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Program extraction for Ramsey's theorem for pairs

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Dipl.-Math. Alexander P. Kreuzer
aus Summit, NJ, USA

Referenten:	Prof. Dr. Ulrich Kohlenbach
	Prof. Jeremy Avigad, PhD
	Prof. Dr. Martin Ziegler
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Introduction

In this thesis we give a proof-theoretic account of the strength of Ramsey's theorem for pairs and related principles. We develop a method to extract programs from proofs using this theorem. Moreover, we consider the strength of different variants of the Bolzano-Weierstraß principle. We show that Ramsey's theorem for pairs implies a variant of the Bolzano-Weierstraß principle and, hence, show that our program extraction method is applicable to proofs using this principle.

Also, we develop a method to extract programs from proofs that use non-principal ultrafilters and along with this we obtain a conservation result for the statement that a non-principal ultrafilter exists. This method is based on the techniques we developed for Ramsey's theorem for pairs.

Ramsey's theorem

Denote the set of unordered k-tuples of the set X by $[X]^k$ and call a mapping into the set $\{0,\ldots,n-1\}$ an n-coloring.

Ramsey's theorem (RT) states that for each n-coloring c of $[\mathbb{N}]^k$ there exists an infinite set $X \subseteq \mathbb{N}$, such that c restricted to $[X]^n$ is constant. In the case of $k = 1$ this is just the infinite pigeonhole principle. In the case of pairs and 2-colorings this statement is equivalent to the assertion that each graph over $\mathbb{N}$ contains an infinite clique or an infinite independent set. We will denote by RT^k_n the restriction of RT to k-tuples and n-colorings and by $\mathsf{RT}^k_{<\infty}$ the statement $\forall n\, \mathsf{RT}^k_n$.[1]

Ramsey's theorem was introduced by Ramsey in 1930. It was then popularized by Erdős and is now one of the fundamental theorems of the—so called—Ramsey theory.

The computational and logical strength of Ramsey's theorem has been investigated since 1971. In particular, it received much attention lately in the context of the reverse mathematics program. See [89, 50, 41, 83, 16, 86, 38, 39, 20], to name only some of the most important references.

Ramsey's theorem for singletons ($\mathsf{RT}^1_{<\infty}$) is equivalent to the—so called—Π^0_1-bounded collection principle and its strength is therefore well established, see [41].

[1] There is also a finite variant of Ramsey's theorem, which states roughly that for colorings c of large enough sets there is always a set X of a given size, such that c is constant on the tuples of X. We do not consider this variant in this thesis, since it provable in the systems we consider and therefore trivial from our perspective.

Ramsey's theorem for triples and bigger k-tuples (of fixed size) is equivalent to arithmetical comprehension (ACA_0) and thus its strength is also well established, see [86].

For Ramsey's theorem for pairs the situation is more difficult. It is know that on the one hand there are computable instance having no computable (even in the first Turing jump) solutions, on the other hand this principle does *not* compute the Turing jump or arithmetical comprehension, see [50, 83]. In [16] it was show that it has low_2 solutions, this means that the second Turing jump of a solution is not harder to compute than the second Turing jump. As consequence, Ramsey's theorem for pairs is not provable in systems like RCA_0 or WKL_0. However, it is also weaker than arithmetical comprehension.

For the first order consequences of RT^2_2 it is known that they are provable by Σ^0_2-induction, but it is not known whether Σ^0_2-induction is implied by RT^2_2. The best known lower bound is the Π^0_1-bounded collection principle, i.e. $\mathsf{RT}^1_{<\infty}$, see [16]. It is not known where exactly between these principles the first order consequences of RT^2_2 lie. In particular, it is not known whether RT^2_2 implies the totality of the Ackermann function (which Σ^0_2-induction does and the Π^0_1-bounded collection principle does not), see [75].

Reverse Mathematics

Reverse mathematics is a field of logic which determines what set existence axioms are needed to prove theorems occurring in (everyday-) mathematics. For instance, it was shown that the Bolzano-Weierstraß principle in the form "every bounded sequence of $\mathbb{R}$ has a cluster point" requires arithmetical comprehension (ACA_0). In fact it was show this principle is equivalent to ACA_0, see [30].

One of the most important findings of the Reverse Mathematics program is that nearly all theorems occurring in mathematics are equivalent over a weak basis theory to one of the so called "big five"-systems. These are RCA_0, WKL_0, ACA_0, ATR_0, Π^1_1-CA_0. In this thesis we will only be concerned with the first three systems. The system RCA_0 is the fragment of second order arithmetic based on recursive comprehension. Roughly this system corresponds to computable mathematics. The system WKL_0 is RCA_0 plus the statement that every infinite 0/1-tree has an infinite branch and ACA_0 is RCA_0 plus arithmetical comprehension. Some examples of theorems equivalent to those systems, its corresponding foundational status and the position of Ramsey's theorem is shown in Figure 1. For details, see [86].

The interest in RT^2_2 is based on the fact that it is not equivalent to one of the big five systems, i.e. it is the exception of this rule, and because it—despite of the huge efforts—resists a clear classification.

Predicativism (Weyl, Feferman)	Sequential compactness, Ascoli lemma, Countable fields over $\mathbb{Q}$ have a basis	
Finitistic Reductionism (Hilbert)	Heine-Borel compactness, Riemann integration, Countable fields posses a *unique* closure	
Recursive analysis with full classical logic	Intermediate value theorem, Countable fields posses a closure	

$$\begin{tikzcd}
\mathsf{ACA}_0 \arrow[r, leftrightarrow] \arrow[dd] & \mathsf{RT}^3_2 \arrow[d] \\
& \mathsf{RT}^2_2 \arrow[dl, leftrightarrow, "/" marking] \arrow[dd] \\
\mathsf{WKL}_0 \arrow[d] & \\
\mathsf{RCA}_0 & \mathsf{RT}^1_{<\infty} \arrow[l]
\end{tikzcd}$$

Figure 1.: Status of Ramsey's theorem for pairs

Program extraction

By program extraction we mean methods that systematically extract terms from proofs, such that these terms witness the existential content of the proved statement. Typically one starts with a proof of a statement of the form $\forall x\,\exists y\,A(x,y)$ in a system $\mathcal{T}$. Then one would like to extract a term t witnessing this y, i.e. a term t such that $A(x,t(x))$. Of course this term t should reflect (depending on the given proof) the strength of the system $\mathcal{T}$. Often one wants that t is provably recursive in $\mathcal{T}$ or a weaker system and that the statement $A(x,t(x))$ is also provable in this system.

To obtain these programs we use Gödel's functional interpretation. The rough idea of Gödel's functional interpretation is that one can assign to each formula a $\forall\exists$-formula which is provably equivalent to the original formula (using only quantifier free axiom of choice) and if it is already a $\forall\exists$-formula then it does not change.[2] For instance, for a Π^0_3-formula

$$B \equiv \forall x\,\exists y\,\forall z\,B_{qf}(x,y,z)$$

one obtains a functional interpretation B^{ND} by building the choice function for z relative to y and obtains

$$B^{ND} \equiv \forall x\,\forall f\,\exists y\,B_{qf}(x,y,f(y)).$$

It is immediately clear that B implies B^{ND}. To see the reverse direction assume that B fails, i.e.

$$\exists x\,\forall y\,\exists z\,\neg B_{qf}(x,y,z)$$

[2]Strictly speaking we are here describing the Shoenfield variant of Gödel's functional interpretation.

and thus that one can define a mapping $f(y)$ yielding a z such that $\neg B_{qf}(x, y, z)$ holds. This contradicts B^{ND}.

This translation is made in such a way that it is closed under the modus ponens rule, which means that if one has terms witnessing the $\exists$-quantifiers of A^{ND} and $(A \to B)^{ND}$ one can build a term witnessing the $\exists$-quantifier of B^{ND} by application.

Now to show that from a system $\mathcal{T}$ one can extract terms in a given term system (or fragment of it) one only has to find witnesses for the interpretation of the axioms.

In order to be able to define the functional interpretation, one needs a system which not only contains second-order variables and quantification, but also variable and quantification for all—so called—*finite types*. This means roughly that the system contains types for $\mathbb{N}$, $\mathbb{N}^{\mathbb{N}}$, $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$, $\mathbb{N}^{\mathbb{N}\times\mathbb{N}}$, etc. The finite type variants of RCA_0, WKL_0, ACA_0 will be denoted by RCA_0^ω, WKL_0^ω, ACA_0^ω, respectively.

The most notable results in program extraction are that

- from proofs in RCA_0^ω one can extract primitive recursive (in the sense of Kleene) terms,
- from proofs in ACA_0^ω one can extract terms primitive recursive in the bar recursor $B_{0,1}$, which are primitive recursive in the sense of Gödel (if the type is not to high), and that
- from proofs in WKL_0^ω one can extract primitive recursive (in the sense of Kleene) bounds on realizers, which (in the case the witnessed quantifiers are numbers) yield actual realizer by bounded search.

The first results is due to Gödel [36], the second due to Spector [90], and the third due to Kohlenbach [57].

Our results

The purpose of this thesis is to develop a program extraction method for proofs that use Ramsey's theorem for pairs, which reflects the fact that Ramsey's theorem for pairs does not imply arithmetical comprehension.

We provide such a method and also consider some consequences of Ramsey's theorem for pairs. In detail, we show that from proofs using RT^2_2 one can extract Ackermann type programs and from proofs using $\mathsf{RT}^2_{<\infty}$ one can extract programs provably total by Σ^0_3-induction (Chapter 3).

For proofs using the cohesive principle (COH) and the atomic model theorem (AMT)—both consequences of RT^2_2—we show with the help of the elimination of monotone Skolem functions ([60]) that one can extract primitive recursive programs (Chapter 2). Definitions of COH, AMT, will be given below; but it is important to note that COH is equivalent to the variant of the Bolzano-Weierstraß principle ($\mathsf{BW}_{\mathsf{weak}}$), which states that every bounded sequence of $\mathbb{R}$ contains a Cauchy-subsequence, and thus is a widely used principle.

Later we strengthen this program extraction result and show that one can even extract primitive recursive programs from proofs that use the chain antichain principle (CAC) (Chapter 4). For the proof of this result we refined Howard's ordinal analysis of bar recursion to terms in the Grzegorczyk hierarchy (Section 4.1). A definition of CAC will also be given later, here it is important to mention that it implies the even stronger variant of the Bolzano-Weierstraß principle, which states that every bounded sequence of $\mathbb{R}$ contains a *monotone* subsequence. Since CAC implies AMT and COH, this result gives a different proof of the previous result.

Furthermore, we analyze the strength of variants of the Bolzano-Weierstraß principle. We already mentioned that we showed that COH is equivalent to the variant of the Bolzano-Weierstraß principle ($\mathsf{BW}_{\mathsf{weak}}$), which states that every bounded sequence of $\mathbb{R}$ contains a Cauchy-subsequence. This principle is weaker than the usual formulation of the Bolzano-Weierstraß principle (BW) which states the every bounded sequence of $\mathbb{R}$ has a cluster point. Since in reverse mathematics real numbers are defined to be Cauchy-sequences of rational numbers with Cauchy-rate 2^{-n}, BW is equivalent to the statement that every bounded sequence of $\mathbb{R}$ contains a Cauchy-subsequence *with Cauchy-rate* 2^{-n}. This equivalence yields a tight classification of $\mathsf{BW}_{\mathsf{weak}}$ and using this we can refine a result by Le Roux and Ziegler. See Chapter 6.

We then consider the Bolzano-Weierstraß principle for weak compactness on a Hilbert space ($\mathsf{weak}\text{-}\mathsf{BW}$) and show that this principle corresponds to two Turing jumps and is thus instance-wise stronger than BW. Using this we can show that Kohlenbach's analysis of the weak compactness functional Ω^* is optimal with respect to the uses of bar recursion. See Chapter 7.

In Chapter 8 we provide a program extraction and conservativity result for non-principal ultrafilters. We show that the existence of non-principal ultrafilters is strictly stronger than ACA_0^ω but is Π^1_2-conservative over ACA_0^ω. Furthermore this conservativity result can be used for program extraction. This result is based on the techniques we developed in the first part of this thesis.

In the last Chapter we formalize a proof of a generalization of the Banach contraction mapping principle and show that it is provable in RT^2_2 resp. $\mathsf{RT}^2_{<\infty}$. For this we use some ultrafilter like properties of cohesive sets.

Publications

The Chapters 1 to 3 have been published in

> Alexander P. Kreuzer and Ulrich Kohlenbach, *Term extraction and Ramsey's theorem for pairs*, forthcoming in the Journal of Symbolic Logic.

The Chapters 4 and 5 have been published in

> Alexander P. Kreuzer, *Primitive recursion and the chain antichain principle*, Notre Dame J. Formal Logic **53** (2012), no. 2, 245–265.

The results of Chapter 6 have been published in

> Alexander P. Kreuzer, *The cohesive principle and the Bolzano-Weierstraß principle*, Math. Log. Quart. **57** (2011), no. 3, 292–298.

The Chapter 8 has been published in

> Alexander P. Kreuzer, *Non-principal ultrafilters, program extraction and higher-order reverse mathematics*, forthcoming in the Journal of Mathematical Logic.

First program extraction results for RT^2_2 have been obtained in

> Alexander P. Kreuzer, Ulrich Kohlenbach, *Ramsey's Theorem for pairs and provably recursive functions*, Notre Dame J. Formal Logic **50** (2009), no. 4, 427–444.

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Zusammenfassung

In dieser Arbeit untersuchen wir die beweistheoretische Stärke des Satzes von Ramsey für Paare. Wir entwickeln eine Methode, um Programme aus Beweisen, die diesen Satz verwenden, zu extrahieren. Des weiteren analysieren wir die Stärke von Varianten des Bolzano-Weierstraß Prinzips. Wir zeigen, dass der Satz von Ramsey eine Variante impliziert und dass damit unsere Programmextraktionsmethode anwendbar ist. Wir entwickeln auch eine Methode zur Extraktion von Programm aus Beweisen, die freie Ultrafilter verwenden. Die Methode basiert auf Techniken, die wir für die Extraktion von Programmen aus Beweisen, die den Satz von Ramsey verwenden, entwickelt haben.

Part I.

Ramsey's theorem for pairs

1. Introduction to Ramsey's theorem for pairs

In this part of the thesis we develop a technique of program extraction for proofs that use Ramsey's theorem for pairs, the cohesive principle, the chain antichain principle and other principles weaker than Ramsey's theorem for pairs. As a consequence it also gives a proof theoretic account of conservation results for those principles.

Ramsey's theorem for pairs (RT^2_n) is the statement that every coloring of pairs of natural numbers ($[\mathbb{N}]^2$) with n colors has an infinite homogeneous set. A simples colorblindness argument shows that

$$\mathsf{RT}^2_2 \leftrightarrow \mathsf{RT}^2_n \quad \text{for every fixed } n.$$

Ramsey's theorem for pairs and arbitrary large colorings ($\mathsf{RT}^2_{<\infty}$) is defined as $\forall n\, \mathsf{RT}^2_n$. This principle is proof-theoretically stronger than RT^2_2, whereas from the viewpoint of computation there is no difference in strength.

A coloring c of pairs is called *stable* if $c(\{x, \cdot\})$ eventually becomes constant for every x. The restriction of RT^2_n to stable colorings is denoted by SRT^2_n. Here a similar colorblindness argument can be applied.

A set G is called *cohesive* for a sequence $(R_i)_{i\in\mathbb{N}}$ of subsets of $\mathbb{N}$ if

$$\forall i \left(G \subseteq^* R_i \vee G \subseteq^* \overline{R_i}\right),$$

where $X \subseteq^* Y :\equiv (X \setminus Y \text{ is finite})$. The *cohesive principle* (COH) states that for every $(R_i)_{i\in\mathbb{N}}$ an infinite cohesive set exists. It is in some way the counterpart to SRT^2_n since

$$\mathsf{RCA}_0 \vdash \mathsf{RT}^2_n \leftrightarrow \mathsf{SRT}^2_n \wedge \mathsf{COH}$$

for $2 \leq n$ or $n =$ "$<\infty$", see [16, 17].

The *chain antichain principle* (CAC) states the each partial ordering of the natural numbers contains either an infinite chain or an infinite antichain. It is easy to see that this principle is a consequence of RT^2_2. Like RT^2_2 this principle also splits into a stable principle, the—so called—*stable chain antichain principle* (SCAC), and COH. The definition of SCAC will be given in Chapter 4.

We also consider the *atomic model theorem* (AMT) which states that every atomic theory has an atomic model, see [40]. This principle is also a consequence of RT^2_2, it is even a consequence of SCAC. A detailed definition of this principle will also follow later.

The computational strength of Ramsey's theorem has been investigated since the early 70's. Specker showed 1971 that there exists a computable coloring of $[\mathbb{N}]^2$ that has no computable homogeneous set, see [89]. Jockusch improved this 1972 by showing that in general there is not even a Σ^0_2 infinite homogeneous set. He also provided an upper bound on the strength of Ramsey's theorem for pairs and showed that each computable coloring of pairs admits an infinite homogeneous set H with $H' \leq_T 0''$, see [50]. Seetapun and Slaman showed in [83] that RT^2_2 does not solve the halting problem. Cholak, Jockusch and Slaman improved both results by showing that an infinite homogeneous low_2 set exists for every computable coloring of pairs, i.e. a set H satisfying $H'' \leq_T 0''$, see [16].

From Specker's results it is clear that $\mathsf{RCA}_0 \nvdash \mathsf{RT}^2_2$. Seetapun's and Slaman's results immediately yield an upper bound on the proof-theoretic strength, it implies that RT^2_2 does not prove Π^0_1-comprehension or—equivalently—ACA_0. Hirst showed 1987 that RT^2_2 implies the infinite pigeonhole principle ($\mathsf{RT}^1_{<\infty}$) which is equivalent to the Π^0_1-bounded collection principle (Π^0_1-CP). Cholak, Jockusch and Slaman showed along their recursion theoretic proof that RT^2_2 is Π^1_1-conservative over $\mathsf{RCA}_0 + \Sigma^0_2$-$\mathsf{IA}$.

This leaves the question whether RT^2_2 implies Σ^0_2-IA. Despite of many efforts in the last years this question could not be settled yet.

Ramsey's theorem for triples and bigger tuples is equivalent to ACA_0 and hence fully classified in the sense of reverse mathematics, see [86].

The cohesive principle has been originally considered in recursion theory, see for instance [87]. Its computational strength has been fully determined in [48]. Cholak, Jockusch and Slaman observed in [16] that Ramsey's theorem for pairs splits nicely into a stable part and the cohesive principle. They also showed that it is Π^1_1-conservative over RCA_0 and $\mathsf{RCA}_0 + \Sigma^0_2$-$\mathsf{IA}$. In the course of the classification of Ramsey's theorem the logical strength of the cohesive principle received attention in the last years, see for instance [18] and [20]. In [20] it was shown that the cohesive principle is Π^1_1-conservative over $\mathsf{RCA}_0{+}\Pi^0_1$-CP. We will show in Chapter 6 that over $\mathsf{RCA}_0{+}\Pi^0_1$-CP the cohesive principle is equivalent to a weak form of the Bolzano-Weierstraß principle. Thus the cohesive principle also shows up in analytic proofs.

The chain antichain principle is interesting in this context because it is only slightly weaker than RT^2_2 but Chong, Slaman and Yang were able to show that it is Π^1_1-conservative over $\mathsf{RCA}_0 + \Pi^0_1$-CP and that is does not implies Σ^0_2-induction, see [20]. Moreover, from the perspective of proof mining, i.e. the extraction of bounds from actual proofs in mathematics, the principle CAC is in most cases sufficient.

For an extensive survey on the current status of Ramsey's theorem for pairs and weaker principles, see [39] and [84].

The purpose of this part is to give an account to the above mentioned conservation results from the perspective of proof mining and program extraction. We provide new proofs for these conservation results which additionally yield realizing terms. Since the types of these terms raise with the complexity of the formula the conservation result is naturally bounded to Π^0_3-sentences.

Proofwise low

Define Π^0_1-comprehension as

$$(\Pi^0_1\text{-}\mathsf{CA})\colon \forall X\, \exists Y\, \forall u\ (u \in Y \leftrightarrow \forall v\ \langle u, v\rangle \in X)\,.$$

This covers the full strength of Π^0_1-comprehension since $\forall v\ \langle u, v\rangle \in X$ is a universal Π^0_1-statement (relative to the parameter u). Full arithmetical comprehension (ACA_0) follows by iteration. For a primitive recursive term t we will write Π^0_1-$\mathsf{CA}(t)$ if X is instantiated with the set $\{n \mid t(n) = 0\}$.[1] For a closed term t the principle Π^0_1-$\mathsf{CA}(t)$ is also called an *instance of* Π^0_1*-comprehension*.

The union of Π^0_1-$\mathsf{CA}(t)$ for all terms t containing only number variables free is the same as light-face Π^0_1-comprehension. In particular, this does not prove ACA_0.

Let $\mathcal{P}$ be a second order principle stating the existence of a set G relative to a set parameter S—that is a principle of the form

$$(\mathcal{P})\colon \forall S\, \exists G\ P(S, G).$$

Definition 1.1 (proofwise low)**.** Call a principle of the form $\mathcal{P}$ *proofwise low* over a system $\mathcal{T}$ if for every provably continuous[2] term φ a provably continuous term ξ exists such that

$$\mathcal{T} \vdash \forall S\ \Big(\Pi^0_1\text{-}\mathsf{CA}(\xi S) \to \exists G\ \Big(P(S, G) \land \Pi^0_1\text{-}\mathsf{CA}(\varphi S G)\Big)\Big)\,. \tag{1.1}$$

If we additionally can prove this for a sequence of solutions, i.e.

$$\mathcal{T} \vdash \forall (S_i)_{i\in\mathbb{N}}\ \Big(\Pi^0_1\text{-}\mathsf{CA}(\xi(S_i)_i) \to \exists (G_i)_{i\in\mathbb{N}}\ \Big(\forall i\, P(S_i, G_i) \land \Pi^0_1\text{-}\mathsf{CA}(\varphi(S_i)_i(G_i)_i)\Big)\Big) \tag{1.2}$$

then we call $\mathcal{P}$ *proofwise low in sequence* over the system $\mathcal{T}$. Here $(S_i)_i$ is (a code of) the sequence of sets S_i. It is given by the set $\{\langle i, x\rangle \mid x \in S_i\}$.

[1] Strictly speaking RCA_0 does not contain terms. Here and in the following we silently assume that we work in the conservative extension of RCA_0 by all primitive recursive functions.

[2] Continuous means here continuous in the sense of Baire space, i.e. φ is continuous if

$$\forall f\, \exists n\, \forall g\ (\forall x < n\, f(x) = g(x) \to \varphi(f) = \varphi(g))\,.$$

Such functionals can be coded into primitive recursive functions. For details see Definitions 1.6 and 1.7 below.

The notion of proofwise low is comparable to low_2 in the recursion theoretic setting: take for instance $\mathcal{T} = \mathsf{WKL}_0$, then a proofwise low statement in $\mathcal{T}$ satisfies

$$\mathsf{RCA}_0 \vdash \forall S \left(\mathsf{WKL} \wedge \Pi^0_1\text{-}\mathsf{CA}(\xi S) \to \exists G \left(\mathcal{P}(S,G) \wedge \Pi^0_1\text{-}\mathsf{CA}(\varphi SG)\right)\right).$$

The analogous recursion theoretic statement would be that relative to an oracle of Turing degree $d \gg 0'$—this resembles the premise—a set G satisfying the statement $P(S,G)$ and its Turing jump G' can be computed. From this follows that $G'' \equiv_T 0''$ or in other word that G is low_2.

The main results of these chapters can be divided into two parts:

1. We show roughly that
   a) RT^2_2 is proofwise low over WKL_0,
   b) COH, AMT are proofwise low in sequence over WKL^*_0,[3] and that
   c) CAC is proofwise low over WKL^*_0.
2. We show for principles $\mathcal{P}$ that
   a) if $\mathcal{P}$ is proofwise low over WKL_0 and P is Π^0_1, the system $\mathsf{WKL}_0 + \Sigma^0_2\text{-}\mathsf{IA} + \mathcal{P}$ is Π^0_3-conservative over Σ^0_2-induction. (This covers RT^2_2.)
   b) if $\mathcal{P}$ is proofwise low in sequence over WKL^*_0 and P is Π^0_3, the system $\mathsf{WKL}_0 + \Pi^0_1\text{-}\mathsf{CP} + \mathcal{P}$ is Π^0_3-conservative over RCA_0. (This covers COH and AMT.)
   c) if $\mathcal{P}$ is proofwise low over WKL^*_0 and P is Π^0_1, then $\mathsf{WKL}_0 + \Pi^0_1\text{-}\mathsf{CP} + \mathcal{P}$ is Π^0_3-conservative over RCA_0. (This covers CAC.)

This simplifies the results slightly. The actual results require a suitable finite type extension of WKL_0 and WKL^*_0, see below.

The proofs of 1.a), 1.b) are based on the proofs by "first jump control" for SRT^2_2 and COH of Cholak, Jockusch and Slaman, see [16], showing that these principles have low_2 solutions. To our knowledge these proofs have not been used before to obtain conservativity results for RT^2_2. Cholak, Jockusch and Slaman developed in this paper a different, more complicated proof needing Π^0_2-comprehension that can be used in a forcing construction to show conservativity of RT^2_2 over Σ^0_2-induction.

The proof of 1.c) is based on the proof of the Π^1_1-conservativity of CAC over RCA_0 $+\Pi^0_1\text{-}\mathsf{CP}$ by Chong, Slaman, Yang, see [20].

[3] The system RCA^*_0 is defined to be RCA_0 where Σ^0_1-induction is replaced by quantifier-free-induction plus the exponential function. The system WKL^*_0 is RCA^*_0 plus weak König's lemma. See [86, X.4.1].

For the second part we use Gödel's functional interpretation (always combined with a negative translation) to extract a term t from a proof of an arbitrary statement of the following form

$$\mathcal{P} \to \forall x\, \exists y\, A(x, y),$$

where A is quantifier-free and $\mathcal{P}$ is a proofwise low principle. For an oracle solution $\mathcal{P}$ of the functional interpretation of $\mathcal{P}$ this term will then satisfy

$$\forall x\, A(x, t(\mathcal{P}, x)).$$

We normalize t so that every application of $\mathcal{P}$ in the proof is of a specific form and one can read off from the term and the proof how much of $\mathcal{P}$ is used. The functional $\mathcal{P}$ is then eliminated from t by interpreting every specific application of $\mathcal{P}$. This is done either by (1.2) (in the case of b) or the functional interpretation of (1.1) (in the cases of a), c)) in a way that retains the instance of comprehension. If this retained instance of comprehension is used for the next interpretation of $\mathcal{P}$ then an inductive treatment of every application of $\mathcal{P}$ yields that

(i) in the first case one instance of the functional interpretation of Π^0_1-CA suffices to prove to totality of t and hence $\forall x\, \exists y\, A(x, y)$,

(ii) in the second case one instance of Π^0_1-CA proves the totality of t and hence $\forall x\, \exists y\, A(x, y)$.

The instance of comprehension is then eliminated in favor of induction:
In (i) the solution to this functional interpreted instance of comprehension is provided by an instance of Spector's bar recursion (in fact by an application of the rule of bar recursion). This usage of bar recursion is then eliminated using Howard's ordinal analysis of bar recursion in favor of Σ^0_2-induction (section 3.4). To obtain conservativity over RCA_0 in the case of c) we refined this ordinal analysis.

In (ii) the instance of comprehension is eliminated through elimination of Skolem functions for monotone formulas, see [60], yielding that $\forall x\, \exists y\, A(x, y)$ is provable in primitive recursive arithmetic.

For this application of the elimination of Skolem function and for the refined ordinal analysis it is crucial that $\mathcal{P}$ is proofwise low over a system that does *not* contain Σ^0_1-induction, for instance WKL^*_0.

These techniques of elimination of instances of comprehension can be viewed as a proof-theoretic refinement of the arithmetical conservativity of ACA_0 over PA, see [9], [29], [92] and [86, IX.1.6].

Comparison to conservation results by syntactic forcing

Syntactic forcing is a method to prove conservativity result. It is commonly used in reverse mathematics.

To show that a second order principle $\mathcal{P}$ is conservative over $\mathcal{T}$ it proceeds by first taking an arbitrary countable model of $\mathcal{T}$. This model is then extended through a forcing argument to include sets solving all instances of $\mathcal{P}$ without altering the first order part. The conservativity then follows by Gödel's completeness theorem. For details and further information see [5].

The elimination of monotone Skolem functions and Howard's elimination of bar recursion are constructive: a careful analysis of the proofs would yield a uniform method to obtain a term of $\mathcal{T}$ for each function provable total using $\mathcal{P}$. Whereas the forcing argument essentially uses a reductio ad absurdum argument (if $\mathcal{P}$ would not be conservative then by the completeness theorem there would be a model that could not be extended). Hence it admits no construction.

Forcing yields in many cases full Π^1_1-conservativity whereas the functional interpretation usually stops at Π^0_3-conservativity. This is a consequence of the way the functional interpretation works: it transforms every statement in a functional, where for every additional quantifier alternation the type-level rises, making it more complex to analyze. For instance, Π^0_3-statements correspond to type 2 functionals (i.e. functionals essentially of the form $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$).

This makes it easier to handle principles implying the Π^0_1-bounded collection principle (Π^0_1-CP). Due to the well-known fact that Π^0_1-CP is Π^0_3-conservative over Σ^0_1-IA the base theory for the functional interpretation does not change. This circumvents the problems forcing experiences when proving conservativity over Π^0_1-CP, see [39, §6].

The original proof that RT^2_2 or COH is Π^1_1-conservative over Σ^0_2-induction uses syntactic forcing, also the proof that COH is Π^1_1-conservative over Σ^0_1-induction uses it, see [16]. The original proof of the fact that COH is Π^1_1-conservative over Π^0_1-CP is done using a complicated double forcing, see [20]. Our proof of the fact $\mathsf{COH} + \Pi^0_1$-CP is Π^0_3-conservative over Σ^0_1-IA is similar to the proof of [16] since we show conservativity over RCA_0 (without Π^0_1-CP) and therefore do not face the problems forcing experiences with Π^0_1-CP and that Chong, Slaman and Yang in [20] deal with. For the proof of the fact that CAC is Π^1_1-conservative over $\mathsf{RCA}_0 + \Pi^0_1$-CP Chong, Slaman and Yang use forcing over non-standard models. Since we eliminated Π^0_1-CP, we can replaces this non-standard construction in our proof by a bounded iteration which is provably total in RCA^*_0.

Additionally, our proof is open for proof mining that means it provides a method for program extraction.

The question arises whether RT^2_2 is also proofwise low over WKL^*_0 or some weak extension of this system such that one can show that it does not prove more than primitive recursive growth. In Chapter 5 we discuss the Erdős-Moser principle (EM). Bovykin and Weiermann observed that this principle is complementary to CAC in the sense that

$$\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \leftrightarrow \mathsf{CAC} \wedge \mathsf{EM},$$

see [12]. We give some low bounds on the strength of EM.

1.1. Logical systems

We will work in a setting based on fragments of Heyting and Peano arithmetic in all finite types introduced in [96], for details see also [67].

1.1.1. Finite types

The set of all finite types $\mathbf{T}$ is inductively defined as

$$0 \in \mathbf{T}, \qquad \rho, \tau \in \mathbf{T} \Rightarrow \tau(\rho) \in \mathbf{T},$$

where 0 denotes the type of natural numbers and $\tau(\rho)$ the type of functions from ρ to τ. The set of pure types $\mathbf{P} \subset \mathbf{T}$ is defined as

$$0 \in \mathbf{P}, \qquad \rho \in \mathbf{P} \Rightarrow 0(\rho) \in \mathbf{P}.$$

They will often be denoted by natural numbers:

$$0(n) := n + 1,$$

e.g. $0(0) = 1$. The degree $deg(\rho)$ of a type ρ is inductively defined as

$$deg(0) := 0, \qquad deg(\tau(\rho)) := \max(deg(\tau), deg(\rho) + 1).$$

We will often denote the type of a term or variable by a superscribed index. For two types ρ, τ we will write $\rho \leq \tau$ if $deg(\rho) \leq deg(\tau)$.

Equality $=_0$ for type 0 objects will be added as primitive notion to the systems. Higher type equality $=_{\tau\rho}$ will be treated as abbreviation:

$$x^{\tau\rho} =_{\tau\rho} y^{\tau\rho} :\equiv \forall z^\rho \, xz =_\tau yz.$$

1.1.2. Gödel's system T

Define the λ-combinators $\Pi_{\rho,\sigma}, \Sigma_{\rho,\sigma,\tau}$ to be the functionals satisfying

$$\Pi_{\rho,\sigma} x^\rho y^\sigma =_\rho x, \qquad \Sigma_{\rho,\sigma,\tau} x^{\tau\sigma\rho} y^{\sigma\rho} z^\rho =_\tau xz(yz).$$

Similar define the recursor R_ρ of type ρ to be the functional satisfying

$$R_\rho 0yz =_\rho y, \qquad R_\rho(Sx^0)yz =_\rho z(R_\rho xyz)x.$$

Let *Gödel's system T* be the $\mathbf{T}$-sorted set of closed terms that can be build up from 0^0, the successor function S^1, the λ-combinators and, the recursors R_ρ for all finite types ρ. Using the λ-combinators one easily sees that T is closed under λ-abstraction, see [96].

T_n denotes the subsystem of Gödel's system T, where primitive recursion is restricted to recursors R_ρ with $deg(\rho) \leq n$. The system T_0 corresponds to the extension of Kleene's primitive recursive functionals to mixed types, see [55], whereas full system T corresponds to Gödel's primitive recursive functionals, see [36].

1.1.3. Heyting and Peano arithmetic

Define the *neutral Heyting/Peano arithmetic* ($\mathsf{N\text{-}HA}^\omega$, $\mathsf{N\text{-}PA}^\omega$) to be the extension of the term system T to a $\mathbf{T}$-sorted intuitionistic resp. classical logical system plus the schema of full induction and the equality axioms for type 0, i.e.

- $x =_0 x, \;\; x =_0 y \to y =_0 x, \;\; x =_0 y \land y =_0 z \to x =_0 z,$
- $x_1 =_0 y_1 \land \cdots \land x_n =_0 y_n \to t(x_1, \ldots, x_n) =_0 t(y_1, \ldots, y_n)$ for any n-ary term t of suitable type,

and substitution schemata for λ-combinators and the recursors, i.e.

$$(\mathsf{SUB})\colon \begin{cases} t[\Pi xy] =_0 t[x] \\ t[\Sigma xyz] =_0 t[xz(yz)] \\ t[R0yz] =_0 t[y] \\ t[R(Sx)yz] =_0 t[z(Rxyz)x] \end{cases} \quad \text{for all } t \text{ of type } 0.$$

For a formal definition see [97, I.1.6.15] (there $\mathsf{N\text{-}HA}^\omega$ is called HA^ω).

These theories are neutral with respect to an intensional or an extensional interpretation of higher type objects. However, for type 0 objects the usual equality axioms hold. Higher type equality is of no effect except for the SUB -rule. Later we will add functionals yielding cohesive and homogeneous set which are not extensional (in the presence of extensionality they would prove full arithmetical comprehension, see [65]) and therefore can only be analyzed in a neutral context.

Let *weakly extensional Heyting/Peano arithmetic* ($\mathsf{WE\text{-}HA}^\omega$, $\mathsf{WE\text{-}PA}^\omega$) be $\mathsf{N\text{-}HA}^\omega$ resp. $\mathsf{N\text{-}PA}^\omega$ plus the quantifier-free rule of extensionality, i.e.

$$(\mathsf{QF\text{-}ER})\colon \frac{A_{qf} \to s =_\rho t}{A_{qf} \to r[s/x^\rho] =_\tau r[t/y^\rho]},$$

where A_{qf} is quantifier-free and s^ρ, t^ρ, r^τ are terms of $\mathsf{WE\text{-}HA}^\omega$. Note that the addition of SUB here is redundant, since $\mathsf{QF\text{-}ER}$ together with the axioms for Π, Σ, R proves it. The systems with *full extensionality*, i.e. $\mathsf{N\text{-}HA}^\omega$, $\mathsf{N\text{-}PA}^\omega$ plus the extensionality axioms

$$(\mathsf{E}_{\rho,\tau})\colon \forall z^{\tau\rho}, x^\rho, y^\rho\, (x =_\rho y \to zx =_\tau zy)$$

for all $\tau, \rho \in \mathbf{T}$, will be denoted by $\mathsf{E\text{-}HA}^\omega$ and $\mathsf{E\text{-}PA}^\omega$. For a detailed definition of these systems, see [67, section 3].

The weakly extensional and neutral theories allow functional interpretation in themselves, which is not possible in the presence of full extensionality. Later we will eliminate the usage of extensionality (see Proposition 1.5 below), hence neither the interpretation of constants yielding cohesive/homogeneous sets nor the functional interpretation will lead to problems. For a discussion of these systems and the connection to functional interpretation we refer to [96].

It is also important to note that in presence of only $\mathsf{QF\text{-}ER}$ the deduction theorem in general fails, see [67, Theorem 9.11]. To overcome this we will restrict the use of principles in premises of $\mathsf{QF\text{-}ER}$. This will be denote by the $\oplus$-sign, e.g. $\mathsf{WE\text{-}PA}^\omega \oplus \mathsf{WKL}$ denote the system $\mathsf{WE\text{-}PA}^\omega + \mathsf{WKL}$, where WKL may not be used in the premise of $\mathsf{QF\text{-}ER}$. The weak extensional systems satisfy the deduction theorem with respect to $\oplus$.

We now introduce fragments of neutral and (weakly) extensional Heyting/Peano arithmetic corresponding to T_n:
Define $\mathsf{N\text{-}HA}^\omega_n{\restriction}$ to be the logical system extending T_n plus Σ^0_{n+1}-IA and plus the case-distinction functionals $(\mathrm{Cond}_\rho)_{\rho\in\mathbf{T}}$ and its substitution axioms

$$(\mathsf{SUB}_{\mathrm{Cond}})\colon \begin{cases} t[\mathrm{Cond}_\rho(0^0, x^\rho, y^\rho)] =_0 t[x] \\ t[\mathrm{Cond}_\rho(Su, x^\rho, y^\rho)] =_0 t[y] \end{cases} \qquad \text{for all } t \text{ of type } 0.$$

These case distinction functionals are needed for the functional interpretation and cannot be defined in these fragments of $\mathsf{N\text{-}HA}^\omega$, see [81, 7]. In the full system T they can be simulated by the recursors. Instead of $\mathsf{N\text{-}HA}^\omega_0{\restriction}$ we also write $\widehat{\mathsf{N\text{-}HA}}^\omega{\restriction}$. The classical systems $\mathsf{N\text{-}PA}^\omega_n{\restriction}$, $\widehat{\mathsf{N\text{-}PA}}^\omega{\restriction}$ are defined similarly. In the same way also the (weakly) extensional systems $\mathsf{(W)E\text{-}HA}^\omega_n{\restriction}$, $\widehat{\mathsf{(W)E\text{-}HA}}^\omega{\restriction}$, $\mathsf{(W)E\text{-}PA}^\omega_n{\restriction}$, $\widehat{\mathsf{(W)E\text{-}PA}}^\omega{\restriction}$ are defined.[4] However for the classical systems defined here one does not need to add Cond to the system since it is provably definable with the λ-combinators and R_0, see [81]. Note that Σ^0_{n+1}-induction is provable with the recursor R_n and quantifier-free induction and full $\mathsf{QF\text{-}AC}$ in all types (definition below) over the classical systems defined here. Hence the addition of it to the classical systems is actually superfluous. This follows from [81] and Kreisel's characterization theorem, see [67, proposition 10.13].

[4] For a formal definition let $\widehat{\mathsf{(W)E\text{-}HA}}^\omega{\restriction}$ be defined as in [67, section 3.4] and define $\mathsf{(W)E\text{-}HA}^\omega_n{\restriction}$ to be $\widehat{\mathsf{(W)E\text{-}HA}}^\omega{\restriction}$ plus Σ^0_{n+1}-IA and the defining axioms and constants for the recursors R_ρ with $deg(\rho) \leq n$. The neutral variants are defined in the same way but without the rule of extensionality.

1.1.4. Grzegorczyk arithmetic

We moreover need weaker fragments of Heyting and Peano arithmetic containing only quantifier-free induction.

Let *weakly extensional Grzegorczyk arithmetic of level n in all finite types* $\mathsf{G}_n\mathsf{A}^\omega_{(\mathsf{i})}$ be the (intuitionistic) system containing $=_0$-axioms, $\mathsf{QF\text{-}ER}$, λ-abstraction, the n-th branch of the Ackermann-function, bounded search and bounded primitive recursion. For a detailed definition see [58].[5] The neutral variant will be denoted by $\mathsf{N\text{-}G}_n\mathsf{A}^\omega$, the extensional one by $\mathsf{E\text{-}G}_n\mathsf{A}^\omega$.

Let $\mathsf{G}_\infty\mathsf{A}^\omega$ be the union of all these systems. This system contains all primitive recursive functions but not all primitive recursive functionals (in the sense of Kleene). For instance R_0 is not contained in $\mathsf{G}_\infty\mathsf{A}^\omega$. Thus it also contains no Σ^0_1-induction. The set of all closed terms of $\mathsf{G}_n\mathsf{A}^\omega$ is called $\mathsf{G}_n\mathsf{R}^\omega$. See [58] and [67, Chapter 3] for all of this.

1.1.5. Quantifier-free axiom of choice

Let $\mathsf{QF\text{-}AC}$ be the schema

$$\forall x\, \exists y\, A_{qf}(x,y) \to \exists f\, \forall x\, A_{qf}(x, f(x)),$$

where A_{qf} is a quantifier-free formula. If the types of x, y are restricted to α, β we write $\mathsf{QF\text{-}AC}^{\alpha,\beta}$.

The scheme $\mathsf{QF\text{-}AC}^{0,0}$ corresponds to recursive comprehension (Δ^0_1-CA) in a second order context. Thus $\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction + \mathsf{QF\text{-}AC}^{1,0}$ and RCA_0 share the same proof theoretic strength. RCA_0 can easily be embedded into $\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction + \mathsf{QF\text{-}AC}^{1,0}$ and $\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction + \mathsf{QF\text{-}AC}^{1,0}$ is conservative over RCA_0 modulo this embedding, see [65]. For this reason $\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction + \mathsf{QF\text{-}AC}^{1,0}$ is called RCA^ω_0. The system $\mathsf{RCA}^\omega_0 + \mathsf{WKL}$ is also called WKL^ω_0.

The system RCA^*_0 is RCA_0, where Σ^0_1-induction is replaced by quantifier-free-induction and the exponential function, see [86, X.4.1]. This system can be embedded into $\mathsf{G}_3\mathsf{A}^\omega + \mathsf{QF\text{-}AC}^{1,0}$ and both systems are Π^0_2-conservative over Kalmar elementary arithmetic.

In ordinary mathematics higher types usually do not occur and second order arithmetic is sufficient to formalize most of it. We require here a system containing all finite types to be able to carry out a functional interpretation and thus cannot use a second order system.

[5] In [67] the system $\mathsf{G}_n\mathsf{A}^\omega$ is defined to include all $\mathbb{N}, \mathbb{N}^\mathbb{N}, \mathbb{N}^{\mathbb{N}^\mathbb{N}}$-true $\forall$-sentences. In a pure proof-mining context these sentences do not matter because they have no impact on the provable recursive functions in the system. We only add quantifier-free induction ($\mathsf{QF\text{-}IA}$), to be able to later establish conservativity over PRA.

1.1.6. The quantifier-free subsystems

In order to exploit the full subtlety of the functional interpretation we will also need the *quantifier-free subsystems of* $\mathsf{N}\text{-}\mathsf{G}_n\mathsf{A}_\mathsf{i}^\omega$ *and* $\mathsf{N}\text{-}\mathsf{HA}_n^\omega\!\restriction$. The quantifier-free subsystems are denote by $\mathsf{qf}\text{-}\mathsf{N}\text{-}\mathsf{G}_n\mathsf{A}^\omega$ resp. $\mathsf{qf}\text{-}\mathsf{N}\text{-}\mathsf{PA}_n^\omega\!\restriction$. (The quantifier free subsystems satisfy the law of excluded middle and are therefore classical.)

They are obtained from the full systems as follows:

- The quantifier-rules and -axioms are dropped from logic.
- For all axioms of the form $A(x_1^{\rho_1}, \ldots, x_n^{\rho_n})$, where A is quantifier-free, the following axioms are added to the system:
$$A(t_1^{\rho_1}, \ldots, t_n^{\rho_n}),$$
where t_i are arbitrary terms.
- The induction schema is replaced by the (quantifier-free) induction rule:
$$\frac{A(0^0), \quad A(x^0) \to A(Sx^0)}{A(t^0)},$$
where A is quantifier-free, x does not occur free in the assumption and t is an arbitrary term.

These quantifier-free systems contain only prime formulas of the form
$$t_0 =_0 t_1,$$
where t_0, t_1 are terms in $\mathsf{N}\text{-}\mathsf{G}_n\mathsf{A}_\mathsf{i}^\omega$ resp. $\mathsf{N}\text{-}\mathsf{HA}_n^\omega\!\restriction$. Formulas are logical combinations of these predicates. Obviously, $\mathsf{qf}\text{-}\mathsf{N}\text{-}\mathsf{G}_n\mathsf{A}^\omega$ and $\mathsf{qf}\text{-}\mathsf{N}\text{-}\mathsf{PA}_n^\omega\!\restriction$ are subsystems of $\mathsf{N}\text{-}\mathsf{G}_n\mathsf{A}_\mathsf{i}^\omega$ resp. $\mathsf{N}\text{-}\mathsf{HA}_n^\omega\!\restriction$. (For a detailed discussion of these systems we also refer the reader to [96, 1.6.5]. For technical reason we use here the variant of the systems described in Remark 1.5.8.)

Observe, that in these system we can only instantiate type 0 variables (via the induction rule) and not higher type variables, hence we immediately obtain the following lemma:

Lemma 1.2. *Let A be a sentence and*
$$\mathcal{T} \vdash A,$$
where $\mathcal{T} = \mathsf{qf}\text{-}\mathsf{N}\text{-}\mathsf{G}_n\mathsf{A}^\omega, \mathsf{qf}\text{-}\mathsf{N}\text{-}\mathsf{PA}_n^\omega\!\restriction$.

Then there exists a derivation of A in $\mathcal{T}$ that contains only the variables of A plus some fresh variables of type 0.

Proof. In a derivation of A in $\mathcal{T}$ replace every variable not of type 0 and not occurring in A by constant 0^ρ of suitable type. Since higher type variables cannot be instantiated the derivation remains valid. □

1.1.7. Functional interpretation

Functional interpretation will denote in this paper a negative translation followed by Gödel's Dialectica translation.

Gödel's Dialectica translation is a proof interpretation that translates proofs from (a fragment of) $\mathsf{WE\text{-}HA}^\omega$ or $\mathsf{N\text{-}HA}^\omega$ into its quantifier-free subsystem, see [36].

