Constant Mean Curvature Surfaces bifurcating from Nodoids

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Zusammenfassung

In der vorliegenden Arbeit werden Familien von Flächen konstanter mittleren Krümmung (kurzzz: CMC-Fläche, CMC für Constant Mean Curvature) konstruiert, die von gewissen wohlbekannten Rotationsflächen abzweigen. Das grundlegende Konstruktionsprinzip dabei ist die Lawson-Korrespondenz, welche eine eindeutige Beziehung zwischen einfach zusammenhängenden Minimalflächen in einer Raumform der Krümmung K und dazu isometrischen CMC Flächen der konstanten mittleren Krümmung c in einer Raumform der Krümmung $K-c^2$ herstellt.

Zwei verschiedene Fälle sollten in der Arbeit behandelt werden. In einem Fall geht es um neue CMC-Flächen, die von den immersierten Rotationsflächen konstanter mittleren Krümmung im dreidimensionalen euklidischen Raum, also Nodoiden, abzweigen. Mazzeo und Pacard haben die lokale (d.h. nah an den Nodoiden) Existenz derartiger Flächen gezeigt. Das Ziel in der vorliegenden Arbeit war, mit Konjugiertenmethoden die kompletten Familien bis hin zur Degeneration zu konstruieren. In dem anderen Fall geht es um eine 1-Parameter Familie von einfach-periodischen Minimalflächen, die vom Helikoid abzweigen.

Entsprechend der Aufgabenstellung gliedert sich die Arbeit in zwei Teile.

Im Teil 1 führen wir die Randkonturen (geodätische Vierecke) des Fundamentalstücks der zu konstruierenden Fläche in der 3-Sphäre ein. Das Plateauproblem lässt sich für die neuen geodätischen Vierecke lösen. Man benutzt die Überlagerungszylinder des soliden Clifford-Torus und Hemisphäre als Barrieren um die Regularität der Flächen beim Fortsetzen durch Spieglungen zu gewährleisten. Wir verallgemeinern das Rado-Argument für die 3-Sphäre und somit lässt die Plateaulösung als Graph über die 2-Sphäre bezüglich einer Hopf-Faserung. Daraus folgt ein Eindeutigkeitssatz für die Plateaulösung und die Stetigkeit der Abzweigungsfamilie. Die neuen einfach periodischen CMC-Flächen sind immersierte 2-Sphäre mit zwei herausgenommen Punkten und besitzen diskrete Symmetrie.

Im Teil 2 verwenden wir die Konjugiertenmethode für den Fall hyperbolischer Flächen mit konstanter mittleren Krümmung 1, um Abzweigungsminimalflächen vom Helikoid in euklidischen Raum zu konstruieren. Der entscheidende Punkt hier ist die Lösung eines Plateauproblems für eine nichtkompakte Randkurve. Der Deformationsparameter ist die Flächennormale im Unendlichen. Die nichtkompakten Minimalflächen gewinnen wir durch Approximation mit kompakten Minimalflächen. Die gewünschte Asymptotik der Flächen ergibt sich aus Krümmungsabschätzungen für die Minimalflächengleichung. Die neuen Minimalflächen sind einfach periodisch und bilden eine 1-parameter Familie.

Abstract

In this work we construct families of CMC (Constant Mean Curvature) surfaces which bifurcate from certain well-known rotational surfaces. The elementary principle of construction in doing so is the Lawson's correspondence which establishes a 1 to 1 relation between simply connected minimal surfaces in one space form of curvature K and the isometric CMC surfaces with mean curvature c in another space form with curvature K-c^2.

Two different cases are discussed in this work. In the first case we construct new CMC surfaces, which bifurcate from the immersed rotational CMC surfaces in the 3-dimensional Euclidean space, namely the nodoids. Mazzeo and Pacard have showed a local (i.e. near nodoids) existence of such surfaces. In this work we shall use conjugate surfaces method to construct the complete family of bifurcating CMC surfaces to the point of degeneration. In the second case we shall construct a 1-parameter family of single periodic minimal surfaces which bifurcating from the helicoid.

According to its scope the work was divided into two parts.

In part 1 we introduce the boundary arcs (geodesic quadrilateral) of the fundamental patch in the 3-sphere. The Plateau problems to the new quadrilaterals are solvable. To guarantee the regularity of the Plateau solution in extending by the Schwarz reflection across its geodesic boundary arcs we use the covering solid cylinder of the solid Clifford torus and hemi-sphere as barriers. We generalize the argument of Rado for the 3-sphere and with that we show that the Plateau solution is a Graph over the 2-sphere in a Hopf fibration. It follows a uniqueness result for the Plateau solution and the continuity of the bifurcation family. The new singly periodic CMC surfaces are immersed 2-sphere with two punctures and has discrete symmetry.

In part 2 we shall apply the conjugate surfaces method for hyperbolic CMC-1 surfaces to construct family of bifurcating minimal surfaces from helicoid in Euclidean space. The key issue here is the solution of Plateau problems for non-compact boundary curves. The deformation parameter is the surface normal at infinity. By a exhaustion process using compact minimal surfaces we get the existence of a non-compact minimal surface. A standard curvature estimate for minimal surface equation yields the desired asymptotic property of the normal vector. The new minimal surfaces are singly periodic and constitute a 1-parameter family.

Contents

1 Introduction

The mean curvature of a surface at a point is the arithmetic mean of the principal curvatures at that point. Surfaces with nonzero **c**onstant **m**ean **c**urvature everywhere are usually referred to as *CMC surfaces*. Surfaces with zero mean curvature everywhere are *minimal surfaces*. A CMC surface is a critical point of the area functional for variations preserving volume that have compact support and fix their boundaries. As early as the 18th century it was known that many concrete problems arising in physics, chemistry and biology could be reduced to the analysis of CMC surfaces (see [Tho61]).

The Delaunay surfaces (discovered by Delaunay in 1841) form a one-parameter family of CMC-1 surfaces in $\mathbb{R}^3$ which are complete, noncompact surfaces of revolution. The family of Delaunay surfaces comes in two families, each parametrized by the *necksize* n (the length of the shortest closed geodesics). The embedded Delaunay surfaces are called *unduloids* and have necksizes $n \in (0, \pi]$. The non-embedded ones are called *nodoids* and have necksizes $n \in (0, \infty)$. For both nodoids and unduloids the limiting surface as $n \to 0$ is a chain of mutually tangent unit spheres arranged along a common axis.

In this work we shall be concerned with the nodoids. Using the Lawson correspondence and Hopf fibration of $\mathbb{S}^3$ we construct entire one-parameter families of CMC-1 surfaces $(M_{m,t})$ in $\mathbb{R}^3$, $m \in \mathbb{N} \setminus \{1\}$, $t \in [0, \frac{\pi}{2})$, which bifurcate from certain nodoids (Fig. 1.1 and Fig. 1.2). The new CMC-1 surfaces $M_{m,t}$ are singly periodic, but have m-fold dihedral symmetry instead of the full rotational symmetry. At the singular limit $t = \frac{\pi}{2}$ the bifurcating surfaces degenerate to a chain of touching spheres.

Main Theorem 1. *For every $m \in \mathbb{N}$ with $m \geq 2$ there is a one-parameter family of complete immersed* CMC-1 *surfaces $M_{m,t}$ in $\mathbb{R}^3$, $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$ with the following properties:*

[1.] Every surface $M_{m,t}$ represents a proper immersion of $\mathbb{S}^2 \setminus \{p_1, p_2\}$, where $p_1 \neq p_2$.

[2.] For each surface it is valid $M_{m,t} = M_{m,-t}$.

[3.] Continuity: Each family of surfaces $(M_{m,t})$ is continuous in t.

[4.] Bifurcation: The surface $M_{m,0}$ is a nodoid with necksize $n = (m-1)\pi$.

[5.] Symmetry: Every surface $M_{m,t}$ is simply periodic and has a $\frac{\pi}{m}$- rotational symmetry.

[6.] Degenerate case: As $|t| \to \frac{\pi}{2}$ principal curvatures of $M_{m,t}$ tend to infinity. Compact subsets of the surfaces $M_{m,t}$ converge in distance to a covering of m mutually touching spheres with centers on a circle.

We will prove the results listed in this theorem in Sect.6.3 and Sect.7.1. It should be noted that for any rational angle $\frac{p}{q} \cdot \pi$ with p, q reduced there are bifurcation CMC-1 surfaces as well. We must reflect the fundamental patch q times to close. Thus the surfaces have multiplicity p.

Let us mention some previous work on these surfaces. R. Mazzeo and F. Pacard [MP02] established the existence of new CMC-1 surfaces in $\mathbb{R}^3$ bifurcating from the nodoids with necksize lying in a non-stable range $r > r_0$. They analyzed the spectrum of the linearized mean curvature operator, i.e. the *Jacobi operator* and hence obtained families corresponding to our surfaces $(M_{m,t})$ for small $|t|$. They conjectured that these branches can be continued to smooth connected branches of bifurcations. In addition they described bifurcation branches to surfaces with screw symmetry. Rossman [Ros05] confirmed numerically that the smallest bifurcation radius r_0 is equal to $\frac{1}{2}$, that is, it has necksize π. M. Jleli [Jle09] showed the existence of bifurcation surfaces from the family of immersed CMC hypersurfaces of revolution in $\mathbb{R}^{n+1}$.

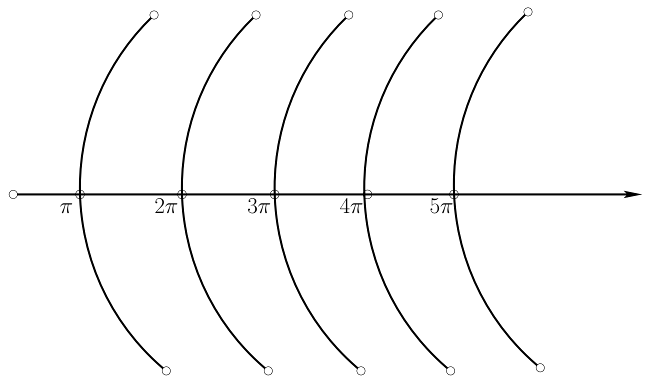

Figure 1.1: The horizontal line represents the family of nodoids parametrized by the necksize $n > 0$. At each necksize $n = (m-1)\pi$, $m > 1$, we construct families $(M_{m,t})$ of CMC-1 surfaces bifurcating from the nodoids. These families are entire in the sense that the surfaces degenerate when $|t|$ approaches an extremal value $\frac{\pi}{2}$.

While minimal surfaces can be constructed in forms of Plateau solutions via Weierstrass data, the situation for CMC surfaces is much more involved. In his paper [Law70] Lawson proved a correspondence between simply connected surfaces in 3-dimensional space forms. In particular, up to isometry there is a one to one correspondence between simply connected minimal surfaces in $\mathbb{S}^3$ and complete CMC-1 surfaces in $\mathbb{R}^3$, as well as minimal surfaces in $\mathbb{R}^3$ and complete CMC-1 surfaces in $\mathbb{H}^3$. This allows us to construct CMC cousin surfaces in $\mathbb{R}^3$ by solving Plateau problems for fundamental geodesic polygons in $\mathbb{S}^3$ and then extend the Plateau solution by Schwarz reflections.

It was observed by Karcher in [Kar89] that for the case that the CMC-1 fundamental patch in $\mathbb{R}^3$ is bounded by planar symmetry lines, the required Plateau contours in $\mathbb{S}^3$ can be described in terms of Hopf vector fields determined by symmetry properties alone. The advantage of this method is that we reduce the free boundary problem for the fundamental patch of a desired CMC-1 surface in $\mathbb{R}^3$ to a Plateau problem for an explicitly known great circle polygon Γ in $\mathbb{S}^3$.

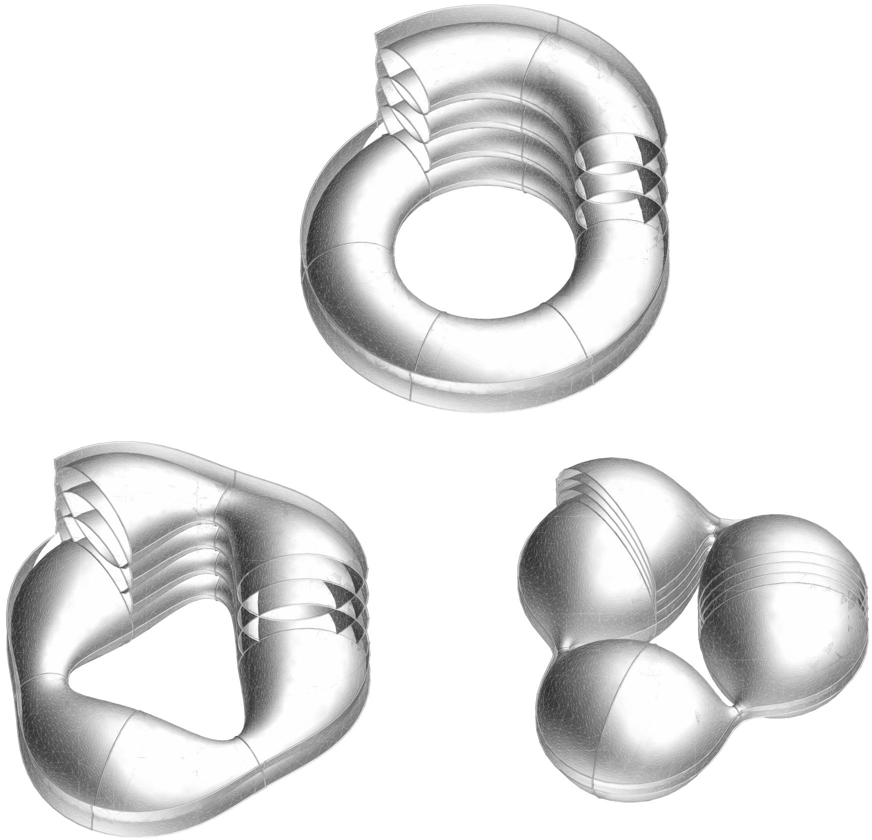

Figure 1.2: Three surfaces in the nodoid bifurcation family $(M_{m,t})$ where $m = 3$. The upper one is a standard nodoid with necksize $n = 2\pi$. Lower left: A CMC-1 bifurcating Nodoid with $t = \frac{\pi}{3}$. Lower right: A CMC-1 bifurcating Nodoid with t close to $\frac{\pi}{2}$.

A similar bifurcation phenomenon can be observed for the helicoids in $\mathbb{R}^3$. In this work we also construct a one-parameter family of minimal surfaces bifurcating from helicoids and the single limit is a Euclidean plane with multiplicity.

Main Theorem 2. *There is a one-parameter family M_φ of properly immersed minimal surfaces in $\mathbb{R}^3$, where $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ with the following properties:*

[1.] Each surface M_φ is singly periodic.

[2.] For $\varphi = 0$ the minimal surface M_φ is a standard helicoid.

[3.] For each surface it is valid: $M_\varphi = M_{-\varphi}$.

[4.] For $|\varphi| \to \frac{\pi}{2}$ the minimal surface M_φ tends to a plane with multiplicity.

[5.] The fundamental patch of each surface M_φ has an asymptotic normal vector $(0, 0, 1)$.

The motivation to study these surfaces comes again from a conjugate cousin perspective: This time relating simply connected minimal surfaces in $\mathbb{R}^3$ to CMC-1 surfaces in hyperbolic 3-space $\mathbb{H}^3$. Being ruled minimal surfaces in $\mathbb{R}^3$ the helicoids have CMC-1 cousin surfaces of revolution in $\mathbb{H}^3$, they are the so-called catenoid cousins. Our bifurcation family of minimal surfaces gives bifurcation families of the catenoid cousins. In fact, differently scaled helicoids lead to non-congruent catenoid cousins, and only for a discrete set of scalings does the bifurcation occur. This gives the following consequence of Main Thm. 2:

Corollary *For every $m \in \mathbb{N}$ with $m \geq 2$ there is a one-parameter family $M^*_{m,\varphi}$ of CMC-1 surfaces in $\mathbb{H}^3$, where $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ with the following properties:*

1) [Bifurcation] The surface $M^*_{m,0}$ is a catenoid cousin with necksize $n = 2(m-1)\pi$.
2) [Limit case] Fixing the symmetry of the surfaces $(M^*_{m,\varphi})$ as $\varphi \to \frac{\pi}{2}$, each surface $M^*_{m,\varphi}$ converges to a horosphere.

The terminology *cousin* was introduced by Bryant [Bry87]. He showed how to parametrize CMC-1 surfaces in hyperbolic 3-space, by meromorphic data. He also calculated several examples of cousins of minimal surfaces in $\mathbb{R}^3$: The catenoid cousins and Enneper cousins. Umehara and Yamada gave techniques to construct examples in terms of certain differential equations and they gave the existence of the so-called *warped catenoid cousins* [UY93], which presumably coincide with our families $M^*_{m,\varphi}$. Umehara, Yamada and Rossman [RUY04], [RUY03] classified the CMC-1 surfaces of low total curvature in $\mathbb{H}^3$ and they gave an explicit meromorphic representation of warped catenoid cousins. Sa Earp and Toubiana [ET04] gave a meromorphic data (different to that of Bryant, Umehara and Yamada) for a CMC-1 conformal immersion in $\mathbb{H}^3$ by the appoach of Kenmotsu [Ken79]. With this representation they also constructed warped catenoid cousins independently from Umehara, Yamada and Rossman. Using the conjugate surface method Karcher [Kar05a] constructed CMC-1 surfaces in $\mathbb{H}^3$. These surfaces have the symmetry group of a Platonic tessellation of $\mathbb{H}^3$ and therefore compact fundamental domain.

Our work consists of two parts. In Part I we use the conjugate surface approach due to Lawson [Law70], Karcher [Kar05b], Große-Brauckmann [GBKS03] to construct bifurcating nodoids in $\mathbb{R}^3$. Chapter 2 to Chapter 5 we introduce prerequisites for our work. In Chapter 2 we describe the Hopf fibration of $\mathbb{S}^3$ with the unit quaternions. In Chapter 3 we explain the Lawson correspondence and its geometrical properties precisely. A great deal of our reasoning depends on the maximum principle. In Chapter 4 we state and derive a maximum principle for minimal surfaces in $\mathbb{S}^3$. In general Plateau solutions are not unique. But with additional geometric condition for the boundary contour Γ we can apply the maximum principle to ensure uniqueness. The Schwarz reflection principle in $\mathbb{S}^3$ (Chapter 5) extends a Plateau solution across its geodesic boundaries. We show that the extension by reflections lets no branch points occur at the boundaries of the fundamental patch.

Chapter 6 is devoted to explaining our construction of bifurcating nodoids. Let $m > 1$. The boundary of a $\frac{\pi}{m}$-fundamental patch of a nodoid is a quadrilateral Γ^* with equal length of a pair of opposite edges. Its conjugate boundary Γ in $\mathbb{S}^3$ is a quadrilateral, which consists of four great circle arcs. We deform the boundary Γ by shortening one of the two equal length edges by $t \in (0, \frac{\pi}{2})$ along the great circle direction and extending the other one with t. We show the existence of a Plateau solution for the new quadrilateral $\Gamma(t)$. Then by using Schwarz reflection we extend this solution to a nice regular complete surface. In Chapter 7 we show a transversality property of the Plateau solution by applying the maximum principle of Chapter 4. Using the transversality we show that each one-parameter family of minimal surfaces in $\mathbb{S}^3$ is continuous in t and that the period of each $M_{m,t}$ is nonzero.

In Part II we carry the conjugate construction to the relation between minimal surface in $\mathbb{R}^3$ and CMC-1 surface in $\mathbb{H}^3$ to study bifurcating helicoids in $\mathbb{R}^3$. In Chapter 8 we use the conjugate method to construct a one-parameter family of singly periodic complete minimal surfaces from standard helicoids in $\mathbb{R}^3$. The boundary Γ of a fundamental patch of a helicoid is non-closed, it consists of two infinite parallel rays and one finite arc connecting them perpendicularly. We deform the boundary Γ to a new boundary Γ_φ by tilting the finite arc from the z-axis with an angle $\varphi \in (0, \frac{\pi}{2})$. Using an exhaustion and a uniform local area bound we show the solvability of the Plateau problem for Γ_φ. The singular limits of the constructed minimal surfaces are Euclidean planes with multiplicity while fixing one of their symmetric vertices and merge to infinity while fixing the whole symmetric arc. The CMC-1 cousins in $\mathbb{H}^3$ degenerate to horospheres respectively.

Part I

Nodoid Bifurcation Families in $\mathbb{R}^3$

2 Geometry of $\mathbb{S}^3$

2.1 Quaternions and Clifford Parallelism.

Quaternions $\mathbb{H}$ are a non-commutative extension of complex numbers, which was devised by Sir Hamilton in 1843. With respect to the basis $\{1, i, j, k\}$ the set of the quaternions $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ is a 4-dimensional $\mathbb{R}-$algebra (or 2-dimensional $\mathbb{C}-$algebra), where

$$i^2 = j^2 = k^2 = ijk = -1, \quad ij = -ji = k. \tag{2.1}$$

For every quaternion $q = a + bi + cj + dk$, where $a, b, c, d \in \mathbb{R}$, we shall define the *conjugation* (in analogy with conjugation in $\mathbb{C}$) by

$$q = a - bi - cj - dk.$$

The product of two quaternions $q = a + bi + cj + dk$ and $p = \alpha + \beta i + \gamma j + \delta k$ is given by

$$\begin{aligned} pq =& (a\alpha - b\beta - c\gamma - d\delta) + (a\beta + b\alpha + c\delta - d\gamma)i \\ &+ (a\gamma - b\delta + c\alpha + d\beta)j + (a\delta + b\gamma - c\beta + d\alpha)k \end{aligned} \tag{2.2}$$

The vector subspace

$$\operatorname{Im}\mathbb{H} := \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$$

of $\mathbb{H}$ is called – in analogy to the complex numbers - the imaginary space of $\mathbb{H}$. For imaginary quaternions $p, q \in \mathrm{Im}\mathbb{H}$ with $q = bi + cj + dk$ and $p = \beta i + \gamma j + \delta k$ the quaternion product (2.2) reduces to

$$pq = (-b\beta - c\gamma - d\delta) + (c\delta - d\gamma)i + (-b\delta + d\beta)j + (b\gamma - c\beta)k.$$

By identifying the imaginary quaternions with $\mathbb{R}^3$ the "scalar part" is the cannonical Euclidean scalar product $\langle p, q\rangle$ of the vectors $p = (\beta, \gamma, \delta)$, $q = (b, c, d) \in \mathbb{R}^3$ and the "vectorial part" of pq is the vector product of these two vectors. We thus obtain

$$pq = \mathrm{Re}(pq) + \mathrm{Im}(pq) = -\langle p, q\rangle + p \times q.$$

Let $\mathbb{H}_1 := \{q \in \mathbb{H} \colon \langle q, q\rangle_{\mathbb{R}^4} = 1\}$. Then every unit quaternion $q \in \mathbb{H}_1 \setminus \{\pm 1\}$ has a unique representation in the form

$$q = \cos\alpha + u \sin\alpha \text{ with } u \in \mathrm{Im}\mathbb{H} \cap \mathbb{H}_1,\ \alpha \in (0, \pi). \tag{2.3}$$

If we write a quaternion as 4×4 real matrices

$$\begin{bmatrix} \alpha & -\beta & -\gamma & -\delta \\ \beta & \alpha & \delta & -\gamma \\ \gamma & -\delta & \alpha & \beta \\ \delta & \gamma & -\beta & \alpha \end{bmatrix} =$$

$$\alpha \cdot \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{=1} + \beta \cdot \underbrace{\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}}_{=i} + \gamma \cdot \underbrace{\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{=j} + \delta \cdot \underbrace{\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{=k} \tag{2.4}$$

then the quaternion multiplication agrees with the matrix product. The unit 3-sphere is defined by the following

$$\mathbb{S}^3 := \left\{ (x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1 \right\}.$$

We can identify the unit quaternions with the 3-sphere $\mathbb{S}^3$. Because the set of unit quternions is closed under the quaternionic multiplication $|pq| = |p||q|$, the 3-sphere $\mathbb{S}^3$ takes on the structure of a group. Moreover, since the quaternionic multiplication is smooth, $\mathbb{S}^3$ is a nonabelian, compact Lie group of dimension 3.

