Global Existence and Fast-Reaction Limit in Reaction-Diffusion Systems with Cross Effects

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Zusammenfassung

Special thanks to André Fischer to whom I owe the following lines.

Die vorliegende Dissertation beschäftigt sich mit Reaktions-Diffusions-Systemen, die in der Populationsdynamik, der Chemie und der Theorie der Elektromigation auftreten. Wir gehen der Frage der globalen Existenz starker und schwacher Lösungen und deren Eindeutigkeit und Regularität nach und untersuchen für chemische Systeme, die vom Massenwirkungsgesetz herrühren, das *fast reaction limit*, den Grenzübergang für schnelle Reaktionen.

In dieser Zusammenfassung stellen wir den Typ der uns interessierenden Evolutionssysteme vor. Anschließend wird ein Überblick über die Arbeit gegeben.

Reaktions-Diffusions-Systeme können wie folgt von Massenerhaltungsbilanzen hergeleitet werden. Ein Multikomponentensystem enthalte P extensive Größen $C_1, \dots, C_P$ (z.B. Populationen, chemische Reaktionsmittel, Ionen,...), deren Dichte durch einen Vektor

$$c(t,x) = (c_1(t,x), \dots, c_P(t,x)),\ t \geq 0,\ x \in \Omega,$$

ausgedrückt werden kann, wobei Ω ein beschränktes glattes Gebiet im $\mathbb{R}^N$ sei. Wir bezeichnen mit J_i und f_i jeweils die Flussdichte und die Produktionsrate der Spezies C_i. Für beliebige glatte beschränkte $A \subset \Omega$ besagt die Massenerhaltung für C_i innerhalb A, dass

$$\frac{d}{dt}\int_A c_i + \int_{\partial A} J_i \cdot \nu = \int_A f_i, \quad i \in \{1, \dots, P\},$$

wobei wir ν für den nach außen gerichteten Normalenvektor an ∂A schreiben. Mit Hilfe des Satzes von Gauß-Green bedeutet das

$$\frac{d}{dt}\int_A c_i + \int_A \operatorname{div} J_i = \int_A f_i\,, \quad i \in \{1, \dots, P\}.$$

Da A beliebig ist, erhalten wir die klassische Massenerhaltungsgleichung

$$\partial_t c_i + \operatorname{div} J_i = f_i\,, \quad i \in \{1, \dots, P\}.$$

Die Flussdichten J_i und Produktionsraten f_i müssen nun über konstituierende Gesetze modelliert werden. Für f_i betrachten wir Funktionen vom Typ $f_i(t,x,c)$, wobei die Abhängigkeit in c meist nichtlinear sein wird.

Im Folgenden stellen wir die verschiedenen Arten von Flussdichten vor, die in dieser Arbeit behandelt werden.

Verschiedene Arten von Flussdichten

Fick'sche Diffusion

Für den Fall, wenn Diffusion die einzige treibende Kraft ist, wurde von Fick 1855 [49] folgendes Standardmodell vorgestellt:

$$J_i = -d_i(t,x,c)\nabla c_i,$$

wobei $d_i > 0$ entsprechend dem zweiten Gesetz der Thermodynamik [35]. Wir arbeiten praktisch ausschließlich mit nicht-entarteten Diffusionskoeffizienten, d.h. diese Koeffizienten sind von unten beschränkt durch positive Konstanten. Unter diesen Annahmen führt die Massenerhaltung für jede Spezies zum Reaktions-Diffusions-System

$$\left.\begin{array}{rcl} \partial_t c_1 - \operatorname{div}(d_1(t,x,c)\nabla c_1) & = & f_1(t,x,c), \\ & \vdots & \\ \partial_t c_P - \operatorname{div}(d_P(t,x,c)\nabla c_1) & = & f_P(t,x,c) \end{array}\right\}, \quad t \in (0,+\infty),\ x \in \Omega, \tag{1}$$

welches mit entsprechenden Randbedingungen und nicht-negativen Anfangsdaten versehen wird. Zeitlich lokale Existenz von starken Lösungen für derartige Systeme ist wohlbekannt für ausreichend reguläre Anfangsdaten, aber die Existenz globaler Lösungen ist im Allgemeinen offen; ohne entsprechende strukturelle Forderungen an f_i kann sie auch gar nicht erwartet werden. Bevor wir zur Beschreibung weiterer Diffusionstypen kommen, wollen wir die Beschaffenheit der Nichtlinearitäten etwas weiter erläutern.

Zunächst wollen wir annehmen, dass die Nichtnegativität der Lösungen c_i bereits durch das Modell sichergestellt ist. Es ist wohlbekannt, dass ein notwendiges und hinreichendes Kriterium hierfür die Quasi-Positivität von $f = (f_1,\dots,f_P)$ darstellt, d.h.

$$(\mathbf{H}_1)\ \forall i \in \{1\dots,P\},\ f_i(t,x,c) \geq 0 \text{ für alle } (t,x,c) \in (0,+\infty)\times\Omega\times[0,+\infty)^P \text{ falls } c_i = 0.$$

Um die Existenz zeitlich globaler Lösungen erwarten zu können, müssen wir jedoch noch mehr Voraussetzungen an f stellen. Zusätzliche Annahmen kommen typischerweise aus dem zugrunde liegenden Modell. Beispielsweise entspricht die Erhaltung der Gesamtmasse der Prämisse, dass $\sum_{i=1}^P f_i = 0$. Im allgemeinen Fall tritt Massendissipation auf, falls

$$(\mathbf{H}_2)\ \sum_{i=1}^P f_i \leq 0.$$

Es ist einfach nachzurechnen, dass Forderungen $(\mathbf{H}_1) - (\mathbf{H}_2)$ mit homogenen Neumann-Randbedingungen sicherstellen, dass Lösungen von (1) in $L^1(\Omega)$ gleichmäßig beschränkt sind, da

$$\forall t > 0, \qquad \int_\Omega \sum_{i=1}^P c_i(t,x)dx \leq \int_\Omega \sum_{i=1}^P c_i(0,x)dx$$

und $\|c_i(t)\|_{L^1(\Omega)} = \int_\Omega c_i(t,x)dx$ wegen der Nichtnegativität der c_i gilt. Im homogenen Fall, wenn die Funktionen c_i nicht von x abhängen, stellen sie Lösungen des zugehörigen gewöhnlichen Differentialgleichungssystem

$$\left.\begin{array}{rcl} \frac{d}{dt}c_1 & = & f_1(t,c), \\ & \vdots & \\ \frac{d}{dt}c_P & = & f_P(t,c), \end{array}\right\}, \quad t \in (0,+\infty)$$

dar. Für nichtnegative Anfangsdaten bleibt die Lösung nichtnegativ und wegen

$$\sum_{i=1}^{P} c_i(t) \leq \sum_{i=1}^{P} c_i(0)$$

ist sie gleichmäßig beschränkt auf dem maximalen Existenzintervall. Folglich existieren globale Lösungen in diesem Spezialfall.

In natürliche Weise stellt sich die Frage, ob $(\mathbf{H}_1) - (\mathbf{H}_2)$ bereits die Existenz globaler Lösungen für das partielle Differentialgleichungssystem (1) garantieren. In [91] wird diese Frage negativ beantwortet: Es werden Lösungen zu einem System vom Typ (1) konstruiert, die in endlicher Zeit einen *Blow up* in $L^\infty(\Omega)$ entwickeln. In diesem Beispiel sind die Diffusivitäten konstant und die Nichtlinearitäten polynomial beschränkt. Dieser *Blow up* kann sogar für Raumdimension $N = 1$ auftreten, falls der Grad der Nichtlinearitäten groß genug ist. Somit benötigen wir für globale Lösungen weitere Forderungen an $(f_1, \dots, f_P)$. In der mathematischen Literatur gibt es eine Vielzahl an Arbeiten zur globalen Existenz für verschiedene zusätzliche Annahmen an $(f_1, \dots, f_P)$ [7,...,94]. Für einen aktuellen Übersichtsartikel verweisen wir auf [90].

Die Existenz globaler schwacher Lösungen stellt leichter überwindbare Hürden. Zum Beispiel wird für konstante Diffusionskoeffizienten und Nichtlinearitäten, die *a priori* für alle $T > 0$ in $L^1((0,T) \times \Omega)$ beschränkt sind, in [90] die Existenz schwacher Lösungen bewiesen. Dieses Resultat impliziert die globale Existenz schwacher Lösungen unter Bedingungen $(\mathbf{H}_1) - (\mathbf{H}_2)$, falls das Wachstum von f_i in *c höchstens quadratisch* ist. Dies beruht stark auf einer L^2-Abschätzung, die während der gesamten Arbeit ausgenutzt werden wird: Im Falle von konstanten Diffusivitäten d_i beispielsweise besagt sie, dass unter Annahmn $(\mathbf{H}_1) - (\mathbf{H}_2)$ die Lösungen von (1) folgender *a priori*-Abschätzung genügen:

$$\forall T > 0,\ \exists C = C(T, \|c(0)\|_{L^2(\Omega)^P}, d_i) > 0\ :\ \|c\|_{L^2((0,T)\times\Omega)^P} \leq C.$$

Kapitel 2 der vorliegenden Arbeit widmet sich der Erweiterung der oben erwähnten Ergebnisse auf allgemeinere Situationen, die noch nicht in der Literatur behandelt worden sind. Insbesondere zeigen wir die Existenz von

- globalen starken Lösungen für elementare chemische Reaktionsnetzwerke und allgemeine nichtlineare Diffusionskoeffizienten in kleinen (aber $N \geq 3$) Raumdimensionen,
- globalen schwachen Lösungen für Systeme, deren Nichtlinearitäten höchstens quadratisches Wachstum besitzen, mit nichtlinearen Diffusionskoeffizienten vom Typ $d_i(c_i)$ und Anfangsdaten, die "nur" in $L^1(\Omega)$ liegen.

Cross-Diffusion

Die Annahme, dass die treibenden Kräfte für eine Spezies unabhängig von den Gradienten der Konzentrationen anderer Spezies sind, stellt in gewissen Situationen eine zu starke Vereinfachung dar. Cross-Diffusion, das Phänomen, bei dem der Gradient einer Konzentration den Massenfluss einer anderen chemischen Spezies induziert, wurde bereits von Onsager und Fuoss [88] in den 1930er Jahren in einer Arbeit über Elektrolyte vorhergesagt. Experimentell wurden diese Cross-Effekte 1955 durch Gosting und Dunlop [44] und später im klassischen Experiment von Duncan und Toor [43] 1962 bestätigt. Während der letzten Jahrzehnte wurde das Phänomen der Cross-Diffusion gründlich untersucht, siehe [100] für einen Überblick über ihre Bedeutung für die physikalische Chemie.

Wir betrachten zunächst ein Modell aus der Populationsdynamik, bei der Cross-Diffusion ursprünglich eingesetzt wurde, um Reibungsphänomene zu modellieren, die zu räumlicher Trennung führen können. In dieser Situation haben die Flussdichten die Form

$$J_i = \nabla(a_i(c_1,\ldots,c_P)c_i).$$

Die Frage nach globaler Existenz von Lösungen für Reaktions-Diffusions-Systeme mit Flussdichten wie oben ist im Allgemeinen offen, sogar in Abwesenheit von Reaktionstermen. In der vorliegenden Arbeit betrachten wir Flussdichten

$$J_i = \nabla(a_i(\tilde{c}_1,\ldots,\tilde{c}_P)c_i),$$

wobei $\tilde{c}_i$ eine regularisierte Versionen von c_i darstellt. Für diesen Fall können wir globale Existenz von Lösungen für beliebige Raumdimensionen und für beliebige positive stetige a_i zeigen. Eindeutigkeit wird für den Fall von lokal Lipschitz-stetigen Funktionen a_i gezeigt. Diese Ergebnisse finden sich in Kapitel 1.

Im gleichen Kapitel wenden wir uns indirekt der globalen Existenz für ein anderes nichtlineares Cross-Diffusions-Problem zu, das in der Massenwirkungskinetik aus einem asymptotischen Grenzübergang für ein System vom Typ (1) resultiert. Es betrifft die typische reversible Reaktion $C_1 + C_2 \rightleftharpoons C_3$ mit der Reaktionsrate $k(c_1c_2 - c_3)$ für den Fall, wenn die Reaktionsgeschwindigkeit k gegen unendlich strebt. Im Limes ist die chemische Reaktion lokal im Equilibrium, d.h. es gilt $c_1c_2 = c_3$. Somit kann das Limessystem mit Hilfe der neuen Variablen $x_1 = c_1 + c_1c_2$, $x_2 = c_2 + c_1c_2$ umgeschrieben werden. Die resultierenden Flussdichten für x_1 und x_2 sind vom Typ

$$J_i = \nabla\Psi(x_1,x_2),$$

wobei Ψ nichtlinear ist, so dass wir ein nichtlineares Cross-Diffusions-System in den neuen Unbekannten x_1,x_2 erhalten. Für $k \to +\infty$ zeigen wir detailliert, dass die Lösungen zum System mit Reaktionsgeschwindigkeit k gegen eine Lösung des Limessystems konvergieren. Folglich erhalten wir auf diese Weise die Existenz einer globalen schwachen Lösung für das Cross-Diffusions-System, während die Arbeiten von H. Amann [2,4] die Existenz von starken Lösungen sicherstellen, allerdings nur lokal in der Zeit. Dies führt uns auf interessante Fragen zur Eindeutigkeit von schwachen Lösungen, welche teilweise beantwortet werden.

Fick'sche Diffusion mit Konvektion

Schließlich beleuchten wir Situationen, in denen Diffusion nicht die einzige für Massentransport verantwortliche Kraft ist.

Betrachten wir ein Stoffgemisch mit nichtverschwindendem Geschwindigkeitsfeld u und berücksichtigen wir auch Fick'sche Diffusion, so sind die Flussdichten vom Typ

$$J_i = -d_i\nabla c_i + c_i u\ ;\ i \in \{1,\ldots,P\}.$$

Als ersten Schritt in Richtung komplexerer Modelle betrachten wir Flussdichten mit gegebenem Datum u. Wir untersuchen dabei globale Existenz von Lösungen zu Reaktions-Diffusions-Systemen, deren Reaktionsterme eine trianguläre Struktur besitzen, d.h. für ein System vom Typ (1) mit rechter Seite $f = (f_1,\ldots,f_P)$ setzen wir die Existenz einer invertierbaren unteren Dreicksmatrix $Q = (q_{ij})_{1\le i,j\le P}$ mit nichtnegativen Einträgen voraus, so dass

$$\exists b \in (0,+\infty)^P : \forall (t,x,c) \in (0,\infty)\times\Omega\times[0,+\infty)^P, \quad Qf(t,x,c) \le \left(1+\sum_{i=1}^{P} c_i\right) b. \tag{2}$$

Falls das Gemisch einen Elektrolyten und die Konzentrationen $c_1, \dots, c_P$ die Konzentrationen von geladenen Spezies mit Ladungszahl $z_i \in \mathbb{Z}$ darstellen, so ist die Ladungsdichte gegeben durch $\sum_{i=1}^P z_i c_i$ und das elektrische Potential ist die Lösung der Poisson-Gleichung

$$-\Delta\Phi = \sum_{i=1}^{P} z_i c_i$$

mit entsprechenden Randbedingungen. Hier sind die physikalischen Parameter ε, F auf 1 gesetzt, wobei F die Faraday-Konstante und ε die Permittivität des Mediums darstellen. Wegen des nichtverschwindenden elektrischen Felds $-\nabla\Phi$ ist die Massenflussdichte von der Form

$$J_i = -d_i \nabla c_i - d_i z_i c_i \nabla\Phi.$$

Das Problem der globalen Existenz für das resultierende sog. "Diffusions-Elektromigrations-System" wird im letzten Abschnitt behandelt.

Unser Beitrag ist wie folgt organisiert.

Überblick

Die Arbeit gliedert sich in drei Kapitel.

Das erste Kapitel enthält zwei bereits veröffentliche Gemeinschaftsarbeiten, wobei die beiden Unterabschnitte 1.6. und 2.4.4. zusätzlich hinzugefügt worden sind. Die zwei anderen Kapitel enthalten jeweils zwei Papers, die bald eingereicht werden. Drei von diesen stellen ebenfalls Gemeinschaftsarbeiten dar.

◇ Kapitel 1 widmet sich zwei Cross-Diffusions-Systemen aus der Populationsdynamik und der Massenwirkungskinetik.

Das erste zu untersuchende Model ist ein relaxiertes Cross-Diffusions-System, das ursprünglich in [11] vorgestellt wurde, um zu zeigen, dass Cross-Diffusions-Systeme ohne Reaktionen zu räumlicher Segregation führen können. In jener Arbeit untersuchen die Autoren Operatoren der Form

$$u = (u_1, \dots, u_P) \mapsto (-\Delta(a_1(\tilde{u})u_1), \dots, -\Delta(a_P(\tilde{u})u_P)),$$

wobei $\tilde{u}$ eine regularisierte Version von u bezeichnet. Globale Existenz starker Lösungen wurde in Raumdimension 2 und für Funktionen a_i mit polynomialem Wachstum gezeigt. In Kapitel 1 beweisen wir die Existenz globaler klassischer Lösungen für dieses Model in beliebiger Raumdimension für Funktionen a_i, die lediglich als stetig und positiv vorausgesetzt werden. Darüberhinaus beweisen wir die Eindeutigkeit für den Fall, wenn a_i lokal Lipschitz-stetig sind.

Das zweite Model kommt aus der Massenwirkungskinetik: Beim Studium des *fast reaction limits* in der reversiblen Reaktion $C_1 + C_2 \rightleftharpoons C_3$ bei Fick'scher Diffusion, siehe oben, erhalten wir als Limessystem ein nichtlineares Cross-Diffusions-System. In Abschnitt 2 beweisen wir, dass die Lösungen des Systems mit endlicher Reaktionsgeschwindigkeit k für $k \to \infty$ gegen eine globale schwache Lösung dieses Cross-Diffusions-Systems konvergieren. Unter gewissen Einschränkungen an die Diffusionskoeffizienten zeigen wir, dass schwache Lösungen eindeutig sind. Dieses Resultat verallgemeinert frühere Arbeiten von D. Bothe [18] auf den Fall von verschiedenen (aber konstanten) Diffusionskoeffizienten.

◇ Da die entwickelten Methoden zur Untersuchung des *fast raction limits* in der Reaktion $C_1 + C_2 \rightleftharpoons C_3$ sehr robust sind, sind wir nun daran interessiert, die erzielten Ergebnisse aus

Kapitel 1 auf realistischere Modelle zu verallgemeinern. Insbesondere haben die folgenden Situationen aus der Sicht der Chemie größere Bedeutung:

- Die Diffusionskoeffizienten hängen von Zeit, Ort und den Konzentrationen ab ($d_i = d_i(t,x,c)$).
- Die Anfangsdaten liegen lediglich in $L^1(\Omega)$.
- Neben der schnellen Reaktion $C_1 + C_2 \rightleftharpoons C_3$ erfolgen gleichzeitig noch weitere langsame chemische Reaktionen.

Kapitel 2 enthält in diesem Zusammenhang mehrere Resultate zur globalen Existenz.

- Wir beweisen die Existenz globaler starker Lösungen für Systeme vom Typ (1) für mehrere verschiedene strukturelle Voraussetzungen an f für kleine Raumdimensionen ($N \leq 5$ für $d_i = d_i(t,x,c)$, $N \leq 9$ für $d_i = d_i(c_i)$). Wir heben dabei den speziellen Fall $C_1 + C_2 \rightleftharpoons C_3$ hervor, doch wir betrachten auch den Fall von mehreren Reaktionen der Form
$$C_i + C_j \rightleftharpoons C_k.$$
- Wir beweisen globale Existenz schwacher Lösungen für allgemeinere Quellterme, für die lediglich quadratisches Wachstum vorausgesetzt wird. Dies beinhaltet den Fall von $L^1(\Omega)$-Anfangsdaten. Aus der Sicht der Modellierung ist dies der natürlichste Rahmen, doch stellt er größere mathematische Hindernisse, da die Kontrolle der Lösung in der Nähe von $t = 0$ delikate Abschätzungen erfordert.

◇ In den vorangehenden Kapiteln haben wir den Fall studiert, wenn Diffusion das einzige für Massentransport verantwortliche Phänomen ist. Wir beachten nun weitere Triebkräfte; im Einzelnen sind dies: Advektion für den Fall nichtverschwindender Fluidbewegung und Elektromigration, wenn das Fluid ein Elektrolyt und die Spezies $c_1, \ldots, c_P$ Ionen sein können.

Falls das Vektorfeld u, das die Fluidbewegung beschreibt, ein gegebenes Datum des Problems ist, verallgemeinern wir das globale Wohlgestelltheitsresultat von M. Pierre [90] für Reaktions-Diffusions-Systeme, deren Reaktionen eine sog. “trianguläre” Struktur (2) haben. Im Wesentlichen bedeutet das, dass wir Reaktionsterme vorliegen haben, für die $f_1, f_1 + f_2, \ldots, f_1 + \ldots + f_P$ durch lineare Funktionen von $c_1, \ldots, c_P$ kontrolliert werden können. Der Beweis von [90] wird auf den Fall von Flussdichten mit Advektionsanteilen und zeit- und ortsabhängigen Fick’schen Diffusionskoeffizienten verallgemeinert.

Im Spezialfall erhalten wir damit globale Existenz für die chemische Reaktion $C_1 + C_2 \rightleftharpoons C_3$ unabhängig von der Reaktionsgeschwindigkeit. Somit sind wir in der Lage, auch hier das *fast reaction limit* dieses Systems zu analysieren. Wir zeigen, dass die Techniken, die wir in Kapitel 1 entwickelt haben, ausreichend stabil sind, um sie auf den Fall von variablen Diffusionskoeffizienten und Advektion anzuwenden.

Im letzten Teil von Kapitel 3 befassen wir uns mit der Existenz von Lösungen für ein Diffusions-Elektromigrations-System *in beliebiger Raumdimension*. Über ein Approximationsverfahren, das die “Entropiestruktur” des Ausgangsproblems aufrecht erhält, zeigen wir die Existenz globaler schwacher Lösungen. Die Ergebnisse des ersten Abschnitts dieses Kapitels, bei denen die Advektion vorgegeben ist, dienen für ein Schauder-Fixpunkt-Argument, um die Existenz von Lösungen für das approximative System zu zeigen.

Résumé en français

Cette thèse est consacrée à l'étude de systèmes de réaction-diffusion qui sont issus de modèles de dynamique des populations, de cinétique chimique et de la théorie de l'électromigration. On étudie des questions d'existence globale, d'unicité des solutions, leur régularité, ainsi que la limite de réaction rapide pour des systèmes issus de la cinétique chimique.

On commence dans ce résumé par introduire brièvement les équations auxquelles on s'intéresse. On présente ensuite la strucure de la thèse, qui s'articule autour de trois chapitres. Enfin, on décrit plus précisément le contenu de chaque chapitre.

Les systèmes de réaction-diffusion peuvent être obtenus à partir d'équations de conservation de la masse comme suit. Supposons qu'on étudie un système contenant P quantités extensives $C_1,\ldots,C_P$ (qui peuvent représenter des densités de population, des concentrations de réactifs chimiques, des ions, etc.), dont les densités sont représentées par un vecteur

$$c(t,x) = (c_1(t,x),\ldots,c_P(t,x)),\ t \geq 0,\ x \in \Omega,$$

où Ω est un domaine borné et régulier de $\mathbb{R}^N$. On note J_i le flux de l'espèce C_i et f_i son taux de création volumique horaire. Pour tout $A \subset \Omega$ borné, régulier, la conservation de la masse pour C_i à l'intérieur de A s'écrit

$$\frac{d}{dt}\int_A c_i + \int_{\partial A} J_i \cdot \nu = \int_A f_i\,, \qquad i \in \{1,\ldots,P\},$$

où ν est la dérivée normale extérieure sur la frontière ∂A de A. D'après le théorème de Gauss-Green,

$$\frac{d}{dt}\int_A c_i + \int_A \operatorname{div} J_i = \int_A f_i\,, \qquad i \in \{1,\ldots,P\}.$$

Comme A est quelconque, on obtient l'équation de conservation de la masse

$$\partial_t c_i + \operatorname{div} J_i = f_i\,, \quad i \in \{1,\ldots,P\}.$$

Les flux J_i et les fonctions f_i doivent maintenant être modélisés par des lois de comportement adéquates.

Nous considérerons des fonctions f_i de la forme $f_i(t,x,c)$, la dépendance en c étant souvent non-linéaire.

Les différents types de flux qui sont étudiés dans cette thèse sont présentés ci-dessous.

Différents types de flux

Diffusion de Fick

Lorsque le transport de masse est seulement lié à la diffusion, un modèle simple a été introduit par Fick en 1855 [49]. Il consiste à poser

$$J_i = -d_i(t,x,c)\nabla c_i,$$

où $d_i > 0$ de façon à respecter le second principe de la thermodynamique [35]. En pratique, on ne considèrera ici que des coefficients de diffusion non dégénérés, *i.e.* des coefficients bornés inférieurement par une constante strictement positive. Sous ces hypothèses, si l'on écrit l'équation de conservation de la masse pour chaque espèce, on obtient un *système de réaction-diffusion*

$$\left.\begin{array}{rcl} \partial_t c_1 - \operatorname{div}(d_1(t,x,c)\nabla c_1) & = & f_1(t,x,c), \\ & \vdots & \\ \partial_t c_P - \operatorname{div}(d_P(t,x,c)\nabla c_1) & = & f_P(t,x,c) \end{array}\right\}, \quad t \in (0,+\infty),\ x \in \Omega, \tag{1}$$

qu'on complète par des conditions au bord et des conditions initiales. Lorsque les données initiales sont suffisamment régulières, l'existence locale de solutions pour les systèmes de la forme (1) est bien connue. L'existence globale est un problème ouvert en général, et ont sait qu'elle ne peut avoir lieu sans hypothèses supplémentaires sur les f_i. Avant de poursuivre la description des différents flux, faisons quelques commentaires sur la structure des nonlinéarités f_i.

Tout d'abord, on supposera toujours que le modèle préserve la positivité des solutions. Il est bien connu que cela revient à supposer que $f = (f_1, \dots, f_P)$ est quasi-positive, ce qui signifie

$(\mathbf{H}_1)$ $\forall i \in \{1\dots,P\}$, $f_i(t,x,c) \geq 0$ pour tout $(t,x,c) \in (0,+\infty) \times \Omega \times [0,+\infty)^P$ tel que $c_i = 0$.

Ensuite, pour espérer l'existence de solutions globales en temps, f doit satisfaire des hypothèses supplémentaires. Ces hypothèses viennent souvent du modèle qu'on étudie. Par exemple, la conservation de la masse correspond à supposer $\sum_{i=1}^P f_i = 0$. Plus généralement, la masse totale décroît si

$(\mathbf{H}_2)$ $\displaystyle\sum_{i=1}^P f_i \leq 0.$

On vérifie facilement que les hypothèses $(\mathbf{H}_1) - (\mathbf{H}_2)$, avec des conditions de Neumann homogènes au bord, garantissent que les solutions de (1) sont uniformément bornées dans $L^1(\Omega)$, étant donné que pour tout $t > 0$,

$$\int_\Omega \sum_{i=1}^P c_i(t,x)dx \leq \int_\Omega \sum_{i=1}^P c_i(0,x)dx,$$

et $\|c_i(t)\|_{L^1(\Omega)} = \int_\Omega c_i(t,x)dx$ puisque c_i est positive. Dans le cas homogène, où les fonctions c_i ne dépendent pas de x, on peut remarquer qu'elles sont aussi solutions du système d'équations aux dérivées ordinaires associé

$$\left.\begin{array}{rcl} \frac{d}{dt}c_1 & = & f_1(t,c), \\ & \vdots & \\ \frac{d}{dt}c_P & = & f_P(t,c) \end{array}\right\}, \quad t \in (0,+\infty).$$

Pour des données initiales positives, les solutions restent positives. Étant donné que

$$\sum_{i=1}^{P} c_i(t) \leq \sum_{i=1}^{P} c_i(0),$$

elles sont uniformément bornées sur leur intervalle maximal de définition. Par conséquent, dans ce cas particulier, les solutions maximales sont globales.

Il est alors naturel de se demander si les hypothèses $(\mathbf{H}_1)-(\mathbf{H}_2)$ sont suffisantes pour assurer l'existence de solutions globales fortes pour le système d'équations aux dérivées partielles (1). La réponse est non : des solutions explicites d'un système du type (1) avec les propriétés $(\mathbf{H}_1)-(\mathbf{H}_2)$ ont été construites dans [91], et ces solutions explosent en norme $L^\infty(\Omega)$ en temps fini. Dans ce dernier exemple, les coefficients de diffusion sont pourtant constants, et les nonlinéarités sont bornées par des expressions polynomiales. L'explosion peut même avoir lieu en dimension $N=1$, à condition que la croissance des nonlinéarités soit assez rapide. Ceci prouve que lorsqu'on s'intéresse à l'existence globale de solutions fortes, on doit faire des hypothèses supplémentaires sur $(f_1,\ldots,f_P)$. Il existe de nombreuses références sur les problèmes d' existence globale pour ces systèmes, où diverses hypothèses sur les fonctions $(f_1,\ldots,f_P)$ sont examinées, cf. [7, 31, 41, 62, 84, 92, 94]. Pour une vue d'ensemble sur ce sujet, voir [90].

L'existence de solutions globales faibles est plus facile à obtenir. Par exemple, dans le cas de coefficients de diffusion constants et pour des nonlinéarités *a priori* bornées pour tout $T>0$ dans $L^1((0,T)\times\Omega)$, l'existence de solutions globales faibles est prouvée dans [90]. Ce résultat implique que si la croissance de f_i par-rapport à c est *au plus quadratique*, on a l'existence de solutions globales faibles sous les hypothèses $(\mathbf{H}_1)-(\mathbf{H}_2)$. Ce résultat repose de façon essentielle sur une estimation L^2 qui sera exploitée tout au long de ce travail : par exemple, dans la cas de coefficients de diffusion d_i constants, cette estimation garantit que sous les hypothèses $(\mathbf{H}_1)-(\mathbf{H}_2)$, les solutions de (1) satisfont les estimations *a priori*

$$\forall T>0,\ \exists\, C=C(T,\|c(0)\|_{L^2(\Omega)^P},d_i)>0\ :\ \|c\|_{L^2((0,T)\times\Omega)^P}\leq C.$$

Le chapitre 2 est consacré à l'extension des résultats rappelés ci-dessus à des situations plus générales, pour lesquelles l'existence globale n'a pas encore été démontrée. En particulier, on prouve l'existence de

– solutions globales *fortes* pour des réseaux de réactions chimiques élémentaires, pour des coefficients de diffusion généraux et pour des dimensions en espace petites (mais $N\geq 3$).
– solutions globales *faibles* pour des systèmes dont les nonlinéaritiés ont une croissance au plus quadratique, pour des coefficients de diffusion non linéaires du type $d_i(c_i)$, et pour des données initiales dans $L^1(\Omega)$ “seulement”.

Diffusion croisée

Considérer que le flux pour une espèce donnée est indépendant des gradients des concentrations des autres espèces est parfois une hypothèse trop simple. L'existence de phénomènes de diffusion croisée, *i.e.* le fait qu'un gradient de concentration non nul pour une espèce induit un flux de masse pour une autre espèce, a été suggérée dans une étude de Onsager et Fuoss sur des électrolytes dans les années 1930 [88]. L'existence de ces effets croisés a ensuite été vérifiée expérimentalement en 1955 par Gosting et Dunlop [44], et plus tard dans une expérience désormais classique de Duncan et Toor en 1962 [43]. Ces dernières années, les diffusions croisées ont donné lieu à de nombreux travaux de recherche. Pour un exposé général sur leur importance en physique, voir [100].

On commence par considérer un modèle de dynamique des populations, où les diffusions croisées ont d'abord été introduites pour modéliser des phénomènes de friction qui peuvent amener à des ségrégations spatiales. Dans ce cas, les flux sont de la forme

$$J_i = \nabla(a_i(c_1,\ldots,c_P)c_i).$$

L'existence globale pour les systèmes de réaction-diffusion avec les flux ci-dessus est un problème ouvert en général, même en l'absence de termes de réaction. Dans cette thèse, on considère des flux du type

$$J_i = \nabla(a_i(\tilde{c}_1,\ldots,\tilde{c}_P)c_i),$$

où les $\tilde{c}_i$ sont des versions régularisées des c_i. On prouve alors l'existence de solutions globales, indépendamment de la dimension de l'espace et pour des fonctions a_i seulement supposées positives et continues. On prouve aussi l'unicité lorsque les a_i sont localement lipschitziennes. Ces résultats sont démontrés dans le chapitre 1.

Dans ce même chapitre, on étudie indirectement la question de l'existence de solutions globales pour un autre système non-linéaire avec des diffusions croisées. Ce système est la limite asymptotique d'une famille de systèmes du type (1) issus des lois la cinétique chimique, où on considère la réaction réversible $C_1 + C_2 \rightleftharpoons C_3$ lorsque la vitesse de réaction k tend vers $+\infty$ dans le terme de réaction $k(c_1c_2 - c_3)$. À la limite, la réaction chimique est localement à l'équilibre, ce qui signifie que la relation $c_1c_2 = c_3$ est vérifiée. Le système limite peut être réécrit avec comme variables principales $x_1 = c_1 + c_1c_2$, $x_2 = c_2 + c_1c_2$. Dans ce cas, les flux résultant pour x_1 et x_2 sont de la forme

$$J_i = \nabla\Psi(x_1,x_2),$$

où Ψ est non-linéaire. On est ainsi ramenés à un système non-linéaire avec des diffusions croisées par rapport aux nouvelles variables x_1,x_2. On prouve alors rigoureusement la convergence lorsque $k \to +\infty$ des solutions du système avec vitesse de réaction k vers une solution globale de ce système limite. Rappellons que la théorie générale de H. Amann [2, 4] garantit l'existence de solutions fortes au problème limite, mais seulement localement en temps. Ceci mène naturellement à des questions d'unicité des solutions faibles. On répond partiellement à ces questions.

Diffusion de Fick avec convection

Dans le dernier chapitre, on se place dans des situations où la diffusion n'est pas le seul phénomène responsable du transport de masse.

Lorsqu'on considère un fluide dont la vitesse u est non nulle, on est amené à étudier des flux de masse du type suivant :

$$J_i = -d_i\nabla c_i + c_iu\ ;\ i \in \{1,\ldots,P\}.$$

On considère ici que u est une donnée du problème. On s'intéresse alors à l'existence globale pour des systèmes de réaction-diffusion dont les termes de réaction ont une structure "triangulaire" : plus précisément, pour un système du type (1) avec $f = (f_1,\ldots,f_P)$, on suppose qu'il existe une matrice triangulaire inférieure inversible $Q = (q_{ij})_{1\le i,j\le P}$ à coefficients positifs, telle que

$$\exists b \in (0,+\infty)^P : \forall (t,x,c) \in (0,+\infty)\times\Omega\times[0,+\infty)^P, \quad Q\,f(t,x,c) \le \Big(1+\sum_{i=1}^{P} c_i\Big)b. \qquad (2)$$

Lorsque le fluide qu'on étudie est un électrolyte et que $c_1,\dots,c_P$ sont les concentrations d'espèces ioniques portant $z_i \in \mathbb{Z}$ charges élémentaires, la densité de charge est $\sum_{i=1}^P z_i c_i$ et le potentiel électrique est la solution Φ de l'équation de Poisson

$$-\Delta\Phi = \sum_{i=1}^{P} z_i c_i$$

avec des conditions de bord adaptées. À cause de la présence d'un champ électrique non nul $-\nabla\Phi$, les flux de masse sont maintenant de la forme

$$J_i = -d_i \nabla c_i - d_i z_i c_i \nabla\Phi.$$

On étudie dans la dernière partie du troisième chapitre la question de l'existence globale pour le système de "diffusion-électromigration" correspondant.

Les contributions de cette thèse sont organisées comme suit.

Plan de la thèse

La thèse est divisée en trois chapitres.

Dans le premier chapitre sont reproduits deux articles déjà publiés, réalisés lors de collaborations. On y a ajouté les sous-parties 1.6 et 2.4.4. Chacun des deux chapitres suivants contient deux articles sur le point d'être soumis pour révision, dont trois sont le résultat de collaborations.

◇ Le premier chapitre est consacré à l'étude de deux systèmes aux diffusions croisées, issus de modèles de dynamique des populations et de cinétique chimique.

  Le premier modèle auquel on s'intéresse est un système aux diffusions croisées relaxé. Il a été introduit dans [11] afin de montrer que les diffusions croisées peuvent, même en l'absence de réaction, induire de la ségrégation spatiale. Dans [11], les auteurs s'intéressent à des opérateurs du type
  $$u = (u_1,\dots,u_P) \mapsto (-\Delta(a_1(\tilde{u})u_1),\dots,-\Delta(a_P(\tilde{u})u_P)),$$
  où $\tilde{u}$ est une version régularisée de u. L'existence globale de solutions fortes est prouvée en dimension 2 pour des fonctions a_i à croissance au plus polynomiale. Dans le chapitre 1, on prouve l'existence de solutions fortes pour ce modèle en toute dimension et pour des fonctions a_i seulement supposées continues et positives. Si on suppose en plus que les a_i sont localement lipschitziennes, on prouve leur unicité.

  Le second modèle auquel on s'intéresse est issu de la cinétique chimique : lorsqu'on étudie la limite de réaction rapide dans la réaction réversible $C_1 + C_2 \rightleftharpoons C_3$, en utilisant la loi d'action de masse et en prenant en compte des diffusions de Fick, comme on l'a expliqué ci-dessus, le système limite est non linéaire, avec des diffusions croisées. Dans une deuxième partie, on prouve que la solution du système avec vitesse de réaction finie k converge lorsque $k \to +\infty$ vers une solution globale de ce système limite. Sous certaines restrictions sur les coefficients de diffusion, on prouve que cette solution est unique. Ces résultats étendent des travaux de D. Bothe [18] au cas où les coefficients de diffusion ne sont pas égaux.

◇ Étant donné que les techniques utilisées pour étudier la limite de réaction rapide dans la réaction $C_1 + C_2 \rightleftharpoons C_3$ sont assez générales, on a alors cherché à généraliser les résultats du chapitre 1 à des modèles plus réalistes. En particulier, les situations suivantes sont plus pertinentes du point de vue de la chimie :

- Les coefficients de diffusion dépendent du temps, de la variable d'espace et des concentrations ($d_i = d_i(t,x,c)$).
- Les données initiales sont dans $L^1(\Omega)$ "seulement".
- Des réactions chimiques lentes ont lieu en même temps que la réaction $C_1 + C_2 \rightleftharpoons C_3$.

Le chapitre 2 est consacré à des preuves d'existence globale dans ces situations. En particulier

- On prouve l'existence de solutions fortes globales pour des systèmes du type (1) pour divers termes de réaction f et pour des dimensions en espace petites ($N \leq 5$ pour $d_i = d_i(t,x,c)$, $N \leq 9$ pour $d_i = d_i(c_i)$). On commence par traiter le cas d'une réaction chimique $C_1 + C_2 \rightleftharpoons C_3$, puis on généralise la méthode à des réseaux de réactions du type

$$C_i + C_j \rightleftharpoons C_k.$$

- On prouve l'existence de solutions faibles globales pour des termes de réaction généraux, qui sont seulement supposés avoir une croissance quadratique. On traite aussi le cas de données initiales dans $L^1(\Omega)$. Bien qu'elle soit naturelle du point de vue de la modélisation, cette hypothèse entraîne d'importantes difficultés dans le traitement mathématique des équations, le contrôle des solutions dans un voisinage de $t = 0$ faisant appel à des estimations délicates.

◇ Dans les travaux des chapitres précédents, on a supposé que la diffusion était le seul phénomène responsable du transport de masse. On prend maintenant en compte d'autres phénomènes de transport, qui sont : soit l'advection lorsque le fluide est en mouvement, soit le phénomène d'électromigration, lorsque le fluide est un électrolyte et que les espèces $c_1, \ldots, c_p$ sont des ions.

En considérant que le champ de vecteurs u décrivant le mouvement du fluide est une donnée du problème, on commence par généraliser un résultat d'existence globale et d'unicité de M. Pierre [90] sur des systèmes de réaction-diffusion dont les termes de réaction ont la structure "triangulaire" (2). Cela revient essentiellement à considérer des termes de réaction où $f_1, f_1 + f_2, \ldots, f_1 + \ldots + f_P$ sont bornés supérieurement par une fonction affine de $c_1, \ldots, c_P$. La preuve de [90] est généralisée au cas où les coefficients de diffusion de Fick dépendent du temps et de la variable d'espace, et où on prend aussi en compte les termes d'advection.

En particulier, le résultat de la partie précédente donne l'existence de solutions globales pour le système associé à la réaction chimique $C_1 + C_2 \rightleftharpoons C_3$, indépendamment de la vitesse de réaction. On peut donc s'intéresser à nouveau à la limite de réaction rapide, mais dans un contexte plus général. On prouve alors que les techniques du chapitre 1 sont suffisamment robustes pour être étendues au cas de systèmes de réaction-diffusion-advection avec des coefficients de diffusion dépendant du temps et de la variable d'espace.

Dans la dernière partie du chapitre 3, on s'intéresse à l'existence de solutions pour un système de diffusion-électromigration, *sans restriction sur la dimension de l'espace*. En utilisant un procédé d'approximation qui respecte la structure "entropique" du problème initial, on prouve l'existence de solutions globales faibles. Le résultat de la première partie de ce chapitre sur les systèmes de réaction-diffusion-advection est utilisé dans la mise en œuvre d'une technique de point fixe de Leray-Schauder pour obtenir l'existence de solutions au problème approché.

On décrit maintenant de façon plus détaillée le contenu de chaque chapitre.

1 Deux systèmes aux diffusions croisées

1.1 Existence globale et unicité pour un système conservatif relaxé aux diffusions croisées

À l'exception du paragraphe 1.6, le contenu de cette partie est issu d'une collaboration avec T. Lepoutre et M. Pierre, et est publié dans [72].

Les diffusions croisées ont été utilisées en dynamique des populations par Shigesada, Kawasaki et Teramoto [97] pour décrire les interactions entre plusieurs espèces lors de leurs mouvements. L'objectif initial était de trouver un modèle permettant d'expliquer les phénomènes de ségrégation spatiale.

Pour fixer les idées, dans le cas de deux populations, un système général s'écrit

$$\left\{\begin{array}{rcll} \partial_t u_1 - \Delta[u_1(d_1 + d_{11}u_1^p + d_{12}u_2^p)] &=& f_1(u_1,u_2) & \text{sur } (0,+\infty)\times\Omega, \\ \partial_t u_2 - \Delta[u_2(d_2 + d_{21}u_1^p + d_{22}u_2^p)] &=& f_2(u_1,u_2) & \text{sur } (0,+\infty)\times\Omega, \\ \partial_\nu[u_i(d_i + d_{i1}u_1^p + d_{i2}u_2^p)] &=& 0 & \text{sur } (0,+\infty)\times\partial\Omega. \end{array}\right. \tag{3}$$

Pour deux populations se partageant des ressources limitées, les termes $\Delta[u_i(d_{i1}u_1^p + d_{i2}u_2^p)]$ modélisent les frictions sociales et la compétition. Dans le cas d'un système proie-prédateur, ces termes modélisent le fait que les prédateurs ont tendance à aller vers les régions où se concentrent les proies, tandis que les proies se déplacent vers les régions où les prédateurs sont rares.

Pour le système (3) avec $p = 1$ et des termes de réaction du type Lotka-Volterra, de nombreux travaux ont été publiés avec des hypothèses supplémentaires garantissant le caractère parabolique des opérateurs, ou encore avec des diffusions croisées pour une espèce seulement (voir par ex. Wang [103], ainsi que les nombreuses références qui s'y trouvent). Un résultat général sur l'existence de solutions globales faibles a été publié par Chen et Jüngel [32], où des fonctions de Lyapunov sont utilisées. Concernant les solutions fortes, on pourra se référer par exemple aux articles [75, 103] de Wang et Li-Zhao. En dynamique des populations, les diffusions croisées peuvent faire apparaître des états stationnaires non homogènes qui n'existent pas dans le cas de diffusions de Fick (voir Iida-Mimura-Ninomyia [63] par exemple). Cependant, dans l'article mentionné, l'existence de solutions stationnaires non homogènes utilise le fait que les termes de réaction sont non nuls (la convergence vers des états stationnaires homogènes en l'absence de réaction étant prouvée dans [32]).

Pour montrer que des forces dispersives non linéaires peuvent générer de la ségrégation spatiale en l'absence de termes de réaction, un modèle conservatif relaxé (et donc non-local) a été introduit dans [11]. Le système

$$\left\{\begin{array}{l} \partial_t u_i - \Delta[a_i(u)u_i] = 0 \text{ sur } (0,+\infty)\times\Omega, \\ \partial_\nu[a_i(u)u_i] = 0 \text{ sur } (0,+\infty)\times\partial\Omega, \\ u = (u_1,\ldots,u_I)\ ;\ u(0,\cdot) = u^0 \text{ donné }, \end{array}\right.$$

où $a_i : [0,\infty)^I \to [\underline{a},\infty)$ pour un $\underline{a} > 0$ donné, est remplacé par le modèle relaxé suivant :

$$\left\{\begin{array}{l} \partial_t u_i - \Delta[a_i(\tilde{u})u_i] = 0,\ \text{sur } (0,+\infty)\times\Omega, \\ \tilde{u}_i - \delta_i\Delta\tilde{u}_i = u_i,\ \text{sur } (0,+\infty)\times\Omega, \\ \partial_\nu u_i = \partial_\nu\tilde{u}_i = 0 \text{ sur } (0,+\infty)\times\partial\Omega, \\ u = (u_1,\ldots u_I)\ ;\ \delta_i > 0,\ u(0,\cdot) = u^0 \text{ donné.} \end{array}\right. \tag{4}$$

Les effets de la relaxation sur la stabilité des équilibres homogènes ont été étudiés dans [11,70,71]. Remarquons que ce modèle non-local prend en compte le fait que chaque individu mesure les densités des autres espèces dans un voisinage de sa position, avec une échelle spatiale δ_i. Ceci est particulièrement intéressant en vue des applications en dynamique des populations. Des modèles avec des diffusions non-locales sont aussi étudiés dans [12], où les coefficients de diffusion pour une population donnée dépendent de sa population totale.

Un premier résultat d'existence et d'unicité pour le système relaxé (4) a été obtenu dans [11, 70] pour des dimensions en espace $N = 1,2$ et avec des restrictions sur la structure des non-linéarités a_i : les a_i sont supposés C^2, avec une croissance polynomiale par-rapport à u. Dans cette première partie, on considère une version intégrée en temps de (4) :

$$\left\{\begin{array}{rcll} u_i - \Delta \int_0^t [a_i(\tilde{u})u_i] &=& u_i^0 & \text{sur } (0,+\infty)\times\Omega, \\ \tilde{u}_i - \delta_i \Delta \tilde{u}_i &=& u_i & \text{sur } (0,+\infty)\times\Omega, \\ \partial_\nu u_i = \partial_\nu \tilde{u}_i &=& 0 & \text{sur } (0,+\infty)\times\partial\Omega. \end{array}\right. \tag{5}$$

On prouve le résultat suivant, où par solution “forte”, on entend que chacune des dérivées qui apparaît dans le système est une fonction mesurable, et que les équations sont satisfaites presque partout.

Théorème 1. *Supposons que a_i est une fonction continue et bornée inférieurement par une constante strictement positive. Alors le système* (5) *a une solution globale forte. Si en plus les a_i sont supposées localement lipschitziennes, cette solution est unique et c'est une solution forte du système* (4).

On commence par prouver l'existence de solutions faibles à partir d'estimations L^2 inspirées de [90]. Un point important est qu'on parvient alors à prouver que $\tilde{u}$ est uniformément bornée, indépendamment de la dimension de l'espace. On doit ensuite gérer des opérateurs du type $u_i \to \partial_t u_i - \Delta(a_i(\tilde{u})u_i)$. Ils ne sont pas sous forme divergentielle, mais sont quand même uniformément paraboliques, puisque $a_i(\tilde{u})$ est borné inférieurement et supérieurement. En utilisant la théorie C^α de Krylov-Safonov (voir [42, 67]) sur les opérateur duaux $U_i \to \partial_t U_i - a_i(\tilde{u})\Delta U_i$, on prouve que $\tilde{u}$ est en fait Höldérienne. Cela prouve que les coefficients $a_i(\tilde{u})$ des opérateurs ci-dessus sont réguliers. Il est alors facile d'en déduire des estimations L^p sur les solutions. Lorsque les a_i sont localement lipschitziennes, on montre que la solution est unique. On prouve aussi que $\partial_t u_i$ et $\Delta(a_i(\tilde{u})u_i)$ sont dans des espaces L^p : la solution est donc une solution forte du système (4).

1.2 Limite de réaction rapide pour une réaction chimique réversible

À l'exception du paragraphe 2.4.4, le contenu de cette partie est issu d'une collaboration avec D. Bothe et M. Pierre, et est publié dans dans [28].

La seconde partie de ce chapitre est consacrée à l'étude de la limite de réaction rapide dans un modèle classique pour la réaction chimique

$$C_1 + C_2 \rightleftharpoons C_3 \,. \tag{6}$$

On suppose aussi que les espèces chimiques diffusent suivant une loi de Fick dont les coefficients de diffusion sont supposés constants, mais peuvent être différents. Plus précisément, on suppose que la vitesse de la réaction est donnée par la loi d'action de masse (voir [46] pour plus de détails sur les mécanismes de réaction). Le domaine Ω est supposé borné et régulier. Si c_i est la concentration de l'espèce C_i, on obtient le système

$$(R^k)\left\{\begin{array}{ll} \left.\begin{array}{l}\partial_t c_1 - d_1\Delta c_1 = -k(c_1c_2 - \kappa c_3)\\ \partial_t c_2 - d_2\Delta c_2 = -k(c_1c_2 - \kappa c_3)\\ \partial_t c_3 - d_3\Delta c_3 = +k(c_1c_2 - \kappa c_3)\end{array}\right\} & \text{sur } (0,+\infty)\times\Omega,\\ \partial_\nu c_1 = \partial_\nu c_2 = \partial_\nu c_3 = 0 & \text{sur } (0,+\infty)\times\partial\Omega,\\ c_1(0,\cdot) = c_1^0,\ c_2(0,\cdot) = c_2^0,\ c_3^0(0,\cdot) = c_3^0 & \text{sur } \Omega,\end{array}\right.$$

où $k > 0$ est la vitesse de réaction et $\kappa > 0$ est la constante d'équilibre. Pour $k < +\infty$ et des données initiales $c^0 \in L^\infty(\Omega)^3_+$, il est bien connu que le système (R^k) a une solution forte globale, pour toute dimension d'espace. Ce résultat est, par exemple, un corollaire du résultat d'existence et d'unicité de solutions globales fortes de M. Pierre [90] pour les systèmes dont la réaction a la structure triangulaire (2).

La raison pour laquelle on étudie le comportement de la solution c^k de (R^k) lorsque $k \to +\infty$ est la suivante : une analyse non-dimensionnelle montre la présence de deux échelles temporelles :

- La diffusion dans des liquides, et à plus forte raison dans les solides, est un processus relativement lent. Par exemple, dans un liquide, même agité, une échelle de temps typique pour la diffusion est
$$\tau_{\text{diff}} \geq 10^{-3}s.$$
Dans des systèmes non agités, ce temps peut être encore beaucoup plus grand.
- Au contraire, les réactions chimiques peuvent être extrêmement rapides, leur vitesse dépendant du mécanisme de réaction. Par exemple, dans la cas de la neutralisation $H^+ + OH^- \rightleftharpoons H_2O$, la réaction dans le sens direct peut avoir une échelle de temps de l'ordre de
$$\tau^f_{\text{reac}} \simeq 10^{-11}s.$$

Vient alors la question de l'écriture d'un système limite pour (R^k) lorsque $k \to +\infty$. On montrera dans la suite qu'il existe un distribution f telle que

$$k(c_1^kc_2^k - \kappa c_3^k) \overset{k\to+\infty}{\longrightarrow} f\ .$$

Par conséquent, on peut raisonnablement s'attendre à ce qu'à la limite $k \to +\infty$, la vecteur de composition chimique c reste sur la variété $\{c_1c_2 = \kappa c_3\}$ sur laquelle la réaction est à l'équilibre. Remarquons aussi que les termes de réaction s'annulent lorsqu'on considère $c_1^k + c_3^k$ et $c_2^k + c_3^k$.

Le résultat principal est le suivant :

Théorème 2. *Supposons $k_n \to +\infty$ et soit c^n la solution correspondante de (R^{k_n}). À une sous-suite près, pour tout $T > 0$, c^n converge fortement dans $L^2(Q_T)$ et faiblement dans $L^{4/3}(0,T;W^{1,4/3}(\Omega))$ vers une solution faible de*

$$(R^\infty)\left\{\begin{array}{ll} \left.\begin{array}{l}\partial_t(c_1+c_3) - \Delta(d_1c_1 + d_3c_3) = 0\\ \partial_t(c_2+c_3) - \Delta(d_2c_2 + d_3c_3) = 0\\ c_1c_2 = \kappa c_3\end{array}\right\} & \textit{sur } (0,+\infty)\times\Omega,\\ \partial_\nu(d_1c_1 + d_3c_3) = \partial_\nu(d_2c_2 + d_3c_3) = 0 & \textit{sur } (0,+\infty)\times\partial\Omega,\\ (c_1+c_3)(0,\cdot) = c_1^0 + c_3^0;\ (c_2+c_3)(0,\cdot) = c_2^0 + c_3^0 & \textit{sur } \Omega.\end{array}\right.$$

La convergence de c^k lorsque $k \to +\infty$ vers une solution du système limite a été prouvée pour des coefficients de diffusion égaux dans [18]. Cette situation est beaucoup plus simple car dans ce cas, $c_1^k + c_3^k$ et $c_2^k + c_3^k$ sont solutions de l'équation de la chaleur, ce qui permet de les estimer uniformément et indépendamment de k dans $L^\infty(\Omega)$ en utilisant le principe du maximum. Dans le cas de coefficients de diffusion différents, cette remarque ne s'applique plus et on peut seulement faire appel à des estimations *a priori* dans $L^2((0,T)\times\Omega)$ pour tout $T>0$, inspirées de [90], qui restent valides pour $c_1^k + c_3^k$ et $c_2^k + c_3^k$. L'autre ingrédient pour obtenir la compacité relative de c^k est une fonction de Lyapunov, communément appelée “estimation entropique” : elle permet de contrôler les gradients et fournit les arguments essentiels pour prouver la convergence ponctuelle de c^k.

On peut réécrire (R^∞) comme un système de réaction-diffusion avec diffusions croisées 2×2 comme suit : on utilise la relation algébrique $c_1c_2 = \kappa c_3$ pour introduire le nouveau couple de fonctions inconnues

$$x_1(c_1,c_2) = c_1 + \kappa c_1 c_2 \ ; \ x_2(c_1,c_2) = c_2 + \kappa c_1 c_2 \, .$$

Des calculs élémentaires (utilisant la positivité de c_1 et c_2) donnent $(c_1,c_2) = (\varphi(x_1,x_2), \overline{\varphi}(x_1,x_2))$, où

$$\varphi(\alpha,\beta) = \frac{1}{2}\sqrt{\kappa^2 + (\alpha-\beta)^2 + 2\kappa(\alpha+\beta)} - (\kappa+\beta-\alpha) \, ; \, \overline{\varphi}(\alpha,\beta) = \varphi(\beta,\alpha).$$

Par conséquent, (R^∞) est équivalent à

$$(\tilde{R}^\infty)\begin{cases} \left.\begin{array}{r} \partial_t x_1 - \Delta\psi_1(x_1,x_2) = 0 \\ \partial_t x_2 - \Delta\psi_2(x_1,x_2) = 0 \end{array}\right\} & \text{sur } (0,+\infty)\times\Omega, \\ \partial_\nu\left(\psi_1(x_1,x_2)\right) = \partial_\nu\left(\psi_2(x_1,x_2)\right) = 0 & \text{sur } (0,+\infty)\times\partial\Omega, \\ x_1(0,\cdot) = x_1^0,\ x_2(0,\cdot) = x_2^0 & \text{sur } \Omega, \end{cases}$$

où $\psi_1 = d_1\varphi + d_3\kappa\varphi\overline{\varphi}$, $\psi_2 = d_2\overline{\varphi} + d_3\kappa\varphi\overline{\varphi}$.

Il n'est pas difficile de voir que les opérateurs sous-jacents dans $(\tilde{R}^\infty)$ sont "normalement elliptiques". Ceci permet d'appliquer la théorie de H. Amann [2, 4] : pour des données initiales suffisamment régulières, $(\tilde{R}^\infty)$ a une unique solution classique sur un intervalle de temps maximal $[0,T^*)$, $T^* \le +\infty$. On a existence globale si on sait estimer la solution uniformément en temps dans un espace de Sobolev approprié. Cependant, on ne sait pas prouver ces estimations.

Les questions suivantes sont naturelles une fois qu'on a prouvé le théorème 2.

- Est-ce que la solution faible coïncide avec la solution classique d'Amann ? C'est une question d'unicité des solutions faibles.
- La solution faible est globale en temps, tandis que celle d'Amann n'existe *a priori* que sur un intervalle de temps $[0,T^*)$, où T^* peut être fini. Est-ce qu'il peut arriver que les solutions faibles soient régulières pendant un certain temps, puis deviennent singulières ?

On fournit des résultats partiels à la première question. Bien que nos solutions soient assez faibles, on parvient à prouver qu'elles sont uniques à condition que (d_1,d_2,d_3) satisfasse la condition

$$\left(\frac{d_1}{d_3}-1\right)^2\left(\frac{d_2}{d_3}-1\right)^2 < 16\frac{d_1d_2}{d_3^2}.$$

Dans ce cas et pour des données initiales régulières, la solution faible coïncide avec celle d'Amann sur l'intervalle où cette dernière existe.

On prouve aussi que si $|d_1 - d_2|$ appartient à un petit intervalle dont la taille dépend de la norme $L^\infty((0,T)\times\Omega)$ de la solution régulière, alors la solution faible coïncide avec la solution régulière sur $[0,T]$. Mais cela ne prouve pas l'unicité des solutions faibles sur des intervalles de temps arbitrairement grands.

Dans la mesure où les techniques qu'on introduit pour prouver le théorème 2 sont assez robustes, elles peuvent être réutilisées pour passer à la limite de réaction rapide dans des systèmes bien plus complexes que (R^k). Il se trouve que la principale difficulté pour traiter le cas de systèmes chimiques plus complexes est de connaître l'existence de solutions globales pour les systèmes avec une vitesse de réaction finie. C'est la raison pour laquelle le chapitre 2 est consacré à des questions d'existence globale.

2 Résultats d'existence globale pour des systèmes aux diffusions non-linéaires

2.1 Existence globale pour des systèmes de réaction-diffusion issus de la cinétique chimique avec des diffusions dépendant des concentrations

Les résultats de cette partie on été obtenus en collaboration avec D. Bothe, et seront publiés dans [29].

Lorsqu'on modélise les flux de masse avec la loi de Fick $J_i = -d_i\nabla c_i$, les d_i sont des fonctions des variables d'état thermodynamiques du système. En particulier, les d_i dépendent du temps, de la variable d'espace et de la composition chimique.

Commençons par considérer à nouveau le système chimique du chapitre précédent, mais cette fois avec des coefficients de diffusion variables :

$$\left\{\begin{array}{ll}\left.\begin{array}{l}\partial_t c_1 - \mathrm{div}(d_1(t,x,c)\nabla c_1) = -c_1c_2 + c_3\\ \partial_t c_2 - \mathrm{div}(d_2(t,x,c)\nabla c_2) = -c_1c_2 + c_3\\ \partial_t c_3 - \mathrm{div}(d_3(t,x,c)\nabla c_3) = +c_1c_2 - c_3\end{array}\right\} & \text{sur } (0,+\infty)\times\Omega,\\ \partial_\nu c_1 = \partial_\nu c_2 = \partial_\nu c_3 = 0 & \text{sur } (0,+\infty)\times\partial\Omega,\\ c(0,\cdot) = (c_1^0, c_2^0, c_3^0) & \text{sur } \Omega.\end{array}\right. \tag{7}$$

On suppose que d_i satisfait $\underline{d} \le d_i$ pour un $\underline{d} > 0$ donné, ainsi qu'une des deux propriétés suivantes :

(a) $d_i \in C^2([0,+\infty)\times\Omega\times\mathbb{R}^3,\mathbb{R}^+)$ lorsque $d_i = d_i(t,x,c)$.

(b) $d_i \in C^2(\mathbb{R},\mathbb{R}^+)$ lorsque $d_i = d_i(c_i)$.

L'existence de solutions globales fortes pour (7) est connue pour des coefficients de diffusion d_i *constants*. Dans ce cas, il a été montré dans [94] que pour des données initiales positives bornées et pour des dimensions en espace $N \le 5$, (7) a une unique solution forte globale, positive, et qu'elle est uniformément bornée. L'existence globale a ensuite été montrée dans [48] pour toute dimension d'espace dans le cas de domaines Ω de classe $C^{2+\alpha}$, $\alpha \in (0,1)$ et pour des données initiales régulières. Ces deux approches sont basées sur la théorie des semi-groupes, et exploitent la structure semi-linéaire des équations. Le système (7) a aussi la structure "triangulaire" (2) pour laquelle l'existence globale de solutions fortes est prouvée dans [90], pour toute dimension d'espace et pour des données initiales bornées. Cette approche utilise la théorie de la régularité maximale [37] sur les équations duales, et fait appel de façon cruciale à la linéarité des opérateurs de diffusion.

Pour des coefficients de diffusion généraux, la question de l'existence de solutions globales classiques est largement ouverte. Le seul résultat proche dont on a connaissance est [84], où le cas de flux du type $d_i(c_i)\nabla c_i$ est traité, avec des réseaux de réactions satisfaisant une structure "quadratique triangulaire " appropriée. L'existence globale est prouvée dans le cas de la dimension $N = 2$.

Dans cette partie, on s'appuie sur la théorie de H. Amann pour l'existence de solutions classiques pour le système (7) sur un intervalle maximal $[0,T^*)$, $0 < T^* \leq +\infty$. On prouve alors que cette solution est uniformément bornée dans $L^\infty(\Omega)$ sur tous les intervalles compacts $[0,T]$, $T \leq T^*$, et on fait appel au critère d'existence globale de Amann pour en déduire $T^* = +\infty$, *i.e.* que les solutions maximales sont globales. Notre méthode est basée sur des estimations classiques combinées avec une technique de bootstrap. On peut la résumer comme suit : étant donnée une estimation intiale sur la solution dans un espace L^p, comme les termes de réaction pour c_1 et c_2 sont bornés supérieurement par c_3, on peut améliorer les exposants p des estimations pour c_1 et c_2, le nouvel exposant dépendant de la dimension de l'espace. Cela fournit une nouvelle estimation sur c_1c_2, et comme le terme de réaction pour c_3 est bornée supérieurement par c_1c_2, une nouvelle estimation sur $c_3\ldots$ Pour des dimensions en espace suffisamment petites, cette technique peut être "bootstrapée" pour obtenir des bornes dans $L^\infty((0,T)\times\Omega)$ pour tout $T > 0$, et donc l'existence globale.

La raison pour laquelle on considère deux hypothèses différentes sur les coefficients de diffusion est la suivante : pour des coefficients satisfaisant (a), la seule estimation disponible pour démarrer la procédure de bootstrap est dans $L^\infty(0,T;L^1(\Omega))$, ce qui correspond à la conservation de la masse totale. Dans le cas plus restrictif des coefficients satisfaisant (b), on dispose d'une estimation initiale dans $L^2(Q_T)$, qui permet de faire fonctionner la procédure de bootstrap pour des dimensions en espace plus grandes.

Pour la réaction chimique $C_1 + C_2 \rightleftharpoons C_3$, le principal résultat est le suivant :

Théorème 3. *Pour des données initiales suffisamment régulières, le système* (7) *a une unique solution forte globale dans les situations suivantes :*

(i) $N \leq 5$ *et les coefficients de diffusion* $d_i(t,x,c)$ *satisfont* (a).

(ii) $N \leq 9$ *et les coefficients de diffusion* $d_i(c_i)$ *satisfont* (b).

On généralise alors ce théorème au cas d'un système de P espèces chimiques $C_1,\ldots,C_P$ impliquées dans un réseau de R réactions du type

$$C_{j_1} + C_{j_2} \;\rightleftharpoons\; C_{j_3}\,; \quad j \in \{1,\ldots,R\}\,;\, j_1, j_2, j_3 \in \{1,\ldots,P\}.$$

Comme précédemment, c_i est la concentration de l'espèce C_i. En utilisant la loi d'action de masse, la vitesse de réaction pour la j-ième réaction est donnée par

$$r_j(c) = c_{j_1}c_{j_2} - c_{j_3},$$

où pour simplifier les écritures, on a omis les constantes de réaction. Soit $(\varepsilon_1,\ldots,\varepsilon_P)$ la base canonique de $\mathbb{R}^P$, on définit alors les *vecteurs stœchiométriques* $\alpha_j := \varepsilon_{j_1} + \varepsilon_{j_2}$, $\beta_j := \varepsilon_{j_3}$ et $\nu_j := \beta_j - \alpha_j$. En utilisant les notations ci-dessus, le taux de création de $c = (c_1,\ldots,c_P)$ est

$$f(c) := \begin{pmatrix} f_1(c) \\ \vdots \\ f_P(c) \end{pmatrix} = \left(\begin{array}{c|c|c} \nu_1^1 & & \nu_R^1 \\ \vdots & \ldots & \vdots \\ \nu_1^P & & \nu_R^P \end{array}\right) \begin{pmatrix} r_1(c) \\ \vdots \\ r_R(c) \end{pmatrix}. \tag{8}$$

En supposant les diffusions comme ci-dessus, l'évolution en temps de $c = (c_1, \ldots, c_P)$ est alors déterminée par les équations

$$\left\{\begin{array}{ll} \begin{pmatrix} \partial_t c_1 - & \mathrm{div}\,(d_1(t,x,c)\nabla c_1) \\ & \vdots \\ \partial_t c_P - & \mathrm{div}\,(d_P(t,x,c)\nabla c_P) \end{pmatrix} = \begin{pmatrix} f_1(c) \\ \vdots \\ f_P(c) \end{pmatrix} & \text{sur } (0,+\infty)\times\Omega, \\ \partial_\nu c = 0 & \text{sur } (0,+\infty)\times\partial\Omega, \\ c(0,\cdot) = c^0 & \text{sur } \Omega. \end{array}\right. \tag{9}$$

Après avoir réarrangé les espèces chimiques et les réactions, en utilisant une procédure de bootstrap analogue à celle du théorème 3, on prouve

Théorème 4. *Pour des données initiales suffisamment régulières et en supposant que la masse totale est conservée, le système* (9) *a une unique solution forte globale dans chacune des situations suivantes :*

(i) $N \leq 3$ *et les coefficients de diffusion* $d_i(t,x,c)$ *satisfont* (a).

(ii) $N \leq 5$ *et les coefficients de diffusion* $d_i(c_i)$ *satisfont* (b).

Pour finir, remarquons que notre technique utilise de façon essentielle le fait que des termes de réaction sont bornées supérieurement par des fonctions linéaires. Par exemple, elle ne permet pas de traiter le cas de la réaction chimique

$$C_1 + C_2 \rightleftharpoons C_3 + C_4,$$

dont le système de réaction-diffusion associé est

$$\left.\begin{array}{l} \partial_t c_1 - \mathrm{div}(d_1(t,x,c)\nabla c_1) = -c_1c_2 + c_3c_4 \\ \partial_t c_2 - \mathrm{div}(d_2(t,x,c)\nabla c_2) = -c_1c_2 + c_3c_4 \\ \partial_t c_3 - \mathrm{div}(d_3(t,x,c)\nabla c_3) = +c_1c_2 - c_3c_4 \\ \partial_t c_4 - \mathrm{div}(d_4(t,x,c)\nabla c_4) = +c_1c_2 - c_3c_4 \end{array}\right\}, \quad (t,x) \in (0,+\infty)\times\Omega. \tag{10}$$

Même pour des coefficients de diffusion constants, la question de l'existence de solutions globales classiques pour (10) est un problème ouvert pour des dimensions en espace $N \geq 3$. La dimension de Hausdorff de l'ensemble des points pouvant être singuliers a été estimée dans [58], où l'existence globale est aussi montrée dans le cas $N = 2$. La théorie de la Régularité Maximale a également été utilisée avec succès par J. Prüss [92] dans le cas $N = 2$ pour obtenir des solutions globales. Ce résultat a été étendu au cas de coefficients de diffusion variables du type $d_i(c_i)$ dans [84].

2.2 Solutions globales faibles avec diffusions non linéaires, réactions quadratiques et données initiales dans L^1

Le contenu de cette partie sera publié dans [93].

Dans cette partie, on prouve l'existence de solutions faibles globales pour des systèmes de la forme

$$\left\{\begin{array}{lll} \partial_t c_i - \mathrm{div}(d_i(c_i)\nabla c_i) = f_i(t,x,c) & \text{sur } (0,+\infty)\times\Omega, & i \in \{1,\ldots,P\}, \\ d_i(c_i)\partial_\nu c_i = g_i & \text{sur } (0,+\infty)\times\partial\Omega, & i \in \{1,\ldots,P\}, \\ c(0,\cdot) = c^0 & \text{sur } \Omega. & \end{array}\right. \tag{11}$$

On examine successivement deux hypothèses sur les données initiales, à savoir $c^0 \in L^2(\Omega, \mathbb{R}_+^P)$ et $c^0 \in L^1(\Omega, \mathbb{R}_+^P)$. On suppose aussi que

(i) Les fonctions f_i ont une croissance au plus quadratique par-rapport à c.

(ii) $(f_1, \dots, f_P)$ est quasi-positive (cf. $(\mathbf{H}_2)$).

(iii) $\exists \underline{d}, \overline{d} > 0$ tels que $\underline{d} \leq d_i \leq \overline{d}$.

Comme on l'a rappelé plus haut, l'existence locale de solutions fortes positives pour le système (11) pour des données initiales régulières est bien connue, mais la question de l'existence globale est ouverte en général, même pour des solutions faibles. Bien qu'elle soit naturelle du point de vue de la modélisation, l'hypothèse que les données initiales sont dans $L^1(\Omega)$ n'a été que peu étudiée. M. Pierre a prouvé dans [90] l'existence globale pour des systèmes avec des données initiales dans $L^1(\Omega)$ et pour des non-linéarités *a priori* bornées dans $L^1(Q_T)$. Ce résultat inclut par exemple les systèmes ayant la structure "triangulaire" (2). Si les non-linéarités f_i ont en plus une croissance polynomiale, il a été montré dans [17] que les solutions sont classiques sur $(0, +\infty) \times \Omega$. Si les f_i sont en fait bornées par des expressions polynomiales de degré $p < \frac{N+2}{N}$, où N est la dimension de l'espace, l'existence de solutions pour des données initiales dans l'espace des mesures de Radon est aussi prouvée.

Dans le résultat qui suit, la principale différence avec la situation étudiée dans [90] est que lorsque $c^0 \in L^1(\Omega, \mathbb{R}_+^P)$, on ne contrôle plus les termes de réaction dans L^1 jusqu'en $t = 0$.

Théorème 5. *Sous les hypothèses $(i) - (iii)$, le système (11) a une unique solution globale faible $c : (0, +\infty) \times \Omega \to \mathbb{R}_+^P$ telle que :*

(i) Si $c^0 = (c_1^0, \dots, c_P^0) \in L^2(\Omega, \mathbb{R}_+^P)$, c satisfait une formulation variationnelle de (11) sur $(0, T) \times \Omega$ pour tout $T > 0$.

(ii) Si $c^0 = (c_1^0, \dots, c_P^0) \in L^1(\Omega, \mathbb{R}_+^P)$, c satisfait une formulation variationnelle de (11) sur $(\tau, T) \times \Omega$ pour $0 < \tau < T < +\infty$ et $c(t) \underset{t \to 0}{\longrightarrow} c^0$ au sens des mesures de Radon.

Pour des données initiales dans $L^2(\Omega)^P$, l'argument central de la preuve est une estimation indépendante de la dimension en espace dans $L^2((0, T) \times \Omega)$. Comme les f_i ont une croissance au plus quadratique, les termes de réaction sont alors contrôlés dans $L^1((0, T) \times \Omega)$, et on peut utiliser des résultats classiques de la théorie des équations paraboliques. Lorsqu'on prend des données initiales dans $L^1(\Omega)^P$, la nouvelle difficulté est que les estimations précédentes dans L^2 ne sont plus valides jusqu'en $t = 0$. On parvient seulement, en combinant des techniques L^2 avec les propriétés régularisantes du Laplacien, à contrôler les solutions dans $L^2((\tau, T) \times \Omega)$ pour tout $\tau \in (0, T)$. Les termes de réaction ne sont donc plus contrôlés dans L^1 jusqu'en $t = 0$. Pour contourner cette difficulté, on s'inspire de [41, 90] pour borner les solutions successivement supérieurement et inférieurement dans un voisinage de $t = 0$. On prouve ainsi la convergence de $c(t)$ vers c^0 dans l'espace des mesures de Radon.

3 Systèmes de réaction-diffusion avec advection-migration

Les résultats de ce chapitre sont issus d'une collaboration avec D. Bothe, A. Fischer et M. Pierre, et seront publiés dans [22] et [23].

3.1 Existence globale et unicité pour des systèmes de réaction-diffusion avec une réaction "triangulaire"

On considère le système

$$\left\{\begin{array}{rcll} \partial_t c_i + \mathrm{div}[-d_i(t,x)\nabla c_i + c_i u_i(t,x)] & = & f_i(t,x,c) & \text{sur } (0,+\infty)\times\Omega, \\ -d_i(t,x)\nabla c_i\cdot\nu + c_i u_i(t,x)\cdot\nu & = & 0 & \text{sur } (0,+\infty)\times\partial\Omega, \\ c_i(0,\cdot) & = & c_i^0 & \text{sur } \Omega, \end{array}\right. \tag{12}$$

où $i\in\{1,\dots,P\}$ et dont l'inconnue est $c=(c_1,\dots,c_P)$. On suppose la donnée initiale $c^0=(c_1^0,\dots,c_P^0)$ dans $L^\infty(\Omega,\mathbb{R}_+^P)$, les termes de réaction réguliers, quasi-positifs, et avec la structure triangulaire (2). Enfin, on suppose les d_i continus, bornées inférieurement par une constante strictement positive et

$$\nabla d_i, u_i \in L^\infty_{loc}([0,+\infty);L^r(\Omega)^N) \text{ pour un } r>\max(2,N).$$

Sous les hypothèses ci-dessus, on prouve le théorème suivant :

Théorème 6. *Le système* (12) *a une unique solution globale forte.*

Ce résultat, ainsi que sa preuve, s'inspirent du Théorème 3.5 de M. Pierre [90], où l'existence globale et l'unicité sont prouvés dans le cas de coefficients de diffusion constants et en l'absence des termes de transport u_i. Dans la situation présente, la principale difficulté est de prendre en compte les u_i, qui sont dans l'espace assez général $L^\infty_{loc}([0,+\infty);L^r(\Omega)^N)$. On doit aussi gérer la dépendance en (t,x) des d_i, et il semble que ce même espace $L^\infty_{loc}([0,+\infty);L^r(\Omega)^N)$ soit celui qu'il convient de choisir pour ∇d_i.

La preuve est basée sur les deux estimations suivantes :

(i) Soit $T>0$, on suppose que w et z sont des fonctions régulières satisfaisant pour un $\theta\in\mathbb{R}$ donné

$$\partial_t w + \mathrm{div}(-d_1\nabla w + w u_1) \le \theta[\partial_t z + \mathrm{div}(-d_2\nabla z + z u_2)],$$

avec des données initiales dans $L^\infty(\Omega)$ et des conditions de Neumann homogènes. Alors pour tout $p\in(1,+\infty)$, la norme L^p de z contrôle la norme L^p de w comme suit :

$$\exists C>0 : \forall t\in(0,T), \quad \|\max(0,w)\|_{L^p((0,t)\times\Omega)} \le C\left(1+\|z\|_{L^p((0,t)\times\Omega)}\right).$$

(ii) Soit $T>0$ et soit c la solution de

$$\partial_t c + \mathrm{div}(-d\nabla c + cu) = f \text{ sur } (0,T)\times\Omega,$$

avec des conditions de Neumann homogènes et des données initiales bornées. Alors il existe $C>0$ tel que

$$\forall t\in(0,T), \quad \|c(t)\|^p_{L^p(\Omega)} \le C\left(1+\int_0^t \|f(s)\|^p_{L^p(\Omega)}ds\right).$$

L'énoncé (i) est au coeur de la preuve du théorème 6, et sa preuve fait appel à la théorie de la régularité maximale [38] pour obtenir des estimation sur le problème dual. L'utilisation de cette théorie requiert l'uniforme continuité des coefficients de diffusion, ainsi que l'hypothèse $\nabla d_i, u_i \in L^\infty((0,T);L^r(\Omega)^N)$ pour $r>\max(2,N)$.

Pour expliquer le principe de la preuve, on se place dans le cas simplifié de deux équations

$$\left\{\begin{array}{lcl}\partial_t c_1 + \mathrm{div}[-d_1(t,x)\nabla c_1 + c_1 u_1(t,x)] & = & f_1(t,x,c) \\ \partial_t c_2 + \mathrm{div}[-d_2(t,x)\nabla c_2 + c_2 u_2(t,x)] & = & f_2(t,x,c)\end{array}\right. , \tag{13}$$

et on suppose

$$f_1 \leq 0\,;\; f_1 + f_2 \leq 0.$$

Comme $c_1 \geq 0$, $f_1 \ \leq 0$ et $u_1 \in L^\infty_{loc}([0,+\infty);L^r(\Omega)^N)$, on sait que c_1 est bornée dans $L^\infty(Q_T)$ pour tout $T > 0$. Alors, en utilisant $f_1 + f_2 \leq 0$, on a

$$\partial_t c_2 + \mathrm{div}(-d_2\nabla c_2 + c_2 u_2) \leq -[\partial_t c_1 + \mathrm{div}(-d_1\nabla c_1 + c_1 u_1)]$$

et on peut utiliser (i) pour obtenir des bornes dans $L^p(Q_T)$ sur c_2 pour tout $p < +\infty$. Comme f a une croissance au plus polynomiale, les deux équations dans (13) ont un terme source borné dans $L^p(Q_T)$ pour tout $p < +\infty$, et en utilisant un résultat classique de O. A. Ladyženskaja, V. A. Solonnikov et N. N Ural'ceva (voir [69], voir aussi la partie 5.5 p.151), (c_1, c_2) est borné dans $L^\infty(Q_T)^2$. Le critère d'existence globale de H. Amann [4] permet alors de conclure que les solutions maximales de (13) sont globales. Comme on ne suppose pas les u_i réguliers, on travaille d'abord sur un système où les données ont été régularisées.

Comme corollaire du théorème 6, on montre l'existence globale et l'unicité de solutions pour un ensemble de systèmes de réaction-diffusion-advection issus de la chimie. Si $c_1, \ldots, c_P$ sont les concentrations de P espèces chimiques $C_1, \ldots, C_P$, on suppose que R réactions de la forme

$$\alpha_j^1 C_1 + \ldots + \alpha_j^P C_P \rightleftharpoons C_{i_j}\;;\; j \in \{1, \ldots, R\} \tag{14}$$

ont lieu simultanément, où $\alpha_j^i \in \mathbb{N}$, $i_j \in \{1, \ldots, P\}$. En utilisant la loi d'action de masse, la vitesse de réaction pour la j-ième réaction est

$$r_j(c) = k_j^f \Pi_{k=1}^P c_k^{\alpha_j^k} - k_j^b c_{i_j},$$

où $k_j^f, k_j^b > 0$ sont les constantes de réaction. Si β_j est le i_j-ième vecteur de la base canonique de $\mathbb{R}^P$ et $\alpha_j = (\alpha_j^1, \ldots, \alpha_j^P)$, le vecteur stœchiométrique de la j-ième réaction est

$$\nu_j = \beta_j - \alpha_j.$$

Le terme source associé au réseau de réactions (14) s'écrit alors

$$(f_1, \ldots, f_P) = \sum_{j=1}^R r_j(c)\nu_j. \tag{15}$$

En réarrangeant convenablement les espèces chimiques et les réactions, on prouve alors qu'une telle réaction a la structure triangulaire (2), et comme conséquence du théorème 6, on a :

Corollaire 1. *Supposons* $(\mathbf{H}_2)$*, alors le système* (12) *avec la réaction* (15) *a une solution globale forte, et elle est unique.*

3.2 Limite de réaction rapide pour $C_1 + C_2 \rightleftharpoons C_3$ avec advection

Comme cas particulier du corollaire 1, il existe des solutions globales pour tout $k > 0$ pour le système

$$\left\{\begin{array}{rcll} \partial_t c_1 + \mathrm{div}[-d_1(t,x)\nabla c_1 + c_1 u(t,x)] &=& -k(c_1c_2 - c_3) & \\ \partial_t c_2 + \mathrm{div}[-d_2(t,x)\nabla c_2 + c_2 u(t,x)] &=& -k(c_1c_2 - c_3) & \text{sur } (0,+\infty)\times\Omega, \\ \partial_t c_3 + \mathrm{div}[-d_3(t,x)\nabla c_3 + c_3 u(t,x)] &=& +k(c_1c_2 - c_3) & \\ -d_i(t,x)\nabla c_i\cdot\nu + c_i u(t,x)\cdot\nu &=& 0 & \text{sur } (0,+\infty)\times\partial\Omega, \\ c_i(0,\cdot) &=& c_i^0 & \text{sur } \Omega,\ i\in\{1,2,3\}. \end{array}\right. \tag{16}$$

Dans ce modèle, les espèces C_1, C_2, C_3 sont dans un fluide dont le mouvement est décrit par le champ de vecteurs u. L'objet de cette partie est de déterminer si la présence de termes d'advection et de coefficients de diffusion variables est un obstacle à l'utilisation des idées du chapitre 1 pour passer à la limite de réaction rapide $k \to +\infty$ dans le système (16).