Let $\mathcal{T}$ be such a system. The Dialectica translation associates to each formula A of $\mathcal{T}$ a $\exists\forall$-formula

$$A^D :\equiv \exists x\, \forall y\, A_D(x, y),$$

where A_D is quantifier-free. In particular, for a Σ^0_2 sentence A the formula A_D is the quantifier-free matrix of A.

From a proof of A one then can extract a term t, such that

$$\mathsf{qf}\text{-}\mathcal{T} \vdash A_D(t, x).$$

A *negative translation* is a proof translation that translates classical proofs into intuitionistic proofs. It also proceeds by associating each formula A a formula A^N such that

$$\mathcal{S} \vdash A \leftrightarrow A^N \qquad \text{and} \qquad \mathcal{S} \vdash A \quad \Longrightarrow \quad \mathcal{S}_{\mathrm{i}} \vdash A^N.$$

Here $\mathcal{S}$ is any of $\mathsf{(W)E\text{-}PA}^\omega$, $\widehat{\mathsf{(W)E\text{-}PA}}^\omega{\restriction}$, $\mathsf{G}_n\mathsf{A}^\omega$ or its neutral variants and $\mathcal{S}_{\mathrm{i}}$ is its intuitionistic counterpart. (To be specific, Kuroda's negative translation A^N is obtained from A by inserting $\neg\neg$ after each $\forall$ and in front of the whole formula.)

Thus we denote by functional interpretation a proof translation from (a fragment of) $\mathsf{WE\text{-}PA}^\omega$ or $\mathsf{N\text{-}PA}^\omega$ into its quantifier-free part. We abbreviate the functional interpretation by ND. The ND-translation of a formula A will be denoted by A^{ND} and the quantifier-free matrix of it by A_{ND}.

The functional interpretation in particular has the property to extract a term for each provable recursive function, i.e. from a proof of a $\forall\exists$-statement (in $\mathsf{WE\text{-}PA}^\omega$ or any other fragment for which the functional interpretation holds)

$$\mathsf{WE\text{-}PA}^\omega \vdash \forall u\, \exists v\, A_{qf}(u, v)$$

it extracts a term t such that

$$\mathsf{qf\text{-}WE\text{-}HA}^\omega \vdash \underbrace{A_{qf}(u, tu)}_{\equiv A_{ND}(t,u)}.$$

For an introduction to the functional interpretation see [67, 7, 96].

1.1.8. Additional notation and definitions

We denote sets by capital letters. Unless otherwise noted they are represented by characteristic functions. Sometimes capital letters also denote higher type functionals. It will be clear from the context what is meant.

It is important to note that in systems not containing Σ^0_1-induction it is in general not provable that every infinite set — that is a set X satisfying $\forall k\,\exists n > k\;\; n \in X$ — can be strictly increasingly enumerated, i.e. there exists a strictly monotone function f such that $rng(f) = X$. The system $\widehat{\mathsf{WE\text{-}HA}}^{\omega}\!\restriction + \mathsf{QF\text{-}AC}^{0,0}$ proves that the first statement implies the second. The converse — every strictly increasingly enumerable set is infinite — is already provable without Σ^0_1-induction, for instance $\mathsf{G_3A}^{\omega}$ suffices.

Sequence codes are denoted by $\langle x_0, \dots, x_n\rangle$. The corresponding projection functions and length function are denoted by $(\cdot)_i$ and $\mathrm{lth}(\cdot)$. We encode sequences using a bijective and monotone (in each component) sequence-coding based on the Cantor pairing, see [67, definition 3.30]. This coding is definable in every system containing $\mathsf{qf\text{-}N\text{-}G_3A}^{\omega}$.

We model in our systems n-colorings of $[\mathbb{N}]^2$ as functions $c\colon \mathbb{N}\times\mathbb{N} \to n$ with $c(x,y) = c(y,x)$.

Further we define the following notions:

- $\bar{f}$ denotes the course-of-value function of f^1, i.e. $\bar{f}(n) = \langle f(0), \dots, f(n-1)\rangle$.
- $x \sqsubset X$ iff x is an initial segment of a strictly monotone enumeration of X.
- $x \subseteq^{fin} X$ iff x is an code for a finite subset of X.

Definition 1.3 (Bounded type 1 recursor, $\tilde{R}_1$)**.** The bounded type 1 recursor $\tilde{R}_1$ is defined as

$$\begin{aligned} \tilde{R}_1 0 yzhu &=_0 \min(y(u), h(0,u)) \\ \tilde{R}_1 (x+1) yzhu &=_0 \min(z(\tilde{R}_1 xyzh)xu, h(x,u)). \end{aligned}$$

We will denote by $(\tilde{R}_1)$ the defining axioms. Note that they are purely universal and that $\tilde{R}_1$ can be trivially majorized.

Definition 1.4 (Uniform weak König's lemma, UWKL, [64])**.** Uniform weak König's lemma is the statement

$$\exists \Phi \;\leq_{1(1)}\; 1 \forall f\, (T^{\infty}(f) \to \forall x^0\, f(\overline{\Phi f}x) = 0),$$

where T^{∞} expresses that f describes an infinite 0/1-tree.

We can modify (in $\mathsf{G_{\infty}A}^{\omega}$) every function f such that it describes an infinite 0/1-tree and is not altered if it already described such a tree. We will write $\check{f}$ for this modification, see [57, 67].

With this we can restate UWKL equivalently as

$$\exists \Phi \ \leq_{1(1)} 1 \forall f^1 \, \forall x^0 \, \check{f}(\overline{\Phi \check{f}} x) = 0.$$

Note that the condition $\leq_{1(1)}$ is superfluous because the modified tree contains only 0/1-sequences.

By Skolemization we add a weak König's Lemma functional constant $\mathcal{B}$ described by the (purely universal) axiom

$$\forall f \, \forall x^0 \, \check{f}(\overline{\mathcal{B} \check{f}} x) = 0. \tag{1.3}$$

This axiom will be denoted by $(\mathcal{B})$. Note that $\mathcal{B}$ can be trivially majorized.

In a system containing full extensionality UWKL implies Π^0_1-CA, see [64], hence it is too strong for our purpose. But in a weakly extensional system it often can be handled like WKL, for instance it vanishes under a monotone functional interpretation like WKL and can be added to the elimination of monotone Skolem functions, see [64].

Proposition 1.5 (Elimination of extensionality, [72])**.** *Let A be a formula containing only free variables and quantification of degree ≤ 1.*

If

$$\mathsf{E\text{-}PA}^\omega + \mathsf{QF\text{-}AC}^{0,1} + \mathsf{QF\text{-}AC}^{1,0} \vdash A$$

then

$$\mathsf{N\text{-}PA}^\omega + \mathsf{QF\text{-}AC}^{0,1} + \mathsf{QF\text{-}AC}^{1,0} \vdash A.$$

The same holds also for the fragments $\widehat{\mathsf{N\text{-}PA}}^\omega\!\restriction$ *and* $\mathsf{N\text{-}G}_n\mathsf{A}^\omega$*.*

Proof. Proposition 10.45 and Lemma 10.41 of [67]. These lemma and proposition actually do not make use of weak extensionality and therefore show conservativity even over a neutral theory. ☐

Recall that a type 2 functional φ is continuous if

$$\forall g^1 \, \exists n^0 \, \forall h^1 \, \left(\bar{g}n = \bar{h}n \rightarrow \varphi(g) = \varphi(h)\right). \tag{1.4}$$

Definition 1.6 (Associate, [54, 69])**.** For every continuous type 2 functional φ we will denote by α_φ an associate of φ, i.e. a type 1 function with the properties

$$\begin{gathered} \forall f \, \exists n \ \alpha_\varphi(\bar{f}n) \neq 0, \\ \forall f, n \, \left(\alpha_\varphi(\bar{f}n) \neq 0 \rightarrow \varphi(f) = \alpha_\varphi(\bar{f}n) \dot{-} 1\right). \end{gathered} \tag{1.5}$$

The value of φ is uniquely determined through α_φ. For every continuous functional there exists an associate, though it is not uniquely determined. For details see also [76].

Definition 1.7. A functional given by a closed term φ^ρ of $\mathcal{T}$ is called *provably continuous* if for some term α_φ (containing at most the free variables of φ) of type 1 (if $\rho > 0$) resp. 0 (if $\rho = 0$), the following holds:

$$\mathcal{T} \vdash \varphi \approx_\rho \alpha_\varphi.$$

Here, for general x^ρ and $\alpha^{0/1}$, the relation $x \approx_\rho \alpha$ is defined by induction on ρ:

$$\begin{aligned} x \approx_0 \alpha &:\equiv x =_0 \alpha, \\ x \approx_{\tau\rho} \alpha &:\equiv \alpha \in \mathsf{ECF}_{\tau\rho} \wedge \forall y^\rho \forall \beta \in \mathsf{ECF}_\rho \left(y \approx_\rho \beta \rightarrow xy \approx_\tau \alpha \upharpoonright \beta \right), \end{aligned}$$

where ECF is the model of extensional hereditarily continuous functionals formalized in $\mathcal{T}$ and $\upharpoonright$ denotes the application in ECF. (See [55, 69, 96], for a definition see also [67, Definitions 3.58, 3.59].)

Especially, in the case of $\rho = 2$ a functional φ is provably continuous in $\mathcal{T}$ if it has an associate α_φ in $\mathcal{T}$ and (1.5) is provable.

Proposition 1.8. *For every term $t^2 \in \mathsf{G}_n\mathsf{R}^\omega, T_0, T_1$ there exists provably in $\mathsf{G}_n\mathsf{A}^\omega$ resp. $\widehat{\mathsf{WE\text{-}PA}}^\omega\upharpoonright$, $\mathsf{WE\text{-}PA}^\omega_1\upharpoonright$ a (primitive recursive) associate α_t. In other words t is provably continuous.*

Proof. We first consider the case of $\widehat{\mathsf{WE\text{-}PA}}^\omega\upharpoonright = \mathsf{WE\text{-}PA}^\omega_0\upharpoonright$ and $\mathsf{G}_n\mathsf{A}^\omega$. Here the only functional constants having no trivial associate are the λ-combinators and R_0 (in the case of $\widehat{\mathsf{WE\text{-}PA}}^\omega\upharpoonright$) and the course-of-value functional (in the case of $\mathsf{G}_n\mathsf{A}^\omega$). The associates of R_0 and the course-of-value functional can easily be computed and (1.5) be proven in the respective systems. By normalization one can find a term $\tilde{t} =_2 t$ that does not include λ-abstraction of type ≥ 1. The proposition for $\widehat{\mathsf{WE\text{-}PA}}^\omega\upharpoonright$ and $\mathsf{G}_n\mathsf{A}^\omega$ follows from this.

In the case of $\mathsf{WE\text{-}PA}^\omega_1\upharpoonright$ we prove by induction over the structure of t that t is provably continuous. For this it is sufficient to prove that every functional constant is provably continuous and to observe that this property is retained under composition. The associates for the λ-combinators are easily definable and provable in these systems, see [96].

Here we only show that the existence of an associate for R_1 is provable in $\mathsf{WE\text{-}PA}^\omega_1\upharpoonright$, since we are only interested in this case. For the other recursors the proof is similar. Let

$$\alpha_{R_1}(0, y', z', u) := \begin{cases} (y')_u + 1 & \text{if } u < \operatorname{lth} y', \\ 0 & \text{otherwise,} \end{cases}$$

$$\alpha_{R_1}(x+1, y', z', u) := \begin{cases} (z')_{\langle x, \overline{(\lambda k.\alpha_{R_1}(x,y',z',k) \dot{-} 1)k} \rangle} & \begin{array}{l} \text{if } \exists k < \operatorname{lth} y', \text{ such that} \\ \alpha_{R_1}(x, y', z', k) > 0 \\ \text{and this is} > 0, \end{array} \\ 0 & \text{otherwise.} \end{cases}$$

Using Π^0_2-induction one shows that

$$\forall x\ (\forall u\, \exists n\, \alpha_{R_1}(x, \bar{y}n, \overline{\alpha_{\lambda r.\overline{zr}n}}n, u) = R_1(x,y,z,u) + 1)$$

and hence that α_{R_1} is an associate of R_1. □

1.1.9. Properties of instances of comprehension

Remark 1.9. A sequence of Π^0_1-comprehension instances $\left(\Pi^0_1\text{-}\mathsf{CA}(f_i)\right)_i$ may be reduced to the single instance of Π^0_1-$\mathsf{CA}(f')$ with $f'xy := f_{(x)_1}(x)_2 y$, see [60, Remark 3.8].

Lemma 1.10 ([60, 61])**.** *For suitable terms ξ_i of $\mathsf{G_3A}^\omega$ we have*

(i) $\mathsf{G_3A}^\omega + \mathsf{QF}\text{-}\mathsf{AC}^{0,0} \vdash \forall f\ \left(\Pi^0_1\text{-}\mathsf{CA}(\xi_1 f) \to \Pi^0_1\text{-}\mathsf{AC}(f)\right)$,

(ii) $\mathsf{G_3A}^\omega + \mathsf{QF}\text{-}\mathsf{AC}^{0,0} \vdash \forall f\ \left(\Pi^0_1\text{-}\mathsf{CA}(\xi_2 f) \to \Delta^0_2\text{-}\mathsf{CA}(f)\right)$,

(iii) $\mathsf{G_3A}^\omega + \mathsf{QF}\text{-}\mathsf{AC}^{0,0} \vdash \forall f\ \left(\Pi^0_1\text{-}\mathsf{CA}(\xi_3 f) \to \Delta^0_2\text{-}\mathsf{IA}(f)\right)$,

(iv) $\mathsf{G_3A}^\omega + \mathsf{QF}\text{-}\mathsf{AC}^{0,0} \vdash \forall f\ \left(\Pi^0_1\text{-}\mathsf{CA}(\xi_4 f) \to \Pi^0_1\text{-}\mathsf{CP}(f)\right)$,

(v) $\mathsf{G_3A}^\omega + \mathsf{QF}\text{-}\mathsf{AC}^{0,0} + \mathsf{WKL} \vdash \forall f\ \left(\Pi^0_1\text{-}\mathsf{CA}(\xi_5 f) \to \Pi^0_2\text{-}\mathsf{WKL}(f)\right)$.

Here the principle $\mathcal{K}$-AC denotes the scheme of axiom of choice, where the base formula is of type $\mathcal{K}$. Similarly $\mathcal{K}$-WKL denotes weak König's lemma where the tree is given by a predicate of type $\mathcal{K}$. The principles $\mathcal{K}$-IA and $\mathcal{K}$-CA are defined likewise.

If $\mathcal{K} = \Pi^0_n, \Sigma^0_n$ then an instance of those principles is given by a function f coding the quantifier-free part of the Π^0_n resp. Σ^0_n formula. For instance

$$\Pi^0_1\text{-}\mathsf{AC}(f) \equiv \forall x\, \exists y\, \forall z\, f(x,y,z) = 0 \to \exists Y\, \forall x\, \forall z\, f(x, Y(x), z) = 0.$$

Similar a Δ^0_2-formula is given by an f coding a function for a Π^0_n and a function for a Σ^0_n formula.

Proof of Lemma 1.10. For (i), (ii) see [61, lemma 4.2]. The statements (iii), (iv) are immediate consequences of these. Note that we require here $\mathsf{G_3A}^\omega$ and not only $\mathsf{G_2A}^\omega$ as in the reference, since we do not add the true universal sentences to the system, see footnote 5.

For (v) let ξ_5 be such that the instance of Π^0_1-CA yields the comprehension function for the innermost quantifier of the tree predicate reducing Π^0_2-WKL to Π^0_1-WKL. This is equivalent to WKL and thus included in the system, see for instance [86]. □

For the ordinal analysis of terms we will need the following abbreviation:

$$\omega^\mu_0 = \mu \quad \text{and} \quad \omega^\mu_{k+1} = \omega^{\omega^\mu_k},$$

where $k \in \mathbb{N}$ and μ is an ordinal.

Lemma 1.11. *Let $n \in \mathbb{N}$ and let $t[g]$ be a type 1 term with the only free variable g such that $\lambda g.t[g] \in T_n$. Then for every term φ in T_{n-1} or in $\mathsf{G}_\infty\mathsf{R}^\omega$ if $n = 0$ there exists a term ξ in the same system such that $\mathsf{WE\text{-}PA}^\omega_{n-1}\restriction + \mathsf{QF\text{-}AC}$ or $\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF\text{-}AC}$ in the case of $n = 0$ proves*

$$\forall g \left(\Pi^0_1\text{-}\mathsf{CA}(\xi g) \rightarrow \exists f^1 \, \Big(f \textit{ satisfies the defining axioms of } t[g] \wedge \Pi^0_1\text{-}\mathsf{CA}(\varphi f g)\Big)\right).$$

Defining axioms of $t[g]$ are a formula A, such that $\forall g, x, y \; (A(g, x, y) \leftrightarrow t[g]x = y)$. (Since t^1 can be defined by (unnested) ordinal recursion of order $< \omega^\omega_{n+1}$, one can take for A the formula describing this recursion.)

Proof. First fix a suitable encoding for ordinals smaller than ε_0 in this system, see for instance [37].

Every term $t^1 \in T_n$ can be defined through (unnested) ordinal recursion of order $< \omega^\omega_{n+1}$; the totality of such a recursion can be proven using a suitable instance of Σ^0_{n+1}-IA, see [80] and Theorem 3.13 below. Such an instance is included in the system because a suitable instance of Π^0_1-CA reduces it to Σ^0_n-IA. This proves the claim that there is a total function f satisfying the definition of $t[g]$.

For the second part note that the defining axioms of unnested ordinal primitive recursion of order type α are given by

$$\begin{aligned} f(0) &:= f_0, \\ f(n) &:= h(n, f(l(n))), \end{aligned} \tag{1.6}$$

where l satisfies

$$l(n) \prec n \qquad \text{for } n > 0 \tag{1.7}$$

and $\prec$ defines a well-ordering on $\mathbb{N}$ of order type α.

We say a finite sequence s *satisfies the defining axioms* (1.6) *up to n* if

$$(s)_0 = f_0, \qquad (s)_i = h(i, (s)_{l(i)}) \qquad \text{for all } i \in \bigcup_{n' \leq n} \bigcup_k \left\{ l^k(n') \right\} \setminus \{0\}$$

For notational ease we assume here that $l(0) = 0$. Note that because of (1.7) the set $\bigcup_k \left\{ l^k(n') \right\}$ defines an $\prec$-descending chain and is therefore provably finite.

For the second part we have to prove a comprehension of the form

$$\exists H \, \forall k \; (k \in H \leftrightarrow \forall x \, \varphi(f, g, k, x) = 0) \,. \tag{1.8}$$

We use the imposed instance of comprehension to prove the following comprehension

$$\exists H \, \forall k \left(k \in H \leftrightarrow \forall x \, \forall s, n \left(s \text{ satisfies the defining axioms of } t[g] \text{ up to } n \right.\right.$$
$$\left.\left. \rightarrow \alpha_{\lambda f.\varphi(f,g,k,x)}(s) \leq 1\right)\right).$$

Note that this comprehension is equivalent to (1.8) if f is total. $\square$

The proof of the comprehension above is similar to the construction of a 1-generic set: If the statement

$$\forall x\, \varphi(f, g, k, x) = 0$$

for a fixed k fails, then there is an x such that $\varphi(f, g, k, x) \neq 0$. Since φ is continuous this depends only on an initial segment of f. We express this by using associates, i.e. this statement is equivalent to

$$\exists n\, \alpha_{\lambda f.\varphi(f,g,k,x)}(\bar{f}n) > 1.$$

Hence it suffices to consider only finite initial segments.

We will use this technique in most proofs of instances of comprehension in this paper. This is the reason why we require φ to be provably continuous in the definition of proofwise low.

2. The cohesive principle and the atomic model theorem

2.1. The cohesive principle (COH)

Let $(R_n)_{n\in\mathbb{N}}$ be a sequence of subsets of $\mathbb{N}$. A set G is *cohesive* for $(R_n)_{n\in\mathbb{N}}$ if $\forall n \left(G \subseteq^* R_n \vee G \subseteq^* \overline{R_n}\right)$, i.e.

$$\forall n\, \exists s\, (\forall j \geq s\ (j \in G \to j \in R_n) \vee \forall j \geq s\ (j \in G \to j \notin R_n))\,.$$

A set G is *strongly cohesive* for $(R_n)_{n\in\mathbb{N}}$ if

$$\forall n\, \exists s\, \forall i < n\ (\forall j \geq s\ (j \in G \to j \in R_i) \vee \forall j \geq s\ (j \in G \to j \notin R_i))\,.$$

The *cohesive principle* (COH) is the statement that for every sequence of sets an infinite cohesive set exists. Similarly the *strong cohesive principle* (StCOH) is the statement that for every sequence of sets an infinite strongly cohesive set exists.

We denote by $(\mathsf{St})\mathsf{COH}(r, G)$ the statement that G is a set that satisfies the (strong) cohesive principle for the sets given by the characteristic functions $(\lambda x.r(i,x))_i$ where $r\colon \mathbb{N}\times\mathbb{N} \to 2$.

In Chapter 6 below we will show that StCOH is equivalent to a variant of the Bolzano-Weierstraß principle.

Proposition 2.1 ([39, 4.4])**.**

(i) $\mathsf{G_3A}^\omega \vdash \mathsf{StCOH} \to \mathsf{COH}$

(ii) $\mathsf{G_3A}^\omega \vdash \mathsf{StCOH} \to \Pi^0_1\text{-}\mathsf{CP}$

(iii) $\mathsf{G_3A}^\omega \vdash \mathsf{StCOH} \leftrightarrow \mathsf{COH} \wedge \Pi^0_1\text{-}\mathsf{CP}$

Proof. The first statement is clear and the third statement is an immediate consequence of the first and second.

For the second we prove the infinite pigeonhole principle $\mathsf{RT}^1_{<\infty}$ from StCOH. The infinite pigeonhole principle is equivalent to Π^0_1-CP, over Σ^0_1-induction. This was shown in [41]. The proof can even be carried out in $\mathsf{G_3A}^\omega$, see [70]:

Let $f\colon \mathbb{N} \to n$ be a coloring. Define $R_i := \{x \mid f(x) = i\}$. Let G be an infinite, strongly cohesive set for R_i. By definition there is an s with

$$\forall i < n\ (\forall j \geq s\ (j \in G \to j \in R_i) \vee \forall j \geq s\ (j \in G \to j \notin R_i))\,.$$

By the totality of f there is exactly one i such that the first disjunction holds, i.e. the color i occurs infinitely often on G and thus on $\mathbb{N}$. ☐

Lemma 2.2. $\mathsf{G}_3\mathsf{A}^\omega$ *proves that a countable number of instances of* $(\mathsf{St})\mathsf{COH}$ *is uniformly equivalent to a single instance of* $(\mathsf{St})\mathsf{COH}$.

Proof. Let $(R_{j,i})_{j,i\in\mathbb{N}}$ be sequence of sequences of sets. A set which is (strongly) cohesive for all of this sets is obviously also (strongly) cohesive for the sets $(R_{j,i})_{i\in\mathbb{N}}$ for each j. Hence a single application of $(\mathsf{St})\mathsf{COH}$ is sufficient to solve the sequence of instance of $(\mathsf{St})\mathsf{COH}$ given by $(R_{j,i})_{i\in\mathbb{N}}$. ☐

Proposition 2.3.

$$\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF\text{-}AC} \oplus \mathsf{WKL} \vdash \forall r\colon \mathbb{N}\times\mathbb{N} \to 2\ \left(\Pi^0_1\text{-}\mathsf{CA}(\xi r) \to \exists G\, \mathsf{StCOH}(r,G)\right),$$

where ξ is a suitable term.

Proof. Define

$$R_n := \lambda x.r(n,x), \qquad\qquad R^x := \bigcap_{i<\mathrm{lth}(x)} \begin{cases} R_i & \text{if } x_i = 0,\\ \overline{R_i} & \text{otherwise.}\end{cases}$$

Here and in the following let x be the code of the sequence $\langle x_0, \ldots, x_{\mathrm{lth}(x)-1}\rangle$.

For every n the set (of sets) $\{R^x \mid x \in 2^n\}$ is a partition of $\mathbb{N}$, i.e.

$$\forall n\, \forall z\ \exists! x \in 2^n\ \ z \in R^x. \tag{2.1}$$

This statement can be proved with an instance of quantifier-free induction (the tuple $\langle x_0, \ldots, x_{n-1}\rangle$ is bounded by $\overline{1}n$ and z is a parameter).

We construct an infinite Π^0_2-0/1-tree T deciding at level n whether for the solution set G either $G \subseteq^* \overline{R_n}$ or $G \subseteq^* R_n$ holds: Let

$$T(\langle x_0, \ldots, x_n\rangle) \quad\text{iff}\quad R^{\langle x_0,\ldots,x_n\rangle} \text{ is infinite.}$$

The statement "R^x is infinite" is Π^0_2. The predicate T clearly defines a tree. The tree is infinite because otherwise

$$\exists n\, \forall x \in 2^n\, \exists y\ \forall z > y\ \ z \notin R^x$$

and this together with an instance of Π^0_1-CP yields a contradiction to (2.1). (x can be bounded by $\bar{1}n$.)

With an application of an instance of Σ^0_1-induction we prove

$$\forall x\,(R^x \text{ infinite} \to \forall n\,\exists\langle l_0,\dots,l_{n-1}\rangle\ (\forall i<n-1\ \ l_i<l_{i+1}\wedge\forall i<n\ \ l_i\in R^x))$$

and then conclude

$$\begin{aligned}&\forall n\,\forall x\,(\mathrm{lth}(x)=n\\&\qquad\wedge R^x \text{ infinite} \to \exists\langle l_0,\dots,l_{n-1}\rangle\,\forall i<n-1\ \ l_i<l_{i+1}\wedge\forall i<n\ \ l_i\in R^x).\end{aligned} \tag{2.2}$$

An instance of Π^0_2-WKL yields an infinite branch b of T, i.e. $\forall n\ \left(R^{\bar{b}(n)}\text{ infinite}\right)$. Using (2.2) we obtain

$$\forall n\,\exists\langle l_0,\dots,l_{n-1}\rangle\ \left(\forall i<n-1\ \ l_i<l_{i+1}\wedge\forall i<n\ \ l_i\in R^{\bar{b}n}\subseteq R^{\bar{b}i}\right). \tag{2.3}$$

An application of QF-AC yields an enumeration $n\mapsto\langle l_0,\dots,l_{n-1}\rangle$ of finite tuples. Searching for the least code of a tuple and the properties of (2.3) assure that every tuple is extended by the following. Hence we may diagonalize to obtain an the set $G:=\{l_0,l_1,\dots\}$. This set is strongly cohesive and solves the proposition.

Note that the instances of Σ^0_1-IA, Π^0_1-CP and Π^0_2-WKL can be reduced to an instance of Π^0_1-CA using Lemma 1.10 and Remark 1.9 yielding a suitable term ξ. ☐

We now strengthen this proposition to

Proposition 2.4. *For every term φ one can construct a term ξ such that*

$$\begin{aligned}&\mathsf{G}_\infty\mathsf{A}^\omega+\mathsf{QF}\text{-}\mathsf{AC}\oplus\mathsf{WKL}\vdash\\&\qquad\forall r\colon\mathbb{N}\times\mathbb{N}\to 2\ \left(\Pi^0_1\text{-}\mathsf{CA}(\xi r)\to\exists G\ \left(\mathsf{StCOH}(r,G)\wedge\Pi^0_1\text{-}\mathsf{CA}(\varphi rG)\right)\right).\end{aligned}$$

Proof. We construct an infinite Π^0_2-0/1-tree, in which we decide at level

- $2n$ whether $G\subseteq^*\overline{R_n}$ or $G\subseteq^* R_n$ and at level,
- $2n+1$ the n-th value of the instance of Π^0_1-comprehension, i.e. whether $\forall k\,(\varphi rG)nk=0$ is true.

We assign to every element of the tree a finite (potential) initial segment L^x of G. At level $2n$ we add—as in the previous proposition—the next element of R^x; at level $2n+1$ we only add the smallest counterexample (extending our old initial segment of

G with elements from R^x) to the statement $\forall k\,(\varphi r G) n k = 0$ if it is false and nothing otherwise. Define:

$$
\begin{aligned}
T(\langle x_0, \ldots, x_{2n}\rangle) \quad &\text{iff} \quad R^{\langle x_0, x_2, \ldots, x_{2n}\rangle} \text{ is infinite},\\
T(\langle x_0, \ldots, x_{2n}, 0\rangle) \quad &\text{iff} \quad \forall l \subseteq^{fin} R^{\langle x_0, x_2, \ldots, x_{2n}\rangle}\, \forall k\ \alpha_\varphi(L^{\langle x_0, \ldots, x_{2n}\rangle} * l, n, k) \leq 1,\\
T(\langle x_0, \ldots, x_{2n}, 1\rangle) \quad &\text{iff} \quad \exists l \subseteq^{fin} R^{\langle x_0, x_2, \ldots, x_{2n}\rangle}\, \exists k\ \alpha_\varphi(L^{\langle x_0, \ldots, x_{2n}\rangle} * l, n, k) > 1,\\
L^{\langle\rangle} &:= \langle\rangle,\\
L^{\langle x_0, \ldots, x_{2n}\rangle} &:= L^{\langle x_0, \ldots, x_{2n-1}\rangle} * \left\langle \min\left\{ x \in R^{\langle x_0, x_2, \ldots, x_{2n}\rangle} \;\middle|\; x > \max L^{\langle x_0, \ldots, x_{2n-1}\rangle} \right\}\right\rangle,\\
L^{\langle x_0, \ldots, x_{2n}, 0\rangle} &:= L^{\langle x_0, \ldots, x_{2n}\rangle},\\
L^{\langle x_0, \ldots, x_{2n}, 1\rangle} &:= L^{\langle x_0, \ldots, x_{2n}\rangle} * l,\\
k^{\langle x_0, \ldots, x_{2n}, 1\rangle} &:= k,\\
k^x &:= 0 \quad \text{for all } x \text{ not of this form},
\end{aligned}
$$

where $\langle l, k\rangle$ minimal with

$$l \sqsubset R^{\langle x_0, x_2, \ldots, x_{2n}\rangle} \wedge \alpha_\varphi(L^{\langle x_0, \ldots, x_{2n}\rangle} * l, n, k) > 1.$$

For notational simplification we omitted the requirements to make T closed under prefix, but we can simply add the conditions of the previous levels to the definition of T making it a tree.

L^x and k^x is clearly defined if $T(x)$ is true (use an instance of Σ^0_1-induction to show this — weaken the Π^0_2-statement "R^x is infinite" in the definition of T to $\exists z \in R^x$).

Using the same argument as in the previous proposition we see that the tree is infinite. But we cannot apply Σ^0_1-$\mathsf{WKL}(\xi r)$, because this instance contains L, which is in general not computable in r (in the sense of $\mathsf{G}_\infty\mathsf{A}^\omega$).

The graph of $x \mapsto (L^x, k^x)$ is definable and Δ^0_1. For notational easy we define the graph of its course-of-value function:

$$(\langle x_0, \ldots, x_n\rangle, \langle L_0, \ldots, L_n\rangle, \langle k_0, \ldots, k_n\rangle) \in \mathcal{G}_{\bar{L}, \bar{k}} \qquad \text{iff}$$

$n = 0$ $L_n = \langle\rangle$, $k = 0$,

n **even** $L_n = L_{n-1} * \langle y\rangle$, $k_n = 0$ where y is minimal with $y \in R^{\langle x_0, \ldots, x_{2m-1}\rangle}$ and $y > \max(L_{n-1})$,

n **odd and** $x_n = 0$ $L_n = L_{n-1}$, $k_n = 0$,

n **odd and** $x_n = 1$ $L_n = L_{n-1} * l$ and $\langle l, k_n\rangle$ is minimal with $l \sqsubset R^{\langle x_0, x_2, \ldots, x_{2n}\rangle}$ and $\alpha_\varphi(L_n * l, (n-1)/2, k_n) > 1$.

(Note that equations like $L_n = L_{n-1} * l$ we omitted for notational ease the bounded quantifier $\exists l < L_n$ for l.) So we can replace every reference to L^x in the definition of T by

$$\exists k, y\,(x,(y,k)) \in \mathcal{G}_{L,k} \qquad \text{or} \qquad \forall k, y\,(x,(y,k)) \in \mathcal{G}_{L,k}.$$

The resulting tree is still Π^0_2 so we may apply an instance of Π^0_2-WKL and obtain an infinite branch b.

Setting $G := \bigcup_n L^{\bar{b}(n)}$ now enumerates an infinite strongly cohesive set and from b we can decide $\forall k\,(\varphi r G)nk = 0$ for every n. □

Corollary 2.5 (to the proof)**.** *For every system $\mathcal{T}$ containing $\mathsf{G}_\infty\mathsf{A}^\omega$ and every provably continuous term φ there exists a term ξ, such that*

$$\begin{aligned}&\mathcal{T} + \mathsf{QF\text{-}AC} \oplus \mathsf{WKL} \vdash \\ &\qquad\qquad \forall r\colon \mathbb{N}\times\mathbb{N} \to 2\, \Big(\Pi^0_1\text{-}\mathsf{CA}(\xi r) \to \exists G\, \Big(\mathsf{COH}(r,G) \wedge \Pi^0_1\text{-}\mathsf{CA}(\varphi r G)\Big)\Big).\end{aligned}$$

Corollary 2.6. $(\mathsf{St})\mathsf{COH}$ *is proofwise low in sequence over* $\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF\text{-}AC} \oplus \mathsf{WKL}$.

Proof. Lemma 2.2 and Proposition 2.4. □

2.2. Atomic model theorem (AMT)

We first discuss the principle Π^0_1 Generic ($\Pi^0_1 G$), a generalization of the atomic model theorem. For details about the recursion-theoretic and model-theoretic strength of $\Pi^0_1 G$ we refer the reader to [21, 40].

Definition 2.7. $\Pi^0_1 G$ is the statement that for every uniformly Π^0_1 collection of sets D_i each of which is dense in $2^{<\omega}$ there is a set G such that $\forall i\, \exists s\, G{\upharpoonright}s \in D_i$.

Here uniformly Π^0_1 collection of sets means that the D_i are of the form

$$D_i := \{x \in 2^{<\omega} \mid \forall z\, A_{qf}(x,i,z)\},$$

where $A_{qf}(x,i,y)$ is a quantifier-free formula. Dense means that

$$\forall x \in 2^{<\omega}\, \exists y\, (x \sqsubseteq y \wedge y \in D_i)\,. \tag{2.4}$$

Proposition 2.8. *For every term φ there is a term ξ with*

$$\begin{aligned}&\widehat{\mathsf{WE\text{-}PA}}^\omega{\upharpoonright} + \mathsf{QF\text{-}AC} \vdash \\ &\qquad\qquad \forall (D_i)_i\, \Big(\Pi^0_1\text{-}\mathsf{CA}(\xi(D_i)_i) \to \exists G\, \Big(\Pi^0_1 G((D)_i, G) \wedge \Pi^0_1\text{-}\mathsf{CA}(\varphi(D)_i G)\Big)\Big).\end{aligned}$$

In other words $\Pi^0_1 G$ is proofwise low over $\widehat{\mathsf{WE\text{-}PA}}^\omega{\upharpoonright} + \mathsf{QF\text{-}AC}$.

Proof. We will define the characteristic function of G using bounded simultaneous primitive recursion. In this recursion we will one after another satisfy the requirements $\exists s\, G{\upharpoonright}s \in D_i$ for each i and decide the i-th value of the comprehension.

The function $f_1(n)$ will code a sequence of initial segments of G. The second function $f_2(n)$ will keep track of which requirement we are currently satisfying. We will only add one element to f_1 at each step to be able to bound the recursion. This is done in view of the proof of Proposition 2.13 below where only bounded recursion is included in the system.

Let $S(i,x)$ be the choice-function for y in (2.4). The existence of this function is provable using an instance of Π^0_1-AC, see Lemma 1.10.

Define now f_1, f_2 by bounded primitive recursion: Let

$$f_1(0) := \langle\rangle, \quad f_2(0) := 0.$$

In the recursion step we make the following case distinction:

Case $f_2(n) = 2i$ We are currently satisfying $\exists s\, G{\upharpoonright}s \in D_i$. If we would not care about bounds for the recursion, we would just extend $f_1(n)$ to $S(f_1(n), i)$ and thus satisfy the requirement. But to be able to bound the recursion we only extend $f_1(n)$ by one element of $S(f_1(n), i)$ at each step n and pass to the next requirement after this is done. Thus we set

$$f_1(n+1) := f_1(n) * \langle (S(f_1(\min\{k \le n \mid f_2(k) = 2i\}), i))_{\mathrm{lth}(f_1(n))} \rangle,$$

$$f_2(n+1) := \begin{cases} 2i+1 & \text{if } S(f_1(\min\{k \le n \mid f_2(k) = 2i\}), i) \sqsubseteq f_1(n+1), \\ 2i & \text{if } S(f_1(\min\{k \le n \mid f_2(k) = 2i\}), i) \not\sqsubseteq f_1(n+1). \end{cases}$$

Case $f_2(n) = 2i+1$ We are trying to decide the comprehension at i, i.e. to find an extension $s \in 2^{<\omega}$ of $f_1(n)$ such that

$$\exists k\, \alpha_\varphi(f_1(n) * s, i, k) > 1 \tag{2.5}$$

or

$$\forall s \in 2^{<\omega}\, \forall k\, \alpha_\varphi(f_1(n) * s, i, k) \le 1. \tag{2.6}$$

If (2.6) is true the comprehension at i is true on every extension of G, if not we extend G by the minimal counterexample s making the comprehension at i false on all further extensions of G.

Just like in the other case if we would not care about bounding the recursion, we could append s directly to $f_1(n)$. But in order to be able to bound the recursion, we only append an element of s at each step. Thus set

$$f_1(n+1) := f_1(n) * \begin{cases} \langle 0 \rangle & \text{if (2.6) is true,} \\ \langle (s)_{n+1} \rangle & \text{if (2.5) is true,} \\ & \text{where } \langle s, k \rangle \text{ is minimal satisfying (2.5),} \end{cases}$$

$$f_2(n+1) := \begin{cases} 2i+2 & \text{if (2.6) is true or (2.5) is true for } s = \langle\rangle, \\ 2i+1 & \text{otherwise.} \end{cases}$$

Note that deciding between (2.5) and (2.6) is constructive relative to the imposed instance of comprehension.

The recursion is obviously bounded by

$$f_1(n) \leq \underbrace{\langle 1, \ldots, 1 \rangle}_{n+1 \text{ times}} \quad \text{and} \quad f_2(n) \leq n.$$

Let G be the set with the characteristic function $(f_1(n))_n$.

To verify this construction we have to show that every requirement is met, that is f_2 eventually takes every value ($\forall k \, \exists n \, f_2(n) = k$). This can be easily proven using Σ^0_1-induction.

Note that this induction cannot be reduced to the instance of Π^0_1-comprehension and quantifier-free induction since itself contains an instance of Π^0_1-comprehension. This is the only usage of Σ^0_1-induction in this proof. $\square$

In the previous proposition $\widehat{\mathsf{WE\text{-}PA}}^\omega{\restriction}$ cannot be replaced by $\mathsf{G}_\infty\mathsf{A}^\omega$. If this would be possible we could prove with the methods below that $\mathsf{RCA}_0 + \Pi^0_1\text{-}\mathsf{CP} + \Pi^0_1 G$ is Π^0_2-conservative over PRA which contradicts the following proposition:

Proposition 2.9 ([40, Theorem 4.3])**.** $\mathsf{RCA}_0 + \Pi^0_1\text{-}\mathsf{CP} \vdash \Pi^0_1 G \to \Sigma^0_2\text{-}\mathsf{IA}$

Treatment of AMT

In this section let T denote a deductively closed decidable theory in a language $\mathcal{L}$.

Definition 2.10.

- A formula $\varphi(x_1, \ldots, x_n)$ is called an *atom* of T if for every formula $\psi(x_1, \ldots, x_n)$ either $T \vdash \varphi \to \psi$ or $T \vdash \varphi \to \neg\psi$.
- A theory T is called *atomic* if for every formula $\psi(x_1, \ldots, x_n)$ consistent with T there is an atom $\varphi(x_1, \ldots, x_n)$ extending ψ, i.e. $T \vdash \varphi \to \psi$.
- A model $\mathcal{A}$ is called *atomic* if every sequence of elements $a_1, \ldots, a_n$ in the universe of $\mathcal{A}$ satisfies an atom of the theory of $\mathcal{A}$.

Definition 2.11 (AMT)**.** The atomic model theorem states that every atomic theory has an atomic model.

Proposition 2.12 ([40, 21])**.**

(i) $\Pi^0_1 G \to \mathsf{AMT}$,

(ii) $\mathsf{SADS} \to \mathsf{AMT}$ *and thus* $\mathsf{RT}^2_2 \to \mathsf{AMT}$,

(iii) $\mathsf{AMT} \wedge \Sigma^0_2\text{-}\mathsf{IA} \to \Pi^0_1 G$,

(iv) $\mathsf{AMT} \nrightarrow \Pi^0_1\text{-}\mathsf{CP}, \mathsf{SADS}, \mathsf{WKL}, \mathsf{RT}^2_2$.

Proposition 2.13. *The principle* AMT *is proofwise low over* $\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF\text{-}AC}$*, i.e. for every term* φ *there is a term* ξ *with*

$$\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF\text{-}AC} \vdash \forall T \left(\Pi^0_1\text{-}\mathsf{CA}(\xi T) \to \exists M \left(\mathsf{AMT}(T, M) \wedge \Pi^0_1\text{-}\mathsf{CA}(\varphi T M)\right)\right).$$

Just like in [40] or [86, Section II.8] we will construct a Henkin-Model M of T. To do so first define the tree of all possible standard Henkin constructions of models of T:

Definition 2.14 ([40, 3.5], [86])**.** Let $\mathcal{L}'$ be the extension of $\mathcal{L}$ adding countably many (Henkin) constants c_i and let $(\varphi_i)_i$ be an enumeration of all sentences of $\mathcal{L}'$. The *tree* $\mathcal{F} \subseteq 2^{<\omega}$ *of all possible standard Henkin constructions of models of* T is defined by recursion. To each node $\sigma \in \mathcal{F}$ we will associate a set S_σ of sentences of $\mathcal{L}'$.

Let $\langle\rangle \in \mathcal{F}$ and set $S_{\langle\rangle} := \emptyset$. Assume that $\sigma \in \mathcal{F}$ and $n = \mathrm{lth}(\sigma)$. If $S_\sigma \cup \{\varphi_n\}$ is consistent with T, let $\sigma * \langle 1\rangle \in \mathcal{F}$. If φ_n is of the form $\exists x\, \psi(x)$ then set $S_{\sigma*\langle 1\rangle} := S_\sigma \cup \{\varphi_n, \psi(c_i)\}$, where i is the smallest number such that c_i does not occur in S_σ; otherwise set $S_{\sigma*\langle 1\rangle} := S_\sigma \cup \{\varphi_n\}$. If $S_\sigma \cup \{\neg\varphi_n\}$ is consistent with T, then let $\sigma * \langle 0\rangle \in \mathcal{F}$ and set $S_{\sigma*\langle 0\rangle} := S_\sigma \cup \{\neg\varphi_n\}$.

Evidently every infinite branch of $\mathcal{F}$ yields a model of T.

Along every branch of $\mathcal{F}$ infinitely many splits occur, i.e. there are incompatible extensions of the branch. These splits are given at least by the sentences determining if various constants are equal. Hence by Lemma 2.15 bellow the tree $\mathcal{F}$ is isomorphic to the full binary tree $2^{<\omega}$. (Note that the instance of Σ^0_1-IA in the lemma is implied by an instance of Π^0_1-CA.) Since the isomorphism is given by a primitive recursive term we may assume that $\mathcal{F} = 2^{<\omega}$.

Let $D_i \subseteq \mathcal{F}$ be the set of all finite Henkin constructions containing an atom for $c_0, \dots, c_i$. For a sentence being atomic is a Π^0_1-property (if the theory is decidable), thus the sets D_i are uniformly Π^0_1-sets. If one assumes that the theory T is atomic then the $(D_i)_i$ are dense. Hence AMT is a special case of $\Pi^0_1 G$.

Since an atom for $c_0, \dots, c_{i+1}$ is also an atom for $c_0, \dots, c_i$ the sets D_i form a descending chain, i.e. $D_i \supseteq D_{i+1}$. The following proof will crucially depend on this property.

Proof of Proposition 2.13. We proceed like in the proof of Proposition 2.8 but we will block the requirement to overcome the need for Σ^0_1-induction. This is a proof-theoretic version of Shore blocking.

By the preceding discussion it is sufficient to consider only cofinitely many D_i.

In a similar way we can block the comprehension requirements, i.e. deciding between (2.5) and (2.6), see Lemma 2.16 below. The application of this lemma leads to an instance of Δ^0_2-comprehension, which is included in the system, see Lemma 1.10.

The functions f_1, f_2 will be defined like in the proof of Proposition 2.8, with the following exceptions

- (2.6) is replaced by (2.8) on page 42,
- if $f_2(n)$ changes its value it is set to $2n+1$ resp. $2n+2$.

The function f_2 is still bounded but now by the function $2n+2$. To verify the construction we only have to show that the image of f_2 cofinal, since we only have to meet the requirement for cofinally many D_i and cofinally many blocked comprehension decisions. Precisely we show

$$\forall k\, \exists n\, f_2(n) \geq k.$$

Let n be a number $> k$, where f_2 changes its value, then $f_2(n) \geq k$. Such an n exists for the same reasons as in Proposition 2.8 (but there $f_2(n)$ only is incremented by 1 and not set to $2n+1$ resp. $2n+2$).

This completes the proof. Note that it does not involve Σ^0_1-induction. $\square$

Lemma 2.15. *Let $\mathcal{F} \subseteq 2^{<\omega}$ be a tree containing infinitely many splits along each path, i.e. a tree satisfying*

$$\forall x \in \mathcal{F}\, \exists y_0, y_1 \in \mathcal{F}\ (x \sqsubseteq y_0, y_1 \wedge y_0 \not\sqsubseteq y_1 \wedge y_1 \not\sqsubseteq y_0)\,. \tag{2.7}$$

Then $\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF\text{-}AC}^{0,0} \oplus \Pi^0_1\text{-}\mathsf{IA}(\xi\mathcal{F})$, for a suitable closed term ξ, proves that $\mathcal{F}$ is isomorphic to $2^{<\omega}$.[1]

Moreover the isomorphism is given by a fixed term.

Proof. We show that $2^{<\omega}$ can be embedded into $\mathcal{F}$.

Clearly we can make y_0, y_1 unique in (2.7) be searching for the shortest split and assume that $y_0 < y_1$.

Now iterating this split-building process with the instance of induction we can prove the existence of a 0/1-tree of splits, i.e.

$$\begin{aligned}\forall n\, \exists y\, \forall k < n\, \forall z \in 2^k\ \big((y)_{z*\langle 0\rangle} < (y)_{z*\langle 1\rangle}\\ \wedge\ (y)_{z*\langle 0\rangle}, (y)_{z*\langle 1\rangle} \text{ splits } \mathcal{F} \text{ at } (y)_z \text{ and is a shortest split}\big)\end{aligned}$$

[1] Two trees are isomorphic if they can be embedded into each other retaining the order $\sqsubseteq$.