Let $\Pi \subset \mathbb{R}^4$ be a two-dimensional subspace, then the intersection $C := \Pi \cap \mathbb{S}^3$ is just the unit circle in Π. In $\mathbb{S}^3$ this circle C is called a *great circle*. For $p, q \in \mathbb{S}^3$ we build the dot product $\langle p, q \rangle$ in $\mathbb{R}^4$, from it we obtain an intrinsic metric $d_{\mathbb{S}^3} \colon \mathbb{S}^3 \times \mathbb{S}^3 \to [0, \pi]$ with $d_{\mathbb{S}^3}(p, q) = \arccos(\langle p, q \rangle)$. Let $C \in \mathbb{S}^3$ be a great circle, we define the distance $d(p, C) := \inf\{d_{\mathbb{S}^3}(p, x) \colon x \in C\}$. Two great circles $C_1, C_2 \in \mathbb{S}^3$ are said to be *Clifford parallel* if $d(m, C_2)$ does not depend on $m \in C_1$.

We view $1 \in \mathbb{H}$ as the north pole of $\mathbb{S}^3$ and for every $p \in \mathrm{Im}\mathbb{H} \cap \mathbb{H}_1$ there is $d_{\mathbb{S}^3}(p, 1) = \frac{\pi}{2}$. Hence the equator of $\mathbb{S}^3$ with respect to the north pole 1 is the set of the purely imaginary unit quaternions. In this sense we will use the identification $\mathbb{S}^2 = \mathrm{Im}\mathbb{H} \cap \mathbb{H}_1$ in the later part of our work.

2.2 Left Invariant Vector Fields on $\mathbb{S}^3$.

Left translation L_p by $p \in \mathbb{S}^3$ in the Lie group $\mathbb{S}^3$ is defined by the quaternion product

$$L_p \colon \mathbb{S}^3 \to \mathbb{S}^3, \quad L_p(q) := p \cdot q.$$

A vector field V is *left invariant* if $V \circ L_p = dL_p(V)$. The evaluating map $V \to V(1)$ defines a one to one correspondence between the left invariant vector fields on $\mathbb{S}^3$ and the tangent space $T_1(\mathbb{S}^3)$. We consider an identification $T_1\mathbb{S}^3 = \mathrm{Im}\mathbb{H} = \mathbb{R}^3$. Then at the point $1 \in \mathbb{S}^3$ the left translation has the differential

$$dL_p\colon T_1\mathbb{S}^3 \to T_p\mathbb{S}^3, \quad dL_p(V) := p \cdot V.$$

We identify the pure imaginary unit quaternion with the 2-sphere $\mathbb{S}^2 = \mathrm{Im}\mathbb{H} \cap \mathbb{S}^3$. Each vector $v \in \mathbb{S}^2$ can be extended by left translation to a smooth vector field on $\mathbb{S}^3$. The integral curve of v through a point $p \in \mathbb{S}^3$ is the great circle

$$c(t) = p(\cos t + v \sin t),\ t \in [0, 2\pi].$$

Using $v^2 = -|v|^2 = -1$ the following calculation can be verified

$$c'(t) = p(-\sin t + v\cos t) = p(\cos t + v \sin t)v = c(t)v \tag{2.5}$$

How does a left invariant vector field v behave along a great circle? We calculate the covariant derivative of v along $c(t)$:

$$\frac{\nabla}{dt} L_{c(t)}(v) = (c(t)\cdot v)')^\top = (c'(t)\cdot v)^\top \tag{2.6}$$

The vector $c'(t)\cdot v$ can be written as the sum of the normal component and the tangential component:

$$c'(t)\cdot v = (c'(t)\cdot v)^\perp + (c'(t)\cdot v)^\top.$$

The normal component of $c'(t)\cdot v$ is proportional to $c(t) \in \mathbb{S}^3 = \mathbb{H}_1$ and $c(t)\cdot c^{-1}(t) = c^{-1}(t)\cdot c(t) = 1 \in \mathbb{S}^3$. We thus have

$$c'(t)\cdot v = c(t)\frac{(c'(t)\cdot v)}{c(t)} = c(t)\cdot(\underbrace{\frac{(c'(t)\cdot v)^\perp}{c(t)}}_{:=\lambda\in\mathbb{R}} + \frac{(c'(t)\cdot v)^\top}{c(t)}) = c(t)\cdot(\lambda + \frac{(c'(t)\cdot v)^\top}{c(t)}). \tag{2.7}$$

By the differential of the left translation $L_{c^{-1}(t)}$ the vector $c'(t) \in T_{c(t)}\mathbb{S}^3$ can be mapped to $T_1\mathbb{S}^3$

$$dL_{c^{-1}(t)}\colon T_{c(t)}\mathbb{S}^3 \to T_1\mathbb{S}^3, \quad dL_{c^{-1}(t)}(v) = c^{-1}(t)\cdot(c'(t)\cdot v)^\top.$$

This together with Equation 2.6 means $\frac{(c'(t)\cdot v)^\top}{c(t)}$ is purely imaginary and consequently by 2.7

$$(\lambda + \frac{(c'(t)\cdot v)^\top}{c(t)}) = \frac{c'(t)}{c(t)}\cdot v.$$

Since $\frac{c'(t)}{c(t)}, v \in \mathrm{Im}\mathbb{H}$, it follows

$$(c'(t)\cdot v)^\top = c(t)\cdot \mathrm{Im}\big(\frac{c'(t)}{c(t)}\cdot v\big) = c(t)\cdot\big(\frac{c'(t)}{c(t)}\times v\big).$$

This formula indicates: Along a geodesic circle $c(t)$ the vector field v also changes in the direction $\frac{c'(t)}{c(t)} \times v$. This force results a rotation around $c'(t)$ towards the right during v moving along $c(t)$. The length of the covariant derivative is

$$|\frac{\nabla}{dt} L_{c(t)}(v)| = |c(t)(\frac{c'(t)}{c(t)} \times v)| = |c(t)\mathrm{Im}(\frac{c'(t)}{c(t)} \cdot v)| = 1, \tag{2.8}$$

this means: the velocity of the rotation is 1. Or in other words, moving along a great circle $c(t)$ rotates $L_{c(t)}(v)$ around $c(t)$ once with respect to the parallel transport.

Definition 1. (Orientation of $\mathbb{S}^3$) The vectors $u, v, w \in T_p\mathbb{S}^3$ are called positively oriented, if u, v, w, p are positively oriented (as vectors in $\mathbb{R}^4$). Let γ be a geodesic in $\mathbb{S}^3$ and v a vector field along γ. Then we say v right (left) rotates with respect to the axis γ' if $\gamma', v, \nabla_{\gamma'} v$ are positively (negatively) oriented.

Thus for $\beta^2 + \gamma^2 + \delta^2 = 1$ a linear combination $\beta \cdot i + \gamma \cdot j + \delta \cdot k$ is a right rotating vector field.

2.3 The Hopf Fibration of $\mathbb{S}^3$

Definition 2. Each $u \in \mathbb{S}^2$ can be extended to a vector field on $\mathbb{S}^3$ by left translation. We call this left invariant vector field the *u-Hopf vector field.* An integral curve of the u-Hopf vector field is said to be a *u-Hopf circle.* The *u-Hopf map*[1], also called the *u-Hopf fibration,* is defined by $h_u : \mathbb{S}^3 \to \mathbb{S}^2$, $h_u(p) := p \cdot u \cdot \overline{p}$, where $\overline{p}$ is the conjugate of p.

By Hamilton's theorem ([ea95] p.216) every mapping $\mathbb{H} \to \mathbb{H}$, $x \mapsto \pm a \cdot x \cdot \overline{a}$, $a \in \mathbb{S}^3$, maps the subspace Im$\mathbb{H}$ onto itself. The length of the image of a u-Hopf map is $|h_u(p)| = |p \cdot u \cdot \overline{p}| = 1$. Thus we have $h_u(p) \in \mathbb{S}^2$. A direct calculation shows that the u-Hopf circles are projected to points of $\mathbb{S}^2$ and the fiber $h_u^{-1}(p)$ of a point $p \in \mathbb{S}^2$ is exactly a u-Hopf circle passing through the point p and vice versa.

$$\begin{aligned} h_u(p(\cos t + u \sin t)) &= p(\cos t + u \sin t)u(\cos t - u \sin t)\overline{p} \\ &= p(u \cos^2 t - u^3 \sin^2 t)\overline{p} \\ &= h_u(p) \end{aligned}$$

We call the great circle C_u: $\cos t + u \sin t$ *the soul.* It is the z-axis in $\mathbb{R}^3$ plus ∞ under the stereographic projection. All the left cosets $pC_u := p \cdot (\cos t + i \sin t)$, $p \in \mathbb{S}^3$ are again u-Hopf circles, any two of them are either identical or disjoint, and they partition the $\mathbb{S}^3$. Thus all the pairwise disjoint u-Hopf circles pC_u build a circle fibration of $\mathbb{S}^3$ - the u-Hopf fibration.

[1] In fact, there are several forms of the Hopf map form $\mathbb{S}^3$ to $\mathbb{S}^2$ depending on the context (see appendix for more details).

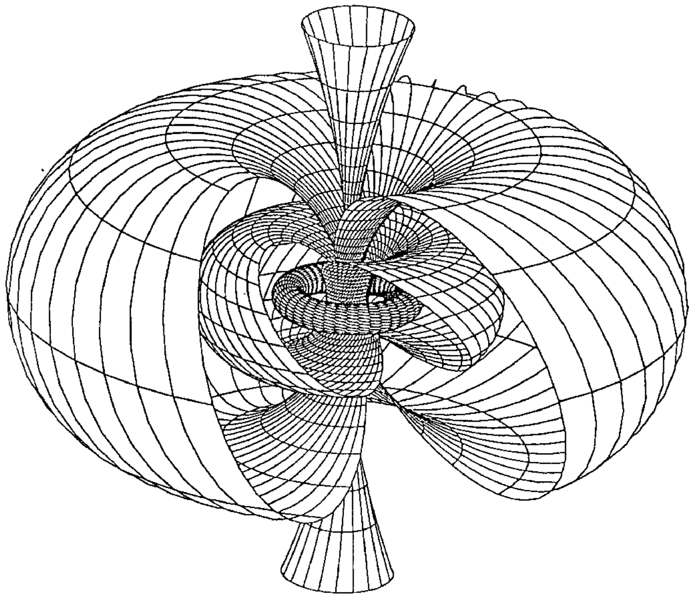

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Figure 2.1: The u-Hopf fibration illustrated by the images of distance tori under stereographic projection. The surface T_θ is a tubular neighborhood of the great circle $\cos t + u \sin t$ with a fixed distance $\theta \in [0, \frac{\pi}{2}]$. It is topologically a torus in $\mathbb{S}^3$. The surface T_0 is the z-axis in $\mathbb{R}^3$ plus ∞ and the surface $T_{\frac{\pi}{2}}$ shrinks to a circle in the xy-plane with center 0. Four of the tori are shown, with two of them cut along a pair of u-Hopf circles.

Choosing the point $(1, 0, 0, 0)$ as the north pole of $\mathbb{S}^3$ the *stereographic projection* is given by the following

$$st\colon \mathbb{S}^3 \to \mathbb{R}^3 \cup \{\infty\},\ (x_0, x_1, x_2, x_3) \mapsto \frac{1}{1 - x_0}(x_1, x_2, x_3).$$

It is well known that the stereographic projection st is a *conformal map*[2]. Another good property of the stereographic projection is that it preserves circles, i.e. the image of a circle on the 3-sphere is also a circle in $\mathbb{R}^3$. Using the stereographic projection we can visualize the Hopf fibration of $\mathbb{S}^3$ as shown in Figure 2.1.

[2]Here we mean the conformality in the Riemannian sence, i.e. a diffeomorphism between two Riemannian manifolds is called a *conformal map* if the pull-back metric is conformally equivalent to the original one.

Proposition 3. *A u-Hopf vector field has the following properties:*

(1) A u-Hopf vector field has a constant angle with a great circle.

(2) If a u-Hopf vector field is not tangent to a great circle it rotates with constant speed and once around the great circle (with respect to the parallel transport).

Proof. Let $u, v \in \mathbb{S}^2$ and $c(t)$ be a v-Hopf circle. By the Equation (2.5) we have $c'(t) = c(t) \cdot v$ and the u-Hopf vector field is $c(t) \cdot u$. Let g be the conformal Riemanian metric on $\mathbb{S}^3$ induced by the Euclidean metric. Then

$$g(c(t) \cdot u, c'(t)) = g(c(t) \cdot u, c(t) \cdot v) \overset{\text{(left\_inv.)}}{=} g(u, v).$$

In particular $g(u, v)$ does not depend on t, thus the angle between the u-Hopf field and the v-Hopf circle is constant.

The derivation of the Equation (2.8) in the last section proofs the second part of the proposition. $\square$

Let us assume $p \in \mathbb{H}$ and $u, v, w \in \mathbb{S}^2$ such that $p, p \cdot u, p \cdot v, p \cdot w$ are positively oriented. Because of property (1) and (2) in the last proposition the angle between two Hopf vector fields can be defined by their values at a point. By the identification $\mathbb{H} = \mathbb{R}^4$ we get a scalar product for quaternions. Evaluating each vector field at p yields a vector in $\mathbb{R}^4$. Hence

$$\angle(p \cdot u, p \cdot v) := \angle(u, v) = \alpha = \arccos g(u, v) \text{ with } \alpha \in [0, \pi]$$

Now assume that the Hopf vector field w satisfies $w \perp u$ and $w \perp v$. Then the oriented angle between two Hopf vector fields u, v with respect to v is defined by

$$\angle_w(u, v) = \pm\angle(u, v).$$

It has a positive (negative) sign if $p, p \cdot u, p \cdot v, p \cdot w$ are positively (negatively) oriented. We define the orientated angle of two Hopf vector fields u, v with respect to a linear independent field w by the orthogonal projection of u, v to $w^\perp$:

$$\angle_w(u, v) := \angle_w(u - g(w, u)w, v - g(w, v)w) \tag{2.9}$$

geometrically the angle $\angle_w(u, v)$ is the rotation angle from u to v along an integral curve of w.

2.4 The Clifford Torus.

Every quaternion $p \in \mathbb{S}^3$ has the representation given in Equation (2.3), we write this symbolically[3] $p = e^{u\theta}$ for $u \in \mathbb{S}^2$ and $\theta \in \mathbb{R}$. Set $u = i$. Then as for complex numbers,

[3]We can in fact define a exponential function on quaternion. The rule of multiplications is, however, not as simple as the case of $\mathbb{C}$. For our purpose we do not need to refer the exponential function.

the transformation on $\mathbb{S}^3$, $f\colon \mathbb{S}^3 \to \mathbb{S}^3$, $x \mapsto e^{i\alpha} \cdot x$ is an orthogonal transformation of $\mathbb{R}^4$

$$\begin{aligned} f(x) &= f(a+bi+cj+dk) = (\cos\alpha + i\sin\alpha)(a+bi+cj+dk) \\ &= a\cos\alpha + (b\cos\alpha)i + (c\cos\alpha)j + (d\cos\alpha)k + (a\sin\alpha)i - b\sin\alpha + (c\sin\alpha)k - (d\sin\alpha)j \\ &= a\cos\alpha - b\sin\alpha + (a\sin\alpha + b\cos\alpha)i + (c\cos\alpha - d\sin\alpha)j + (c\sin\alpha + d\cos\alpha)k. \end{aligned}$$

In matrix notation we have the following for f

$$\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & -\sin\alpha \\ 0 & 0 & \sin\alpha & \cos\alpha \end{pmatrix},$$

which comprises two 2×2 blocks. The upper one is a rotation counterclockwise by an angle of α in the $1i$ plane and the lower block is a rotation clockwise by an angle of α in the jk plane. Consider the i-Hopf fibration. Since by setting $1 \in \mathbb{H}$ as the north pole of $\mathbb{S}^3$ the set of purely imaginary unit quaternions is the equator of $\mathbb{S}^3$, the north pole of $\mathbb{S}^2$ in the i-Hopf fibration is the point i. Thus geometrically f rotates the equator of $\mathbb{S}^2$(the great circle joining j, k) to the right by an angle α, and at the same time pushes the axis i upward by α.
Similarly the map $g\colon \mathbb{S}^3 \to \mathbb{S}^3$, $x \mapsto x \cdot e^{-i\beta}$ is an orthogonal transformation of $\mathbb{R}^4$ with the following matrix form

$$\begin{pmatrix} \cos\beta & -\sin\beta & 0 & 0 \\ \sin\beta & \cos\beta & 0 & 0 \\ 0 & 0 & \cos\beta & \sin\beta \\ 0 & 0 & -\sin\beta & \cos\beta \end{pmatrix}.$$

Geometrically g turns the equator of $\mathbb{S}^2$, the great circle joining j, k to the right by β, and at the same time pushes the axis i backward by β.

For a $\theta \in [0, \frac{\pi}{2}]$ we define the map

$$F_\theta\colon [0,\pi] \times [0, 2\pi] \to \mathbb{S}^3,\ (x,y) \mapsto e^{ix} \cdot e^{j\theta} \cdot e^{iy}. \tag{2.10}$$

and componentwise we have

$$F_\theta(x,y) = \begin{pmatrix} \cos\theta\cos(x+y) \\ \cos\theta\sin(x+y) \\ \sin\theta\cos(x-y) \\ \sin\theta\sin(x-y) \end{pmatrix}.$$

Then a straightforward calculation leads to

$$F_\theta(x+2\pi, y) = F_\theta(x,y) \text{ and } F_\theta(x, y+\pi) = F_\theta(x+\pi, y). \tag{2.11}$$

Let $C_0\colon \cos t + i\sin t$ be the great circle connecting 1 and i. For a given θ, the great circle $C_\theta\colon y \mapsto e^{j\theta} \cdot e^{iy}$ is also an i-Hopf circle through the point $e^{j\theta}$. Since C_θ is obtained from C_0 by a left translation about $e^{j\theta}$, the great circles C_0 and C_θ are Clifford parallel with the distance $d_{\mathbb{S}^3}(C_\theta, C_0) = \theta$. On the other hand the orthogonal transformation e^{ix} is a group action of $\mathbb{S}^1$ on $\mathbb{S}^3$. Thus the set $\partial T_\theta := F_\theta([0,\pi] \times [0, 2\pi])$ is topological a torus $\mathbb{S}^1 \times \mathbb{S}^1$. For each point $p \in \partial T_\theta$ the distance is equal a constant $d(p, C_0) = \inf\{d_{\mathbb{S}^3}(p,x) \colon x \in C_0\} = \theta$. Then the set ∂T_θ is called the *distance torus* (with distance θ to C_0, see Figure 2.1) of the i-Hopf fibration. The *Clifford torus* is the torus ∂T_θ of distance $\theta = \frac{\pi}{4}$. The *solid Clifford torus* ($T_{\frac{\pi}{4}} := \bigcup_{\theta < \frac{\pi}{4}} \partial T_\theta$) and its covering space serve as important barriers by showing existence result for the Plateau problem in later parts of our work. The tangent vectors of $F_{\frac{\pi}{4}}$ are

$$\frac{\partial}{\partial x} F_\theta = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} F_\theta; \qquad \frac{\partial}{\partial y} F_\theta = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} F_\theta = i \cdot F_\theta.$$

After a direct calculation we then have

$$\langle \frac{\partial}{\partial x} F_{\frac{\pi}{4}}, \frac{\partial}{\partial y} F_{\frac{\pi}{4}} \rangle_{\mathbb{R}^4} = 0.$$

Consequently the Clifford torus has orthogonal asymptotic lines[4] and thus a minimal surface in $\mathbb{S}^3$. By the definition 1 a vector field V on $\mathbb{S}^3$ right rotates along the great circle $p \cos t + q \sin t$, $p \perp q \in \mathbb{S}^3$ if $q, V(p), V(q), p$ is positively oriented. It can be verified that $\frac{\partial}{\partial y} F_\theta$ is a right rotating i-Hopf vector field and $\frac{\partial}{\partial x} F_\theta$ is a left rotating vector field. We describe the left rotating field $\frac{\partial}{\partial x} F_{\frac{\pi}{4}}$ with the right rotating vector fields. Along an i-line the field $\frac{\partial}{\partial x} F_\theta$ rotates left once with respect to parallel fields, but the reference vector field $\frac{\partial}{\partial y} F_\theta$ is a right rotating field. So along an i-Hopf circle the vector field $\frac{\partial}{\partial x} F_\theta$ rotates twice in the jk-plane. On the Clifford torus we have

$$\frac{\partial}{\partial x} F_{\frac{\pi}{4}} = (j \cos 2y - k \sin 2y) \cdot F_{\frac{\pi}{4}}.$$

The inner normal of the solid Clifford torus is

$$n(x,y) = (k \cos 2y + j \sin 2y) \cdot F_{\frac{\pi}{4}}.$$

[4]A regular curve $c = f \circ \gamma$ is an *asymptotic line*, if the normal curvature $g(S\gamma', \gamma')$ vanishes for all t.