Le point crucial consiste à estimer la solution c^k de (16) dans $L^2((0,T)\times\Omega)^P$ indépendamment de k. Dans le chapitre 1, cette estimation découle de l'étude de $\partial_t(c_1^k + c_3^k)$ et $\partial_t(c_2^k + c_3^k)$. Cette fois, on a pour $i \in \{1,2\}$,

$$\left\{\begin{array}{rcll} \partial_t(c_i^k + c_3^k) + \mathrm{div}[-d_i\nabla c_i^k - d_3\nabla c_3^k + (c_i^k + c_3^k)u] &=& 0 & \text{sur } (0,+\infty)\times\Omega, \\ -[d_i\nabla c_i^k + d_3\nabla c_3^k]\cdot\nu + (c_i^k + c_3^k)u\cdot\nu &=& 0 & \text{sur } (0,+\infty)\times\partial\Omega, \\ (c_i^k + c_3^k)(0,\cdot) &=& c_i^0 + c_3^0 & \text{sur } \Omega. \end{array}\right.$$

On peut réécrire ces équations en posant $W_i^k = c_i^k + c_3^k$ pour $i \in \{1,2\}$:

$$\left\{\begin{array}{rll} \partial_t W_i^k + \mathrm{div}(-\nabla(A_i^k W_i^k) + W_i^k\tilde{u}) = 0 & \text{sur} & Q_T\ ; \\ -\nabla(A_i^k W_i^k)\cdot\nu + W_i^k\tilde{u}\cdot\nu = 0 & \text{sur} & \Sigma_T\ ; \\ W_i^k(0,\cdot) = W^0 & \text{sur} & \Omega, \end{array}\right. \tag{17}$$

où $0 < \underline{a} \le A_i^k \le \overline{a} < +\infty$ pour des constantes $\underline{a}, \overline{a}$ indépendantes de k, et où $\tilde{u}$ a la même régularité que u. À cause du nouveau terme $W^k\tilde{u}$, on doit ici utiliser une technique différente de celle du chapitre 1 pour obtenir des estimations dans L^2. Pour $\Theta \in C_c^\infty(Q_T, \mathbb{R}_+)$, on introduit le problème dual de (17)

$$-[\partial_t\Psi + A_i^k\Delta\Psi + \tilde{u}\cdot\nabla\Psi] = \Theta \text{ sur } Q_T\ ;\ \partial_\nu\Psi = 0 \text{ sur } \Sigma_T\ ;\ \Psi(T,\cdot) = 0 \text{ sur } \Omega.$$

Des méthodes classiques permettent alors d'estimer $\|\Psi(0)\|_{L^2(\Omega)}$ et $\|\Psi\|_{L^2(Q_T)}$ en fonction de $\|\Theta\|_{L^2(Q_T)}$, avec des constantes dépendant seulement de $\underline{a}$ et $\overline{a}$. On obtient ainsi des estimations sur W^k dans $L^2(Q_T)$ par dualité, independamment de k. Cependant, la méthode utilisée au chapitre 1 pour obtenir la compacité forte de c^k dans $L^2(Q_T)$ ne semble pas pouvoir s'étendre facilement à cette situation plus complexe. Ici, on prouve cependant la compacité de c^k dans $L^p(Q_T)^3$ pour $p \in [1,2)$. Avec des estimations similaires à celles du chapitre 1 pour contrôler les gradients, ceci est suffisant pour passer à la limite $k \to +\infty$, et la limite $c = (c_1, c_2, c_3)$ est une solution faible du problème

$$\left\{\begin{array}{ll} \left.\begin{array}{l} c_1c_2 = c_3 \\ \partial_t(c_1+c_3) - \mathrm{div}[-d_1c_1 - d_3c_3 + (c_1+c_3)u] = 0 \\ \partial_t(c_2+c_3) - \mathrm{div}[-d_2c_2 - d_3c_3 + (c_1+c_3)u] = 0 \end{array}\right\} & \text{sur } Q_T, \\ \partial_\nu(c_1+c_3) = \partial_\nu(c_2+c_3) = 0 & \text{sur } \Sigma_T, \\ (c_1+c_3)(0,\cdot) = c_1^0 + c_3^0\ ;\ (c_2+c_3)(0,\cdot) = c_2^0 + c_3^0 & \text{sur } \Omega. \end{array}\right. \tag{18}$$

Dans le cas où $\operatorname{div} u$ est dans $L^\infty(Q_T)$, il se trouve que les fonctions

$$c_i^k \log c_i^k + c_i^k + c_3^k \log c_3^k + c_3^k \; ; \; i \in \{1,2\}$$

sont solutions d'équations similaires à (17). Par conséquent, elle sont bornées dans $L^2(Q_T)$, ce qui garantit l'"uniforme intégrabilité" de c_i^k dans $L^2(Q_T)$, et permet de recouvrer la compacité forte des c^k dans $L^2(Q_T)^3$ avec un argument du type Vitali.

Pour résumer, si on suppose que les données de (16) satisfont aux mêmes hypothèses de régularité que celles du théorème 6, on a :

Théorème 7. *Soit $k_n \to +\infty$, soit c^n la solution globale de (16) correspondante. À une sous-suite près, c^n converge dans $L^p(Q_T)$ pour tout $p \in [1,2)$ et tout $T > 0$ vers une solution faible de (18). Si on suppose en plus $\operatorname{div} u \in L^\infty(Q_T)$ pour tout $T > 0$, alors la convergence de c^n a lieu dans $L^2(Q_T)$.*

3.3 Un système de diffusion-électromigration

Dans la dernière partie, on s'intéresse à l'existence globale de solutions pour le système de diffusion-électromigration suivant, sans restriction sur la dimension en espace :

$$\left\{ \begin{array}{rcll} \partial_t c_i - \operatorname{div}(d_i \nabla c_i + d_i z_i c_i \nabla \Phi) & = & 0 & \text{sur } (0,+\infty) \times \Omega, \\ \partial_\nu c_i + z_i c_i \partial_\nu \Phi & = & 0 & \text{sur } (0,+\infty) \times \partial\Omega, \quad i \in \{1,\dots,P\}, \\ -\Delta\Phi - \sum_{i=1}^P z_i c_i & = & 0 & \text{sur } (0,+\infty) \times \Omega, \\ \partial_\nu \Phi + \tau\Phi & = & \xi & \text{sur } (0,+\infty) \times \partial\Omega, \\ c(0,\cdot) & = & c^0 & \text{sur } \Omega. \end{array} \right. \tag{19}$$

Ce système décrit l'évolution en temps d'un électrolyte. L'inconnue est $(c_1,\dots,c_P,\Phi)$, où $c_1,\dots,c_P$ sont les concentrations de P espèces chimiques pouvant être chargées avec un nombre de charges z_i, et Φ est le potentiel électrique. La condition de bord pour Φ peut être motivée en considérant localement la frontière comme un condensateur plan : $\tau > 0$ représente alors sa capacité, et la fonction ξ est la donnée d'un potentiel extérieur.

Dans le cas de la dimension $N = 2$, l'existence globale, l'unicité et le comportement asymptotique de (19) sont déjà bien connus : dans [15] est prouvée l'existence et l'unicité de solutions globales faibles, ainsi que la convergence vers un état stationnaire unique. Pour des données initiales suffisamment régulières, il est montré dans [33] qu'il existe une unique solution globale classique. Ces résultats ont été améliorés dans [14], où sont calculées des vitesses de convergence explicites à l'aide d'inégalités de Sobolev logarithmiques. Dans les articles [51,55,56,57], les auteurs enrichissent le modèle en y rajoutant des termes de réaction issus de la cinétique chimique, et prouvent l'existence globale, l'unicité et la convergence exponentielle vers un état stationnaire. Le système (19) a aussi été couplé avec les équations de Navier-Stokes, cf. [24,36,95,96].

Jusqu'à présent, l'existence globale en dimension $N = 3$ n'a été montrée que sous des hypothèses supplémentaires, qui consistent par exemple à prendre des données initiales proches de l'état stationnaire, cf. [15], ou encore à supposer que la solution c est bornée dans $L^\infty(0,T;L^2(\Omega))$ indépendamment de $T > 0$, cf. [33]. Dans [64], l'existence de solutions globales faibles pour des coefficients de diffusion constants est montrée dans le cadre plus général des équations de Navier-Stokes-Nernst-Planck-Poisson, mais pour $P = 2$, ce qui fournit des estimations supplémentaires.

On montre ici l'existence de solutions globales dans le cas de coefficients de diffusions dépendant du temps et de la variable d'espace, sans restriction sur la dimension de l'espace ni sur le nombre d'espèces chimiques présentes. Notre preuve est basée sur une fonction de Lyapunov du système (19).

On suppose que les données satisfont

(i) Pour $i = 1,\dots,P,\ d_i \in L^\infty_{loc}([0,+\infty);L^\infty(\Omega))$.
Pour tout $T > 0$, il existe $\underline{d_i}(T), \overline{d_i}(T) > 0$ tels que

$$0 < \underline{d_i}(T) \le d_i \le \overline{d_i}(T) < +\infty \ \text{ sur } Q_T.$$

(ii) $c^0 \in L^\infty(\Omega,[0,+\infty)^P)$.

(iii) $\xi \in C^\infty(\partial\Omega)$ est une fonction indépendante du temps.

On peut résumer notre résultat comme suit :

Théorème 8. *Sous les hypothèses ci-dessus,* (19) *a une solution globale faible pour toute dimension d'espace.*

Pour prouver ce théorème, on commence par étudier une version approchée de (19) où la charge électrique totale $\sum_{i=1}^P z_i c_i$ est régularisée. On introduit les notations

$$\varepsilon > 0\ ;\ B_\varepsilon = I - \varepsilon\Delta\ ;\ m = 2N\ ;\ k \in \{0,\dots,m\},$$

et on considère

$$\left.\begin{array}{rcll} \partial_t c_i - \operatorname{div}(d_i \nabla c_i + d_i z_i c_i \nabla\Phi) & = & 0 & \text{sur } (0,+\infty)\times\Omega \\ \partial_\nu c_i + z_i c_i \partial_\nu \Phi & = & 0 & \text{sur } (0,+\infty)\times\partial\Omega \\ c_i(0) & = & c_i^0 & \text{sur } \Omega \end{array}\right\}, \tag{20}$$

$$\left.\begin{array}{rcll} B_\varepsilon^{m+1}\Psi - \sum_{i=1}^P z_i c_i & = & 0 & \text{sur } (0,+\infty)\times\Omega \\ \partial_\nu[B_\varepsilon^k \Psi] + \tau B_\varepsilon^k \Psi & = & 0 & \text{sur } (0,+\infty)\times\partial\Omega \end{array}\right\}, \tag{21}$$

$$\left.\begin{array}{rcll} -\Delta\Phi & = & \Psi & \text{sur } (0,+\infty)\times\Omega \\ \partial_\nu\Phi + \tau\Phi & = & \xi & \text{sur } (0,+\infty)\times\partial\Omega \end{array}\right\}. \tag{22}$$

On prouve l'existence et l'unicité de solutions à ce système en utilisant le théorème de point fixe de Leray-Schauder, où le théorème 6 est utilisé pour définir la fonction dont le point fixe est solution du système. L'intérêt principal de ce procédé d'approximation est qu'il préserve la structure "entropique" du système (19) : il existe encore une fonction de Lyapunov pour $(20)-(22)$. On parvient même à écrire explicitement la fonction de dissipation. Cela fournit des estimations indépendantes de ε, qu'on exploite pour obtenir la compacité des solutions approchées et passer à la limite $\varepsilon \to 0$.

Introduction

This thesis is devoted to the study of reaction-diffusion systems arising in population dynamics, chemistry and electromigration theory. We investigate global existence issues for strong and weak solutions, uniqueness, regularity, and study the fast reaction limit for systems from mass-action kinetics chemistry.

In this introduction, we first present the kind of evolution systems we are interested in. Next, we give the outline of this work and explain how the results will be presented in three different chapters. Finally, we describe in more detail the main results of each chapter.

Let us briefly recall how reaction-diffusion systems may be derived from mass conservation balances: assume we are studying a multicomponent system containing P extensive quantities $C_1,\ldots,C_P$ (that may represent populations, chemical reactants, ions...), whose densities are represented by a vector

$$c(t,x) = (c_1(t,x),\ldots,c_P(t,x)),\ t \geq 0,\ x \in \Omega,$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$. Let J_i denote the flux of species C_i and f_i denote its production rate density. For any smooth bounded subset A of Ω, mass conservation for C_i inside A reads

$$\frac{d}{dt}\int_A c_i + \int_{\partial A} J_i \cdot \nu = \int_A f_i\,, \quad i \in \{1,\ldots,P\},$$

where ν is the normal exterior derivative on the boundary ∂A of A. Using the Gauss-Green theorem,

$$\frac{d}{dt}\int_A c_i + \int_A \operatorname{div} J_i = \int_A f_i\,, \quad i \in \{1,\ldots,P\}.$$

Since A is arbitrary, we get the classical mass conservation equation

$$\partial_t c_i + \operatorname{div} J_i = f_i\,, \quad i \in \{1,\ldots,P\}.$$

Now we need to model the fluxes J_i and the production rates f_i with appropriate constitutive laws.

We will consider functions f_i of the type $f_i = f_i(t,x,c)$, the dependence in c being most often nonlinear.

Let us introduce the different fluxes that will be considered in this work.

Different types of fluxes

Fickian diffusion

When diffusion is the only driving force, a standard model was introduced by Fick in 1855 [49], which reads

$$J_i = -d_i(t,x,c)\nabla c_i,$$

where $d_i > 0$ due to the second law of thermodynamics [35]. In practice, we will only consider non-degenerate diffusion coefficients, *i.e.* coefficients which are bounded below by positive constants. Under these assumptions, mass conservation for each species leads to the so-called reaction-diffusion system

$$\left.\begin{array}{rcl} \partial_t c_1 - \mathrm{div}(d_1(t,x,c)\nabla c_1) & = & f_1(t,x,c), \\ & \vdots & \\ \partial_t c_P - \mathrm{div}(d_P(t,x,c)\nabla c_1) & = & f_P(t,x,c) \end{array}\right\}, \quad t \in (0,+\infty),\ x \in \Omega, \tag{1}$$

which is complemented with appropriate boundary conditions and nonnegative initial data. Local existence in time of strong solutions for such systems is well-known for regular enough initial data, but the existence of global solutions remains open in general, and cannot hold without adequate structure assumptions on the f_i. Before describing other types of diffusion, let us comment on the structure of these nonlinearities f_i.

First, we will always assume that the nonnegativity of the solutions c_i is guaranteed in the model. It is well known that the necessary and sufficient condition for this is to require that $f = (f_1,\dots,f_P)$ is quasi-positive, which means

$$(\mathbf{H}_1)\ \forall i \in \{1\dots,P\},\ f_i(t,x,c) \geq 0 \text{ for any } (t,x,c) \in (0,+\infty)\times\Omega\times[0,+\infty)^P \text{ such that } c_i = 0.$$

Next, to expect the existence of global solutions in time, more structure must be required on f. Additional assumptions usually come from the underlying model. For instance, the conservation of the total mass will correspond to the assumption that $\sum_{i=1}^P f_i = 0$. More generally, dissipation of mass will hold if

$$(\mathbf{H}_2)\ \sum_{i=1}^{P} f_i \leq 0.$$

One easily checks that assumptions $(\mathbf{H}_1)-(\mathbf{H}_2)$ with homogeneous Neumann boundary conditions imply that the solutions of (1) are uniformly bounded in $L^1(\Omega)$, since

$$\forall t > 0, \qquad \int_\Omega \sum_{i=1}^P c_i(t,x)dx \leq \int_\Omega \sum_{i=1}^P c_i(0,x)dx$$

and $\|c_i(t)\|_{L^1(\Omega)} = \int_\Omega c_i(t,x)dx$ due to the nonnegativity of c_i. Remark that in the homogeneous case, where functions c_i do not depend on x, they are solutions of the associated ODE system

$$\left.\begin{array}{rcl} \frac{d}{dt}c_1 & = & f_1(t,c), \\ & \vdots & \\ \frac{d}{dt}c_P & = & f_P(t,c) \end{array}\right\}, \quad t \in (0,+\infty).$$

For nonnegative initial data, the solutions remain nonnegative, and since $\sum_{i=1}^P c_i(t) \leq \sum_{i=1}^P c_i(0)$, they are uniformly bounded on the maximum time interval of existence. Therefore, existence of global solutions holds for this special case.

It is then natural to wonder if $(\mathbf{H}_1)-(\mathbf{H}_2)$ guarantee the existence of global strong solutions for the PDE system (1). A negative answer has been given in [91], where explicit solutions of a system of the type (1) with properties $(\mathbf{H}_1)-(\mathbf{H}_2)$ are constructed, and these solutions do blow-up in $L^\infty(\Omega)$ in finite time. In the latter example, the diffusion coefficients are constant, and the nonlinearities are polynomially bounded. This blow-up may even occur for space dimension $N=1$, provided the degree of the nonlinearities is high enough. This proves that when looking for global strong solutions, additional assumptions must be done on $(f_1,\ldots,f_P)$. There exists a wide literature on global existence issues for these systems, for various additional structural assumptions on $(f_1,\ldots,f_P)$, see *e.g.* [7,31,41,62,84,92,94]. For a recent survey on this issue, we refer to [90].

The existence of global weak solutions is less demanding. For instance, for constant diffusion coefficients and nonlinearities that are *a priori* bounded in $L^1((0,T)\times\Omega)$ for any $T>0$, global existence of weak solutions is proved in [90]. This result implies that if the growth of the f_i with respect to c is *at most quadratic*, then global existence of weak solutions holds under assumptions $(\mathbf{H}_1)-(\mathbf{H}_2)$. This strongly relies on an L^2-estimate that will be present and exploited all along this work: for instance, in the case of constant coefficients d_i, it says that under assumptions $(\mathbf{H}_1)-(\mathbf{H}_2)$, the solutions of (1) satisfy the *a priori* estimate

$$\forall T>0,\ \exists C=C(T,\|c(0)\|_{L^2(\Omega)^P},d_i)>0\ :\ \|c\|_{L^2((0,T)\times\Omega)^P}\le C.$$

Chapter 2 of the present work is devoted to the extension of the above recalled results to more general situations not yet covered in the literature. In particular, we prove the existence of

- global *strong* solutions for networks of elementary chemical reaction, for general nonlinear diffusion coefficients and for small (but $N\ge 3$) space dimensions.
- global *weak* solutions for systems whose nonlinearities have at most quadratic growth, with nonlinear diffusion coefficients of the type $d_i(c_i)$, and for initial data “only” in $L^1(\Omega)$.

Cross-diffusion

The fact that the driving forces for one species are independent of the gradients of the concentrations of the other species sometimes happens to be an oversimplification. Cross-diffusion, the phenomenon in which a gradient in the concentration of one species induces a flux of another species, has been suggested in a study of Onsager and Fuoss on electrolytes in the 1930s [88]. The presence of these cross effects was experimentally confirmed in 1955 by Gosting and Dunlop [44], and later by the classical experiment of Duncan and Toor in 1962 [43]. Cross diffusion has been widely investigated during the last decades: for a survey on its importance in physical chemistry, see [100].

We first consider population dynamics model, where cross-diffusion has originally been introduced to take into account friction phenomena that might lead to spatial segregation. In this situation, the fluxes have the form

$$J_i=\nabla(a_i(c_1,\ldots,c_P)c_i).$$

Global existence of solutions for reaction-diffusion systems with the above fluxes is open in general, even without reaction terms. In the present work, we consider fluxes

$$J_i=\nabla(a_i(\tilde c_1,\ldots,\tilde c_P)c_i),$$

where functions $\tilde{c}_i$ are regularized versions of c_i. In this case, we can prove global existence of solutions in any space dimension and for general positive continuous a_i. Uniqueness is also proved for locally Lipschitz continuous a_i. This will be done in Chapter 1.

In the same chapter, we indirectly address the global existence issue for another nonlinear cross-diffusion system, which arises in mass-action kinetics chemistry as an asymptotic limit of systems of type (1). It concerns the typical reversible reaction $C_1 + C_2 \rightleftharpoons C_3$, when the reaction speed k tends to infinity in the reaction rate $k(c_1c_2 - c_3)$. In the limit, the chemical reaction is locally in equilibrium, *i.e.* the relation $c_1c_2 = c_3$ holds. Then the limit system may be rewritten with $x_1 = c_1 + c_1c_2$, $x_2 = c_2 + c_1c_2$ as the main variables. The resulting fluxes for x_1 and x_2 are of the type

$$J_i = \nabla\Psi(x_1, x_2),$$

where Ψ is nonlinear, so that one is led to a nonlinear cross-diffusion system with respect to the new unknowns x_1, x_2. We rigorously prove the convergence as $k \to +\infty$ of the solutions with reaction speed k to a solution of this limit system. As a consequence, we prove in this way the existence of weak global solutions of the cross-diffusion system, while the general theory of H. Amann [2, 4] guarantees the existence of strong solutions, but only locally in time. This actually leads to interesting questions on uniqueness of weak solutions. We provide partial results in this direction.

Fickian diffusion with convection

Finally, we consider situations when diffusion is not the only phenomenon responsible for mass transport.

When considering a mixture whose velocity u is nonzero, taking also into account Fickian diffusion, the mass fluxes are of the type

$$J_i = -d_i\nabla c_i + c_i u \;\; ; \;\; i \in \{1, \ldots, P\}.$$

As a first step towards more complex models, we consider such fluxes where u is assumed to be a data of the problem. We investigate global existence of solutions for reaction-diffusion systems whose reaction terms have a "triangular" structure, *i.e.* for a system of type (1) with $f = (f_1, \ldots, f_P)$, we assume the existence of a lower triangular invertible matrix $Q = (q_{ij})_{1\le i,j\le P}$ with nonnegative diagonal entries, such that

$$\exists b \in (0, +\infty)^P : \forall (t, x, c) \in (0, +\infty) \times \Omega \times [0, +\infty)^P, \quad Q f(t, x, c) \le \Big(1 + \sum_{i=1}^{P} c_i\Big) b. \qquad (2)$$

In the situation when the mixture is an electrolyte and $c_1, \ldots, c_P$ are the concentrations of charged species, with charge number $z_i \in \mathbb{Z}$, the total charge density is $\sum_{i=1}^{P} z_i c_i$ and the electrical potential is the solution of the Poisson equation

$$-\Delta\Phi = \sum_{i=1}^{P} z_i c_i$$

with appropriate boundary data. Here, the physical parameters ε, F are set to 1, where F is the Faraday constant and ε the permittivity of the medium. Due to the presence of a nonzero electrical field $-\nabla\Phi$, the mass fluxes are of the type

$$J_i = -d_i\nabla c_i - d_i z_i c_i \nabla\Phi.$$

Global existence issues for the resulting so-called "diffusion-electromigration" systems are covered in the last section.

Let us now summarize how our contributions are organized.

Outline

This work is divided in three chapters.

In the first chapter are reproduced two already published papers which are collaborative works, within two extra subsections, namely subsections 1.6 and 2.4.4. Each of the two other chapters contains two papers that will be submitted for publication soon. Three of them are collaborative works.

- ◇ Chapter 1 is devoted to the study of two cross-diffusion systems, arising in population dynamics and mass-action kinetics chemistry.

  The first model we investigate is a relaxed cross-diffusion system which was originally introduced in [11] to prove that cross-diffusion systems without reaction may lead to spatial segregation. In the latter work, the authors investigate operators of the type

  $$u = (u_1, \ldots, u_P) \mapsto (-\Delta(a_1(\tilde{u})u_1), \ldots, -\Delta(a_P(\tilde{u})u_P)),$$

  where $\tilde{u}$ is a regularized version of u. Global existence of strong solutions was proved in space dimension 2 and for functions a_i with polynomial growth. In Chapter 1, we prove existence of global classical solutions for this model in any space dimension and for functions a_i that are only assumed to be continuous and positive. If moreover the a_i are locally Lipschitz continuous, we prove that uniqueness holds.

  The second model comes from mass-action kinetics chemistry: when studying the fast reaction limit in the reversible reaction $C_1 + C_2 \rightleftharpoons C_3$ in the presence of Fickian diffusion, as explained above, the limit system is a nonlinear cross-diffusion system. In Section 2, we prove that the solution of the system with finite reaction speed k converges when $k \to +\infty$ to a weak global solution of this cross-diffusion system. Under some restrictions on the diffusion coefficients, we prove that weak solutions are unique. This result extends earlier works of D. Bothe [18] to the case of different (but constant) diffusion coefficients.

- ◇ Considering that the techniques developed to investigate the fast reaction limit in the reaction $C_1 + C_2 \rightleftharpoons C_3$ are rather robust, we have been interested in generalizing the results of Chapter 1 to more realistic models. In particular, the following situations are more relevant from the point of view of chemical engineering:

  - The diffusion coefficients depend on time, space and on the concentrations ($d_i = d_i(t,x,c)$).
  - The initial data are in $L^1(\Omega)$ "only".
  - Other slow chemical reactions occur at the same time as the fast reaction $C_1 + C_2 \rightleftharpoons C_3$.

  Chapter 2 contains several global existence results covering these situations. In particular

  - We prove existence of global strong solutions for systems of type (1) with various structural assumptions on f for small space dimensions ($N \leq 5$ for $d_i = d_i(t,x,c)$, $N \leq 9$ for $d_i = d_i(c_i)$). We emphasize the particular case of the chemical reaction $C_1 + C_2 \rightleftharpoons C_3$, but we also consider networks of reactions of the type

    $$C_i + C_j \rightleftharpoons C_k.$$

– We prove global existence of weak solutions for more general source terms which are only assumed to have at most a quadratic growth. This result includes the case of initial data in $L^1(\Omega)$. This framework is the most natural from a modeling point of view, but it is mathematically more difficult since the control of the solutions in a neighborhood of $t = 0$ requires delicate estimates.

◇ In the works of the previous chapters, we considered that diffusion was the only phenomenon responsible for mass transport. We now take into account other driving forces, that will be either advection when the fluid's motion is nonzero, or electromigration when the fluid is an electrolyte and species $c_1, \dots, c_P$ might be ions.

Assuming that the vector field u describing the fluid's motion is a given data, we generalize a global well-posedness result of M. Pierre [90] for reaction-diffusion systems whose reaction has the "triangular" structure (2). This essentially means that we consider reaction terms where $f_1, f_1 + f_2, \dots, f_1 + \dots + f_P$ are bounded above by a linear function of $c_1, \dots, c_P$. The proof of [90] is generalized to the case of fluxes with advection terms and Fickian diffusion coefficients depending on time and space.

As a particular case of the previous results, we get global existence for the reaction-diffusion-advection system associated with the chemical reaction $C_1 + C_2 \rightleftharpoons C_3$, independently of the reaction speed. Consequently, we can investigate once more the fast-reaction limit. We prove that the techniques developed in Chapter 1 are robust enough to carry over to variable diffusion coefficients and advection.

In the last part of Chapter 3, we investigate the existence of global weak solutions for a diffusion-electromigration system, *in any space dimension*. Using an approximation procedure which respects the "entropic structure" of the initial problem, we prove the existence of global solutions. The results of the first section of this chapter, where advection is prescribed, are used in a Leray-Schauder's fixed point argument to derive the existence of solutions for the approximate system.

We will now explain in more details the results obtained in each chapter.

1 Two cross-diffusion systems

1.1 Global well-posedness of a conservative relaxed cross diffusion system

Except for Subsection 1.6, this section is a joint work with T. Lepoutre and M. Pierre, published in [72].

Cross diffusion models have been used in population dynamics by Shigesada Kawasaki and Teramoto [97] to describe the interaction between species not only through reaction, but also through motion. The original aim of the introduction of nonlinear dispersive forces in the models was to describe pattern formation between competitive species.

A general system reads, in the simplified case of two populations,

$$\begin{cases} \partial_t u_1 - \Delta[u_1(d_1 + d_{11}u_1^p + d_{12}u_2^p)] &= f_1(u_1,u_2) \quad \text{on } (0,+\infty)\times\Omega, \\ \partial_t u_2 - \Delta[u_2(d_2 + d_{21}u_1^p + d_{22}u_2^p)] &= f_2(u_1,u_2) \quad \text{on } (0,+\infty)\times\Omega, \\ \partial_\nu[u_i(d_i + d_{i1}u_1^p + d_{i2}u_2^p)] &= 0 \qquad\qquad\;\; \text{on } (0,+\infty)\times\partial\Omega. \end{cases} \tag{3}$$

For populations sharing limited resources, the terms $\Delta[u_i(d_{i1}u_1^p + d_{i2}u_2^p)]$ model social friction and competition. In the case of predator-prey systems, these terms may take into account the fact that predators tend to move towards higher concentrations of prey, whereas prey move towards regions where predators are rare.

For system (3) with $p = 1$ and Lotka-Volterra-type reaction, there exists a wide literature, studying specific cases where an additional structure keeps the system parabolic, or with cross diffusion pressure on only one of the species (see e.g. Wang [103] and the many references therein). The most general result on global weak solutions might be found in Chen and Jüngel [32], where the entropy structure of the model is used. For existence of classical solutions the reader might consult for instance [75, 103] by Wang and Li-Zhao. In population dynamics, one of the most interesting features of cross diffusion is its effect on steady states: cross diffusion pressure might yield the appearance of nonconstant steady states when the reaction structure does not drive to segregation (see Iida-Mimura-Ninomyia [63] for instance). However, in these cases, the pattern formation relies on the reaction terms (for instance, the convergence to homogeneous steady states in the absence of reaction is proved in [32]).

To prove that nonlinear dispersive forces can drive spatial segregation and create patterns without any additional reaction terms, a relaxed conservative nonlocal cross-diffusion system was introduced in [11], replacing

$$\begin{cases} \partial_t u_i - \Delta[a_i(u)u_i] = 0 \text{ on } (0,+\infty)\times\Omega, \\ \partial_\nu [a_i(u)u_i] = 0 \text{ on } (0,+\infty)\times\partial\Omega, \\ u = (u_1,\dots,u_I)\ ;\ u(0,\cdot) = u^0 \text{ given }, \end{cases}$$

where $a_i : [0,\infty)^I \to [\underline{a},\infty)$ for some $\underline{a} > 0$, by the following relaxed model:

$$\begin{cases} \partial_t u_i - \Delta[a_i(\tilde{u})u_i] = 0, \text{ on } (0,+\infty)\times\Omega, \\ \tilde{u}_i - \delta_i\Delta\tilde{u}_i = u_i, \text{ on } (0,+\infty)\times\Omega, \\ \partial_\nu u_i = \partial_\nu \tilde{u}_i = 0 \text{ on } (0,+\infty)\times\partial\Omega, \\ u = (u_1,\dots u_I)\ ;\ \delta_i > 0,\ u(0,\cdot) = u^0 \text{ given.} \end{cases} \tag{4}$$

The effects of the relaxation on the stability of the homogeneous equilibria is investigated in [11, 70, 71]. Remark that this nonlocal model takes into account the fact that the individuals measure the densities of all the other species in a neighborhood of their position, with a characteristic spatial length δ_i. This might be more relevant in view of concrete applications to population dynamics. Models with nonlocal diffusion coefficients can also be seen in [12], where diffusion operators of the type $a_i(\int_\Omega u_i)\Delta u_i$ are studied.

A first well-posedness result for the relaxed system (4) was derived in [11, 70] for space dimensions $N = 1,2$ and with some restrictions on the structure of the nonlinearities a_i: basically, the a_i are C^2 and have at most a polynomial growth in u. In this first section, we consider the "integrated-in-time" version of (4)

$$\begin{cases} u_i - \Delta\int_0^t [a_i(\tilde{u})u_i] &= u_i^0 \quad \text{on } (0,+\infty)\times\Omega, \\ \tilde{u}_i - \delta_i\Delta\tilde{u}_i &= u_i \quad \text{on } (0,+\infty)\times\Omega, \\ \partial_\nu u_i = \partial_\nu\tilde{u}_i &= 0 \quad \text{on } (0,+\infty)\times\partial\Omega. \end{cases} \tag{5}$$

We prove the following result, where by "strong" solution, we mean a solution for which each derivative involved in the PDE is in some L^p space, and where the boundary and the initial data are satisfied in a pointwise sense:

Theorem 1.1. *Assume that a_i are continuous and bounded from below by a positive constant. Then system* (5) *has a global nonnegative strong solution. If moreover the a_i are assumed to be locally Lipschitz continuous, this solution is unique and it is actually a strong solution of* (4).

We first prove the existence of weak solutions, using some L^2-estimates in the spirit of [90]. A main point is that $\tilde{u}$ is proved to be uniformly bounded for these weak solutions, and this is valid in any space dimension. Next, one has to deal with parabolic operators $u_i \to \partial_t u_i - \Delta(a_i(\tilde{u})u_i)$. They are not of divergence form, but they are uniformly parabolic since $a_i(\tilde{u})$ is then bounded from above and below. In the spirit of Krylov-Safonov [42, 67], using the C^α-theory for the duals of these operators, namely $U_i \to \partial_t U_i - a_i(\tilde{u})\Delta U_i$, we prove that $\tilde{u}$ is even Hölder-continuous. This proves that the coefficients $a_i(\tilde{u})$ of the above operators are regular. Then L^p-estimates classically follow for the solution. When the a_i are locally Lipschitz continuous, $\partial_t u_i$ and $\Delta(a_i(\tilde{u})u_i)$ are proved to be in some L^p-spaces, so that the solution is strong. Moreover, weak solutions are then proved to be unique.

1.2 Cross-diffusion limit for a reaction-diffusion system with fast reversible reaction

Except for Subsection 2.4.4, the results of this section will appear in [28] in a joint work with D. Bothe and M. Pierre.

The second section of this chapter is devoted to the study of the fast-reaction limit in a classical model for the chemical reaction

$$C_1 + C_2 \rightleftharpoons C_3\,, \tag{6}$$

where in addition to the reaction, Fickian diffusive fluxes are taken into account, with constant but possibly different coefficients. More precisely, we assume that the reaction mechanism is modeled with mass-action kinetics (see [46] for more details on chemical reaction mechanisms). The reactants are placed in a bounded isolated domain, represented by Ω. If c_i denotes the concentration of species C_i, we are led to the system

$$(R^k)\begin{cases} \left.\begin{aligned} \partial_t c_1 - d_1\Delta c_1 &= -k(c_1c_2 - \kappa c_3)\\ \partial_t c_2 - d_2\Delta c_2 &= -k(c_1c_2 - \kappa c_3)\\ \partial_t c_3 - d_3\Delta c_3 &= +k(c_1c_2 - \kappa c_3)\end{aligned}\right\} & \text{on } (0,+\infty)\times\Omega,\\ \partial_\nu c_1 = \partial_\nu c_2 = \partial_\nu c_3 = 0 & \text{on } (0,+\infty)\times\partial\Omega,\\ c_1(0,\cdot) = c_1^0,\ c_2(0,\cdot) = c_2^0,\ c_3^0(0,\cdot) = c_3^0 & \text{on } \Omega,\end{cases}$$

where $k > 0$ is the reaction speed and $\kappa > 0$ is the so-called equilibrium constant. For finite k, global existence and uniqueness of a strong nonnegative solution c^k for (R^k) is known, for initial data in $L^\infty(\Omega)^3_+$ and for any space dimension. This is, for instance, a special case in the global wellposedness result of M. Pierre [90] for systems with the "triangular" structure (2).

The study of the behaviour of the solution c^k of (R^k) in the limit $k \to +\infty$ may be motivated by a non-dimensional analysis, which reveals the presence of two different time scales:

- Diffusion in liquids, or especially in solids, is a relatively slow process. For example, even in an actively mixed aqueous system, typical times scales for diffusion are

$$\tau_{\text{diff}} \geq 10^{-3}s.$$

In systems without agitation, it will be several magnitudes larger.

- Chemical transformations can be extremely fast, depending on the reaction mechanism. For instance, in case of the neutralization $H^+ + OH^- \rightleftharpoons H_2O$, the forward reaction can have a time scale as small as

$$\tau^f_{\mathrm{reac}} \simeq 10^{-11} s.$$

Now comes the question of writing a limit system for (R^k) when $k \to +\infty$.We will prove that there exists a distribution f such that

$$k(c_1^k c_2^k - \kappa c_3^k) \overset{k\to+\infty}{\longrightarrow} f\,.$$

Consequently, it is reasonable to expect that in the limit $k \to +\infty$, the chemical composition c will remain on the manifold $\{c_1 c_2 = \kappa c_3\}$ on which the reaction is in equilibrium. Another important point is that the reaction terms cancel when considering the sums $c_1^k + c_3^k$ and $c_2^k + c_3^k$.

Our main result is the following:

Theorem 1.2. *Let $k_n \to +\infty$ and c^n be the corresponding solution of (R^{k_n}). Up to a subsequence and for any $T > 0$, c^n converges strongly in $L^2(Q_T)$ and weakly in $L^{4/3}(0,T;W^{1,4/3}(\Omega))$ to a weak solution of*

$$(R^\infty)\begin{cases} \left.\begin{array}{l} \partial_t(c_1+c_3) - \Delta(d_1c_1 + d_3c_3) = 0 \\ \partial_t(c_2+c_3) - \Delta(d_2c_2 + d_3c_3) = 0 \\ c_1c_2 = \kappa c_3 \end{array}\right\} & \text{on } (0,+\infty)\times\Omega, \\ \partial_\nu(d_1c_1 + d_3c_3) = \partial_\nu(d_2c_2 + d_3c_3) = 0 & \text{on } (0,+\infty)\times\partial\Omega, \\ (c_1+c_3)(0,\cdot) = c_1^0 + c_3^0;\ (c_2+c_3)(0,\cdot) = c_2^0 + c_3^0 & \text{on } \Omega. \end{cases}$$

The convergence of c^k when $k \to +\infty$ to a solution of the limit system has been proven for equal diffusion coefficients in [18]. The latter situation is much more simple since $c_1^k + c_3^k$ and $c_2^k + c_3^k$ are solution of the heat equation, which provides uniform-in-time *a priori* bounds independent of k in $L^\infty(\Omega)$ by the maximum principle. For different diffusion coefficients, these bounds are no longer valid and one can only use estimates in $L^2((0,T)\times\Omega)$ for any $T > 0$ in the spirit of [90], that remain valid for $c_1^k + c_3^k$ and $c_2^k + c_3^k$. The other main ingredient to derive the relative compactness of c^k is a Lyapunov function, commonly refered to as "entropy estimate", which provides a control on the gradients and crucial arguments for the proof of the pointwise convergence of c^k.

We may rewrite (R^∞) as a 2×2 cross-diffusion system as follows: using the algebraic relation $c_1c_2 = \kappa c_3$, we introduce the new unknown functions

$$x_1(c_1,c_2) = c_1 + \kappa c_1 c_2\ ;\ x_2(c_1,c_2) = c_2 + \kappa c_1 c_2\,.$$

Basic computations (relying on the nonnegativity of c_1 and c_2) yield $(c_1,c_2) = (\varphi(x_1,x_2), \overline{\varphi}(x_1,x_2))$, where

$$\varphi(\alpha,\beta) = \frac{1}{2}\sqrt{\kappa^2 + (\alpha-\beta)^2 + 2\kappa(\alpha+\beta)} - (\kappa + \beta - \alpha)\ ;\ \overline{\varphi}(\alpha,\beta) = \varphi(\beta,\alpha).$$

As a consequence, (R^∞) is equivalent to

$$(\tilde{R}^\infty)\begin{cases} \left.\begin{array}{r} \partial_t x_1 - \Delta\psi_1(x_1,x_2) = 0 \\ \partial_t x_2 - \Delta\psi_2(x_1,x_2) = 0 \end{array}\right\} & \text{on } (0,+\infty)\times\Omega, \\ \partial_\nu\ \psi_1(x_1,x_2)\ =\ \partial_\nu\ \psi_2(x_1,x_2)\ =\ 0 & \text{on } (0,+\infty)\times\partial\Omega, \\ x_1(0,\cdot) = x_1^0,\ x_2(0,\cdot) = x_2^0 & \text{on } \Omega, \end{cases}$$

where $\psi_1 = d_1\varphi + d_3\kappa\varphi\overline{\varphi}$, $\psi_2 = d_2\overline{\varphi} + d_3\kappa\varphi\overline{\varphi}$.

Simple analysis indicates that the underlying operators in $(\tilde{R}^\infty)$ are "normally elliptic". This allows to apply H. Amann's results [2, 4]: for regular enough initial data, $(\tilde{R}^\infty)$ has a unique classical nonnegative solution on a maximal time interval $[0,T^*)$, $T^* \leq +\infty$. Global existence of classical solutions would follow from uniform-in-time estimates in an appropriate Sobolev space. However, the existence of these bounds remains an open problem.

Having Theorem 1.2 at hand, the following questions arise naturally:

– Does our solution coincide with Amann's classical solution? This is a uniqueness question for weak solutions.
– Our weak solutions are global in time, whereas Amann's solution is proved to exist only on some interval $[0,T^*)$, where T^* may be finite. Can it happen that weak solutions are regular for some time, but become singular after some finite time?

We provide partial answers to the first question. Despite our solutions are rather weak, we are able to prove that they are unique provided (d_1,d_2,d_3) satisfies the condition

$$\left(\frac{d_1}{d_3}-1\right)^2\left(\frac{d_2}{d_3}-1\right)^2 < 16\frac{d_1d_2}{d_3^2}\,.$$

In this case and for smooth initial data, our solution coincides with Amann's solution on its maximum time interval of existence.

We also prove that if $|d_1-d_2|$ belongs to some small interval depending on the $L^\infty((0,T)\times\Omega)$-norm of the regular solution, then our weak solution coincides with the regular one on $[0,T]$. But this does not say anything about uniqueness of weak global solutions for large time.

Since our approach in Theorem 1.2 is rather robust, it may be applied to pass to the fast-reaction limit in much more general systems than (R^k). It happens that the main difficulty to deal with more complex chemical systems is to know the existence of global solutions for systems with finite reaction speed. This is the reason why Chapter 2 is devoted to global existence issues.

2 Global existence for some systems with nonlinear diffusions

2.1 Global existence for a class of reaction-diffusion systems with mass action kinetics and concentration-dependent diffusivities

The results of this section will appear in [29] in a joint work with D. Bothe.

When modeling mass fluxes with Fick's diffusion law $J_i = -d_i\nabla c_i$, the d_i are functions of the system's thermodynamic state variables. In particular, they may depend on time, space and on the mixture composition.

As a simple example, we may consider the chemical system of the previous chapter, but with nonconstant diffusion coefficients:

$$\left\{\begin{array}{ll} \left.\begin{array}{l} \partial_t c_1 - \mathrm{div}(d_1(t,x,c)\nabla c_1) = -c_1c_2 + c_3 \\ \partial_t c_2 - \mathrm{div}(d_2(t,x,c)\nabla c_2) = -c_1c_2 + c_3 \\ \partial_t c_3 - \mathrm{div}(d_3(t,x,c)\nabla c_3) = +c_1c_2 - c_3 \end{array}\right\} & \text{on } (0,+\infty)\times\Omega, \\ \partial_\nu c_1 = \partial_\nu c_2 = \partial_\nu c_3 = 0 & \text{on } (0,+\infty)\times\partial\Omega, \\ c(0,\cdot) = (c_1^0,c_2^0,c_3^0) & \text{on } \Omega. \end{array}\right. \tag{7}$$

We assume the coefficients d_i to satisfy $\underline{d} \leq d_i$ for some $\underline{d} > 0$, and one of the following properties:

(a) $d_i \in C^2([0,+\infty) \times \Omega \times \mathbb{R}^3, \mathbb{R}^+)$, for the case $d_i = d_i(t,x,c)$.

(b) $d_i \in C^2(\mathbb{R}, \mathbb{R}^+)$, for the case $d_i = d_i(c_i)$.

Global existence of strong solutions for (7) is known for *constant* diffusivities d_i. In that case, it was shown in [94] that for bounded initial data and space dimension $N \leq 5$, (7) has a unique global nonnegative strong solution, which is uniformly bounded. Global existence and boundedness in any space dimension for smooth Ω (namely, Ω is of class $C^{2+\alpha}$ for some $\alpha \in (0,1)$) and smooth initial data has later been shown in [48]. Both these approaches are based on semigroup theory and hence exploit the semilinear structure. This prototype system also has the particular "triangular" structure (2) for which global existence of strong solutions is proved in [90] for any space dimension and bounded initial data. This approach uses Maximal Regularity theory [37] on the dual equations, and strongly relies on the linearity of the diffusion operators.

For general variable diffusion coefficients, the question of the existence of global classical solutions is widely open. The only closely related work we are aware of is [84], where the case of fluxes of the type $d_i(c_i)\nabla c_i$ is investigated, together with reaction networks satisfying an appropriate "quadratic triangular structure". Global existence is obtained in case of space dimension $N = 2$.

In the present work, we rely on H. Amann's theory for the existence of a classical solution to (7) on a maximum time interval $[0,T^*)$, $0 < T^* \leq +\infty$. Then we prove that this solution is uniformly bounded in $L^\infty(\Omega)$ on any compact time interval $[0,T]$, $T \leq T^*$, and use Amann's global existence criterion to deduce $T^* = +\infty$, *i.e.* the maximal solution is global. Our method relies on classical bootstrap estimates and may be summarized as follows: given an initial estimate of the solution in some L^p-space, since the reaction terms for c_1 and c_2 are linearly bounded above, we may improve the exponent p for c_1 and c_2, the new exponent depending on the space dimension. This provides a new estimate on c_1c_2, and since the reaction term for c_3 is bounded above by c_1c_2, a new estimate on c_3, and so on. For sufficiently small space dimensions, this procedure can be bootstraped to get bounds in $L^\infty((0,T) \times \Omega)$ on the solution for any $T > 0$, whence global existence.

The reason why we consider two different assumptions on the diffusivities is the following: for diffusivities satisfying (i), the only available initial estimate to start the bootstrap procedure is in $L^\infty(0,T;L^1(\Omega))$, which corresponds to the conservation of the total mass. For the more restrictive case of diffusivities (ii), an estimate in $L^2(Q_T)$ is available, and it allows to make the bootstrap procedure work for higher space dimensions.

For the reaction $C_1 + C_2 \rightleftharpoons C_3$, our main result reads:

Theorem 2.1. *For sufficiently smooth initial data, system* (7) *has a unique global strong solution provided one of the following conditions is satisfied:*

(i) $N \leq 5$ *and the diffusivities* $d_i(t,x,c)$ *satisfy* (a).

(ii) $N \leq 9$ *and the diffusivities* $d_i(c_i)$ *satisfy* (b).

This theorem is then generalized to the case of P chemical species $C_1,\dots,C_P$, whose concentrations are $c_1,\dots,c_P$, involved in a network of R chemical reactions of the type

$$C_{j_1}+C_{j_2} \rightleftharpoons C_{j_3}\ ; \quad j\in\{1,\dots,R\}\ ;\ j_1,j_2,j_3\in\{1,\dots,P\}.$$

On the basis of mass action kinetics, the reaction rate for the j^{th} reaction is given by

$$r_j(c)=c_{j_1}c_{j_2}-c_{j_3},$$

where for clarity reasons, we omitted the forward and backward rate constants. Let $(\varepsilon_1,\dots,\varepsilon_P)$ be the canonical basis of $\mathbb{R}^P$, we define the so-called *stoichiometric vectors* as $\alpha_j:=\varepsilon_{j_1}+\varepsilon_{j_2}$, $\beta_j:=\varepsilon_{j_3}$ and $\nu_j:=\beta_j-\alpha_j$. Using the above notations, the creation rate of $c=(c_1,\dots,c_P)$ reads

$$f(c):=\begin{pmatrix} f_1(c)\\ \vdots\\ f_P(c)\end{pmatrix}=\left(\begin{array}{c|c|c} \nu_1^1 & & \nu_R^1\\ \vdots & \dots & \vdots\\ \nu_1^P & & \nu_R^P\end{array}\right)\begin{pmatrix} r_1(c)\\ \vdots\\ r_R(c)\end{pmatrix}. \tag{8}$$

Assuming the same diffusion laws as above, the time-evolution of $c=(c_1,\dots,c_P)$ is now governed by the equations

$$\left\{\begin{array}{ll} \begin{pmatrix} \partial_t c_1- & \mathrm{div}\,(d_1(t,x,c)\nabla c_1)\\ & \vdots\\ \partial_t c_P- & \mathrm{div}\,(d_P(t,x,c)\nabla c_P)\end{pmatrix}=\begin{pmatrix} f_1(c)\\ \vdots\\ f_P(c)\end{pmatrix} & \text{on } (0,+\infty)\times\Omega,\\ \partial_\nu c=0 & \text{on } (0,+\infty)\times\partial\Omega,\\ c(0,\cdot)=c^0 & \text{on } \Omega.\end{array}\right. \tag{9}$$

We assume the conservation of the number of atoms, which holds in real chemistry and provides a uniform control on the total mass. After rearranging the chemical reactions and species, and using a similar bootstrap procedure as for Theorem 2.1, we prove the following:

Theorem 2.2. *For sufficiently smooth initial data and assuming the conservation of atoms, system* (9) *has a unique global solution provided one of the following conditions is satisfied:*

(i) $N\le 3$ *and the diffusivities* $d_i(t,x,c)$ *satisfy* (a).

(ii) $N\le 5$ *and the diffusivities* $d_i(c_i)$ *satisfy* (b).

Finally, remark that our techniques strongly rely on the upper linear bound on the reaction term for c_1 and c_2, and cannot be applied to the more complex case of the chemical reaction

$$C_1+C_2\rightleftharpoons C_3+C_4,$$

whose corresponding reaction-diffusion system is

$$\left.\begin{array}{l} \partial_t c_1-\mathrm{div}(d_1(t,x,c)\nabla c_1)=-c_1c_2+c_3c_4\\ \partial_t c_2-\mathrm{div}(d_2(t,x,c)\nabla c_2)=-c_1c_2+c_3c_4\\ \partial_t c_3-\mathrm{div}(d_3(t,x,c)\nabla c_3)=+c_1c_2-c_3c_4\\ \partial_t c_4-\mathrm{div}(d_4(t,x,c)\nabla c_4)=+c_1c_2-c_3c_4\end{array}\right\},\quad (t,x)\in(0,+\infty)\times\Omega. \tag{10}$$

Even for constant diffusivities, global existence of classical solutions for (10) is an open problem for space dimensions $N\ge 3$. The Hausdorff dimension of the set of possible singular points has been estimated in [58], where global existence is also derived for $N=2$. Maximal Regularity theory has also been succesfully applied for $N=2$ by J. Prüss [92] to get global strong solutions. This result has been extended to the case of variable diffusion coefficients of the type $d_i(c_i)$ in [84].

2.2 Global weak solutions with nonlinear diffusions, quadratic reactions and L^1 initial data

The contribution of this section is the content of the article [93].

In this section, we prove the existence of global weak solutions for systems of the type

$$\begin{cases} \partial_t c_i - \operatorname{div}(d_i(c_i)\nabla c_i) = f_i(t,x,c) & \text{on } (0,+\infty)\times\Omega, \quad i\in\{1,\dots,P\}, \\ d_i(c_i)\partial_\nu c_i = g_i & \text{on } (0,+\infty)\times\partial\Omega, \quad i\in\{1,\dots,P\}, \\ c(0,\cdot) = c^0 & \text{on } \Omega. \end{cases} \tag{11}$$

We will successively study two different assumptions for the initial data: $c^0 \in L^2(\Omega,\mathbb{R}_+^P)$ and $c^0 \in L^1(\Omega,\mathbb{R}_+^P)$. In addition to this, our main requirements are:

(i) The functions f_i have at most a quadratic growth with respect to c.

(ii) $(f_1,\dots,f_P)$ is quasi-positive (see $(\mathbf{H}_2)$).

(iii) $\exists \underline{d},\overline{d}>0$ such that $\underline{d}\le d_i \le \overline{d}$.

As mentioned above, local existence of strong, nonnegative solutions for system (11) for smooth initial and boundary data is well known, but the question of global existence of solutions is open in general, even for weak solutions. Although natural from the modeling point of view, few results for L^1-initial data are available. Amongst them, in [90], M. Pierre investigated this situation and proved global existence for systems whose nonlinearities are *a priori* bounded in $L^1(Q_T)$. This is the case for instance for nonlinearities with the "triangular" structure (2). If in addition the functions f_i have a polynomial growth, it is shown in [17] that the solutions are classical on $(0,+\infty)\times\Omega$. If moreover the f_i are bounded by a polynomial expression of degree $p<\frac{N+2}{N}$, where N is the space dimension, existence of solutions with Radon measure initial data is also proved.

In the subsequent study, the main difference with the situation investigated in [90] is that when $c^0 \in L^1(\Omega,\mathbb{R}_+^P)$, we do not control the reaction terms in L^1 up to $t=0$.

Theorem 2.3. *Under assumptions $(i)-(iii)$, system (11) has a weak global nonnegative solution $c:(0,+\infty)\times\Omega\to\mathbb{R}_+^P$ such that:*

(i) *If $c^0=(c_1^0,\dots,c_P^0)\in L^2(\Omega,\mathbb{R}_+^P)$, c satisfies a variational formulation of (11) on $(0,T)\times\Omega$ for any $T>0$.*

(ii) *If $c^0=(c_1^0,\dots,c_P^0)\in L^1(\Omega,\mathbb{R}_+^P)$, c satisfies a variational formulation of (11) on $(\tau,T)\times\Omega$ for any $0<\tau<T<+\infty$ and $c(t)\underset{t\to0}{\longrightarrow} c^0$ in the sense of Radon measures.*

For initial data in $L^2(\Omega)^P$, the core argument of the proof is a dimension-independent L^2-estimate. Together with the quadratic growth assumption on f_i, the reaction terms are controlled in $L^1((0,T)\times\Omega)$, and we can use classical results on parabolic equations. When considering initial data in $L^1(\Omega)^P$, the main difficulty is that the previous L^2-estimate is no longer valid up to $t=0$. Instead, we have to combine the L^2-techniques with the regularizing properties of the Laplacian to control the solution in $L^2((\tau,T)\times\Omega)$ for any $\tau\in(0,T)$. The reaction terms are not controlled any more in L^1 up to $t=0$. To get round this difficulty, we use a two-sided approach (inspired from [41,90]) to estimate the solutions in a neighborhood of $t=0$ from above and below, and prove the convergence of $c(t)$ to c^0 in the space of Radon measures.

3 Reaction-diffusion systems with advection-migration

The results of this chapter will appear in the articles [22] and [23], in a joint work with D. Bothe, A. Fischer and M. Pierre.

3.1 Global wellposedness for reaction-diffusion-advection systems with a "triangular" reaction

We consider the system

$$\left\{\begin{array}{rcll} \partial_t c_i + \mathrm{div}[-d_i(t,x)\nabla c_i + c_i u_i(t,x)] &=& f_i(t,x,c) & \text{on } (0,+\infty)\times\Omega, \\ -d_i(t,x)\nabla c_i\cdot\nu + c_i u_i(t,x)\cdot\nu &=& 0 & \text{on } (0,+\infty)\times\partial\Omega, \\ c_i(0,\cdot) &=& c_i^0 & \text{on } \Omega, \end{array}\right. \tag{12}$$

where $i\in\{1,\dots,P\}$ and whose unknown is $c=(c_1,\dots,c_P)$. We assume the initial data $c^0=(c_1^0,\dots,c_P^0)$ to be in $L^\infty(\Omega,\mathbb{R}_+^P)$, the reaction terms to be regular, quasi-positive, and with the triangular structure (2). Finally, we assume that the functions d_i are continuous, bounded below by a positive constant, and

$$\nabla d_i, u_i \in L^\infty_{loc}([0,+\infty);L^r(\Omega)^N) \text{ for some } r>\max(2,N).$$

Under the above assumptions, we prove

Theorem 3.1. *System* (12) *has a unique global strong solution.*

This result and its proof are inspired by Theorem 3.5 in [90], where global well-posedness is shown in the case of constant diffusion coefficients d_i and zero individual velocities u_i. The main new difficulty in our proof is to take into account the velocity terms u_i in the rather large space $L^\infty_{loc}([0,+\infty);L^r(\Omega)^N)$. One also has to deal with the (t,x)-dependence of d_i and it seems that the same space $L^\infty_{loc}([0,+\infty);L^r(\Omega)^N)$ is the right one for ∇d_i.

The proof is based on two estimates:

(i) Let $T>0$, if w and z are smooth functions such that for some $\theta\in\mathbb{R}$,

$$\partial_t w + \mathrm{div}(-d_1\nabla w + wu_1) \le \theta[\partial_t z + \mathrm{div}(-d_2\nabla z + zu_2)],$$

together with initial data in $L^\infty(\Omega)$ and homogeneous Neumann boundary conditions, then for any $p\in(1,+\infty)$, the L^p-norm of z controls the L^p-norm of w as follows:

$$\exists C>0 : \forall t\in(0,T),\quad \|\max(0,w)\|_{L^p((0,t)\times\Omega)} \le C\left(1+\|z\|_{L^p((0,t)\times\Omega)}\right).$$

(ii) Let $T>0$ and c be the solution of

$$\partial_t c + \mathrm{div}(-d\nabla c + cu) = f \text{ on } (0,T)\times\Omega,$$

with homogeneous Neumann boundary conditions and bounded initial data. Then there exists $C>0$ such that

$$\forall t\in(0,T),\quad \|c(t)\|^p_{L^p(\Omega)} \le C\left(1+\int_0^t \|f(s)\|^p_{L^p(\Omega)}ds\right).$$

Statement (i) is the core argument of the proof of Theorem 3.1, and its proof uses Maximal Regularity theory [38] to get estimates on the dual problem. The use of this theory requires the continuity of the diffusion coefficients and the assumption $\nabla d_i, u_i \in L^\infty((0,T);L^r(\Omega)^N)$ for $r > \max(2,N)$.

To explain the principle of the proof, we now consider the case of 2 equations

$$\left\{\begin{array}{rcl} \partial_t c_1 + \mathrm{div}[-d_1(t,x)\nabla c_1 + c_1 u_1(t,x)] & = & f_1(t,x,c) \\ \partial_t c_2 + \mathrm{div}[-d_2(t,x)\nabla c_2 + c_2 u_2(t,x)] & = & f_2(t,x,c) \end{array}\right., \tag{13}$$

and assume

$$f_1 \leq 0\ ;\ f_1 + f_2 \leq 0.$$

Using $c_1 \geq 0$, $f_1 \ \leq 0$ and $u_1 \in L^\infty_{loc}([0,+\infty);L^r(\Omega)^N)$, we know that c_1 is bounded in $L^\infty(Q_T)$ for any $T > 0$. Then, using $f_1 + f_2 \leq 0$,

$$\partial_t c_2 + \mathrm{div}(-d_2\nabla c_2 + c_2 u_2) \leq -\left[\partial_t c_1 + \mathrm{div}(-d_1\nabla c_1 + c_1 u_1)\right]$$

and we may use (i) to get some bounds in $L^p(Q_T)$ on c_2 for any $p < +\infty$. Since f has polynomial growth, both equations in (13) have a right-hand side bounded in $L^p(Q_T)$ for any $p < +\infty$, and using classical results of O. A. Ladyženskaja, V. A. Solonnikov and N. N Ural'ceva (see [69] and also Section 5.5 p.151), (c_1, c_2) is bounded in $L^\infty(Q_T)^2$. Then a global existence criterion of H. Amann [4] guarantees that the maximal solutions of (13) are global. Since we do not assume u_i to be regular, we have to work first on an approximate problem with smooth data.

As an application of Theorem 3.1, we also prove global well-posedness for a class of reaction-diffusion-advection systems from chemistry. If $c_1,\ldots,c_P$ are the concentrations of P chemical species $C_1,\ldots,C_P$, we assume that R reactions of the type

$$\alpha_j^1 C_1 + \ldots + \alpha_j^P C_P \rightleftharpoons C_{i_j}\ ;\ j \in \{1,\ldots,R\}, \tag{14}$$

are taking place simultaneously, where $\alpha_j^i \in \mathbb{N}$, $i_j \in \{1,\ldots,P\}$. On the basis of mass-action kinetics, the reaction speed for the j^{th} reaction is

$$r_j(c) = k_j^f \Pi_{k=1}^P c_k^{\alpha_j^k} - k_j^b c_{i_j},$$

where $k_j^f, k_j^b > 0$ are the so-called forward and backward reaction rates. If β_j is the $(i_j)^{th}$ vector of the canonical basis of $\mathbb{R}^P$ and $\alpha_j = (\alpha_j^1,\ldots,\alpha_j^P)$, the stoechiometric vector for the j^{th} reaction is

$$\nu_j = \beta_j - \alpha_j.$$

Then the reaction rate associated with the network of reactions (14) is

$$(f_1,\ldots,f_P) = \sum_{j=1}^R r_j(c)\nu_j. \tag{15}$$

With a convenient rearrangement of the chemical species and reactions, we then prove that such a reaction has the triangular structure (2), and as a consequence of Theorem 3.1, we get

Corollary 3.1. *Assuming* $(\mathbf{H}_2)$*, system* (12) *with reaction* (15) *has a unique global strong solution.*

3.2 The fast reaction limit for $C_1 + C_2 \rightleftharpoons C_3$ with advection

As a special case of Corollary 3.1, global existence of solutions holds for any $k > 0$ in the system

$$\left\{\begin{array}{rcll} \partial_t c_1 + \mathrm{div}[-d_1(t,x)\nabla c_1 + c_1 u(t,x)] &=& -k(c_1c_2 - c_3) & \\ \partial_t c_2 + \mathrm{div}[-d_2(t,x)\nabla c_2 + c_2 u(t,x)] &=& -k(c_1c_2 - c_3) & \text{on } (0,+\infty)\times\Omega, \\ \partial_t c_3 + \mathrm{div}[-d_3(t,x)\nabla c_3 + c_3 u(t,x)] &=& +k(c_1c_2 - c_3) & \\ -d_i(t,x)\nabla c_i\cdot\nu + c_i u(t,x)\cdot\nu &=& 0 & \text{on } (0,+\infty)\times\partial\Omega, \\ c_i(0,\cdot) &=& c_i^0 & \text{on } \Omega,\ i\in\{1,2,3\}. \end{array}\right. \tag{16}$$

In this situation, species C_1, C_2, C_3 are in a fluid whose motion is described by the vector field u. The main point of this section is to know if the presence of advection and variable diffusion coefficients is an obstacle to use the ideas of Chapter 1 to pass to the fast reaction limit $k \to +\infty$ in system (16).

The main point is to estimate the solution c^k of (16) in $L^2((0,T)\times\Omega)^P$ independently of k. In Chapter 1, we derive these estimates from the study of $\partial_t(c_1^k + c_3^k)$ and $\partial_t(c_2^k + c_3^k)$. This time, we have for $i \in \{1,2\}$,

$$\left\{\begin{array}{rcll} \partial_t(c_i^k + c_3^k) + \mathrm{div}[-d_i\nabla c_i^k - d_3\nabla c_3^k + (c_i^k + c_3^k)u] &=& 0 & \text{on } (0,+\infty)\times\Omega, \\ -[d_i\nabla c_i^k + d_3\nabla c_3^k]\cdot\nu + (c_i^k + c_3^k)u\cdot\nu &=& 0 & \text{on } (0,+\infty)\times\partial\Omega, \\ (c_i^k + c_3^k)(0,\cdot) &=& c_i^0 + c_3^0 & \text{on } \Omega. \end{array}\right.$$

This can be rewritten, setting $W_i^k = c_i^k + c_3^k$,

$$\left\{\begin{array}{rll} \partial_t W_i^k + \mathrm{div}(-\nabla(A_i^k W_i^k) + W_i^k\tilde{u}) = 0 & \text{on} & Q_T\ ; \\ -\nabla(A_i^k W_i^k)\cdot\nu + W_i^k\tilde{u}\cdot\nu = 0 & \text{on} & \Sigma_T\ ; \\ W_i^k(0,\cdot) = W^0 & \text{on} & \Omega, \end{array}\right. \tag{17}$$

where $0 < \underline{a} \le A_i^k \le \overline{a} < +\infty$ for some constants $\underline{a}, \overline{a}$ independent of k and $\tilde{u}$ has the same regularity as u. Due to the new term $W_i^k\tilde{u}$, we have to use a different technique than in Chapter 1 to derive the expected L^2-estimate: for $\Theta \in C_c^\infty(Q_T, \mathbb{R}_+)$, we introduce the dual problem of (17)

$$-[\partial_t\Psi + A_i^k\Delta\Psi + \tilde{u}\cdot\nabla\Psi] = \Theta \text{ on } Q_T\ ;\ \partial_\nu\Psi = 0 \text{ on } \Sigma_T\ ;\ \Psi(T,\cdot) = 0 \text{ on } \Omega.$$

Then standard energy methods allow to estimate $\|\Psi(0)\|_{L^2(\Omega)}$ and $\|\Psi\|_{L^2(Q_T)}$ in terms of $\|\Theta\|_{L^2(Q_T)}$ with constants depending only on $\underline{a}, \overline{a}$, whence the $L^2(Q_T)$-estimate of W^k by duality, independently of k. However, the method we used in the previous chapters to derive the strong compactness of c^k in $L^2(Q_T)$ could not be extended to this more complex situation. Here, we prove the strong compactness of c^k in $L^p(Q_T)^3$ for $p \in [1,2)$ "only". Together with similar estimates as in Chapter 1, this is sufficient to pass to the limit $k \to +\infty$, and the limit c is a weak solution of

$$\left\{\begin{array}{ll} \left.\begin{array}{l} \partial_t(c_1 + c_3) - \mathrm{div}[-d_1c_1 - d_3c_3 + (c_1 + c_3)u] = 0 \\ \partial_t(c_2 + c_3) - \mathrm{div}[-d_2c_2 - d_3c_3 + (c_1 + c_3)u] = 0 \\ c_1c_2 = c_3 \end{array}\right\} & \text{on } Q_T, \\ \partial_\nu(c_1 + c_3) = \partial_\nu(c_2 + c_3) = 0 & \text{on } \Sigma_T, \\ (c_1 + c_3)(0,\cdot) = c_1^0 + c_3^0\ ;\ (c_2 + c_3)(0,\cdot) = c_2^0 + c_3^0 & \text{on } \Omega. \end{array}\right. \tag{18}$$

In the case when $\operatorname{div} u$ is assumed to be in $L^\infty(Q_T)$, it happens that the functions

$$c_i^k \log c_i^k + c_i^k + c_3^k \log c_3^k + c_3^k \ ; \ i \in \{1,2\}$$

satisfy similar equations as (17). Therefore, they are bounded in $L^2(Q_T)$, which provides a "uniform integrability" property of c_i^k in $L^2(Q_T)$, and allows to recover the strong compactness of c^k in $L^2(Q_T)^3$ with a Vitali-type argument.

All in all, if we assume that the data of (16) satisfy the same assumptions as in Theorem 3.1, we have

Theorem 3.2. *Let $k_n \to +\infty$ and c^n be the corresponding global solution of* (16). *Up to a subsequence, c^n converges in $L^p(Q_T)$ for any $p \in [1,2)$ and any $T > 0$ to a weak solution of* (18). *If in addition* $\operatorname{div} u \in L^\infty(Q_T)$ *for any $T > 0$, then c^n also converges in $L^2(Q_T)$.*

3.3 A diffusion-electromigration system

In the last section, we are interested in the existence of global solutions in any space dimension for the diffusion-electromigration system

$$\begin{cases} \partial_t c_i - \operatorname{div}(d_i \nabla c_i + d_i z_i c_i \nabla \Phi) &= 0 \quad \text{on } (0,+\infty)\times\Omega, \\ \partial_\nu c_i + z_i c_i \partial_\nu \Phi &= 0 \quad \text{on } (0,+\infty)\times\partial\Omega, \quad i \in \{1,\dots,P\}, \\ -\Delta\Phi - \sum_{i=1}^P z_i c_i &= 0 \quad \text{on } (0,+\infty)\times\Omega, \\ \partial_\nu \Phi + \tau\Phi &= \xi \quad \text{on } (0,+\infty)\times\partial\Omega, \\ c(0,\cdot) &= c^0 \quad \text{on } \Omega. \end{cases} \tag{19}$$

This system describes the evolution of an electrolyte. The unknown is $(c_1,\dots,c_P,\Phi)$, where $c_1,\dots,c_P$ are the concentrations of P chemical species which may be charged with charge number z_i, and Φ is the electrical potential. The boundary condition for Φ may be motivated by considering locally the boundary as a plate capacitor: $\tau > 0$ denotes its capacity, and the function ξ, which is a data of the problem, is connected with some exterior potential.

For space dimension $N = 2$, well-posedness and long-time behaviour of (19) is already well-understood: in [15] existence and uniqueness of global weak solutions is shown, as well as convergence to uniquely determined steady states. For sufficiently smooth data, it is proved in [33] that there is a unique global classical solution. These results are improved in [14] by computing an explicit exponential convergence rate with the help of logarithmic Sobolev inequalities. In the papers [51, 55, 56, 57] the authors supplement the model with quite general reactions terms coming from mass-action kinetics chemistry, and prove global well-posedness and exponential convergence to the steady state. System (19) has also been complemented by the Navier-Stokes equations modeling the fluid flow, see e.g. [24, 36, 95, 96].

So far, global well-posedness in dimension $N = 3$, even for time and space independent diffusivities, has only been shown under additional assumptions. These include initial data lying close to the steady state [15], or the *a priori* knowledge that the solution c is bounded in $L^\infty(0,T;L^2(\Omega))$ independently of $T > 0$ [33]. In [64], existence of global weak solutions for constant diffusivities is shown in the more general setting of the Navier-Stokes-Nernst-Planck-Poisson system, but for $P = 2$, which provides additional structure and estimates.

In the present work, we prove the existence of global solutions in the case of time and space dependent diffusivities and without any restriction on the number of chemical species. Our proof is based on the energy method: it relies on the physical structure of the equations, and exploits the available Lyapunov functional for system (19).

For the data, we assume the following

(i) For $i = 1,\dots,P,\ d_i \in L^\infty_{loc}([0,+\infty);L^\infty(\Omega))$.
For all $T > 0$, there exist $\underline{d_i}(T), \overline{d_i}(T) > 0$ such that

$$0 < \underline{d_i}(T) \leq d_i \leq \overline{d_i}(T) < +\infty \ \text{ on } Q_T.$$

(ii) $c^0 \in L^\infty(\Omega,[0,+\infty)^P)$.

(iii) $\xi \in C^\infty(\partial\Omega)$ is a time-independent function.

Our main result may be summarized as

Theorem 3.3. *Under the above assumptions, there exists a global weak solution to* (19) *in any space dimension.*

To prove this theorem, we first study an approximate version of (19), where the total charge density $\sum_{i=1}^P z_i c_i$ is regularized: letting $\varepsilon > 0$, B_ε denote the differential operator $I - \varepsilon\Delta$, $m = 2N$, $k \in \{0,\dots,m\}$, we consider

$$\left.\begin{array}{rcll} \partial_t c_i - \operatorname{div}(d_i \nabla c_i + d_i z_i c_i \nabla\Phi) &=& 0 & \text{on } (0,+\infty)\times\Omega \\ \partial_\nu c_i + z_i c_i \partial_\nu \Phi &=& 0 & \text{on } (0,+\infty)\times\partial\Omega \\ c_i(0) &=& c_i^0 & \text{on } \Omega \end{array}\right\}, \tag{20}$$

$$\left.\begin{array}{rcll} B_\varepsilon^{m+1}\Psi - \sum_{i=1}^P z_i c_i &=& 0 & \text{on } (0,+\infty)\times\Omega \\ \partial_\nu[B_\varepsilon^k \Psi] + \tau B_\varepsilon^k \Psi &=& 0 & \text{on } (0,+\infty)\times\partial\Omega \end{array}\right\}, \tag{21}$$

$$\left.\begin{array}{rcll} -\Delta\Phi &=& \Psi & \text{on } (0,+\infty)\times\Omega \\ \partial_\nu\Phi + \tau\Phi &=& \xi & \text{on } (0,+\infty)\times\partial\Omega \end{array}\right\}. \tag{22}$$

We prove the well-posedness of this system using a Leray-Schauder fixed point argument, where Theorem 3.1 is strongly used as a first step to define the right mapping and to prove its necessary properties. The advantage of this approximation method lies in the fact that it preserves the natural "entropy" structure of system (19), and there exists a Lyapunov function for (20) – (22). Actually, it is even possible to state the corresponding dissipation rate explicitly. This provides estimates independent of ε that are exploited to derive the compactness of the approximate solutions and to pass to the limit $\varepsilon \to 0$.

Part I

Two cross-diffusion systems

1

Global well-posedness of a conservative relaxed cross diffusion system

Except for Subsection 1.6, this section is a joint work with T. Lepoutre and M. Pierre, published in [72].

We prove global existence in time of solutions to relaxed conservative cross diffusion systems governed by nonlinear operators of the form $u_i \to \partial_t u_i - \Delta(a_i(\tilde{u})u_i)$ where the $u_i, i = 1, \ldots, I$ represent I density-functions, $\tilde{u}$ is a spatially regularized form of $(u_1, \ldots, u_I)$ and the nonlinearities a_i are merely assumed to be continuous and bounded from below. Existence of global weak solutions is obtained in any space dimension. Solutions are proved to be regular and unique when the a_i are locally Lipschitz continuous.

1.1 Introduction

Introduced by Shigesada *et al.* [97], cross diffusion models try to represent the effect of the interaction between species through motion, and not only as usual through reaction. These models have been studied by Levin [74], Levin and Segel [73], Okubo [87], Mimura and Murray [81], Mimura and Kawasaki [80], Mimura and Yamaguti [82], Andreianov *et al.* [6], Bendahmane and Langlais [10] and many other authors: a survey by A. Jüngel may be found in [65] for applications to population dynamics. In those references, a general system is the following:

$$\begin{cases} \partial_t u_1 - \Delta[u_1(d_1 + d_{11}u_1^p + d_{12}u_2^p)] = r_1(u_1, u_2), \\ \partial_t u_2 - \Delta[u_2(d_2 + d_{21}u_1^p + d_{22}u_2^p)] = r_2(u_1, u_2), \\ \partial_n[u_1(d_1 + d_{11}u_1^p + d_{12}u_2^p)] = \partial_n[u_2(d_2 + d_{21}u_1^p + d_{22}u_2^p)] = 0. \end{cases} \tag{1.1}$$

For the system (1.1) with $p = 1$ and Lotka-Volterra-type reaction, there exists a wide literature, studying specific cases of the system where an additional structure keeps it parabolic or with cross diffusion pressure only on one of the species (see e.g. Wang [103] and the many references therein, especially in the introduction). To our knowledge, the most general result on global weak solutions might be found in Chen and Jüngel [32] where the entropy structure of the model is used. For existence of classical solutions the reader might consult [75, 103] by Wang and Li-Zhao for instance. In population dynamics, one of the most interesting features of cross diffusion is its effect on steady states: cross diffusion pressure might help the appearance of nonconstant steady states when the reaction structure does not drive to segregation (see Iida-Mimura-Ninomyia [63] for instance). However, in these cases, the pattern formation relies on the reaction term (for instance, the convergence to homogeneous steady states in absence of reaction is proved in [32]).

In [11], [70], T. Lepoutre and his collaborators introduced a relaxation of conservative cross diffusion systems, replacing

$$\begin{cases} \partial_t u_i - \Delta[a_i(u)u_i] = 0, \text{ on } (0,+\infty)\times\Omega,\ \Omega\subset\mathbb{R}^N, \text{ bounded,} \\ u = (u_1,\dots,u_I), \\ \partial_n[a_i(u)u_i] = 0 \text{ on } (0,+\infty)\times\partial\Omega, \quad u(0,\cdot) = u^0 \text{ given,} \end{cases}$$

where $a_i : [0,\infty)^I \to [0,\infty)$, by the following relaxed model:

$$\begin{cases} \partial_t u_i - \Delta[a_i(\tilde{u})u_i] = 0, \text{ on } (0,+\infty)\times\Omega, \\ u = (u_1,\dots u_I), \\ \tilde{u}_i - \delta_i\Delta\tilde{u}_i = u_i, \text{ on } (0,+\infty)\times\Omega, \qquad \delta_i > 0, \\ \partial_n u_i = \partial_n\tilde{u}_i = 0 \text{ on } (0,+\infty)\times\Omega, \quad u(0,\cdot) = u^0 \text{ given.} \end{cases} \tag{1.2}$$

This model was introduced in order to investigate the effect of non classical cross diffusion pressure on the segregative behavior (and $a_i(\cdot)$ is often truly nonlinear). One of the purposes was to drive spatial segregation only through motion. Its effects on the stability of the homogeneous equilibria is investigated in [11,70,71]. This relaxed version is also relevant in some applications: it takes into account that the intensity of the underlying Brownian motion depends on the density of the population measured with a spatial length δ_i and not exactly at the exact location x. It takes therefore into account the fact that a species can react to the presence of another species in a neighborhood.
Models with nonlocal diffusion coefficients can be seen also in [12] (where the self-diffusion coefficients depend on the total population). Nonlocal reaction terms can also be considered, see [13, 52, 86] for instance, but the goal of our model is more to create patterns only through motion.

A first well-posedness result for the relaxed model was derived in [11, 70] in dimension $N = 1,2$ and with some restrictions on the structure of the nonlinearities a_i (basically, the a_i are C^2 and have at most a polynomial growth in u). In this section, we prove existence of solutions for this system *in any dimension and for general nonlinearities a_i*, which are only assumed to be continuous and bounded from below. Weak solutions are obtained in general and they are proved to be strong and unique as soon as the a_i are locally Lipschitz continuous. Some L^2-estimates are exploited in the spirit of [90] to prove existence of weak solutions. A main point is that $\tilde{u}$ is uniformly bounded in any dimension for these weak solutions. Next, one has to deal with parabolic operators of the form $u_i \to \partial_t u_i - \Delta(a_i(\tilde{u})u_i)$: they are not of divergence form, but they are uniformly parabolic since $a_i(\tilde{u})$ is then bounded from above and from below. Using the C^α-theory for the duals of these operators, namely $U_i \to \partial_t U_i - a_i(\tilde{u})\Delta U_i$, in the spirit of Krylov-Safonov [67], [42] (see also the book by Lieberman [77]), we prove that $\tilde{u}$ is even Hölder-continuous. This provides continuous coefficients $a_i(\tilde{u})$ for the above operators, and then, L^p-estimates classically follow for the solution. When the a_i are locally Lipschitz continuous, even $\partial_t u_i, \Delta(a_i(\tilde{u})u_i)$ are proved to be in L^p so that the solution is strong: moreover, weak solutions are then proved to be unique.

Let us fix the notations and state the main result. We assume that $\Omega\subset\mathbb{R}^N$ is a bounded subset with a C^2-boundary. The exterior normal derivative operator on $\partial\Omega$ is denoted by ∂_n. For all $T > 0$, we denote $Q_T = (0,T)\times\Omega, \Sigma_T = (0,T)\times\partial\Omega$. For $\alpha\in(0,1]$, we denote

$$C^\alpha(Q_T) = \{v\in L^\infty(Q_T); \|v\|_T^{(\alpha)} < +\infty\},$$

$$\|v\|_T^{(\alpha)} = \|v\|_{L^\infty(Q_T)} + \sup\left\{ \frac{|v(t,x)-v(s,y)|}{[|t-s|+|x-y|^2]^{\frac{\alpha}{2}}},\ (t,x),(s,y)\in Q_T \right\}.$$

We will at least assume that

$$\forall i=1,\dots,I,\ \ a_i:[0,\infty)^I \to [0,\infty) \textit{ is continuous and} :\ \inf_{r\in[0,\infty)^I} a_i(r) \geq \underline{d} > 0. \tag{1.3}$$

And we are given $\delta_i \in (0,\infty), \forall i=1,\dots,I$.

Theorem 1.1. *Assume* (1.3) *and* $u^0=(u_1^0,\dots,u_I^0)\in L^\infty(\Omega,[0,\infty))^I$. *Then, there exists a nonnegative solution* $u=(u_1,\dots,u_I)$ *to the following problem:*

$$\begin{cases} \forall T\in(0,\infty), \forall i=1,\dots,I,\ \forall p\in[1,\infty), \\ u_i\in L^p(Q_T); \tilde{u}_i \in C^\alpha(Q_T)\cap L^p\left(0,T;W^{2,p}(\Omega)\right)\ \textit{for some}\ \alpha\in(0,1], \\ \int_0^t a_i(\tilde{u})u_i \in L^p\left(0,T;W^{2,p}(\Omega)\right), \\ u_i(t)-\Delta[\int_0^t a_i(\tilde{u})u_i] = u_i^0\ \textit{in}\ Q_T, \\ \tilde{u}_i - \delta_i\Delta\tilde{u}_i = u_i\ \textit{in}\ Q_T \\ \partial_n\left(\int_0^t a_i(\tilde{u})u_i\right) = 0 = \partial_n\tilde{u}_i\ \textit{on}\ \Sigma_T. \end{cases} \tag{1.4}$$

If moreover

$$\forall i=1,\dots,I,\ a_i:[0,\infty)^I\to[0,\infty)\ \textit{is locally Lipschitz continuous} \tag{1.5}$$

then, $\forall i=1,\dots,I,\ \forall T>0, \forall p\in[1,\infty)$,

$$u_i\in L^\infty(Q_T), \forall\tau\in(0,T), \partial_t u_i, \Delta(a_i(\tilde{u})u_i)\in L^p\left((\tau,T)\times\Omega\right)$$

and $\partial_t u_i - \Delta(a_i(\tilde{u})u_i)=0, \partial_n(a_i(\tilde{u})u_i)=0$ *in a pointwise sense. Finally, under assumption* (1.5), *solutions of* (1.4) *are unique.*

The section is organized as follows.

Section 1.2 first assumes that the nonlinearities a_i are also bounded from above. We prove existence of a weak solution to the system (1.4) by a standard Leray-Schauder fixed-point argument. The underlying space is an adequate subspace of $L^2(Q_T)$ and the required compactness follows essentially from Lemma 1.4.

Section 1.3 is devoted to the proof of the L^∞-estimate on $\tilde{u}$. Then, the assumption of the bound from above on the a_i may be dropped.

Section 1.4 exploits this L^∞-estimate to prove that the weak solution is actually rather regular, and existence as stated in Theorem 1.1 follows. The C^α-theory for non-divergence parabolic operators is used there. An alternative more elementary proof of the regularity is also given when monotonicity properties hold for the a_i together with locally Lipschitz continuity.

The uniqueness stated in Theorem 1.1 is proved in Section 1.5. It is based on solving an original dual problem, interesting for itself.

A short Section 1.6 indicates without proof a complementary approach which provides a constructive and alternative way of proving existence of a solution and which may be used to compute it numerically.

1.2 Global existence when a_i is bounded

In this section, we first prove existence of *weak-solutions* of (1.4) on a given interval $[0,T]$ when, besides (1.3), the nonlinearities a_i also satisfy

$$\exists \overline{d} > 0,\ \forall i = 1,\dots,I,\ \sup_{r\in[0,\infty)^I} a_i(r) \le \overline{d}. \tag{1.6}$$

Proposition 1.2. *Let $T > 0$. Assume* (1.3), (1.6) *and* $\forall i = 1,\dots,I, u_i^0 \in L^2(\Omega;[0,\infty))$. *Then, there exists a nonnegative solution $u = (u_1,\dots,u_I)$ to the system*

$$\begin{cases} \forall i = 1,\dots,I, \\ u_i \in L^2(Q_T),\ \int_0^t a_i(\tilde{u})u_i \in L^2\left(0,T;H^2(\Omega)\right), \\ \tilde{u}_i \in L^2\left(0,T;H^2(\Omega)\right),\ \tilde{u}_i - \delta_i\Delta\tilde{u}_i = u_i \text{ on } Q_T,\ \tilde{u}_i \ge 0 \\ u_i - \Delta(\int_0^t a_i(\tilde{u})u_i) = u_i^0 \text{ on } Q_T, \\ \partial_n\tilde{u}_i = 0 = \partial_n(\int_0^t a_i(\tilde{u})u_i) \text{ on } \Sigma_T. \end{cases} \tag{1.7}$$

To prove Proposition 1.2, we will use the classical Leray-Schauder's approach, namely (see e.g. [53], Theorem 11.3)

Lemma 1.3 (Leray-Schauder)**.** *Let $(X,\|\cdot\|_X)$ be a Banach space and $\mathcal{T} : X \to X$ a continuous compact mapping. Suppose that*

$$\exists M > 0,\ \forall\sigma \in [0,1],\ [\, u \in X,\ u = \sigma\mathcal{T}u\,]\ \Rightarrow\ [\ \|u\|_X \le M\,].$$

Then, there exists $u \in X$ such that $u = \mathcal{T}u$.

To define the mapping $\mathcal{T}$, we will use the following lemma.

Lemma 1.4. *Let $T > 0$, $w_0 \in L^2(\Omega;[0,+\infty))$, $A \in L^\infty(Q_T)$, $\underline{a}$, $\overline{a} \in (0,\infty)$ such that $0 < \underline{a} \le A \le \overline{a} < +\infty$. Then there exists a unique nonnegative solution $w = w(A,w_0)$ to*

$$\begin{cases} w \in L^2(Q_T),\ \int_0^t Aw \in L^2(0,T;H^2(\Omega)), \\ w - \Delta\left(\int_0^t Aw\right) = w_0 \text{ on } Q_T,\ \partial_n\left(\int_0^t Aw\right) = 0 \text{ on } \Sigma_T. \end{cases} \tag{1.8}$$

Moreover, if

$$A^n \in L^\infty(Q_T),\ 0 < \underline{a} \le A^n \le \overline{a} < \infty,\ A^n \to A \text{ a.e.},\ w_0^n \to w_0 \text{ in } L^2(\Omega),$$

then $w(A^n,w_0^n)$ converges strongly in $L^2(Q_T)$ to $w(A,w_0)$.