Let Y be the choice function for y. Then $x \in 2^{<\omega} \mapsto (Y(\operatorname{lth} x))_x$ defines an isomorphism between $2^{<\omega}$ and $\mathcal{F}$.

Since this is provable in $\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction$ a functional interpretation yields a fixed primitive recursive term t_Y realizing Y. $\square$

Lemma 2.16 (Blocking of instances of comprehension)**.** *Let t^0 be a code of a finite sequence and $(A_i)_{i<n}$ be a finite set of formulas of the form*

$$A_i(q) \equiv \forall k\, \alpha_\varphi(q, i, k) \leq 1 \qquad \text{for } i < n,$$

where φ is a provably continuous term and α_φ its associate.

Over $\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF\text{-}AC}$ there exists an instance of Δ^0_2-comprehension that proves that there is an extension $s \sqsupseteq t$ such that for each $i < n$ either $A_i(q)$ is true for all $q \sqsupseteq s$ or $A_i(s)$ already fails.

Proof. The idea of the proof is to successively decide A_i and extend if possible t with a counterexample. At each step we will use our knowledge of the preceding steps to obtain the so far constructed s.

Technically we proceed by using the instance of Δ^0_2-comprehension to find a tuple $\langle e_0, \dots, e_{n-1}\rangle \in 2^n$ satisfying the Δ^0_2-sentence

$$\begin{aligned}
&(e_0 = 0 \to \forall s_0, k_0\ \alpha_\varphi(t * s_0, 0, k_0) \leq 1)\\
\wedge\ &(e_0 \neq 0 \to \exists s_0, k_0\ \alpha_\varphi(t * s_0, 0, k_0) > 1)\\
\wedge\ &\Big(e_1 = 0 \to\\
&\quad\Big(\quad e_0 = 0 \to \forall s_1, k_1\ \alpha_\varphi(t * s_1, 1, k_1) \leq 1\\
&\quad\ \wedge\ e_0 \neq 0 \to \forall s_0, k_0 \begin{pmatrix} s_0, k_0 \text{ minimal with } \alpha_\varphi(t * s_0, 0, k_0) > 1\\ \to \forall s_1, k_1\, \alpha_\varphi(t * s_0 * s_1, 0, k_1) \leq 1\end{pmatrix}\Big)\Big)\\
&\quad\vdots
\end{aligned} \tag{2.8}$$

Set $s := t * s_0 * \cdots * s_{n-1}$. This proves the lemma. $\square$

Remark 2.17. The Propositions 2.8 and 2.13 are also true for sequences of dense sets resp. theories, because in the construction of G, M no Π^0_1-LEM is involved, which would become a comprehension. Hence AMT is proofwise low in sequence over $\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF\text{-}AC}$.

2.3. Term-normalization

Denote by $T_{(k)}[F_0, \dots, F_{n-1}]$ the extension of the system T_k resp. T with the constants $F_0, \dots, F_{n-1}$. Further we treat here R_ρ as an unspecified constant (without R_ρ axioms) in the case of $\mathsf{qf\text{-}G}_n\mathsf{A}^\omega$.

In the following we will call the reduction of

$$\mathrm{Cond}_{\rho(\tau)}(x, y, z)u^\tau \quad \text{to} \quad \mathrm{Cond}_\rho(x, yu, zu)$$

a Cond-*reduction*. These Cond-reductions are provably valid in $\mathsf{qf}\text{-}\mathsf{N}\text{-}\mathsf{G}_n\mathsf{A}^\omega$.

Theorem 2.18 (term-normalization for degree 2)**.** *Let F_i be constants of degree ≤ 2.*

For every term $t^1 \in T_0[(\mathrm{Cond}_\rho)_{\rho\in\mathbf{T}}, F_0, \dots, F_{n-1}]$ there is provably in $\mathsf{qf}\text{-}\mathsf{N}\text{-}\mathsf{G}_3\mathsf{A}^\omega$ a term $\tilde{t} \in T_0[\mathrm{Cond}_0, F_0, \dots, F_{n-1}]$ for which

$$\forall x \; tx =_0 \tilde{t}x$$

and where every occurrence of an F_i is of the form

$$F_i(\tilde{t}_0[y^0], \dots, \tilde{t}_{k-1}[y^0]).$$

Here k is the arity of F_i, and $\tilde{t}_j[y^0]$ are fixed terms whose only free variable is y^0.

Proof. Without loss of generality we take the system $T_0[F]$ where F is of type 2. For notational simplification we assume that the recursor R_0 can be obtained from F. This can always be achieved with coding.

Let t^1 be a term in $T_0[F]$. The term tx, where x is a fresh variable, is $=_0$-equal to a term $t'[x]$ where t' results from tx by carrying out all possible Π-, Σ-, and Cond-reductions. The outermost symbol of t' cannot be Π, Σ, or Cond_ρ with $\rho \neq 0$, since otherwise in t' either not all Π-, Σ-reductions had been carried out or t' would not be of type 0.

Hence one of the following holds:
1) $t'[x] = 0^0$
2) $t'[x] = S(t_a^0[x])$
3) $t'[x] = F(t_b^1[x])$
4) $t'[x] = \mathrm{Cond}_0(t_c^0[x], t_d^1[x], t_e^1[x])$

In the first case we are done, $\lambda x.t'[x]$ satisfies the theorem. In the second case we proceed the same way with the term t_a. In the third case we proceed with the term $t_b y^0$ where y^0 is a new variable making t_b to type 0 and in the fourth case we proceed with the terms $t_c, t_d y^0, t_e y^0$. Note that we can code the variables x and y in on type 0 variable. Also note that since we applied all Cond-reductions only Cond_0 occurs.

By the strong normalization theorem this process stops, yielding the desired term, see e.g. [33]. □

Theorem 2.19 (term-normalization for degree 3)**.** *Now let G_i be constants of degree $\leq$ 3. For every term $t^1 \in T_0[(\mathrm{Cond}_\rho)_{\rho\in\mathbf{T}}, G_0, \dots, G_{n-1}]$ there is provably in $\mathsf{qf}\text{-}\mathsf{N}\text{-}\mathsf{G}_3\mathsf{A}^\omega$ a term $\tilde{t} \in T_0[\mathrm{Cond}_0, G_0, \dots, G_{n-1}]$ for which*

$$\forall x \; tx =_0 \tilde{t}x$$

and where every occurrence of an G_i is of the form

$$G_i(\tilde{t}_0[f^1], \dots, \tilde{t}_{k-1}[f^1]).$$

Here k is the arity of G_i, and $\tilde{t}_j[f^1]$ are fixed terms whose only free variable is f^1.

Proof. Analogous to proof of Theorem 2.18. See also [62, proof of Proposition 4.2]. □

Note that the equality between $t, \tilde{t}$ is only pointwise. Therefore one needs (weak) extensionality to conclude that $s[t] =_0 s[\tilde{t}]$ for an arbitrary term s.

Application to proofs in quantifier-free systems

For a term t call the term where every maximal type 0 subterm (i.e. every subterm of type 0 which is not included in a different subterm of type 0) is replaced by a fresh type 0 variable *skeleton.* Obviously, t can be regained from its skeleton by substitution of type 0 terms.

Lemma 2.20. *Let $\mathcal{T}$ be $\mathsf{qf}\text{-}\mathsf{N}\text{-}\mathsf{G}_n\mathsf{A}^\omega$ with $n \geq 3$ or $\mathsf{qf}\text{-}\widehat{\mathsf{N}\text{-}\mathsf{PA}}^\omega\!\restriction$ augmented with arbitrary constants $H_0, H_1, \dots,$ let $t_0, t_1 \in T_0[\mathrm{Cond}_0, H_0, H_1, \dots]$ and in t_0, t_1 all possible Π-, Σ-reductions have been carried out.*

Then the following are equivalent:

(i) *The terms t_0, t_1 are provable equal in every term context ($\mathcal{T} \vdash s[t_0] =_0 s[t_1]$ for every term s).*

(ii) *$\mathcal{T} \vdash P(t_0) =_0 P(t_1)$, where P is a variable of suitable type.*

(iii) *The terms t_0, t_1 have the same skeleton (modulo renaming of type 0 variables) and t_0, t_1 are obtained from the skeleton by substitution of $=_0$-equal terms.*

Proof. (i) ⇒ (ii) is clear. (ii) ⇒ (i) follows from the fact that one can replace P by any term in the derivation and so in particular by $\lambda x.s[x]$. By definition of the axioms of a quantifier-free system the axioms of this new derivation are also in $\mathcal{T}$. (iii) ⇒ (i) follows from the $=_0$-axioms.

For (ii) ⇒ (iii) observe that the only way to prove the equality in (ii) are the SUB rule, the $\mathsf{SUB}_{\mathrm{Cond}}$ rule for Cond_0, or the $=_0$-axioms. The Π-, and Σ-reductions commute with applications of $=_0$-axioms and in t_0, t_1 all possible Π- and Σ-reductions have been carried out we may assume that only the $=_0$-axioms, $\mathsf{SUB}_{\mathrm{Cond}}$-axioms for Cond_0, and the SUB-axioms for R_0 are used. These axioms only change type 0 values and, therefore, the skeletons have to be the same. The lemma follows. □

Proposition 2.21. *Let $\mathcal{T}$ be $\mathsf{qf}\text{-}\mathsf{N}\text{-}\mathsf{G}_n\mathsf{A}^\omega$ where $n \geq 3$ or $\mathsf{qf}\text{-}\widehat{\mathsf{N}\text{-}\mathsf{PA}}^\omega\!\restriction$ augmented by a type 2 constant F. Further let A be a formula containing only type 0 variables free and satisfying $\mathcal{T} \vdash A$.*

Then there exists a formula $\tilde{A}$ such that the weakly extensional intuitionistic system $\mathcal{T}_{\mathrm{WE}}$ corresponding to $\mathcal{T}$ (i.e. $\mathsf{G}_n\mathsf{A}^\omega_\mathsf{i}$ or $\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction$) proves $A \leftrightarrow \tilde{A}$ and that there is a derivation $\tilde{\mathcal{D}}$ of $\mathcal{T} \vdash \tilde{A}$ where every occurring term is normalized according to Theorem 2.18, i.e. each occurrence of F is of the form $F(t_i[x])$.

Moreover, these applications of F can be chosen independently from each other in the sense that

$$\mathcal{T} \nvdash P[F(t')] =_0 P[F(t'')] \qquad \textit{for a fresh variable } P$$

for all type 0 substitution instances t', t'' of t_i resp. t_j with $i \neq j$. (In other words, the theory $\mathcal{T}$ does not see that the $F(t'), F(t'')$ are applications of F and not just an arbitrary term of suitable type and with the same free variables. Hence they may be replaced independently.)

Using coding we may also allow finitely many constants F_i of degree 2.

Proof. Let $\mathcal{D}$ be a derivation of $\mathcal{T} \vdash A$. By Lemma 1.2 we may assume that only the variables of A and some free type 0 variables occur in $\mathcal{D}$. Hence every term showing up in $\mathcal{D}$ satisfies the premise of Theorem 2.18.

We obtain a new derivation $\tilde{\mathcal{D}}$ by replacing every term in $\mathcal{D}$ with its normal form as defined in the proof of Theorem 2.18 (in particular all possible Π-, and Σ-reductions have been carried out and only Cond_0 occurs in $\tilde{t}$). The derivation $\tilde{\mathcal{D}}$ is still valid because the used logical axioms and rules, SUB-axioms for the recursor and Cond, $=_0$-axioms, and quantifier-free induction rule are translated into other instances of themselves. The used SUB-axioms for Π and Σ become trivial since in all terms all possible Π- and Σ-reductions have been carried out.

Let $\tilde{A}$ be the result of $\tilde{\mathcal{D}}$. Each term occurring in $\tilde{A}$ is just the normal form of the term at the same position in A and therefore weakly extensional equal to it. Hence

$$\mathcal{T}_{\mathrm{WE}} \vdash A \leftrightarrow \tilde{A}.$$

Obviously, the derivation $\tilde{\mathcal{D}}$ contains only finitely many applications t_i of F. Each of the t_i contains only type 0 variables free. However, these applications of F are not independent from each other because there might be equalities between them provable in $\mathcal{T}$.

Passing to the skeletons of t_i we obtain applications of F which are by Lemma 2.20 pairwise independent or literally equal and which still contain only type 0 parameters. □

Remark 2.22. If in the above theorem one adds a type 3 constant G instead of F to the system and uses Theorem 2.19 instead of 2.18 one obtains a similar result with the exception that the applications t_i now also depend on function variables f_i. (These variables result from the normalization defined in Theorem 2.19. They can be coded together into one variable f such that the derivation $\tilde{\mathcal{D}}$ may be contains only the variables occurring in $\tilde{A}$ plus some fresh type 0 variables.)

2.4. Elimination of monotone Skolem functions

Let Δ be a set of sentences of the form $\forall a\, \exists b < ra\, \forall c^0\, B_{qf}(a,b,c)$, where r is a closed term and B_{qf} is quantifier-free and contains any further free variables than those shown. Let $\tilde{\Delta}$ be the corresponding set of Skolem normal form of the sentence of Δ, i.e. the corresponding formulas of the form $\exists B < r\, \forall a, c^0\, B_{qf}(a, Ba, c)$.

Theorem 2.23 ([60, 3.8])**.** *Let γ be an arbitrary type and let A_{qf} be a quantifier-free statement where only the shown variables are free and let s be a term in $\mathsf{G}_\infty\mathsf{R}^\omega$. If*

$$\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF\text{-}AC} \oplus \Delta \vdash \forall u^1\, \forall v \leq_\gamma su\, \Big(\Pi^0_1\text{-}\mathsf{CA}(\xi uv) \to \exists w^0\, A_{qf}(u,v,w)\Big)$$

then one can extract from a proof a term $t \in T_0$ such that

$$\widehat{\mathsf{WE\text{-}HA}}^\omega{\upharpoonright} \oplus \tilde{\Delta} \vdash \forall u^1\, \forall v \leq_\gamma su\, \exists w \leq_0 tu\ A_{qf}(u,v,w).$$

Especially, in case that $A_{qf} \in \mathcal{L}(\mathsf{PRA})$, u of type 0, v absent and $\Delta = \emptyset$ we have

$$\mathsf{PRA} \vdash \forall u^0\, A_{qf}(u, tu).$$

Corollary 2.24. *Let γ, ξ, s, A_{qf} be as in Theorem 2.23. However ξ may contain $\mathcal{B}$ but s and A_{qf} must not. Then the following holds: If*

$$\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF\text{-}AC} \oplus (\mathcal{B}) \vdash \forall u\, \forall v \leq_\gamma su\, \Big(\Pi^0_1\text{-}\mathsf{CA}(\xi uv) \to \exists w^0\, A_{qf}(u,v,w)\Big)$$

then one can extract from a proof a term $t \in T_0$ such that

$$\widehat{\mathsf{WE\text{-}HA}}^\omega{\upharpoonright} \vdash \forall u^1\, \forall v \leq_\gamma su\, \exists w \leq_0 tu\ A_{qf}(u,v,w).$$

Proof. First note that due to [60, Remark 2.10] we may add the (majorizable) constant $\mathcal{B}$ to $\mathsf{G}_\infty\mathsf{A}^\omega$ in Theorem 2.23.

Apply this theorem to $\Delta := \Big\{\forall f\, \forall x\, \check{f}(\mathcal{B}\check{f}x) = 0\Big\}$, cf. Definition 1.4 and (1.3) on p. 28. The premise of the corollary implies that

$$\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF\text{-}AC} \oplus \Delta \vdash \forall u\, \forall v \leq_\gamma su\, \Big(\Pi^0_1\text{-}\mathsf{CA}(\xi uv) \to \exists w^0\, A_{qf}(u,v,w)\Big)\,.$$

Theorem 2.23 and noticing that $\Delta \equiv \tilde{\Delta}$ yields

$$\widehat{\mathsf{WE\text{-}HA}}^\omega{\upharpoonright} \oplus \Delta \vdash \forall u^1\, \forall v \leq_\gamma su\, \exists w \leq_0 tu\ A_{qf}(u,v,w)$$

and so

$$\widehat{\mathsf{WE\text{-}HA}}^\omega{\upharpoonright} \vdash \Delta \to \forall u^1\, \forall v \leq_\gamma su\, \exists w \leq_0 tu\ A_{qf}(u,v,w).$$

Since the constant $\mathcal{B}$ only occurs in Δ, we may replace it with a new variable and so replace Δ with UWKL. The corollary now follows from [67, Corollary 10.34]. $\square$

2.5. Results for the cohesive principle and the atomic model theorem

Our goal is now to interpret consequences (of the form $\forall x^1\, \exists y^0\, A_{qf}(x,y)$) of a principle $\mathcal{P}$ that is proofwise low in sequence. For this we will strengthen $\mathcal{P}$ to the statement that there exists a uniform solution functional $\mathcal{P}$ for $\mathcal{P}$. The functional $\mathcal{P}$ must be of degree ≤ 2, such that after extracting terms using the functional interpretation one can normalizing them with the tools of Section 2.3. With this we will see that $\mathcal{P}$ is only used finitely many times and can be replaced using the lowness property in favor of an instance of Π^0_1-CA.

The properties of the solution functional $\mathcal{P}$ must be axiomatizable universally, since they will become an implicative assumption. After prenexation they will become purely existential and the functional interpretation will extract terms witnessing them. Existential quantifier in the axiomitation of $\mathcal{P}$ would become universal after prenexation and therefore would need to be presented afterward.

If $\mathcal{P}$ is of the form

$$\forall S\, \exists G\; \underbrace{\forall x\, P_{qf}(S,G,x)}_{\equiv: P(S,G)}, \tag{2.9}$$

where P_{qf} is quantifier-free. Then one can take for $\mathcal{P}$ the Skolem functional for G, i.e. a functional $\mathcal{P}$ satisfying

$$\forall S\, \forall x\, P_{qf}(S, \mathcal{P}(S), x).$$

With the help of the following lemma we can obtain a functional for $\mathcal{P}$ where P is a Π^0_3 formula. This is sufficient for StCOH and AMT.

Lemma 2.25. *Let $\mathcal{P}$ be a principle proofwise low in sequence over $\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF}\text{-}\mathsf{AC} \oplus \mathsf{WKL}$, that has the form*

$$(\mathcal{P})\colon \forall S\, \exists G\; \underbrace{\forall x\, \exists y\, \forall z\, P_{qf}(S,G,x,y,z)}_{\equiv: P(S,G)}, \tag{2.10}$$

where P_{qf} is quantifier-free.

Then the principle

$$(\mathcal{P}')\colon \forall S\, \exists G, Y\, \forall Z^1\, \forall x\;\, P_{qf}(S,G,x,Y(x,Z),Z(Y(x,Z))) \tag{2.11}$$

is proofwise low in sequence, in the sense that for every closed term φ a closed term ξ exists, such that Π^0_1-$\mathsf{CA}(\xi(S_i)_i(Z_i)_i)$ proves

$$\exists (G_i)_i, (Y_i)_i\, (\forall i, Z', x\;\, P_{qf}(S_i, G_i, x, (Y)_i(x,Z'), Z'(Y_i(x,Z'))) \;\wedge \\ \Pi^0_1\text{-}\mathsf{CA}(\varphi(S_i)(Z_i)(G_i)(\lambda x. Y_i(x,Z_i)_i))).$$

Proof. The lowness of $\mathcal{P}$ provides that for every term φ' an instance of Π^0_1-comprehension Π^0_1-$\mathsf{CA}(\xi SZ)$ proves

$$\exists G\ \Big(\forall x^0\, \exists y^0\, \forall z^0\, P_{qf}(S,G,x,y,z) \wedge \Pi^0_1\text{-}\mathsf{CA}(\varphi' SZG)\Big)\,.$$

Hence it also proves

$$\exists G\ \Big(\forall x, Z\, \exists y\, P_{qf}(S,G,x,y,Z(y)) \wedge \Pi^0_1\text{-}\mathsf{CA}(\varphi' SZG)\Big)\,.$$

By searching for the least y we may assume that there exists a unique y for each x, Z. Let $Y(x,Z)$ be the choice function for y obtained using $\mathsf{QF\text{-}AC}$. To show that $\mathcal{P}'$ is proofwise low it suffices to show for every closed φ that there is a closed φ' (and thus a closed ξ) such that Π^0_1-$\mathsf{CA}(\varphi SZG(\lambda x.Y(x,Z)))$ is provable from Π^0_1-$\mathsf{CA}(\varphi SZ)$.

Since Y is computable in S, G a suitable φ can easily be constructed with the same generic construction used in the proof of Lemma 1.11.

One also easily verifies that the whole argumentation is stable under sequences and hence that $\mathcal{P}'$ is proofwise low in sequence. ∎

It is easy to see that $\mathcal{P}'$ is equivalent to $\mathcal{P}$ over $\mathsf{QF\text{-}AC}^{0,0}$. For such principle we could then use a solution functional $\mathcal{P} = (\mathcal{P}_G, \mathcal{P}_Y)$ that codes together the Skolem functions for G, Y in (2.11), i.e.

$$\forall S\, \forall Z\, \forall x\ \underbrace{P_{qf}(S, \mathcal{P}_G(S), x, \mathcal{P}_Y(G,x,Z), Z(\mathcal{P}_G(G,x,Z))}_{\equiv: P_S(\mathcal{P},(Z,x))}\,. \tag{2.12}$$

Proposition 2.26. *Let $A_{qf} \in \mathcal{L}(\mathsf{G_\infty A}^\omega)$ be a quantifier-free formula that contains only the shown variables free and let $\mathcal{P}$ be a principle proofwise low in sequence over $\mathsf{G_\infty A}^\omega + \mathsf{QF\text{-}AC} \oplus \mathsf{WKL}$ of the form* (2.10). *If*

$$\widehat{\mathsf{E\text{-}PA}}^\omega{\restriction} + \mathsf{QF\text{-}AC}^{0,1} + \mathsf{QF\text{-}AC}^{1,0} + \Pi^0_1\text{-}\mathsf{CP} + \mathcal{P} + \mathsf{WKL} \vdash \forall x^1\, \exists y^0\, A_{qf}(x,y),$$

then one can find a term ξ such that

$$\mathsf{G_\infty A}^\omega + \mathsf{QF\text{-}AC} \oplus (\mathcal{B}) \vdash \forall x^1\ \Big(\Pi^0_1\text{-}\mathsf{CA}(\xi x) \rightarrow \exists y^0\, A_{qf}(x,y)\Big)\,.$$

Proof. We first prove the proposition without Π^0_1-CP.

Note that due to

- the deduction theorem (which holds for $\widehat{\mathsf{E\text{-}PA}}^\omega{\restriction}$),
- the elimination of extensionality (Proposition 1.5),
- the strengthening of WKL to UWKL and

- the strengthening of $\mathcal{P}$ to the Skolem normal-form of $\mathcal{P}'$, i.e. the statement there exists an $\mathcal{P}$ satisfying (2.12)

we obtain

$$\mathsf{N}\text{-}\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF}\text{-}\mathsf{AC} \vdash (\exists \mathcal{P}\, \forall u^1\, P_S(\mathcal{P}, u)) \wedge (R_0) \wedge (\mathcal{B}) \rightarrow \forall x^1\, \exists y^0\, A_{qf}(x, y),$$

where u codes the pair (Z, x) from (2.12) and (R_0) are the defining axioms for the recursor R_0. Note that also the formulas (R_0), $(\mathcal{B})$ can be written in the form $\exists R_0\, \forall u^1\, (R_0)_{qf}(R_0, u)$ resp. $\exists \mathcal{B}\, \forall u^1\, (\mathcal{B})_{qf}(\mathcal{B}, u)$ for quantifier free $(R_0)_{qf}$, $(\mathcal{B})_{qf}$.

Applying the functional interpretation to this yields terms $t_y, t_P, t_{R_0}, t_{\mathcal{B}} \in \mathsf{G}_\infty\mathsf{R}^\omega$ such that

$$\begin{aligned}\mathsf{qf}\text{-}\mathsf{N}\text{-}\mathsf{G}_\infty\mathsf{A}^\omega \vdash (&P_S(\mathcal{P}, t_P(x, \mathcal{P}, R_0, \mathcal{B})) \wedge \\ &(R_0)_{qf}(R_0, t_{R_0}(x, \mathcal{P}, R_0, \mathcal{B})) \wedge (\mathcal{B})_{qf}(\mathcal{B}, t_{\mathcal{B}}(x, \mathcal{P}, R_0, \mathcal{B})) \\ &\rightarrow A_{qf}(x, t_y(x, \mathcal{P}, R_0, \mathcal{B}))),\end{aligned} \quad (2.13)$$

see [96, 58].

The terms $t_P(x, \mathcal{P}, R_0, \mathcal{B})$, $t_{R_0}(x, \mathcal{P}, R_0, \mathcal{B})$, $t_{\mathcal{B}}(x, \mathcal{P}, R_0, \mathcal{B})$, $t_y(x, \mathcal{P}, R_0, \mathcal{B})$ have degree ≤ 1. By Proposition 2.21 we obtain a new derivation in $\mathsf{qf}\text{-}\mathsf{N}\text{-}\mathsf{G}_\infty\mathsf{A}^\omega$ of a sentence which is equivalent to (2.13) over $\mathsf{qf}\text{-}\mathsf{G}_\infty\mathsf{A}^\omega$ and where each application of $\mathcal{P}$ is of the form $\mathcal{P}(t_i[z^0])$ or a substitution instance of $\mathcal{P}(t_i[z^0])$ and $\mathcal{P}(t_i[z^0])$ and $\mathcal{P}(t_j[z^0])$ are independent (in the sense of Proposition 2.21). Same for $R_0, \mathcal{B}$.

Our goal is now to replace these occurrences of $\mathcal{P}$, R_0, and $\mathcal{B}$ in the normalized derivation of (2.13) by a low solution to those principles, such that the premise of (2.13) becomes provable.

We proceed by inductively over the nesting-depth of $\mathcal{P}$, R_0, and $\mathcal{B}$ replacing the applications (and their substitution instances) with low solutions retaining the instance of comprehension. This operation leaves the derivation valid since the different applications are independent. Concretely we replace $\mathcal{P}, R_0, \mathcal{B}$ by the following:

- $R_0(t_i[z^0])$ simply defines a primitive recursive function, which is provably total using an instance of Σ^0_1-induction. This instance can be obtained from $\mathsf{QF}\text{-}\mathsf{IA}$ and an instance of Π^0_1-comprehension. Then Lemma 1.11 yields a new instance of comprehension (which allows $R_0(t_i[z^0])$ as parameter).
- $\mathcal{P}(t_i[z^0])$ can be handled using the assumption that $\mathcal{P}$ is proofwise low in sequence (Lemma 2.25)
- $\mathcal{B}(t_i[z^0])$ can trivially be handled because it is present in the verifying system.

For the construction of these replacements we work in the system $\mathsf{G}_\infty\mathsf{A}^\omega$, i.e. with weak extensionality and quantifiers. After this the premise of (2.13) becomes provable.

Quantifying over all x and coding x, z together into a new variable x, yields the proposition without Π^0_1-CP.

To prove the full proposition note that we can add StCOH to the system since it is proofwise low in sequence, see Corollary 2.6, and that StCOH implies Π^0_1-CP, see Proposition 2.1. This completes the proof. □

Theorem 2.27 (Conservation for proofwise low in sequence)**.** *Let $\mathcal{P}$ be a principle of the form* (2.10) *that is proofwise low in sequence over $\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF\text{-}AC} \oplus \mathsf{WKL}$. In particular, this includes all principles of this form proofwise low in sequence over WKL^*_0.*

If

$$\widehat{\mathsf{E\text{-}PA}}^\omega\!\restriction + \mathsf{QF\text{-}AC}^{0,1} + \mathsf{QF\text{-}AC}^{1,0} + \Pi^0_1\text{-}\mathsf{CP} + \mathcal{P} + \mathsf{WKL} \vdash \forall x^1\, \exists y^0\, A_{qf}(x, y)$$

then one can extract a primitive recursive term t such that

$$\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction \vdash \forall x^1\, A_{qf}(x, tx).$$

In particular, if $A_{qf} \in \mathcal{L}(\mathsf{PRA})$ and x is of type 0 *we have $\mathsf{PRA} \vdash \forall x\, A_{qf}(x, tx)$.*

Proof. We may assume that $A_{qf} \in \mathcal{L}(\mathsf{G}_\infty\mathsf{A}^\omega)$. Otherwise it would contain R_0. If this is the case we normalize every term occurring in A_{qf} and replace every occurrence of R_0uvw by a fresh variable that will be $\exists$-quantified. There are no other occurrence of R_0 in A_{qf} since it contains (beside Π, Σ) no constant of type > 2. These fresh variables will hold the value of R_0uvw. This values exists provably with Σ^0_1-IA and can be expressed in a quantifier-free way.

Apply now elimination of Skolem function for monotone formulas (Corollary 2.24) to the result of Proposition 2.26. □

Corollary 2.28. *Especially from a proof of*

$$\widehat{\mathsf{E\text{-}PA}}^\omega\!\restriction + \mathsf{QF\text{-}AC}^{0,1} + \mathsf{QF\text{-}AC}^{1,0} + \Pi^0_1\text{-}\mathsf{CP} + \mathsf{COH} + \mathsf{AMT} + \mathsf{WKL} \vdash \forall x^1\, \exists y^0\, A_{qf}(x, y)$$

one can extract a primitive recursive term t such that

$$\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction \vdash \forall x^1\, A_{qf}(x, tx).$$

Proof. Theorem 2.27, Corollary 2.6, and Remark 2.17. □

Corollary 2.29. *The system*

$$\mathsf{WKL}^\omega_0 + \Pi^0_1\text{-}\mathsf{CP} + \mathsf{COH} + \mathsf{AMT}$$

is Π^0_2-conservative over PRA. Additionally, for every Π^0_2-sentence one can extract uniformly a primitive recursive (provably) realizing term.

Further $\mathsf{WKL}^\omega_0 + \Pi^0_1\text{-}\mathsf{CP} + \mathsf{COH} + \mathsf{AMT}$ is Π^0_3-conservative over RCA^ω_0.

Proof. The first statement is clear from the preceding corollary and the definition of WKL_0. The second statement follows also from this corollary by noting that over $\mathsf{QF\text{-}AC}^{0,0}$ a Π^0_3-formula

$$\forall x^0 \, \exists y^0 \, \forall z^0 \, A_{qf}(x, y, z)$$

is equivalent to

$$\forall x^0, Z^1 \, \exists y^0 \, A_{qf}(x, y, Zy).$$

□

This in some sense is the best possible result since $\mathsf{RCA}_0 + \Pi^0_1\text{-}\mathsf{CP}$ is not Σ^0_3-conservative over a theory containing only Σ^0_1-induction, see [4].

3. Ramsey's theorem for pairs

3.1. Stable Ramsey's theorem for pairs (SRT^2_2)

An n-coloring $c\colon [\mathbb{N}]^2 \to n$ is called *stable* if

$$\forall x \, \exists k \, \forall y > k \, c(x,k) = c(x,y).$$

The point k is called a *stability point for x*.

We call an n-coloring *strongly stable* if

$$\forall x \, \exists k \, \forall y > k \, \forall x' \le x \, c(x',k) = c(x',y).$$

Over Π^0_1-CP strongly stable and stable coincide. Even an instance of the collection principle of the form Π^0_1-$\mathsf{CP}(\xi c)$ where ξ is a suitable term and c the coloring suffices to prove this equivalence.

Let SRT^2_n be the statement expressing that every *stable* n-coloring of pairs has an infinite homogeneous set and let $\mathsf{SRT}^2_{<\infty} :\equiv \forall n \, \mathsf{SRT}^2_n$. For a stable n-coloring c the statement $\mathsf{SRT}^2_n(c,H)$ denotes that H is a homogeneous set for c.

The principle SRT^2_2 is over Σ^0_1-induction equivalent to the statement that for every Δ^0_2-set X there exists an infinite set Y with $Y \subseteq X$ or $Y \subseteq \overline{X}$, see [16, 17].

Before we go on with the main result we need some auxiliary lemmata:

Lemma 3.1 ([16, Lemma 4.2])**.** *For every fixed n, let $(\xi_{k,i})_{k<n, i\in\mathbb{N}}$ be a sequence of Π^0_1-sentences of the form $\xi_{k,i} \equiv \forall x \, A(k,i,x)$ for a quantifier-free A such that*

$$\forall i \, \exists k < n \; \xi_{k,i}.$$

Then WKL proves that there exists a choice function $g\colon \mathbb{N} \to n$ satisfying $\forall i \, \xi_{g(i),i}$.

If WKL is replaced by Σ^0_1-WKL the same holds for Π^0_2-sentences.

Proof. Define

$$f(\langle x_0, \ldots, x_n \rangle) = 0 \quad \text{iff} \quad \bigwedge_{i \le n} \xi_{x_i,i}.$$

The function f clearly defines a Π^0_1-0–n-tree and is by assumption infinite.

Via the equivalence of 0–n-trees and 0/1-trees and of Π^0_1-WKL and WKL (see [86]), weak König's lemma yields a infinite branch g solving the lemma. $\square$

Lemma 3.2 (and definition, Π^0_1-class, [51])**.** *A Π^0_1-class $\mathcal{A}$ of 2^ω is a set of functions of the form*

$$\mathcal{A} = \{f \in 2^\omega \mid \forall n\, A(\bar{f}n)\},$$

where A is a quantifier-free formula.

WKL *proves that a Π^0_1-class $\mathcal{A}$ is not empty if*

$$\forall n^0\, \exists s \in 2^n\, \forall s' \sqsubseteq s\ A(s'). \tag{3.1}$$

(The definition of Π^0_1-class induces an infinite tree in which every $f \in \mathcal{A}$ codes an infinite path through it.) The statement (3.1) *is equivalent to a Π^0_1-statement.*

Note that one may also allow A to be a Π^0_1-formula as the $\forall$-quantifier can be coded into the quantification over n (see for instance [86]).

Remark 3.3 (Treatment of Π^0_1-0/1-trees)*.* Let $T(w) \equiv (\forall k\, T_{qf}(w,k) = 0)$ be a Π^0_1-predicate. Using the UWKL functional $\mathcal{B}$ one can define the functional

$$\mathcal{B}_{\Pi^0_1}(T_{qf}) := \mathcal{B}\left(\min_{w' \sqsubseteq w, k \leq \mathrm{lth}\, w} T_{qf}(w',k)\right)$$

that yields an infinite branch of T, if T defines an infinite 0/1-tree.

Furthermore, an instance of Π^0_1-CA decides whether the tree T is infinite, since

$$\forall n\, \exists w \in 2^n\, \forall k\, T_{qf}(w,k)$$

is equivalent a Π^0_1-statement (over $\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF\text{-}AC}$).

Hence one can treat Π^0_1-0/1-trees mostly like quantifier free trees.

Proposition 3.4.

$$\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF\text{-}AC} \vdash \forall c\colon \mathbb{N}\times\mathbb{N} \to 2\ \left(\Pi^0_1\text{-}\mathsf{CA}(\xi c) \to \exists H\, \mathsf{SRT}^2_2(c,H)\right),$$

where ξ is a suitable term.

Proof. Assume that the coloring c is stable. Define for $i < 2$

$$A_i := \{x \mid \forall k\, \exists y \geq k\ \ c(x,y) = i\}.$$

By stability

$$A_i\ = \{x \mid \exists k\, \forall y \geq k\ \ c(x,y) = i\}.$$

Hence each A_i is a Δ^0_2-set.

At least for one i the set A_i is infinite (by RT^1_2). Fix such an i. With an instance of Π^0_1-CP we obtain strong stability, i.e.

$$\forall x\, \exists k\, \forall y > k\, \forall x' \leq x\ \ c(x',k) = c(x',y).$$

This instance of Π^0_1-CP follows from a suitable instance of Π^0_1-CA, see Lemma 1.10.(iv).

Together with the infinity of A_i we get

$$\forall x\, \exists k \in A_i\, \forall x' \leq x\ \left(x' \in A_i \to c(x', k) = i\right).$$

Define the set H inductively by

$$x \in H \quad \text{iff} \quad x \in A_i \text{ and } c(x', x) = i \text{ for all } x' < x \text{ with } x' \in H.$$

This definition only uses bounded course-of-value recursion in the characteristic function of A_i which can be obtained from a suitable instance of Π^0_1-CA, see Lemma 1.10.(ii). (The characteristic function χ_H of H is clearly bounded and hence also its course-of-value function $\overline{\chi_H}$, which is actually defined in the recursion.)

The set H is clearly infinite and homogeneous. (The two instances of Π^0_1-CA can be coded into one term ξ, see remark 1.9.) $\square$

Proposition 3.5. *Let $\varphi c H$ be a term that is provably continuous in H, where the function $\alpha_{\varphi c}(\cdot, n, k)$ is an associate for $\lambda H.\varphi(c, H, n, k)$. Then there exists a term ξ, such that*

$$\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction + \mathsf{QF\text{-}AC} \oplus (\mathcal{B}) \oplus (\tilde{R}_1) \vdash \\ \forall c \colon \mathbb{N} \times \mathbb{N} \to 2\ \Big(\Pi^0_1\text{-}\mathsf{CA}(\xi c) \to \exists H\, \mathsf{SRT}^2_2(c, H) \wedge \Pi^0_1\text{-}\mathsf{CA}(\varphi c H)\Big).$$

If $\varphi c H$ is moreover provably continuous in c the term ξ can be chosen such that it is provably continuous.

Sketch of proof. We assume that each A_i is unbounded, otherwise we are done.

We will build a set G such that $G \cap A_0$ and $G \cap A_1$ are infinite, homogeneous and at least for one $i < 2$ the comprehension Π^0_1-$\mathsf{CA}(\varphi c(G \cap A_i))$ is decided. The set $H := G \cap A_i$ then solves this proposition.

We will construct the set G in steps such that at each step n we will assure that

$$|G \cap A_i| \geq n \quad \text{for every } i < 2$$

and for some $i < 2$ the comprehension for $G \cap A_i$ at the position $(n)_i$ will be decided, i.e. whether the statement

$$\forall k\, (\varphi c(G \cap A_i)(n)_i)k = 0 \tag{3.2}$$

holds. More precisely, we will construct functions $I, J \colon \mathbb{N} \to 2$, such that

$$\exists I, J\, \forall n\ \Big(\forall k\, (\varphi c(G \cap A_{I(n)})(n)_{I(n)})k = 0 \leftrightarrow J(n) = 0\Big).$$

With these functions we can then obtain a comprehension function for one of the sets $G \cap A_i$, because either

$$\forall m \, \exists n \, \Big(m = (n)_{I(n)} \wedge I(n) = 0 \Big) \tag{3.3}$$

and then $J(N(m))$, where $N(m)$ is some choice function for n obtained by $\mathsf{QF\text{-}AC}$, decides the comprehension for $G \cap A_0$ or

$$\exists m \, \forall n \, \Big(m \neq (n)_{I(n)} \vee I(n) = 1 \Big) . \tag{3.4}$$

By choosing $n = \langle m, m' \rangle$ we obtain $\forall m' \, I(\langle m, m' \rangle) = 1$ and therefore the function $\lambda m'.J(\langle m, m' \rangle)$ decides the comprehension for $G \cap A_1$.

The set G and the functions I, J will be constructed by recursion. We will first give a sketch of the argument and later show that R_0 and the imposed comprehension suffice for the construction.

By induction we construct (d_n, L_n), such that the sequence (d_n) is an ascending sequence of finite sets and (L_n) is a descending sequence of infinite sets of possible candidates to extend d_n (i.e. $d_{n+1} \setminus d_n \subset L_n$ and $\min(L_n)$ is greater than the stability point of d_n). Each set L_n is *low*, in the sense that it can be described by a term containing $\mathcal{B}$ and $\tilde{R}_1$. The set G will be given by $\bigcup_n d_n$.

We start with $(\emptyset, \mathbb{N})$. Assume (d_n, L_n) is already defined. We distinguish two cases:

Case i) A partition Z_0 and Z_1 of L_n exists such that

$$\forall z \subseteq^{fin} Z_i \, \Big(z \text{ is } i\text{-homogeneous} \rightarrow \forall k \;\, \alpha_{\varphi c}(d_n^i \cup z)(n)_i k \leq 1 \Big) , \tag{3.5}$$

where $d_n^i = d_n \cap A_i$, holds for all $i < 2$. (If we extend the initial segment d_n with elements from Z_i the comprehension remains true.)

At least one of Z_0 and Z_1 is infinite because L_n is infinite. We take this set as L_{n+1}, forcing (3.2) to be true for this i on all further extensions and let $d_{n+1} := d_n$.

Case ii) No partition satisfying (3.5) exists.
We know then that especially $L_n \cap A_0$ and $L_n \cap A_1$ is no such partition. So we can find for one i a finite i-homogeneous set $d' \subseteq^{fin} A_i$ such that

$$\exists k \;\, \alpha_{\varphi c}(d_n^i \cup d')(n)_i k > 1.$$

Setting $d_{n+1} := d_n \cup d'$ and $L_{n+1} := \{x \in L_n \mid x > \max d'\}$ forces the comprehension function to be $\neq 0$ at $(n)_i$.

Note that (3.5) defines a Π^0_1-class of 2^ω. (We view here a partition of $\mathbb{N}$ into two sets Z_0, Z_1 as a function $f \in 2^\omega$ with $f(n) = i$ iff $n \in Z_i$.) Thus we may assume that the Z_i are low and we can decide which case holds by asking if a certain 0/1-tree is infinite (this is a Π^0_1-statement).

The size requirements are met by extending d_{n+1} with suitable elements of L_n.

The set $G := \bigcup_n d_n$ then satisfies the proposition. □

Proof. Define

$$
\begin{aligned}
L^{\langle\rangle}(w) &:= 0, \\
L^{\langle x_0,\dots,x_{n-1},(d,k,y)\rangle}(w) &:= \begin{cases} 1 & \text{if } w \leq y, \\ sg\left|\mathcal{B}_{\Pi^0_1}(\theta(L^{\langle x_0,\dots,x_{n-1}\rangle}, d)) - (k-1)\right| & \text{if } k \geq 1 \text{ and } w > y, \\ L^{\langle x_0,\dots,x_{n-1}\rangle} w & \text{if } k = 0 \text{ and } w > y. \end{cases}
\end{aligned}
\tag{3.6}
$$

(d is just an auxiliary parameter used to build the tree, it will be set to d_{n-1} defined below; k denotes the case, $k = 0$ for case ii), $k \geq 1$ for case i) and Z_{k-1} infinite in the sketch; y is a lower bound for L.)

Here $\theta(B, d^0, d^1)wk$ will be the characteristic function of the predicate

$$
\begin{aligned}
&\forall i < 2 \;\; \forall y \subseteq^{fin} B \cap \{x < \mathrm{lth}(w) \mid (w)_x = i\} \\
&\qquad\qquad \Big(y \text{ is } i\text{-homogeneous} \rightarrow \alpha_{\varphi c}(d^i \cup y)(n)_i k \leq 1\Big),
\end{aligned}
\tag{3.7}
$$

where the variables w, y are numerals coding finite sets. The statement

$$T_{B,d^0,d^1}(w) :\equiv \forall k\, \theta(B, d^0, d^1)wk = 0$$

defines the Π^0_1-0/1-tree build in (3.5) in the sketch.

We will write $T_{B,d}$ and $\theta(B,d)wk$ for T_{B,d^0,d^1} resp. $\theta(B, d \cap A_0, d \cap A_1)wk$. This will not lead to problems because $d \cap A_i$ is just a number computable from d relative to the imposed instance of comprehension. Note that L^x can be defined in $\mathcal{B}$ and θ using the bounded iterator $\tilde{R}_1$. Thus the function L^x can be described by a term in this system.

We assume that for all x and i the set $L^x \cap A_i$ is infinite if L^x is infinite. Otherwise the set $L^x \cap [k, \infty]$ for a suitable k would be an infinite subset of A_{1-i} and therefore solve the proposition.

Using this and an instance of Δ^0_2-comprehension (over L) we generate functions g_i such that

$$g_i(x) := \min(L^x \cap A_i). \tag{3.8}$$

With an application of an instance of Π^0_1-AC and taking a maximum we obtain a function $h(\langle x_1, \dots, x_n\rangle)$ giving a common stability point of $x_1, \dots, x_n$.

We now define (d_n, l_n) by recursion. (L^{l_n} should match L_n from sketch above.) We use primitive recursion in the sense of Kleene, i.e. the recursion can be defined with the recursor R_0.

Let

$$d_0 := \langle\rangle \quad \text{and} \quad l_0 := \langle\rangle.$$

For the recursion step we distinguish the cases:

Case i) The tree $T_{L^{l_n},d_n}(w)$ is infinite, i.e.

$$\forall m\, \exists w \in 2^m\, \forall k\ \theta(L^{l_n}, d_n)wk = 0.$$

By RT^1_2 there is at least one $j < 2$ such that $\{x \in \mathbb{N} \mid \mathcal{B}_{\Pi^0_1}(\theta(L^{l_n}, d_n))x = j\}$ is infinite. An index j can be chosen constructively relative to Σ^0_1-WKL, see Lemma 3.1. Set

$$d'_{n+1} := d_n \quad \text{and} \quad k'_{n+1} := j + 1.$$

Case ii) The tree $T_{L^{l_n},d_n}(w)$ is finite, i.e.

$$\exists m\, \forall w \in 2^m\, \exists k\ \theta(L^{l_n}, d_n)wk \neq 0.$$

Then especially the set A_0 does not code a path through the tree, i.e. for this m

$$\exists k\, \theta(L^{l_n}, d_n)(\overline{\chi_{A_0}}m)k \neq 0,$$

where χ_{A_0} is the characteristic function of A_0. So there is an i and a finite i-homogeneous set $y \subseteq^{fin} A_i \cap \{0, \dots, m-1\} \cap L^{l_n}$ such that

$$\exists k\, \alpha_{\varphi c}(d^i \cup y)(n)_i k > 1.$$

Set

$$d'_{n+1} := d \cup y \quad \text{and} \quad k'_{n+1} := 0.$$

Note that this case distinction is constructive relative to the given instance of comprehension (the second quantifier of the formula is bounded).

Now we extend d'_{n+1} with suitable elements, such that the size requirements are met:

$$\begin{aligned} d_{n+1} &:= d_n \cup \bigcup_{i<2} \{g_i(l_n * \langle d_n, l', h(d'_{n+1}) + 1\rangle)\} \\ l_{n+1} &:= l_n * \langle d_n, k'_{n+1}, h(d_{n+1}) + 1\rangle \end{aligned}$$

Applying RT^1_2 yields an i such that all comprehension instances are decided. From the d_n and the given comprehension one can easily obtain a enumeration of the set $G \cap A_i =: H$.

This solves the proposition. The term ξc is continuous in c because the only discontinuous functional in this system is $\mathcal{B}$ but it is only used to define L^x and to prove WKL. Hence ξ can be chosen such that c does not occur as a parameter to $\mathcal{B}$. More precisely ξc is of the form $\xi'[t_1 c, \lambda x.L^x]$ with $\xi', t \in T_0$ and therefore continuous. $\square$

Proposition 3.6. *Let φcH be a term that is provably continuous in H and let α_{φ_c} be as in Proposition 3.5. Then there exists a term ξ such that*

$$\widehat{\mathsf{WE\text{-}PA}}^\omega{\restriction} + \Sigma^0_2\text{-}\mathsf{IA} + \mathsf{QF\text{-}AC} \oplus (\mathcal{B}) \oplus (\tilde{R}_1) \vdash \\ \forall c\colon \mathbb{N} \times \mathbb{N} \to n \left(\Pi^0_1\text{-}\mathsf{CA}(\xi c) \to \exists H\, \mathsf{SRT}^2_{<\infty}(c, H) \wedge \Pi^0_1\text{-}\mathsf{CA}(\varphi cH)\right).$$

If φ is moreover provably continuous in c the term ξ can be chosen such that it is provably continuous in c.