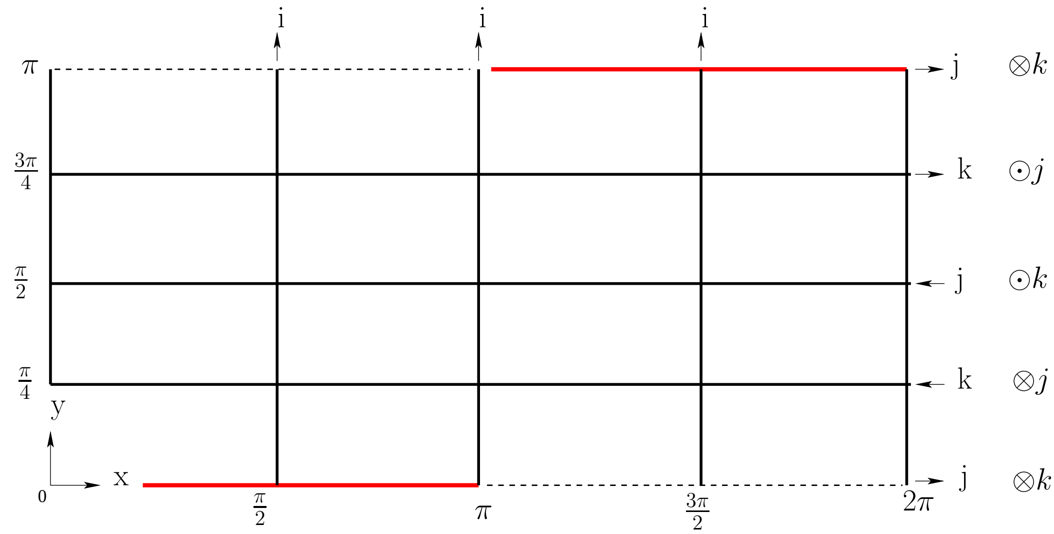


Figure 2.2: In the i-Hopf fibration the Hopf field $\frac{\partial}{\partial x}F_{\frac{\pi}{4}}$ on the Clifford Torus are represented in the basis $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ of right rotating vector fields. All the **i**-lines are Clifford parallel. By Equation 2.11 we identify the red half j-Hopf circular arcs with each other and the two dotted half j-Hopf circular arcs with each other. On the right we indicate the normal of the Clifford Torus.

3 Minimal Surfaces and Their CMC-1 Cousins

In $\mathbb{R}^3$ to each simply connected minimal surface $f\colon U \to \mathbb{R}^3$ there is a conjugate minimal surface f^* as well. This was generalized by Lawson in [Law70] to surfaces of constant mean curvature in space forms, the so-called Lawson correspondence. Using this correspondence he constructed doubly periodic CMC-1 surfaces in $\mathbb{R}^3$ from some minimal surfaces in $\mathbb{S}^3$ by solving Plateau problems for geodesic polygons in $\mathbb{S}^3$ and then using Schwarz reflections to get regular complete minimal surfaces. Following this schema we will construct entire one-parameter families of new CMC-1 surfaces in $\mathbb{R}^3$, which bifurcate from the nodoids family.

3.1 Lawson Correspondence

Let $f\colon \Sigma \to M^3(K)$ be an immersion of class C^2, where Σ is a simply connected, oriented Riemannian surface and $M^3(K)$ is a three-dimensional *space form,* i.e. an simply connected, oriented three-dimensional Riemannian manifold of constant sectional curvature $K \in \mathbb{R}$. Denote by g the first fundamental form, h the second fundamental form, and S the shape operator of f.

The Gauss equation and the Codazzi equation are the integrability conditions of hypersurfaces, which guarantee the existence of an immersed surface. More precisely, given a metric g on a simply connected Riemannian surface Σ and a $(1,1)$-tensor field S satisfying

$$k = \det S + K \qquad (\textit{Gauss equation}) \tag{3.1}$$

$$S([X,Y]) = \nabla_X S(Y) - \nabla_Y S(X) \qquad (\textit{Codazzi equation}) \tag{3.2}$$

there exists an isometric immersion $f\colon \Sigma \to M^3(K)$ with the first fundamental form g and shape operator S. Here κ is the Gauss curvature of Σ and X, Y are smooth tangent vector fields of Σ.

Suppose now that $f\colon \Sigma \to M^3(K)$ is a minimal surface, i.e. the mean curvature of f is $H = \operatorname{tr}(S) = 0$, in $M^3(K)$ with $K = 1$ or 0. We define for $c \in \mathbb{R}$ a new shape operator

$$S^* := J \circ S + c \cdot \mathrm{id}.$$

Here J is the $\frac{\pi}{2}$- rotation in the tangent space $T_p\Sigma$, that is, with respect to an oriented orthonormal basis of $T_p\Sigma$ it has the matrix form

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then with respect to such a basis, the shape operator S^* has the following matrix form

$$S^*_+ = J \circ S + c \cdot \mathrm{id} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \circ \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} + \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} = \begin{pmatrix} -s_{12} + c & -s_{22} \\ s_{11} & s_{12} + c \end{pmatrix}$$

Thus we have $H^* := \frac{\mathrm{tr}(S^*)}{2} = c$ and $\det(S^*) = \det(S) + c^2$. This together with (3.1) gives

$$k = \det(S^*) + K - c^2.$$

Define $g^* := g$. Then the pair (g^*, S^*) satisfies the Gauss equation in $M^3(K - c^2)$. Since the $\frac{\pi}{2}$- rotation J maps parallel vector fields to parallel vector fields, it commutes with the Levi-Civita connection ∇

$$J\nabla_X Y = \nabla_X J(Y)$$

the pair (g^*, S^*) also satisfies the Codazzi equation

$$\begin{aligned} \nabla_X S^*(Y) - \nabla_Y S^*(X) &= \nabla_X J \circ S(Y) - \nabla_Y J \circ S(X) + c(\nabla_X Y - \nabla_Y X) \\ &= J \circ S([X, Y]) + c[X, Y] \\ &= S^*([X, Y]) \end{aligned}$$

Hence there exists an isometric immersion in $M^3(K - c^2)$ with the first fundamental form $g^* = g$ and the shape operator S^*, which has constant mean curvature $H^* = c$.

Theorem 4. *[[Law70],p. 364] Assume $M^3(K)$ is an oriented 3-dimensional Riemannian manifold with sectional curvature K. Let $f\colon \Sigma \to M^3(K)$ be a minimal immersion of a simply connected oriented Riemannian surface into $M^3(K)$ with the induced metric g and the second fundamental tensor S. Then there exists an associated CMC surface $f^*\colon \Sigma \to M^3(K - c^2)$ with mean curvature c in the space $M^3(K - c^2)$. The CMC surface has the first fundamental form $g^* = g$ and the second fundamental tensor $S^* = S + c \cdot id$.*

The relation of a minimal immersion f in $M(K)$ and its associated CMC surface f^* in $M^3(K - c^2)$ in Thm.4 is called *Lawson correspondence* and the surfaces f, f^* are called *cousin surfaces.*

Remark 5. Set $K = 1$ and $c = 1$ we have the correspondence between minimal surfaces in $\mathbb{S}^3$ and CMC-1 surfaces in $\mathbb{R}^3$. Similarly by setting $K = 0$ and $c = 1$ we have the correspondence between minimal surfaces in $\mathbb{R}^3$ and CMC-1 surfaces in $\mathbb{H}^3$, the 3-dimensional hyperbolic space.

Remark 6. Examples: 1. The unit 2-sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ is a CMC-1 surface with $S^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Its minimal cousin in $\mathbb{S}^3$ is the great sphere in $\mathbb{S}^3$ with $S = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

2. The standard cylinder in $\mathbb{R}^3$ is a CMC-1 surface with the second fundamental tensor $S^* = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$. Its minimal cousin in $\mathbb{S}^3$ is the Clifford torus with $S = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

3. The plane $\mathbb{R}^2$ is a minimal surface in $\mathbb{R}^3$ with $S = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Its CMC-1 cousin in $\mathbb{H}^3$ is the horosphere with $S^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

It should be noted that in the case $\mathbb{S} - \mathbb{R}^3$ there is also a nice first order description of the cousin relation. It says that the differentials of a minimal surface f in $\mathbb{S}^3$ and its CMC-1 cousin f^* satisfy

$$df = f \cdot df^* \circ J. \tag{3.3}$$

and

$$\nu = f \cdot \nu^*$$

where ν,ν^* are the Grauss maps of f and f^* respectively. The dot product means the multiplication of quaternions[1]. If a CMC-1 immersion f^* in $\mathbb{R}^3$ is given, then Equation (3.3) has a solution f, which is a minimal immersion in $\mathbb{S}^3$ (see [GB05] p. 751 for more details).

3.2 Symmetries under Lawson's Correspondence

Definition 7. Let $\gamma \subset \Omega$ be a curve parameterized by arc length in the simply connected domain Ω and $f\colon \Omega \to \mathbb{R}^3$ a CMC-1 immersion. A vector $df(b) \in T_{f\circ\gamma(t)}f(\Omega)$ is called *conormal* if $b = J\gamma'(t)$.

Notation 8. Since $g = g^*$ we can assume that $c := f \circ \gamma$, $c^* = f^* \circ \gamma$ are geodesics of the surfaces f and f^* in the spaces $\mathbb{S}^3$ with (g, S) and $\mathbb{R}^3$ with (g^*, S^*) respectively. We denote the covariant derivative $\nabla_{c'}$ along a geodesic by $\frac{d}{dt}$ in $\mathbb{R}^3$ and by $\frac{D}{dt}$ in $\mathbb{S}^3$. Let us

define the *(normal) curvature* of a regular curve c with normal vector ν in a Riemannian manifold (g, S) by

$$\kappa := g(\nabla_{c'} c', \nu)$$

and the *torsion* by

$$\tau := g(\nabla_{c'} \nu, J \cdot c').$$

[1] Here we use the identification $\mathbb{R}^3 = \mathrm{Im}\mathbb{H}$ and consider $\mathbb{S}^3$ as the set of unit quaternions

Then using the identity $S := -(df)^{-1} \circ dv$ we have

$$\begin{aligned} \tau^* &= g^*(\frac{d}{dt}\nu^*, J \cdot (c^*)') \\ &= -g^*(S^* \cdot \gamma', J \cdot \gamma') \\ &= -g^*((J \circ S + \mathrm{id}) \cdot \gamma', J \cdot \gamma') \\ &= -g(S \cdot \gamma', \gamma') \\ &= g(\frac{D}{dt}\nu, \gamma') \\ &= -g(v, \frac{D}{dt}\gamma') \\ &= -\kappa \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \kappa^* &= g^*(\frac{d}{dt}(c^*)', \nu^*) \\ &= g^*(S^* \cdot \gamma', \gamma') \\ &= g^*((J \circ S + \mathrm{id})(\gamma)', (\gamma)') \\ &= g(J \circ S(\gamma'), \gamma') + 1 \\ &= -g(S \cdot \gamma', J \cdot \gamma') + 1 \\ &= g(\frac{D}{dt}\nu, J \cdot \gamma') + 1 \\ &= \tau + 1 \end{aligned} \tag{3.5}$$

In the last step of the derivation in Equation (3.4) we used

$$0 = g(\nu, \gamma') = g(\frac{D}{dt}\nu, \gamma') + g(\nu, \frac{D}{dt}\gamma').$$

Hence the curve c^* has constant conormal iff the torsion of c^* vanishes.

Lemma 9. *The geodesic c on the minimal immersion f is a geodesic in the space form $M(K)$ iff the conjugate geodesic curve c^* on the surface f^* has a constant conormal in $M(K-1)$.*

Proof. It follows directly from Equation (3.4). □

Remark 10. We call a curve on a surface *planar*, if its torsion is 0 and normal of the surface is the principle curvature normal of the curve. Note that curves are principal curvature lines iff $\tau = 0$. Then it follows that, the geodesic c on a minimal immersion f is a straight line in $\mathbb{R}^3$ iff the conjugate geodesic curve c^* on the CMC-1 surface f^* lies in a 2-dimensional totally geodesic subspace in $\mathbb{H}^3$.

Remark 11. In the case $K = 1$ we have the first order description (3.3) and the following equality holds

$$df(\gamma'(t)) = f(\gamma(t))(df^* \circ J)\gamma'(t) \tag{3.6}$$

If $f \circ \gamma$ is a geodesic arc in $\mathbb{S}^3$, then $f(\gamma(t)) = p(\cos t + u \sin t)$ for some $p \in \mathbb{S}^3$ and a $u \in \mathrm{Im}\mathbb{H} \cap \mathbb{S}^3$. Using $-u^2 = |u|^2 = 1$ we have

$$(f \circ \gamma(t))' = df(\gamma'(t)) = f(\gamma(t))u.$$

We conclude that the geodesic $f^* \circ \gamma(t)$ has a constant conormal u.

Proposition 12. *A curve $f^* \circ \gamma(t)$ is a planar curve of reflection with conormal $u \in \mathbb{S}^2$ iff $f \circ \gamma(t)$ is a u-Hopf circle.*

It can be shown that if a CMC-1 surface in $\mathbb{R}^3$ is a surface of revolution, its minimal cousin surface in $\mathbb{S}^3$ is a ruled surface (ruled by great circles).

The curvature of a geodesic in a Riemannian manifold with constant curvature is the tilting speed of the normal and the torsion of the geodesic is the rotation velocity of the normal. Then we can define the *tilting angle* t^* of the normal ν^*. If the geodesic arc c^* of length $|c^*| = l$ has constant conormal, then the tilting angle t^* of the normal ν^* is given by

$$t^* = \int_0^l g^*(\frac{D}{dt}\nu^*, (c^*)')ds$$

and the rotation angle of normal ν by

$$r = -\int_0^l g(\frac{D}{dt}\nu, J \cdot \gamma')ds.$$

Then we have

$$\begin{aligned} t^* &= \int_0^l g^*(\frac{D}{dt}\nu^*, (c^*)')ds \\ &= -\int_{c^*} \kappa^* ds \\ &= -\int_c (\tau + 1)ds \\ &= -\int_0^l g(\frac{D}{dt}\nu, J \cdot \gamma')ds - l \\ &= r - l \end{aligned} \tag{3.7}$$

4 A Maximum Principle for Minimal Surfaces in $\mathbb{S}^3$

The maximum principle for minimal surfaces is also valid in Riemannian manifolds, but its proof is not easy to locate in the literature. For this reason we will provide a proof here, where we specialize to the case that the ambient space is the 3-sphere. In the first section we derive a mean curvature equation for graphs in $\mathbb{S}^3$ using the stereographic projection ([Rei70] §1). Through the mean curvature equation we have a second order quasilinear elliptic operator. In the second section we prove the maximum principle for this quasilinear elliptic operator. Applying this maximum principle we can exclude one-sided interior contacts between two minimal surfaces unless they coincide.

4.1 Mean Curvature Equation for Graphs in $\mathbb{S}^3$

Since the auxiliary calculations in this subsection are quite long and technical, we put these calculations in the appendix to increase readability.

Set $S = \mathbb{S}^m \setminus \{N\}$ with N as the north pole of $\mathbb{S}^m$. The stereographic projection is defined by

$$\pi\colon S \to \mathbb{R}^m \times \{0\},\ x \mapsto \pi(x) = N + h(x)(x - N) \tag{4.1}$$

where

$$h(x) := \frac{1}{1 - \langle x, N\rangle}. \tag{4.2}$$

Then we have

$$\begin{aligned} \frac{2}{h(x)} &= 2 - 2\langle x, N\rangle \\ &= \langle N, N\rangle + \langle x, x\rangle - \langle x, N\rangle - \langle x, N\rangle \\ &= \langle x - N, x - N\rangle \\ &= |x - N|^2 \end{aligned} \tag{4.3}$$

An analogous calculation gives us

$$h(x) = \frac{1}{2}(1 + |\pi(x)|^2) \tag{4.4}$$

Let $e_1, \cdots, e_m$ be an local orthonormal frame on $\mathbb{S}^m$ with dual frame $\omega_1, \cdots, \omega_m$. Set $v_a = \langle N, e_a\rangle_{\mathbb{R}^{m+1}}$ we write the differential of x and h as 1-forms

$$dx = \sum_a \omega_a e_a, \quad \text{and} \quad dh = \sum_a h^2 v_a \omega_a, \tag{4.5}$$

With (4.1) and (4.5) we have

$$d\pi = dh(x - N) + hdx = \sum_a h^2 v_a \omega_a (x - N) + \sum_b h\omega_b e_b \tag{4.6}$$

Finally we get (for the derivation see appendix)

$$\langle d\pi, d\pi \rangle_{\mathbb{R}^{m+1}} = \sum_a h^2 \omega_a^2 \tag{4.7}$$

Equation (4.7) shows that the stereographic projection π is a conformal map with conformality factor h. Through the map π we get the pullback fields

$$\theta_a = h\omega_a. \tag{4.8}$$

Then (θ_a) constitute orthonormal coframe fields for the Riemannian flat metric on S induced by the immersion π. Let ω_{ab} be the connection forms on S corresponding to the frame field ω_a defined by the Cartan structure equations (dC76 Ch.5 Prop.1).

$$d\omega_a = \sum_b \omega_{ab} \wedge \omega_b, \quad \omega_{ab} = -\omega_{ba}; \tag{4.9}$$

$$d\omega_{ab} = \sum_c \omega_{ac} \wedge \omega_{cb}. \tag{4.10}$$

The differential of the Equation (4.8) together with the Cartan structure equations gives

$$d\theta_a = dh \wedge \omega_a + hd\omega_a = \sum_b (\omega_{ab} - v_b\theta_a) \wedge \theta_b. \tag{4.11}$$

The induced connection form is

$$\theta_{ab} = \omega_{ab} - v_b\theta_a + v_a\theta_b. \tag{4.12}$$

Assume that M is an $n-$dimensional differentiable manifold with $2 \leq n < m$ and $X : M \to S$ is an immersion. Then the composition $Y = \pi \circ X \colon M \to \mathbb{R}^m \times \{0\}$ maps the submanifold $X(M) \subset S$ through the stereographic projection π to $\mathbb{R}^m \times \{0\}$ and Y is again an immersion. We assume that $M \subset S$ and X is the inclusion map. Let the fields $e_1, ..., e_n$ be tangent to M and $e_{n+1}, ..., e_m$ be normal to M. The differentials $f_a = \frac{1}{h} d\pi(e_a)$, $a \in \{1, \ldots, m\}$, constitute an orthonormal frame in $\mathbb{R}^m \times \{0\}$ which is adapted along Y.

We shall make use of the following convention on the ranges of indices:

$$1 \leq i, j, k \leq n; \qquad n + 1 \leq r, s \leq m.$$

If we restrict the adapted forms $\omega_a, \theta_a, \omega_{ab}, \theta_{ab}$ to the tangent spaces of M the normal frame fields vanish

$$\omega_r = \theta_r = 0.$$

Hence by the structure equation (4.9) we get

$$\begin{aligned} 0 &= d\omega_r = \sum_a \omega_{ra} \wedge \omega_a = \sum_j \omega_{rj} \wedge \omega_j = \sum_j \omega_j \wedge \omega_{jr} \\ 0 &= d\theta_r = \sum_a \theta_{ra} \wedge \theta_a = \sum_j \theta_{rj} \wedge \theta_j = \sum_j \theta_j \wedge \theta_{jr} \end{aligned}$$

We introduce the Cartan Lemma ([dC76] (p. 80))

Lemma 13. *(Cartan's lemma) Let V^n be a vector space of dimension n, and let $\omega_1, \dots, \omega_n \colon V^n \to \mathbb{R}$, $r \leq n$, be linear forms in V that are linearly independent. Assume that there exist forms $\theta_1, \dots, \theta_r \colon V \to \mathbb{R}$ such that $\sum_{i=1}^r \omega_i \wedge \theta_i = 0$. Then*

$$\theta_i = \sum_j a_{ij}\omega_j, \quad \textit{with } a_{ij} = a_{ji}.$$

By Cartan's lemma we can write

$$\omega_{jr} = \sum_i b_{rij}\omega_j, \tag{4.13}$$

$$\theta_{jr} = \sum_i \beta_{rij}\theta_i \tag{4.14}$$

where the symmetric matrices b_{rij} and β_{rij} are second fundamental forms in direction e_r of (M, X) and (M, Y) respectively.

By Equation (4.12) it follows that (see appendix for the details of derivation)

$$\beta_{rij} = \frac{1}{h} b_{rij} - v_r\delta_{ij}. \tag{4.15}$$

A direct calculus using Equation (4.1), (4.3), (4.6) and (4.4) we verify the equality (for details see the appendix)

$$v_r = \frac{1}{h}\langle f_r, Y\rangle. \tag{4.16}$$

The mean curvatures of the immersions (M, X) and (M, Y) in direction e_r and f_r respectively are given by

$$\overline{H_r} = \overline{H}(e_r) = \frac{1}{n}\sum_i b_{rii}$$

and

$$H_r = H(f_r) = \frac{1}{n} \sum_i \beta_{rii}.$$

By (4.15) we have the following relation between the two mean curvatures

$$\overline{H}_r = hH_r + \langle Y, f_r \rangle. \tag{4.17}$$

Locally a regular surface in the Euclidean space can be written as a graph with respect to its tangent plane. We cite the following lemma from [GB]

Lemma 14. *Let $(f, \nu): U \to \mathbb{R}^n$ be a hypersurface and $p \in U$. Then there is a domain $V \subset T_p f$ with $0 \in V$, a change variables $\varphi: V \to \varphi(V) \subset U$ with $\varphi(0) = p$, and a smooth function $u: V \to \mathbb{R}$ (the height function) such that $\tilde{f} := f \circ \varphi$ satisfies*

$$\tilde{f}(x) = f(p) + x + u(x)\nu(p) \qquad \textit{for all } x \in V \subset T_p f.$$

Now consider (M, X) to be a simply connected regular hypersurface in $\mathbb{S}^3$ with mean curvature $\overline{H}$. Its image (M, Y) in $\mathbb{R}^3$ under the stereographic projection is again a regular hypersurface with mean curvature H. By lemma (14) the surface Y has a local representation as a graph $(x_1, x_2, u(x_1, x_2))$ over its tangent plane.

We use the notation

$$\begin{aligned} \nabla u &:= g^{ij} \partial_j u \frac{\partial}{\partial x_i}, \\ W &:= 1 + |\nabla u|^2, \\ \nu &:= \frac{(-\nabla u, 1)}{1 + |\nabla u|^2}. \end{aligned}$$

The mean curvature of a graph in $\mathbb{R}^3$ is given by

$$H = \frac{1}{2} \mathrm{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right).$$

By (4.17) we obtain an expression in divergence form for the mean curvature of the surface (M, X),

$$\overline{H} = h \cdot H + \langle Y, \nu \rangle = \frac{1}{2} \mathrm{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) + \langle Y, \nu \rangle.$$

We can write

$$\overline{H} = a^{ij}(x, u, \nabla u) \partial_{ij} u + b(x, u, \nabla u) \tag{4.18}$$

where

$$\begin{aligned}
a^{11} &= \frac{h}{2\left(\sqrt{1+u_1^2+u_2^2}\right)^3}(1+u_2^2)\\
a^{12} &= -\frac{h}{2\left(\sqrt{1+u_1^2+u_2^2}\right)^3}(u_1u_2)\\
a^{21} &= a^{12}\\
a^{22} &= \frac{h}{2\left(\sqrt{1+u_1^2+u_2^2}\right)^3}(1+u_1^2)
\end{aligned}$$

and

$$b(x,u,\nabla u) = \langle Y,\nu\rangle.$$

Equation (4.18) gives us a second order operator $f \to \mathcal{H}[f]$, which is quasilinear.

