3. Theory

3.1. Elementary Theory of Earth's Gravity Field

3.1.1. Newton's Law

The gravitational force **F** is the force acting on a mass point m_1 due to the mass of another mass point m_2 given by *Newton's Law of Gravitation* (Telford, 1981):

$$\mathbf{F} = -G\frac{m_1 m_2}{r^2}\hat{r} \tag{3.1}$$

where:

F = gravitational force acting on m_2 (N)
G = gravitational constant (6.67×10^{-11} $kg^{-1}m^3s^{-2}$)
m_1, m_2 = mass of the two mass points (kg)
r = distance between m_1 and m2 (m)
$\hat{r}$ = unit vector directed from m_1 towards m_2.

The force exerted on the mass point m_2 by the mass point m_1, is given by *Newton's Law of Motion*:

$$\mathbf{F} = m_2\,\mathbf{a} \tag{3.2}$$

where:

a = acceleration of mass point m_2 due to the gravitational force of the mass point m_1 (m/s^2).
Combining equations (3.1) and (3.2) yields:

$$\mathbf{a} = \frac{\mathbf{F}}{m_2} = -G\frac{m_1}{r^2}\hat{r} \tag{3.3}$$

For the gravity field of the Earth, the acceleration of gravity **g** is:

$$\mathbf{g} = -G\frac{M_e}{R_e^2}\hat{r} \tag{3.4}$$

where:

g = acceleration of gravity observed on or above earth's surface
M_e = Earth mass (5.97378×10^{24} kg)
R_e = distance from the observation point to Earth's centre of mass
$\hat{r}$ = unit vector extending outward from the centre of the Earth along radius.

The value of gravity acceleration **g** on the Earth's surface varies between 9.78 m/s^2 at the equator and 9.83 m/s^2 at the poles, because of combination of three factors (Lillie, 1999):

1. Centrifugal acceleration **z** caused by the Earth rotation.
2. The Earth's outward bulging, whereby the radius (R_e) to the centre is increased.
3. The added mass of the bulge amplifies the acceleration.

Notice that the first two factors decrease the acceleration **g** at the equator, while the third increases it.
Gravity (gravitational acceleration) is commonly expressed in units of milligals (mgal), where:

$$1 \text{ gal} = 1 \text{ cm/sec}^2 = 0.01 \text{ m/sec}^2,$$

so that

$$1 \text{ mgal} = 1*10^{-5} \text{ m/sec}^2.$$

3.1.2. Gravitational potential

Since gravitational forces are conservative, work is path independent. The gravitational force is a central force whose direction is along the line joining the centres of two masses.
Conservative forces **F** are derived from a scalar potential function U according

$$\nabla U_r = \mathbf{F(r)}/m_2 = \mathbf{g(r)} \tag{3.5}$$

The gravity potential U(r) can be calculated therefore by

$$U(r) = \int_{\infty}^{R} \mathbf{g}.d\mathbf{r} = -GM\int_{\infty}^{R} \frac{dr}{r^2} = \frac{GM}{R} \tag{3.6}$$

Equation (3.6) gives the work which is necessary to move an unit mass from a point in infinity along any path to a point in distant R from the centre of mass M.
The potential U at point P(x,y,z) caused by a three-dimensional mass can be calculated by taking a mass element (fig.3.1) *dm* at a distance *r* from P according (3.6),

$$dU = G\, dm / r = G\rho\, dx\, dy\, dz / r$$

where:
ρ = density [kg/m^3] and
$r^2 = x^2 + y^2 + z^2$.

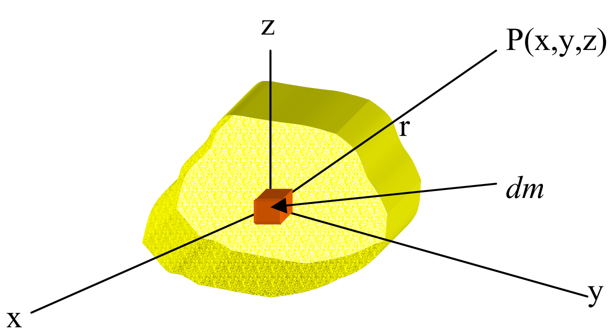


Fig. 3.1 Potential of a three-dimensional mass.

