int_0^t u \,\mathrm{d}\tau + (m - \Delta_{\Gamma_0^+})^{-1} F(u)$$

then we obtain the following two equations on Γ_0^+:

$$\begin{aligned}
&\nu_0 \cdot S(u,\theta)\nu_0 - \sigma\Delta_{\Gamma_0^+}\int_0^t \nu_0 \cdot u \,\mathrm{d}\tau + \sigma(m - \Delta_{\Gamma_0^+})\eta \\
&\quad = \sigma m\nu_0 \cdot \int_0^t u \,\mathrm{d}\tau + \nu_0 \cdot [Eu - E_u u]\nu_0 + \nu_0 \cdot S_u(u,\theta)[\nu_0 - \nu_u] \\
&\quad - 2\sigma\left(\nabla\int_0^t u \,\mathrm{d}\tau\right) : \left(\nabla_{\Gamma_0^+}\nu_0\right) + \sigma\kappa_0 \\
&\quad =: G_\perp^+(u,\theta)
\end{aligned}$$

and

$$\begin{aligned}
\partial_t\eta - \nu_0 \cdot u &= (m - \Delta_{\Gamma_0^+}\partial_t F(u) \\
&= (\Delta_{\Gamma_0^+}\nu_0) \cdot u + \nu_0 \cdot \dot{\Delta}_{\Gamma^+(t)}\int_0^t u \,\mathrm{d}\tau + \nu_0 \cdot [\Delta_{\Gamma^+(t)} - \Delta_{\Gamma_0^+}]u + \nu_0 \cdot \dot{\Delta}_{\Gamma^+(t)}\xi \\
&=: K^+(u).
\end{aligned}$$

Here we write $\dot{\Delta}_{\Gamma^+(t)}$ for the operator $\partial_t\Delta_{\Gamma^+(t)}$. Thus we obtain the equation (3) with

$$\begin{aligned}
F_1(u,\theta) &= (\Delta_u - \Delta)u + (\nabla - \nabla_u)\theta \\
F_d(u) &= (\mathrm{div} - \mathrm{div}_u)u \\
G_\parallel^+(u) &= \Pi_0 Eu(\nu_0 - \nu_u) + \Pi_0(Eu - E_u u)\nu_u + \Pi_0(\Pi_0 - \Pi_u)E_u u\nu_u \\
G_\perp^+(u) &= \sigma m\nu_0 \cdot \int_0^t u \,\mathrm{d}\tau + \nu_0 \cdot [Eu - E_u u]\nu_0 + \nu_0 \cdot S_u(u,\theta)[\nu_0 - \nu_u] \\
&\quad - 2\sigma\left(\nabla\int_0^t u \,\mathrm{d}\tau\right) : \left(\nabla_{\Gamma_0^+}\nu_0\right) + \sigma\kappa_0 \\
K^+(u) &= (\Delta_{\Gamma_0^+}\nu_0) \cdot u + \nu_0 \cdot \dot{\Delta}_{\Gamma^+(t)}\int_0^t u \,\mathrm{d}\tau + \nu_0 \cdot [\Delta_{\Gamma^+(t)} - \Delta_{\Gamma_0^+}]u + \nu_0 \cdot \dot{\Delta}_{\Gamma^+(t)}\xi.
\end{aligned}$$

We will occasionally write Γ_u^+ instead of $\Gamma^+(t)$ to highlight the dependence on u, for we will need to estimate e. g. differences $\Delta_{\Gamma_u^+} - \Delta_{\Gamma_v^+}$. In order to deal with the operators $\Delta_{\Gamma_u^+}$ and $\dot{\Delta}_{\Gamma_u^+}$ we will need the following Lemma.

LEMMA 3.1. *Let $R > 0$ and $n < p < \infty$. Then there is $T' > 0$ such that for all $0 < T < T'$, and $u, v, w \in \mathbb{E}_u(J)$ with $J = (0,T)$ and $\|u\|_{\mathbb{E}_u(J)}, \|v\|_{\mathbb{E}_u(J)} \le R$ as well as $h \in W_p^{2-1/p}(\mathbb{R}^{n-1})$ with $\|h\|_{W_p^{2-1/p}(\mathbb{R}^{n-1})} \le 1$ the estimates*

$$\begin{aligned}
\left\|\left[\Delta_{\Gamma_u^+} - \Delta_{\Gamma_v^+}\right] w\right\|_{L_p(J;W_p^{-1/p}(\Gamma_0^+))} &\le C_R T^{1-1/p}\|u - v\|_{\mathbb{E}_u(J)}\|w\|_{\mathbb{E}_u(J)} \\
\left\|\left[\dot{\Delta}_{\Gamma_u^+} - \dot{\Delta}_{\Gamma_v^+}\right]\int_0^t w\right\|_{L_p(J;W_p^{-1/p}(\Gamma_0^+))} &\le C_R T^{1-1/p}\|u - v\|_{\mathbb{E}_u(J)}\|w\|_{\mathbb{E}_u(J)} \\
\left\|\nu_0 \cdot \left[\dot{\Delta}_{\Gamma_u^+} - \dot{\Delta}_{\Gamma_v^+}\right]\xi\right\|_{L_p(J;W_p^{-1/p}(\Gamma_0^+))} &\le C_R\|u - v\|_{\mathbb{E}_u(J)}\|h\|_{W_p^{2-1/p}(\Gamma_0^+)}
\end{aligned}$$

hold with a constant $C_R > 0$ independent of T, u, v, w, h.

We postpone the proof of this result to the end of this Chapter.

4. The linear problem

Linearising the operators $F_1, F_d, G_\parallel^+, G_\perp^+$ and K^+ in (3) around zero we are led to study the linear evolution equation

$$(6)\qquad \begin{cases} \partial_t u - \Delta u + \nabla\theta = f_1 & \text{in } J\times\Omega_0 \\ \operatorname{div} u = f_d & \text{in } J\times\Omega_0 \\ \Pi_0 E u \nu_0 = g_\parallel^+ & \text{on } J\times\Gamma_0^+ \\ \nu_0\cdot S(u,\theta)\nu_0 - \sigma(m-\Delta_{\Gamma_0^+})\eta = g_\perp^+ & \text{on } J\times\Gamma_0^+ \\ \partial_t\eta - u\cdot\nu_0 = k^+ & \text{on } J\times\Gamma_0^+ \\ u = 0 & \text{on } J\times\Gamma^- \\ u(0) = u_0 & \text{in } \Omega_0 \\ \eta(0) = 0 & \text{on } \Gamma_0^+ \end{cases}$$

or equivalently

$$(7)\qquad \begin{cases} \partial_t u - \Delta u + \nabla\theta = f_1 & \text{in } J\times\Omega_0 \\ \operatorname{div} u = f_d & \text{in } J\times\Omega_0 \\ S(u,\theta)\nu_0 - \sigma(m-\Delta_{\Gamma_0^+})\eta\nu_0 = g^+ & \text{on } J\times\Gamma_0^+ \\ \partial_t\eta - u\cdot\nu_0 = k^+ & \text{on } J\times\Gamma_0^+ \\ u = 0 & \text{on } J\times\Gamma^- \\ u(0) = u_0 & \text{in } \Omega_0 \\ \eta(0) = 0 & \text{on } \Gamma_0^+. \end{cases}$$

In addition to the notation introduced in the previous section let us define the following spaces for the solution and data of (7):

$$\begin{aligned} f_1 &\in \mathbb{F}_1(J) = L_p(J; L_p(\Omega_0)) \\ f_d &\in \mathbb{F}_d(J) = H_p^1(J; {}_0\hat{H}_p^{-1}(\Omega_0)) \cap H_p^{1/2}(J; L_p(\Omega_0)) \cap L_p(J; H_p^1(\Omega_0)) \\ g_\parallel^+, g_\perp^+, g^+ &\in \mathbb{G}^+(J) = W_p^{1/2-1/2p}(J; L_p(\Gamma_0^+)) \cap L_p(J; W_p^{1-1/p}(\Gamma_0^+)) \\ k^+ &\in \mathbb{K}^+(J) = L_p(J; W_p^{2-1/p}(\Gamma_0^+)). \end{aligned}$$

Here we write ${}_0\hat{H}_p^{-1}(\Omega_0)$ for the dual space of

$${}^0\hat{H}_{p'}^1(\Omega_0) = \left\{ v \in \hat{H}_{p'}^1(\Omega_0) \colon v\big|_{\Gamma_0^+} = 0 \right\}$$

where we write

$$\hat{H}_{p'}^1(\Omega_0) = \{ v \in L_{p',\mathrm{loc}}(\Omega_0) \colon \nabla v \in L_{p'}(\Omega_0) \}$$

for the homogeneous Sobolev space with norm $\|v\|_{\hat{H}_{p'}^1(\Omega_0)} = \|\nabla v\|_{L_{p'}(\Omega_0)}$ as in [**Gal11**]. We identify two elements of $\hat{H}_{p'}^1(\Omega_0)$ if their gradients coincide. Observe that by Poincaré's inequality the space ${}^0\hat{H}_{p'}^1(\Omega_0)$ coincides with its inhomogeneous counterpart

$${}^0H_{p'}^1(\Omega_0) = \left\{ v \in H_{p'}^1(\Omega_0) \colon v\big|_{\Gamma_0^+} = 0 \right\}$$

and thus also their dual spaces ${}_0\hat{H}_p^{-1}(\Omega_0)$ and ${}_0H_p^{-1}(\Omega_0)$ coincide.

We will write

$$\mathbb{F}(J) = \mathbb{F}_1(J) \times \mathbb{F}_d(J) \times \mathbb{G}^+(J) \times \mathbb{K}^+(J)$$

for the space containing the functions on the right hand side of (7).

For the remainder of this section we will assume the data $(f_1, f_d, g^+, k^+, u_0) \in \mathbb{F}(J) \times \mathbb{I}_u$ to satisfy the compatibility conditions

$$\begin{cases} \operatorname{div} u_0 = f_d\big|_{t=0} & \text{in } \Omega_0 \\ \Pi_0 E u_0 \cdot \nu_0 = g_\parallel^+\big|_{t=0} & \text{on } \Gamma_0^+ \\ u_0 = 0 & \text{on } \Gamma^-. \end{cases} \tag{8}$$

The main result of this section is the following.

THEOREM 4.1. *Let $n \geq 2$, $n < p < \infty$ and $\sigma^* > 0$. Let $J = (0,T)$ denote an arbitrary finite time interval. Assume that Ω_0 is parametrised as in (2) with $h \in W_p^{3-1/p}(\mathbb{R}^{n-1})$ of sufficiently small norm. Then for $0 \leq \sigma \leq \sigma^*$ and data $(f_1, f_d, g^+, k^+, u_0)$ belonging to $\mathbb{F}(J) \times \mathbb{I}_u$ satisfying the compatibility conditions (8) there is a unique solution $(u, \theta, \eta) \in \mathbb{E}^\sigma(J)$ of (7). Moreover, there is a constant $C > 0$ independent of σ and the data such that the estimate*

$$\|u\|_{\mathbb{E}_u(J)} + \|\theta\|_{\mathbb{E}_\theta(J)} + \|\eta\|_{\mathbb{E}_\eta^\sigma(J)} \leq C \left\|(f_1, f_d, g^+, k^+, u_0)\right\|_{\mathbb{F}(J)\times\mathbb{I}_u} \tag{9}$$

holds.

Furthermore, denoting the solution for a given $\sigma \geq 0$ and data $F^\sigma = (f_{1,\sigma}, f_{d,\sigma}, g_\sigma^+, k_\sigma^+, u_{0,\sigma})$ with $\sup_\sigma \|F^\sigma\|_{\mathbb{F}(J)} < \infty$ with $(u^\sigma, \theta^\sigma, \eta^\sigma)$, we have

$$\lim_{\sigma\to 0^+} \sigma \,\|\eta^\sigma\|_{L_p(J; W_p^{3-1/p}(\Gamma_0^+))} = 0 \tag{10}$$

and if $F^\sigma \to F^0$ in $\mathbb{F}(J)$ as $\sigma \to 0$ then

$$\lim_{\sigma\to 0^+} \left\|(u^\sigma, \theta^\sigma, \eta^\sigma) - (u^0, \theta^0, \eta^0)\right\|_{\mathbb{E}^0(J)} = 0. \tag{11}$$

This will be the basis for a proof of the main results Theorem 2.1 and Theorem 2.2. If we assume that k^+ has additional time regularity, namely $k^+ \in L^p(J; W^{2-1/p,p}(\Gamma^+)) \cap W_p^{1-1/2p}(J; L_p(\Gamma_0^+))$, then $\eta \in \mathbb{E}_\eta^\sigma(J) \cap W_p^{2-1/2p}(J; L_p(\Gamma_0^+))$ and a statement analogous to Theorem 4.1 holds.

We will begin with the proof of Theorem 4.1 in the case that Ω_0 is a flat layer of the form $\mathbb{R}^{n-1} \times (0, \delta)$, i. e. we begin with the case $h \equiv 0$. In this case we will write $\bar{\Omega}$ instead of Ω_0 as well as $\bar{\Gamma}^\pm$ instead of Γ_0^+ and Γ^-. Once Theorem 4.1 is established for $h \equiv 0$ we can use perturbation methods to show that the result carries over to $h \in W_p^{3-1/p}(\mathbb{R}^{n-1})$ of sufficiently small norm.

As a first step we will show existence of solutions to (7) in the case $h \equiv 0$, i. e. in the case of a flat layer $\bar{\Omega} = \mathbb{R}^{n-1} \times (0, \delta)$. We begin with the reduced equation

$$\begin{cases} \partial_t u - \Delta u + \nabla\theta = 0 & \text{in } J \times \bar{\Omega} \\ \operatorname{div} u = 0 & \text{in } J \times \bar{\Omega} \\ S(u,\theta)\nu + \sigma(m - \Delta')\eta\nu = 0 & \text{on } J \times \bar{\Gamma}^+ \\ \partial_t \eta - u_n = k^+ & \text{on } J \times \bar{\Gamma}^+ \\ u = 0 & \text{on } J \times \bar{\Gamma}^- \\ u(0) = 0 & \text{in } \bar{\Omega} \\ \eta(0) = 0 & \text{on } \bar{\Gamma}^+. \end{cases} \tag{12}$$

This corresponds to (7) in the case where h, f_1, f_d, g^+ and u_0 are identically zero.

Applying the Laplace transform $\mathscr{L}$ in time t to (12) we are led to study the system

$$\begin{cases} \lambda\tilde{u} - \Delta\tilde{u} + \nabla\tilde{\theta} = 0 & \text{in } \bar{\Omega} \\ \operatorname{div}\tilde{u} = 0 & \text{in } \bar{\Omega} \\ S(\tilde{u},\tilde{\theta})\nu + \sigma(m-\Delta')\tilde{\eta}\nu = 0 & \text{on } \bar{\Gamma}^+ \\ \lambda\tilde{\eta} - \tilde{u}_n = \tilde{k}^+ & \text{on } \bar{\Gamma}^+ \\ \tilde{u} = 0 & \text{on } \bar{\Gamma}^-. \end{cases} \tag{13}$$

Here we write $\tilde{u} = \mathscr{L}u$, $\tilde{\theta} = \mathscr{L}\theta$, $\tilde{\eta} = \mathscr{L}\eta$, and $\tilde{k}^+ = \mathscr{L}k^+$ for the transformed functions. We have derived in Chapter II a representation formula (formula II.2.3) for the function $\tilde{\eta}$, namely

$$\mathcal{F}\tilde{\eta} = \frac{\mathbf{M}^\sharp_{7,7}}{\det\mathbf{M}}\mathcal{F}\tilde{k}^+,$$

with $\mathbf{M}^\sharp_{7,7}$ and $\det\mathbf{M}$ as in formula II.2.3. Combining Lemma II.2.1 and Lemma II.2.2 we see that there are $\pi/2 < \rho < \pi$, $0 < \varepsilon < \pi - \rho$, and $\lambda_0 > 0$ such that the function

$$\mathcal{S}\colon \Sigma_\varepsilon \times \{\lambda \in \Sigma_\rho\colon\ |\lambda| > \lambda_0\} \to \mathbb{C}, \qquad (z,\lambda) \mapsto \frac{\mathbf{M}^\sharp_{7,7}(z,\lambda)}{\det\mathbf{M}(z,\lambda)}$$

is holomorphic and satisfies for $|z| \le 1$ the estimate

$$|\mathcal{S}(z,\lambda)| \lesssim \frac{1}{|\lambda|},$$

and for $|z| \ge 1$ we have

$$|\mathcal{S}(z,\lambda)| \lesssim \frac{(|z|^2 + |\lambda|)^{3/2}}{(|z|^2 + |\lambda|)^{3/2}\,|\lambda| + \sigma\,|z|^2},$$

both uniformly in $0 \le \sigma \le \sigma^*$ and $|\lambda| \ge \lambda_0$. These estimates show that the functions

$$(z,\lambda) \mapsto \lambda\mathcal{S}(z,\lambda), \qquad (z,\lambda) \mapsto \sigma\frac{z^2}{\sqrt{1+z^2}}\mathcal{S}(z,\lambda)$$

are holomorphic and bounded on $\Sigma_\varepsilon \times \{\lambda \in \Sigma_\rho\colon\ |\lambda| > \lambda_0\}$, uniformly in $0 \le \sigma \le \sigma^*$. Then the functions

$$(z,\lambda) \mapsto \lambda\mathcal{S}(z,\mu+\lambda), \qquad (z,\lambda) \mapsto \sigma\frac{z^2}{\sqrt{1+z^2}}\mathcal{S}(z,\mu+\lambda)$$

are holomorphic and bounded on $\Sigma_\varepsilon \times \Sigma_\rho$ if we choose $\mu \ge \lambda_0$ sufficiently large. Thus we can use the joint $\mathcal{H}_\infty$-calculus as defined in [**KW01**], see also [**vB11**] for a detailed account, of the operators $-\mu + \partial_t$ and $-\Delta'$ to obtain on any finite time interval $J = (0,T)$ the estimate

$$\|\partial_t\eta\|_{\mathbb{K}^+(J)} + \sigma\left\|\sqrt{1-\Delta'}^{-1}\Delta'\eta\right\|_{\mathbb{K}^+(J)} \lesssim \left\|k^+\right\|_{\mathbb{K}^+(J)}$$

uniformly in $0 \le \sigma \le \sigma^*$. This is possible since $-\Delta'$ and $-\mu + \partial_t$ each admit a bounded $\mathcal{H}_\infty$-calculus on $\mathbb{K}^+(J)$. This in turn follows for $-\Delta'$ from e. g. Proposition 8.3.4 of [**Haa06**], and for $-\mu + \partial_t$ this can be shown along the lines of the proof of Theorem 8.5.8 of [**Haa06**] or Satz 5.1.5 of [**vB11**]. Then these operators have a joint bounded $\mathcal{H}_\infty$-calculus by [**KW01**]. But then we also have

$$\|\eta\|_{H^1_p(J;W^{2-1/p}_p(\bar{\Gamma}^+))} + \sigma\,\|\Delta'\eta\|_{L_p(J;W^{1-1/p}_p(\bar{\Gamma}^+))} \lesssim \left\|k^+\right\|_{\mathbb{K}^+(J)},$$

and thus we obtain the estimate

$$\|\eta\|_{\mathbb{E}^\sigma_\eta(J)} \le C\left\|k^+\right\|_{\mathbb{K}^+}$$

uniformly in $0 \le \sigma \le \sigma^*$. It remains to estimate the functions u and θ, which necessarily satisfy the system

$$\begin{cases} \partial_t u - \Delta u + \nabla\theta = 0 & \text{in } J \times \bar{\Omega} \\ \operatorname{div} u = 0 & \text{in } J \times \bar{\Omega} \\ S(u,\theta)\nu = -\sigma(m - \Delta')\eta\nu & \text{on } J \times \bar{\Gamma}^+ \\ u = 0 & \text{on } J \times \bar{\Gamma}^- \\ u(0) = 0 & \text{in } \bar{\Omega}. \end{cases}$$

This system has been treated in Theorem 3.3 of [**Abe05a**], where it was shown that u and θ satisfy

$$\|u\|_{\mathbb{E}_u(J)} + \|\theta\|_{\mathbb{E}_\theta} \lesssim \sigma \left\|(m - \Delta')\eta\right\|_{\mathbb{G}^+(J)}.$$

We will show that we have indeed

$$\sigma \left\|(m - \Delta')\eta\right\|_{\mathbb{G}^+(J)} \lesssim \|\eta\|_{\mathbb{E}^\sigma_\eta(J)}.$$

To this end it suffices to show that

$$\sigma \left\|\Delta'\eta\right\|_{W_p^{1/2-1/2p}(J;L_p(\bar{\Gamma}^+))} \lesssim \|\eta\|_{H_p^1(J;W_p^{2-1/p}(\bar{\Gamma}^+))} + \sigma \left\|\Delta'\eta\right\|_{L_p(J;W_p^{1-1/p}(\bar{\Gamma}^+))}$$

uniformly in $\sigma > 0$, but this follows immediately from the interpolation results in Lemma 4.3 of [**DSS08**]. It is contained in Theorem 4.1 of [**SS11**] that solutions to (14) are unique in the case $h \equiv 0$, and thus we obtain the following Lemma.