Proof of Lemma 1.4. Using convolution, we approximate A by a sequence of smooth functions $(A^n)_{n\in\mathbb{N}} \in C^\infty(\overline{Q_T})$ such that $\underline{a} \le A^n \le \overline{a}$ and $A^n \to A$ a.e.. Let also w_0^n be a regular approximation of w_0. There exists a classical regular nonnegative solution w^n of (see e.g. [69], Theorem V.7.4, applied to the unknown $A^n w^n$)

$$\partial_t w^n - \Delta(A^n w^n) = 0 \text{ on } Q_T,\ \partial_n(A^n w^n) = 0 \text{ on } \Sigma_T, w^n(0,\cdot) = w_0^n. \tag{1.9}$$

Integrating (1.9) in time gives

$$w^n(t) - \Delta\left(\int_0^t A^n w^n\right) = w_0^n \text{ on } Q_T, \partial_n\left(\int_0^t A^n w^n\right) = 0 \text{ on } \Sigma_T. \tag{1.10}$$

We multiply by $A^n w^n$ and use the following identity, valid for $z^n = A^n w^n$:

$$-\int_\Omega z^n \Delta \int_0^t z^n = \int_\Omega \nabla z^n \nabla \int_0^t z^n = \int_\Omega \frac{1}{2}\partial_t |\nabla \int_0^t z^n|^2. \tag{1.11}$$

We obtain the following estimate after integration in time

$$\int_{Q_T} A^n (w^n)^2 + \int_\Omega \frac{1}{2}|\nabla \int_0^T A^n w^n|^2 = \int_{Q_T} w_0^n A^n w^n. \tag{1.12}$$

In particular

$$\underline{a}\int_{Q_T} (w^n)^2 \le \bar{a}\sqrt{T}\left(\int_\Omega (w_0^n)^2\right)^{1/2}\left(\int_{Q_T} (w^n)^2\right)^{1/2} \Rightarrow \underline{a}\|w^n\|_{L^2(Q_T)} \le \bar{a}\sqrt{T}\|w_0^n\|_{L^2(\Omega)}. \tag{1.13}$$

Now, up to a subsequence, w^n converges weakly in $L^2(Q_T)$ to some w. By the pointwise and uniformly bounded convergence of A^n to A, for all $\psi \in L^2(Q_T)$, ψA^n converges strongly in $L^2(Q_T)$ to ψA (using the dominated convergence theorem). Thus, $\int_{Q_T} \psi A^n w^n$ converges to $\int_{Q_T} \psi A w$. In other words, $z^n = A^n w^n$ also converges weakly in $L^2(Q_T)$ to $z = Aw$.

By (1.10), $\Delta \int_0^t z^n$ is bounded in $L^2(Q_T)$; since $\int_0^t z^n$ is bounded in $L^2(Q_T)$ as well, this implies that $\int_0^t z^n$ is bounded in $L^2(0,T;H^2(\Omega))$. We now may pass to the weak limit in (1.10) to deduce that w is solution of (1.8).

For the uniqueness, let w be the difference of two solutions of (1.8) (then $w(0)=0$). We denote $S(t) = \int_0^t Aw$. *Formally*, the idea is to multiply the equation $w - \Delta S = 0$ by $S' = Aw$. Then, after integration

$$\int_{Q_T} Aw^2 = \int_{Q_T} S'\Delta S = -\int_{Q_T} \nabla S' \nabla S = -\int_{Q_T} \frac{1}{2}\partial_t |\nabla S(t)|^2 = -\int_\Omega \frac{1}{2}|\nabla S(T)|^2 \le 0.$$

Whence $w \equiv 0$ since $A > 0$.

Since we do not know whether $\nabla S' \in L^2(Q_T)$, we have to justify this computation in an approximate way. For $h \in (0,T)$, let us denote

$$\forall h \in (0,T),\ \ S_h(t) := \tfrac{S(t+h)-S(t)}{h} = \tfrac{1}{h}\int_t^{t+h} (Aw)(s)ds. \tag{1.14}$$

Note that

$$S_h \in L^2\left(0, T-h; H^2(\Omega)\right),\ \|S_h - Aw\|_{L^2(Q_{T-h})} \to 0 \; as \; h \to 0. \tag{1.15}$$

We have

$$\forall t \in [0, T-h),\ w(t+h) + w(t) - \Delta[S(t) + S(t+h)] = 0.$$

We multiply by $S_h(t)$ and integrate over Ω to obtain

$$\int_\Omega [w(t+h)+w(t)]S_h(t) = -\int_\Omega \nabla S_h(t)[\nabla S(t+h) + \nabla S(t)] = -\int_\Omega \frac{1}{h}\left\{|\nabla S(t+h)|^2 - |\nabla S(t)|^2\right\}.$$

After integration on $[0,T-h]$ and an easy change of variable, we have:

$$\int_{Q_{T-h}} [w(\cdot+h)+w]S_h = -\frac{1}{h}\int_{(T-h,T)\times\Omega} |\nabla S|^2 + \frac{1}{h}\int_{(0,h)\times\Omega} |\nabla S|^2 \le \frac{1}{h}\int_{Q_h} |\nabla S|^2. \tag{1.16}$$

To pass to the limit as $h \to 0$, we use

$$\int_{Q_h} |\nabla S|^2 = \int_{Q_h} -Sw = \int_\Omega -\int_0^h \left[w(t)\int_0^t (Aw)(\sigma)d\sigma\right] dt \le \|A\|_{L^\infty(Q_T)} h \int_{Q_h} w^2 dt.$$

Letting h decrease to 0 in (1.16) and using that $S_h \to Aw$ in L^2 (see (1.15)) leads to $\int_{Q_T} 2wAw \le 0$, whence $w \equiv 0$.

Let us now prove the continuity result. Let us first note that, for any solution of (1.8), we have the identity

$$\int_{Q_T} Aw^2 + \int_\Omega \frac{1}{2}|\nabla \int_0^T Aw|^2 = \int_{Q_T} w_0 Aw. \tag{1.17}$$

This may be justified as we did above for the uniqueness (namely in the case $w_0 = 0$) by passing to the limit in the following identity where $S(t) = \int_0^t Aw, S_h(t) = [S(t+h) - S(t)]/h$:

$$\int_{Q_{T-h}} [w(\cdot+h)+w]S_h + \nabla S_h \nabla[S(\cdot+h)+S] = 2\int_{Q_{T-h}} w_0 S_h, \tag{1.18}$$

$$\int_{Q_{T-h}} [w(\cdot+h)+w]S_h + \frac{1}{h}\int_{(T-h,T)\times\Omega} |\nabla S|^2 - \frac{1}{h}\int_{(0,h)\times\Omega} |\nabla S|^2 = 2\int_{Q_{T-h}} w_0 S_h, \tag{1.19}$$

and we pass to the limit as above as $h \to 0$ to obtain (1.17) (at least for a.e.T).

Let $w^n = w(A^n, w_0^n)$. As in the beginning of this proof (see (1.13),(1.17)), the relation

$$\int_{Q_T} A^n (w^n)^2 + \int_\Omega \frac{1}{2}|\nabla \int_0^T A^n w^n|^2 = \int_{Q_T} w_0^n A^n w^n \tag{1.20}$$

proves that w^n is bounded in $L^2(Q_T)$. From equation (1.10), we deduce that $\int_0^t A^n w^n$ is bounded in $L^2(0,T;H^2(\Omega))$. A subsequence of $\left(w^n, \Delta\int_0^t A^n w^n\right)$ converges weakly in $L^2(Q_T)^2$ to $\left(w, \Delta\int_0^t Aw\right)$ and w is solution of the limit problem (1.8). By uniqueness, the full sequence converges. Since $A^n \to A$ a.e., $\sqrt{A^n}w^n$ converges also weakly in $L^2(Q_T)$ to $\sqrt{A}w$ and, by the estimate (1.20), $\nabla\int_0^T A^n w^n$ converges weakly in $L^2(\Omega)$, the limit being necessarily $\nabla\int_0^T Aw$. In particular

$$\int_{Q_T} Aw^2 \le \liminf_{n\to\infty} \int_{Q_T} A^n (w^n)^2, \quad \int_\Omega |\nabla \int_0^T Aw|^2 \le \liminf_{n\to\infty} \int_\Omega |\nabla \int_0^T A^n w^n|^2. \tag{1.21}$$

But, since $\lim_{n\to\infty} \int_{Q_T} w_0^n A^n w^n = \int_{Q_T} w_0 Aw$, and since the identity (1.17) is true for w, it follows from (1.20), (1.17) that *equality holds* in the two inequalities (1.21). In particular, the norm of $\sqrt{A^n}w^n$ in $L^2(Q_T)$ converges to the norm of $\sqrt{A}w$; this implies that the $L^2(Q_T)$-weak convergence of $\sqrt{A^n}w^n$ to $\sqrt{A}w$ *is actually strong*. Using again the pointwise convergence of A^n, we deduce that w^n converges strongly in $L^2(Q_T)$ as well.

□

Remark 1.5. *As a consequence of* (1.17)*, there is a constant* $C = C(\underline{a}, \overline{a}, \|w_0\|_{L^2(\Omega)})$ *such that for any solution* w *of* (1.8)*,*

$$\|w\|_{L^2(Q_T)} \le \sqrt{T}C. \tag{1.22}$$

The next step is the definition of a compact continuous mapping $\mathcal{T}$ whose fixed points are solutions of (1.7). We introduce the Hilbert space

$$X = \Pi_{1\le i\le I} X_i,\ X_i = \{v \in L^2(Q_T) : \partial_t(J_{\delta_i} v) \in L^2(Q_T)\}, \tag{1.23}$$

where the Hilbert norm $\|\cdot\|_i$ is defined on X_i by

$$\|v\|_i^2 := \|v\|_{L^2(Q_T)}^2 + \|\partial_t(J_{\delta_i} v)\|_{L^2(Q_T)}^2,$$

and where $J_\delta = (I - \delta\Delta)^{-1}$ is the resolvent of the Laplace operator on $L^2(\Omega)$ with homogeneous Neumann boundary conditions, that is

$$[f \in L^2(\Omega),\ Z = J_\delta f] \Leftrightarrow [Z \in H^2(\Omega),\ Z - \delta\Delta Z = f,\ \partial_n Z = 0 \text{ on } \partial\Omega]. \tag{1.24}$$

Definition 1.6. *We fix $u^0 \in L^2(\Omega,[0,\infty))^I$. Let $v=(v_1,\dots,v_I)\in X$ and let $\tilde u=(\tilde u_1,\dots,\tilde u_I)$ be the solution of (see [30], Proposition 9.24 and Theorem 9.26):*

$$\forall i=1,\dots,I,\ \tilde u_i\in L^2\big(0,T;H^2(\Omega)\big),\ \tilde u_i-\delta_i\Delta\tilde u_i=v_i \text{ on } Q_T,\ \ \partial_n\tilde u_i=0 \text{ on } \Sigma_T.$$

Next, we define

$$\mathcal{T}:X\to X \text{ by } \mathcal{T}(v):=u=(u_1,\dots,u_I),$$

where u_i is the solution w of (1.8) with $A=a_i([\tilde u]^+), w_0=u_i^0$; $[\tilde u]^+=([\tilde u_1]^+,\dots,[\tilde u_I]^+)$ and $[\tilde u_i]^+$ is the positive part of $\tilde u_i$.

Proposition 1.7. *Assume* (1.3), (1.6) *and $\forall i=1,\dots,I, u_i^0\in L^2(\Omega;[0,\infty))$. Then the mapping $\mathcal{T}$ is continuous and compact from X into itself.*

Proof of Proposition 1.7. First, remark that for $v\in X$, $u=\mathcal{T}(v)\in X$. Indeed, since u_i is solution of (1.8) with $A=a_i([\tilde u]^+)$ and $w_0=u_i^0$, we may write

$$J_{\delta_i}u_i=J_{\delta_i}\Delta\int_0^t Au_i+J_{\delta_i}u_i^0=\int_0^t \Delta J_{\delta_i}(Au_i)+J_{\delta_i}u_i^0\Rightarrow \partial_t(J_{\delta_i}u_i)=\Delta J_{\delta_i}(Au_i)\in L^2(Q_T).$$

Let v^n be a bounded sequence in X. Up to a subsequence, me may assume that v_i^n converges weakly to v_i in $L^2(Q_T)$. Then

$$\tilde u_i^n-\delta_i\Delta\tilde u_i^n=v_i^n \text{ on } Q_T,\ \partial_n\tilde u_i^n=0 \text{ on } \Sigma_T\ \ \Rightarrow \partial_t\tilde u_i^n=\partial_t(J_{\delta_i}v_i^n).$$

Thus $\tilde u_i^n$ is bounded in $L^2\left(0,T;H^2(\Omega)\right)$ and $\partial_t\tilde u_i^n=\partial_t\left(J_{\delta_i}v_i^n\right)=J_{\delta_i}(\partial_t v_i^n)$ is bounded in $L^2(Q_T)$. As a consequence, $\tilde u_i^n$ is relatively compact in $L^2(Q_T)$, and so is $[\tilde u_i^n]^+$. Up to a subsequence again, we may assume that they converge strongly in $L^2(Q_T)$ and a.e. in Q_T. By continuity of a_i, $a_i([\tilde u^n]^+)$ converges a.e. and $0<\underline{d}\le a_i([\tilde u^n]^+)\le\overline{d}<\infty$. By Lemma 1.4, $u^n:=\mathcal{T}(v^n)$ converges (up to a subsequence) strongly in $L^2(Q_T)$. Moreover

$$u_i^n=\Delta\left(\int_0^t a_i\left([\tilde u^n]^+\right)u_i^n\right)+u_i^0\ \Rightarrow\ \partial_t(J_{\delta_i}u_i^n)=\Delta J_{\delta_i}\left[\left(a_i[\tilde u^n]^+\right)u_i^n\right].$$

But the Yosida approximation ΔJ_{δ_i} is Lipschitz continuous on $L^2(Q_T)$, and $a_i([\tilde u^n]^+)u_i^n$ converges in $L^2(Q_T)$. Therefore, $\partial_t(J_{\delta_i}u_i^n)$ converges also in $L^2(Q_T)$. Finally, this proves that u^n converges in X (at least up to a subsequence), whence the compactness of $\mathcal{T}$.

For the continuity of $\mathcal{T}$, let $v^n\to v$ in X as $n\to\infty$. If $\tilde u^n=(\tilde u_1^n,\dots,\tilde u_I^n)$ is the solution of

$$\forall\, i=1,\dots,I,\ \ \tilde u_i^n-\delta_i\Delta\tilde u_i^n=v_i^n \text{ on } Q_T,\ \partial_n\tilde u_i^n=0 \text{ on } \Sigma_T,$$

then $\tilde u_i^n$ converges in $L^2\left(0,T;H^2(\Omega)\right)$ to the solution $\tilde u_i$ of

$$\tilde u_i-\delta_i\Delta\tilde u_i=v_i \text{ on } Q_T,\ \partial_n\tilde u_i=0 \text{ on } \Sigma_T.$$

By definition, $u^n=\mathcal{T}(v^n)=(u_1^n,\dots,u_I^n)$ is the solution of

$$\left\{\begin{array}{l} u_i^n\in L^2(Q_T),\ \int_0^t a_i([\tilde u^n]^+)u_i^n\in L^2\left(0,T;H^2(\Omega)\right),\\ u_i^n-\Delta\left(\int_0^t a_i([\tilde u^n]^+)u_i^n\right)=u_i^0 \text{ on } Q_T,\ \partial_n\left(\int_0^t a_i([\tilde u^n]^+)u_i^n\right)=0 \text{ on } \Sigma_T.\end{array}\right.\tag{1.25}$$

Using the compactness of $\mathcal{T}$ proven above, the sequence $(u^n)_{n\in\mathbb{N}}$ is relatively compact in X. Let $u^\infty=\lim_{p\to\infty}u^{n_p}$ be a limit point. Up to a subsequence, $\tilde u_i^{n_p}$ converges a.e. to $\tilde u_i$. By continuity of a_i, $a_i([\tilde u^{n_p}]^+)\to A_i:=a_i([\tilde u]^+)$ almost everywhere, and it is uniformly bounded from above and from below. According to Lemma 1.4, we can pass to the limit as $n_p\to+\infty$ in (1.25). By the uniqueness result in Lemma 1.4 with $A=A_i$, we necessarily have $u^\infty=\mathcal{T}(v)$. The sequence $(u^n)_{n\in\mathbb{N}}$ lies in a compact set and has a unique possible limit point, so $u^n=\mathcal{T}(v^n)\to\mathcal{T}(v)$ and $\mathcal{T}$ is continuous on X.

□

Proof of Proposition 1.2. Let $T \in (0,\infty)$ and $\sigma \in [0,1]$. Suppose that $u \in X$ is a solution of $u = \sigma\mathcal{T}(u)$. By definition of $\mathcal{T}$, we have

$$\begin{cases} \forall i=1,\ldots,I, u_i \in L^2(Q_T),\ u_i \geq 0, \\ \tilde{u}_i, \int_0^t a_i(\tilde{u})u_i \in L^2(0,T;H^2(\Omega)), \\ \tilde{u}_i - \delta_i\Delta\tilde{u}_i = u_i \text{ on } Q_T, \partial_n\tilde{u}_i = 0 \text{ on } \Sigma_T, \\ u_i - \Delta\int_0^t a_i(\tilde{u})u_i = \sigma u_i^0 \text{ on } Q_T, \partial_n(\int_0^t a_i(\tilde{u})u_i) = 0 \text{ on } \Sigma_T. \end{cases} \tag{1.26}$$

The initial conditions σu_i^0 are uniformly bounded in $L^2(\Omega)$ for $\sigma \in [0,1]$. Therefore, by the estimate (1.22), the function u_i remains bounded in $L^2(Q_T)$, independently of σ. We also have $\partial_t(J_{\delta_i}u_i) = \Delta J_{\delta_i}(a_i(\tilde{u})u_i)$, so u is bounded in X independently of σ. Using Proposition 1.7 and Leray-Schauder's Lemma 1.3, we can conclude that $\mathcal{T}$ has a fixed point, which is a nonnegative solution of (1.7) (the nonnegativity of $\tilde{u}_i$ is a consequence of $u_i \geq 0$ and of the maximum principle property of $(I-\delta_i\Delta)^{-1}$ with homogeneous Neumann boundary conditions, see e.g. [30], Proposition 9.30).

1.3 $\mathbf{L^\infty}$ - estimate of $\tilde{\mathbf{u}}$ in Proposition 1.2

A main estimate in the proof of Theorem 1.1 is given in the next proposition.

Proposition 1.8. *Assume $u^0 \in L^\infty(\Omega,[0,+\infty))^I$ and* (1.3), (1.6) *as in Proposition 1.2. Let us define*

$$\forall k \geq 0,\ \ G(k) = \max_i\{\sup_{r\in[0,k]^I} a_i(r)\}. \tag{1.27}$$

Then, for any solution $u, \tilde{u}$ of Proposition 1.2, we have

$$\max_{1\leq i\leq I}\left\{\delta_i\|\tilde{u}_i\|_{L^\infty(Q_T)} + \|\int_0^t a_i(\tilde{u})u_i\|_{L^\infty(Q_T)}\right\} \leq M_0 + M_1\, T\, G(k_0), \tag{1.28}$$

where M_0, M_1 and k_0 depend only on $u^0, \underline{\delta} := \min_i \delta_i, \overline{\delta} := \max_i \delta_i$.

The proof of Proposition 1.8 uses the following classical lemma.

Lemma 1.9. *Let $f \in L^\infty(\Omega)$ and let w satisfy*

$$w \in H^2(\Omega),\ w \geq 0,\ \ -\Delta w \leq f \text{ on } \Omega,\ \partial_n w = 0 \text{ on } \partial\Omega.$$

Then there exists $C = C(\Omega)$ such that

$$\|w\|_{L^\infty(\Omega)} \leq C\left(\|f\|_{L^\infty(\Omega)} + \int_\Omega w\right). \tag{1.29}$$

Proof. First, we rewrite the equation as $w - \Delta w \leq f + w$. Let us fix $p \in (N/2,\infty)$. Using $w \geq 0$, the comparison principle and elliptic regularity theory, we know (see e.g. [53], Theorem 8.15) the existence of $C = C(\Omega,p)$ such that

$$\begin{aligned} \|w\|_{L^\infty} &\leq C(\|f+w\|_{L^p}) \leq C(\|f\|_{L^p} + \|w\|_{L^p}), \\ &\leq C\left(\|f\|_{L^p} + \|w\|_{L^\infty}^{(p-1)/p}(\int_\Omega w)^{1/p}\right), \\ &\leq C\left(\|f\|_{L^p} + \varepsilon\|w\|_{L^\infty} + c(\varepsilon)\int_\Omega w\right) \text{ (Young's inequality)} \end{aligned}$$

and we conclude choosing ε small enough.

□

Remark 1.10. Obviously, the conclusion of Lemma 1.9 would be the same when assuming only $f \in L^p(\Omega), p > N/2$.

Proof of Proposition 1.8. We rewrite the equations in $u_i, \tilde u_i$ of Proposition 1.2 as

$$\tilde u_i - \Delta\left(\delta_i \tilde u_i + \int_0^t a_i(\tilde u)u_i\right) = u_i^0, \quad \tilde u_i - \Delta w_i = u_i^0, \quad w_i = \delta_i \tilde u_i + \int_0^t a_i(\tilde u)u_i. \tag{1.30}$$

We sum up the equations (1.30), denoting $\tilde U = \sum_i \tilde u_i, W = \sum_i w_i$:

$$\tilde U - \Delta W = U^0 := \sum_i u_i^0. \tag{1.31}$$

Next, we apply Lemma 1.9 with $w = W(t), a.e.t, f = U^0$ (note that $-\Delta W(t) \le U^0$). It gives

$$a.e.t,\ \|W(t)\|_{L^\infty(\Omega)} \le C\left(\|U^0\|_{L^\infty(\Omega)} + \int_\Omega W(t)\right). \tag{1.32}$$

By nonnegativity of $\tilde u_i, a_i(\tilde u)u_i$, we also have (see the definitions of W, w_i):
$\forall i = 1,\dots,I,\ a.e.t \in [0,T]$:

$$\delta_i \|\tilde u_i(t)\|_{L^\infty(\Omega)}, \|\int_0^t a_i(\tilde u)u_i\|_{L^\infty(\Omega)} \le \|W(t)\|_{L^\infty(\Omega)}.$$

Then, to end the proof of Proposition 1.8, it is sufficient to prove the following lemma.

Lemma 1.11.

$$a.e.t \in [0,T],\ \int_\Omega W(t) \le C_0 + C_1 T G(k_0),$$

where C_0, C_1, k_0 depend only on $u^0, \underline{\delta}, \overline{\delta}$ and G is defined in (1.27).

Proof of Lemma 1.11. By integrating the equations on u_i and $\tilde u_i$ in Proposition 1.2, we get:

$$\forall t \ge 0, \qquad \int_\Omega u_i(t) = \int_\Omega \tilde u_i(t) = \int_\Omega u_i^0. \tag{1.33}$$

Recall that $\tilde u_i, w_i \in L^2(0,T;H^2(\Omega)), a_i(\tilde u)u_i \in L^2(Q_T)$. We also have $\partial_t \tilde u_i = \Delta J_{\delta_i}(a_i(\tilde u)u_i) \in L^2(Q_T)$. From (1.30), we may write, with $\partial_t w_i = \delta_i \partial_t \tilde u_i + a_i(\tilde u)u_i \in L^2(Q_T)$,

$$\partial_t w_i - \delta_i \Delta(\partial_t w_i) = a_i(\tilde u)u_i. \tag{1.34}$$

Differentiating $\partial_n w_i = 0$ with respect to t on $\partial\Omega$ leads formally to $\partial_n(\partial_t w_i) = 0$. Let us check that $\partial_t w_i = \theta(t)$ where $\theta(t)$ is the unique solution of

$$\theta \in L^2\big(0,T;H^2(\Omega)\big),\ a.e.t \in [0,T],\ \theta(t) - \delta_i \Delta\theta(t) = \big(a_i(\tilde u)u_i\big)(t),\ \partial_n \theta(t) = 0 \text{ on } \partial\Omega. \tag{1.35}$$

Using also $a_i(\tilde u)u_i \ge 0$, it will then follow that

$$\partial_t w_i \ge 0,\ \partial_t w_i \in L^2(0,T;H^2(\Omega)),\ \|\partial_t w_i\|_{L^2(Q_T)} \le \|a_i(\tilde u)u_i\|_{L^2(Q_T)}. \tag{1.36}$$

By integration in time of (1.35), and with $\Theta(t) = \int_0^t \theta(s)ds$, we have

$$\Theta(t) - \delta_i \Delta\Theta(t) = \int_0^t \big(a_i(\tilde u)u_i\big)(t),\ \partial_n \Theta(t) = 0 \text{ on } \partial\Omega.$$

Comparing with $w_i - \delta_i \Delta w_i = \delta_i u_i^0 + \int_0^t a_i(\tilde{u})u_i, \partial_n w_i = 0$ implies by uniqueness that: $\Theta(t) = w_i + (I - \delta_i \Delta)^{-1} u_i^0$, whence $\Theta'(t) = \theta = w_i$ after differentiating in t.

We denote

$$\widetilde{V} = \sum_i \delta_i \widetilde{u}_i, B = \sum_i a_i(\tilde{u})u_i.$$

Recall also that

$$\widetilde{U} = \sum_i \tilde{u}_i, W = \sum_i w_i, w_i = \delta_i \tilde{u}_i + \int_0^t a_i(\tilde{u})u_i.$$

Summing the I equations in $u_i, \tilde{u}_i$ as in (1.31), we have

$$\overline{\delta}^{-1} \widetilde{V} - \Delta W \leq \widetilde{U} - \Delta W = U^0. \tag{1.37}$$

We multiply this equation by $\partial_t W = \sum_i \partial_t w_i = \partial_t \widetilde{V} + B \geq 0$ (see (1.36)) and get

$$\overline{\delta}^{-1} \int_\Omega \widetilde{V}(\partial_t \widetilde{V} + B) + \frac{1}{2} \int_\Omega \partial_t |\nabla W|^2 \leq \int_\Omega U^0(\partial_t \widetilde{V} + B).$$

We integrate in time to obtain (we denote $\widetilde{V}^0 := \widetilde{V}(0) = W(0)$)

$$\int_\Omega \widetilde{V}^2(T) + \int_{Q_T} 2B\widetilde{V} + \overline{\delta} \int_\Omega |\nabla W(T)|^2 \leq \int_\Omega (\widetilde{V}^0)^2 + \overline{\delta} |\nabla \widetilde{V}^0|^2 + 2\overline{\delta} U^0(\widetilde{V}(T) - \widetilde{V}^0) + \int_{Q_T} 2\overline{\delta} U^0 B. \tag{1.38}$$

Since we have by definition

$$\bar{\delta} U^0 = \bar{\delta} \widetilde{U}^0 - \bar{\delta} \Delta \widetilde{V}^0 \geq \widetilde{V}^0 - \bar{\delta} \Delta \widetilde{V}^0,$$

we have

$$\int_\Omega (\widetilde{V}^0)^2 + \bar{\delta} |\nabla \widetilde{V}^0|^2 - 2\bar{\delta} U^0 \widetilde{V}^0 \leq - \int_\Omega (\widetilde{V}^0)^2 + \bar{\delta} |\nabla \widetilde{V}^0|^2 \leq 0,$$

so that (1.38) becomes

$$\int_\Omega \widetilde{V}^2(T) + \int_{Q_T} 2B\widetilde{V} + \overline{\delta} \int_\Omega |\nabla W(T)|^2 \leq 2\overline{\delta} \int_\Omega U^0 \widetilde{V}(T) + \int_{Q_T} 2\overline{\delta} U^0 B. \tag{1.39}$$

We have in particular, with $\|U^0\|_\infty = \|U^0\|_{L^\infty(\Omega)}$, and by using (1.33):

$$\int_{Q_T} B\widetilde{V} \leq \overline{\delta} \|U^0\|_\infty \left(\int_\Omega \widetilde{V}^0 + \int_{Q_T} B \right), \tag{1.40}$$

Thus, we have for any $k > 0$

$$k \int_{Q_T \cap \{\widetilde{V} \geq k\}} B \leq \overline{\delta} \|U^0\|_\infty \left(\int_\Omega \widetilde{V}^0 + \int_{Q_T \cap \{\widetilde{V} < k\}} B + \int_{Q_T \cap \{\widetilde{V} \geq k\}} B \right). \tag{1.41}$$

Note that, $\{\widetilde{V} < k\} \subset \cap_i \{\tilde{u}_i \leq k\underline{\delta}^{-1}\}$. Thanks to the L^1 estimate (1.33), we have

$$\int_{Q_T \cap \{\widetilde{V} < k\}} B = \int_{Q_T \cap \{\widetilde{V} < k\}} \sum_i a_i(\tilde{u})u_i \leq T \left[\int_\Omega U^0 \right] G(k\underline{\delta}^{-1}),$$

where G is defined in (1.27). Finally choosing $k = 2\overline{\delta}\|U^0\|_\infty$ in (1.41), we obtain

$$\int_{Q_T\cap\{\widetilde{V}\geq k\}} B \leq \left(2\int_\Omega \widetilde{V}^0 + T\left[\int_\Omega U^0\right] G(k_0)\right),\ k_0 = 2\underline{\delta}^{-1}(\overline{\delta}\|U^0\|_\infty).$$

Adding the two last inequalities gives

$$\int_{Q_T} B \leq C_0 + C_1 T\, G(k_0), \tag{1.42}$$

where C_1 depends only on $u^0, \overline{\delta}, \underline{\delta}$.

To end the proof of Lemma 1.11, we use that $W(t) = \sum_i \delta_i \tilde{u}_i(t) + \int_0^t B(s)ds$ so that

$$\forall t \in [0,T],\ \int_\Omega W(t) \leq \int_\Omega \widetilde{U}^0 + \int_{Q_T} B.$$

□

From the L^∞-estimate of Proposition 1.8, we may now deduce that the problem (1.4) in Theorem 1.1 has at least a *weak solution* under the only assumption of continuity of the a_i's.

Corollary 1.12. *Assume* (1.3) *(only) and* $\forall i = 1,\ldots,I, u_i^0 \in L^\infty(\Omega;[0,\infty))$. *Then, there exists a nonnegative solution* $u = (u_1,\ldots,u_I)$ *to the system*

$$\begin{cases} \forall T > 0,\ \forall i = 1,\ldots,I, \\ u_i, a_i(\tilde{u})u_i \in L^2(Q_T),\ \int_0^t a_i(\tilde{u})u_i \in L^2\left(0,T;H^2(\Omega)\right), \\ \tilde{u}_i \in L^\infty(Q_T)\cap L^2\left(0,T;H^2(\Omega)\right),\ \tilde{u}_i - \delta_i\Delta\tilde{u}_i = u_i \text{ on } Q_T, \\ u_i - \Delta(\int_0^t a_i(\tilde{u})u_i) = u_i^0 \text{ on } Q_T, \\ \partial_n\tilde{u}_i = 0 = \partial_n(\int_0^t a_i(\tilde{u})u_i) \text{ on } \Sigma_T. \end{cases} \tag{1.43}$$

Proof. Here, a_i is assumed to satisfy only (1.3) (and not (1.6)). Let $T > 0$. We introduce $M_2 := \underline{\delta}^{-1}[M_0 + M_1 T G(k_0)]$ where the function G is defined in (1.27) of Proposition 1.8 and M_0, M_1, k_0 are defined in (1.28) of the same proposition. We define

$$\forall r \in [0,M_2]^I,\ \overline{a}_i(r) := a_i(r), \forall r \in [0,\infty)^I \setminus [0,M_2]^I, \overline{a}_i(r) = \min\{a_i(r), G(M_2)\}. \tag{1.44}$$

Then, $\overline{a}_i$ is continuous on $[0,\infty)^I$ and

$$0 < \underline{d} \leq \overline{a}_i \leq G(M_2) < \infty,\ \overline{a}_i \leq a_i.$$

Therefore, we may apply Proposition 1.2 with a_i replaced by $\overline{a}_i$. By Proposition 1.8, the corresponding $\tilde{u}$ satisfies

$$\forall i = 1,\ldots,I,\ \|\tilde{u}_i\|_{L^\infty(Q_T)} \leq \underline{\delta}^{-1}\left[M_0 + M_1 T\overline{G}(k_0)\right],$$

where $\overline{G}$ is defined as in (1.27) with a_i replaced by $\overline{a}_i$. But $\overline{G}(k_0) \leq G(k_0)$, so that

$$\forall i = 1,\ldots,I,\ 0 \leq \tilde{u}_i \leq M_2,\ \overline{a}_i(\tilde{u}) = a_i(\tilde{u}).$$

Therefore, the solution obtained with the data $\overline{a}_i$ is also solution with the data a_i.

This provides a solution of (1.43) in Corollary 1.12 with the estimate (1.28), but only on $[0,T]$ and it may depend on T. To construct a global solution on $(0,\infty)$, we may argue as follows: let T_p be an increasing sequence of times with $\lim_{p\to+\infty} T_p = +\infty$. Let u^p be a solution of our problem on the interval $[0,T_p]$ given by the above proof. For $k \in \mathbb{N}$, we denote by X^k the space X as defined in (1.23) with T replaced by T_k and we denote by $\mathcal{T}^k : X^k \to X^k$ the operator $\mathcal{T}$ with $T = T_k$. For $p \geq k$, we denote $u^{p,k} := u^p_{[0,T_k]}$ so that $\mathcal{T}^k(u^{p,k}) = u^{p,k}$. We will prove that

$$\forall k \in \mathbb{N},\ (u^{p,k})_{p\geq k} \text{ is relatively compact in } X^k. \tag{1.45}$$

Thus, using a diagonal process, we obtain a sequence $p_m \to \infty$ as $m \to \infty$ and some limit u defined on $(0,\infty)$ so that, for all $k \in \mathbb{N}$, $u^{p_m,k}$ converges to $u_{[0,T_k]}$ in X^k as $m \to \infty$. Then, $\mathcal{T}_k(u_{[0,T_k]}) = u_{[0,T_k]}$ and u is a global solution of (1.43).

Let k be fixed in $\mathbb{N}$ and let us prove (1.45). By the L^∞-estimate (1.28) in Proposition 1.8,

$$\forall\, p \geq k,\ \ \|\tilde{u}^p_i\|_{L^\infty(Q_{T_k})} \leq \frac{1}{\delta_i}[M_0 + M_1 T_k G(k_0)]. \tag{1.46}$$

Thus, $a_i(\tilde{u}^p)$ is uniformly bounded on Q_{T_k}. This implies by (1.22) that u^p is bounded in $L^2(Q_{T_k})^I$ and so is $\partial_t \tilde{u}^p$ since by (1.36)

$$\delta_i \|\partial_t \tilde{u}^p_i\|_{L^2(Q_{T_k})} \leq 2\|a_i(\tilde{u}^p)u^p_i\|_{L^2(Q_{T_k})} \leq C(k).$$

Thus, $u^{p,k}$ is bounded in X^k and, by compactness of $\mathcal{T}^k$, it is relatively compact in X^k, whence (1.45).

□

1.4 Proof of existence in Theorem 1.1

Existence of a *weak solution* to (1.4) is already proved in Corollary 1.12. It only remains to prove that this solution is actually as regular as stated in Theorem 1.1. This will mainly be a consequence of the L^∞-estimate on $\tilde{u}$ proved in the previous section, namely

$$\forall i = 1,\dots,I,\ \|\tilde{u}_i\|_{L^\infty(Q_T)} \leq C_0 + C_1 T,\ \ \|a_i(\tilde{u})\|_{L^\infty(Q_T)} \leq C(T),$$

where C_0, C_1 depend only on the data and $C(T) = G(C_0 + C_1 T)$.

We begin by the following simple estimates.

Proposition 1.13. *Let $w_i = \delta_i \tilde{u}_i + \int_0^t a_i(\tilde{u})u_i$ where $u, \tilde{u}$ is solution of (1.43) in Corollary 1.12. Assume $u^0 \in L^\infty(\Omega,[0,\infty))^I$. Then,*

$$\forall T > 0,\ \nabla w_i \in L^\infty(Q_T)^N,\ w_i, \partial_t w_i \in L^\infty(Q_T),\ \partial_t w_i \geq 0. \tag{1.47}$$

Proof. The fact that $w_i \in L^\infty(Q_T)$ is a consequence of (1.32) and Lemma 1.11. We recall the two equations (see (1.30), (1.34)):

$$\tilde{u}_i - \Delta w_i = u^0_i,\ \partial_t w_i - \delta_i \Delta(\partial_t w_i) = a_i(\tilde{u})u_i.$$

Since $w_i, \Delta w_i \in L^\infty(Q_T)$ and $\partial_n w_i = 0$ on Σ_T, we deduce that $\nabla w_i \in L^\infty(Q_T)^N$ (at least). We have already seen that $\partial_t w_i \geq 0$ comes directly from the second equation and the nonnegativity of $a_i(\tilde{u})u_i$. Now we rewrite this equation as

$$(\partial_t w_i - C(T)\tilde{u}_i) - \delta_i \Delta(\partial_t w_i - C(T)\tilde{u}_i) = (a_i(\tilde{u}) - C(T))u_i \leq 0.$$

Together with $\partial_n(\partial_t w_i - C(T)\tilde{u}_i) = 0$ on Σ_T, this implies

$$\partial_t w_i - C(T)\tilde{u}_i \leq 0,\ \textit{so that}\ 0 \leq \partial_t w_i \leq C(T)[C_0 + C_1 T].$$

□

We will now prove that $U_i(t,x) := \int_0^t [a_i(\tilde{u})u_i](s,x)ds$ is in $C^\alpha(\overline{Q_T})$ so that, since $\tilde{u}_i = w_i - U_i$, it will follow that $\tilde{u}_i$ is not only bounded, but Hölder-continuous (at least).

To prove it, we rely on the C^α-regularity theory of Krylov-Safonov for the solutions of non-divergence parabolic equations with bounded coefficients. We actually use them in the rather particular case of the operator $-A\Delta$ where A is bounded from above and from below. We may state the result we need as follows:

Lemma 1.14. *Let $A \in C(\overline{Q_T}), g \in L^\infty(Q_T), \underline{a}, \overline{a} \in (0,\infty)$ with $0 < \underline{a} \le A \le \overline{a} < \infty$. Let $w \in C^{2,1}(Q_T) \cap C^{1,1}(\overline{Q_T})$ solution of*

$$\begin{cases} \partial_t w - A\Delta w = g \text{ in } Q_T \\ \partial_n w = 0 \text{ on } \Sigma_T,\ w(0) = 0. \end{cases} \tag{1.48}$$

Then, there exists $\alpha \in (0,1), C > 0$ such that

$$\|w\|_T^{(\alpha)} \le C \tag{1.49}$$

where α, C depend only on $\underline{a}, \overline{a}, T, \|g\|_{L^\infty(Q_T)}, \Omega$.

Remark 1.15. Note that an estimate in L^∞ for w is easy by a comparison argument (valid here thanks to the a priori regularity of w and of A): we remark that the function $W(t,x) := t \sup g$ is a supersolution of the problem (1.48), so that $W \ge w$. Doing the same from below, we obtain

$$\|w\|_{L^\infty(Q_T)} \le T\|g\|_{L^\infty(Q_T)}. \tag{1.50}$$

Next, we may use the Krylov-Safonov result: the global estimate with homogeneous Neumann boundary conditions as stated above may, for instance, be found in [42, Lemma 2.2] (in a quite more general setting). We more generally refer to [42, 67, 77] for this kind of results.

We apply this result to prove the regularity of $U_i = \int_0^t a_i(\tilde{u})u_i$.

Proposition 1.16. *Let $T > 0$ and $u^0 \in L^\infty(\Omega, [0,\infty))^I$. There exists $\alpha \in (0,1), C > 0$ such that*

$$\|U_i\|_T^{(\alpha)} + \|\tilde{u}_i\|_T^{(\alpha)} \le C.$$

Proof. Let $u, \tilde{u}$ be the solution of (1.43) in Corollary 1.12. Recall that $0 < \underline{d} \le a_i(\tilde{u}) \le C(T)$. Since Lemma 1.14 a priori applies to regular solutions only, we will use a convenient approximation of u. For this, let A^n be a smooth approximation of $a_i(\tilde{u})$ such that

$$0 < \underline{d} \le A^n \le C(T),\ A^n \to a_i(\tilde{u})\, a.e.$$

Let also v^n be a smooth approximation of u_i^0 such that

$$0 \le v^n \le \|u_i^0\|_{L^\infty(\Omega)},\ v^n \to u_i^0 \text{ in } L^2(\Omega).$$

Let u_i^n be the solution of

$$\partial_t u_i^n - \Delta(A^n u_i^n) = 0,\ \partial_n u_i^n = 0 \text{ on } \Sigma_T, u_i^n(0) = v^n.$$

Then, after integration in time, we see that $U_i^n = \int_0^t A^n u_i^n$ satisfies

$$\partial_t U_i^n - A^n \Delta U_i^n = A^n v^n,\ \partial_n U_i^n = 0 \text{ on } \Sigma_T,\ U_i^n(0) = 0. \tag{1.51}$$

By Lemma 1.14, there exists α, C independent of n such that $\|U_i^n\|_T^{(\alpha)} \le C$. By Lemma 1.4, u_i^n converges to u_i in $L^2(Q_T)$ which implies that U_i^n also converges to U_i in $L^2(Q_T)$. Whence the estimate of Proposition 1.16 on U_i. The estimate on $\tilde{u}_i = w_i - U_i$ follows by combining with Proposition 1.13 which says that w_i is even Lipschitz continuous.

□

Now that we know that the coefficient $a_i(\tilde{u})$ is not only bounded but also continuous, we may continue improving the regularity of u.

Proposition 1.17. *Assume $u^0 \in L^\infty(\Omega, [0,\infty))^I$. Then,*

$$\forall p \in [1,\infty),\ \forall T > 0,\ \forall i = 1,\dots,I,\ u_i, \partial_t U_i, \Delta U_i \in L^p(Q_T).$$

Proof. We may formally write

$$\partial_t U_i - a_i(\tilde{u})\Delta U_i = a_i(\tilde{u})u_i^0,\ \partial_n U_i = 0,\ U_i(0) = 0. \tag{1.52}$$

Here $a_i(\tilde{u})$ is continuous on $\overline{Q_T}$ so that, $a_i(\tilde{u})$ being given, this equation has a unique solution : let us call it V_i. We set $v_i := \partial_t V_i / a_i(\tilde{u})$. Then

$$v_i - \Delta V_i = u_i^0,\ V_i = \int_0^t a_i(\tilde{u})v_i,\ \partial_n\Big(\int_0^t a_i(\tilde{u})v_i\Big) = 0.$$

Thus, v_i coincides with our u_i (and V_i coincides with our U_i) thanks to the uniqueness result of Lemma 1.4.

Moreover, L^p-maximal regularity holds for the equation (1.52) since $a_i(\tilde{u})$ is continuous (see e.g. [69, Theorem 9.1], or [79, Theorem 2.5.2]) so that, as $a_i(\tilde{u})u_i^0 \in L^\infty(Q_T) \subset L^p(Q_T)$, we have

$$\forall p \in (1,\infty),\ \|\partial_t U_i\|_{L^p(Q_T)}, \|\Delta U_i\|_{L^p(Q_T)} \le C,$$

where C depends on $p, \|a_i(\tilde{u})u_i^0\|_{L^\infty(Q_T)}$ and on the modulus of continuity of the function $a_i(\tilde{u})$.

Next, from $0 \le u_i \le \underline{d}^{-1}a_i(\tilde{u})u_i = \underline{d}^{-1}|\partial_t U_i|$, we deduce that $u_i \in L^p(Q_T)$ as well. And $p = 1$ is also included since Q_T is bounded.

□

With Proposition 1.17, the first part of the existence result in Theorem 1.1 is now complete. We will now assume that a_i is locally Lipschitz continuous.

Proposition 1.18. *Besides (1.3), assume a_i is locally Lipschitz continuous for all $i = 1,\dots,I$. Assume also $u^0 \in L^\infty(\Omega, [0,\infty))^I$. Then*

$$\forall i = 1,\dots,I,\ \forall T > 0,\ u_i \in L^\infty(Q_T),\ \forall p \in [1,\infty), \forall \tau \in (0,T),\ \partial_t u_i, \Delta(a_i(\tilde{u})u_i) \in L^p((\tau,T)\times\Omega),$$

and

$$\partial_t u_i - \Delta(a_i(\tilde{u})u_i) = 0 \text{ on } Q_T,\ \partial_n(a_i(\tilde{u})u_i) = 0 \text{ on } \Sigma_T,$$

is satisfied pointwise.

Proof. The equation in u_i may also be written (at least formally to start):

$$\partial_t(a_i(\tilde{u})u_i) - a_i(\tilde{u})\Delta(a_i(\tilde{u})u_i) = u_i D a_i(\tilde{u}) \cdot \partial_t \tilde{u}. \tag{1.53}$$

We know that $\delta_i \partial_t \tilde{u}_i + a_i(\tilde{u})u_i \in L^\infty(Q_T)$ (see Proposition 1.13), and $a_i(\tilde{u})u_i \in L^p(Q_T)$, for all $p < \infty$, so that $\partial_t \tilde{u}_i \in L^p(Q_T)$ for all $p < \infty$. The right hand side of this equation $F := u_i D a_i(\tilde{u}) \cdot \partial_t \tilde{u}$ is therefore in $L^p(Q_T)$ for all $p < \infty$ since also $Da_i(\tilde{u}) \in L^\infty(Q_T)^N$ (as a_i is locally Lipschitz and $\tilde{u}$ is bounded).

Since $a_i(\tilde{u})$ is continuous on $\overline{Q_T}$, we know (see again e.g. [69, Theorem 9.1], or [79, Theorem 2.5.2]), there exists a (unique) solution θ to

$$\begin{cases} \forall p < \infty,\ \theta \in C(0,T; L^p(\Omega)), \forall \tau \in (0,T), \partial_t \theta, \Delta\theta \in L^p((\tau,T)\times\Omega) \\ \partial_t \theta - a_i(\tilde{u})\Delta\theta = F,\ \partial_n \theta = 0 \text{ on } \Sigma_T,\ \theta(0) = a_i(\tilde{u}^0)u_i^0, \end{cases} \tag{1.54}$$

and we have

$$\|\theta\|_{L^\infty(Q_T)} + \|\partial_t \theta\|_{L^p((\tau,T)\times\Omega)} + \|\Delta\theta\|_{L^p((\tau,T)\times\Omega)} \le C[\|F\|_{L^p(Q_T)} + \|u_i^0\|_{L^\infty(\Omega)}], \tag{1.55}$$

where C depends on τ, T, p, Ω and of the modulus of continuity of $a_i(\tilde{u})$.

If we knew that $\theta = a_i(\tilde{u})u_i$, then the proof of Proposition 1.18 would be complete using moreover:

$$\partial_t u_i = a_i(\tilde{u})^{-1}[\partial_t(a_i(\tilde{u})u_i) - u_i Da_i(\tilde{u})\cdot\partial_t\tilde{u}] \in L^p((\tau,T)\times\Omega).$$

To prove it, we recall (see the proof of Lemma 1.4) that u_i is the limit of the approximate solutions u^n of

$$\partial_t u^n - \Delta(A^n u^n) = 0,\ \partial_n u^n = 0 \text{ on } \Sigma_T,\ u^n(0) = u_i^0,$$

where A^n is smooth and converges pointwise to $a_i(\tilde{u})$ with $0 < \min a_i(\tilde{u}) \le A^n \le \max a_i(\tilde{u}) < +\infty$. Moreover, u^n is bounded in $L^p(Q_T)^I$ for all $p < \infty$ by the analysis in Proposition 1.17. Here, we choose such an approximation A^n which moreover satisfies

$$A^n \to a_i(\tilde{u}) \text{ in } L^\infty(Q_T),\ \partial_t A^n \to \partial_t a_i(\tilde{u}) = Da_i(\tilde{u})\cdot\partial_t\tilde{u} \text{ in } L^p(Q_T)\ \forall p < \infty.$$

Then, we apply the estimates (1.55) to $A^n u^n$ which satisfies

$$\partial_t(A^n u^n) - A^n\Delta(A^n u^n) = u^n\partial_t A^n,\ \partial_n A^n = 0 \text{ on } \Sigma_T,\ A^n u^n(0) = a_i(\tilde{u}^0)u_i^0,$$

and they are preserved at the limit. Whence $\theta = a_i(\tilde{u})u_i$ by uniqueness in (1.54).

□

Proof of the Existence in Theorem 1.1. It is a consequence of Corollary 1.12 and of Propositions 1.16, 1.17, 1.18.

□

Remark 1.19. Note that, not only we proved existence of a solution with the announced regularity, but we even proved that any *weak solution* as in Corollary 1.12 has actually the announced regularity. This will be useful in the proof of uniqueness

Remark 1.20. The assumption that the a_i are bounded from below is essential in our proof of existence, first for the L^2-estimate, next to apply the Krylov-Safonov regularity theory. In the case when the a_i degenerate ($a_i \ge 0$), the L^2 a priori estimate is to be replaced by $\sqrt{a_i(\tilde{u})}u_i \in L^2(Q_T)$. However, we loose the L^2-compactness of the approximate solutions and also most regularity properties of the solution as well. It would however be interesting to study the possibility of existence of weak solutions.

Remark 1.21. The above analysis relies on the use of the C^α-Krylov-Safonov estimates. However, it is interesting to notice that one can prove directly, by an elementary estimate, that $u \in L^\infty(Q_T)$, without using these estimates in the (rather general situation) where, besides (1.3), a_i satisfies

$$\forall i = 1,\dots,I,\ a_i \text{ is locally Lipschitz continuous},\ \forall j = 1,\dots,I,\ \partial_{\tilde{u}_j} a_i \ge 0. \tag{1.56}$$

Once the L^∞-estimate is proved on u_i, the full regularity follows by the same arguments as in Proposition 1.18.

We indicate below (at least formally) the computations which leads to $u \in L^\infty(Q_T)$.

Proof of $\mathbf{u} \in \mathbf{L}^\infty(\mathbf{Q_T})$ under assumption (1.56).

We write a_i for $a_i(\tilde{u})$ and $a_{ij} = \partial_{\tilde{u}_j} a_i$. We multiply the equation $\partial_t u_i - \Delta(a_i u_i) = 0$ by $p(a_i u_i)^{p-1}$ and we integrate over Ω:

$$\frac{d}{dt}\int_\Omega a_i^{p-1} u_i^p + \int_\Omega p(p-1)(a_i u_i)^{p-2} |\nabla(a_i u_i)|^2 = (p-1)\sum_j \int_\Omega a_i^{p-2} u_i^p a_{ij} \partial_t \tilde{u}_j. \tag{1.57}$$

We proved in Proposition 1.13 that $\partial_t \tilde{u}_j + a_j u_j \le C(T) < \infty$. This implies $\partial_t \tilde{u}_j \le C(T)$. Plugging this into (1.57), using $a_{ij} \ge 0$, a_{ij} bounded and $a_i \ge \underline{d}$ leads with some C_T independent of p to:

$$\frac{d}{dt}\int_\Omega a_i^{p-1} u_i^p + \int_\Omega p(p-1)(a_i u_i)^{p-2} |\nabla(a_i u_i)|^2 \le C_T(p-1)\sum_j \int_\Omega a_i^{p-1} u_i^p.$$

Summing over i and using Gronwall's lemma on the term $\sum_i \int_\Omega a_i^{p-1} u_i^p$, we then have

$$\sum_i \int_\Omega a_i^{p-1} u_i^p(t) \le e^{ITC_T(p-1)} \sum_i \int_\Omega a_i^{p-1} u_i^p(0).$$

Using the lower and upper bounds on a_i, we have with A, C_T^1 both independent of p:

$$\sum_i \int_\Omega a_i^p u_i^p(t) \le A e^{C_T^1 p} (1 + \sum_i \|a_i u_i(0)\|_\infty)^p.$$

This implies

$$\|(a_i u_i)(t)\|_p \le A^{1/p} e^{C_T^1} (1 + \sum_i \|a_i u_i(0)\|_\infty),$$

whence the L^∞-estimate on $a_i u_i$ by letting $p \to \infty$, and then on u_i itself by using the lower bound on a_i.

□

1.5 Proof of uniqueness in Theorem 1.1

Actually, we will prove the following more general result:

Proposition 1.22. *Let $u^0 \in L^\infty(\Omega, [0,\infty))^I$. Assume that for all $i = 1,\ldots,I$, a_i satisfies* (1.3) *and is locally Lipschitz continuous. Then there exists a unique solution to the system* (1.43) *in Corollary 1.12.*

Proof. By Remark 1.19, we already know that any solution of (1.43) satisfies the regularity stated in Proposition 1.18 and Theorem 1.1. Let u, v be two such solutions. We denote $a_i = a_i(\tilde{u}), b_i = a_i(\tilde{v})$. By difference,

$$\partial_t(u_i - v_i) - \Delta[a_i(u_i - v_i) + v_i(a_i - b_i)] = 0.$$

We set

$$U_i = u_i - v_i, \tilde{U}_i = \tilde{u}_i - \tilde{v}_i, \widetilde{U} = \tilde{u} - \tilde{v}, A_i = \int_0^1 Da_i(t\tilde{u} + (1-t)\tilde{v})dt,$$

so that $a_i - b_i = A_i \cdot (\tilde{u} - \tilde{v}) = \sum_j A_{ij} \tilde{U}_j$. Note that $\|A_i\|_{L^\infty} < \infty$. Then

$$\partial_t U_i - \Delta\left[a_i U_i + v_i A_i \cdot \widetilde{U}\right] = 0, \ \ \partial_n(a_i U_i + v_i A_i \cdot \widetilde{U}) = 0. \tag{1.58}$$

Lemma 1.23. *Let $F \in C_0^\infty(Q_T)^I$. There exists a solution to the dual problem*

$$\begin{cases} \forall i = 1,\dots,I,\ \varphi_i, \partial_t \varphi_i, \Delta\varphi_i \in L^2(Q_T), \\ \partial_t \varphi_i + a_i \Delta\varphi_i + J_{\delta_i}(B_i \cdot \Delta\varphi) = F_i \text{ on } Q_T, \\ \varphi = (\varphi_1,\dots,\varphi_I), \partial_n \varphi_i = 0 \text{ on } \Sigma_T,\ \varphi_i(T) = 0, \end{cases} \tag{1.59}$$

where $B_i = (B_{i1},\dots,B_{iI}), B_{ij} = v_j A_{ji}$.

Assuming this lemma, we multiply each equation (1.58) by φ_i and we obtain after integration on Q_T (the integrations by parts are allowed, thanks to the regularity of $u, v, \tilde{u}, \tilde{v}, \varphi_i$ and the boundary conditions; we also use $\int_{Q_T} U_i J_{\delta_i}(B_i \cdot \Delta\varphi) = \int_{Q_T} \widetilde{U}_i B_i \cdot \Delta\varphi$):

$$0 = \int_{Q_T} U_i[\partial_t \varphi_i + a_i \Delta\varphi_i] + \Delta\varphi_i v_i A_i \cdot \tilde{U} = \int_{Q_T} U_i F_i - \tilde{U}_i B_i \cdot \Delta\varphi + \Delta\varphi_i v_i A_i \cdot \tilde{U}.$$

Summing these I identities gives $\sum_i \int_{Q_T} U_i F_i = 0$ which implies $U \equiv 0$ by arbitrarity of the F_i, whence uniqueness.

□

Proof of Lemma 1.23. To solve the dual problem (actually interesting for itself), we may start with a_i replaced by regular approximations A_i^n converging in the usual way to a_i (which means a.e. and uniformly bounded from above and from below), and we first solve

$$\partial_t \theta_i^n + \Delta(A_i^n \theta_i^n) + \Delta J_{\delta_i}(B_i \cdot \theta^n) = \Delta F_i,\ \partial_n(A_i^n \theta_i^n) = 0,\ \theta_i(T) = 0.$$

This is possible since $\theta \in L^2(Q_T)^I \to \left(\Delta J_{\delta_i}(B_i \cdot \theta)\right)_{1 \le i \le I} \in L^2(Q_T)^I$ is a Lipschitz perturbation (recall that $B_i \in L^\infty$ and ΔJ_{δ_i} is the Yosida approximation of the operator $-\Delta$ with homogeneous Neumann boundary conditions). Note that $\int_\Omega \theta_i^n(t) = 0$. Next, we solve

$$\Delta\varphi_i^n = \theta_i^n \text{ in } \Omega,\ \partial_n(\varphi_i^n) = 0 \text{ on } \partial\Omega, \int_\Omega \phi_i^n = 0,$$

so that, "by applying Δ^{-1}" to the equation in θ_i^n, we obtain

$$\partial_t \varphi_i^n + A_i^n \Delta\varphi_i^n + J_{\delta_i}(B_i \cdot \Delta\varphi^n) = F_i,\ \partial_n(\varphi_i^n) = 0 \text{ on } \Sigma_T,\ \varphi_i^n(T) = 0. \tag{1.60}$$

Next, multiplying by $\Delta\varphi_i^n$ gives

$$\int_\Omega -\frac{1}{2}\partial_t |\nabla\varphi_i^n|^2 + A_i^n (\Delta\varphi_i^n)^2 + \Delta\varphi_i^n J_{\delta_i}(B_i \cdot \Delta\varphi^n) = \int_\Omega F_i \Delta\varphi_i^n \le \int_\Omega \varepsilon(\Delta\varphi_i^n)^2 + C_\varepsilon F_i^2.$$

We choose $\varepsilon := \underline{d}/2$ and deduce

$$\int_\Omega -\frac{1}{2}\partial_t |\nabla\varphi_i^n|^2 + \frac{d}{2}(\Delta\varphi_i^n)^2 \le C\int_\Omega F_i^2 + \int_\Omega \nabla Z \nabla\varphi_i^n \le C\int_\Omega F_i^2 + \int_\Omega \varepsilon|\nabla Z|^2 + C_\varepsilon |\nabla\varphi_i^n|^2, \tag{1.61}$$

where $Z - \delta_i \Delta Z = B_i \cdot \Delta\varphi^n$, $\partial_n Z = 0$. Multipling this by Z gives

$$\int_\Omega Z^2 + \delta_i |\nabla Z|^2 = \int_\Omega Z B_i \cdot \Delta\varphi^n \le \|B_i\|_{L^\infty} \int_\Omega \varepsilon Z^2 + C_\varepsilon |\Delta\varphi^n|^2 \Rightarrow \int_\Omega |\nabla Z|^2 \le C\int_\Omega |\Delta\varphi^n|^2.$$

Summing the equations in (1.61) and choosing adequately ε leads to (with a different C)

$$-\partial_t \int_\Omega \sum_i |\nabla\varphi_i^n|^2 + \frac{d}{2}\sum_i \int_\Omega (\Delta\varphi_i^n)^2 \le C\left[\int_\Omega \sum_i [F_i^2 + |\nabla\varphi_i^n|^2]\right].$$

Integrating the Gronwall estimate in $\sum_i |\nabla\varphi_i^n|^2$ and plugging back the terms in $\Delta\varphi_i^n$ yield

$$\sup_{0\le t\le T}\int_\Omega \sum_i |\nabla\varphi_i^n|^2 + \frac{d}{2}\int_{Q_T} |\Delta\varphi^n|^2 \le C\int_{Q_T} |F|^2.$$

By going back to (1.60), we also obtain that $\partial_t\varphi_i^n$ is bounded in $L^2(Q_T)$. Now, we can pass to the limit as $n\to\infty$, weakly in $L^2(Q_T)$ in each term of (1.60), to prove the existence result of Lemma 1.23.

□

Remark 1.24. We do not know whether uniqueness holds without assuming Lipschitz continuity of the a_i. The above proof indicates that uniqueness is essentially equivalent to solving the "dual" problem (1.59). The fact that $B_i \in L^\infty(Q_T)$ (which is equivalent to the Lipschitz continuity of a_i) is strongly used in the estimates to solve (1.59). It is not clear how to weaken it.

1.6 A constructive approximation procedure

In this subsection, we give an alternative proof of the existence of solutions as stated in Corollary 1.12, which follows the ideas of [70]. It relies on an approximation procedure, built on a time semi-discretization with an explicit treatment of $\tilde u$ and an implicit treatment of u_i in $a_i(\tilde u)u_i$. An interesting point is that it provides a *constructive approach* which may be used to provide numerical approximations.

Let $T>0$, $n_0\in\mathbb{N}^*$ and $\tau = T/n_0 > 0$ be the time step. We introduce the following approximate system: for $n\in\{1,\dots,n_0-1\}$ and $i\in\{1,\dots,I\}$,

$$\begin{cases} \frac{u_i^{n+1}-u_i^n}{\tau} - \Delta[a_i(\tilde u^n)u_i^{n+1}] &= 0 \qquad \text{in } \Omega,\\ -\delta_i\Delta\tilde u_i^n + \tilde u_i^n &= u_i^n \qquad \text{in } \Omega,\ \delta_i\in(0,+\infty),\\ \partial_n\tilde u_i^n = \partial_n[a_i(\tilde u^n)u_i^{n+1}] &= 0 \qquad \text{on } \partial\Omega. \end{cases} \tag{1.62}$$

The point is to get the existence of solutions by proving the convergence of the above sequence rather that using the Leray-Schauder fixed point theorem. As before, we start by assuming that besides (1.3), the a_i are bounded above, *i.e.* there exist $\underline{d},\overline{d}>0$ such that

$$0<\underline{d}\le a_i\le\overline{d}<+\infty.$$

Wellposedness of the scheme, nonnegativity.

Let $u_i^0\in L^\infty(\Omega)$, set $\tilde u_i^{-1}=0$ by convention and let us prove by induction that for $n\in\{0,\dots,n_0\}$, the following property holds:

(P_n) $\quad u_i^n,\tilde u_i^{n-1}$ are uniquely determined by (1.62) ; $u_i^n,\tilde u_i^{n-1}\in L^\infty(\Omega,[0,+\infty))$.

(P_0) is obviously true. Assume (P_n) is true for $n\in\{0,\dots,n_0-1\}$, then elliptic regularity theory guarantees that equations (1.62) define a unique function $\tilde u_i^n$ that belongs to $W^{2,p}(\Omega,[0,+\infty))$ for any $p<\infty$. Then $\tilde u^n = (\tilde u_1^n,\dots,\tilde u_I^n)\in C(\overline{\Omega},[0,+\infty)^I)$ and we can define v_i^{n+1} as the solution of

$$v_i^{n+1} - \tau a_i(\tilde u^n)\Delta v_i^{n+1} = a_i(\tilde u^n)u_i^n \text{ in } \Omega\,;\ \partial_n v_i^{n+1} = 0 \text{ on } \partial\Omega.$$

Since $a_i(\tilde u_i^n)\in C(\overline{\Omega})$, v_i^{n+1} is uniquely determined and $v_i^{n+1}\in W^{2,p}(\Omega,[0,+\infty))$ for any $p<\infty$ (see e.g. [53]). Then $u_i^{n+1} := a_i(\tilde u^n)^{-1}v_i^{n+1}$ satisfies the first and last equation in (1.62) and using the lower bound on a_i, $u_i^{n+1}\in L^\infty(\Omega,[0,+\infty))$. By induction, (P_n) is true for all $n\in\{1,\dots,n_0\}$, which proves the wellposedness of the scheme (1.62).

A priori-estimates: discrete versions.

We now derive the discrete analogs of the *a priori* estimates from Section 1.2. Remember that a_i is still supposed to be bounded above by $\overline{d}$.

Notations.

$$a_i^n = a_i(\tilde{u}_i^n)\ ; \quad A^{n+1} = \sum_{i=1}^I a_i^n u_i^{n+1}\ ; \quad U^n = \sum_{i=1}^I u_i^n\ ; \quad \tilde{U}^n = \sum_{i=1}^I \tilde{u}_i^n\ ; \quad \tilde{V}^n = \sum_{i=1}^I \delta_i \tilde{u}_i^n\ ;$$

$$w_i^{n+1} = \delta_i \tilde{u}_i^{n+1} + \tau \sum_{k=0}^n a_i^k u_i^{k+1}\ ; \quad W^{n+1} = \sum_{i=1}^I w_i^{n+1}\ ; \quad \overline{\delta} = \max_i \delta_i\ ; \quad \underline{\delta} = \min_i \delta_i.$$

Mass Conservation. The first thing to notice is that due to the homogeneous Neumann boundary conditions, for all $n \in \mathbb{N}$ and all $i \in \{1,\dots,I\}$,

$$\int_\Omega u_i^n = \int_\Omega \tilde{u}_i^n = \int_\Omega u_i^0. \tag{1.63}$$

Analog of the $\mathbf{L^\infty(Q_T)}$-estimate on $\tilde{U}$.
Let us first prove that

$$\forall n \in \mathbb{N}, \quad w_i^{n+1} - w_i^n \geq 0. \tag{1.64}$$

We have

$$\frac{w_i^{n+1} - w_i^n}{\tau} = a_i^n u_i^{n+1} + \delta_i \left(\frac{\tilde{u}_i^{n+1} - \tilde{u}_i^n}{\tau} \right).$$

Applying the Laplacian on both sides of the above equality and using (1.62),

$$\Delta \left[\frac{w_i^{n+1} - w_i^n}{\tau} \right] = \frac{u_i^{n+1} - u_i^n}{\tau} + \delta_i \Delta \left(\frac{\tilde{u}_i^{n+1} - \tilde{u}_i^n}{\tau} \right) = \frac{\tilde{u}_i^{n+1} - \tilde{u}_i^n}{\tau}.$$

Therefore,

$$\frac{w_i^{n+1} - w_i^n}{\tau} - \delta_i \Delta \left(\frac{w_i^{n+1} - w_i^n}{\tau} \right) = a_i^n u_i^{n+1} \geq 0 \text{ in } \Omega\ ;\ \partial_n w_i^n = \partial_n w_i^{n+1} = 0 \text{ on } \partial\Omega,$$

and the maximum principle yields (1.64).
Consider now the first equation in (1.62), take the sum from 0 to k and sum over i to get

$$\overline{\delta}^{-1} \tilde{V}^{k+1} - \Delta W^{k+1} \leq \tilde{U}^{k+1} - \Delta W^{k+1} = U^0. \tag{1.65}$$

Now we multiply (1.65) by $0 \leq \frac{W^{k+1} - W^k}{\tau} = \frac{\tilde{V}^{k+1} - \tilde{V}^k}{\tau} + A^{k+1}$ and integrate by parts. Recall that for any $x, y \in \mathbb{R}$, for any $\tau > 0$, $\frac{y^2 - x^2}{2\tau} \leq y \frac{y-x}{\tau}$, so

$$\int_\Omega \frac{(\tilde{V}^{k+1})^2 - (\tilde{V}^k)^2}{2\tau} \leq \int_\Omega \tilde{V}^{k+1} \left(\frac{\tilde{V}^{k+1} - \tilde{V}^k}{\tau} \right),$$

$$\int_\Omega \frac{|\nabla W^{k+1}|^2 - |\nabla W^k|^2}{2\tau} \leq \int_\Omega \nabla W^{k+1} \cdot \left(\frac{\nabla W^{k+1} - \nabla W^k}{\tau} \right) = \int_\Omega -\Delta W^{k+1} \left(\frac{W^{k+1} - W^k}{\tau} \right).$$

We get, after summation from $k=0$ to n:

$$\frac{1}{\overline{\delta}}\int_\Omega \frac{(\tilde V^{n+1})^2}{2}+\frac{1}{\overline{\delta}}\int_\Omega \tau\sum_{k=0}^n A^{k+1}\tilde V^{k+1}+\int_\Omega\frac{|\nabla W^{n+1}|^2}{2}$$
$$\leq \frac{1}{\overline{\delta}}\int_\Omega \frac{(\tilde V^0)^2}{2}+\int_\Omega\frac{|\nabla\tilde V^0|}{2}+\int_\Omega U^0(\tilde V^{n+1}-\tilde V^0)+\int_\Omega U^0\tau\sum_{k=0}^n A^{k+1}.$$

Using Young's inequality to control $\int_\Omega U^0\tilde V^{n+1}$ with $\overline{\delta}^{-1}\int_\Omega \tilde(V^{n+1})^2$, there exists $C>0$ depending only on $\|U^0\|_{L^\infty(\Omega)},\overline{\delta}$, such that

$$\int_\Omega \tau\sum_{k=0}^n A^{k+1}\tilde V^{k+1}\leq C(1+\int_\Omega\tau\sum_{k=0}^n A^{k+1}). \tag{1.66}$$

Then for any $\alpha>0$, we have

$$\alpha\tau\sum_{k=0}^n\int_{\Omega\cap\{\tilde V^{k+1}\geq\alpha\}}A^{k+1}\leq C\left(1+\tau\sum_{k=0}^n\int_{\Omega\cap\{\tilde V^{k+1}\geq\alpha\}}A^{k+1}+\tau\sum_{k=0}^n\int_{\Omega\cap\{\tilde V^{k+1}<\alpha\}}A^{k+1}\right). \tag{1.67}$$

Since $\delta_i\tilde u_i^k\leq\alpha$ on $\{\tilde V^{k+1}<\alpha\}$ and using (1.63), we have

$$\tau\sum_{k=0}^n\int_{\Omega\cap\{\tilde V^{k+1}<\alpha\}}A^{k+1}\leq\tau G(\frac{\alpha}{\underline{\delta}})\|U^0\|_{L^1(\Omega)},$$

where G is defined in (1.27). Choosing $\alpha=2C$ in (1.67), there exists $\tilde C=\tilde C(\|U^0\|_{L^\infty(\Omega)},\overline{\delta},\underline{\delta})>0$ such that for any $n\in\{0,\dots,n_0-1\}$,

$$\tau\sum_{k=0}^n\int_\Omega A^{k+1}\leq\tilde C(1+TG(\frac{2C}{\underline{\delta}})), \tag{1.68}$$

which is the discrete analog of (1.42).
Similarly to what we did in Lemma 1.9, we have

$$\tilde U^{n+1}-\Delta W^{n+1}=U^0 \text{ in } \Omega\,;\quad \partial_n W^{n+1}=0 \text{ on } \partial\Omega\,;\quad \tilde U^{n+1},W^{n+1}\geq 0,$$

so there exists $C=C(\Omega)>0$ such that

$$\forall n\in\mathbb{N},\quad \|W^{n+1}\|_{L^\infty(\Omega)}\leq C\left(\|U^0\|_{L^\infty(\Omega)}+\int_\Omega W^{n+1}\right). \tag{1.69}$$

Recall that $W^{n+1}=\tilde V^{n+1}+\tau\sum_{k=0}^n A^{k+1}$, so combining (1.63), (1.68) and (1.69), we get the existence of $C=C(\|U^0\|_{L^\infty(\Omega)},\overline{\delta},\underline{\delta},T)>0$ such that for all $n\in\{0,\dots,n_0-1\}$,

$$\|W^{n+1}\|_{L^\infty(\Omega)}\leq C\,;\quad \|\tilde U^{n+1}\|_{L^\infty(\Omega)}\leq C. \tag{1.70}$$

Note that C does not depend on the upper bound on a_i. Moreover, we obtain

$$\|a_i^n\|_{L^\infty(\Omega)}\leq G(C). \tag{1.71}$$

Analog of the $\mathbf{L^2(Q_T)}$-estimate on U. There are two ways to get the discrete analog of (1.13). It can be proven using a discrete dual problem (which is done in [70]), or directly as follows: consider

$$u_i^{n+1} - \tau\Delta\sum_{k=0}^{n}(a_i^k u_i^{k+1}) = u_i^0, \tag{1.72}$$

multiply it by $\tau a_i^n u_i^{n+1}$, integrate on Ω and sum from $n=0$ to n_0-1 to get

$$\int_\Omega \tau\sum_{n=0}^{n_0-1} a_i^n(u_i^{n+1})^2 - \tau^2\int_\Omega\sum_{n=0}^{n_0-1} a_i^n u_i^{n+1}\Delta\sum_{k=0}^{n} a_i^k u_i^{k+1} = \tau\int_\Omega u_i^0\sum_{n=0}^{n_0-1} a_i^n u_i^{n+1}. \tag{1.73}$$

Integrating by parts, we have

$$\begin{aligned} -\int_\Omega\sum_{n=0}^{n_0-1} a_i^n u_i^{n+1}\Delta\sum_{k=0}^{n} a_i^k u_i^{k+1} &= \int_\Omega \nabla\sum_{n=0}^{n_0-1} a_i^n u_i^{n+1}\cdot\nabla\sum_{k=0}^{n} a_i^k u_i^{k+1} \\ &= \frac{1}{2}\left|\sum_{n=0}^{n_0-1}\nabla(a_i^n u_i^{n+1})\right|^2 + \frac{1}{2}\sum_{n=0}^{n_0-1}|\nabla(a_i^n u_i^{n+1})|^2 \geq 0. \end{aligned}$$

Recall that $\underline{d}$ (resp. $\overline{d}$) denotes the lower (resp. upper) bound on a_i. Going back to (1.73),

$$\underline{d}\tau\sum_{n=0}^{n_0-1}\int_\Omega (u_i^{n+1})^2 \leq \tau\sum_{n=0}^{n_0-1}\int_\Omega u_i^0 a_i^n u_i^{n+1} \leq \tau G(C)\sum_{n=0}^{n_0-1}\|u_i^0\|_{L^2(\Omega)}\|u_i^{n+1}\|_{L^2(\Omega)},$$

where $G(c)$ is defined in (1.71). Finally, using Young's inequality, we get the existence of $C = C(\underline{d}, u^0, T) > 0$ such that

$$\tau\sum_{n=0}^{n_0-1}\int_\Omega (u_i^{n+1})^2 \leq C. \tag{1.74}$$

Discrete $\mathbf{L^2(0,T;H^2(\Omega))}$-estimates.
Since $-\delta_i\Delta\tilde{u}_i^n + \tilde{u}_i^n = u_i^n$ with homogeneous Neumann boundary conditions, using (1.74) and elliptic regularity theory, there exists $C > 0$ depending only on the data (including T), such that

$$\tau\sum_{n=0}^{n_0}\|\tilde{u}_i^n\|^2_{H^2(\Omega)} \leq C. \tag{1.75}$$

Similarly, considering equation (1.72), using (1.74) and elliptic regularity, there exists $C > 0$ depending only on the data, such that

$$\tau\sum_{n=0}^{n_0}\left\|\tau\sum_{k=0}^{n} a_i^k u_i^{k+1}\right\|^2_{H^2(\Omega)} \leq CT. \tag{1.76}$$

Existence proof.

We first work with the assumption that $0 < \underline{d} \leq a_i(\tilde{u}^n) \leq \overline{d} < +\infty$. Since $\tilde{u}^n$ is uniformly bounded in $L^\infty(\Omega)^I$ independently of $\overline{d}$, this assumption will be dropped using a truncation of a_i "above" $\sup_{n\in\{0,n_0\}}\|\tilde{u}^n\|_{L^\infty(\Omega)^I}$.

Set $v=(v_1,\dots,v_I)$, $\tilde v=(\tilde v_1,\dots,\tilde v_I)$, where $v_i,\tilde v_i : Q_T\to\mathbb{R}_+$ are defined as follows:

$$\forall n\in\{0,\dots,n_0-1\},\ \forall t\in[n\tau,(n+1)\tau[,\quad v_i(t,\cdot)=u_i^{n+1}\ ;\quad \tilde v_i(t,\cdot)=\tilde u_i^n.$$

The goal is to show that when $n_0=T/\tau\to+\infty$, $v\to v^\infty$, where v^∞ is a solution of (1.43).
With the above notations, it is clear that for all $t\in[n\tau,(n+1)\tau[$,

$$v_i(t)-\int_0^t\Delta[a_i(\tilde v)v_i]=u_i^0+\int_t^{(n+1)\tau}\Delta[a_i(\tilde v)v_i], \tag{1.77}$$

and we want to pass to the limit $n_0\to+\infty$. As a consequence of $(1.70)-(1.71)$, $\tilde v$ and $a_i(\tilde v)$ are bounded in $L^\infty(Q_T)$ independently of n_0. In terms of $v,\tilde v$, the previous discrete estimates $(1.74),(1.75),(1.76)$ now read, where "bounded" means "bounded independently of n_0":

$$\begin{cases} v_i, a_i(\tilde v)v_i & \text{are bounded in } L^2(Q_T);\\ \tilde v_i & \text{is bounded in } L^2(0,T;H^2(\Omega));\\ \int_0^t a_i(\tilde v)v_i & \text{is bounded in } L^2(0,T;H^2(\Omega)).\end{cases} \tag{1.78}$$

Remark that $\int_0^t\Delta[a_i(\tilde v)v_i]$ is bounded in $W^{1,2}(0,T;H^{-2}(\Omega))$. Using Sobolev's embeddings,

$$\exists M>0:\quad \|\int_0^t\Delta[a_i(\tilde v)v_i]\|_{C^{1/2}(0,T;H^{-2}(\Omega))}:=\sup_{x\neq y}\left(\frac{|\int_x^y\Delta[a_i(\tilde v)v_i]|_{H^{-2}(\Omega)}}{|x-y|^{1/2}}\right)\le M.$$

Then

$$\forall t\in[n\tau,(n+1)\tau[,\quad \|\int_t^{(n+1)\tau}\Delta[a_i(\tilde v)v_i]\|_{H^{-2}(\Omega)}\le M\tau^\alpha$$

and therefore $\int_t^{(n+1)\tau}\Delta[a_i(\tilde v)v_i]\to 0$ in $L^\infty(0,T;H^{-2}(\Omega))$. Let us now prove the convergence of the other terms in (1.77): all the sequences mentioned in (1.78) are weakly relatively compact in the corresponding spaces: there exist v_i^∞, $\tilde v_i^\infty$, A_i, B_i such that when $n_0\to+\infty$, up to a subsequence,

$$\begin{cases} v_i\to v_i^\infty & \text{weakly in } L^2(Q_T);\\ \tilde v_i\to\tilde v_i^\infty & \text{weakly in } L^2(0,T;H^2(\Omega));\\ a_i(\tilde v)v_i\to A_i & \text{weakly in } L^2(Q_T);\\ \int_0^t a_i(\tilde v)v_i\to B_i & \text{weakly in } L^2(0,T;H^2(\Omega)).\end{cases} \tag{1.79}$$

We need to check that $A_i=a_i(\tilde v^\infty)v_i^\infty$ and $B_i=\int_0^t a_i(\tilde v^\infty)v_i^\infty$.
To that purpose, define $v_i^c\in C([0,T];L^2(\Omega))$ and $\tilde v_i^c\in C([0,T];H^2(\Omega))$ as

$$\forall t\in[n\tau,(n+1)\tau[\ ,\quad \begin{cases} v_i^c(t,\cdot)=\dfrac{t-n\tau}{\tau}u_i^{n+1}+\dfrac{(n+1)\tau-t}{\tau}u_i^n,\\ \tilde v_i^c(t,\cdot)=\dfrac{t-n\tau}{\tau}\tilde u_i^{n+1}+\dfrac{(n+1)\tau-t}{\tau}\tilde u_i^n.\end{cases}$$

We have

$$\partial_t\tilde v_i^c-\delta_i\Delta(\partial_t\tilde v_i^c)=\Delta[a_i(\tilde v)v_i]\text{ in }Q_T\ ;\quad \partial_n(\partial_t\tilde v_i^c)=0\text{ on }\Sigma_T, \tag{1.80}$$

which can be rewritten, with the notations (1.24),

$$\partial_t\tilde v_i=J_{\delta_i}\Delta[a_i(\tilde v)v_i]=\Delta J_{\delta_i}[a_i(\tilde v)v_i]\text{ in }Q_T.$$

Since $a_i(\tilde{v})v_i$ is bounded in $L^2(Q_T)$, using elliptic regularity, $\partial_t \tilde{v}_i^c$ is bounded in $L^2(Q_T)$. As $\tilde{v}_i^c$ is also bounded in $L^2(0,T;H^2(\Omega))$, using the Aubin-Simon compactness results (see [98, Corollary 4]), $\tilde{v}_i^c$ is relatively compact in $L^2(Q_T)$. Let us look now how close $\tilde{v}_i^c$ is to $\tilde{v}_i$:

$$\|\tilde{v}_i^c - \tilde{v}_i\|^2_{L^2(Q_T)} = \sum_{n=0}^{n_0-1} \int_{n\tau}^{(n+1)\tau} \left(\frac{t-n\tau}{\tau}\right)^2 \int_\Omega (\tilde{u}_i^{n+1} - \tilde{u}_i^n)^2 = \frac{\tau}{3} \sum_{n=0}^{n_0-1} \int_\Omega (\tilde{u}_i^{n+1} - \tilde{u}_i^n)^2. \tag{1.81}$$

The fact that $\partial_t \tilde{v}_i^c$ is bounded in $L^2(Q_T)$ reads

$$\tau \sum_{n=0}^{n_0-1} \int_\Omega \left(\frac{\tilde{u}_i^{n+1} - \tilde{u}_i^n}{\tau}\right)^2 \leq C.$$

Combined with (1.81), we get

$$\|\tilde{v}_i^c - \tilde{v}_i\|^2_{L^2(Q_T)} \leq C\tau^2 \underset{n_0 \to +\infty}{\longrightarrow} 0,$$

so $\tilde{v}_i$ is also relatively compact in $L^2(Q_T)$. In particular, up to a subsequence, it converges a.e. to $\tilde{v}$ in Q_T, and therefore $a_i(\tilde{v})$ converges to $a_i(\tilde{v}^\infty)$ in $L^p(Q_T)$ for any $p < \infty$. Then it is clear that $A_i = a_i(\tilde{v}^\infty)v_i^\infty$ and $B_i = \int_0^t a_i(\tilde{v}^\infty)v_i^\infty$. All together, we can pass to the limit $n_0 \to +\infty$ in (1.77) and v_i^∞ is a solution of (1.43).

Similarly as what was done in Section 1.3, assumption (1.6) may be dropped as follows: we first prove the convergence of $v, \tilde{v}$ as above using the truncated functions $\overline{a}_i$ defined in (1.44). Then using that the $L^\infty(Q_T)$ estimate on $\tilde{u}$ does not depend on the upper bound on a_i and is uniform in n_0, using a truncation high enough, we see that the solution for (1.43) with functions $\overline{a}_i$ is also a solution for (1.43) with functions a_i, which ends the proof of Corollary 1.12.

1.7 Other relaxation procedures

We have studied above the situation of diffusive fluxes of the type

$$\nabla(a_i(\tilde{u})u_i), \text{ where } \tilde{u}_k - \delta_k \Delta \tilde{u}_k = u_k,\ k \in \{1,\dots,I\},\ \delta_k > 0. \tag{1.82}$$

$\tilde{u}_k$ can be interpreted as a "spatial average" of u_k with a space characteristic length $\sqrt{\delta_k}$. Amongst the possible ways to generalize this, one might think about the following situations:

1. The characteristic spatial length depend on each species: (1.82) is replaced by

$$\nabla(a_i(\tilde{u}^i)u_i), \text{ where } \tilde{u}_k^i - \delta_k^i \Delta \tilde{u}_k^i = u_k,\ k \in \{1,\dots,I\},\ \delta_k^i > 0. \tag{1.83}$$

2. Several characteristic spatial lengths can influence the behaviour of each species. For instance, for two spatial lengths, we get diffusivities of the type

$$\nabla(a_i(\tilde{u}^1, \tilde{u}^2)u_i), \text{ where } \tilde{u}_k^1 - \delta_k^1 \Delta \tilde{u}_k^1 = u_k,\ \tilde{u}_k^2 - \delta_k^2 \Delta \tilde{u}_k^2 = u_k,\ k \in \{1,\dots,I\},\ \delta_k^1, \delta_k^2 > 0. \tag{1.84}$$

The existence and regularity results as stated in Theorem 1.1 carry over to the cases of diffusivities of types (1.83) and (1.84) with only slight modifications of the above proof. Uniqueness also holds by solving the dual problem, which should be modified as follows: if $(u,\tilde{u})$ and $(v,\tilde{v})$ are two solutions, we get: for $F \in C_0^\infty(Q_T)$, find a function $\varphi = (\varphi_1,\dots,\varphi_I)$ such that for $i \in \{1,\dots,I\}$, $\varphi_i, \partial_t \varphi_i, \Delta \varphi_i \in L^2(Q_T)$; $\partial_n \varphi_i = 0$ *on* Σ_T ; $\varphi_i(T) = 0$ and

- For the relaxation (1.83): if $A_i := \int_0^t Da_i(t\tilde{u}^i + (1-t)\tilde{v}^i)dt$,

$$\partial_t \varphi_i + a_i(\tilde{u}^i)\Delta\varphi_i + \sum_{j=1}^{I} J_{\delta_i^j}(v_j A_{ji} \Delta\varphi_j) = F_i \text{ on } Q_T.$$

- For the relaxation (1.84): if $A_i := \int_0^t Da_i(t(\tilde{u}^1, \tilde{u}^2) + (1-t)(\tilde{v}^1, \tilde{v}^2))dt$, for $k \in \{1,2\}$, $B_i^k = (B_{i,1}^k, \dots, B_{i,I}^k)$, $B_{i,j}^1 = v_j A_{ji}$, $B_{i,j}^2 = v_j A_{j,i+I}$.

$$\partial_t \varphi_i + a_i(\tilde{u}^1, \tilde{u}^2)\Delta\varphi_i + J_{\delta_i^1}(B_i^1 \cdot \Delta\varphi) + J_{\delta_i^2}(B_i^2 \cdot \Delta\varphi) = F_i \text{ on } Q_T.$$

For both cases, the resolutions of the dual problems are similar to what was done in Section 1.5 and therefore uniqueness still holds as in Theorem 1.1.