4.2 A Maximum Principle for Quasilinar Elliptic Operators

For a domain $\Omega \subset \mathbb{R}^n$ and $U \subset \Omega\times\mathbb{R}\times\mathbb{R}^n$, we call a second order operator Q *quasi-linear* if by applying it to a function $u \in C^2(\Omega)\cap C^0(\overline{\Omega})$, it gives

$$Q(u) = \sum_{i,j=1}^{n} a^{ij}(x,u,Du)\partial_{ij}u + b(x,u,Du)$$

where $a^{ij}, b \in C^{\mathbf{1}}(U)$, $a^{ij} = a^{ji}$ and $\partial_{ij} = \frac{\partial^2}{\partial u^i \partial u^j}$.
For sake of simplicity let us write $a^{ij}(x,z,p)$ instead of $a^{ij}(x,u,Du)$ and $b(x,z,p)$ instead of $b(x,u,Du)$. We say the operator Q is *elliptic* if for each $(x,z,p) \in U$ and for all $\xi \in \mathbb{R}^n$, $\xi \neq 0$ the following holds

$$\sum_{i,j} a^{ij}(x,z,p)\xi^i\xi^j > 0.$$

That is, if $\lambda(x), \Lambda(x)$ denote respectively the minimum and maximum eigenvalues of (a^{ij}) then

$$0 < \lambda(x,z,p)|\xi|^{\mathbf{2}} \leq a^{ij}(x,z,p)\xi_i\xi_j \leq \Lambda(x,z,p)|\xi|^2.$$

If $\frac{\lambda}{\Lambda}$ is bounded in Ω, we shall say that Q is *uniformly elliptic*.
Then we can state a strong maximum principle[1] for second order quasi-linear elliptic operators.

Lemma 15. *Let $Q = Q(u)$ be a second order quasi-linear elliptic operator. Suppose that the functions $u, v \in C^2(\Omega)\cap C^0(\overline{\Omega})$ satisfy*

[1] there is also a comparison principle for quasilinear elliptic operators which need additional requirements on the coefficients a^{ij} and b (see theorem 10.2 in [GT01]) .

(1) $u \leq v$ *on* Ω *and* $u(x_0) = v(x_0)$ *for a* $x_0 \in \Omega$*, and*
(2) $Q(v) \leq Q(u)$ *on* Ω*.*
Then $u \equiv v$ *on* Ω*.*

Proof. By convention we write (x, z, p, r) instead of (x, u, Du, D^2u) and we set

$$\begin{aligned} w &= u - v \\ u_t &= tu - (1-t)v \qquad \text{for } t \in [0,1] \end{aligned}$$

By condition 2) we have

$$\begin{aligned} 0 &\leq Q(w) \\ &= Q(u) - Q(v) \\ &= Q(u_t)\Big|_0^1 \\ &= \int_0^1 \frac{d}{dt} Q(u_t) dt. \end{aligned} \tag{4.19}$$

The total derivative is

$$\begin{aligned} \frac{d}{dt} Q(u_t) &= \frac{d}{dt} Q(x, u_t, Du_t, D^2 u_t) \\ &= \sum_{i,j} \frac{\partial Q}{\partial r_{ij}} \underbrace{(\partial_{ij} u - \partial_{ij} v)}_{\partial_{ij} w} + \sum_i \frac{\partial Q}{\partial p_i} \underbrace{(\partial_i u - \partial_i v)}_{\partial_i w} + \frac{\partial Q}{\partial z} \underbrace{(u - v)}_{w}. \end{aligned}$$

Therefore

$$\begin{aligned} \int_0^1 \frac{d}{dt} Q(u_t) dt &= \sum_{i,j} (\int_0^1 \frac{\partial}{\partial r_{ij}} Q(x, u_t, Du_t, D^2 u_t) dt) \partial_{ij} w \\ &\quad + \sum_i (\int_0^1 \frac{\partial}{\partial p_i} Q(x, u_t, Du_t, D^2 u_t) dt) \partial_i w \\ &\quad + (\int_0^1 \frac{\partial}{\partial z} Q(x, u_t, Du_t, D^2 u_t) dt) w. \end{aligned}$$

Now we set

$$\begin{aligned} a^{ij}(x) &= \int_0^1 \frac{\partial}{\partial r_{ij}} Q(x, u_t, Du_t, D^2 u_t) dt \\ b^i(x) &= \int_0^1 \frac{\partial}{\partial p_i} Q(x, u_t, Du_t, D^2 u_t) dt \\ c(x) &= \int_0^1 \frac{\partial}{\partial z} Q(x, u_t, Du_t, D^2 u_t) dt. \end{aligned}$$

This yields a linear uniformly elliptic equation:

$$Lw := \sum_{i,j} a^{ij}(x)\partial_{ij}w(x) + \sum_i b^i(x)\partial_i w(x) + c(x)w(x)$$

By (4.19) we know that

$$Lw \geq 0 \qquad \text{in } \Omega.$$

The function w achieves its maximum 0 at x_0, which is in the interior of Ω. Then the set Ω^- on which $w < 0$ satisfies $\Omega^- \subset \Omega$ and $\partial\Omega^- \cap \Omega \neq \emptyset$. Let x_1 be a point in Ω^- that is closer to $\partial\Omega^-$ than to $\partial\Omega$. We consider the largest ball $B \subset \partial\Omega^-$ having x_1 as center. Then $u(y) = 0$ for some point $y \in \partial B$ while $w < 0$ in B. Hopf's boundary point lemma[2,3] implies that $Dw(y) \neq 0$, which is impossible at the interior maximum y unless $w \equiv 0$ on Ω. □

Using the lemma above we can show a strong maximum principle for minimal surfaces.

Theorem 16. *Let M_1, M_2 be two minimal immersions into the 3-sphere $\mathbb{S}^3$. Assume that M_1 has a common point p with M_2, and M_1 lies locally to one side of M_2 near this common point. Then M_1 and M_2 coincide in a neighbourhood of p.*

Proof. Let $st(M_1)$ and $st(M_2)$ be the images of M_1 and M_2 in $\mathbb{R}^3$ under the stereographic projection. Since the stereographic projection is a diffeomorphism[4] the surface $st(M_1)$ and $st(M_2)$ are immersions in $\mathbb{R}^3$ and can locally, near the point $st(p)$, be written as graphs over a neighbourhood $U(st(p))$. By applying the stereographic projection backwards of the graph representation of $st(M_1)$ and $st(M_2)$ we get graph representations for M_1 and M_2 in $\mathbb{S}^3$. Let $z_1(u, v)$ and $z_2(u, v)$ be the graph functions over a neighbourhood $U(p)$

$$M_1\colon \begin{pmatrix} u \\ v \\ z_1(u,v) \\ 0 \end{pmatrix}, \quad M_2\colon \begin{pmatrix} u \\ v \\ z_2(u,v) \\ 0 \end{pmatrix}.$$

Since M_1 lies locally to one side of M_2 we can assume that the height function h satisfies: $z_1 \leq 0 \leq z_2$ on U and $z_1(p) = z_2(p) = 0$.
Then both z_1 and z_2 are minimal graphs, which satisfy the quasilinear elliptic equation (4.18). By lemma (15) we get $z_1 = z_2$ on U. □

We will apply the maximum principle to show the uniqueness of Plateau solution.

[2]here we have a point $y \in \text{int}\Omega$ with $u(y) = 0$ and the conclusion of Hopf's boundary point lemma holds independent of the sign of c (see lemma 3.4 [GT01]).

[3]it is not possible to apply the strong maximum principle directly in our situation, which requires in addition the coefficient $c = 0$ or $c \lesseqgtr 0$ with $\frac{c}{\lambda}$ to be bounded, where λ is the minimum eigenvalue of (a_{ij}).

[4]The stereographic projection is however not an isometry, the images of minimal surfaces are no more minimal surfaces in $\mathbb{R}^3$.

5 The Schwarz Reflection Principle in $\mathbb{S}^3$

5.1 The Schwarz Reflection Principle

Let γ be a geodesic in $\mathbb{S}^3$ which lies in the x_1x_2 plane, and let S be the geodesic 2-sphere in $\mathbb{S}^3$ given by $x_4 = 0$. By geodesic reflection across γ we mean the map $\sigma_\gamma\colon \mathbb{S}^3 \to \mathbb{S}^3$ given by

$$\sigma_\gamma(x_1, x_2, x_3, x_4) = (x_1, x_2, -x_3, -x_4).$$

By geodesic reflection across S we mean the map $\sigma_S\colon \mathbb{S}^3 \to \mathbb{S}^3$ given by

$$\sigma_S(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, -x_4).$$

We denote the open disk with $D := \{z \in \mathbb{C}\colon |z| \leq 1\}$, the half disk with $D^+ = D \cap \{\mathrm{Im} z \geq 0\}$ and their common boundary with $I = \{\mathrm{Im} z = 0, |\mathrm{Re} z| \leq 1\}$. We now derive a Schwarz reflection principle for minimal surfaces in $\mathbb{S}^3$ [Law70, p. 339].

Lemma 17. *Let M be a minimal surface which is of class C^2 at the boundary ∂M. Then:*

[1] If ∂M contains a geodesic arc γ, the surface M can be continued as an analytic minimal surface across each non-trivial component of $\partial M \cap \gamma$ by geodesic reflection.

[2] If part of ∂M lies in a geodesic 2-sphere S and if S meets M orthogonally there, then M can be extended to an analytic minimal surface across each non-trivial component of $\partial M \cap S$ by geodesic reflection.

Proof. For part [1] let $\gamma \subset \mathbb{S}^3$ be given by $x_3 = x_4 = 0$ and a point p in the interior of $\partial M \cap \gamma$. Then by ([Mor66], p. 366) there is a conformal map $f\colon D^+ \to \mathbb{S}^3$ such that in a neighborhood of p the following are satisfied

$$f(0,0) = p,\ f_3(u,0) = f_4(u,0) = 0 \text{ for all } (u,0) \in D^+.$$

Since f is conformal and represents a minimal surface, it is a harmonic map from D^+ to $\mathbb{S}^3$ satisfying:

$$\Delta f = -\langle \nabla f, \nabla f \rangle f, \tag{5.1}$$

where

$$\langle \nabla f, \nabla f \rangle = \sum_{i=1}^{4} \left((\frac{\partial f_i}{\partial u})^2 + (\frac{\partial f_i}{\partial v})^2 \right)$$

and

$$\Delta f = \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}.$$

We extend f to the entire unit disk $D = \{(u, v) \in \mathbb{R}^2 \colon u^2 + v^2 \leq 1\}$ by setting

$$f\colon D^- \to \mathbb{S}^3, f(u,v) = \begin{pmatrix} f_1(u,-v) \\ f_2(u,-v) \\ -f_3(u,-v) \\ -f_4(u,-v) \end{pmatrix} \tag{5.2}$$

From Equation (5.1) one can immediately infer that $f_3 \in C^2(D)$ and $f_4 \in C^2(D)$. It is also easy to see that $f_{1,u}$,$f_{2,u}$,$f_{1,uu}$,$f_{1,vv}$ are continuous functions on D. It remains to show that $f_{1,v}$ and $f_{2,v}$ vanish at the boundary $v = 0$. From $\langle f, f\rangle = 1$ it follows that on $y = 0$ we have the following identities:

$$\langle f, f_u\rangle = f_1 f_{1,u} + f_2 f_{2,u} = 0 \tag{5.3}$$

$$\langle f, f_v\rangle = f_1 f_{1,v} + f_2 f_{2,v} = 0 \tag{5.4}$$

On the other hand since branch points can occur at most at isolated points on the arc $y = 0$, we have $f_{1,u}^2 + f_{2,u}^2 = |f_u|^2 > 0$ for almost all x when $y = 0$. This is true because otherwise we would have $f_{1,u} = f_{2,u} = 0$ on $y = 0$ which means that the boundary arc degenerates to a point, but we have chosen $f(I) \cap \gamma$ to be non-trivial. By Equations (5.3), (5.4) we have $f_u \perp f$ and $f_v \perp f$ on the x_1x_2 plane, and $\langle f_u, f_v\rangle = 0$ shows $f_u \neq \pm f_v$ in the x_1x_2 plane. Hence we have $f_{1,v} = f_{2,v} = 0$ and $f_{1,uv} = f_{2,uv} = 0$. Therefore $f \in C^2(D)$ and it follows that Equation (5.1) holds for the entire unit disk. □

Remark 18. Differentiability can be defined of set with boundary. A bounded domain $\Omega \subset \mathbb{R}^n$ and its boundary are of class C^2, if at each point $x_0 \in \partial\Omega$ there is a ball $B = B(x_0)$ and a one-to-one mapping ϕ of B onto $D \subset \mathbb{R}^n$ such that

$$(i)\ \phi(B \cap \Omega) \subset \mathbb{R}^n_+; \quad (ii)\ \phi(B \cap \partial\Omega) \subset \partial\mathbb{R}^n_+ \quad (iii)\ \phi \in C^2(B) \text{ and } \phi^{-1} \in C^2(D).$$

It is important to note that if $\partial\Omega$ is C^2 then a function $f \in C^2(\Omega)$ can be extended to a function in $C^2(\overline{\Omega})$. Thus the strong regularity assumption in Lemma 17 of the boundary is well defined.

5.2 Solvability of Plateau Problems and Boundary Regularity.

In [Mor66, p.389] Morrey considered the case of homogenously regular manifolds without boundary. A Riemannian 3-manifold M is *homogenously regular* if there exist positive constants k, K such that every point of the manifold lies in the image of a chart φ with the unit ball in $\mathbb{R}^3$ as domain such that

$$k||v||^2 \leq g_{ij}(\varphi(x))v_i v_j \leq K||v||^2 \text{ for all } x \text{ in } B_1(0),$$

where v is any tangent vector to x and $g = (g_{ij})$ is the metric on M. This can be shown to be equivalent to M having sectional curvature bounded above and injectivity radius bounded away from zero. Note that any closed manifold is automatically homogenously regular and hence so any manifold which covers a closed manifold. In [Mee82] Meeks and Yau generalized Morrey's result to the case of closed manifolds with boundary, by introducing additional conditions analogous to convexity needed for ∂M in order to obtain existence theorems for least area surfaces lying properly in M.

Definition 19. Let M is a compact manifold with boundary. The boundary ∂M is said to be *piecewise smooth,* if there exists a Riemannian manifold N such that M is isometric to a submanifold of N and after a suitable triangulation of N, ∂M is a two-dimensional subcomplex of N consisting of smooth two-dimensional simplexes $\{S_1, ..., S_n\}$.

Definition 20. A Riemannian manifold M with boundary is said to be *mean convex*, if the following conditions hold:

[(i)] ∂M is piecewise smooth,

[(ii)] Each smooth subsurface S_i of ∂M has non-negative mean curvature with respect to the inner normal,

[(iii)] Each smooth subsurface S_i of ∂M extends to a smooth embedded surface $\bar{S}_i$ in N such that $\bar{S}_i \cap M = S_i$, where N is the manifold of Definition 19.

We call the sets S_i *barriers.*

Theorem 21. *[Mee82] Let M be a compact mean convex Riemannian 3-manifold, and let Γ be a simple closed curve in M which is null-homotopic in M. Then Γ bounds a least area disk in M. If γ is embedded in ∂M, then any such least area disk D is either properly embedded in M or is embedded in ∂M.*

Remark 22. In [Mee82] Meeks and Yau require the boundary only to be mean convex, which doesn't exclude the case when two barriers meet tangentially. However, in their proof they assume that the barriers S_i intersect each other transversally. It is not clear if in the most general case Theorem 21 is still applicable. In our situation this problem does not arise. We will show in the proof of Lemma 29 the barriers in our work intersect each other transversally.

Remark 23. Tomi [Tom78] showed that if a Jordan curve in $\mathbb{R}^3$ lies on the boundary of a mean convex body, it is the boundary of some embedded minimal surface of the type of the disk.

Let $f\colon D \to \mathbb{S}^3$ be a generalized minimal immersion, i.e. a harmonic and almost conformal map of the closed unit disk D into $\mathbb{S}^3$. Almost conformal means conformal at all points where the derivative is nonzero. The map f satisfies $f(\partial D) = \Gamma$. A point $p \in \overline{D}$ with $df(p) = 0$ is called a *branch point.* We use the barriers to exclude branch points at the vertices and at the interior of the edges of the polygon Γ in doing Schwarz reflections.

Consider a point $p \in \Gamma^*$. Then the $180°$ rotations about those edges which contain p generate a group $G_p \subset \mathrm{Isom}(\mathbb{S}^3)$. In the case p is an interior point of an edge we have $G_p \cong \mathbb{Z}_2$. In the case p is a vertex with angle $\frac{\pi}{k}$, $k \in \mathbb{N}$, the group is isomorphic to the dihedral group D_k which is of order $2k$.

Lemma 24. *[GBK10] Let M be a solution to the Plateau problem for a spherical polygonal Jordan curve Γ, which is contained in the boundary of a mean convex domain $\Omega \subset \mathbb{S}^3$. Suppose at any point $p \in \Gamma$, the images of Ω under the group G_p are disjoint. Then the extension of M across Γ by the Schwarz reflection group G_p gives a smoothly immersed surface.*

Proof. The solution M of the Plateau problem is parameterized by a weakly conformal harmonic map from the disk to $\mathbb{S}^3$, continuous up to the boundary.

For any $p \in \Gamma$, Schwarz reflections in G_p extend the parameterization to some neighborhood of p. This local extension is a continuous map from the disk to $\mathbb{S}^3$, carrying the origin to p. It is again a weakly conformal harmonic map except possibly at the origin, in case p is a vertex of Γ. In dimension two, however, such point singularities are removable.

A weakly harmonic conformal harmonic map is a branched immersion. A branch point is a zero of the differential; however, the surface still has a tangent plane, which is attained with multiplicity at least 2. In the interior, the Plateau solution is known to be free of branch points.

Now consider the case of a boundary point $p \in \Gamma$ with angle π/k, $k \geq 2$. We first deal with the case where the tangent cone $C_p\Omega$ is not half space. Then by definition of a mean convex set, there exist two barriers ∂N_1, ∂N_2 with distinct tangent planes at p. Thus they are transverse to one another at p, and in particular one of these barriers is transverse to T_pM. The tangent cone C_pM to M at p is a wedge bounded by the two tangent rays of Γ at p and thus has positive angle α congruent to $\pm\frac{\pi}{k}$ mod 2π. But this wedge C_pM must be contained in the tangent cone of N_p at p, which is a half space, and so α can be at most π. This means $\alpha = \frac{\pi}{k}$ and hence p is not a branch point.

Otherwise, the tangent cone $C_p\Omega$ is a half space. Since the images of the tangent cone $C_p\Omega$ under the group G_p are disjoint, its order is at most 2, meaning that p can only be an interior point of an edge of Γ. To rule out that p is a branch point, we can assume that M is tangent to $\partial\Omega$ at p since otherwise we are done by the claim above. By the Hopf boundary maximum principle the image of the Plateau solution must then be contained in $\partial\Omega \cap \partial N_i$ for some barrier ∂N_i. If p is a (false) boundary branch point, the surface $M \subset \partial N_i$ must cover a disk-type component of $N_i \setminus \Gamma$ with multiplicity at least 2, and hence M does not minimize area. This contradiction completes the proof that the extension of M by Schwarz reflection represents an immersed surface without branch points. $\square$

Remark 25. There is also another approach to achieve regularity of the Plateau solution. Following the result of [Gul73] there are no branch points of the interior of M. The result of Hildebrandt [Hil69] shows that f is analytic at each point of the boundary which is mapped to the interior of an analytic subarc of Γ.

6 Bifurcation Surfaces of Nodoids

We recall from the introduction that the nonembedded Delaunay surfaces are called *nodoids.* The meridian of a nodoid is the trace made by the focus of a hyperbola when it is rolled along a fixed line. In this chapter we use conjugate surface method to construct one-parameter families of surfaces bifurcating from nodoids.