Proof. Analogous to Proposition 3.5.

The applications of RT^1_2 become applications of $\mathsf{RT}^1_{<\infty}$, which is equivalent to Π^0_1-CP and thus provable using Σ^0_2-IA. The 0/1-trees will become 0–n-trees; but these trees can be constructively transformed into 0/1-trees, see [86].

The only difficult part is adopt the assumption that

$$\forall x\, \forall i < n\ (L^x \text{ infinite} \to L^x \cap A_i \text{ infinite}), \tag{3.9}$$

which leads to the definition of g_i in (3.8) because we cannot simply deduce the existence of a solution from the failure of (3.9).

First note that (3.9) due to the minimal element parameter (y in (3.6)) is equivalent to

$$\forall x\, \forall i < n\ (L^x \text{ infinite} \to L^x \cap A_i \text{ not empty}). \tag{3.10}$$

If (3.9) resp. (3.10) does not hold, our goal is to find a set L^x on which — provided we neglect colors that do not occur — the assumption holds. This can be done by finding a maximal set $K \subseteq n$, such that there is an x with $L^x \cap \bigcup_{k \in K} A_k$ is empty. Then for all $x' \sqsupseteq x$ and $i \notin K$ the sets $A_i \cup L^x$ are not empty. Thus if we relativize our argumentation to L^x and the colors $n \setminus K$ the condition (3.9) holds.

To find such a K and x define

$$\eta(\langle s_0, \dots, s_{n-1}\rangle) :\equiv \exists x \left(L^x \text{ infinite} \wedge \bigwedge_i (s_i = 0 \to L^x \cap A_i = \emptyset)\right).$$

η is clearly Σ^0_3. Finding a minimal tuple $\langle s_0, \dots, s_{n-1}\rangle$ satisfying η yields a suitable solution. A minimal tuple can be obtained using an instance of Σ^0_3-induction, which is provable from Σ^0_2-IA and an instance of Π^0_1-comprehension. $\square$

Corollary 3.7. *Let $\varphi c H$ be a term that is provably continuous in H. Then there exists a term ξ such that*

$$\begin{aligned}&\widehat{\mathsf{WE\text{-}PA}}^\omega{\restriction} + \mathsf{QF\text{-}AC} \oplus (\mathcal{B}) \oplus (\tilde{R}_1) \vdash \\ &\qquad \forall c\colon \mathbb{N}\times\mathbb{N} \to n\ \Big(\Pi^0_1\text{-}\mathsf{CA}(\xi c) \to \exists H\, \mathsf{RT}^2_2(c,H) \wedge \Pi^0_1\text{-}\mathsf{CA}(\varphi c H)\Big).\end{aligned} \tag{3.11}$$

The term ξ can be chosen such that c does not occur as a subterm of a parameter of $\mathcal{B}$.

If Σ^0_2-IA is added to the system, RT^2_2 may be replaced by $\mathsf{RT}^2_{<\infty}$.

Hence RT^2_2 is proofwise low over $\widehat{\mathsf{WE\text{-}PA}}^\omega{\restriction} + \mathsf{QF\text{-}AC} \oplus (\mathcal{B}) \oplus (\tilde{R}_1)$ and $\mathsf{RT}^2_{<\infty}$ is proofwise low over $\widehat{\mathsf{WE\text{-}PA}}^\omega{\restriction} + \mathsf{QF\text{-}AC} \oplus (\mathcal{B}) \oplus (\tilde{R}_1) + \Sigma^0_2$-$\mathsf{IA}$.

Proof. Let $R_i = \{x \in \mathbb{N} \mid c(i,x) = 0\}$ and let g be a strictly increasing enumeration of a cohesive set for R_i. The coloring $c'(x,y) := c(gx, gy)$ is stable and for each homogeneous set H' of c' the set gH' is homogeneous for c. See [16].

Hence the corollary follows from Corollary 2.5 and Proposition 3.5 resp. Proposition 3.6. □

3.2. ND-Interpretation

Later we want to use the previous theorem to interpret the usages of RT^2_2 in proofs of $\forall\exists$-statements. The problem is that with Corollary 3.7 we can only interpret applications of $\mathcal{R}$ to fixed closed terms.

The principle Π^0_2-LEM is needed in the proof of Proposition 3.5 to show either (3.3) or (3.4) holds. But if we go to the functional interpretation (i.e. ND-interpretation) the need for Π^0_2-LEM vanishes and we can interpret the solutions to the functional interpretation if it is applied to terms that *may contain free variable of type* 1. By Remark 2.22 this suffices.

For notational simplification we sometimes will not apply the last application of $\mathsf{QF\text{-}AC}$ to the ND-interpretation. This corresponds to the so-called Shoenfield translation, see [91]. For RT^2_2 we use the formalization

$$\mathsf{RT}^2_2 :\equiv \forall c\colon [\mathbb{N}]^2 \to 2\, \exists H\, \forall u < v\ c(Hu, Hv) = c(H0, H1).$$

The ND-interpretation then yields

$$\mathsf{RT}^{2\,ND}_2 :\equiv \forall c\colon [\mathbb{N}]^2 \to 2\, \forall U < V\, \exists H\ \underbrace{c(H(UH), H(VH)) = c(H0, H1)}_{\equiv: \mathsf{RT}^2_{2ND}(H; c, U, V)}. \tag{3.12}$$

Here the set H is given as an enumeration, i.e. H is strictly monotone and Hn is the n-th element of H, and $U < V$ is defined pointwise.[1] Sometimes the parameters c, U, V in $\mathsf{RT}^2_{2ND}(H; c, U, V)$ will be coded into a single parameter.

For the ND-interpretation of Π^0_1-comprehension we use an ε-calculus like formulation:

$$\Pi^0_1\text{-}\widehat{\mathsf{CA}}(\varphi) :\equiv \exists f\, \forall x, y\, \underbrace{(\varphi(x, f(x)) = 0 \vee \varphi(x, y) \neq 0)}_{\equiv: \left(\Pi^0_1\text{-}\widehat{\mathsf{CA}}(\varphi)\right)_{QF}(f,x,y)}. \tag{3.13}$$

This leads to following ND-interpretation (modulo a last application of $\mathsf{QF\text{-}AC}$)

$$\left(\Pi^0_1\text{-}\widehat{\mathsf{CA}}(\varphi)\right)^{ND} \equiv \forall X, Y\, \exists f\, \left(\varphi(Xf, f(Xf) = 0) \vee \varphi(Xf, Yf) \neq 0\right).$$

Because RT^2_2 and Π^0_1-$\widehat{\mathsf{CA}}(\varphi)$ are only $\forall\exists\forall$-statements, the ND-interpretation coincides with the no-counterexample interpretation. So one might view a solution to RT^{2ND}_2, i.e. a term $t(c, U, V)$ that yields for every c, U, V a set H that may not be homogeneous in total but for which $c(H0, H1) = c(H(UH), H(VH))$ holds, as a procedure that disproves every possible counterexample to RT^2_2. Same for Π^0_1-$\widehat{\mathsf{CA}}(\varphi)$.

Proposition 3.8 ([90], [67, 82])**.** *The solution to $\left(\Pi^0_1\text{-}\widehat{\mathsf{CA}}(\varphi)\right)^{ND}$ can be defined with a single use of Φ_0, this is Spector's bar recursor for type* 0*:*

$$t_f := \Phi_0 X u 0(\lambda k^0.0), \qquad unv := \begin{cases} 1 & \textit{if } \varphi(n, Y(v1)), \\ Y(v1) & \textit{otherwise.} \end{cases}$$

The bar recursor Φ_0 is defined as in [67]. It is primitive recursively and instance-wise definable in the bar recursor $B_{0,1}$, see definition 3.15 below.

The statement from Corollary 3.7 spelled out is

$$\begin{aligned}
&\widehat{\mathsf{WE\text{-}PA}}^{\omega}\!\restriction + \mathsf{QF\text{-}AC} \oplus (\mathcal{B}) \oplus (\tilde{R}_1) \vdash \\
&\qquad\qquad \forall c \Big(\exists f_\xi\, \forall x_\xi, y_\xi\, \left(\Pi^0_1\text{-}\widehat{\mathsf{CA}}(\xi c)\right)_{QF}(f_\xi, x_\xi, y_\xi) \to \\
&\qquad\qquad\quad \exists H \Big(\forall u < v\, c(Hu, Hv) = c(H0, H1) \\
&\qquad\qquad\qquad\qquad \wedge \exists f_\varphi\, \forall x_\varphi, y_\varphi\, \left(\Pi^0_1\text{-}\widehat{\mathsf{CA}}(\varphi c H)\right)_{QF}(f_\varphi, x_\varphi, y_\varphi)\Big)\Big).
\end{aligned}$$

An ND-interpretation leads then to

[1] Officially, quantification over functions like $c\colon [\mathbb{N}]^2 \to 2$ or strictly monotone increasing functions like H are not included in our system as primitive notions, but we can enforce the same behavior by quantifying over $c\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $H\colon \mathbb{N} \to \mathbb{N}$ and replacing every occurrence of c, H with

$$\tilde{c}(x, y) := \min\left(1, \begin{cases} c(x, y) & \text{if } x < y \\ c(y, x) & \text{otherwise} \end{cases}\right), \qquad \tilde{H}(x) := x + \Sigma_{k \leq x} H(k).$$

Theorem 3.9 (ND-interpretation of Corollary 3.7)**.** *For every provably continuous (in c, H) term $\varphi \in T_0[\mathcal{B}, \tilde{R}_1]$ a term $\xi \in T_0[\mathcal{B}, \tilde{R}_1]$ (that is continuous in c) exists such that*

$$\begin{aligned} \widehat{\mathsf{WE\text{-}HA}}^{\omega}\!\restriction \oplus (\mathcal{B}) \oplus (\tilde{R}_1) \vdash \forall c\, \forall f_\xi\, \forall U < V\, \forall X_\varphi, Y_\varphi\, \exists x_\xi, y_\xi\, \exists H\, \exists f_\varphi \\ \Big((\Pi^0_1\text{-}\widehat{\mathsf{CA}}(\xi c))_{QF}(f_\xi, x_\xi, y_\xi) \to (c(H(UHf_\varphi), H(VHf_\varphi)) = c(H0, H1) \\ \wedge\, \Pi^0_1\text{-}\widehat{\mathsf{CA}}(\varphi c H))_{QF}(f_\varphi, X_\varphi H f_\varphi, Y_\varphi H f_\varphi)\Big). \end{aligned} \tag{3.14}$$

Moreover, there exist terms $t_{x_\xi}, t_{y_\xi}, t_H, t_{f_\varphi} \in T_0[\mathcal{B}, \tilde{R}_1]$ (with the given parameters) satisfying this formula.

Proof. The system $\widehat{\mathsf{WE\text{-}PA}}^{\omega}\!\restriction + \mathsf{QF\text{-}AC}$ has an ND-interpretation into $\widehat{\mathsf{WE\text{-}HA}}^{\omega}\!\restriction$. This also extends to additions of new constants and universal axioms. See e.g. [7, 67]. ☐

The term t_H and t_{f_φ} can be seen as procedures transforming the no-counterexample interpretation of the premise to the no-counterexample interpretation of the conclusion; the terms t_{x_ξ} and t_{y_ξ} yield which instance of the premise is needed to prove the conclusion.

Note that the counter-functions of RT^2_2 and Π^0_1-$\widehat{\mathsf{CA}}$ have access to both t_H and t_{f_φ}. The proof of Proposition 3.19 bellow will use this.

To show that the no-counterexample interpretation of the conclusion (and hence the conclusion) holds we have to provide an f_ξ that satisfies $(\Pi^0_1\text{-}\widehat{\mathsf{CA}}(\xi c))_{QF}(f_\xi, t_{x_\xi}, t_{y_\xi})$. This can be done using $B_{0,1}$, see Proposition 3.8.

Note that here the application of $(\Pi^0_1\text{-}\widehat{\mathsf{CA}}(\varphi))^{ND}$ in the premise is not fully interpreted. We obtain this form by applying logical simplifications after the negative translation. This leads to fixed terms in the second and third parameter of the premise and will reduce the need for the bar recursor $B_{0,1}$ to the rule of $B_{0,1}$.

3.3. Majorizing the bar recursor

Definition 3.10 (bar induction of type 0)**.** Let bar induction of type 0 be

$$(\mathsf{BI}_0)\colon \begin{cases} \forall x^1\, \exists n_0^0\, \forall n \geq n_0\, Q(\overline{x,n}; n) \,\wedge \\ \forall x^1, n^0\ (\forall d\, Q(\overline{x,n} * d; n+1) \to Q(\overline{x,n}; n)) \\ \to \forall x^1, n^0\, Q(\overline{x,n}; n), \end{cases}$$

where

$$(\overline{x,n})k := \begin{cases} x(k), & \text{if } k < n, \\ 0, & \text{otherwise,} \end{cases} \qquad (\overline{x,n} * d)k := \begin{cases} x(k), & \text{if } k < n, \\ d, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

If Q is restricted to formulas in $\mathcal{K}$, we write $\mathcal{K}$-BI_0.

Lemma 3.11.

$$\widehat{\mathsf{WE}\text{-}\mathsf{PA}}^{\omega}\restriction + \mathsf{QF}\text{-}\mathsf{AC}^{0,0} \vdash \Pi^0_1\text{-}\mathsf{BI}_0$$

Proof. Let $Q(\overline{x,n};n) \equiv \forall k\, Q_{qf}(\overline{x,n};n;k)$. Suppose that Π^0_1-BI_0 does not hold, i.e. the premises of Π^0_1-BI_0 are true and

$$\exists x^1_0, n^0_0\, \neg\forall k^0_0\, Q_{qf}(\overline{x_0,n_0};n_0;k_0),$$

which is equivalent to

$$\exists x^1_0, n^0_0, k^0_0\, \neg Q_{qf}(\overline{x_0,n_0};n_0;k_0). \tag{3.15}$$

The second premise yields

$$\forall x^1, n^0, k^0\, \exists d, k'\ \left(\neg Q_{qf}(\overline{x,n};n;k) \to \neg Q_{qf}(\overline{x,n} * d; n+1; k')\right).$$

Since the whole statement only depends on an initial segment of x^1, it can be coded in a type 0 object x'^0. For instance let $x' := \bar{x}n$ then $\overline{\lambda i.(x')_i, n} = \overline{x,n}$.

Using $\mathsf{QF}\text{-}\mathsf{AC}^{0,0}$ we then obtain functions $D(x,n,k)$, $K(x,n,k)$ with

$$\forall x^0, n, k\ \left(\neg Q_{qf}(\overline{\lambda i.(x)_i, n};n;k) \to \neg Q_{qf}(\overline{\lambda i.(x)_i, n} * D(x,n,k); n+1; K(x,n,k)\right). \tag{3.16}$$

Then define using simultaneous course-of-value recursion (n_0, x_0, k_0 are from (3.15)) the functions D_0, K_0:

$$\left.\begin{aligned} D_0(n) &:= x_0(n) \\ K_0(n) &:= k_0 \end{aligned}\right\} \quad \text{for } n \le n_0,$$

$$\left.\begin{aligned} D_0(n) &:= D(\overline{D_0,n}, n, K_0(n-1)) \\ K_0(n) &:= K(\overline{D_0,n}, n, K_0(n-1)) \end{aligned}\right\} \quad \text{for } n > n_0.$$

The definition of D_0 and (3.15),(3.16) yield

$$\forall n \ge n_0\, \neg Q(\overline{D_0,n};n)$$

and hence a contradiction to the first premise of Π^0_1-BI_0. □

Proposition 3.12. $\widehat{\mathsf{WE}\text{-}\mathsf{PA}}^{\omega}\restriction + \mathsf{QF}\text{-}\mathsf{AC}^{0,0}$ *proves that there exists a majorant* $B^*_{0,1}$ *of* $B_{0,1}$.

Proof. Define $B^*_{0,1}$ like in [67, proof of Theorem 11.17]. By the cited proof it suffices to show Π^0_1-BI_0. (Note that in that proof Q is a Π^0_1 formula in the case where $\rho = 0$.) Hence the proposition is an immediate consequence of Lemma 3.11. See also [11]. □

3.4. Ordinal analysis of terms

Ordinal Peano/Heyting arithmetic

In this section we will investigate the strength of induction along ordinals the systems $\widehat{\mathsf{WE\text{-}HA}}^\omega\restriction$, $\widehat{\mathsf{WE\text{-}PA}}^\omega\restriction$.

We will code ordinals using the ordinal coding of [37, II.3.a]. (This coding uses the Cantor normal form for ordinals to define primitive recursive codes for ordinals.) For convenience we repeat the definition of ω_k^μ:

$$\omega_0^\mu = \mu \quad \text{and} \quad \omega_{k+1}^\mu = \omega^{\omega_k^\mu}$$

Here $k \in \mathbb{N}$ and μ is an arbitrary ordinal number.

Theorem 3.13 ([80], [98]). *The functions and functionals of level 2 that are ordinal recursive (unnested) in an ordering $< \omega_{k+1}^\omega$ are exactly the functions and functionals in T_k.*

Theorem 3.14 ([37, II.3.18]).

$$\widehat{\mathsf{WE\text{-}HA}}^\omega\restriction + \Sigma^0_{m+k-1}\text{-}\mathsf{IA} \vdash \Sigma^0_m\text{-}\mathsf{LNP}(\omega_k^\omega)$$

for every $m, k \in \mathbb{N}$, where LNP denotes the least number principle.
In particular, $\widehat{\mathsf{WE\text{-}PA}}^\omega\restriction + \Sigma^0_{k+1}\text{-}\mathsf{IA}$ proves the totality of $< \omega_{k+1}^\omega$-recursive functionals of type ≤ 2.

Proof. See [37, II.3.18] and [80]. $\square$

Application to bar recursion

Our goal is now to use the equivalences between ordinal induction and Σ^0_k-induction and an ordinal analysis of bar recursion to establish conservation results of bar recursion over induction along ω.

Definition 3.15 (Howard's bar recursor). Define the bar recursor $B_{\rho,\tau}$ as

$$B_{\rho,\tau}AFGt :=_\tau \begin{cases} Gt, & \text{if } A[t] < \operatorname{lth} t, \\ Ft(\lambda u^\rho . B_{\rho,\tau}AFG(t * u)), & \text{otherwise,} \end{cases}$$

where $[t] := \lambda x.(t)_x$.

Definition 3.16 (restricted bar recursor).

$$\Phi'_\tau AFGt :=_\tau \begin{cases} Gt, & \text{if } A[t] < \operatorname{lth} t, \\ Ft(\Phi'_\tau AFG(t * 0))(\Phi'_\tau AFG(t * 1)), & \text{otherwise.} \end{cases}$$

The bar recursor Φ'_0 can be used to solve the functional interpretation of WKL, see [44]. (Φ'_τ is the restricted bar recursor schema 1 from there.)

We call a term *semi-closed* if it contains only variables of degree ≤ 1 free. Howard introduced the notion of computational size for semi-closed terms, see [43, 44]. Roughly speaking the computation size of a semi-closed term of type 0 is an upper bound on the number of term reductions on has to apply to obtain a numeral. The computational size of a degree 1 term is the computational size of $t(H_0, \dots, H_n)$, where H_i are fresh variables such that the terms is of type 0.

Theorem 3.17 ([44, 2.2, 2.3])**.** *Let $\Phi'_0 AFGc$ resp. $B_{0,1}AFGc$ be a semi-closed term and let A, F, G have the computational sizes a, f, g then*

(i) *$\Phi'_0 AFGc$ has computational size $\sigma := (f + g + h)\omega + \omega(h + 1)$, where $h := \omega a + \omega$ and,*

(ii) *$B_{0,1}AFGc$ has computational size $\sigma := \omega^{g+f2h}$, where $h := \omega a + \omega$.*

This equivalence can be proven in Σ^0_1-$\mathsf{LNP}(\sigma)$.

Proof. See the proofs in [44, 2.2, 2.3]. Note that these proofs actually define a counting function for the computation-tree through transfinite recursion. This recursion is essentially a transfinite primitive recursion over σ. Hence this proof may be carried out in Σ^0_1-$\mathsf{LNP}(\sigma)$. $\square$

Remark 3.18. If we apply the rule of bar recursion to semi-closed, primitive recursive terms (in the sense of Kleene, i.e. terms of computation size ω^n for $n \in \omega$) we obtain a term with computation size $< \omega^{m\omega}$ for an $m \in \omega$ and therefore a term that is provably definable already in $\widehat{\mathsf{WE\text{-}HA}}^\omega\restriction_{\omega_2^l}$ for an $l \in \omega$ or in $\widehat{\mathsf{WE\text{-}HA}}^\omega\restriction + \Sigma^0_2$-$\mathsf{IA}$. We can carried out the proof of the equivalence, Theorem 3.17, in the same system, see Theorem 3.13. Hence in each of these systems we can also proof the equivalence of both terms.

If we apply the rule of restricted bar recursion to primitive recursive terms, which contain only free variable of type 0, we even end up with a primitive recursive term.

3.5. Application to Ramsey's theorem

Proposition 3.19. *Let $t^1[g]$ be a term such that $\lambda g.t^1[g] \in T_0[\mathcal{R}]$, where $\mathcal{R}$ is a functional solving $\mathsf{RT}^{2\,ND}_2$, and every occurrence of $\mathcal{R}$ is of the form*

$$\mathcal{R}(t_c[g], t_u[g], t_v[g]).$$

Then there exist terms $t_x, t_y, \xi \in T_0[\tilde{R}_1, \mathcal{B}]$, such that one can inductively replace every occurrence of $\mathcal{R}$ in t with a new term

$$r(f, g; \tilde{t}_c[g], \tilde{t}_u[g], \tilde{t}_v[g])$$

(here r is a term and $\tilde{t}_c[g], \tilde{t}_u[g], \tilde{t}_v[g]$ are the results of replacing $\mathcal{R}$ in $t_c[g], t_u[g], t_v[g]$), such that

$$\widehat{\mathsf{WE}\text{-}\mathsf{HA}}^\omega\restriction + \mathsf{QF}\text{-}\mathsf{AC} \oplus (\mathcal{B}) \oplus (\tilde{R}_1) \vdash \forall g^1, f\ (\Pi^0_1\text{-}\widehat{\mathsf{CA}}(\xi g))_{QF}(f, t_x g, t_y g) \\ \to \mathsf{RT}^2_{2ND}(r(f, g; \tilde{t}_c[g], \tilde{t}_u[g], \tilde{t}_v[g]);\ \tilde{t}_c[g], \tilde{t}_u[g], \tilde{t}_v[g]).$$

The formula RT^2_{2ND} denotes the quantifier-free part of RT^{2ND}_2, see (3.12) on p. 60.

Proof. We use Theorem 3.9 to inductively interpret the term t. For convenience we repeat (3.14), the existential quantified variables are replaced by their realizing terms constructed in that theorem:

$$\widehat{\mathsf{WE}\text{-}\mathsf{HA}}^\omega\restriction \oplus (\mathcal{B}) \oplus (\tilde{R}_1) \vdash \forall c\, \forall f_\xi\, \forall U < V\, \forall X_\varphi, Y_\varphi \\ \Big((\Pi^0_1\text{-}\widehat{\mathsf{CA}}(\xi c))_{QF}(f_\xi, t_{x_\xi}, t_{y_\xi}) \to c(t_H(U t_H t_{f_\varphi}), t_H(V t_H t_{f_\varphi})) = c(t_H 0, t_H 1) \\ \wedge\ (\Pi^0_1\text{-}\widehat{\mathsf{CA}}(\varphi c t_H))_{QF}(t_{f_\varphi}, X_\varphi(t_H t_{f_\varphi}), Y_\varphi(t_H t_{f_\varphi}))\Big) \quad (3.17)$$

It is clear that in case of $t_c, t_u, t_v \in T_0$, i.e. there are no nested applications of $\mathcal{R}$, every application of $\mathcal{R}$ in the term t can be interpreted using (3.17). (Just set $c = t_c$, $U = \lambda f_\varphi.t_u$, $V = \lambda f_\varphi.t_v$ and the others variable to 0.) Using contraction of Π^0_1-comprehension, see Remark 1.9, a term containing multiple such occurrence of $\mathcal{R}$ can be interpreted.

If the term t_c contains a single occurrence of $\mathcal{R}$ then we first interpret this inner $\mathcal{R}$ but now we will take advantage of φ and set $\varphi, X_\varphi, Y_\varphi$ so that the resulting instance of ND-comprehension suffices to interpret the outer occurrence of $\mathcal{R}$ in t.

Iterating this process allows us to interpret all terms $t \in T_0[\mathcal{R}]$ where every occurrence of $\mathcal{R}$ is of the form $\mathcal{R}(t_c[g], t_u[g], t_v[g])$ with $t_u, t_v \in T_0$.

Now inductively assume that t_u, t_v are terms for which this proposition holds, i.e. there exists terms $\tilde{t}_u, \tilde{t}_v$ equal to t_u, t_v modulo a given instance of ND-comprehension with the parameter H. The problem is now that the instances of comprehension cannot be generated parallel to t_c because they include the parameter H. But we take advantage of the argument t_{f_φ} of U and V. Coding the instances of ND-comprehension together (ND-interpretation of Remark 1.9) we can find $\varphi', X'_\varphi, Y'_\varphi$ such that

$$(\Pi^0_1\text{-}\widehat{\mathsf{CA}}(\varphi' c H))_{QF}(f_\varphi, X'_\varphi(H f_\varphi), Y'_\varphi(H f_\varphi))$$

proves the original ND-instance of Π^0_1-$\widehat{\mathsf{CA}}$ for φ and those needed for t_u, t_v.

This proves the proposition. □

Corollary 3.20 (Extension to R_1, Φ'_0)**.** *The statement of Proposition 3.19 also holds for terms $t^1[g]$ with $\lambda g.t[g] \in T_0[\mathcal{R}, R_1, \Phi'_0] = T_1[\mathcal{R}, \Phi'_0]$, where every occurrence of $\mathcal{R}$*

is of the form required in Proposition 3.19 and every occurrence of R_1 or Φ'_0 is of the form

$$R_1(t_1[g], t_2[g], t_3[g]) \quad \textit{resp.} \quad \Phi'_0(t_1[g], t_2[g], t_3[g]).$$

Proof. The proof proceeds like in Proposition 3.19:

To interpret R_1 while retaining the instance of ND-comprehension, we will essentially use a functional interpretation of the proof of Lemma 1.11 (for $n = 1$). First note that $s := R_1(t_1[g], t_2[g], t_3[g])$ defines a type 1 function in $T_1[g]$. Arguing as in Lemma 1.11, it is clear that over $\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction$ a suitable instance of Π^0_1-CA with the parameter g proves that s is total ($\forall x\, \exists y\, \langle x, y\rangle \in \mathcal{G}_s[g]$, where $\mathcal{G}_s$ is the graph of s). An ND-interpretation of this statement yields that even an instance of the ND-interpretation of Π^0_1-CA is sufficient to prove that s is total. Another instance of ND-comprehension proves the ND-interpretation of the Π^0_1-CA -instance in (1.8) on p. 31. This instance is modulo the totality of s equivalent to an instance of ND-comprehension with the parameter s. The two instances of ND-comprehension used can be coded together, see Remark 1.9.

The functional Φ'_0 can be replaced by a function in $T_1[g]$, see Theorem 3.17 and Remark 3.18, and hence can also be interpreted. □

Proposition 3.21. *Let A_{qf} be a quantifier-free formula that contains only the shown variables free. If*

$$\mathsf{N\text{-}PA}^\omega_1\!\restriction + \mathsf{QF\text{-}AC} + \mathsf{RT}^2_2 + \mathsf{WKL} \vdash \forall x^1\, \exists y^0\, A_{qf}(x, y) \tag{3.18}$$

then one can find a terms $t_y, t_u, t_v, \xi \in T_0[\mathcal{B}, \tilde{R}_1]$ such that

$$\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction \oplus (\mathcal{B}) \oplus (\tilde{R}_1)$$
$$\vdash \forall x^1\, \forall f\, \Big(\big(\Pi^0_1\text{-}\widehat{\mathsf{CA}}(\xi x)\big)_{QF}(f, t_u f x, t_v f x) \to A_{qf}(x, t_y f x) \Big).$$

Proof. We may assume that $A_{qf}(x, y)$ does not contain R_1. Otherwise we could write A_{qf} as $t(x, y, R_1) = 0$ for a term $t \in T_0$. Using associates this could be rewritten as $\exists n\, \alpha_t(\overline{x}n, y, \overline{\alpha_{R_1}}n) = 1$. By Proposition 1.8 this is in T_0. Now coding n into y yields a A_{qf} without R_1.

A functional interpretation of the statement (3.18) yields closed terms resp. term tuples $t_y, t_{R_1}, t_{\mathcal{R}}, t_{\Phi'_0} \in T_0$, such that

$$\mathsf{qf\text{-}}\widehat{\mathsf{N\text{-}PA}}^\omega\!\restriction \vdash ((R_1)_{ND}(R_1, t_{R_1} R_1 \mathcal{R} \Phi'_0 x) \wedge \mathsf{RT}^2_{2ND}(\mathcal{R}, t_{\mathcal{R}} R_1 \mathcal{R} \Phi'_0 x)$$
$$\wedge\, \mathsf{WKL}_{ND}(\Phi'_0, t_{\Phi'} R_1 \mathcal{R} \Phi'_0 x)) \to A_{qf}(x, t_y R_1 \mathcal{R} \Phi'_0 x).$$

Here we use that $\widehat{\mathsf{N\text{-}PA}}^\omega\!\restriction + (R_1) + \Sigma^0_2$-$\mathsf{IA}$ is the same as $\mathsf{N\text{-}PA}^\omega_1\!\restriction$ and that R_1 solves the functional interpretation of Σ^0_2-IA, see [81].

Apply now Proposition 2.21 and Remark 2.22 to this derivation to normalize it such that only finitely many independent applications of $\mathcal{R}, R_1, \Phi_0'$ occur, where each of them is of the form

$$\mathcal{R}^*(t_1[g], t_2[g], t_3[g]) \quad \text{resp.} \quad R_1(t_1[g], t_2[g], t_3[g]), \quad \Phi_0'(t_1[g], t_2[g], t_3[g])$$

and t_1, t_2, t_3 are semi-closed.

The terms occurring in this normalized derivation can be interpreted using Corollary 3.20. (Applications to literally equal terms are replaced by the same interpretation.)

The instances of ND-comprehension needed for Corollary 3.20 can be coded together in one instance using Remark 1.9. ☐

Remark 3.22. One may also interpret $\Pi^0_1 G$ like RT^2_2 in Proposition 3.21. But this is superfluous because $\mathsf{AMT} \wedge \Sigma^0_2\text{-}\mathsf{IA} \to \Pi^0_1 G$, see Proposition 2.12, and thus such results are already implied by Proposition 3.21.

The application of $\Pi^0_1\text{-}\mathsf{CA}$ can be interpreted by a non-iterated use R-$(B_{0,1})$ of the rule of bar-recursion—this means we substitute f with a solution t_f to $(\Pi^0_1\text{-}\widehat{\mathsf{CA}})^{ND}$:

$$\begin{aligned}&\widehat{\mathsf{WE\text{-}HA}}^\omega{\restriction} \oplus (\mathcal{B}) \oplus (\tilde{R}_1) \oplus \text{R-}(B_{0,1}) \\ &\qquad\qquad \vdash \forall x^1 \ \Big(\big(\Pi^0_1\text{-}\widehat{\mathsf{CA}}(\xi x)\big)_{QF}(t_f[x], t_u t_f[x]x, t_v t_f[x]x) \to A_{qf}(x, t_y t_f[x]x)\Big)\end{aligned}$$

The term $t_f \in T_0[\mathcal{B}, \tilde{R}_1, B_{0,1}]$ is defined as in Proposition 3.8. Note that t_f depends on ξ, t_u, t_v and that it is of type 2 containing only *one application* of $B_{0,1}$ to semi-closed terms defining a type 2 object.

Since t_f solves the instance of comprehension we obtain:

$$\widehat{\mathsf{WE\text{-}HA}}^\omega{\restriction} \oplus (\mathcal{B}) \oplus (\tilde{R}_1) \oplus \text{R-}(B_{0,1}) \vdash \forall x^1 \, A_{qf}(x, t_y t_f[x]x).$$

The term $t := \lambda x. t_y t_f[x]x \in T_0[\mathcal{B}, \tilde{R}_1, B_{0,1}]$, contains only majorizable constants; the majorants to $\mathcal{B}$, $\tilde{R}_1$ are trivial and $B_{0,1}$ is essentially majorized by itself, see Proposition 3.12, hence we can find a majorant $t^* \in T_0[B_{0,1}]$ to t containing also only one application of $B_{0,1}$ to semi-closed terms. Now we can apply bounded search to obtain a new realizer t' for y not containing $\mathcal{B}$ or $\tilde{R}_1$:

$$t'x := \begin{cases} \text{minimal } y \le t^*x \text{ with } A_{qf}(x,y), & \text{if such a } y \text{ exists,} \\ 0, & \text{otherwise.} \end{cases}$$

Since t' now does not contain $\mathcal{B}$ anymore we may weaken $(\mathcal{B})$ to UWKL and then eliminate it from the system using a monotone functional interpretation, see

[57, 67]. Hence we obtain a term $t' \in T_0[B_{0,1}]$ containing after normalization only one occurrence of $B_{0,1}$ defining a type 2 object, such that with the rule R-$(B_{0,1})$ of $B_{0,1}$

$$\widehat{\mathsf{WE\text{-}HA}}^{\omega}\!\upharpoonright \oplus (\tilde{R}_1) \oplus \mathrm{R}\text{-}(B_{0,1}) \vdash \forall x^1\, A_{qf}(x, t'x).$$

Using ordinal analysis of the $B_{0,1}$-rule (cf. Theorem 3.17 and Remark 3.18) yields a new term t'' definable with ordinal primitive recursion up to ω_2^{ω} such that

$$\widehat{\mathsf{WE\text{-}HA}}^{\omega}\!\upharpoonright_{\omega_2^{\omega}} \oplus (\tilde{R}_1) \vdash \forall x^1\, A_{qf}(x, t''x).$$

Combining this with Theorem 3.13 and noting that $\tilde{R}_1$ is included in $\mathsf{WE\text{-}HA}_1^{\omega}\!\upharpoonright$ and that $\widehat{\mathsf{WE\text{-}PA}}^{\omega}\!\upharpoonright + \Sigma_2^0\text{-}\mathsf{IA}$ has an ND-interpretation in $\mathsf{WE\text{-}HA}_1^{\omega}\!\upharpoonright$ we obtain the following theorem:

Theorem 3.23 (Conservation for RT_2^2)**.** *If*

$$\mathsf{N\text{-}PA}_1^{\omega}\!\upharpoonright + \mathsf{QF\text{-}AC} + \mathsf{RT}_2^2 + \mathsf{WKL} \vdash \forall x^1\, \exists y^0\, A_{qf}(x, y)$$

then one can extract a term $t \in T_1$ such that

$$\mathsf{WE\text{-}HA}_1^{\omega}\!\upharpoonright \vdash \forall x^1\, A_{qf}(x, tx).$$

Extension to $\mathsf{RT}_{<\infty}^2$

Proposition 3.19 holds analogously for $\mathsf{RT}_{<\infty}^2$ if one adds R_1 and Σ_2^0-IA to the verifying system; Corollary 3.20 holds if one replaces R_1 by R_2.

But in contrast to the previous the technique used in remark 2.22 to extract terms that meet the requirements of these propositions can only be applied to terms in $T_1[\mathcal{R}_\infty]$ and not to terms $T_2[\mathcal{R}_\infty]$, because $deg(R_2) = 4$ and therefore we could not apply the term normalization. The mathematical reason is that R_2 is strong enough to iterate $B_{0,1}$ and $\mathcal{R}_\infty$.

This will hinder us to achieve full conservativity for full Σ_3^0-IA over a system in all finite types but a restricted variant of Σ_3^0-induction can be handled. Define the rule of Σ_3^0-induction Σ_3^0-IR as

$$(\Sigma_3^0\text{-}\mathsf{IR})\colon \frac{\exists x\, \forall y\, \exists z\, A_{qf}(0, x, y, z, \underline{a}) \qquad \begin{array}{l} \forall n\, (\exists x\, \forall y\, \exists y\, A_{qf}(n, x, y, z, \underline{a}) \\ \qquad \to \exists u\, \forall v\, \exists w\, A_{qf}(n+1, u, v, w, \underline{a})) \end{array}}{\forall n\, \exists x\, \forall y\, \exists z\, A_{qf}(n, x, y, z, \underline{a})},$$

where A_{qf} is quantifier-free and contains only the variables shown, u, v, w, x, y, z, n are type 0 variables and $\underline{a}$ denotes an arbitrary tuple of parameters. Let Σ_3^0-IR_2 be the restriction of Σ_3^0-IR to parameters $\underline{a}$ of type ≤ 2 then

Theorem 3.24 (Conservation for $\mathsf{RT}^2_{<\infty}$)**.** *If*

$$\mathsf{N\text{-}PA}^\omega_1{\restriction} + \mathsf{QF\text{-}AC} + \Sigma^0_3\text{-}\mathsf{IR}_2 + \mathsf{RT}^2_{<\infty} + \mathsf{WKL} \vdash \forall x^1\, \exists y^0\, A_{qf}(x,y) \tag{3.19}$$

then one can extract a term $t \in T_2$ such that

$$\mathsf{WE\text{-}HA}^\omega_2{\restriction} \vdash \forall x^1\, A_{qf}(x, tx).$$

Proof. The ND-interpretation of the conclusion of Σ^0_3-IR_2 is given by

$$\forall n^0\, \forall Y^2\, \exists x^0, Z^1\, A_{qf}(n, x, YxZ, Z(YxZ), \underline{a}^2).$$

One immediately see that Σ^0_3-IR_2 introduces only type 3 terms (t_Z, t_x ranging over $n^0, Y^2, \underline{a}^2$). Hence we can ND-interpret (3.19) in

$$\mathsf{qf\text{-}N\text{-}PA}^\omega_1{\restriction} + (G_1) + \cdots + (G_n)$$

where (G_i) are defining axioms and constants of degree ≤ 3 introduced by the rule Σ^0_3-IR_2. The terms occurring in the derivation can be viewed as terms in $T_1[\mathcal{R}_\infty, \Phi'_0, G_1, \ldots, G_n]$. The requirements of Theorem 2.19 in Remark 2.22 are met and we obtain a normalized derivation.

By [81], $(\Sigma^0_3\text{-}\mathsf{IA})^{ND}$ can be solved by R_2. Since Σ^0_3-IA implies Σ^0_3-IR_2 the constants G_i may be chosen to be in $T_2[\mathcal{R}_\infty, \Phi'_0]$. These terms can be handled like in Proposition 3.21.

This completes the proof. ☐

Corollary 3.25. *If*

$$\mathsf{E\text{-}PA}^\omega_1{\restriction} + \mathsf{QF\text{-}AC}^{0,1} + \mathsf{QF\text{-}AC}^{1,0} + \mathsf{RT}^2_2 + \mathsf{WKL} \vdash \forall x^1\, \exists y^0\, A_{qf}(x,y)$$

one can extract a term $t \in T_1$ such that

$$\mathsf{WE\text{-}HA}^\omega_1{\restriction} \vdash \forall x^1\, A_{qf}(x, tx).$$

If $\mathsf{RT}^2_{<\infty} + \Sigma^0_3$-$\mathsf{IR}_2$ is added to the above system then one can extract a term $t \in T_2$ realizing y provably in $\mathsf{WE\text{-}HA}^\omega_2{\restriction}$ instead of $\mathsf{WE\text{-}HA}^\omega_1{\restriction}$.

Proof. Apply elimination of extensionality (Proposition 1.5) and use Theorem 3.23.

For the second statement use Theorem 3.24. To be able to use the elimination of extensionality the induction rule Σ^0_3-IR_2 has to be altered to include the premise that the parameters are extensional. Since this is a formula of the form $\forall u^1\, \exists v^0\, B_{qf}(u,v)$, the functional interpretation does not introduce terms of degree > 3 and the rule which still follows from Σ^0_3-IA can be interpreted like in the proof of Theorem 3.24. ☐

Corollary 3.26.

- $\mathsf{WKL}_0^\omega + \Sigma_2^0\text{-}\mathsf{IA} + \mathsf{RT}_2^2$ *is conservative over* $\mathsf{RCA}_0^\omega + \Sigma_2^0\text{-}\mathsf{IA}$ *for sentences of the form* $\forall x^1 \, \exists y^0 \, A_{qf}(x, y)$. *Moreover one can extract a term* $t \in T_1$ *realizing* y.
- $\mathsf{WKL}_0^\omega + \Sigma_2^0\text{-}\mathsf{IA} + \Sigma_3^0\text{-}\mathsf{IR}_2 + \mathsf{RT}_{<\infty}^2$ *is conservative over* $\mathsf{RCA}_0^\omega + \Sigma_3^0\text{-}\mathsf{IA}$ *for sentences of the form* $\forall x^1 \, \exists y^0 \, A_{qf}(x, y)$. *Moreover one can extract a term* $t \in T_2$ *realizing* y.

Since every sentence of the form $\forall x^1 \, \exists y^0 \, \forall z^0 \, B_{qf}(x, y, z)$ is over $\mathsf{QF\text{-}AC}^{0,0}$ equivalent to a sentence of the form $\forall x^1 \, \exists y^0 \, A_{qf}(x, y)$ also Π_3^0-conservativity is obtained.

4. The chain antichain principle

In this chapter we refine the techniques from the previous chapter and show that one can obtain primitive recursive realizers if one can prove the proofwise low property using only quantifier-free induction and not Σ^0_1-induction. We show that CAC is proofwise low in such a system and apply this result to this principle.

We already mentioned that Chong, Slaman, Yang in [20] recently proved that CAC is Π^1_1-conservative over $\mathsf{RCA}_0 + \Pi^0_1\text{-}\mathsf{CP}$ which implies that CAC does not yield Σ^0_2-induction. We use here some of there ideas to show the proofwise lowness and provide a different, purely syntactical and constructive proof of the fact that CAC does not imply Σ^0_2-induction. We show that CAC even together with WKL is Π^0_2-conservative over PRA.

We start by refining Howard's ordinal analysis of the bar recursor $B_{0,1}$, see [44] and Section 3.4. We show that applications of $B_{0,1}$ to terms in $\mathsf{RCA}^{\omega*}_0$ (actually even in $\mathsf{G}_\infty\mathsf{A}^\omega$) yield only primitive recursive functions. Crucial for this analysis is the structure of higher order functionals of $\mathsf{RCA}^{\omega*}_0$. Most important is that this system does not contain a function iterator constant (which in this system is equivalent to Σ^0_1-IA). Our refined ordinal analysis mentioned above corresponds to the fact that QF-IA plus an instance of Π^0_1-CA implies each instance of Σ^0_1-IA and hence the totality of all primitive recursive functions but not of the Ackermann function.

Using this refinement of Howard's ordinal analysis of $B_{0,1}$ we can improve a result from the previous sections and show that for each principle P which is proofwise low over $\mathsf{WKL}^{\omega*}_0$ the system $\mathsf{WKL}^{\omega}_0 + \Pi^0_1\text{-}\mathsf{CP} + \mathsf{P}$ is Π^0_3-conservative over RCA^{ω}_0 and that one can extract primitive recursive realizing terms.

This chapter is organized as follows. First we refine Howard's ordinal analysis of bar recursion. In Section 4.3 we use this result to refine our techniques from the last chapter and in Section 4.4 we show that CAC is proofwise low over a suitable system not containing Σ^0_1-induction and conclude that CAC is Π^0_3-conservative over RCA^{ω}_0.

4.1. Ordinal analysis of bar recursion of terms in $\mathsf{G}_\infty\mathsf{R}^\omega$

The goal of this section is to show that a single application of the bar recursor $B_{0,1}$ to terms in $\mathsf{G}_\infty\mathsf{R}^\omega$ does only lead to primitive recursive terms (in the sense of Kleene), i.e. terms with computational size $< \omega^\omega$. We use here the definition of computational size from Howard [43, 44]. Recall that the computational size of a term t of type 0 is roughly an upper bound on the number of term reductions one has to apply to obtain a numeral. The computational size of a higher type term t is defined to be the computational size of $t(H_0, \dots, H_n)$ where H_i are fresh variables such that the term is of type 0. Like Howard we assume that a term t has $deg(t) \leq 2$ and is semi-closed (i.e. contains only variables of degree 1 free) whenever we speak about the computational size of a term t.

Recall that the bar recursor $B_{0,1}$ is defined to be

$$B_{0,1}AFGc :=_1 \begin{cases} Gc & \text{if } A[c] < \operatorname{lth} c, \\ Fc(\lambda u^0.B_{0,1}(AFG(c * \langle u \rangle))) & \text{otherwise,} \end{cases}$$

where $[c] := \lambda i.(c)_i$.

Howard uses for technical reasons an extension of the term system. This extension is conservative and hence does not lead to any problems. Since we are only going to modify his analysis we will follow this approach:
For each type 1 variable α and terms c, t of type 0 add a new term $\{\alpha, c, t\}$ to the system. The term $\{\alpha, c, t\}$ has the same type as $B_{0,1}A$. The subterms of it consist only of the subterms of t. The purpose of this extension is to bind all occurrences of α in t. The term $B_{0,1}AFGc$ is equal to $\{\alpha, c, A\alpha\}FGc$ and can also be contracted to this term. The term $\{\alpha, c, t\}$ satisfies following contractions:

$$\begin{array}{rll} \{\alpha, c, t\} & \textbf{contr} \;\; \{\alpha, c, t'\} & \text{if } t \textbf{ contr } t' \\ \{\alpha, c, i\}FGc & \textbf{contr} \;\; Gc & \text{if } i \text{ is numeral} < \operatorname{lth}(c) \\ \{\alpha, c, t\}FGc & \textbf{contr} \;\; M & \\ \{\alpha, c, t\}FG(c * \langle n \rangle) & \textbf{contr} \;\; \{\alpha, c * \langle n \rangle, t\}FG(c * \langle n \rangle) & \end{array}$$

where

$$M := \begin{cases} Gc & \text{if } t[\lambda i.(c)_i/\alpha] < \operatorname{lth}(c), \\ Fc(\lambda u.\{\alpha, c, t\}FG(c * \langle u \rangle)) & \text{otherwise.} \end{cases} \tag{4.1}$$

For details we refer the reader to [44]. Note that $\{\alpha, c, t\}$ is there defined for bar recursors of arbitrary types and not only for $B_{0,1}$.

We now state a modified version of Theorem 2.3 of [44]. The proof of the following theorem differs from Howard's proof only in using other ordinal estimates. The result of it is more suitable for terms which have finite computational size because it shows in this case that the resulting term has computational size $< \omega^\omega$, whereas in Howard's theorem the computational size is always $\geq \omega^\omega$. For parameters which have computational size of an infinite ordinal Howard's theorem yields better results.