The potential U of the total mass *m* will be

$$U = G\rho \iiint_{x\,y\,z} \frac{1}{r}\, dx\, dy\, dz \tag{3.7}$$

The acceleration $\mathbf{g_z}$ in the direction of the z-axis is given by:

$$g_z = \frac{\partial U}{\partial z} = -G\rho \iiint_{x\,y\,z} \frac{z}{r^3}\, dx\, dy\, dz \tag{3.8}$$

In cylinder coordinates, we write $dx\ dy\ dz = r\ dr\ d\phi\ dz$ and obtain

$$U = G\rho \iiint_{r\,\phi\,z} dr\, d\phi\, dz \tag{3.9}$$

$$g_z = -G\rho \iiint_{r\,\varphi\,z} \frac{z}{r^2}\, dr\, d\varphi\, dz \tag{3.10}$$

In spherical coordinates, we write $dx\ dy\ dz = r^2 \sin\theta\, dr\ d\phi\ d\theta$ and obtain

$$U = G\rho \iiint_{r\,\phi\,\theta} r \sin\theta\, dr\, d\phi\, d\theta \tag{3.11}$$

$$g_z = -G\rho \iiint_{r\,\varphi\,\theta} \sin\theta\, \cos\theta\, dr\, d\varphi\, d\theta \tag{3.12}$$

3.1.3. Gravity gradient

The vertical gravity gradient can be found by calculating the first vertical derivative of g_z:

$$\frac{\partial g_z}{\partial z} = \frac{\partial^2 U}{\partial z^2} = -G\rho \iiint_{x\,y\,z} \left(\frac{1}{r^3} - \frac{3z^2}{r^5} \right) dx\, dy\, dz \tag{3.13}$$

The second derivative of g_z has been employed considerably in gravity interpretation work for upward and downward continuation and for the enhancement of small anomalies at the expense of large-scale effects. The second derivative of (3.8) is:

$$\frac{\partial^2 g_z}{\partial z^2} = \frac{\partial^3 U}{\partial z^3} = -3G\rho \iiint_{x\,y\,z} \left(\frac{5z^3}{r^7} - \frac{3z}{r^5} \right) dx\, dy\, dz \tag{3.14}$$

Horizontal gravity gradient (HGG) can be computed from gravity profiles or contours, as the slope or rate of change of g_z in horizontal directions *dx* or *dy*. By taking, the derivative of g_z along the x-axis we obtain:

$$U_{xz} = \frac{\partial^2 U}{\partial x \partial z} = 3G\rho \iiint_{x\,y\,z} \frac{xz}{r^5}\, dx\, dy\, dz \qquad (3.15)$$

The sharpness of gravity profile is an indication of the depth of the anomalies; this parameter is very significant in gravity interpretation.
Another second derivative of potential is *horizontal directive tendency* (HDT) – *differential curvature*. The horizontal gradient of a horizontal component of gravity, e.g., from (3.7):

$$U_{xx} = \frac{\partial g_x}{\partial x} = \frac{\partial^2 U}{\partial x^2} = G\rho \iiint_{x\,y\,z} \left(\frac{3x^2}{r^5} - \frac{1}{r^3} \right) dx\, dy\, dz \quad (3.16)$$

Other components are $U_{yy} = \partial^2 U/\partial y^2$ and $U_{xy} = \partial^2 U/\partial x \partial y$. Differential curvature can only be measured directly by using the torsion balance or modern gravity gradiometers.

3.2. Gravity Anomalies

Gravity anomaly is the difference between the real gravity value g and the theoretical gravity value γ of an ideal homogeneous Earth of a particular station on the geoid. Gravity depends on three factors:

1. the latitude φ of the observation point, accounted for by the theoretical gravity formula;
2. the elevation *h*, the vertical distance from sea-level datum (geoid) to the observation point, which changes the distance R from the observation point to the centre of the earth;
3. the subsurface mass distribution M.