6.1 Nodoids

Let us consider a nodoid with necksize radius r. Since nodoids are surfaces of revolution we take a φ-segment around the rotation axis with $\varphi = \frac{\pi}{m}$, $m \in \mathbb{N} \setminus \{0, 1\}$ of a half period as depicted in Fig.6.1. We denote the simply connected fundamental patch by M, the boundary of M is a quadrilateral $\Gamma^* := \{\gamma_1^*, \gamma_2^*, \gamma_3^*, \gamma_4^*\}$. Then each boundary arc γ_n^* $(n = 1, 2, 3, 4)$ is a curve of planar reflection and each γ_n^* meets the other two boundary arcs at vertices P_n and P_{n+1} orthogonally. Let us put the boundary Γ^* in the standard coordinate system $\mathbf{ijk}$ so that γ_1^* has exterior conormal $\mathbf{i}$, i.e. the reflection plane to γ_1^* has normal $\mathbf{i}$. Successively we can label normals of the other three reflection planes and we have

$$
\begin{aligned}
&\gamma_1^* \text{ is a planar curve of reflection with conormal } \mathbf{i},\\
&\gamma_2^* \text{ is a planar curve of reflection with conormal } \mathbf{j}\cos\varphi + \mathbf{k}\sin\varphi,\\
&\gamma_3^* \text{ is a planar curve of reflection with conormal } \mathbf{i},\\
&\gamma_4^* \text{ is a planar curve of reflection with conormal } -\mathbf{j}.
\end{aligned}
$$

By the symmetry results in Proposition 12 the conjugate boundary Γ in $\mathbb{S}^3$ consists of four great circle arcs $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$. The Hopf fields of these arcs are then given explicitly by the oriented conormal of the corresponding planar curves in $\mathbb{R}^3$:

$$
\begin{aligned}
\gamma_1 &\text{ is a geodesic arc of a } \mathbf{i}\text{ -Hopf circle}\\
\gamma_2 &\text{ is a geodesic arc of a } (\mathbf{j}\cos\varphi + \mathbf{k}\sin\varphi)\text{-Hopf circle,}\\
\gamma_3 &\text{ is a geodesic arc of an } \mathbf{i}\text{- Hopf circle ,}\\
\gamma_4 &\text{ is a geodesic arc of a } (-\mathbf{j})\text{-Hopf circle.}
\end{aligned}
\tag{6.1}
$$

We notice that at each vertex the normal is explicitly determined by the two adjacent edges. Each pair of adjacent edges makes an angle of $\frac{\pi}{2}$. Let us assume that $|\gamma_1| = |\gamma_1^*| = l > 0$. Then by Theorem 2.1 in [GB93] we can choose the length of γ_3^* to be $|\gamma_3| = |\gamma_3^*| = l + \varphi$. Since γ_2^* and γ_4^* are meridians of the nodoid, they have equal length

$|\gamma_2| = |\gamma_4| = x$. To determine the value of x we use the condition that the spherical quadrilateral Γ is closed. We assume that $P_1 = 1 \in \mathbb{S}^3$. Then the vertices of Γ must satisfy the following equations:

$$\begin{aligned}
P_1 &= 1 \in \mathbb{S}^3 \\
P_2 &= P_1 e^{\mathbf{i}l} = \cos l + \mathbf{i} \sin l \\
P_3 &= P_2 e^{(\mathbf{j}\cos\varphi + \mathbf{k}\sin\varphi)x} \\
&= (\cos l + \mathbf{i}\sin l)(\cos x + (\mathbf{j}\cos\varphi + \mathbf{k}\sin\varphi)\sin x) \\
&= (\cos l + \mathbf{i}\sin l)(\cos x + \mathbf{j}\cos\varphi\sin x + \mathbf{k}\sin\varphi\sin x) \\
&= \cos l\cos x + \mathbf{j}\cos l\cos\varphi\sin x + \mathbf{k}\cos l\sin\varphi\sin x + \\
&\quad + i\sin l\cos x + \mathbf{k}\sin l\cos\varphi\sin x - \mathbf{j}\sin l\sin\varphi\sin x \\
&= \cos l\cos x + i\sin l\cos x + \mathbf{j}\sin x\cos(l+\varphi) + \mathbf{k}\sin x\sin(l+\varphi) \qquad (6.2) \\
P_4 &= P_1 e^{\mathbf{j}x} \\
&= \cos x + \mathbf{j}\sin x \\
P_3 &= P_4 e^{-\mathbf{i}(l+\varphi)} \\
&= (\cos x + \mathbf{j}\sin x)(\cos(l+\varphi) - \mathbf{i}\sin(l+\varphi)) \\
&= \cos x\cos(l+\varphi) - \mathbf{i}\sin(l+\varphi)\cos x + \\
&\quad + \mathbf{j}\sin x\cos(l+\varphi) + \mathbf{k}\sin(l+\varphi)\sin x \qquad (6.3)
\end{aligned}$$

By comparing Equation 6.2 and Equation 6.3 we have

$$\begin{aligned}
\cos l\cos x &= \cos x\cos(l+\varphi) \\
\sin l\cos x &= -\sin(l+\varphi)\cos x
\end{aligned}$$

It follows that $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$. We choose the length $|\gamma_2| = |\gamma_4| = \frac{\pi}{2}$. Then the length of each edge of Γ is known:

$$|\gamma_1| = |\gamma_1^*| = l, \quad |\gamma_2| = |\gamma_2^*| = \frac{\pi}{2}, \quad |\gamma_3| = |\gamma_3^*| = \varphi + l, \quad |\gamma_4| = |\gamma_4^*| = \frac{\pi}{2} \qquad (6.4)$$

Since the symmetry arcs γ_1^* and γ_3^* are contained in two parallel planes with non-zero distance, their conjugate arcs γ_1 and γ_3 are contained in two Clifford parallel **i**-Hopf circle with distance $\frac{\pi}{2}$. The normal at each vertex can be read off directly from the adjacent boundary arcs.

We take the solid Clifford tori T described in Sect. 2.4. Then for $l < \frac{\pi}{2}$ and $2l + \varphi = \pi$ we can embed the spherical quadrilateral into ∂T (after appropriate translation and rotation), as illustrated through the top figure in Fig. 6.3. Along γ_2 the Hopf vector field **i** rotates right with double velocity and after $\frac{\pi}{2}$ it arrives γ_3. In later sections Clifford torus serves as an important mean convex barrier by excluding boundary branch points of our Plateau solutions.

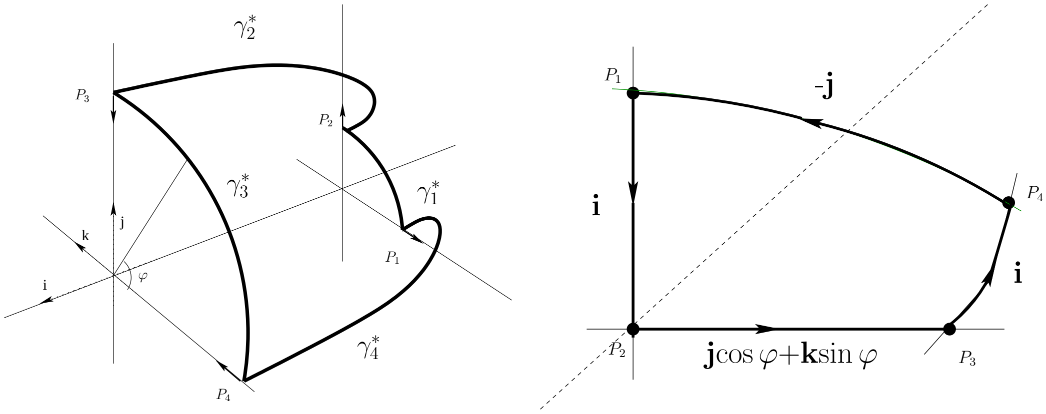


Figure 6.1: Left: The boundary Γ^* of a fundamental patch of a nodoid in $\mathbb{R}^3$. We choose $\varphi = \frac{\pi}{2}$ and the normal at the vertices are directly known from the symmetry. Right: The boundary Γ of the fundamental patch of the conjugate minimal patch in $\mathbb{S}^3$ under stereographic projection. The edges were labeled by their Hopf fields respectively.

6.2 Spherical Quadrilaterals Associated to Bifurcation Surfaces

To generate a fundamental patch of a new CMC surface we deform the boundary polygon Γ of a nodoid in a symmetric way: for any $t \in (0, \frac{\pi}{2})$ we extend γ_4 about t along the (-**j**)-Hopf circle and shorten γ_2 about t along the (**j**cos φ+**k**sin φ)-Hopf circle (as shown in Fig.(6.2) and Fig. (6.3)). We then get a new quadrilateral.

Lemma 26. *Let $t \in [0, \frac{\pi}{2})$. Let Γ be the quadrilateral in $\mathbb{S}^3$ given in the previous section with $\varphi = \frac{\pi}{m}$, where $m \in \mathbb{N}$ and $m \geq 2$. Let $|\gamma_1| = l \in (0, \frac{\pi}{2})$. Then there is a continuous one-parameter family of quadrilaterals $\Gamma_m(t) \subset \mathbb{S}^3$ with the following properties:*

1. $\Gamma_m(0) = \Gamma$.
2. $\Gamma_m(t)$ has the same Hopf-fields as Γ and the lengths of its boundary arcs are
$$|\gamma_1| = l, \quad |\gamma_2| = \frac{\pi}{2} - t, \quad |\gamma_3| = l + \varphi, \quad |\gamma_4| = \frac{\pi}{2} + t. \tag{6.5}$$
3. $l = \frac{m-1}{2m}\pi$.

Proof. For any $t \in (0, \frac{\pi}{2})$ we deform the quadrilateral Γ in a symmetric way: We extend γ_4 about t along the (-**j**)-Hopf circle and shorten γ_2 about t along the (**j**cos φ+**k**sin φ)-Hopf circle. The vertices P_3 and P_4 are moved to $P_3(t)$ and $P_4(t)$ respectively.

We claim that joining the vertices $P_3(t)$ and $P_4(t)$ with a (**i**)-Hopf circle arc of the length $l + \varphi$ yields a new geodesic quadrilateral $\Gamma_m(t)$ with the same Hopf fields as Γ and the lengths of its boundary arcs are given by Equation (6.5).

Since by deforming Γ the arc γ_1 remains unchanged and the arcs γ_2,γ_4 only change the lengths, the crucial point in proving the claim is to show that we can connect the vertices $P_3(t)$ and $P_4(t)$ with a **i**-Hopf circle arc. We shall show this in two steps.

Step1. We show that there is a **i**-Hopf circle through the vertex $P_4(t)$. Like the case of nodoids we can describe the vertices through the following equations

$$\begin{aligned}
P_1 &= 1 \in S^3 \\
P_2 &= P_1 e^{\mathbf{i}l} = \cos l + \mathbf{i}\sin l \\
P_3(t) &= P_2 e^{(\mathbf{j}\cos\varphi + \mathbf{k}\sin\varphi)(\frac{\pi}{2}-t)} \\
&= (\cos l + \mathbf{i}\sin l)\big(\cos(\frac{\pi}{2}-t) + (\mathbf{j}\cos\varphi + \mathbf{k}\sin\varphi)\sin(\frac{\pi}{2}-t)\big) \\
&= (\cos l + \mathbf{i}\sin l)(\sin t + (\mathbf{j}\cos\varphi + \mathbf{k}\sin\varphi)\cos t) \\
&= \cos l\sin t + \mathbf{i}\sin l\sin t + (\mathbf{j}\cos\varphi + \mathbf{k}\sin\varphi)\cos l\cos t + \mathbf{k}\sin l\cos\varphi\cos t - \mathbf{j}\cos t\sin l\sin\varphi \\
&= \cos l\sin t + \mathbf{i}\sin l\sin t + \mathbf{j}\cos t\cos(l+\varphi) + \mathbf{k}\cos t\sin(l+\varphi)
\end{aligned}$$

and tracing from P_1 along the quadrilateral backwards we have

$$\begin{aligned}
P_4(t) &= P_1 e^{j(\frac{\pi}{2}+t)} \\
&= -\sin t + \mathbf{j}\cos t
\end{aligned}$$

Let us denote the **i**-Hopf circle containing γ_1 by C_1. Then by 18.2.2.5 in [Ber96] there is a **i**-Hopf circle through $P_4(t)$, which is Clifford parallel to C_1. Let $s \in [0, 2\pi]$. We can consider a **i**-Hopf circle C through $P_4(t)$ with the following parametrization

$$\begin{aligned}
c(s) &= P_4(t)e^{-is} \\
&= (-\sin t + \mathbf{j}\cos t)(\cos s - \mathbf{i}\sin s) \\
&= -\sin t\cos s + \mathbf{i}\sin t\sin s + \mathbf{j}\cos t\cos s + \mathbf{k}\cos t\sin s
\end{aligned}$$

Step2. We show that the Hopf-circle C intersects the arc γ_2 at exactly $P_3(t)$. Observe that $d(C_1, C) = \frac{\pi}{2} - t$. We denote the great circle containing γ_2 by C_2. The essential question whether the great circle C intersects the great circle C_2 at exactly $P_3(t)$ is equivalent to the question if $P_3(t) \in c(s)$ or if the following system of equations with s as variable and t, l, φ as parameters

$$\begin{aligned}
\cos l\sin t &= -\cos s\sin t \\
\sin l\sin t &= \sin t\sin s \\
\cos t\cos(l+\varphi) &= \cos t\cos s \\
\cos t\sin(l+\varphi) &= \cos t\sin s
\end{aligned} \tag{6.6}$$

has a solution.

If $0 < t < \frac{\pi}{2}$, the system of equations can be simplified to

$$\begin{aligned} \cos l &= -\cos s \\ \sin l &= \sin s \\ \cos(l+\varphi) &= \cos s \\ \sin(l+\varphi) &= \sin s \end{aligned}$$

Since $l \in (0, \frac{\pi}{2})$ the equations above have the solution

$$s = l + \varphi$$

only if

$$2l + \varphi = \pi \tag{6.7}$$

If the parameter m is given and if the arc length of γ_1 and the angle φ satisfy $2l+\varphi = \pi$, then for every $t \in (0, \frac{\pi}{2})$ the system of equations (6.6) has the same solution $s = l + \varphi$. Geometrically, this means that for every $t \in (0, \frac{\pi}{2})$ the great circle C through $P_4(t)$ intersects the arc γ_2 at exactly the vertex $P_3(t)$ with a constant length $l + \varphi$. Thus for every $t \in (0, \frac{\pi}{2})$ only if the parameters l, φ satisfy Equation (6.7) there exists a quadrilateral $\Gamma_m(t)$ with the same Hopf fields as Γ and its boundary arcs have the lengths given by Equation (6.5). Since the vertices $P_3(t)$ and $P_4(t)$ are different for different $t \in (0, \frac{\pi}{2})$ the quadrilateral $\Gamma_m(t)$ to each t is unique. Thus the family of quadrilaterals $\Gamma_m(t)$ is continuous in t.
From Equation (6.7) we derive an important fact:

$$2l + \varphi = \pi \Longrightarrow 2l + \frac{\pi}{m} = \pi \Longrightarrow l = \frac{m-1}{2m}\pi$$

□

Lemma 26 reveals a simple but important fact for the deformation of quadrilaterals in $\mathbb{S}^3$: Given $m \in \mathbb{N}$ with $m \geq 2$, the length of γ_1 and γ_3 are already determined by m and hence the related nodoid. In other words, for each $m \in \mathbb{N}$ with $m \geq 2$ there is exactly one nodoid, namely the one with necksize $(m-1)\pi$, from which we can use the conjugate surface method to construct a family of bifurcating CMC surfaces. We will discuss this more thoroughly later in Theorem 31.

In Lemma 29 we will show the solvability of the Plateau problem of each quadrilateral $\Gamma_m(t)$. Each arc of $\Gamma_m(t)$ is a great circle arc in $\mathbb{S}^3$ and hence an arc of reflection. The parameter m in Lemma 26 is fundamentally important for part 1. Since whether our conjugate construction in generating CMC surfaces works depends on the choice of φ which is the size of a fundamental patch from the initial nodoid. Only when φ is a rational multiple of π, i.e. $\varphi = \frac{p}{q}\pi$ with p, q reduced, the resulting surface will close smoothly with multiplicity p after q times Schwarz reflections. Here we choose $\varphi = \frac{\pi}{m}$, where $m \in \mathbb{N}$ with $m \geq 2$.

Remark 27. We should note that without the requirement $\varphi = \frac{\pi}{m}$ (or $\varphi = \frac{p}{q}\pi$) in Lemma 26 a continuous one-parameter family of deformation quadrilaterals would still exist. But the Plateau solutions to these quadrilaterals would not necessarily close after finite Schwarz reflections.

Equation (6.1) and (6.5) yield an interesting geometrical feature: The deformation of boundary shifts the arc γ_3 towards γ_1. Because of antipodality in $\mathbb{S}^3$, any two *Clifford parallel* great circles have at most distance $\frac{\pi}{2}$. Hence the deformation decreases the distance between γ_1 and γ_3 by t. All Hopf vector fields of the boundary arcs remain unchanged compared with the original quadrilateral Γ (as shown in Fig. 6.2 and Fig. 6.3). Hence the new quadrilateral $\Gamma(t)$ is contained in a Clifford torus ∂T (as shown in Fig.6.2 and Fig. 6.3).

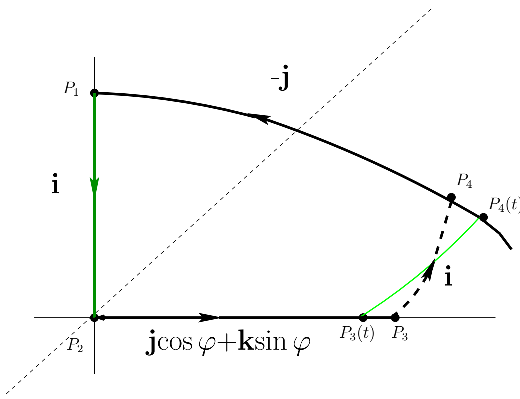


Figure 6.2: The new quadrilateral in $\mathbb{S}^3$ under the stereographic projection. The edges are labeled by their Hopf fields respectively.

The Clifford torus ∂T divides the 3-sphere $\mathbb{S}^3$ into two components. In the part the inner normal of ∂T pointing to we measure the rotational angles of the Hopf fields in the sense of Equation (2.9). Along γ_4 the inner normal of the torus is k. Therefore the Hopf field -i rotates through k to i and $\angle_{-j}(i,i) = -\pi$. Analogously we obtain the rotational angles for other edges

$$\begin{aligned} \gamma_1 : \angle_i(-\mathbf{j}, \mathbf{j}\cos\varphi + \mathbf{k}\sin\varphi) &= -2l \\ \gamma_2 : \angle_{\mathbf{j}\cos\varphi+\mathbf{k}\sin\varphi}(\mathbf{i},\mathbf{i}) &= \pi \\ \gamma_3 : \angle_{\mathbf{i}}(\mathbf{j}\cos\varphi+\mathbf{k}\sin\varphi, -\mathbf{j}) &= 2l \\ \gamma_3 : \angle_{-\mathbf{j}}(\mathbf{i},\mathbf{i}) &= -\pi \end{aligned} \tag{6.8}$$

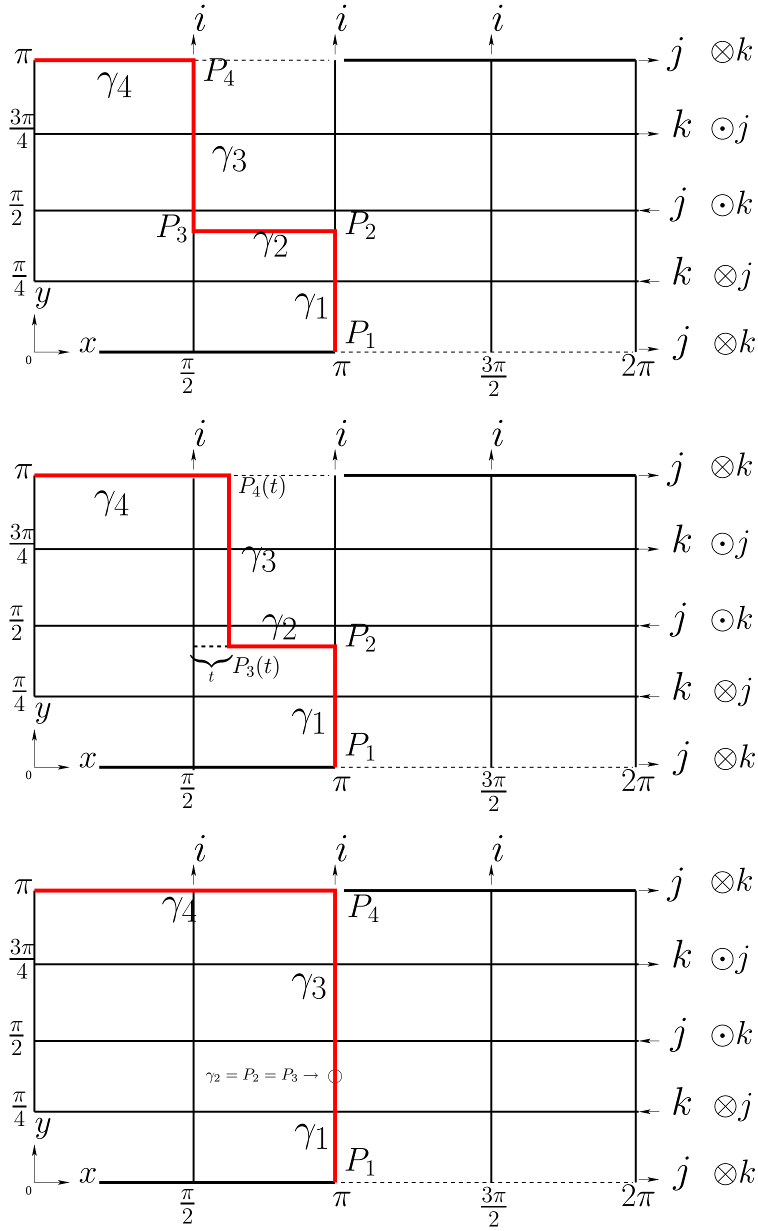


Figure 6.3: Three boundary arcs in the Clifford torus ∂T with $l = \frac{\pi}{3}$ and $\varphi = \frac{\pi}{3}$.
Top: The boundary polygon for $t = 0$ which corresponds to a $\frac{\pi}{3}$-fundamental patch of a nodoid.
Middle: The boundary polygon $\Gamma(t)$ for $t = \frac{\pi}{6}$.
Bottom: The degenerate case $t = \frac{\pi}{2}$. The arc γ_2 shrinks to a point and the two arcs γ_1, γ_3 merge into a half **i**-circle. The quadrilateral Γ degenerates to a 2-gon in $\mathbb{S}^3$ consisting of two orthogonal great circle arcs of equal length π.

6.3 Existence of Plateau Solutions

In this section we will show the solvability of the Plateau problem to the spherical quadrilateral $\Gamma_m(t)$ constructed in the previous section. Every arc $\gamma_n, (n = 1, ..., 4)$ is a geodesic arc in the smooth manifold $\mathbb{S}^3$, consequently real analytic. The Plateau solution can be analytically continued as a regular minimizing surface in $\mathbb{S}^3$ across each of its boundary arcs $\gamma_1, \ldots, \gamma_4$ by Schwarz reflections. To exclude vertex branch points we embed the boundary curve $\Gamma_m(t)$ into the boundary of a mean convex set. Then by Lemma 24 the surface bounded by $\Gamma_m(t)$ is free of branch points and can be extended as an immersion across each boundary. Once we get a regular complete minimal surface $\tilde{M}_\Gamma$ in $\mathbb{S}^3$ we lift its metric to the universal covering surface U_Γ of $\tilde{M}_\Gamma$. By the Lawson correspondence we get its CMC-1 cousin M_Γ. Since $\tilde{M}_\Gamma$ is regular, its second fundamental tensor is bounded, so is also its CMC-1 cousin. The analyticity of M_Γ follows from the mean curvature equation.