Theorem 4.1. *Let F, G and t have computational sizes f, g and $size(t)$. Then the term $\{\alpha, c, t\}FGc$ has computational size 2^{g+f4h}, where $h = \omega + \omega size(t) + \omega$.*

Proof. We assume that $f, g \geq 1$.

Like Howard, we say for a term $\{\alpha, d, s\}$ that the sequence d is m-critical in s if the term to be contracted in s is of the form αm and $m \geq \mathrm{lth}(d)$. We define $\mathit{ord}(\alpha, d, s)$ to be $\omega + \omega \mathit{size}(s) + 1$ if d is not critical in s and s is not a numeral. If d is m-critical we let $\mathit{ord}(\alpha, d, s) = \omega + \omega \mathit{size}(s) + m - \mathrm{lth}(d) + 3$. If s is a numeral n, we let $\mathit{ord}(\alpha, d, s) = \omega + (n \dot{-} \mathrm{lth}(d)) + 2$.

Like in [44, Theorem 2.3] we prove by transfinite induction on $b = \mathit{ord}(\alpha, c, t)$ that $\{\alpha, c, t\}FGc$ has computational size 2^{g+f4b}.

We consider the following cases:

- If t is not a numeral and c is not critical then executing a computation step reduces t to t' such that $\mathit{size}(t') < \mathit{size}(t)$ and hence $\mathit{ord}(\alpha, c, t') < \mathit{ord}(\alpha, c, t)$ and so $2^{g+f4\,\mathit{ord}(\alpha,c,t')} < 2^{g+f4b}$.

- If t is a numeral that is $< \mathrm{lth}(c)$ then $\{\alpha, c, t\}FGc$ reduces to Gc which has computation size $g \leq 2^g < 2^{g+f4b}$.

- The cases where c is critical or t is a numeral $\geq \mathrm{lth}(c)$ remain. We treat here at first the former case, the later will follow from a slight modification of this.

  We can reduce $\{\alpha, c, t\}FGc$ to M from (4.1) in one step. For the case distinction in M we have to compute $t[\lambda i.(c)_i/\alpha]$. By Theorem 2.1 from [44] we can compute it in $\omega \mathit{size}(t)$ steps. By finitely many steps j we then arrive at either

  $$Gc \qquad \text{or} \qquad \underbrace{Fc(\lambda u.\{\alpha, c, t\}FG(c * \langle u \rangle))}_{M_2}.$$

  In the case of Gc additionally g more computation steps are needed. In total this yields

  $$g + \underbrace{j + \omega \mathit{size}(t) + 1}_{<b} < 2^{g+f4b}. \tag{4.2}$$

  In the case of M_2 we reduce

  $$\lambda u.\{\alpha, c, t\}FG(c * \langle u \rangle)x \qquad \text{to} \qquad \{\alpha, c * \langle n \rangle, t\}FG(c * \langle n \rangle)$$

  in 3 steps. Let $a = \mathit{ord}(\alpha, c * \langle n \rangle, t)$. By definition of ord we have $a < b$. By induction hypothesis $\{\alpha, c * \langle n \rangle, t\}FG(c * \langle n \rangle)$ has computational size 2^{g+f4a}. The term c has computational size $\omega \leq 2^{g+f4a}$. Together with Theorem 2.1

from [44] this show that M_2 has computation size

$$\begin{aligned}(2^{g+f4a}+3)f &\leq (2^{g+f4a}+2^{g+f4a})f && (a \geq \omega)\\ &\leq 2^{g+f4a+1}\cdot f \\ &< 2^{g+f4a+1}\cdot 2^{f+1} && (f < 2^{f+1})\\ &= 2^{g+f4a+1+f+1}\\ &\leq 2^{g+f4a+f3} && (f \geq 1)\end{aligned}$$

Together with the steps for the cases distinction we obtain the following computational size

$$\begin{aligned}(2^{g+f4a}+3)f+\underbrace{j+\omega size(t)+1}_{=:z} &< 2^{g+f4a+f3}+2^{z+1}\\ &\leq 2^{\max(g+f4a+f3,z+1)}\cdot 2\\ &\leq 2^{g+f4b}\end{aligned}$$

The last $\leq$ holds since $\max(g+f4a+f3, z+1) < g+f4b$ and therefore $\max(g+f4a+f4, z+1)+1 \leq g+f4b$.

The case where t is a numeral $\geq \mathrm{lth}(c)$ can be treated similarly. Here $t[\lambda i.(c)_i/\alpha]$ does not need to be computed. Hence, the equation (4.2) becomes

$$g+j+1 < 2^{g+f4b}.$$

Since $j+1 < \omega < b$ this is still valid. The rest of the argument remains the same because also $a < b$ holds.

This proves the theorem. □

Remark 4.2. Define Bezem's bar recursor $B^B_{0,1}$ to be

$$B^B_{0,1}AFGc :=_1 \begin{cases} Gc & \text{if } A[c]^B < \mathrm{lth}\, c,\\ Fc(\lambda u^0.B^B_{0,1}(AFG(c*\langle u\rangle))) & \text{otherwise,}\end{cases}$$

where $[c]^B := \begin{cases}(c)_i & \text{if } i < \mathrm{lth}(c)\\ (c)_{\mathrm{lth}(c)\dot{-}1} & \text{otherwise.}\end{cases}$

This bar recursor differs from Howard's bar recursor only in the definition of $[\cdot]$. Hence, Theorem 4.1 also holds for $B^B_{0,1}$.

We will use this bar recursor in Theorem 4.5 below to define a majorant for $B_{0,1}$.

In the following we will treat $B^{(B)}_{0,1}$ as a constant satisfying the defining equations of the bar recursor, but which is *not* provably total.

Theorem 4.3. *The system* $\widehat{\mathsf{WE\text{-}PA}}^\omega\!\upharpoonright$ *proves that for all semi-closed terms* A, F, G, c *with provably finite computational size* $B_{0,1}AFGc$ *is total, i.e. there exists a term that provably satisfies the defining equations. The same holds for* $B^B_{0,1}AFGc$.

Proof. Let f, g, a be the computational sizes of F, G, A.

The proof of Theorem 4.1 for $\{\alpha, c, A\alpha\}FGc$ can be formalized in a system containing the Σ^0_1-least number principle for sets containing elements $< 2^{g+f4(\omega+\omega a+\omega)}$. Since

$$2^{g+f4(\omega+\omega a+\omega)} = 2^{\omega(a+2)} = \omega^{a+2} < \omega^\omega$$

this principle is equivalent to Σ^0_1-induction (over $\mathbb{N}$), see [37, II.3.18] and also Theorem 3.14. Hence the system $\widehat{\mathsf{WE\text{-}PA}}^\omega\!\upharpoonright$ suffices.

The conservativity of Howard's extended term system can also be formalized in $\widehat{\mathsf{WE\text{-}PA}}^\omega\!\upharpoonright$. Therefore this systems also proves the totality of $B_{0,1}AFGc$. $\square$

For the analysis of terms in $\mathsf{G}_\infty\mathsf{R}^\omega$ we use the following property:

Proposition 4.4 ([58, Proposition 2.2.22], [67, Corollary 3.42])**.** *Let* $\rho = 0\rho_k \ldots \rho_1$ *with* $deg(\rho_i) \le 1$. *For each term* $t^\rho \in \mathsf{G}_\infty\mathsf{R}^\omega$ *there exists a term* $t^*[x_1^{\rho_1}, \ldots, x_k^{\rho_k}]$ *such that*

- $t^*[x_1, \ldots, x_k]$ *contains at most* $x_1, \ldots, x_k$ *as free variables,*
- $t^*[x_1, \ldots, x_k]$ *is build up only from* $x_1, \ldots, x_k, 0^0, A_0, A_1, \ldots,$ *where* A_i *is the* i*-th branch of the Ackermann function,*
- $\mathsf{G}_\infty\mathsf{A}^\omega_\mathsf{i} \vdash \lambda x_1, \ldots, x_k.t^*[x_1, \ldots, x_k]$ maj t.

In particular, every term $t \in \mathsf{G}_\infty\mathsf{R}^\omega$ *of degree* ≤ 2 *is provably majorized by a term that has provably finite computational size.*

Theorem 4.5. *Let* $A[x^1], F[x], G[x], c[x]$ *be terms of appropriated type such that* $B_{0,1}AFGc$ *is well-formed and such that* $\lambda x^1.A[x], F[x], G[x], c[x] \in \mathsf{G}_\infty\mathsf{R}^\omega$. *Then* $\widehat{\mathsf{WE\text{-}HA}}^\omega\!\upharpoonright$ *proves that* $f := \lambda x^1.\lambda y^0.B_{0,1}AFGcy$ *is total. Moreover this system proves that there exists a majorant to* f.

Proof. First observe that the totality of the bar recursor in f can be proven using Π^0_2-bar induction of type 0 (Π^0_2-BI_0). (Use the bar induction to prove the statement $\forall u\, \exists v\, B_{0,1}AFGcu = v$.) To make use of the properties described in Proposition 4.4 we will first show that a majorant to f exists. With this we can bound the $\exists$-quantifier in the bar induction and obtain that Π^0_1-bar induction (Π^0_1-BI_0) suffices. By Lemma 3.11 this is included in $\widehat{\mathsf{WE\text{-}PA}}^\omega\!\upharpoonright + \mathsf{QF\text{-}AC}$.

We now show that there exists majorant to f and that it is total. Let

$$\begin{aligned} B^\times_{0,1} &:= \lambda A, F, G, c.B^B_{0,1}AF_GGc\,, \\ B^*_{0,1} &:= \lambda A, F, G, c.(B^\times_{0,1}AFGc)^M, \end{aligned} \tag{4.3}$$

where

$$F_G tf := \max(Gt, Ftf_{(\mathrm{lth}(t) \dot{-} 1)}), \qquad f_i(x) := f(\max(i, x))$$
$$\text{and} \qquad (f)^M x := \max_{y \leq x} f(x).$$

We have $B^*_{0,1}$ maj $B_{0,1}$ provably in $\widehat{\mathsf{WE\text{-}PA}}^\omega\restriction + \mathsf{QF\text{-}AC}$, see Proposition 3.12 and also [11]. In Proposition 3.12 we use a different majorant but mutatis mutandis the proof also shows that $B^*_{0,1}$ as defined in (4.3) majorizes $B_{0,1}$.[1]

Applying Proposition 4.4 we obtain majorizing semi-closed terms A^*, F^*, G^*, c^* for A, F_G, G, c with finite computational size. Since $B^*_{0,1}$ is a specific application of $B^B_{0,1}$, we can apply Theorem 4.3 to $B^*_{0,1}A^*F^*G^*c^*$ to obtain its totality. With this the totality of f and the existence of a majorant is proven in the system $\widehat{\mathsf{WE\text{-}PA}}^\omega\restriction + \mathsf{QF\text{-}AC}$.

Since this statement is $\forall\exists$, the functional translates this proof into a proof in $\widehat{\mathsf{WE\text{-}HA}}^\omega\restriction$. This provides the theorem. □

Corollary 4.6. *The term $B_{0,1}AFGc$ where A, F, G, c are semi-closed terms of $\mathsf{G}_\infty\mathsf{A}^\omega$ is provably equal to a term in T_0 (i.e. the fragment of Gödel's T where the recursor is restricted to recursion of type 0).*

Proof. Apply the functional interpretation (combined with a negative translation) to the result of Theorem 4.5, see [67, Proposition 10.53]. The term extracted using this satisfies the corollary. □

This result can be used to reprove the following result from Parsons [79, Lemma 4].

Corollary 4.7. *Let R_1 be the recursor for type 1 objects, i.e. $R_1 0 f G x = fx$ and $R_1(n+1)fGx = G(R_1 n f G)nx$, where x, n, fx are of type 0. (Note that R_1 cannot be reduced to primitive recursion, since G takes an element of $\mathbb{N}^\mathbb{N}$ as first parameter.)*

Then the term $R_1 n f G$ where G is a semi-closed term of $\mathsf{G}_\infty\mathsf{A}^\omega$ is provably equal to a term in T_0.

Proof. Corollary 4.6 and the fact that R_1 is elementarily definable from $B_{0,1}$. □

4.2. Dependent Choice

The bar recursor $B_{0,1}$ interprets not only the functional interpretation of Π^0_1-CA but also of Π^0_1-dependent choice (Π^0_1-DC). In a context with Σ^0_1-induction Π^0_1-CA and

[1] We do not use here the majorant of $B_{0,1}$ as defined in Definition 3.10 which would build internally paths through the tree A which are *not* monotone. Before applying the majorant A^* to such paths they have to be made monotone such that they are majorants. But this cannot be done using only terms with finite computational size.

Π^0_1-DC are equivalent and the interpretation using $B_{0,1}$ is optimal with respect to the provable total functions, see [67].

We take this opportunity to discuss the differences between instances of Π^0_1-CA and instances Π^0_1-DC and the strength of the finite analog of Π^0_1-DC, finite Π^0_1-dependent choice (Π^0_1-fDC).

We will show that instances of Π^0_1-CA are strictly weaker than instances of Π^0_1-DC in a context without Σ^0_1-induction. However, Corollary 4.6 shows that the interpretation of the non-iterated rule of Π^0_1-CA and the non-iterated rule of Π^0_1-DC over $\mathsf{G}_\infty\mathsf{A}^\omega$ using $B_{0,1}$ does not prove more than primitive recursion. Thus, the interpretation of the rule of Π^0_1-CA over $\mathsf{G}_\infty\mathsf{A}^\omega$ using $B_{0,1}$ is still optimal despite the fact that $B_{0,1}$ does interpret more than that.

We also discuss Π^0_1-fDC. The functional interpretation of this principle can be solved using the finite bar recursor B_{fin}, for a definition see [67, Section 11.4]. The finite bar recursor B_{fin} has been used by Oliva to solve the functional interpretation of IPP, see [77]. We will show that Π^0_1-fDC is equivalent to Σ^0_2-induction and thus that from B_{fin} can define all functions which are provable total relative to Σ^0_2-induction.

Dependent choice Let *dependent choice for natural numbers* (DC^0) be the following schema

$$\forall n^0\, \forall x^0\, \exists y^0\, A(n,x,y) \to \exists g^1\, \forall n\, A(n, g(n), g(n+1)).$$

If A is restricted to Π^0_1 formulas, we will write Π^0_1-DC.

It is easy to see that over—say RCA^ω_0—the principles Π^0_1-DC and Π^0_1-CA are equivalent.

We will now consider instances of these principles. Recall, that an instance of Π^0_1-comprehension is defined to be

$$(\Pi^0_1\text{-}\mathsf{CA}(f))\colon \exists g\, \forall n^0\, \Big(g(n) = 0 \leftrightarrow \forall u^0\, f(n,u) = 0\Big)$$

In the same way we define an instance of Π^0_1-dependent choice to be

$$\begin{aligned}(\Pi^0_1\text{-}\mathsf{DC}(f))\colon \forall n^0\, \forall x^0\, \exists y^0\, \forall u^0\ &(f(n,x,y,u) = 0)\\ &\to \exists g\, \forall n\, \forall u\ (f(n, g(n), g(n+1), u) = 0)\end{aligned}$$

Instances of Π^0_1-DC are in general stronger than instances of Π^0_1-CA. We will show that instances of Π^0_1-DC imply over a weak basis theory without Σ^0_1-induction (e.g. $\mathsf{G}_\infty\mathsf{A}^\omega$ or $\mathsf{RCA}^{\omega *}_0$) instances of Π^0_2-induction and therefore the totality of the Ackermann function, which is not the case for instances of Π^0_1-CA, see [60].

Proposition 4.8. *There is a closed term φ such that*

$$\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF\text{-}AC} + \Pi^0_1\text{-}\mathsf{DC}(\varphi) \vdash \Pi^0_2\text{-}\mathsf{IA}^-.$$

Here Π^0_2-IA^- *denotes* Π^0_2*-induction restricted to formulas having only type* 0 *parameter.*

A proof of this proposition is given below.

The reason for the strength of instances of Π^0_1-DC is the implicative assumption, which may be proven using for instance Σ^0_1-induction, which then is iterated. However if one restrict A to a formula where this is not the case it becomes weak.

We will show now that instances of Π^0_1-DC restricted to formulas given by

$$A(n,x,y) :\equiv B_{qf}(n,x,y) \lor \forall u\, \neg B_{qf}(n,x,u) \tag{4.4}$$

i.e. Π^0_1-$\mathsf{DC}(f)$, where $f(n,x,y,u) = \min(f'(n,x,y), 1 \dot{-} sg(f'(n,x,u)))$ for an f', is not stronger than a suitable instance of Π^0_1-CA. In this cases the premise of the instance of depended choice is provable by the law of exclude middle, hence in any classical system.

Lemma 4.9. *There exists a closed term φ such that for each f of the form give above that following holds*

$$\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF\text{-}AC} \vdash \forall f' \left(\Pi^0_1\text{-}\mathsf{CA}(\varphi(f')) \to \Pi^0_1\text{-}\mathsf{DC}(\min(f'(n,x,y), 1 \dot{-} sg(f'(n,x,u))))\right).$$

Proof. Fix an f'. By searching for the minimal y, we may assume that y is unique.

Now let $h(e)$ be the comprehension function for the following formula

$$\exists w \left((w)_0 = 0 \land \bigwedge_{i<\mathrm{lth}(e)} ((e)_i = 0 \to f'(n,(w)_i,(w)_{i+1}) \land (e)_i \neq 0 \to (w)_{i+1} = 0)\right) \tag{4.5}$$

The function h exists by a suitable instance of comprehension, which is given by a closed term having f' as parameter.

Now define by bounded recursion

$$h'(0) := \langle\rangle$$
$$h'(n+1) := h'(n) * \langle h(h'(n) * \langle 0\rangle)\rangle = h'(n) * \begin{cases} \langle 0\rangle & \text{if } h(h'(n) * \langle 0\rangle) = 0, \\ \langle 1\rangle & \text{otherwise.} \end{cases}$$

We claim that

(i) $h(h'(n)) = 0$ for each n and that

(ii) $h'(n)$ is the smallest in the lexicographic order element of 2^n with $h(e) = 0$.

The properties (i) and (ii) follow by quantifier-free induction. For $n = 0$ is clear that these properties are true. For the induction step note that if $h(h'(n)) = 0$ then also $h(h'(n) * \langle 1\rangle) = 0$. Hence, property (i) follows from the definition of h'. The property (ii) follows in a similar way.

Now by (i) together with the fact that h is a comprehension function for (4.5) we obtain

$$\forall n\, \exists w\, \Big((w)_0 = 0 \wedge \bigwedge_{i<n} ((h'(n+1))_i = 0 \to f'(n, (w)_i, (w)_{i+1})\Big).$$

The property (ii) strengthens this to

$$\forall n\, \exists w\, \Big((w)_0 = 0 \wedge \bigwedge_{i<n} ((h'(n+1))_i = 0 \to f'(n, (w)_i, (w)_{i+1}) \\ \wedge\, (h'(n+1))_i \neq 0 \to \big((w)_{i+1} = 0 \wedge \forall u\, f'(n, (w)_i, u)\big)\,)\Big). \quad (4.6)$$

Since this w is unique, we can obtain a solution f to Π^0_1-$\mathsf{DC}(f)$ by an instance of Π^0_1-AC, which follows from a suitable instance of Π^0_1-CA and QF-AC, and diagonalization.

Coding these instances of comprehension together yields a suitable term φ, see [60, Remark 3.8]. $\square$

Finite dependent choice We will denote by finite depended choice (fDC) that statement that arbitrary finite approximations of the depended choice function exists, i.e.

$$\forall n^0\, \forall x^0\, \exists y^0\, A(n, x, y) \to \forall k^0\, \exists s^0\, \forall n < k\ A(n, (s)_n, (s)_{n+1})).$$

Again, if A is restricted to Π^0_1-formulas we will write Π^0_1-fDC. It is clear that DC^0 implies fDC and Π^0_1-DC implies Π^0_1-fDC.

We will now show that Π^0_1-fDC is equivalent Σ^0_2-induction.

Proposition 4.10. $\mathsf{EA} \vdash \Pi^0_1$-$\mathsf{fDC} \leftrightarrow \Pi^0_2$-$\mathsf{IA}$.

Proof. A formula $B(x, y)$ describes (the graph of) a function if $\forall x\, \exists! y\, B(x, y)$. Let T_B be the statement that for each n the n-fold iteration of the function described by a formula B exists, i.e.

$$\forall x\, \exists! y\, B(x, y) \to \forall k^0\, \exists s^0\ ((s)_0 = 0 \wedge \forall n < k\ B((s)_n, (s)_{n+1}))\,.$$

Further, let $T\Pi^0_1$ be T_B for all $B \in \Pi^0_1$. Hájek and Pudlák showed that $T\Pi^0_1$ is equivalent to Π^0_2-IA, see Theorem I.2.24 and Lemma I.2.12 in [37]. Thus, it is sufficient to show Π^0_1-$\mathsf{fDC} \leftrightarrow T\Pi^0_1$.

The left-to-right direction follows immediately. For the right-to-left direction fix a Π^0_1-formula $A(n, x, y)$ and assume that the premise of fDC for this formula holds ($\forall n, x\, \exists y\, A(n, x, y)$). By Σ^0_1-induction, which follows from $T\Pi^0_1$, we may assume that y is minimal and thus unique. Now coding n, x together and applying $T\Pi^0_1$, yields Π^0_1-fDC. $\square$

From this proof follows Proposition 4.8. The left-to-right direction was first shown by Avigad (personal communication).

If one restricts A also to formulas of the form (4.4), as we did for the infinite case, then also Π^0_1-fDC becomes considerably weaker as we will show in the following proposition.

Proposition 4.11. *The principle* Π^0_1-fDC*, where A is restricted to formulas of the form given in* (4.4)*, is equivalent to* Σ^0_1*-induction.*

Proof. For the left to right direction note that Π^0_1-fDC proves finite comprehension for Π^0_1-formulas, i.e. the statement that

$$\forall k\, \exists s \in 2^k\, \forall n < k\ ((s)_n = 0 \rightarrow A(n) \wedge (s)_n \neq 0 \rightarrow \neg A(k))\,.$$

By [79] this implies Σ^0_1-IA and by [81] it is actually equivalent to Σ^0_1-IA. (In these articles this principle is called AS^{Π}_1.)

For the right-to-left direction proceed in the same way as in the proof of Lemma 4.9 for the equation (4.6) but use finite comprehension instead of comprehension. This is sufficient since h is only used for 0-1-sequence of length $\leq n$ to prove (4.6). □

4.3. Proofwise low relative to $\mathsf{G}_\infty\mathsf{A}^\omega$

In Section 2.5 we showed that principles P of the form

$$(\mathsf{P})\colon \forall c^1\, \exists g^1\, \underbrace{\forall u^0\, P_{qf}(c, g, u)}_{\equiv: P(c,g)}, \tag{4.7}$$

where P_{qf} is quantifier free, which are proofwise low relative to $\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction\! + \mathsf{QF\text{-}AC} \oplus \mathsf{WKL}$ are conservative over $\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction + \Sigma^0_2$-$\mathsf{IA}$ for sentences of the form $\forall x^1\, \exists y^0\, A_{qf}(x, y)$.

We now show that for principles P which are proofwise low relative to $\mathsf{G}_\infty\mathsf{A}^\omega$ + $\mathsf{QF\text{-}AC} \oplus \mathsf{WKL}$ the system $\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction + \mathsf{QF\text{-}AC} \oplus \mathsf{WKL} \oplus \mathsf{P}$ is conservative over $\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction$ for sentences of the form $\forall x^1\, \exists y^0\, A_{qf}(x, y)$. (Actually we only treated the case of RT^2_2 but mutatis mutandis this works for each principle of this form.)

Let now P be a principle that is proofwise low over $\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF\text{-}AC} \oplus \mathsf{WKL}$ (a fortiori it is sufficient that P is proofwise low over $\mathsf{WKL}_0^{\omega *}$ since this system can be embedded into the other). This means we have for each provably continuous term φ a provably continuous term ξ such that

$$\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF\text{-}AC} \oplus \mathsf{WKL} \vdash \forall c\, \left(\Pi^0_1\text{-}\mathsf{CA}(\xi c) \rightarrow \exists g\, \left(P(c, g) \wedge \Pi^0_1\text{-}\mathsf{CA}(\varphi c g)\right)\right).$$

A functional interpretation of this statement yields

$$\begin{aligned}&\mathsf{G}_\infty\mathsf{A}^\omega_\mathsf{i} \oplus (\mathcal{B}) \vdash \\ &\qquad \forall c\, \forall U\, \forall f_\xi\, \forall X_\varphi, Y_\varphi\, \exists x_\xi, y_\xi\, \exists g\, \exists f_\varphi \left((\Pi^0_1\text{-}\widehat{\mathsf{CA}}(\xi f))_{qf}(f_\xi, x_\xi, y_\xi) \right. \\ &\qquad\qquad \left. \to (P(c, g, Ugf_\varphi) \wedge \Pi^0_1\text{-}\widehat{\mathsf{CA}}(\varphi f g))_{qf}(f_\varphi, X_\varphi g f_\varphi, Y_\varphi g f_\varphi)) \right), \end{aligned} \tag{4.8}$$

and that there exist terms in $\mathsf{G}_\infty\mathsf{R}^\omega$ realizing $x_\xi, y_\xi, g, f_\varphi$, cf. to Theorem 3.9.

Using (4.8) in the proof of Proposition 3.19 instead of Theorem 3.9 we obtain a variant of Proposition 3.19 where $\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction$ is replaced by $\mathsf{G}_\infty\mathsf{A}^\omega_\mathsf{i}$, RT^2_2 is replaced by P and $T_0[\mathcal{R}]$ is replaced by $\mathsf{G}_\infty\mathsf{R}^\omega[\mathcal{R}]$ (here $\mathcal{R}$ is now a solution functional for P^{ND}). In the same way we obtained Corollary 3.20 from Proposition 3.19 we can extend the previous statement to terms in $\mathsf{G}_\infty\mathsf{R}^\omega[\mathcal{R}, R_0, \Phi'_0]$ (which is equal to $T_0[\mathcal{R}, \Phi'_0]$) but of course *not* to terms containing R_1. As consequence we obtain the following modification of Proposition 3.21:

Proposition 4.12. *Let A_{qf} be a quantifier-free formula that contains only the shown variables free and let P be a principle of the form (4.7) which is proofwise low over $\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF\text{-}AC} \oplus \mathsf{WKL}$. If*

$$\widehat{\mathsf{N\text{-}PA}}^\omega\!\restriction + \mathsf{QF\text{-}AC} + \mathsf{WKL} + \mathsf{P} \vdash \forall x^1\, \exists y^0\, A_{qf}(x, y)$$

then one can find terms $t_y, t_u, t_v, \xi \in \mathsf{G}_\infty\mathsf{R}^\omega[\mathcal{B}]$ such that

$$\mathsf{G}_\infty\mathsf{A}^\omega_\mathsf{i} \oplus (\mathcal{B}) \vdash \forall x^1\, \forall f\, \left((\Pi^0_1\text{-}\widehat{\mathsf{CA}}(\xi x))_{QF}(f, t_u f x, t_v f x) \to A_{qf}(x, t_y f x) \right).$$

Similarly to the discussion preceding Theorem 3.23, we interpret $\Pi^0_1\text{-}\widehat{\mathsf{CA}}(\xi x)$ with a single application of $B_{0,1}$ (or in other words using a single application of the rules of bar recursion). With this we obtain

$$\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction \oplus (\mathcal{B}) + \mathsf{R}\text{-}(B_{0,1}) \vdash \forall x^1\, A_{qf}(x, tx),$$

where $t \in \mathsf{G}_\infty\mathsf{R}^\omega[\mathcal{B}, B_{0,1}]$ and t contains only a single application of $B_{0,1}$ to semi-closed terms $A[x], F[x], G[x], c[x]$ and $\mathsf{R}\text{-}(B_{0,1})$ is the rule of $B_{0,1}$ which states that applications of $B_{0,1}$ to semi-closed term of $\mathsf{G}_\infty\mathsf{R}^\omega$ exists. We strengthened the verifying theory to $\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction$ because we do not know whether one can show without $\Sigma^0_1\text{-}\mathsf{IA}$ that an application of $B_{0,1}$ solves the functional interpretation of an instance of $\Pi^0_1\text{-}\mathsf{CA}$.

We now build a majorant t^* of t. The application of $B_{0,1}$ will be majorized like in the proof of Theorem 4.5. By Proposition 3.12 and the fact that the the theory used in this Proposition has a functional interpretation in $\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction$, we obtain that $B^*_{0,1}$ applied to majorants of A, F, G, c majorizes $B_{0,1}AFGc$ Hence we obtain

$$\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction \oplus (\mathcal{B}) + \mathsf{R}\text{-}(B_{0,1}) \vdash \forall x^1\, \exists y \le t^* x\ A_{qf}(x, y),$$

where $t^* \in \mathsf{G}_\infty\mathsf{R}^\omega[B_{0,1}]$ and t^* contains only a single application of $B_{0,1}$ to semi-closed terms with finite computational size.

Applying bounded search we obtain a new realizer t' for y:

$$t'x := \begin{cases} \text{minimal } y \leq t^*x \text{ with } A_{qf}(x,y), & \text{if such a } y \text{ exists,} \\ 0 & \text{otherwise.} \end{cases}$$

Now using the ordinal analysis of $B_{0,1}$ we obtain a term t'' that is provably equal to t' and that is definable using transfinite primitive recursion up to $< \omega^\omega$ and hence in $\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction$, see [37, II.3.18] and also Theorem 3.14. So that

$$\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction \oplus (\mathcal{B}) \vdash \forall x^1\, A_{qf}(x, t''x).$$

The principle $(\mathcal{B})$ may be eliminate from the system with a monotone functional interpretation like we did it in Chapter 3, see [57], [67, section 10.3]. We obtain

$$\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction \vdash \forall x^1\, A_{qf}(x, t''x).$$

Combining this discussion with Proposition 3.21 we obtain the following theorem:

Theorem 4.13. *Let $A_{qf}(x^1, y^0)$ be a quantifier-free formula with only x, y free and P a principle of the form (4.7) which is proofwise low over $\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF\text{-}AC} \oplus \mathsf{WKL}$. If*

$$\widehat{\mathsf{N\text{-}PA}}^\omega\!\restriction + \mathsf{QF\text{-}AC} + \mathsf{WKL} + \mathsf{P} \vdash \forall x^1\, \exists y^0\, A_{qf}(x,y)$$

then one can extract a term $t \in T_0$ such that

$$\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction \vdash \forall x^1\, A_{qf}(x, tx).$$

Together with elimination of extensionality (see [72], [67, section 10.4] and also Proposition 1.5) we obtain:

Corollary 4.14. *If*

$$\widehat{\mathsf{E\text{-}PA}}^\omega\!\restriction + \mathsf{QF\text{-}AC}^{0,1} + \mathsf{QF\text{-}AC}^{1,0} + \mathsf{WKL} + \mathsf{P} \vdash \forall x^1\, \exists y^0\, A_{qf}(x,y)$$

then one can extract a term $t \in T_0$ such that

$$\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction \vdash \forall x^1\, A_{qf}(x, tx).$$

Corollary 4.15. *Let P be a principle of the form (4.7) that is proofwise low over $\mathsf{WKL}_0^{\omega *}$. Then the system $\mathsf{WKL}_0^\omega + \mathsf{P}$ is conservative over RCA_0^ω for sentences of the form $\forall x^1\, \exists y^0\, A_{qf}(x,y)$. Moreover, one can extract from a proof of this statement a term $t \in T_0$ realizing y (that is a primitive recursive functional in the sense of Kleene).*

In particular, $\mathsf{WKL}_0^\omega + \mathsf{P}$ is Π^0_3-conservative over RCA_0^ω and Π^0_2-conservative over PRA.

Proof. The first part of this corollary is just a reformulation of the previous corollary. The second part follows from the observation that over RCA_0^ω each Π^0_3-sentence is equivalent to a sentence of the form $\forall x^1\,\exists y^0\,A_{qf}(x,y)$. The last statement follows from the fact that RCA_0^ω is Π^0_2-conservative over PRA. □

Remark 4.16. With the techniques from Section 2.5 one may also allow principles P where $P(c,g)$ is Π^0_3. However we will not need this here.

4.4. Application to the chain antichain principle

Let the chain antichain principle (CAC) be the principle that states that every partial order on $\mathbb{N}$ has an infinite chain or antichain. For notational ease we assume here that each (anti)chain is also ordered by the ordering of $\mathbb{N}$. We formalize CAC in the following way:

$$\begin{aligned}(\mathsf{CAC})\colon \forall \chi_P\,\exists H\Big(\quad &\forall u,v\in H\,(u<v\to u\leq_P v)\\ &\lor \forall u,v\in H\,(u<v\to u\geq_P v)\\ &\lor \forall u,v\in H\,(u<v\to u\,|_P\, v)\Big),\end{aligned}$$

where the set H is a given as strictly increasing enumeration, i.e. H is a function such that Hn is the n-th element of H.[2] The partial order P is given by its characteristic function χ_P. The relations $\leq_P$, $|_P$ are defined to be

$$\begin{aligned}u\leq_P v &:\equiv \begin{cases}\chi_P(u,v)=0 & \text{The relation defined by } \chi_P \text{ forms a partial order} \\ & \text{on the set } [0;\max(u,v)],\\ x=y & \text{otherwise,}\end{cases}\\ u\,|_P\,v &:\equiv \neg\,(u\leq_P v)\land\neg\,(v\leq_P u)\,.\end{aligned}$$

(We assume here that the paring $\langle u,v\rangle$ is monotone in both components.) With this any function χ_P describes a partial order.

Hirschfeldt and Shore observed in [39] that CAC splits into the cohesive principle and the, so called, stable chain antichain principle. The *stable chain antichain principle* (SCAC) is the restriction of CAC to stable partial orderings, where we call a partial ordering $\leq_P$ *stable* if one of the following holds

[2] Strictly speaking we cannot quantify over strictly monotone functions. Officially, we quantify over all functions from $\mathbb{N}\to\mathbb{N}$ and replace every occurrence of $H(n)$ by

$$\tilde H(n) := \begin{cases}H(n) & \text{if } n=0 \text{ or } H(n)>\tilde H(n\dot-1),\\ \tilde H(n\dot-1)+1 & \text{otherwise.}\end{cases}$$

(i) For all x either $x \leq_P y$ for all but finitely many y or $x \mid_P y$ for all but finitely many y.

(ii) For all x either $x \geq_P y$ for all but finitely many y or $x \mid_P y$ for all but finitely many y.

We will show in this section that CAC is proofwise low over $\mathsf{G_\infty A^\omega} + \mathsf{QF\text{-}AC} \oplus \mathsf{WKL}$ and hence that Theorem 4.13 and the Corollaries 4.14 and 4.15 apply to it. This strengthens our result of Chapter 2, where we were only able to handle COH.

Our proof is based on [20]. The non-standard construction is replaced by the following argument.

4.4.1. Building infinite sets without Σ^0_1-induction

Recall a set X is

- *infinite* or *unbounded* if
$$\forall k\, \exists n > k\, n \in X$$
and

- *strictly increasingly enumerable* if there
$$\text{exists a strictly monotone function } f \text{ such that } rng(f) = X.$$

Further, recall that a strictly increasingly enumerable set is also unbounded. However, to construct a strictly increasing enumeration for an unbounded set in general requires Σ^0_1-IA (e.g. RCA_0 or $\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction + \mathsf{QF\text{-}AC}$).

We will now discuss a way to build unbounded sets in a system that does not contain Σ^0_1-IA. Let f be a function that maps (codes of) finite subsets of $\mathbb{N}$ into (codes of) finite subsets of $\mathbb{N}$ and that is monotone in the sense of

$$x \subsetneq f(x), \qquad f(x) \setminus x \subseteq [\max(x) + 1, \infty[\,, \tag{4.9}$$

where $\max(\emptyset) := -1$.

Define now $X \subseteq \mathbb{N}$ by
$$X := \bigcup_{n\in\mathbb{N}} f^n(\emptyset),$$
where f^n is the n-th iteration of f.

The properties of f ensure that

$$n \in X \quad \longleftrightarrow \quad n \in f^{n+1}(\emptyset). \tag{4.10}$$

Hence, the function $g(n) := \big[n\text{-th element of } f^{n+1}(\emptyset)\big]$ defines a strictly increasing enumeration of X that is definable for instance in RCA_0 or $\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction + \mathsf{QF\text{-}AC}$ (if f is).

In a system without Σ^0_1-IA (e.g. RCA_0^* or $\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF\text{-}AC}$) it is a priori not clear whether X is well defined since one cannot build the n-th iterate of the unbounded function f.

To define a set that is provably equal to X let

$$\tilde{f}_k(x) := \begin{cases} f(x) & \text{if } f(x) \subseteq [0,k[, \\ x & \text{otherwise.} \end{cases}$$

The function $\tilde{f}_k$ is bounded and therefore can be iterated using bounded recursion. For $\tilde{f}_k$ we have the following equivalence

$$n \in X \quad \longleftrightarrow \quad n \in f^{n+1}(\emptyset) \quad \longleftrightarrow \quad n \in f\left((\tilde{f}_n)^n(\emptyset)\right).$$

To see that the last equivalence holds let m' be the least $m \le n+1$ with $f^m(\emptyset)\cap[n,\infty[\neq \emptyset$. By (4.9) we have $f^{(m' \dot{-} 1)}(\emptyset) \subseteq [0,n[$ and hence $(\tilde{f}_n)^n(\emptyset) = f^{(m' \dot{-} 1)}(\emptyset)$ and $f(\tilde{f}_n)^n(\emptyset) = f^{m'}(\emptyset)$.

Therefore, we can define that characteristic function χ_X by

$$\chi_X(n) := \begin{cases} 0 & \text{if } n \in f\left((\tilde{f}_n)^n(\emptyset)\right), \\ 1 & \text{otherwise.} \end{cases}$$

To show now that X is unbounded assume for a contradiction that X is bounded by b. By the definition of X we then have that $(\tilde{f}_{b+1})^n(\emptyset) = f^n(\emptyset)$. Hence f is also bounded (at least along the iteration). Therefore bounded recursion suffices to iterate the function and the strictly increasing enumeration g of the set X can be defined. But this contradicts the boundedness of X. Hence X is unbounded.

4.4.2. Proofwise low

We will use the ideas of the preceding section to show that CAC is proofwise low over $\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF\text{-}AC} \oplus \mathsf{WKL}$. To apply these ideas let uCAC be the CAC with the exception that it only require an unbounded (anti)chain, i.e.

$$\begin{aligned}
(\mathsf{uCAC})\colon \forall \chi_P\, \exists H = \chi_H, f_H \Big(\forall n\, \max(f_H(n), n) \in H \\
\wedge \Big(\quad & \forall u, v \in H\, (u < v \to u \le_P v) \\
\vee\ & \forall u, v \in H\, (u < v \to u \ge_P v) \\
\vee\ & \forall u, v \in H\, (u < v \to u \mid_P v)\Big)\Big).
\end{aligned}$$

Here H is given as a characteristic function χ_H plus a witness for the unboundedness f_H (i.e. $f_H(n) \geq n$ and its range is included in H). Let uSCAC be the restriction of uCAC to stable partial orderings.

For a partial order $\leq_P$ define

$$A_\Box := \{x \mid x \mathrel{\Box} y \text{ for all but finitely many } y\},$$

where $\Box \in \{\leq_P, \geq_P, |_P\}$. If $\leq_P$ is stable then these sets are disjoint and either $A_{\leq_P} \cup A_{|_P} = \mathbb{N}$ or $A_{\geq_P} \cup A_{|_P} = \mathbb{N}$. Hence these sets are Δ^0_2. One can easily establish that each infinite chain, antichain is a subset of $A_{\leq_P}$ resp. $A_{\geq_P}$, $A_{|_P}$.

We will write in the following $y \subseteq^{fin} X$ for y being a code for a finite subset of a set X and $y \sqsubseteq X$ for y is an initial segment of the strictly increasing enumeration of the set X.

Proposition 4.17. *For every closed term φ there exists a closed term ξ such that*

$$\begin{aligned}&\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF}\text{-}\mathsf{AC}\\ &\qquad \vdash \forall \chi_P \left(\Pi^0_1\text{-}\mathsf{CA}(\xi\chi_P) \to \exists H, f_H \left(\mathsf{uSCAC}(\chi_P, H) \land \Pi^0_1\text{-}\mathsf{CA}(\varphi\chi_P H f_H)\right)\right).\end{aligned}$$

Here $\mathsf{uSCAC}(\chi_P, H, f_H)$ expresses that H, f_H is a solution to uSCAC and the partial order described by χ_P.

In other words uSCAC is proofwise low over $\mathsf{G}_\infty\mathsf{A}^\omega + \mathsf{QF}\text{-}\mathsf{AC}$.

Proof. Let χ_P be the characteristic function of a stable partial ordering. Without loss of generality we assume that (i) from the definition of stability holds, the case (ii) can be handled analogously.

We will start with the following claim:

Claim: Let Y be an infinite Σ^0_1-set whose characteristic function is given by a term t which contains only χ_P and type 0 variables free. This means $n \in Y$ iff $\exists x\, tnx = 0$. Then Y either has an element in $A_{\leq_P}$ or one can define an infinite antichain that solves the lemma.

Proof of the claim: Suppose that Y does not contain an element of $A_{\leq_P}$ i.e. $Y \subseteq A_{|_P}$. By an instance of Π^0_1-CP (which follows from the instance of Π^0_1-CA), we know that for each finite set $y \subseteq^{fin} A_{|_p}$ there is common point of stability, i.e. a k, such that for all $z > k$ each element of y is incomparable. Together with the properties of Y this yields

$$\forall y \subseteq^{fin} Y\ (y \text{ is an antichain} \to \exists z \in Y\ y \cup \{z\} \text{ is an antichain}).$$

This is equivalent to

$$\begin{aligned}&\forall y\, \forall x\ (\forall i < \mathrm{lth}(y)\ t(y)_i(x)_i = 0 \land y \text{ is an antichain})\\ &\qquad\qquad\qquad \to \exists z, x'\ (tzx' = 0 \land y \cup \{z\} \text{ is an antichain}).\end{aligned}$$

Now let f be the choice function that chooses the unique z (and x') extending y (and x). For instance f could choose the minimal pair $\langle z, x' \rangle$.

Iterating f using an instance of Σ^0_1-IA (which also follows from the instance of Π^0_1-CA) yields an infinite antichain H. The instance of comprehension Π^0_1-$\mathsf{CA}(\varphi\chi_P H)$ can be reduced to the imposed instance of comprehension using the following equivalence

$$\forall n\ (\forall k\, \varphi\chi_P Hnk \leftrightarrow \forall k\, \forall h \sqsubseteq H\, \alpha_{\varphi\chi_P}(h, n, k) \leq 1)$$

and the fact that $h \sqsubseteq H$ can be expressed using a quantifier-free formula depending only on t, h. (This formula just expresses that h, x are the result of the iteration of f.) The function $\alpha_{\varphi\chi_P}(h, n, k)$ here is an associate to the function $\lambda H.\varphi\chi_P Hnk$. For notational ease we assume here that H is given as strictly increasing enumeration. Since one can define from this a characteristic function for H and f_H by a term in $\mathsf{G}_\infty\mathsf{A}^\omega$ this does not lead to any problems. This proves the claim.

We assume from now on that there is no Σ^0_1-set $Y \subseteq A_{|P}$ given by such a term t. Otherwise we would be done. The assumption implies that $A_{\leq_P}$ has infinitely many elements. (If not the set $Y := [\max(A_{\leq_P}) + 1, \infty[$ would be an infinite subset of $A_{|P}$ which could be easily described by a term.) We will show that we can construct an unbounded $\leq_P$-chain $H \subseteq A_{\leq_P}$ for which we can prove the instance of Π^0_1-CA.

First we define a function $g_1(n, h)$ that for a given n extends a given finite $\leq_P$-chain $h \subseteq^{fin} A_{\leq_P}$ to a finite $\leq_P$-chain $h' \subseteq^{fin} A_{\leq_P}$ such that for all $\leq_P$-chains X with $h' \sqsubseteq X$ and $X \subseteq A_{\leq_P}$ the following holds

$$\forall n' < n\ (\forall k\ \varphi\chi_P Xn'k = 0 \leftrightarrow \forall k\ \alpha_{\varphi\chi_P}(h', n', k) \leq 1)\,. \tag{4.11}$$

In other words we extend the initial segment h to h' such that the instance of comprehension Π^0_1-$\mathsf{CA}(\varphi\chi_P H)$ is decided up to the index n.

Define for each $D \subseteq [0, n]$ the set

$$S_{D,h} := \{h' \mid h' \text{ is a finite } \leq_P\text{-chain} \wedge h \sqsubseteq h' \wedge \forall n' \in D\, \exists k\, \alpha_{\varphi\chi_P}(h', n', k) > 1\}.$$

The elements of this set are those extensions of h which make the comprehension Π^0_1-$\mathsf{CA}(\varphi\chi_P H)$ for the indices in D false. This set is Σ^0_1 and can be defined by a fixed term containing only the parameters χ_P, D, h.

The statement that there is no extension of h in $S_{D,h}$ whose elements are in $A_{\leq_P}$ is

$$\forall y\ \left(y \notin S_{D,h} \cap \mathcal{P}^{fin}(A_{\leq_P})\right). \tag{4.12}$$

This formula is Π^0_2. We will show that there exists a Σ^0_2 formula that is equivalent and hence that the statement is Δ^0_2.

Consider the set $M_{D,h} := \{\max_P(y) \mid y \in S_{D,h}\}$, where $\max_P(y)$ is the $\leq_P$ maximum of the chain y. This set is also Σ^0_1 and again does only depend on χ_P and the type 0 objects D, h.

We will distinguish the following cases:

- The set $M_{D,h}$ is infinite. In this case there exists by the assumption and the claim an element of $M_{D,h}$ that is also in $A_{\leq_P}$. This means that there exits a $\leq_P$-chain y in $S_{D,h}$ whose $\max_P$ is in $A_{\leq_P}$ and hence the whole $\leq_P$-chain is in $A_{\leq P}$. Therefore (4.12) fails.

- The set $M_{D,h}$ is finite. Each chain in $S_{D,h}$ contains only elements which are $\leq_P x$ for some $x \in M_{D,h}$. By stability for each $x \in M_{D,h}$ there are only finitely many elements y with $x \geq_P y$. Applying Π^0_1-CP to this yields that there are only finitely elements y with $\exists x \in M_{D,h}\, y \leq_P x$ and hence that $S_{D,h}$ is finite.

In total (4.12) is equivalent to

$$\begin{aligned}\exists x \Big(\quad & \forall y \left(y \text{ is } \leq_P\text{-chain} \wedge \max_P(y) > x \to y \notin S_{D,h}\right) \\ & \wedge \forall y \left(y \text{ is } \leq_P\text{-chain} \wedge \max_P(y) \leq x \to y \notin S_{D,h} \cap \mathcal{P}^{fin}(A_{\leq_P})\right)\Big)\end{aligned}$$

where the second quantification over y can be bounded and hence (4.12) is Δ^0_2.