3.2.1. Normal gravity γ

The normal gravity formula (see table 3.1) describes gravity as a function of the geodetic latitude φ and the orthometric height *h* for a homogenous layered earth model. The latitude dependency is given by a series expansion, extended to the order of f^2 (Torge, 1989):

$$\gamma_\varphi = \gamma_e (1 + \beta \sin^2 \varphi - \beta_1 \sin^2 2\varphi) \qquad (3.17a)$$

where:

γ_φ = gravity for the latitude of the observation point
γ_e = equatorial gravity (ms^{-2})
φ = latitude of the observation point
β = gravity flattening

$$\beta = \frac{\gamma_p - \gamma_e}{\gamma_e} \qquad (3.17b)$$

γ_p gravity at pole= 9.8321864 ms^{-2}

β_1 is a function of the geometrical flattening *f*, the ratio m of equatorial centrifugal acceleration and the equatorial gravity γ_e:

$$\beta_1 = -\frac{1}{8} f^2 + \frac{5}{8} f\, m \tag{3.17c}$$

$$f = \frac{a-b}{a} \tag{3.17d}$$

$$m = \frac{\omega^2 a}{\gamma_e} \tag{3.17e}$$

with
a = semi major axis = 6 378 137 m
b = semi minor axis = 6 356 752.3 m
ω = angular velocity of the earth (7.292115 x 10^{-5} rad s^{-1}).

Tab. 3.1 Coefficients of Normal Gravity Formulas

Name	γ_e (ms^{-2})	β	β_1	f
Helmert (1901)	9.780 30	0.005 302	0.000 007	1:298.3
U.S. Coast and Geodetic Survey (Bowie, 1971)	9.780 39	0.005 294	0.000 007	1:297.4
Intern. Gravity Formula (Cassinis, 1930)	9.780 49	0.005 2884	0.000 0059	1:297.0
Geod. Ref. System 1967 (Int. Ass. Geod., 1971)	9.780 318 (incl. atmosph. mass)	0.005 3024	0.000 0059	1:298.274
Geod. Ref. System 1980 (Moritz, 1984)	9.780 327 (incl. atmosph. mass)	0.005 3024	0.000 0058	1:298.257

At the 7th general assembly of the International Union of Geodesy and Geophysics (IUGG) December, 1979 in Canberra, the International Association of Geodesy (IAG) has adopted the Geodetic Reference System 1980 (GRS80); the normal gravity γ_φ, at the latitude φ is computed according

$$\gamma_\varphi = 978032.7(1+0.0053024 \sin^2\varphi - 0.0000058 \sin^2 2\varphi) \text{ (mgal)} \tag{3.18}$$

The normal gravity γ around Merapi (φ =- 7.5°) amounts therefore

$$\gamma_\varphi = 978120.673 \text{ mgal.}$$

The World Geodetic System 1984 (WGS84) is an approximation to a geocentric reference system, and includes a spherical harmonic gravitational model complete up

to degree and order 180 (DMA, 1987). The WGS 84 constants are (see http//www.nima.mil)
- semi major axis of WGS 84 Ellipsoid: a = 6378137.0 m
- flattening of WGS 84 Ellipsoid: f = 1.0/298.2572235630
- Earth's Gravitational Constant w/ atmosphere: GM= 0.3986004418 x 10^{15} m^3/s^2
- Earth's angular velocity: ω = 7.292115 * 10^{-5} rad/sec

The normal gravity formula for the WGS84 is therefore identical to the formula of the GRS 1980.