LEMMA 4.2. *Let $h \equiv 0$. Under the assumptions of Theorem 4.1 there is $C > 0$ such that for every $k^+ \in \mathbb{K}^+$ and $0 \le \sigma \le \sigma^*$ there is a unique solution (u, θ, η) of (12) which satisfies the estimate*

$$\|u\|_{\mathbb{E}_u(J)} + \|\theta\|_{\mathbb{E}_\theta(J)} + \|\eta\|_{\mathbb{E}^\sigma_\eta(J)} \le C \left\|k^+\right\|_{\mathbb{K}^+}.$$

In order to obtain a corresponding existence result for the full linear problem (7) in the case $h \equiv 0$ let $(v, \pi) \in \mathbb{E}_u(J) \times \mathbb{E}_\theta(J)$ denote the unique solution of the equation

$$\begin{cases} \partial_t v - \Delta v + \nabla\pi = f_1 & \text{in } J \times \bar{\Omega} \\ \operatorname{div} v = f_d & \text{in } J \times \bar{\Omega} \\ S(v,\pi)\nu = g^+ & \text{on } J \times \bar{\Gamma}^+ \\ v = 0 & \text{on } J \times \bar{\Gamma}^- \\ v(0) = u_0 & \text{in } \bar{\Omega} \end{cases}$$

for functions $f_1 \in \mathbb{F}_1(J)$, $f_d \in \mathbb{F}_d(J)$, $g^+ \in \mathbb{G}^+(J)$, and $u_0 \in \mathbb{I}_u$ satisfying the compatibility conditions (8). This solution is known to exist due to Theorem 3.3 of [**Abe05a**]. For a given function $k^+ \in \mathbb{K}^+(J)$ let (u, θ, η) denote the solution to (12) with data $k^+ + v_n$. Then (w, ϑ, η) with $w = u + v$ and $\vartheta = \theta + \pi$ satisfies the full linear problem (14) for $h \equiv 0$ and we obtain the following Lemma.

LEMMA 4.3. *Let $h \equiv 0$. Under the assumptions of Theorem 4.1 there is $C > 0$ such that for every $(f_1, f_d, g^+, k^+, u_0) \in \mathbb{F}(J) \times \mathbb{I}_u$ satisfying the compatibility conditions (8) and $0 \le \sigma \le \sigma^*$ there is a unique solution (u, θ, η) of (12) which satisfies the estimate*

$$\|u\|_{\mathbb{E}_u(J)} + \|\theta\|_{\mathbb{E}_\theta(J)} + \|\eta\|_{\mathbb{E}^\sigma_\eta(J)} \le C \left\|(f_1, f_d, g^+, k^+, u_0)\right\|_{\mathbb{F}(J) \times \mathbb{I}_u}.$$

Now that we have established existence of solutions to (7) in the case of a flat layer, we turn to convergence of solutions as $\sigma \to 0^+$. We will show the following result.

LEMMA 4.4. *Let $h \equiv 0$. Under the assumptions of Theorem 4.1 let*

$$F^\sigma = (f_{1,\sigma}, f_{d,\sigma}, g_\sigma^+, k_\sigma^+, u_{0,\sigma}) \in \mathbb{F}(J) \times \mathbb{I}_u$$

satisfying the compatibility conditions (8) *for $0 \le \sigma \le \sigma^*$, and write $(u^\sigma, \theta^\sigma, \eta^\sigma) \in \mathbb{E}^\sigma(J)$ for the corresponding solution to* (14).

If $\sup_\sigma \|F^\sigma\|_{\mathbb{F}(J)\times\mathbb{I}_u} < \infty$ then

$$\lim_{\sigma\to 0^+} \sigma \|\eta^\sigma\|_{L_p(J;W_p^{3-1/p}(\bar\Gamma^+))} = 0,$$

and if $F^\sigma \to F^0$ in $\mathbb{F}(J) \times \mathbb{I}_u$ then

$$\lim_{\sigma\to 0^+} \left\|(u^\sigma, \theta^\sigma, \eta^\sigma) - (u^0, \theta^0, \eta^0)\right\|_{\mathbb{E}^0(J)} = 0.$$

PROOF. We begin with the case where $F^\sigma = (0, 0, 0, k_\sigma^+, 0)$. Then, as above, η^σ admits the representation

$$\eta^\sigma = \left\{ \left((z,\lambda) \mapsto \frac{\mathbf{M}^\sharp_{7,7}}{\det \mathbf{M}}(z,\lambda) \right) \Big|_{\lambda=\partial_t, z=\sqrt{-\Delta'}} \right\} k_\sigma^+.$$

Given $\varepsilon > 0$ we can find $k^+_{\sigma,\varepsilon} \in C_c^\infty(J \times \bar\Gamma^+)$ such that $\left\|k_\sigma^+ - k^+_{\sigma,\varepsilon}\right\|_{\mathbb{K}^+(J)} < \varepsilon$ as well as $\left\|k^+_{\sigma,\varepsilon}\right\|_{L_p(J;W_p^{3-1/p}(\bar\Gamma^+))} \le C_\varepsilon$ uniformly in $\sigma \ge 0$. We can infer from the proof of Lemma 4.2 that

$$\begin{aligned}
\sigma \|\Delta' \eta^\sigma\|_{L_p(J;W_p^{1-1/p}(\bar\Gamma^+))} &= \left\| \left\{ \sigma \frac{z^2 \mathbf{M}^\sharp_{7,7}}{\det \mathbf{M}} (\sqrt{-\Delta'}, \partial_t) \right\} k_\sigma^+ \right\|_{L_p(J;W_p^{1-1/p}(\bar\Gamma^+))} \\
&\le \left\| \left\{ \sigma \frac{z^2 \mathbf{M}^\sharp_{7,7}}{\det \mathbf{M}} (\sqrt{-\Delta'}, \partial_t) \right\} k^+_{\sigma,\varepsilon} \right\|_{L_p(J;W_p^{1-1/p}(\bar\Gamma^+))} \\
&\quad + \left\| \left\{ \sigma \frac{z^2 \mathbf{M}^\sharp_{7,7}}{\det \mathbf{M}} (\sqrt{-\Delta'}, \partial_t) \right\} (k_\sigma^+ - k^+_{\sigma,\varepsilon}) \right\|_{L_p(J;W_p^{1-1/p}(\bar\Gamma^+))} \\
&\le \sigma \left\|k^+_{\sigma,\varepsilon}\right\|_{L_p(J;W_p^{3-1/p}(\bar\Gamma^+))} + \left\|k_\sigma^+ - k^+_{\sigma,\varepsilon}\right\|_{L_p(J;W_p^{2-1/p}(\bar\Gamma^+))} \\
&\le C_\varepsilon \sigma + \varepsilon.
\end{aligned}$$

For $\sigma \to 0$ we obtain

$$\limsup_{\sigma\to 0^+} \sigma \|\Delta' \eta^\sigma\|_{L_p(J;W_p^{1-1/p}(\bar\Gamma^+))} \le \varepsilon$$

and since $\varepsilon > 0$ was chosen arbitrarily the assertion follows. For $F^\sigma = (f_{1,\sigma}, f_{d,\sigma}, g_\sigma^+, k_\sigma^+, u_{0,\sigma}) \in \mathbb{F}(J) \times \mathbb{I}_u$ it was established in the proof of Lemma 4.2 that the function η^σ can be obtained as above if we replace k_σ^+ with $k_\sigma^+ + v_n^\sigma$ where (v^σ, π^σ) denote the solution of

$$\begin{cases}
\partial_t v^\sigma - \Delta v^\sigma + \nabla \pi^\sigma = f_{1,\sigma} & \text{in } J \times \bar\Omega \\
\operatorname{div} v^\sigma = f_{d,\sigma} & \text{in } J \times \bar\Omega \\
S(v^\sigma, \pi^\sigma)\nu = g_\sigma^+ & \text{on } J \times \bar\Gamma^+ \\
v^\sigma = 0 & \text{on } J \times \bar\Gamma^- \\
v^\sigma(0) = u_{0,\sigma} & \text{in } \bar\Omega
\end{cases}$$

Then the same proof as above with k_σ^+ replaced by $k_\sigma^+ + v_n^\sigma$ shows the first assertion.

It remains to prove convergence of solutions as $\sigma \to 0$ in case that F^σ converges to F^0 as $\sigma \to 0$. To this end write $w^\sigma = u^\sigma - u^0$ and $\vartheta^\sigma = \theta^\sigma - \theta^0$. Then $(w^\sigma, \vartheta^\sigma, \eta^\sigma)$ satisfies the equations

$$\begin{cases} \partial_t w^\sigma - \Delta w^\sigma + \nabla \vartheta^\sigma = f_{1,\sigma} - f_{1,0} & \text{in } J \times \bar{\Omega} \\ \operatorname{div} w^\sigma = f_{d,\sigma} - f_{d,0} & \text{in } J \times \bar{\Omega} \\ S(w^\sigma, \vartheta^\sigma)\nu = -\sigma(m - \Delta')\eta^\sigma \nu + g^+_\sigma - g^+_0 & \text{on } J \times \bar{\Gamma}^+ \\ \partial_t \eta^\sigma - w^\sigma{}_n = k^+_\sigma - k^+_0 & \text{on } J \times \bar{\Gamma}^+ \\ w^\sigma = 0 & \text{on } J \times \bar{\Gamma}^- \\ w^\sigma(0) = u_{0,\sigma} - u_{0,0} & \text{in } \bar{\Omega} \\ \eta^\sigma(0) = 0 & \text{in } \bar{\Gamma}^+. \end{cases}$$

As a consequence of Lemma 4.3 we obtain the estimate

$$\|(w^\sigma, \vartheta^\sigma, \eta^\sigma)\|_{\mathbb{E}^0(J)} \lesssim \left\|F^\sigma - F^0\right\|_{\mathbb{F}(J)\times\mathbb{I}_u} + \sigma \left\|(m - \Delta')\eta\right\|_{\mathbb{G}^+(J)}.$$

As in the proof of Lemma 4.2 we can invoke Lemma 4.3 of [**DSS08**] to obtain the estimate

$$\|(w^\sigma, \vartheta^\sigma, \eta^\sigma)\|_{\mathbb{E}^0(J)} \lesssim \left\|F^\sigma - F^0\right\|_{\mathbb{F}(J)\times\mathbb{I}_u} + \sigma \left\|\eta^\sigma\right\|_{\mathbb{E}^0_\eta(J)} + \sigma \left\|\Delta'\eta^\sigma\right\|_{L_p(J;W^{1-1/p}_p(\bar{\Gamma}^+))}$$

whence it follows, together with our previous considerations, that the right hand side converges indeed to zero as $\sigma \to 0$. □

Combining Lemma 4.4 with Lemma 4.3 we obtain Theorem 4.1 in the case $h \equiv 0$. In order to treat the general case of a perturbed layer

$$\Omega_0 = \{x \in \mathbb{R}^n_+ : 0 < x_n < \delta + h(x')\}$$

for a given function $h\colon \mathbb{R}^{n-1} \to \mathbb{R}$, which we assume to be sufficiently regular and of sufficiently small modulus, we will transform the equations in a perturbed layer Ω_0 to a set of equations in a flat layer $\bar{\Omega} = \mathbb{R}^{n-1} \times (0, \delta)$ by means of the mapping $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\Phi(y) = (y', y_n(\delta + h(y'))/\delta).$$

If we continue to write $\bar{\Gamma}^+ = \mathbb{R}^{n-1} \times \{\delta\}$ and $\bar{\Gamma}^- = \mathbb{R}^{n-1} \times \{0\} = \Gamma^-$ then the mapping Φ satisfies

$$\Omega_0 = \Phi(\bar{\Omega}), \qquad \Gamma^+_0 = \Phi(\bar{\Gamma}^+), \qquad \Gamma^- = \Phi(\bar{\Gamma}^-) = \mathbb{R}^{n-1} \times \{0\}.$$

Using the coordinate transformation $u \rightsquigarrow u \circ \Phi$ we can transform the equations (7) to equations on the flat layer $\bar{\Omega}$. Let $\bar{u} = u \circ \Phi$, $\bar{\theta} = \theta \circ \Phi$ and $\bar{\eta} = \eta \circ \Phi$. The quantities in (7) transform as follows:

$$
\begin{array}{lll}
\partial_t u & \rightsquigarrow & \partial_t \bar{u} \\
\partial_j u & \rightsquigarrow & \partial_j \bar{u} - \frac{x_n}{h+\delta}\partial_n \bar{u}\partial_j h \\
\partial_n u & \rightsquigarrow & \partial_n \bar{u} - \frac{h}{h+\delta}\partial_n \bar{u} \\
\nabla u & \rightsquigarrow & \nabla \bar{u} - \frac{1}{h+\delta}\begin{pmatrix} x_n \nabla' h \\ h \end{pmatrix} \otimes \partial_n \bar{u} \\
Eu & \rightsquigarrow & E\bar{u} - \frac{1}{h+\delta}\left[\begin{pmatrix} x_n \nabla' h \\ h \end{pmatrix} \otimes \partial_n \bar{u} + \partial_n \bar{u} \otimes \begin{pmatrix} x_n \nabla' h \\ h \end{pmatrix}\right] \\
\operatorname{div} u & \rightsquigarrow & \operatorname{div} \bar{u} - \frac{1}{h+\delta}\partial_n \bar{u} \cdot \begin{pmatrix} x_n \nabla' h \\ h \end{pmatrix} \\
\Delta u & \rightsquigarrow & \Delta \bar{u} - \frac{h^2 + 2h\delta}{(h+\delta)^2}\partial_n^2 \bar{u} - 2\frac{x_n}{h+\delta}\nabla' \partial_n \bar{u} \nabla' h + \frac{x_n^2}{(h+\delta)^2}\left|\nabla' h\right|^2 \partial_n^2 \bar{u} \\
 & & \quad - \frac{x_n}{h+\delta}\partial_n \bar{u} \Delta' h + 2\frac{x_n}{(h+\delta)^2}\partial_n \bar{u}\left|\nabla' h\right|^2 \\
\nabla \theta & \rightsquigarrow & \nabla \bar{\theta} - \frac{1}{h+\delta}\partial_n \bar{\theta}\begin{pmatrix} x_n \nabla' h \\ h \end{pmatrix}.
\end{array}
$$

The normal vector ν_0 at a point $\xi \in \Gamma_0^+$ is given by

$$
\nu_0(\xi) = \frac{\begin{pmatrix} -\nabla' h(\xi') \\ 1 \end{pmatrix}}{\sqrt{1 + \left|\nabla' h(\xi')\right|^2}}.
$$

We thus obtain the following set of equations in a flat layer $\bar{\Omega} = \mathbb{R}^{n-1} \times (0,\delta)$, where $\bar{f}_1 = f_1 \circ \Phi$, $\bar{f}_d = f_d \circ \Phi$, $\bar{g}^+ = (g^+ \circ \Phi)\sqrt{1 + |\nabla' h|^2}$, $\bar{k}^+ = k^+ \circ \Phi$, and $\bar{u}_0 = u_0 \circ \Phi$.

$$
(14) \qquad \left\{
\begin{array}{rll}
\partial_t \bar{u} - \Delta \bar{u} + \nabla \bar{\theta} = \bar{f}_1 + \bar{F}_1(\bar{u}, \bar{\theta}, h) & & \text{in } J \times \bar{\Omega} \\
\operatorname{div} \bar{u} = \bar{f}_d + \bar{F}_d(\bar{u}, h) & & \text{in } J \times \bar{\Omega} \\
S(\bar{u}, \bar{\theta})\nu + \sigma(m - \Delta')\bar{\eta}\nu = \bar{g}^+ + \bar{G}^+(\bar{u}, \bar{\eta}, h) & & \text{on } J \times \bar{\Gamma}^+ \\
\partial_t \bar{\eta} - \bar{u}_n = \bar{k}^+ + \bar{K}^+(\bar{u}, \bar{\eta}, h) & & \text{on } J \times \bar{\Gamma}^+ \\
\bar{u} = 0 & & \text{on } J \times \bar{\Gamma}^- \\
\bar{u}(0) = \bar{u}_0 & & \text{in } \bar{\Omega} \\
\bar{\eta}(0) = 0 & & \text{in } \bar{\Gamma}^+
\end{array}
\right.
$$

with mappings

$$\bar F_1(\bar u,\bar\theta,h) = -\frac{h^2+2h\delta}{(h+\delta)^2}\partial_n^2\bar u - 2\frac{x_n}{h+\delta}\nabla'\partial_n\bar u\nabla' h + \frac{x_n^2}{(h+\delta)^2}\left|\nabla' h\right|^2\partial_n^2\bar u + \frac{1}{h+\delta}\partial_n\bar\theta\begin{pmatrix}x_n\nabla' h\\ h\end{pmatrix}$$

$$\bar F_d(\bar u,h) = \frac{1}{h+\delta}\partial_n\bar u\cdot\begin{pmatrix}x_n\nabla' h\\ h\end{pmatrix}$$

$$\begin{aligned}\bar G_\sigma^+(\bar u,\bar\theta,\bar\eta,h) = &-E\bar u\begin{pmatrix}-\nabla' h\\ 0\end{pmatrix} + \frac{1}{h+\delta}\left[\begin{pmatrix}x_n\nabla' h\\ h\end{pmatrix}\otimes\partial_n\bar u + \partial_n\bar u\otimes\begin{pmatrix}x_n\nabla' h\\ h\end{pmatrix}\right]\begin{pmatrix}-\nabla' h\\ 1\end{pmatrix}\\ &-\bar\theta\begin{pmatrix}\nabla' h\\ 0\end{pmatrix} + \sigma(m-\Delta')\bar\eta\begin{pmatrix}\nabla' h\\ 0\end{pmatrix} + \sigma\left[\Delta_{\Gamma_0^+}-\Delta'\right]\bar\eta\begin{pmatrix}-\nabla' h\\ 1\end{pmatrix}\end{aligned}$$

$$\bar K^+(\bar u,h) = \bar u_n\left(1-\frac{1}{\sqrt{1+\left|\nabla' h\right|^2}}\right) + \bar u\cdot\begin{pmatrix}\nabla' h\\ 0\end{pmatrix}\frac{1}{\sqrt{1+\left|\nabla' h\right|^2}}$$

Here we write $v\otimes w$ for the matrix $(v_iw_j)_{ij}$. The diffeomorphism Φ induces isomorphisms between $\mathbb{E}^\sigma(J,\bar\Omega)$ and $\mathbb{E}^\sigma(J,\Omega_0)$ as well as between $\mathbb{F}(J,\bar\Omega)$ and $\mathbb{F}(J,\Omega_0)$, and $\mathbb{I}_u(\bar\Omega)$ and $\mathbb{I}_u(\Omega_0)$ if $n<p<\infty$. Let us write

$$\mathcal{R}^\sigma(\bar u,\bar\theta,\bar\eta,h) = \begin{pmatrix}\bar F_1(\bar u,\bar\theta,h)\\ \bar F_d(\bar u,h)\\ \bar G_\sigma^+(\bar u,\bar\theta,\bar\eta,h)\\ \bar K^+(\bar u,h)\\ 0\end{pmatrix} \tag{15}$$

for the perturbation terms appearing on the right hand side of (14). Then, for fixed h, $\mathcal{R}^\sigma(\cdot,h)$ is a linear operator, which turns out to be continuous from $\mathbb{E}^\sigma(J)$ to $\mathbb{F}(J)$. For the remainder of this section we will drop the bars and write e. g. u instead of $\bar u$ again. We will need the following Lemmata to deal with the perturbation $\mathcal{R}^\sigma$.