Therefore an instance of Δ^0_2-CA (which is provable from an instance of Π^0_1-CA, see Lemma 1.10.(ii)) is sufficient to prove that there exists a maximal $D' \subseteq [0,n]$ for which $S_{D,h} \cap \mathcal{P}^{fin}(A_{\leq_P})$ is not empty, i.e.

$$\begin{aligned}\exists D' \subseteq [0,n]\, \exists h' \, (h' \in S_{D',h} \cap \mathcal{P}^{fin}(A_{\leq_P}) \\ \wedge \forall E \left(D' \subsetneq E \subseteq [0,n] \to \forall h' \left(h' \notin S_{E,h} \cap \mathcal{P}^{fin}(A_{\leq_P})\right)\right)).\end{aligned}$$

Since D' is maximal each $h' \in S_{D',h} \cap \mathcal{P}^{fin}(A_{\leq_P})$ satisfies (4.11).

Hence taking for $g_1(n,h)$ the function that chooses for h and n an $h' \in S_{D',h} \cap \mathcal{P}^{fin}(A_{\leq_P})$ for a maximal D' has the desired properties. This choice function exists by an instance of Σ^0_2-AC which is also provable from an instance of Π^0_1-CA.

Now define g_2 to be a function which extends each chain $h \subseteq^{fin} A_{\leq_P}$ by one element in $A_{\leq_P}$, for instance

$$g_2(h) := h \cup \{\, \min\{x \in A_{\leq_P} \mid \max(h) < x \wedge \max_P(h \cap A_{\leq_P}) \leq_P x\}\}.$$

This function exists also by an instance of Δ^0_2-CA and Σ^0_1-AC (which follows from QF-AC).

The function $f(h) := g_2(g_1(\max(h), h))$ now satisfies the properties in (4.9) on page 86. By the discussion in the previous section the set $H := \bigcup_n f^n(\emptyset)$ is definable in this system and provably unbounded. The values of f are finite $\leq_P$-chains that are included in $A_{\leq_P}$. Hence H defines an unbounded $\leq_P$-chain.

Furthermore, one can prove Π^0_1-$\mathsf{CA}(\varphi\chi_P H)$: To decide whether

$$\forall k\, \varphi\chi_P H n k = 0 \tag{4.13}$$

holds for an n take an element $x \in H$ with $x > n$. By the unboundedness this exists. In particular there exists a smallest m such that $x \in f^m(\emptyset)$ (or equivalently $x \in f\left((\tilde{f}_x)^{m \dot{-} 1}(\emptyset)\right)$). For this we have $f^m(\emptyset) = f\left((\tilde{f}_x)^x(\emptyset)\right)$. By the definition g_1 and (4.11) we have that (4.13) is true iff

$$\forall k\, \alpha_{\varphi\chi_P}(g_1(\max(f^m(\emptyset)), f^m(\emptyset)), n, k) \leq 1.$$

This is again by the definition of g_1 true iff

$$\forall k\, \alpha_{\varphi\chi_P}(f^{m+1}(\emptyset), n, k) \leq 1.$$

Then

$$\forall k\, \alpha_{\varphi\chi_P}(ff((\tilde{f}_x)^x(\emptyset)), n, k) \leq 1$$

and thus can be computed using the imposed instance of comprehension by computing the comprehension function with the parameters x, n for $\forall k\, \alpha_{\varphi\chi_P}(x, n, k) \leq 1$ and in parallel the function f.

The different instances of Π^0_1-CA can be coded together into a term ξ, see Remark 1.9. This solves the proposition. $\square$

Corollary 4.18. CAC *is proofwise low over* $\mathsf{G_\infty A}^\omega + \mathsf{QF\text{-}AC} \oplus \mathsf{WKL}$.

Proof. Lemma 1.11 for $n = 0$ shows that one can iterate f_H in the results of Proposition 4.17 while retaining the instance of comprehension. With this one can define an strictly increasing enumeration of H and hence shows that SCAC is proofwise low over $\mathsf{G_\infty A}^\omega + \mathsf{QF\text{-}AC}$.

The result follows from the fact that COH is proofwise low of $\mathsf{G_\infty A}^\omega + \mathsf{QF\text{-}AC} \oplus \mathsf{WKL}$ (Corollary 2.5) and from noting that the proof

$$\mathsf{SCAC} + \mathsf{COH} \rightarrow \mathsf{CAC}$$

in [39, Proposition 3.7] can be carried out in $\mathsf{G_\infty A}^\omega$ while retaining the proofwise low property. $\square$

Theorem 4.19. *The system*

$$\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction + \mathsf{QF\text{-}AC} \oplus \mathsf{WKL} \oplus \mathsf{CAC}$$

is conservative over $\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction$ *for sentences of the form* $\forall x^1\, \exists y^0\, A_{qf}(x, y)$. *Moreover one can extract a primitive recursive realizing term* $t[x]$ *for* y.

In particular,

$$\mathsf{WKL}_0^\omega + \mathsf{CAC}$$

is conservative for sentences of the from $\forall x^1\, \exists y^0\, A_{qf}(x, y)$ *and a fortiori* Π^0_3*-conservative over* RCA_0^ω.

Proof. Corollary 4.18 and Corollaries 4.14, 4.15. □

This result raises the question whether one can extend it and show that RT^2_2 is proofwise low over a system like $\mathsf{WKL}_0^{\omega *}$ or any other system without Σ^0_1-induction and thus can show that RT^2_2 does not imply Σ^0_2-induction.

Let the Erdős-Moser principle (EM) be the principle that states that every tournament on $\mathbb{N}$ contains an infinite transitive subgraph. A tournament is a directed graph $\langle \mathbb{N}, \to \rangle$ such that for each pairs of nodes x, y either $x \to y$ or $x \leftarrow y$. The principle RT^2_2 is equivalent to $\mathsf{CAC} + \mathsf{EM}$ (in fact even to $\mathsf{ADS} + \mathsf{EM}$), see Chapter 5. Corollary 4.18 shows that is sufficient to show the EM is proofwise low over a system without Σ^0_1-induction in order to show that RT^2_2 does not imply Σ^0_2-induction.

5. The Erdős-Moser principle

A tournament is a directed graph $\langle E, \to \rangle$ such that for each pairs of nodes x, y with $x \neq y$ either $x \to y$ or $x \leftarrow y$ but not both. The Erdős-Moser principle (EM) states that each tournament on $\mathbb{N}$ contains an infinite transitive subtournament. It is easy to see that EM follows from RT^2_2 if one identifies the tournament with the following 2-coloring of pairs of $\mathbb{N}$: For $x < y$ let

$$\begin{aligned} c(\{x, y\}) = 0 \quad &\text{iff} \quad x \to y, \\ c(\{x, y\}) = 1 \quad &\text{iff} \quad x \leftarrow y. \end{aligned} \tag{5.1}$$

On any homogeneous set of c the relation $\to$ is transitive. Hence RT^2_2 yields an infinite transitive subtournament.

In the other direction EM and ADS (the principle CAC restricted to linear orderings) imply RT^2_2. To see this let for some coloring c the relation $\to$ be defined by (5.1). Using EM one finds an infinte subset on which $\to$ is a linear ordering. The principle ADS yields an infinite $\to$-chain. By definition c is constant on this chain.

The principle EM was introduced by Bovykin and Weiermann in [12]. They also proved the above stated equivalence.

We now give some lower bounds on the strength of EM:

Proposition 5.1.

$$\mathsf{RCA}_0 \vdash \mathsf{EM} \to \Pi^0_1\text{-}\mathsf{CP}$$

Proof. We show that EM proves the infinite pigeonhole principle. The result follows from this by [41].

Let $f \colon \mathbb{N} \to n$ be coloring of $\mathbb{N}$ with n colors. We consider the following infinite tournament. For $x < y$ let

$$\begin{aligned} x \to y \quad &\text{iff} \quad f(x) = f(y), \\ x \leftarrow y \quad &\text{iff} \quad f(x) \neq f(y). \end{aligned}$$

Applying EM yields and an infinite set X on which $\to$ is transitive. We claim that f restricted to X eventually becomes constant. Suppose not, then

$$\forall k \in X\, \exists x \in X\ (k < x \wedge f(k) \neq f(x))$$

which is by definition of $\to$

$$\forall k \in X \, \exists x \in X \; (k < x \wedge k \leftarrow x)$$

Now applying Σ^0_1-induction we obtain $n+1$ elements $x_1, \ldots, x_{n+1} \in X$ with

$$x_1 < x_2 < \cdots < x_{n+1} \qquad \text{and} \qquad x_1 \leftarrow x_2 \leftarrow \cdots \leftarrow x_{n+1}.$$

By transitivity and definition of $\to$ we obtain that $f(x_i)$ are pairwise different. But this contradicts the fact that f is bounded by n.

The infinite pigeonhole principle for f and hence the proposition follows from this. □

Proposition 5.2. *There exists a computable tournament $\langle \mathbb{N}, \to \rangle$ that has no low infinite transitive subtournament, i.e. no set X such that $\to$ is transitive on X and $X' \leq_T 0'$.*

Proof. By [25] there exists a computable stable 2-coloring of pairs c, such that there is no low homogeneous set. Let $\to$ be the corresponding tournament as described by (5.1).

Suppose now that there is a low set X on which $\to$ is transitive and hence a linear ordering. Since c is stable this ordering is also stable. By Theorem 2.11 of [39] there exists an infinite chain Y that is low relative to X and hence low. Since on this chain the coloring c is homogeneous, this contradict the choice of c. □

Part II.

The Bolzano-Weierstraß principle

6. The Bolzano-Weierstraß principle and the cohesive principle

In this chapter, we show that BW and Σ^0_1-WKL are *instance-wise* equivalent. Instance-wise means here that for every instance of BW, i.e. every bounded sequence, one can compute, uniformly, an instance of Σ^0_1-WKL, i.e. a code for an infinite Σ^0_1-0/1-tree, such that from a solution of this instance of Σ^0_1-WKL one can compute, uniformly, an accumulation point and vice versa. *Instance-wise equivalence* refines the usual logical equivalence where the full second order closure of the principles may be used—e.g. Π^0_∞-CA and Π^0_1-CA are equivalent but they are not instance-wise equivalent. As consequence we obtain that the Turing degrees containing solutions to all instances of Σ^0_1-WKL (i.e. the degrees d with $d \gg 0'$, see below) are exactly those containing an accumulation point for each computable bounded sequence.

Furthermore, we show that $\mathsf{BW}_{\mathsf{weak}}$ is instance-wise equivalent to the strong cohesive principle. Using this one can apply classification results obtained for the (strong) cohesive principle. In particular, this shows also that $\mathsf{BW}_{\mathsf{weak}}$ does not lead to more than primitive recursive growth when added to RCA_0.

6.1. Cohesive Principle

We call a set (*p-cohesive*) *r-cohesive* if it is cohesive for all (primitive) recursive sets. We will denote by $(\mathsf{St})\mathsf{COH}(X)$ the statement that for the sequence of sets $(R_n)_n$ coded by X an infinite (strongly) cohesive set exists.

Recall that StCOH is equivalent to $\mathsf{COH} \wedge \Pi^0_1$-$\mathsf{CP}$. Since Π^0_1-CP follows from Σ^0_2-induction, there is no recursion theoretic difference between StCOH and COH.

To state the recursion theoretic strength of COH we will need following notation. Denote by $a \gg b$ that the Turing degree a contains an infinite computable branch for every b-computable 0/1-tree, see [85]. In particular, the degrees $d \gg 0'$ are exactly those which contain an infinite path for every Σ^0_1-0/1-tree. By the low basis theorem for every b there exists a degree $a \gg b$ which is *low* over b, i.e. $a' \equiv b'$, see [51].

Theorem 6.1 ([48, 49], see also [16, Theorem 12.4])**.** *For any degree d the following are equivalent:*

- *There is an r-cohesive (p-cohesive) set with jump of degree d,*

- $d \gg 0'$.

In particular, there exists a low$_2$ r-cohesive set.

Theorem 6.2. COH *is* Π^1_1*-conservative over* RCA_0*,* $\mathsf{RCA}_0 + \Pi^0_1\text{-}\mathsf{CP}$*,* $\mathsf{RCA}_0 + \Sigma^0_2\text{-}\mathsf{IA}$.

This result for RCA_0 and $\mathsf{RCA}_0 + \Sigma^0_2\text{-}\mathsf{IA}$ is due to Cholak, Jockusch, Slaman, see [16], the result for $\mathsf{RCA}_0 + \Pi^0_1\text{-}\mathsf{CP}$ is due to Chong, Slaman, Yang, see [20].

In Chapter 2 we proved $\mathsf{RCA}_0 + \mathsf{StCOH}$ is Π^0_2-conservative over PRA and admits a program-extraction of primitive recursive term.

6.2. Bolzano-Weierstraß principle

Let BW be the statement that every sequence $(y_i)_{i\in\mathbb{N}}$ of rational numbers in the interval $[0,1]$ admits a fast converging subsequence, that is a subsequence converging with the rate 2^{-n} or equivalently any other rate given by a computable function resp. by a function in the theory. This principle covers the full strength of Bolzano-Weierstraß, i.e. one can take a bounded sequence of real numbers.

Let $\mathsf{BW}_{\mathsf{weak}}$ be the statement that every sequence $(y_i)_{i\in\mathbb{N}}$ of rational numbers in the interval $[0,1]$ admits a Cauchy subsequence (a sequence converging but not necessarily fast), more precisely

$$(\mathsf{BW}_{\mathsf{weak}}):$$
$$\forall (y_i)_{i\in\mathbb{N}} \subseteq \mathbb{Q}\cap[0,1]\, \exists f \text{ strictly monotone} \forall n\, \exists s\, \forall v,w \ge s\ \ |y_{f(v)} - y_{f(w)}| <_{\mathbb{Q}} 2^{-n}.$$

The statement $\mathsf{BW}_{\mathsf{weak}}$ also implies that every bounded sequence of real numbers contains a Cauchy subsequence. Just continuously map the bounded sequence into $[0,1]$ and take a diagonal sequence of rational approximations of the elements of the original sequence.

We will denote by $\mathsf{BW}(Y)$ and $\mathsf{BW}_{\mathsf{weak}}(Y)$ the statement that the bounded sequence coded by Y contains a (slowly) converging subsequence.

The principles BW and $\mathsf{BW}_{\mathsf{weak}}$ also imply the corresponding Bolzano-Weierstraß principle for the Cantor space $2^{\mathbb{N}}$:

Lemma 6.3. *Over* RCA_0

- BW *implies the Bolzano-Weierstraß principle for the Cantor space* $2^{\mathbb{N}}$ *and*
- $\mathsf{BW}_{\mathsf{weak}}$ *implies the weak Bolzano-Weierstraß principle for the Cantor space* $2^{\mathbb{N}}$*, i.e. for every sequence in* $2^{\mathbb{N}}$ *there exists a slowly converging Cauchy subsequence.*

Moreover these implications are instance-wise*, i.e. there exists an* e *such that over* RCA_0 *the (weak) Bolzano-Weierstraß principles for a sequence* $(x_i)_{i\in\mathbb{N}} \subseteq 2^{\mathbb{N}}$ *coded by* X *is implied by* $\mathsf{BW}_{(\mathsf{weak})}(\{e\}^X)$.

Proof. Define the mapping $h\colon 2^{\mathbb{N}} \to [0,1]$ as

$$h(x) = \sum_{i=0}^{\infty} \frac{2x(i)}{3^{i+1}}.$$

The image of h is the Cantor middle-third set.

One easily establishes

$$dist_{2^{\mathbb{N}}}(x,y) < 2^{-n} \quad \text{iff} \quad dist_{\mathbb{R}}(h(x),h(y)) < 3^{-(n+1)}.$$

Therefore (slow) Cauchy sequences of $2^{\mathbb{N}}$ primitive recursively correspond to (slow) Cauchy sequences of the Cantor middle-third set.

For $\{e\}$ choose the function mapping $(x_i)_{i\in\mathbb{N}}$ to $(h(x_i))_{i\in\mathbb{N}}$. The lemma follows. □

The full Bolzano-Weierstraß principle (BW) results from $\mathsf{BW}_{\mathsf{weak}}$, if we additionally require an effective Cauchy-rate, e.g. $s = 2^{-n}$ in the above definition of $\mathsf{BW}_{\mathsf{weak}}$. One also obtains full BW if one uses an instance of Π^0_1-comprehension (or Turing jump) to thin out the Cauchy sequence making it fast converging.

The weak version of the Bolzano-Weierstraß principle is for instance considered in computational analysis, see [71, Section 3].

$\mathsf{BW}_{\mathsf{weak}}$ is also interesting in the context of proof-mining or "hard analysis", i.e. the extraction of quantitative information for analytic statements. For an introduction to hard analysis see [93, §1.3], for proof-mining see [67]. For instance if one uses $\mathsf{BW}_{\mathsf{weak}}$ to prove that a sequence converges, by Theorem 6.7 below one can expect a primitive recursive rate of metastability, in the sense of Tao [93, §1.3]. Such proofs occur in fixed-point theory, for example Ishikawa's fixed-point theorem uses such an argument, see [66, 45].

Note that in this case only a single instance of the Bolzano-Weierstraß principle is used and the accumulation point is not used in a Σ^0_1-induction, therefore one obtains the same results using Kohlenbach's elimination of Skolem functions for monotone formulas, see for instance [63, Theorem 1.2]. Nested uses of BW imply arithmetic comprehension and thus lead to non-primitive recursive growth. In contrast to that, we will show that even nested uses of $\mathsf{BW}_{\mathsf{weak}}$ in a context with full Σ^0_1-induction do not result in more than primitive recursive growth.

6.3. Results

Theorem 6.4. *Over* RCA_0 *the principles* BW *and* Σ^0_1-WKL *are instance-wise equivalent. More precisely*

$$\mathsf{RCA}_0 \vdash \exists e_1\, \forall X\, \left(\Sigma^0_1\text{-}\mathsf{WKL}(\{e_1\}^X) \to \mathsf{BW}(X)\right),$$
$$\mathsf{RCA}_0 \vdash \exists e_2\, \forall Y\, \left(\mathsf{BW}(\{e_2\}^Y) \to \Sigma^0_1\text{-}\mathsf{WKL}(Y)\right),$$

where Σ^0_1-$\mathsf{WKL}(Y)$ *is weak König's lemma for a* Σ^0_1*-tree coded by* Y*.*

In language with higher order functionals $\{e_1\}$ *and* $\{e_2\}$ *could be given by fixed primitive recursive functionals.*

Proof. For the first implication see [82] and [59, Section 5.4].

For the converse implication note that Σ^0_1-WKL is instance-wise equivalent to Σ^0_2-separation, i.e. the statement that for two Σ^0_2-sets A_0, A_1 with $A_0 \cap A_1 = \emptyset$ there exists a set S, such that $A_0 \subseteq S \subseteq \overline{A_1}$. This is for instance a consequence of [86, lemma IV.4.4] relativized to Δ^0_2-sets. This proof of this lemma also yields a construction of the sets A_0, A_1, i.e. an e' such that $\{e'\}^Y$ yields a set coding A_0, A_1.

Thus is suffices to prove Σ^0_2-separation of two Σ^0_2-sets A_0, A_1.

Let B_i for $i < 2$ be a quantifier free formula such that

$$n \in \overline{A_i} \equiv \forall x\, \exists y\, B_i(x, y; n).$$

We assume that y is unique; one can always achieve this by requiring y to be minimal. Note that by assumption $\forall x\, \exists y\, B_0(x, y; n) \vee \forall x\, \exists y\, B_1(x, y; n)$.

Then define

$$f_i(n, k) := \max \{ s < k \mid \forall x < \mathrm{lth}\, s\ (B_i(x, (s)_x; n))\} .$$

We use here a sequence coding that is monotone in each component, i.e. for two sequences s, t with the same length we have $s \leq t$ if $(s)_x \leq (t)_x$ for all $x < \mathrm{lth}(s)$, see for instance [67, definition 3.30].

If for fixed n, i the statement $\forall x\, \exists y\, B_i(x, y; n)$ holds and f_y is the choice function for y, i.e. the function satisfying $\forall x\, B_i(x, f_y(x); n)$, then for the course-of-value function $\bar{f}_y$ of f_y

$$f_i(n, \bar{f}_y(m) + 1) = \bar{f}_y(m).$$

If $\forall x\, \exists y\, B_i(x, y; n)$ does not hold then $\lambda k.f_i(n, k)$ is bounded. Define $g_i(n, k) := \mathrm{lth}(f_i(n, k))$ and for each n let $g_{i,n} := \lambda k.g_i(n, k)$. Then for each i

$$\text{the range of } g_{i,n} \text{ is } \mathbb{N} \quad \text{iff} \quad \forall x\, \exists y\, B_i(x, y; n).$$

Therefore it is sufficient to find a set S obeying

$$\forall n\ (rng(g_{0,n}) \neq \mathbb{N} \rightarrow n \in S \wedge rng(g_{1,n}) \neq \mathbb{N} \rightarrow n \notin S)\,. \tag{6.1}$$

Define a sequence $(h_k)_{k \in \mathbb{N}} \subseteq 2^{\mathbb{N}}$ by

$$h_k(n) := \begin{cases} 0 & \text{if } g_0(n, k) \geq g_1(n, k), \\ 1 & \text{otherwise.} \end{cases}$$

By hypothesis, for each n there is at least one $i < 2$ such that the range of $g_{i,n}$ is $\mathbb{N}$. For a fixed n, if there is exactly one $i < 2$, such that the range of $g_{i,n}$ is $\mathbb{N}$ then $\lim_{k\to\infty} h_k(n) = i$. In this case (6.1) is satisfied for this n if

$$n \in S \quad \text{iff} \quad \lim_{k\to\infty} h_k(n) = 1.$$

If for each $i < 2$ the range $g_{i,n}$ is $\mathbb{N}$ then (6.1) is trivially satisfied for this n.

Applying BW to h_k, yields an accumulation point h. For h then

$$h(n) = \lim_{k\to\infty} h_k(n) \quad \text{if the limit exists.}$$

Hence h describes a characteristic function of a set S obeying (6.1).

A number e_2 of a Turing machine such that $\{e_2\}^Y$ yields the Cantor middle-third set belonging to $(h_k)_k$ can easily be computed using e from Lemma 6.3 and e'.

This proves the theorem. □

Since

$$\mathsf{RCA}_0 \vdash \Sigma^0_1\text{-}\mathsf{WKL} \leftrightarrow \Pi^0_1\text{-}\mathsf{CA}$$

one obtains as consequence of this theorem that well known result that BW is equivalent to ACA_0 over RCA_0, see [86, theorem I.9.1].

Notice that in Theorem 6.4 the use of Σ^0_1-WKL could neither be replaced by Π^0_1-CA nor Π^0_2-CA.

Theorem 6.5. *Over* RCA_0 *the principles* $\mathsf{BW}_{\mathsf{weak}}$ *and* StCOH *are instance-wise equivalent. More precisely*

$$\mathsf{RCA}_0 \vdash \exists e_1\, \forall X\, \left(\mathsf{StCOH}(\{e_1\}^X) \to \mathsf{BW}_{\mathsf{weak}}(X)\right),$$
$$\mathsf{RCA}_0 \vdash \exists e_2\, \forall Y\, \left(\mathsf{BW}_{\mathsf{weak}}(\{e_2\}^Y) \to \mathsf{StCOH}(Y)\right).$$

In a language with higher order functionals $\{e_1\}$ *and* $\{e_2\}$ *could be given by fixed primitive recursive functionals.*

Proof. To prove $\mathsf{BW}_{\mathsf{weak}}$ for a sequence $(x_i)_{i\in\mathbb{N}}$ coded by X define

$$R_i := \left\{ j \in \mathbb{N} \;\middle|\; x_j \in \bigcup_{k \text{ even}} \left[\frac{k}{2^i}, \frac{k+1}{2^i}\right] \right\}$$

and

$$R^y := \bigcap_{i<\mathrm{lth}(y)} \begin{cases} R_i & \text{if } (y)_i = 0, \\ \overline{R_i} & \text{otherwise.} \end{cases}$$

Let f be a strictly increasing enumeration of a strongly cohesive set for $(R_i)_i$. Then by definition it follows, that

$$\forall i \, \exists y, s \; (\mathrm{lth}(y) = i \wedge \forall w > s \;\; f(w) \in R^y) \,.$$

This statement is equivalent to

$$\forall i \, \exists k, s \, \forall w > s \; \left(x_{f(w)} \in \left[\frac{k}{2^i}, \frac{k+1}{2^i} \right] \right) ,$$

which implies $\mathsf{BW_{weak}}$. Clearly there exists a number e_1 of a Turing machine computing $(R_i)_i$. The first part of the theorem follows.

For the other direction, let $(R_i)_{i\in\mathbb{N}}$ be a sequence of sets coded by Y. Let $(x_i)_{i\in\mathbb{N}} \subseteq 2^{\mathbb{N}}$ be the sequence defined by

$$x_i(n) := \begin{cases} 1 & \text{if } i \in R_n, \\ 0 & \text{if } i \notin R_n. \end{cases}$$

Applying $\mathsf{BW_{weak}}$ and Lemma 6.3 to $(x_i)_i$ yields a slowly converging subsequence $(x_{f(i)})_{i\in\mathbb{N}}$, i.e.

$$\forall n \, \exists s \, \forall j, j' \geq s \, \mathit{dist}(x_{f(j)}, x_{f(j')}) < 2^{-n}.$$

By spelling out the definition of dist and x_i we obtain

$$\forall n \, \exists s \, \forall j, j' \geq s \, \forall i < n \; (f(j) \in R_i \leftrightarrow f(j') \in R_i) \,,$$

which implies that the set strictly monotone enumerated by f is strongly cohesive.

The number e_2 can be easily computed using the construction in Lemma 6.3. $\square$

As immediate corollary we obtain:

Corollary 6.6.

$$\mathsf{RCA_0} \vdash \mathsf{StCOH} \leftrightarrow \mathsf{BW_{weak}}$$

Hence the conservativity results for COH and our term-extraction result from Chapter 2 for StCOH carry over to $\mathsf{BW_{weak}}$ and we obtain the following.

Theorem 6.7.

(i) $\mathsf{BW_{weak}}$ *is* Π^1_1*-conservative over* $\mathsf{RCA_0} + \Pi^0_1\text{-}\mathsf{CP}$, $\mathsf{RCA_0} + \Sigma^0_2\text{-}\mathsf{IA}$. *Especially* $\mathsf{RCA_0} + \mathsf{BW_{weak}}$ *is* Π^0_2*-conservative over* PRA.

(ii) *The theories* $\widehat{\mathsf{E\text{-}PA}}^\omega\!\restriction + \mathsf{QF\text{-}AC}^{0,1} + \mathsf{QF\text{-}AC}^{1,0} + \Pi^0_1\text{-}\mathsf{CP} + \mathsf{COH} + \mathsf{AMT} + \mathsf{WKL} + \mathsf{BW_{weak}}$ *and* $\mathsf{WKL}_0^\omega + \Pi^0_1\text{-}\mathsf{CP} + \mathsf{COH} + \mathsf{AMT} + \mathsf{BW_{weak}}$ *admit program extraction of primitive recursive terms, cf. Corollaries 2.28 and 2.29.*

Proof. Theorems 6.5 and 6.2 and Corollaries 2.28 and 2.29. □

Remark 6.8. The principle ADS, which is CAC restricted to linear orders, is equivalent to the statement that every sequence in $\mathbb{R}$ has a monotone subsequence. If the sequence is bounded then the monotone subsequence is a fortiori converging (possible slowly). Hence ADS and CAC can be seen as generalizations of this variant of the Bolzano-Weierstraß principle and one can strengthen Theorem 6.7 to include this Bolzano-Weierstraß principle instead of $\mathsf{BW}_{\mathsf{weak}}$, cf. Theorem 4.19.

To see that ADS implies that the sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ has an monotone subsequence one has take some care since equality on $\mathbb{R}$ and hence also $\leq_{\mathbb{R}}$ is not decidable. To prove the statement one has to make the following case distinction. Either (x_n) has a constant subsequence or there exists a subsequence of pairwise different elements. The solution to the former case is trivial and the latter case can be solved by applying ADS since $\leq_{\mathbb{R}}$ coincides with $<_{\mathbb{R}}$ on this sequence and is therefore decidable.

For the other direction it suffices to show that each countable linear ordering can be embedded into a subset of $\mathbb{Q}$. This follows from the construction described in the proof of [24, Theorem 2.1] and by noting that it can be carried out in RCA_0.

Here it is also interesting to mention that de Smet and Weiermann did a fine grain analysis of a density variant of this principle restricted to natural numbers in [22, 23].

Theorem 6.9.

(i) *Every bounded recursive sequence of real numbers contains a low*$_2$ *Cauchy subsequence (a sequence converging but not necessarily fast).*

(ii) *There exists a bounded recursive sequence of real numbers containing no computable Cauchy subsequence.*

(iii) *There exists a bounded recursive sequence of real numbers containing no converging subsequence computable in* $0'$*.*

Proof. Theorem 6.5 and Theorem 6.1. For (iii) note that the jump of a slowly converging Cauchy sequence computes a fast converging subsequence. □

Theorem 6.4 gives rise to another proof of this theorem and Theorem 6.1: Let d be a degree containing solutions to all recursive instances of BW. Since BW is equivalent to Σ^0_1-WKL any degree $d \gg 0'$ suffices. Thus we may assume that d is *low* over $0'$, i.e. $d' \equiv 0''$. Now let e be a degree containing solutions to all recursive instances of $\mathsf{BW}_{\mathsf{weak}}$. Since the choice of a fast convergent subsequence of a slow convergent subsequence is equivalent to the halting problem, e may be chosen such that $e' \equiv d$. Thus $e'' \equiv 0''$ or in other words e is *low*$_2$.

Theorem 6.9.(i) improves a result obtained by Le Roux and Ziegler in [71, section 3], which only considers full Turing jumps.

7. The Bolzano-Weierstraß principle for weak compactness

In this chapter we investigate the computational and logical strength of weak sequential compactness in the separable Hilbert space ℓ_2.

The strength of weak compactness has so far only been studied in the context of proof mining, see [68, 56]. There general Hilbert spaces in a more general logical system are considered. It is straightforward to deduce from this analysis that weak compactness for ℓ_2 is equivalent to ACA_0 over RCA_0.

Here we refine this result and show that weak compactness on ℓ_2 is instance-wise equivalent to Π^0_2-CA over RCA_0. This means that for each bounded sequence in ℓ_2 one can uniformly compute a function f such that from a comprehension function for $\forall x\, \exists y\, f(x, y, n) = 0$ one can compute a weak cluster point and vice versa.

As consequence we obtain that the degrees $d \geq_T 0''$ are exactly those degrees that compute a weak cluster point for each computable bounded sequence in ℓ_2 and that there is a computable bounded sequence in ℓ_2 such that from a cluster point of this sequence one can compute $0''$.

This shows that instances of the Bolzano-Weierstraß principle for weak compactness are strictly stronger than instances of the usual Bolzano-Weierstraß principle.

This chapter is organized as follows: first the Hilbert space ℓ_2 is defined. This definition follows [86, 8]. Then the actual results are proven (Theorems 7.8 and 7.12) and we show that the result can also be formulated for abstract Hilbert spaces, in the sense of Kohlenbach [67] (Theorem 7.10). As corollary of this we obtain that Kohlenbach's analysis of the weak compactness functional Ω^* in [56] is optimal (Corollary 7.11). At the end, we reformulate the result of the analysis in terms of the Weihrauch lattice (Remark 7.14).

Definition 7.1 (vector space, [86, II.10])**.** A *countable vector space A over a countable field K* consists of a set $|A| \subseteq \mathbb{N}$ with operations $+\colon |A| \times |A| \to |A|$ and $\cdot\colon |K| \times |A| \to |A|$ and a distinguished element $0 \in |A|$ such that $(|A|, +, \cdot, 0)$ satisfies the usual axioms for a vector space over K.

Definition 7.2 (Hilbert space, [8, Definition 9.3])**.** A *(real) separable Hilbert space H* consists of a countable vector space A over $\mathbb{Q}$ together with a function $\langle \cdot, \cdot \rangle\colon A \times A \to \mathbb{R}$ satisfying

(i) $\langle x, x\rangle \geq 0$,

(ii) $\langle x, y\rangle = \langle y, x\rangle$,

(iii) $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$,

for all $x, y, z \in A$ and $a, b \in \mathbb{Q}$.

The inner product on H induces a pseudo-norm $\|x\| := \sqrt{\langle x, x\rangle}$. We think of the Hilbert space H as the completion of A under the pseudometric $d(x, y) = \|x - y\|$. Thus an element of H consists of a sequence $(x_n)_{n\in\mathbb{N}} \subseteq A$, such that $d(x_n, x_m) < 2^{-n}$ for all $m > n$. The inner product $\langle \cdot, \cdot\rangle$ is continuously extended to the whole space H.

A Hilbert space is finite dimensional if it is spanned by finitely many vectors. If this is not the case we say that it is infinite dimensional.

Avigad, Simic showed in [8, Theorem 10.9] that every Hilbert space H in the sense of Definition 7.2 has an orthonormal basis. Since each such Hilbert space is separable this basis is at most countable.

As consequence of this each two infinite dimensional (separable) Hilbert spaces are isomorphic over RCA_0, see [8, Corollary 10.11]. Thus we many restrict our attention to ℓ_2, as given by the following definition.

Definition 7.3 (ℓ_2, [86, II.10.2])**.** Let $A = (|A|, +, \cdot, 0)$ be a vector space over $\mathbb{Q}$, where $|A|$ is the set of all finite sequences of rational numbers $\langle r_0, \dots, r_m\rangle$, such that either $m = 0$ or $r_m \neq 0$. Addition is defined by putting $\langle r_0, \dots, r_m\rangle + \langle s_0, \dots, s_n\rangle = \langle r_0 + s_0, \dots, r_k + s_k\rangle$ where $r_i, s_i = 0$ for $i > m, n$ and $k = \max\{i \mid i = 0 \vee r_i + s_i \neq 0\}$. For scalar multiplication put $q \cdot \langle r_0, \dots, r_m\rangle = \langle 0\rangle$ if $q = 0$ and $\langle q \cdot r_0, \dots, q \cdot r_m\rangle$ otherwise.

The space ℓ_2 is defined to be the Hilbert space consisting of A with the inner product

$$\big\langle \langle r_0, \dots, r_m\rangle, \langle s_0, \dots, s_n\rangle \big\rangle = \sum_{i=0}^{\max(n,m)} r_i s_i.$$

The canonical orthonormal basis $(e_n)_n$ of ℓ_2 is given by

$$e_n = \langle \underbrace{0, \dots, 0}_{n \text{ times}}, 1\rangle.$$

Definition 7.4 (projection)**.** Let M be a closed linear subspace of a Hilbert space H. A point $y \in M$ is called *projection* of $x \in H$ if $x - y$ is orthogonal to (each element of) M.

A bounded linear operator P_M on H that maps each point of H to its projection on M is called *projection function for M*.

Usually projections are defined differently, see e.g. [8, Definition 12.1]. Avigad, Simic showed that this definition is over RCA_0 equivalent to the usual definition, see [8, Lemma 12.2].

We immediately obtain the following lemma:

Lemma 7.5. *Let $N \subset \mathbb{N}$ and M be the subspace of ℓ_2 that is spanned by $\{e_n \mid n \in N\}$. Then RCA_0 proves that the projection P_M of ℓ_2 onto the space M exists.*

Proof. The projection of an element $\langle r_0, \dots, r_m\rangle$ of the space $|A|$ is given by the vector $\langle r'_0, \dots, r'_{m'}\rangle$, where $r'_i = r_i$ if $n \in N$ and $r'_i = 0$ if $n \notin N$ and $m' = \max\{i \leq m \mid r_i \neq 0 \vee i = 0\}$.

It is easy to show that P_M is linear and that it is bounded by 1 (at least on $|A|$). From this one can deduce that P_M is continuous and continuously extend it to the full space ℓ_2. $\square$

Definition 7.6 (weak convergence)**.** We say that a sequence $(x_i)_{i\in\mathbb{N}}$ of elements of a Hilbert space H *converges weakly* to a point x if

$$\forall y \in H \ \lim_{i\to\infty} \langle y, x_i\rangle = \langle y, x\rangle. \tag{7.1}$$

The *Bolzano-Weierstraß principle for weak convergence* is defined to be the statement that for every bounded sequence $(x_i)_{i\in\mathbb{N}}$ of elements of H there exists a point x such that a subsequence of $(x_i)_i$ converges weakly to x. This principle is abbreviated by $\mathsf{weak\text{-}BW}$. The restriction of this principle to a fixed sequence $(x_i)_{i\in\mathbb{N}}$ is denoted by $\mathsf{weak\text{-}BW}((x_i)_i)$.

If H has an orthonormal basis it is sufficient to have (7.1) only for all y in the basis.

Lemma 7.7. *Projections are weakly continuous in the sense that if x is the weak limit of a sequence $(x_i)_{i\in\mathbb{N}}$, then Px is the weak limit of $(Px_i)_{i\in\mathbb{N}}$ for any projection P.*

Proof. Follows from the definition of the projection and the continuity of $\langle\cdot,\cdot\rangle$. $\square$

Recall that Π^0_2-$\mathsf{CA}(h)$ denotes the instance of Π^0_2-comprehension given by the formula $A(n) \equiv \forall x\, \exists y\, h(x, y, n) = 0$, i.e. that statement

$$\exists g\, \forall n\ (g(n) = 0 \leftrightarrow \forall x\, \exists y\, h(x, y, n) = 0)\,.$$

Theorem 7.8. *Each instance $\mathsf{A}(n) \equiv [\forall x\, \exists y\, h(x, y, n) = 0]$ of Π^0_2-CA is uniformly implied by an instance of $\mathsf{weak\text{-}BW}$. More precisely, there exists a closed term F in RCA^ω_0, such that*

$$\mathsf{RCA}^\omega_0 \vdash \forall h\, \Big(\mathsf{weak\text{-}BW}(F(h)) \to \Pi^0_2\text{-}\mathsf{CA}(h)\Big)\,.$$

Proof. Fix an h and define

$$f(n,i) := \max\{x \leq i \mid \forall x' < x \, \exists y < i \, (h(x',y,n) = 0)\}.$$

It is clear that $\lambda i.f(n,i)$ is increasing for each n.
Claim 1.

$$\mathsf{A}(n) \quad \text{iff} \quad \lambda i.f(n,i) \text{ is unbounded, i.e. } \forall k \, \exists i \, (f(n,i) > k).$$

Proof of Claim 1.

- The right to left direction follows immediately from the definition of f.
- For the left to right direction fix an n. We will show that not the right side implies not the left side.

  Hence assume that $\lambda i.f(n,i)$ is bounded by k, i.e.

  $$\forall i \, (f(n,i) \leq k). \tag{7.2}$$

  By Σ^0_1-induction we may assume that k is minimal and thus

  $$\exists i \, (f(n,i) = k).$$

  From the definition of f we obtain

  $$\forall x < k \, \exists y \, (h(x,y,n) = 0).$$

  Together with (7.2) we obtain that

  $$\forall y \, (h(k,y,n) \neq 0)$$

  and hence $\neg\mathsf{A}(n)$.

This proves the claim.
Let

$$y_{n,i} := e_{\langle n, f(n,i)\rangle}.$$

The sequence $(y_{n,i})_{i\in\mathbb{N}}$ is obviously bounded by 1 and hence possesses for each n a weak cluster point y_n.
Claim 2.

- $\|y_n\| =_{\mathbb{R}} 0$, if $\mathsf{A}(n)$ and
- $\|y_n\| =_{\mathbb{R}} 1$, if $\neg\mathsf{A}(n)$.

Proof of Claim 2.

- If $\mathsf{A}(n)$ is true, then $\lambda i.f(n,i)$ is unbounded and hence $\langle e_j, y_{n,i}\rangle$ eventually becomes 0. Therefore $y_{n,i}$ converges weakly to 0.

- If $\mathsf{A}(n)$ is false, then $\lambda i.f(n,i)$ is bounded. By Σ^0_1-induction we obtain a smallest upper bound k and since $\lambda i.f(n,i)$ is increasing we obtain that $\lim_{i\to\infty} f(n,i) = k$. As consequence we obtain that $y_{n,i}$ eventually becomes constant $e_{\langle n,k\rangle}$ and hence that $y_n = e_{\langle n,k\rangle}$ and $\|y_n\| =_{\mathbb{R}} 1$.

This proves the claim.

We parallelize this process to obtain the comprehension function for $\mathsf{A}(n)$. For this let

$$x_i := \sum_{n=0}^{i} 2^{-\frac{n+1}{2}} y_{n,i}.$$

Since the $y_{n,i}$ are orthogonal for different n, we obtain by Pythagoras that

$$\|x_i\|^2 = \sum_{n=0}^{i} 2^{-(n+1)} \|y_{n,i}\|^2 \leq 1$$

and thus that (x_i) is bounded.

It is also clear that there exists a closed term F such that $x_i = F(h,i)$.

By $\mathsf{weak\text{-}BW}(F(h))$ there exists a weak cluster point x of (x_i). Let now M_n be the closed linear space spanned by $\{e_{\langle n,k\rangle} \mid k \in \mathbb{N}\}$. By definition the subspaces M_n are disjoint (except for the 0 vector) for different n, and $y_{n,i} \in M_n$ for all i, n.

By Lemma 7.5 the projections P_{M_n} onto the spaces M_n exist. For this projections we have

$$P_{M_n}(x_i) = 2^{-\frac{n+1}{2}} y_{n,i} \quad \text{for } n \geq i.$$

Since P_{M_n} is weakly continuous, see Lemma 7.7, we get

$$P_{M_n}(x) = 2^{-\frac{n+1}{2}} y_n.$$

Now Claim 2 yields that $\|P_{M_n}(x)\| =_{\mathbb{R}} 0$ if $\mathsf{A}(n)$ and $\|P_{M_n}(x)\| =_{\mathbb{R}} 2^{-\frac{n+1}{2}}$ if $\neg\mathsf{A}(n)$. Hence the function

$$g(n) := \begin{cases} 0 & \text{if } \|P_{M_n}(x)\|(n+1) <_{\mathbb{Q}} 2^{-\frac{n+1}{2}}, \\ 1 & \text{otherwise,} \end{cases}$$

where $\|P_{M_n}(x)\|(n+1)$ is a $2^{-(n+1)}$ good rational approximation of $\|P_{M_n}(x)\|$, provides a comprehension function and solves the theorem. $\square$

In second-order arithmetic one can circumscribe F in the statement of Theorem 7.8 by a code for a Turing machine and obtains the following:

$$\mathsf{RCA}_0 \vdash \exists e\, \forall h\, \Big(\mathsf{weak\text{-}BW}(\{e\}^h) \to \Pi^0_2\text{-}\mathsf{CA}(h)\Big)\,.$$

As immediate consequence of Theorem 7.8 we obtain the following corollary:

Corollary 7.9. *There is a sequence $(x_i)_i$ of elements in ℓ_2 such that from a cluster point x of this sequence one can compute any element of the second Turing jump $0''$.*

Proof. Take for $\mathsf{A}(n)$ in Theorem 7.8 the Π^0_2 statement that the Turing machine $\{n\}^{0'}(n)$ halts. □

Kohlenbach studies weak compactness in the context of arbitrary abstract Hilbert spaces, see [67, 68]. By abstract Hilbert space we mean that the Hilbert space is added as a new type to the system together with the Hilbert space axioms and that the space is not coded as sequences of numbers. With this one can analyze Hilbert spaces without referring to a concrete space like ℓ_2 and one does not automatically obtain a separable Hilbert space but can analyze general Hilbert spaces.

We do not introduce the notation for abstract Hilbert spaces here but refer the reader to [67, Chapter 17]. We show now that the statement of Theorem 7.8 is also applicable in this context:

Theorem 7.10. *Let $\widehat{\mathsf{PA}}^\omega{\restriction}[X, \langle\cdot,\cdot\rangle]$ be the extension of $\widehat{\mathsf{PA}}^\omega{\restriction}$ by the abstract Hilbert space X with the scalar product $\langle\cdot,\cdot\rangle$ and let $\mathsf{weak\text{-}BW}_X$ denote the Bolzano-Weierstraß principle for weak compactness in X.*

Then there is a closed term F, such that

$$\widehat{\mathsf{PA}}^\omega{\restriction}[X, \langle\cdot,\cdot\rangle] + \Pi^0_1\text{-}\mathsf{CP} \vdash \forall h\, \forall (e_i)_{i\in\mathbb{N}}\ (\forall i, j\ \langle e_i, e_j\rangle = \delta_{ij}) \\ \to \Big(\mathsf{weak\text{-}BW}_X(F((e_i)_i, h)) \to \Pi^0_2\text{-}\mathsf{CA}(t)\Big)\,.$$

In other words, if X is provably infinite dimensional and $(e_i)_i$ is a witness for that, then Theorem 7.8 also holds with ℓ_2 replaced by X.

Proof. The only step in the proof of Theorem 7.8 that does not formalize in the system $\widehat{\mathsf{PA}}^\omega{\restriction}[X, \langle\cdot,\cdot\rangle]$ is projection of x onto M_n, i.e. Lemma 7.5, since this depends on the coding of ℓ_2.

We show now how to obtain this projection of x in this system. For this consider

$$\begin{aligned}\|x\|^2 &= \langle x, x\rangle = \lim_{i\to\infty} \langle x, x_i\rangle \\ &= \lim_{i\to\infty} \sum_{n=0}^{i} 2^{-(n+1)} \langle x, y_{n,i}\rangle \\ &\le \lim_{i\to\infty} \sum_{n=0}^{k} 2^{-(n+1)} \langle x, y_{n,i}\rangle + 2^{-k} \qquad \text{for each } k \\ &= \sum_{n=0}^{k} 2^{-(n+1)} \lim_{i\to\infty} \langle x, y_{n,i}\rangle + 2^{-k}.\end{aligned}$$

Now

$$\langle x, y_{n,i}\rangle = \lim_{j\to\infty} \langle x_j, y_{n,i}\rangle = 2^{-(n+1)} \lim_{j\to\infty} \langle y_{n,j}, y_{n,i}\rangle. \tag{7.3}$$

Thus, by the definition of $y_{n,i}$ the term $\langle x, y_{n,i}\rangle$ is monotone in i and in particular for each n there is an m, such that

$$\lim_{i\to\infty} \langle x, y_{n,i}\rangle = \langle x, y_{n,i'}\rangle \quad \text{for } i' \ge m.$$

By Π^0_1-CP there is now an m which does it for all $n \le k$. Hence, we obtain

$$\forall k\, \exists i\ \|x\|^2 \le \sum_{n=0}^{k} 2^{-(n+1)} \langle x, y_{n,i}\rangle + 2^{-k}.$$

By (7.3) the term $\langle x, y_{n,i}\rangle$ is either 0 or $2^{-(n+1)}$, hence

$$= \sum_{n=0}^{k} \langle x, y_{n,i}\rangle^2 + 2^{-k}.$$

Thus, $\sum_{n=0}^{k} \langle x, y_{n,i}\rangle y_{n,i}$ is a $2^{-k/2}$ good approximation of x consisting of finite linear combinations of (e_i). Using an application of QF-AC one easily obtains a sequence of approximations converging to x at the rate 2^{-k}. Using this one can obtain $P_{M_n}(x)$ like in Lemma 7.5.