2

Cross-diffusion limit for a reaction-diffusion system with fast reversible reaction

Except for subsection 2.4.4, the results of this section will appear in [28] in a joint work with D. Bothe and M. Pierre.

We consider a reaction-diffusion system which models a fast reversible reaction of type $C_1 + C_2 \rightleftharpoons C_3$ between mobile reactants inside an isolated vessel. Assuming mass action kinetics, we study the limit when the reaction speed tends to infinity in case of unequal diffusion coefficients and prove convergence of a subsequence of solutions to a weak solution of an appropriate limiting pde-system, where the limiting problem turns out to be of cross-diffusion type. The proof combines the L^2-approach to reaction-diffusion systems having at most quadratic reaction terms with a thorough exploitation of the entropy functional for mass action systems. The limiting cross-diffusion system has unique local strong solutions for sufficiently regular initial data, while uniqueness of weak solutions is in general open but is shown to be valid under restrictions on the diffusivities.

2.1 Introduction

The main goal of this section is to identify the limit as $k \to +\infty$ for the following reaction-diffusion system

$$(R^K)\begin{cases} \left.\begin{aligned} \partial_t c_1 - d_1\Delta c_1 &= -k(c_1c_2 - \kappa c_3) \\ \partial_t c_2 - d_2\Delta c_2 &= -k(c_1c_2 - \kappa c_3) \\ \partial_t c_3 - d_3\Delta c_3 &= +k(c_1c_2 - \kappa c_3) \end{aligned}\right\} \text{ on } (0,+\infty)\times\Omega, \\ \partial_\nu c_1 = \partial_\nu c_2 = \partial_\nu c_3 = 0 \text{ on } (0,+\infty)\times\partial\Omega, \\ c_1(0,\cdot) = c_1^0,\ c_2(0,\cdot) = c_2^0,\ c_3^0(0,\cdot) = c_3^0. \end{cases} \tag{2.1}$$

where Ω is a bounded regular subset of $\mathbb{R}^N$ (we assume throughout the section that $\partial\Omega$ is of class C^2), ∂_ν denotes the exterior normal derivative to $\partial\Omega$, $\kappa > 0, d_i > 0$ and the initial data c_i^0 are nonnegative. We denote $K = (k,\kappa)$.

This system is a classical model for the chemical reaction

$$C_1 + C_2 \underset{k^b}{\overset{k^f}{\rightleftharpoons}} C_3\,, \tag{2.2}$$

when the reaction takes place in an isolated domain represented by Ω where diffusive transport of the species C_i occurs. We assume that the reaction follows the law of mass action with positive rate constants k^f and k^b for the forward and backward reaction, respectively, and that linear Fickian diffusion applies. We also impose no-flux conditions at the boundary. This leads to system (R^K), where $c_i(t,x)$ represents the molar concentration of the species C_i at time t and position $x \in \Omega$.

To understand the reason and the meaning of letting $k \to +\infty$ in this system, let us look at the time scales for both mechanisms diffusion and reaction. For this purpose, we need to consider the reaction-diffusion system (2.1) in its dimensionless form. The latter is of the same type as (2.1), but with differently defined model parameters: the c_i then denote dimensionless concentrations, obtained by normalizing the molar concentrations with a characteristic reference value c_0. The independent variables time and space are also normalized by appropriate characteristic values τ and l, respectively. Then, in the non-dimensional form of (2.1), the model parameters are

$$d_i = \frac{D_i}{D_0}, \qquad k = \frac{k^f c_0 l^2}{D_0}, \qquad \kappa = \frac{k^b}{k^f c_0},$$

where we have already chosen the diffusion time scale $\tau_{\text{diff}} = l^2/D_0$ as the characteristic time τ with D_0 denoting a characteristic diffusivity. Note that both k and κ are time scale ratios, namely

$$k = \frac{\tau_{\text{diff}}}{\tau^f_{\text{reac}}}, \qquad \kappa = \frac{\tau^f_{\text{reac}}}{\tau^b_{\text{reac}}}.$$

The quantity k^f/k^b is called the equilibrium constant of the reversible reaction. Let us note in passing that for fixed equilibrium constant, one can always assume $\kappa = 1$ by choosing $c_0 = k^f/k^b$.

Now, diffusion in liquids or especially in solids is a relatively slow process. For example, even in an actively mixed aqueous system the smallest achievable concentration length scales are typically about $l \simeq 10^{-6} m$, often considerably larger. Therefore, with typical diffusivities in water of about $D_0 \simeq 10^{-9} m^2 s^{-1}$, a conservative estimate for τ_{diff} is given by $\tau_{\text{diff}} \geq 10^{-3} s$. In systems without agitation it will be several magnitudes larger. On the other hand, chemical transformations can be extremely fast, depending on the reaction mechanism. For instance in case of the neutralization $H^+ + OH^- \rightleftharpoons H_2O$, the forward reaction can have a time scale as small as $\tau^f_{\text{reac}} \simeq 10^{-11} s$. Other examples for fast reversible reactions include dissociations, other ionic as well as radical reactions; cf. [46] for more details on chemical reaction mechanisms and rates. Therefore, in many actual experiments one or several reactions are much faster than the diffusive transport processes.

For concrete reversible reactions the equilibrium constants can often be obtained from the literature or by means of measurements, while the individual rate constants are usually unknown, especially for fast reactions. On the other side, it is reasonable to expect that during the evolution, according to (R^K), the chemical composition $c(t,\cdot)$ will be close to the manifold on which the fast reversible reaction is in equilibrium, driven by the diffusive transport processes. This is the motivation to study rigorously what happens at the limit as $k \to +\infty$ in system (R^K). More precisely, we are interested in the slightly more general limit $K = (k, \kappa) \to (+\infty, \kappa^\infty)$, where $\kappa^\infty > 0$.

To understand better what may happen at the limit, let us first recall *what happens for the associated O.D.E.*, that is the same system as above, but without diffusion. Let $c = (c_1, c_2, c_3)$ be the solution and let us set $c_i(t) = \kappa \tilde{c}_i(k\kappa t)$. We are led to the system

$$(\tilde{C}) \quad \begin{cases} \frac{d}{dt}\begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \\ \tilde{c}_3 \end{pmatrix} = \begin{pmatrix} -\tilde{c}_1\tilde{c}_2 + \tilde{c}_3 \\ -\tilde{c}_1\tilde{c}_2 + \tilde{c}_3 \\ \tilde{c}_1\tilde{c}_2 - \tilde{c}_3 \end{pmatrix}, \\ \tilde{c}(0) = \kappa(c_1^0, c_2^0, c_3^0) \quad \in \mathbb{R}^3_+. \end{cases}$$

It is easy to check that this system has a global nonnegative and uniformly bounded solution on $[0,\infty)$ (note that $\tilde{c}_1(t)+\tilde{c}_2(t)+2\tilde{c}_3(t)=\kappa(c_1^0+c_2^0+2c_3^0)$). If we assume for simplicity that κ is fixed (say $\kappa=1$), then the limit as $k\to+\infty$ of the original system is exactly described through the asymptotic behavior of $\tilde{c}(t)$ as $t\to+\infty$. It is well known (and easy to check) that the entropy function

$$V(\tilde{c}):=\sum_{i=1}^{3}\tilde{c}_i\log(\frac{\tilde{c}_i}{\tilde{c}_i^*})+(\tilde{c}_i-\tilde{c}_i^*)$$

is a Lyapunov function for $(\tilde{C})$, where $\tilde{c}_1^*$, $\tilde{c}_2^*$, $\tilde{c}_3^*$ are positive numbers such that $\tilde{c}_1^*\tilde{c}_2^*=\tilde{c}_3^*$. From this, the compactness of the trajectories and La Salle's invariance principle, we deduce that $\tilde{c}_i(t), i=1,2,3$ converge as $t\to+\infty$ to the unique nonnegative solution $(c_1^\infty,c_2^\infty,c_3^\infty)$ of

$$\begin{cases} c_1^\infty c_2^\infty &= c_3^\infty,\\ c_1^\infty+c_3^\infty &= c_1^0+c_3^0,\\ c_2^\infty+c_3^\infty &= c_2^0+c_3^0. \end{cases}$$

Going back to the solution $c=(c_1,c_2,c_3)$ of the first system, this implies that

$$\forall\alpha>0,\ \forall i=1,2,3,\quad \|c_i-c_i^\infty\|_{L^\infty([\alpha,+\infty))}\underset{k\to+\infty}{\longrightarrow}0.$$

In other words, the limit system is "constant", which means that a constant equilibrium is reached very quickly when k is large. Note that there is a boundary layer at t=0 if $c_1^0c_2^0\neq\kappa c_3^0$.

For the treatment of more general O.D.E.-systems with several fast reversible reactions and additional slow processes, see [19].

The mathematical analysis is quite more involved for the limit of the full reaction-diffusion system. As we will see, global existence of classical solutions still holds for each (k,κ). In the case $d_1=d_2=d_3=d$ of equal diffusion coefficients, some of the features of the O.D.E. system remain also valid. In particular, if we set $U=c_1+c_2+2c_3$, then $\partial_tU-d\Delta U=0$, and by maximum principle

$$\|c_1(t)+c_2(t)+2c_3(t)\|_{L^\infty(\Omega)}\leq\|c_1^0+c_2^0+2c_3^0\|_{L^\infty(\Omega)}. \tag{2.3}$$

Together with positivity, this implies a uniform bound on the solution, uniformly in time. This property was exploited in [18], together with the Lyapunov property of the entropy function – which remains also valid here– to prove convergence in some adequate sense of the solution of (R^K) as $k\to+\infty$ to the solution of the limit system

$$\begin{cases} \partial_t(c_1+c_3)-d\Delta(c_1+c_3)=0 \text{ in } (0,\infty)\times\Omega,\\ \partial_t(c_2+c_3)-d\Delta(c_2+c_3)=0 \text{ in } (0,\infty)\times\Omega,\\ \partial_\nu(c_1+c_3)=\partial_\nu(c_2+c_3)=0 \text{ on } (0,\infty)\times\partial\Omega,\\ c_1(0)+c_3(0)=c_1^0+c_3^0,\ c_2(0)+c_3(0)=c_2^0+c_3^0 \text{ in } \Omega,\\ c_1c_2=\kappa^\infty c_3 \text{ in } \Omega. \end{cases} \tag{2.4}$$

Note that the first four lines of this system completely determine the two sums c_1+c_3 and c_2+c_3. Coupling with the fifth equation and the positivity of the c_i's, this implies uniqueness of classical solutions for the above system.

Now the situation when the diffusion coefficients are different from each other is quite more difficult to analyze and this is the main purpose of the present section. In particular, the uniform estimate (2.3) is no longer valid, although a global classical solution, bounded for all $T > 0$, does exist for (R^K); for the readers convenience, this is recalled in Section 2.2. Moreover, the limit system is quite more difficult to understand.

The following is one of the main results of this section, where we employ the common notation $Q_T = (0,T) \times \Omega$, $\Sigma_T = (0,T) \times \partial\Omega$.

Theorem 2.1. *Let $K^n := (k_n, \kappa_n) \overset{n\to+\infty}{\longrightarrow} (+\infty, \kappa^\infty)$ with $\kappa^\infty > 0$ and let $c^n = (c_1^n, c_2^n, c_3^n)$ be the solution of (R^{K_n}) on $[0,\infty)$ with initial data $c^0 = (c_1^0, c_2^0, c_3^0) \in L^\infty(\Omega, \mathbb{R}_+^3)$. Then, up to a subsequence, $(c^n)_{n\in\mathbb{N}}$ converges for all $T > 0$ in $L^2(Q_T)^3$ to a limit $c = (c_1, c_2, c_3)$, solution of the following for all $T > 0$:*

$$\begin{cases} \forall i = 1,2,3,\ c_i \in L^2(Q_T),\ \nabla c_i \in L^{\frac{4}{3}}(Q_T)^N,\ \ c_i \geq 0,\ \ c_1c_2 = \kappa^\infty c_3, \\ \forall \psi \in C^\infty(\overline{Q_T}) \text{ such that } \psi(T) = 0, \\ \quad \begin{cases} -\int_\Omega \psi(0)(c_1^0 + c_3^0) + \int_{Q_T} -\psi_t(c_1 + c_3) + \nabla\psi.\nabla(d_1c_1 + d_3c_3) = 0, \\ -\int_\Omega \psi(0)(c_2^0 + c_3^0) + \int_{Q_T} -\psi_t(c_2 + c_3) + \nabla\psi.\nabla(d_2c_2 + d_3c_3) = 0. \end{cases} \end{cases} \tag{2.5}$$

System (2.5) is a weak formulation of

$$\begin{cases} \partial_t(c_1 + c_3) - \Delta(d_1c_1 + d_3c_3) = 0 & \text{in } Q_T, \\ \partial_t(c_2 + c_3) - \Delta(d_2c_2 + d_3c_3) = 0 & \text{in } Q_T, \\ \partial_\nu(d_1c_1 + d_3c_3) = \partial_\nu(d_2c_2 + d_3c_3) = 0 & \text{on } \Sigma_T, \\ (c_1 + c_3)(0,\cdot) = c_1^0 + c_3^0;\ (c_2 + c_3)(0,\cdot) = c_2^0 + c_3^0 & \text{in } \Omega, \\ c_1c_2 = \kappa^\infty c_3 & \text{in } Q_T. \end{cases} \tag{2.6}$$

It couples a cross-diffusion system with an algebraic equation. This system is quite harder to understand than (2.4) which was built of two classical heat equations for the sums $c_1 + c_3, c_2 + c_3$. As we will see in Section 2.3, this limit system can be rewritten in a different way as a 2×2 nonlinear reaction-cross-diffusion system. Using known results (in particular in [1,3,4]), we may then prove that it has a classical regular solution, at least on some time-interval $[0, T^*)$, $T^* \leq +\infty$ and for regular enough initial data, and this solution is unique among classical solutions . However, two questions remain open in general:

- Does the solution of (2.5) coincide with this classical solution on $[0, T^*)$? This is a uniqueness question for the (weak) solutions of (2.5).
- The solution obtained in (2.5) is global in time, while the classical regular solution is proved to exist only on some interval $[0, T^*)$, where T^* may be finite. Can it happen that the solution of (2.5) is regular for some time, but becomes singular after some finite time?

We give in Section 2.3 some interesting partial answer to the first question: we prove that, if d_1, d_2 are both close enough to d_3 (with an explicit range), then uniqueness holds for the global (weak) solution of (2.5). This implies that the whole sequence of approximate solutions c^n converges and not only a subsequence. Moreover, the unique global weak solution of (2.5) necessarily coincides with the regular one on the interval where this regular solution exists. But even in this restricted range of values for d_1, d_2, d_3, we do not know if the global weak solution is regular for all time.

We also provide another type of uniqueness result: if $|d_1 - d_2|$ belongs to some small interval depending on the $L^\infty((0,T) \times \Omega)$-norm of the regular solution, then the (weak) solution of (2.5) coincides with this regular one on $[0,T]$. Thus, the whole sequence c^n converges on $[0,T]$. But, this does not say anything about uniqueness of the weak global solution of (2.5) for large time.

We focus here on the specific reaction (2.2). However, our approach is rather general and applies for instance to reactions of the type

$$\sum_{i=1}^{p-1} \alpha_i C_i \rightleftharpoons C_p \,. \tag{2.7}$$

This is discussed in Section 2.4 together with some further remarks on possible extensions of the tools introduced here to various chemical systems.

Let us finally mention some related work. The case of a single fast reversible reaction of type $A \rightleftharpoons B$ has been treated in [25]. For the resulting RD-system, a priori L^∞-estimates independent of k are available from flow invariance properties which considerably simplify the analysis of convergence of solutions. Using again invariant sets independent of k, a first result on convergence of solutions of (R^K) has been obtained in [18]; note that this approach to (R^K) is restricted to the case of equal diffusivities. In [16] and [26], a coupled system of two reversible reactions of type $A+B \rightleftharpoons C \rightleftharpoons D+E$ is studied. There, in contrast to the present study, the species C is considered highly reactive, modeling the case of a so-called intermediate. For the somewhat less related topic of RD-systems with fast irreversible reactions we refer to [61], [27] and the references therein.

2.2 Proof of the main theorem

First, let us recall the arguments that prove the global existence of a unique strong solution for the problem (R^K). The local existence of strong solutions is a consequence of a classical result (see e.g. [4, 59, 94]):

Lemma 2.2. *Let us consider the following $m \times m$-system: for all $i = 1, ..., m$,*

$$\partial_t u_i - d_i \Delta u_i = f_i(u_1, ..., u_m) \text{ in } \mathbb{R}_+ \times \Omega,\ \partial_\nu u_i = 0 \text{ on } \partial\Omega,\ u_i(0) = u_{i0}, \tag{2.8}$$

where $d_i \in (0, +\infty)$, $f = (f_1, ..., f_m) : \mathbb{R}^m \to \mathbb{R}^m$ is C^1 and $u_{i0} \in L^\infty(\Omega)$. Then, there exist $T > 0$ and a unique classical solution of (2.8) on $[0, T)$. If T^* denotes the greatest of these T's, then

$$\left[\sup_{t \in [0,T^*), 1 \le i \le m} \|u_i(t)\|_{L^\infty(\Omega)} < +\infty \right] \Rightarrow [T^* = +\infty]. \tag{2.9}$$

If the nonlinearity $(f_i)_{1 \le i \le m}$ is moreover **quasi-positive***, which means*

$$\forall i = 1, ..., m,\ \forall u_1, ..., u_m \ge 0,\ f_i(u_1, ..., u_{i-1}, 0, u_{i+1}, ..., u_m) \ge 0,$$

then

$$[\forall i = 1, ..., m,\ u_{i0} \ge 0] \Rightarrow [\forall i = 1, ..., m,\ \forall t \in [0, T^*),\ u_i(t) \ge 0] \,.$$

In the case of system (R^K), the nonlinearity is quasi-positive and the initial data are in $L^\infty(\Omega, \mathbb{R}_+^3)$, so the previous lemma yields the local existence and uniqueness of classical, nonnegative solutions. To show that these solutions are global, according to (2.9), we need an *a priori* estimate for c in $L^\infty((0, T^*) \times \Omega)$. This is not as standard as the local existence result. We may use the following result proved in [90] (see also [50, 78, 83, 92] for earlier proofs).

Lemma 2.3. *Using the same notations and hypotheses as in Lemma 2.2, suppose moreover that f has at most polynomial growth and that there exist $\mathbf{b} \in \mathbb{R}^m$ and a lower triangular invertible matrix P with nonnegative entries such that*

$$\forall r \in [0,+\infty)^m,\ Pf(r) \leq [1+\sum_{i=1}^{m} r_i]\,\mathbf{b}\,.$$

Then, for $u_0 \in L^\infty(\Omega, \mathbb{R}_+^m)$, system (2.8) has a global strong solution.

In the case of system (R^K), the existence of such a matrix P is obvious thanks to the linear dependence in c_3. Indeed, we may choose for instance

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},\ \mathbf{b} = \begin{pmatrix} k\kappa \\ k\kappa \\ 0 \end{pmatrix}.$$

Therefore, (R^K) has a unique global strong solution for every K.

Notation. *Throughout the rest of this section, the solution of (R^K) will be denoted by $c^K = (c_1^K, c_2^K, c_3^K)$.*

We are now interested in the fast-reaction limit.

The scheme of the proof of Theorem 2.1 is the following :

1) The sum of the first and the third equations of (R^K) yields a zero right-hand side: using the L^2-compactness result of Lemma 2.7, we will deduce that $c_1^K + c_3^K$ is relatively compact in $L^2(Q_T)$ for all $T < \infty$ (see Lemma 2.6). To check the full assumptions of Lemma 2.7, we will first use the estimates provided by the entropy inequality of Lemma 2.4 and use Aubin-Simon type compactness result [98].

2) Similarly $c_2^K + c_3^K$ is relatively compact in $L^2(Q_T)$ as $K \to (+\infty, \kappa^\infty)$.

3) If we knew that, along some subsequence, the c_i^K were converging a.e. on $(0,\infty)\times\Omega$ for each $i = 1,2,3$, then, by dominated convergence based on

$$0 \leq c_1^K \leq c_1^K + c_3^K,\ 0 \leq c_2^K \leq c_2^K + c_3^K,$$

we would deduce that each c_i^K actually converges in $L^2(Q_T)$ for all $T > 0$ (along the considered subsequence). This convergence a.e. would for instance hold if we knew that $k(c_1^K c_2^K - \kappa c_3^K)$ was bounded in $L^1(Q_T)$ (this would indeed imply the relative compactness of the c_i in $L^1(Q_T)$, see e.g. [9]). This L^1-bound is proved to be valid in [18] when $d_1 = d_2 = d_3$. But, we are not able to prove it in general.

4) However, we are able to exploit the entropy inequality (see Lemma 2.4) to prove that c_i does converges a.e. up to a subsequence. Whence the expected convergence of c_i in $L^2(Q_T)$ for all $T < \infty$.

5) To pass to the limit in the weak version (2.5) of the system, we still need some control on ∇c_i. Again, this is provided by the entropy inequality that we state next.

Lemma 2.4. *Let $K = (k,\kappa)$ and let $c^K = (c_1^K, c_2^K, c_3^K)$ be the solution of (R^K). Let J be a compact subset of $(0,+\infty)$. Then there exists $C > 0$ independent of $K \in (0,\infty)\times J$ such that, for all $T > 0$,*

$$k\int_{Q_T} (c_1^K c_2^K - \kappa c_3^K)\left[\log(c_1^K c_2^K) - \log(\kappa c_3^K)\right] + \sum_{i=1}^{3} d_i \int_{Q_T} \frac{|\nabla c_i^K|^2}{c_i^K} \leq C\,. \tag{2.10}$$

Proof. We define the nonnegative functions

$$W_i^K = c_i^K \log\left(\frac{c_i^K}{c_i^{K^*}}\right) - (c_i^K - c_i^{K^*}),\ \ W^K = \sum_{i=1}^3 W_i^k,\ \ Z^K = \sum_{i=1}^3 d_i W_i^K,$$

where $c_1^{K^*}, c_2^{K^*}$ and $c_3^{K^*}$ are positive numbers such that $c_1^{K^*} c_2^{K^*} = \kappa c_3^{K^*}$. A straightforward computation yields

$$\begin{aligned}
&\partial_t W^K - \Delta Z^K \\
&= -\left(\sum_{i=1}^3 d_i \frac{|\nabla c_i^K|^2}{c_i^K}\right) - k(c_1^K c_2^K - \kappa c_3^K)\left(\log(\frac{c_1^K}{c_1^{K^*}}) + \log(\frac{c_2^K}{c_2^{K^*}}) - \log(\frac{c_3^K}{c_3^{K^*}})\right), \\
&= -\left(\sum_{i=1}^3 d_i \frac{|\nabla c_i^K|^2}{c_i^K}\right) - k(c_1^K c_2^K - \kappa c_3^K)(\log(c_1^K c_2^K) - \log(\kappa c_3^K)),
\end{aligned}$$

where we used the relation $c_1^{K^*} c_2^{K^*} = \kappa c_3^{K^*}$ to get the last equality. Using the nonnegativity of W^K and the fact that $\int_\Omega \Delta Z^K = \int_{\partial\Omega} \partial_\nu Z^K = 0$, we get after integration on Q_T:

$$\begin{aligned}
k\int_{Q_T} (c_1^K c_2^K - \kappa c_3^K)(\log(c_1^K c_2^K) - \log(\kappa c_3^K)) + \sum_{i=1}^3 d_i \int_{Q_T} \frac{|\nabla c_i^K|^2}{c_i^K} \\
= \int_\Omega W^K(0,\cdot) - \int_\Omega W^K(T,\cdot) \le \int_\Omega W^K(0,\cdot).
\end{aligned}$$

It is easy to see that the right-hand side of the above inequality is bounded independently of $K \in (0,\infty)\times J$:

$$\int_\Omega W^K(0,\cdot) = \sum_{i=1}^3 \int_\Omega W_i^K(0,.) = \sum_{i=1}^3 \int_\Omega c_i^0 \log(\frac{c_i^0}{c_i^{K^*}}) - (c_i^0 - c_i^{K^*}).$$

By assumption, $c_i^0 \in L^\infty(\Omega)^+$. The right member is bounded if the $c_i^{K^*}$ remain in a compact set of $(0,+\infty)$, and this is the case if we choose for instance $c_1^{K^*} = c_2^{K^*} = 1$ and $c_3^{K^*} = 1/\kappa$, $\kappa \in J$. Therefore, there exists a constant C independent of K, T such that (2.10) holds.

Remark 2.5. Note that it is sufficient to assume that $c_i^0|\log c_i^0| \in L^1(\Omega)$ to obtain the above bound C and consequently to get the estimate (2.10).

Lemma 2.6. *The families $(c_1^K + c_3^K)_{K\in(0,+\infty)^2}$, $(c_2^K + c_3^K)_{K\in(0,+\infty)^2}$ are relatively compact in $L^2(Q_T)$ for all $T > 0$.*

Proof. By definition of c^K, $(c_1^K + c_3^K)$ and $(c_2^K + c_3^K)$ are classical solutions of

$$\begin{cases}
\left.\begin{aligned} \partial_t(c_1^K + c_3^K) - \Delta(d_1 c_1^K + d_3 c_3^K) = 0 \\ \partial_t(c_2^K + c_3^K) - \Delta(d_2 c_2^K + d_3 c_3^K) = 0 \end{aligned}\right\} \text{ on } (0,T)\times\Omega, \\
\partial_\nu(c_1^K + c_3^K) = \partial_\nu(c_2^K + c_3^K) = 0 \text{ on } (0,T)\times\partial\Omega, \\
(c_1^K + c_3^K)(0,\cdot) = c_1^0 + c_3^0,\ (c_2^K + c_3^K)(0,\cdot) = c_2^0 + c_3^0.
\end{cases} \tag{2.11}$$

For $j \in \{1,2\}$, we define

$$\tilde W_j^K = c_j^K + c_3^K,\ \tilde Z_j^K = d_j c_j^K + d_3 c_3^K,\ d_j^{\min} = \min(d_j, d_3),\ d_j^{\max} = \max(d_j, d_3).$$

Using the nonnegativity of c^K, we see that

$$d_j^{\min}\tilde{W}_j^K \le \tilde{Z}_j^K \le d_j^{\max}\tilde{W}_j^K \quad \text{with } 0 < d_j^{\min} \le d_j^{\max} < +\infty,$$

and $(\tilde{W}_j^K, \tilde{Z}_j^K)$ is a solution of

$$\begin{cases} \partial_t \tilde{W}_j^K - \Delta \tilde{Z}_j^K = 0 & \text{on } (0,T)\times\Omega, \\ \partial_\nu \tilde{W}_j^K = \partial_\nu \tilde{Z}_j^K = 0 & \text{on } (0,T)\times\partial\Omega, \\ \tilde{W}_j^K(0,\cdot) = \tilde{W}_j^0 := c_j^0 + c_3^0 & \text{on } \Omega\,. \end{cases}$$

After integration in time, we see that $(\tilde{W}_j^K, \tilde{Z}_j^K)$ is solution of (2.12) in the next Lemma 2.7 with $(W,Z) = (\tilde{W}_j^K, \tilde{Z}_j^K)$ and with $W(0) = c_1^0 + c_j^0$. According to this lemma, to prove the relative $L^2(Q_T)$-compactness of $\tilde{W}_j^K$, it is sufficient to prove that, up to a subsequence, it converges a.e. as $K \to (\infty, \kappa)$.

For this, we will prove that $\zeta_j^K := (1+\tilde{W}_j^K)^{1/2} = (1 + c_j^k + c_3^K)^{1/2}$ is relatively compact in $L^2(Q_T)$. Indeed

$$2\,|\nabla\zeta_j^K| = \left|\frac{\nabla c_j^K + \nabla c_3^K}{\zeta_j^K}\right| \le \frac{|\nabla c_j^K|}{(c_j^K)^{1/2}} + \frac{|\nabla c_3^K|}{(c_3^K)^{1/2}}.$$

By (2.10) in Lemma 2.4, $\nabla\zeta_j^K$ is bounded in $L^2(Q_T)^N$. Now

$$2\,\partial_t\zeta_j^K = \frac{\partial_t(c_j^K + c_3^K)}{\zeta_j^K} = \frac{\Delta(d_j c_j^K + d_3 c_3^K)}{\zeta_j^K} = \nabla\cdot f_j^K + g_j^K,$$

$$f_j^K := \frac{\nabla(d_j c_j^K + d_3 c_3^K)}{\zeta_j^K}, \;\; g_j^K := \frac{\nabla(d_j c_j^K + d_3 c_3^K)\nabla(c_j^K + c_3^K)}{2(\zeta_j^K)^3}.$$

Again, by (2.10) in Lemma 2.4, we have that f_j^K is bounded in $L^2(Q_T)^N$ and g_j^K is bounded in $L^1(Q_T)$. Therefore, $\partial_t\zeta_j^K$ is bounded in

$$L^2(0,T:H^{-1}(\Omega)) + L^1(Q_T) \subset L^1(0,T:Y),\; Y := H^{-1}(\Omega) + L^1(\Omega).$$

simon86nce ζ_j^K is also bounded in $L^2(0,T;H^1(\Omega))$ where $H^1(\Omega)$ is compactly embedded into $L^2(\Omega) \subset Y$, by the Aubin-Simon compactness results (see [98, Corollary 4]), ζ_j^K is compact in $L^2(Q_T)$. This ends the proof of Lemma 2.6, based on the following lemma, inspired from the results in [90] and whose proof is given in the Appendix.

Lemma 2.7. *Let $0 < d^{\min} \le d^{\max} < +\infty$ and let $\mathcal{G}$ be a bounded subset of $L^2(\Omega)^+$. We denote by $\mathcal{F}$ the family of functions $(W,Z) \in H^1(Q_T)^2$ such that $W(0) \in \mathcal{G}$ and*

$$\begin{cases} W(t) - \Delta\int_0^t Z(s)\,ds = W(0) \text{ on } Q_T,\; \partial_\nu\int_0^t Z(s)\,ds = 0 \text{ on } \Sigma_T, \\ W, Z \ge 0, \\ d^{\min} \le Z/W \le d^{\max}. \end{cases} \tag{2.12}$$

Then the family $\mathcal{F}$ is bounded in $L^2(Q_T)^2$ by a constant depending only on $d^{\min}, d^{\max}, \mathcal{G}, T$. Next, let $(W^p, Z^p)_{p\ge 0}$ be a sequence in $\mathcal{F}$ converging to (W,Z) weakly in $L^2(Q_T)^2$. Assume that $A^p := Z^p/W^p$ converges to $A := Z/W$ for the weak*-$L^\infty(Q_T)$ convergence, namely

$$\forall\psi \in L^1(Q_T),\; \lim_{p\to\infty}\int_{Q_T}\psi A^p = \int_{Q_T}\psi A. \tag{2.13}$$

Then W^p converges strongly to W in $L^2(Q_T)$. Property (2.13) holds in particular if W^p converges a.e. or if A^p converges a.e. on Q_T.

Remark 2.8. More generally, we could choose initial data $c_i^0 \in L^2(\Omega)$. Approximating them in $L^2(\Omega)$ by bounded data c_i^n, we could still apply Lemma 2.7 with $\mathcal{G} = \{(c_j^n + c_3^n)_{n\geq 0}\}$ and obtain the same $L^2(Q_T)$ compactness.

Using Lemma 2.4 and 2.6, we are now able to show a convergence result for the approximate solutions.

Lemma 2.9. *Let $\kappa^\infty > 0$ and $K_n := (k_n, \kappa_n) \overset{n\to+\infty}{\longrightarrow} (+\infty, \kappa^\infty)$. We denote by c^n the solution of (R^{K_n}). Then, up to a subsequence, c^n converges to a limit $c = (c_1, c_2, c_3)$ in $L^2(Q_T)^3$ for all $T > 0$ and $c_1 c_2 = \kappa^\infty c_3$ holds a.e. in Q_T.*

Proof. The entropy inequality (2.10) yields, with the notations of Lemma 2.4,

$$\|(c_1^n c_2^n - \kappa_n c_3^n)(\log(c_1^n c_2^n) - \log(\kappa_n c_3^n))\|_{L^1(Q_T)} \leq \frac{C}{k_n} \underset{n\to+\infty}{\longrightarrow} 0,$$

and Lemma 2.6 guarantees that $(c_1^n + c_3^n)_{n\in\mathbb{N}}$ and $(c_2^n + c_3^n)_{n\in\mathbb{N}}$ are relatively compact in $L^2(Q_T)$. Using a diagonal process, we may assume that this holds for all $T > 0$. Therefore, up to a subsequence,

$$\begin{cases} c_1^n + c_3^n & \underset{n\to+\infty}{\longrightarrow} & \alpha & \text{in } L^2(Q_T) \text{ and a.e.} \\ c_2^n + c_3^n & \underset{n\to+\infty}{\longrightarrow} & \beta & \text{in } L^2(Q_T) \text{ and a.e.} \\ (c_1^n c_2^n - \kappa_n c_3^n)(\log(c_1^n c_2^n) - \log(\kappa_n c_3^n)) & \underset{n\to+\infty}{\longrightarrow} & 0 & \text{in } L^1(Q_T) \text{ and a.e.} \end{cases} \tag{2.14}$$

for all $T > 0$ with $\alpha, \beta \in L^2((0,\infty) \times \Omega; \mathbb{R}_+)$. From now on, we work with this subsequence. Let $(t,x) \in Q_T$ such that the three pointwise convergence above hold. The sequence $(c^n(t,x))_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}_+^3$, so it has a limit point $l = (l_1, l_2, l_3) \in \mathbb{R}_+^3$. Using (2.14), we easily see that l is a solution of the system

$$l_1 + l_3 = \alpha, \; l_2 + l_3 = \beta, \; l_1 l_2 = \kappa^\infty l_3, \tag{2.15}$$

where we omitted the dependence in (t,x) for $\alpha(t,x)$ and $\beta(t,x)$. Actually, this system has a unique solution in $\mathbb{R}_+^3$, given by

$$(l_1, l_2, l_3) = (\varphi(\alpha,\beta), \varphi(\beta,\alpha), \varphi(\alpha,\beta)\varphi(\beta,\alpha)/\kappa^\infty), \tag{2.16}$$

where

$$\varphi(\alpha,\beta) := \frac{1}{2}\sqrt{(\kappa^\infty)^2 + (\alpha-\beta)^2 + 2\kappa^\infty(\alpha+\beta)} - (\kappa^\infty + \beta - \alpha).$$

The bounded sequence $(c^n(t,x))_{n\in\mathbb{N}}$ has a unique possible limit point, so it converges to this limit point. This holds for almost all $(t,x) \in Q_T$, so up to a subsequence, c^n converges pointwise to a limit c with $c_1 c_2 = \kappa^\infty c_3$. Finally, we have

$$\begin{cases} c_1^n(t,x) \underset{n\to+\infty}{\longrightarrow} c_1(t,x) & \text{for almost every } (t,x) \in Q_T \\ 0 \leq c_1^n \leq c_1^n + c_3^n \underset{n\to+\infty}{\longrightarrow} \alpha \in L^2(Q_T). \end{cases}$$

By dominated convergence, the sequence $(c_1^n)_{n\in\mathbb{N}}$ converges to c_1 in $L^2(Q_T)$. We do the same for c_2^n and c_3^n, which proves the $L^2(Q_T)$ convergence of the subsequence c^n.

Proof of Theorem 2.1. Lemma 2.9 guarantees that, up to a subsequence, c^n goes to a limit $c = (c_1, c_2, c_3)$ in $L^2(Q_T)^3$ for all $T > 0$ with $c_1 c_2 = \kappa^\infty c_3$. Using the estimate on the gradients in Lemma 2.4, for $i = 1,2,3$, we get a bound on $\int_{Q_T} \frac{|\nabla c_i^n|^2}{c_i^n}$ independent of n. This bound can be exploited together with the $L^2(Q_T)$-bound on c^n to get an estimate on ∇c^n. Letting $l,m > 0$, we have

$$\int_{Q_T} |\nabla c_i^n|^l = \int_{Q_T} \frac{|\nabla c_i^n|^l}{(c_i^n)^m}(c_i^n)^m \leq \left(\int_{Q_T} \frac{|\nabla c_i^n|^{lp}}{(c_i^n)^{mp}}\right)^{1/p} \left(\int_{Q_T} (c_i^n)^{mp'}\right)^{1/p'} \text{(Hölder's inequality)},$$

where $p,\ p' \in [1,+\infty]$, $\frac{1}{p} + \frac{1}{p'} = 1$. We know that the right-hand side is bounded independently of n for

$$lp = 2,\ mp = 1,\ mp' = 2,$$

so taking $(l,m,p) = (\frac{4}{3}, \frac{2}{3}, \frac{3}{2})$, we get that $\|\nabla c^n\|_{L^{\frac{4}{3}}(Q_T)}$ is bounded independently of n. Since $L^{\frac{4}{3}}(Q_T)$ is a reflexive space, up to a subsequence, $\forall i = 1,2,3$,

$$\forall T \in (0,\infty),\ \nabla c_i^n \rightharpoonup \nabla c_i \quad \text{for the weak topology } \sigma(L^{\frac{4}{3}}(Q_T)^N, L^4(Q_T)^N).$$

To use this result, let us write a weaker formulation for system (2.11) which involves only the first-order derivatives of c: for all $n \in \mathbb{N}$, c^n is a solution of

$$(R^n) \begin{cases} \forall i = 1,2,3,\ c_i^n \in L^2(Q_T),\ \nabla c_i^n \in L^{\frac{4}{3}}(Q_T)^N, \\ \forall \psi \in C^\infty(\overline{Q_T}) \text{ such that } \psi(T) = 0, \\ \quad \begin{cases} -\int_\Omega \psi(0)(c_1^0 + c_3^0) + \int_{Q_T} -\psi_t(c_1^n + c_3^n) + \nabla\psi \cdot \nabla(d_1 c_1^n + d_3 c_3^n) = 0 \\ -\int_\Omega \psi(0)(c_2^0 + c_3^0) + \int_{Q_T} -\psi_t(c_2^n + c_3^n) + \nabla\psi \cdot \nabla(d_2 c_2^n + d_3 c_3^n) = 0. \end{cases} \end{cases}$$

Using $c_i^n \overset{L^2(Q_T)}{\underset{n\to+\infty}{\longrightarrow}} c$ and $\nabla c_i^n \overset{L^{\frac{4}{3}}(Q_T)^N}{\underset{n\to+\infty}{\rightharpoonup}} \nabla c$, we can pass to the limit in this formulation and obtain that c is solution of (2.5).

Remark 2.10. Actually, we can prove somewhat more regularity of the limit solution. Namely, if we set

$$\mathcal{C}_i(t,x) = \int_0^t c_i(s,x)ds,\ Z_1 = d_1\mathcal{C}_1 + d_3\mathcal{C}_3,\ Z_2 = d_2\mathcal{C}_2 + d_3\mathcal{C}_3,$$

then for all $T < \infty$, $\mathcal{C}_i \in L^\infty(Q_T)$ and

$$Z_1, Z_2 \in L^2\left((0,T); H^2(\Omega)\right) \cap L^4\left((0,T); W^{1,4}(\Omega)\right) \cap L^\infty\left((0,T); H^1(\Omega)\right). \tag{2.17}$$

Indeed, if we set $\mathcal{C}_i^n(t,x) = \int_0^t c_i^n(s,x)ds$, we have after integrating (2.11) in time

$$c_1^n(t) + c_3^n(t) - \Delta(d_1\mathcal{C}_1^n + d_3\mathcal{C}_3^n) = c_1^0 + c_3^0. \tag{2.18}$$

Using $c_1^n + c_3^n \geq \mu(d_1 c_1^n + d_3 c_3^n)$ with $\mu = \min\{d_1^{-1}, d_3^{-1}\}$, we see that $Z^n = d_1\mathcal{C}_1^n + d_3\mathcal{C}_3^n$ satisfies the inequality

$$\mu\partial_t Z^n - \Delta Z^n \leq c_1^0 + c_3^0.$$

Therefore, $\|Z^n(t)\|_{L^\infty(\Omega)} \leq t\mu^{-1}\|c_1^0 + c_3^0\|_{L^\infty(\Omega)}$. The same is valid for $d_2\mathcal{C}_2^n + d_3\mathcal{C}_3^n$. By positivity, all three $\mathcal{C}_i^n$ are bounded in $L^\infty(Q_T)$. This estimate is preserved at the limit for the $\mathcal{C}_i$.

Going back to (2.18), we see that $\|\Delta Z^n\|_{L^2(Q_T)}$ is bounded independently of n. As a consequence,

$\Delta Z_1 \in L^2(Q_T)$ and similarly $\Delta Z_2 \in L^2(Q_T)$. Together with the boundary conditions and the regularity of Ω, we deduce that $Z_1, Z_2 \in L^2\left((0,T);H^2(\Omega)\right)$. We may then use the Gagliardo-Nirenberg inequality, namely $\|\nabla Z\|^4_{L^4(\Omega)} \leq C\|Z\|^2_{L^\infty(\Omega)}\|Z\|^2_{H^2(\Omega)}$, to obtain that $Z_1, Z_2 \in L^4\left((0,T);W^{1,4}(\Omega)\right)$.

Finally, let us multiply (2.18) by $d_1c_1^n + d_3c_3^n$ and integrate on Q_t. We obtain

$$\int_{Q_t}(c_1^n + c_3^n)(d_1c_1^n + d_3c_3^n) + \frac{1}{2}\int_\Omega |\nabla Z_1^n(t)|^2 = \int_{Q_T}(d_1c_1^n + d_3c_3^n)(c_1^0 + c_3^0),$$

and the right-hand side is bounded independently of n. It provides the last estimate for (2.17).

2.3 Study of the limit problem

This section is devoted to an independent study of the non-standard limit problem (2.5). *Throughout the rest of this section, we assume for simplicity that $d_3 = 1$ and $\kappa_\infty = 1$.* This can be done without loss of generality: indeed, by setting $c_i(t,x) = \kappa^\infty \tilde{c}_i(d_3 t, x)$, we have for instance

$$\partial_t(\tilde{c}_1 + \tilde{c}_3) = \Delta(\frac{d_1}{d_3}\tilde{c}_1 + \tilde{c}_3),\ \tilde{c}_1\tilde{c}_2 = \tilde{c}_3.$$

Then any result with $d_3 = 1, \kappa^\infty = 1$ carries over to the general case by replacing d_i by d_i/d_3 and changing $\tilde{c}$ into c.

2.3.1 Existence of strong local solutions

Let us consider the limit system in its explicit version (2.6). We may rewrite it as a 2×2 cross-diffusion system as follows. Let us introduce new unknown functions as

$$x(c_1,c_2) := c_1 + c_1c_2;\ \ y(c_1,c_2) := c_2 + c_1c_2\,. \tag{2.19}$$

As seen in (2.15), (2.16), we have $(c_1,c_2) = \phi(x,y) = (\varphi(x,y),\varphi(y,x))$, where ϕ defines a C^∞-diffeomorphism from $(0,\infty)^2$ onto itself, which extends to a C^∞-homeomorphism from $[0,\infty)^2$ onto itself. The function $\psi : (0,+\infty)^2 \to (0,+\infty)^2$ with

$$\begin{pmatrix} \psi_1(x,y) \\ \psi_2(x,y) \end{pmatrix} := \begin{pmatrix} d_1c_1 + c_1c_2 \\ d_2c_2 + c_1c_2 \end{pmatrix}(x,y)$$

is also C^∞. The limit problem (2.6) can be rewritten as

$$\left\{\begin{aligned} \partial_t x - \Delta\psi_1(x,y) &= 0 \quad \text{in } Q_T, \\ \partial_t y - \Delta\psi_2(x,y) &= 0 \quad \text{in } Q_T, \\ \partial_\nu\left(\psi_1(x,y)\right) = \partial_\nu\left(\psi_2(x,y)\right) &= 0 \quad \text{on } \Sigma_T, \\ x(0,\cdot) = x^0,\ y(0,\cdot) &= y^0 \quad \text{in } \Omega. \end{aligned}\right. \tag{2.20}$$

For the boundary condition, we used that

$$\nabla\left(\psi_i(x,y)\right) = \nabla\left(d_ic_i + c_1c_2\right) = d_i\nabla c_i + c_1\nabla c_2 + c_2\nabla c_1.$$

The new system is a nonlinear cross-diffusion system. We may apply Amann's local existence theory [3,4]. For this purpose we need to study the spectrum of the Jacobian matrix $D\psi$ of ψ. Let us denote

$$g : \mathbb{R}^2 \to \mathbb{R}^2,\ (c_1,c_2) \mapsto (d_1c_1 + c_1c_2, d_2c_2 + c_1c_2).$$

With the above notations, we have

$$\forall (x,y) \in (0,+\infty)^2, \quad \psi(x,y) = g \circ \phi(x,y).$$

Differentiating this expression, we get

$$\begin{aligned} D\psi(x,y) &= Dg(\phi(x,y))D\phi(x,y) = Dg(c_1,c_2)D\phi(\phi^{-1}(c_1,c_2)) \\ &= Dg(c_1,c_2)[D\phi^{-1}(c_1,c_2)]^{-1}, \end{aligned}$$

hence

$$(1+c_1+c_2)D\psi\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} d_1+c_2 & c_1 \\ c_2 & d_2+c_1 \end{pmatrix}\begin{pmatrix} 1+c_1 & -c_1 \\ -c_2 & 1+c_2 \end{pmatrix}.$$

We have

$$\begin{aligned} &0 < (1+c_1+c_2)\mathrm{trace}(D\psi) = d_1+d_2+(d_1+1)c_1+(d_2+1)c_2, \\ &0 < \det(D\psi) = d_1d_2+d_1c_1+d_2c_2\,. \end{aligned}$$

Thus, the spectrum of $D\psi(x,y)$ is in $\{z \in \mathbb{C} : \mathrm{Re}\ z > 0\}$ for all $(x,y) \in [0,+\infty)^2$. Therefore, the operator $(x,y) \mapsto \Delta(\psi(x,y))$ with homogeneous Neumann boundary conditions is *normally elliptic* in the sense of [3,4]. Moreover, $\partial_2\Psi_1(0,y) = \partial_1\Psi_2(x,0) = 0$ for $x,y \geq 0$, so we have

Proposition 2.11. *Let $s > 0$ and $p \in (\max\{N,N/s\},+\infty)$. For $c^0 \in W^{s,p}(\Omega,\mathbb{R}_+^2)$, there exists a unique classical and nonnegative solution $c \in C([0,T^*)\times\overline{\Omega})\cap C^\infty((0,T^*)\times\Omega)$ for the problem (2.20) on a maximal time interval $[0,T^*)$.*

Remark 2.12. Note that the above result applies with $s = 1$ and all $p > N$. Global existence would follow from a uniform bound in $W^{1,p}(\Omega)$ on $[0,T^*)$. This question is open here. However, the existence result of Theorem 2.1 does provide a *global* weak solution to system (2.20). We do not know in general if it coincides with the regular one obtained in Proposition 2.11, even on the interval $[0,T^*)$. The following paragraph gives, however, a partial answer to this question.

2.3.2 A uniqueness result

Let $D = \{(d_1,d_2) \in \mathbb{R}_+^2 : (d_1-1)^2(d_2-1)^2 < 16d_1d_2\}$.

Theorem 2.13. *There exists a unique solution to (2.5) for $(d_1,d_2) \in D$.*

Remark 2.14. This uniqueness result is interesting since it applies to very weak solutions. An interesting consequence is that, in Theorem 2.1, the whole sequence c^n converges as $n \to +\infty$ to the unique solution of the limit system on the whole interval $[0,\infty)$. It also proves that, for regular enough initial data, the solution obtained in Theorem 2.1 coincides with the regular solution of Proposition 2.11 on $[0,T^*)$. However, we do not know if it stays regular for all time (or whether $T^* = +\infty$).

Proof of Theorem 2.13. Let $c = (c_1,c_2,c_3)$ and $\hat{c} = (\hat{c}_1,\hat{c}_2,\hat{c}_3)$ be two solutions of (2.5) on $[0,T)$. We define $U = c_1-\hat{c}_1$, $V = c_2-\hat{c}_2$, $W = c_3-\hat{c}_3$. Using the relations $c_1c_2 = c_3$ and $\hat{c}_1\hat{c}_2 = \hat{c}_3$, we have $W = \hat{c}_2U + c_1V$, so that (U,V) is a solution of

$$\begin{cases} \forall \psi_1,\psi_2 \in C^\infty(\overline{Q_T}) \ \textit{with} \ \ \psi_1(T) = 0 = \psi_2(T), \\ \quad \int_{Q_T} -\partial_t\psi_1[(1+\hat{c}_2)U+c_1V]+\nabla\psi_1\cdot\nabla[(d_1+\hat{c}_2)U+c_1V] = 0, \\ \quad \int_{Q_T} -\partial_t\psi_2[\hat{c}_2U+(1+c_1)V]+\nabla\psi_2\cdot\nabla[\hat{c}_2U+(d_2+c_1)V] = 0. \end{cases} \tag{2.21}$$

We may rewrite this in a vectorial way with the scalar product $<\cdot,\cdot>$ in $\mathbb{R}^2$, namely

$$\int_{Q_T} -<\partial_t\Psi, AX> + <\nabla\Psi, \nabla BX> = 0,$$

where we set

$$X = \begin{pmatrix} U \\ V \end{pmatrix}, \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, A = \begin{pmatrix} 1+\hat{c}_2 & c_1 \\ \hat{c}_2 & 1+c_1 \end{pmatrix}, B = \begin{pmatrix} d_1+\hat{c}_2 & c_1 \\ \hat{c}_2 & d_2+c_1 \end{pmatrix}.$$

Choosing $\Psi = \int_t^T \Phi = \int_t^T (\Phi_1, \Phi_2)$ where $\Phi_1, \Phi_2 \in C^\infty(\overline{Q_T})$, this leads, after an integration by parts in time, to

$$\forall \Phi \in C^\infty(\overline{Q_T})^2, \int_{Q_T} <\Phi, AX> + <\nabla\Phi, \nabla \int_0^t BX> = 0. \tag{2.22}$$

Note that

$$AX, BX \in L^2(Q_T)^2, \ \nabla(BX) \in L^{4/3}(Q_T)^{2N},$$

since $U, V, W \in L^2(Q_T), \nabla U, \nabla V, \nabla W \in L^{4/3}(Q_T)^N$. Property (2.22) implies that, a.e. on $[0,T]$, $\int_0^t BX$ is solution in a variational sense of

$$\Delta\left(\int_0^t BX\right) = AX \text{ in } \Omega, \ \partial_\nu\left(\int_0^t BX\right) = 0 \text{ on } \partial\Omega. \tag{2.23}$$

Since Ω is assumed to be bounded and with a C^2-boundary, this solution is in $H^2(\Omega)$ for a.e. $t \in (0,T)$ (see Remark 2.15 below) and even in $L^2\left((0,T); H^2(\Omega)\right)^2$ since $AX \in L^2(Q_T)^2$. Moreover, the boundary condition is valid in a strong sense. Then (2.22) leads to

$$\forall \Phi \in L^2(Q_T)^2, \ \int_{Q_T} <\Phi, AX> - <\Phi, \Delta \int_0^t BX> = 0, \tag{2.24}$$

where we used the density of $C^\infty(\overline{Q_T})^2$ in $L^2(Q_T)^2$.

Let M be a symmetric positive definite matrix. Then, choosing $\Phi = MBX$ in (2.24) leads to

$$\int_{Q_T} <MBX, AX> = \int_{Q_T} <MBX, \Delta \int_0^t BX> = \int_{Q_T} <M^{\frac{1}{2}}BX, \Delta \int_0^t M^{\frac{1}{2}}BX>.$$

The last integral above is nonnegative. Indeed, if we set $F(t) = \int_0^t M^{\frac{1}{2}}BX$, we have, *at first only formally*,

$$\int_{Q_T} <\partial_t F, \Delta F> = -\int_{Q_T} <\nabla\partial_t F, \nabla F> = -\frac{1}{2}\int_\Omega \left|\nabla \int_0^T F\right|^2 \le 0. \tag{2.25}$$

Actually this computation is not justified since conditions in (2.5) do not imply $\nabla\partial_t F \in L^2(Q_T)^{2N}$. However, we will prove below (see (2.28)) that nevertheless

$$\int_{Q_T} <\partial_t F, \Delta F> \le 0, \tag{2.26}$$

so that we do have

$$\int_{Q_T} <MBX, AX> \le 0. \tag{2.27}$$

Let us continue by proving that we can choose $M = \begin{bmatrix} m_1 & 1 \\ 1 & m_2 \end{bmatrix}$ in such a way that the scalar product $< MBY, AY >$ is positive for all $Y \in \mathbb{R}^2 \setminus \{0\}$. Then (2.27) will imply $X = 0$, whence uniqueness. This happens if and only if MBA^{-1} has a symmetric part which is positive definite and which we denote by $\mathrm{Sym}(MBA^{-1})$. We may write

$$(1 + c_1 + \hat{c}_2) MBA^{-1} = P_0 + c_1 P_1 + \hat{c}_2 P_2$$

where

$$P_0 := \begin{pmatrix} d_1 m_1 & d_2 \\ d_1 & d_2 m_2 \end{pmatrix}, \; P_1 := \begin{pmatrix} d_1 m_1 & m_1 - d_1 m_1 + 1 \\ d_1 & -d_1 + m_2 + 1 \end{pmatrix},$$

$$P_2 := \begin{pmatrix} m_1 - d_2 + 1 & d_2 \\ -m_2 d_2 + m_2 + 1 & m_2 d_2 \end{pmatrix}.$$

Considering the symmetric parts, we have

$$(1 + c_1 + \hat{c}_2)\mathrm{Sym}(MBA^{-1}) = \mathrm{Sym}(P_0) + c_1 \mathrm{Sym}(P_1) + \hat{c}_2 \mathrm{Sym}(P_2),$$

so that $\mathrm{Sym}(MBA^{-1})$ is positive definite for any $c_1 \geq 0,\ \hat{c}_2 \geq 0$ if and only if $\mathrm{Sym}(P_0)$ is positive definite and $\mathrm{Sym}(P_1), \mathrm{Sym}(P_2)$ are positive. Using the traces and the determinants, this is equivalent to the conditions

$$\left\{ \begin{array}{rcl} m_1 m_2 & > & \max\{1, \frac{(d_1+d_2)^2}{4d_1 d_2}\} \\ 0 & \leq & d_1(m_1 - 1) + m_2 + 1 \\ 0 & \leq & d_2(m_2 - 1) + m_1 + 1 \\ m_1 & \geq & \frac{(d_2-1)^2}{4d_2} m_2 + \frac{(d_2-1)^2}{2d_2} + \frac{(d_2+1)^2}{4d_2} \frac{1}{m_2} \\ m_2 & \geq & \frac{(d_1-1)^2}{4d_1} m_1 + \frac{(d_1-1)^2}{2d_1} + \frac{(d_1+1)^2}{4d_1} \frac{1}{m_1} \end{array} \right. .$$

The first three inequalities are satisfied for m_1, m_2 large enough. The two last inequalities may also be satisfied for m_1, m_2 large enough if

$$\Delta_1 \Delta_2 < 1, \quad \text{where } \Delta_1 := \frac{(d_1 - 1)^2}{4d_1}, \, \Delta_2 := \frac{(d_2 - 1)^2}{4d_2}.$$

Indeed, we may then choose

$$m_1 = \lambda m_2 \quad \textit{with} \quad \Delta_2 < \lambda < \Delta_1^{-1},$$

and the two last inequalities are satisfied for m_1, m_2 large enough. The condition $\Delta_1 \Delta_2 < 1$ exactly means that $(d_1, d_2) \in D$.

To end the proof of Theorem 2.13 we need to justify (2.26). We denote by L the Laplace operator in $L^2(\Omega)$ with Neumann boundary conditions, namely

$$D(L) = \{u \in H^2(\Omega);\, \partial_\nu u = 0 \text{ on } \partial\Omega\}, \;\; \forall u \in D(L), Lu = -\Delta u.$$

For $\varepsilon > 0$, we denote $J_\varepsilon = (I + \varepsilon L)^{-1}$ its resolvent and we recall that, for all $v \in L^2(\Omega)$, $J_\varepsilon v \to v$ in $L^2(\Omega)$ as $\varepsilon \to 0$. Consequently, if $w \in L^2(Q_T)$, then $J_\varepsilon w$ converges in $L^2(Q_T)$ to w.

We set $F_\varepsilon(t) = J_\varepsilon F(t)$ where $F(t) = \int_0^t M^{\frac{1}{2}} BX$. Recall that $F(t) \in D(L)$ (see (2.23-2.24)) and $\partial_t F \in L^2(Q_T)$. We have the commutations

$$\partial_t F_\varepsilon(t) = J_\varepsilon(\partial_t F(t)), \, LF_\varepsilon(t) = J_\varepsilon LF(t).$$

Consequently, $\partial_t F_\varepsilon, LF_\varepsilon$ converge in $L^2(Q_T)^2$ to $\partial_t F, LF$. Since F_ε is regular enough to make the computation (2.25), we have

$$\int_{Q_T} < \partial_t F_\varepsilon, LF_\varepsilon > \geq 0, \tag{2.28}$$

and this inequality remains valid in the limit as $\varepsilon \to 0$, whence (2.26).

Remark 2.15. It is not so easy to find references for the uniqueness (up to a constant) of the solution to the variational problem

$$u \in W^{1,p}(\Omega), \forall \psi \in C^\infty(\overline{\Omega}), \int_\Omega \nabla\psi \cdot \nabla u = 0, \tag{2.29}$$

when $p \in [1,2)$ "only". Since Ω is regular, the above relation is valid by density for all $\psi \in W^{1,p'}(\Omega)$, $p' = p/(p-1)$. If $p = 2$, we may choose $\psi = u$ in (2.29), which easily yields uniqueness. But if $p \in [1,2)$, we need a different approach, for instance the following.

Let $\theta : \Omega \to \mathbb{R}$ be a C^∞-function with compact support and $\int_\Omega \theta = 0$. We introduce the solution (unique up to a constant) of

$$v \in H^2(\Omega) \cap W^{1,\infty}(\Omega), \ -\Delta v = \theta \text{ in } \Omega, \ \partial_\nu v = 0 \text{ on } \partial\Omega,$$

where the regularity $H^2(\Omega) \cap W^{1,\infty}(\Omega)$ is due to the regularity of Ω. Then, we may choose $\psi = v$ in (2.29). Next, we need to justify the integration by parts

$$\int_\Omega \nabla v \cdot \nabla u = \int_\Omega (-\Delta v)\, u.$$

For this, we approximate u in $W^{1,p}(\Omega)$ by a sequence of regular functions u_n. The integration by parts is valid for u_n. Then, we pass to the limit. Finally, the relation $0 = \int_\Omega \theta\, u$, for all θ as above, implies that u is a constant function.

2.3.3 Extra remarks on uniqueness

We just saw that the weak solution of the limit-problem (2.5) is unique if d_1, d_2 are close enough to d_3. The above sufficient condition may be written as

$$(\frac{d_1}{d_3} - 1)^2 (\frac{d_2}{d_3} - 1)^2 < 16 \frac{d_1 d_2}{d_3^2}$$

in general. This does not include the full case $d_1 = d_2 = d$. Uniqueness can nevertheless be proved directly in this case as follows: going back to system (2.21) for the difference of two solutions, and taking the difference of the two equations, we obtain that $U - V$ satisfies the heat equation in a weak sense:

$$\partial_t (U - V) - d\Delta(U - V) = 0, \ or \ (U - V)(t) - d\Delta \int_0^t (U - V)(s) ds = 0.$$

Multiplying by $(U - V)(t)$ and integrating on Q_T yields $\int_{Q_T} (U - V)^2 \leq 0$ (taking into account that we start with a weak solution, this may be justified by regularization as in the proof of Theorem 2.13). Hence $U = V$. Now, using $W = \hat{c}_2 U + c_1 V = (\hat{c}_2 + c_1) U$, the first equation gives

$$\partial_t [(1 + c_1 + \hat{c}_2) U] - \Delta[((d + c_1 + \hat{c}_2) U] = 0.$$

Integration on $(0,t)$, multiplication by $(d+c_1+\hat{c}_2)U$ and integration on Q_T leads to

$$\int_{Q_T}(1+c_1+\hat{c}_2)(d+c_1+\hat{c}_2)U^2 \leq 0.$$

Whence $U=0$ and then $V=0=W$, i.e. the solution is unique.

More generally, we may expect some uniqueness if d_1 and d_2 are close enough to each other. We may indeed prove the following.

Proposition 2.16. *Assume the initial data is regular. If d_1 is close enough to d_2, the limit-solution of (2.5) coincides with the regular solution of Proposition 2.11.*

Proof. We only indicate the main computations (justifications are the same as in the proof of Theorem 2.13). By difference of the two equations of (2.21), we have

$$\partial_t(U-V)-d_2\Delta(U-V)=(d_1-d_2)\Delta U. \tag{2.30}$$

From this, we first deduce

$$\|U-V\|_{L^2(Q_T)} \leq \frac{|d_1-d_2|}{d_2}\|U\|_{L^2(Q_T)}. \tag{2.31}$$

This may be proved by duality, by introducing the solution of

$$\begin{cases} -[\partial_t\phi+d_2\Delta\phi]=U-V \text{ on } Q_T, \\ \phi(T)=0,\ \partial_\nu\phi=0 \text{ on } \Sigma_T. \end{cases} \tag{2.32}$$

Multiplying equation (2.30) by ϕ and integrating by parts gives

$$\int_{Q_T}(U-V)^2=(d_1-d_2)\int_{Q_T}U\Delta\phi \leq |d_1-d_2|\|U\|_{L^2(Q_T)}\|\Delta\phi\|_{L^2(Q_T)}. \tag{2.33}$$

Multiplying the equation in ϕ by $-\Delta\phi$ leads to

$$\begin{aligned} -\int_{Q_T}\frac{1}{2}\partial_t|\nabla\phi|^2+d_2\int_{Q_T}(\Delta\phi)^2 &= \int_{Q_T}(V-U)\Delta\phi, \\ &\leq \frac{d_2}{2}\int_{Q_T}(\Delta\phi)^2+\frac{1}{2d_2}\int_{Q_T}(U-V)^2. \end{aligned}$$

Integrating in time the first integral and using its positivity, we deduce

$$\int_{Q_T}(\Delta\phi)^2 \leq \frac{1}{d_2^2}\int_{Q_T}(U-V)^2.$$

Whence (2.31) using also (2.33). Next, using the first equation in (2.21) and "multiplying" it by d_1U+W leads to:

$$\int_{Q_T}(U+W)(d_1U+W) \leq 0.$$

Setting $\zeta=V-U$ and using $W=\hat{c}_2U+c_1(U+\zeta)$, this may be rewritten as

$$\int_{Q_T}[(1+c_1+\hat{c}_2)U+c_1\zeta][(d_1+c_1+\hat{c}_2)U+c_1\zeta] \leq 0.$$

This implies

$$\int_{Q_T} (1+c_1+\hat{c}_2)(d_1+c_1+\hat{c}_2)U^2 + (c_1\zeta)^2 \leq \int_{Q_T} c_1|\zeta U|[d_1+1+2(c_1+\hat{c}_2))],$$

$$\leq \alpha \int_{Q_T} (c_1\zeta)^2 + \frac{1}{4\alpha}\int_{Q_T} [d_1+1+2(c_1+\hat{c}_2))]^2 U^2,$$

where we choose $\alpha = \max\left\{2, \frac{(d_1+1)^2}{2d_1}\right\}$ so that, for all $\theta \geq 0$,

$$\frac{1}{4\alpha}[d_1+1+2\theta]^2 \leq \frac{1}{2}(1+\theta)(d_1+\theta).$$

Finally, we may write

$$\int_{Q_T} (1+c_1+\hat{c}_2)(d_1+c_1+\hat{c}_2)U^2 \leq 2\alpha \int_{Q_T} (c_1\zeta)^2. \tag{2.34}$$

Now, we assume that $c = (c_1, c_2)$ is the regular solution so that, for $T < T^*$, $\|c_1\|_{L^\infty(Q_T)} < +\infty$ and we use (2.31):

$$\int_{Q_T} (1+c_1+\hat{c}_2)[d_1+c_1+\hat{c}_2]U^2 \leq 2\alpha\|c_1\|^2_{L^\infty(Q_T)} \frac{(d_1-d_2)^2}{d_2^2}\int_{Q_T} U^2.$$

If

$$d_1 > 2\alpha\|c_1\|^2_{L^\infty(Q_T)}\frac{(d_1-d_2)^2}{d_2^2},$$

we deduce that $U \equiv 0$. It follows that $V = U = 0 = W$.

Remark 2.17. This uniqueness result is not as "good" as the one obtained in Theorem 2.13: first, in the latter theorem, uniqueness is obtained for the global weak solution; moreover, it holds for a fixed region of values for d_1, d_2, d_3. Here, the distance required between d_1, d_2 depends on the L^∞-norm of the regular solution. It might tend to zero if the solution becomes singular in finite time. And it may then happen that a bifurcation appears with multiple weak solutions. This is an open question.

2.3.4 A third way to write the limit system

It turns out that there is still one more "formal" way to write the limit problem. We are not able to derive more information with it than we already did, but it seems nevertheless worth being mentioned.

Let us make the following computation for the limit of $c^K = (c_1^K, c_2^K, c_3^K)$. Let f be the distribution such that

$$k(c_1^K c_2^K - \kappa c_3^K) \overset{K\to(+\infty,\kappa^\infty)}{\longrightarrow} f.$$

If c is a solution of the limit problem satisfying $c_1c_2 = \kappa c_3$, we can differentiate in time this relation:

$$c_2\partial_t c_1 + c_1\partial_t c_2 = \kappa\partial_t c_3$$
$$c_2(d_1\Delta c_1 - f) + c_1(d_2\Delta c_2 - f) = \kappa(d_3\Delta c_3 + f).$$

Therefore, there is a unique possible choice for f, namely:

$$f = \frac{d_1 c_2 \Delta c_1 + d_2 c_1 \Delta c_2 - \kappa d_3 \Delta c_3}{c_1 + c_2 + \kappa}\ .$$

Replacing $k(c_1 c_2 - \kappa c_3)$ by f in (R^K) suggests the new form of the limit system:

$$(R^\infty)\begin{cases} \partial_t c &= (I - P(c))D\Delta c \quad \text{for } t > 0\ , x \in \Omega \\ \partial_\nu c &= 0 \quad \text{for } t > 0\ , x \in \partial\Omega \\ c(0) &= c^0 \quad \text{for } x \in \Omega;\ c^0 \in L^\infty(\Omega, \mathbb{R}^3_+) \end{cases}$$

where $D = \operatorname{diag}(d_1, d_2, d_3)$ and

$$P(c) = \frac{1}{c_1 + c_2 + \kappa}\begin{bmatrix} c_2 & c_1 & -\kappa \\ c_2 & c_1 & -\kappa \\ -c_2 & -c_1 & \kappa \end{bmatrix}.$$

Thus, we are led to a new nonlinear reaction-cross-diffusion system. Unfortunately, it is not possible to use it for the weak solutions expected in the limit since we do not know how to make sense of products like $c_i \Delta c_j$ when the c_i are not regular.

However, a simple analysis indicates that the matrix involved in (R^∞) has its spectrum in the closed right half-plane of the complex plane. Thus, the operator is "normally elliptic", up to adding a positive factor of the identity. Applying again H. Amann's results [3, 4], we obtain existence of local classical solutions for all given regular initial data. A difference with the previous 2×2 system (2.20) is that it is more general in the sense we do not require the initial conditions to satisfy $c_1^0 c_2^0 = \kappa c_3^0$. It is built into the system that the solutions must satisfy $[c_1 c_2 - \kappa c_3](t) = [c_1^0 c_2^0 - \kappa c_3^0]$, but this expression is not necessarily equal to zero. On the other hand, system (R^∞) does not preserve positivity while its restriction to the manifold $c_1 c_2 = \kappa c_3$ does.

2.4 Extensions

As explained in the introduction, the goal of this section is mainly to understand what happens in a reaction-diffusion system when a reversible reaction is considerably faster than diffusion. We chose to focus on the specific system (2.1) in order to concentrate on the main difficulties without being disturbed by other technical aspects. However, the techniques we have developed are rather general and can be applied to quite more general situations. Below, we discuss some explicit examples.

2.4.1 Extension to the chemical reaction $\sum_{i=1}^{p-1} \alpha_i C_i \rightleftharpoons C_p$

We indicate what should be added in the proof of Theorem 2.1 to extend it to the more general reaction of type

$$\sum_{i=1}^{p-1} \alpha_i C_i \underset{k\kappa}{\overset{k}{\rightleftharpoons}} C_p, \quad \alpha_i \in \mathbb{N}. \tag{2.35}$$

In the following, the concentration of C_i is denoted by c_i and the reaction term is supposed to be of the form $r(c) = k(\Pi_{i=1}^{p-1} c_i^{\alpha_i} - \kappa c_p)$, according to the mass action law, where $c = (c_1, ..., c_p)$. The associated reaction-diffusion system can be written as

$$(R_0^K)\begin{cases} \partial_t c - D\Delta c &= r(c)\nu \quad \text{on } (0, +\infty) \times \Omega, \\ \partial_\nu c &= 0 \quad \text{on } (0, +\infty) \times \partial\Omega, \\ c(0, \cdot) &= c^0 \quad \text{on } \Omega \end{cases} \tag{2.36}$$

with $D = \mathrm{diag}(d_1,...,d_p)$, $d_i > 0$, $\nu = (-\alpha_1,...,-\alpha_{p-1},1)$. The reaction term is quasi-positive and, for $c^0 \in L^\infty(\Omega,\mathbb{R}_+^p)$, we have the local existence of nonnegative classical solutions. The global existence still holds since the growth of the reaction term with respect to c_p is linear (Theorem 3.5 in [90] applies as well).

Again, we want to let $(k,\kappa) \mapsto (+\infty,\kappa^\infty)$. Note that

$$\forall i = 1,...,p-1,\ \partial_t(c_i + \alpha_i c_p) - \Delta(d_i c_i + \alpha_i d_p c_p) = 0. \tag{2.37}$$

There is a similar entropy inequality as (2.10) which provides estimates independent on $K = (k,\kappa)$ on the gradients in $L^{4/3}(Q_T)^N$ and shows that

$$\|(\Pi c_i^{\alpha_i} - \kappa c_p)(\log(\Pi c_i^{\alpha_i}) - \log(\kappa c_p))\|_{L^1(Q_T)} \to 0 \text{ when } (k,\kappa) \to (+\infty,\kappa^\infty).$$

The scheme of the proof is the same as what we did in the proof of Lemma 2.4: we only need to redefine

$$W_i = \alpha_i(c_i \log(c_i/c_i^*) - (c_i - c_i^*)),\ W = \textstyle\sum_{i=1}^p W_i,\ Z = \sum_{i=1}^p d_i W_i, \text{ with } c_1^{*\alpha_1}...c_{p-1}^{*\alpha_{p-1}} = \kappa c_p^* \neq 0.$$

Thanks to (2.37) and to the estimates coming from the entropy inequality, it is also possible to use Lemma 2.7 to get the compactness in $L^2(Q_T)$ of $c_i^K + \alpha_i c_p^K, 1 \leq i \leq p-1$.

Let $K_n := (k_n,\kappa_n) \to (+\infty,\kappa^\infty)$ and let c^n be the classical solution of $(R_0^{K_n})$ on Q_T. Up to a subsequence, $(c_i^n + \alpha_i c_p^n)_{1\leq i\leq p-1}$ converges to a limit $(a_i)_{1\leq i\leq p-1} \in L^2(Q_T)^{p-1}$ for all $T > 0$ and almost everywhere, and $(\Pi_{i=1}^{p-1} c_i^{n\alpha_i} - \kappa_n c_p^n)_{n\in\mathbb{N}}$ converges to 0 almost everywhere. Let $(t,x) \in Q_T$ such that this pointwise convergence holds. The sequence $(c^n(t,x))_{n\in\mathbb{N}}$ is bounded and a limit point $l = (l_1,...,l_p)$ for this sequence is a solution in $\mathbb{R}_+^p$ of the system

$$(s)\left\{\begin{array}{ccc} l_1 + \alpha_1 l_p & = & a_1(t,x) \\ \vdots & & \\ l_{p-1} + \alpha_{p-1} l_p & = & a_{p-1}(t,x) \\ l_1^{\alpha_1}\cdots l_{p-1}^{\alpha_{p-1}} & = & \kappa^\infty l_p \end{array}\right. ,\qquad (a_1,...,a_{p-1})(t,x) \in [0,\infty)^{p-1}. \tag{2.38}$$

Lemma 2.18. *The system (s) has a unique solution $l \in [0,\infty)^p$.*

Proof. Let l, l' be two solutions. Suppose first that $\forall i, a_i(t,x) > 0$. This implies: $\forall i, l_i > 0, l_i' > 0$. Then, taking the logarithm in the last equality of (s), we see that $< \log l - \log l', \nu >= 0$, where $< \cdot,\cdot >$ is the usual scalar product in $\mathbb{R}^p$. The linear relations in system (s) can be rewritten as $< L_i, l - l' >= 0$ for some $p-1$ independent vectors $L_i \in \mathbb{R}_+^p$. It is easy to check that $\forall i, < L_i, \nu >= 0$. Therefore, $l - l'$ is parallel to ν. Finally, we have

$$< \log l - \log l', l - l' >= 0 = \sum_{i=1}^p (\log l_i - \log l_i')(l_i - l_i') = 0.$$

Since the function log is increasing on $(0,+\infty)$, we deduce $l = l'$.

Suppose now that $I = \{i \in \{1,...,p-1\} : a_i = 0\}$ is not empty. If l is a solution of (s), we have $l_i = 0$ for $i \in I \cup \{p\}$ and for $j \notin I \cup \{p\}$, $l_j = a_j$, so l is unique.

From here on, everything works like in the previous proof: for almost every $(t,x) \in Q_T$, a subsequence of $c^n(t,x)$ is bounded and has a unique limit point, so it converges to this limit point. This shows the pointwise convergence of a subsequence of c^n. Since each c_i^n is dominated by an

$L^2(Q_T)$-convergent subsequence, the convergence of the subsequence of c_i^n holds also in $L^2(Q_T)$. Finally, the limit is a solution of the problem

$$\begin{cases} \forall i=1,...,p,\ c_i \in L^2(Q_T), \nabla c_i \in L^{\frac{4}{3}}(Q_T)^N,\ c_i \geq 0,\ c_1^{\alpha_1} c_2^{\alpha_2} ... c_{p-1}^{\alpha_{p-1}} = \kappa^\infty c_p, \\ \forall i=1,...,p-1,\ \forall \psi \in C^\infty(\overline{Q_T}) \text{ such that } \psi(T)=0, \\ \ -\int_\Omega \psi(0)(c_i^0+\alpha_i c_p^0) + \int_{Q_T} -\psi_t(c_i+\alpha_i c_p) + \nabla\psi.\nabla(d_i c_i + \alpha_i d_p c_p) = 0. \end{cases}$$

2.4.2 The case of a general chemical reaction

One may wonder what happens for a general chemical reaction

$$\sum_{i=1}^p \alpha_i C_i \underset{k\kappa}{\overset{k}{\rightleftharpoons}} \sum_{i=1}^p \beta_i C_i\ .$$

The corresponding system is similar to (2.36) except that

$$r(c) = k\left(\Pi_{i=1}^p c_i^{\alpha_i} - \kappa \Pi_{i=1}^p c_i^{\beta_i}\right).$$

We still have $p-1$ independent positive linear relations between the equations which will provide compactness of $p-1$ linearly independent combinations of the solution. Thanks to the reversibility, the entropy inequality will still hold and helps to pass to the limit a.e. and in $L^2(Q_T)$ for all components.

However, a main difference is that the existence of global solutions for (k,κ) fixed is still an open problem in general (see e.g. [90] for more comments). One can nevertheless say that, if we are in a situation where global existence holds for all (k,κ), then passing to the limit as $(k,\kappa) \to (+\infty, \kappa^\infty)$ will be essentially the same as for the previous examples. Some specific features may provide global existence of classical solutions (see e.g. [68]). Recall also that global weak solutions exist for the (k,κ)-system when $\sum_i \beta_i \leq 2$ (or $\sum_i \alpha_i \leq 2$) (see [90]). Our approach can very likely be extended to cases when one starts with weak solutions for the (k,κ) system.

2.4.3 Fast reaction limit with additional linearly bounded slow processes

We may also consider the case where the reaction

$$C_1 + C_2 \underset{k\kappa}{\overset{k}{\rightleftharpoons}} C_3 \tag{2.39}$$

is coupled with some other *slow processes* which would lead to a system

$$\begin{cases} \partial_t c - D\Delta c &= kr(c)\nu + g(c), \\ c(0) &= c^0 \in L^\infty(\Omega, [0,\infty)^p), \end{cases}$$

where $D = \operatorname{diag}(d_1,...,d_p), d_i > 0,\ r(c) = c_1c_2 - \kappa c_3,\ \nu = (-1,-1,1,0,...,0)$ and $g: \mathbb{R}^p \to \mathbb{R}^p$ is Lipschitz continuous and quasi-positive. Let us indicate how this situation can be also treated by using the same tools.

Thanks to the linear growth of g and to Lemma 2.3, the above system has global classical solutions for $K=(k,\kappa)$ fixed. Moreover, the $L^1(\Omega)^p$-norm of $c(t)$ is bounded on any interval. Setting

$$W = \sum_i c_i + c_3,\ Z = \sum_i d_i c_i + d_3 c_3, G = \sum_i g_i + g_3,$$

we have $\partial_t W - \Delta Z = G$.
Using that $G \leq k_1 W + k_2, k_1, k_2 \in (0,\infty)$, we obtain an $L^2(Q_T)$-bound on W as in Lemma 5. At this step, we need an alternative for the entropy inequality (2.10). The same computation as in the proof of Lemma 2.4 leads to an inequality where the right-hand side "C" of (2.10) is to be replaced by $\int_{Q_T} \sum_i g_i(c) \log c_i$ which is also bounded for each T (due to the $L^2(Q_T)$-bound on c_i and the Lipschitz continuity of g; see also [19]).

From a slight extension of Lemma 2.7 to $\partial_t W - \Delta Z = G$, we deduce the $L^2(Q_T)$-compactness of W as $K \to (+\infty, \kappa^\infty)$ as before. From the $L^2(Q_T)$-bound on the c_i and the linear growth of g, we deduce the compactness of each $c_i, i \geq 4$ in $L^2(Q_T)$.

Now, we are left with checking what happens for c_1, c_2, c_3. We still have $L^2(Q_T)$-compactness of $c_1 + c_3, c_2 + c_3$. The rest of the proof is the same and we are led to the limit system

$$\left\{ \begin{array}{l} \partial_t(c_1+c_3) - \Delta(d_1c_1+d_3c_3) = g_1+g_3 \\ \partial_t(c_2+c_3) - \Delta(d_2c_2+d_3c_3) = g_2+g_3 \\ c_1c_2 = \kappa^\infty c_3 \\ \forall i \geq 4, \ \partial_t c_i - d_i \Delta c_i = g_i(c) \end{array} \right. \qquad \text{on } Q_T.$$

together with initial and boundary conditions.

Note that the above applies in particular to the famous Michaelis-Menten reaction for enzymatic catalysis:

$$C_1 + C_2 \underset{k\kappa}{\overset{k}{\rightleftharpoons}} C_3 \xrightarrow{k_2} C_1 + C_4. \tag{2.40}$$

In this situation, $g_1 = k_2c_3, g_2 = k_2c_3, g_3 = -k_2c_3$. We identify as above the limit system as $(k,\kappa) \to (+\infty, \kappa^\infty)$. Note that it does not directly lead to the famous Michaelis-Menten homographic limit model which would require one more asymptotics, taking into account small initial concentrations of the enzyme C_1.

2.4.4 Fast-reaction limit with additional slow reactions

We now analyse the situation when P chemically reacting species $C_1, \ldots, C_P$ are present, R chemical reactions happen simultaneously, and amongst them, the reaction $C_1 + C_2 \rightleftharpoons C_3$ is supposed to be much faster.
More precisely, we consider

$$\left\{ \begin{array}{rcll} \partial_t c - D\Delta c &=& \sum_{j=1}^R k_j r_j(c) \nu_j & \text{on } (0,+\infty)\times\Omega, \\ \partial_\nu c &=& 0 & \text{on } (0,+\infty)\times\partial\Omega, \\ c(0) &=& c^0, & \text{on } \Omega. \end{array} \right. \tag{2.41}$$

As before, $c^0 \in L^\infty(\Omega, \mathbb{R}_+^P)$; $D = \mathrm{diag}(d_1, \ldots, d_P),\ d_i > 0$ is the diffusion matrix. The indices $j = 1, \ldots, R$ refer to the different chemical reactions

$$\alpha_j^1 C_1 + \ldots + \alpha_j^P C_P \underset{k_j\kappa_j}{\overset{k_j}{\rightleftharpoons}} \beta_j^1 C_1 + \ldots + \beta_j^P C_P,$$

where $\alpha_j = (\alpha_j^1, \ldots, \alpha_j^P)$, $\beta_j = (\beta_j^1, \ldots, \beta_j^P)$ are nonzero vectors of $\mathbb{N}^n$. The so-called stoechiometric vectors are defined as $\nu_j = \beta_j - \alpha_j \in \mathbb{Z}^n$. The constants $k_j, \kappa_j > 0$ denote the reaction speed and the equilibrium constant of the j^{th} reaction. The reaction terms are modeled with Mass Action Kinetics, which reads, with the notation $c^\gamma = \Pi_{i=1}^P c_i^{\gamma_i}$ for $\gamma \in \mathbb{N}^P$,

$$\forall j = 1,\dots,R, \quad r_j(c) = c^{\alpha_j} - \kappa_j c^{\beta_j}.$$

The indice 1 will refer to the chemical reaction $C_1 + C_2 \rightleftharpoons C_3$, so $\nu_1 = (-1,-1,1,0,\dots,0)$ and $r_1(c) = c_1c_2 - \kappa_1 c_3$.

It is known that (2.41) has a unique classical solution on a maximum time interval $[0,T^*)$, $T^* \le +\infty$, and since the reaction terms are quasi-positive, it remains nonnegative. However, it is not known in general whether or not this solution is global. In the following, we assume

(i) The reaction-diffusion system (2.41) has some global classical solutions. This is the case, for instance, if all the reactions are of the type $\sum_{i=1}^{p-1} \alpha_i C_i \rightleftharpoons C_p$ (see Corollary 5.6).

(ii) The polynomials r_j are of degree at most two.

(iii) The vectors $\nu_i \in \mathbb{R}^P$ are linearly independent.