LEMMA 4.5. *Let $n<p<\infty$. Then we have the following estimates.*

$$\begin{aligned}\left\|[\Delta_{\Gamma_0^+}-\Delta']w\right\|_{W_p^{1-1/p}(\mathbb{R}^{n-1})} &\lesssim \left\{1+\|h\|^3_{W_p^{3-1/p}(\mathbb{R}^{n-1})}\right\}\|h\|_{W_p^{3-1/p}(\mathbb{R}^{n-1})}\|w\|_{W_p^{3-1/p}(\mathbb{R}^{n-1})}\\ \left\|[\Delta_{\Gamma_0^+}-\Delta']w\right\|_{L_p(\mathbb{R}^{n-1})} &\lesssim \left\{1+\|h\|^3_{W_p^{3-1/p}(\mathbb{R}^{n-1})}\right\}\|h\|_{W_p^{3-1/p}(\mathbb{R}^{n-1})}\|w\|_{H_p^2(\mathbb{R}^{n-1})}.\end{aligned}$$

PROOF. The upper surface Γ_0^+ is parametrised with the function Φ and thus the associated Riemannian metric is given by

$$g = \nabla'\Phi^T\nabla'\Phi = \mathrm{Id}_{n-1} + (\nabla' h)\otimes(\nabla' h)$$

with determinant

$$\det g = \det[\mathrm{Id}_{n-1} + (\nabla' h)\otimes(\nabla' h)] = 1+\left|\nabla' h\right|^2$$

and inverse

$$g^{-1} = \mathrm{Id} - \frac{(\nabla' h)\otimes(\nabla' h)}{1+\left|\nabla' h\right|^2}$$

due to Lemma I.2.1. Thus we have

$$\begin{aligned}[\Delta_{\Gamma_0^+}-\Delta']w = &-\left[1-\frac{1}{\sqrt{1+\left|\nabla' h\right|^2}}\right]\operatorname{div}\left\{\sqrt{1+\left|\nabla' h\right|^2}g^{-1}\nabla w\right\}\\ &+\operatorname{div}\left\{\left[\sqrt{1+\left|\nabla' h\right|^2}g^{-1}-\mathrm{Id}\right]\nabla w\right\}.\end{aligned}$$

It is a consequence of Theorem 5.4.3.1 of [**RS96**] that we have

$$\left\| |\nabla' h|^2 \right\|_{W_p^{2-1/p}(\mathbb{R}^{n-1})} \lesssim \|\nabla' h\|_{W_p^{2-1/p}(\mathbb{R}^{n-1})} \|\nabla' h\|_{L_\infty(\mathbb{R}^{n-1})}$$

and due to the Sobolev embedding Theorem as stated in e. g. Theorem 2.4.4.1 of [**RS96**] we have

$$\left\| |\nabla' h|^2 \right\|_{W_p^{2-1/p}(\mathbb{R}^{n-1})} \lesssim \|\nabla' h\|^2_{W_p^{2-1/p}(\mathbb{R}^{n-1})}.$$

We can write

$$\sqrt{1 + |\nabla' h|^2} g^{-1} - \mathrm{Id} = (\sqrt{1 + |\nabla' h|^2} - 1)\,\mathrm{Id} - \frac{(\nabla' h) \otimes (\nabla' h)}{\sqrt{1 + |\nabla' h|^2}}.$$

Now we can use the power series expansion of $\sqrt{1+x}$ and the fact that $W_p^{2-1/p}(\mathbb{R}^{n-1})$ is an algebra with respect to multiplication to show

$$\left\| \sqrt{1 + |\nabla' h|^2} - 1 \right\|_{W_p^{2-1/p}(\mathbb{R}^{n-1})} \lesssim \|\nabla' h\|_{W_p^{2-1/p}(\mathbb{R}^{n-1})} \lesssim \|h\|_{W_p^{3-1/p}(\mathbb{R}^{n-1})}.$$

Theorem 5.5.1.1 of [**RS96**] implies

$$\left\| 1 - \frac{1}{\sqrt{1 + |\nabla' h|^2}} \right\|_{W_p^{1-1/p}(\mathbb{R}^{n-1})} \lesssim \|\nabla' h\|_{W_p^{1-1/p}(\mathbb{R}^{n-1})}$$

and Theorem 5.5.1.2 of [**RS96**] implies

$$\left\| \frac{\nabla' h}{\sqrt{1 + |\nabla' h|^2}} \right\|_{W_p^{2-1/p}(\mathbb{R}^{n-1})} \lesssim \|\nabla' h\|_{W_p^{2-1/p}(\mathbb{R}^{n-1})} (1 + \|\nabla' h\|_{L_\infty(\mathbb{R}^{n-1})}).$$

This shows

$$\begin{aligned} \left\| \sqrt{1 + |\nabla' h|^2} g^{-1} - \mathrm{Id} \right\|_{W_p^{2-1/p}(\mathbb{R}^{n-1})} &\lesssim \|\nabla' h\|_{W_p^{2-1/p}(\mathbb{R}^{n-1})} (1 + \|\nabla' h\|^2_{W_p^{2-1/p}(\mathbb{R}^{n-1})}) \\ &\lesssim \|h\|_{W_p^{3-1/p}(\mathbb{R}^{n-1})} (1 + \|h\|^2_{W_p^{3-1/p}(\mathbb{R}^{n-1})}) \end{aligned}$$

The algebra property of $W_p^{1-1/p}(\mathbb{R}^{n-1})$ and $W_p^{2-1/p}(\mathbb{R}^{n-1})$ now implies

$$\begin{aligned}
\left\|[\Delta_{\Gamma_0^+}-\Delta']w\right\|_{W_p^{1-1/p}(\mathbb{R}^{n-1})} &\lesssim \|\nabla' h\|_{W_p^{1-1/p}(\mathbb{R}^{n-1})}\left\|\operatorname{div}\left\{\sqrt{1+|\nabla' h|^2}g^{-1}\nabla w\right\}\right\|_{W_p^{1-1/p}(\mathbb{R}^{n-1})}\\
&\quad+\left\|\operatorname{div}\left\{\left[\sqrt{1+|\nabla' h|^2}g^{-1}-\mathrm{Id}\right]\nabla w\right\}\right\|_{W_p^{1-1/p}(\mathbb{R}^{n-1})}\\
&\lesssim \|\nabla' h\|_{W_p^{1-1/p}(\mathbb{R}^{n-1})}\left\|\sqrt{1+|\nabla' h|^2}g^{-1}\nabla w\right\|_{W_p^{2-1/p}(\mathbb{R}^{n-1})}\\
&\quad+\left\|\left[\sqrt{1+|\nabla' h|^2}g^{-1}-\mathrm{Id}\right]\nabla w\right\|_{W_p^{2-1/p}(\mathbb{R}^{n-1})}\\
&\lesssim \left\{1+\|\nabla' h\|_{W_p^{1-1/p}(\mathbb{R}^{n-1})}\right\}\left\|\left[\sqrt{1+|\nabla' h|^2}g^{-1}-\mathrm{Id}\right]\nabla w\right\|_{W_p^{2-1/p}(\mathbb{R}^{n-1})}\\
&\quad+\|\nabla' h\|_{W_p^{1-1/p}(\mathbb{R}^{n-1})}\|\nabla w\|_{W_p^{2-1/p}(\mathbb{R}^{n-1})}\\
&\lesssim \left\{1+\|h\|^3_{W_p^{3-1/p}(\mathbb{R}^{n-1})}\right\}\|h\|_{W_p^{3-1/p}(\mathbb{R}^{n-1})}\|\nabla w\|_{W_p^{2-1/p}(\mathbb{R}^{n-1})}\\
&\quad+\|h\|_{W_p^{3-1/p}(\mathbb{R}^{n-1})}\|\nabla w\|_{W_p^{2-1/p}(\mathbb{R}^{n-1})}\\
&\lesssim \left\{1+\|h\|^3_{W_p^{3-1/p}(\mathbb{R}^{n-1})}\right\}\|h\|_{W_p^{3-1/p}(\mathbb{R}^{n-1})}\|w\|_{W_p^{3-1/p}(\mathbb{R}^{n-1})}.
\end{aligned}$$

The second assertion can be shown along the same lines. □

LEMMA 4.6. *Under the assumptions of Theorem 4.1 the mapping $\mathcal{R}^\sigma$ as given in* (15) *maps set of functions (u,θ,η) in $\mathbb{E}^\sigma(J)$ that satisfy $u\big|_{\bar\Gamma^-}=0$ boundedly into $\mathbb{F}(J)$, and there is $C>0$ such that*

$$\|\mathcal{R}^\sigma(u,\theta,\eta,h)\|_{\mathbb{F}(J)}\le C\,\|h\|_{W_p^{3-1/p}(\mathbb{R}^{n-1})}\|(u,\theta,\eta)\|_{\mathbb{E}^\sigma(J)}$$

for all $0\le\sigma\le\sigma^*$, $(u,\theta,\eta)\in\mathbb{E}^\sigma(J)$ satisfying $u\big|_{\bar\Gamma^-}=0$, and $h\in W_p^{3-1/p}(\mathbb{R}^{n-1})$ of sufficiently small norm.

PROOF. This proof is essentially an application of the embedding results in Chapter 2 of [**RS96**] and the product estimates in Chapter 4 of [**RS96**]. First of all choose $r>0$ such that $\|h\|_{L_\infty(\mathbb{R}^{n-1})}\le\delta/2$ whenever $\|h\|_{W_p^{3-1/p}(\mathbb{R}^{n-1})}\le r$. The existence of this constant $r>0$ follows from the Sobolev Embedding Theorem as in e. g. Theorem 2.2.4 of [**RS96**]. For $p>n$ the space $W_p^{1-1/p}(\mathbb{R}^{n-1})$ is embedded into $L_\infty(\mathbb{R}^{n-1})$ and thus forms an algebra. Using the geometric series representation of $(h+\delta)^{-1}$ we obtain immediately the estimate

$$\left\|\bar F_1(u,\theta,h)\right\|_{\mathbb{F}_1(J)}\lesssim\|h\|_{W_p^{2-1/p}(\mathbb{R}^{n-1})}\|(u,\theta)\|_{\mathbb{E}_u(J)\times\mathbb{E}_\theta(J)}.$$

We turn to estimates for $\bar F_d(u,h)$. The estimates in the space $L_p(J;H_p^1(\bar\Omega))$ and $H_p^{1/2}(J;L_p(\bar\Omega))$ follow as above from Theorem 4.7.1 and Theorem 2.2.4 of [**RS96**]. The estimates in the space $H_p^1(J;{}_0\hat H_p^{-1}(\bar\Omega))$ are slightly more involved. Take a function $g\in\overset{0}{\hat H}{}^1_{p'}(\bar\Omega)$. Then integration by parts shows

$$\begin{aligned}
\left\langle g,\bar F_d(u,h)\right\rangle &= \int_{\bar\Omega}\frac{g}{h+\delta}\partial_n u\cdot\begin{pmatrix}x_n\nabla' h\\ h\end{pmatrix}\,\mathrm{d}x\\
&= -\int_{\bar\Omega}u\frac{\partial_n g}{h+\delta}\cdot\begin{pmatrix}x_n\nabla' h\\ h\end{pmatrix}+u\frac{g}{h+\delta}\cdot\begin{pmatrix}\nabla' h\\ 0\end{pmatrix}\,\mathrm{d}x
\end{aligned}$$

and from Hölder's inequality we can infer

$$\left|\left\langle g, \bar{F}_d(u,h)\right\rangle\right| \lesssim \|u\|_{L_p(\bar{\Omega})} \|\partial_n g\|_{L_{p'}(\bar{\Omega})} \|h\|_{W_p^{2-1/p}(\mathbb{R}^{n-1})} + \|u\|_{L_p(\bar{\Omega})} \|g\|_{L_{p'}(\bar{\Omega})} \|h\|_{W_p^{2-1/p}(\mathbb{R}^{n-1})}.$$

Then Poincaré's inequality shows

$$\left|\left\langle g, \bar{F}_d(u,h)\right\rangle\right| \lesssim \|h\|_{W_p^{2-1/p}(\mathbb{R}^{n-1})} \|u\|_{L_p(\bar{\Omega})} \|g\|_{{}^0\hat{H}^1_{p'}(\bar{\Omega})}.$$

This shows the estimate

$$\left\|\bar{F}_d(u,h)\right\|_{{}_0\hat{H}_p^{-1}(\bar{\Omega})} \lesssim \|h\|_{W_p^{2-1/p}(\mathbb{R}^{n-1})} \|u\|_{L_p(\bar{\Omega})},$$

and in the same way one can prove

$$\left\|\partial_t \bar{F}_d(u,h)\right\|_{{}_0\hat{H}_p^{-1}(\bar{\Omega})} \lesssim \|h\|_{W_p^{2-1/p}(\mathbb{R}^{n-1})} \|\partial_t u\|_{L_p(\bar{\Omega})}.$$

Taking the L_p-norm with respect to the time variable t this shows

$$\left\|\bar{F}_d(u,h)\right\|_{\mathbb{F}_d(J)} \lesssim \|h\|_{W_p^{2-1/p}(\mathbb{R}^{n-1})} \|u\|_{\mathbb{E}_u(J)}.$$

In order to estimate the remaining operators $\bar{G}_\sigma^+$ and $\bar{K}^+$ we will need the following estimates which are consequences of the trace theorem as presented e. g. in Lemma 3.5 of [**DHP07**]. We have for $u \in \mathbb{E}_u(J)$ the estimates

$$\left\|u\big|_{\Gamma^+}\right\|_{\mathbb{K}^+(J)} \lesssim \|u\|_{\mathbb{E}_u(J)} \qquad \left\|\nabla u\big|_{\Gamma^+}\right\|_{\mathbb{G}^+(J)} \lesssim \|u\|_{\mathbb{E}_u(J)}.$$

Using again Theorem 4.7.1 and Theorem 2.2.4 as well as Theorem 4.6.4.1 of [**RS96**] we immediately obtain the estimates

$$\begin{aligned}\left\|\bar{G}_\sigma^+(u,\theta,\eta,h)\right\| \lesssim{}& \|h\|_{W_p^{2-1/p}(\mathbb{R}^{n-1})} \|u\|_{\mathbb{E}_u(J)} + \|h\|_{W_p^{2-1/p}(\mathbb{R}^{n-1})} \|\theta\|_{\mathbb{E}_u(J)} \\ &+ \|h\|_{W_p^{2-1/p}(\mathbb{R}^{n-1})} \|\sigma(m-\Delta')\eta\|_{\mathbb{G}^+(J)} + \left\|\sigma(\Delta_{\Gamma_0^+} - \Delta')\eta\right\|_{\mathbb{G}^+(J)}.\end{aligned}$$

As in the proof of Lemma 4.4 we can estimate $(m-\Delta')\eta$ by means of the embedding from Lemma 4.3 of [**DSS08**], and $(\Delta_{\Gamma_0^+} - \Delta')\eta$ using Lemma 4.5 to obtain

$$\|h\|_{W_p^{2-1/p}(\mathbb{R}^{n-1})} \|\sigma(m-\Delta')\eta\|_{\mathbb{G}^+(J)} + \left\|\sigma(\Delta_{\Gamma_0^+} - \Delta')\eta\right\|_{\mathbb{G}^+(J)} \lesssim \|h\|_{W_p^{3-1/p}(\mathbb{R}^{n-1})} \|\eta\|_{\mathbb{E}_\eta^\sigma(J)}$$

and thus

$$\left\|\bar{G}_\sigma^+(u,\theta,\eta,h)\right\| \lesssim \|h\|_{W_p^{3-1/p}(\mathbb{R}^{n-1})} \|(u,\theta,\eta)\|_{\mathbb{E}^\sigma(J)}.$$

It remains to estimate $\bar{K}^+(u,h)$, but this can be done in essentially the same way using the trace theorem and the cited results from [**RS96**]. We obtain

$$\left\|\bar{K}_\sigma^+(u,h)\right\| \lesssim \|h\|_{W_p^{3-1/p}(\mathbb{R}^{n-1})} \|u\|_{\mathbb{E}_u(J)}$$

and hence the assertion follows. □

If we write for $\sigma \geq 0$

$$\mathbb{X}^\sigma(J) = \left\{(u,\theta,\eta) \in \mathbb{E}^\sigma(J) \colon \eta(0) = 0 \text{ and } u\big|_{\Gamma^-} = 0\right\} \subset \mathbb{E}^\sigma(J)$$

and

$$\mathbb{Y}(J) = \left\{(f_1, f_d, g^+, k^+, u_0) \in \mathbb{F}(J) \times \mathbb{I}_u \colon (8) \text{ is satisfied}\right\} \subset \mathbb{F}(J) \times \mathbb{I}_u$$

then we can consider the operator

$$\Lambda^\sigma \colon \mathbb{X}^\sigma(J) \to \mathbb{Y}(J), \qquad (u,\theta,\eta) \mapsto (f_1, f_d, g^+, k^+, u_0)$$

associated to the linear problem (7). From our considerations above we know that in the case $h \equiv 0$ the operator $\Lambda^\sigma \colon \mathbb{X}^\sigma \to \mathbb{Y}$ is an isomorphism for every $\sigma \geq 0$. Writing $F = (f_1, f_d, g^+, k^+, u_0)$ the equation (14) is equivalent to the operator equation

$$\Lambda^\sigma(u, \theta, \eta) = F + \mathcal{R}^\sigma(u, \theta, \eta, h). \tag{16}$$

We turn to the proof of Theorem 4.1.

Proof of Theorem 4.1. The diffeomorphism $\Phi \colon \Omega_0 \to \bar{\Omega}$ induces an isomorphism between the spaces $\mathbb{E}^\sigma(J, \Omega_0)$ and $\mathbb{E}^\sigma(J, \bar{\Omega})$, the spaces $\mathbb{F}(J, \Omega_0)$ and $\mathbb{F}(J, \bar{\Omega})$, as well as $\mathbb{I}_u(\Omega_0)$ and $\mathbb{I}_u(\bar{\Omega})$. This follows e. g. from the substitution rule combined with the integral characterisation of the spaces in question.

Thus it suffices to show existence, uniqueness and convergence of solutions for the system (14). Note that the diffeomorphism Φ preserves the compatibility conditions (8), i. e. the compatibility conditions are satisfied in (7) if and only if they are satisfied in the transformed system (14).