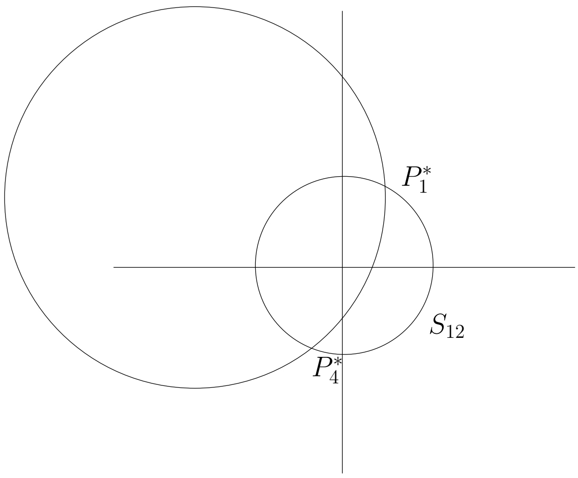


Figure 6.4: A great circle meets the geodesic 2-sphere S_{12}

Lemma 28. *Let Γ be a spherical quadrilateral with vertices $P_1, ..., P_4$ and edges $\gamma_1, ..., \gamma_4$ such that the length of each edge is not greater than π . Then Γ is contained in a closed hemisphere of $\mathbb{S}^3$.*

Proof. We denote S_{12} as the unique geodesic 2-sphere containing γ_1, γ_2 and P_1, P_2, P_4. Let us take S_{12} as the equator of $\mathbb{S}^3$. Then S_{12} divides $\mathbb{S}^3$ into two components: the upper open hemisphere H^+ which contains the north pole and the lower hemisphere H^- which contains the south pole. So $S_{12} = \partial H^{\pm}$. In $\mathbb{S}^3$ there are three possibilities to set the vortex P_3 of Γ:

(1) $P_3 \in S_{12}$. For the position of the arc γ_3 with respect to S_{12} there are two possibilities. The first possibility: γ_3 is contained in S_{12} and we are finished. The second

possibility: The arc γ_3 is not contained in S_{12}. In this case the great circle containing γ_3 meets the equator S_{12} exactly at two antipodal points P_2 and P_3 (see figure 6.4) and γ_3 must have the length $(2k+1)\pi$, $k \in \mathbb{N}$. But according to the assumption we have $|\gamma_3| < \pi$, so the geodesic arc γ_3 can only be contained in S_{12}. Since $|\gamma_4| < \pi$ an analogous argument gives γ_4 is also contained in S_{12}. Hence Γ is contained in S_{12}.

(2) $P_3 \in H^+$. Since $|\gamma_3|, |\gamma_4| < \pi$ it follows that $\gamma_3, \gamma_4 \subset H^+$ and hence $\Gamma_m(t) \subset \overline{H^+}$.

(3) $P_3 \in H^-$. Because of the symmetry of $\mathbb{S}^3$ this case is the same as case 2 and we have $\Gamma \subset \overline{H^-}$.

Therefore the quadrilateral Γ is contained in one of the closed hemispheres determined by S_{12}. □

Note that ∂T divides $\mathbb{S}^3$ into two components. We have chosen T to be the solid Clifford Torus such that the inner normal vector of ∂T points to T and such that $\Gamma(t) \subset \partial T$ for $t \in (0, \frac{\pi}{2})$. By endowing $\Gamma(t)$ a orientation we determine how the Hopf fields (see fig. 6.3) along each γ_i $(i = 1, ..., 4)$ rotate, this again determines the inner normal vector of ∂T.

Lemma 29. *Let $\Gamma_m(t)$ be the spherical quadrilateral constructed in Lemma26 with $t < \frac{\pi}{2}$ and let the length of γ_1 be shorter than $\frac{\pi}{2}$. Then the Plateau problem to $\Gamma_m(t)$ is solvable.*

Proof. The idea of the proof is to solve the Plateau problem in a closed mean-convex manifold. For that purpose we consider the space T^c, which is the universal covering of the solid Clifford Torus T. Topologically T^c is a solid cylinder and hence mean-convex.

1. Γ **bounds a disk in** T^c. This is equivalent to show that $\Gamma_m(t)$ is contractible in T^c. Recall the map F_θ defined in Sect. 2.4,

$$[0, \frac{\pi}{4}] \to T, \; \theta \mapsto F_\theta(x, \pi - x)$$

As shown in Figure 6.5, for $\theta = \frac{\pi}{4}$ the dotted curve $F_\theta(x, \pi - x)$ (image of the straight line $x + y = \pi$) is a geodesic arc on ∂T. Moreover $F_{\frac{\pi}{4}}(x, \pi - x)$ is contractible in T by the homotopic equivalence $F_{\frac{\pi}{4}}(x, \pi - x) \sim F_0(x, \pi - x) = (1, 0, 0, 0)^t$. At the same time the dotted curve $F_{\frac{\pi}{4}}(x, \pi - x)$ is homotopic to $\Gamma_m(t)$ on the Clifford cylinder ∂T^c and hence $\Gamma_m(t)$ bounds a disk in T^c.

2. **Constructing mean convex barriers.** As shown in Fig. 6.3 each quadrilateral is contained in the covering cylinder, $\Gamma_m(t) \subset \partial T^c$ for $m > 1$. We construct a mean convex Riemannian 3-manifold by defining two quarter great spheres as additional barriers (see Fig. 6.5): Extending γ_3 to π yields a new endpoint P_3' on the **j**-Hopf circle containing γ_4. Denote by $\gamma_{4,1}$ the half **j**-Hopf circle through P_1 joining P_3' and P_4. Then $\gamma_{4,1}$ and the extended γ_3 (which is a half **i**-Hopf) bound a quarter sphere Q_1. In a similar way, we extend the arc γ_1 to a half **i**-Hopf with endpoint P_2' and the extended arcs γ_1 and γ_4 bound a quarter sphere Q_2. Then the connected compact component of $N^k := (T^c \setminus Q_1) \setminus Q_2$ is a mean convex set.

Since the boundary of Q_1 are two asymptotic lines of the Clifford Torus ∂T at P_4 and

$P_3 \in Q_1 \cap \partial T$, then by Corollary 2.8 [GB93] we conclude that $\Gamma_m(t)$ lies entirely on one side of $T^c \setminus Q_1$ and the interior intQ_i $(i = 1, 2)$ does not contain any point of $\Gamma_m(t)$. Thus, $\Gamma_m(t)$ is contained in the compact component of N^k and all vertices of $\Gamma_m(t)$ are contained in two different mean convex barriers.

By Theorem 21 there exists a solution to the Plateau problem of $\Gamma_m(t)$. $\square$

Remark 30. We remark that the Plateau solution M obtained through Lemma 29 is completely contained in the interior of the mean convex set N^k. Otherwise by Theorem 21 M would be embedded in ∂N^k, which is not a minimal disk in N^k.

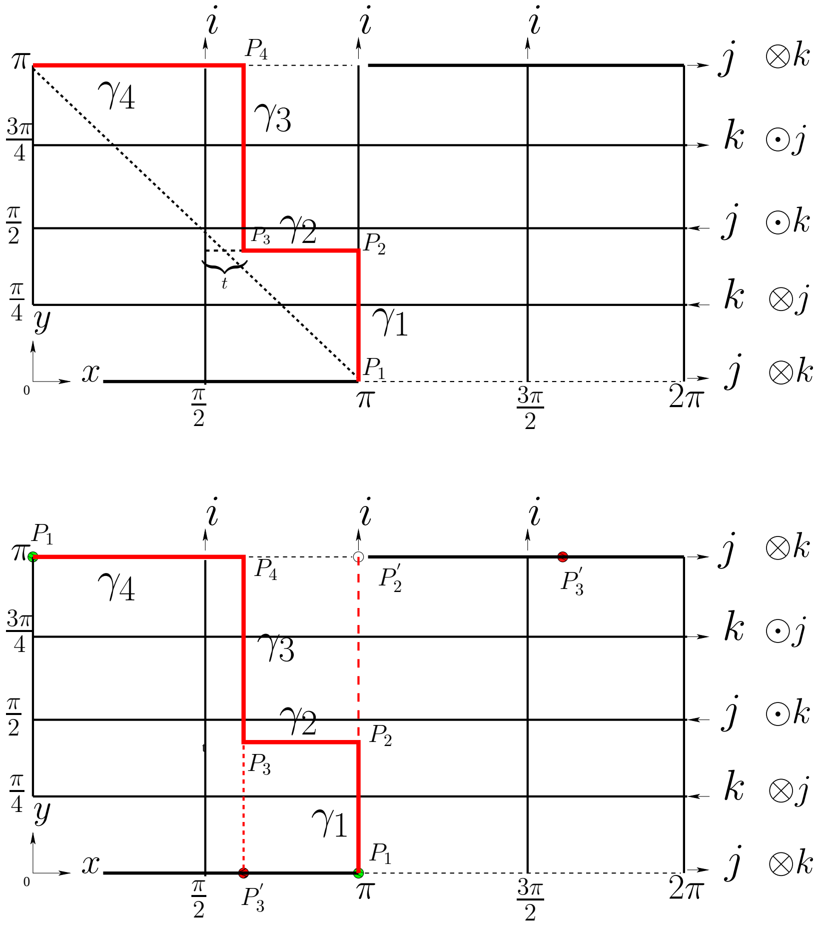


Figure 6.5: Top: The boundary polygon $\Gamma(t)$ for $t = \frac{\pi}{6}$ consists of the red lines. The dotted curve is the image of the straight line $x + y = \pi$ under F_θ for $\theta = \frac{\pi}{4}$. Bottom: Extending γ_3 to P_3' and then joining P_3' and P_4 with a geodesic arc orthogonal to γ_3 yields the half **j**-Hopf circle $\gamma_{4,1}$. In a similar way, we extend the arc γ_1 to a half **i**-Hopf with endpoint P_2'.

6.4 Existence of Bifurcation Families

Now we prove the following theorem

Theorem 31. *For every $m \in \mathbb{N}$ with $m \geq 2$ there is a family of* CMC-1 *surfaces $M_{m,t}$ in $\mathbb{R}^3$, $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$ with the following properties:*

1. *Bifurcation: The surface $M_{m,0}$ is a nodoid with necksize $n = (m-1)\pi$.*
2. *Symmetry: Every surface $M_{m,t}$ has a $\frac{\pi}{m}$- rotational symmetry.*

3. Degenerate case: As $|t| \to \frac{\pi}{2}$ the principal curvatures of $M_{m,t}$ tend to infinity. The surfaces $M_{m,t}$ converge to a chain of m spheres.

Proof. 1. By Equation (6.7) we have

$$2l + \varphi = \pi.$$

It follows that

$$2l + \frac{\pi}{m} = \pi$$

and thus

$$l = \frac{m-1}{2m}\pi. \tag{6.9}$$

Since by the construction the arc γ_1^* is taken to be the arc of the smallest circle of a nodoid, the necksize radius satisfies

$$r\varphi = l.$$

Thus by Equation (6.9) we have

$$r = \frac{m-1}{2}. \tag{6.10}$$

Hence as shown in Fig. Figure 1.1 on page 2 the bifurcation points are discrete: $2\pi \cdot r = \pi, 2\pi, 3\pi, \ldots$

2. **The $\frac{\pi}{m}$-rotational symmetry of $M_{m,t}$.** Consider the fundamental patch M bounded by $\Gamma_m(t)$ with $t \in [0, \frac{\pi}{2})$. The angle between the Hopf circles γ_2 and γ_4 is $\varphi = \frac{\pi}{m}$. By doing reflections across the edges γ_2 and γ_4 successively $2m-1$ times (restricted to the fundamental patch) we return to the original surface M. Through these $2m-1$ reflections we extend M to a minimal surface M_{2m} bounded by two disjoint closed **i**-Hopf circles containing γ_1 and γ_3 respectively, since by reflections we eliminate the boundary γ_2 and γ_4, the arcs γ_1 and γ_3 are extended to two disjoint closed **i**-Hopf circles. The conjugate fundamental patch is bounded by $\Gamma_m^*(t)$ (the corresponding quadrilateral of $\Gamma_m(t)$ in $\mathbb{R}^3$) with $t \in [0, \frac{\pi}{2})$. The dihedral angle between the symmetry planes of γ_2^* and γ_4^* is $\varphi = \frac{\pi}{m}$. We choose the ijk coordinates such that the symmetry planes of γ_2^* and γ_4^* intersect at i-axis. The dihedral angle φ between the symmetry planes containing γ_2 and γ_4 is $\frac{\pi}{m}$. After doing Schwarz reflections across the edges γ_2^* and γ_4^* successively $2m$ times we return to the original fundamental patch. All these $2m$ symmetry planes intersect at the i-axis. Thus the conjugate CMC-1 surface of M_{2m} has a m-fold rotational geometry (i.e. it is invariant under a rotation of $\frac{2\pi}{m}$).

3. In the limit $t = \frac{\pi}{2}$, the quadrilateral $\Gamma_m(t)$ degenerates to a quarter sphere in $\mathbb{S}^3$: The edge γ_2 degenerates to a point P_2 and the edges γ_1, γ_3 merge to one long edge, i.e. a half **i** circle. The other long edge is γ_4, which is a half **-j** circle. In $\mathbb{R}^3$ one reflection across the arc $\gamma_1 + \gamma_3$ extends the CMC-1 surface to a hemisphere. Assume $t = \frac{\pi}{2} - \varepsilon$, then $|\gamma_2| = |\gamma_2^*| = \varepsilon$. As discussed in the part 2 of the proof the dihedral angle $\varphi = \frac{\pi}{m}$ between the symmetry planes of γ_2^* and γ_4^* does not depend on the value t. Therefore as $\varepsilon \to 0$ the symmetric plane of γ_2^* turns out to be the tangent plane of γ_1^* at the vertex P_2. Reflecting the fundamental CMC-1 patch across this tangent plane and across the arc γ_4^* successively $2m$ times gives rise to a chain of m touching spheres. □

7 Properties of the CMC-1 Surface Families

In general even in $\mathbb{R}^3$ one can not expect uniqueness of a Plateau solution. However, certain additional geometric conditions for a Jordan curve Γ ensure the uniqueness. The idea is due to Rado [Rad33]: if Γ has a one-to-one projection onto a planar convex curve γ, then Γ bounds at most one disk-type minimal surface. We want to show that the Plateau solution obtained in the last chapter is unique by generalizing the Rado argument to $\mathbb{S}^3$. By uniqueness we mean every two minimal surfaces bounded by the same boundary polygon Γ with $t \in [0, \frac{\pi}{2})$ and have the same barriers must coincide.

7.1 A Uniqueness Result

In this section let $\Gamma \subset H$ be a Jordan curve contained in a 3-dimensional open hemisphere H. Note that the Hopf bundle is not trivial, i.e. $\mathbb{S}^3$ is not a product $\mathbb{S}^2 \times \mathbb{S}^1$. However, with one puncture of $\mathbb{S}^2$ we get a trivial bundle: for each point $p \in \mathbb{S}^2$ we remove one point $q \in \mathbb{S}^2$ with $p \neq q$. The set $U := \mathbb{S}^2 \setminus \{q\}$ is a neighbourhood of p whose preimage $\Pi_u^{-1}(U)$ in $\mathbb{S}^3$ can be identified with a product of U and a u-Hopf circle, $\Pi_u^{-1}(U) = U \times \mathbb{S}^1$. Let q be the north pole of $\mathbb{S}^2$ and $u \in \mathbb{S}^2$. We denote $S := \mathbb{S}^3 \setminus \{C_q\}$ and $U := \mathbb{S}^2 \setminus \{q\}$ where $C_q = \Pi_u^{-1}(q)$. Then the restricted Hopf-fibration $\xi = (S, U, \Pi_u)$ is a trivial fibration.

Let $\widetilde{\Gamma} := \Pi_u(\Gamma)$ be the image of Γ under the projection and let $\Omega \subset \mathbb{S}^2$ be the interior of $\widetilde{\Gamma}$. By the notation "interior" here we mean the domain enclosed by the convex curve $\widetilde{\Gamma}$, which does not contain the north pole of $\mathbb{S}^2$. Note that for the interior of M the image $\Pi_u(\mathrm{int}M)$ may approach its boundary from both sides. We ensure that the image of our Plateau-solution under the u-Hopf projection lies on one side of the boundary by showing in the following lemma a transversality property of the Plateau solution M of Sect.6.3.

Lemma 32. *Let M be a solution of the Plateau problem for $\Gamma_m(t)$, $(0 \leq t < \frac{\pi}{2})$, where $\Gamma_m(t)$ is the spherical quadrilateral in Section 6.3. Then M is transverse to the restricted Hopf fibration $\xi = (S, U, \Pi_i)$ by i-Hopf circles and the surface $\Pi_i(M)$ immerses into $\mathbb{S}^2$.*

Proof. We lift the solid Clifford torus to its universal covering - the solid cylinder T^c. The $\mathbb{S}^1$- bundle on the Clifford torus T lifts to a $\mathbb{R}$-bundle in T^c. The vertical i - lines in T^c induce a Killing field Z. Let $\Phi_t \colon T^c \to T^c$, $t \in (0, \infty)$ be the flow of the field Z. We denote the interior of M by M^0. Then as mentioned in Remark 30 we have $M^0 \cap \partial T^c = \emptyset$.
We show that M^0 is transverse to the i-lines by using an indirect approach. Assume that

M^0 is not transverse to the i-lines. Since M is a least area disc in T^c and $\partial M \subset \partial T^c$ there are $p, q \in M^0$ with $p \neq q$ and $t \in (0, \infty)$ such that $\Phi_t(p) = q$ (see Figure 7.1).

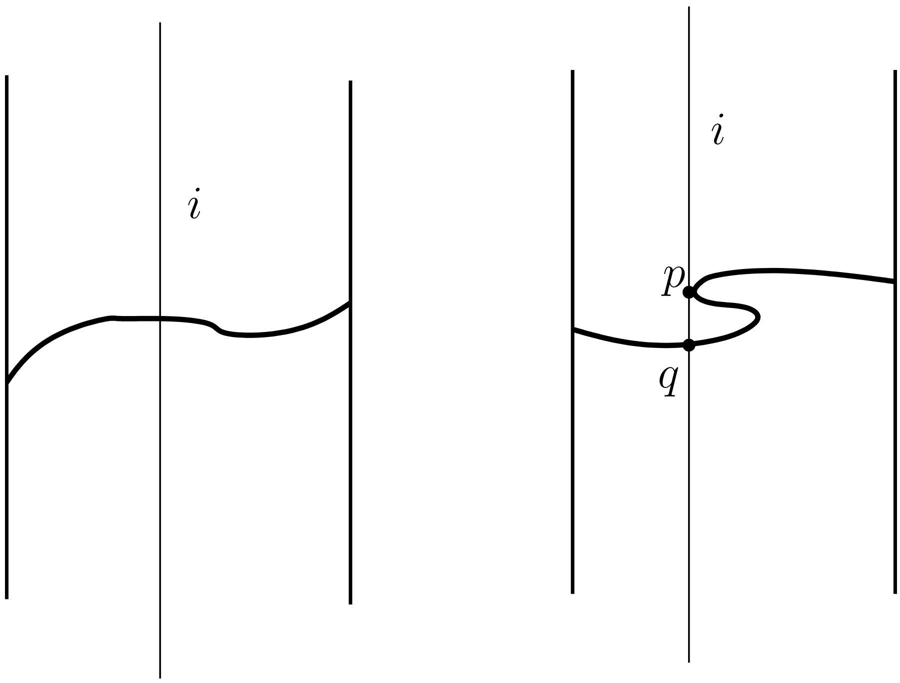


Figure 7.1: Two cross sections of the Plateau solution M in the Clifford-Cylinder T^c. The left hand side figure is a vertical section (i-direction) in the case where M is transverse to the i-lines. The right hand figure is a vertical section (i-direction) in the case where the transversality of M to the i-lines does not hold.

For all $x \in U$ which has more than one preimage in M there is a minimum $t_{\min} := \inf\{t > 0\colon M^0 \cap \Phi_t(M^0) = \emptyset\}$. Such $t_{\min}$ exists because of the compactness of M. Then denote by $M_1 := \Phi_{t_{\min}}(M)$ the surface obtained by a vertical translation $\Phi_{t_{\min}}$ along i-lines of the surface M. Clearly M and M_1 have an one-sided contact.

We consider the case that both surfaces meet at their interior points. But this is impossible by Theorem 16.

The case that the common point is an interior point of one surface and a boundary point of the other is also not possible because $\mathrm{int} M \cap \partial T = \emptyset$.

We consider the case that the common point is an interior boundary point of both surfaces. This leads to a contradiction to Hopf boundary point lemma: let $f\colon \overline{B} \to \mathbb{S}^3$ and $g\colon \overline{B} \to \mathbb{S}^3$ be the parametrizations of M and M_1 and $x_0 \in \partial B$ be the common point $f(x_0) = g(x_0)$. Then we can write M and M_1 as graphs locally near x_0 (this is possible because M, M_1 can be extended across their boundaries by Schwarz reflections). We denote by f_3 and g_3 the vertical components of the graphs. Then the normal derivative at the point x_0 is $\frac{\partial(f_3-g_3)}{\partial n}(x_0) = \langle \nabla f_3(x_0) - \nabla g_3(x_0), n\rangle = 0$. This is a contradiction to the Hopf boundary point lemma.

The case that the common point is a boundary vertex of both surfaces is not possible. At the corner the tangent space of the surfaces is determined by the two adjacent arcs, and they do not have any interior contact before meeting at the corner. Thus the conormal along the i-line must have a rotational angle greater than π. That is not possible because each vertex of $\Gamma_m(t)$ is contained in two different hemispheres.

It remains to deal with the case that the common point is an interior boundary point of one surface (e.g. M_1) and boundary vertex of the other one (e.g. M). Again the tangent plane at the vertex is the same as that of the cylinder ∂T^c. Then applying the Hopf boundary point lemma on ∂T^c and M_1 we get a contradiction.

The same arguments work also for the case $t < 0$ and from that the statement of transversality follows. And since M is transverse to the i-lines, $\Pi_i\colon M \to \mathbb{S}^2$ is an immersion. $\square$

Theorem 33. *Let M_1, M_2 be two solutions to the Plateau problem of $\Gamma_m(t)$, where $m \geq 2$ and $0 \leq t < \frac{\pi}{2}$. Then $M_1 \equiv M_2$.*

Proof. Idea of the proof: By Lemma 32 both M_1 and M_2 are transverse to the i-lines, hence we can write M_1 and M_2 as graphs over the domain Ω^* in the restricted bundle ξ, where Ω^* is the domain bounded by $\tilde{\Gamma} := \Pi_i(\Gamma)$. Then the unicity can be achieved by applying the maximum principle (Lem.15) of the stereographic projections of these minimal graphs.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be non-parametric representations of the stereographic projection of M_1,M_2. We can assume that the open set $\Omega^+ := x \in \Omega\colon (u - v) \geq 0$ is not empty (if $\Omega^+ = \emptyset$ we consider $v - u$ and the following arguments work as well). Let $m \geq 0$ be the maximum of $u - v$. Then $u \leq v + m$ on Ω and $u = v + m$ at a point $x_0 \in \Omega$. The first requirement of Lemma 15 is satisfied. Through the mean curvature equation 4.18 we induce a second order quasilinear elliptic operator Q. Because M_1 and M_2 are minimal surface we have $Q(u) = Q(v) = 0$ on Ω, which satisfies the second requirement of Lemma 15. Then by Lemma 15 we have $u \equiv v + m$ on Ω and the boundary condition $u \equiv v$ on $\partial\Omega$ gives $m = 0$. Thus $u \equiv v$ on $\overline{\Omega}$ and M_1, M_2 coincide. $\square$

Theorem 34. *The family* $(M_{m,t})$ *of* CMC-1 *surfaces from Theorem 31 is continuous in* t *and it is valid* $M_{m,t} = M_{m,-t}$.