In this situation, the main new difficulty is to deal with the quadratic reaction terms. We now generalize Theorem 2.1 as follows:

Proposition 2.19. *Let $k_1^n \to +\infty$ and c^n be the corresponding solution of (2.41) on $(0,+\infty)\times\Omega$. Under assumptions $(i),\dots,(iii)$, up to a subsequence, $c^n \to c$ in $L^2(Q_T)^P$ for any $T>0$, where $c = (c_1,\dots,c_P)$ satisfies*

$c_1c_2 = \kappa_1 c_3$; $\forall T > 0$, $c \in L^2(Q_T)$, $\nabla c \in L^{4/3}(Q_T)$;

$\forall \psi \in C_c^\infty(Q_T)$ s.t. $\psi(T) = 0$, $\forall i \in \{4,\dots,P\}$,

$$\begin{cases} \int_{Q_T} -\partial_t \psi\,(c_1+c_3) + \nabla\psi\nabla(d_1c_1+d_3c_3) = \int_{Q_T} \psi\left(\sum_{j=1}^R k_j r_j(\nu_j^1+\nu_j^3)\right) + \int_\Omega \psi(0)(c_1^0+c_3^0), \\ \int_{Q_T} -\partial_t \psi\,(c_2+c_3) + \nabla\psi\nabla(d_2c_2+d_3c_3) = \int_{Q_T} \psi\left(\sum_{j=1}^R k_j r_j(\nu_j^2+\nu_j^3)\right) + \int_\Omega \psi(0)(c_2^0+c_3^0), \\ \int_{Q_T} -\partial_t \psi c_i + d_i\nabla\psi\nabla c_i = \int_{Q_T} \psi\left(\sum_{j=1}^R k_j r_j \nu_j^i\right) + \int_\Omega \psi(0)c_i^0. \end{cases}$$

Proof. As an easy consequence of assumption (iii), there exists $c^* = (c_1^*,\dots,c_P^*) \in (0,+\infty)^P$ satisfying

$$\forall j \in \{1,\dots,R\}, \quad c^{*\alpha_j} = \kappa_j c^{*\beta_j}.$$

This may be seen by taking the logarithm of the above expressions, see [18]. Let

$$W_i^n = c_i^n \log\left(\frac{c_i^n}{c_i^*}\right) - (c_i^n - c_i^*) \ge 0\ ;\ W^n = \sum_{i=1}^P W_i^n\ ;\ Z^n = \sum_{i=1}^P d_i W_i^n\ .$$

Similarly as in Lemma 2.4, a straightforward computation yields the "entropy equality"

$$\partial_t W^n - \Delta Z^n + \sum_{i=1}^P d_i \frac{|\nabla c_i^n|^2}{c_i^n} + \sum_{j=1}^R k_j (c^{n\,\alpha_j} - \kappa_j c^{n\,\beta_j})(\log(c^{n\,\alpha_j}) - \log(\kappa_j c^{n\,\beta_j})) = 0. \tag{2.42}$$

We use two different techniques to exploit this equation: on the one side, integration of (2.42) on Q_T yields, using the homogeneous Neumann boundary conditions,

$$\int_\Omega W^n(T) + \sum_{i=1}^P \int_{Q_T} d_i \frac{|\nabla c_i^n|^2}{c_i^n} + \sum_{j=1}^R k_j \int_{Q_T} (c^{n\,\alpha_j} - \kappa_j c^{n\,\beta_j})(\log(c^{n\,\alpha_j}) - \log(\kappa_j c^{n\,\beta_j})) = \int_\Omega W(0). \tag{2.43}$$

Since $\int_\Omega W(0)$ does not depend on n and all the terms on the left-hand side are nonnegative, they are all bounded independently of n. On the other side, remark that if $\underline{d} = \min\{d_i\}$, $\overline{d} = \max\{d_i\}$, then $\underline{d}W^n \le Z^n \le \overline{d}W^n$ and according to (an easy generalization of) Lemma 2.7 ,

$$\forall i \in \{1,\dots,P\},\ c_i^n \log^+ c_i^n \text{ is bounded in } L^2(Q_T). \tag{2.44}$$

Let $4 \le i \le P$. Since the constant k_1 is only present in the reaction terms for c_1, c_2 and c_3, using Lemma 4.4 (ii) (see Section 4), $(c_i)_{n\in\mathbb{N}}$ is relatively compact in $L^p(Q_T)$ for $p \in [1,2)$. Since $c_i^n \log^+ c_i^n$ is bounded in $L^2(Q_T)$, c_i^n is uniformly integrable in $L^2(Q_T)$ and using the Vitali theorem, $(c_i^n)_{n\in\mathbb{N}}$ is relatively compact in $L^2(Q_T)$. To recover the compactness of $c_1^n + c_3^n$ and $c_2^n + c_3^n$, we may argue as in Lemma 2.6 : for $i \in \{1,2\}$, if $\zeta_i^n = (1 + c_i^n + c_3^n)^{1/2}$, we have

$$2\,|\nabla \zeta_i^n| = \left| \frac{\nabla c_i^n + \nabla c_3^n}{\zeta_i^n} \right| \le \frac{|\nabla c_i^n|}{(c_i^n)^{1/2}} + \frac{|\nabla c_3^n|}{(c_3^n)^{1/2}}.$$

Using (2.43), $\nabla \zeta_i^n$ is bounded in $L^2(Q_T)^N$. Now

$$2\,\partial_t \zeta_i^n = \frac{\partial_t(c_i^n + c_3^n)}{\zeta_i^n} = \frac{\Delta(d_i c_i^n + d_3 c_3^n) + \sum_{j=1}^P k_j r_j(c^n)(\nu_j^i + \nu_j^3)}{\zeta_i^n} = \nabla \cdot f_i^n + g_i^n + h_i^n,$$

$$f_i^n = \frac{\nabla(d_i c_i^n + d_3 c_3^n)}{\zeta_i^n},\ \ g_i^n = \frac{\nabla(d_i c_i^n + d_3 c_3^n)\nabla(c_i^n + c_3^n)}{2(\zeta_i^n)^3},\ \ h_i = \frac{\sum_{j=1}^P k_j r_j(c^n)(\nu_j^i + \nu_j^3)}{\zeta_i^n}.$$

Using once more (2.43), f_i^n is bounded in $L^2(Q_T)^N$ and g_i^n is bounded in $L^1(Q_T)$. Using (2.44), h_i is bounded in $L^1(Q_T)$, so $\partial_t \zeta_i^n$ is bounded in

$$L^2(0,T : H^{-1}(\Omega)) + L^1(Q_T) \subset L^1(0,T : Y),\ Y := H^{-1}(\Omega) + L^1(\Omega).$$

Since ζ_j^n is also bounded in $L^2(0,T;H^1(\Omega))$ where $H^1(\Omega)$ is compactly embedded into $L^2(\Omega) \subset Y$, by the Aubin-Simon compactness results (see [98, Corollary 4]), $(\zeta_j^n)_{n\in\mathbb{N}}$ is relatively compact in $L^2(Q_T)$. Since $(c_i^n + c_3^n)\log(c_i^n + c_3^n)$ is bounded in $L^2(Q_T)$, the Vitali theorem guarantees that $(c_i^n + c_3^n)_{n\in\mathbb{N}}$ is relatively compact in $L^2(Q_T)$. The rest of the proof is similar to what has been done for Theorem 2.1 : we show the pointwise convergence of c^n as in Lemma 2.9 , then we get the compactness of c_1^n, c_2^n, c_3^n in $L^2(Q_T)$. This allows to prove the convergence of the (quadratic) reaction terms, and using (2.43), (2.44), we can pass to the limit in a variational formulation, which ends the proof of Proposition 2.19.

2.5 Appendix

Proof of Lemma 2.7.

Multiplying the equation in W, Z of Lemma 2.7 by Z and integrating on Q_T leads to

$$\int_{Q_T} \left[W Z + \nabla Z \cdot \nabla \int_0^t Z(s)ds \right] = \int_\Omega W(0) \int_0^T Z(s)ds,$$

or

$$\int_{Q_T} W Z + \frac{1}{2} \int_\Omega \left| \nabla \int_0^t Z(s)ds \right|^2 = \int_\Omega W(0) \int_0^T Z(s)ds. \tag{2.45}$$

We deduce

$$d^{\min} \int_{Q_T} W^2 \le d^{\max} \sqrt{T} \left[\int_\Omega W(0)^2 \right]^{1/2} \left[\int_{Q_T} W^2 \right]^{1/2}.$$

The announced $L^2(Q_T)$-bound on W, and therefore on Z, follows.

Now, let (W^p, Z^p) be a sequence in $\mathcal{F}$ such that $W^p(0) \in \mathcal{G}$ and (W^p, Z^p) converges weakly in $L^2(Q_T)^2$ to (W, Z). Let us pass to the limit as $p \to +\infty$ in

$$W^p(t) - \Delta \int_0^t Z^p(s)\, ds = W^p(0),\ \partial_\nu \int_0^t Z^p(s)\, ds = 0 \text{ on } \Sigma_T.$$

Note that $\Delta \int_0^t Z^p(s)\, ds$ is bounded in $L^2(Q_T)$ so that $\int_0^t Z^p(s)\, ds$ is bounded in $L^2\left(0,T;H^2(\Omega)\right)$. Thus, we may pass to the limit (weakly in L^2) to get

$$W(t) - \Delta \int_0^t Z(s)\, ds = W_0,\ \partial_\nu \int_0^t Z(s)\, ds = 0\ on\ \Sigma_T,$$

where W_0 is the weak limit in $L^2(\Omega)$ of $W^p(0)$. Now, we multiply the identity

$$W^p(t) - W(t) - \Delta \int_0^t [Z^p - Z](s)\, ds = W^p(0) - W_0,$$

by $Z^p - Z$. As in the computation leading to (2.45), we will use that

$$\int_{Q_T} -(Z^p - Z)\Delta \int_0^t [Z^p - Z](s)\, ds \geq 0. \tag{2.46}$$

This may be justified by introducing $Z_h(t) = h^{-1} \int_t^{t+h} [Z^p - Z](s) ds$. Then $Z_h, \Delta \int_0^t Z_h$ converge in $L^2(Q_T)$ respectively to $Z^p - Z, \Delta \int_0^t (Z^p - Z)$. Moreover, $Z_h \in L^2\left(0,T;H^1(\Omega)\right)$ so that the following computation is allowed

$$\int_{Q_T} -Z_h \Delta \int_0^t Z_h(s) ds = \int_{Q_T} \nabla Z_h \nabla \int_0^t Z_h(s) ds = \frac{1}{2} \int_\Omega \left| \nabla \int_0^T Z_h(s) ds \right|^2 \geq 0.$$

And we may pass to the limit as $h \to 0$ to recover (2.46). It implies

$$\int_{Q_T} (W^p - W)(Z^p - Z) \leq \int_\Omega (W^p(0) - W_0) \int_0^T [Z^p - Z](s)\, ds. \tag{2.47}$$

Next, let $H^p(t) := \int_0^t Z^p(s)\, ds$. We have

$$\int_\Omega |\nabla H^p(t)|^2 = \int_{Q_t} 2\, \partial_t (\nabla H^p) \nabla H^p = -\int_{Q_t} 2\, \partial_t H^p \Delta H^p,$$

so that, since ΔH^p and $\partial_t H^p = Z^p$ are bounded in $L^2(Q_T)$,

$$\sup_{t \in [0,T]} \int_\Omega |\nabla H^p(t)|^2 \leq 2 \left[\int_{Q_T} (\partial_t H^p)^2 \right]^{1/2} \left[\int_{Q_T} (\Delta H^p)^2 \right]^{1/2} \leq C < +\infty.$$

It follows that H^p is bounded in $L^\infty\left(0,T;H^1(\Omega)\right) \cap H^1\left(0,T;L^2(\Omega)\right)$: by compact embedding of H^1 into L^2 and by Ascoli's Theorem, H^p is compact in $C\left([0,T];L^2(\Omega)\right)$. We deduce that it converges strongly in $C([0,T];L^2(\Omega))$ and the limit is necessarily $\int_0^t Z(s) ds$. Thus the right hand side of (2.47) tends to zero.

Now, using $Z^p - Z = A^p W^p - A^p W + A^p W - Z$ in the left-hand side of (2.47), we may write

$$\limsup_{p \to \infty} \int_{Q_T} A^p (W^p - W)^2 + Z^p W - W^p Z - A^p W^2 + WZ \leq 0. \tag{2.48}$$

Using assumption (2.13) and the weak-L^2 convergence of (W^p, Z^p) towards (W,Z), the integral $\int_{Q_T} Z^p W - W^p Z - A^p W^2 + WZ$ converges to zero as $p \to \infty$. Using $A^p \geq d^{\min} > 0$, we deduce that $W^p - W$ converges to zero strongly in $L^2(Q_T)$.

Let us now show that, if (W^p, Z^p) converges weakly in $L^2(Q_T)$ to (W,Z) and if moreover W^p converges a.e., then (2.13) holds. Let W^∞ be the a.e. limit of W^p. Thanks to the $L^2(Q_T)$-bound on W^p, by a Vitali-type argument, it is classical that W^p converges in $L^1(Q_T)$ to W^∞ (and even in $L^q(Q_T)$ for all $q \in [1,2)$). In particular, $W^\infty = W$. Then, note that this implies that any weak$^* - L^\infty(Q_T)$ limit-point A^∞ of A^p is equal to Z/W. Indeed, if $\psi \in C_0^\infty(\Omega)$, ψW^p converges strongly in $L^1(Q_T)$ to ψW, so that, up to convenient subsequences

$$\forall \psi \in C_0^\infty(Q_T), \ \int_{Q_T} \psi W^p A^p = \int_{Q_T} \psi Z^p \to \int_{Q_T} \psi W A^\infty = \int_{Q_T} \psi Z.$$

The last equality, valid for all $\psi \in C_0^\infty(Q_T)$, implies that $A^\infty = Z/W$, and it follows that the full sequence A^p converges to $A = Z/W$ in the sense of (2.13).

Finally, if A^p converges a.e., and if A^∞ denotes its a.e. limit, by dominated convergence (recall that A^p is uniformly bounded), A^p converges in any $L^q(Q_T), q < \infty$ towards A^∞ (and also in weak*-$L^\infty(Q_T)$). To see that $A^\infty = A = Z/W$, we pass to the limit in the identity $Z^p = A^p W^p$ where

$$(Z^p, W^p) \to (Z, W) \ \textit{in weak} - L^2(Q_T)^2, \ A^p \to A^\infty \ \textit{strongly in} \ L^2(Q_T),$$

so that $Z = A^\infty W$.

Part II

Global existence for some systems with nonlinear diffusions

3

Global existence for a class of reaction-diffusion systems with mass action kinetics and concentration-dependent diffusivities

The results of this section will appear in [29] in a joint work with D. Bothe.

In this work, we study the existence of global classical solutions for a class of reaction-diffusion systems with quadratic growth naturally arising in mass action chemistry when studying networks of reactions of the type $C_i + C_j \rightleftharpoons C_k$ with Fickian diffusion, where the diffusion coefficients might depend on time, space and on all the concentrations c_i of the chemical species. In the case of one single reaction, we prove global existence for space dimensions $N \leq 5$. In the more restrictive case of diffusion coefficients of the type $d_i(c_i)$, we use an L^2-approach to prove global existence for $N \leq 9$. For space dimensions $N = 2$ and $N = 3$, global existence holds for more than quadratic reactions terms, with an explicit dependence between N and the admissible exponents. Finally, we investigate the general case of networks of reactions and extend the previous method to get global solutions for $N \leq 3$ and general diffusivities and for $N \leq 5$ and diffusivities $d_i(c_i)$.

3.1 Introduction

Chemical reaction-diffusion systems (RD-systems for short) consist of mass balances, often given in terms of molar mass densities c_i of certain chemical species C_i, where $i = 1, \ldots, P$ in case of P involved chemical components. This leads to PDE-systems of the form

$$\partial_t c_i + \operatorname{div} J_i = r_i \qquad (i = 1, \ldots, P), \tag{3.1}$$

where J_i is the (molar) mass flux of species C_i and the source term r_i models the rate of change of C_i due to chemical reactions.

While transport of C_i is usually mediated by several parallel mechanisms like convection, diffusion or migration, the fluxes in (3.1) are commonly considered to be of diffusive type in case of RD-systems. These diffusive fluxes are most often modeled by the classical Fick's law, i.e. constitutive relations of the type

$$J_i = -d_i \nabla c_i \qquad (i = 1, \ldots, P) \tag{3.2}$$

are employed for this purpose, where the diffusivities d_i are nonnegative due to the second law of thermodynamics [35]. In (3.2), the d_i will be (complicated) functions of the system's thermodynamic state variables, in particular the diffusivities depend significantly on the mixture composition, i.e. on the concentration vector $c := (c_1, \ldots, c_P)$. A flux of Fickian type (3.2) can either model so-called molecular diffusion caused by the random thermal motion of all molecules, or an effective diffusive flux due to other stochastic particle motions such as random convective motions of fluid parcels in a turbulent velocity field. In the latter case one also speaks of dispersive mixing or dispersion instead of diffusion; cf. [8].

We consider systems of type (3.1) in bounded domains $\Omega \subset \mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$ under the homogeneous Neumann boundary condition

$$J_i \cdot \nu = 0 \quad \text{on } \partial\Omega \qquad (i = 1, \ldots, P), \tag{3.3}$$

where ν denotes the outward unit normal to Ω. We also impose the initial conditions

$$c_i(0, \cdot) = c_{0,i} \quad \text{on } \Omega \qquad (i = 1, \ldots, P), \tag{3.4}$$

where the initial concentrations $c_{0,i}$ are non-negative and sufficiently regular, at least in $L^\infty(\Omega)$.

One main emphasis of the present section lies on the investigation of RD-systems from physico-chemical backgrounds. Typical applications come from Chemical Reaction Engineering, say reactions in liquid systems under isobaric conditions (such that no convective flow occurs) or diffusion of reactive species into solids. There is a large amount of measurement data from such applications, showing the dependence of the Fickian diffusivities on the concentrations; see in particular [34]. Instead of going into further details on measured dependencies, we prefer to include a brief theoretical explanation. For such systems, the Maxwell-Stefan equations provide a more fundamental and thermodynamically consistent approach to model diffusive multicomponent transport; cf. [54], [66]. The Maxwell-Stefan equations form a reduced set of partial momentum balances for the involved constituents, relying on a scale-separation argument which is a very accurate approximation for diffusion velocities far below the speed of sound [21]. To avoid cases with additional migrative transport, we also assume that the species are uncharged, which rules out certain cases with ionic species especially appearing in aqueous solutions. Furthermore, we assume isothermal conditions to avoid thermal diffusion processes and, more important, severe complications due to the usually significant temperature dependence of chemical reactions. Finally, we assume that no convective transport occurs in the mixture. In case of a fluid system this corresponds to isobaric conditions, since any pressure gradient will cause the mixture to flow. In the resulting isobaric and isothermal case without species-dependent body forces, the Maxwell-Stefan equations read

$$-\sum_{j \neq i} \frac{x_j J_i - x_i J_j}{c_{\text{tot}}\, Đ_{ij}} = \frac{x_i}{RT} \nabla \mu_i \quad \text{for } i = 1, \ldots, P. \tag{3.5}$$

Here $c_{\text{tot}} := \sum_i c_i$ is the total concentration, $x_i := c_i / c_{\text{tot}}$ are the molar fractions, R is the universal gas constant, T the absolute temperature and μ_i the chemical potential of species C_i. Moreover, the $Đ_{ij}$ are the so-called Maxwell-Stefan diffusivities which are symmetric, where the latter is either seen as a consequence of Onsager's reciprocal relations, or can be deduced under the assumption of binary interactions; cf. [21]. Like the Fickian diffusivities, the $Đ_{ij}$ are not constant but depend on the thermodynamic state variables - especially, $Đ_{ij} = Đ_{ij}(c)$. The set of equations (3.5) is complemented by the constraint

$$\sum_{i=1}^{P} J_i = 0, \tag{3.6}$$

expressing the fact that diffusive fluxes are taken relative to a common mixture velocity, where the latter is assumed to be zero throughout this section.

The system of equations (3.5) and (3.6) can be inverted to obtain the diffusive fluxes J_i; see [54], [20]. The resulting fluxes account both for direct cross-effects due to friction between the components as expressed by the left-hand side in (3.5), and for non-idealities due to complex material behavior which enters via the chemical potentials on the right-hand side of (3.5). In the general case of a multicomponent system with diffusive fluxes modeled by (3.5) and (3.6), a fully coupled RD-system with fluxes of type

$$J_i = -\sum_{j=1}^{P} d_{ij}\,\nabla c_j \qquad (i = 1,\dots,P) \tag{3.7}$$

results, where the non-diagonal diffusion matrix $[d_{ij}]$ depends on the composition c. Without chemical reactions, the pure diffusion system (3.1), (3.3) – (3.6) is locally in time wellposed for sufficiently regular initial data as shown in [20]. But for the chemically reactive case no results on global existence of solutions are currently known.

The present section investigates the complications due to non-constant diffusivities, but possible diffusive cross-effects are ignored. To motivate these particular class of RD-systems with concentration-dependent diffusivities but without cross-diffusion, let us briefly discuss two important special cases in which the Maxwell-Stefan equations can be explicitly inverted. For a *binary system*, i.e. a system with two components, it follows from $x_1 + x_2 = 1$ and $J_1 + J_2 = 0$ that

$$J_1\,(= -J_2) = -\frac{Đ_{12}}{RT} c_1\,\mathrm{grad}\,\mu_1. \tag{3.8}$$

The chemical potential of C_1, say, is of the form $\mu_1 = \mu_1^0 + RT\ln(\gamma_1 x_1)$ with a reference chemical potential μ_1^0 which only depends on pressure and temperature and the so-called activity coefficient $\gamma_1 = \gamma_1(x_1)$; note that the additional variable c_{tot} of γ_1 is constant in the considered isobaric case. This yields

$$J_1 = -Đ_{12}\Big(1 + \frac{x_1\,\gamma_1'(x_1)}{\gamma_1(x_1)}\Big)\,\nabla c_1, \tag{3.9}$$

where $Đ_{12}$ is a function of x_1. Inserting this into (3.1) leads to the nonlinear diffusion equation

$$\partial_t c_1 - \Delta\phi(c_1) = r(c_1), \tag{3.10}$$

where the function $\phi : \mathbb{R} \to \mathbb{R}$ satisfies $\phi'(s c_{\mathrm{tot}}) = Đ_{12}(s)(1 + s\gamma'(s)/\gamma(s))$ and, say, $\phi(0) = 0$. Equation (3.10) is also known as the filtration equation (or, the generalized porous medium equation) in other applications. Note that (3.10) is locally wellposed in $L^1(\Omega)$ as soon as ϕ is continuous and nondecreasing which will also be used below; cf., e.g., [101]. For constant $Đ_{12}$, the monotonicity of ϕ holds if $s \to s\gamma(s)$ is increasing. This means that the chemical potential μ_1 should be an increasing function of x_1, which characterizes systems without phase separation.

A *dilute system* is a system in which one component, say C_P, satisfies $x_P \approx 1$ and acts as a solvent, while the other components are solutes and only appear in small concentrations, i.e. $x_i \ll 1$ for $i = 1,\dots,P-1$. In this case the chemical potential of the dilute species is given by

$$\mu_i = \mu_i^0 + RT\ln x_i.$$

This leads to the diffusive fluxes

$$J_i = -\frac{Đ_{iP}}{RT} c_{\mathrm{tot}}\nabla x_i = -\frac{Đ_{iP}}{RT}\nabla c_i. \tag{3.11}$$

Here the basic assumption is that interactions only occur between individual solutes and the solvent, but not between different solutes. Hence $Đ_{iP}$ depends only on x_i and x_P. Since x_P is almost constant equal 1, it is essentially a function of x_i, i.e. of c_i. This leads to Fick's law with diffusivities $d_i = d_i(c_i)$.

Combining the above prototype cases leads to a large class of mixtures in which two components are present in large amounts, while all other components are dilute. This applies to many concrete cases in Chemical Reaction Engineering, in which one species (e.g., water) acts as a solvent, one further species is the main feed into the process and the other constituents are further reactants, catalysts, initiators, intermediates or products. This case leads to diffusivities d_i which not only depend on c_i, but also on at least one further c_j, while still no cross-diffusion appears.

Let us note that other chemical applications as well as completely different motivations also lead to RD-systems with concentration-dependent diffusivities. Besides reactive turbulent flows (cf. [8]), let us only mention reactive transport in the underground, i.e. inside porous media (cf. [76]). A common approach to model multicomponent transport in porous media employs an extension of the Maxwell-Stefan equations, the so-called dusty gas model. The latter is based on adding another species, modeling the pore walls, which is immobile. For a dilute species in a porous medium this again leads to diffusivities of type $d_i(c_i)$, as sketched above.

More general, system (3.1) can represent a set of population balances (cf., e.g., [85]), in which case c_i denotes a number density of individuals of the i-th population. Then the diffusive fluxes correspond to stochastic motions of the individuals, while additional migrative fluxes might also occur in such situations. Again, the d_i will be non-constant as well as non-negative.

The mass production terms r_i are nonlinear functions of the composition, with superlinear growth except in rare cases like isomerizations of type $C_1 \rightleftharpoons C_2$. Hence, while local-in-time existence of even classical solutions usually follows from known results on quasi-linear parabolic PDE-systems (like the theory from [2], [4]), the issue of global existence of solutions can be a much more difficult one, depending on the structure of the reaction terms. To this end, in order to have reliable information about the form of the r_i at all, we focus on the case of (networks of) elementary reactions. These are chemical reactions which run in a single step without the formation of intermediate species. In other words, if intermediate steps occur, they have to be fully modeled by an appropriate reaction network. In this case the rate functions for the elementary reactions are accurately modeled by so-called mass action kinetics. To be more specific, the rate function r for the single reversible reaction of type

$$\alpha_1 C_1 + \ldots + \alpha_P C_P \rightleftharpoons \beta_1 C_1 + \ldots + \beta_P C_P$$

with stoichiometric coefficients $\alpha_i, \beta_i \in \mathbb{N}_0$ is given as $r = r^f - r^b$ with the forward and backward rates

$$r^f(c) = k^f \prod_{i=1}^{P} c_i^{\alpha_i} \qquad \text{and} \qquad r^b(c) = k^b \prod_{i=1}^{P} c_i^{\beta_i},$$

respectively. The so-called rate constants k^f, k^b are not constant but depend especially on the temperature. Still, considering only isothermal systems, we will assume them to be constants below.

RD-systems with mass action kinetics, or more general rate functions of polynomial type, say, but with *constant* Fickian diffusivities have been studied in many papers for long time. Concerning global existence of solutions, already for constant diffusion coefficients the situation is complicated unless all d_i's are the same. A recent survey about the subject can be found in [90]. Here, let us only emphasize that the main elementary reactions which occur in chemical reaction networks are of the form

$$C_1 + C_2 \rightleftharpoons C_3, \tag{3.12}$$

or

$$C_1 + C_2 \rightleftharpoons C_3 + C_4, \tag{3.13}$$

i.e. at most two reaction partners appear on each side since (reactive) collisions of more than two molecules are very rare events. Note that we left out reactions of the form $C_1 \rightleftharpoons C_2$ which are considered trivial due to their linear rate functions, while $C_1 = C_2$ or $C_3 = C_4$ is allowed in the reaction mechanism (3.12), respectively (3.13). Reactions of type (3.12) occur for example if double bonds are opened in halogenizations, hydrations, sulfonizations etc., while mechanism (3.13) is typical for exchange reactions, where one reactant breaks into two parts, one of them being replaced by the reaction partner.

Let us note that a reaction which is formally of type (3.13) might involve an intermediate species C_5, such that the elementary steps are rather

$$C_1 + C_2 \rightleftharpoons C_5 \rightleftharpoons C_3 + C_4, \tag{3.14}$$

instead. In this case, the reaction is build from blocks of type (3.12). Let us also note that even without occurrence of an intermediate form C_5, the reaction from $C_1 + C_2$ to $C_3 + C_4$ proceeds via a so-called transition state, but the latter has a very limited life time of about $10^{-13}s$, only. Compared to any transport process by diffusion, the transition hence is so fast that the transition state need not be separately accounted for in the model. Indeed, the rigorous limit of the RD-system modeling (3.14) as the intermediate's life time approaches zero turns out to be the RD-system for (3.13); cf. [26]. For more information about chemical kinetics and reaction mechanisms see [46].

Global existence of solutions is known for a single reaction of type (3.12) in the case of *constant* diffusivities. Indeed, it was shown in [94] that for bounded initial data and space dimensions $N \leq 5$, the system (3.1), (3.3), (3.4) has a unique nonnegative classical solution, which is uniformly bounded. Global existence and boundedness in any space dimension for smooth Ω (of class $C^{2+\alpha}$, $0 < \alpha < 1$) and smooth initial data has been shown in [48]. Both these approaches are based on semigroup theory and hence exploit the semilinear structure. This prototype RD-system has a particular triangular structure for which global existence of strong solutions is proved in [90] for more general systems, for any space dimension and bounded initial data. This approach uses maximal regularity theory (see [37]) on the dual equations, and strongly relies on the linearity of the diffusion operators.

For a single reaction of type (3.13), still with constant diffusion coefficients, the question of global existence of solutions has an affirmative answer only for $N = 2$ so far, while the physically more interesting case $N \geq 3$ is open; see [92] and also [58], where the Hausdorff dimension of the set of possible singularities is estimated.

For *non-constant* diffusivities, the issue of global existence for such RD-systems is widely open. The only closely related result which we are aware of is [84], where the case $d_i(c_i)$ and reaction networks with at most quadratic terms and an appropriate triangular-type structure ("intermediate sum"-condition) are considered and global existence is obtained in case $N = 2$.

In the present section, we consider reaction networks with building blocks of type (3.12) and with diffusivities which depend on time, space and composition. We obtain global existence of solutions for initial values from an appropriate Sobolev space, the regularity index of which is optimal in a certain sense. The core point of our approach is a thorough analysis of the RD-system with a single reversible reaction of type (3.12). We first derive an initial estimate on the solutions from the conservation of the total mass for general diffusivities, and from L^2-techniques in the case of diffusivities $d_i(c_i)$. Since the solutions are nonnegative and the reaction terms for some equations are linearly bounded above, this initial estimate may be improved for the corresponding c_i. For small space dimensions, this provides new estimates on some quadratic reaction terms, which allows to improve the regularity for other c_i's. Bootstraping this procedure, we may estimate the solution in $L^p((0,T\times\Omega))$ for any $T>0$, $p<+\infty$, and then in $L^\infty((0,T)\times\Omega)$ by classical results on parabolic equations [69]. Global existence follows from a global existence criterion from [4].

3.2 The main results

We are interested in the well-posedness of the reaction-diffusion system

$$\begin{cases} \left.\begin{array}{l} \partial_t c_1 - \operatorname{div}(d_1(t,x,c)\nabla c_1) = -k^f c_1 c_2 + k^b c_3 \\ \partial_t c_2 - \operatorname{div}(d_2(t,x,c)\nabla c_2) = -k^f c_1 c_2 + k^b c_3 \\ \partial_t c_3 - \operatorname{div}(d_3(t,x,c)\nabla c_3) = +k^f c_1 c_2 - k^b c_3 \end{array}\right\} \text{ on } (0,+\infty)\times\Omega, \\ \partial_\nu c_1 = \partial_\nu c_2 = \partial_\nu c_3 = 0 \text{ on } (0,+\infty)\times\partial\Omega, \\ c=(c_1,c_2,c_3);\ c(0,\cdot)=(c_{0,1},c_{0,2},c_{0,3}) \text{ on } \Omega,\ c_{0,i}\ge 0. \end{cases} \tag{3.15}$$

Throughout the section, Ω denotes an open and bounded subset of $\mathbb{R}^N$, whose boundary $\partial\Omega$ is supposed to be at least of class C^2. The normal exterior derivative of a function c on $\partial\Omega$ is denoted by $\partial_\nu c$. As mentioned in the introduction, the system (3.15) represents the time-evolution of the concentration $c=(c_1,c_2,c_3)$ of three chemical species taking part in the reaction

$$C_1 + C_2 \underset{k^b}{\overset{k^f}{\rightleftharpoons}} C_3,$$

where $k^f, k^b > 0$ are the rate constants for the forward and backward reaction. The reaction rates are modeled by mass action kinetics, which is usually relevant for such an elementary reaction. The transport of species is assumed to be driven only by diffusion, with mass fluxes of the type $d_i(t,x,c)\nabla c_i$. Remark that indirect cross-effects can occur, since the diffusion coefficients depend on all species. This simple system is interesting since it contains most mathematical difficulties to treat the case of larger systems of reactions of the type $C_i + C_j \rightleftharpoons C_k$ (see Section 3.4).

The aim of this work is to prove the well-posedness of system (3.15) for nonlinear diffusivities and smooth initial data. More precisely, we assume that the diffusion coefficient for the i^{th} species $d_i = d_i(t,x,c)$ depends on all the concentrations and

$$d_i \in C^{2-}([0,+\infty)\times\Omega\times\mathbb{R}^3,\mathbb{R}^+)\ ;\ \exists \underline{d}>0 \text{ such that } \underline{d}\le d_i, \tag{3.16}$$

where for $k\ge 1$, C^{k-} is the space of $(k-1)$ times continuously differentiable functions whose derivatives of order $k-1$ are locally Lipschitz continuous. The special situation when d_i only depends on the i^{th} variable (i.e. $d_i = d_i(c_i)$) is also interesting since it allows to use some recent L^2-techniques, which are not available in general. In this case, we write $d_i(c_i)$ instead of $d_i(t,x,c)$ and assume

$$d_i \in C^{2-}(\mathbb{R},\mathbb{R}^+)\ ;\ \exists \underline{d}>0 \text{ such that } \underline{d}\le d_i. \tag{3.17}$$

The first step in the proof is of course the local existence of solutions which is based on a local well-posedness result from Amann [4], where the following notion of weak solution is used: consider the general reaction-diffusion system

$$\begin{cases} \partial_t c_i - \operatorname{div}(d_i(t,x,c)\nabla c_i) &=& f_i(c) & \text{on } (0,+\infty)\times\Omega, \\ \partial_\nu c_i &=& 0 & \text{on } (0,+\infty)\times\partial\Omega, \\ c_i(0,\cdot) &=& c_{0,i} & \text{on } \Omega, \end{cases} \tag{3.18}$$

where $c = (c_1, \dots, c_P)$ and $f_i \in C^{1-}(\mathbb{R}^P)$.
Definition (weak-W^s_p solution). *Let $T \in (0, +\infty]$, $p > 1$, $p' = p/(p-1)$, $s > 0$ satisfying*

$$\frac{N}{p} < s < \min(1 + \frac{1}{p}, 2 - \frac{N}{p}) \,, \tag{3.19}$$

and assume $c_{0,i} \in W^s_p(\Omega)$. A weak-$W^s_p$ solution of system (3.18) on $[0,T)$ is a function $c = (c_1, \dots, c_P) : [0,T) \times \Omega \to \mathbb{R}^P$ such that

$$c \in C([0,T); W^s_p(\Omega)^P) \cap C^1((0,T); W^{s-2}_p(\Omega)^P),$$

$c(0) = c_0$ and for all $t \in (0,T)$, $v \in W^{2-s}_{p'}(\Omega)$, $i \in \{1, \dots, P\}$,

$$\langle \partial_t c_i(t), v \rangle_{W^{s-2}_p, W^{2-s}_{p'}} + \langle d_i(t,x,c) \nabla c_i(t), \nabla v \rangle_{W^{s-1}_p, W^{1-s}_{p'}} = \langle f_i(c), v \rangle_{L^\infty, W^{2-s}_{p'}}.$$

Throughout the rest of the section, by a *classical solution* we denote a function that belongs (at least) to $C([0,T) \times \overline{\Omega}) \cap C^1((0,T); C(\overline{\Omega})) \cap C((0,T); C^2(\overline{\Omega}))$ and satisfies the equations pointwise.

To guarantee that system (3.15) preserves the nonnegativity of the solutions, it is easy to check that its reaction terms necessarily satisfy the following condition:
Definition (quasi-positivity). *A vector field $f = (f_1, \dots, f_P) : \mathbb{R}^P \to \mathbb{R}^P$, $r = (r_1, \dots, r_P) \mapsto f(r)$ is quasi-positive if*

$$\forall i \in \{1, \dots, P\}, \forall r \in \mathbb{R}^P_+, \quad r_i = 0 \Rightarrow f_i(r) \geq 0. \tag{3.20}$$

We can now state the main theorem.

Theorem 3.1. *Let $p > 1$, $s > 0$ satisfying* (3.19) *and $c_0 \in W^s_p(\Omega, \mathbb{R}^3_+)$. System* (3.15) *has a unique global weak-W^s_p solution $c = (c_1, c_2, c_3) : [0, +\infty) \times \Omega \to \mathbb{R}^3$ provided one of the following conditions is satisfied:*

(i) $N < 6$ and the diffusivities $d_i(t,x,c)$ satisfy (3.16).

(ii) $N < 10$ and the diffusivities $d_i(c_i)$ satisfy (3.17).

This solution is nonnegative. It is actually a classical solution and (3.15) *is satisfied in a pointwise sense. If, in addition d_i and $\partial\Omega$ are smooth, then $c \in C^\infty((0, +\infty) \times \overline{\Omega}\,; \mathbb{R}^P_+)$.*

For any $T > 0$, we will use the common notations $Q_T = (0,T) \times \Omega$, $\Sigma_T = (0,T) \times \partial\Omega$.
Outline of the Proof. According to Amann's theory [4], local well-posedness and nonnegativity holds for (3.15). The solution is global provided it is *a priori* bounded in $L^\infty(Q_T)$ for any $T < +\infty$. The conservation of the total mass gives a first estimate on c in $L^\infty(0,T; L^1(\Omega))$, and actually the reaction terms in (3.15) are bounded in $L^1(Q_T)$. Then we use the theory on scalar parabolic equations to estimate c in $L^{\frac{N+2}{N} - \varepsilon}(Q_T)$ for any $\varepsilon > 0$. The reaction term for c_1 and c_2 is (linearly) bounded above by c_3, so c_1 and c_2 can be estimated in a better $L^p(Q_T)$-space (p depending on N). Then the reaction term for c_3 is bounded above by $c_1 c_2$, and for small enough space dimensions, the previous estimates are sufficient to improve the regularity on c_3. Bootstrapping this procedure, we get estimates in $L^p(Q_T)$ for any p and whence in $L^\infty(Q_T)$ for any $T > 0$ by classical results from [69]. In the special case of diffusivities $d_i(c_i)$, we can directly start with estimates in $L^2(Q_T)$, which allows for higher space dimensions.

Section 3.3 contains the proof of Theorem 3.1. For clarity reasons, some technical details are postponed to an Appendix. At the end of Section 3.3, we generalize Theorem 3.1 to the case

of reaction terms of the type $k^f c_1^\alpha c_2^\beta - k^b c_3^\gamma$ for some $\alpha, \beta, \gamma \geq 1$ depending on N, with explicit examples in dimensions 2 and 3.
Section 3.4 is devoted to the case of P chemically reacting species $C_1, \ldots, C_P$, where the chemical reactions are assumed to be of the type $C_i + C_j \rightleftharpoons C_k$ and the total mass of involved atoms is preserved. After re-sorting the reactions and chemical species to get a block-triangular structure, the ideas of the proof of Theorem 3.1 can be adapted to this case, but under stronger restrictions on the space dimensions.

Finally, let us mention some related works, which all concern the case of constant diffusivities: asymptotics has first been studied by Rothe in [94], where it is proved that $c(t)$ converges to a uniquely determined homogeneous stationary state when $t \to +\infty$. In [40], Desvillettes and Fellner used the entropy method to give explicit convergence rates to the equilibrium. The fast-reaction limit $k^f, k^b \to +\infty$ for the RD-system (3.15) has first been studied in [18], in the special case when the diffusion coefficients are equal, and then in [28] for the case of different but constant diffusivities. Note in passing that the techniques developed in the latter paper carry over with only slight modifications to the case of nonlinear diffusions of the type $d_i(c_i)\nabla c_i$. Then using the above global existence result, Theorem 1 in [28] can be extended to the case of diffusivities (3.17) for space dimensions $N \leq 9$.

3.3 Proof of Theorem 3.1

Using the rescaling

$$(t,x) \mapsto \frac{k^f}{k^b} c(\frac{t}{k^b}, x),$$

we can assume, without loss of generality, that $k^f = k^b = 1$. As mentioned above, the reaction term in (3.15) satisfies quasi-positivity assumption (3.20), so according to Amann's theory (see [4], Theorems 14.4 and 15.1, [2] for the proofs), (3.15) has a unique nonnegative weak W_p^s-solution c, defined on a maximum time interval $[0, T^*)$, $T^* \leq +\infty$. The additional regularity properties in Theorem 3.1 are consequences of Theorem 14.6 and Corollary 14.7 in [4].

It remains to prove that the solution is global and, according to Theorem 16.3 in [4], it suffices to prove that c is *a priori* bounded in $L^\infty(Q_T)$ for any $T > 0$. For this purpose, we first estimate the solution in $L^p(Q_T)$ spaces for finite p. The subsequent Lemma is the main tool to improve these estimates by a bootstrap procedure: given a bound in $L^r(Q_T)$ on the positive part of a reaction term f_i, it shows in which $L^q(Q_T)$ space c_i is bounded. The proof is given in the Appendix.

Lemma 3.2. *Let $d \in C(Q_T)$ with $0 < \underline{d} \leq d$, let $f \in L^r(Q_T)$ for $1 \leq r < +\infty$ and u be a nonnegative classical solution of*

$$\partial_t u - \operatorname{div}(d(t,x)\nabla u) \leq f(t,x) \text{ in } Q_T, \quad \partial_\nu u = 0 \text{ on } \Sigma_T, \quad u(0) = u_0 \in L^\infty(\Omega). \tag{3.21}$$

Then $\|u\|_{L^q(Q_T)}$ is bounded by a constant depending only on $T, \underline{d}, \|f\|_{L^r(Q_T)}$ and $\|u_0\|_{L^\infty(\Omega)}$, provided $1 \leq q < +\infty$ and (r,q) satisfies

$$(i)\ \ r = 1 \text{ and } \begin{cases} 1 - \frac{2}{N+2} < \frac{1}{q} & \text{for } N \geq 2, \\ q < 2 & \text{for } N = 1. \end{cases}$$

$$(ii)\ \ r > 1 \text{ and } \begin{cases} \frac{1}{r} - \frac{2}{N+2} \leq \frac{1}{q} & \text{for } N \geq 3, \\ \frac{1}{r} - \frac{1}{2} < \frac{1}{q} & \text{for } N = 2, \\ \frac{1}{r} - \frac{1}{2} \leq \frac{1}{q} & \text{for } N = 1. \end{cases}$$

Step 1. The initial estimate.

Let $0 < T < +\infty$, $T \leq T^*$. We estimate c on Q_T as follows:

For diffusivities $d_i(t,x,c)$ satisfying (3.16).

Let $r_0 \in [1,(N+2)/N)$ if $N \geq 2$, $r_0 \in [1,2)$ if $N = 1$, and let us prove that

$$\exists C = C(T, \underline{d}, \|c_0\|_{L^\infty(\Omega)^3}) > 0 \ : \quad \|c\|_{L^{r_0}(Q_T)^3} \leq C. \tag{3.22}$$

Using the homogeneous Neumann boundary conditions in (3.15), it is clear that

$$\frac{d}{dt}\int_\Omega c_1(t) + c_2(t) + 2c_3(t) = 0.$$

As c is nonnegative,

$$\forall i \in \{1,2,3\}, \qquad \sup_{t\in[0,T^*)} \|c_i(t)\|_{L^1(\Omega)} \leq \|c_{0,1}\|_{L^1(\Omega)} + \|c_{0,2}\|_{L^1(\Omega)} + 2\|c_{0,3}\|_{L^1(\Omega)}. \tag{3.23}$$

After integration of the first equation in (3.15) on Q_T and integration by parts,

$$\int_{Q_T} c_1c_2 = \int_{Q_T} c_3 + \int_\Omega c_{0,1} - \int_\Omega c_1(T).$$

All the integrals on the right-hand side are bounded, so c_1c_2 is bounded in $L^1(Q_T)$, and the reaction terms in (3.15) are bounded in $L^1(Q_T)$. Then (3.22) is a consequence of Lemma 3.2 (i).

With diffusivities $d_i(c_i)$ satisfying (3.17).

Let us prove that

$$\exists C = C(T, \underline{d}, \|c_0\|_{L^2(\Omega)^3}) > 0 : \quad \|c\|_{L^2(Q_T)^3} \leq C. \tag{3.24}$$

In this case, (3.15) can be rewritten

$$\partial_t c_i - \Delta D_i(c_i) = \varepsilon_i(c_1c_2 - c_3) \text{ on } Q_T \ ; \quad \partial_\nu D_i(c_i) = 0 \text{ on } \Sigma_T \ ; \quad c_i(0) = c_{0,i} \text{ on } \Omega, \tag{3.25}$$

where $i \in \{1,2,3\}$, $\varepsilon = (-1,-1,1)$, $D_i(y) = \int_0^y d_i(s)ds$. Using assumption (3.17), $\underline{d}\,y \leq D_i(y)$ for $y \geq 0$. Then (3.24) is a straightforward consequence of the following lemma (applied to $(c_1,c_2,2c_3)$), which generalizes Proposition 6.1 in [90] to the case of nonlinear diffusions:

Lemma 3.3. *Let $T > 0$, $c = (c_1,\dots,c_P)$ be a nonnegative solution of*

$$\partial_t c_i - \Delta D_i(c_i) = f_i \textit{ on } Q_T, \ ; \quad \partial_\nu D_i(c_i) = 0 \textit{ on } \Sigma_T, \ ; \quad c_i(0) = c_{0,i} \in L^2(\Omega,\mathbb{R}_+), \tag{3.26}$$

where $i \in \{1,\dots,P\}$, $f_i : Q_T \to \mathbb{R}$ is measurable, $D_i : \mathbb{R}_+ \to \mathbb{R}_+$ and

$$\exists \underline{d} > 0 : \ \forall y \geq 0, \ \underline{d}\,y \leq D_i(y) \ ; \quad \sum_{i=1}^P f_i \in L^2(Q_T). \tag{3.27}$$

Then there exists $C = C(T, \underline{d}, \|\sum_{i=1}^P f_i\|_{L^2(Q_T)}, \|c_0\|_{L^2(\Omega)^P}) > 0$ such that

$$\|c\|_{L^2(Q_T)^P} \leq C. \tag{3.28}$$

Proof of Lemma 3.3. Set

$$W := \sum_{i=1}^{P} c_i \; ; \quad W^0 := \sum_{i=1}^{P} c_i^0 \; ; \quad A := \frac{\sum_{i=1}^{P} D_i(c_i)}{\sum_{i=1}^{P} c_i} \; ; \quad F := \sum_{i=1}^{P} f_i,$$

and note that $A \geq \underline{d}$. Let $t \in (0,T)$ and integrate (3.26) on $(0,t)$ to get, for $i \in \{1,\dots,P\}$,

$$c_i - \Delta \int_0^t D_i(c_i) = c_i^0 + \int_0^t f_i \text{ on } Q_T \; ; \quad \partial_\nu D_i(c_i) = 0 \text{ on } \Sigma_T \; ; \quad c_i(0) = c_{0,i} \text{ on } \Omega. \tag{3.29}$$

Summing these equations over i yields

$$W - \Delta \int_0^t AW = W^0 + \int_0^t F \text{ on } Q_T \; ; \quad \partial_\nu(AW) = 0 \text{ on } \Sigma_T \; ; \quad W(0) = W^0 \text{ on } \Omega. \tag{3.30}$$

After multiplication by AW, integration on Q_T and integration by parts, we get

$$\begin{aligned}
\int_{Q_T} AW^2 + \int_{Q_T} \nabla(AW)\nabla \int_0^t AW &= \int_{Q_T} W^0 AW + \int_{Q_T} \left(\int_0^t F\right) AW, \\
\int_{Q_T} AW^2 + \frac{1}{2}\int_\Omega \left|\nabla \int_0^T AW\right|^2 &= \int_\Omega W^0 \int_0^T AW + \int_{Q_T} F \int_t^T AW \\
&\leq \|W^0\|_{L^2(\Omega)} \|\int_0^T AW\|_{L^2(\Omega)} + \sqrt{T}\|F\|_{L^2(Q_T)}\|\int_0^T AW\|_{L^2(\Omega)} \\
&\leq C\|\int_0^T AW\|_{L^2(\Omega)},
\end{aligned} \tag{3.31}$$

where $C > 0$ denotes a constant depending only on $\|F\|_{L^2(Q_T)}, \|c_0\|_{L^2(\Omega)^P}$, $\underline{d}$ and T. Using the Poincaré-Wirtinger inequality,

$$\exists C > 0 : \quad \int_{Q_T} AW^2 + \frac{1}{2}\int_\Omega \left|\nabla \int_0^T AW\right|^2 \leq C\left(\|\nabla \int_0^T AW\|_{L^2(\Omega)} + \int_{Q_T} AW\right).$$

Then Young's inequality yields

$$\exists C > 0 : \quad \int_{Q_T} AW^2 + \frac{1}{2}\int_\Omega \left|\nabla \int_0^T AW\right|^2 \leq C + \frac{1}{4}\int_\Omega \left|\nabla \int_0^T AW\right|^2 + C\int_{Q_T} AW. \tag{3.32}$$

Letting $\alpha > 0$, $\{W > \alpha\} := \{(t,x) \in Q_T : W(t,x) > \alpha\}$ and $\{W \leq \alpha\} := Q_T \setminus \{W > \alpha\}$, we have

$$\begin{aligned}
\int_{Q_T} AW &= \int_{\{W>\alpha\}} AW + \int_{\{W\leq\alpha\}} AW \\
&\leq \frac{1}{\alpha}\int_{Q_T} AW^2 + \int_{\{W\leq\alpha\}} \sum_{i=1}^{p} D_i(c_i) \\
&\leq \frac{1}{\alpha}\int_{Q_T} AW^2 + M_\alpha,
\end{aligned} \tag{3.33}$$

where we used the fact that $c_i \leq \alpha$ on $\{W \leq \alpha\}$ and

$$M_\alpha := |\Omega|\, T \left[\sum_{i=1}^{P} \max_{0\leq x\leq\alpha} D_i(x)\right].$$

Choosing $\alpha = 2C$, where C is defined in (3.32), we get

$$\underline{d}\int_{Q_T} W^2 \leq \int_{Q_T} AW^2 + \frac{1}{2}\int_{\Omega}\left|\nabla \int_0^T AW\right|^2 \leq 2C(M_{2C}+1).$$

Using $c_i \geq 0$ and $W = \sum_{i=1}^{P} c_i$, this proves the desired bound on c in $L^2(Q_T)^P$.

□

Step 2. The bootstrap procedure.

Let us prove that the maximal solution of (3.15) is bounded in $L^p(Q_T)$ for any $p < +\infty$ and any $T \leq T^*$, $T < +\infty$. The idea is to exploit the fact that the reaction terms for c_1 and c_2 are linearly bounded from above to get new estimates on c_1 and c_2. For small space dimensions, we get a better estimate on c_1c_2, which is an upper bound for the reaction term for c_3, so we might improve the estimate on c_3. Then we go back to the equations in c_1 and c_2 and bootstrap this procedure.

Assume first that $N = 1$. For diffusivities satisfying (3.16) or (3.17), according to (3.22), c is bounded in $L^{r_0}(Q_T)^3$ for $r_0 < 2$. Using Lemma 3.2, c_1 and c_2 are bounded in $L^p(Q_T)$ for any $p < +\infty$, so c_1c_2 is also bounded in any $L^p(Q_T)$ and using once more Lemma 3.2, c_3 is bounded in any $L^p(Q_T)$.

For $N \geq 2$, let $r_0 > 1$ be such that c is bounded in $L^{r_0}(Q_T)$. According to Lemma 3.2,

c_1, c_2 are bounded in $L^{q_1}(Q_T)$, where $\frac{1}{r_0} - \frac{2}{N+2} < \frac{1}{q_1}$;

c_1c_2 is bounded in $L^{q_2}(Q_T)$, where $\frac{2}{r_0} - \frac{4}{N+2} < \frac{1}{q_2}$. We can choose $q_2 \geq 1$ provided

$$\frac{2}{r_0} - \frac{4}{N+2} < 1. \tag{3.34}$$

c_3 is bounded in $L^{r_1}(Q_T)$, where

$$\frac{2}{r_0} - \frac{6}{N+2} < \frac{1}{r_1}. \tag{3.35}$$

The initial estimate is improved if we can choose $r_0 < r_1$, i.e. if

$$\frac{1}{r_0} < \frac{6}{N+2}. \tag{3.36}$$

Suppose r_0 satisfies conditions (3.34) and (3.36). Then c is bounded in $L^{r_1}(Q_T)^3$ for some $r_1 > r_0$, which also satisfies (3.34) and (3.36). Then it is clear that we can build by induction an increasing sequence $(r_n)_{n\in\mathbb{N}}$ such that c is bounded in $L^{r_n}(Q_T)^3$ and

$$\frac{2}{r_n} - \frac{6}{N+2} < \frac{1}{r_{n+1}}.$$

Let us prove that $(r_n)_{n\in\mathbb{N}}$ can be built such that $r_n \to +\infty$. Let $0 < \varepsilon < \frac{6}{N+2} - \frac{1}{r_0}$. We define $r_{n+1} > r_n$ by

If $\frac{2}{r_n} - \frac{6}{N+2} < 0$, $\quad r_{n+1} = r_n + 1$.

If $\frac{2}{r_n} - \frac{6}{N+2} \geq 0$, $\quad \frac{1}{r_{n+1}} = \frac{2}{r_n} - \frac{6}{N+2} + \varepsilon$.

Suppose that $\frac{2}{r_n} - \frac{6}{N+2} \geq 0$ for all $n \in \mathbb{N}$. Then the sequence $u_n := \frac{1}{r_n} \in (0,1]$ is decreasing and satisfies $u_{n+1} = 2u_n - \frac{6}{N+2} + \varepsilon$. This yields $u_n \to -\infty$, a contradiction, so there exists $n_0 \in \mathbb{N}$ such that $\frac{2}{r_{n_0}} - \frac{6}{N+2} < 0$. Then for all $n \geq n_0$, $r_n = r_{n_0} + n - n_0$ and therefore $r_n \to +\infty$. Consequently, c is bounded in $L^p(Q_T)^3$ for any $p < +\infty$.

It remains to give some explicit sufficient conditions so that we can choose r_0 satisfying (3.34) and (3.36):

- For diffusivities $d_i(t,x,c)$ satisfying (3.16): according to (3.22), c is bounded in $L^{r_0}(Q_T)$ for $r_0 < \frac{N+2}{N}$ if $N \geq 2$, $r_0 < 2$ if $N = 1$. hence equations (3.34) and (3.36) can be satisfied if and only if $N < 6$.
- For diffusivities $d_i(c_i)$ satisfying (3.17): according to (3.24), c is bounded in $L^{r_0}(Q_T)$ with $r_0 = 2$. Hence equations (3.34) and (3.36) are satisfied if and only if $N < 10$.

Step 3. Once we know that c is bounded in $L^p(Q_T)$ for any $p < +\infty$, we can use a classical result from [69] on parabolic equations (see Theorem III.7.1) to say that for all i, c_i is bounded in $L^\infty(Q_T)$. This is valid for any $T \leq T^*$, $T < +\infty$, so using Theorem 16.3 in [4], $T^* = +\infty$, i.e. c is a global solution. □

Remarks:

- In [69], Theorem III.7.1 is stated for Dirichlet boundary conditions, but the result also holds for Neumann boundary conditions, with a similar proof (see the Appendix of Section 5).
- In [4], the results we used from Chapters 14, 15 and 16 are stated for time-independent operators. To see that they are still valid for the time-dependent case, it is sufficient to "artificially" add the time in the equations, replacing $c = (c_1, \dots, c_P)$ by $\tilde{c} = (c_1, \dots, c_P, s)$ in (3.18), where s satisfies $\partial_t s - \Delta s = 1$ with homogeneous Neumann boundary conditions. Note that $s(t,x) \equiv t$, then.
- In the case of Michaelis-Menten-Henri (MMH) enzymatic reaction

$$C_1 + C_2 \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} C_3 \underset{k_{-2}}{\overset{k_2}{\rightleftharpoons}} C_1 + C_4\ ; \quad k_1, k_{-1}, k_2, k_{-2} \geq 0,$$

we are led to the equations

$$\begin{cases} \left.\begin{array}{llllll} \partial_t c_1 - \mathrm{div}(d_1(t,x,c)\nabla c_1) = & -k_1 c_1 c_2 & +k_{-1} c_3 & +k_2 c_3 & -k_{-2} c_1 c_4 \\ \partial_t c_2 - \mathrm{div}(d_2(t,x,c)\nabla c_2) = & -k_1 c_1 c_2 & +k_{-1} c_3 & & \\ \partial_t c_3 - \mathrm{div}(d_3(t,x,c)\nabla c_3) = & k_1 c_1 c_2 & -k_{-1} c_3 & -k_2 c_3 & +k_{-2} c_1 c_4 \\ \partial_t c_4 - \mathrm{div}(d_4(t,x,c)\nabla c_3) = & & & +k_2 c_3 & -k_{-2} c_1 c_4 \end{array}\right\} \text{ on } (0,+\infty) \times \Omega, \\ \partial_\nu c_1 = \partial_\nu c_2 = \partial_\nu c_3 = 0 \ \text{ on } \ (0,+\infty) \times \partial\Omega, \\ c_i(0,\cdot) = c_{0,i}, \ c_{0,i} \in L^\infty(\Omega, \mathbb{R}_+). \end{cases}$$

Similarly as in (3.15), the reaction terms for c_1, c_2 and c_4 are linearly bounded above, and it is clear that with obvious modifications in the above proof, the results from Theorem 3.1 also hold for this system, whith the same space dimension restrictions. In the literature on MMH reaction systems, the second reaction is usually assumed to be irreversible with $k_{-2} = 0$. Note that this case is included in our analysis.
- The estimate in $L^2(Q_T)$ from Lemma 3.3 may be improved in an estimate in $L^{2+\varepsilon}(Q_T)$ for some $\varepsilon > 0$ depending on the space dimension N (see [79], [39] in the case of smooth domains and [60] for convex domains). Using this $L^{2+\varepsilon}$-estimate, Theorem 3.1 could be extended to the limit case $N = 10$ for diffusivities $d_i(c_i)$.
- In the special case of bounded diffusivities of type $d_i(c_i)$, we can relax the initial regularity to $c_0 \in L^\infty(\Omega)^P_+$. The main reason is that the estimations in $L^\infty(Q_T)$ only depend on $\|c_0\|_{L^\infty(\Omega)^P}$, and therefore the solutions may be estimated in some $W^s_p(\Omega)$-space for a.e. $t \in (0,T)$. Using a regularization procedure, we get the existence of a solution that becomes a weak W^s_p-solution

in the sense of Amann on $(t,+\infty)$ for arbitrary small t, and therefore the solution is regular on $(0,+\infty)$. Uniqueness will follow from the uniform bounds and the Lipschitz continuity of the reaction terms on bounded subsets.

Generalization to reaction terms of the type $\mathbf{c}_1^\alpha \mathbf{c}_2^\beta - \mathbf{c}_3^\gamma$.

In Chemical Kinetics there also appears mass action kinetics with fractional orders, obtained as empirical rate laws from experimental measurements. In several cases this can be theoretically understood as a time scale limit when an intermediate species is highly reactive, for instance if radicals are involved. The classical reaction for which this was accomplished is the formation of hydrogen bromide (HBr). As the simplest prototype example, consider a reaction mechanism of type

$$A \rightleftharpoons 2R, \qquad B+R \rightleftharpoons P,$$

where R stands for a radical. From conservation of mass, the overall conversion of A and B into the product P is of the form

$$\frac{1}{2}A+B \rightleftharpoons P$$

which formally would lead to a rate function of the type

$$k^f c_A^{1/2} c_B - k^b c_P.$$

Interestingly, by application of the so-called quasi-stationary-state-approximation, this kind of rate function indeed results in a certain regime in which c_B and c_P are in a way small compared to c_A. In more complicated cases, other fractional orders can appear; cf. [46] for more information.

Motivated by such examples we consider system (3.15) once more, but replace the reaction term $c_1c_2 - c_3$ by rate functions of the type $c_1^\alpha c_2^\beta - c_3^\gamma$. In order to obtain uniqueness of solutions, we focus on the case $\alpha,\beta,\gamma \geq 1$, while (without uniqueness) the same estimates would apply for $\alpha,\beta,\gamma > 0$. We are looking for some sufficient conditions on α,β,γ to extend Theorem 3.1 to this case, especially for space dimensions 2 and 3.

The first step of the proof of Theorem 3.1 carries over to these generalized reaction terms for diffusivities $d_i(c_i)$, and we get the same estimate (3.24). For diffusivities $d_i(t,x,c)$, we have to choose $\gamma \leq 1$ to recover (3.22). Now Step 2 must be adapted as follows: let $r_0 > 1$ such that c is bounded in $L^{r_0}(Q_T)^3$, using Lemma 3.2, for $N \geq 2$,

c_3^γ is bounded in $L^{\frac{r_0}{\gamma}}(Q_T)$, where γ satisfies

$$\gamma \leq r_0. \tag{3.37}$$

c_1, c_2 are bounded in $L^{q_1}(Q_T)$, where $\frac{\gamma}{r_0} - \frac{2}{N+2} < \frac{1}{q_1}$.

$c_1^\alpha c_2^\beta$ are bounded in $L^{q_2}(Q_T)$, where $\frac{\gamma(\alpha+\beta)}{r_0} - \frac{2(\alpha+\beta)}{N+2} < \frac{1}{q_2}$. We can choose $q_2 \geq 1$ provided

$$\frac{\gamma(\alpha+\beta)}{r_0} - \frac{2(\alpha+\beta)}{N+2} < 1. \tag{3.38}$$

c_3 is bounded in $L^{r_1}(Q_T)$, where $\frac{\gamma(\alpha+\beta)}{r_0} - \frac{2(\alpha+\beta+1)}{N+2} < \frac{1}{r_1}$.

The initial estimate can be improved if we can choose $r_1 > r_0$, i.e. if

$$\frac{\gamma(\alpha+\beta)-1}{r_0} < \frac{2(\alpha+\beta+1)}{N+2}. \tag{3.39}$$

If r_0 satisfies (3.37), (3.38) and (3.39), the same arguments as in Step 2 in the proof of Theorem 3.1 show that c in bounded in $L^p(Q_T)^3$ for any $p < +\infty$. In the case $N = 1$, similar computations provide *a priori* bounds on c_3 in $L^p(Q_T)$ for any $p > 1$ provided r_0 satisfies (3.37), (3.38) and (3.39), where N is replaced by 2.

Finally recall that for diffusivities $d_i(t,x,c)$ satisfying (3.16), $r_0 < \frac{N+2}{N}$ if $N \geq 2$ and $r_0 < 2$ if $N = 1$. Then r_0 can satisfy inequalities (3.37), (3.38) and (3.39) if and only if

$$\gamma < \frac{N+2}{N} \text{ and } (\alpha+\beta)(\gamma N - 2) < N+2 \text{ for } N \geq 2\,; \quad \gamma < 2 \text{ for } N = 1. \tag{3.40}$$

For diffusivities $d_i(c_i)$ satisfying (3.17), $r_0 = 2$, so r_0 satisfies inequalities (3.37), (3.38) and (3.39) if and only if

$$\gamma \leq 2 \text{ and } (\alpha+\beta)\left(\tfrac{\gamma}{2} - \tfrac{2}{N+2}\right) < \tfrac{N+6}{2N+4}\,; \quad \gamma \leq 2 \text{ for } N = 1. \tag{3.41}$$

Step 3 carries over these new reaction terms without modifications, so we get

Corollary 3.4. *Theorem 3.1 extends to the case of reaction terms of the type $c_1^{\alpha} c_2^{\beta} - c_3^{\gamma}$, provided $\alpha, \beta, \gamma \geq 1$ and one of the following conditions is satisfied:*

(i) The diffusivities $d_i(t,x,c)$ are of the type (3.16) *and N, α, β, γ satisfy* (3.40) *with $\gamma = 1$.*

(ii) The diffusivities $d_i(c_i)$ are of the type (3.17) *and N, α, β, γ satisfy* (3.41).

Here are some examples of possible choices for α, β, γ in space dimensions $N = 2$ and $N = 3$:

	Diffusivities $d_i(t,x,c)$	Diffusivities $d_i(c_i)$
$N=2 \quad \gamma=1$	$\alpha,\beta < +\infty$	$\alpha,\beta < +\infty$
$N=3 \quad \gamma=1$	$\alpha+\beta < 5$	$\alpha+\beta < 9$
$N=2 \quad \gamma=3/2$		$\alpha+\beta < 4$
$N=3 \quad \gamma=3/2$		$\alpha+\beta < 18/7$

3.4 Systems of elementary reactions

In this section, we suppose that P chemical species $C_1, \ldots, C_P$ are present, and that they are involved in R chemical reactions of the type

$$C_{j_1} + C_{j_2} \underset{k_j^b}{\overset{k_j^f}{\rightleftharpoons}} C_{j_3}\,; \quad j \in \{1,\ldots,R\}\,;\ j_1, j_2, j_3 \in \{1,\ldots,P\}\,;\ k_j^f, k_j^b \geq 0.$$

Remark that j_1 and j_2 are not necessarily distinct, so that reactions of the type $2C_{j_1} \rightleftharpoons C_{j_3}$ are included, as well as the irreversible reactions $C_{j_1} + C_{j_2} \to C_{j_3}$ and $C_{j_3} \to C_{j_1} + C_{j_2}$, which are obtained by taking $k_j^b = 0$, respectively $k_j^f = 0$.

As before, c_i denotes the concentration of species C_i. Let $(\varepsilon_1, \ldots, \varepsilon_P)$ be the canonic basis of $\mathbb{R}^P$ and define the so-called *stoichiometric vectors* as $\alpha_j := \varepsilon_{j_1} + \varepsilon_{j_2}$, $\beta_j := \varepsilon_{j_3}$ and $\nu_j := \beta_j - \alpha_j$. The *stoichiometric matrix* $M \in \mathcal{M}_{P,R}(\mathbb{R})$ is the matrix whose columns are $\nu_1, \ldots, \nu_R$. On the basis of mass action kinetics, the reaction rate for the j^{th} reaction is given by $r_j(c) = k_j^f c_{j_1} c_{j_2} - k_j^b c_{j_3}$. We also assume the existence of an *atomic conservation law* (see [45], Chap. 3): if $\langle \cdot, \cdot \rangle$ denotes the usual scalar product on $\mathbb{R}^P$, we impose the condition

$$\exists e \in (0,+\infty)^P : \forall i \in \{1,\ldots,R\}, \quad \langle e, \nu_i \rangle = 0. \tag{3.42}$$

Remark that assumption (3.42) excludes chemical reactions of the type $C_{j_1} + C_{j_2} \rightleftharpoons C_{j_1}$. Using the above notations, the creation rate of $c = (c_1, \ldots, c_P)$ reads

$$f(c) := \begin{pmatrix} f_1(c) \\ \vdots \\ f_P(c) \end{pmatrix} = \left(\begin{array}{c|c|c} \nu_1^1 & & \nu_R^1 \\ \vdots & \ldots & \vdots \\ \nu_1^P & & \nu_R^P \end{array} \right) \begin{pmatrix} r_1(c) \\ \vdots \\ r_R(c) \end{pmatrix} = M \begin{pmatrix} r_1(c) \\ \vdots \\ r_R(c) \end{pmatrix}. \tag{3.43}$$

Note that the vector field f is quasi-positive: indeed, we have for all $i \in \{1, \ldots, P\}$,

$$f_i(c) = \sum_{j=1}^{R} \nu_j^i r_j(c) = \sum_{j:\, \nu_j^i > 0}^{R} \nu_j^i r_j(c) + \sum_{j:\, \nu_j^i < 0}^{R} \nu_j^i r_j(c),$$

and for $c \in \mathbb{R}_+^P$, $\nu_j^i > 0$, $c_i = 0$ implies $r_j(c) = \nu_j^i k_j^+ c_{j_1} c_{j_2} \geq 0$; in case $\nu_j^i < 0$, $c_i = 0$ implies $r_j(c) = -\nu_j^i k_j^- c_{j_3} \geq 0$.
Assuming the same diffusion laws as above, the time-evolution of $c = (c_1, \ldots, c_P)$ is now governed by the equations

$$\begin{cases} \begin{pmatrix} \partial_t c_1 - & \text{div } (d_1(t,x,c)\nabla c_1) \\ & \vdots \\ \partial_t c_P - & \text{div } (d_P(t,x,c)\nabla c_P) \end{pmatrix} = \begin{pmatrix} f_1(c) \\ \vdots \\ f_P(c) \end{pmatrix} \text{ on } (0,+\infty) \times \Omega, \\ \partial_\nu c = 0 \text{ on } (0,+\infty) \times \partial\Omega, \\ c(0,\cdot) = c_0 \text{ on } \Omega. \end{cases} \tag{3.44}$$

Theorem 3.5. *Let $p > 1$, $s > 0$ satisfying* (3.19) *and $c_0 \in W_p^s(\Omega, \mathbb{R}_+^P)$. System* (3.44) *has a unique global nonnegative weak-W_p^s solution $c = (c_1, \ldots, c_P) : [0,+\infty) \times \Omega \to \mathbb{R}^P$ provided one of the following conditions is satisfied:*

(i) $N < 4$ and the diffusivities $d_i(t,x,c)$ satisfy (3.16).

(ii) $N < 6$ and the diffusivities $d_i(c_i)$ satisfy (3.17).

This solution is actually classical and (3.15) *is satisfied in a pointwise sense. If, in addition, $d_i \in C^\infty([0,+\infty) \times \overline{\Omega} \times \mathbb{R}_+^P, \mathbb{R})$ and Ω is C^∞, then $c \in C^\infty((0,+\infty) \times \overline{\Omega}, \mathbb{R}_+^P)$.*

As for Theorem 3.1, the proof consists in showing that c is uniformly *a priori* bounded. After deriving a first *a priori* estimate from the conservation law (3.42), or in $L^2(Q_T)$ in the case of diffusivities of the type(3.17), we use Lemma 3.2 to improve the regularity of those c_i's whose reaction terms are linearly bounded above. This gives estimates on some quadratic terms, and hence estimates on some other c_i's. Then we can estimate some new quadratic terms, and so on; here, the atomic conservation law guarantees that we obtain improved estimates for all constituents c_i. Once we have improved the estimates on all the c_i's, we bootstrap this procedure to get estimates in $L^p(Q_T)$ for any $p < +\infty$, and finally in $L^\infty(Q_T)$.

Such a procedure requires that the reactions and the chemical components have been previously sorted. Notice that a permutation of the chemical species corresponds to a permutation of the rows of the stoichiometric matrix M, and a permutation of the chemical reactions corresponds to a permutation of its columns. The concrete way to bring the species and reactions in an appropriate order is based on the following idea: a row in the stoichiometric matrix with only zeros and ones corresponds to a chemical species that is always a product for all of the chemical reactions $C_{j_1} + C_{j_2} \to C_{j_3}$. If such a species exists, the matrix has a certain block structure. But as we assume an atomic mass conservation law, any chemical species whose molar mass is maximal amongst the molar masses of $C_1, \dots, C_P$ leads to such a row. Indeed, if it would appear a reactant in $C_{j_1} + C_{j_2} \to C_{j_3}$, the product C_{j_3} would be heavier - a contradiction.

Lemma 3.6. *Assuming* (3.42)*, up to a permutation of its rows and columns, the stoichiometric matrix M reads*

$$M = \left(\begin{array}{cccc} \boxed{\begin{array}{c} N_1 \\ \hline 1 \ \dots \ 1 \end{array}} & \boxed{\begin{array}{c} N_2 \\ \hline 1 \ \dots \ 1 \end{array}} & & \boxed{\begin{array}{c} N_k \\ \hline 1 \ \dots \ 1 \end{array}} \\ & & \ddots & \\ & 0 & & \end{array}\right), \tag{3.45}$$

where the submatrices N_i have nonpositive entries.

Proof. We denote by m_{ij} the coefficient in the i^{th} row and j^{th} column of M. By construction, the columns of M are permutations of the vectors $(-1,-1,1,0,\dots,0)$ and $(-2,1,0,\dots,0)$. In particular, there is exactly one coefficient equal to 1 in each column. Suppose that we have proved the existence of a nonzero row with nonnegative entries. Then, after an appropriate permutation of its rows and columns, M reads

$$M = \left(\begin{array}{cc} \boxed{\begin{array}{c} M_1 \\ \hline 0 \ \dots \ 0 \end{array}} & \boxed{\begin{array}{c} N \\ \hline 1 \ \dots \ 1 \end{array}} \end{array}\right),$$

where N has nonpositive entries and M_1 satisfies the same hypothesis as M. By induction, it is then clear that M can be put into the form (3.45).

Consequently, the proof comes down to find a nonzero row with nonnegative entries. Let $q \geq 1$, $L_{i_1}, \dots, L_{i_q}$ be the rows containing at least one positive entry, and suppose that amongst $L_{i_1}, \dots, L_{i_q}$, every row also has a negative entry. Let $e = (e_1, \dots, e_P) \in (0, +\infty)^P$ defined in (3.42). We build by induction a sequence $(u_n)_{n \in \mathbb{N}}$ with values in $\{e_{i_1}, \dots, e_{i_q}\}$ as follows: $u_0 = e_{i_1}$; let $n \geq 0$ and assume that $u_0, \dots, u_n$ are built such that $u_0 < \dots < u_n$, $u_i \in \{e_{i_1}, \dots, e_{i_q}\}$. By construction, there exists $l \in \{1, \dots, q\}$ such that $u_n = e_{i_l}$. The i_l^{th} row of M has a negative entry by assumption, so there exists $r \in \{1, \dots, R\}$ such that $m_{i_l r} \in \{-1, -2\}$. According to (3.42), the r^{th} column of M satisfies $\langle v_r, e \rangle = 0$, which reads

$$\exists j \in \{1, \dots, m\}, \exists k \in \{i_1, \dots, i_q\}: \quad e_{i_l} + e_j = e_k \text{ if } m_{i_l r} = -1, \quad 2e_{i_l} = e_k \text{ if } m_{i_l r} = -2.$$

Then we set $u_{n+1} = e_k$, and by induction, $(u_n)_{n \in \mathbb{N}}$ is a strictly increasing sequence with values in $\{e_{i_1}, \dots, e_{i_q}\}$: contradiction, so there exists one row amongst $L_{i_1}, \dots, L_{i_q}$ that contains only zeros and ones. $\square$

Remark 1. *According to* (3.42), *$P > R$, so the matrix N_1 in* (3.45) *is nonempty. Let $s \geq 1$ be the number of rows in N_1. The point in permuting the rows and columns of M is the following: suppose that M satisfies* (3.45)*; using the above definition of the reaction terms, there exists $C > 0$ depending only on k_j^f, k_j^b, such that*

$$1 \leq k \leq s \;\Rightarrow\; f_k(c) \leq C \sum_{i=1}^{P} c_i, \tag{3.46}$$

$$s+1 \leq k \leq P \;\Rightarrow\; f_k(c) \leq C \left(\sum_{i=1}^{P} c_i + \sum_{i=1}^{k-1} c_i^2 \right). \tag{3.47}$$

Proof of Theorem 3.5. As for Theorem 3.1, the existence of a unique maximal nonnegative weak-W_p^s solution $c = (c_1, \ldots, c_P) : [0, T^*) \times \Omega \to \mathbb{R}^P$ and the regularity results are a consequence of Amann's theory [4]. To prove that $T^* = +\infty$, we have to find *a priori* bounds in $L^\infty(Q_T)$ for any $T \leq T^*$, $T < +\infty$. Similarly as for Step 1 in the proof of Theorem 3.1, the first estimates are consequences of the atomic conservation law: using the no-flux boundary conditions,

$$\forall t \in (0,T), \quad \sum_{i=1}^{P} \int_\Omega e_i\, c_i(t) = \sum_{i=1}^{P} \int_\Omega e_i\, c_{0,i}.$$

Then, using Lemma 3.2 (i), c is bounded in $L^{r_0}(Q_T)$ for $r_0 \in [1, (N+2)/N)$ if $N \geq 2$, $r_0 \in [1,2)$ if $N = 1$. For diffusivities $d_i(c_i)$, we write (with $D_i(y) = \int_0^y d_i(s)ds$)

$$\partial_t \sum_{i=1}^{P} e_i c_i + \Delta \sum_{i=1}^{P} e_i D_i(c_i) = 0 \text{ on } Q_T \;;\; \partial_\nu \sum_{i=1}^{P} e_i D_i(c_i) = 0 \text{ on } \Sigma_T \;;\; \sum_{i=1}^{P} e_i c_i(0,\cdot) = \sum_{i=1}^{P} e_i c_{0,i}.$$

Then Lemma 3.3 guarantees that c is bounded in $L^2(Q_T)$.

To improve these estimates, using Lemma 3.6, we go down without loss of generality to the case when M has the form given in (3.45). Assuming first $N = 1$, we know that c is bounded in $L^{r_0}(Q_T)$ for $r_0 < 2$. Using the notations of Remark 1, (3.46) and Lemma 3.2 guarantee that $c_1, \ldots, c_s$ are bounded in $L^p(Q_T)$ for any $p < +\infty$. Then, using (3.47), c_{s+1} is bounded in $L^p(Q_T)$ for any $p < +\infty$ and, by induction, for any $k \in \{s+1, \ldots, P\}$, c_k is bounded in $L^p(Q_T)$ for any $p < +\infty$.

Suppose $N \geq 2$ and let $r_0 > 1$ be such that c is bounded in $L^{r_0}(Q_T)$. Using (3.46), (3.47) and Lemma 3.2,

$c_1, \ldots, c_s$ are bounded in $L^{q_1}(Q_T)$, where $\frac{1}{r_0} - \frac{2}{N+2} < \frac{1}{q_1}$.

$c_1^2, \ldots, c_s^2$ are bounded in $L^{q_2}(Q_T)$, where $\frac{2}{r_0} - \frac{4}{N+2} < \frac{1}{q_2}$, and $q_2 \geq 1$ provided $\frac{2}{r_0} - \frac{4}{N+2} < 1$.

c_{s+1} is bounded in $L^{q_3}(Q_T)$, where $\frac{2}{r_0} - \frac{6}{N+2} < \frac{1}{q_3}$.

Then it is possible to continue improving the estimates for $c_{s+2}, \ldots, c_P$ if $q_3 \geq q_1$, i.e. if

$$\frac{2}{r_0} - \frac{6}{N+2} < \frac{1}{r_0} - \frac{2}{N+2}. \tag{3.48}$$

Note that $\frac{2}{r_0} - \frac{4}{N+2} < 1$ is a consequence of (3.48). For diffusivities $d_i(t,x,c)$ satisfying (3.16), $r_0 < \frac{N+2}{N}$ and (3.48) can be satisfied if and only if $N < 4$. For diffusivities $d_i(c_i)$ satisfying (3.17), $r_0 = 2$ and (3.48) can be satisfied if and only if $N < 6$. Once we have (3.48), it is clear that

$c_{s+1},\dots,c_P$ are bounded in $L^{q_1}(Q_T)$ by induction. Then, similarly as for Theorem 3.1, we bootstrap this procedure to show that c is bounded in $L^p(Q_T)^P$ for any $p<+\infty$.

Finally, we use Theorem III.7.1 in [69] to show that c_i is bounded in $L^\infty(Q_T)$ for all i, whence global existence in Theorem 3.5. $\square$

Example: for the prototype chain-growth polymerization process, the chemical reaction network reads as

$$C_r + C_1 \underset{k_r^b}{\overset{k_r^f}{\rightleftharpoons}} C_{r+1} \ ; \ r\in\{1,\dots,R\},\ k_r^f, k_r^b \geq 0.$$

Typical values for R are large, say about 100 or more. As an example, we write below the equations for $R=4$:

$$\begin{pmatrix} \partial_t c_1 - \mathrm{div}(d_1(t,x,c)\nabla c_1) \\ \partial_t c_2 - \mathrm{div}(d_2(t,x,c)\nabla c_2) \\ \partial_t c_3 - \mathrm{div}(d_3(t,x,c)\nabla c_3) \\ \partial_t c_4 - \mathrm{div}(d_4(t,x,c)\nabla c_4) \end{pmatrix} = \begin{pmatrix} -2 & -1 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} k_1^f c_1^2 - k_1^b c_2 \\ k_2^f c_1 c_2 - k_2^b c_3 \\ k_3^f c_1 c_3 - k_3^b c_4 \end{pmatrix}.$$

Remark that the stoichiometric matrix is naturally “well sorted” in the sense of Lemma 3.6. Theorem 3.5 guarantees the global existence of classical solution for any R in dimension $N=3$ for general diffusivities, and in dimensions $N\leq 5$ for diffusivities $d_i(c_i)$.

3.5 Appendix

Notations. Let $\mathcal{M}=\mathcal{M}(Q_T,\mathbb{R})$ be the set of measurable functions on Q_T and for $p\geq 1$, let

$L^\infty(0,T;L^p(\Omega)) = \{u\in\mathcal{M} : \underset{t\in(0,T)}{\mathrm{supess}}\|u(t)\|_{L^p(\Omega)} < +\infty\}$, endowed with

$$\|u\|_{L^\infty(0,T;L^p(\Omega))} := \mathrm{supess}\|u(t)\|_{L^p(\Omega)}.$$

$L^p(0,T;H^1(\Omega)) = \{u\in\mathcal{M} : u\in L^p(0,T;L^2(\Omega))\ ,\ \nabla u\in L^p(0,T;L^2(\Omega)^N)\}$, endowed with

$$\|u\|_{L^p(0,T;H^1(\Omega))} := \left(\int_0^T [\|u(t)\|^p_{L^2(\Omega)} + \|\nabla u(t)\|^p_{L^2(\Omega)^N}]dt\right)^{\frac{1}{p}}.$$

$V_2(Q_T) = L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))$, endowed with

$$\|u\|_{V_2(Q_T)} := \left(\|u\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u\|^2_{L^2(0,T;H^1(\Omega))}\right)^{\frac{1}{2}}.$$

To prove Lemma 3.2, we will use the following interpolation result:

Lemma 3.7. *Let $T>0$, Ω be a bounded domain of $\mathbb{R}^N$ whose boundary $\partial\Omega$ is at least C^1, let $1\leq p<+\infty$ and $u\in L^\infty(0,T;L^p(\Omega))\cap L^2(0,T;H^1(\Omega))$. There exists a constant $C>0$ depending only on Ω, such that*

$$\|u\|_{L^q(Q_T)} \leq C\|u\|^{1-\alpha}_{L^\infty(0,T;L^p(\Omega))}\|u\|^{\alpha}_{L^2(0,T;H^1(\Omega))}, \tag{3.49}$$

where $\alpha=\frac{2}{q}$ and q satisfies

$$q = 2+\frac{2p}{N}\ \textit{for}\ N\geq 3\ ; \quad 2\leq q<2+p\ \textit{for}\ N=2\ ; \quad q=2+p\ \textit{for}\ N=1. \tag{3.50}$$

We first recall some classical results: we have the embedding

$$H^1(\Omega) \hookrightarrow L^s(\Omega), \tag{3.51}$$

where $s \geq 1$ satisfies $\frac{1}{s} = \frac{1}{2} - \frac{1}{N}$ if $N \geq 3$; $s < +\infty$ if $N = 2$; $s = +\infty$ if $N = 1$. As a consequence of Hölder's inequality, for $u : \Omega \to \mathbb{R}$ measurable, $q, r, s \in [1, +\infty]$ and $\alpha \in [0, 1]$,

$$\|u\|_{L^q(\Omega)} \leq \|u\|_{L^r(\Omega)}^{1-\alpha} \|u\|_{L^s(\Omega)}^{\alpha}, \quad \text{where} \quad \frac{1}{q} = \frac{1-\alpha}{r} + \frac{\alpha}{s}. \tag{3.52}$$

Combining (3.51) and (3.52), we get the following "Gagliardo-Nirenberg"-type inequality: there exists $C > 0$ depending only on Ω, such that

$$\|u\|_{L^q(\Omega)} \leq C \|u\|_{L^p(\Omega)}^{1-\alpha} \|u\|_{H^1(\Omega)}^{\alpha}, \tag{3.53}$$

where $p, q \in [1, +\infty]$, $\alpha \in [0, 1]$ and

$$\frac{1}{q} = (1-\alpha)\frac{1}{p} + \alpha\left(\frac{1}{2} - \frac{1}{N}\right) \text{ if } N \geq 3 \text{ ;} \quad \frac{1-\alpha}{p} < \frac{1}{q} \text{ if } N = 2 \text{ ;} \quad \frac{1-\alpha}{p} = \frac{1}{q} \text{ if } N = 1. \tag{3.54}$$

Proof of Lemma 3.7. As $u \in L^\infty(0, T; L^p(\Omega)) \cap L^2(0, T; H^1(\Omega))$, for a.e. $t \in (0, T)$, $u(t) \in L^p(\Omega) \cap H^1(\Omega)$. Using (3.53), we get

$$\begin{aligned} \int_0^T \|u(t)\|_{L^q(\Omega)}^q dt &\leq C^q \int_0^T \|u(t)\|_{L^p(\Omega)}^{q(1-\alpha)} \|u(t)\|_{H^1(\Omega)}^{q\alpha} dt, \\ &\leq C^q \|u\|_{L^\infty(0,T;L^p(\Omega))}^{q(1-\alpha)} \int_0^T \|u(t)\|_{H^1(\Omega)}^{q\alpha} dt, \end{aligned} \tag{3.55}$$

where α and q satisfy (3.54). Now we choose $q \geq 2, \alpha > 0$ such that $q\alpha = 2$. It is easy to see that conditions (3.54) with $q\alpha = 2$ are equivalent to conditions (3.50). Taking the $(1/q)^{th}$ power in (3.55), we get (3.49).