We will use the operator equation notation (16) introduced above, i. e. we investigate the equation

$$\Lambda^\sigma(u, \theta, \eta) = F + \mathcal{R}^\sigma(u, \theta, \eta, h).$$

Due to Lemma 4.6 we know that for fixed h the mapping $\mathcal{R}^\sigma(\cdot, h)$ is linear and has small norm whenever h is small enough in $W_p^{3-1/p}(\mathbb{R}^{n-1})$. Thus, choosing h sufficiently small, invertibility of Λ^σ implies invertibility of $\Lambda^\sigma - \mathcal{R}^\sigma$. It remains to show convergence of solutions as $\sigma \to 0^+$. Let us write $(u^\sigma, \theta^\sigma, \eta^\sigma)$ for the solution corresponding to some right hand side $F^\sigma = (f_{1,\sigma}, f_{d,\sigma}, g_\sigma^+, k_\sigma^+, u_{0,\sigma}) \in \mathbb{F}(J) \times \mathbb{I}_u$ that satisfies the compatibility conditions (8), i. e.

$$\Lambda^\sigma(u^\sigma, \theta^\sigma, \eta^\sigma) = F^\sigma + \mathcal{R}^\sigma(u^\sigma, \theta^\sigma, \eta^\sigma, h).$$

We begin with the assertion (10), but this follows immediately from Lemma 4.4 if we replace F^σ with $F^\sigma + \mathcal{R}^\sigma(u^\sigma, \theta^\sigma, \eta^\sigma, h)$. This is possible since the latter is uniformly bounded due to Lemma 4.6.

We turn to the proof of (11). The difference $(u^\sigma, \theta^\sigma, \eta^\sigma) - (u^0, \theta^0, \eta^0)$ of two solutions satisfies

$$\begin{aligned}
\Lambda^0[(u^\sigma, \theta^\sigma, \eta^\sigma) - (u^0, \theta^0, \eta^0)] &= \Lambda^0(u^\sigma, \theta^\sigma, \eta^\sigma) - \Lambda^0(u^0, \theta^0, \eta^0) \\
&= \Lambda^0(u^\sigma, \theta^\sigma, \eta^\sigma) - F^0 - \mathcal{R}^0(u^0, \theta^0, \eta^0, h) \\
&= \Lambda^\sigma(u^\sigma, \theta^\sigma, \eta^\sigma) - [\Lambda^\sigma - \Lambda^0](u^\sigma, \theta^\sigma, \eta^\sigma) - F^0 - \mathcal{R}^0(u^0, \theta^0, \eta^0, h) \\
&= F^\sigma + \mathcal{R}^\sigma(u^\sigma, \theta^\sigma, \eta^\sigma) - [\Lambda^\sigma - \Lambda^0](u^\sigma, \theta^\sigma, \eta^\sigma) - F^0 \\
&\qquad - \mathcal{R}^0(u^0, \theta^0, \eta^0, h) \\
&= \mathcal{R}^0[(u^\sigma, \theta^\sigma, \eta^\sigma) - (u^0, \theta^0, \eta^0)] + [\mathcal{R}^\sigma - \mathcal{R}^0](u^\sigma, \theta^\sigma, \eta^\sigma) \\
&\qquad - [\Lambda^\sigma - \Lambda^0](u^\sigma, \theta^\sigma, \eta^\sigma) + F^\sigma - F^0 \\
&= \mathcal{R}^0[(u^\sigma, \theta^\sigma, \eta^\sigma) - (u^0, \theta^0, \eta^0)] + \sigma \mathcal{Q}(u^\sigma, \theta^\sigma, \eta^\sigma) + F^\sigma - F^0
\end{aligned}$$

with

$$\sigma \mathcal{Q}(u, \theta, \eta) = [\mathcal{R}^\sigma - \mathcal{R}^0 - \Lambda^\sigma + \Lambda^0](u, \theta, \eta) = \sigma \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \tilde{\mathcal{Q}}(\eta, h) \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and

$$\begin{aligned}\tilde{Q}(\eta,h) &= (m-\Delta')\eta\begin{pmatrix}\nabla' h\\ -1\end{pmatrix} + \left[\Delta_{\Gamma_0^+}-\Delta'\right]\eta\begin{pmatrix}-\nabla' h\\ 1\end{pmatrix}\\ &= (m-\Delta_{\Gamma_0^+})\eta\begin{pmatrix}\nabla' h\\ -1\end{pmatrix}.\end{aligned}$$

From Lemma 4.3 and Lemma 4.6 we can infer the estimates

$$\begin{aligned}&\left\|(u^\sigma,\theta^\sigma,\eta^\sigma)-(u^0,\theta^0,\eta^0)\right\|_{\mathbb{E}^0(J)} \le C\left\|h\right\|_{W_p^{3-1/p}(\mathbb{R}^{n-1})}\left\|(u^\sigma,\theta^\sigma,\eta^\sigma)-(u^0,\theta^0,\eta^0)\right\|_{\mathbb{E}^0(J)}\\ &\qquad + \left\|F^\sigma - F\right\|_{\mathbb{F}(J)\times\mathbb{I}_u} + \sigma\left\|(m-\Delta')\eta^\sigma\begin{pmatrix}\nabla' h\\ -1\end{pmatrix}\right\|_{\mathbb{G}^+(J)} + \sigma\left\|\left[\Delta_{\Gamma_0^+}-\Delta'\right]\eta^\sigma\begin{pmatrix}-\nabla' h\\ 1\end{pmatrix}\right\|_{\mathbb{G}^+(J)}\\ &\le C\left\|h\right\|_{W_p^{3-1/p}(\mathbb{R}^{n-1})}\left\|(u^\sigma,\theta^\sigma,\eta^\sigma)-(u^0,\theta^0,\eta^0)\right\|_{\mathbb{E}^0(J)} + \left\|F^\sigma - F\right\|_{\mathbb{F}(J)\times\mathbb{I}_u}\\ &\qquad + C\sigma\left\|(m-\Delta')\eta^\sigma\right\|_{\mathbb{G}^+(J)} + \sigma\left\|\left[\Delta_{\Gamma_0^+}-\Delta'\right]\eta^\sigma\right\|_{\mathbb{G}^+(J)}.\end{aligned}$$

We can use Lemma 4.5 to estimate the term involving the Laplace-Beltrami operator $\Delta_{\Gamma_0^+}$ and thus obtain

$$\begin{aligned}\left\|(u^\sigma,\theta^\sigma,\eta^\sigma)-(u^0,\theta^0,\eta^0)\right\|_{\mathbb{E}^0(J)} &\le C\left\|h\right\|_{W_p^{3-1/p}(\mathbb{R}^{n-1})}\left\|(u^\sigma,\theta^\sigma,\eta^\sigma)-(u^0,\theta^0,\eta^0)\right\|_{\mathbb{E}^0(J)}\\ &\quad + \left\|F^\sigma - F\right\|_{\mathbb{F}(J)\times\mathbb{I}_u} + C\sigma\left\|\eta^\sigma\right\|_{\mathbb{E}^0_\eta(J)}\\ &\quad + C\sigma\left\|\Delta'\eta^\sigma\right\|_{L_p(J;W_p^{1-1/p}(\bar\Gamma))}\\ &\quad + C\sigma\left\|h\right\|_{W_p^{3-1/p}(\mathbb{R}^{n-1})}\left\|\eta^\sigma\right\|_{\mathbb{E}^0_\eta(J)}.\end{aligned}$$

If h is sufficiently small then this shows the estimate

$$\left\|(u^\sigma,\theta^\sigma,\eta^\sigma)-(u^0,\theta^0,\eta^0)\right\|_{\mathbb{E}^0(J)} \lesssim \left\|F^\sigma - F\right\|_{\mathbb{F}(J)\times\mathbb{I}_u} + \sigma\left\|\eta^\sigma\right\|_{\mathbb{E}^0_\eta(J)} + \sigma\left\|\Delta'\eta^\sigma\right\|_{L_p(J;W_p^{1-1/p}(\bar\Gamma))}$$

uniformly in $0\le\sigma\le\sigma^*$. Since

$$\left\|\eta^\sigma\right\|_{\mathbb{E}^0_\eta(J)} \lesssim \left\|F^\sigma\right\|_{\mathbb{F}(J)\times\mathbb{I}_u}$$

we can estimate

$$\left\|(u^\sigma,\theta^\sigma,\eta^\sigma)-(u^0,\theta^0,\eta^0)\right\|_{\mathbb{E}^0(J)} \lesssim \left\|F^\sigma - F\right\|_{\mathbb{F}(J)\times\mathbb{I}_u} + \sigma\left\|F^\sigma\right\|_{\mathbb{F}(J)\times\mathbb{I}_u} + \sigma\left\|\Delta'\eta^\sigma\right\|_{L_p(J;W_p^{1-1/p}(\bar\Gamma))}$$

and now it follows from (10), which we have already shown, and the assumptions that the right hand side vanishes as $\sigma\to 0^+$. ☐

5. The nonlinear problem I: Existence and uniqueness of solutions

The purpose of this section is to establish a proof of Theorem 2.1. This will be accomplished by a fixed point argument based Theorem 4.1 and a thorough understanding of the nonlinearities on the right hand side of (3).

PROPOSITION 5.1. *Let $n\ge 2$, $n<p<\infty$ and $\sigma^*>0$. Let $\Omega\subset\mathbb{R}^n$ be parametrised as in (2) with a function $h\in W_p^{3-1/p}(\mathbb{R}^{n-1})$ of sufficiently small norm, and let $0\le\sigma\le\sigma^*$. Let $\varepsilon>0$*

and $R > 0$. Then there is a time interval $J = (0, T)$ such that the nonlinear operator

$$\mathcal{N}^\sigma(u,\theta)\colon B_R(0) \cap \mathbb{E}^\sigma(J) \to \mathbb{F}(J), \qquad (u,\theta) \mapsto \begin{pmatrix} F_1(u,\theta) \\ F_d(u) \\ G_\parallel^+(u) \\ G_\perp^+(u) \\ K^+(u) \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{17}$$

is well-defined, i. e. $\mathcal{N}^\sigma$ maps the ball of radius R in $\mathbb{E}^\sigma(J)$ around zero into $\mathbb{F}(J)$. Furthermore the estimate

$$\begin{aligned} \|\mathcal{N}^\sigma(u,\theta) - \mathcal{N}^\sigma(v,\vartheta)\|_{\mathbb{F}(J)} &\le \varepsilon \,\|(u,\theta)-(v,\vartheta)\|_{\mathbb{E}^\sigma(J)} + \varepsilon \,\|u(0)-v(0)\|_{L_p(\Omega_0)} \\ &\quad + C_R \,\|h\|_{W_p^{2-1/p}(\mathbb{R}^{n-1})} \|u-v\|_{\mathbb{E}_u(J)} \end{aligned}$$

holds for all $0 \le \sigma \le \sigma^*$, and all $(u,\theta),(v,\vartheta) \in \mathbb{E}^\sigma(J')$ with norm not exceeding R.

Proof. In [**Abe05a**, Lemma 4.3] it has already been established that this is the case for F_1, F_d, $G_\parallel^+$ and the major part of $G_\perp^+$. It remains to estimate K^+ and a part of $G_\perp^+$. We start with the missing part of $G_\perp^+$, namely

$$\tilde{G}_\perp^+(u) = \sigma m \nu_0 \cdot \int_0^t u \,\mathrm{d}\tau - 2\sigma \left(\nabla \int_0^t u\,\mathrm{d}\tau\right) : \left(\nabla_{\Gamma_0^+} \nu_0\right) + \sigma\kappa_0$$

Observe that $\tilde{G}_\perp^+(u)$ is linear in u, except for the term $\sigma\kappa_0$. Using the fact that $W_p^{1-1/p}(\Gamma^+)$ is an algebra for $p > n$ and the trace theorem we estimate

$$\begin{aligned} \left\|\tilde{G}_\perp^+(u)\right\|_{\mathbb{G}^+(J)} &\le 2\sigma \left\|\nabla \int_0^t u\,\mathrm{d}\tau\right\|_{\mathbb{G}^+(J)} \left\|\nabla_{\Gamma_0^+}\nu_0\right\|_{W_p^{1-1/p}(\Gamma_0^+)} \\ &\quad + \sigma m \,\|\nu\|_{W^1_\infty(\Gamma_0^+)} \left\|\int_0^t u\,\mathrm{d}\tau\right\|_{\mathbb{G}^+(J)} + \sigma T^{1/p}\,\|\kappa_0\|_{W_p^{1-1/p}(\Gamma_0^+)}. \end{aligned}$$

We can use the trace theorem and interpolation theory to estimate the terms

$$\nabla \int_0^t u\,\mathrm{d}\tau, \qquad \int_0^t u\,\mathrm{d}\tau$$

in $\mathbb{G}^+(J)$ and obtain

$$\begin{aligned} \left\|\nabla \int_0^t u\,\mathrm{d}\tau\right\|_{\mathbb{G}^+(J)} &\le \left\|\int_0^t \nabla u\,\mathrm{d}\tau\right\|_{H_p^{1/2}(J;L_p(\Omega_0))\cap L_p(J;H_p^1(\Omega_0))} \\ &\le \left\|\int_0^t u\,\mathrm{d}\tau\right\|_{H_p^{1/2}(J;H_p^1(\Omega_0))} + \left\|\int_0^t u\,\mathrm{d}\tau\right\|_{L_p(J;H_p^2(\Omega_0))} \\ &\lesssim \left\|\int_0^t u\,\mathrm{d}\tau\right\|^{1/2}_{L_p(J;H_p^1(\Omega_0))} \left\|\int_0^t u\,\mathrm{d}\tau\right\|_{H_p^1(J;H_p^1(\Omega_0))} + \left\|\int_0^t u\,\mathrm{d}\tau\right\|_{L_p(J;H_p^2(\Omega_0))} \\ &\lesssim T^{1/2}\,\|u\|_{L_p(J;H_p^2(\Omega_0))} \lesssim T^{1/2}\,\|u\|_{\mathbb{E}_u(J)} \end{aligned}$$

and in the same way we obtain

$$\left\|\int_0^t u\,\mathrm{d}\tau\right\|_{\mathbb{G}^+(J)} \lesssim T^{1/2}\,\|u\|_{\mathbb{E}_u(J)}.$$

Furthermore, since $\kappa_0 = \nu_0 \cdot \Delta_{\Gamma_0^+}\xi$, we obtain

$$\|\kappa_0\|_{W_p^{1-1/p}(\Gamma_0^+)} \lesssim \|h\|^2_{W_p^{3-1/p}(\mathbb{R}^{n-1})}.$$

Combining this with Lemma 4.3 in [**Abe05a**] we see that $G_\perp^+$ maps $\mathbb{E}_u(J)$ into $\mathbb{G}^+(J)$ for sufficiently small $h \in W_p^{3-1/p}(\mathbb{R}^{n-1})$ and, choosing the time interval J small enough, we obtain

$$\left\|G_\perp^+(u) - G_\perp^+(v)\right\|_{\mathbb{G}^+(J)} \le \varepsilon \|u-v\|_{\mathbb{E}_u(J)}$$

for $u, v \in \mathbb{E}_u(J)$ of norm $\le R$, and then also

$$\left\|G_\perp^+(u)\right\|_{\mathbb{G}^+(J)} \le \varepsilon \|u\|_{\mathbb{E}_u(J)} + \varepsilon \|h\|_{W_p^{3-1/p}(\mathbb{R}^{n-1})}.$$

Let us turn to

$$K^+(u) = (m - \Delta_{\Gamma_0^+})^{-1}\left[u\cdot(\Delta_{\Gamma_0^+}\nu_0) - \nu_0\cdot\dot{\Delta}_{\Gamma_u^+}\int_0^t u\,\mathrm{d}\tau + \nu_0\cdot(\Delta_{\Gamma_0^+} - \Delta_{\Gamma_u^+})u - \nu_0\cdot\dot{\Delta}_{\Gamma_u^+}\xi\right].$$

Observe that K^+ is a nonlinear operator due to the appearance of u in the operators $\Delta_{\Gamma_u^+}$ and $\dot{\Delta}_{\Gamma_u^+}$. Since $K^+(0) = 0$ it suffices to estimate differences

$$\begin{aligned} K^+(u) - K^+(v) = (m - \Delta_{\Gamma_0^+})^{-1}\Big\{&[u-v]\cdot(\Delta_{\Gamma_0^+}\nu_0) - \nu_0\cdot\dot{\Delta}_{\Gamma_u^+}\int_0^t u - v\,\mathrm{d}\tau \\ &+\nu_0\cdot\left[\dot{\Delta}_{\Gamma_u^+} - \dot{\Delta}_{\Gamma_v^+}\right]\int_0^t v\,\mathrm{d}\tau + \nu_0\cdot\left[\Delta_{\Gamma_v^+} - \Delta_{\Gamma_u^+}\right]v + \nu_0\cdot(\Delta_{\Gamma_0^+}\cdot \\ &-\Delta_{\Gamma_u^+})[u-v] - \nu_0\cdot\left[\dot{\Delta}_{\Gamma_u^+} - \dot{\Delta}_{\Gamma_v^+}\right]\xi\Big\}. \end{aligned}$$

We wish to estimate

$$\begin{aligned} \left\|K^+(u) - K^+(v)\right\|_{\mathbb{K}^+} \le{} & \left\|[u-v]\cdot(\Delta_{\Gamma_0^+}\nu_0)\right\|_{L_p(J;W_p^{-1/p}(\Gamma_0^+))} \\ &+ \left\|\nu_0\cdot\dot{\Delta}_{\Gamma_u^+}\int_0^t u - v\,\mathrm{d}\tau\right\|_{L_p(J;W_p^{-1/p}(\Gamma_0^+))} \\ &+ \left\|\nu_0\cdot\left[\dot{\Delta}_{\Gamma_u^+} - \dot{\Delta}_{\Gamma_v^+}\right]\int_0^t v\,\mathrm{d}\tau\right\|_{L_p(J;W_p^{-1/p}(\Gamma_0^+))} \\ &+ \left\|\nu_0\cdot\left[\Delta_{\Gamma_u^+} - \Delta_{\Gamma_v^+}\right]v\right\|_{L_p(J;W_p^{-1/p}(\Gamma_0^+))} \\ &+ \left\|\nu_0\cdot(\Delta_{\Gamma_0^+} - \Delta_{\Gamma_u^+})[u-v]\right\|_{L_p(J;W_p^{-1/p}(\Gamma_0^+))} \\ &+ \left\|\nu_0\cdot\left[\dot{\Delta}_{\Gamma_u^+} - \dot{\Delta}_{\Gamma_v^+}\right]\xi\right\|_{L_p(J;W_p^{-1/p}(\Gamma_0^+))}. \end{aligned}$$

First observe that one can estimate

$$\begin{aligned} \|\nu_0 w\|_{L_p(J;W_p^{-1/p}(\Gamma_0^+))} &\le \|e_n w\|_{L_p(J;W_p^{-1/p}(\Gamma_0^+))} + \|(\nu_0 - e_n)w\|_{L_p(J;W_p^{-1/p}(\Gamma_0^+))} \\ &\le C\left(1 + \|\nabla' h\|_{W_p^{1-1/p}(\mathbb{R}^{n-1})}\right)\|w\|_{L_p(J;W_p^{-1/p}(\Gamma_0^+))} \\ &\le C\|w\|_{L_p(J;W_p^{-1/p}(\Gamma_0^+))}. \end{aligned}$$

Thus we obtain

$$\begin{aligned}\left\|K^+(u)-K^+(v)\right\|_{\mathbb{K}^+} \le{}& \left\|[u-v]\cdot(\Delta_{\Gamma_0^+}\nu_0)\right\|_{L_p(J;W_p^{-1/p}(\Gamma_0^+))} + \left\|\dot{\Delta}_{\Gamma_u^+}\int_0^t u-v\,\mathrm{d}\tau\right\|_{L_p(J;W_p^{-1/p}(\Gamma_0^+))}\\ &+\left\|\left[\dot{\Delta}_{\Gamma_u^+}-\dot{\Delta}_{\Gamma_v^+}\right]\int_0^t v\,\mathrm{d}\tau\right\|_{L_p(J;W_p^{-1/p}(\Gamma_0^+))}\\ &+\left\|\left[\Delta_{\Gamma_u^+}-\Delta_{\Gamma_v^+}\right]v\right\|_{L_p(J;W_p^{-1/p}(\Gamma_0^+))}\\ &+\left\|(\Delta_{\Gamma_0^+}-\Delta_{\Gamma_u^+})[u-v]\right\|_{L_p(J;W_p^{-1/p}(\Gamma_0^+))}\\ &+\left\|\nu_0\cdot\left[\dot{\Delta}_{\Gamma_u^+}-\dot{\Delta}_{\Gamma_v^+}\right]\xi\right\|_{L_p(J;W_p^{-1/p}(\Gamma_0^+))}.\end{aligned}$$