3.2.2. Free air anomaly

The normal gravity in proximity of the Earth's surface can be calculated by Taylor series expansion with respect to orthometric height *h* (Torge, 1989):

$$\gamma(\varphi,h) = \gamma_\varphi + \left(\frac{\partial\gamma_\varphi}{\partial h}\right)h + \frac{1}{2}\left(\frac{\partial^2\gamma_\varphi}{\partial h^2}\right)h^2 + ... \qquad (3.19a)$$

with:

$$\left(\frac{\partial\gamma_\varphi}{\partial h}\right) = -\frac{2\gamma_\varphi}{a}\left(1 + f - 2f\sin^2\varphi + \frac{3}{2}f^2 - 2f^2\sin^2\varphi + \frac{1}{2}f^2\sin^4\varphi\right) - 2\omega^2 \qquad (3.19b)$$

and

$$\left(\frac{\partial^2\gamma_\varphi}{\partial h^2}\right) = \frac{6\gamma_\varphi}{a^2\left(1 - f\sin^2\varphi\right)^2} \qquad (3.19c)$$

The height dependency of g in proximity of the ellipsoid can be computed by (3.19a). An expansion to the order of *f* yields:

$$\gamma(\varphi,h) = \gamma_\varphi - 3.0877x10^{-6}\left(1 - 0.00142\sin^2\varphi\right)h + 0.75x10^{-12}\ h^2 \quad (ms^{-2}) \qquad (3.20)$$

where *h* = orthometric height in m.

Mostly, only the linear approximation $\left(\frac{\partial\gamma_\varphi}{\partial h}\right)$ is used as free air correction:

$$FAC_{mgal} = -\left(\frac{\partial\gamma_\varphi}{\partial h}\right) * h = -0.3087 * h_m \qquad (3.21)$$

.

For φ = ±7.5°, $\left(\frac{\partial\gamma_\varphi}{\partial h}\right)$ amounts –0.3087625 (mgal/m).

To compare gravity observations at stations with different elevations, the *free air correction* FAC must be added to the observed values.

The free air anomaly Δg_{fa} is the observed gravity, corrected for the latitude and elevation of the station:

$$\Delta g_{fa} = g_{ob} - \gamma_{\varphi} + FAC \qquad (3.22)$$

where:
Δg_{fa} = free air anomaly
g_{ob} = gravity observed at the station.
γ_{φ} = normal gravity at the station latitude φ

3.2.3. Bouguer anomaly

The Bouguer correction considers the gravitational attraction of masses above a reference level by approximating the mass as an infinite slab, with thickness *h* equal to the elevation of the station. Relative to areas near sea level, mountainous areas would have extra mass, tending to increase the gravity. The attraction of such a slab is:

$$BC = 2\pi\rho Gh = 0.0419\rho h \qquad (3.23)$$

where:
BC = Bouguer correction in (mgal)
ρ = density of slab (g/cm^3)
G = universal gravitational constant (6.67 x 10^{-11} kg^{-1}m^{3}s^{-2})
h = thickness of the slab (station elevation (m))

For regions above sea level, the simple Bouguer anomaly Δg_B results from subtracting the vertical gravitational attraction of the infinite slab (*BC*) from the free air anomaly:

$$\Delta g_B = \Delta g_{fa} - BC \qquad (3.24a)$$

To determine the Bouguer correction, the density of the infinite slab (ρ) must be assumed; commonly the average density of the Earth's crust

$$\rho = 2670 \text{ kg/m}^3$$

is taken into account. Using the standard correction, the simple Bouguer gravity anomaly on land is computed from the free air gravity anomaly according to:

$$\Delta g_B = \Delta g_{fa} - 0.112\, h \text{ (mgal)} \qquad (3.24b)$$

The Bouguer correction at the sea can be envisioned as an infinite slab, equal to the depth of the water as

$$BC_s = 0.0419(\rho_w - \rho_c)h_w \qquad (3.25a)$$

Where:
BC_s = Bouguer correction at sea (mgal)
ρ_w = density of sea water (g/cm^3)