This proves the theorem. □

By applying the functional interpretation to this we obtain the following corollary:

Corollary 7.11. *Let Ω be a solution of the functional interpretation of* weak-BW_X *then for every $n \ge 1$ there are terms in T_n, such that the application of Ω to these terms is (extensionally) equal to a function definable in the T_{n+2} but not in T_{n+1}.*

Proof. Let A be the statement that the function $f_{\omega_{n+1}}$ from the fast growing hierarchy is total. It is well known that the statement A cannot be proven in Σ^0_{n+2}-IA but can be proven using a suitable instance of Σ^0_{n+3}-IA, see [37, II.3.(d)]. Thus a solution of the functional interpretation of A cannot be found in T_{n+1} but can be found in T_{n+2}.

Let $\widehat{\mathsf{PA}}^\omega\restriction[X, \langle\cdot,\cdot\rangle, (e_i)_{i\in\mathbb{N}}]$ be the extension of $\widehat{\mathsf{PA}}^\omega\restriction[X, \langle\cdot,\cdot\rangle]$ by the constant $(e_i)_i$, which can be majorized by $\lambda i.1$, and the axiom $\forall i, j \in \mathbb{N}\, \langle e_i, e_j\rangle =_{\mathbb{R}} \delta_{ij}$. For this system the metatheorem [67, Theorem 17.69.2)], see also [35],

- relativized to the fragment $\widehat{\mathsf{PA}}^\omega\restriction$ of $\mathcal{A}^\omega$, cf. [67, Section 17.1, p. 382] and
- extended by the constant $(e_i)_i$ and the purely universal axiom for it, cf. [67, Section 17.5]

holds.

By Theorem 7.10 a suitable instance of $\mathsf{weak\text{-}BW}_X$ can reduce an instance of Σ^0_{n+3}-IA to Σ^0_{n+1}-IA. Thus the system $\widehat{\mathsf{PA}}^\omega\restriction[X, \langle\cdot,\cdot\rangle, (e_i)_{i\in\mathbb{N}}] + \Sigma^0_{n+1}$-$\mathsf{IA}$ proves that a suitable instance of $\mathsf{weak\text{-}BW}_X$ implies A. Applying the metatheorem to this statement yields terms in T_n such that an application of these terms to Ω yields a solution of the functional interpretation of A.

This proves the corollary. $\square$

This shows that Kohlenbach's analysis of Ω^* (a majorant of a solution of the functional interpretation of $\mathsf{weak\text{-}BW}_X$) in [56] is optimal.

This analysis and actually even his proof of weak compactness for abstract Hilbert spaces [68, Theorem 11] shows that only two nested instances of Π^0_1-CA (plus some uses of WKL) are needed to prove an instance of $\mathsf{weak\text{-}BW}_X$. Thus, the lower bound on the strength of instances of $\mathsf{weak\text{-}BW}_X$ from the Theorems 7.8 and 7.10 is strict in the sense that there is no instance of Π^0_3-CA which is implied by an instance of $\mathsf{weak\text{-}BW}_X$.

We now give a reversal for the special case of ℓ_2 and analyze the exact computational content:

Theorem 7.12. *Each instance of* $\mathsf{weak\text{-}BW}$ *given by a bounded sequence* $(x_i)_{i\in\mathbb{N}}$ *in* ℓ_2 *is over* RCA^ω_0 *uniformly provable from a suitable instance of* Π^0_2-CA*. More precisely, there is a term* F *of* RCA^ω_0 *such that*

$$\mathsf{RCA}^\omega_0 \vdash \forall (x_i)\ \left(\Pi^0_2\text{-}\mathsf{CA}(F((x_i))) \to \mathsf{weak\text{-}BW}((x_i)_{i\in\mathbb{N}})\right).$$

In particular, each bounded and computable sequences of ℓ_2 *has a weak cluster point computable in* $0''$.

Proof. We show that provably in RCA^ω_0 a cluster point of $(x_i)_i$ can be computed in the second Turing jump. The result follows then from the fact that any function computable in the second Turing jump is recursive in a suitable instance of Π^0_2-CA.

We assume that $(x_i)_i$ is bounded by 1.

Note that the Bolzano-Weierstraß theorem for the space $[-1,1]^{\mathbb{N}}$ (with the product metric $d((x_i)_i,(y_i)_i) = \sum_{i=0}^{\infty} \frac{\min(|x_i - y_i|,1)}{2^{i+1}}$) is instance-wise equivalent to the Bolzano-Weierstraß theorem for $[-1,1]$. This can easily be seen from the fact that the Bolzano-Weierstraß theorem for $[-1,1]$ is instance-wise equivalent to the theorem for the Cantor space $2^{\mathbb{N}}$ and the fact that $2^{\mathbb{N}}$ is isomorphic to $(2^{\mathbb{N}})^{\mathbb{N}}$.

Hence by Lemma 6.3 and Theorem 6.4 one can find a cluster point of the sequence

$$y_i := (\langle e_0, x_i\rangle, \langle e_1, x_i\rangle, \dots)$$

in $[-1,1]^{\mathbb{N}}$ by computing an infinite path trough a Σ^0_1-tree. Call this cluster point $c = (c_0, c_1, \dots) \in [-1,1]^{\mathbb{N}}$.
Claim. $\sum_{j=0}^{\infty} c_j \le 1$
Proof of claim. Since the elements of y_i are elements of a Hilbert space and are norm bounded by 1 we have that $\sum_{j=0}^{k} (y_i)_j^2 \le 1$. Now for each k and for each ε there is an y_i such that $|c_j - (y_i)_j| \le \varepsilon$ for $j \le k$ and hence

$$\sum_{j=0}^{k} (c_j)^2 \le \sum_{j=0}^{k} ((y_i)_j + \varepsilon)^2 \le 1 + 3(k+1)\varepsilon.$$

From this follows the claim.

Now one easily checks that the sequence $(z_i)_{i\in\mathbb{N}}$ with $z_i := \langle c_0, \dots, c_i\rangle$ converges in the ℓ_2-norm to a weak cluster point x of $(x_i)_i$. This convergence is monotone in the sense that $\|z_i\| \le \|z_{i+1}\|$ thus the limit point x can be computed in the Turing jump of $(z_i)_i$.

The point x is provably uniformly computable in the second Turing jump of $(x_i)_i$ because c is by the low basis theorem ([51]) computable in a degree provably low over the first Turing jump. The proof of the low basis theorem is effective and uniform and it formalizes in RCA_0. Therefore the jump of $(z_i)_i$ and thus x is computable in the second Turing jump and one can find a suitable term F. $\square$

Again, we can circumscribe the functional F by a code of a Turing machine obtain

$$\mathsf{RCA}_0 \vdash \exists e\, \forall (x_i) \left(\Pi^0_2\text{-}\mathsf{CA}(\{\{e\}^{(x_i)}) \rightarrow \text{weak-}\mathsf{BW}((x_i)_{i\in\mathbb{N}}) \right).$$

The Theorems 7.8 and 7.12 yield a classify of the computational strength of weak compactness on ℓ_2:

Corollary 7.13. *For a Turing degree d the following are equivalent:*

- *$d \ge_T 0''$ and*
- *d computes a weak cluster point for each computable, bounded sequence in ℓ_2.*

As consequence we obtain that the Bolzano-Weierstraß principle for weak compactness is instance-wise strictly stronger than the Bolzano-Weierstraß principle for the unit interval $[0, 1]$.

Remark 7.14 (Weihrauch lattice). The proofs of the Theorems 7.8 and 7.12 can also be used to classify the Bolzano-Weierstraß principle for weak compactness in ℓ_2 in the Weihrauch lattice. We do not introduce the notation for the Weihrauch lattice but refer the reader to [14].

Let $\mathsf{BWT}_{\text{weak-}\ell_2} :\subseteq (\ell_2)^{\mathbb{N}} \rightrightarrows \ell_2$ be the partial multifunction which maps bounded sequences of ℓ_2 to a weak cluster point of that sequence.

The proof of Theorem 7.8 immediately yields that

$$\mathsf{BWT}_{\text{weak-}\ell_2} \geq_{\mathrm{W}} \widehat{\mathsf{LPO}} \circ \widehat{\mathsf{LPO}} \equiv_{\mathrm{W}} \lim^{(2)}.$$

Whereas the proof of Theorem 7.12 yields that

$$\mathsf{BWT}_{\text{weak-}\ell_2} \leq_{\mathrm{W}} \mathsf{MCT} * \mathsf{BWT}_{\mathbb{R}^{\mathbb{N}}}.$$

The function $\mathsf{BWT}_{\mathbb{R}^{\mathbb{N}}}$ is used to compute the cluster point $c \in \mathbb{R}^{\mathbb{N}}$, the function MCT is used for the convergence of $(\|z_i\|)_i$. By the same argument as in the proof $\mathsf{BWT}_{\mathbb{R}} \equiv_{\mathrm{W}} \mathsf{BWT}_{\mathbb{R}^{\mathbb{N}}}$. Since all of these multifunctions are cylinders one may also strengthen the reducibility to strong Weihrauch reducibility. Thus

$$\begin{aligned} \mathsf{BWT}_{\text{weak-}\ell_2} &\leq_{\mathrm{sW}} \mathsf{MCT} *_{\mathrm{s}} \mathsf{BWT}_{\mathbb{R}} \\ &\leq_{\mathrm{sW}} \lim *_{\mathrm{s}} \mathfrak{L}' \\ &\leq_{\mathrm{sW}} \lim *_{\mathrm{s}} \mathfrak{L}_{1,1} \\ &\equiv_{\mathrm{sW}} \lim \circ \lim. \end{aligned}$$

(For the last equivalence see [14, Corollary 8.8], which is a consequence of an analysis of the low basis theorem in the Weihrauch lattice, see [13].)

In total we obtain that

$$\mathsf{BWT}_{\text{weak-}\ell_2} \equiv_{\mathrm{sW}} \lim^{(2)}.$$

As consequence we also obtain that $\mathsf{BWT}_{\text{weak-}\ell_2} >_{\mathrm{sW}} \mathsf{BWT}_{\mathbb{R}}$.

Part III.

Non-principal ultrafilters

8. Non-principal ultrafilters, program extraction and higher order reverse mathematics

In this chapter we will investigate the strength of the existence of a non-principal ultrafilter over fragments of higher order arithmetic. We will classify the consequences of this statement in the spirit of reverse mathematics. Furthermore, we will provide a program extraction method.

Let $(\mathcal{U})$ be the statement that a non-principal ultrafilter on $\mathbb{N}$ exists. Let ACA_0^ω be $\mathsf{RCA}_0^\omega + \Pi^0_1\text{-}\mathsf{CA}$. This system corresponds to ACA_0 like RCA_0^ω to RCA_0. In these systems the statement $(\mathcal{U})$ can be formalized using an object of type 2.

Further, let Feferman's μ be a functional of type 2 satisfying

$$f(\mu(f)) = 0 \quad \text{if} \quad \exists x\, f(x) = 0$$

and let (μ) be the statement that such a functional exists. It is clear that (μ) implies arithmetical comprehension.

We will show that

- over RCA_0^ω the statement $(\mathcal{U})$ implies (μ) and therefore also ACA_0^ω, and that
- $\mathsf{ACA}_0^\omega + (\mu) + (\mathcal{U})$ is Π^1_2-conservative over ACA_0^ω and therefore also conservative over PA. Moreover, we will show that from a proof of $\forall f\, \exists g\, \mathsf{A}_{qf}(f,g)$ in $\mathsf{ACA}_0^\omega + (\mu) + (\mathcal{U})$, where A_{qf} is quantifier free, one can extract a realizing term t in Gödel's system T, i.e. a term such that $\forall f\, \mathsf{A}_{qf}(f, t(f))$.

The system $\mathsf{ACA}_0^\omega + (\mu) + (\mathcal{U})$ is strong in the sense that one can carry out nearly all ultralimit and non-standard arguments. For instance one can carry out in this theory the construction of Banach limits and many Loeb measure constructions. Our result shows that this system is weak with respect to Π^1_2 sentences. Moreover, our program extraction result shows that one can still obtain constructive (even primitive recursive in the sense of Gödel) realizers and bounds from proofs using highly non-constructive objects like non-principal ultrafilters.

Using this technique it is possible to extract bounds from proofs using ultralimits and non-standard techniques. Such proofs do occur in mathematics, for instance in metric fixed point theory, see [1] and [53]. In [34], Gerhardy extracted a rate of proximity from such a proof by eliminating the ultrafilter by hand. Our result here shows that this can be done with similar uses of ultrafilters.

Comparison with other approaches

Solovay first used partial ultrafilter. He constructed a filter which acts on the hyperarithmetical sets like a non-principal ultrafilter. With this he showed an effective version of the Galvin-Prikry theorem, see [88]. His construction of the partial ultrafilter is similar to ours. Avigad analyzed his result in terms of reverse mathematics and formalized this particular proof in ATR_0, see [3]. However, this result does not follow from our meta-theorem, since it not only uses a non-principal ultrafilter but also a substantial amount of transfinite recursion.

Using our approach one also obtains upper bounds on the strength of non-standard analysis and program extraction methods. This can be done by constructing an ultrapower model of non-standard analysis in $\mathsf{ACA}_0^\omega + (\mu) + (\mathcal{U})$. If one is not interested in the ultrafilter but only in the axiomatic treatment of non-standard analysis one can obtain refined results by interpreting it directly, see for instance [6], [52] and for program extraction [10].

Palmgren used in [78] an approach similar to ours to interpret non-standard arithmetic. He builds (partial) non-principal ultrafilters for the definable sets of a fixed level in the arithmetical hierarchy and obtains a conservation results similar to ours. However he cannot treat ultrafilter nor obtains program extraction.

In reverse mathematics idempotent ultrafilters are considered in the context of Hindman's theorem, which can be proven using an idempotent ultrafilter (or at least a countable part of it), see Hirst [42] and Towsner [95]. We code an ultrafilter over $\mathbb{N}$ like Hirst does. However, our construction of ultrafilters is different since we are not aiming for idempotent ultrafilters. An idempotent ultrafilter is a very special ultrafilter and it seems that even the construction of countable parts of an idempotent ultrafilter requires a system that is proof theoretically stronger than $\mathsf{ACA}_0^\omega + (\mu)$ and is therefore beyond our method.

Recently, Towsner considered the addition of an ultrafilter to fragments of second order arithmetic. He adds the ultrafilter as a predicate over sets. Independently he also obtains conservation results for non-principal ultrafilter related to ours but using different methods. However, in the context he considers, the non-principal ultrafilter is weaker in the sense that one cannot use it to define other higher-order objects like μ and thus cannot use the ultrafilter to prove statements beyond ACA_0. See [94].

Enayat considered which kind of non-principal ultrafilters can be defined on the second order part of models of say ACA_0, see [26].

We already mentioned that the system RCA_0^ω has a functional interpretation (always combined with elimination of extensionality and a negative translation) in T_0. The system ACA_0^ω has a functional interpretation in $T_0[\mu]$ if one interprets comprehension using μ or in $T_0[B_{0,1}]$ if one interprets comprehension using the bar recursor of lowest type $B_{0,1}$. See [65] and [7] for the interpretation using μ and [67, Chapter 11] for the interpretation using $B_{0,1}$.

Definition 8.1 (non-principal ultrafilter, $(\mathcal{U})$)**.** Let $(\mathcal{U})$ be the statement that there exists a non-principal ultrafilter (on $\mathbb{N}$):

$$
(\mathcal{U}): \begin{cases} \exists \mathcal{U}^2 \, (\, \forall X \, \left(X \in \mathcal{U} \vee \overline{X} \in \mathcal{U} \right) \\ \qquad \wedge \forall X^1, Y^1 \, (X \cap Y \in \mathcal{U} \to Y \in \mathcal{U}) \\ \qquad \wedge \forall X^1, Y^1 \, (X, Y \in \mathcal{U} \to (X \cap Y) \in \mathcal{U}) \\ \qquad \wedge \forall X^1 \, (X \in \mathcal{U} \to \forall n \, \exists k > n \, (k \in X)) \\ \qquad \wedge \forall X^1 \, (\mathcal{U}(X) =_0 \mathrm{sg}(\mathcal{U}(X), 1) =_0 \mathcal{U}(\lambda n.\, \mathrm{sg}(X(n))))\,) \end{cases}
$$

Here $X \in \mathcal{U}$ is an abbreviation for $\mathcal{U}(X) =_0 0$. The type 1 variables X, Y are viewed as characteristic function of sets, where $n \in X$ is defined to be $X(n) = 0$. The operation $\cap$ is defined as taking the pointwise maximum of the characteristic functions. With this the intersection of two sets can be expressed in a quantifier-free way. The last line of the definition states that $\mathcal{U}$ yields the same value for different characteristic functions of the same set and that $\mathcal{U}(X) \leq 1$.

For notational ease we will usually add a Skolem constant $\mathcal{U}$ and denote this also with $(\mathcal{U})$.

The second line in the definition of $(\mathcal{U})$ is equivalent to the following axiom usually found in the axiomatization of (ultra)filters:

$$\forall X, Y \, (X \subseteq Y \wedge X \in \mathcal{U} \to Y \in \mathcal{U}) \, .$$

We avoided this statement in $(\mathcal{U})$ since $\subseteq$ cannot be expressed in a quantifier free way.

Lemma 8.2 (finite partition property)**.** *The ultrafilter $\mathcal{U}$ satisfies the finite partition property over RCA_0^ω. This means that for each finite partition $(X_i)_{i<n}$ of $\mathbb{N}$ the system RCA_0^ω proves that there exists a unique $i < n$ with $X_i \in \mathcal{U}$.*

Proof. We prove by quantifier-free induction on m the statement

$$\exists! i \leq m \left(\left(i < m \to X_i \in \mathcal{U} \right) \wedge \, \left(i = m \to \bigcup_{j=m}^{n-1} X_j \in \mathcal{U} \right) \right). \tag{8.1}$$

In the cases $m < 2$ the statement follows directly from $(\mathcal{U})$. For the induction step we assume that the statement for m holds. This means there exists an i as stated in (8.1). If $i < m$ then this i also satisfies (8.1) with m replaced by $m + 1$ and we are done. Otherwise we have $\bigcup_{j=m}^{n-1} X_j \in \mathcal{U}$.

The axiom $(\mathcal{U})$ yields

$$\bigcup_{j=0}^{m} X_j \in \mathcal{U} \ \vee \ \bigcup_{j=m+1}^{n-1} X_j \in \mathcal{U}.$$

If the left side of the disjunction holds then

$$X_m = \bigcup_{j=0}^{m} X_j \cap \bigcup_{j=m}^{n-1} X_j \in \mathcal{U}$$

and $i := m$ satisfies the (8.1) with m replaced by $m+1$. If the right side of the disjunction holds $i := m+1$ satisfies (8.1).

The lemma follows from (8.1) by taking $m := n$. □

Theorem 8.3.

$$\mathsf{RCA}_0^\omega + (\mathcal{U}) \vdash (\mu)$$

In particular $\mathsf{RCA}_0^\omega + (\mathcal{U}) \vdash \mathsf{ACA}_0^\omega$.

Proof. Let $f \colon \mathbb{N} \to \mathbb{N}$ be a function. The set $X_f := \{x \in \mathbb{N} \mid \exists x' < x\, f(x') = 0\}$ is cofinal if $\exists x\, f(x) = 0$, if not then the set X_f is empty. Hence

$$X_f \in \mathcal{U} \quad \text{iff} \quad \exists x\, f(x) = 0.$$

From this it follows that

$$\forall f\, \exists x\, (X_f \in \mathcal{U} \to f(x) = 0)\,.$$

An application of $\mathsf{QF\text{-}AC}^{1,0}$ now yields a functional satisfying (μ). □

Theorem 8.4 (Program extraction)**.** *Let* $\mathsf{A}_{qf}(f,g)$ *be a quantifier free formula of* RCA_0^ω *containing only* f, g *free. In particular* A_{qf} *must not contain* μ *or* $\mathcal{U}$*.*

If

$$\mathsf{ACA}_0^\omega + (\mu) + (\mathcal{U}) \vdash \forall f^1\, \exists g^1\, \mathsf{A}_{qf}(f,g)$$

then one can extract a closed term $t \in T$ *such that*

$$\forall f\, \mathsf{A}_{qf}(f, tf).$$

The proof of this theorem proceeds in five steps:

1. Using the functional interpretation and some methods we developed in Chapter 2 we show that a proof of the statement

$$\mathsf{ACA}_0^\omega + (\mu) + (\mathcal{U}) \vdash \forall f\, \exists g\, \mathsf{A}_{qf}(f,g)$$

can be normalized in such a way that each application of the functional $\mathcal{U}$ that occurs in the proof has the form $\mathcal{U}(t[n^0])$, where t is a term that contains only n free and with $\lambda n.t \in T_0[\mathcal{U}]$. (We do not have to consider μ here, since it can be defined from $\mathcal{U}$ by Theorem 8.3.) In particular, this shows that the ultrafilter $\mathcal{U}$ is used only on countable many sets.

2. We show that we can construct in $\mathsf{RCA}_0^\omega + (\mu)$ a *partial ultrafilter*, that is an object that behaves like an ultrafilter on the sets that occur in the proof. We then replace $\mathcal{U}$ by this partial ultrafilter and obtain a proof of $\forall f\, \exists g\, \mathsf{A}_{qf}(f,g)$ in $\mathsf{RCA}_0^\omega + (\mu)$.

3. By [7, 27], the theory $\mathsf{RCA}_0^\omega + (\mu)$ is conservative over ACA_0^ω for such sentences. Thus, we obtain $\mathsf{ACA}_0^\omega \vdash \forall f\, \exists g\, A_{qf}(f,g)$.

4. Applying the functional interpretation to this statement and interpreting the comprehension using $B_{0,1}$ yields a term $t^2 \in T_0[B_{0,1}]$, such that
$$\forall f\, A_{qf}(f, tf).$$

5. Since this term t is only of type 2, one can use an ordinal analysis of the bar recursor to eliminated it and obtain a new term $t' \in T$, such that $t' =_2 t$ and hence that
$$\forall f\, A_{qf}(f, t'f).$$

Before we prove this theorem we show how to construct a partial ultrafilter and provide some proof theoretic lemmata.

Partial ultrafilter

Definition 8.5 (partial ultrafilter)**.**

- Call a set $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ of subsets of natural numbers, that is closed under complement, finite unions and finite intersections, an *algebra*.
- Let $\mathcal{A}$ be an algebra. Call a set $\mathcal{F} \subseteq \mathcal{A}$ a *partial non-principal ultrafilter* for $\mathcal{A}$ iff $\mathcal{F}$ satisfies the non-principal ultrafilter axioms in Definition 8.1 relativized to $\mathcal{A}$, i.e.
$$\begin{cases} \quad \forall X \in \mathcal{A}\, \left(X \in \mathcal{F} \vee \overline{X} \in \mathcal{F}\right) \\ \wedge \forall X, Y \in \mathcal{A}\, (X \cap Y \in \mathcal{F} \to Y \in \mathcal{F}) \\ \wedge \forall X, Y \in \mathcal{A}\, (X, Y \in \mathcal{F} \to (X \cap Y) \in \mathcal{F}) \\ \wedge \forall X \in \mathcal{A}\, (X \in \mathcal{F} \to \forall n\, \exists k > n\, k \in X) \\ \wedge \forall X^1\, (\mathcal{F}(X) =_0 \mathrm{sg}(\mathcal{F}(X)) =_0 \mathcal{F}(\lambda n.\, \mathrm{sg}(X(n)))\,. \end{cases}$$

The sets $\mathcal{A}$ and $\mathcal{F}$ are given here —like $\mathcal{U}$—as characteristic functions. In the following we will also refer to algebras and filters given by a countable sequence of sets, i.e. $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$ resp. $\mathcal{F} = (F_i)_{i\in\mathbb{N}}$. In this case the characteristic function $\chi_{\mathcal{A}}$ of $\mathcal{A}$ is given by

$$\chi_{\mathcal{A}} = \begin{cases} 0 & \text{if } \exists i\ (A_i = B), \\ 1 & \text{otherwise.} \end{cases}$$

The last equality is a set equality and definable using μ. The characteristic function for $\mathcal{F}$ is defined likewise.

Note that in RCA_0^ω every sequence of sets can be extended to a countable algebra. Further note that a partial non-principal ultrafilters for countable algebras are also countable.

A partial ultrafilter $\mathcal{F}$ can be viewed as the closed subset $\{\mathcal{U} \in \beta\mathbb{N} \mid \mathcal{U} \supseteq \mathcal{F}\}$ of the Stone-Čech compactification $\beta\mathbb{N}$.

Proposition 8.6. *Let $\mathcal{A}$ be a countable algebra and let $\mathcal{F} = (F_i)_{i\in\mathbb{N}}$ be a countable partial non-principal ultrafilter for $\mathcal{A}$. Then $\mathsf{RCA}_0^\omega + (\mu)$ proves that for each countable extension $\tilde{\mathcal{A}} = (\tilde{A}_i)_{i\in\mathbb{N}} \supseteq \mathcal{A}$ there exists a countable partial non-principal ultrafilter $\tilde{\mathcal{F}} = (\tilde{F}_i)_{i\in\mathbb{N}} \supseteq \mathcal{F}$.*

Proof. In the following let x be the code for a tuple $\langle x_0, \dots, x_{\mathrm{lth}(x)-1}\rangle$ in $2^{<\mathbb{N}}$. Let

$$\tilde{A}^x := \bigcap_{i<\mathrm{lth}\, x} \begin{cases} \tilde{A}_i & \text{if } x_i = 0, \\ \overline{\tilde{A}_i} & \text{if } x_i = 1. \end{cases}$$

Using quantifier free induction one easily sees that for every n the set $\left\{\tilde{A}^x \;\middle|\; x \in 2^n\right\}$ defines a partition of $\mathbb{N}$, i.e. for all z

$$\forall n\, \exists! x \in 2^n \left(z \in \tilde{A}^x\right). \tag{8.2}$$

Define a Π^0_2-0/1-tree T by

$$T(x) \quad \text{iff} \quad \forall j \left(\tilde{A}^x \cap F_j \text{ is infinite}\right).$$

The tree T is infinite because otherwise we would have

$$\exists n\, \forall x \in 2^n\, \exists j\, \exists y\, \forall z > y \left(z \notin \tilde{A}^x \cap F_j\right).$$

The bounded collection principle Π^0_1-CP yields

$$\exists n\, \exists j^*, y^*\, \forall x \in 2^n\, \forall z > y^* \left(z \notin \tilde{A}^x \cap \bigcap_{j \le j^*} F_j\right). \tag{8.3}$$

The set $\bigcap_{j\leq j^*} F_j$ is in $\mathcal{F}$ and is, therefore, infinite. In particular, it contains an element z which is bigger than y^*. Because $\tilde{A}^x$ with $x \in 2^n$ defines a partition of $\mathbb{N}$ there is an x such that $z \in \tilde{A}^x$. This contradicts (8.3) and therefore the tree T is infinite.

Hence we obtain using Π^0_2-WKL (which is provable in ACA_0^ω and hence using μ) an infinite branch b of T. We claim that the set

$$\tilde{\mathcal{F}} = \left\{ \tilde{A}_i \;\middle|\; b(i) = 0 \right\}$$

defines then a partial non-principal ultrafilter for $\tilde{\mathcal{A}}$ which contains $\mathcal{F}$.

It is clear that each set $\tilde{A}_i \in \tilde{\mathcal{F}}$ is infinite since $\tilde{A}^{\overline{b}(i+1)}$ is a subset of $\tilde{A}_i$ and is infinite by definition of the tree. For each set $\tilde{A}_i$ exactly one of $\tilde{A}_i$, $\overline{\tilde{A}_i} = \tilde{A}_j$ is in $\tilde{\mathcal{F}}$ since otherwise the set $\tilde{A}^{\overline{b}(\max(i,j)+1)}$ is empty and therefor not infinite, which contradicts the definition of the tree. Now suppose $\tilde{A}_i, \tilde{A}_{i'} \in \tilde{\mathcal{F}}$ then the intersection $\tilde{A}_j$ is also in $\tilde{\mathcal{F}}$ since $\tilde{A}^{\overline{b}(\max(i,i',j)+1)} \subseteq \tilde{A}_i \cap \tilde{A}_{i'} = \tilde{A}_j$, which rules out the case that $\overline{\tilde{A}_j}$ is in $\tilde{\mathcal{F}}$. For a similar reason also supersets of set in $\tilde{\mathcal{F}}$ are also in the filter.

Each set $\tilde{A}_i = F_i \in \mathcal{F}$ is also in $\tilde{\mathcal{F}}$ since otherwise the complement $\overline{\tilde{A}_i} = \tilde{A}_j$ would be in $\mathcal{F}$ and then $\tilde{A}^{\overline{b}(j+1)} \subseteq \tilde{A}_j$ which has empty intersection with F_i and this contradicts the definition of the tree. $\square$

Proof theory

The system RCA_0^ω contains full extensionality. Since functional interpretation cannot handle this directly, the use of full extensionality have to be eliminated beforehand, see Proposition 1.5. Since we added a new higher order constant $\mathcal{U}$ this proposition is not applicable and we have to extend it by checking that this constant is extensional. This will be done in the following lemma.

Lemma 8.7 (Elimination of extensionality). *The system* $\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction + (\mathcal{U})$ *proves that* $\mathcal{U}$ *is extensional, i.e.*

$$\forall X, Y\, \big(\forall k\ (k \in X \leftrightarrow k \in Y) \to (X \in \mathcal{U} \leftrightarrow Y \in \mathcal{U})\big).$$

In particular, the elimination of extensionality is applicable to $\mathsf{RCA}_0^\omega + (\mathcal{U})$*. This means the following rule holds: If* A *is a sentence that contains only quantification over variables of degree* ≤ 1 *and*

$$\mathsf{RCA}_0^\omega \vdash (\mathcal{U}) \to \mathsf{A}$$

then

$$\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction + \mathsf{QF\text{-}AC}^{1,0} \vdash (\mathcal{U}) \to \mathsf{A}.$$

Proof. Suppose that $\mathcal{U}$ is not extensional. Then there exist two sets X, Y, such that

$$\forall k\ (k \in X \leftrightarrow k \in Y)) \quad \text{and} \quad X \in \mathcal{U} \wedge Y \notin \mathcal{U}.$$

By the axiom $(\mathcal{U})$ we obtain that $\overline{Y} \in \mathcal{U}$ and with this

$$X \cap \overline{Y} \in \mathcal{U}.$$

By the second to last line of $(\mathcal{U})$ there exists an $n \in X \cap \overline{Y}$. This contradicts the assumption and we conclude that $\mathcal{U}$ is extensional.

For the elimination of extensionality we use the techniques presented in Section 10.4 of [67], see also Proposition 1.5. We will also use the notation introduced in this section for the rest of this proof.

The extensionality of $\mathcal{U}$ translates into $\mathcal{U} =^e \mathcal{U}$. Since $(\mathcal{U})$ is (after the Skolemization) analytic and the constant $\mathcal{U}$ is extensional, we obtain $(\mathcal{U})_e \leftrightarrow (\mathcal{U})$. Because A does not contain quantification of degree > 1 we also obtain that A_e is equivalent to A. Hence $(\mathcal{U}) \to \mathsf{A}$ does not change under the $(\cdot)_e$ relativization.

The lemma now follows from Proposition 10.45 in [67] relativized according to [67, Section 10.5] to RCA_0^ω. □

The axiom $(\mathcal{U})$ can be prenexted into a statement of the form

$$\begin{aligned} \exists \mathcal{U}^2\, \forall X^1, Y^1\, \forall n\, \exists k\ (\quad & \left(X \in \mathcal{U} \vee \overline{X} \in \mathcal{U}\right) \\ & \wedge (X \cap Y \in \mathcal{U} \to Y \in \mathcal{U}) \\ & \wedge (X, Y \in \mathcal{U} \to (X \cap Y) \in \mathcal{U}) \\ & \wedge (X \in \mathcal{U} \to (k > n \wedge k \in X)) \\ & \wedge (\mathcal{U}(X) =_0 \mathcal{U}(\lambda n.\min(Xn, 1)))\,). \end{aligned}$$

By coding the sets X, Y together into one set Z and calling the quantifier free matrix of the above statement $(\mathcal{U})_{qf}$ we arrive at

$$\exists \mathcal{U}^2\, \forall Z^1\, \forall n\, \exists k\, (\mathcal{U})_{qf}(\mathcal{U}, Z, n, k).$$

Applying $\mathsf{QF\text{-}AC}^{1,0}$ yields

$$\exists \mathcal{U}^2\, \exists K^2\, \forall Z^1\, \forall n\, (\mathcal{U})_{qf}(\mathcal{U}, Z, n, KnZ). \tag{8.4}$$

Note that $\mathcal{U}$ and K are only of degree 2. This will be crucial for the following proof.

For K one may always choose

$$K'(n, X) := \begin{cases} \min\{k \in X \mid k > n\} & \text{if exists,} \\ 0 & \text{otherwise.} \end{cases} \tag{8.5}$$

The functional K' is definable using μ. Therefore the real difficulty lies in finding a solution for $\mathcal{U}$.

We are now in the position to give a proof of Theorem 8.4.

Proof of Theorem 8.4. In the light of Theorem 8.3 it is sufficient to prove only that $\mathsf{RCA}_0^\omega + (\mathcal{U})$ is conservative.

Let $\mathsf{A}_{qf}(f, g)$ be a quantifier-free statement not containing $\mathcal{U}$, such that

$$\mathsf{RCA}_0^\omega + (\mathcal{U}) \vdash \forall f^1 \, \exists g^1 \, \mathsf{A}_{qf}(f, g).$$

By the deduction theorem we obtain

$$\mathsf{RCA}_0^\omega \vdash (\mathcal{U}) \to \forall f \, \exists g \, \mathsf{A}_{qf}(f, g).$$

Using Lemma 8.7 we obtain

$$\widehat{\mathsf{WE\text{-}PA}}^\omega{\restriction} + \mathsf{QF\text{-}AC}^{1,0} \vdash (\mathcal{U}) \to \forall f \, \exists g \, \mathsf{A}_{qf}(f, g).$$

Reintroducing a variable $\mathcal{U}$ for the ultrafilter together with (8.4) gives

$$\left(\exists \mathcal{U}^2 \, \exists K^2 \, \forall Z^1 \, \forall n \, (\mathcal{U})_{qf}(\mathcal{U}, Z, n, KnZ)\right) \to \forall f \, \exists g \, \mathsf{A}_{qf}(f, g)$$

which is equivalent to

$$\forall f \, \forall \mathcal{U}^2 \, \forall K^2 \, \exists Z^1, n \, \exists g \, \left((\mathcal{U})_{qf}(\mathcal{U}, Z, n, KnZ) \to \mathsf{A}_{qf}(f, g)\right).$$

A functional interpretation yields terms $t_Z, t_n, t_g \in T_0[\mathcal{U}, K, f]$ such that

$$\widehat{\mathsf{WE\text{-}PA}}^\omega{\restriction} \vdash \forall f \, \forall \mathcal{U}^2 \, \forall K^2 \, \left((\mathcal{U})_{qf}(\mathcal{U}, t_Z, t_n, K t_n t_Z) \to \mathsf{A}_{qf}(f, t_g)\right), \tag{8.6}$$

see for instance Theorem 10.53 in [67]. Now by Theorem 2.18 applied to t_Z, t_n, t_g we obtain normalized term t'_Z, t'_n, t'_g which are provably (relative to $\widehat{\mathsf{WE\text{-}PA}}^\omega{\restriction}$) equal and such that every occurrence of $\mathcal{U}$ and K is of the form

$$\mathcal{U}(t[j^0]) \qquad \text{resp.} \qquad K(n^0, t[j^0]),$$

where t is a term in $T_0[\mathcal{U}, K, f]$.

Let $(t_i)_{i<n}$ be the list of all of these terms t to which $\mathcal{U}$ and K are applied. Assume that this list is partially ordered according to the subterm ordering, i.e. if t_i is a subterm of t_j then $i < j$.

We now build for each f a partial non-principal ultrafilter $\mathcal{F}$ which acts on these occurrences like a real non-principal ultrafilter. For this fix an arbitrary f.

The filter $\mathcal{F}$ is build by iterated applications of Proposition 8.6:
To start the iteration let $\mathcal{A}_{-1}$ be the trivial algebra $\{\emptyset, \mathbb{N}\}$ and $\mathcal{F}_{-1} = \{\mathbb{N}\}$ be the partial non-principal ultrafilter for $\mathcal{A}_{-1}$.
Let $\mathcal{A}_i$ be the algebra spanned by $\mathcal{A}_{i-1}$ and the sets described by t_i where $\mathcal{U}, K$ are replaced by $\mathcal{F}_{i-1}$ and K' from (8.5), i.e. $\left(t_i[\mathcal{U}/\mathcal{F}_{i-1}, K/K'](j)\right)_{j\in\mathbb{N}}$. Let $\mathcal{F}_i$ be an extension of $\mathcal{F}_{i-1}$ to the new algebra $\mathcal{A}_i$ as constructed in Proposition 8.6.

Obviously in a term t_i the functional $\mathcal{F}$ is only applied to subterms of t_i. Since the (t_i) is sorted according to the subterm ordering the partial non-principal ultrafilter is already fixed for this applications.

For the resulting partial non-principal ultrafilter $\mathcal{F} := \mathcal{F}_n$ we then get

$$\mathsf{RCA}_0^\omega + (\mu) \vdash \forall f\, \exists \mathcal{F}\, (\mathcal{U})_{qf}(\mathcal{F}, t_Z[\mathcal{F}, K', f], t_n[\mathcal{F}, K', f], K' t_n[\mathcal{F}, K', f] t_Z[\mathcal{F}, K', f]).$$

Combining this with (8.6) yields

$$\mathsf{RCA}_0^\omega + (\mu) \vdash \forall f\, \exists \mathcal{F}\, \mathsf{A}_{qf}(f, t_g[\mathcal{F}, K', f])$$

and hence

$$\mathsf{RCA}_0^\omega + (\mu) \vdash \forall f\, \exists g\, \mathsf{A}_{qf}(f, g).$$

With this we have eliminated the use of $(\mathcal{U})$ in the proof.

By Theorem 8.3.4 of [7] the theory $\mathsf{RCA}_0^\omega + (\mu)$ is conservative for sentences of this form over ACA_0^ω and therefore

$$\mathsf{ACA}_0^\omega \vdash \forall f\, \exists g\, \mathsf{A}_{qf}(f, g).$$

To obtain a realizer for g we apply the functional interpretation to the last statement. This extracts a realizer $t \in T_0[B_{0,1}]$. Since t_g is only a term of type 2 one can find a term $t' \in T$ which is equal to t, see [62, Corollary 4.4.(1)]. This t' solves the theorem. $\square$

If one is not interested in the extracted program then one can obtain a stronger conservation result:

Theorem 8.8 (Conservation)**.** *The system* $\mathsf{ACA}_0^\omega + (\mu) + (\mathcal{U})$ *is* Π^1_2*-conservative over* ACA_0^ω *and therefore also conservative over* PA*.*

Proof. Let $\forall f\, \exists g\, \mathsf{A}(f, g)$ be an arbitrary Π^1_2 statement which is provable in $\mathsf{ACA}_0^\omega + (\mu) + (\mathcal{U})$ and does not contain μ or $\mathcal{U}$. We will show that this statement is provable in ACA_0^ω and if it is arithmetical also in PA.

Relative to (μ) each arithmetical formula is equivalent to a quantifier free formula. Hence there exists a quantifier free formula A'_{qf} such that

$$\mathsf{RCA}_0^\omega + (\mu) \vdash \mathsf{A}(f, g) \leftrightarrow \mathsf{A}'_{qf}(f, g).$$

This gives

$$\mathsf{RCA}_0^\omega + (\mu) + (\mathcal{U}) \vdash \forall f\, \exists g\, \mathsf{A}'_{qf}(f, g).$$

Since the system $\mathsf{RCA}_0^\omega + (\mu)$ has a functional interpretation in $T_0[\mu]$, see [7, 8.3.1], one can now apply the same argument as in the proof of Theorem 8.4 with T_0 replaced by $T_0[\mu]$, and obtains that

$$\mathsf{RCA}_0^\omega + (\mu) \vdash \forall f\, \exists g\, \mathsf{A}'_{qf}(f, g)$$

and therefore also

$$\mathsf{RCA}_0^\omega + (\mu) \vdash \forall f\, \exists g\, \mathsf{A}(f,g).$$

The result follows now also from Theorem 8.3.4 of [7]. □

8.1. Elimination of Skolem functions for monotone formulas

We will show in this section that uses of a partial non-principal ultrafilter for an algebra given by a fixed term over a weak basis theory does not lead to more than primitive recursive growth. For this we will make use of Kohlenbach's elimination of Skolem functions for monotone formulas.

Let $\mathcal{U}(\mathcal{A})$ be the principle that states that for the algebra $\mathcal{A} = (A_n)_{n\in\mathbb{N}}$ given by $(f(n))_{n\in\mathbb{N}}$ there exists a set $F \subseteq \mathbb{N}$, such that

$$\mathcal{F} = \{A \mid \exists n \in F\, (A = A_i)\}$$

satisfies $(\mathcal{U})$ relativized to $\mathcal{A}$. This means that

$$\begin{cases} \quad \forall i,j\, \left(A_i = \overline{A_j} \rightarrow (i \in F \vee j \in F)\right) \\ \wedge \forall i,j\, ((A_i \subseteq A_j \wedge i \in F) \rightarrow j \in F) \\ \wedge \forall i,j,k\, ((i,j \in F \wedge A_k = A_i \cap A_j) \rightarrow k \in F) \\ \wedge \forall i\, (i \in F \rightarrow \forall n\, \exists k > n\, (k \in A_i))\,. \end{cases}$$

We obtain the following theorem:

Theorem 8.9. *Let $\mathsf{A}_{qf}(f,x)$ be a quantifier free formula that contains only f, x free and let t_1, t_2 be terms in $\mathsf{WKL}_0^{\omega *}$. If*

$$\mathsf{WKL}_0^{\omega *} \vdash \forall f\, \left(\Pi_1^0\text{-}\mathsf{CA}(t_1 f) \wedge \mathcal{U}(t_2 f) \rightarrow \exists x\, \mathsf{A}_{qf}(f,x)\right)$$

then one can extract a primitive recursive (in the sense of Kleene) functional Φ such that

$$\mathsf{RCA}_0^\omega \vdash \forall f\, \mathsf{A}_{qf}(f, \Phi(f)).$$

In particular if f is only of type 0 one obtains that there exists a primitive recursive function g such that

$$\mathsf{PRA} \vdash \forall x\, \mathsf{A}_{qf}(x, g(x)).$$

Proof. We will show, by formalizing the construction of b in the proof of Proposition 8.6, that there exists a term t' such that

$$\forall h\, \left(\Pi_1^0\text{-}\mathsf{CA}(t'h) \rightarrow \mathcal{U}(h)\right).$$

The theorem follows then from the elimination of Skolem functions for monotone formulas, Section 2.4, and and the fact that one can code the two instances of Π^0_1-CA given by $t_1 f$ and $t' t_2 f$ into one, see Remark 1.9.

In the construction of b in the proof of Proposition 8.6 only two steps cannot be formalized in $\mathsf{WKL}_0^{\omega *}$. The first step is the application of Π^0_1-CP and the second is the use of Π^0_2-WKL. The use of Π^0_1-CP can be reduced to a suitable instance of Π^0_1-CA (with the parameters $\mathcal{F}, \tilde{\mathcal{A}}$) and QF-$\mathsf{AC}^{1,0}$. The use of Π^0_2-WKL follows from Π^0_1-WKL and another instance of Π^0_1-CA (also with the parameters $\mathcal{F}, \tilde{\mathcal{A}}$). Since Π^0_1-WKL is equivalent to WKL and one can code the two instances of comprehension together one obtains in total that the index function b can be constructed in $\mathsf{WKL}_0^{\omega *} + \Pi^0_1$-$\mathsf{CA}(t\mathcal{F}\tilde{\mathcal{A}})$ for a suitable t. (Note that the set $\tilde{\mathcal{F}}$ cannot be defined since it involves μ.)

Using this one can extend the partial ultrafilter $\mathcal{F} = \{\mathbb{N}\}$ on the trivial algebra $\mathcal{A} = \{\emptyset, \mathbb{N}\}$ to an (index set of an) ultrafilter satisfying $\mathcal{U}(h)$. From this one can easily construct a term t'. This provides the theorem. $\square$

Remark 8.10. Although the restriction of $\mathcal{U}$ to an algebra given by a term seems to be weak, it is strong enough to prove instances of ultralimit, i.e. that the ultralimit exists for (a sequence of) sequences or given by one fixed term.

To see this let $(x_n)_{n\in\mathbb{N}}$ be a sequence in the interval $[0,1]$. Without loss of generality we may assume that $(x_n) \subseteq \mathbb{Q}$. We will prove that the ultralimit of this sequence exists using $(\mathcal{U})(t[(x_n)])$ for a term t. For this let

$$A_{i,k} := \left\{ n \in \mathbb{N} \,\middle|\, x_n \in \left[\frac{i}{2^k}, \frac{i+1}{2^k}\right[\right\}.$$

Let $\mathcal{A}$ be the algebra created by this sets. It is clear that $\mathcal{A}$ can be described by a term $t[(x_n)]$.

Observe that the proof of Lemma 8.2 can also be carried out in RCA_0^*. Since $(A_{i,k})_{i\le 2^k}$ defines a finite partition of $\mathbb{N}$, Lemma 8.2 provides

$$\forall k \, \exists! i \le 2^k \ (A_{i,k} \in \mathcal{U}),$$

(strictly speaking we obtain that the index of $A_{i,k}$ is in an index set of $\mathcal{U}$) and QF-$\mathsf{AC}^{1,0}$ yields a choice function $f(k)$ for i. Note that the ultrafilter properties provide that each $A_{f(k),k}$ is infinite and that

$$\forall k \, \forall k' > k \ \left(A_{f(k'),k'} \subseteq A_{f(k),k}\right).$$

Let $g(k)$ be the k-th element of $A_{f(k),k}$ then the sequence $\left(x_{g(k)}\right)_k$ defines a Cauchy-sequence with Cauchy-rate 2^{-k} which converges to $\lim_{n\to\mathcal{U}} x_n$.

9. The generalized Banach contraction mapping principle

In this chapter we show that a generalization of the Banach contraction mapping principle follows from Ramsey's theorem for pairs over a weak basis theory.

Let $(\mathcal{X}, d)$ be a metric space, let $m \in \mathbb{N}$ be > 0 and let $\mu \in [0,1[$. We call a function $T\colon \mathcal{X} \to \mathcal{X}$ a (m,γ)-*g-contraction* if for all $x, y \in \mathcal{X}$ there is an $i \in [1, m]$, such that $d(T^i x, T^i y) <_{\mathbb{R}} \gamma^i d(x,y)$.

The ordinary Banach contraction mapping theorem states that every (m,μ)-g-contraction with $m = 1$ has a fixed point. The generalized Banach contraction mapping principle states that *every* (m,μ)-g-contraction has a fixed point.