□

Proof of Lemma 3.2.

The case $r = 1$.

Integration of (3.21) on $\Omega \times (0, t)$ for $t \in (0, T)$ yields, after integration by parts and using the homogeneous Neumann boundary conditions,

$$\|u\|_{L^\infty(0,T;L^1(\Omega))} \leq \|f\|_{L^1(Q_T)} + \|u_0\|_{L^1(\Omega)}. \tag{3.56}$$

Let $e = \exp(1)$ and define

$$j : \mathbb{R}_+ \to [0, 1),\ y \mapsto 1 - \frac{1}{\log(e+y)} \quad ; \quad J : \mathbb{R}_+ \to \mathbb{R}_+,\ y \mapsto \int_0^y j(s) ds.$$

Multiplication of (3.21) by $j(u)$ and integration by parts on Q_T yields

$$\int_\Omega J(u(T)) + \int_{Q_T} \frac{d|\nabla u|^2}{(e+u)\log(e+u)^2} \leq \int_\Omega J(u_0) + \int_{Q_T} f j(u),$$

hence

$$\underline{d} \int_{Q_T} \frac{|\nabla u|^2}{(e+u)\log(e+u)^2} \leq \|u_0\|_{L^1(\Omega)} + \|f\|_{L^1(Q_T)}. \tag{3.57}$$

Let $\beta \in (0, \frac{1}{2})$ and set

$$G : \mathbb{R}_+ \to \mathbb{R}_+,\ y \mapsto \frac{\log(e+y)^2}{(e+y)^{1-2\beta}}\ ;\ \|G\|_\infty := \sup_{y\in\mathbb{R}_+} G(y) < +\infty.$$

Then, for $v = (e+u)^\beta$,

$$\begin{aligned}\int_{Q_T} |\nabla v|^2 &= \beta^2 \int_{Q_T} \frac{|\nabla u|^2}{(e+u)^{2-2\beta}},\\ &= \beta^2 \int_{Q_T} \frac{\log(e+u)^2}{(e+u)^{1-2\beta}} \frac{|\nabla u|^2}{(e+u)\log(e+u)^2},\\ &\le \frac{\|G\|_\infty}{4\underline{d}} \left(\|u_0\|_{L^1(\Omega)} + \|f\|_{L^1(Q_T)} \right), \end{aligned} \tag{3.58}$$

where we used (3.57) in the last inequality. According to (3.56), v is bounded in $L^\infty(0,T;L^{1/\beta}(\Omega))$, so together with (3.58), v is bounded in $L^\infty(0,T;L^{1/\beta}(\Omega)) \cap L^2(0,T;H^1(\Omega))$ and Lemma 3.7 guarantees that v is bounded in $L^r(Q_T)$, with

$$r = 2 + \frac{2}{\beta N} \text{ for } N \ge 3\ ; \quad r < 2 + \frac{1}{\beta} \text{ for } N = 2\ ; \quad r = 2 + \frac{1}{\beta} \text{ for } N = 1.$$

Then u is bounded in $L^q(Q_T)$ with $q = \beta r$, which means

$$q = 2\beta + \frac{2}{N} \text{ for } N \ge 3\ ; \quad q < 2\beta + 1 \text{ for } N = 2\ ; \quad q = 2\beta + 1 \text{ for } N = 1.$$

Since β can be chosen arbitrarily close to $1/2$, u is bounded in $L^q(Q_T)$, where q satisfies conditions (i) in Lemma 3.2.

The case $r > 1$.

Let $p > 1$, $t \in (0,T)$. Multiplication of (3.21) by $pu^{p-1} \ge 0$ and integration by parts on Q_t yields

$$\begin{aligned}&\int_{Q_t} \partial_t u^p + 4(1-\frac{1}{p}) \int_{Q_t} d|\nabla(u^{p/2})|^2 \le p \int_{Q_t} f u^{p-1},\\ &\int_\Omega u^p(t) + 4(1-\frac{1}{p}) \int_{Q_t} d|\nabla(u^{p/2})|^2 \le \int_\Omega u_0^p + p \int_{Q_t} f u^{p-1}. \end{aligned} \tag{3.59}$$

Here and below, C denotes appropriate constants depending only on $p, \underline{d}, T$ and $\|u_0\|_{L^\infty(\Omega)}$. Evidently, (3.59) yields

$$\|u^{p/2}\|^2_{V_2(Q_T)} \le C \left(1 + \int_{Q_T} |f| u^{p-1} \right). \tag{3.60}$$

According to Lemma 3.7, we have the continuous embedding $V_2(Q_T) \hookrightarrow L^s(Q_T)$, where

$$s = \frac{2(N+2)}{N} \text{ for } N \ge 3\ ; \quad s < 4 \text{ for } N = 2\ ; \quad s = 4 \text{ for } N = 1. \tag{3.61}$$

Assuming s satisfies (3.61), inequality (3.60) yields

$$\exists C > 0 : \quad \|u^{p/2}\|^2_{L^s(Q_T)} \le C \left(1 + \int_{Q_T} |f| u^{p-1} \right).$$

Recall that $f \in L^r(Q_T)$, so Hölder's inequality yields

$$\|u\|^p_{L^{\frac{ps}{2}}(Q_T)} \leq C\left(1 + \|f\|_{L^r(Q_T)} \|u\|^{p-1}_{L^{\frac{r(p-1)}{r-1}}(Q_T)}\right). \tag{3.62}$$

We choose $p > 1$ such that

$$1 \leq \frac{r(p-1)}{r-1} \leq \frac{ps}{2}, \tag{3.63}$$

which is equivalent to

$$1 + \frac{s}{2r} - \frac{s}{2} \leq \frac{1}{p} \leq \frac{r}{2r-1}. \tag{3.64}$$

Such a choice is possible if

$$1 + \frac{s}{2r} - \frac{s}{2} < 1 \quad \text{and} \quad 1 + \frac{s}{2r} - \frac{s}{2} \leq \frac{r}{2r-1}. \tag{3.65}$$

It is easy to check that both inequalities in (3.65) are satisfied for $s \geq 2$, which will be assumed in the following; note that this is compatible with (3.61). As p satisfies (3.63), using Young's inequality in (3.62) and $L^{\frac{ps}{2}}(Q_T) \hookrightarrow L^{\frac{r(p-1)}{r-1}}(Q_T)$ it follows that

$$\exists C > 0: \quad \|u\|^p_{L^{\frac{ps}{2}}(Q_T)} \leq C\left(1 + \|f\|^p_{L^r(Q_T)} + \frac{1}{2}\|u\|^p_{L^{\frac{ps}{2}}(Q_T)}\right), \tag{3.66}$$

and hence u is bounded in $L^{\frac{ps}{2}}(Q_T)$. To get the best estimate, we choose p as large as possible: combining (3.61) with (3.64), we see that the condition on p becomes

$$\frac{N+2}{N}\frac{1}{r} - \frac{2}{N} \leq \frac{1}{p} \text{ for } N \geq 3\,; \quad \frac{2}{r} - 1 < \frac{1}{p} \text{ for } N = 2\,; \quad \frac{2}{r} - 1 \leq \frac{1}{p} \text{ for } N = 1. \tag{3.67}$$

Since u is bounded in $L^{\frac{ps}{2}}(Q_T)$ with p satisfying (3.67) and s satisfying (3.61), altogether, u is bounded in $L^q(Q_T)$, where q satisfies (ii) in Lemma 3.2.

□

4

Global existence for a class of quadratic reaction-diffusion systems with nonlinear diffusions and L^1 initial data

The contribution of this section is the content of the article [93].

In this work, we prove the existence of global weak solutions for a class of reaction-diffusion systems with nonlinear diffusions and with at most quadratic reaction terms, in any space dimension. The proof relies on a dimension-independent L^2 estimate, based on a total mass control assumption. If the initial data are in L^2, this estimate provides a control of the quadratic nonlinearities in L^1. We prove that in the case when initial data are only in L^1, the L^2-estimate can be localized in time, which allows to pass to the limit in an approximate system for $t > 0$. We are also able to prove that the initial data are preserved at the limit.

4.1 Introduction

We are interested in the existence of global solutions in time for the system

$$\left\{ \begin{array}{rcll} \partial_t c_i - \Delta D_i(c_i) & = & f_i(t,x,c) & \text{on } (0,+\infty)\times\Omega, \quad i \in \{1,\dots,P\}, \\ \partial_\nu D_i(c_i) & = & g_i & \text{on } (0,+\infty)\times\partial\Omega, \quad i \in \{1,\dots,P\}, \\ c(0,\cdot) & = & c_0 & \text{on } \Omega. \end{array} \right. \tag{4.1}$$

The unknown is $c = (c_1,\dots,c_P)$. Throughout the section, Ω is an open, bounded subset of $\mathbb{R}^N$, endowed with the Lebesgue measure λ. Its boundary $\partial\Omega$ is supposed to be at least of class C^2, and $\partial_\nu D_i(c_i)$ is the normal exterior derivative of $D_i(c_i)$ on $\partial\Omega$.

We assume that the data satisfy

$(H1)$ $\quad g_i \in L^2_{loc}([0,+\infty);L^2(\partial\Omega)^+)$.

$(H2)$ $\quad \forall i \in \{1,\dots,P\},\ f_i \in C^1((0,+\infty)\times\Omega\times\mathbb{R}^P,\mathbb{R})$; $\forall (t,x,r) \in (0,+\infty)\times\Omega\times[0,+\infty)^P$,

$$f_i(t,x,r_1,\dots,r_{i-1},0,r_{i+1},\dots,r_P) \geq 0 \quad \text{(quasi-positivity)}.$$

$(H3)$ $\quad \exists \gamma \in C([0,+\infty),\mathbb{R}_+) : \forall (t,x,r) \in (0,+\infty)\times\Omega\times[0,+\infty)^P,\ \forall i \in \{1,\dots,P\}$,

$$|f_i(t,x,r)| \leq \gamma(t)(1+\textstyle\sum_{j=1}^P r_j^2).$$

$(H4)$ $D_i \in C^2([0,+\infty))\,; D_i(0)=0\,;\ \exists \underline{d},\overline{d}>0\ :\ \underline{d} \le D_i' \le \overline{d}$.

$(H5)$ $\exists a_1,\dots,a_P>0, \exists F \in L^2_{loc}([0,+\infty);L^2(\Omega))\ :\ \forall (t,x,r) \in (0,+\infty)\times\Omega\times[0,+\infty)^P$,

$$\sum_{i=1}^P a_i f_i(t,x,r) \le F(t,x).$$

For the initial data, we investigate two different situations. We first deal with initial data $c_0=(c_{01},\dots,c_{0P}) \in L^2(\Omega,[0,+\infty)^P)$, which are "compatible" with L^2-techniques, and then treat the more difficult case $c_0 \in L^1(\Omega,[0,+\infty)^P)$. The latter choice is motivated by the fact that systems of the type (4.6) usually arise in ecology or in chemistry, and c_i may represent population densities or concentrations of chemical species. Having these applications in mind, it is more natural to require that initially, the total mass of the chemical species, or the total population, is bounded, i.e. to work with L^1-initial data.

Although assumptions $(H2)$ and $(H5)$ guarantee the nonnegativity of the solutions and a uniform control on the total mass, it has been shown in [91] that they are not sufficient to prevent a blow-up of the solutions in finite time in $L^\infty(\Omega)$ or in $L^p((0,T)\times\Omega)$ for some finite p and $T>0$, even in the case of linear diffusion. This blow-up may even occur for space dimension 1, provided the degree of the nonlinearities is high enough. However, for initial data in L^2, it can be proved that the solutions remain bounded in $L^2((0,T)\times\Omega)$ for any $T>0$. This estimate, together with the quadratic growth assumption (3) which guarantees that the reaction terms remain bounded in $L^1((0,T)\times\Omega)$, is the core argument of the proof of the existence of global weak solutions. When considering initial data in $L^1(\Omega)$, the main new difficulty is that the previous L^2-estimate is no longer valid up to $t=0$. Instead, we manage to use the regularizing properties of the Laplacian and then localize the L^2-estimate to control the solution in $L^2((\tau,T)\times\Omega)$ for $\tau\in(0,T)$. The reaction terms are not estimated any more in L^1 up to $t=0$. To get round this difficulty, we use a two-sided approach (inspired from [41, 90]) to estimate the solutions in a neighborhood of $t=0$ from above and below and prove that the initial data remain satisfied, but in a weaker sense.

Throughout the section and for any $0<\tau<T<+\infty$, we use the common notations

$$Q_T=(0,T)\times\Omega\,;\ \Sigma_T=(0,T)\times\partial\Omega\,;\ Q_{\tau,T}=(\tau,T)\times\Omega\,;\ \Sigma_{\tau,T}=(\tau,T)\times\partial\Omega.$$

For initial data in $L^2(\Omega,[0,+\infty)^P)$, we prove the following

Theorem 4.1. *Assume $c_0=(c_{01},\dots,c_{0P}) \in L^2(\Omega,[0,+\infty)^P)$ and $(H1)-(H5)$. Then system (4.1) has a global solution in the following sense:*

$\exists c=(c_1,\dots,c_P):[0,+\infty)\times\Omega\to[0,+\infty)^P\ :\ \forall i\in\{1,\dots,P\}, \forall T>0,$

(i) $c_i \in L^2(Q_T)\cap C([0,T];L^1(\Omega))\,;\ \forall \eta\in[1,4/3),\ \nabla D_i(c_i)\in L^\eta(Q_T)^N\,;$

(ii) $\forall \varphi_i \in C^\infty(\overline{Q_T})$ *such that* $\varphi_i(T)=0$,

$$\int_{Q_T} -c_i\partial_t\varphi_i + \nabla D_i(c_i)\cdot\nabla\varphi_i = \int_\Omega c_{0i}\varphi_i(0) + \int_{Q_T} f_i(t,x,c)\varphi_i + \int_{\Sigma_T} g_i\varphi_i. \tag{4.2}$$

In the following, we denote by $[h]^- = \max(0,-h)$ the negative part of a real-valued function h. We also define the projections

$$\begin{array}{rcl} p_i : [0,+\infty)^P & \to & [0,+\infty)^P, \\ c=(c_1,\dots,c_P) & \mapsto & (c_1,\dots,c_{i-1},0,c_{i+1},\dots,c_P). \end{array} \tag{4.3}$$

For initial data in $L^1(\Omega,[0,+\infty)^P)$, we also require the reaction terms to satisfy

$(H6)$ $\exists \beta \in C([0,+\infty),\mathbb{R})$ such that $\forall c \in [0,+\infty)^P$, $\forall i \in \{1,\ldots,P\}$,

$$[f_i(c) - f_i(p_i(c))]^- \leq \beta(c_i)\Big(1 + \sum_{j=1}^{P} c_j\Big).$$

This assumption will be used to control the solutions from below in a neighborhood of $t = 0$. In view of applications, this is not a strong restriction: polynomial functions of degree 2 satisfy $(H6)$, as well as many other nonlinearities. However, it is possible to build reaction terms that satisfy $(H2),(H3),(H5)$ but not $(H6)$: this is the case for

$$(f_1, f_2)(c_1, c_2) = (c_2^2 \sin(c_1 c_2^2), -c_2^2 \sin(c_1 c_2^2)).$$

More generally, assumption $(H6)$ may not be satisfied for functions f_i such that the growth of $\partial_i f_i$ is more than linear with respect to $p_i(c)$.

Our main result is the following

Theorem 4.2. *Assume $c_0 \in L^1(\Omega,[0,+\infty)^P)$ and $(H1)-(H6)$. Then system (4.1) has a global weak solution in the following sense:*

$\exists c \in L^\infty(0,+\infty; L^1(\Omega)_+^P) \cap L^2_{loc}(0,+\infty; L^2(\Omega)^P)$ such that

(i) $\forall i \in \{1,\ldots,P\}$, $\forall T > 0$, $c_i \in L^1(Q_T)$; $\forall \eta \in [1,4/3)$, $\nabla c_i \in L^\eta_{loc}(0,+\infty; L^\eta(\Omega))^N$.

(ii) $\forall i \in \{1,\ldots,P\}$, for a.e. $0 < \tau < T < +\infty$, $\forall \varphi_i \in C^\infty(\overline{Q_{\tau,T}})$ such that $\varphi_i(\tau) = \varphi_i(T) = 0$,

$$\int_{Q_{\tau,T}} \big(-c_i \partial_t \varphi_i + \nabla D_i(c_i) \cdot \nabla \varphi_i\big) = \int_{Q_{\tau,T}} f_i(t,x,c)\varphi_i + \int_{\Sigma_{\tau,T}} g_i \varphi_i. \tag{4.4}$$

(iii) $c \in C(0,+\infty; L^1(\Omega)^P)$ and $c(t) \underset{t\to 0}{\longrightarrow} c_0$ for the weak topology on Radon measures, i.e.

$$\forall i \in \{1,\ldots,P\},\ \forall \varphi \in C(\overline{\Omega}),\ \int_\Omega c_i(t)\varphi \underset{t\to 0}{\longrightarrow} \int_\Omega c_{0i}\varphi. \tag{4.5}$$

Let us finally mention some related works. In [41], similar L^2-estimates are used to prove the existence of global weak solutions for particular versions of system (4.1): the case of diffusive fluxes of the type $-d_i \nabla c_i$ with time, space dependent and possibly degenerate diffusion coefficients d_i is investigated, but they are independent of c_i. Additional structure on the reaction terms is also required so that there exists a Lyapunov function, and the initial data are assumed to satisfy $c_{0i}\log(|c_{0i}|) \in L^2(\Omega)$. In [92], Maximal Regularity theory has been succesfully applied to obtain new global existence results for quadratic systems arising in mass-action kinetics chemistry, for small space dimensions and constant diffusivities. For space dimension 2 and smooth initial data, global existence and uniqueness of solutions is shown for systems similar to (4.1) (for nonlinear diffusions) in [84], where (3) is replaced by a "triangular structure" assumption, which allows to deal with more general polynomially bounded reaction terms. For global existence results for systems with quadratic nonlinearities, see also [48,58,94]. For a survey on global existence issues for reaction-diffusion system with nonnegative solutions and control of the total mass (i.e. with $(H2)$ and $(H5)$), we refer to [90].

For initial data in $L^1(\Omega)$, global existence of weak solutions has been shown in [89,90] for systems with constant diffusion coefficients and whose nonlinearities are known to be *a priori* bounded in $L^1(Q_T)$ for any $T > 0$. In these works, the nonlinearities are not assumed to have a polynomial growth. For systems which satisfy an additional "triangular structure", the regularity of the solutions has been investigated in [17]: if the nonlinearities are polynomially bounded, the solutions are shown to be classical for $t > 0$. If the nonlinearities are bounded with a polynomial expression of degree $p < \frac{N+2}{N}$ (N being the space dimension), the existence of solutions with Radon measure initial data is also proved.

In the present work, we emphasize that such a control on the nonlinearities up to $t = 0$ is not available, and the reaction terms are only known to be bounded in $L^1(Q_{\tau,T})$ for $0 < \tau < T < +\infty$.

4.2 Proof of Theorem 4.1.

Outline of the proof. *We build a solution of* (4.1) *as a limit of a sequence* $(c^n)_{n\in\mathbb{N}}$ *of solutions of an approximate problem, where the data are regularized and the reaction terms are truncated. We rely on H. Amann's theory for the existence of such a sequence. Quasi-positivity assumption* $(H2)$ *and the nonnegativity of the boundary conditions guarantee that* c^n *remains nonnegative. To prove the relative compactness of* $\{c^n,\ n \in \mathbb{N}\}$*, the main tool is an estimate in* $L^2(Q_T)$*, inspired from the techniques of [90], and which strongly relies on* $(H5)$*. Using the quadratic growth of* f_i*, the reaction terms are then controlled in* $L^1(Q_T)$*. This allows to estimate the gradients* $\nabla D_i(c_i^n)$ *in* $L^\eta(Q_T)$ *for* $\eta \in (1,4/3)$*, and using Aubin-Simon compactness results, we are able to prove the a.e. convergence of* c^n *in* Q_T *for any* $T > 0$*. Then we prove the compactness of* $(c^n)_{n\in\mathbb{N}}$ *in* $L^2(Q_T)$ *and the convergence of the approximate reaction terms in* $L^1(Q_T)$*. Finally, we use a diagonal extraction to pass to the limit* $n \to +\infty$ *for any* $T > 0$ *in the variational formulation* (4.2).

Let $n \in \mathbb{N}$, $\alpha^n : \mathbb{R} \to \mathbb{R}$ be a smooth nondecreasing function satisfying

$$\alpha^n(x) = x \text{ for } x \in [0,n] \quad ; \quad \sup_{x\in\mathbb{R}} |\alpha^n(x)| \le n+1.$$

Let $T^n : \mathbb{R}^P \to \mathbb{R}^P$, $r = (r_1,\dots,r_P) \mapsto (\alpha^n(r_1),\dots,\alpha^n(r_P))$, we define truncated reaction terms as $f_i^n(t,x,r) = f_i(t,x,T^n(r))$. For $n \in \mathbb{N}$, let $c_0^n \in C_c^\infty(\Omega,[0,+\infty)^P)$, $g^n \in C^\infty([0,+\infty)\times\partial\Omega,[0,+\infty)^P)$ such that

$$c_0^n \to c_0 \text{ in } L^2(\Omega)^P \ ; \ g^n \to g \text{ in } L^2_{loc}([0,+\infty);L^2(\partial\Omega)^P).$$

We consider the following approximate problem, whose reaction terms are now bounded with respect to c :

$$\left\{\begin{array}{rcll} \partial_t c_i - \Delta D_i(c_i) &=& f_i^n(t,x,c) & \text{on } (0,+\infty)\times\Omega, \quad i \in \{1,\dots,P\}, \\ \partial_\nu D_i(c_i) &=& g_i^n & \text{on } (0,+\infty)\times\partial\Omega \quad i \in \{1,\dots,P\}, \\ c(0,\cdot) &=& c_0^n & \text{on } \Omega. \end{array}\right. \tag{4.6}$$

For $i \in \{1,\dots,P\}$, f_i^n satisfies the quasi-positivity assumption $(H2)$, and using $(H3)$, it is bounded on $(0,T)\times\Omega\times\mathbb{R}^P$ for any $T > 0$. Under assumptions $(H1)-(H5)$, H. Amann's theory on parabolic systems guarantees that (4.6) has a unique nonnegative solution $c^n \in C^0([0,+\infty)\times\overline{\Omega}) \cap C^1(0,+\infty;C^2(\overline{\Omega}))$ (see [4], Theorems 14.4 and 14.6, see [2] for the proofs). Now using assumption $(H5)$, we have

$$\left\{\begin{array}{rcll} \partial_t[\sum_{i=1}^P a_i c_i^n] - \Delta[\sum_{i=1}^P a_i D_i(c_i^n)] &\le& F & \text{on } (0,+\infty)\times\Omega, \\ \partial_\nu[\sum_{i=1}^P a_i D_i(c_i^n)] &=& \sum_{i=1}^P a_i g_i^n & \text{on } (0,+\infty)\times\partial\Omega, \\ \sum_{i=1}^P a_i c_i^n(0,\cdot) &=& \sum_{i=1}^P a_i c_{0i}^n & \text{on } \Omega. \end{array}\right. \tag{4.7}$$

Set

$$W^n = \sum_{i=1}^{P} a_i c_i^n \; ; \; W_0^n = \sum_{i=1}^{P} a_i c_{0i}^n \; ; \; A^n = \frac{\sum_{i=1}^{P} a_i D_i(c_i^n)}{\sum_{i=1}^{P} a_i c_i^n} \; ; \; G^n = \sum_{i=1}^{P} a_i g_i^n. \tag{4.8}$$

Then (4.7) reads

$$\begin{cases} \partial_t W^n - \Delta[A^n W^n] & \leq & F & \text{on } (0,+\infty)\times\Omega, \\ \partial_\nu [A^n W^n] & = & G^n & \text{on } (0,+\infty)\times\partial\Omega, \\ W^n(0,\cdot) & = & W_0^n & \text{on } \Omega. \end{cases} \tag{4.9}$$

The subsequent lemma is inspired from Proposition 6.1 in [90]:

Lemma 4.3. *For any $T > 0$, $(W^n)_{n\in\mathbb{N}}$ is bounded in $L^2(Q_T)$.*

Proof. Let $t \in (0,T)$, integrate (4.9) on $(0,t)$ to get

$$W^n - \Delta \int_0^t A^n W^n \leq W_0^n + \int_0^t F \text{ on } Q_T, \;\; \partial_\nu [\int_0^t A^n W^n] = \int_0^t G^n \text{ on } \Sigma_T.$$

After multiplication by $A^n W^n \geq 0$ and integration on Q_T,

$$\int_{Q_T} A^n (W^n)^2 - \left(\Delta \int_0^t A^n W^n\right) A^n W^n \leq \int_\Omega W_0^n \left(\int_0^T A^n W^n\right) + \int_{Q_T} \left(\int_0^t F\right) A^n W^n. \tag{4.10}$$

Remark that

$$\begin{aligned} -\int_{Q_T} \left(\Delta \int_0^t A^n W^n\right) A^n W^n &= \frac{1}{2} \int_{Q_T} \frac{d}{dt} \left|\nabla \int_0^t A^n W^n\right|^2 - \int_{\Sigma_T} \partial_\nu \left(\int_0^t A^n W^n\right) A^n W^n \\ &= \frac{1}{2} \int_\Omega \left|\nabla \int_0^T A^n W^n\right|^2 - \int_{\Sigma_T} \left(\int_0^t G^n\right) A^n W^n \\ &= \frac{1}{2} \int_\Omega \left|\nabla \int_0^T A^n W^n\right|^2 - \int_{\Sigma_T} G^n \int_t^T A^n W^n. \end{aligned}$$

Then (4.10) becomes

$$\begin{aligned} \int_{Q_T} A^n (W^n)^2 &+ \frac{1}{2} \int_\Omega \left|\nabla \int_0^T A^n W^n\right|^2 \\ &\leq \int_\Omega W_0^n \left(\int_0^T A^n W^n\right) + \int_{Q_T} F \int_t^T A^n W^n + \int_{\Sigma_T} G^n \int_t^T A^n W^n \\ &\leq \int_\Omega W_0^n \left(\int_0^T A^n W^n\right) + \int_{Q_T} |F| \int_0^T A^n W^n + \int_{\Sigma_T} G^n \int_0^T A^n W^n \\ &\leq \left(\|W_0^n\|_{L^2(\Omega)} + \sqrt{T}\|F\|_{L^2(Q_T)}\right) \left\|\int_0^T A^n W^n\right\|_{L^2(\Omega)} \\ &\quad + \sqrt{T}\|G^n\|_{L^2(\Sigma_T)} \left\|\int_0^T A^n W^n\right\|_{L^2(\partial\Omega)}. \end{aligned} \tag{4.11}$$

In the following, we denote by $C > 0$ any constant depending only on the data of (4.1) and T. Using the Poincaré-Wirtinger inequality and the continuity of the trace operator from $H^1(\Omega)$ into $L^2(\partial\Omega)$, there exists $C > 0$ such that

$$\left\|\int_0^T A^n W^n\right\|_{L^2(\Omega)} + \left\|\int_0^T A^n W^n\right\|_{L^2(\partial\Omega)} \leq C \left(\left\|\nabla \int_0^T A^n W^n\right\|_{L^2(\Omega)} + \int_\Omega \int_0^T A^n W^n\right).$$

Consequently, using $\underline{d} \le A^n$ and Young's inequality in (4.11), there exists $C > 0$ such that

$$\underline{d} \int_{Q_T} (W^n)^2 \le \int_{Q_T} A^n (W^n)^2 + \frac{1}{4} \int_\Omega \left| \nabla \int_0^T A^n W^n \right|^2 \le C \left(1 + \int_{Q_T} A^n W^n \right). \tag{4.12}$$

By integration of (4.9) on (Q_t) for any $t \in (0,T)$, using the nonnegativity of W^n, we easily get the existence of $C > 0$ such that

$$\forall n \in \mathbb{N}, \quad \|W^n\|_{L^\infty(0,+\infty;L^1(\Omega))} \le C. \tag{4.13}$$

Combined with $A^n \le \overline{d}$, (4.13) yields that the right-hand side in (4.12) is bounded independently of n, so $(W^n)_{n\in\mathbb{N}}$ is bounded in $L^2(Q_T)$.

□

Since $W^n = \sum_{i=1}^P a_i c_i^n$ and $a_i > 0$, $c_i^n \ge 0$, Lemma 4.3 yields

$$\exists C > 0 : \forall n \in \mathbb{N}, \quad \|c^n\|_{L^2(Q_T)^P} \le C. \tag{4.14}$$

Combined with the restriction (H3) on the growth of functions f_i, we get

$$\exists C > 0 : \forall i \in \{1, \ldots, P\}, \forall n \in \mathbb{N}, \quad \|f_i^n(t,x,c^n)\|_{L^1(Q_T)} \le C. \tag{4.15}$$

We now consider the equations of system (4.6) separately: for all $i \in \{1, \ldots, P\}$, c_i^n is bounded in $L^2(Q_T)$, $(t,x) \mapsto f_i^n(t,x,c^n(t,x))$ is bounded in $L^1(Q_T)$, so we are in position to apply the subsequent result:

Lemma 4.4. *Let $T > 0$, $n \in \mathbb{N}$ and u^n be a strong nonnegative solution of*

$$\left\{ \begin{array}{rcll} \partial_t u^n - \mathrm{div}(d^n \nabla u^n) &=& f^n & \text{on } Q_T, \\ d^n\, \partial_\nu u^n &=& g^n & \text{on } \Sigma_T, \\ u^n(0,\cdot) &=& u_0^n & \text{on } \Omega, \end{array} \right. \tag{4.16}$$

where $d^n \in L^\infty(Q_T)$, $0 < \underline{d} \le d^n \le \overline{d} < +\infty$, $(f^n)_{n\in\mathbb{N}}$ is bounded in $L^1(Q_T)$, $(u_0^n)_{n\in\mathbb{N}}$ is bounded in $L^1(\Omega)$, $(g^n)_{n\in\mathbb{N}}$ is bounded in $L^1(\Sigma_T)$. If $(u^n)_{n\in\mathbb{N}}$ is bounded in $L^2(Q_T)$, then

(i) $(u^n)_{n\in\mathbb{N}}$ is bounded in $L^p(0,T;W^{1,p}(\Omega))$ for any $1 \le p < \frac{4}{3}$.

(ii) $(u^n)_{n\in\mathbb{N}}$ is relatively compact in $L^p(Q_T)$ for any $1 \le p < 2$.

Proof. Let $e = \exp(1)$ and define

$$j : [0,+\infty) \to [0,1),\ x \mapsto 1 - \frac{1}{\log(e+x)} \quad ; \quad J : [0,+\infty) \to [0,+\infty),\ x \mapsto \int_0^x j(s)ds.$$

Multiplication of (4.16) by $j(u^n)$ and integration by parts on Q_T yields

$$\begin{aligned} \int_{Q_T} \partial_t(J(u^n)) + \int_{Q_T} \frac{d^n |\nabla u^n|^2}{(e+u^n)\log(e+u^n)^2} &= \int_{Q_T} f^n j(u^n) + \int_{\Sigma_T} g^n j(u^n), \\ \int_\Omega J(u^n(T)) + \int_{Q_T} \frac{d^n |\nabla u^n|^2}{(e+u^n)\log(e+u^n)^2} &= \int_\Omega J(u_0^n) + \int_{Q_T} f^n j(u^n) + \int_{\Sigma_T} g^n j(u^n), \\ \underline{d} \int_{Q_T} \frac{|\nabla u^n|^2}{(e+u^n)\log(e+u^n)^2} &\le \|u_0^n\|_{L^1(\Omega)} + \|f^n\|_{L^1(Q_T)} + \|g^n\|_{L^1(\Sigma_T)}, \end{aligned} \tag{4.17}$$

and the right-hand side is bounded by assumption. For $\varepsilon > 0$ small enough, we have

$$\begin{aligned}\int_{Q_T} |\nabla u^n|^{\frac{4}{3}-\varepsilon} &= \int_{Q_T} \left[\frac{|\nabla u^n|^2}{(e+u^n)\log(e+u^n)}\right]^{\frac{2}{3}-\frac{\varepsilon}{2}} [(e+u^n)\log(e+u^n)]^{\frac{2}{3}-\frac{\varepsilon}{2}} \\ &\le \left[\int_{Q_T} \left[\frac{|\nabla u^n|^2}{(e+u^n)\log(e+u^n)}\right]^{1-3\varepsilon}\right]^{\frac{2}{3}} \left[\int_{Q_T} [(e+u^n)\log(e+u^n)]^{2-\frac{3\varepsilon}{2}}\right]^{\frac{1}{3}}.\end{aligned}$$

Using (4.17) and the assumption that $(u^n)_{n\in\mathbb{N}}$ is bounded in $L^2(Q_T)$, the right-hand side is bounded independently of n. Since ε can be chosen arbitrarily small, $|\nabla u^n|$ is bounded in $L^p(Q_T)$ for any $1 \le p < \frac{4}{3}$, which proves (i).

Let $p \in (1, 4/3)$, $X = W^{-1,p}(\Omega) + L^1(\Omega)$. Using (i), $\partial_t u^n = \mathrm{div}(d^n \nabla u^n) + f^n$ is bounded in $L^1(0,T;X)$. Since $(u_n)_{n\in\mathbb{N}}$ is also bounded in $L^1(0,T;W^{1,p}(\Omega))$ and

$$W^{1,p}(\Omega) \overset{c}{\hookrightarrow} L^1(\Omega) \hookrightarrow X,$$

using Corollary 4 in [98], $(u^n)_{n\in\mathbb{N}}$ is relatively compact in $L^1(Q_T)$. We assumed that $(u^n)_{n\in\mathbb{N}}$ is also bounded in $L^2(Q_T)$, so using the Vitali theorem, $(u^n)_{n\in\mathbb{N}}$ is relatively compact in $L^p(Q_T)$ for any $p \in [1,2)$.

□

As a consequence of Lemma 4.4, up to a subsequence, $(c^n)_{n\in\mathbb{N}}$ converges *a.e.* in Q_T to a limit $c = (c_1, \ldots, c_P)$. Using the notations (4.8), since $(A^n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(Q_T)$ and $(W^n)_{n\in\mathbb{N}}$ is bounded in $L^2(Q_T)$, if

$$W = \sum_{i=1}^{P} a_i c_i \;;\; A = \frac{\sum_{i=1}^{P} a_i D_i(c_i)}{\sum_{i=1}^{P} a_i c_i} \,,$$

then $A \in L^\infty(Q_T)$, $W \in L^2(Q_T)$ and up to a subsequence,

$$\begin{aligned} W^n &\underset{n\to+\infty}{\longrightarrow} W \quad \text{a.e. and weakly in } L^2(Q_T)\,, \\ A^n &\underset{n\to+\infty}{\longrightarrow} A \quad \text{a.e. and strongly in } L^p(Q_T) \text{ for any } p < +\infty. \end{aligned} \tag{4.18}$$

Actually, the above convergence of A^n is sufficient to prove the relative compactness of a sequence of supersolutions of (4.9). The following Lemma generalizes Lemma 5 in [28] to the case of inhomogeneous boundary conditions.

Lemma 4.5. *Let $\overline{W}^n$ be the solution of*

$$\partial_t \overline{W}^n - \Delta[A^n \overline{W}^n] = F \text{ on } Q_T,\ \partial_\nu[A^n \overline{W}^n] = G^n \text{ on } \Sigma_T,\ \overline{W}^n(0,\cdot) = W_0^n \text{ on } \Omega. \tag{4.19}$$

Then $(\overline{W}^n)_{n\in\mathbb{N}}$ is relatively compact in $L^2(Q_T)$.

Proof. Performing the same computations as in Lemma 4.3, we know that $(\overline{W}^n)_{n\in\mathbb{N}}$ is bounded in $L^2(Q_T)$, and therefore weakly converges (up to a subsequence) to $\overline{W} \in L^2(Q_T)$. The sequence $(A^n \overline{W}^n)_{n\in\mathbb{N}}$ is also bounded in $L^2(Q_T)$, so using (4.18), up to a subsequence,

$$A^n \overline{W}^n \underset{n\to+\infty}{\longrightarrow} A\overline{W} \quad \text{weakly in } L^2(Q_T)\,.$$

After integration of (4.19) on $(0,t)$, we get

$$\overline{W}^n - \Delta \int_0^t A^n \overline{W}^n = W_0^n + \int_0^t F \text{ on } Q_T,\ \ \partial_\nu[\int_0^t A^n \overline{W}^n] = \int_0^t G^n \text{ on } \Sigma_T. \tag{4.20}$$

Using classical elliptic regularity results, $\int_0^t A^n\overline{W}^n$ is bounded (and whence weakly relatively compact) in $L^2(0,T;H^{\frac{3}{2}}(\Omega))$. Up to a subsequence, we can pass to the limit $n \to +\infty$ in (4.20), so that

$$\overline{W} - \Delta \int_0^t A\overline{W} = W_0 + \int_0^t F \text{ on } Q_T, \ \partial_\nu [\int_0^t A\overline{W}] = \int_0^t G \text{ on } \Sigma_T. \tag{4.21}$$

Taking the difference of (4.20) and (4.21), we get

$$\left\{ \begin{array}{rcll} \overline{W}^n - \overline{W} - \Delta \int_0^t [A^n\overline{W}^n - A\overline{W}] &=& W_0^n - W_0 & \text{on } Q_T, \\ \partial_\nu \int_0^t [A^n\overline{W}^n - A\overline{W}] &=& \int_0^t G^n - G & \text{on } \Sigma_T. \end{array} \right. \tag{4.22}$$

Let us prove that

$$\int_{Q_T} \left(\Delta \int_0^t A^n\overline{W}^n - A\overline{W}\right)(A^n\overline{W}^n - A\overline{W}) \le \int_{\Sigma_T} |G^n - G| \left| \int_t^T A^n\overline{W}^n - A\overline{W} \right|. \tag{4.23}$$

Formally, we have

$$\begin{aligned} &\int_{Q_T} \left(\Delta \int_0^t A^n\overline{W}^n - A\overline{W}\right)(A^n\overline{W}^n - A\overline{W}) \\ &\quad = -\frac{1}{2}\int_\Omega \left|\nabla \int_0^T A^n\overline{W}^n - A\overline{W}\right|^2 + \int_{\Sigma_T} \left(\int_0^t G^n - G\right)(A^n\overline{W}^n - A\overline{W}) \\ &\quad = -\frac{1}{2}\int_\Omega \left|\nabla \int_0^T A^n\overline{W}^n - A\overline{W}\right|^2 + \int_{\Sigma_T} (G^n - G)\left(\int_t^T A^n\overline{W}^n - A\overline{W}\right), \end{aligned} \tag{4.24}$$

whence (4.23). Since we do not know whether $\nabla(A^nW^n - AW) \in L^2(Q_T)$, the above computation must be justified by approximation. For instance, for $h > 0$, we may introduce the time average $Z_h = h^{-1}\int_t^{t+h} A^n\overline{W}^n - A\overline{W}$. Then $Z_h \in L^2(0,T;H^{\frac{3}{2}}(\Omega))$, and since $A^n\overline{W}^n - A\overline{W} \in L^2(Q_T)$ and $\int_0^t A^n\overline{W}^n - A\overline{W} \in L^2(0,T;H^{\frac{3}{2}}(\Omega))$,

$$Z_h \underset{h\to 0}{\longrightarrow} A^n\overline{W}^n - A\overline{W} \text{ in } L^2(Q_T)\,; \ \int_0^t Z_h \underset{h\to 0}{\longrightarrow} \int_0^t (A^n\overline{W}^n - A\overline{W}) \text{ in } L^2(0,T;H^{\frac{3}{2}}(\Omega)).$$

As a consequence, for *a.e.* T,

$$\int_t^T Z_h \underset{h\to 0}{\longrightarrow} \int_t^T (A^n\overline{W}^n - A\overline{W}) \text{ in } L^2(\Sigma_T).$$

Performing the same computation as in (4.24) with Z_h instead of $A^n\overline{W}^n - A\overline{W}$ and passing to the limit $h \to 0$, we get that (4.23) holds for *a.e.* T. Consequently, if we multiply (4.22) by $A^n\overline{W}^n - A\overline{W}$ and integrate on Q_T, we get

$$\begin{aligned} &\int_{Q_T} (\overline{W}^n - \overline{W})(A^n\overline{W}^n - A\overline{W}) \\ &\le \int_{\Sigma_T} |G^n - G| \left|\int_t^T A^n\overline{W}^n - A\overline{W}\right| + \int_\Omega |W_0^n - W_0| \left|\int_0^T A^n\overline{W}^n - A\overline{W}\right|. \end{aligned} \tag{4.25}$$

Since $\int_0^t A^n\overline{W}^n$ is bounded in $L^2(0,T;H^{\frac{3}{2}}(\Omega))$, at least for *a.e.* T, there exists $C > 0$ such that for all $n \in \mathbb{N}$,

$$\| \int_t^T A^n\overline{W}^n - A\overline{W}\|_{L^2(\Sigma_T)} + \| \int_0^T A^n\overline{W}^n - A\overline{W}\|_{L^2(\Omega)} \le C.$$

Using $G^n \to G$ in $L^2(\Sigma_T)$, $W_0^n \to W_0$ in $L^2(\Omega)$ and passing to the limsup in (4.25), we finally get

$$\text{for } a.e. T > 0, \quad \limsup_{n\to+\infty} \int_{Q_T} (\overline{W}^n - \overline{W})(A^n\overline{W}^n - A\overline{W}) \leq 0. \tag{4.26}$$

To derive the strong convergence of $\overline{W}^n$ in $L^2(Q_T)$, we write

$$\int_{Q_T} A^n(\overline{W}^n - \overline{W})^2 = \int_{Q_T} (\overline{W}^n - \overline{W})(A^n\overline{W}^n - A\overline{W}) + \int_{Q_T} (\overline{W}^n - \overline{W})\overline{W}(A - A^n).$$

Using Young's inequality and $\underline{d} \leq A^n$ there exists $C = C(\underline{d}) > 0$ such that

$$\underline{d}\int_{Q_T} (\overline{W}^n - \overline{W})^2 \leq \int_{Q_T} (\overline{W}^n - \overline{W})(A^n\overline{W}^n - A\overline{W}) + \frac{\underline{d}}{2}\int_{Q_T} (\overline{W}^n - \overline{W})^2 + C\int_{Q_T} \overline{W}^2(A - A^n)^2,$$
$$\frac{\underline{d}}{2}\int_{Q_T} (\overline{W}^n - \overline{W})^2 \leq \int_{Q_T} (\overline{W}^n - \overline{W})(A^n\overline{W}^n - A\overline{W}) + C\int_{Q_T} \overline{W}^2(A - A^n)^2.$$

Using $(4.18), (4.26)$ and passing to the limsup as $n \to +\infty$ on both sides,

$$\limsup_{n\to+\infty} \int_{Q_T} (\overline{W}^n - \overline{W})^2 \leq 0,$$

whence the strong convergence of $\overline{W}^n$ to $\overline{W}$ in $L^2(Q_T)$.

□

The comparison principle in equation (4.9) guarantees $0 \leq W^n \leq \overline{W}^n$. Combined with Lemma 4.7, the a.e. convergence of W^n and the Lebesgue convergence theorem, W^n converges to W strongly in $L^2(Q_T)$. Similarly, since c_i^n converges a.e. in Q_T, $0 \leq a_i c_i^n \leq W^n$ and using assumption (H3), we get

$$c^n \to c \text{ strongly in } L^2(Q_T)^P, \;\; f^n(t,x,c^n) \to f(t,x,c) \text{ strongly in } L^1(Q_T)^P.$$

According to Lemma 4.4 (i), $\nabla D_i(c_i^n)$ is bounded (and whence weakly relatively compact) in $L^\eta(Q_T)$ for any $\eta \in (1, 4/3)$. Since $D_i(c_i^n) \to D_i(c_i)$ $a.e.$ in Q_T, up to a diagonal extraction, $\nabla D_i(c_i^n) \to \nabla D_i(c_i)$ weakly in $L^\eta(Q_T)$ for any $\eta \in [1, 4/3)$. Up to another diagonal extraction, we may pass to the limit $n \to +\infty$ in the variational formulation

$$\int_{Q_T} \big(-c_i^n \partial_t \varphi_i + \nabla D_i(c_i^n)\cdot\nabla\varphi_i\big) = \int_\Omega c_{0i}^n \varphi_i(0) + \int_{Q_T} f_i^n(t,x,c^n)\varphi_i + \int_{\Sigma_T} g_i^n \varphi_i$$

for any $T > 0$, so that c satisfies (4.2).

Finally, a consequence of the so-called "L^1-contraction principle" (see $e.g.$ [101]), for all $p, q \in \mathbb{N}$, we have

$$\sup_{t\in(0,T)} \|c_i^p(t) - c_i^q(t)\|_{L^1(\Omega)} \leq \|c_{0i}^p - c_{0i}^q\|_{L^1(\Omega)} + \|f_i^p - f_i^q\|_{L^1(Q_T)} + \|g_i^p - g_i^q\|_{L^1(\Sigma_T)}, \tag{4.27}$$

which proves that $(c^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T]; L^1(\Omega)^P)$. Then $c \in C([0,T]; L^1(\Omega)^P)$ for any $T > 0$, which ends the proof of Theorem 4.1.

Remark 4.6. *Actually, since the functions c^n are regular,* (4.27) *can be easily recovered as follows: let $\alpha \in C^1(\mathbb{R})$ be a nondecreasing function such that $\alpha(0) = 0$, $-1 \le \alpha \le 1$. We multiply*

$$\partial_t(c_i^p - c_i^q) - \Delta(D_i(c_i^p) - D_i(c_i^q)) = f_i^p(t,x,c^p) - f_i^q(t,x,c^q)$$

by $\alpha(D_i(c_i^p) - D_i(c_i^q))$ and integrate by parts on Ω to get

$$\begin{aligned}\int_\Omega \alpha(D_i(c_i^p) - D_i(c_i^q))\partial_t(c_i^p - c_i^q) + \int_\Omega |\nabla(D_i(c_i^p) - D_i(c_i^q))|^2\alpha'(D_i(c_i^p) - D_i(c_i^q))\\ = \int_{\partial\Omega}(g_i^p - g_i^q)\alpha(D_i(c_i^p) - D_i(c_i^q)) + \int_\Omega (f_i^p - f_i^q)\alpha(D_i(c_i^p) - D_i(c_i^q)).\end{aligned}$$

Letting α go to the " sign" function and using $sign(D_i(c_i^p) - D_i(c_i^q)) = sign(c_i^p - c_i^q)$, we get

$$\frac{d}{dt}\int_\Omega |c_i^p - c_i^q| \le \int_\Omega |f_i^p - f_i^q| + \int_{\partial\Omega} |g_i^p - g_i^q|,$$

whence (4.27) *after integration on $(0,t)$ for any $t \in (0,T)$.*

4.3 Proof of Theorem 4.2

Outline of the proof. *Similarly as in the proof of Theorem 4.1, we build a global solution by proving the convergence of a subsequence of approximate solutions c^n from system* (4.6). *Since the initial data are controlled in $L^1(\Omega)$ "only", the proof of the $L^2(Q_T)$-estimate from Lemma 4.3 is no longer valid. The main difficulty is to localize in time this estimate. In Lemma 4.7, we show that the solutions of* (4.6) *remain a priori bounded in $L^2(Q_{\tau,T})$ for any $0 < \tau < T < +\infty$ by combining L^2-techniques inspired from [90] with smoothing effects of the heat equation in $L^\infty(\Omega)$. Then the arguments to prove the convergence of c^n on $Q_{\tau,T}$ to a function c satisfying the variational formulation* (4.4) *are similar to those used in the previous section. In particular, c^n converges a.e. on Q_T. We also prove that $(c^n)_{n\in\mathbb{N}}$ is uniformly integrable, which will provide the strong convergence of c^n in $L^1(Q_T)$ with a Vitali-type argument. It remains to check that the limit solution satisfies the prescribed initial data, and this is not an easy step. To that purpose, we use a two-sided approach inspired from [90]: on the one side, we use a truncation technique and show that there is no "mass loss" at $t = 0$ when passing to the limit $n \to \infty$ in each function $c_i(t)$. On the other side, we use the "total mass control" assumption* (H5) *to bound $\sum_{i=1}^P a_i c_i(t)$ from above. All together, this allows to prove the convergence of $c(t)$ to c_0 when $t \to 0$ in the sense of Radon measures.*

We consider the same truncated problem (4.6) as in the proof of Theorem 4.1, except that in the approximation procedure of the data, we now assume

$$c_0^n \underset{n\to+\infty}{\longrightarrow} c_0 \text{ in } L^1(\Omega)^P .$$

As before, $c^n = (c_1^n, \dots, c_P^n)$ denotes the solution of (4.6) on $[0,+\infty) \times \Omega$.

Step 1. Estimate in $L^2(Q_{\tau,T})$

Using the notations (4.8), we have

$$\partial_t W^n - \Delta(A^n W^n) \le F \text{ on } Q_T,\ \partial_\nu(A^n W^n) = G^n \text{ on } \Sigma_T,\ W^n(0,\cdot) = W_0^n \text{ on } \Omega. \tag{4.28}$$

Let $0 \le W^n \le \overline{W}^n$ be a supersolution of (4.28), satisfying

$$\partial_t \overline{W}^n - \Delta(A^n \overline{W}^n) = F \text{ on } Q_T,\ \partial_\nu(A^n \overline{W}^n) = G^n \text{ on } \Sigma_T,\ \overline{W}^n(0,\cdot) = W_0^n \text{ on } \Omega.$$

Since the above equation is linear, we can split $\overline{W}^n$ into $\overline{W}_1^n + \overline{W}_2^n$, where

$$\left.\begin{array}{l} \partial_t \overline{W}_1^n - \Delta(A^n \overline{W}_1^n) = F \ \text{ on } Q_T, \quad \partial_\nu(A^n \overline{W}_1^n) = G^n \ \text{ on } \Sigma_T, \ \ \overline{W}_1^n(0,\cdot) = 0 \quad \text{ on } \Omega, \\ \partial_t \overline{W}_2^n - \Delta(A^n \overline{W}_2^n) = 0 \ \text{ on } Q_T, \ \ \partial_\nu(A^n \overline{W}_2^n) \ = 0 \quad \text{on } \Sigma_T, \ \ \overline{W}_2^n(0,\cdot) = W_0^n \ \text{ on } \Omega. \end{array}\right\} \tag{4.29}$$

As a consequence of Lemma 4.3, $\overline{W}_1^n$ is bounded in $L^2(Q_T)$. Since the initial data are now in $L^1(\Omega)$, Lemma 4.3 is not applicable to $\overline{W}_2^n$ and it has to be localized as follows:

Lemma 4.7. *There exist a constant $C > 0$ depending only on $\underline{d}, \overline{d}, T$ and $\|W_0\|_{L^1(\Omega)}$, such that*

$$\forall \tau \in (0,T),\ \forall n \in \mathbb{N}, \quad \| \int_\tau^T A^n \overline{W}_2^n \|_{L^\infty(\Omega)} \le \frac{C}{\tau^{N/2}} \ ; \ \|\overline{W}_2^n\|_{L^2(Q_{\tau,T})} \le \frac{C}{\tau^{N/4}}. \tag{4.30}$$

Proof. In the following, any positive constant which appears and only depends on $\underline{d}, \overline{d}, T$ and $\|W_0\|_{L^1(\Omega)}$, will be denoted by C. Remark that since $W_0^n \ge 0$, $F \ge 0$, $G \le 0$ we have $\overline{W}_1^n, \overline{W}_2^n \ge 0$. Let $\tau \in (0,T)$, integration of the second equation in (4.29) on (τ, T) yields

$$-\Delta \int_\tau^T A^n \overline{W}_2^n = \overline{W}_2^n(\tau) - \overline{W}_2^n(T). \tag{4.31}$$

Set $V^n(\tau) = \int_\tau^T A^n \overline{W}_2^n$, then

$$-\Delta V^n(\tau) \le \frac{1}{A^n} A^n \overline{W}_2^n(\tau) - \overline{W}_2^n(T) \le \frac{1}{\underline{d}} A^n \overline{W}_2^n(\tau).$$

Since $\partial_\tau V^n(\tau) = -A^n \overline{W}_2^n(\tau)$, we get

$$\partial_\tau V^n - \underline{d} \Delta V^n \le 0 \text{ on } Q_T,\ \partial_\nu V^n = 0 \text{ on } \Sigma_T,\ V^n(0) = \int_0^T A^n \overline{W}_2^n \text{ on } \Omega.$$

Using the regularizing properties of the heat equation with initial data in $L^1(\Omega)$ (see [47]), the nonnegativity of V^n and the comparison principle, there exists $C > 0$ such that for all $\tau \in (0,T)$,

$$\|V^n(\tau)\|_{L^\infty(\Omega)} \le \frac{C}{\tau^{N/2}} \|V^n(0)\|_{L^1(\Omega)}. \tag{4.32}$$

Using (4.29) and the nonnegativity of $\overline{W}_2^n$, we also have

$$\overline{W}_2^n \text{ is bounded in } L^\infty(0,+\infty; L^1(\Omega)).$$

Since A^n is uniformly bounded, this yields that $V^n(0)$ is bounded in $L^1(\Omega)$ and therefore from (4.32), there exists $C > 0$ such that

$$\|V^n(\tau)\|_{L^\infty(\Omega)} = \| \int_\tau^T A^n \overline{W}_2^n \|_{L^\infty(\Omega)} \le \frac{C}{\tau^{N/2}}. \tag{4.33}$$

For $t \in (\tau,T)$, we integrate (4.29) on (τ,t), multiply by $A^n\overline{W}_2^n(t) \geq 0$ and integrate by parts on $Q_{\tau,T}$ to get

$$\int_{Q_{\tau,T}} A^n(\overline{W}_2^n)^2 + \frac{1}{2}\int_\Omega \left|\nabla \int_\tau^T A^n\overline{W}_2^n\right|^2 \leq \int_\Omega \overline{W}_2^n(\tau)\int_\tau^T A^n\overline{W}_2^n$$
$$\leq \|\overline{W}_2^n(\tau)\|_{L^1(\Omega)}\|V^n(\tau)\|_{L^\infty(\Omega)}.$$

Using (4.32), (4.33) and $\underline{d} \leq A^n$, we finally get

$$\underline{d}\int_{Q_{\tau,T}} (\overline{W}_2^n)^2 \leq \int_{Q_{\tau,T}} A^n(\overline{W}_2^n)^2 + \frac{1}{2}\int_\Omega \left|\nabla \int_\tau^T A^n\overline{W}_2^n\right|^2 \leq \frac{C}{\tau^{N/2}}, \tag{4.34}$$

which ends the proof of Lemma 4.7.

□

Step 2. Convergence in $L^1(Q_T)$ and estimation of the gradients

According to Lemma 4.7, $(c^n)_{n\in\mathbb{N}}$ is bounded in $L^2(Q_{\tau,T})$; using (4.13), it is also bounded in $L^\infty(0,T;L^1(\Omega))$. This actually yields that $(c^n)_{n\in\mathbb{N}}$ is uniformly integrable on Q_T :

Lemma 4.8. *Let $(u^n)_{n\in\mathbb{N}}$ be a bounded sequence of $L^\infty(0,T;L^1(\Omega))$, and assume that $(u^n)_{n\in\mathbb{N}}$ is bounded in $L^2(Q_{\tau,T})$ for any $\tau \in (0,T)$. Let $\mathcal{B}(Q_T)$ be the Borel algebra on Q_T and λ denote the Lebesgue measure on $\mathcal{B}(Q_T)$. Then*

$$\forall \varepsilon > 0, \exists \eta > 0 : \forall A \in \mathcal{B}(Q_T), \quad \lambda(A) < \eta \Rightarrow \forall n \in \mathbb{N}, \int_A |u^n| < \varepsilon. \tag{4.35}$$

Proof. Let $\varepsilon > 0$, $A \in \mathcal{B}(Q_T)$, $\tau \in (0,T)$. Let C_1, $C_2(\tau) > 0$ such that

$$\forall n \in \mathbb{N}, \quad \operatorname*{supess}_{t\in(0,T)} \|u^n(t)\|_{L^1(\Omega)} \leq C_1, \;\; \|u^n\|_{L^2(Q_{\tau,T})} \leq C_2(\tau).$$

We have

$$\begin{aligned}\int_A |u^n| &= \int_{A\cap Q_{0,\tau}} |u^n| + \int_{A\cap Q_{\tau,T}} |u^n| \\ &\leq \tau C_1 + \sqrt{\lambda(A)}\|u^n\|_{L^2(Q_{\tau,T})} \\ &\leq \tau C_1 + \sqrt{\lambda(A)}C_2(\tau).\end{aligned}$$

Choosing $\tau = \frac{\varepsilon}{2C_1}$ and $\eta = (\frac{\varepsilon}{2C_2(\tau)})^2$ yields (4.35).

□

Let $\tau \in (0,T)$, using Lemma 4.7 and the quadratic growth assumption (H3), for all $i \in \{1,\dots,P\}$, $(t,x) \mapsto f_i^n(t,x,c_i^n(t,x))$ is bounded in $L^1(Q_{\tau,T})$. We have

$$\partial_t c_i^n - \Delta D_i(c_i^n) = f_i^n(t,x,c^n) \text{ on } Q_{\tau,T} \; ; \; \partial_\nu D_i(c_i^n) = g_i^n \text{ on } \Sigma_{\tau,T}. \tag{4.36}$$

Since $c_i^n(\tau,\cdot)$ is bounded in $L^1(\Omega)$, we can apply Lemma 4.4 (i) and the fact that

$$(c_i^n)_{n\in\mathbb{N}} \text{ is bounded in } L^p(\tau,T;W^{1,p}(\Omega)) \text{ for any } 1 \leq p < \tfrac{4}{3}. \tag{4.37}$$

Similarly as in the proof of Lemma 4.7 (ii), we can apply Simon's compactness results [98] to get the relative compactness of $(c_i^n)_{n\in\mathbb{N}}$ in $L^1(Q_{\tau,T})$. The choice of $\tau \in (0,T)$ is arbitrary, so up to a subsequence, we can assume that c^n converges a.e. in Q_T. Applying Lemma 4.8, $(c^n)_{n\in\mathbb{N}}$ satisfies the uniform integrability property (4.35), so the Vitali theorem guarantees that it converges in $L^1(Q_T)$.

Step 3. Convergence in $L^2(Q_{\tau,T})$

The main idea of the proof of the convergence of $(W^n)_{n\in\mathbb{N}}$ in $L^2(Q_{\tau,T})$ for the strong topology is similar to what is done in Lemma 4.5. We only indicate what changes should be made in the present situation.

We use the decomposition $\overline{W}^n = \overline{W}_1^n + \overline{W}_2^n$ introduced in (4.29). With a similar proof as what is done in Lemma 4.5, we can prove that $\overline{W}_1^n$ converges strongly in $L^2(Q_T)$. For $\overline{W}_2^n$, we perform the same computations as in the proof of Lemma 4.5, with Q_T replaced by $Q_{\tau,T}$. Equation (4.25) becomes, since $\overline{W}_2^n$ has homogeneous boundary conditions:

$$\int_{Q_{\tau,T}} (\overline{W}_2^n - \overline{W}_2)(A^n\overline{W}_2^n - A\overline{W}_2) \leq \int_\Omega |W_2^n(\tau) - W_2(\tau)| \left| \int_\tau^T A^n\overline{W}_2^n - A\overline{W}_2 \right|. \tag{4.38}$$

Since $\overline{W}^n$ converges in $L^1(Q_T)$ and $\overline{W}_2^n$ converges in $L^2(Q_T)$, $\overline{W}_2^n = \overline{W}^n - \overline{W}_1^n$ converges in $L^1(Q_T)$. In particular, at least for a.e. τ and up to a subsequence, $\overline{W}_2^n(\tau)$ converges in $L^1(\Omega)$ to $\overline{W}_2(\tau)$. Using Lemma 4.7, $\int_\tau^T A^n\overline{W}_2^n$ is uniformly bounded in $L^\infty(\Omega)$, so the right-hand side in (4.38) converges to 0. We may continue as in Lemma 4.5 to prove that $\overline{W}_2^n$ converges in $L^2(Q_{\tau,T})$. Then we use the Lebesgue convergence theorem to prove that $W^n \to W$ in $L^2(Q_{\tau,T})$, $f_i^n(t,x,c^n) \to f_i(t,x,c)$ in $L^1(Q_{\tau,T})$. Together with (4.37), this allows to pass to the limit $n \to +\infty$ in the weak formulation (4.4).

Step 4. Convergence of the initial data

Similarly as in (4.27), up to a subsequence and for a.e. $0 < \tau < T < +\infty$, we have for $p,q \in \mathbb{N}$,

$$\sup_{t\in(\tau,T)} \|c_i^p(t) - c_i^q(t)\|_{L^1(\Omega)} \leq \|c_i^p(\tau) - c_i^q(\tau)\|_{L^1(\Omega)} + \|f_i^p - f_i^q\|_{L^1(Q_{\tau,T})} + \|g_i^p - g_i^q\|_{L^1(\Sigma_{\tau,T})}.$$

Then $(c^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([\tau,T];L^1(\Omega)^P)$ and consequently,

$$c^n \to c \text{ in } C((0,+\infty);L^1(\Omega)^P). \tag{4.39}$$

In the following, $\mathcal{M}(\overline{\Omega})$ denotes the space of Radon measures on $\overline{\Omega}$, *i.e.* the topological dual space of the separable space $(C(\overline{\Omega}), \|\cdot\|_\infty)$. Since $c \in L^\infty(0,+\infty;L^1(\Omega))$, $\{c(t), t \in (0,1)\}$ is relatively compact for the weak-$*$ topology on $\mathcal{M}(\overline{\Omega})$. Let $(t_m)_{m\in\mathbb{N}}$ be a decreasing sequence of positive numbers such that $t_m \to 0$, assume that $c_i(t_m) \to \mu_i$ for the weak-$*$ topology on $\mathcal{M}(\overline{\Omega})$, and let us prove that $\mu_i = c_i^0$.

Using $(H5)$, for all $m \in \mathbb{N}$,

$$\sum_{i=1}^P a_i c_i^n(t_m) - \sum_{i=1}^P a_i c_{0i}^n \leq \Delta \int_0^{t_m} \sum_{i=1}^P a_i D_i(c_i^n) + \int_0^{t_m} F. \tag{4.40}$$

For the left-hand side, using (4.39),

$$\sum_{i=1}^P a_i c_i^n(t_m) - \sum_{i=1}^P a_i c_{0i}^n \underset{n\to+\infty}{\longrightarrow} \sum_{i=1}^P a_i c_i(t_m) - \sum_{i=1}^P a_i c_{0i} \underset{m\to+\infty}{\longrightarrow} \sum_{i=1}^P a_i \mu_i - \sum_{i=1}^P a_i c_{0i}.$$

We now consider the right-hand side of (4.40): let $\varphi \in C^\infty(\overline{\Omega})^+$ such that $\partial_\nu \varphi = 0$ on $\partial\Omega$. Multiplication by φ and integration by parts on Ω yields

$$\int_\Omega [\Delta \int_0^{t_m} \sum_{i=1}^P a_i D_i(c_i^n) + \int_0^{t_m} F]\varphi = \int_{Q_{t_m}} \left(\sum_{i=1}^P a_i D_i(c_i^n) \right) \Delta\varphi + F\varphi + \int_{\Sigma_{t_m}} G^n \varphi.$$

We pass to the limit $n \to +\infty$ to get

$$\int_\Omega [\Delta \int_0^{t_m} \sum_{i=1}^P a_i D_i(c_i) + \int_0^{t_m} F]\varphi = \int_{Q_{t_m}} \left(\sum_{i=1}^P a_i D_i(c_i)\right) \Delta\varphi + F\varphi + \int_{\Sigma_{t_m}} G\varphi,$$

and then it is clear that the right-hand side goes to zero when $m \to +\infty$. Multiplying (4.40) by $\varphi \in \mathcal{A} = \{\varphi \in C^\infty(\overline{\Omega})^+ : \partial_\nu \varphi = 0 \text{ on } \partial\Omega\}$, and passing to the limit $n \to +\infty$ and then $m \to +\infty$, we get

$$\forall \varphi \in \mathcal{A}, \quad \int_\Omega [\sum_{i=1}^P a_i \mu_i]\varphi \le \int_\Omega [\sum_{i=1}^P a_i c_{0i}]\varphi.$$

Since Ω is smooth, $\mathcal{A}$ is a dense subset of $(C(\overline{\Omega})^+, \|\cdot\|)$, and consequently

$$\sum_{i=1}^P a_i \mu_i \le \sum_{i=1}^P a_i c_{0i}. \tag{4.41}$$

To estimate μ_i from below, let $k \in \mathbb{N}$ and define $T_k \in C^\infty([0,+\infty))$ such that

$$T_k(r) = r \text{ for } 0 \le r \le k\,;\ 0 \le T_k' \le 1\,;\ T_k'' \le 0\,;\ \|T_k\|_{L^\infty(\mathbb{R}_+)} \le k+1, \tag{4.42}$$

and

$$\hat{T}_k : [0,+\infty) \to [0,+\infty), \quad r \mapsto \int_0^r T_k' \circ D_i^{-1}(s)\, ds.$$

We will also use the notation $p_i(c) = (c_1, \ldots, c_{i-1}, 0, c_{i+1}, \ldots, c_P)$. Remark that since D_i is an increasing C^1-diffeomorphism, $\hat{T}_k'' \le 0$. As c_i^n is a solution of (4.6), we can write for $i \in \{1, \ldots, P\}$,

$$\begin{aligned}
\partial_t T_k(c_i^n) - \Delta[\hat{T}_k(D_i(c_i^n))] &= T_k'(c_i^n)[\partial_t c_i^n - \Delta(D_i(c_i^n))] - \hat{T}_k''(D_i(c_i^n))|\nabla D_i(c_i^n)|^2 \\
&= T_k'(c_i^n) f_i^n(t,x,c^n) - \hat{T}_k''(D_i(c_i^n))|\nabla D_i(c_i^n)|^2 \\
&\ge T_k'(c_i^n)\, f_i^n(t,x,c^n) \\
&= T_k'(c_i^n)[f_i^n(t,x,c^n) - f_i^n(t,x,p_i(c^n))] + T_k'(c_i^n) f_i^n(t,x,p_i(c^n)) \\
&\ge -T_k'(c_i^n)\beta(c_i^n)(1 + \sum_{j\ne i} c_j^n),
\end{aligned}$$

where we used assumptions $(H2)$ and $(H6)$ for the last inequality. We integrate on $(0,t)$ to get

$$T_k(c_i^n) - T_k(c_{0i}^n) - \int_0^t \Delta[\hat{T}_k(D_i(c_i^n))] + \int_0^t T_k'(c_i^n)\beta(c_i^n)(1 + \sum_{j\ne i} c_j^n) \ge 0.$$

After multiplication by $\varphi \in C^\infty(\overline{\Omega})^+$ and integration by parts on Ω, we get

$$\begin{aligned}
\int_\Omega (T_k(c_i^n) - T_k(c_{0i}^n))\varphi - \int_{Q_t} \hat{T}_k(D_i(c_i^n))\Delta\varphi + \int_{Q_t} T_k'(c_i^n)\beta(c_i^n)(1 + \sum_{j\ne i} c_j^n)\varphi \qquad (4.43)\\
\ge \int_{\Sigma_t} T_k(c_i^n)[g_i^n - \partial_\nu \varphi] \\
\ge -(k+1)t \int_{\partial\Omega} |\partial_\nu \varphi|.
\end{aligned}$$

Recall that T_k' has a compact support, so $T_k'(c_i^n)\beta(c_i^n)$ is bounded in $L^\infty(Q_T)$ independently of n. Since $c^n \to c$ in $L^1(Q_T)^P$ and in $C(0,T;L^1(\Omega)^P)$, we can pass to the limit $n \to +\infty$ in (4.43) to get

$$\int_\Omega (T_k(c_i) - T_k(c_{0i}))\varphi + \int_{Q_t} -\hat{T}_k(D_i(c_i))\Delta\varphi + T_k'(c_i)\beta(c_i)(1+\sum_{j\neq i} c_j)\varphi \geq -(k+1)t\int_{\partial\Omega} |\partial_\nu \varphi|.$$

Choosing $t = t_m$ and using $c_i(t) \geq T_k(c_i(t))$,

$$\int_\Omega (c_i(t_m) - T_k(c_{0i}))\varphi \geq \int_{Q_{t_m}} \hat{T}_k(D_i(c_i))\Delta\varphi - T_k'(c_i)\beta(c_i)(1+\sum_{j\neq i} c_j)\varphi - (k+1)t_m\int_{\partial\Omega} |\partial_\nu \varphi|.$$

The right-hand side goes to 0 as $m \to +\infty$, so

$$\forall k \in \mathbb{N},\ \forall i \in \{1,\dots,P\},\ \forall \varphi \in C^\infty(\overline{\Omega}), \quad \int_\Omega \mu_i \varphi \geq \int_\Omega T_k(c_{0i})\varphi,$$

and since $C^\infty(\overline{\Omega})$ is dense in $C(\overline{\Omega})$, passing to the limit $k \to +\infty$ we finally get

$$\mu_i \geq c_{0i}\ , \quad i \in \{1,\dots,P\}.$$

Combined with (4.41), this yields $\mu = c_0$. Then $\{c(t), t > 0\}$ is relatively compact in the space of Radon measures for the weak-$*$ topology, c_0 is the only possible limit point when $t \to 0$, so $c(t) \to c_0$ for the weak-$*$ topology on $\mathcal{M}(\Omega)$, *i.e.* in the sense (4.5).

4.4 Remarks

- ◇ The main obstacle to have a better control of the solutions in a neighborhood of $t = 0$ is that we are not able to control the quadratic reaction terms $f_i(t,x,c)$ in $L^1(Q_T)$. The question of the continuity of $t \mapsto c(t)$ in $L^1(\Omega)$ up to $t = 0$ remains open for initial data in $L^1(\Omega)$.
- ◇ The estimate $\|W^n\|_{L^2(Q_{\tau,T})} \leq \frac{C}{\tau^{N/4}}$ in Lemma 4.7 may be improved as follows. To simplify the writings, we choose homogeneous Neumann boundary conditions and set $F = 0$. Rewriting equation (4.31) with $\tau = 0$, we have

  $$-\Delta V(0) = W_0 - W(T) \leq W_0 \text{ on } \Omega\ ;\ \partial_\nu V(0) = 0 \text{ on } \partial\Omega.$$

  Using the nonnegativity of V and classical elliptic regularity results, $V(0)$ can be estimated in terms of $\|W_0\|_{L^1(\Omega)}$ not only in $L^1(\Omega)$, but also in $L^p(\Omega)$ for any $p \in [1, \frac{N}{N-2})$. Then we may use the regularizing properties of the heat equation in Lemma 4.7 with initial data in $L^p(\Omega)$ instead of $L^1(\Omega)$, and the exponent $\frac{N}{2}$ in equation (4.33) can be replaced by $p < \frac{N}{2} - 1$ (see *e.g.* [102]), which yields

  $$\|W^n\|_{L^2(Q_{\tau,T})} \leq \frac{C}{\tau^p}\ ;\ p < \frac{N}{4} - \frac{1}{2}.$$
- ◇ Theorems 4.1 and 4.2 are also valid for Dirichlet boundary conditions, with a similar proof. The main reason is that the above lemmas are results on linear equations, which can easily be reduced to homogeneous Dirichlet boundary conditions. Then the same results as above can be recovered with similar computations (without the boundary terms). What should be adapted is the sense in which the boundary conditions are satisfied. For instance, for Theorem 4.1, using the same approximation procedure as above, we get the same compactness results. In particular,

since $D_i(c_i^n)$ is relatively compact in $L^p(0,T;W^{1,p}(\Omega))$ for $p \in [1,4/3)$, we see that we can pass to the limit $n \to +\infty$ for the boundary conditions in a pointwise sense:

$$D_i(c_i^n) \underset{n\to+\infty}{\longrightarrow} D_i(c_i) \text{ weakly in } L^p(0,T;W^{1-1/p,p}(\Omega)),\ p \in (1,\frac{4}{3}).$$

Then the conclusion of Theorem 4.1 remains valid, where assertion (ii) should be rewritten as
$(ii)'$ $D_i(c_i) = g_i$ on Σ_T ; $\forall \varphi_i \in C^\infty([0,T];C_c^\infty(\Omega))$ such that $\varphi_i(T) = 0$,

$$\int_{Q_T} -c_i\partial_t\varphi_i + \nabla D_i(c_i)\cdot\nabla\varphi_i = \int_\Omega c_{0i}\varphi_i(0) + \int_{Q_T} f_i(t,x,c)\varphi_i.$$

Part III

Reaction-diffusion systems with advection-migration

5

Global well-posedness for reaction-diffusion-advection systems with a "triangular" reaction

We prove global existence and uniqueness of global solutions for a class of reaction-diffusion systems whose reactions have a "triangular" structure. Our result generalizes a theorem of M. Pierre to the case when the diffusion coefficients depend on time and space. We also introduce advection terms, where the fluid's motion is a given data. As an application, we derive global existence for a class of reaction-diffusion systems from mass-action kinetics chemistry.

5.1 Introduction

Let Ω be a smooth bounded domain of $\mathbb{R}^N$ and consider the system

$$\left\{\begin{array}{lcl} \partial_t c_1 + \mathrm{div}[-d_1(t,x)\nabla c_1 + c_1 u(t,x)] & = & -c_1c_2 + c_3 \\ \partial_t c_2 + \mathrm{div}[-d_2(t,x)\nabla c_2 + c_2 u(t,x)] & = & -c_1c_2 + c_3 \\ \partial_t c_3 + \mathrm{div}[-d_3(t,x)\nabla c_3 + c_3 u(t,x)] & = & +c_1c_2 - c_3 \end{array}\right. \quad \text{on } (0,+\infty)\times\Omega\,, \tag{5.1}$$

together with bounded nonnegative initial data and no-flux boundary conditions. Due to the presence of quadratic reaction terms, the existence of global solutions for this system is not obvious. However, one can notice that for nonnegative functions c_i, the reaction term for the two first equations are (linearly) bounded above by c_3. Moreover, the reaction terms cancel when considering the sum $c_1 + c_3$. Since the reaction terms for c_1, c_2 and $c_1 + c_3$ are linearly bounded above, this system has what we call a "triangular" structure, which is crucial to derive global existence. Several results are available in the case of constant functions d_i and $u = 0$. It was shown in [94] for space dimensions $N \leq 5$ that (5.1) has a unique nonnegative classical solution. Global wellposedness in any space dimension for smooth Ω ($C^{2+\alpha}$, $0 < \alpha < 1$) and smooth initial data has been shown in [48]. Both these approaches are based on semigroup theory, and do not seem to be easily extendable to the case of time-dependent diffusions and convection. More general systems with the "triangular" structure have been studied in [90], where global existence and uniqueness of strong solutions for any space dimension and bounded initial data is proven. This is the approach we chose to extend: in the present work, we generalize Theorem 3.5 in [90] to the case of more general mass fluxes, where the Fickian diffusion coefficients might depend on time and space, and with advection terms. However, the vector field describing the fluid's motion is assumed to be a given data.

Let us describe in more details the class of systems we are interested in. Throughout this work, let Ω be a bounded subset of $\mathbb{R}^N$, whose boundary $\partial\Omega$ is of class C^2. For $T > 0$, we write $Q_T = \Omega \times (0,T)$, $\Sigma_T = \partial\Omega \times (0,T)$. We denote by ν the normal exterior vector on $\partial\Omega$, $\partial_\nu c$ is the normal exterior derivative of a function c. If (X,d) is a metric space, the modulus of continuity of a function $h : X \to \mathbb{R}$ is defined as

$$\omega_h : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}, \quad \delta \mapsto \sup_{d(x,y)\le\delta} |h(x) - h(y)|. \tag{5.2}$$

For $i \in \{1,\dots,P\}$, consider

$$\left\{\begin{array}{rcll} \partial_t c_i + \operatorname{div}[-d_i(t,x)\nabla c_i + c_i u_i(t,x)] &=& f_i(t,x,c) & \text{on } (0,+\infty)\times\Omega, \\ -d_i(t,x)\nabla c_i \cdot \nu + c_i u_i(t,x)\cdot\nu &=& 0 & \text{on } (0,+\infty)\times\partial\Omega, \\ c_i(0,\cdot) &=& c_i^0 & \text{on } \Omega. \end{array}\right. \tag{5.3}$$

Letting $r > \max(2,N)$, we require

(i) $c^0 = (c_1^0,\dots,c_P^0) \in L^\infty(\Omega,[0,+\infty)^P)$.

(ii) $d_i \in C([0,+\infty)\times\overline{\Omega};(0,+\infty))$; $\nabla d_i \in L^\infty_{loc}([0,+\infty);L^r(\Omega)^N)$.

(iii) $u_i \in L^\infty_{loc}([0,+\infty);L^r(\Omega)^N)$.

(iv) $f \in C^1([0,+\infty)\times\Omega\times\mathbb{R}^P,\mathbb{R}^P)$; f is quasi-positive, *i.e.*

$$f_i(t,x,y) \ge 0 \text{ for any } (t,x,y) \in (0,+\infty)\times\Omega\times[0,+\infty)^P \text{ such that } c_i = 0.$$

(v) There exists a lower triangular invertible matrix $Q = (q_{ij})_{1\le i,j\le P}$ with nonnegative diagonal entries and $b \in \mathbb{R}_+^P$ such that

$$\forall (t,x,y) \in [0,+\infty)\times\Omega\times[0,+\infty)^P, \quad Qf(t,x,y) \le (1+\sum_{j=1}^P y_j)b.$$

(vi) f has at most a polynomial growth with respect to the last variable, *i.e.*

$$\forall T > 0,\ \exists C,p > 0 : \forall i,\ \forall (t,x,y) \in Q_T\times[0,+\infty)^P, \quad |f_i(t,x,y)| \le C(1+|y|^p).$$

Moreover, we define

$$\underline{d}(T) := \min_{i=1,\dots,P} \inf_{(t,x)\in Q_T} d_i(t,x) \quad ; \quad \overline{d}(T) := \max_{i=1,\dots,P} \sup_{(t,x)\in Q_T} d_i(t,x), \tag{5.4}$$

so that using (ii),

$$0 < \underline{d}(T) \le d_i(t,x) \le \overline{d}(T) < +\infty \text{ for } (t,x) \in Q_T.$$

Our main result is the following:

Theorem 5.1. *Under assumptions $(i)-(vi)$, system* (5.3) *has a unique global nonnegative solution $c=(c_1,\dots,c_P)$ in the following sense:*

$$\left.\begin{array}{l} \forall T>0,\ \forall i\in\{1,\dots,P\},\ c_i\in C([0,T];L^2(\Omega))\cap L^\infty(Q_T)\cap L^2(0,T;W^{1,2}(\Omega))\ ; \\ \forall \psi\in C^\infty(\overline{Q_T}) \text{ such that } \psi(T)=0, \\ \qquad -\displaystyle\int_\Omega c_i^0\psi(0)+\int_{Q_T}\big(-c_i\partial_t\psi+(d_i\nabla c_i-c_iu_i)\nabla\psi\big)=\int_{Q_T} f_i\psi. \end{array}\right\} \tag{5.5}$$

Moreover, for any $T>0$, there exists $C>0$ depending only on

$$T, \|c^0\|_{L^\infty(\Omega)}, \underline{d}(T), \overline{d}(T), \omega_{d_i}, \|\nabla d_i\|_{L^\infty(0,T;L^r(\Omega))}, \|u_i\|_{L^\infty(0,T;L^r(\Omega))}, \tag{5.6}$$

such that

$$\|c\|_{L^\infty(Q_T)}+\|c\|_{L^2(0,T;W^{1,2}(\Omega))}+\|\partial_t c\|_{L^2(0,T;W^{-1,2}(\Omega))}\le C. \tag{5.7}$$

Scheme of the proof. We first state a global existence result under extra regularity assumptions on the data, based on a local existence theorem from [2]. This result is interesting in itself since it also provides extra regularity on the solution. Global existence is shown by proving that any solution is *a priori* bounded in $L^\infty(Q_T)$ for any $T>0$. We first derive bounds in $L^p(Q_T)$ for any finite p by a duality method, where Maximal Regularity theory plays a crucial role. Since the $L^\infty(Q_T)$-bounds only require assumptions $(i)-(vi)$ on the data, using an approximation procedure, we get the existence of weak solutions for non-smooth coefficients. The bounds in $L^\infty(Q_T)$ are a consequence of $L^p(Q_T)$-bounds that we get on the solutions for any finite p, obtained by duality. Finally, we prove that these solutions, although rather weak, are unique.

In Section 5.2, we prove global wellposedness for a regularized version of system (5.3). In Section 5.3, we use the result of the previous section to prove Theorem 5.1. As an example, we show in Section 5.4 global well-posedness for a class of reaction-diffusion-convection systems arising in mass-action kinetics chemistry. Finally, since Section 5.2 relies on a theorem on parabolic equations proved in [69] in the case of Dirichlet boundary conditions only, we give a proof of this result for the Neumann case in an Appendix.

5.2 Global existence for an approximate system

Proposition 5.2. *In addition to $(i)-(vi)$, assume*

$$d_i\in C^2([0,+\infty)\times\overline{\Omega},\mathbb{R}_+)\ ; u_i\in C^2([0,+\infty)\times\overline{\Omega},\mathbb{R}^N)\ ;\ c^0\in C^2(\overline{\Omega},\mathbb{R}_+^P).$$

Then system (5.3) *has a unique global classical nonnegative solution*

$$c=(c_1,\dots,c_P)\in C([0,+\infty);C(\overline{\Omega}))\cap C^1((0,+\infty);C(\overline{\Omega}))\cap C((0,+\infty);C^2(\overline{\Omega})),$$

and it satisfies estimates (5.7).