ρ_c = density of Earth's crust (g/cm^3)
h_w = water depth below the observation point (m).
Assuming ρ_w = 1030 kg/m^3 and ρ_c = 2670 kg/m^3, the Bouguer anomaly at the sea Δg_{Bs} is computed from the free air anomaly:

$$\Delta g_{Bs} = \Delta g_{fa} + 0.0687\, h_w \text{ (mgal)} \qquad (3.25b)$$

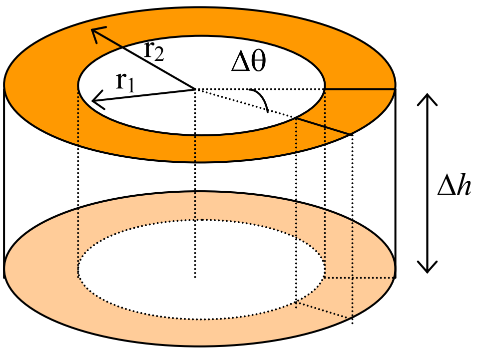
3.2.4. Complete Bouguer anomaly

In areas of smooth relief, the infinite slab correction is normally sufficient to approximate the gravitational effect of masses above the datum near the station (simple Bouguer anomaly). In rough areas, however, may be exist significant effects due to nearby mountains pulling upward on the station, or valleys which do not contain subtracted mass. For such stations, additional terrain corrections (Telford et al., 1981; Dehlinger, 1978) are applied to the simple Bouguer anomaly, yielding the complete Bouguer anomaly:

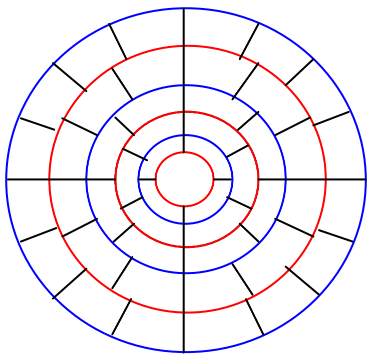
$$\Delta g_{Bc} = \Delta g_B + TC \qquad (3.26)$$

where:
Δg_{Bc} = complete Bouguer anomaly
TC = terrain correction

Terrain corrections can be determined in various ways. The classical procedure is to compute the gravitational attraction produced by segments of hollow right-circular cylinders, with the station on the cylinder axis (Fig.3.2a).



(a)



(b)

Fig. 3.2a Hollow right-circular cylinder with segments of inner and outer radii r_1 and r_2, angle $\Delta\theta$, and height Δh above (below) station.
Fig. 3.2b Schematic representation of zones and compartments used to compute terrain corrections at a station positioned in the centre of the grid.

Cylinders are divided into segments of such size that the sums of corrections for each segment, ΔTC, are within prescribed limits. The height Δh of each segment is the average elevation above or below the height of the station. The terrain correction corresponding to a segment of a hollow cylinder can be written:

$$\Delta TC = G\rho(\Delta\theta)\left(\left(r_1^2 + \Delta h^2\right)^{1/2} - \left(r_2^2 + \Delta h^2\right)^{1/2} + \left(r_2 - r_1\right)\right) \tag{3.27}$$

where:

ρ = density (kg/m^3)
r_1 = inner radius (m)
r_2 = outer radius (m)
Δh = height (m)
$\Delta\theta$ = angle (rad)

The total terrain correction, *TC*, is the sum of individual segment corrections *ΔTC*. Terrain corrections are commonly obtained by dividing a region into segment circular zones (Fig.3.2b) with the station at the centre of the grid. Average terrain height Δh is assigned to each segment.
Terrain correction can also be calculated by rectangular prisms of various size and densities; this procedure offers advantages concerning the Digital Elevation Model (DEM). Only one model has to be generated for all gravity stations, whereas the former method requires the generation of separate models for each station. Digital computer programs using subroutines to calculate the vertical attraction at the station produced by rectangular prisms were developed by different authors for instance Nagy (1966) or Forsberg (1984); Hubbert (1948) developed the calculation of terrain correction with the line-integral method. More details can be found in an extended study of terrain reductions methods in Geodesy and Geophysics by Forsberg (1984).