Proof. The continuity follows directly from the uniqueness result in Theorem 33. The fundamental quadrilateral Γ_{-t} is in fact obtained from Γ_t merely by interchanging the edges γ_2 and γ_4. Therefore the two quadrilaterals Γ_{-t} and Γ_t agree geometrically and their CMC-1 surfaces are also the same (up to a rotation). Again from the uniqueness result Theorem 33 it follows that their Plateau solutions coincide. □

A compact CMC surface M is *Alexandrov-embedded* if M is properly immersed and if there exists a compact three-manifold W with boundary $\partial W =: \Sigma$ and a proper immersion $F: W \to \mathbb{R}^3$ whose boundary restriction $f: \Sigma \to \mathbb{R}^3$ parametrizes M. Alexandrov [Ale62] showed that the round sphere is the only compact Alexandrov-embedded CMC surface in $\mathbb{R}^3$.

Theorem 35. *Every surface* $M_{m,t}$ *is simply periodic and represents an immersion of* $S^2 \setminus \{p_1, p_2\}$, *where* $p_1 \neq p_2$.

Proof. As shown in the proof of Lemma 29 the boundary arcs $\Gamma_m(t)$ are contained in the Clifford cylinder ∂T^c and the Plateau solutions $M_m(t)$ lie completely in the solid Clifford cylinder T^c. By Lemma 32 we know the minimal surface $M_m(t)$ is tranverse to the **i**-lines in $\mathbb{S}^3$. Let $M_m^*(t)$ be the conjugate CMC-1 surface of $M_m(t)$.

We show that the conjugate arcs γ_1^* and γ_3^* lie in different symmetry planes. The following general version of Alexandrov's theorem [Ale62] is needed.

A compact, Alexandrov-embedded CMC *surface must be* $\mathbb{S}^2$.

We claim that γ_1^* and γ_3^* are contained in the same symmetry plane P with the normal vector i, then by Schwarz reflections we continue M^* to an Alexandrov-embedded torus. Firstly it is obvious that by doing Schwarz reflections we continue M^* to a torus T. In fact, γ_2^* and γ_4^* lie in two different vertical (with respect to i) planes with angle $\varphi = \frac{\pi}{m}$, therefore T consists of $2m$ copies of M^* from both sides of P. The symmetry plane P divides T into two components, T_+ and T_-(as open sets). We have $T = T_+ \cup T_- \cup T_0$, where $T_0 := T \cap P = 2m$ copies of γ_1^* and γ_3^*. We show T is Alexandrov-embedded:

a) By Lemma 32 the area minimizing M is transverse to i and by construction γ_2 and γ_4 are i-transverse in $\mathbb{S}^3$. Then by the first order description of the Lawson-correspondence (Equation 3.3)

$$df = f \cdot df^* \cdot J$$

the CMC-1 surface $M_m^*(t)$ in $\mathbb{R}^3$ is transversal to the z-direction. To see this we build the inner product of $e_z := (0, 0, 1)$ and the normal vector ν^* of the surface $M_m^*(t)$

$$\langle \nu^*, e_z \rangle_{\mathbb{R}^3} = \langle f \cdot \nu^*, f \cdot e_z \rangle_{\mathbb{S}^3} = \langle \nu, i \rangle_{\mathbb{S}^3}$$

the first equality is valid because the left multiplication with f is an isometry from $\mathbb{R}^3 = T_1\mathbb{S}^3$ to $T_f\mathbb{S}^3$. The second equality is valid because the i-field at the point f is defined by $f \cdot e_z$. Therefore if the minimal surface $M_m(t)$ in $\mathbb{S}^3$ is transverse to the i-fiber, then its conjugate CMC-1 surface $M_m^*(t)$ in $\mathbb{R}^3$ is transversal to the z-direction.

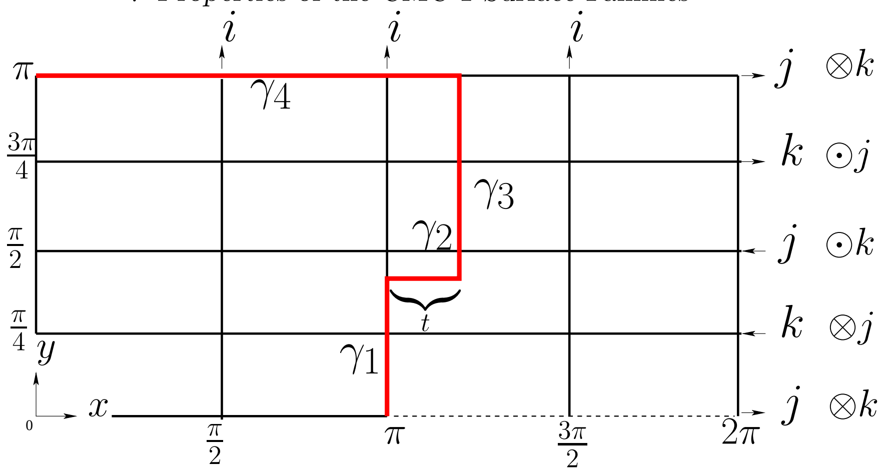
b) Furthermore T_+ lies strictly in one side of P. Along the boundary ∂T_+ the conormal of T_+ is z. Geometrically that means along its boundary T_+ tends to lie in the upper half space of P. So if there were a point in T_+ in the lower half space of P, by the compactness we would have a minimum point. But the mean curvature of T_+ points downwards. Then by sliding a plane from infinity to T_+ there would be an interior contact at this minimum point before touching the boundary. This is however a contradiction to the maximum principle for CMC surfaces.

c) Let $\Pi \colon T_+ \to P$ be the z-projection from T_+ to the symmetry plane and $A := \Pi(T_+)$. Clearly, Π is an immersion. Denote by $\phi(A) := \Pi^{-1}(A)$ the preimage of A under the projection. We construct an immersed 3-manifold bounded by T as the following: For each $p \in A$ we locate by $h(\phi(p))$ the height of the image point in T_+, we then map $(-1, 1)$ diffeomorphically onto $(-h(\phi(p)), h(\phi(p)))$. The 3-manifold $N := \{(-h(\phi(p)), h(\phi(p))) \in \mathbb{R}^3 \mid p \in A\}$ is the desired manifold.

Hence by Alexandrov's theorem T must be $\mathbb{S}^2$. This contradiction shows that γ_1^* and γ_3^* lie in two different parallel planes and the surface can not be closed up by doing Schwarz reflections in the **i**-direction. Therefore the complete CMC-1 surface $M_m^*(t)$ is simply periodic and represents an immersion of $\mathbb{S}^2$ with two punctures. $\square$

7.2 The Case $t > \frac{\pi}{2}$

It is natural to ask if deformations for the case $t > \frac{\pi}{2}$ still lead to the existence of new CMC surfaces. The answer is affirmative: If we continue to extend γ_2 and γ_4 beyond the degenerating limit $\frac{\pi}{2}$ (see Fig.7.2) we get again a quadrilateral in $\mathbb{S}^3$. The geodesic boundary arc γ_4 is a long arc, i.e. longer than π, and the Hopf-vector field along γ_2 changes sign. The long quadrilateral $\Gamma_m(t)$ can be embedded in the universal covering of the Clifford torus. For every $t \in (\frac{\pi}{2}, \frac{3\pi}{2})$ there is a solution to the Plateau problem for the quadrilaterals $\Gamma_m(t) \subset \mathbb{S}^3$. By using the solid covering cylinder together with great hemispheres as barriers a similar argument as for the case $t < \frac{\pi}{2}$ yields the existence of Plateau solutions for the long quadrilateral. However in this case we can not get uniqueness.



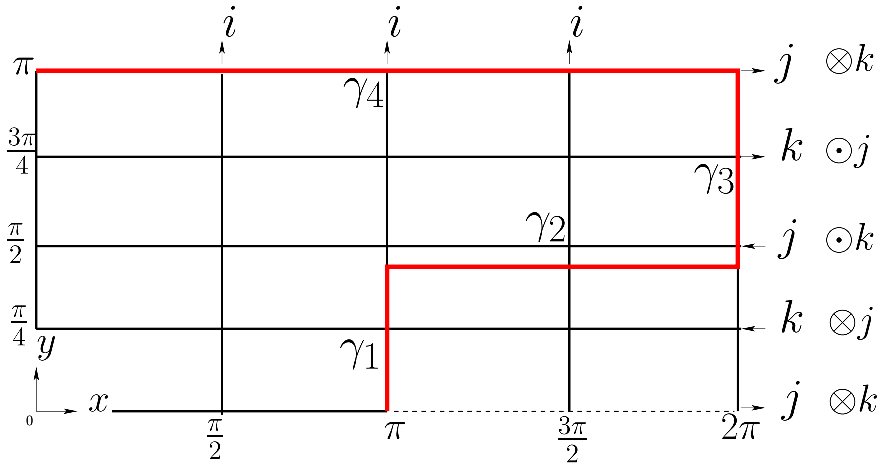


Figure 7.2: Top: A long polygon ($t = \frac{2}{3}\pi$) in the universal covering cylinder
Bottom: The degenerating case of $\Gamma_m(t)$ for $t = \frac{3}{2}\pi$

Remark 36. The case $t > \frac{\pi}{2}$ was already discussed in [GB93] Sect. 5.1 by using the conjugate method to construct CMC surface with cylinder ends. In our work the surfaces arise in the context of discussing bifurcating nodoids. As $t \to \frac{3\pi}{2}$ the CMC surfaces $M_{m,t}$ turns out to be a CMC surface with two cylindrical ends and a lobe of m spheres attached in a constant distance (Fig.7.3). t is then a parameter of this distance. These surfaces present a singly periodic variant of multi-bubbleton in ([PS89, Fig. 8.1]), since as $t \to \infty$ the distance of two lobes of spheres tends to be infinity and the surface turns to be a (m-fold covered) CMC cylinder with a lobe of m spheres attached in the middle.

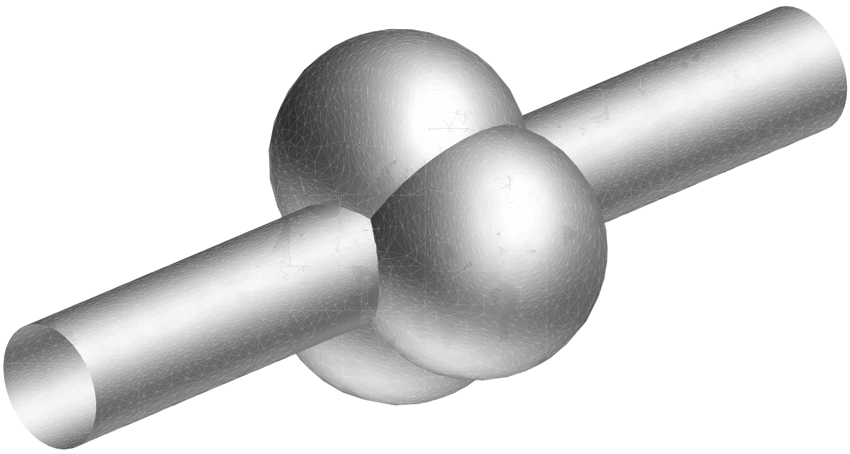


Figure 7.3: One fundamental domain of a singly periodic CMC surface

Remark 37. A *Jacobi field* of a CMC surface is a nonzero solution $u \in C^2(M)$ of the *Jacobi equation*

$$\Delta u + |A|^2 u = 0$$

where Δ is the Laplace-Beltrami operator and $|A|^2$ is the square of the length of the second fundamental form of the surface. We call

$$\Delta + |A|^2$$

the *Jacobi operator*. For a nodoid $M_{m,0}$ with necksize $n = (m-1)\pi$ we denote its Jacobi operator by J_m. The Jacobi equation is obtained from the second variation formula for the nodoid $M_{m,0}$. Each surface branch $(M_{m,t})$ of Theorem 31 corresponds to a Jacobi field. In other words, each deformation of $M_{m,0}$ shown in Fig.1.1 yields an eigenfunction to the eigenvalue $\lambda = 0$ of J_m.

It is also natural to ask if there are bifurcations of the embedded Delaunay surfaces - the unduloids. In fact, by the main Theorem of [KKS89] each embedded CMC surface with 2 ends is rotational symmetric and hence an unduloid.

Part II

The Bifurcating Helicoids in $\mathbb{R}^3$

8 Minimal Surfaces Bifurcating from Helicoids

The second part of this work is devoted to carry the conjugate construction onto the setting of $\mathbb{R}^3 \leftrightarrow \mathbb{H}^3$. In hyperbolic 3-space the *catenoid cousins* are the only CMC-1 surfaces of revolution with two ends. Like the Delaunay surfaces in $\mathbb{R}^3$ the catenoid cousins constitute different components: embedded ones and immersed ones (with self-intersections). The helicoids are the minimal surfaces in $\mathbb{R}^3$ to the catenoid cousins in the Lawson-correspondence.

Let $k \geq 2$. We consider a fundamental patch of a helicoid, which is associated to a non-embedded catenoid cousin. The boundary of the fundamental patch is bounded by two parallel infinite rays connected by a finite straight line arc. We then deform the boundary arc by changing the angle between the finite arc and the z-axis away from 0. We show that the Plateau problem to the new boundary is solvable. Then through an exhaustion argument we show that the Plateau solution of the conjugate contour is a graph over a plane[1]. By Schwarz reflection across the boundary arcs we get a one-parameter family of complete singly periodic minimal surfaces in $\mathbb{R}^3$ bifurcating from the standard helicoid.

8.1 The Helicoids in $\mathbb{R}^3$

In $\mathbb{R}^3$ it is well known that any ruled complete minimal surface in $\mathbb{R}^3$ is part of a plane or a helicoid. Consider the following conformally parametrized of a helicoid $f\colon \mathbb{R}^2 \to \mathbb{R}^3$

$$f(x,y) = \begin{pmatrix} \sinh x \cos y \\ -\sinh x \sin y \\ ay \end{pmatrix},$$

where $2\pi a$ is called the *pitch* of the helicoid. It is the height of one complete helix turn, measured parallel to the z-axis. If $a = 0$ the immersion f parametrizes the xy- plane and for $a = 1$ the standard helicoid. If $a > 0$ the helicoid is right-handed, if $a < 0$ then it is left-handed. Because helicoids are singly periodic, the quantity $2\pi a$ is the height of one sheet of the helicoid f. On the associate catenoids the parameter $2\pi a$ is the *necksize.* Notice that for $\pm a$ we have the same catenoids with reversed normal vectors.

We compute

[1] The fact that such Dirichlet problems have graph solutions implies that the tangent planes along the vertical edges have to rotate in a monotone way, otherwise the surface would not be a graph over the interior. This implies that the corresponding cousin arcs in $\mathbb{H}^3$ are convex arcs.

$$\frac{\partial f}{\partial x}(x,y) = \begin{pmatrix} \cosh x \cos y \\ -\cosh x \sin y \\ 0 \end{pmatrix}, \quad \frac{\partial f}{\partial y}(x,y) = \begin{pmatrix} -\sinh x \sin y \\ -\sinh x \cos y \\ a \end{pmatrix},$$

and then the normal vector of f is given by

$$\nu(x,y) := \frac{\partial_x f \wedge \partial_y f}{|\partial_x f \wedge \partial_y f|} = \frac{1}{\sqrt{a^2 + \sinh^2 x}} \begin{pmatrix} -a \sin y \\ -a \cos y \\ -\sinh x \end{pmatrix}.$$

Along the straight x-line $x \mapsto f(x,y)$ the normal can only rotate so that we expect that Se_1 is a multiple of $df(e_2)$:

$$\begin{aligned} df(Se_1) &= -d\nu(e_1) = -\frac{\partial \nu}{\partial x} \\ &= \frac{a \cosh x}{\left(\sqrt{a^2 + \sinh^2 x}\right)^3} \begin{pmatrix} -\sinh x \sin y \\ -\sinh x \cos y \\ a \end{pmatrix} \\ &= \frac{a \cosh x}{\left(\sqrt{a^2 + \sinh^2 x}\right)^3} df(e_2). \end{aligned}$$

Similarly

$$\begin{aligned} df(Se_2) &= -d\nu(e_2) = -\frac{\partial \nu}{\partial y} \\ &= \frac{1}{\sqrt{a^2 + \sinh^2 x}} \begin{pmatrix} a \cos y \\ -a \sin y \\ 0 \end{pmatrix} \\ &= \frac{a}{\cosh x \cdot \sqrt{a^2 + \sinh^2 x}} df(e_1) \end{aligned}$$

which means that along the helices $y \mapsto f(x,y)$ the normals do not tilt but they rotate purely. Along the straight axis $y \mapsto f(0,y)$ we expect also this behaviour. Set

$$\alpha = \frac{a \cosh x}{\left(\sqrt{a^2 + \sinh^2 x}\right)^3}; \quad \beta = \frac{a}{\cosh x \cdot \sqrt{a^2 + \sinh^2 x}}.$$

We have

$$Se_1 = \alpha e_2; \quad Se_2 = \beta e_1.$$

Hence

$$S(\beta e_1 \pm \alpha e_2) = \alpha\beta(e_1 \pm e_2).$$

Consequently the product of α, β gives the Gauss curvature K of f

$$K = \alpha\beta = \frac{a \cosh x}{\left(\sqrt{a^2 + \sinh^2 x}\right)^3} \cdot \frac{a}{\cosh x \cdot \sqrt{a^2 + \sinh^2 x}} = \frac{a^2}{(a^2 + \sinh^2 x)^2} \tag{8.1}$$

This gives a matrix representation of the shape operator S

$$S_{(x,y)} = \begin{pmatrix} 0 & \alpha\beta \\ \alpha\beta & 0 \end{pmatrix}.$$

The straight axis $y \mapsto f(0, y)$ is a curve of planar reflection on the CMC-1 cousin f^* in $\mathbb{H}^3$ and also all the x-lines are curves of planar reflection on f^* (in Lemma 40). Thus there is a continuous group of reflections and the CMC-1 surfaces are either rotationally symmetric (if $a \neq 0$) catenoid cousins or translation invariant ($a = 0$) horospheres.

Definition 38. A *catenoid cousin* is the associated CMC-1 surface in $\mathbb{H}^3$ of a helicoid $f_a(x, y)$ with $a \in \mathbb{R} \setminus \{0\}$ in $\mathbb{R}^3$.

8.2 Boundary Contours and Mean Convex Barriers.

8.2.1 Schwarz Reflection Principle

The Plateau solution in $\mathbb{R}^3$ with polygonal boundary can be extended across each boundary arc by applying the Schwarz reflection principle. A geodesic on a surface is called *a principal geodesic* if it is a curvature line. The Schwarz reflection principle for minimal surfaces states that minimal surfaces containing principal geodesics must possess Euclidean symmetries. The following theorem of Schwarz reflection principle is from [HK97] (p.16):

Theorem 39. *(Schwarz Reflection Principle). If a minimal surface contains a line segment L, then it is symmetric under rotation by π about L. If a minimal surface is bounded by a line segment L, it may be extended by rotation by π about L to a smooth minimal surface containing L in its interior.*

If a nonplanar minimal surface contains a principle geodesic - necessarily a planar curve - then it is symmetric under reflection in the plane of that curve. If a minimal surface meets a plane orthogonally on its boundary, the surface may be extended by reflection across the plane to a smooth minimal surface with this curve in its interior.

Let $M \setminus \{p_j\}$ be a compact Riemannian surface with finite punctures $\{p_j\}$. We denote by $\Sigma := \widetilde{M \setminus \{p_j\}}$ the universal covering of $M \setminus \{p_j\}$. Given a CMC-1 surface $f^* \colon \Sigma \to \mathbb{H}^3$, we denote by $f \colon \Sigma \to \mathbb{R}^3$ its conjugate minimal cousin. We are now interested in the symmetries between the surface f^* and its conjugate minimal cousin f in $\mathbb{R}^3$.

Lemma 40. *A curve $f^* \circ \gamma$ is a curve of planar reflection (with respect to a hyperbolic plane) on the CMC-1 surface f^* in $\mathbb{H}^3$, iff $f \circ \gamma$ is contained in a straight line.*

Proof. Given a unit speed curve γ in the domain of f^* with normal field ν, which satisfies $\nu(p) \perp T_{f^*(p)} f^*(\Sigma)$. We then choose as frame along the image curve $c := f^* \circ \gamma$ the tangent field $e_1 := Tf^*(\gamma')$, the conormal field $e_2 := Tf^*(J \cdot \gamma')$ and the surface normal field $e_3 := \nu \circ \gamma$. We then have the following expressions for normal curvature κ_n^* and normal torsion τ_n^* for f^*:

$$\begin{aligned} \kappa_n^* &= g^*(e_3', e_1) \\ &= g^*(S^*\gamma', \gamma') \\ &= g(J \circ S(\gamma') \pm \gamma', \gamma') \\ &= g(S(\gamma'), J \cdot \gamma') \pm 1 \\ &= \tau \pm 1 \end{aligned} \tag{8.2}$$

$$\begin{aligned} \tau_n^* &= g^*(e_3', e_2) \\ &= g^*(S^*\gamma', J \cdot \gamma') \\ &= g(J \circ S(\gamma') \pm \gamma', J \cdot \gamma') \\ &= g(S(\gamma'), \gamma') \\ &= \kappa_n \end{aligned} \tag{8.3}$$

where κ_n and τ_n denotes the normal curvature and torsion of the conjugate minimal surface f in $\mathbb{R}^3$. If $c(t)$ is a curve of planar reflection, we have $\tau_n^* = 0$ and by Equation (8.3) the curve $f \circ \gamma$ on the conjugate minimal cousin must be a straight line (since $\kappa_n = \kappa_g = 0$). The other direction can be achieved analogously. $\square$

8.2.2 Mean Convex Barriers.

In this subsection, we will describe the deformation of the boundary Γ in standard coordinates and show the existence of a family of singly periodic minimal surfaces in $\mathbb{R}^3$.