Let us start with the first term:

$$\begin{aligned}\left\|[u-v]\cdot(\Delta_{\Gamma_0^+}\nu_0)\right\|_{L_p(J;W_p^{-1/p}(\Gamma_0^+))} &\le C\left\|u-v\right\|_{L_p(J;W_p^{1-1/p}(\Gamma_0^+))}\left\|\Delta_{\Gamma_0^+}\nu_0\right\|_{W_p^{-1/p}(\Gamma_0^+)}\\ &\le C\left\|u-v\right\|_{L_p(J;W_p^{1-1/p}(\Gamma_0^+))}\left\|\nu_0-e_n\right\|_{W_p^{2-1/p}(\Gamma_0^+)}\\ &\le C\left\|h\right\|_{W_p^{3-1/p}(\mathbb{R}^{n-1})}\left\|u-v\right\|_{L_p(J;W_p^{1-1/p}(\Gamma_0^+))}.\end{aligned}$$

Now we can estimate

$$\begin{aligned}\left\|u\right\|_{L_p(J;W_p^{1-1/p}(\Gamma_0^+)} &\le \left\|u\right\|_{L_p(J;H_p^1(\Omega_0)}\\ &\le C\left\|u\right\|^{1/2}_{L_p(J;L_p(\Omega_0)}\left\|u\right\|^{1/2}_{L_p(J;H_p^2(\Omega_0)}\\ &\le C\left\{T^{1/p}\left\|u(0)\right\|_{L_p(\Omega_0)}+T\left\|\partial_t u\right\|_{L_p(J;L_p(\Omega_0))}\right\}^{1/2}\left\|u\right\|^{1/2}_{L_p(J;H_p^2(\Omega_0)}\\ &\le CT^{1/2}\left\|u\right\|_{L_p(J;H_p^2(\Omega_0)}+CT^{1/2p}\left\|u(0)\right\|^{1/2}_{L_p(\Omega_0)}\left\|u\right\|^{1/2}_{L_p(J;H_p^2(\Omega_0)}\\ &\le CT^{1/2p}\left\|u\right\|_{\mathbb{E}_u(J)}+CT^{1/2p}\left\|u(0)\right\|_{L_p(\Omega_0)}\end{aligned}$$

and thus we obtain

$$\begin{aligned}\left\|[u-v]\cdot(\Delta_{\Gamma_0^+}\nu_0)\right\|_{L_p(J;W_p^{-1/p}(\Gamma_0^+))} &\le C\left\|u-v\right\|_{L_p(J;W_p^{1-1/p}(\Gamma_0^+))}\\ &\le CT^{1/2p}\left\|u-v\right\|_{\mathbb{E}_u(J)}+CT^{1/2p}\left\|u(0)-v(0)\right\|_{L_p(\Omega_0)}.\end{aligned}$$

The estimates for the remaining terms follow from Lemma 3.1. □

This enables us to prove the first of our main results by means of a fixed point iteration.

Proof of Theorem 2.1. Let $R>0$ and u_0 as in the assumptions. Let $0\le\sigma\le\sigma^*$. Write

$$F=(0,\dots,0,u_0,0).$$

Define a space

$$\mathbb{X}_R^\sigma(J)=\left\{(u,\theta,\eta)\in\mathbb{E}^\sigma(J)\colon\ \left\|(u,\theta,\eta)\right\|_{\mathbb{E}^\sigma(J)}\le R \text{ and } u\big|_{t=0}=u_0\right\}$$

and an operator

$$\Xi^\sigma\colon\mathbb{X}_R^\sigma(J)\to\mathbb{E}^\sigma(J),\qquad (u,\theta,\eta)\mapsto(\Lambda^\sigma)^{-1}\left[\mathcal{N}^\sigma(u,\theta)+F\right]$$

where Λ^σ denotes the operator corresponding to the left hand side of (7). From the results above we know that this operator is well-defined, and (u,θ,η) is a solution of (3) if and only if (u,θ,η) is a fixed point of Ξ^σ. In order to invoke Banach's fixed point theorem we show that Ξ^σ maps

$\mathbb{X}_R^\sigma(J)$ into itself and is Lipschitz continuous on $\mathbb{X}_R^\sigma(J)$ with constant smaller than one. Given $(u,\theta,\eta),(v,\vartheta,\rho)\in\mathbb{X}_R^\sigma(J)$ we compute

$$\begin{aligned}\|\Xi^\sigma(u,\theta,\eta)-\Xi^\sigma(v,\vartheta,\rho)\|_{\mathbb{E}^\sigma(J)} &\le C\,\|\mathcal{N}^\sigma(u,\theta,\eta)-\mathcal{N}^\sigma(v,\vartheta,\rho)\|_{\mathbb{F}(J)}\\ &\le C\varepsilon\,\|(u,\theta,\eta)-(v,\vartheta,\rho)\|_{\mathbb{E}^\sigma(J)}+C\,\|h\|_{W_p^{2-1/p}(\mathbb{R}^{n-1})}\,\|u-v\|_{\mathbb{E}_u(J)}\end{aligned}$$

and if we choose $\varepsilon>0$ and h small enough, then Ξ^σ has Lipschitz constant smaller than 1, uniformly in $0\le\sigma\le\sigma^*$. Now

$$\begin{aligned}\|\Xi^\sigma(u,\theta,\eta)\|_{\mathbb{E}^\sigma(J)} &\le \|\Xi^\sigma(u,\theta,\eta)-\Xi^\sigma(0,0,0)\|_{\mathbb{E}^\sigma(J)}+\|\Xi^\sigma(0,0,0)\|_{\mathbb{E}^\sigma(J)}\\ &\le C\,\|\mathcal{N}^\sigma(u,\theta,\eta)-\mathcal{N}_\sigma(0,0,0)\|_{\mathbb{F}(J)}+C\,\|u_0\|_{\mathbb{I}_u}+C\sigma\,\|\kappa_0\|_{\mathbb{G}^+(J)}\\ &\le C\varepsilon\,\|(u,\theta,\eta)\|_{\mathbb{E}^\sigma(J)}+C\,\|h\|_{W_p^{2-1/p}(\mathbb{R}^{n-1})}\,\|u\|_{\mathbb{E}_u(J)}\\ &\qquad +C\,\|u_0\|_{\mathbb{I}_u}+C\sigma\,\|h\|_{W_p^{3-1/p}(\mathbb{R}^{n-1})}\\ &\le C\varepsilon R+C\,\|h\|_{W_p^{2-1/p}(\mathbb{R}^{n-1})}\,R+C\,\|u_0\|_{\mathbb{I}_u}+C\sigma\,\|h\|_{W_p^{3-1/p}(\mathbb{R}^{n-1})}\,.\end{aligned}$$

Since $0<C\varepsilon+C\,\|h\|_{W_p^{2-1/p}(\mathbb{R}^{n-1})}<1$ we can simply choose $R>0$ large enough to ensure that Ξ^σ maps $\mathbb{X}_R^\sigma$ into itself. □

6. The nonlinear problem II: Convergence of solutions

Subject of this section is a proof of our second main result Theorem 2.2.

Proof of Theorem 2.2. Let $0\le\sigma\le\sigma^*$ and let $(u^\sigma,\theta^\sigma,\eta^\sigma)$ and (u^0,θ^0,η^0) denote the corresponding solutions. As above we write (7) as $\Lambda^\sigma(u,\theta,\eta)=F^\sigma+\mathcal{R}^\sigma(u,\theta,\eta)$ with $F^\sigma=(0,\dots,0,u_0^\sigma,0)$. Then

$$\Lambda^\sigma(u,\theta,\eta)=\Lambda^0(u,\theta,\eta)+\sigma\tilde\Lambda(\eta)=\Lambda^0(u,\theta,\eta)+\sigma\begin{pmatrix}0\\0\\0\\(m-\Delta_{\Gamma_u^+})\eta\\0\\0\\0\\0\end{pmatrix}$$

and similarly

$$\mathcal{N}^\sigma(u,\theta,\eta)=\mathcal{N}^0(u,\theta,\eta)+\sigma\tilde{\mathcal{N}}(u)$$

where

$$\tilde{\mathcal{N}}(u)=\begin{pmatrix}0\\0\\0\\-2\left(\nabla\int_0^t u\,\mathrm{d}\tau\right):\left(\nabla_{\Gamma_0^+}\nu_0\right)+m\nu_0\cdot\int_0^t u\,\mathrm{d}\tau+\kappa_0\\0\\0\\0\\0\end{pmatrix}.$$

Thus we see that the difference $(u^\sigma, \theta^\sigma, \eta^\sigma) - (u^0, \theta^0, \eta^0)$ satisfies

$$\begin{aligned}\Lambda^0(u^\sigma - u^0, \theta^\sigma - \theta^0, \eta^\sigma - \eta^0) &= \Lambda^0(u^\sigma, \theta^\sigma, \eta^\sigma) - \Lambda^0(u^0, \theta^0, \eta^0)\\ &= \Lambda^\sigma(u^\sigma, \theta^\sigma, \eta^\sigma) - \sigma\tilde{\Lambda}(\eta^\sigma) - \Lambda^0(u^0, \theta^0, \eta^0)\\ &= \mathcal{N}^\sigma(u^\sigma, \theta^\sigma, \eta^\sigma) - \sigma\tilde{\Lambda}(\eta^\sigma) - \mathcal{N}^0(u^0, \theta^0, \eta^0) + F^\sigma - F^0\\ &= \mathcal{N}^0(u^\sigma, \theta^\sigma, \eta^\sigma) + \sigma\tilde{\mathcal{N}}(u^\sigma) - \sigma\tilde{\Lambda}(\eta^\sigma) - \mathcal{N}^0(u^0, \theta^0, \eta^0) + F^\sigma - F^0\\ &= \mathcal{N}^0(u^\sigma - u^0, \theta^\sigma - \theta^0, \eta^\sigma - \eta^0) + \sigma\left[\tilde{\mathcal{N}}(u^\sigma) - \tilde{\Lambda}(\eta^\sigma)\right] + F^\sigma - F^0.\end{aligned}$$

As the compatibility conditions for the linear problem are satisfied we can invoke Theorem 4.1 to invert Λ^0 and obtain the estimate

$$\begin{aligned}\left\|(u^\sigma - u^0, \theta^\sigma - \theta^0, \eta^\sigma - \eta^0)\right\|_{\mathbb{E}^0(J)} &\le C\left\|\mathcal{N}^0(u^\sigma - u^0, \theta^\sigma - \theta^0, \eta^\sigma - \eta^0)\right\|_{\mathbb{F}(J)} + \left\|u_0^\sigma - u_0^0\right\|_{\mathbb{I}_u}\\ &\quad + C\sigma\left(\left\|\tilde{\Lambda}(\eta^\sigma)\right\|_{\mathbb{F}(J)} + \left\|\tilde{\mathcal{N}}(u^\sigma)\right\|_{\mathbb{F}(J)}\right)\\ &\le \varepsilon\left\|(u^\sigma - u^0, \theta^\sigma - \theta^0, \eta^\sigma - \eta^0)\right\|_{\mathbb{E}^0(J)} + \left\|u_0^\sigma - u_0^0\right\|_{\mathbb{I}_u}\\ &\quad + C\sigma\left\|u^\sigma\right\|_{\mathbb{E}_u(J)} + C\sigma\left\|\eta^\sigma\right\|_{\mathbb{E}^1_\eta(J)}.\end{aligned}$$

For $0 < \varepsilon < 1$ this shows the estimate

$$\begin{aligned}\left\|(u^\sigma - u^0, \theta^\sigma - \theta^0, \eta^\sigma - \eta^0)\right\|_{\mathbb{E}^0(J)} &\le C\sigma\left\|u_0^\sigma\right\|_{\mathbb{I}_u} + C\sigma\left\|h\right\|_{W_p^{3-1/p}(\mathbb{R}^{n-1})}\\ &\quad + C\sigma\left\|\eta^\sigma\right\|_{\mathbb{E}^1_\eta(J)} + \left\|u_0^\sigma - u_0^0\right\|_{\mathbb{I}_u}\end{aligned}$$

and for $\sigma \to 0^+$ the assertion follows from Theorem 4.1. □

7. The nonlinear problem III: Boundary regularity

This section is devoted to investigating the regularity of the free upper surface $\Gamma^+(t)$ for $t > 0$ and its properties in the singular limit of vanishing surface tension. To this end we investigate a certain parametrisation of the moving upper boundary $\Gamma^+(t)$ by means of a height function. We look for a mapping $H\colon J \times \Gamma_0^+ \to \mathbb{R}$ such that

$$\Gamma^+(t) = \left\{\xi + \nu_0 H(t, \xi)\colon \xi \in \Gamma_0^+\right\}.$$

This is equivalent to the existence of a mapping $\psi\colon J \times \Gamma_0^+ \to \Gamma_0^+$ such that

$$\zeta + \nu_0(\zeta)H(t, \zeta) = X_u(t, \xi)$$

with $\zeta = \psi(t, \xi)$ for all $0 < t < T$ and $\xi \in \Gamma_0^+$. Let $\Pi_0(\xi)d = d - \nu_0(\nu_0 \cdot d)$ as before. Then

$$\Pi_0(\zeta)\zeta = \Pi_0(\zeta)X_u(t, \xi) \qquad \text{and} \qquad \nu_0(\zeta)\cdot\zeta + H(t, \zeta) = \nu_0(\zeta)\cdot X_u(t, \xi). \tag{18}$$

We will show the existence of said mapping ψ, and then we can define H to be

$$H(t, \zeta) = -\nu_0(\zeta)\cdot\zeta + \nu_0(\zeta)\cdot X_u(t, \xi).$$

We will only treat the case where the initial domain Ω_0 is a flat layer. In this case $\Gamma_0^+ = \mathbb{R}^{n-1}\times\{\delta\}$, $\nu_0 \equiv e_n$, and $\Pi_0 d = (d_1, \ldots, d_{n-1}, 0)^T$. Then $\psi_n \equiv \delta$ and (18) reduces to

$$\psi_j(t, \xi) = \xi_j + \int_0^t u_j(\tau, \xi)\,\mathrm{d}\tau$$

for $j = 1, \ldots, n-1$, which immediately defines a functions $\psi\colon J\times\Gamma_0^+ \to \Gamma_0^+$ such that $\psi(t, \cdot)$ is a diffeomorphism of Γ_0^+ onto itself. Now we can define H via

$$H(t, \psi(t, \xi)) = \int_0^t u_n(\tau, \xi)\,\mathrm{d}\tau$$

for $t > 0$ and $\xi \in \Gamma_0^+$. It is an immediate consequence of Theorem 2.1 that the mapping ψ can be written as $\psi = \mathrm{Id} + \phi$ with ϕ contained in $W_p^{2-1/2p}(J; L_p(\Gamma_0^+)) \cap H_p^1(J; W_p^{2-1/p}(\Gamma_0^+))$. This in turn shows $H \in W_p^{2-1/2p}(J; L_p(\Gamma_0^+)) \cap H_p^1(J; W_p^{2-1/p}(\Gamma_0^+))$. We will show that the composite mapping

$$(t, \xi) \mapsto H(t, \psi(t, \xi)) = \int_0^t u_n(\tau, \xi)\, \mathrm{d}\tau$$

is not only in $W_p^{2-1/2p}(J; L_p(\Gamma_0^+)) \cap H_p^1(J; W_p^{2-1/p}(\Gamma_0^+))$ but additionally in $L_p(J; W_p^{3-1/p}(\Gamma_0^+))$.

Let us write

$$S\colon J \times \Gamma_0^+ \to \mathbb{R}, \qquad (t, \xi) \mapsto \cdot \int_0^t u_n(\tau, \xi)\, \mathrm{d}\tau \tag{19}$$

and let us take another look at the mapping

$$\begin{aligned}\eta(t, \cdot) = \nu_0 \cdot \int_0^t u\, \mathrm{d}\tau + (m - \Delta_{\Gamma_0^+})^{-1} \Big[&(\Delta_{\Gamma_0^+} \nu_0) \cdot \int_0^t u\, \mathrm{d}\tau \\ &+ \nu_0 \cdot (\Delta_{\Gamma_0^+} - \Delta_{\Gamma_u^+}) \int_0^t u\, \mathrm{d}\tau + \nu_0 \cdot (\Delta_{\Gamma_0^+} - \Delta_{\Gamma_u^+}) \xi \Big].\end{aligned}$$

Appearently,

$$\begin{aligned}\eta(t, \cdot) = S(t, \cdot) + (m - \Delta_{\Gamma_0^+})^{-1} \Big[&(\Delta_{\Gamma_0^+} \nu_0) \cdot \int_0^t u\, \mathrm{d}\tau \\ &+ \nu_0 \cdot (\Delta_{\Gamma_0^+} - \Delta_{\Gamma_u^+}) \int_0^t u\, \mathrm{d}\tau + \nu_0 \cdot (\Delta_{\Gamma_0^+} - \Delta_{\Gamma_u^+}) \xi \Big].\end{aligned}$$

In the case $h \equiv 0$ we have $\nu_0 \equiv e_n$, in particular ν_0 commutes with the appearing differential operators, and η can be written in the form

$$\begin{aligned}\eta(t, \cdot) &= S(t, \cdot) + (m - \Delta_{\Gamma_0^+})^{-1} \Big[\nu_0 \cdot (\Delta_{\Gamma_0^+} - \Delta_{\Gamma_u^+}) \int_0^t u\, \mathrm{d}\tau + \nu_0 \cdot (\Delta_{\Gamma_0^+} - \Delta_{\Gamma_u^+}) \xi \Big] \\ &= S(t, \cdot) + (m - \Delta_{\Gamma_0^+})^{-1} \Big[(\Delta_{\Gamma_0^+} - \Delta_{\Gamma_u^+}) S(t, \cdot) + (\Delta_{\Gamma_0^+} - \Delta_{\Gamma_u^+}) \xi_n \Big] \\ &= (m - \Delta_{\Gamma_0^+})^{-1} \Big[(m - \Delta_{\Gamma_u^+}) S(t, \cdot) + (\Delta_{\Gamma_0^+} - \Delta_{\Gamma_u^+}) \xi_n \Big] \\ &= (m - \Delta_{\Gamma_0^+})^{-1} (m - \Delta_{\Gamma_u^+}) S(t, \cdot) + (m - \Delta_{\Gamma_0^+})^{-1} (\Delta_{\Gamma_0^+} - \Delta_{\Gamma_u^+}) \xi_n.\end{aligned}$$

Also, $\xi_n = \delta$ and thus

$$\eta(t, \cdot) = (m - \Delta_{\Gamma_0^+})^{-1} (m - \Delta_{\Gamma_u^+}) S(t, \cdot).$$

Since $\eta \in L_p(J; W_p^{3-1/p}(\Gamma_0^+))$ by Theorem 2.1 we can infer from the mapping properties of the Laplace-Beltrami operator that we also have $S \in L_p(J; W_p^{3-1/p}(\Gamma_0^+))$.