As in [90], global existence is based on L^p-estimates obtained by duality:

Lemma 5.3. *Let $u_1,u_2\in C([0,T]\times\overline{\Omega},\mathbb{R}^N)$, d_1,d_2 satisfying (ii) and let w, z be smooth functions such that*

$$\begin{cases} \partial_t w+\operatorname{div}(-d_1\nabla w+wu_1)\le\theta_1\partial_t z+\theta_2\operatorname{div}(-d_2\nabla z+zu_2)+\theta_3 z+H & \text{on } Q_T, \\ -d_1\nabla w\cdot\nu+wu_1\cdot\nu=-d_2\nabla z\cdot\nu+zu_2\cdot\nu=0 & \text{on } \Sigma_T, \\ w(0,\cdot)=w^0\ ;\ z(0,\cdot)=z^0 & \text{on } \Omega, \end{cases} \tag{5.8}$$

where $\theta_i \in \mathbb{R}$, $w^0, z^0 \in L^\infty(\Omega)$, $H \in L^p(Q_T)$ for some $\frac{r}{r-1} < p < +\infty$, $r > \max\{2, N\}$. Then for any $T > 0$, there exists $C > 0$ depending only on

$$p,\ T,\ \underline{d}(T),\ \overline{d}(T),\ \omega_{d_1},\ \|\nabla d_i\|_{L^\infty(0,T;L^r(\Omega))},\ \|u_i\|_{L^\infty(0,T;L^r(\Omega))},\ \|w^0\|_{L^\infty(\Omega)} \text{ and } \|z^0\|_{L^\infty(\Omega)}\,, \tag{5.9}$$

such that for all $t \in (0,T)$,

$$\|w^+\|_{L^p(Q_t)} \le C\left(1 + \|z\|_{L^p(Q_t)} + \int_0^t \|H(s)\|_{L^p(\Omega)} ds\right). \tag{5.10}$$

Proof. Let $\Theta \in C_0^\infty(Q_T)^+$ and for $t \in (0,T)$, consider the dual problem

$$-[\partial_t \Psi + \operatorname{div}(d_1 \nabla \Psi) + u_1 \nabla \Psi] = \Theta \text{ on } Q_t \;\; ; \;\; \partial_\nu \Psi = 0 \text{ on } \Sigma_t \;\; ; \;\; \Psi(t,\cdot) = 0 \text{ on } \Omega. \tag{5.11}$$

Let $\frac{r}{r-1} < p < +\infty$, $p' = p/(p-1) < r$. It is known that (5.11) has a unique nonnegative solution Ψ^t, which satisfies the following “Maximal Regularity” estimates (see [38, Theorem 2.1]): there exists $C > 0$, depending only on the parameters indicated in (5.9), such that for all $t \in (0,T]$,

$$\|\Psi^t\|_{W^{1,p'}(0,t;L^{p'}(\Omega))} + \|\Psi^t\|_{L^{p'}(0,t;W^{2,p'}(\Omega))} \le C\|\Theta\|_{L^{p'}(Q_t)}. \tag{5.12}$$

As a consequence, $\Psi^t \in C([0,t]; L^{p'}(\Omega))$, and for $s \in (0,t)$,

$$\|\Psi^t(s)\|_{L^{p'}(\Omega)} = \left\|\int_s^t \partial_t \Psi^t\right\|_{L^{p'}(\Omega)} \le \int_0^T \left\|\partial_t \Psi^t\right\|_{L^{p'}(\Omega)} \le T^{1/p} \left\|\partial_t \Psi^t\right\|_{L^{p'}(Q_t)}. \tag{5.13}$$

Using the Sobolev embedding theorem, we also have

$$\|\nabla \Psi^t\|_{L^{p'}(0,T;L^q(\Omega))} \le C\|\Theta\|_{L^{p'}(Q_t)} \text{ for } \frac{1}{q} = \frac{1}{p'} - \frac{1}{N}. \tag{5.14}$$

Combining (5.12), (5.13), (5.14) and using $u_2, \nabla d_2 \in L^\infty(0,T;L^r(\Omega))$ with $r > N$,

$$\sup_{s\in[0,t]} \|\Psi^t(s)\|_{L^{p'}(\Omega)} + \|u_2 \cdot \nabla \Psi^t\|_{L^{p'}(Q_t)} + \|\operatorname{div}(d_2 \nabla \Psi^t)\|_{L^{p'}(Q_t)} \le C\|\Theta\|_{L^{p'}(Q_t)},$$

where $C > 0$ depends only on the parameters in (5.9), but not on t. After multiplying inequality (5.8) by $\Psi^t \ge 0$ and integrating by parts, we get

$$\begin{aligned}
\int_{Q_t} w\Theta &= \int_\Omega \Psi^t(0) w^0 + \int_{Q_t} \Psi^t[\partial_t w - \operatorname{div}(d_1 \nabla w - w u_1)] \\
&\le \int_\Omega \Psi^t(0) w^0 + \int_{Q_t} \Psi^t[\theta_1 \partial_t z + \theta_2 \operatorname{div}(d_2 \nabla z - z u_2) + \theta_3 z + H] \\
&= \int_\Omega \Psi^t(0)(w^0 - \theta_1 z^0) + \int_{Q_t} [-\theta_1 \partial_t \Psi^t + \theta_2 \operatorname{div}(d_2 \nabla \Psi^t) + \theta_2 u_2 \nabla \Psi^t + \theta_3 \Psi^t] z + \int_{Q_t} \Psi^t H \\
&\le C\|\Theta\|_{L^{p'}(Q_t)}(1 + \|z\|_{L^p(Q_t)} + \int_0^t \|H(s)\|_{L^p(\Omega)} ds),
\end{aligned}$$

and since $\Theta \in C_0^\infty(Q_T)^+$ is arbitrary, (5.10) holds by duality.

□

Remark 5.4. *The surprising condition $\frac{r}{r-1} < p$ or equivalently $p' < r$ in Lemma 5.3 is needed for maximal $L^{p'}$-regularity in problem (5.11) by virtue of [38, Theorem 2.1], cf. condition (SD). This is not restrictive for our purpose since we will apply Lemma 5.3 for p large. Also remark that the conditions $\nabla d_i, u_i \in L^\infty(0,T;L^r(\Omega))$ for some $r > \max(2,N)$ are required to use the results of [38].*

In order to apply a Gronwall argument in the proof of Theorem 5.1, we also need to control $\|c_i(t)\|_{L^p(\Omega)}$ in terms of the $L^p(Q_t)$-norm of the right-hand side, for any $1 < p < +\infty$. In the case of the heat equation $\partial_t c - d\Delta c = f$ with constant diffusivity $d > 0$ and no-flux boundary conditions, the solution can be represented by the so-called "variation-of-constant" formula:

$$c(t) = S(t)c^0 + \int_0^t S(t-s)f(s)ds,$$

where S is the semigroup generated by the operator $A = -d\Delta$ with Neumann boundary conditions, and we have

$$\forall t > 0, \quad \|c(t)\|_{L^p(\Omega)} \le \|c^0\|_{L^p(\Omega)} + \int_0^t \|f\|_{L^p(\Omega)}.$$

In our case, the convection terms and the dependence in (t,x) of the diffusivities prevent us from using semigroup theory to derive such an estimate. Instead, we use the following lemma:

Lemma 5.5. *Let $1 < p < \infty$, $f \in L^p(Q_T)$, $d : Q_T \to \mathbb{R}_+$ measurable and such that $\underline{d} \le d$ for some $\underline{d} > 0$, $r > \max(2,N)$, $u \in L^\infty(0,T;L^r(\Omega)^N)$, $c^0 \in L^\infty(\Omega)$. Let c be a nonnegative classical solution of*

$$\left\{\begin{array}{rcll} \partial_t c + \operatorname{div}(-d\nabla c + cu) &=& f & \text{on } Q_T, \\ -d\nabla c \cdot \nu + cu \cdot \nu &=& 0 & \text{on } \Sigma_T, \\ c(0,\cdot) &=& c^0 & \text{on } \Omega. \end{array}\right. \tag{5.15}$$

Then there exists a constant $C > 0$ depending only on $p, T, \underline{d}$ and $\|u\|_{L^\infty(0,T;L^r(\Omega)^N)}$, such that

$$\forall t \in (0,T), \quad \|c(t)\|^p_{L^p(\Omega)} \le C\left(\|c^0\|^p_{L^p(\Omega)} + \int_0^t \|f(s)\|^p_{L^p(\Omega)} ds\right). \tag{5.16}$$

Proof. Since (5.15) is linear, we can split c^0 and f into their positive and negative parts to go down without loss of generality to the case $c^0, f \ge 0$, and hence $c \ge 0$. Up to a change of c into $e^{-t}c$ and f into $e^{-t}f$, we also go down to

$$\partial_t c + c + \operatorname{div}(-d\nabla c + cu) = f \text{ on } Q_T. \tag{5.17}$$

Multiplying (5.17) by pc^{p-1} and integrating by parts on Ω yields

$$\frac{d}{dt}\int_\Omega c^p + p\int_\Omega c^p + p(p-1)\int_\Omega d|\nabla c|^2 c^{p-2} = p(p-1)\int_\Omega u \cdot \nabla c\, c^{p-1} + p\int_\Omega f c^{p-1}. \tag{5.18}$$

Remark that

$$p(p-1)\int_\Omega d|\nabla c|^2 c^{p-2} = 4(1-\frac{1}{p})\int_\Omega d|\nabla c^{p/2}|^2 \ge 4\underline{d}(1-\frac{1}{p})\int_\Omega |\nabla c^{p/2}|^2.$$

Set

$$2^* = +\infty \text{ if } N = 1\ ;\ 2 < 2^* < +\infty \text{ if } N = 2\ ;\ \frac{1}{2^*} = \frac{1}{2} - \frac{1}{N} \text{ if } N \ge 3. \tag{5.19}$$

Using Sobolev's embedding theorem, $W^{1,2}(\Omega) \hookrightarrow L^{2^*}(\Omega)$, so there exists $\alpha_0 = \alpha_0(\underline{d}, \Omega, p) > 0$ such that

$$\alpha_0\|c^{p/2}\|^2_{L^{2^*}(\Omega)} \le p\int_\Omega c^p + \frac{p(p-1)}{2}\int_\Omega d|\nabla c|^2 c^{p-2}. \tag{5.20}$$

Going back to (5.18), if $\alpha = \min(\underline{d}\frac{p(p-1)}{2}, \alpha_0)$, we have

$$\frac{d}{dt}\int_\Omega c^p + \alpha\|c^{p/2}\|^2_{L^{2^*}(\Omega)} + \alpha\int_\Omega |\nabla c|^2 c^{p-2} \le p(p-1)\int_\Omega u \cdot \nabla c\, c^{p-1} + p\int_\Omega f c^{p-1}. \tag{5.21}$$

To apply Gronwall’s lemma, we estimate the right members as follows: using Hölder’s and Young’s inequalities,

$$p\int_\Omega f c^{p-1} \le p\|f\|_{L^p(\Omega)}\|c\|_{L^p(\Omega)}^{p-1} \le \|f\|_{L^p(\Omega)}^p + (p-1)\int_\Omega c^p. \tag{5.22}$$

Let $k \ge 0$, we have

$$\begin{aligned}\int_\Omega u\cdot\nabla c\, c^{p-1} &\le \int_\Omega |u||\nabla c| c^{\frac{p}{2}-1} c^{\frac{p}{2}} \\ &\le \int_\Omega \left[(|u|-k)^+ + k\right] |\nabla c| c^{\frac{p}{2}-1} c^{\frac{p}{2}}.\end{aligned} \tag{5.23}$$

Let us define $r_0 < r$ by

$$\begin{cases} r_0 = 2 & \text{if} \quad N = 1, \\ \frac{1}{r_0} = \frac{1}{2} - \frac{1}{2^*} & \text{if} \quad N = 2, \qquad (2^* \text{ has to be chosen large enough}) \\ r_0 = N & \text{if} \quad N \ge 3. \end{cases}$$

Remark that $1 = \frac{1}{r_0} + \frac{1}{2} + \frac{1}{2^*}$, so using Hölder’s inequality,

$$\begin{aligned}\int_\Omega (|u|-k)^+ |\nabla c| c^{\frac{p}{2}-1} c^{\frac{p}{2}} &\le \|(|u|-k)^+\|_{L^{r_0}(\Omega)} \left(\int_\Omega |\nabla c|^2 c^{p-2}\right)^{\frac{1}{2}} \|c^{\frac{p}{2}}\|_{L^{2^*}(\Omega)} \\ &\le 2\|(|u|-k)^+\|_{L^{r_0}(\Omega)} \left[\left(\int_\Omega |\nabla c|^2 c^{p-2}\right) + \|c^{\frac{p}{2}}\|_{L^{2^*}(\Omega)}^2\right].\end{aligned} \tag{5.24}$$

Then using once more Hölder’s inequality,

$$\begin{aligned}\int_\Omega (|u|-k)^{+\,r_0} &\le \left[\int_\Omega (|u|-k)^{+\,r}\right]^{\frac{r_0}{r}} \left[\int_\Omega \mathbb{1}_{\{|u|>k\}}\right]^{\frac{r-r_0}{r}} \\ &\le \|u\|_{L^\infty(0,T;L^r(\Omega))}^{r_0} \left[\int_\Omega \mathbb{1}_{\{|u|>k\}}\right]^{\frac{r-r_0}{r}} \\ &\le \frac{1}{k^{r-r_0}} \|u\|_{L^\infty(0,T;L^r(\Omega))}^r\,,\end{aligned} \tag{5.25}$$

where we used for the last inequality

$$k^r \left[\int_\Omega \mathbb{1}_{\{|u|>k\}}\right] \le \|u\|_{L^\infty(0,T;L^r(\Omega))}^r.$$

We choose $k = (\frac{4}{\alpha})^{\frac{r_0}{r-r_0}} \|u\|_{L^\infty(0,T;L^r(\Omega))}^{\frac{r}{r-r_0}}$, which makes the right-hand side of (5.25) be equal to $(\frac{\alpha}{4})^{r_0}$, so that (5.24) yields

$$p(p-1)\int_\Omega (|u|-k)^+ |\nabla c| c^{p-1} \le \frac{\alpha}{2}\left[\left(\int_\Omega |\nabla c|^2 c^{p-2}\right) + \|c^{\frac{p}{2}}\|_{L^{2^*}(\Omega)}^2\right]. \tag{5.26}$$

To estimate the second term in (5.23), we use Young’s inequality: there exists $C = C(\alpha, p, k) > 0$ such that

$$p(p-1)k\int_\Omega |\nabla c| c^{\frac{p}{2}-1} c^{\frac{p}{2}} \le \frac{\alpha}{2}\int_\Omega |\nabla c|^2 c^{p-2} + C\int_\Omega c^p. \tag{5.27}$$

Going back to (5.21) and using (5.22), (5.26) and (5.27), we get

$$\frac{d}{dt}\int_\Omega c^p \le \|f\|_{L^p(\Omega)}^p + (C + p - 1)\int_\Omega c^p.$$

Finally, Gronwall’s lemma yields (5.16).

$\square$

Proof of Proposition 5.2. The existence of a unique nonnegative classical solution c on a maximal time interval $[0,T^*)$, $0 < T^* \leq +\infty$, and the regularity of c, are consequences of Amann's results (see [2, Theorem 1] and [4, Theorem 15.1]). To prove that $T^* = +\infty$, let us assume that $T^* < +\infty$ and find *a priori* bounds on c in $L^\infty(Q_T)$ for any $T \leq T^*$ (see [2, Theorem 3]).

Set $W := \sum_{j=1}^{P} c_j$, using assumption (v), if q_{ij} denotes the coefficient of Q on the i^{th} row and j^{th} column, for $i \in \{1,\dots,P\}$,

$$\begin{aligned} q_{ii}[\partial_t c_i + \operatorname{div}(-d_i\nabla c_i + c_i u_i)] = q_{ii} f_i(t,x,c) &\leq (1+W)b_i - \sum_{j=1}^{i-1} q_{ij} f_j(t,x,c) \\ &= (1+W)b_i - \sum_{j=1}^{i-1} q_{ij}\left[\partial_t c_j + \operatorname{div}(-d_j\nabla c_j + c_j u_j)\right]. \end{aligned} \tag{5.28}$$

Let z_i be the solution of

$$\begin{cases} q_{ii}[\partial_t z_i + \operatorname{div}(-d_i\nabla z_i + z_i u_i)] = (1+W)b_i & \text{on} \quad Q_T, \\ -d_i\nabla z_i\cdot\nu + z_i u_i\cdot\nu = 0 & \text{on} \quad \Sigma_T, \\ z_i(0,\cdot) = 0 & \text{on} \quad \Omega. \end{cases}$$

Inequality (5.28) now reads

$$q_{ii}[\partial_t(c_i - z_i) + \operatorname{div}(-d_i\nabla(c_i - z_i) + (c_i - z_i)u_i] \leq -\sum_{j=1}^{i-1} q_{ij}[\partial_t c_j + \operatorname{div}(-d_j\nabla c_j + c_j u_j)].$$

Using an obvious extension of Lemma 5.3, if $C > 0$ denotes any constant depending only on the data,

$$\|(c_i - z_i)^+\|_{L^p(Q_t)} \leq C\big(1 + \sum_{j=1}^{i-1}\|c_j\|_{L^p(Q_t)}\big).$$

By induction, we get for $i \in \{1,\dots,P\}$,

$$\|c_i^+\|_{L^p(Q_t)} = \|c_i\|_{L^p(Q_t)} \leq C\big(1 + \sum_{j=1}^{i}\|z_j\|_{L^p(Q_t)}\big). \tag{5.29}$$

Taking the p^{th} power and summing over i,

$$\|W\|^p_{L^p(Q_t)} \leq C\big(1 + \sum_{j=1}^{P}\|z_j\|^p_{L^p(Q_t)}\big). \tag{5.30}$$

Applying Lemma 5.5 to z_j, we get for all $j \in \{1,\dots,P\}$, $t \in (0,T)$,

$$\|z_j(t)\|^p_{L^p(\Omega)} \leq C\big(1 + \int_0^t \|W\|^p_{L^p(\Omega)}\big) = C(1 + \|W\|^p_{L^p(Q_t)}).$$

Summing over j and using (5.30), we get

$$\sum_{j=1}^{P}\|z_j(t)\|^p_{L^p(\Omega)} \leq C\big(1 + \int_0^t \sum_{j=1}^{P}\|z_j\|^p_{L^p(\Omega)}\big).$$

Then Gronwall’s lemma guarantees that for all j, z_j is bounded in $L^p(Q_T)$, and so is c by (5.29). Since f has a polynomial growth, the reaction term in system (5.3) is bounded in $L^p(Q_T)$ for any $p<+\infty$, so using Theorem 5.8 (see the Appendix), c is bounded in $L^\infty(Q_T)$ for any $T\leq T^*$. Finally, using [2, Theorem 3], $T^*=+\infty$. Remark that the $L^\infty(Q_T)-$bound on c only depends on the quantities mentioned in (5.6). To get the estimates (5.7), multiply (5.3) by c_i, integrate over Q_T and by parts to get

$$\begin{aligned}
&\frac{1}{2}\|c_i(T)\|^2_{L^2(\Omega)}+\underline{d}(T)\|\nabla c_i\|^2_{L^2(Q_T)}\\
&\leq \frac{1}{2}\|c_i^0\|^2_{L^2(\Omega)}+\overline{d}(T)\int_{Q_T}|c_iu_i\cdot\nabla c_i|+\int_{Q_T}|f_i|(t,x,c)c_i\\
&\leq \frac{1}{2}\|c_i^0\|^2_{L^2(\Omega)}+\frac{\underline{d}(T)}{2}\|\nabla c_i\|^2_{L^2(Q_T)}+C\int_{Q_T}|c_iu_i|^2+\int_{Q_T}|f_i|(t,x,c)c_i.
\end{aligned}$$

Using that $u\in L^\infty(0,T;L^2(\Omega))$ and $c\in L^\infty(Q_T)$, ∇c_i is bounded in $L^2(Q_T)$. Finally, the fact that $\partial_t c$ is bounded in $L^2(0,T;W^{-1,2}(\Omega))$ is a direct consequence of equation (5.3) and the previous bounds on c and ∇c_i.

□

5.3 Proof of the main theorem

Existence. Let $T>0$. We approximate (*e.g.* using mollifiers) c_i^0, d_i and u_i from system (5.3) by smooth functions c_i^{0n}, d_i^n, u_i^n such that

$$c_i^{0n}\underset{n\to+\infty}{\longrightarrow}c_i^0 \text{ in } L^2(\Omega),\ d_i^n\underset{n\to+\infty}{\longrightarrow}d_i \text{ in } L^2(Q_T),\ u_i^n\underset{n\to+\infty}{\longrightarrow}u \text{ in } L^2(Q_T)^P,$$

and such that $(c_i^{0n})_{n\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)^+$, $\omega_{d_i^n}\leq\omega_{d_i}$, $(u_i^n)_{n\in\mathbb{N}}$ and $(\nabla d_i^n)_{n\in\mathbb{N}}$ are bounded in $L^\infty(0,T;L^r(\Omega))$, $\underline{d}\leq d_i^n\leq\overline{d}$. According to Proposition 5.2, system (5.3) with data (c_i^{0n},d_i^n,u_i^n) has a unique solution $c^n:[0,T]\times\Omega\to\mathbb{R}_+^P$. Moreover (5.7) guarantees that for any $T>0$, $\|c^n\|_{L^\infty(Q_T)}$, $\|c^n\|_{L^2(0,T;W^{1,2}(\Omega))}$ and $\|\partial_t c^n\|_{L^2(0,T;W^{-1,2}(\Omega))}$ are bounded independently of n. By virtue of Corollary 4 in [98], we deduce that $(c^n)_{n\in\mathbb{N}}$ is relatively compact in $L^2(Q_T)$, and therefore has a subsequence that converges a.e. in Q_T.

Let $(T_k)_{k\in\mathbb{N}}\in\mathbb{R}_+^{\mathbb{N}}$ be an increasing unbounded sequence. In the following, we denote by $c^n_{i\,|[0,T_k]}$ the restriction of c_i^n on Q_{T_k}. Using the above results, there exists $c:(0,+\infty)\times\Omega\to\mathbb{R}_+^P$ such that, up to a diagonal extraction, we have:

$\forall i\in\{1,\dots,P\}$, $\forall k\in\mathbb{N}$,

$$\left.\begin{array}{rcll}
c^n_{i\,|[0,T_k]} & \underset{n\to+\infty}{\longrightarrow} & c_i & \text{in } L^p(Q_{T_k}) \text{ for any } p<+\infty \text{ and a.e. };\\
f_i(t,x,c^n)_{|[0,T_k]} & \underset{n\to+\infty}{\longrightarrow} & f_i(t,x,c) & \text{in } L^p(Q_{T_k}) \text{ for any } p<+\infty;\\
\nabla c^n_{i\,|[0,T_k]} & \underset{n\to+\infty}{\longrightarrow} & \nabla c_i & \text{weakly in } L^2(Q_{T_k})^N;\\
\partial_t c_i^n & \underset{n\to+\infty}{\longrightarrow} & \partial_t c_i & \text{weakly in } L^2(0,T_k;W^{-1,2}(\Omega)).
\end{array}\right\}\tag{5.31}$$

As c^n is a classical solution of (5.3), for all $T>0$ and all $\psi\in C^\infty(\overline{Q_T})$ such that $\psi(T)=0$, we have

$$-\int_\Omega c_i^{0n}\psi(0)+\int_{Q_T}\big(-c_i^n\partial_t\psi+(d_i^n\nabla c_i^n-c_i^nu_i^n)\nabla\psi\big)=\int_{Q_T}f_i(t,x,c^n)\psi.\tag{5.32}$$

Using (5.31), we can pass to the limit $n \to +\infty$ in (5.32) for any $T > 0$, so c satisfies (5.5). Finally, $c \in L^2(0,T;W^{1,2}(\Omega))$, $\partial_t c \in L^2(0,T;W^{-1,2}(\Omega))$. Thus by [5, Theorem III.4.10.2] it follows that $c \in C([0,T];B^0_{2,2}(\Omega))$, where $B^0_{2,2}(\Omega)$ denotes the standard Besov space, see [99]. Using extension and restriction operators it is possible to show that $B^0_{2,2}(\Omega) = L^2(\Omega)$ and therefore $c_i \in C([0,T],L^2(\Omega))$. This ends the existence proof in Theorem 5.1.

Uniqueness. Let $T > 0$, $c, \hat{c}$ be two solutions of (5.5) on Q_T with the same initial data, and let $w_i := c_i - \hat{c}_i$. In the following, $C > 0$ denotes any constant depending only on T and the data of (5.3).

We first prove that $w_i = 0$ using a formal computation, and justify it afterwards. Formally, we have

$$\left.\begin{array}{l}\partial_t w_i - \operatorname{div}(d_i \nabla w_i - w_i u_i) = f_i(c) - f_i(\hat{c}) \text{ on } Q_T, \\ \partial_\nu w_i = 0 \text{ on } \Sigma_T \text{ ; } w_i(0) = 0 \text{ on } \Omega.\end{array}\right\} \tag{5.33}$$

Let $t_0 \in (0,T)$, multiplying (5.33) by w_i and integrating by parts on Q_{t_0}, we get

$$\frac{1}{2}\|w_i(t_0)\|^2_{L^2(\Omega)} + \int_{Q_{t_0}} d_i |\nabla w_i|^2 = \int_{Q_{t_0}} w_i u_i \nabla w_i + \int_{Q_{t_0}} [f_i(c) - f_i(\hat{c})] w_i. \tag{5.34}$$

As c and $\hat{c}$ are *a priori* bounded in $L^\infty(Q_T)$ and as f_i is locally Lipschitz continuous,

$$\exists C = C(T) > 0 : \int_{Q_{t_0}} [f_i(c) - f_i(\hat{c})] w_i \le C \int_{Q_{t_0}} w_i^2. \tag{5.35}$$

Since $u_i \in L^\infty(0,T;L^r(\Omega)^N)$, if $r^* > 1$ satisfies $\frac{1}{r^*} + \frac{1}{r} + \frac{1}{2} = 1$, we have

$$\begin{aligned}\int_{Q_{t_0}} w_i u_i \nabla w_i &\le \int_0^{t_0} \|w_i\|_{L^{r^*}(\Omega)} \|u_i\|_{L^r(\Omega)^N} \|\nabla w_i\|_{L^2(\Omega)^N} \\ &\le C \int_0^{t_0} \|w_i\|_{L^{r^*}(\Omega)} \|\nabla w_i\|_{L^2(\Omega)^N} \\ &\le \varepsilon \|\nabla w_i\|^2_{L^2(Q_{t_0})^N} + C_\varepsilon \|w_i\|^2_{L^2(Q_{t_0})},\end{aligned} \tag{5.36}$$

where $\varepsilon > 0$ is arbitrarily small and we used that

$$\forall \varepsilon > 0,\ \exists C_\varepsilon > 0 : \forall w \in L^{r^*}(\Omega),\ \|w\|_{L^{r^*}(\Omega)} \le \varepsilon \|\nabla w\|_{L^2(\Omega)^N} + C_\varepsilon \|w\|_{L^2(\Omega)}.$$

This comes from the compact embedding of $W^{1,2}(\Omega)$ into $L^{r^*}(\Omega)$, which holds since $r > N$ implies $-\frac{1}{2} + \frac{1}{N} > -\frac{1}{r^*}$. Using inequalities (5.35) and (5.36) in (5.34), noting $\underline{d} \le d_i$ and choosing ε small enough, we get

$$\frac{1}{2}\|w_i(t_0)\|^2_{L^2(\Omega)} + \frac{\underline{d}}{2} \int_{Q_{t_0}} |\nabla w_i|^2 \le C \int_{Q_{t_0}} w_i^2. \tag{5.37}$$

Then Gronwall's lemma yields $w_i = 0$, *i.e.* $c = \hat{c}$.

Let us now justify this computation on weak solutions: it is clear that (5.36) still holds for weak solutions, we only need to justify (5.34). The starting point is that for all $\psi \in C^\infty(\overline{Q_T})$ such that $\psi(T) = 0$,

$$\int_{Q_T} -w_i \partial_t \psi + (d_i \nabla w_i - w_i u_i) \nabla \psi = \int_{Q_T} [f_i(c) - f_i(\hat{c})] \psi. \tag{5.38}$$

Let $t_0 \in (0,T)$, we would like to choose $\psi = \mathbb{1}_{(0,t_0)} w_i$ in (5.38), but $\mathbb{1}_{(0,t_0)} w_i$ is not regular enough to differentiate in time, so we have to regularize: let $h > 0$, we define $\psi_h(t) = \frac{1}{2h} \int_{t-h}^{t+h} \mathbb{1}_{(0,t_0)} w_i$.

Then $\psi_h \in C([0,T];L^2(\Omega)) \cap W^{1,2}(Q_T)$ and $\psi_h(T) = 0$ for h small enough. By density, (5.5) is valid for ψ_h, and therefore

$$\int_{Q_T} \Big(-\frac{1}{2h} w_i(t)[\mathbb{1}_{(0,t_0)}(t+h)w_i(t+h) - \mathbb{1}_{(0,t_0)}(t-h)w_i(t-h)] + d_i \nabla w_i \nabla \psi_h \Big)$$
$$= \int_{Q_T} w_i u_i \nabla \psi_h + [f_i(c) - f_i(\hat{c})]\psi_h.$$

Remark that $\psi_h \underset{h\to 0}{\longrightarrow} \mathbb{1}_{(0,t_0)} w_i$, $\nabla \psi_h \underset{h\to 0}{\longrightarrow} \mathbb{1}_{(0,t_0)} \nabla w_i$ in $L^2(Q_T)$, so

$$\int_{Q_T} \Big(w_i u_i \nabla \psi_h + [f_i(c) - f_i(\hat{c})]\psi_h \Big) \underset{h\to 0}{\longrightarrow} \int_{Q_{t_0}} w_i u_i \nabla w_i + [f_i(c) - f_i(\hat{c})] w_i$$
$$\int_{Q_T} d_i \nabla w_i \nabla \psi_h \underset{h\to 0}{\longrightarrow} \int_{Q_{t_0}} d_i |\nabla w_i|^2.$$

Moreover, we have

$$\begin{aligned}
&\int_{Q_T} -\frac{1}{2h} w_i(t)[\mathbb{1}_{(0,t_0)}(t+h)w_i(t+h) - \mathbb{1}_{(0,t_0)}(t-h)w_i(t-h)] \\
&= -\frac{1}{2h}\int_\Omega \left[\int_{-h}^{t_0-h} w_i(t)w_i(t+h) - \int_h^{t_0+h} w_i(t-h)w_i(t) \right] \\
&= -\frac{1}{2h}\int_\Omega \left[\int_{-h}^{t_0-h} w_i(t)w_i(t+h) - \int_0^{t_0} w_i(t)w_i(t+h) \right] \\
&= -\frac{1}{2h}\int_\Omega \left[\int_{-h}^{0} w_i(t)w_i(t+h) - \int_{t_0-h}^{t_0} w_i(t)w_i(t+h) \right] \\
&\underset{h\to 0}{\longrightarrow} \frac{1}{2}\int_\Omega w_i(t_0)^2,
\end{aligned}$$

where we used $w_i \in C([0,T];L^2(\Omega))$ to pass to the limit $h \to 0$ and by convention, $w_i(t) = 0$ for $t < 0$. This proves that (5.34) holds for weak solutions, whence uniqueness.

To prove the estimates (5.7), simply remark that we proved that they are valid for smooth solutions in Proposition 5.2. Since the norms can only decrease when passing to the weak limit, they remain valid for the solutions of Theorem 5.1.

□

5.4 Application to a class of systems from chemistry

We now show how the results of the previous sections can be applied to a classical model from mass-action kinetics chemistry. More precisely, we consider the following situation: we study the evolution of the concentrations $c_1, \dots, c_P$ of P chemical species $C_1, \dots, C_P$ placed in a bounded and isolated vessel. We assume that mass transport may be due to both Fickian diffusion (with time and space dependent coefficients) and to the fluid's bulk motion. The vector field describing the fluid's motion is assumed to be a given data. Finally, we assume that R chemical reactions happen simultaneously.

The equations describing the evolution of $(c_1, \dots, c_P)$ are

$$\left.\begin{array}{rcll}
\partial_t c_i + \mathrm{div}[-d_i(t,x)\nabla c_i + c_i u(t,x)] &=& \sum_{j=1}^R k_j r_j(c) \omega_j^i & \text{on } (0,+\infty)\times\Omega, \\
-d_i(t,x)\nabla c_i \cdot \nu + c_i u(t,x)\cdot \nu &=& 0 & \text{on } (0,+\infty)\times\partial\Omega, \\
c_i(0,\cdot) &=& c_i^0 & \text{on } \Omega,\ i \in \{1,\dots,P\}.
\end{array}\right\} \quad (5.39)$$

For $j \in \{1,\dots,R\}$, $k_j \geq 0$ is the reaction speed of the j^{th} chemical reaction. For the chemical reaction

$$\alpha_j^1 C_1 + \dots + \alpha_j^P C_P \rightleftharpoons \beta_j^1 C_1 + \dots + \beta_j^P C_P, \qquad j \in \{1,\dots,R\}\ ,$$

where $\alpha_j^k, \beta_j^k \in \mathbb{N}$, we write $\alpha_j = (\alpha_j^1,\dots,\alpha_j^P)$, $\beta_j = (\beta_j^1,\dots,\beta_j^P)$. The so-called stoechiometric vectors are defined as $\omega_j = \beta_j - \alpha_j \in \mathbb{Z}^P$. We will also use the notation $c^\gamma = \Pi_{i=1}^P c_i^{\gamma_i}$ for $\gamma \in \mathbb{N}^P$. In the following, we assume

(a_1) $\forall j \in \{1,\dots,R\}$, $r_j(c) = c^{\alpha_j} - \kappa_j c^{\beta_j}$ (mass action kinetics) ; $\kappa_j \geq 0$ is given.

(a_2) $\forall j \in \{1,\dots,R\}$, β_j is a permutation of $(1,0,\dots,0) \in \mathbb{N}^P$.

(a_3) $\omega_1,\dots,\omega_R$ are linearly independent in $\mathbb{R}^P$.

(a_4) $\exists e \in (0,+\infty)^P$ such that for all $j \in \{1,\dots,R\}$, $\langle e, \omega_j\rangle = 0$ (conservation of atoms).

Remark that assumption (a_2) means that we only consider reactions of the type

$$\alpha_j^1 C_1 + \dots + \alpha_j^P C_P \rightleftharpoons C_{i_j}\ ,\ i_j \in \{1,\dots,P\}.$$

Corollary 5.6. *Under assumptions $(i)-(iii)$ and $(\mathrm{a}_1)-(\mathrm{a}_4)$, system (5.39) has a unique global solution in the sense (5.5).*

The proof mostly consists in reorganizing the reactions and the chemical components. It is based on the following elementary result on matrices. We denote by $\mathcal{M}_{P,R}(\mathbb{R})$ the space of matrices with real entries and with P columns and R rows, and write $\mathcal{M}_P(\mathbb{R}) = \mathcal{M}_{P,P}(\mathbb{R})$.

Lemma 5.7. *Let $M \in \mathcal{M}_{P,R}(\mathbb{R})$,*

$$M = \left(\begin{array}{c|c|c} \omega_1 & \dots & \omega_R \end{array}\right)\ ; \quad \omega_j = \begin{pmatrix} \omega_j^1 \\ \vdots \\ \omega_j^P \end{pmatrix} \in \mathbb{R}^P,$$

and assume

(a) $\forall j \in \{1,\dots,R\}$, there exists a unique $i \in \{1,\dots,P\}$ such that $\omega_j^i > 0$.

(b) $\omega_1,\dots,\omega_R$ are linearly independent in $\mathbb{R}^P$.

(c) $\exists e \in (0,+\infty)^P$ such that for all $j \in \{1,\dots,R\}$, $\langle e, \omega_j\rangle = 0$.

Then, up to a permutation of its columns and rows,

$$M = \begin{pmatrix} \boxed{N_1} & & & 0 \\ & \boxed{N_2} & & \\ & & \ddots & \\ & & & \boxed{N_k} \\ & * & & \end{pmatrix}, \tag{5.40}$$

where N_i are row-matrices with (strictly) negative entries.

Proof. First, remark that properties $(a)-(c)$ are unchanged when permuting the rows or columns of M. We prove Lemma 5.7 by induction on P. Since the vectors $e,\omega_1,\dots,\omega_R$ are linearly independent, we have $P>R$. Using assumption (a), M has exactly R positive entries, so there exists $N\in\mathcal{M}_{P-R,R}(\mathbb{R})$ with nonpositive entries, $M_0\in\mathcal{M}_R(\mathbb{R})$, such that up to a permutation of the rows,

$$M=\begin{pmatrix} \boxed{N} \\ \boxed{M_0} \end{pmatrix}.$$

If $N=0$, assumption (b) implies that M_0 is invertible, and assumption (c) implies that there exists a nonzero vector with is orthogonal to the column vectors of M_0: contradiction, so $N\neq 0$. Let L_1 be a nonzero row of N: by a permutation of the rows, we put L_1 at the top of M, and by a permutation of the columns, we put the negative entries of L_1 at the top left corner, so that

$$M=\begin{pmatrix} \boxed{N_1} & 0 \\ * & \boxed{M_1} \end{pmatrix},$$

N_1 is nontrivial and has negative entries. It is easy to check that M_1 satisfies assumptions $(a)-(c)$, and then Lemma 5.7 holds by an obvious induction on P.

$\square$

Proof of Corollary 5.6. Let $M\in\mathcal{M}_{P,R}(\mathbb{R})$ be the matrix whose columns are $\omega_1,\dots,\omega_R$. Using assumptions $(\mathrm{a}_1)-(\mathrm{a}_4)$, M satisfies $(a)-(c)$ from Lemma 5.7. Consequently, up to a permutation on the chemical species and on the chemical reactions (which correspond respectively to a permutation on the rows and on the columns of M), we can assume that M satisfies (5.40). To prove that there exists a lower triangular invertible matrix $Q\in\mathcal{M}_P(\mathbb{R})$ with nonnegative entries, such that QM has nonpositive entries, we may proceed as follows: first, recall that the multiplication of M by such a matrix Q corresponds to adding to each row of M a positive linear combination of the above rows. We may define the matrix Q as the product $Q_1\dots Q_k$, where Q_i are lower triangular invertible matrices with nonnegative entries, satisfying

1) The columns of Q_kM corresponding to the block N_k are nonpositive. This is obtained by choosing a matrix Q_k which corresponds to adding convenient positive factors of the k^{th} row to the rows below.
2) The columns of $Q_{k-1}Q_kM$ corresponding to the block N_{k-1} are nonpositive. This is obtained by choosing a matrix Q_{k-1} which corresponds to adding convenient positive factors of the $(k-1)^{th}$ row to the rows below. The crucial point is that this operation leaves the columns corresponding to N_k unchanged.
3) We iterate this procedure to build a sequence of matrices $Q_1,\dots,Q_k$ such that $Q_1\dots,Q_kM$ has nonpositive entries.

Then if we denote by $F=(F_1,\dots,F_P)$ the reaction term in (5.39), remark that

$$\begin{pmatrix} F_1 \\ \vdots \\ F_P \end{pmatrix}=\left(\begin{array}{c|c|c} \omega_1 & \dots & \omega_R \end{array}\right)\cdot\begin{pmatrix} k_1r_1 \\ \vdots \\ k_Pr_P \end{pmatrix} \quad ; \quad QF=QM\begin{pmatrix} k_1r_1 \\ \vdots \\ k_Pr_P \end{pmatrix}.$$

Using assumptions $(\mathrm{a}_1)-(\mathrm{a}_2)$, we have

$$\forall j \in \{1,\dots,P\}, \quad -r_j(c) \leq (\max_{j=1,\dots,R} \kappa_j) \sum_{i=1}^{P} c_i,$$

and since QM has nonpositive entries, Assumption (v) from Theorem 5.1 is satisfied. Consequently, Theorem 5.1 can be applied to system (5.39) and Corollary 5.6 holds.

□

5.5 Appendix

In this section, we prove that if c satisfies the equation

$$\begin{cases} \partial_t c + \mathrm{div}(-d\nabla c + cu) &= f \quad \text{on } Q_T, \\ -d\partial_\nu c + cu\cdot\nu &= 0 \quad \text{on } \Sigma_T, \\ c(0,\cdot) &= c^0 \quad \text{on } \Omega, \end{cases} \tag{5.41}$$

and f is in $L^q(Q_T)$ with q large enough, then c is *a priori* bounded in $L^\infty(Q_T)$. This has been shown in [69] for the case of Dirichlet boundary conditions (and for general parabolic operators). In the following, we adapt the proof of [69] to the case of Neumann boundary conditions.

As before, Ω is an open, bounded subset of $\mathbb{R}^N$, whose boundary is at least C^2. We assume that the data satisfy

(i) $c^0 \in L^\infty(\Omega)^+$.

(ii) $d: Q_T \to \mathbb{R}$ is measurable ; $\exists \underline{d} > 0$ such that $\underline{d} \leq d$.

(iii) $|u|^2 \in L^\infty(0,T;L^r(\Omega))$; $f \in L^q(Q_T)$ and $r,q \geq 1$ satisfy

$$\frac{N}{2r} = 1-\theta_1^u \quad ; \quad \frac{1}{q}\frac{N+2}{2} = 1-\theta_1^f, \tag{5.42}$$

where $\theta_1^u, \theta_1^f \in (0,1)$ for $N \geq 2$ and $\theta_1^u, \theta_1^f \in (0,\frac{1}{2})$ for $N=1$.

Theorem 5.8. *Let c be a classical solution of* (5.41) *on Q_T. Under assumptions $(i)-(iii)$, there exists a constant $M>0$ depending only on the data and T, such that*

$$c(t,x) \leq M \quad \textit{for a.e. } (t,x) \in Q_T.$$

Let us summarize the notations that will be used in the following:

Notations. Let $c:(0,T)\times\Omega \to \mathbb{R}$ be a measurable function and λ denote the Lebesgue measure, we write

$c_k = \max(0, c-k), \quad k \in \mathbb{R}$.

$Q_T(k) = \{(t,x) \in Q_T : c(t,x) > k\}$.

$A_k(t) = \{x \in \Omega \ : \ c(t,x) > k\}$.

For $q,r \in [1,+\infty]$, the norm on $L^r(0,T;L^q(\Omega))$ is denoted by $\|\cdot\|_{r,q,Q_T}$.

$V_2(Q_T) = L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$. For (r,q) such that $V_2(Q_T) \hookrightarrow L^r(0,T;L^q(\Omega))$, $\beta > 0$ is a constant such that

$$\|\cdot\|_{r,q,Q_T} \leq \beta \|\cdot\|_{V_2(Q_T)}.$$

Note that β can be chosen independently of T (see [69] p. 74).

The subsequent result provides a sufficient condition to deduce uniform bounds on a function from estimates in $V_2(Q_T)$ and $L^q(Q_T)$ for finite q.

Lemma 5.9. *Let $c \in V_2(Q_T)$ and assume that*

$$\forall k \geq \hat{k}, \quad \|c_k\|_{V_2(Q_T)} \leq \gamma k \left(\mu_1(k)^{\frac{1+\theta_1}{r_1}} + \mu_2(k)^{\frac{1+\theta_2}{r_2}} \right), \tag{5.43}$$

where

$$\hat{k}, \gamma, \theta_i > 0 \; ; \; \mu_i(k) = \int_0^T \lambda(A_k(t))^{\frac{r_i}{q_i}} \, dt \; ; \; \mu_i(k) \in [0,1] \text{ for } k \geq \hat{k} \; ; \; i \in \{1,2\},$$

and (r_i, q_i) are chosen such that $V_2(Q_T) \hookrightarrow L^{r_i}(0,T;L^{q_i}(\Omega))$. Then there exists $M > 0$ depending only on the data, such that for a.e. $(t,x) \in Q_T$,

$$c(t,x) \leq M. \tag{5.44}$$

We will use the following elementary result on numerical sequences:

Let $C, b, \theta > 0$ and assume that $(y_n)_{n\in\mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}}$ satisfies

$$\forall n \in \mathbb{N}, \quad y_{n+1} \leq C b^n {y_n}^{1+\theta}.$$

Then a straightforward induction on n yields

$$\forall n \in \mathbb{N}, \quad y_n \leq C^{\frac{(1+\theta)^n - 1}{\theta}} b^{\frac{(1+\theta)^n - 1}{\theta^2} - \frac{n}{\theta}} y_0^{(1+\theta)^n}.$$

As a consequence,

$$\left[b > 1 \text{ and } y_0 \leq \frac{1}{C^{\frac{1}{\theta}} b^{\frac{1}{\theta^2}}} \right] \Longrightarrow y_n \underset{n\to+\infty}{\longrightarrow} 0. \tag{5.45}$$

Proof of Lemma 5.9. Let $M > \hat{k}$, $h_k = M(2 - 2^{-k})$ for $k \in \mathbb{N}$. It is easy to check that

$$(h_{k+1} - h_k) \mu_i(h_{k+1})^{\frac{1}{r_i}} \leq \|c_{h_k}\|_{r_i, q_i, Q_T}, \quad i \in \{1,2\}. \tag{5.46}$$

Using $V_2(Q_T) \hookrightarrow L^{r_i}(0,T;L^{q_i}(\Omega))$ and (5.43),

$$\|c_k\|_{r_i,q_i,Q_T} \leq \beta \|c_k\|_{V_2(Q_T)} \leq \beta \gamma k \left(\mu_1(h_k)^{\frac{1+\theta_1}{r_1}} + \mu_2(h_k)^{\frac{1+\theta_2}{r_2}} \right), \quad i \in \{1,2\}.$$

Then

$$\begin{aligned}
\mu_i(h_{k+1})^{\frac{1}{r_i}} &\leq \frac{\|c_{h_k}\|_{r_i,q_i,Q_T}}{h_{k+1} - h_k} \leq \frac{\beta \gamma h_k}{h_{k+1} - h_k} \left(\mu_1(h_k)^{\frac{1+\theta_1}{r_1}} + \mu_2(h_k)^{\frac{1+\theta_2}{r_2}} \right) \\
&\leq 4 \beta \gamma 2^k \left(\mu_1(h_k)^{\frac{1+\theta_1}{r_1}} + \mu_2(h_k)^{\frac{1+\theta_2}{r_2}} \right), \quad i \in \{1,2\}.
\end{aligned} \tag{5.47}$$

Let $\theta = \min(\theta_1, \theta_2)$. Since $\mu_i(h_k) \in [0,1]$, we have

$$\mu_1(h_k)^{\frac{1+\theta_1}{r_1}} + \mu_2(h_k)^{\frac{1+\theta_2}{r_2}} \leq \mu_1(h_k)^{\frac{1+\theta}{r_1}} + \mu_2(h_k)^{\frac{1+\theta}{r_2}} \leq C \left(\mu_1(h_k)^{\frac{1}{r_1}} + \mu_2(h_k)^{\frac{1}{r_2}} \right)^{1+\theta},$$

where $C > 0$ only depends on θ. Going back to (5.47), we have

$$\mu_1(h_{k+1})^{\frac{1}{r_1}} + \mu_2(h_{k+1})^{\frac{1}{r_2}} \leq 8 \beta \gamma C 2^k \left(\mu_1(h_k)^{\frac{1}{r_1}} + \mu_2(h_k)^{\frac{1}{r_2}} \right)^{1+\theta}.$$

According to (5.45), the sequence $(\mu_1(h_k)^{\frac{1}{r_1}} + \mu_2(h_k)^{\frac{1}{r_2}})_{k\in\mathbb{N}}$ converges to 0 as $k \to +\infty$ provided its initial value $\mu_1(M)^{\frac{1}{r_1}} + \mu_2^{(}M)\frac{1}{r_2}$ is small enough. Similarly as in (5.46), we have

$$(M-\hat{k})\mu_i(M)^{\frac{1}{r_i}} \le \|c_{\hat{k}}\|_{r_i,q_i,Q_T}, \quad i \in \{1,2\}.$$

Using (5.43),

$$\begin{aligned}(M-\hat{k})(\mu_1(M)^{\frac{1}{r_1}} + \mu_2(M)^{\frac{1}{r_2}}) &\le 2\beta \|c_{\hat{k}}\|_{V_2(Q_T)} \\ &\le 2\beta\gamma\hat{k}\left(\mu_1(h_{\hat{k}})^{\frac{1+\theta_1}{r_1}} + \mu_2(h_{\hat{k}})^{\frac{1+\theta_2}{r_2}}\right) \\ &\le 4\beta\gamma\hat{k} \quad (\text{since } \mu_i(h_{\hat{k}}) \in [0,1]).\end{aligned}$$

We deduce that $\mu_1(M)^{\frac{1}{r_1}} + \mu_2(M)^{\frac{1}{r_2}}$ can be chosen arbitrarily small provided M is large enough, and then

$$\mu_1(2M)^{\frac{1}{r_1}} + \mu_2(2M)^{\frac{1}{r_2}} \le \mu_1(h_k)^{\frac{1}{r_1}} + \mu_2(h_k)^{\frac{1}{r_2}} \underset{k\to+\infty}{\longrightarrow} 0,$$

whence $c(t,x) \le 2M$ for a.e. $(t,x) \in Q_T$.

□

Proof of Theorem 5.8. Let $k \ge \|c^0\|_{L^\infty(\Omega)}$. We multiply equation (5.41) by c_k, integrate on Q_{t_1} for $t_1 \in (0,T)$ and integrate by parts to get, using the homogeneous Neumann boundary conditions,

$$\begin{aligned}\int_{Q_{t_1}(k)} \frac{1}{2}\partial_t(c_k^2) + \int_{Q_{t_1}(k)} d|\nabla c_k|^2 &= \int_{Q_{t_1}(k)} c\, u\cdot\nabla c_k + f c_k\ , \\ \frac{1}{2}\int_\Omega c_k^2(t_1) + \underline{d}\int_{Q_{t_1}(k)} |\nabla c_k|^2 &\le \int_{Q_{t_1}(k)} |c|\ |u|\ |\nabla c_k| + |f| c_k.\end{aligned}$$

Using Young's inequality to absorb the term ∇c_k in the left-hand side, there exists $\alpha = \alpha(\underline{d}) > 0$ such that

$$\alpha\left[\int_\Omega c_k^2(t_1) + \int_{Q_{t_1}(k)} |\nabla c_k|^2\right] \le \int_{Q_{t_1}(k)} |u|^2|c|^2 + |f|c_k,$$

and consequently

$$\alpha\|c_k\|^2_{V_2(Q_{t_1})} \le \int_{Q_{t_1}(k)} |u|^2|c|^2 + |f|c_k.$$

From now on, we impose $k \ge 1$, so that

$$\alpha\|c_k\|^2_{V_2(Q_{t_1})} \le \int_{Q_{t_1}(k)} |u|^2|c|^2 + |f|c_k \le 2\int_{Q_{t_1}(k)} (|u|^2 + |f|)(c_k^2 + k^2). \tag{5.48}$$

We now estimate the right-hand side as follows:

$$\begin{aligned}\int_{Q_{t_1}(k)} |u|^2(c_k^2+k^2) &\le \||u|^2\|_{\infty,r,Q_{t_1}(k)}\|c_k^2+k^2\|_{1,\frac{r}{r-1},Q_{t_1}(k)} \\ &\le \||u|^2\|_{\infty,r,Q_{t_1}(k)}\big(\|c_k\|^2_{2,\frac{2r}{r-1},Q_{t_1}(k)} + k^2\|1\|_{1,\frac{r}{r-1},Q_{t_1}(k)}\big).\end{aligned}$$

Using Hölder's inequality,

$$\|c_k\|_{2,\bar{r},Q_{t_1}(k)} \le \|c_k\|_{2(1+\theta^u),\hat{r},Q_{t_1}(k)}\ \mu_u(k)^{\frac{\theta^u}{2(1+\theta^u)}},$$

where
$$\mu_u(k) = \int_0^{t_1} \lambda(A_k(t))^{\frac{r-1}{r}} dt \; ; \; \overline{r} = \frac{2r}{r-1} \; ; \; \hat{r} = \overline{r}(1+\theta^u) \; ; \; \theta^u = \frac{2\theta_1^u}{N}.$$
It is easy to check that
$$\frac{1}{2} + \frac{N}{2\overline{r}} = \frac{N}{4} + \frac{\theta_1^u}{2} \quad ; \quad \frac{1}{2(1+\theta^u)} + \frac{N}{2\hat{r}} = \frac{N}{4},$$
and therefore we have the embedding $V_2(Q_{t_1}) \hookrightarrow L^{2(1+\theta^u)}(0,t_1;L^{\hat{r}}(\Omega))$ (see e.g. [69] p.74). As a consequence, there exists $\beta > 0$ (independent of t_1), such that
$$\|c_k\|^2_{2,\overline{r},Q_{t_1}(k)} \leq \beta^2 \|c_k\|^2_{V_2(Q_{t_1})} \mu_u(k)^{\frac{\theta^u}{1+\theta^u}}. \tag{5.49}$$
For the second term, we have
$$k^2 \|1\|_{1,\frac{r}{r-1},Q_{t_1}(k)} = k^2 \left(\int_0^{t_1} \lambda(A_k(t))^{\frac{r-1}{r}} dt \right) = k^2 \mu_u(k). \tag{5.50}$$
Similarly,
$$\begin{aligned}\int_{Q_{t_1}(k)} |f|(c_k^2 + k^2) &\leq \|f\|_{q,Q_{t_1}(k)} \big(\|c_k^2 + k^2\|_{\frac{q}{q-1},Q_{t_1}(k)} \big) \\ &\leq \|f\|_{q,Q_{t_1}(k)} \big(\|c_k\|^2_{\frac{2q}{q-1},Q_{t_1}(k)} + k^2 \|1\|_{\frac{q}{q-1},Q_{t_1}(k)} \big).\end{aligned}$$
Then using Hölder's inequality,
$$\|c_k\|_{\overline{q},Q_{t_1}(k)} \leq \|c_k\|_{\hat{q},Q_{t_1}(k)} \mu_f(k)^{\frac{1}{\overline{q}} - \frac{1}{\hat{q}}},$$
where
$$\mu_f(k) = \int_0^{t_1} \lambda(A_k(t)) dt \; ; \; \overline{q} = \frac{2q}{q-1} \; ; \; \hat{q} = \overline{q}(1+\theta^f) \; ; \; \theta^f = \frac{2\theta_1^f}{N}.$$
One can check that $\frac{1}{\overline{q}} + \frac{N}{2\overline{q}} = \frac{N}{4} + \frac{\theta_1^f}{2}$, $\frac{1}{\hat{q}} + \frac{N}{2\hat{q}} = \frac{N}{4}$, so $V_2(Q_{t_1}) \hookrightarrow L^{\hat{q}}(Q_{t_1})$ and therefore
$$\|c_k\|^2_{\frac{2q}{q-1},Q_{t_1}(k)} \leq \beta^2 \|c_k\|^2_{V_2(Q_{t_1})} \mu_f(k)^{\frac{2\theta^f}{\hat{q}}}. \tag{5.51}$$
The last term is
$$k^2 \|1\|_{\frac{q}{q-1},Q_{t_1}(k)} = k^2 \mu_f(k)^{\frac{2(1+\theta^f)}{\hat{q}}}. \tag{5.52}$$
Going back to (5.48) and using (5.49) – (5.52), there exists $C > 0$ depending only on β, $\||u|^2\|_{\infty,r,Q_T(k)}$ and $\|f\|_{L^q(Q_T)}$ (but not on t_1), such that for all $k \geq \max(\|c^0\|_{L^\infty(\Omega)}, 1)$,
$$\alpha \|c_k\|^2_{V_2(Q_{t_1})} \leq C \left[\|c_k\|^2_{V_2(Q_{t_1})} \big(\mu_u(k)^{\frac{\theta^u}{1+\theta^u}} + \mu_f(k)^{\frac{2\theta^f}{\hat{q}}} \big) + k^2 \big(\mu_u(k) + \mu_f^{\frac{2(1+\theta^f)}{\hat{q}}}(k) \big) \right]. \tag{5.53}$$
We now choose $t_1 \in (0,T)$ small enough so that
$$C\big(\mu_u(k)^{\frac{\theta^u}{1+\theta^u}} + \mu_f(k)^{\frac{2\theta^f}{\hat{q}}} \big) \leq \frac{\alpha}{2} \quad ; \quad t_1 \lambda(\Omega)^{\frac{r-1}{r}} \leq 1 \quad ; \quad t_1 \lambda(\Omega) \leq 1.$$

This is the case provided

$$C\big(t_1^{\frac{\theta^u}{1+\theta^u}}\lambda(\Omega)^{\frac{2\theta^u}{\hat r}}+t_1^{\frac{2\theta^f}{\hat q}}\lambda(\Omega)^{\frac{2\theta^f}{\hat q}}\big)\le\frac{\alpha}{2}\quad;\quad t_1\lambda(\Omega)^{\frac{r-1}{r}}\le 1\quad;\quad t_1\lambda(\Omega)\le 1. \tag{5.54}$$

For t_1 satisfying (5.54), inequality (5.53) yields, if $\hat{k}=\max(\|c^0\|_{L^\infty(\Omega)},1)$,

$$\forall k\ge\hat{k},\quad \frac{\alpha}{2}\|c_k\|^2_{V_2(Q_{t_1})}\le k^2C\big(\mu_u(k)^{\frac{2(1+\theta^u)}{2(1+\theta^u)}}+\mu_f(k)^{\frac{2(1+\theta^f)}{\hat q}}\big). \tag{5.55}$$

Moreover, for all $k\ge\hat{k},\ \mu_u(k),\mu_f(k)\in[0,1]$, so we can apply Lemma 5.9 and c is bounded on Q_{t_1}. Remark that t_1 does not depend on $\hat{k}$. Then we may subdivide $Q_T=\Omega\times(0,T)$ in a finite sequence of cylinders $\Omega\times(t_i,t_{i+1})$, $i=1\ldots,P$, whose altitudes $(t_{i+1}-t_i)$ are subject to the requirement (5.54). Applying the above result on each cylinder, we get that c is bounded on Q_T, which ends the proof of Theorem 5.8.

6

Fast-reaction limit for $\mathbf{C_1 + C_2 \rightleftharpoons C_3}$ with advection

In this section, we investigate the fast-reaction limit in the reaction-diffusion system with reaction $C_1 + C_2 \rightleftharpoons C_3$ in a more complex situation than in Section 2. We take into account the fluid's bulk motion, whose velocity field is assumed to be a data of the problem. Mass fluxes are now the sum of advection and Fickian diffusion terms, where in the latter the diffusion coefficients depend on time and space. For general mixture velocity fields, we prove the convergence of the solution c^k with finite reaction speed k when $k \to +\infty$, in $L^p(Q_T)$ for any $p \in [1,2)$ and $T > 0$. If, in addition, the divergence of the velocity field is assumed to be bounded, we prove the convergence of c^k in $L^2(Q_T)$.

6.1 Introduction

The fast-reaction limit case in the chemical reaction $C_1 + C_2 \rightleftharpoons C_3$ in the presence of Fickian diffusion with constant coefficients has been studied in Section 2, where we prove that if $d_i, \kappa > 0$ are given positive constants, $k > 0$ is the reaction speed, and if c^k is the solution of

$$\begin{cases} \left.\begin{aligned} \partial_t c_1 - d_1 \Delta c_1 &= -k(c_1 c_2 - \kappa c_3) \\ \partial_t c_2 - d_2 \Delta c_2 &= -k(c_1 c_2 - \kappa c_3) \\ \partial_t c_3 - d_3 \Delta c_3 &= +k(c_1 c_2 - \kappa c_3) \end{aligned}\right\} \text{ on } (0,+\infty)\times\Omega, \\ \partial_\nu c = 0 \text{ on } (0,+\infty)\times\partial\Omega, \\ c(0,\cdot) = c^0 \text{ on } \Omega, \end{cases} \tag{6.1}$$

then c^k converges when $k \to +\infty$ in $L^2(Q_T)$ and weakly in $L^{\frac{4}{3}}(0,T;W^{1,\frac{4}{3}}(\Omega))$ for any $T > 0$ to a weak solution of

$$\begin{cases} \left.\begin{aligned} &\partial_t (c_1 + c_3) - \Delta(d_1 c_1 + d_3 c_3) = 0 \\ &\partial_t (c_2 + c_3) - \Delta(d_2 c_2 + d_3 c_3) = 0 \\ &c_1 c_2 = \kappa c_3 \end{aligned}\right\} \text{ on } (0,+\infty)\times\Omega, \\ \partial_\nu c = 0 \text{ on } (0,+\infty)\times\partial\Omega, \\ c(0,\cdot) = c^0 \text{ on } \Omega. \end{cases} \tag{6.2}$$

In this section, we prove that the techniques of Section 2 are robust enough to carry over to the case when the diffusivities d_i depend on time and space, and when convective mass transfer is taken into account, with a given velocity field. For rather general velocity fields, we are able to prove the convergence of the solution of the generalized system (6.1) as $k \to +\infty$, strongly in $L^{\frac{4}{3}}(Q_T)$ and weakly in $L^{\frac{4}{3}}(0,T;W^{1,\frac{4}{3}}(\Omega))$ for any $T>0$. If, in addition, the compressibility of the fluid is assumed to be bounded, we recover the convergence in the same spaces as in Section 2.

More precisely, we consider

$$\left.\begin{array}{rcll} \partial_t c_1 + \mathrm{div}(-d_1\nabla c_1 + c_1 u) &=& -k^n(c_1c_2-\kappa^n c_3) & \text{on } (0,\infty)\times\Omega \\ \partial_t c_2 + \mathrm{div}(-d_2\nabla c_2 + c_2 u) &=& -k^n(c_1c_2-\kappa^n c_3) & \text{on } (0,\infty)\times\Omega \\ \partial_t c_3 + \mathrm{div}(-d_3\nabla c_3 + c_3 u) &=& k^n(c_1c_2-\kappa^n c_3) & \text{on } (0,\infty)\times\Omega \\ -d_i\nabla c_i\cdot\nu + c_i u\cdot\nu &=& 0 & \text{on } (0,\infty)\times\partial\Omega \\ c_i(0,\cdot) &=& c_i^0 & \text{on } \Omega \end{array}\right\}, \qquad (R^n)$$

where $n\in\mathbb{N}$, $k^n,\kappa^n>0$. For the data, we work with the same assumptions as in Section 5, *i.e.* Ω is a smooth bounded domain and we assume that d_i,u,c^0 satisfy $(i)-(iii)$ on p. 140. Remark that the reaction terms in (R^n) also satisfy assumption (v) on p.140, so from Theorem 5.1, for any $n\in\mathbb{N}$, (R^n) has a unique global weak solution $c^n=(c_1^n,c_2^n,c_3^n):(0,+\infty)\times\Omega\to[0,+\infty)^3$.

We are interested in the limit behaviour of c^n when

$$K^n=(k^n,\kappa^n)\to(+\infty,\kappa^\infty), \quad \kappa^\infty\in(0,+\infty). \tag{6.3}$$

In this regard, our main result is the following

Theorem 6.1. *Assume $(i)-(iii)$ on p.140 and (6.3). Then for all $T>0$, up to a subsequence, $(c^n)_{n\in\mathbb{N}}$ converges strongly in $L^{\frac{4}{3}}(Q_T)$ and weakly in $L^{\frac{4}{3}}(0,T;W^{1,\frac{4}{3}}(\Omega))$ to a nonnegative function $c=(c_1,c_2,c_3)$ satisfying*

$$\left.\begin{array}{l} c_1c_2=\kappa^\infty c_3, \\ \forall\psi\in C^\infty(\overline{Q_T}) \text{ such that } \psi(T)=0,\ \forall j\in\{1,2\}, \\ -\displaystyle\int_\Omega (c_j^0+c_3^0)\psi(0)+\int_{Q_T}\Big(-(c_j+c_3)\psi_t+[d_j\nabla c_j+d_3\nabla c_3-(c_j+c_3)u]\cdot\nabla\psi\Big)=0. \end{array}\right\} \tag{6.4}$$

6.2 Proof of the main theorem

The proof is based on *a priori* estimates independent of n, which provide the relative compactness of $(c^n)_{n\in\mathbb{N}}$. To derive these estimates, the fact that Theorem 5.1 provides not only the existence of global solutions for (R^n) but also a good approximation sequence of these solutions by smooth functions will help to avoid technical difficulties: we may first derive estimates on smooth approximations of c^n, and then use the convergence result (5.31) to prove that they remain valid for c^n.

These estimates are the content of two lemmas:

- Lemma 6.2 provides an estimate in $L^2(Q_T)$ for solutions of a certain class of parabolic equations, generalizing a technique from [90] to mass fluxes with advection and variable diffusivities. It is interesting to notice that the same assumption $u\in L^\infty(0,T;L^r(\Omega))$, $r>\max(2,N)$ as in Section 5, is sufficient to extend these L^2-estimates to our situation.
- In Lemma 6.3, we prove that the convection terms do not destroy the usual entropy estimate, provided they are bounded in $L^\infty(0,T;L^r(\Omega))$ for $r>\max(2,N)$.

Lemma 6.2. *Let $A \in C(\overline{Q_T})$ such that $0 < \underline{a} \le A \le \overline{a} < +\infty$, $u \in L^\infty(0,T;L^r(\Omega))$, $H \in L^2(Q_T)$, $r > \max(2,N)$. Let W be a classical solution of*

$$\begin{cases} \partial_t W + \mathrm{div}[-\nabla(AW) + Wu] \le H \text{ on } Q_T \ ; \\ -\nabla(AW)\cdot\nu + Wu\cdot\nu = 0 \text{ on } \Sigma_T \ ; \ W(0,\cdot) = W^0 \text{ on } \Omega. \end{cases} \tag{6.5}$$

Then there exists $C > 0$ depending only on $T, \underline{a}, \overline{a}$ and $\|u\|_{L^\infty(0,T;L^r(\Omega))}$, such that

$$\|W^+\|_{L^2(Q_T)} \le C\left(\|W^0\|_{L^2(\Omega)} + \|H\|_{L^2(Q_T)}\right). \tag{6.6}$$

Proof. Let $\Theta \in C_0^\infty(Q_T, \mathbb{R}_+)$. We consider the dual problem

$$-[\partial_t \Psi + A\Delta\Psi + u\cdot\nabla\Psi] = \Theta \text{ on } Q_T \ ; \ \partial_\nu \Psi = 0 \text{ on } \Sigma_T \ ; \ \Psi(T,\cdot) = 0 \text{ on } \Omega. \tag{6.7}$$

According to [38], Theorem 2.1, (6.7) has a strong solution $\Psi \ge 0$. We multiply (6.5) by Ψ and integrate over Q_T and by parts to obtain

$$\begin{aligned} \int_{Q_T} (\partial_t W + \mathrm{div}(-\nabla(AW) + Wu))\Psi &\le \int_{Q_T} H\Psi, \\ -\int_{Q_T} W(\partial_t \Psi + A\Delta\Psi + u\cdot\nabla\Psi) &\le \int_{Q_T} H\Psi + \int_\Omega W^0\Psi(0), \end{aligned}$$

thus

$$\left|\int_{Q_T} W\Theta\right| \le \|W^0\|_{L^2(\Omega)}\|\Psi(0)\|_{L^2(\Omega)} + \|H\|_{L^2(Q_T)}\|\Psi\|_{L^2(Q_T)}. \tag{6.8}$$

We now estimate $\|\Psi(0)\|_{L^2(\Omega)}$ and $\|\Psi\|_{L^2(Q_T)}$ in terms of $\|\Theta\|_{L^2(Q_T)}$. Multiply (6.7) by $-\Delta\Psi$ and integrate over Ω and by parts to get, using the homogeneous Neumann boundary conditions,

$$\begin{aligned} -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Psi\|^2_{L^2(\Omega)} + \|\sqrt{A}\Delta\Psi\|^2_{L^2(\Omega)} &= -\int_\Omega u\cdot\nabla\Psi\,\Delta\Psi - \int_\Omega \Theta\Delta\Psi \\ &\le \|u\|_{L^r(\Omega)}\|\nabla\Psi\|_{L^p(\Omega)}\|\Delta\Psi\|_{L^2(\Omega)} + \|\Theta\|_{L^2(\Omega)}\|\Delta\Psi\|_{L^2(\Omega)}, \end{aligned}$$

where we used Hölder's inequality and $p > 1$ is defined by $1/r + 1/p = 1/2$. Then the Gagliardo-Nirenberg inequality (see *e.g.* [69]) yields the existence of $C > 0$ such that

$$\|\nabla\Psi\|_{L^p(\Omega)} \le C\|\nabla\Psi\|^{N/r}_{W^{1,2}(\Omega)}\|\nabla\Psi\|^{1-N/r}_{L^2(\Omega)}. \tag{6.9}$$

Since $u \in L^\infty(0,T;L^r(\Omega))$, using Young's inequality and (6.9), for $\varepsilon_i > 0$ there are $C_i > 0$ such that

$$\begin{aligned} -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}&\|\nabla\Psi\|^2_{L^2(\Omega)} + \underline{a}\|\Delta\Psi\|^2_{L^2(\Omega)} \\ &\le \varepsilon_1\|\Delta\Psi\|^2_{L^2(\Omega)} + C_{\varepsilon_1}(\|\nabla\Psi\|^2_{L^p(\Omega)} + \|\Theta\|^2_{L^2(\Omega)}) \\ &\le \varepsilon_2(\|\Delta\Psi\|^2_{L^2(\Omega)} + \|\nabla\Psi\|^2_{W^{1,2}(\Omega)}) + C_{\varepsilon_2}(\|\nabla\Psi\|^2_{L^2(\Omega)} + \|\Theta\|^2_{L^2(\Omega)}) \\ &\le \varepsilon_3\|\Delta\Psi\|^2_{L^2(\Omega)} + C_{\varepsilon_3}(\|\nabla\Psi\|^2_{L^2(\Omega)} + \|\Theta\|^2_{L^2(\Omega)}), \end{aligned}$$

where for the last inequality, we used that $\|\nabla\Psi\|_{W^{1,2}(\Omega)} \le C(\Omega)\|\Delta\Psi\|_{L^2(\Omega)}$: since $\partial_\nu\Psi = 0$ on $\partial\Omega$ and Ω is smooth, this is a consequence of elliptic regularity, see *e.g.* [30]. Thus, if we choose $\varepsilon_3 < \frac{\underline{a}}{2}$ and apply Gronwall's lemma, using $\Psi(T) = 0$,

$$\sup_{0\le t\le T}\|\nabla\Psi(t)\|_{L^2(\Omega)} \le C\|\Theta\|_{L^2(Q_T)} \ ; \ \|\Delta\Psi\|_{L^2(Q_T)} \le C\|\Theta\|_{L^2(Q_T)}. \tag{6.10}$$

Since $A \leq \overline{a}$, we also have $\|A\Delta\Psi\|_{L^2(Q_T)} \leq C\|\Theta\|_{L^2(Q_T)}$. Then, integration of (6.7) on Q_t for any $t \in (0,T)$ yields

$$\|\Psi\|_{L^\infty(0,T;L^1(\Omega))} \leq C\|\Theta\|_{L^2(Q_T)}. \tag{6.11}$$

Finally, combining (6.10), (6.11) and using Poincaré-Wirtinger's inequality, we get

$$\|\Psi(0)\|_{L^2(\Omega)} + \|\Psi\|_{L^2(Q_T)} \leq C\|\Theta\|_{L^2(Q_T)},$$

whence (6.6) by duality.

□

Lemma 6.3. *Let $r > \max(2,N)$, $u \in L^\infty(0,T;L^r(\Omega))$ and let c^n be the solution of (R^n) on Q_T. There exists $C > 0$ depending only on $\|u\|_{L^\infty(0,T;L^r(\Omega))}$, $\|c_i^0 \log^+ c_i^0\|_{L^1(\Omega)}$, but not on n, such that*

$$k^n \int_{Q_T} (c_1^n c_2^n - \kappa^n c_3^n)(\log(c_1^n c_2^n) - \log(\kappa^n c_3^n)) + \sum_{i=1}^3 \frac{d(T)}{2} \int_{Q_T} \frac{|\nabla c_i^n|^2}{c_i^n} \leq C. \tag{6.12}$$

Proof. Set

$$V_i^n = c_i^n \log \frac{c_i^n}{c_i^{n^*}} - (c_i^n - c_i^{n^*}) \geq 0 \;;\; V^n = \sum_{i=1}^3 V_i^n, \tag{6.13}$$

where $c_i^{n^*}$ are positive numbers such that $c_1^{n^*} c_2^{n^*} = \kappa^n c_3^{n^*}$. Assume first that c^n is a classical solution of (R^n). A straightforward computation, taking into account the no-flux boundary conditions, yields

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega V^n &= \sum_{i=1}^3 \int_\Omega \operatorname{div}(d_i \nabla c_i^n - c_i^n u) \log \frac{c_i^n}{c_i^{n^*}} - k^n \int_\Omega (c_1^n c_2^n - \kappa^n c_3^n)(\log(c_1^n c_2^n) - \log(\kappa^n c_3^n)) \\
&= \sum_{i=1}^3 \int_\Omega -d_i \frac{|\nabla c_i^n|^2}{c_i^n} + u \cdot \nabla c_i^n - k^n \int_\Omega (c_1^n c_2^n - \kappa^n c_3^n)(\log(c_1^n c_2^n) - \log(\kappa^n c_3^n)).
\end{aligned} \tag{6.14}$$

Using Hölder's inequality with exponents $1 = \frac{1}{r} + \frac{1}{r'} = \frac{r'}{2} + \frac{2-r'}{2}$ and Young's inequality, since $u \in L^\infty(0,T;L^r(\Omega))$, we have

$$\begin{aligned}
\int_\Omega u \cdot \nabla c_i^n \leq \|u\|_{L^r(\Omega)} \|\nabla c_i^n\|_{L^{r'}(\Omega)} &\leq C\left[\int_\Omega \left(\frac{|\nabla c_i^n|^2}{c_i^n}\right)^{\frac{r'}{2}} (c_i^n)^{\frac{r'}{2}}\right]^{\frac{1}{r'}} \\
&\leq C\left[\int_\Omega \frac{|\nabla c_i^n|^2}{c_i^n}\right]^{\frac{1}{2}} \left[\int_\Omega (c_i^n)^{\frac{r'}{2-r'}}\right]^{\frac{2-r'}{2r'}} \\
&\leq \varepsilon \int_\Omega \frac{|\nabla c_i^n|^2}{c_i^n} + C_\varepsilon \|c_i^n\|_{L^{\frac{r'}{2-r'}}(\Omega)}.
\end{aligned} \tag{6.15}$$

To absorb the last term, we use that since $r > N$, we have $\frac{2-r'}{2r'} > \frac{1}{2} - \frac{1}{N}$ and the embedding $W^{1,2}(\Omega) \hookrightarrow L^{\frac{2r'}{2-r'}}(\Omega)$ is compact: as a consequence, there exists $C'_\varepsilon > 0$ such that

$$\|c_i^n\|_{L^{\frac{r'}{2-r'}}} = \|\sqrt{c_i^n}\|^2_{L^{\frac{2r'}{2-r'}}} \leq \varepsilon \|\nabla \sqrt{c_i^n}\|^2_{L^2(\Omega)} + C'_\varepsilon \|c_i^n\|_{L^1(\Omega)} = \frac{\varepsilon}{4} \int_\Omega \frac{|\nabla c_i^n|^2}{c_i^n} + C'_\varepsilon \|c_i^n\|_{L^1(\Omega)}. \tag{6.16}$$

Note that due to mass conservation in (R^n), c_i^n is bounded independently of n in $L^\infty(0,+\infty;L^1(\Omega))$. Choosing ε small enough, using $0<\underline{d}(T)\le d_i$ and integrating (6.14) on $(0,T)$, it follows from (6.15)-(6.16) that there exists $C>0$ independent of n and T, such that

$$k^n\int_{Q_T}(c_1^nc_2^n-\kappa^nc_3^n)(\log(c_1^nc_2^n)-\log(\kappa^nc_3^n))+\sum_{i=1}^3\int_{Q_T}\frac{d_i}{2}\frac{|\nabla c_i^n|^2}{c_i^n}+\int_\Omega V^n(T)\le\int_\Omega V^n(0)+C. \tag{6.17}$$

Setting for instance $c_1^{n^*}=c_2^{n^*}=c_3^{n^*}=\kappa^n\to\kappa^\infty$, it is easy to see that $\int_\Omega V^n(0)$ only depends on $\|c_i^0\log^+c_i^0\|_{L^1(\Omega)}$. Since $V^n(T)\ge 0$, (6.17) yields (6.12) for smooth c^n. As c^n only has the regularity stated in Theorem 5.1, we approximate c^n by the sequence used in the proof of Theorem 5.1: utilizing the result above, estimate (6.12) is valid for approximate solutions, and applying the convergence result (5.31), it remains valid for c^n.

□

Proof of Theorem 6.1. Throughout the proof, c^n denotes the solution of (R^n) on $(0,+\infty)\times\Omega$. For $i\in\{1,2\}$, we introduce

$$W_i^n=1+c_i^n+c_3^n\ ;\ W_i^0=1+c_i^0+c_3^0\ ;\ A_i^n=\frac{1+d_ic_i^n+d_3c_3^n}{1+c_i^n+c_3^n}\ ;\ \tilde{u}_i^n=\frac{c_i^n\nabla d_i+c_3^n\nabla d_3-u}{1+c_i^n+c_3^n}+u.$$

Since $c^n\ge 0$, using notations (5.4), we have for all $T>0$,

$$\tilde{u}_i^n \text{ is bounded in } L^\infty(0,T;L^r(\Omega)) \text{ for } r>\max(2,N)\ ;$$
$$0<\min(1,\underline{d}(T))\le A_i^n\le\max(1,\overline{d}(T))<+\infty.$$

Remark that

$$0=\partial_tW_i^n+\operatorname{div}(-d_i\nabla c_i^n-d_3\nabla c_3^n-u+W_i^nu), \tag{6.18}$$
$$0=\partial_tW_i^n+\operatorname{div}\left(-\nabla(A_i^nW_i^n)+W_i^n\tilde{u}_i^n\right). \tag{6.19}$$

Similarly as in the proof of Lemma 6.3, we may first consider a regularized version $\tilde{c}^n$ of c^n to smoothen A_i^n. Then Lemma 6.2 is applicable to the corresponding $(\tilde{W}_i^n,\tilde{A}_i^n)$, and therefore $(\tilde{W}_i^n)_{n\in\mathbb{N}}$ is bounded in $L^2(Q_T)$. Using the convergence result (5.31), this bound remains valid for W_i^n and consequently

$$\exists C>0:\forall n\in\mathbb{N},\quad \|W_1^n\|_{L^2(Q_T)}+\|W_2^n\|_{L^2(Q_T)}+\|c^n\|_{L^2(Q_T)^3}\le C. \tag{6.20}$$

According to Lemma 6.3, we know that

$$\exists C>0:\forall n\in\mathbb{N},\forall i\in\{1,2,3\},\quad \int_{Q_T}\frac{|\nabla c_i^n|^2}{c_i^n}\le C.$$

Combined with (6.20) and using Hölder's inequality, this yields

$$\exists C>0:\forall n\in\mathbb{N},\quad \int_{Q_T}|\nabla c_i^n|^{\frac{4}{3}}\le\left(\int_{Q_T}\frac{|\nabla c_i^n|^2}{c_i^n}\right)^{\frac{2}{3}}\left(\int_{Q_T}(c_i^n)^2\right)^{\frac{1}{3}}\le C. \tag{6.21}$$

Since c^n is also bounded in $L^2(Q_T)^P$, $(c^n)_{n\in\mathbb{N}}$ is bounded in $L^{\frac{4}{3}}(0,T;W^{1,\frac{4}{3}}(\Omega)^P)$. Let us now prove that

$$(W_i^n)_{n\in\mathbb{N}} \text{ is relatively compact in } L^{\frac{4}{3}}(Q_T). \tag{6.22}$$

Remark that since $r > \max\{2,N\}$, $W^{1,\frac{4}{3}}(\Omega) \hookrightarrow L^{\frac{4r}{3r-4}}(\Omega)$. As a consequence,

$$\begin{aligned}\|W_i^n u\|_{L^{\frac{4}{3}}(0,T;\,L^{\frac{4}{3}}(\Omega))} &\leq \|W_i^n\|_{L^{\frac{4}{3}}(0,T;\,L^{\frac{4r}{3r-4}}(\Omega))}\|u\|_{L^\infty(0,T;\,L^r(\Omega))}\\ &\leq C\|W_i^n\|_{L^{\frac{4}{3}}(0,T;\,W^{1,\frac{4}{3}}(\Omega))}\|u\|_{L^\infty(0,T;\,L^r(\Omega))},\end{aligned}$$

and the right-hand side is bounded independently of n. Using equation (6.18) , $\partial_t W_i^n$ is bounded in $L^{\frac{4}{3}}(0,T;W^{-1,\frac{4}{3}}(\Omega))$. Since W_i^n is also bounded in $L^{\frac{4}{3}}(0,T;W^{1,\frac{4}{3}}(\Omega))$ and

$$W^{1,\frac{4}{3}}(\Omega) \overset{c}{\hookrightarrow} L^{\frac{4}{3}}(\Omega) \hookrightarrow W^{-1,\frac{4}{3}}(\Omega),$$

Corollary 4 in [98] yields (6.22).

Putting together the above estimates, there exists $c \in L^{\frac{4}{3}}_{loc}([0,+\infty);W^{1,\frac{4}{3}}(\Omega))$, such that up to a diagonal extraction, for all $T > 0$,

$$c_i^n \underset{n\to+\infty}{\longrightarrow} c_i \quad \text{weakly in } L^{\frac{4}{3}}(0,T;W^{1,\frac{4}{3}}(\Omega)),\ i \in \{1,2,3\}. \tag{6.23}$$

Then for all $T > 0$, we may pass to the limit $n \to +\infty$ in the variational formulation

$$\forall \psi \in C^\infty(\overline{Q_T}) \text{ such that } \psi(T) = 0,\ \forall j \in \{1,2\},$$
$$-\int_\Omega (c_j^0 + c_3^0)\psi(0) + \int_{Q_T}\Big(-(c_j^n + c_3^n)\psi_t + \big[d_j\nabla c_j^n + d_3\nabla c_3^n - (c_j^n + c_3^n)u\big]\cdot\nabla\psi\Big) = 0.$$

It remains to prove the strong convergence of c^n in $L^{\frac{4}{3}}(Q_T)^3$ and $c_1c_2 = \kappa^\infty c_3$.

According to Lemma 6.3, $((c_1^n c_2^n - \kappa^n c_3^n)(\log(c_1^n c_2^n) - \log(\kappa^n c_3^n)))_{n\in\mathbb{N}}$ goes to zero in $L^1(Q_T)$ for any $T > 0$ when $n \to +\infty$. Consequently, up to a subsequence, $c_1^n c_2^n - \kappa^n c_3^n \to 0$ *a.e.* in $(0,+\infty)\times\Omega$.

Using (6.22), up to a subsequence, $W_i^n(t,x) = c_i^n(t,x) + c_3^n(t,x)$ converges for all $(t,x) \in Q_T\backslash Z$, where Z is of Lebesgue-measure zero, and therefore $c^n(t,x)$ is bounded for all $(t,x) \in Q_T\backslash Z$.

Let $(t,x) \in Q_T\backslash Z$ and let $\alpha = (\alpha_1,\alpha_2,\alpha_3) \in [0,+\infty)^3$ be a limit point of $(c^n(t,x))_{n\in\mathbb{N}}$. Then one easily check that

$$\alpha_1 + \alpha_3 = c_1(t,x) + c_3(t,x)\ ;\ \alpha_2 + \alpha_3 = c_2(t,x) + c_3(t,x)\ ;\ \alpha_1\alpha_2 = \kappa^\infty\alpha_3. \tag{6.24}$$

The only nonnegative solution of (6.24) is $\alpha = c(t,x)$ (see also Section 2), so $c^n(t,x)$ converges to $c(t,x)$ for all $(t,x) \in Q_T\backslash Z$ and $c_1c_2 = \kappa^\infty c_3$.

Since $0 \leq c_i^n, c_3^n \leq W_i^n$, using the pointwise convergence of c^n and the strong convergence of W_i^n in $L^{\frac{4}{3}}(Q_T)$, the dominated convergence theorem yields the strong convergence of c^n in $L^{\frac{4}{3}}(Q_T)^3$.

This ends the proof of Theorem 6.1.

□

6.3 Convergence in $\mathbf{L^2(Q_T)}$

Note that in the proof of Theorem 6.1, from the boundedness of c^n in $L^2(Q_T)$, the sequence $(c^n)_{n\in\mathbb{N}}$ is weakly relatively compact in $L^2(Q_T)$. Since it converges *a.e.* in Q_T, using the Egorov theorem, it is strongly relatively compact in $L^p(Q_T)$ for $p \in [1,2)$.

In this section, under the additional assumption that the compressibility of the fluid is bounded, we are able to prove the following

Proposition 6.4. *In addition to the hypothesis of Theorem 6.1, assume* $\operatorname{div} u \in L^\infty_{loc}([0,+\infty); L^\infty(\Omega))$. *Then for any* $T > 0$, $(c^n)_{n\in\mathbb{N}}$ *is relatively compact in* $L^2(Q_T)$.

Proof. Using notation (6.13), remark that V_i^n satisfies

$$\partial_t V_i^n + \operatorname{div}(-d_i \nabla V_i^n + V_i^n u) = r_i^n \log(c_i^n / c_i^{n^*}) - d_i \frac{|\nabla c_i^n|^2}{c_i^n} - (c_i^n - c_i^{n^*}) \operatorname{div} u, \tag{6.25}$$

where $r_i^n = \varepsilon_i k^n (c_1^n c_2^n - \kappa_n c_3^n)$, $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (-1,-1,1)$. Similarly as in the proof of Theorem 6.1, setting

$$V^n = 1 + \sum_{i=1}^3 V_i^n \ ; \ A^n = \frac{1+\sum_{i=1}^3 d_i V_i^n}{1+\sum_{i=1}^3 V_i^n} \ ; \ \tilde{u}^n = \frac{\sum_{i=1}^3 V_i^n \nabla d_i - u}{1+\sum_{i=1}^3 V_i^n} + u,$$

and summing (6.25) over i, we get

$$\begin{aligned}
&\partial_t V^n - \operatorname{div}[\nabla(A^n V^n) - V^n \tilde{u}^n] \\
&= -k^n (c_1^n c_2^n - \kappa^n c_3^n)(\log(c_1^n c_2^n) - \log(\kappa^n c_3^n)) - \sum_{i=1}^3 d_i \frac{|\nabla c_i^n|^2}{c_i^n} - \sum_{i=1}^3 (c_i^n - c_i^{n^*}) \operatorname{div} u \\
&\le -\sum_{i=1}^3 (c_i^n - c_i^{n^*}) \operatorname{div} u \\
&\le C \|\operatorname{div} u\|_{L^\infty(Q_T)} \left(1 + \sum_{i=1}^3 c_i^n\right).
\end{aligned}$$

Then applying Lemma 6.2 (on a regularized version of V^n), $(V^n)_{n\in\mathbb{N}}$ is bounded in $L^2(Q_T)$, and consequently $(c_i^n \log c_i^n)_{n\in\mathbb{N}}$ is bounded in $L^2(Q_T)$ by a constant $C > 0$. Let $B \subset Q_T$ be measurable, with $|B| < \eta$. Let $\varepsilon > 0$, for any $K > 1$, we have

$$\begin{aligned}
\int_B (c_i^n)^2 &= \int_{B\cap\{c_i^n \le K\}} (c_i^n)^2 + \int_{B\cap\{c_i^n > K\}} (c_i^n)^2 \\
&\le K^2 \eta + \frac{1}{(\log K)^2} \int_{B\cap\{c_i^n > K\}} (c_i^n \log c_i^n)^2 \\
&\le K^2 \eta + \frac{C}{(\log K)^2},
\end{aligned}$$

so choosing K large enough and η small enough, we have $\int_B (c_i^n)^2 < \varepsilon$, thus $(c_i^n)_{n\in\mathbb{N}}$ is uniformly integrable in $L^2(Q_T)$. We proved in Theorem 6.1 that there exists a subsequence of c^n that converges *a.e.* in Q_T, so using a Vitali-type argument, this subsequence converges strongly in $L^2(Q_T)$. $\square$

7

Global existence for diffusion-electromigration systems in any space dimension

In this section, we prove global existence of weak solutions for a diffusion-electromigration system, in any space dimension. Theorem 5.1 is used as a tool to derive the existence of solutions for an approximate system, where the total charge density is regularized. The crucial point is that the approximation procedure preserves the Lyapunov structure of the original system. It is even possible to compute explicitely the dissipation rate, and the corresponding estimates provide compactness for the approximate solutions.