First results on the generalized Banach contraction mapping principle have been established in Jachymski, Schroder and Stein [46], where it is shown that it true for g-contraction where $m = 2$. In Jachymski and Stein [47] it was show that the principle is true for all m if the g-contraction is uniformly continuous. Later in Merryfield, Rothschild and Stein [73] it was shown, that this principle is true for all continuous g-contractions and for $m = 3$ without this continuity assumption. This proof uses Ramsey's theorem. However, it also uses full arithmetical comprehension,which is—as we will see below—much stronger than this contraction principle. Therefore, this proof is not suitable for a faithful formalization. The principle in its full generality was finally proved in Arvanitakis [2] and independently in Merryfield and Stein [74].

We define this principle in the language of $\mathsf{WE\text{-}PA}^{\omega}_{1}\!\restriction$ in the following way: A complete separable metric space $(\hat{\mathcal{X}}, \hat{d})$ is represented as completion of a countable metric space $(\mathcal{X}, d)$. A point in $\hat{\mathcal{X}}$ is given by a Cauchy sequence of elements of $\mathcal{X}$ having a fixed Cauchy-rate. Thus a point in $\hat{\mathcal{X}}$ is represented by a type 1 object. The metric $\hat{d}$ is the continuous extension of d to $\hat{\mathcal{X}}$. A function $T\colon \mathcal{X} \to \mathcal{X}$ can then be represented by a type 2 object. To build the iteration T^n of T we in general require therefore the recursor R_1, we will therefore work over there basis theory $\mathsf{WE\text{-}PA}^{\omega}_{1}\!\restriction$ (and not over $\widehat{\mathsf{WE\text{-}PA}}^{\omega}\!\restriction$). See [67, Chapter 4].

In the case the function T is continuous in the sense of reverse mathematics, i.e. T has a continuous modulus of continuity, then one can prove the totality of the iteration T^n in Σ^0_2-IA and in fact is equivalent to Σ^0_2-IA, see [31, Theorem 4.3]. Thus if one is only interested in such T one could weaken the basis theory to $\widehat{\mathsf{WE\text{-}PA}}^{\omega}\!\restriction + \Sigma^0_2$-$\mathsf{IA}$. If one additionally assumes that $\mathcal{X}$ is compact then one could also use WKL instead of Σ^0_2-IA, see [31, Theorem 4.5].

We are now in the position to formally define the generalized Banach contraction mapping principle. This definition is relative to $\mathsf{WE\text{-}PA}^\omega_1\!\restriction$.

Definition 9.1. Let GBCC_m be that statement that each presentable complete separable metric space space $(\mathcal{X}, d)$ and each function $T\colon \mathcal{X} \to \mathcal{X}$, which is a (m,γ)-g-contraction for a $\gamma \in [0,1[$ there exists a fixed point of T.

Further, let $\mathsf{GBCC} :\equiv \forall m\, \mathsf{GBCC}_m$ and let GBCC^{cont}_m, GBCC^{cont} be the restriction of those principles to continuous functions T. (GBCC is an abbreviation for "Generalized Banach contraction conjecture".)

We will show the following theorem.

Theorem 9.2.

(i) $\mathsf{WE\text{-}PA}^\omega_1\!\restriction + \mathsf{QF\text{-}AC}^{0,0} \vdash \mathsf{RT}^2_2 \to \mathsf{GBCC}_m$ *for each* m,

(ii) $\mathsf{WE\text{-}PA}^\omega_1\!\restriction + \mathsf{QF\text{-}AC}^{0,0} \vdash \mathsf{RT}^2_{<\infty} \to \mathsf{GBCC}^{cont}$,

(iii) $\mathsf{WE\text{-}PA}^\omega_1\!\restriction + \mathsf{QF\text{-}AC}^{0,0} \vdash (\mathsf{RT}^2_{<\infty} \wedge \Sigma^0_3\text{-}\mathsf{IA}) \to \mathsf{GBCC}$.

This theorem immediately shows that one can extend the program extraction results of Chapter 3, e.g. Corollary 3.25, to include also GBCC_m for each m (in the case of RT^2_2) and GBCC^{cont} (in the case of $\mathsf{RT}^2_{<\infty}$).

Theorem 9.2 is established by formalizing the proof of the generalized Banach contraction mapping principle of Fremlin [28] and by using some ideas of the proof of [2].

We will first prove the case where T is continuous and the extend it to the general case. Before we can do this we provide some combinatorial lemmata.

9.1. Combinatorial lemmata

Proposition 9.3 ([16, Lemmas 7.10, 7.12], [19, 17])**.** *Over* $\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction + \mathsf{QF\text{-}AC}^{0,0}$ *the principle* SRT^2_2 *is equivalent to the statement that for every* Δ^0_2*-set* A *there exists an infinite set* X *such that* $X \subseteq A$ *or* $X \subseteq \overline{A}$*.*

The principle $\mathsf{SRT}^2_{<\infty}$ *is equivalent to the statement that for every finite* Δ^0_2*-partition* $(A_i)_{i<n}$ *of* $\mathbb{N}$ *there exists an* $i<n$ *and an infinite set* X *such that* $X \subseteq A_i$*. (If* n *is uniformly bounded this principle follows from* SRT^2_2 *by induction on the metalevel.)*

Remark 9.4 (COH as partial non-principal ultrafilter). Let $(R_i)_{i\in\mathbb{N}}$ be a sequence of sets $R_i \subseteq \mathbb{N}$ and let S be an infinite cohesive set for this sequence.

Define $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ by

$$X \in \mathcal{F} \quad \text{iff} \quad S \subseteq^* X.$$

Then as long as one is only concerned about sets in $(R_i)_i$ the usual properties of an non-principal ultrafilter hold; i.e. let $i, j \in \mathbb{N}$ then

- $R_i \subseteq R_j \land R_i \in \mathcal{F} \Rightarrow R_j \in \mathcal{F}$,
- $R_i, R_j \in \mathcal{F} \Rightarrow R_i \cap R_j \in \mathcal{F}$,
- $R_i \in \mathcal{F} \lor \overline{R_i} \in \mathcal{F}$ (by cohesiveness of S)
- $R_i \in \mathcal{F} \Rightarrow R_i$ is infinite.

In other words $\mathcal{F}$ defines a non-principal ultrafilter in the algebra of sets created by $(R_i)_i$. Hence, if one can fix in advance a countable number of sets, for which the properties of an non-principal ultrafilter are needed, the ultrafilter may be replaced by the filter $\mathcal{F}$.

Note that the statement $X \in \mathcal{F}$ is $\Delta^0_2(S)$ for $X \in (R_i)$.

Syndetic sets

Definition 9.5 (Syndetic)**.**

- Let $m \geq 1$. A set $I \subseteq \mathbb{N}$ is called *m-syndetic* if for all $k \in \mathbb{N}$ the set $I \cap [k, k+m[$ is not empty.
- A set $I \subseteq \mathbb{N}$ is called *piecewise m-syndetic* if there exists arbitrary large intervals $[j_1, j_2]$, such that for all $k \in [j_1, j_2 - m]$ the set $I \cap [k, k+m[$ is not empty.

Lemma 9.6 ($\widehat{\mathsf{WE\text{-}PA}^\omega}\!\restriction \oplus \mathsf{SRT}^2_{<\infty}$)**.** *Let $n \in \mathbb{N}$ and $(A_i)_{i<n}$ be a finite sequence of disjoint Δ^0_2-subsets of $\mathbb{N}$, such that $I := \bigcup_{i<n} A_i$ is m-syndetic for an m, then there exists an infinite set X such that $X \subseteq A_i$ for an i.*

This lemma requires an instance of SRT^2_2 if n and m are fixed and $\mathsf{SRT}^2_{<\infty}$ otherwise.

Proof. Define a Δ^0_2-function $f\colon \mathbb{N} \to [0, n]$, via a Δ^0_2-formula for its graph, denoting to which set a number belong by

$$f(x) := \begin{cases} i & \text{if } x \in A_i, \\ n & \text{otherwise.} \end{cases}$$

We now divide the natural numbers into blocks of size m, and define the Δ^0_2-function g assigning to each of those blocks the sequence of values f on it:

$$g(x) := \langle f(x \cdot m), \dots, f(x \cdot m + m - 1) \rangle$$

Note that because I is m-syndetic $g(x) \neq \langle n, \dots, n \rangle$ for all x. The function $g(x)$ defines a Δ^0_2-partition $(B_i)_{i<n'}$ of $\mathbb{N}$ with

$$B_i := \{\, x \mid g(x) = i \,\}, \qquad n' := \langle \underbrace{n, \dots, n}_{m \text{ times}} \rangle.$$

By Proposition 9.3 we can find an infinite set Y on which g is constant. Since $g(Y) \neq \langle n, \ldots, n\rangle$ there is an $j < m$ such that $\big(g(Y)\big)_j \neq n$. Let $X := \{x \cdot m + j \mid x \in Y\}$. By definition is f on Y constant and $f(Y) \neq n$ thus $Y \subseteq A_{f(Y)}$. $\square$

The original proof of Arvanitakis uses the well known fact that piecewise syndetic is partition stable. This was proved by Brown in [15] and others later, see for instance [32, Theorem 1.23]. These proofs use the Bolzano-Weierstraß principle for the Cantor space and hence comprehension and are therefore not faithful. Luckily we only need the following weaker facts about partitions of even syndetic sets and not piecewise syndetic sets.

The following two lemmas are based on [15, Lemma 1].

Lemma 9.7 ($\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction$)**.** *Let X be an m-syndetic set. If X is partition into 2 parts $A_0, A_1 = X \setminus A_0$ then either each A_i is piecewise m-syndetic or there are $i < 2$ and k such that A_i is k-syndetic.*

Proof. Suppose that there is no k such that A_0 is k-syndetic then there are intervals I of arbitrary length such that $A_0 \cap I = \emptyset$. But this means that $A_1 \cap I = X \cap I$ hence A_1 is piecewise m-syndetic. Same for A_0. $\square$

Corollary 9.8 ($\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction \oplus \Sigma^0_2\text{-}\mathsf{IA}$)**.** *Let X be an m-syndetic set. If X is partitioned into finitely many parts $(A_i)_{i<n}$ then there is an $I \subseteq [0, n[$ and an k such that each A_i with $i \in I$ is piecewise k-syndetic and $Y := \bigcup_{i \in I} A_i$ is k-syndetic.*

If the numbers of partitions n is uniformly bounded no Σ^0_2-IA is needed.

Proof. Note that being k-syndetic is a Π^0_1-statement $(\forall x\, \exists y < x + k\ (y \in X))$. We search for a $\subseteq$-minimal set $I \subseteq [0, n[$, such that there is a k with $\bigcup_{i \in I} A_i$ is k-syndetic. To do so we build by finite Σ^0_2-comprehension a finite sequence s such that

$$(s)_j = 0 \qquad \text{iff} \qquad \text{there is an } k \text{ such that the set coded by } j \text{ is } k\text{-syndetic}$$

and then search for a minimal set. This finite comprehension requires Σ^0_2-induction if greatest index of a set $I \subseteq [0, n[$ is not fixed, i.e. if n is not uniformly bounded.

If $I = i$ for an i then A_i is by assumption piecewise k-syndetic and the corollary follows. The case $I = \emptyset$ is ruled out because then $Y = \emptyset$ and Y would therefore not be syndetic. If $I = [0, n[$ then, by the argument in the previous lemma every A_i is piecewise m-syndetic. Otherwise, we found a set I such that $Y := \bigcup_{i \in I} A_i$ is k-syndetic, but no A_i is syndetic. Hence by the previous argument every A_i is piecewise k-syndetic. $\square$

Combining Lemma 9.6 and Corollary 9.8 we obtain

Proposition 9.9 ($\widehat{\mathsf{WE\text{-}PA}}^\omega\restriction \oplus \mathsf{SRT}^2_{<\infty} \oplus \Sigma^0_3\text{-}\mathsf{IA}$)**.** *Let X be an m-syndetic set partitioned into Δ^0_2-sets $(A_i)_{i<n}$, then there exists an i such that A_i is piecewise k-syndetic and a infinite set I such that $I \subseteq A_i$. Note that we do not require I to be piecewise syndetic.*

If n is uniformly bounded one needs an instance of SRT^2_2. Otherwise Σ^0_3-IA and an instance of $\mathsf{SRT}^2_{<\infty}$ is needed.

Proof. By Corollary 9.8 we can find a set J such that $(A_i)_{i\in J}$ is syndetic and each A_i with $i \in J$ is piecewise syndetic. Note that Σ^0_3 is needed since the partition is Δ^0_2. An application of Lemma 9.7 now proves the proposition. $\square$

9.2. The proof of GBCC

The continuous case Fix a provably presentable complete separable metric space $(\mathcal{X}, d)$ and a (m, γ)-g-contraction $T\colon \mathcal{X} \to \mathcal{X}$ which is continuous.

Lemma 9.10 ($\mathsf{WE\text{-}PA}^\omega_1\restriction$, [28, Lemma 2])**.** *For all points $x, y \in \mathcal{X}$ the set*

$$I := \left\{ i \in \mathbb{N} \;\middle|\; d(T^i x, T^i y) <_{\mathbb{R}} \gamma^i d(x, y) \right\}$$

is m-syndetic.

Proof. By the g-contraction property $I \cap [1, m] \neq \emptyset$ and for each $i \in I$ there is an $j \in [1, m[$ such that $i + j \in I$. $\square$

Lemma 9.11 ($\mathsf{WE\text{-}PA}^\omega_1\restriction$, [73, Lemma 1], [28, Lemma 5])**.** *For each $x \in \mathcal{X}$ there exists an $M >_{\mathbb{R}} 0$ such that the set*

$$I := \left\{ i \in \mathbb{N} \;\middle|\; d(T^i x, x) <_{\mathbb{R}} M \right\}$$

is m-syndetic.

Proof. Let $M = \frac{2}{1-\gamma} \max_{i\in[0,m]} d(T^i x, x)$. (We assume here that $Tx \neq x$, otherwise we would be done.) It is clear that $0 \in I$. For each $i \in I$ there is $j \in [1, m]$ such that $d(T^{j+i}x, T^j x) <_{\mathbb{R}} \gamma^j d(T^i x, x) <_{\mathbb{R}} \gamma M$ and hence

$$d(T^{i+j}x, x) <_{\mathbb{R}} \gamma M + d(T^j x, x) <_{\mathbb{R}} \gamma M + (1-\gamma)M = M$$

and thus $i + j \in I$. $\square$

Lemma 9.12 ($\widehat{\mathsf{WE\text{-}PA}}^\omega\restriction \oplus \mathsf{RT}^2_{<\infty} \oplus \Sigma^0_3\text{-}\mathsf{IA}$, [2, 4.2], [28, Lemma 4])**.**
Let $R \subseteq \mathbb{N} \times \mathbb{N}$ be such that

(i) *the set $\{ i \mid (i, 0) \in R \}$ is m-syndetic,*

(ii) *for every* $(i,j) \in R$ *the set* $\{ k \mid (i+k, j+k) \in R \}$ *is* m*-syndetic.*

Then there exists an infinite set I *and a piecewise syndetic* Δ^0_2*-set* $\tilde{I}$ *such that* $I \subseteq \tilde{I} \subseteq \mathbb{N}$ *and for every* $i, j \in \tilde{I}$ *there is a* k *with* $(k,i) \in R$ *and* $(k,j) \in R$.

If m *is fixed then* RT^2_2 *suffices. If the existence of* I *is sufficient (in other words the* Δ^0_2*-set* $\tilde{I}$ *is not needed) then* $\mathsf{RT}^2_{<\infty}$ *suffices. Otherwise* $\mathsf{RT}^2_{<\infty}$ *and* Σ^0_3*-*IA *is needed.*

Proof. We claim that R meets $[l, l+2m[\times [k, k+m[$ for all $k, l \in \mathbb{N}$ with $k \leq l$. To prove this claim note that by (i) there is an $i \in [l-k, l-k+m[$ such that $(i, 0) \in R$, and by (ii) there is now an $j \in [k, k+m[$ such that $(i+j, j) \in R$ and that also $(i+j, j) \in [l, l+2m[\times [k, k+m[$.

For each $i \in \mathbb{N}$ and $j < 2m$ let $L_{ij} := \{ l \mid (l+j, i) \in R \}$. Using the cohesive principle (which follows from RT^2_2, see p. 15) we find a cohesive set S for $(L_{ij})_{i,j}$ and a non-principal ultrafilter $\mathcal{F} := \{ X \mid S \subseteq^* X \}$ in the algebra created by (L_{ij}). The ultrafilter is Δ^0_2, see Remark 9.4.

By the claim it follows that

$$\bigcup_{\substack{i \in [k, k+m[\\ j < 2m}} L_{ij} \supseteq [k, \infty[.$$

Hence by the ultrafilter property of $\mathcal{F}$ there is for each k some $i \in [k, k+m[$ and $j < 2m$ such that $L_{ij} \in \mathcal{F}$.

Now for $j < 2m$ define $I_j := \{ i \mid L_{ij} \in \mathcal{F} \}$. Observe that by the previous argument the set $\bigcup_{j<2m} I_j$ is m-syndetic. The sets I_j are Δ^0_2-set, see Remark 9.4.

Using Proposition 9.9 we can find an infinite set I and a j such that I_j is piecewise syndetic and $I \subseteq I_j$.

If $i, i' \in I_j$, then L_{ij} and $L_{i'j}$ belong to $\mathcal{F}$, so the cannot be disjoint, and so there is some l such that $(l+j, i)$ and $(l+j, i')$ belong to R. Hence I and $\tilde{I} = I_j$ satisfies the lemma. If one is only interested in I then Lemma 9.6 instead of Proposition 9.9 suffices. ☐

We are now in the position to show Theorem 9.2 restricted to the continuous case, i.e.

(i) $\mathsf{WE\text{-}PA}^\omega_1\!\restriction + \mathsf{QF\text{-}AC}^{0,0} \vdash \mathsf{RT}^2_2 \rightarrow \mathsf{GBCC}^{cont}_m$ for each m,

(ii) $\mathsf{WE\text{-}PA}^\omega_1\!\restriction + \mathsf{QF\text{-}AC}^{0,0} \vdash \mathsf{RT}^2_{<\infty} \rightarrow \mathsf{GBCC}^{cont}$.

Proof of Theorem 9.2 for the continuous case. Let $x \in \mathcal{X}$ be an arbitrary point. By Lemma 9.11 an $M >_{\mathbb{R}} 0$ exists, such that $\{ i \mid d(T^i x, x) <_{\mathbb{R}} M \}$ is m-syndetic. We further may assume that $M >_{\mathbb{R}} d(Tx, x)$. Let $R \subseteq \mathbb{N} \times \mathbb{N}$ be the relation

$$R := \left\{ (i,j) \;\middle|\; d(T^i x, T^j x) <_{\mathbb{R}} M\gamma^j \right\}$$

By definition $\{\, i \mid (i,0) \in R \,\}$ is m-syndetic. If $(i,j) \in R$ then

$$\{\, k \mid (i+k, j+k) \in R \,\} \supseteq \left\{ k \;\middle|\; d(T^{i+k}x, T^{j+k}x) \leq \gamma^k d(T^i x, T^j x) \right\}$$

and hence is by Lemma 9.10 also m-syndetic.

The set R satisfies the assumptions of Lemma 9.12. But this lemma is not directly applicable since the set R is just a Σ^0_1-set because $<_{\mathbb{R}}$ is a Σ^0_1-statement. However we can easily build an recursive set $R' \subseteq R$ satisfying also the assumptions of Lemma 9.12: By $\mathsf{QF\text{-}AC}^{0,0}$ and the properties of R we can find a function $f_1(i,w)$ such that if w is a witness for $(i,0) \in R$ then $f_1(i,w) = (k,w')$ with $k < m$ and w' witnesses that $(i+k+1, 0) \in R$. Similarly there exists a function $f_2(i,j,w) = (k,w')$ for the second property. Now let w be a witness for the fact that $(1,0)$ is in R. Let

$$\begin{aligned} R'_0 &:= \{(1,0,w)\}, \\ R'_{n+1} &:= \{(i+k+1, 0, w') \mid (i,0,w) \in R'_n \text{ and } f_1(i,w) = (k',w')\} \\ &\qquad \cup \{(i+k+1, j+k+1, w') \mid (i,j,w) \in R'_n \text{ and } f_2(i,j,w) = (k',w')\}, \end{aligned}$$

and let R' be the projection of $\bigcup_n R'_n$ to the first two components. The membership in R' is decidable, since the first component of the elements of the sets (R_n) always increases and thus $(i,j) \in R'$ iff $\exists w\, (i,j,w) \in \bigcup_{n \leq i} R_n$. The $\exists$-quantifier here is decidable since the sets (R_n) are finite. By definition R' satisfies the assumptions of Lemma 9.12 and is a subset of R.

Hence there is an infinite set $I \subseteq \mathbb{N}$ such that for all $i,j \in I$ there is a $k \in \mathbb{N}$ such that $(k,i),(k,j) \in R' \subseteq R$. By definition of R we have

$$d(T^i x, T^j x) \leq_{\mathbb{R}} d(T^k x, T^i x) + d(T^k x, T^j x) \leq_{\mathbb{R}} M\gamma^i + M\gamma^j \underset{i,j\to\infty}{\longrightarrow} 0.$$

Thus, the sequence $(T^i x)_{i\in I}$ is Cauchy with speed $2M\gamma^i$ and admits a limit point, call it z.

Note that by continuity of T for all k we have

$$\lim_{i\in I} T^{i+k}x = T^k z.$$

Since $(1,0) \in R'$, the set $L := \{\, k \mid (1+k,k) \in R' \,\} \subseteq \{\, k \mid (1+k,k) \in R \,\}$ is m-syndetic and so we can find for every $i \in I$ an $j_i \in [0,m[$ such that $i + j_i \in L$, i.e.

$$d(T^{i+j_i+1}x, T^{i+j_i}x) \leq_{\mathbb{R}} M\gamma^{i+j_i}.$$

By the infinite pigeonhole principle there is a j and an infinite set $J \subseteq I$ on which $j_i = j$ is constant. For every $i \in J$ then holds

$$\begin{aligned} d(T^j z, T^{j+1}z) &\leq d(T^j z, T^{i+j}x) + d(T^{i+j}x, T^{i+j+1}x) + d(T^{i+j+1}x, T^{j+1}z) \\ &\leq d(T^j z, T^{i+j}x) + M\gamma^{i+j} + d(T^{i+j+1}x, T^{j+1}z) \end{aligned}$$

The last expression tends to 0 as $i \in J$ tends to infinity. This yields that $T^j z$ is a fixed-point.

The proof formalizes in $\mathsf{WE\text{-}PA}^\omega_1\restriction + \mathsf{QF\text{-}AC}^{0,0}$ except for Lemma 9.12, where we need RT^2_2 is m is uniformly bounded and $\mathsf{RT}^2_{<\infty}$ otherwise. Hence, the statement follows. $\square$

Proof of the general case Now let $T\colon \mathcal{X} \to \mathcal{X}$ be an arbitrary mapping.

Lemma 9.13 ($\mathsf{WE\text{-}PA}^\omega_1\restriction$, [28, Lemma 3])**.** *Let $x \in \mathcal{X}$. If there exists an $n \geq 1$ such that $T^n x = x$ then already $Tx = x$.*

Proof. Assume that n is minimal with $T^n x =_{\mathcal{X}} x$. Since $x =_{\mathcal{X}} y$ is Π^0_1 one can find such an n using Σ^0_1-IA.

If $n \geq 2$ take $i < j \in [1, n[$ such that $d(T^i x, T^j x)$ is minimal. Again Σ^0_1-IA proves that such i, j exists.

By the (m, γ)-g-contraction property there is a $k \in [1, m]$ such that

$$\gamma^k d(T^i x, T^j x) >_{\mathbb{R}} d(T^{i+k} x, T^{j+k} x).$$

By the assumption $T^n(x) = x$ the right side is equal to $d(T^{(i+k) \bmod n} x, T^{(j+k) \bmod n} x)$ which is a contradiction to the minimality.

Hence $n = 1$ and $Tx = x$. $\square$

Lemma 9.14 ($\widehat{\mathsf{WE\text{-}PA}}^\omega\restriction$, [2, Lemma 3.2])**.** *Let N be a given multiple of m. Then for all $u, v \in \mathbb{N}$ there exists a number $p(u, v) \in \mathbb{N}$ such that whenever $R \in [1, p(u, v)] \times [0, \infty[$ is a relation satisfying*

(i) *the set $\{ i \mid (i, 0) \in R \}$ meets every sets $[k+1, k+N] \subseteq [1, p(u, v)]$,*

(ii) *if $i + m \leq p(u, v)$ and $(i, j) \in R$, then there are $1 \leq i', j' \leq J$ such that $(i + i', j + j') \in R$,*

then there exits a subinterval $[k+1, k+N] \subseteq [1, p(u, v)]$ and $k_1, \ldots, k_u \in \mathbb{N}$ such that

(i) *$k_{r+1} - k_r \geq m$ for $1 \leq r < u$,*

(ii) *for every k_r there exists a $q \in [k+1, k+N]$ such that $(q, k_r) \in R$.*

Proof. The proof of Arvanitakis in [2, Lemma 3.2] uses only quantifier free induction and can be formalized even in elementary arithmetic. $\square$

Lemma 9.15 ($\mathsf{WE\text{-}PA}^\omega_1\restriction$, [2, Lemma 3.1])**.** *Assume that no power of T has a fixed-point, then for every $N \in \mathbb{N}$ there exists a $p(N) \in \mathbb{N}$ such that for every point $z \in X$ there exists an $\varepsilon \geq 0$ with the property that for every $y \in X$ one finds N successive iterates of T in the set $y, Ty, \ldots, T^{p(N)-1} y$ whose distance to z is bigger than ε.*

Proof. This lemma is an elementary application of the previous lemma. The proof of Arvanitakis ([2, Lemma 3.1]) can also be formalized in this system. □

Proof of Theorem 9.2. Like in the continuous case we construct using Lemma 9.12 an infinite set I. We now use that this lemma also provides a piecewise N-syndetic Δ^0_2-set $\tilde{I}$, such that $I \subseteq \tilde{I} \subseteq \mathbb{N}$. Like in the continuous case $(T^i x)_{i \in \tilde{I}}$ is Cauchy sequence with Cauchy-rate $2M\gamma^i$. We call the limit point z. Note that the sequence restricted to the elements in I converges to z, too, hence z is definable in the system.

Assume for a contradiction that T has no fixed point. By Lemma 9.13 no power of T has a fixed point and hence by Lemma 9.15 for a given N there are $p(N), \varepsilon$, such that for every point $y \in \mathcal{X}$ in $(T^i y)_{i \in [1,p(N)]}$ there are N successive elements, which are more than ε apart from z.

By the convergence of $(T^i x)_{i \in \tilde{I}}$ there exists an i_0 such that

$$d(T^i x, z) < \varepsilon \quad \text{for } i \in \tilde{I} \text{ and } i \geq i_0.$$

The Δ^0_2-set $\tilde{I}_0 := \tilde{I} \cap [i_0, \infty[$ is evidently also piecewise N-syndetic.

Using the piecewise N-syndetic property of $\tilde{I}_0$ one can find a subset of size $p(N)$ where at least every N-th element is ε-close to z, contradicting the conclusion of Lemma 9.15 and thus the assumption that T has no fixed point.

This proves the theorem.

Again, the proof formalizes in $\mathsf{WE}\text{-}\mathsf{PA}^\omega_1\restriction + \mathsf{QF}\text{-}\mathsf{AC}^{0,0}$ except for Lemma 9.12, where we need RT^2_2 if m is uniformly bounded and $\mathsf{RT}^2_{<\infty}$ and $\Sigma^0_3\text{-}\mathsf{IA}$ otherwise. Hence, the statement follows. □

Bibliography

[1] Asuman G. Aksoy and Mohamed A. Khamsi, *Nonstandard methods in fixed point theory*, Universitext, Springer-Verlag, New York, 1990, With an introduction by W. A. Kirk. MR 1066202

[2] Alexander D. Arvanitakis, *A proof of the generalized Banach contraction conjecture*, Proc. Amer. Math. Soc. **131** (2003), no. 12, 3647–3656 (electronic). MR 1998170

[3] Jeremy Avigad, *An effective proof that open sets are Ramsey*, Arch. Math. Logic **37** (1998), no. 4, 235–240. MR 1635557

[4] ———, *Notes on Π^1_1-conservativity, ω-submodels, and collection schema*, Tech. report, Carnegie Mellon Department of Philosophy, 2002.

[5] ———, *Forcing in proof theory*, Bull. Symbolic Logic **10** (2004), no. 3, 305–333. MR 2083287

[6] ———, *Weak theories of nonstandard arithmetic and analysis*, Reverse mathematics 2001, Lect. Notes Log., vol. 21, Assoc. Symbol. Logic, La Jolla, CA, 2005, pp. 19–46. MR 2185426

[7] Jeremy Avigad and Solomon Feferman, *Gödel's functional ("Dialectica") interpretation*, Handbook of proof theory, Stud. Logic Found. Math., vol. 137, North-Holland, Amsterdam, 1998, pp. 337–405. MR 1640329

[8] Jeremy Avigad and Ksenija Simic, *Fundamental notions of analysis in subsystems of second-order arithmetic*, Ann. Pure Appl. Logic **139** (2006), no. 1-3, 138–184. MR 2206254

[9] Jon Barwise and John Schlipf, *On recursively saturated models of arithmetic*, Model theory and algebra (A memorial tribute to Abraham Robinson), Springer, Berlin, 1975, pp. 42–55. Lecture Notes in Math., Vol. 498. MR 0409172

[10] Benno van den Berg, Eyvind Briseid, and Pavol Safarik, *A functional interpretation for nonstandard arithmetic*, to appear in Ann. Pure Appl. Logic, `arXiv:1109.3103`.

[11] Marc Bezem, *Strongly majorizable functionals of finite type: a model for bar recursion containing discontinuous functionals*, J. Symbolic Logic **50** (1985), no. 3, 652–660. MR 805674

[12] Andrey Bovykin and Andreas Weiermann, *The strength of infinitary Ramseyan principles can be accessed by their densities*, accepted for publication in Ann. Pure Appl. Logic, `http://logic.pdmi.ras.ru/~andrey/research.html`, 2005.

[13] Vasco Brattka, Matthew de Brecht, and Arno Pauly, *Closed choice and a uniform low basis theorem*, Ann. Pure Appl. Logic **163** (2012), no. 8, 986 – 1008.

[14] Vasco Brattka, Guido Gherardi, and Alberto Marcone, *The Bolzano-Weierstrass theorem is the jump of weak König's lemma*, Ann. Pure Appl. Logic **163** (2012), no. 6, 623–655, Computability in Europe 2010.

[15] Tom C. Brown, *An interesting combinatorial method in the theory of locally finite semigroups*, Pacific J. Math. **36** (1971), 285–289. MR 0280627

[16] Peter A. Cholak, Carl G. Jockusch, Jr., and Theodore A. Slaman, *On the strength of Ramsey's theorem for pairs*, J. Symbolic Logic **66** (2001), no. 1, 1–55. MR 1825173

[17] ______, *Corrigendum to: "On the strength of Ramsey's theorem for pairs"*, J. Symbolic Logic **74** (2009), no. 4, 1438–1439. MR 2583829

[18] Chitat T. Chong, *Nonstandard methods in Ramsey's theorem for pairs*, Computational prospects of infinity. Part II. Presented talks, Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap., vol. 15, World Sci. Publ., Hackensack, NJ, 2008, pp. 47–57. MR 2449459

[19] Chitat T. Chong, Steffen Lempp, and Yue Yang, *On the role of the collection principle for Σ^0_2-formulas in second-order reverse mathematics*, Proc. Amer. Math. Soc. **138** (2010), no. 3, 1093–1100. MR 2566574

[20] Chitat T. Chong, Theodore Slaman, and Yue Yang, Π^0_1*-conservation of combinatorial principles weaker than Ramsey's theorem for pairs*, forthcomming in Advances in Mathematics.

[21] Chris J. Conidis, *Classifying model-theoretic properties*, J. Symbolic Logic **73** (2008), no. 3, 885–905. MR 2444274

[22] Michiel de Smet and Andreas Weiermann, *Phase transitions for weakly increasing sequences*, Logic and theory of algorithms, Lecture Notes in Comput. Sci., vol. 5028, Springer, Berlin, 2008, pp. 168–174. MR 2507015

[23] ———, *Sharp thresholds for a phase transition related to weakly increasing sequences*, J Logic Computation **22** (2012), no. 2, 207–211.

[24] Rod Downey, *Computability theory and linear orderings*, Handbook of recursive mathematics, Vol. 2, Stud. Logic Found. Math., vol. 139, North-Holland, Amsterdam, 1998, pp. 823–976. MR 1673590

[25] Rod Downey, Denis R. Hirschfeldt, Steffen Lempp, and Reed Solomon, *A Δ^0_2 set with no infinite low subset in either it or its complement*, J. Symbolic Logic **66** (2001), no. 3, 1371–1381. MR 1856748

[26] Ali Enayat, *From bounded arithmetic to second order arithmetic via automorphisms*, Logic in Tehran, Lect. Notes Log., vol. 26, Assoc. Symbol. Logic, La Jolla, CA, 2006, pp. 87–113. MR 2262315

[27] Solomon Feferman, *Theories of finite type related to mathematical practice*, Handbook of mathematical logic (Jon Barwise, ed.), North-Holland, Amsterdam, 1977, pp. 913–971. Studies in Logic and the Foundations of Math., Vol. 90.

[28] David Fremlin, *The generalized Banach contraction theorem*, `http://www.essex.ac.uk/maths/people/fremlin/n02k16.ps`, 2002.

[29] Harvey Friedman, *Some systems of second order arithmetic and their use*, Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1, Canad. Math. Congress, Montreal, Que., 1975, pp. 235–242. MR 0429508

[30] ———, *Systems of second order arithmetic with restricted induction, I, II*, J. Symbolic Logic **41** (1976), 557–559, abstract.

[31] Harvey Friedman, Stephen G. Simpson, and Xiaokang Yu, *Periodic points and subsystems of second-order arithmetic*, Ann. Pure Appl. Logic **62** (1993), no. 1, 51–64, Logic Colloquium '89 (Berlin). MR 1229168

[32] Hillel Furstenberg, *Recurrence in ergodic theory and combinatorial number theory*, Princeton University Press, Princeton, N.J., 1981, M. B. Porter Lectures. MR 603625

[33] Robin O. Gandy, *Proofs of strong normalization*, To H. B. Curry: essays on combinatory logic, lambda calculus and formalism, Academic Press, London, 1980, pp. 457–477. MR 592815

[34] Philipp Gerhardy, *A quantitative version of Kirk's fixed point theorem for asymptotic contractions*, J. Math. Anal. Appl. **316** (2006), no. 1, 339–345. MR 2201765

[35] Philipp Gerhardy and Ulrich Kohlenbach, *General logical metatheorems for functional analysis*, Trans. Amer. Math. Soc. **360** (2008), no. 5, 2615–2660. MR 2373327

[36] Kurt Gödel, *Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes*, Dialectica **12** (1958), 280–287. MR 0102482

[37] Petr Hájek and Pavel Pudlák, *Metamathematics of first-order arithmetic*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1998, Second printing. MR 1748522

[38] Denis R. Hirschfeldt, Carl G. Jockusch, Jr., Bjørn Kjos-Hanssen, Steffen Lempp, and Theodore A. Slaman, *The strength of some combinatorial principles related to Ramsey's theorem for pairs*, Computational Prospects of Infinity, Part II: Presented Talks (Chitat Chong, Qi Feng, Theodore A. Slaman, W. Hugh Woodin, and Yue Yang, eds.), Lecture Notes Series, Institute for Mathematical Sciences, National University of Singapore, World Scientific, 2008, pp. 143–162.

[39] Denis R. Hirschfeldt and Richard A. Shore, *Combinatorial principles weaker than Ramsey's theorem for pairs*, J. Symbolic Logic **72** (2007), no. 1, 171–206. MR 2298478

[40] Denis R. Hirschfeldt, Richard A. Shore, and Theodore A. Slaman, *The atomic model theorem and type omitting*, Trans. Amer. Math. Soc. **361** (2009), no. 11, 5805–5837. MR 2529915

[41] Jeffry L. Hirst, *Combinatorics in subsystems of second order arithmetic*, Ph.D. thesis, Pennsylvania State University, 1987.

[42] ———, *Hindman's theorem, ultrafilters, and reverse mathematics*, J. Symbolic Logic **69** (2004), no. 1, 65–72. MR 2039345

[43] William A. Howard, *Ordinal analysis of terms of finite type*, J. Symbolic Logic **45** (1980), no. 3, 493–504. MR 583368

[44] ———, *Ordinal analysis of simple cases of bar recursion*, J. Symbolic Logic **46** (1981), no. 1, 17–30. MR 604874

[45] Shiro Ishikawa, *Fixed points and iteration of a nonexpansive mapping in a Banach space*, Proc. Amer. Math. Soc. **59** (1976), no. 1, 65–71. MR 0412909

[46] Jacek R. Jachymski, Bernd Schroder, and James D. Stein, Jr., *A connection between fixed-point theorems and tiling problems*, J. Combin. Theory Ser. A **87** (1999), no. 2, 273–286. MR 1704262

[47] Jacek R. Jachymski and James D. Stein, Jr., *A minimum condition and some related fixed-point theorems*, J. Austral. Math. Soc. Ser. A **66** (1999), no. 2, 224–243. MR 1671960

[48] Carl Jockusch and Frank Stephan, *A cohesive set which is not high*, Math. Logic Quart. **39** (1993), no. 4, 515–530. MR 1270396

[49] ———, *Correction to: "A cohesive set which is not high"*, Math. Logic Quart. **43** (1997), no. 4, 569. MR 1477624

[50] Carl G. Jockusch, Jr., *Ramsey's theorem and recursion theory*, J. Symbolic Logic **37** (1972), 268–280. MR 0376319

[51] Carl G. Jockusch, Jr. and Robert I. Soare, Π^0_1 *classes and degrees of theories*, Trans. Amer. Math. Soc. **173** (1972), 33–56. MR 0316227

[52] H. Jerome Keisler, *Nonstandard arithmetic and reverse mathematics*, Bull. Symbolic Logic **12** (2006), no. 1, 100–125. MR 2209331

[53] Mohamed A. Khamsi and Brailey Sims, *Ultra-methods in metric fixed point theory*, Handbook of metric fixed point theory, Kluwer Acad. Publ., Dordrecht, 2001, pp. 177–199. MR 1904277

[54] Stephen C. Kleene, *Countable functionals*, Constructivity in mathematics: Proceedings of the colloquium held at Amsterdam, 1957 (edited by A. Heyting), Studies in Logic and the Foundations of Mathematics, North-Holland Publishing Co., Amsterdam, 1959, pp. 81–100. MR 0112837

[55] ———, *Recursive functionals and quantifiers of finite types. I*, Trans. Amer. Math. Soc. **91** (1959), 1–52. MR 0102480

[56] Ulrich Kohlenbach, *Gödel functional interpretation and weak compactness*, Ann. Pure Appl. Logic, published online 24 December 2011, available at `http://dx.doi.org/10.1016/j.apal.2011.12.009`.

[57] ———, *Effective bounds from ineffective proofs in analysis: an application of functional interpretation and majorization*, J. Symbolic Logic **57** (1992), no. 4, 1239–1273. MR 1195271

[58] ———, *Mathematically strong subsystems of analysis with low rate of growth of provably recursive functionals*, Arch. Math. Logic **36** (1996), no. 1, 31–71. MR 1462200

[59] ———, *Arithmetizing proofs in analysis*, Logic Colloquium '96 (San Sebastián) (J. M. Larrazabal, D. Lascar, and G. Mints, eds.), Lecture Notes in Logic, vol. 12, Springer, Berlin, 1998, pp. 115–158. MR 1674949

[60] ______, *Elimination of Skolem functions for monotone formulas in analysis*, Arch. Math. Logic **37** (1998), 363–390. MR 1634279

[61] ______, *On the arithmetical content of restricted forms of comprehension, choice and general uniform boundedness*, Annals of Pure and Applied Logic **95** (1998), 257–285. MR 1650651

[62] ______, *On the no-counterexample interpretation*, J. Symbolic Logic **64** (1999), no. 4, 1491–1511. MR 1780065

[63] ______, *Things that can and things that cannot be done in PRA*, Ann. Pure Appl. Logic **102** (2000), no. 3, 223–245. MR 1740484

[64] ______, *On uniform weak König's lemma*, Ann. Pure Appl. Logic **114** (2002), no. 1-3, 103–116, Commemorative Symposium Dedicated to Anne S. Troelstra (Noordwijkerhout, 1999). MR 1879410

[65] ______, *Higher order reverse mathematics*, Reverse mathematics 2001, Lect. Notes Log., vol. 21, Assoc. Symbol. Logic, La Jolla, CA, 2005, pp. 281–295. MR 2185441

[66] ______, *Some computational aspects of metric fixed-point theory*, Nonlinear Anal. **61** (2005), no. 5, 823–837. MR 2130066

[67] ______, *Applied proof theory: Proof interpretations and their use in mathematics*, Springer Monographs in Mathematics, Springer Verlag, 2008. MR 2445721

[68] ______, *On the logical analysis of proofs based on nonseparable Hilbert space theory*, Proofs, Categories and Computations. Essays in Honor of Grigori Mints (S. Feferman and W. Sieg, eds.), College Publications, 2010.

[69] Georg Kreisel, *Interpretation of analysis by means of constructive functionals of finite types*, Constructivity in mathematics: Proceedings of the colloquium held at Amsterdam, 1957 (edited by A. Heyting), Studies in Logic and the Foundations of Mathematics, North-Holland Publishing Co., Amsterdam, 1959, pp. 101–128. MR 0106838

[70] Alexander P. Kreuzer, *Der Satz von Ramsey für Paare und beweisbar rekursive Funktionen*, Diploma Thesis, Technische Universität Darmstadt, 2009.

[71] Stéphane Le Roux and Martin Ziegler, *Singular coverings and non-uniform notions of closed set computability*, Math. Log. Quart. **54** (2008), no. 5, 545–560. MR 2451914

[72] Horst Luckhardt, *Extensional Gödel functional interpretation. A consistency proof of classical analysis*, Lecture Notes in Mathematics, Vol. 306, Springer-Verlag, Berlin, 1973. MR 0337512

[73] James Merryfield, Bruce Rothschild, and James D. Stein, Jr., *An application of Ramsey's theorem to the Banach contraction principle*, Proc. Amer. Math. Soc. **130** (2002), no. 4, 927–933 (electronic). MR 1873763

[74] James Merryfield and James D. Stein, Jr., *A generalization of the Banach contraction principle*, J. Math. Anal. Appl. **273** (2002), no. 1, 112–120. MR 1933019

[75] Antonio Montalbán, *Open questions in Reverse Mathematics*, Bull. Symbolic Logic **17** (2011), no. 3, pp. 431–454.

[76] Dag Normann, *Recursion on the countable functionals*, Lecture Notes in Mathematics, vol. 811, Springer, Berlin, 1980. MR 583016

[77] Paulo Oliva, *Understanding and using Spector's bar recursive interpretation of classical analysis*, Proceedings of CiE'2006, Lecture Notes in Computer Science, Springer 3988, 2006, pp. 423–434.

[78] Erik Palmgren, *An effective conservation result for nonstandard arithmetic*, Math. Log. Q. **46** (2000), no. 1, 17–23. MR 1736646

[79] Charles Parsons, *On a number theoretic choice schema and its relation to induction*, Intuitionism and Proof Theory (Proc. Conf., Buffalo, N.Y., 1968) (A. Kino, J. Myhill, and R. E. Vesley, eds.), North-Holland, Amsterdam, 1970, pp. 459–473. MR 0280330

[80] ———, *Proof-theoretic analysis of restricted induction schemata*, Meetings of the Association for Symbolic Logic (Gaisi Takeuti, ed.), Association for Symbolic Logic, 1971, abstract, p. 361.

[81] ———, *On n-quantifier induction*, J. Symbolic Logic **37** (1972), 466–482. MR 0325365

[82] Pavol Safarik and Ulrich Kohlenbach, *On the computational content of the Bolzano-Weierstraß principle*, Math. Log. Q. **56** (2010), no. 5, 508–532. MR 2742886

[83] David Seetapun and Theodore A. Slaman, *On the strength of Ramsey's theorem*, Notre Dame J. Formal Logic **36** (1995), no. 4, 570–582, Special Issue: Models of arithmetic. MR 1368468

[84] Richard A. Shore, *Reverse mathematics: the playground of logic*, Bull. Symbolic Logic **16** (2010), no. 3, 378–402. MR 2731250

[85] Stephen G. Simpson, *Degrees of unsolvablity*, Handbook of mathematical logic (John Barwise, ed.), North-Holland Publishing Co., Amsterdam, 1977.

[86] ———, *Subsystems of second order arithmetic*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1999. MR 1723993

[87] Robert I. Soare, *Recursively enumerable sets and degrees*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1987, A study of computable functions and computably generated sets. MR 882921

[88] Robert M. Solovay, *Hyperarithmetically encodable sets*, Trans. Amer. Math. Soc. **239** (1978), 99–122. MR 0491103

[89] Ernst Specker, *Ramsey's theorem does not hold in recursive set theory*, Logic Colloquium '69 (Proc. Summer School and Colloq., Manchester, 1969), North-Holland, Amsterdam, 1971, pp. 439–442. MR 0278941

[90] Clifford Spector, *Provably recursive functionals of analysis: a consistency proof of analysis by an extension of principles formulated in current intuitionistic mathematics*, Proc. Sympos. Pure Math., Vol. V, American Mathematical Society, Providence, R.I., 1962, pp. 1–27. MR 0154801

[91] Thomas Streicher and Ulrich Kohlenbach, *Shoenfield is Gödel after Krivine*, Math. Log. Q. **53** (2007), no. 2, 176–179. MR 2308497

[92] Gaisi Takeuti, *Two applications of logic to mathematics*, Iwanami Shoten, Publishers, Tokyo, 1978, Kanô Memorial Lectures, Vol. 3, Publications of the Mathematical Society of Japan, No. 13. MR 0505474

[93] Terence Tao, *Structure and randomness*, American Mathematical Society, Providence, RI, 2008, Pages from year one of a mathematical blog. MR 2459552

[94] Henry Towsner, *Ultrafilters in reverse mathematics*, `arXiv:1109.3902`.

[95] ———, *Hindman's theorem: an ultrafilter argument in second order arithmetic*, J. Symbolic Logic **76** (2011), no. 1, 353–360.

[96] Anne S. Troelstra (ed.), *Metamathematical investigation of intuitionistic arithmetic and analysis*, Lecture Notes in Mathematics, Vol. 344, Springer-Verlag, Berlin, 1973. MR 0325352

[97] ———, *Note on the fan theorem*, J. Symbolic Logic **39** (1974), 584–596. MR 0384494

[98] Andreas Weiermann, *How is it that infinitary methods can be applied to finitary mathematics? Gödel's* T*: a case study*, J. Symbolic Logic **63** (1998), no. 4, 1348–1370. MR 1665727

Wissenschaftlicher Werdegang

2003–2009 Studium der Mathematik, Technische Universität Darmstadt.
Diplomarbeit: Der Satz von Ramsey für Paare und beweisbar rekursive Funktionen (betreut von Prof. Ulrich Kohlenbach).

2009–2012 Promotionsstudium, Technische Universität Darmstadt.