An alternative point of view on this result is the following: The free upper surface $\Gamma^+(t)$ admits a parametrisation

$$\Gamma^+(t) = X_u(t, \Gamma_0^+) = \left\{ \xi + \nu_0 \int_0^t u_n(\tau, \xi)\, \mathrm{d}\tau + \Pi_0 \int_0^t u(\tau, \xi)\, \mathrm{d}\tau \colon \xi \in \Gamma_0^+ \right\}$$

where the first part $\xi \mapsto \xi$ is obviously smooth, the second part is in addition to the standard regularity also contained in $L_p(J; W_p^{3-1/p}(\Gamma_0^+))$, and for the remaining part given by

$$\int_0^t u_j(\tau, \xi)\, \mathrm{d}\tau, \qquad j = 1, \dots, n-1$$

it seems to be unclear whether it has regularity $L_p(J; W_p^{3-1/p}(\Gamma_0^+))$. This should be contrasted with the results obtained in [**DGH$^+$11**], where a similar system was investigated in a Eulerian formulation. There it was shown that the free upper boundary can be parametrised by a height function in $W_p^{2-1/2p}(J; L_p(\Gamma_0^+)) \cap H_p^1(J; W_p^{2-1/p}(\Gamma_0^+)) \cap L_p(J; W_p^{3-1/p}(\Gamma_0^+))$. However, it seems unclear whether the Eulerian approach allows a derivation of estimates that are uniform in the surface tension parameter $\sigma \geq 0$.

8. Proof of Lemma 3.1

In this section we will show the estimates for the operators $\Delta_{\Gamma_u^+}$ and $\dot{\Delta}_{\Gamma_u^+}$ stated in Lemma 3.1, i. e. given functions $u, v, w \in \mathbb{E}_u(J)$ and $n < p < \infty$ we wish to estimate the quantities

$$\left[\Delta_{\Gamma_u^+} - \Delta_{\Gamma_v^+}\right] w, \qquad \left[\dot{\Delta}_{\Gamma_u^+} - \dot{\Delta}_{\Gamma_v^+}\right] \int_0^t w, \qquad \left[\dot{\Delta}_{\Gamma_u^+} - \dot{\Delta}_{\Gamma_v^+}\right] \xi$$

in the space $L_p(J; W_p^{-1/p}(\Gamma_0^+))$. As above let $\bar{\Omega} = \mathbb{R}^{n-1} \times (0, \delta)$, $\bar{\Gamma}^+ = \mathbb{R}^{n-1} \times \{\delta\}$, and

$$\Phi \colon \mathbb{R}^{n-1} \times [0, \delta] \to \mathbb{R}^n, \qquad x \mapsto (x', x_n(\delta + h(x'))/\delta).$$

Then $\Phi(\bar{\Omega}) = \Omega_0$, and $\Phi(\bar{\Gamma}^+) = \Gamma_0^+$, and we have $\Omega(t) = X_u(t, \cdot) \circ \Phi(\bar{\Omega})$ and $\Gamma_u^+(t) = X_u(t, \cdot) \circ \Phi(\bar{\Gamma}^+)$. The Laplace-Beltrami operator $\Delta_{\Gamma_u^+} w$ has the representation

$$\left\{\Delta_{\Gamma_u^+} w\right\} \circ \Phi = |\det g_u|^{-1/2} \operatorname{div} \left\{|\det g_u|^{1/2} g_u^{-1} \nabla \tilde{w}\right\}$$

with $\tilde{w} = w \circ \Phi$ and

$$g_u = \nabla' \Phi^T \left\{\left[\nabla X_u^T \nabla X_u\right] \circ \Phi\right\} \nabla' \Phi.$$

Observe that the matrix $\nabla' \Phi$ has full rank regardless of the size of h and thus g_u has full rank whenever ∇X_u has full rank. This in turn is the case if e. g.

$$\left| \int_0^t u(\tau, \xi) \, \mathrm{d}\tau \right| \leq C T^{1-1/p} \|u\|_{\mathbb{E}_u(J)} < 1.$$

In particular for any $R > 0$ there is $T' > 0$ such that whenever $0 < T < T'$ and $\|u\|_{\mathbb{E}_u(0,T)} \leq R$ then g_u is regular, and then also positive definite. The operator $\dot{\Delta}_{\Gamma_u^+}$ is given by

$$\begin{aligned}\left\{\dot{\Delta}_{\Gamma_u^+} w\right\} \circ \Phi = {} & [\partial_t |\det g_u|^{-1/2}] \operatorname{div} \left\{|\det g_u|^{1/2} g_u^{-1} \nabla \tilde{w}\right\} \\ & + |\det g_u|^{-1/2} \operatorname{div} \left\{[\partial_t |\det g_u|^{1/2} g_u^{-1}] \nabla \tilde{w}\right\}.\end{aligned}$$

Since the diffeomorphism Φ induces an isomorphism between between $L_p(J; W_p^{-1/p}(\Gamma_0^+))$ and $L_p(J; W_p^{-1/p}(\bar{\Gamma}^+))$ it is sufficient to estimate the quantities

$$\left\{\left[\Delta_{\Gamma_u^+} - \Delta_{\Gamma_v^+}\right] w\right\} \circ \Phi, \qquad \left\{\left[\dot{\Delta}_{\Gamma_u^+} - \dot{\Delta}_{\Gamma_v^+}\right] \int_0^t w\right\} \circ \Phi, \qquad \left\{\left[\dot{\Delta}_{\Gamma_u^+} - \dot{\Delta}_{\Gamma_v^+}\right] \xi\right\} \circ \Phi$$

in the space $L_p(J; W_p^{-1/p}(\bar{\Gamma}^+))$. To this end we will need the following Lemmata.

LEMMA 8.1. *Let $1 < p < \infty$ and $u, v \in \mathbb{E}_u(J)$. Then we have the estimates*

$$\begin{aligned}\|\nabla X_u - \nabla X_v\|_{L_\infty(J; W_p^{1-1/p}(\Gamma^+))} &\lesssim T^{1-1/p} \|u - v\|_{\mathbb{E}_u(J)} \\ \|\nabla X_u - \nabla X_v\|_{H_p^1(J; W_p^{1-1/p}(\Gamma^+))} &\lesssim (1 + T) \|u - v\|_{\mathbb{E}_u(J)}.\end{aligned}$$

PROOF. We have

$$\nabla X_u(t,\xi) - \nabla X_v(t,\xi) = \int_0^t \nabla u(\tau,\xi) - \nabla v(\tau,\xi) \,\mathrm{d}\tau$$

and

$$\partial_t \nabla X_u(t,\xi) - \partial_t \nabla X_v(t,\xi) = \nabla u(t,\xi) - \nabla v(t,\xi)$$

and thus we can use the trace theorem to estimate

$$\begin{aligned}\|\nabla X_u - \nabla X_v\|_{L_\infty(J;W_p^{1-1/p}(\Gamma^+))} &\le \sup_{t\in J} \int_0^t \|\nabla u(\tau,\cdot) - \nabla v(\tau,\cdot)\|_{W_p^{1-1/p}(\Gamma^+)} \,\mathrm{d}\tau \\ &\lesssim T^{1-1/p} \|u - v\|_{\mathbb{E}_u(J)}\end{aligned}$$

as well as

$$\begin{aligned}\|\nabla X_u - \nabla X_v\|_{H_p^1(J;W_p^{1-1/p}(\Gamma^+))} &\le \|\nabla X_u - \nabla X_v\|_{L_p(J;W_p^{1-1/p}(\Gamma^+))} \\ &\quad + \|\partial_t \nabla X_u - \partial_t \nabla X_v\|_{L_p(J;W_p^{1-1/p}(\Gamma^+))} \\ &\lesssim T \|u - v\|_{\mathbb{E}_u(J)} + \|u - v\|_{\mathbb{E}_u(J)} .\end{aligned}$$

□

LEMMA 8.2. *Let $n < p < \infty$, $T' > 0$ and $R > 1$. Let $J = (0,T)$ with $0 < T \le T'$. Then for every $u, v \in \mathbb{E}_u(J)$ of norm $\le R$ and $h \in W_p^{3-1/p}(\mathbb{R}^{n-1})$ of norm ≤ 1 we have the estimates*

$$\begin{aligned}\|g_0 - \mathrm{Id}_{n-1}\|_{W_p^{1-1/p}(\bar\Gamma^+)} &\le \|h\|^2_{W_p^{2-1/p}(\mathbb{R}^{n-1})} \\ \|g_u - g_v\|_{L_\infty(J;W_p^{1-1/p}(\bar\Gamma^+))} &\le C_R T^{1-1/p} \|u - v\|_{\mathbb{E}_u(J)} \\ \|g_u - g_v\|_{H_p^1(J;W_p^{1-1/p}(\bar\Gamma^+))} &\le C_R \|u - v\|_{\mathbb{E}_u(J)}\end{aligned}$$

uniformly in $0 < T \le T_0$.

PROOF. The first assertion follows immediately from

$$g_0 - \mathrm{Id} = \nabla'\Phi^T \nabla'\Phi - \mathrm{Id} = (\nabla' h) \otimes (\nabla' h)$$

and the algebra property of $W_p^{1-1/p}(\mathbb{R}^{n-1})$. For the second assertion and third we compute

$$\begin{aligned}g_u - g_v &= \nabla'\Phi^T \nabla X_u^T \nabla X_u \nabla'\Phi - \nabla'\Phi^T \nabla X_v^T \nabla X_v \nabla'\Phi \\ &= \nabla'\Phi^T \left[\nabla X_u^T \nabla X_u - \nabla X_v^T \nabla X_v\right] \nabla'\Phi \\ &= \nabla'\Phi^T \left[(\nabla X_u - \nabla X_v)^T \nabla X_u + \nabla X_v^T (\nabla X_u - \nabla X_v)\right] \nabla'\Phi\end{aligned}$$

and thus we can estimate

$$
\begin{aligned}
&\|g_u - g_v\|_{L_\infty(J;W_p^{1-1/p}(\bar\Gamma^+))}\\
&\quad\lesssim \left(1 + \|\nabla'\Phi - \mathrm{Id}\|_{W_p^{1-1/p}(\bar\Gamma^+)}\right)^2\\
&\qquad\cdot \left\| (\nabla X_u - \nabla X_v)^T \nabla X_u + \nabla X_v^T (\nabla X_u - \nabla X_v) \right\|_{L_\infty(J;W_p^{1-1/p}(\bar\Gamma^+))}\\
&\quad\lesssim \left(1 + \|\nabla'\Phi - \mathrm{Id}\|_{W_p^{1-1/p}(\bar\Gamma^+)}\right)^2\\
&\qquad\cdot \left(1 + \|\nabla X_u - \mathrm{Id}\|_{L_\infty(J;W_p^{1-1/p}(\bar\Gamma^+))} + \|\nabla X_v - \mathrm{Id}\|_{L_\infty(J;W_p^{1-1/p}(\bar\Gamma^+))}\right)\\
&\qquad\cdot \|\nabla X_u - \nabla X_v\|_{L_\infty(J;W_p^{1-1/p}(\bar\Gamma^+))}\\
&\quad\lesssim \left(1 + \|h\|_{W_p^{2-1/p}(\bar\Gamma^+)}\right)^2 \left(1 + T^{1-1/p}\|u\|_{\mathbb{E}_u(J)} + T^{1-1/p}\|v\|_{\mathbb{E}_u(J)}\right)\\
&\qquad\cdot T^{1-1/p}\|u - v\|_{\mathbb{E}_u(J)}
\end{aligned}
$$

where the last inequality is a consequence of Lemma 8.1. This shows the second assertion. For the third assertion it suffices to estimate $\partial_t g_u - \partial_t g_v$ in $L_p(J; W_p^{1-1/p}(\bar\Gamma^+))$. We have

$$
\begin{aligned}
\partial_t g_u - \partial_t g_v &= \nabla'\Phi^T \left\{\partial_t \left[(\nabla X_u - \nabla X_v)^T \nabla X_u + \nabla X_v^T (\nabla X_u - \nabla X_v)\right]\right\} \nabla'\Phi\\
&= \nabla'\Phi^T \left[(\nabla u - \nabla v)^T \nabla X_u + (\nabla X_u - \nabla X_v)^T \nabla u\right] \nabla'\Phi\\
&\quad + \nabla'\Phi^T \left[\nabla X_v^T (\nabla u - \nabla v) + \nabla v^T (\nabla X_u - \nabla X_v)\right] \nabla'\Phi
\end{aligned}
$$

and thus we obtain

$$
\begin{aligned}
\|\partial_t g_u - \partial_t g_v\|_{L_p(J;W_p^{1-1/p}(\bar\Gamma^+))} &\lesssim \left(1 + \|\nabla X_u - \mathrm{Id}\|_{L_\infty(J;W_p^{1-1/p}(\bar\Gamma^+)} + \|\nabla X_v - \mathrm{Id}\|_{L_\infty(J;W_p^{1-1/p}(\bar\Gamma^+)}\right)\\
&\quad\cdot \|\nabla u - \nabla v\|_{L_p(J;W_p^{1-1/p}(\bar\Gamma^+))}\\
&\lesssim \left(1 + RT^{1-1/p}\|u\|_{\mathbb{E}_u(J)} + RT^{1-1/p}\|v\|_{\mathbb{E}_u(J)}\right)\\
&\quad\cdot \|\nabla u - \nabla v\|_{L_p(J;W_p^{1-1/p}(\bar\Gamma^+))}
\end{aligned}
$$

and now the third assertion follows from the trace theorem. □

LEMMA 8.3. *Let $n < p < \infty$ and $R \geq 1$. Assume the functions*

$$
F\colon \mathbb{C}^{(n-1)\times(n-1)} \to \mathbb{C} \qquad \text{and} \qquad G\colon \mathbb{C}^{(n-1)\times(n-1)} \times \mathbb{C}^{(n-1)\times(n-1)} \to \mathbb{C}
$$

are twice continuously differentiable in a neighbourhood of Id_{n-1} and $(\mathrm{Id}_{n-1}, 0)$, respectively. Assume further that G is linear in the second entry. Then there is $T' > 0$ such that for any $0 < T < T'$ and $J = (0, T)$ we have

$$
\begin{aligned}
\|F(g_0) - F(\mathrm{Id})\|_{W_p^{1-1/p}(\bar\Gamma^+)} &\lesssim \|h\|^2_{W_p^{2-1/p}(\mathbb{R}^{n-1})}\\
\|F(g_u) - F(g_v)\|_{L_\infty(J;W_p^{1-1/p}(\bar\Gamma^+))} &\leq C_R T^{1-1/p}\|u - v\|_{\mathbb{E}_u(J)}\\
\|G(g_u, \partial_t g_u) - G(g_v, \partial_t g_v)\|_{L_p(J;W_p^{1-1/p}(\bar\Gamma^+))} &\leq C_R \|u - v\|_{\mathbb{E}_u(J)}
\end{aligned}
$$

for every $u, v \in \mathbb{E}_u(J)$ of norm $\leq R$.

PROOF. Since the first assertion can be shown in the same way as the second we will only show the second and third assertion. For any twice continuously differentiable function

$F\colon \mathbb{C}^{(n-1)\times(n-1)} \to \mathbb{C}$ we can write

$$\begin{aligned} F(g_u) - F(g_v) &= \left(\int_0^1 F'(g_v + s(g_u - g_v))\, \mathrm{d}s \right) (g_u - g_v) \\ &= \left(F'(\mathrm{Id}) + \int_0^1 F'(g_v + s(g_u - g_v)) - F'(\mathrm{Id})\, \mathrm{d}s \right) (g_u - g_v) \\ &= F'(\mathrm{Id})(g_u - g_v) + \Lambda(g_v - \mathrm{Id}, g_u - g_v)(g_u - g_v) \end{aligned}$$

with a continuously differentiable function

$$\Lambda\colon \mathbb{C}^{(n-1)\times(n-1)} \times \mathbb{C}^{(n-1)\times(n-1)} \to \mathbb{C}, \qquad \Lambda(a,b) = \int_0^1 F'(a + sb + \mathrm{Id}) - F'(\mathrm{Id})\, \mathrm{d}s.$$

Then we can use Theorem 5.5.1.1 of [**RS96**] to obtain

$$\begin{aligned} &\|F(g_u) - F(g_v)\|_{L_\infty(J;W_p^{1-1/p}(\bar\Gamma^+))} \lesssim |F'(\mathrm{Id})| \, \|g_u - g_v\|_{L_\infty(J;W_p^{1-1/p}(\bar\Gamma^+))} \\ &\qquad + \|\Lambda(g_v - \mathrm{Id}, g_u - g_v)\|_{L_\infty(J;W_p^{1-1/p}(\bar\Gamma^+))} \|g_u - g_v\|_{L_\infty(J;W_p^{1-1/p}(\bar\Gamma^+))} \\ &\lesssim \Big(1 + \|g_v - \mathrm{Id}\|_{L_\infty(J;W_p^{1-1/p}(\bar\Gamma^+))} + \|g_u - g_v\|_{L_\infty(J;W_p^{1-1/p}(\bar\Gamma^+))} \Big) \\ &\qquad \cdot \|g_u - g_v\|_{L_\infty(J;W_p^{1-1/p}(\bar\Gamma^+))} \\ &\lesssim \Big(1 + \|g_v - g_0\|_{L_\infty(J;W_p^{1-1/p}(\bar\Gamma^+))} + \|g_0 - \mathrm{Id}\|_{W_p^{1-1/p}(\bar\Gamma^+)} + \|g_u - g_v\|_{L_\infty(J;W_p^{1-1/p}(\bar\Gamma^+))} \Big) \\ &\qquad \cdot \|g_u - g_v\|_{L_\infty(J;W_p^{1-1/p}(\bar\Gamma^+))}. \end{aligned}$$

Now, as a consequence of Lemma 8.2, we have the estimate

$$\|F(g_u) - F(g_v)\|_{L_\infty(J;W_p^{1-1/p}(\bar\Gamma^+))} \lesssim C_R T^{1-1/p} \|u - v\|_{\mathbb{E}_u(J)}.$$

In a similar fashion we can show the estimates involving the function G. To this end we write

$$\begin{aligned} &G(g_u, \partial_t g_u) - G(g_v, \partial_t g_v) = G'(\mathrm{Id}, 0)(g_u - g_v, \partial_t g_u - \partial_t g_v) \\ &\qquad + \left(\int_0^1 G'(g_v + s(g_u - g_v), \partial_t g_v + s(\partial_t g_u - \partial_t g_v)) - G'(\mathrm{Id}, 0)\, \mathrm{d}s \right) (g_u - g_v, \partial_t g_u - \partial_t g_v) \\ &\quad = G'(\mathrm{Id}, 0)(g_u - g_v, \partial_t g_u - \partial_t g_v) \\ &\qquad + \left(\int_0^1 G'(g_v + s(g_u - g_v), \partial_t g_u - \partial_t g_v) - G'(\mathrm{Id}, \partial_t g_u - \partial_t g_v)\, \mathrm{d}s \right) (g_u - g_v) \\ &\quad = G'(\mathrm{Id}, 0)(g_u - g_v, \partial_t g_u - \partial_t g_v) \\ &\qquad + \Xi(g_v - \mathrm{Id}, g_u - g_v, \partial_t g_u - \partial_t g_v)(g_u - g_v) \end{aligned}$$

with a continuously differentiable function

$$\Xi\colon \mathbb{C}^{(n-1)\times(n-1)} \times \mathbb{C}^{(n-1)\times(n-1)} \times \mathbb{C}^{(n-1)\times(n-1)} \to \mathbb{C}$$

given by

$$\Xi(a,b,c) = \int_0^1 G'(a + sb + \mathrm{Id}, c) - G'(\mathrm{Id}, c)\, \mathrm{d}s.$$