7.1 Introduction

Let Ω be a smooth open bounded subset of $\mathbb{R}^N$. We are interested in the system

$$\left\{ \begin{array}{rcll} \partial_t c_i - \mathrm{div}(d_i \nabla c_i + d_i z_i c_i \nabla \Phi) & = & 0 & \text{on } (0,+\infty)\times\Omega,\ i\in\{1,\dots,P\}, \\ \partial_\nu c_i + z_i c_i \partial_\nu \Phi & = & 0 & \text{on } (0,+\infty)\times\partial\Omega,\ i\in\{1,\dots,P\}, \\ -\Delta\Phi - \sum_{i=1}^P z_i c_i & = & 0 & \text{on } (0,+\infty)\times\Omega, \\ \partial_\nu \Phi + \tau\Phi & = & \xi & \text{on } (0,+\infty)\times\partial\Omega, \\ c(0,\cdot) & = & c^0 & \text{on } \Omega, \end{array} \right. \tag{7.1}$$

whose unknowns are $(c_1,\dots,c_P,\Phi)$. This system - commonly referred to as Debye-Hückel system or Nernst-Planck-Poisson system - describes the evolution of the concentrations c_i of P chemical species placed in an electrolyte. These species may be charged, with charge number $z_i \in \mathbb{Z}$. The function Φ represents the electrical potential inside the electrolyte. The boundary condition for Φ may be motivated by considering locally the boundary $\partial\Omega$ as a plate capacitor: $\tau > 0$ represents the so-called "capacity" of the boundary, and the function ξ is connected with some exterior potential. For more details, we refer to [24].

The mathematical treatment of this problem has gained quite some attention during the past two decades. For space dimension $N = 2$, well-posedness and long-time behaviour of (7.1) is already well-understood: in [15] existence and uniqueness of global weak solutions is shown, as well as convergence to uniquely determined steady states. For sufficiently smooth data, it is proven in [33] that there is a unique global classical solution. These results have been improved in [14] by computing an explicit exponential convergence rate with the help of logarithmic Sobolev inequalities. In the papers [51, 55, 56, 57] the authors include in the model quite general reaction terms coming from mass-action kinetics chemistry, and prove global well-posedness and exponential convergence to the steady state. In recent years system (7.1) has been supplemented by the Navier-Stokes equations modeling the fluid flow, see e.g. [24, 36, 95, 96].

So far, global well-posedness in dimension $N = 3$, even for time and space independent diffusivites, has only been shown under additional assumptions. These include initial data lying close to the steady state [15], or the *a priori* knowledge that the solution c is bounded in $L^\infty(0,T;L^2(\Omega))$ independently of $T > 0$ [33]. In [64], existence of global weak solutions for constant diffusivities is shown in the more general setting of the Navier-Stokes-Nernst-Planck-Poisson system, but for $P = 2$, which provides additional structure and estimates. In the present work, we prove the existence of global solutions in the case of time and space dependent diffusivities and without any restriction on the number of chemical species. Our proof is based on the energy method: it relies on the physical structure of the equations, and exploits the available Lyapunov functional for system (19).

Throughout the section, $\Omega \subset \mathbb{R}^N$ is an open, bounded domain, whose boundary $\partial\Omega$ is assumed to be smooth. The normal exterior vector (resp. the normal exterior derivative) on $\partial\Omega$ is denoted by ν (resp. ∂_ν). We also use the common notations $Q_T = \Omega \times (0,T)$, $\Sigma_T = \Omega \times (0,T)$ for $T > 0$.

Our requirements on the data are

(i) For $i = 1,\dots,P$, $d_i \in L^\infty_{loc}([0,+\infty);L^\infty(\Omega))$; for any $T > 0$, there exist $\underline{d}(T), \overline{d}(T) > 0$ such that

$$0 < \underline{d}(T) \le d_i \le \overline{d}(T) < +\infty \text{ on } Q_T. \tag{7.2}$$

(ii) $c^0 \in L^\infty(\Omega\,;\,\mathbb{R}^P_+)$.

(iii) $\xi \in C^\infty(\partial\Omega\,;\,\mathbb{R})$ is a time-independent function.

Independently of the space dimension, we can prove the following:

Theorem 7.1. *Assume* $(i)-(iii)$*. Then there exist* $c \in L^\infty(0,+\infty;L^1(\Omega)^P)$, $\Phi \in L^\infty(0,+\infty;W^{1,2}(\Omega))$, *such that* (7.1) *is satisfied in the following sense:*

For all $T > 0$, $c_i \in L^1(0,T;W^{1,1}_{loc}(\Omega))$, $d_i\nabla c_i + d_i z_i c_i \nabla\Phi \in L^1(Q_T)$ *and for all* $\psi \in C^\infty(\overline{Q_T})$ *such that* $\psi(T) = 0$,

$$\int_{Q_T}\Big(-c_i\partial_t\psi + (d_i\nabla c_i + d_i z_i c_i\nabla\Phi)\cdot\nabla\psi\Big) = \int_\Omega c_i^0\psi(0). \tag{7.3}$$

For all $\varphi \in C^\infty(\overline{\Omega})$, *for a.e.* $t \in \mathbb{R}_+$,

$$\int_\Omega \nabla\Phi(t)\cdot\nabla\varphi + \int_{\partial\Omega}(\tau\Phi(t) - \xi)\varphi = \int_\Omega \sum_{i=1}^P z_i c_i(t)\varphi. \tag{7.4}$$

In the particular case of space dimension $N = 3$, it is possible to use some Sobolev embeddings to get round technical difficulties due to the generality of the above setting, and to derive additional regularity on the solutions: this will be contained in [23].

This section is organized as follows.

In Subsection 7.2, we prove the well-posedness of an approximate version of system (7.1), where the total charge density $\sum_{i=1}^{P} z_i c_i$ is regularized, as well as the diffusion coefficients d_i. Our proof is based on a Leray-Schauder fixed point argument, and uses Theorem 5.1 in Section 5. Recall (see *e.g.* [24,56]) that system (7.1) admits a Lyapunov function. The main point with our regularization is that it preserves the Lyapunov structure, and it is possible to state the corresponding dissipation rate explicitly: this is the content of Subsection 7.3, where we also derive the *a priori* estimates that will provide compactness of the approximate solutions. Finally, Subsection 7.4 contains the proof of Theorem 7.1.

7.2 Well-posedness of an approximate system

In this section, we prove existence and uniqueness of solutions on Q_T for any $T > 0$ for an approximate problem, where the total charge density $\sum_{i=1}^{P} z_i c_i$ and the diffusion coefficients d_i have been regularized.

Let $\varepsilon > 0$, B_ε denote the differential operator $I - \varepsilon\Delta$, $m = 2N$, and consider

$$\left.\begin{array}{rcll} \partial_t c_i - \operatorname{div}(d_i^\varepsilon \nabla c_i + d_i^\varepsilon z_i c_i \nabla\Phi) &=& 0 & \text{on } (0,+\infty)\times\Omega \\ \partial_\nu c_i + z_i c_i \partial_\nu \Phi &=& 0 & \text{on } (0,+\infty)\times\partial\Omega \\ c_i(0,\cdot) &=& c_i^0 & \text{on } \Omega \end{array}\right\},\ i\in\{1,\ldots,P\}, \tag{7.5}$$

$$\left.\begin{array}{rcll} B_\varepsilon^{m+1}\Psi - \sum_{i=1}^{P} z_i c_i &=& 0 & \text{on } (0,+\infty)\times\Omega \\ \partial_\nu[B_\varepsilon^k \Psi] + \tau B_\varepsilon^k \Psi &=& 0 & \text{on } (0,+\infty)\times\partial\Omega \end{array}\right\},\ k\in\{0,\ldots,m\}, \tag{7.6}$$

$$\left.\begin{array}{rcll} -\Delta\Phi &=& \Psi & \text{on } (0,+\infty)\times\Omega \\ \partial_\nu\Phi + \tau\Phi &=& \xi & \text{on } (0,+\infty)\times\partial\Omega \end{array}\right\}, \tag{7.7}$$

where $d_i^\varepsilon \in C([0,+\infty);C(\overline{\Omega}))$, $\nabla d_i^\varepsilon \in L^\infty_{loc}([0,+\infty);L^r(\Omega)^N)$ for some $r > \max(2,N)$ and

$$\left\{\begin{array}{ll} d_i^\varepsilon(t,x) \overset{\varepsilon\to 0}{\longrightarrow} d_i(t,x) & \text{for a.e. } (t,x)\in(0,+\infty)\times\Omega, \\ \underline{d}(T) \le d_i^\varepsilon \le \overline{d}(T) & \text{for } (t,x)\in(0,T)\times\Omega. \end{array}\right. \tag{7.8}$$

Proposition 7.2. *For any $T > 0$, there exists a solution $(c^\varepsilon,\Psi^\varepsilon,\Phi^\varepsilon)$ of $(7.5)-(7.7)$ on $(0,T)\times\Omega$ in the sense*

(i) $c^\varepsilon \in L^\infty(Q_T)\cap L^2(0,T;W^{1,2}(\Omega))\cap C([0,T];L^2(\Omega))$, $\partial_t c^\varepsilon \in L^2(0,T;W^{-1,2}(\Omega))$, $\Psi^\varepsilon \in C([0,T],W^{2m+2,2}(\Omega))$; $\Phi^\varepsilon \in C([0,T],W^{2m+4,2}(\Omega))$.

(ii) *For all $\psi \in C^\infty(\overline{Q_T})$ such that $\psi(T) = 0$,*

$$\int_{Q_T}\Big(-c_i^\varepsilon\partial_t\psi + (d_i^\varepsilon\nabla c_i^\varepsilon + d_i^\varepsilon z_i c_i^\varepsilon\nabla\Phi^\varepsilon)\cdot\nabla\psi\Big) = \int_\Omega c_i^0\psi(0) \tag{7.9}$$

and $(7.6),(7.7)$ are satisfied in a pointwise sense.

Proof. We will use the Leray-Schauder fixed point theorem (see Lemma 1.3) in the space

$$X = L^\infty(0,T;W^{1,\infty}(\Omega)).$$

Let $\Phi \in X$. According to Theorem 5.1, there exists

$$c \in L^\infty(Q_T)\cap L^2(0,T;W^{1,2}(\Omega))\cap C([0,T];L^2(\Omega))$$

such that $\partial_t c_i \in L^2(0,T;W^{-1,2}(\Omega))$ and satisfying (7.5) with data Φ (in the sense (7.9)). Then using classical elliptic regularity results (see e.g [30]), we can define $\Psi \in C([0,T];W^{2m+2,2}(\Omega))$ as the solution of (7.6) with data c, and finally $\hat{\Phi} \in C([0,T];W^{2m+4,2}(\Omega))$ as the solution of (7.7) with data Ψ. Since $m = 2N$, $W^{2m+4,2}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ and we can define

$$\mathcal{T} : X \to X,\ \Phi \mapsto \hat{\Phi}.$$

Let $(\Phi^n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ be a bounded sequence and $(c^n, \Psi^n, \hat{\Phi}^n)$ be the corresponding solution of (7.5) – (7.7). Using Theorem 5.1, $\partial_t c^n$ is bounded in $L^2(0,T;W^{-1,2}(\Omega))$, c^n is bounded in $L^\infty(Q_T)$. Differentiating (7.6) and (7.7) in time (recall that the boundary conditions are time-independent) and using elliptic regularity theory in L^2, $\partial_t\Psi^n$ is bounded in $L^2(0,T;W^{2m+1,2}(\Omega))$, $\partial_t\hat{\Phi}^n$ is bounded in $L^2(0,T;W^{2m+3,2}(\Omega))$. Since c^n is also bounded in $L^\infty(Q_T)$, using L^p-elliptic regularity theory in equations (7.6) – (7.7), Ψ^n and $\hat{\Phi}^n$ are bounded in $L^\infty(0,T;W^{2,p}(\Omega))$ for any $p < +\infty$ (see *e.g.* [53]). We choose p large enough so that the embedding $W^{2,p}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ is compact. Then, using Corollary 4 in [98], $\{\hat{\Phi}^n, n \in \mathbb{N}\}$ is relatively compact in X, whence the compactness of $\mathcal{T}$.

To prove the continuity of $\mathcal{T}$, let $\Phi^n \to \Phi$ in X. Since $\mathcal{T}$ is compact, $\{\hat{\Phi}^n = \mathcal{T}\Phi^n,\ n \in \mathbb{N}\}$ is relatively compact in X. Let $\hat{\Phi}$ be a limit point. Recall that c^n is the solution of (7.5) with data Φ^n. Similarly as before, the estimates from Theorem 5.1 and Corollary 4 in [98] guarantee that $(c^n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(Q_T) \cap L^2(0,T;W^{1,2}(\Omega))$ and relatively compact in $L^2(Q_T)$. Therefore, we may extract a subsequence that converges a.e. and in $L^p(Q_T)$ for any $p < +\infty$ to a limit c, and such that $\nabla c^n \to \nabla c$ weakly in $L^2(Q_T)$. Then we may pass to the limit $n \to +\infty$ in (7.9) and using uniqueness from Theorem 5.1, c is *the* solution of (7.5) with data Φ. Then we pass to the limit $n \to +\infty$ in equation (7.6), so that $\Psi^n \to \Psi$, where Ψ is the solution of (7.6) with data c, and finally in equation (7.7), which yields $\hat{\Phi} = \mathcal{T}\Phi$. The only possible limit point for $(\mathcal{T}\Phi^n)_{n\in\mathbb{N}}$ is $\mathcal{T}\Phi$ and $(\mathcal{T}\Phi^n)_{n\in\mathbb{N}}$ lies in a compact subset of X, so $\mathcal{T}\Phi^n \to \mathcal{T}\Phi$, whence the continuity of $\mathcal{T}$.

Let $\sigma \in [0,1]$, $\Phi \in X$, (c, Ψ, Φ) be the corresponding solution of (7.5) – (7.7), and assume $\Phi = \sigma\mathcal{T}\Phi$. Remark that

$$\Phi \text{ satifies (7.7) with data } (\sigma\xi, \sigma\Psi) \text{ instead of } (\xi, \Psi). \tag{7.10}$$

By integration of (7.5) on Q_t for any $t \in (0,T)$, we have

$$\forall i \in \{1,\dots,P\},\ \forall t \in [0,T], \qquad \int_\Omega c_i(t) = \int_\Omega c_i^0. \tag{7.11}$$

Using the nonnegativity of c, c is bounded in $L^\infty(0,T;L^1(\Omega)^P)$ independently of σ. Using L^1 elliptic regularity theory [1] in (7.6), Ψ is bounded in $L^\infty(0,T;W^{2m,1}(\Omega))$ independently of σ. Using (7.10), so is Φ in

$$L^\infty(0,T;W^{2m,1}(\Omega)) \hookrightarrow L^\infty(0,T;W^{1,\infty}(\Omega)) \qquad (\text{ recall that } m = 2N)\ .$$

Therefore, any solution of $\Phi = \sigma\mathcal{T}\Phi$ is *a priori* bounded in X, so according to the Leray-Schauder theorem, $\mathcal{T}$ has a fixed point Φ^ε, and the corresponding $(c^\varepsilon, \Psi^\varepsilon, \Phi^\varepsilon)$ satisfies (7.5) – (7.7) in the sense of Proposition 7.2. By construction (see the definition of $\mathcal{T}$), $\Psi^\varepsilon \in C([0,T];W^{2m+2,2}(\Omega))$,

1. For instance, we may use that $(I - \varepsilon\Delta)^{-1}$ (with homogeneous Robin boundary conditions) is continuous from $L^1(\Omega)$ into $L^{\frac{N}{N-2}-\varepsilon}(\Omega)$ for arbitrarily small $\varepsilon > 0$, and then apply elliptic regularity theory in $L^p(\Omega)$, $p > 1$.

$\Phi^\varepsilon \in C([0,T];W^{2m+4,2}(\Omega))$, so using Sobolev's embedding theorem, $\Psi^\varepsilon, \Phi^\varepsilon \in C([0,T],W^{2,\infty}(\Omega))$ and $(7.6)-(7.7)$ are satisfied in a strong sense.

$\square$

Remark. Actually, the solution of Proposition 7.2 is unique. This may be shown by considering two solutions $(c^\varepsilon, \Psi^\varepsilon, \Phi^\varepsilon)$ and $(\tilde{c}^\varepsilon, \tilde{\Psi}^\varepsilon, \tilde{\Phi}^\varepsilon)$ of $(7.5)-(7.7)$ with the same initial data. For all $\psi \in C^\infty(\overline{Q_T})$ such that $\psi(T)=0$, we have

$$\int_{Q_T} \left[-(c_i^\varepsilon - \tilde{c}_i^\varepsilon)\psi_t + \Big(d_i^\varepsilon \nabla (c_i^\varepsilon - \tilde{c}_i^\varepsilon) + d_i^\varepsilon z_i \Big((c_i^\varepsilon - \tilde{c}_i^\varepsilon)\nabla\Phi^\varepsilon + \tilde{c}_i^\varepsilon \nabla(\Phi^\varepsilon - \tilde{\Phi}^\varepsilon)\Big)\Big) \cdot \nabla\psi \right] = 0.$$

Formally, choosing $\psi = (c_i^\varepsilon - \tilde{c}_i^\varepsilon)\mathbb{1}_{(0,t_0)}$ for $t_0 \in (0,T)$, we get

$$\frac{1}{2}\int_\Omega (c_i^\varepsilon - \tilde{c}_i^\varepsilon)^2(t_0) + \int_{Q_{t_0}} d_i^\varepsilon |\nabla(c_i^\varepsilon - \tilde{c}_i^\varepsilon)|^2 + \left[d_i^\varepsilon z_i (c_i^\varepsilon - \tilde{c}_i^\varepsilon)\nabla\Phi^\varepsilon \ + \tilde{c}_i^\varepsilon \nabla(\Phi^\varepsilon - \tilde{\Phi}^\varepsilon)\right] \cdot \nabla (c_i^\varepsilon - \tilde{c}_i^\varepsilon) = 0.$$

Recall that d_i^ε is bounded from above and below by positive constants, independently of ε (see (7.8)), and that c_i^ε, $\tilde{c}_i^\varepsilon$, $\nabla\Phi^\varepsilon$, $\nabla\tilde{\Phi}^\varepsilon$ are in $L^\infty(Q_T)$. Applying Hölder's and Young's inequalities, there exists $C>0$ such that

$$\frac{1}{2}\int_\Omega (c_i^\varepsilon - \tilde{c}_i^\varepsilon)^2(t_0) + \frac{d(T)}{2}\int_{Q_{t_0}} |\nabla(c_i^\varepsilon - \tilde{c}_i^\varepsilon)|^2 \le C\int_0^{t_0}\int_\Omega (c_i^\varepsilon - \tilde{c}_i^\varepsilon)^2(s)ds.$$

Since $(c_i^\varepsilon - \tilde{c}_i^\varepsilon)(0) = 0$, the Gronwall inequality implies $c_i^\varepsilon = \tilde{c}_i^\varepsilon$, whence uniqueness.
The above computation can be made rigorous by choosing test functions

$$\psi_h(t) := \frac{1}{2h}\int_{t-h}^{t+h} \mathbb{1}_{(0,t_0)}(c_i^\varepsilon - \tilde{c}_i^\varepsilon)(s)ds, \quad h>0,$$

and passing to the limit $h \to 0$ (see e.g. the proof of Theorem 5.1).

7.3 Energy estimates

In the following, we derive *a priori* estimates that will provide compactness when $\varepsilon \to 0$ in equations $(7.5)-(7.7)$. It is well known (see *e.g.* [24, 56]) that there exists a Lyapunov function for system (7.1), namely

$$V_0(t) = \sum_{i=1}^P \int_\Omega c_i \log c_i + \frac{1}{2}\int_\Omega |\nabla\Phi|^2 + \frac{\tau}{2}\int_{\partial\Omega}\Phi^2,$$

which physically describes the total energy of the system. In the subsequent lemma, we show that our approximation procedure does preserve this "energetic" structure:

Lemma 7.3. *Let* (c,Ψ,Φ) *satisfy* $(7.5)-(7.7)$ *in the sense of Proposition 7.2. For* $k \in \mathbb{N}$, *define* $A^k \ge 0$ *by:*

If $k = 2n$, $\qquad A^k = \frac{1}{2}\int_\Omega (B_\varepsilon^n \Psi)^2,$

If $k = 2n+1$, $\qquad A^k = \frac{1}{2}\int_\Omega (B_\varepsilon^n\Psi)^2 + \frac{\varepsilon}{2}\int_\Omega |\nabla B_\varepsilon^n \Psi|^2 + \frac{\varepsilon\tau}{2}\int_{\partial\Omega}(B_\varepsilon^n\Psi)^2,$

and set

$$V(t)=\sum_{i=1}^{P}\int_{\Omega}c_i\log c_i+\frac{1}{2}\int_{\Omega}|\nabla\Phi|^2+\frac{\tau}{2}\int_{\partial\Omega}\Phi^2+\varepsilon\sum_{k=0}^{m}A^k+\varepsilon\int_{\partial\Omega}\xi(B_\varepsilon^m+\ldots+B_\varepsilon+I)\Psi. \quad (7.12)$$

Then

(i)

$$\frac{d}{dt}V(t)=-\int_{\Omega}\sum_{i=1}^{P}\frac{1}{d_i^\varepsilon c_i}|d_i^\varepsilon\nabla c_i+d_i^\varepsilon z_i c_i\nabla\Phi|^2. \quad (7.13)$$

(ii)

$$\forall k\in\{0,\ldots,m\},\ \varepsilon\int_{\partial\Omega}|B_\varepsilon^k\Psi|\le\frac{1}{\tau}\int_{\Omega}|\sum_{i=1}^{P}z_ic_i|. \quad (7.14)$$

(iii) *There exists* $C>0$ *such that for all* $t\in(0,+\infty)$, $\varepsilon\in(0,1)$, $i\in\{1,\ldots,P\}$, $k\in\{0,\ldots,m\}$,

$$\int_{\Omega}c_i\log^+c_i+\frac{1}{2}\int_{\Omega}|\nabla\Phi|^2+\int_{\partial\Omega}\Phi^2+\varepsilon A^k+\varepsilon\int_{\partial\Omega}|B_\varepsilon^k\Psi|\le C. \quad (7.15)$$

Proof . We give here a formal proof, and indicate how the computations can be made rigorous afterwards. Set $J_i=d_i^\varepsilon\nabla c_i+d_i^\varepsilon z_ic_i\nabla\Phi$, we have

$$\begin{aligned}\frac{d}{dt}\sum_{i=1}^{P}\int_{\Omega}c_i\log c_i&=\sum_{i=1}^{P}\int_{\Omega}\mathrm{div}(d_i^\varepsilon\nabla c_i+d_i^\varepsilon z_ic_i\nabla\Phi)\log c_i\\&=-\sum_{i=1}^{P}\int_{\Omega}J_i\frac{\nabla c_i}{c_i}\\&=-\sum_{i=1}^{P}\int_{\Omega}\frac{1}{d_i^\varepsilon c_i}J_i(J_i-d_i^\varepsilon z_ic_i\nabla\Phi)\\&=-\sum_{i=1}^{P}\int_{\Omega}\frac{1}{d_i^\varepsilon c_i}|J_i|^2+\sum_{i=1}^{P}\int_{\Omega}z_iJ_i\nabla\Phi. \end{aligned}\quad (7.16)$$

$$\sum_{i=1}^{P}\int_{\Omega}z_iJ_i\nabla\Phi=-\sum_{i=1}^{P}\int_{\Omega}z_i(\mathrm{div}J_i)\Phi=-\int_{\Omega}\partial_t(\sum_{i=1}^{P}z_ic_i)\Phi=-\int_{\Omega}(B_\varepsilon^{m+1}\Psi_t)\Phi.$$

Remark that if F and G are smooth functions on Ω satisfying $\partial_\nu F+\tau F=0$ on $\partial\Omega$, $\partial_\nu G+\tau G=g$ on $\partial\Omega$, two integrations by parts yield

$$\int_{\Omega}B_\varepsilon F\,G=\int_{\Omega}F\,B_\varepsilon G+\varepsilon\int_{\partial\Omega}Fg. \quad (7.17)$$

Using (7.17),

$$\begin{aligned}\int_{\Omega}[B_\varepsilon^{m+1}\Psi_t]\Phi&=\int_{\Omega}[B_\varepsilon^m\Psi_t]B_\varepsilon\Phi+\varepsilon\int_{\partial\Omega}[B_\varepsilon^m\Psi_t]\xi\\&=\int_{\Omega}[B_\varepsilon^m\Psi_t]\Phi+\varepsilon\int_{\Omega}[B_\varepsilon^m\Psi_t]\Psi+\varepsilon\int_{\partial\Omega}[B_\varepsilon^m\Psi_t]\xi.\end{aligned}\quad (7.18)$$

Let $k \in \mathbb{N}$, if $k = 2n$,

$$\int_\Omega [B_\varepsilon^k \Psi_t]\Psi = \int_\Omega [B_\varepsilon^{2n}\Psi_t]\Psi = \int_\Omega [B_\varepsilon^n \Psi_t] B_\varepsilon^n \Psi = \frac{d}{dt}\left[\frac{1}{2}\int_\Omega [B_\varepsilon^n \Psi]^2\right] = \frac{d}{dt}A^k.$$

If $k = 2n+1$,

$$\begin{aligned}
\int_\Omega [B_\varepsilon^k \Psi_t]\Psi &= \int_\Omega [B_\varepsilon^{2n+1}\Psi_t]\Psi = \int_\Omega [B_\varepsilon^n \Psi_t] B_\varepsilon^{n+1}\Psi \\
&= \int_\Omega B_\varepsilon^n \Psi_t [B_\varepsilon^n \Psi - \varepsilon \Delta B_\varepsilon^n \Psi] \\
&= \int_\Omega B_\varepsilon^n \Psi_t \, B_\varepsilon^n \Psi + \varepsilon \int_\Omega \nabla B_\varepsilon^n \Psi_t \, \nabla B_\varepsilon^n \Psi + \varepsilon\tau \int_{\partial\Omega} B_\varepsilon^n \Psi_t \, B_\varepsilon^n \Psi \\
&= \frac{d}{dt}\left[\frac{1}{2}\int_\Omega (B_\varepsilon^n \Psi)^2 + \frac{\varepsilon}{2}\int_\Omega |\nabla B_\varepsilon^n \Psi|^2 + \frac{\varepsilon\tau}{2}\int_{\partial\Omega} (B_\varepsilon^n \Psi)^2\right] = \frac{d}{dt}A^k.
\end{aligned}$$

Going back to (7.18), a straightforward induction yields

$$\begin{aligned}
\int_\Omega [B_\varepsilon^{m+1}\Psi_t]\Phi &= \int_\Omega \Psi_t \Phi + \varepsilon \frac{d}{dt}\sum_{k=0}^m A^k + \varepsilon \int_{\partial\Omega} \xi (B_\varepsilon^m + \ldots + B_\varepsilon + I)\Psi_t \\
&= \int_\Omega -(\Delta\Phi_t)\Phi + \varepsilon \frac{d}{dt}\sum_{k=0}^m A^k + \varepsilon \int_{\partial\Omega} \xi (B_\varepsilon^m + \ldots + B_\varepsilon + I)\Psi_t \\
&= \int_\Omega \nabla\Phi_t \nabla\Phi + \tau \int_{\partial\Omega} \Phi_t \Phi + \varepsilon \frac{d}{dt}\sum_{k=0}^m A^k + \varepsilon \int_{\partial\Omega} \xi (B_\varepsilon^m + \ldots + B_\varepsilon + I)\Psi_t \\
&= \frac{d}{dt}\left[\frac{1}{2}\int_\Omega |\nabla\Phi|^2 + \frac{\tau}{2}\int_{\partial\Omega} \Phi^2 + \varepsilon \sum_{k=0}^m A^k + \varepsilon \int_{\partial\Omega} \xi (B_\varepsilon^m + \ldots + B_\varepsilon + I)\Psi\right]. \qquad (7.19)
\end{aligned}$$

Then (7.13) is a consequence of (7.16) and (7.19).

To prove (7.14), we introduce the notation $\Psi_k = B_\varepsilon^k \Psi$ for $k \in \{0, \ldots, m+1\}$, *i.e.*

$$\left\{\begin{array}{rcrcl}
 & - & \Delta\Phi & = & \Psi_0 = \Psi, \\
\Psi_0 & - & \varepsilon\Delta\Psi_0 & = & \Psi_1, \\
 & & \vdots & & \\
\Psi_{m-1} & - & \varepsilon\Delta\Psi_{m-1} & = & \Psi_m, \\
\Psi_m & - & \varepsilon\Delta\Psi_m & = & \sum_{i=1}^P z_i c_i = \Psi_{m+1}.
\end{array}\right. \qquad (7.20)$$

Let p be a smooth increasing function such that $p(0) = 0$, $-1 \le p \le 1$. Multiplying equation $\Psi_k - \varepsilon\Delta\Psi_k = \Psi_{k+1}$ by $p(\Psi_k)$ and integrating by parts on Ω, we get (using the boundary conditions from (7.6)),

$$\int_\Omega p(\Psi_k)\Psi_k + \varepsilon \int_\Omega |\nabla\Psi_k|^2 p'(\Psi_k) + \varepsilon\tau \int_{\partial\Omega} \Psi_k p(\Psi_k) = \int_\Omega \Psi_{k+1} p(\Psi_k).$$

Letting p go to the "sign" function,

$$\int_\Omega |\Psi_k| + \varepsilon\tau \int_{\partial\Omega} |\Psi_k| \le \int_\Omega |\Psi_{k+1}|.$$

Then by an obvious induction,

$$\forall k \in \{0,\dots,m\}, \quad \int_\Omega |\Psi_k| + \varepsilon\tau \int_{\partial\Omega} |\Psi_k| \le \int_\Omega |\sum_{i=1}^P z_i c_i|,$$

whence (7.14).

Let us prove (iii). Using that $(I-\varepsilon\Delta)^{-1}$ is a contraction in $L^\infty(\Omega)$, one can easily check that $V(0)$ is bounded independently of $\varepsilon \in (0,1)$. Using the nonnegativity of c and the homogeneous boundary conditions in (7.5), c is bounded in $L^\infty(0,+\infty;L^1(\Omega)^P)$, independently of $\varepsilon \in (0,1)$. Then (iii) is a straightforward consequence of (i) and (ii).

□

Remark 7.4. *Note that since $x \mapsto x\log x$ is not differentiable in 0 we cannot differentiate (7.12) directly. Replacing $\int c_i \log c_i$ by $\int (c_i+\delta)\log(c_i+\delta)$ for $\delta > 0$ and passing to the limit $\delta \to 0$, the previous computation can be made rigorous . This is done for instance in [24, Lemma 3.7].*

As a consequence, we get the following *a priori* estimates on the solutions:

Lemma 7.5. *Let (c,Ψ,Φ) satisfy $(7.5)-(7.7)$ in the sense of Proposition 7.2, let $C>0$ denote any constant depending on the data of $(7.5)-(7.7)$ but not on ε and T.*

(i) *Given $T>0$, there exists $C>0$ such that*

$$\sum_{i=1}^P \int_{Q_T} \frac{1}{d_i^\varepsilon c_i} |d_i^\varepsilon \nabla c_i + d_i^\varepsilon z_i c_i \nabla\Phi|^2 \le C. \tag{7.21}$$

(ii) *Given $k \in \mathbb{N}$, $x \in \Omega$ and $t>0$, let*

$$|D^k\Psi(t,x)| = \max_{k_1+\ldots+k_P=k} |\partial_{x_1}^{k_1}\cdots\partial_{x_N}^{k_N}\Psi(t,x)|. \tag{7.22}$$

Then there exists $C>0$ such that

$$\sum_{k=0}^m \varepsilon^{k+1} \int_\Omega |D^k\Psi|^2 \le C. \tag{7.23}$$

(iii) *Given $T>0$, $\zeta \in C_c^\infty(\Omega,[0,1])$, there exists $C=C(T)>0$ such that*

$$\int_{Q_T} \frac{|\nabla c_i|^2}{c_i}\zeta^2 + c_i|\nabla\Phi|^2\zeta^2 \le C. \tag{7.24}$$

Proof. We first integrate (7.13) on $(0,T)$ to get

$$\sum_{i=1}^P \int_{Q_T} \frac{1}{d_i^\varepsilon c_i} |d_i^\varepsilon \nabla c_i + d_i^\varepsilon z_i c_i \nabla\Phi|^2 \le V(0) - V(T).$$

Recall that $c^0 \in L^\infty(\Omega)$. As mentioned in the proof of Lemma 7.3 (iii), $V(0)$ is bounded independently of $\varepsilon \in (0,1)$. Moreover, using (7.12), (7.14) and the fact that c is bounded in $L^\infty(0,T;L^1(\Omega)^P)$, $V(T)$ is bounded below independently of ε, whence (7.21).

Throughout the rest of the proof, C denotes a positive constant that depends on the data of

(7.5) – (7.7), but not on ε.

We will use the following basic result: let $\varepsilon, \tau > 0$ and $u \in W^{2,2}(\Omega)$, $f \in L^2(\Omega)$ satisfying

$$u - \varepsilon \Delta u = f \text{ on } \Omega \ ; \ \partial_\nu u + \tau u = 0 \text{ on } \partial\Omega.$$

Then there exists $C > 0$ such that

$$\|u\|_{L^2(\Omega)} + \varepsilon^{1/2}\|u\|_{W^{1,2}(\Omega)} + \varepsilon\|u\|_{W^{2,2}(\Omega)} \le C\|f\|_{L^2(\Omega)}. \tag{7.25}$$

This can be justified by multiplying $u - \varepsilon\Delta u = f$ by u and integrating by parts to get

$$\int_\Omega u^2 + \varepsilon|\nabla u|^2 + \varepsilon\tau\int_{\partial\Omega} u^2 = \int_\Omega fu \le \frac{1}{2}\int_\Omega f^2 + \frac{1}{2}\int_\Omega u^2.$$

Consequently, u, $\varepsilon^{1/2}\nabla u$ and then $\varepsilon\Delta u$ are bounded in $L^2(\Omega)$ independently of ε. Finally, we use elliptic regularity theory to deduce regularity on u in $W^{2,2}(\Omega)$ from the regularity of Δu (see e.g. [30]).

Recall that $m = 2N$. Since $V(t)$ is the sum of nonnegative terms (except for $\sum_{i=1}^P (c_i - 1)$) and of $\varepsilon\int_{\partial\Omega}\xi(B_\varepsilon^m + \ldots + B_\varepsilon + I)\Psi$, which is bounded independently of ε according to (7.14) *(iii)*, there exists $C > 0$ such that $\varepsilon^{1/2}B_\varepsilon^N\Psi$ is bounded in $L^2(\Omega)$ independently of ε. Then Lemma 7.5 *(ii)* is a consequence of the following fact, applied with $n = N$:

(P_n) $\begin{cases}$ Assume $\psi \in W^{2n,2}(\Omega)$, $\partial_\nu(B_\varepsilon^k\psi) + \tau B_\varepsilon^k\psi = 0$ on $\partial\Omega$ for $k \in \{0, \ldots, n-1\}$ and $\varepsilon^{1/2}B_\varepsilon^n\psi$ is bounded in $L^2(\Omega)$ independently of ε. Then there exists $C > 0$, independent of ε, such that
$$\sum_{k=0}^{2n}\varepsilon^{k+1}\int_\Omega |D^k\psi|^2 \le C.$$

We prove (P_n) by induction: (P_1) is a consequence of (7.25) applied to

$$\varepsilon^{1/2}\psi - \varepsilon^{3/2}\Delta\psi = \varepsilon^{1/2}B_\varepsilon\psi \text{ on } \Omega \ ; \ \partial_\nu\psi + \tau\psi = 0 \text{ on } \partial\Omega.$$

Let $n \ge 2$ and assume (P_{n-1}) is true. Let $\psi \in W^{2n,2}(\Omega)$ such that $\varepsilon^{1/2}B_\varepsilon^n\psi$ is bounded in $L^2(\Omega)$ independently of ε. According to (7.25), there exists $C > 0$ such that

$$\varepsilon^{\frac{1}{2}}\|B_\varepsilon^{n-1}\psi\|_{L^2(\Omega)} + \varepsilon\|B_\varepsilon^{n-1}\psi\|_{W^{1,2}(\Omega)} + \varepsilon^{\frac{3}{2}}\|B_\varepsilon^{n-1}\psi\|_{W^{2,2}(\Omega)} \le C.$$

Then $\varepsilon^{1/2}B_\varepsilon^{n-1}(\varepsilon\Delta\psi)$ is bounded in $L^2(\Omega)$, $\partial_\nu(B_\varepsilon^k\Delta\psi) + \tau B_\varepsilon^k\Delta\psi = 0$ on $\partial\Omega$ for $k \in \{0, \ldots, n-2\}$, so using (P_{n-1}) on $\varepsilon\Delta\psi$, we get the existence of $C > 0$ such that

$$\sum_{k=0}^{2n-2}\varepsilon^{k+1}\int_\Omega |D^k(\varepsilon\Delta\psi)|^2 \le C,$$

and therefore

$$\varepsilon^{k+3}\|\Delta\psi\|^2_{W^{k,2}(\Omega)} \le C\,, \quad k \in \{0, \ldots, 2n-2\}.$$

Using elliptic regularity theory in $W^{k,2}(\Omega)$-spaces (see *e.g.* [30, Theorem 9.26]), this yields

$$\varepsilon^{k+3}\|\psi\|^2_{W^{k+2,2}(\Omega)} \le C \text{ for } k \in \{0, \ldots, 2n-2\}.$$

Using (P_1) to estimate $\varepsilon\|\psi\|_{L^2(\Omega)}$ and $\varepsilon^2\|\psi\|_{W^{1,2}(\Omega)}$, this proves (P_n). By induction, (P_n) is true for $n = N$.

We now prove assertion (iii). Let $\zeta \in C_c^\infty(\Omega,[0,1])$. Since the integrand in (7.21) is nonnegative, there exists $C > 0$ (independent of T) such that

$$\sum_{i=1}^{P}\int_{Q_T}\frac{1}{d_i^\varepsilon c_i}|d_i^\varepsilon\nabla c_i + d_i^\varepsilon z_i c_i\nabla\Phi|^2\zeta^2 \le C.$$

Recall that d_i^ε is bounded below and above on Q_T by positive constants $\underline{d}(T), \overline{d}(T)$ (see (7.8)), so there exists $C = C(T) > 0$ such that

$$\sum_{i=1}^{P}\int_{Q_T}\frac{1}{c_i}|\nabla c_i + z_i c_i\nabla\Phi|^2\zeta^2 = \sum_{i=1}^{P}\int_{Q_T}\frac{|\nabla c_i|^2}{c_i}\zeta^2 + z_i^2 c_i|\nabla\Phi|^2\zeta^2 + 2z_i\nabla c_i\nabla\Phi\,\zeta^2 \le C.$$

To prove (iii), it is sufficient to show that $I := \sum_{i=1}^{P}\int_{Q_T} z_i\nabla c_i\nabla\Phi\,\zeta^2$ is bounded below independently of ε. Let us first do it in the limit case $\varepsilon = 0$, to show how the local estimates (7.24) are natural and easy to obtain. We write, after integration by parts and using Young's inequality,

$$\begin{aligned} I &= \int_{Q_T}(\Delta\Phi)^2\zeta^2 + 2\int_{Q_T}\Delta\Phi\nabla\Phi\zeta\nabla\zeta \\ &\ge \frac{1}{2}\int_{Q_T}(\Delta\Phi)^2\zeta^2 - 2\int_{Q_T}|\nabla\Phi|^2|\nabla\zeta|^2 \\ &\ge -C, \end{aligned}$$

where we use (7.14) to estimate $|\nabla\Phi|$ in the last inequality.
We now go back to the case $\varepsilon > 0$. Using notations (7.20),

$$\begin{aligned} \sum_{i=1}^{P}\int_{Q_T} z_i\nabla c_i\nabla\Phi\,\zeta^2 &= -\int_{Q_T}\sum_{i=1}^{P} z_i c_i\Delta\Phi\,\zeta^2 - \int_{Q_T}\sum_{i=1}^{P} z_i c_i\nabla\Phi\,\nabla\zeta^2 \\ &= \int_{Q_T}(\Psi_m - \varepsilon\Delta\Psi_m)\Psi\,\zeta^2 - \int_{Q_T}(\Psi_m - \varepsilon\Delta\Psi_m)\nabla\Phi\,\nabla\zeta^2 \\ &= \int_{Q_T}\Psi_m\Psi\,\zeta^2 + \int_{Q_T} -\Psi_m\nabla\Phi\,\nabla\zeta^2 + \varepsilon\int_{Q_T} -\Delta\Psi_m\Psi\zeta^2 + \varepsilon\int_{Q_T}\Delta\Psi_m\nabla\Phi\,\nabla\zeta^2 \\ &= I^1 + I^2 + \varepsilon I\!I^1 + \varepsilon I\!I^2. \end{aligned}$$

The subsequent computations are rather technical, but the idea which is behind is simple: we first integrate by parts I^1 to get a nonnegative term, namely $\int_{Q_T}\Psi_N^2\zeta^2$ (recall that $m = 2N$). Then we use this term and Lemma 7.5 (ii) to control all the other terms, and I^2. Similarly, after integration by parts of $I\!I^1$, we get the nonnegative term $\int_{Q_T}|\nabla\Psi_N|^2\zeta^2$, from which we can control (together with Lemma 7.5 (ii)) all the other terms and $I\!I^2$.
Let us introduce some notations: for $f \in W^{1,2}(\Omega)$ and $g \in W^{2,2}(\Omega)$,

$$A_1(f,g) := 2\nabla f\nabla g + f\Delta g.$$

Remark that

$$B_\varepsilon(fg) = fg - \varepsilon\Delta(fg) = fg - \varepsilon\Delta f\,g - \varepsilon\nabla f\nabla g - \varepsilon f\Delta g = [B_\varepsilon f]g - \varepsilon A_1(f,g).$$

By induction, for $k \in \mathbb{N}$ we have

$$B_\varepsilon^k(fg) = [B_\varepsilon^k f]g - \varepsilon \sum_{j=0}^{k-1} B_\varepsilon^j A_1(B_\varepsilon^{k-1-j} f, g). \tag{7.26}$$

We extend A_1 to an operator on vectors by setting

$$A_1(F,g) := (A_1(f_1,g),\dots,A_1(f_N,g))\ ;\ A_1(f,G) = (A_1(f,g_1),\dots,A_1(f,g_N)) \tag{7.27}$$

for $F = (f_1,\dots,f_N) \in W^{1,2}(\Omega)^N$, $G = (g_1,\dots,g_N) \in W^{2,2}(\Omega)^N$. We will also use the $(m+2)^{th}$ root of ζ: using the notations (7.22), we define

$$\zeta = \sigma^{m+2}\ ,\ \sigma \in C_c^\infty(\Omega,[0,+\infty))\ ;\ M_\sigma = \left(\sum_{j=0}^{m+2} \sup_{x\in\Omega} |D^j\sigma(x)|\right)^{m+2}.$$

We start with

$$\begin{aligned} I\!I^1 &= -\int_{Q_T} \Delta\Psi_m \Psi \zeta^2 = -\int_{Q_T} [B_\varepsilon^m \Delta\Psi]\Psi\zeta^2 \\ &= \int_{Q_T} [B_\varepsilon^m \nabla\Psi]\nabla\Psi\zeta^2 + \int_{Q_T} [B_\varepsilon^m \nabla\Psi]\Psi\nabla\zeta^2 = I\!I_1^1 + I\!I_2^1. \end{aligned}$$

Using (7.26) and (7.27),

$$\begin{aligned} \varepsilon I\!I_1^1 &= \varepsilon \int_{Q_T} [B_\varepsilon^{2N}\nabla\Psi]\nabla\Psi\zeta^2 = \varepsilon \int_{Q_T} [B_\varepsilon^N \nabla\Psi][B_\varepsilon^N\ \nabla\Psi\zeta^2] \\ &= \varepsilon \int_{Q_T} [B_\varepsilon^N\nabla\Psi]^2\zeta^2 - \int_{Q_T} \varepsilon^2 \left[B_\varepsilon^N\nabla\Psi\right] \left[\sum_{j=0}^{N-1} B_\varepsilon^j A_1(B_\varepsilon^{N-1-j}\nabla\Psi, \zeta^2)\right]. \end{aligned} \tag{7.28}$$

Let us analyze the last bracket: A_1 is a second order operator with respect to the second variable, so the highest order of derivation of ζ^2 is m. When applying $B_\varepsilon = I - \varepsilon\Delta$, we see that a derivation of Ψ of order 1 or 2 always comes with a multiplication by ε. As A_1 is a first order operator with respect to the first variable, the highest order of derivation of Ψ is m, and there exists $C > 0$ such that

$$\left|\sum_{j=0}^{N-1} B_\varepsilon^j A_1(B_\varepsilon^{N-1-j}\nabla\Psi, \zeta^2)\right| \le CM_\sigma|\zeta| \sum_{j=0}^{N-1} \varepsilon^j \left(|D^{2j+2}\Psi| + |D^{2j+1}\Psi|\right).$$

Note that the appearance of $|\zeta|$ as a factor comes from the choice of $\zeta = \sigma^{m+2}$: when computing $D^k(\zeta^2)$ for $k \le m$, we check that ζ can be kept as a factor of all terms in the expansion. As a consequence

$$\begin{aligned} &\int_{Q_T} \varepsilon^2 \left|B_\varepsilon^N\nabla\Psi\right| \left|\sum_{j=0}^{N-1} B_\varepsilon^j A_1(B_\varepsilon^{N-1-j}\nabla\Psi,\zeta^2)\right| \\ &\le CM_\sigma \int_{Q_T} \left[\varepsilon^{1/2}|B_\varepsilon^N\nabla\Psi|\ |\zeta|\right] \left[\sum_{j=0}^{N-1} \varepsilon^{j+3/2}(|D^{2j+2}\Psi| + |D^{2j+1}\Psi|)\right]. \end{aligned} \tag{7.29}$$

According to (7.23), the right bracket is bounded in $L^\infty(0,T;L^2(\Omega))$ independently of ε, and using Young's inequality, the left bracket can be absorbed into $\varepsilon\int_{Q_T}[B_\varepsilon^N\nabla\Psi]^2\zeta^2$ in (7.28). As a consequence, there exists $R\in\mathbb{R}$ depending only on the data, such that

$$\varepsilon I\!I_1^1 \geq \frac{\varepsilon}{2}\int_{Q_T}[B_\varepsilon^N\nabla\Psi]^2\zeta^2 + R. \tag{7.30}$$

Now we estimate

$$\begin{aligned}\varepsilon I\!I_2^1 &= \varepsilon\int_{Q_T}[B_\varepsilon^m\nabla\Psi]\Psi\nabla\zeta^2 = \varepsilon\int_{Q_T}[B_\varepsilon^N\nabla\Psi][B_\varepsilon^N(\Psi\nabla\zeta^2)]\\ &= \int_{Q_T}2\varepsilon^{1/2}[B_\varepsilon^N\nabla\Psi]\zeta\left[\varepsilon^{1/2}[B_\varepsilon^N\Psi]\nabla\zeta\right] - \int_{Q_T}\varepsilon^{1/2}[B_\varepsilon^N\nabla\Psi]\left[\varepsilon^{3/2}\sum_{j=0}^{N-1}B_\varepsilon^jA_1(B_\varepsilon^{N-1-j}\Psi,\nabla\zeta^2)\right].\end{aligned}$$

According to Lemma 7.3 (iii), $\varepsilon^{1/2}[B_\varepsilon^N\Psi]\nabla\zeta$ is bounded in $L^\infty(0,T;L^2(\Omega))$. Similarly as above,

$$\varepsilon^{3/2}\sum_{j=0}^{N-1}|B_\varepsilon^jA_1(B_\varepsilon^{N-1-j}\Psi,\nabla\zeta^2)| \leq CM_\sigma|\zeta|\left[\sum_{j=0}^{N-1}\varepsilon^{j+3/2}\left(|D^{2j+1}\Psi|+|D^{2j}\Psi|\right)\right], \tag{7.31}$$

the right bracket being bounded independently of ε by (7.23). Using Young's inequality and (7.30), we get the existence of $R\in\mathbb{R}$ such that

$$\varepsilon I\!I^1 = \varepsilon I\!I_1^1 + \varepsilon I\!I_2^1 \geq \frac{\varepsilon}{4}\int_{Q_T}[B_\varepsilon^N\nabla\Psi]^2\zeta^2 + R. \tag{7.32}$$

Now we study

$$\begin{aligned}\varepsilon I\!I^2 = \varepsilon\int_{Q_T}\Delta\Psi_m\nabla\Phi\nabla\zeta^2 &= -\varepsilon\int_{Q_T}[B_\varepsilon^{2N}\nabla\Psi]\,\nabla(\nabla\Phi\nabla\zeta^2)\\ &= -\int_{Q_T}[\varepsilon^{1/2}B_\varepsilon^N\nabla\Psi]\,[\varepsilon^{1/2}B_\varepsilon^N\nabla(\nabla\Phi\nabla\zeta^2)].\end{aligned} \tag{7.33}$$

Using elliptic regularity theory in (7.7), there exists $C>0$ such that

$$\forall k\in\{1,\dots,m\},\ \|\Phi\|_{W^{k+2,2}(\Omega)} \leq C\|\Psi\|_{W^{k,2}(\Omega)}.$$

Combined with (7.23), and using Lemma 7.3 to control $\nabla\Phi$ in $L^2(\Omega)$, this yields

$$\int_\Omega|\nabla\Phi|^2 + \sum_{j=0}^m\varepsilon^{j+1}\int_\Omega|D^{j+2}\Phi|^2 \leq C. \tag{7.34}$$

Then by the same computations as in (7.29),

$$|\varepsilon^{\frac{1}{2}}B_\varepsilon^N\nabla(\nabla\Phi\nabla\zeta^2)| \leq CM_\sigma|\zeta|\left(\sum_{j=0}^N\varepsilon^{j+\frac{1}{2}}\left(|D^{2j+2}\Phi|+|D^{2j+1}\Phi|\right)\right),$$

and the right-hand side is bounded in $L^\infty(0,T;L^2(\Omega))$, independently of ε using (7.34). As a consequence, using (7.32) and Young's inequality in (7.33), there exists $R\in\mathbb{R}$ such that

$$\varepsilon I\!I^1 + \varepsilon I\!I^2 \geq \frac{\varepsilon}{8}\int_{Q_T}[B_\varepsilon^N\nabla\Psi]^2\zeta^2 + R. \tag{7.35}$$

For I^1, we have

$$I^1 = \int_{Q_T} \Psi_m \Psi \zeta^2 = \int_{Q_T} B_\varepsilon^N \Psi \, B_\varepsilon^N (\Psi \zeta^2)$$
$$= \int_{Q_T} (B_\varepsilon^N \Psi)^2 \zeta^2 - \int_{Q_T} (B_\varepsilon^N \Psi) \varepsilon \sum_{j=0}^{N-1} B_\varepsilon^j A_1 (B_\varepsilon^{N-1-j} \Psi, \zeta^2),$$

and

$$\varepsilon \sum_{j=0}^{N-1} |B_\varepsilon^j A_1 (B_\varepsilon^{N-1-j} \Psi, \zeta^2)| \leq C M_\sigma |\zeta| \sum_{j=0}^{N-1} \varepsilon^{j+1} (|D^{2j} \Psi| + |D^{2j+1} \Psi|),$$

which is bounded in $L^\infty(0,T;L^2(\Omega))$ by (7.23). Using Young's inequality, there exists $R \in \mathbb{R}$ such that

$$I^1 \geq \frac{1}{2} \int_{Q_T} (B_\varepsilon^N \Psi)^2 \zeta^2 + R.$$

Finally,

$$I^2 = - \int_{Q_T} \Psi_m \nabla \Phi \nabla \zeta^2 = - \int_{Q_T} B_\varepsilon^N \Psi \, B_\varepsilon^N (\nabla \Phi \nabla \zeta^2), \tag{7.36}$$

$$|B_\varepsilon^N (\nabla \Phi \nabla \zeta^2)| \leq C M_\sigma |\zeta| \left[|\nabla \Phi| + \sum_{j=1}^{N} \varepsilon^j (|D^{2j} \Phi| + |D^{2j+1} \Phi|) \right].$$

According to (7.34), the right member is bounded in $L^\infty(0,T;L^2(\Omega))$. Using Young's inequality in (7.36), we get the existence of $R \in \mathbb{R}$ such that

$$I^1 + I^2 \geq \frac{1}{4} \int_{Q_T} (B_\varepsilon^N \Psi)^2 + R. \tag{7.37}$$

Combining (7.35) and (7.37), $I^1 + I^2 + \varepsilon(II^1 + II^2)$ is bounded below independently of ε, which ends the proof of *(iii)*.

7.4 Proof of Theorem 7.1

Let $T > 0$, let $(\varepsilon_n)_{n \in \mathbb{N}}$ be a decreasing sequence of positive numbers such that $\varepsilon_n \to 0$. According to Proposition 7.2, there exists a solution of (7.5) – (7.7) on Q_T with parameter ε_n, denoted by (c^n, Ψ^n, Φ^n) in the following. We will also use the notation

$$d_i^n := d_i^{\varepsilon_n} \quad ; \quad J_i^n := d_i^n \nabla c_i^n + d_i^n z_i c_i^n \nabla \Phi^n.$$

We now derive the compactness results that will allow to pass to the limit as $n \to +\infty$ in a weak formulation of (7.5) – (7.7).

Using the mass conservation (7.11), $(c_i^n)^{\frac{1}{2}}$ is bounded in $L^\infty(0,T;L^2(\Omega))$. From Lemma 7.5 *(iii)*, $\nabla (c_i^n)^{\frac{1}{2}}$ is bounded in $L^2(0,T;L^2_{loc}(\Omega)^N)$, so $(c_i^n)^{\frac{1}{2}}$ is bounded in $L^2(0,T;W^{1,2}_{loc}(\Omega))$. Lemma 7.5 *(i)* implies that $(c_i^n)^{-\frac{1}{2}} J_i^n$ is bounded in $L^2(Q_T)^N$. Using Schwarz's inequality, J_i^n is bounded in $L^1(0,T;L^1_{loc}(\Omega)^N)$. To avoid singularities in 0, let us introduce $\delta > 0$ and compute

$$2 \partial_t (c_i^n + \delta)^{\frac{1}{2}} = \frac{\partial_t c_i^n}{(c_i^n + \delta)^{\frac{1}{2}}} = \frac{\mathrm{div} J_i^n}{(c_i^n + \delta)^{\frac{1}{2}}},$$

which is bounded in $L^1(0,T;W^{-1,1}_{loc}(\Omega))$. Since $(c_i^n+\delta)^{\frac{1}{2}}$ is bounded in $L^2(0,T;W^{1,2}_{loc}(\Omega))$, according to Simon's compactness results (see Corollary 4 in [98]), $(c_i^n+\delta)^{\frac{1}{2}}$ is relatively compact in $L^2(0,T;L^2_{loc}(\Omega))$. Consequently, c_i^n is relatively compact in $L^1(0,T;L^1_{loc}(\Omega))$ and up to a diagonal extraction, we can assume that c^n converges a.e. in Q_T to a limit c. According to Lemma 7.3 (iii), $c_i^n \log c_i^n$ is bounded in $L^\infty(0,T;L^1(\Omega))$, so the Vitali theorem guarantees that c_i^n is relatively compact in $L^1(Q_T)$. Since c^n is bounded in $L^\infty(0,T;L^1(\Omega)^P)$, up to a subsequence, we may assume

$$c^n \to c \text{ strongly in } L^1(Q_T)^P,\ c \in L^\infty(0,T;L^1(\Omega)^P). \tag{7.38}$$

We know from Lemma 7.3 (iii) that Φ^n is bounded in $L^\infty(0,T;W^{1,2}(\Omega))$, so up to a subsequence, we can assume

$$\Phi^n \to \Phi \text{ weakly in } L^p(0,T;W^{1,2}(\Omega)) \text{ for any } p<+\infty,\ \Phi \in L^\infty(0,T;W^{1,2}(\Omega)). \tag{7.39}$$

Since $(c_i^n)^{-\frac{1}{2}} J_i^n$ is weakly relatively compact in $L^2(Q_T)^N$ and $(c_i^n)^{\frac{1}{2}}$ converges in $L^2(Q_T)$ (using (7.38)), up to a subsequence,

$$J_i^n \to J_i \text{ weakly in } L^1(Q_T)^N. \tag{7.40}$$

To identify the limit, let us analyse the convergence of the two terms in $J_i^n = d_i^n \nabla c_i^n + d_i^n z_i c_i \nabla \Phi^n$. First, $(c_i^n)^{\frac{1}{2}}$ converges to $(c_i)^{\frac{1}{2}}$ in $L^2(Q_T)$. Since $d_i^n \to d_i$ a.e. and $\underline{d}(T) \le d_i^n \le \overline{d}(T)$, we have $d_i^n (c_i^n)^{\frac{1}{2}} \to d_i (c_i)^{\frac{1}{2}}$ in $L^2(Q_T)$. As recalled above, up to a subsequence, $\nabla (c_i^n)^{\frac{1}{2}}$ weakly converges to $\nabla (c_i)^{\frac{1}{2}}$ in $L^2(0,T;L^2_{loc}(\Omega)^N)$, so $\nabla c_i^n = 2(c_i^n)^{\frac{1}{2}} \nabla (c_i^n)^{\frac{1}{2}}$ weakly converges to $2 c_i^{\frac{1}{2}} \nabla c_i^{\frac{1}{2}} = \nabla c_i$ in $L^1(0,T;L^1_{loc}(\Omega)^N)$. Note in particular that

$$c \in L^1(0,T;W^{1,1}_{loc}(\Omega)^P). \tag{7.41}$$

The second term is $d_i^n c_i^n \nabla \Phi^n = d_i^n (c_i^n)^{\frac{1}{2}}\ (c_i^n)^{\frac{1}{2}} \nabla \Phi^n$. We have $d_i^n (c_i^n)^{\frac{1}{2}} \to d_i (c_i)^{\frac{1}{2}}$ in $L^2(Q_T)$, and using Lemma 7.5 (iii), $(c_i^n)^{\frac{1}{2}} \nabla \Phi^n$ is weakly relatively compact in $L^2(0,T;L^2_{loc}(\Omega)^N)$. To identify the limit, we use $\Phi^n \to \Phi$ weakly in $L^2(0,T;W^{1,2}(\Omega))$ (see (7.39)) and $(c_i^n)^{\frac{1}{2}} \to (c_i)^{\frac{1}{2}}$ strongly in $L^2(Q_T)$, so that $(c_i^n)^{\frac{1}{2}} \nabla \Phi^n \to (c_i)^{\frac{1}{2}} \nabla \Phi$, weakly in $L^1(Q_T)^N$. Since the convergence also occurs weakly in $L^2(0,T;L^2_{loc}(\Omega))$, we get $d_i^n c_i^n \nabla \Phi^n \to d_i c_i \nabla \Phi$ weakly in $L^1(0,T;L^1_{loc}(\Omega)^N)$. All together, this yields

$$J_i = d_i \nabla c_i + d_i z_i c_i \nabla \Phi.$$

Let $(T_k)_{k\in\mathbb{N}}$ be an increasing unbounded sequence of positive numbers, let $(c^{k,n}, \Psi^{k,n}, \Phi^{k,n})$ be a solution of (7.5) – (7.7) on Q_{T_k} with parameter ε_n, and let $(c^{k,p,n}, \Psi^{k,p,n}, \Phi^{k,p,n})$ denote the restriction of $(c^{k,n}, \Psi^{k,n}, \Phi^{k,n})$ on Q_{T_p} for $p \le k$. Let $J_i^{k,p,n} = d_i^n \nabla c_i^{k,p,n} + d_i^n z_i c_i^{k,p,n} \nabla \Phi^{k,p,n}$. Remark that the above compactness results (7.38) – (7.41) hold for $(c^{k,p,k}, \Psi^{k,p,k}, \Phi^{k,p,k})_{k\in\mathbb{N}}$ on Q_{T_p}. As a consequence, there exists (c,Φ) such that, up to a diagonal extraction, for all $p \in \mathbb{N}$,

$$c^{k,p,k} \underset{k\to+\infty}{\longrightarrow} c \quad \text{strongly in } L^1(Q_{T_p}),\ \text{weakly in } L^1(0,T_p;W^{1,1}_{loc}(\Omega))). \tag{7.42}$$

$$J_i^{k,p,k} \underset{k\to+\infty}{\longrightarrow} d_i \nabla c_i + d_i z_i c_i \nabla \Phi \quad \text{weakly in } L^1(Q_{T_p})^N. \tag{7.43}$$

$$c^{k,p,k}(t) \underset{k\to+\infty}{\longrightarrow} c(t) \quad \text{strongly in } L^1(\Omega),\ \text{for a.e. } t \in (0,T_p). \tag{7.44}$$

$$\Phi^{k,p,k} \underset{k\to+\infty}{\longrightarrow} \Phi \quad \text{weakly in } L^q(0,T_p;W^{1,2}(\Omega)) \text{ for any } q<+\infty. \tag{7.45}$$

Then (7.42) and (7.43) allow to pass to the limit $k \to +\infty$ in the following weak formulations of (7.5), namely:

For all $\psi \in C^\infty(\overline{Q_T})$ such that $\psi(T) = 0$, for $i \in \{1,\dots,P\}$,

$$\int_{Q_T} \Big(-c_i^{k,p,k}\partial_t \psi + (d_i^k \nabla c_i^{k,p,k} + d_i^k z_i c_i^{k,p,k} \nabla \Phi^{k,p,k}) \cdot \nabla \psi \Big) = \int_\Omega c_i^0 \psi(0),$$

so (c,Φ) satisfies (7.3). Let $\varphi_1 \in C_c^\infty(0,T_p\,;\mathbb{R})$ and $\varphi_2 \in C^\infty(\overline{\Omega}\,;\mathbb{R})$. For $k \geq p$, we may write a variational formulation of (7.7) as

$$\int_{Q_{T_p}} \varphi_1 \nabla \Phi^{k,p,k} \nabla \varphi_2 + \int_{\Sigma_{T_p}} \varphi_1 (\tau \Phi^{k,p,k} - \xi) \varphi_2 = \int_{Q_{T_p}} \varphi_1 \Psi^{k,p,k} \varphi_2.$$

Using (7.45), we can pass to the limit $k \to +\infty$ in both terms of the left-hand side. Using that $(B_\varepsilon)^{-1}$ is an L^1-contraction in equations (7.6) and (7.44), we obtain

$$\Psi(t)^{k,p,k} \underset{k\to+\infty}{\longrightarrow} \sum_{i=1}^P z_i c_i(t) \text{ in } L^1(\Omega), \text{ for } a.e.\ t \in (0,T_p).$$

Since $\Psi^{k,p,k}$ is bounded in $L^\infty(0,T_p;L^1(\Omega))$, using the Dominated Convergence theorem, we have

$$\int_{Q_{T_p}} \varphi_1 \Psi^{k,p,k} \varphi_2 = \int_0^{T_p} \varphi_1 \left[\int_\Omega \Psi^{k,p,k} \varphi_2 \right] \underset{k\to+\infty}{\longrightarrow} \int_{Q_{T_p}} \varphi_1 \left[\sum_{i=1}^P z_i c_i\ \varphi_2 \right].$$

All together, we have for all $\varphi_1 \in C_c^\infty(0,T_p)$, $\varphi_2 \in C^\infty(\overline{\Omega})$,

$$\int_0^{T_p} \varphi_1 \left[\int_\Omega \nabla \Phi \nabla \varphi_2 - \int_{\partial\Omega} (\xi - \tau\Phi)\varphi_2 \right] = \int_0^{T_p} \varphi_1 \left[\int_\Omega \sum_{i=1}^P z_i c_i\ \varphi_2 \right].$$

As a consequence, (7.4) holds for a.e. $t \in (0,T_p)$, and since p is arbitrary and $T_p \to +\infty$, (7.4) holds for $a.e.\ t \in (0,+\infty)$. Finally, the fact that $c \in L^\infty(0,+\infty;L^1(\Omega)^P)$ and $\Phi \in L^\infty(0,+\infty;W^{1,2}(\Omega))$ is a consequence of the time-independent estimates (7.11) and $(7.13)-(7.14)$.

$\square$

Bibliography

[1] H. Amann, *Global existence for semilinear parabolic systems*, J. Reine Angew. Math., 360 (1985), pp. 47–83.

[2] ——, *Dynamic theory of quasilinear parabolic systems. III. Global existence*, Math. Z., 202 (1989), pp. 219–250.

[3] ——, *Dynamic theory of quasilinear parabolic equations. II. Reaction-diffusion systems*, Differential Integral Equations, 3 (1990), pp. 13–75.

[4] ——, *Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems*, Function Spaces, Differential Operators and Nonlinear Analysis, (1993).

[5] ——, *Linear and Quasilinear Parabolic Problems - Abstract Linear Theory*, vol. 1 of Monographs in mathematics, Birkhäuser Verlag Basel, 1994.

[6] B. Andreianov, M. Bendahmane, and R. Ruiz Baier, *Analysis of a finite volume method for a cross-diffusion model in population dynamics*, M3AS Math. Models Methods Appl. Sci., 21 (2011), pp. 307–344.

[7] S. Aniţa, W. E. Fitzgibbon, and M. Langlais, *Global existence and internal stabilization for a reaction-diffusion system posed on non coincident spatial domains*, Discrete Contin. Dyn. Syst. Ser. B, 11 (2009), pp. 805–822.

[8] J. Baldyga and J. Bourne, *Turbulent mixing and chemical reactions*, John Wiley & Sons, New York, 1999.

[9] P. Baras and M. Pierre, *Problèmes paraboliques semi-linéaires avec données mesures*, Applicable Anal., 18 (1984), pp. 111–149.

[10] M. Bendahmane and M. Langlais, *A one-dimensional model of cell diffusion and aggregation, incorporating volume filling and cell-to-cell adhesion*, J. of Evol. Equ., 10 (2010), pp. 883–904.

[11] M. Bendahmane, T. Lepoutre, A. Marrocco, and B. Perthame, *Conservative cross diffusions and pattern formation through relaxation*, Journal de Mathématiques Pures et Appliquées, 92 (2009), pp. 651 – 667.

[12] M. Bendahmane and M. Sepulveda, *Convergence of a finite volume scheme for nonlocal reaction-diffusion systems modelling an epidemic disease*, Discrete and Cont. Dyn. Systems, Series B, 11 (2009), pp. 823–853.

[13] H. Berestycki, G. Nadin, B. Perthame, and L. Ryzhik, *The non-local Fisher-KPP equation: travelling waves and steady states*, Nonlinearity, 22 (2009), p. 2813.

[14] P. Biler and J. Dolbeault, *Long time behavior of solutions of Nernst-Planck and Debye-Hückel drift-diffusion systems*, Ann. Henri Poincaré, 1 (2000), pp. 461–472.

[15] P. Biler, W. Hebisch, and T. Nadzieja, *The Debye system: existence and large time behavior of solutions*, Nonlinear Anal., 23 (1994), pp. 1189–1209.

[16] M. BISI, F. CONFORTO, AND L. DESVILLETTES, *Quasi-steady-state approximation for reaction-diffusion equations*, Bull. Inst. Math. Acad. Sin. (N.S.), 2 (2007), pp. 823–850.

[17] S. BONAFEDE AND D. SCHMITT, *"Triangular" reaction-diffusion systems with integrable initial data*, Nonlinear Anal., 33 (1998), pp. 785–801.

[18] D. BOTHE, *Nonlinear Evolutions in Banach Spaces – Existence and Qualitative Theory with Applications to Reaction-Diffusion Systems*, Habilitation thesis, Univ. of Paderborn, 1999.

[19] ——, *Instantaneous limits of reversible chemical reactions in presence of macroscopic convection*, J. Differential Equations, 193 (2003), pp. 27–48.

[20] ——, *On the Maxwell-Stefan approach to multicomponent diffusion*, in Progress in Nonlinear Differential Equations and their Applications, Springer, Basel, 2011, pp. 81–93.

[21] D. BOTHE AND W. DREYER, *Continuum-thermodynamics of chemically reacting fluid mixtures and the Maxwell-Stefan equations of multicomponent mass transport*, Submitted.

[22] D. BOTHE, A. FISCHER, M. PIERRE, AND G. ROLLAND, *Global existence and fast-reaction limit for reaction-diffusion-advection systems with a triangular structure*, In preparation.

[23] ——, *Global existence for diffusion-electromigration systems in any space dimension*, In preparation.

[24] D. BOTHE, A. FISCHER, AND J. SAAL, *Global well-posedness and stability of electrokinetic flows*, arXiv:1206.1457v2, (2012).

[25] D. BOTHE AND D. HILHORST, *A reaction-diffusion system with fast reversible reaction*, J. Math. Anal. Appl., 286 (2003), pp. 125–135.

[26] D. BOTHE AND M. PIERRE, *Quasi-steady-state approximation for a reaction-diffusion system with fast intermediate*, J. Math. Anal. Appl., 368 (2010), pp. 120–132.

[27] ——, *The instantaneous limit for reaction-diffusion systems with a fast irreversible reaction*, Discrete Contin. Dyn. Syst. Ser. S, 5 (2012), pp. 49–59.

[28] D. BOTHE, M. PIERRE, AND G. ROLLAND, *Cross-diffusion limit for a reaction-diffusion system with fast reversible reaction*, To appear in Communications in Partial Differential Equations, (2012).

[29] D. BOTHE AND G. ROLLAND, *Global existence for a class of reaction-diffusion systems with mass action kinetics and concentration-dependent diffusivities*, (In preparation).

[30] H. BREZIS, *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, Springer, New York, 2011.

[31] M. C. CAPUTO AND A. VASSEUR, *Global regularity of solutions to systems of reaction-diffusion with sub-quadratic growth in any dimension*, Comm. Partial Differential Equations, 34 (2009), pp. 1228–1250.

[32] L. CHEN AND A. JÜNGEL, *Analysis of a parabolic cross-diffusion population model without self-diffusion*, J. Differential Equations, 224 (2006), pp. 39–59.

[33] Y. CHOI AND R. LUI, *Multi-dimensional electrochemistry model*, Arch. Rational Mech. Anal., 130 (1995), pp. 315–342.

[34] E. CUSSLER, *Diffusion: Mass Transfer in Fluid Systems*, Cambridge University Press, 2 ed., 1997.

[35] S. R. DE GROOT AND P. MAZUR, *Nonequilibrium thermodynamics*, Dover Publications Inc., New York, 1984. Reprint of the 1962 original.

[36] C. DENG, J. ZHAO, AND S. CUI, *Well-posedness for the Navier-Stokes-Nernst-Planck-Poisson system in Triebel-Lizorkin space and Besov space with negative indices*, J. Math. Anal. Appl., 377 (2011), pp. 392–405.

[37] R. DENK, M. HIEBER, AND J. PRÜSS, *$\mathcal{R}$-Boundedness, Fourier Multipliers and Problems of Elliptic and Parabolic type*, Mem. Amer. Math. Soc., 166 (2003), pp. 1–114.

[38] ——, *Optimal $L^p - L^q$-regularity for parabolic problems with inhomogeneous boundary data*, Math. Z., 257 (2007), pp. 193–224.

[39] L. DESVILLETTES, J. A. CANIZO, K. FELLNER, AND F. OTTO, *Improved duality estimates and applications to reaction-diffusion equations*, In preparation.

[40] L. DESVILLETTES AND K. FELLNER, *Exponential decay toward equilibrium via entropy methods for reaction-diffusion equations*, Journal of Mathematical Analysis and Applications, 319 (2006), pp. 157–176.

[41] L. DESVILLETTES, K. FELLNER, M. PIERRE, AND J. VOVELLE, *Global existence for quadratic systems of reaction-diffusion*, Adv. Nonlinear Stud., 7 (2007), pp. 491–511.

[42] G. DONG, *Initial and nonlinear oblique boundary value problem for fully nonlinear parabolic equations*, Journal of Partial Differential Equations, 1 (1988), pp. 12–42.

[43] J. B. DUNCAN AND H. L. TOOR, *An experimental study of three component gas diffusion*, AIChE Journal, 8 (1962), pp. 38–41.

[44] P. DUNLOP AND L. GOSTING, J. Am. Chem. Soc., 77 (1955), pp. 5238–5249.

[45] P. ÉRDI AND J. TÓTH, *Mathematical Models of Chemical Reactions*, Nonlinear Science: Theory and Applications, Princeton University Press, Princeton, NJ, 1989. Theory and applications of deterministic and stochastic models.

[46] J. ESPENSON, *Chemical Kinetics and Reaction Mechanisms*, McGraw-Hill, 2nd ed., 1995.

[47] L. C. EVANS, *Regularity properties for the heat equation subject to nonlinear boundary constraints*, Nonlinear Anal., 1 (1976/77), pp. 593–602.

[48] W. FENG, *Coupled system of reaction-diffusion equations and applications in carrier facilitated diffusion*, Nonlinear Anal., 17 (1991), pp. 285–311.

[49] A. FICK, *Über Diffusion.*, Poggendorff's Annalen der Physik und Chemie, 94 (1855), pp. 59–86.

[50] W. B. FITZGIBBON, S. L. HOLLIS, AND J. J. MORGAN, *Stability and Lyapunov functions for reaction-diffusion systems*, SIAM J. Math. Anal., 28 (1997), pp. 595–610.

[51] H. GAJEWSKI, *On existence, uniqueness and asymptotic behaviour of solutions of the basic equations for carrier transport in semiconductors*, Z. Angew. Math. u. Mech, 65 (1985), pp. 101–108.

[52] S. GENIEYS, V. VOLPERT, AND P. AUGER, *Pattern and waves for a model in population dynamics with nonlocal consumption of resources*, Mathematical Modelling of Natural Phenomena, 1 (2006), pp. 63–80.

[53] D. GILBARG AND N. TRUDINGER, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[54] V. GIOVANGIGLI, *Multicomponent flow modeling*, Modeling and Simulation in Science, Engineering and Technology, Birkhäuser Boston Inc., Boston, MA, 1999.

[55] A. GLITZKY, K. GRÖGER, AND R. HÜNLICH, *Free energy and dissipation rate for reaction diffusion processes of electrically charged species*, Appl. Anal., 60 (1995), pp. 201–217.

[56] A. GLITZKY AND R. HÜNLICH, *Electro-reaction-diffusion systems for heterostructures*, Z. Angew. Math. Mech., 77 (1997), pp. 823–832.

[57] ——, *Global estimates and asymptotics for electro-reaction-diffusion systems in heterostructures*, Appl. Anal., 66 (1997), pp. 205–226.

[58] T. GOUDON AND A. VASSEUR, *Regularity analysis for systems of reaction-diffusion equations*, Ann. Sci. Ec. Norm. Sup., 43 (2010), pp. 117–142.

[59] D. HENRY, *Geometric theory of semilinear parabolic equations*, vol. 840 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1981.

[60] M. HIEBER AND I. WOOD, *The Dirichlet problem in convex bounded domains for operators in non-divergence form with L^∞-coefficients*, Differential Integral Equations, 20 (2007), pp. 721–734.

[61] D. HILHORST, M. MIMURA, AND H. NINOMIYA, *Fast reaction limit of competition-diffusion systems*, in Handbook of differential equations: evolutionary equations. Vol. V, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2009, pp. 105–168.

[62] S. L. HOLLIS, R. H. MARTIN, JR., AND M. PIERRE, *Global existence and boundedness in reaction-diffusion systems*, SIAM J. Math. Anal., 18 (1987), pp. 744–761.

[63] M. IIDA, M. MIMURA, AND H. NINOMIYA, *Diffusion, cross-diffusion and competitive interaction*, J. Math. Biol., 53 (2006), pp. 617–641.

[64] J. JEROME AND R. SACCO, *Global weak solutions for an incompressible charged fluid with multi-scale couplings: Initial-boundary-value problem*, Nonlinear Analysis: Theory, Methods & Applications, 71 (2009), pp. e2487 – e2497.

[65] A. JÜNGEL, *Diffusive and nondiffusive population models*, in: G. Naldi, L. Paresch and G. Toscani (eds.), Mathematical Modeling of Collective Behavior in Socio-Economic and Life Sciences, (2010).

[66] R. KRISHNA AND R. TAYLOR, *Multicomponent mass transfer*, Wiley (New York), 1993.

[67] N. KRYLOV, *Nonlinear elliptic and parabolic equations of the second order*, Mathematics and its applications (D. Reidel Publishing Company): Soviet series, D. Reidel Pub. Co., 1987.

[68] E. LAAMRI, *Global existence of classical solutions for a class of reaction-diffusion systems*, Acta Appl. Math., 115 (2011), pp. 153–165.

[69] O. A. LADYŽENSKAJA, V. A. SOLONNIKOV, AND N. N. URAL'CEVA, *Linear and quasilinear equations of parabolic type*, Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23, American Mathematical Society, Providence, R.I., 1967.

[70] T. LEPOUTRE, *Analyse et modélisation de phénomènes de croissance et mouvement issus de la biologie*, PhD thesis, Université Pierre et Marie Curie, November 2009.

[71] T. LEPOUTRE AND S. MARTINEZ, *Steady states of relaxed cross diffusion models*. In preparation.

[72] T. LEPOUTRE, M. PIERRE, AND G. ROLLAND, *Global well-posedness of a conservative relaxed cross diffusion system.*, SIAM Journal on Mathematical Analysis, 44 (2012), pp. 1674–1693.

[73] S. LEVIN AND L. SEGEL, *Hypothesis for origin of planktonic patchiness*, Nature, 259 (1976), p. 659.

[74] S. A. LEVIN, *A more functional response to predator-prey stability*, The American Naturalist, 108 (1977), pp. 207–228.

[75] Y. LI AND C. ZHAO, *Global existence of solutions to a cross-diffusion system in higher dimensional domains*, Discrete Contin. Dyn. Syst., 12 (2005), pp. 185–192.

[76] P. LICHTNER, C. I. STEEFEL, AND E. OELKERS, *Reactive Transport in Porous Media*, vol. 34 of Reviews in Mineralogy, Mineralogical Society of America, 1996.

[77] G. LIEBERMAN, *Second order parabolic differential equations*, World Scientific, 2005.

[78] R. H. MARTIN, JR. AND M. PIERRE, *Nonlinear reaction-diffusion systems*, in Nonlinear equations in the applied sciences, vol. 185 of Math. Sci. Engrg., Academic Press, Boston, MA, 1992, pp. 363–398.

[79] A. MAUGERI, D. K. PALAGACHEV, AND L. G. SOFTOVA, *Elliptic and parabolic equations with discontinuous coefficients*, vol. 109 of Mathematical Research, Wiley-VCH Verlag Berlin GmbH, Berlin, 2000.

[80] M. MIMURA AND K. KAWASAKI, *Spatial segregation in competitive interaction-diffusion equations*, J. Math. Biol., 9 (1980), pp. 49–64.

[81] M. MIMURA AND J. D. MURRAY, *On a diffusive prey-predator model which exhibits patchiness*, J. Theoret. Biol., 75 (1978), pp. 249–262.

[82] M. MIMURA AND M. YAMAGUTI, *Pattern formation in interacting and diffusing systems in population biology*, Adv Biophys, 15 (1982), pp. 19–65.

[83] J. MORGAN, *Global existence for semilinear parabolic systems*, SIAM J. Math. Anal., 20 (1989), pp. 1128–1144.

[84] J. MORGAN AND S. WAGGONNER, *Global existence for a class of quasilinear reaction-diffusion systems*, Commun. Appl. Anal., 8 (2004), pp. 153–166.

[85] J. MURRAY, *Mathematical Biology*, Springer, Berlin, 1989.

[86] G. NADIN, B. PERTHAME, AND M. TANG, *Can a traveling wave connect two unstable states? The case of the nonlocal Fisher equation*, Comptes Rendus Mathématiques, 349 (2011), pp. 553 – 557.

[87] A. OKUBO, *Diffusion and ecological problems: mathematical models*, vol. 10 of Biomathematics, Springer-Verlag, Berlin, 1980. An extended version of the Japanese edition, Translated by G. N. Parker.

[88] L. ONSAGER AND R. FUOSS, *Irreversible processes in electrolytes. Diffusion, conductance, and viscous flow in arbitrary mixtures of strong electrolytes.*, J. Phys. Chem, 36 (1932), pp. 2686–2778.

[89] M. PIERRE, *Weak solutions and supersolutions in L^1 for reaction-diffusion systems*, J. Evol. Equ., 3 (2003), pp. 153–168. Dedicated to Philippe Bénilan.

[90] ——, *Global existence in reaction-diffusion systems with control of mass: a survey*, Milan J. Math., 78 (2010), pp. 417–455.

[91] M. PIERRE AND D. SCHMITT, *Blowup in reaction-diffusion systems with dissipation of mass*, SIAM Rev., 42 (2000), pp. 93–106 (electronic).

[92] J. PRÜSS, *Maximal regularity for evolution equations in L_p-spaces*, Conf. Semin. Mat. Univ. Bari, (2002), pp. 1–39 (2003).

[93] G. ROLLAND, *Global existence for quadratic reaction-diffusion systems with nonlinear diffusions and L^1 initial data*, (In preparation).

[94] F. ROTHE, *Uniform bounds from bounded-L_p-functionals in reaction-diffusion equations*, J. Differential Equations, 45 (1982), pp. 207–233.

[95] R. RYHAM, *Existence, uniqueness, regularity and long-term behavior for dissipative systems modeling electrohydrodynamics*, arXiv:0910.4973v1, (2009).

[96] M. SCHMUCK, *Analysis of the Navier-Stokes-Nernst-Planck-Poisson system*, Mathematical Models and Methods in Applied Sciences, 19 (2009), pp. 993–1014.

[97] N. SHIGESADA, K. KAWASAKI, AND E. TERAMOTO, *Spatial segregation of interacting species*, J. Theoret. Biol., 79 (1979), pp. 83–99.

[98] J. SIMON, *Compact sets in the space $L^p(0,T;B)$*, Annali di Matematica Pura ed Applicata, 146 (1986), pp. 65–96.

[99] H. TRIEBEL, *Interpolation Theory, Function Spaces, Differential Operators*, North Holland, 1978.

[100] V. K. VANAG AND I. R. EPSTEIN, *Cross-diffusion and pattern formation in reaction-diffusion systems*, Phys. Chem. Chem. Phys., 11 (2009), pp. 897–912.

[101] J. VAZQUEZ, *The Porous Medium Equation - Mathematical Theory*, Clarendon-Press, Oxford, 2007.

[102] L. VÉRON, *Coercivité et propriétés régularisantes des semi-groupes non linéaires dans les espaces de Banach*, Université de Tours, Faculté de sciences, Mathématiques, 1977.

[103] Y. WANG, *The global existence of solutions for a cross-diffusion system*, Acta Math. Appl. Sin. Engl. Ser., 21 (2005), pp. 519–528.

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