3.3. Interpreting Gravity Anomalies

All gravity anomalies are generated by lateral inhomogeneous density distributions. If the Earth consists of layers with horizontally uniform density, there would be no gravity anomalies. The layers with different densities are disturbed in such way that gravity anomalies due to mass concentration are produced. The magnitude and form of gravity effects depend on the densities involved, their magnitudes, vertical relief, depth, and horizontal extent (Nettleton, 1971).

3.3.1. Analytical interpretation methods

Location, distribution and density of buried masses can be obtained by inversion of gravity anomalies. The mass distribution cannot be determined uniquely. For the inversion normally assumptions are made, if either the density or shape of the body is known; the other parameters can be obtained then directly under somewhat simplified conditions.
For analyzing structures, indirect and direct methods have been developed. Indirect methods use hypothetical models, for which corresponding attractions are computed and compared with observed (“forward trial and error modelling”), whereas direct methods attempt to solve equations that represent the anomaly.

3.3.2. Integration methods for mathematically described bodies

The vertical component of gravitational attraction Δg_z and the gravitational potential ΔU produced by an arbitrary mass (fig.3.3) can be written (Dehlinger, 1978):

$$\Delta g_z = G\int_v \frac{\Delta\rho \cos\varsigma \, dv}{q^2} \tag{3.28a}$$

$$\Delta U = G\int_v \frac{\Delta\rho \, dv}{q} \tag{3.28b}$$

where:
$\Delta\rho$ = density contrast
G = gravitational constant (6.67×10^{-11} $kg^{-1}m^3s^{-2}$)
q = vector from P to a point Q of the body
ζ = angle which q makes with the vertical
dv = volume element of body
integration is carried out over the volume v of the body.

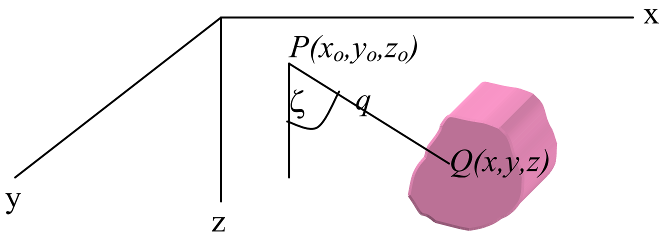


Fig.3.3. Three-dimensional body producing attractions at point P; ζ is the angle that the vector r (from P to Q) makes with the vertical.

Equations 3.28a and 3.28b are commonly integrated in the coordinate system most appropriate to the shape of the body.
Three-dimensional formulas for computing gravitational acceleration and potential produced by anomalous masses (Dehlinger, 1978) are:
- rectangular coordinate system

$$\Delta g_z = G\iiint_{z\,y\,x} \frac{\Delta\rho(z-z_0)\,dx\,dy\,dz}{\left[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right]^{3/2}} \tag{3.29a}$$

$$\Delta U = G\iiint_{z\,y\,x} \frac{\Delta\rho \; dx\,dy\,dz}{\left[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right]^{1/2}} \tag{3.29b}$$

- cylinder coordinate system

$$\Delta g_z = G\iiint_{z\,\theta\,r} \frac{\Delta\rho(z-z_0)\,rdr\,d\theta\,dz}{\left[(r-r_0)^2+(z-z_0)^2\right]^{3/2}} \tag{3.30a}$$

$$\Delta U = G\iiint_{z\,\theta\,r} \frac{\Delta\rho \; r dr\, d\theta\, dz}{\left[(r-r_0)^2+(z-z_0)^2\right]^{1/2}} \tag{3.30b}$$

- spherical coordinate system

$$\Delta g_z = G\iiint_{z\,y\,x} \frac{\Delta\rho(r\;\cos\phi - r_0\;\cos\phi)r^2\sin\phi\; dr\, d\phi\, d\theta}{q^3} \tag{3.31a}$$

$$\Delta U = G\iiint_{z\,y\,x} \frac{\Delta\rho\, r^2\sin\phi\; dr\, d\phi\, d\theta}{q} \tag{3.31b}$$

Applications of these formulas are given in chapter 4.