The mapping Ξ satisfies $\Xi(0,0,0) = 0$ and thus we can apply Theorem 5.5.1.1 of [**RS96**] to obtain the estimate

$$\begin{aligned}
&\|G(g_u, \partial_t g_u) - G(g_v, \partial_t g_v)\|_{L_p(J;W_p^{1-1/p}(\bar\Gamma^+))} \lesssim \|(g_u - g_v, \partial_t g_u - \partial_t g_v)\|_{L_p(J;W_p^{1-1/p}(\bar\Gamma^+))} \\
&\qquad + \|(g_v - \mathrm{Id}, g_u - g_v, \partial_t g_u - \partial_t g_v)\|_{L_p(J;W_p^{1-1/p}(\bar\Gamma^+))} \|(g_u - g_v)\|_{L_\infty(J;W_p^{1-1/p}(\bar\Gamma^+))} \\
&\quad \lesssim \|g_u - g_v\|_{H_p^1(J;W_p^{1-1/p}(\bar\Gamma^+))} \\
&\qquad + \|g_v - \mathrm{Id}\|_{L_p(J;W_p^{1-1/p}(\bar\Gamma^+))} \|g_u - g_v\|_{H_p^1(J;W_p^{1-1/p}(\bar\Gamma^+))} \|g_u - g_v\|_{L_\infty(J;W_p^{1-1/p}(\bar\Gamma^+))} \\
&\quad \lesssim C_R \|u - v\|_{\mathbb{E}_u(J)}
\end{aligned}$$

where the last inequality follows from Lemma 8.2. □

This puts us in a position to show the first assertion of Lemma 3.1. We can write

$$\begin{aligned}
[\Delta_{\Gamma_u^+} - \Delta_{\Gamma_v^+} w] \circ \Phi &= |\det g_u|^{-1/2} \operatorname{div}\left\{|\det g_u|^{1/2} g_u^{-1} \nabla \tilde w\right\} \\
&\quad - |\det g_v|^{-1/2} \operatorname{div}\left\{|\det g_v|^{1/2} g_v^{-1} \nabla \tilde w\right\} \\
&= \left[|\det g_u|^{-1/2} - |\det g_v|^{-1/2}\right] \operatorname{div}\left\{|\det g_u|^{1/2} g_u^{-1} \nabla \tilde w\right\} \\
&\quad + |\det g_v|^{-1/2} \operatorname{div}\left\{\left[|\det g_u|^{1/2} g_u^{-1} - |\det g_v|^{1/2} g_v^{-1}\right] \nabla \tilde w\right\}.
\end{aligned}$$

Now we can use Hölder's inequality and Theorem 4.6.1.2 of [**RS96**] to estimate

$$\begin{aligned}
&\left\|\left\{[\Delta_{\Gamma_u^+} - \Delta_{\Gamma_v^+}] w\right\} \circ \Phi\right\|_{L_p(J;W_p^{-1/p}(\bar\Gamma^+))} \\
&\quad \lesssim \left\||\det g_u|^{-1/2} - |\det g_v|^{-1/2}\right\|_{L_\infty(J;W_p^{1-1/p}(\bar\Gamma^+)} \\
&\qquad \cdot \left\||\det g_u|^{1/2} g_u^{-1} \nabla \tilde w\right\|_{L_p(J;W_p^{1-1/p}(\bar\Gamma^+))} \\
&\qquad + \left(\left\||\det g_v|^{-1/2} - |\det g_0|^{-1/2}\right\|_{L_\infty(J;W_p^{1-1/p}(\bar\Gamma^+)} + \left\||\det g_0|^{-1/2} - 1\right\|_{W_p^{1-1/p}(\bar\Gamma^+)} + 1\right) \\
&\qquad\quad \cdot \left\|\left[|\det g_u|^{1/2} g_u^{-1} - |\det g_v|^{1/2} g_v^{-1}\right] \nabla \tilde w\right\|_{L_p(J;W_p^{1-1/p}(\bar\Gamma^+))} \\
&\quad \lesssim \left\||\det g_u|^{-1/2} - |\det g_v|^{-1/2}\right\|_{L_\infty(J;W_p^{1-1/p}(\bar\Gamma^+)} \\
&\qquad \cdot \left(\left\||\det g_u|^{1/2} g_u^{-1} - |\det g_0|^{1/2} g_0^{-1}\right\|_{L_\infty(J;W_p^{1-1/p}(\bar\Gamma^+))}\right. \\
&\qquad\quad \left. + \left\||\det g_0|^{1/2} g_0^{-1} - \mathrm{Id}\right\|_{W_p^{1-1/p}(\bar\Gamma^+)} + 1\right) \|\tilde w\|_{L_p(J;W_p^{2-1/p}(\bar\Gamma^+))} \\
&\qquad + \left(\left\||\det g_v|^{-1/2} - |\det g_0|^{-1/2}\right\|_{L_\infty(J;W_p^{1-1/p}(\bar\Gamma^+)} + \left\||\det g_0|^{-1/2} - 1\right\|_{W_p^{1-1/p}(\bar\Gamma^+)} + 1\right) \\
&\qquad\quad \cdot \left\||\det g_u|^{1/2} g_u^{-1} - |\det g_v|^{1/2} g_v^{-1}\right\|_{L_\infty(J;W_p^{1-1/p}(\bar\Gamma^+))} \|\tilde w\|_{L_p(J;W_p^{2-1/p}(\bar\Gamma^+))}.
\end{aligned}$$

Choosing $T' > 0$ small enough we can infer that for any $0 < T < T'$ the matrices g_u and g_v are regular, and thus we obtain from Lemma 8.3 the estimates

$$\begin{aligned}
&\left\| \left\{ [\Delta_{\Gamma_u^+} - \Delta_{\Gamma_v^+}] w \right\} \circ \Phi \right\|_{L_p(J; W_p^{-1/p}(\bar{\Gamma}^+))} \\
&\quad \lesssim_R \left(1 + T^{1-1/p} \|u\|_{\mathbb{E}_u(J)} + T^{1-1/p} \|v\|_{\mathbb{E}_u(J)} + \|h\|^2_{W_p^{2-1/p}(\mathbb{R}^{n-1})} \right) \\
&\qquad \cdot T^{1-1/p} \|u - v\|_{\mathbb{E}_u(J)} \|\tilde{w}\|_{L_p(J; W_p^{2-1/p}(\bar{\Gamma}^+))} \\
&\quad \leq C_R T^{1-1/p} \|u - v\|_{\mathbb{E}_u(J)} \|\tilde{w}\|_{L_p(J; W_p^{2-1/p}(\bar{\Gamma}^+))}.
\end{aligned}$$

This shows the first assertion of Lemma 3.1. We turn to the second assertion, i. e. we will estimate $\left\{ [\dot{\Delta}_{\Gamma_u^+} - \dot{\Delta}_{\Gamma_v^+}] W \right\} \circ \Phi$ with $W = \int_0^t w$ and $\dot{\Delta}_{\Gamma_u^+}$ given by

$$\begin{aligned}
\left\{ \dot{\Delta}_{\Gamma_u^+} W \right\} \circ \Phi = [\partial_t |\det g_u|^{-1/2}] \operatorname{div} \left\{ |\det g_u|^{1/2} g_u^{-1} \nabla \tilde{W} \right\} \\
+ |\det g_u|^{-1/2} \operatorname{div} \left\{ [\partial_t |\det g_u|^{1/2} g_u^{-1}] \nabla \tilde{W} \right\}.
\end{aligned}$$

Then, writing $\tilde{W}$ for $W \circ \Phi$, we have

$$\begin{aligned}
\left\{ [\dot{\Delta}_{\Gamma_u^+} - \dot{\Delta}_{\Gamma_v^+}] W \right\} \circ \Phi = &\left[\partial_t |\det g_u|^{-1/2} - \partial_t |\det g_v|^{-1/2} \right] \operatorname{div} \left\{ |\det g_u|^{1/2} g_u^{-1} \nabla \tilde{W} \right\} \\
&+ [\partial_t |\det g_v|^{-1/2}] \operatorname{div} \left\{ \left[|\det g_u|^{1/2} g_u^{-1} - |\det g_v|^{1/2} g_v^{-1} \right] \nabla \tilde{W} \right\} \\
&+ \left[|\det g_u|^{-1/2} - |\det g_v|^{-1/2} \right] \operatorname{div} \left\{ [\partial_t |\det g_u|^{1/2} g_u^{-1}] \nabla \tilde{W} \right\} \\
&+ |\det g_v|^{-1/2} \operatorname{div} \left\{ \left[\partial_t |\det g_u|^{1/2} g_u^{-1} - \partial_t |\det g_v|^{1/2} g_v^{-1} \right] \nabla \tilde{W} \right\}
\end{aligned}$$

and we can use Hölder's inequality and Theorem 4.6.1.2 of [**RS96**] to obtain the estimate

$$\begin{aligned}
&\left\| \left\{ [\dot{\Delta}_{\Gamma_u^+} - \dot{\Delta}_{\Gamma_v^+}] W \right\} \circ \Phi \right\|_{L_p(J; W_p^{1-1/p}(\bar{\Gamma}^+))} \lesssim \left\| \partial_t |\det g_u|^{-1/2} - \partial_t |\det g_v|^{-1/2} \right\|_{L_p(J; W_p^{1-1/p}(\bar{\Gamma}^+))} \\
&\qquad \cdot \left\| \operatorname{div} \left\{ |\det g_u|^{1/2} g_u^{-1} \nabla \tilde{W} \right\} \right\|_{L_\infty(J; W_p^{-1/p}(\bar{\Gamma}^+))} \\
&\quad + \left\| \partial_t |\det g_v|^{-1/2} \right\|_{L_p(J; W_p^{1-1/p}(\bar{\Gamma}^+))} \\
&\qquad \cdot \left\| \operatorname{div} \left\{ \left[|\det g_u|^{1/2} g_u^{-1} - |\det g_v|^{1/2} g_v^{-1} \right] \nabla \tilde{W} \right\} \right\|_{L_\infty(J; W_p^{-1/p}(\bar{\Gamma}^+))} \\
&\quad + \left\| |\det g_u|^{-1/2} - |\det g_v|^{-1/2} \right\|_{L_\infty(J; W_p^{1-1/p}(\bar{\Gamma}^+))} \\
&\qquad \cdot \left\| \operatorname{div} \left\{ [\partial_t |\det g_u|^{1/2} g_u^{-1}] \nabla \tilde{W} \right\} \right\|_{L_p(J; W_p^{-1/p}(\bar{\Gamma}^+))} \\
&\quad + \left(\left\| |\det g_v|^{-1/2} - |\det g_0|^{-1/2} \right\|_{L_\infty(J; W_p^{1-1/p}(\bar{\Gamma}^+))} + \left\| |\det g_0|^{-1/2} - 1 \right\|_{W_p^{1-1/p}(\bar{\Gamma}^+)} + 1 \right) \\
&\qquad \cdot \left\| \operatorname{div} \left\{ \left[\partial_t |\det g_u|^{1/2} g_u^{-1} - \partial_t |\det g_v|^{1/2} g_v^{-1} \right] \nabla \tilde{W} \right\} \right\|_{L_p(J; W_p^{-1/p}(\bar{\Gamma}^+))}.
\end{aligned}$$

Some of these terms have already been treated in the course of the proof of the first assertion of Lemma 3.1. We will estimate the remaining quantities separately. First of all observe that $\det g_u$ is positive, and thus we can compute

$$\partial_t |\det g_u| = \partial_t \det g_u = (\det g_u) \operatorname{tr}\{g_u^{-1} \partial_t g_u\}$$

and

$$\partial_t g_u^{-1} = -g_u^{-1}(\partial_t g_u)g_u^{-1}.$$

In particular we obtain

$$\partial_t |\det g_u|^{-1/2} = -\frac{1}{2} |\det g_u|^{-1/2} \operatorname{tr}\{g_u^{-1}\partial_t g_u\}$$

and

$$\partial_t |\det g_u|^{1/2} g_u^{-1} = \frac{1}{2}(\det g_u) \operatorname{tr}\{g_u^{-1}\partial_t g_u\}g_u^{-1} - |\det g_u|^{1/2} g_u^{-1}(\partial_t g_u)g_u^{-1}.$$

This shows

$$\partial_t |\det g_u|^{-1/2} - \partial_t |\det g_v|^{-1/2} = -\frac{1}{2} |\det g_u|^{-1/2} \operatorname{tr}\{g_u^{-1}\partial_t g_u\} + \frac{1}{2} |\det g_v|^{-1/2} \operatorname{tr}\{g_v^{-1}\partial_t g_v\}.$$

We can write this as

$$\partial_t |\det g_u|^{-1/2} - \partial_t |\det g_v|^{-1/2} = G(g_u, \partial_t g_u) - G(g_v, \partial_t g_v)$$

with G satisfying the assumptions of Lemma 8.3, and thus we obtain the estimate

$$\left\| \partial_t |\det g_u|^{-1/2} - \partial_t |\det g_v|^{-1/2} \right\|_{L_p(J;W_p^{1-1/p}(\bar{\Gamma}^+))} \leq C_R \|u - v\|_{\mathbb{E}_u(J)} .$$

The same reasoning shows

$$\left\| \partial_t |\det g_u|^{1/2} g_u^{-1} - \partial_t |\det g_v|^{1/2} g_u^{-1} \right\|_{L_p(J;W_p^{1-1/p}(\bar{\Gamma}^+))} \leq C_R \|u - v\|_{\mathbb{E}_u(J)}$$

and since we can write

$$\partial_t |\det g_u|^{1/2} g_u^{-1} = \partial_t |\det g_u|^{1/2} g_u^{-1} - \partial_t |\det g_0|^{1/2} g_0^{-1}$$

we also obtain

$$\left\| \partial_t |\det g_u|^{1/2} g_u^{-1} \right\|_{L_p(J;W_p^{1-1/p}(\bar{\Gamma}^+))} \leq C_R \|u\|_{\mathbb{E}_u(J)} \leq C_R$$

and similarly

$$\left\| \partial_t |\det g_v|^{-1/2} \right\|_{L_p(J;W_p^{1-1/p}(\bar{\Gamma}^+))} \leq C_R \|v\|_{\mathbb{E}_u(J)} \leq C_R.$$

So far this shows the estimate

$$\begin{aligned}
&\left\| \left\{ [\dot{\Delta}_{\Gamma_u^+} - \dot{\Delta}_{\Gamma_v^+}]W \right\} \circ \Phi \right\|_{L_p(J;W_p^{1-1/p}(\bar{\Gamma}^+))} \\
&\quad \leq C_R \|u - v\|_{\mathbb{E}_u(J)} \left\| |\det g_u|^{1/2} g_u^{-1} \nabla \tilde{W} \right\|_{L_\infty(J;W_p^{1-1/p}(\bar{\Gamma}^+))} \\
&\qquad + C_R \|v\|_{\mathbb{E}_u(J)} \left\| \left[|\det g_u|^{1/2} g_u^{-1} - |\det g_v|^{1/2} g_v^{-1} \right] \nabla \tilde{W} \right\|_{L_\infty(J;W_p^{1-1/p}(\bar{\Gamma}^+))} \\
&\qquad + C_R T^{1-1/p} \|u - v\|_{\mathbb{E}_u(J)} \left\| [\partial_t |\det g_u|^{1/2} g_u^{-1}] \nabla \tilde{W} \right\|_{L_p(J;W_p^{1-1/p}(\bar{\Gamma}^+))} \\
&\qquad + C_R \left(T^{1-1/p} \|v\|_{\mathbb{E}_u(J)} + 1 \right) \\
&\qquad\qquad \cdot \left\| \left[\partial_t |\det g_u|^{1/2} g_u^{-1} - \partial_t |\det g_v|^{1/2} g_v^{-1} \right] \nabla \tilde{W} \right\|_{L_p(J;W_p^{1-1/p}(\bar{\Gamma}^+))} .
\end{aligned}$$

We can estimate the remaining quantities as follows:

$$\begin{aligned}
&\left\| |\det g_u|^{1/2}\, g_u^{-1} \nabla \tilde{W} \right\|_{L_\infty(J; W_p^{1-1/p}(\bar{\Gamma}^+))} \\
&\quad \lesssim \left(1 + \left\| |\det g_0|^{1/2}\, g_0^{-1} - \mathrm{Id} \right\|_{W_p^{1-1/p}(\bar{\Gamma}^+)} \right. \\
&\qquad \left. + \left\| |\det g_u|^{1/2}\, g_u^{-1} - |\det g_0|^{1/2}\, g_0^{-1} \right\|_{L_\infty(J; W_p^{1-1/p}(\bar{\Gamma}^+))} \right) \left\| \nabla \tilde{W} \right\|_{L_\infty(J; W_p^{1-1/p}(\bar{\Gamma}^+))} \\
&\quad \leq C_R \left\| \nabla \tilde{W} \right\|_{L_\infty(J; W_p^{1-1/p}(\bar{\Gamma}^+))} \leq C_R T^{1-1/p} \left\| w \right\|_{\mathbb{E}_u(J)}
\end{aligned}$$

where the last inequality is due to the trace theorem. Similarly we can obtain

$$\begin{aligned}
&\left\| \left[|\det g_u|^{1/2}\, g_u^{-1} - |\det g_v|^{1/2}\, g_v^{-1} \right] \nabla \tilde{W} \right\|_{L_\infty(J; W_p^{1-1/p}(\bar{\Gamma}^+))} \\
&\quad \lesssim \left\| |\det g_u|^{1/2}\, g_u^{-1} - |\det g_v|^{1/2}\, g_v^{-1} \right\|_{L_\infty(J; W_p^{1-1/p}(\bar{\Gamma}^+))} \left\| \nabla \tilde{W} \right\|_{L_\infty(J; W_p^{1-1/p}(\bar{\Gamma}^+))} \\
&\quad \leq C_R T^{2-2/p} \left\| u - v \right\|_{\mathbb{E}_u(J)} \left\| w \right\|_{\mathbb{E}_u(J)}
\end{aligned}$$

as well as

$$\begin{aligned}
&\left\| [\partial_t \, |\det g_u|^{1/2}\, g_u^{-1}] \nabla \tilde{W} \right\|_{L_p(J; W_p^{1-1/p}(\bar{\Gamma}^+))} \\
&\quad \lesssim \left\| \partial_t \, |\det g_u|^{1/2}\, g_u^{-1}] \right\|_{L_p(J; W_p^{1-1/p}(\bar{\Gamma}^+))} \left\| \nabla \tilde{W} \right\|_{L_\infty(J; W_p^{1-1/p}(\bar{\Gamma}^+))} \\
&\quad \leq C_R T^{1-1/p} \left\| w \right\|_{\mathbb{E}_u(J)}
\end{aligned}$$

and

$$\begin{aligned}
&\left\| \left[\partial_t \, |\det g_u|^{1/2}\, g_u^{-1} - \partial_t \, |\det g_v|^{1/2}\, g_v^{-1} \right] \nabla \tilde{W} \right\|_{L_p(J; W_p^{1-1/p}(\bar{\Gamma}^+))} \\
&\quad \lesssim \left\| \partial_t \, |\det g_u|^{1/2}\, g_u^{-1} - \partial_t \, |\det g_v|^{1/2}\, g_v^{-1} \right\|_{L_p(J; W_p^{1-1/p}(\bar{\Gamma}^+))} \left\| \nabla \tilde{W} \right\|_{L_\infty(J; W_p^{1-1/p}(\bar{\Gamma}^+))} \\
&\quad \leq C_R T^{1-1/p} \left\| u - v \right\|_{\mathbb{E}_u(J)} \left\| w \right\|_{\mathbb{E}_u(J)} .
\end{aligned}$$

All in all this shows

$$\left\| \left\{ [\dot{\Delta}_{\Gamma_u^+} - \dot{\Delta}_{\Gamma_v^+}] W \right\} \circ \Phi \right\|_{L_p(J; W_p^{1-1/p}(\bar{\Gamma}^+))} \lesssim C_R T^{1-1/p} \left\| u - v \right\|_{\mathbb{E}_u(J)} \left\| w \right\|_{\mathbb{E}_u(J)}$$

and this is the second assertion. We turn to the third assertion. To this end we will estimate

$$\begin{aligned}
\left\{ \nu_0 \cdot [\dot{\Delta}_{\Gamma_u^+} - \dot{\Delta}_{\Gamma_v^+}] \xi \right\} \circ \Phi &= - \frac{e_n}{\sqrt{1 + |\nabla' h|^2}} \cdot \left\{ [\dot{\Delta}_{\Gamma_u^+} - \dot{\Delta}_{\Gamma_v^+}] \xi \right\} \circ \Phi \\
&\quad + \frac{\begin{pmatrix} \nabla' h \\ 0 \end{pmatrix}}{\sqrt{1 + |\nabla' h|^2}} \cdot \left\{ [\dot{\Delta}_{\Gamma_u^+} - \dot{\Delta}_{\Gamma_v^+}] \xi \right\} \circ \Phi \\
&= - \frac{1}{\sqrt{1 + |\nabla' h|^2}} \cdot \left\{ [\dot{\Delta}_{\Gamma_u^+} - \dot{\Delta}_{\Gamma_v^+}] \xi_n \right\} \circ \Phi \\
&\quad + \frac{\begin{pmatrix} \nabla' h \\ 0 \end{pmatrix}}{\sqrt{1 + |\nabla' h|^2}} \cdot \left\{ [\dot{\Delta}_{\Gamma_u^+} - \dot{\Delta}_{\Gamma_v^+}] \xi \right\} \circ \Phi
\end{aligned}$$

in the space $L_p(J; W_p^{-1/p}(\bar{\Gamma}^+))$. We have

$$\begin{aligned}\Big\{[\dot{\Delta}_{\Gamma_u^+} - \dot{\Delta}_{\Gamma_v^+}]\xi\Big\} \circ \Phi = &\Big[\partial_t \left|\det g_u\right|^{-1/2} - \partial_t \left|\det g_v\right|^{-1/2}\Big] \operatorname{div}\Big\{\left|\det g_u\right|^{1/2} g_u^{-1} \nabla\Phi\Big\} \\ &+ [\partial_t \left|\det g_v\right|^{-1/2}] \operatorname{div}\Big\{\Big[\left|\det g_u\right|^{1/2} g_u^{-1} - \left|\det g_v\right|^{1/2} g_v^{-1}\Big] \nabla\Phi\Big\} \\ &+ \Big[\left|\det g_u\right|^{-1/2} - \left|\det g_v\right|^{-1/2}\Big] \operatorname{div}\Big\{[\partial_t \left|\det g_u\right|^{1/2} g_u^{-1}]\nabla\Phi\Big\} \\ &+ \left|\det g_v\right|^{-1/2} \operatorname{div}\Big\{\Big[\partial_t \left|\det g_u\right|^{1/2} g_u^{-1} - \partial_t \left|\det g_v\right|^{1/2} g_v^{-1}\Big] \nabla\Phi\Big\}\end{aligned}$$

and then also

$$\begin{aligned}\Big\{[\dot{\Delta}_{\Gamma_u^+} - \dot{\Delta}_{\Gamma_v^+}]\xi_n\Big\} \circ \Phi = &\Big[\partial_t \left|\det g_u\right|^{-1/2} - \partial_t \left|\det g_v\right|^{-1/2}\Big] \operatorname{div}\Big\{\left|\det g_u\right|^{1/2} g_u^{-1} \nabla' h\Big\} \\ &+ [\partial_t \left|\det g_v\right|^{-1/2}] \operatorname{div}\Big\{\Big[\left|\det g_u\right|^{1/2} g_u^{-1} - \left|\det g_v\right|^{1/2} g_v^{-1}\Big] \nabla' h\Big\} \\ &+ \Big[\left|\det g_u\right|^{-1/2} - \left|\det g_v\right|^{-1/2}\Big] \operatorname{div}\Big\{[\partial_t \left|\det g_u\right|^{1/2} g_u^{-1}]\nabla' h\Big\} \\ &+ \left|\det g_v\right|^{-1/2} \operatorname{div}\Big\{\Big[\partial_t \left|\det g_u\right|^{1/2} g_u^{-1} - \partial_t \left|\det g_v\right|^{1/2} g_v^{-1}\Big] \nabla' h\Big\}.\end{aligned}$$

Most of the involved quantities have already been encountered in the proof of the first and second assertion, so we immediately obtain the estimates

$$\begin{aligned}&\Big\|\Big\{[\dot{\Delta}_{\Gamma_u^+} - \dot{\Delta}_{\Gamma_v^+}]\xi\Big\} \circ \Phi\Big\|_{L_p(J; W_p^{-1/p}(\bar{\Gamma}^+))} \\ &\le C_R \left\|u - v\right\|_{\mathbb{E}_u(J)} \Big\|\operatorname{div}\Big\{\left|\det g_u\right|^{1/2} g_u^{-1} \nabla\Phi\Big\}\Big\|_{L_\infty(J; W_p^{-1/p}(\bar{\Gamma}^+))} \\ &\qquad + C_R T^{1-1/p} \left\|u - v\right\|_{\mathbb{E}_u(J)} (1 + \|h\|^2_{W_p^{2-1/p}(\mathbb{R}^{n-1})}) \\ &\qquad + C_R(1 + \|h\|^2_{W_p^{2-1/p}(\mathbb{R}^{n-1})}) \\ &\le C_R \left\|u - v\right\|_{\mathbb{E}_u(J)}\end{aligned}$$

and

$$\Big\|\Big\{[\dot{\Delta}_{\Gamma_u^+} - \dot{\Delta}_{\Gamma_v^+}]\xi_n\Big\} \circ \Phi\Big\|_{L_p(J; W_p^{-1/p}(\bar{\Gamma}^+))} \lesssim C_R \left\|u - v\right\|_{\mathbb{E}_u(J)} \|h\|_{W_p^{2-1/p}(\mathbb{R}^{n-1})}.$$

This shows

$$\Big\|\Big\{\nu_0 \cdot [\dot{\Delta}_{\Gamma_u^+} - \dot{\Delta}_{\Gamma_v^+}]\xi\Big\} \circ \Phi\Big\|_{L_p(J; W_p^{-1/p}(\bar{\Gamma}^+))} \lesssim C_R \left\|u - v\right\|_{\mathbb{E}_u(J)} \|h\|_{W_p^{2-1/p}(\mathbb{R}^{n-1})}$$

which is the third assertion of Lemma 3.1.

Bibliography

[Abe04] Takayuki Abe. On a resolvent estimate of the Stokes equation with Neumann-Dirichlet-type boundary condition on an infinite layer. *Math. Methods Appl. Sci.*, 27(9):1007–1048, 2004.

[Abe05a] Helmut Abels. The initial-value problem for the Navier-Stokes equations with a free surface in L^q-Sobolev spaces. *Adv. Differential Equations*, 10(1):45–64, 2005.

[Abe05b] Helmut Abels. Reduced and generalized Stokes resolvent equations in asymptotically flat layers. I. Unique solvability. *J. Math. Fluid Mech.*, 7(2):201–222, 2005.

[Abe05c] Helmut Abels. Reduced and generalized Stokes resolvent equations in asymptotically flat layers. II. H_∞-calculus. *J. Math. Fluid Mech.*, 7(2):223–260, 2005.

[Abe06] Helmut Abels. Generalized Stokes resolvent equations in an infinite layer with mixed boundary conditions. *Math. Nachr.*, 279(4):351–367, 2006.

[AG13] Ken Abe and Yoshikazu Giga. Analyticity of the Stokes semigroup in spaces of bounded functions. *Acta Math.*, 211(1):1–46, 2013.

[AG14] Ken Abe and Yoshikazu Giga. The L^∞-Stokes semigroup in exterior domains. *J. Evol. Equ.*, 14(1):1–28, 2014.

[AGSS14a] Ken Abe, Yoshikazu Giga, Katharina Schade, and Tomoyuki Suzuki. On the Stokes resolvent estimates for cylindrical domains with several exits to infinity. In preparation, 2014.

[AGSS14b] Ken Abe, Yoshikazu Giga, Katharina Schade, and Tomoyuki Suzuki. On the Stokes semigroup in some non-Helmholtz domain. Submitted, 2014.

[Ama95] Herbert Amann. *Linear and quasilinear parabolic problems. Vol. I*, volume 89 of *Monographs in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1995. Abstract linear theory.

[Ama97] Herbert Amann. Operator-valued Fourier multipliers, vector-valued Besov spaces, and applications. *Math. Nachr.*, 186:5–56, 1997.

[AS03a] Takayuki Abe and Yoshihiro Shibata. On a resolvent estimate of the Stokes equation on an infinite layer. *J. Math. Soc. Japan*, 55(2):469–497, 2003.

[AS03b] Takayuki Abe and Yoshihiro Shibata. On a resolvent estimate of the Stokes equation on an infinite layer. II. $\lambda = 0$ case. *J. Math. Fluid Mech.*, 5(3):245–274, 2003.

[AW05] Helmut Abels and Michael Wiegner. Resolvent estimates for the Stokes operator on an infinite layer. *Differential Integral Equations*, 18(10):1081–1110, 2005.

[AY10] Takayuki Abe and Masao Yamazaki. On a stationary problem of the Stokes equation in an infinite layer in Sobolev and Besov spaces. *J. Math. Fluid Mech.*, 12(1):61–100, 2010.

[BCD11] Hajer Bahouri, Jean-Yves Chemin, and Raphaël Danchin. *Fourier analysis and nonlinear partial differential equations*, volume 343 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011.

[Bea81] J. Thomas Beale. The initial value problem for the Navier-Stokes equations with a free surface. *Comm. Pure Appl. Math.*, 34(3):359–392, 1981.

[Bea84] J. Thomas Beale. Large-time regularity of viscous surface waves. *Arch. Rational Mech. Anal.*, 84(4):307–352, 1983/84.

[vB11] Lorenz von Below. H^∞-Kalkül für Familien von sektoriellen Operatoren. Diploma thesis, TU Darmstadt, 2011.

[Ber09] Dennis S. Bernstein. *Matrix mathematics*. Princeton University Press, Princeton, NJ, second edition, 2009. Theory, facts, and formulas.

[BN85] J. Thomas Beale and Takaaki Nishida. Large-time behavior of viscous surface waves. In *Recent topics in nonlinear PDE, II (Sendai, 1984)*, volume 128 of *North-Holland Math. Stud.*, pages 1–14. North-Holland, Amsterdam, 1985.

[DGH$^+$11] Robert Denk, Matthias Geissert, Matthias Hieber, Jürgen Saal, and Okihiro Sawada. The spin-coating process: Analysis of the free boundary value problem. *Communications in Partial Differential Equations*, 36(7):1145–1192, 2011.

[DHP01] Wolfgang Desch, Matthias Hieber, and Jan Prüss. L^p-theory of the Stokes equation in a half space. *J. Evol. Equ.*, 1(1):115–142, 2001.

[DHP07] Robert Denk, Matthias Hieber, and Jan Prüss. Optimal L^p-L^q-estimates for parabolic boundary value problems with inhomogeneous data. *Math. Z.*, 257(1):193–224, 2007.

[DSS08] R. Denk, J. Saal, and J. Seiler. Inhomogeneous symbols, the Newton polygon, and maximal L^p-regularity. *Russ. J. Math. Phys.*, 15(2):171–191, 2008.

[DU77] J. Diestel and J. J. Uhl, Jr. *Vector measures*. American Mathematical Society, Providence, R.I., 1977. With a foreword by B. J. Pettis, Mathematical Surveys, No. 15.

[Gal11] G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations*. Springer Monographs in Mathematics. Springer, New York, second edition, 2011. Steady-state problems.

[GMS99] Yoshikazu Giga, Shin'ya Matsui, and Yasuyuki Shimizu. On estimates in Hardy spaces for the Stokes flow in a half space. *Math. Z.*, 231(2):383–396, 1999.

[Gra08] Loukas Grafakos. *Classical Fourier analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2008.

[Gra09] Loukas Grafakos. *Modern Fourier analysis*, volume 250 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2009.

[GV92] S. Gindikin and L. R. Volevich. *The method of Newton's polyhedron in the theory of partial differential equations*, volume 86 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1992. Translated from the Russian manuscript by V. M. Volosov.

[Haa06] Markus Haase. *The functional calculus for sectorial operators*, volume 169 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 2006.

[Han77] R. Hanks. Interpolation by the real method between BMO, $L^\alpha(0 < \alpha < \infty)$ and $H^\alpha(0 < \alpha < \infty)$. *Indiana Univ. Math. J.*, 26(4):679–689, 1977.

[Hör60] Lars Hörmander. Estimates for translation invariant operators in L^p spaces. *Acta Math.*, 104:93–140, 1960.

[Hör83] Lars Hörmander. *The analysis of linear partial differential operators. I*, volume 256 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1983. Distribution theory and Fourier analysis.

[HP57] Einar Hille and Ralph S. Phillips. *Functional analysis and semi-groups*. American Mathematical Society Colloquium Publications, vol. 31. American Mathematical Society, Providence, R. I., 1957. rev. ed.

[HS69] Edwin Hewitt and Karl Stromberg. *Real and abstract analysis. A modern treatment of the theory of functions of a real variable*. Second printing corrected. Springer-Verlag, New York-Berlin, 1969.

[HW06] Tuomas Hytönen and Lutz Weis. Singular integrals on Besov spaces. *Math. Nachr.*, 279(5-6):581–598, 2006.

[Hyt04] Tuomas Hytönen. Fourier embeddings and Mihlin-type multiplier theorems. *Math. Nachr.*, 274/275:74–103, 2004.

[Kir34] M.D. Kirszbraun. Über die zusammenziehenden und Lipschitzschen Transformationen. *Fundam. Math.*, 22:77–108, 1934.

[KMP99] Natan Krugljak, Lech Maligranda, and Lars-Erik Persson. The failure of the Hardy inequality and interpolation of intersections. *Ark. Mat.*, 37(2):323–344, 1999.

[Koz98] Hideo Kozono. L^1-solutions of the Navier-Stokes equations in exterior domains. *Math. Ann.*, 312(2):319–340, 1998.

[KW01] N. J. Kalton and L. Weis. The H^∞-calculus and sums of closed operators. *Math. Ann.*, 321(2):319–345, 2001.

[LP64] J.-L. Lions and J. Peetre. Sur une classe d'espaces d'interpolation. *Inst. Hautes Études Sci. Publ. Math.*, (19):5–68, 1964.

[Mar08] Sorin Mardare. On Poincaré and de Rham's theorems. *Rev. Roumaine Math. Pures Appl.*, 53(5-6):523–541, 2008.

[Mar09] Paolo Maremonti. Pointwise asymptotic stability of steady fluid motions. *J. Math. Fluid Mech.*, 11(3):348–382, 2009.

[Maz85] Vladimir G. Maz'ja. *Sobolev spaces*. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1985. Translated from the Russian by T. O. Shaposhnikova.

[Miy96] Tetsuro Miyakawa. Hardy spaces of solenoidal vector fields, with applications to the Navier-Stokes equations. *Kyushu J. Math.*, 50(1):1–64, 1996.

[Miz87] Takahiro Mizuhara. On Fourier multipliers of homogeneous Besov spaces. *Math. Nachr.*, 133:155–161, 1987.

[MS91] I. Sh. Mogilevskiĭ and V. A. Solonnikov. On the solvability of a free boundary problem for the Navier-Stokes equations in the Hölder space of functions. In *Nonlinear analysis*, Sc. Norm. Super. di Pisa Quaderni, pages 257–271. Scuola Norm. Sup., Pisa, 1991.

[PS10] Jan Prüss and Gieri Simonett. On the two-phase Navier-Stokes equations with surface tension. *Interfaces Free Bound.*, 12(3):311–345, 2010.

[PSS12] J. Pruess, J. Saal, and G. Simonett. Singular limits for the two-phase Stefan problem. *ArXiv e-prints*, December 2012.

[dR84] Georges de Rham. *Differentiable manifolds*, volume 266 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984. Forms, currents, harmonic forms, Translated from the French by F. R. Smith, With an introduction by S. S. Chern.

[RS73] N. M. Rivière and Y. Sagher. Interpolation between L^∞ and H^1, the real method. *J. Functional Analysis*, 14:401–409, 1973.

[RS75] Michael Reed and Barry Simon. *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975.

[RS96] Thomas Runst and Winfried Sickel. *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, volume 3 of *de Gruyter Series in Nonlinear Analysis and Applications*. Walter de Gruyter & Co., Berlin, 1996.

[Saa07] Jürgen Saal. The Stokes operator with Robin boundary conditions in solenoidal subspaces of $L^1(\mathbb{R}^n_+)$ and $L^\infty(\mathbb{R}^n_+)$. *Comm. Partial Differential Equations*, 32(1-3):343–373, 2007.

[Sch66] Laurent Schwartz. *Théorie des distributions*. Publications de l'Institut de Mathématique de l'Université de Strasbourg, No. IX-X. Nouvelle édition, entiérement corrigée, refondue et augmentée. Hermann, Paris, 1966.

[Sch69] J. T. Schwartz. *Nonlinear functional analysis*. Gordon and Breach Science Publishers, New York-London-Paris, 1969. Notes by H. Fattorini, R. Nirenberg and H. Porta, with an additional chapter by Hermann Karcher, Notes on Mathematics and its Applications.

[Sol86] V. A. Solonnikov. Unsteady flow of a finite mass of a fluid bounded by a free surface. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 152(Kraev. Zadachi Mat. Fiz. i Smezhnye Vopr. Teor. Funktsii18):137–157, 183–184, 1986.

[Sol89] V. A. Solonnikov. Unsteady motions of a finite isolated mass of a self-gravitating fluid. *Algebra i Analiz*, 1(1):207–249, 1989.

[Sol91] V. A. Solonnikov. Solvability of a problem on the evolution of a viscous incompressible fluid, bounded by a free surface, on a finite time interval. *Algebra i Analiz*, 3(1):222–257, 1991.

[SS07] Yoshihiro Shibata and Senjo Shimizu. On a free boundary problem for the Navier-Stokes equations. *Differential Integral Equations*, 20(3):241–276, 2007.

[SS11] Yoshihiro Shibata and Senjo Shimizu. Report on a local in time solvability of free surface problems for the Navier-Stokes equations with surface tension. *Appl. Anal.*, 90(1):201–214, 2011.

[Ste93] Elias M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[Syl90] Donna Lynn Gates Sylvester. Large time existence of small viscous surface waves without surface tension. *Comm. Partial Differential Equations*, 15(6):823–903, 1990.

[Tan96] Atusi Tani. Small-time existence for the three-dimensional Navier-Stokes equations for an incompressible fluid with a free surface. *Arch. Rational Mech. Anal.*, 133(4):299–331, 1996.

[Tay06] Michael E. Taylor. *Measure theory and integration*, volume 76 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2006.

[Tre73] Walter Trebels. *Multipliers for (C,α)-bounded Fourier expansions in Banach spaces and approximation theory*. Lecture Notes in Mathematics, Vol. 329. Springer-Verlag, Berlin-New York, 1973.

[Tri78] Hans Triebel. *Interpolation theory, function spaces, differential operators*, volume 18 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam-New York, 1978.

[Tri83] Hans Triebel. *Theory of function spaces*, volume 78 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1983.

[Tri92] Hans Triebel. *Theory of function spaces. II*, volume 84 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1992.

[TT95] Atusi Tani and Naoto Tanaka. Large-time existence of surface waves in incompressible viscous fluids with or without surface tension. *Arch. Rational Mech. Anal.*, 130(4):303–314, 1995.

[Yos74] Kôsaku Yosida. *Functional analysis*. Springer-Verlag, New York, fourth edition, 1974. Die Grundlehren der mathematischen Wissenschaften, Band 123.

[Zad04] Ewa Zadrzyńska. Free boundary problems for nonstationary Navier-Stokes equations. *Dissertationes Math. (Rozprawy Mat.)*, 424:135, 2004.

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