Let M be a π-piece of a helicoid in $\mathbb{R}^3$. Let $D := [0, \infty) \times [0, \pi]$. A parametrization of M is then given by $f \colon D \to \mathbb{R}^3$

$$f(x, y) = \begin{pmatrix} \sinh x \cos y \\ -\sinh x \sin y \\ \frac{l_0 y}{\pi} \end{pmatrix}.$$

The boundary of M is $\Gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$, where

$$\begin{aligned} \gamma_1 &:= \{x \geq 0,\ y = z = 0\} \\ \gamma_2 &:= \{x = y = 0,\ 0 \leq z \leq l_0\} \\ \gamma_3 &:= \{x \leq 0,\ y = 0,\ z = l_0\} \end{aligned}$$

We describe a mean convex set S such that $\Gamma \subset \partial S$ and $M \subset S$

$$S := \{(x, y, z) \in \mathbb{R}^3 \colon 0 \leq y,\ 0 \leq z \leq l_0\}$$

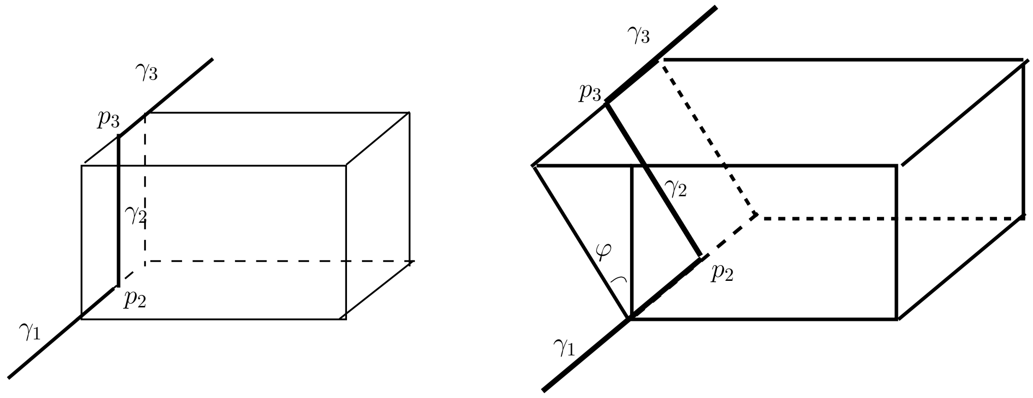


Figure 8.1: Left: The boundary of the fundamental patch of a helicoid which contained in the infinite square box S. The boundary Γ consists of two parallel rays γ_1, γ_3 and one finite straight line arc γ_2. Right: The deformation of an angle $\varphi \in (0, \frac{\pi}{2})$. The straight line arc γ_2 makes an angle φ to the z-axis.

As one can see from the parametrization above, the minimal surface M is contained in S and for $x \to \infty$ the normal of f tends to $(0, 0, 1)$. We want to deform M to get different asymptotics. We will do this by rotation the boundary Γ about the x-axis while keeping M asymptotically fixed. Since the boundary contains infinite rays we use barriers to illustrate the deformation as shown in Figure (8.1), whereas the left one is the boundary of a fundamental patch of a standard helicoid and the right one is the deformation of the boundary by tilting the arc γ_2 around the x-axis about the angle $\varphi \in (0, \frac{\pi}{2})$ and prolong the arc γ_2.

Notation 41. Let Γ_φ be a curve consisting of two parallel rays and a straight line arc joining the endpoints of the rays. Specifically, we choose one ray horizontal to run along the positive x-axis to $(\infty, 0, 0)$ and the other horizontal to be parallel to the negative x-axis starting from the point $p_3 := (0, -l \sin\varphi, l \cos\varphi)$ where $\varphi \in [0, \pi/2)$ and l is a strictly monotonic smooth function of φ with $l(0) = l_0 > 0$ and $l(\frac{\pi}{2}) = \infty$.

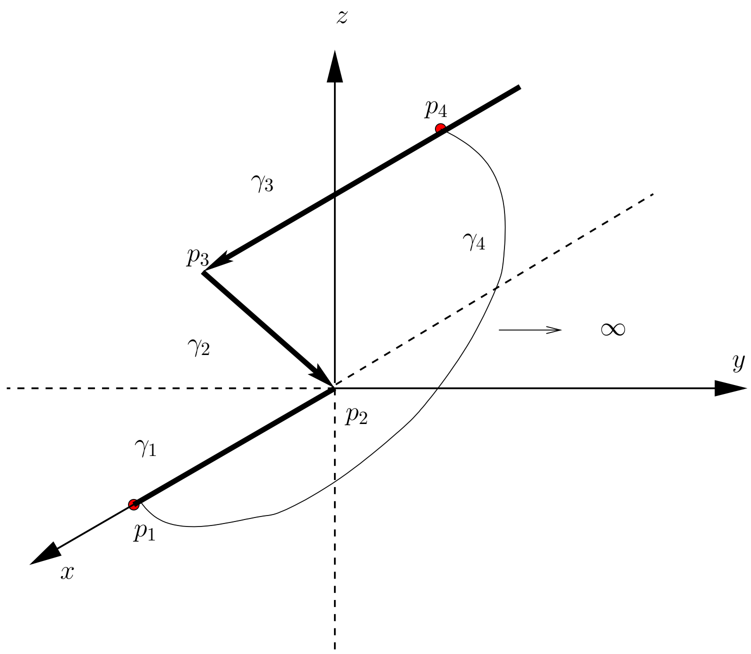


Figure 8.2: boundary curve: $\Gamma(n)$

Definition 42. A sequence $(M_k)_{k\in\mathbb{N}}$ of minimal surfaces satisfies *uniform local area bounds* if there exists $C \in \mathbb{R}$ and $r > 0$ such that for each ball $B_\rho(x)$ with $B_\rho(x) \cap \partial M_k = \emptyset$ and for all $\rho < r$

$$\forall x \in M. \forall k \in \mathbb{N}\colon |M_k \cap B_\rho(x)| < C$$

holds.

We want solve the Plateau problem to Γ_φ. The idea is: we truncate the infinite rays of Γ_φ to get a sequence of closed Jordan curves. Each of the Jordan curve possesses a convex orthogonal projection and by the well-known Rado's argument the Plateau problem is solvable. We then get a sequence of Plateau solutions. This sequence converges to a minimal graph as $r \to \infty$ if it satisfies the uniform curvature estimate. Finally by doing Schwarz reflections we extend the limit surface to a complete minimal surface bifurcating from a helicoid.

Notation 43. (**Boundary contour**) Let (r_n), $r_n > 0$ for all $n \in \mathbb{N}$ be a strictly monotonically increasing sequence with $\lim_{n\to\infty} r_n = \infty$. We truncate Γ_φ at the length r_n at $p_1(n) := (r_n, 0, 0)$ and $p_4(n) := (-r_n, -l\sin\varphi, l\cos\varphi)$. Now we connect $p_1(n), p_4(n)$ with an arc $\gamma_4(n)$ to get a sequence of closed curves $\Gamma_{n,\varphi} := \gamma_1(n) \cup \gamma_2(n) \cup \gamma_3(n) \cup \gamma_4(n)$, where

$$
\begin{aligned}
\gamma_1(n) &:= \{0 \le x \le r_n,\ y = z = 0\}, \\
\gamma_2(n) &:= \{x = 0,\ z = -\cot\varphi \cdot y\colon\ -l\sin\varphi \le y \le 0\}, \\
\gamma_3(n) &:= \{-r_n \le x \le 0,\ y = -l\sin\varphi,\ z = l\cos\varphi\}, \\
\gamma_4(n) &:= \left\{x = \lambda_n \cos t,\ y = \lambda_n \sin t - \frac{1}{2} l \sin\varphi,\ z = \frac{l\cos\varphi}{2\pi} t\colon\ 0 \le t \le \pi\right\}.
\end{aligned}
$$

and $\lambda_n^2 := r_n^2 + (\frac{1}{2} l \sin\varphi)^2$. The following properties of $\gamma_4(n)$ are true:

a) $\gamma_4(n)$ is a smooth curve with no self-intersection.

b) The arc $\gamma_4(n)$ is strictly monotone in the z direction.

The sequence $(\Gamma_{n,\varphi})$ is a sequence of Jordan curves.

Notation 44. (**Barrier**) Consider the following three half spaces

$$
\begin{aligned}
&H_1 = \{z \ge 0\}, \\
&H_2 = \{z \le l\cos\varphi\}, \\
&H_3 = \{v \in \mathbb{R}^3\colon w = (0, \cos\varphi, \sin\varphi), \langle v, w\rangle \ge 0\}
\end{aligned}
$$

Throughout the following let S_φ be the intersection of the three half spaces, which is a slab in $\mathbb{R}^3$ with an angle $\varphi \in [0, \frac{\pi}{2})$.

$$S_\varphi := H_1 \cap H_2 \cap H_3.$$

We denote by $S_{n,\varphi}$ the truncated slab obtained from S_φ by the following

$$S_{n,\varphi} := \left\{(x,y,z) \in \mathbb{R}^3\colon (x - \frac{1}{2} l \sin\varphi)^2 + y^2 \le \lambda_n^2,\ 0 \le z \le l\cos\varphi\right\} \cap S_\varphi$$

In the following section we will show that the Plateau solution $M_{n,\varphi}$ to the quadrilateral $\Gamma_{n,\varphi}$ exists as a graph over the plane perpendicular to the vector $(0, -\sin\varphi, \cos\varphi)$.

8.3 Convergence of a Sequence of Plateau Solutions.

Proposition 45. *To each truncated curve $\Gamma_{n,\varphi}$ there exists a minimal disk $M(n)$ which is an embedded immersion and has a non-parametric representation.*

Proof. We project the closed curve $\Gamma_{n,\varphi}$ to the plane P_φ in $\mathbb{R}^3$ which contains the origin and is perpendicular to γ_2. According to the construction of $\Gamma_{n,\varphi}$ in subsection 8.2.2 we get a convex component in P_φ. Then by § 401 [Nit75] each curve $\Gamma_{n,\varphi}$ bounds a Plateau solution $M_{n,\varphi}$ uniquely, which has a non-parametric representation. □

Because Γ_φ contains two infinite rays, we need to solve improper Plateau problems, i.e. problems with non-closed boundary curves with infinite length. We will show, the sequence of minimal surfaces obtained in Proposition 45 converges and the limit surface presents a Plateau solution for Γ_φ. To ensure the desirable convergence of the minimal graphs a theorem from [PR05] (Thm. 4.34) is useful.

Let $\Omega \subset \mathbb{R}^2$ be an open set and $u \in C^\infty(\Omega)$. The minimal surface equation is given by

$$(1+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)u_{yy} = 0 \tag{8.4}$$

Theorem 46. *Consider a sequence $\{u_n\} \subset C^\infty(\Omega)$ of solutions of the minimal surface equation, satisfying*

1. There exists $p \in \Omega$ such that $\{u_n(p)\}$ is bounded.

2. $\{|\nabla u_n|\}$ is uniformly bounded on each compact subset of Ω.

Then, there exists a subsequence $\{u_k\}$ and a solution $u \in C^\infty(\Omega)$ of the minimal surface equation such that $u_k \to u$ in the C^m-topology, for all m.

Proof. The result follows by Corollary 16.7 in [GT01] and Ascoli-Arzela's Theorem. Take a subdomain $\Omega' \subset\subset \Omega$ such that $p \in \Omega'$. Then hypotheses 1 and 2 together with the Mean Value Theorem give that $\{\sup_{\Omega'} |u_n|\}$ is bounded. Corollary 16.7 in [GT01] gives that for all multi-index α, the sequence of partial derivatives $\{D_\alpha u_n\}$ is uniformly bounded in Ω'. In this situation, Ascoli-Arzela's Theorem implies that a subsequence of $\{u_n\}$ converges to a function $u \in C^\infty(\Omega')$ in $C^m(\Omega')$, for all m. A standard diagonal process using an increasing exhaustive sequence of relative compact domains gives a subsequence $\{u_k\} \subset \{u_n\}$ that converges to a function $u \in C^\infty(\Omega)$ in the C^m-topology in Ω, for all m. Clearly, u satisfies the minimal surface equation. □

We denote by

$$R_x(\varphi) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$$

the counterclockwise rotation through the angle φ about the x-axis. The new coordinates after this rotation are $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R_x(\varphi) \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Through the construction we have

$$S_\varphi = \bigcup_{n\in\mathbb{N}} S_{n,\varphi}, \quad \Gamma_{n,\varphi} \subset S_{n,\varphi}, \quad \Gamma_{k,\varphi} \cap S_{n,\varphi} = \Gamma_\varphi \cap S_{n,\varphi} \text{ for all } k > n$$

By Proposition 45 the Plateau solution $M_{n,\varphi}$ to the quadrilateral $\Gamma_{n,\varphi}$ exists and is a graph $(x', y', u_{n,\varphi}(x', y'))$ over the plane P_φ which is the $x'y'$-plane in the new coordinates. Here we have $u_{n,\varphi} \in C^2(D^+) \cap C^0(D^+ \cup I)$. Our setting $\{\Gamma_{n,\varphi}, \Gamma_\varphi, S_{n,\varphi}, S_\varphi\}$ satisfies the requirements of Theorem 46. With the barriers $S_{n,\varphi}$ we can control the boundary behavior of sequence of minimal graphs. Thus the Plateau problem for Γ_φ is solvable and we have

Theorem 47. *Let D^+ be the open upper half-disc in the $x'y'$-plane and let I be the unit interval $I := \{(x',0) : x' \in (-1,1)\}$. Let $u_{n,\varphi}(x',y') \in C^2(D^+) \cap C^0(D^+ \cup I)$ be the Plateau solution for $\Gamma_{n,\varphi}$ introduced in Notation 43. Then there exists a subsequence $(u_{k,\varphi})$ such that $u_\varphi := \lim_{k\to\infty} u_{k,\varphi}$ and the normal vector of the limit surface at infinity is $(0, \sin\varphi, \cos\varphi)$ (which is $(0,0,1)$ in the xyz-coordinates).*

Proof. (idea of F. Tomi). Since each $\Gamma_{n,\varphi}$ is compact and its Plateau solution is contained in the compact set $S_{n,\varphi}$, requirement 1 of Theorem 46 is satisfied. With Corollary 16.7 in [GT01] we have the interior gradient bound as required in Theorem 46. Thus there exists a convergent subsequence $(u_{k,\varphi})$ with limit $u_\varphi := \lim_{k\to\infty} u_{k,\varphi}$. Then $u_\varphi \in C^2(D^+) \cap C^0(D^+ \cup I)$ solves the minimal surface equation. Let M_φ be the Plateau solution represented by $(x', y', u(x', y'))$. Pick a point $p := (x'_p, y'_p, z'_p) \in M_\varphi$. Let us assume $x'_p \leq 0$. In this case we extend the surface M_φ by doing Schwarz reflection across γ_3. The extended minimal surface M^2_φ is also a minimal graph over the $x'y'$-plane and hence stable. Let $r = \sqrt{(x'_p)^2 + (y'_p)^2}$. Then the open geodesic disc $B(p,r)$ with center p and radius r does not touch γ_2. Moreover the Euclidean distance of any point on γ_1 (the positive x-axis) and the point (x'_p, y'_p) (the projection of the point p onto the $x'y'$-plane) is strictly greater than r. Thus the geodesic disc $B(p,r)$ in M^2_φ does not touch any boundary arc of M^2_φ. For $x'_p > 0$ we extend the surface M_φ across γ_1 and the same argument works. By pushing p to infinity the x'-coordinate of the point p may be either $x'_p > 0$ or $x'_p \leq 0$ and the geodesic disc $B(p,r)$ may have common points either with γ_1 or with γ_3. In such cases we extend the surface M_φ by doing Schwarz reflection across γ_1 or γ_3, the extended surface M^2_φ is also a minimal graph and hence stable. Then by Heinz's curvature estimate result [Hei52] for stable minimal surfaces there is an absolute constant C such that $|A|(p) \leq \frac{C}{r^2}$, where A is the second fundamental form of M_φ. As $p \to \infty$ the radius $r \to \infty$ and Heinz's curvature estimate gives $|A|(p) \to 0$. That means, the larger the geodesic disc $B(p,r)$ is, the less the normal vector (or the tangent plane T_pM_φ) of M_φ oscillates on the geodesic disc. The surface M_φ is contained in the barrier slab S_φ with finite thickness $l\cos\varphi$ and the normal vector of the two parallel horizontal half-planes of S_φ is $(0, \sin\varphi, \cos\varphi)$. If the asymptotically normal vector of M_φ is not equal to $(0, \sin\varphi, \cos\varphi)$, there would be a sequence of points (p_n) with $\lim_{n\to\infty} p_n = \infty$ such that the normal vector $\nu(p_n) \to \nu_1 \neq (0, \sin\varphi, \cos\varphi)$ as $n \to \infty$. By the uniform graph lemma 4.4 in [KKS89] there exists for each p_n a $r^*_n > 0$ such that the geodesic disc $B(p_n, r^*_n) \subset M_\varphi$ (or M^2_φ) is a graph above the tangent plane $T_{p_n}M_\varphi$ of a height function u, such that $\lim_{n\to\infty} u = 0$. Hence sufficient large geodesic discs $B(p_n, r^*_n)$ would exceed the constant barrier height $l\cos\varphi$ (or $2l\cos\varphi$). This is a contradiction to $M_\varphi \subset S_\varphi$. Therefore the surface M_φ has the asymptotically normal vector $(0, \sin\varphi, \cos\varphi)$. □

For $\varphi \in [0, \frac{\pi}{2})$ extending the surface M_φ by Schwarz reflections across the finite boundary arc γ_2 we get a regular complete minimal surface $\tilde{M}_\varphi$ bounded by two parallel straight lines. As shown in Fig. 8.4 the surface $\tilde{M}_\varphi$ is contained in a union of two slabs with a common face.

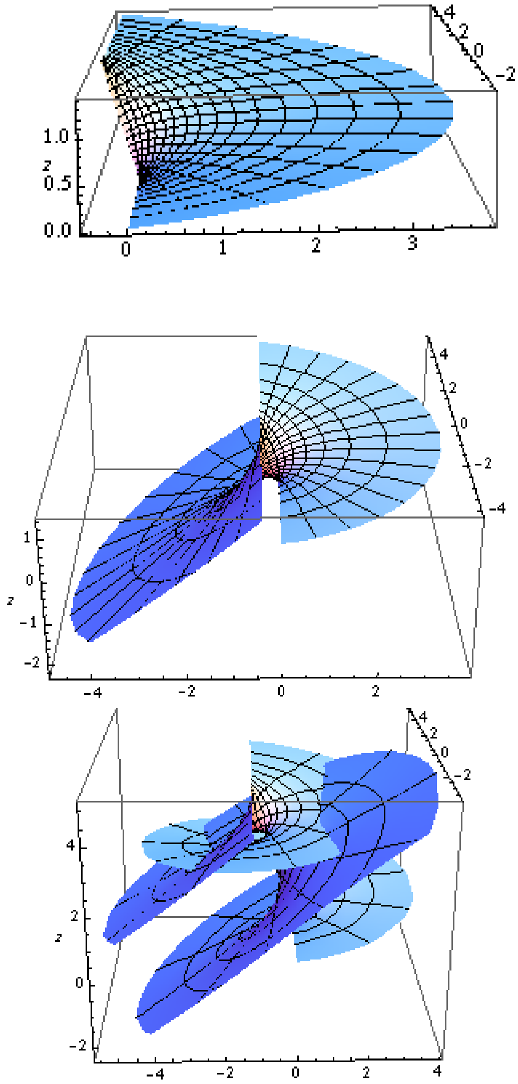


Figure 8.3: In the figure the first picture shows a part of M_φ, $\varphi = \frac{\pi}{6}$, the second one shows the resulting surface by geodesic reflection across γ_2 and finally the third one shows the reflection about γ_1 and γ_2.

8.4 Regularity of Extending by Schwarz Reflection

The surface M_φ constructed in the previous section can be analytically continued to a complete minimal immersion across each of its boundary arcs $\gamma_1, \gamma_2, \gamma_3$ by Schwarz reflections. After two times successive reflections at a vertex, say p_2, we return to the original surface M_φ. Hence the Schwarz reflections generate an analytic minimal surface near p_2 with possible branch points at the boundary γ_1. It remains to show that the surface produced by Schwarz reflections is in fact non singular on the boundary. The smooth local extension by Schwarz reflections 39 allows us to define that $p \in \partial B$ is a *boundary branch point* if the extended conformal map satisfies $df_p = 0$.

Definition 48. We say that a plane $P \subset \mathbb{R}^3$ is a *barrier* for the set $M \subset \mathbb{R}^3$ at $p \in M$, if $p \in P$ and M does not meet one of the two connected components of $\mathbb{R}^3 \setminus P$.

For any rational angle with $\alpha = \frac{p}{q}\pi$ with p, q reduced we must reflect $2q$ times for the surface to close. However, we necessarily create a branch point: the tangent plane will be covered p times! Since by the construction each edge is contained in a barrier (a Euclidean plane), we assert that our Plateau solution M_φ has only good angles, which admits extensions without branch points at the boundary.

Lemma 49. *There are no boundary branch points on M_φ.*

Proof. The straight line arc γ_1 is contained in the plane P_{12} determined by γ_1 and γ_2. Taking P_{12} as a barrier, by Schwarz reflection principle there are no branch points at the interior of γ_1. The same argument works for γ_3.

To exclude branch points on the vertices and on γ_2 we apply a curvature estimate result of [GBK10] Prop. 4.4, which requires that, along the vertical arcs, the dihedral angle of the barrier must be strictly smaller than π.

We place the boundary curve Γ_φ together with S_φ such that γ_2 is vertical in z-direction, i.e. the barrier S_φ is inclined to the xy-plane at an angle $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$. On γ_2 we consider a quarter standard helicoid H_1 bounded by Γ_φ. Then H_1 is a upper barrier of M_φ. Restricted at each compact segment of γ_2 the surface H_1 meets S_φ at an angle $< \pi - \varepsilon$. Therefore the curvature of M_φ is bounded along compact segments of γ_2.

It remains to exclude branch point at the vertices p_2 and p_3. In vertex p_3 the curvature estimate of [GBK10] works. Because at p_3 the surface H_1 meets S_φ tangentially with the common edge γ_3. In vertex p_2 we consider a Plateau solution $M_{\varphi+\varepsilon}$ with $(\varphi+\varepsilon) < \frac{\pi}{2}$ as a lower barrier. Since the normal vector of $M_{\varphi+\varepsilon}$ rotates along γ_2 monotonically from p_2 to p_3, in which the angle turns from 0 to π. Therefore on each compact segment of γ_2 in $M_{\varphi+\varepsilon}$, which starts at p_2, the angle between $M_{\varphi+\varepsilon}$ and S_φ is strictly smaller than π. Hence the approach of [GBK10] is applicable. $\square$

Finally we remark that the new minimal surfaces are different from the helicoid we started from. The arc γ_2 makes an angle φ to the $z-$axis and the surface M_φ is contained in the wedge W_φ. The asymptotically normal of M_φ is $\nu_0 = (0, 0, 1)$ and the asymptotically normal vector of its mirror image is $\nu_1 = (0, \sin 2\varphi, \cos 2\varphi)$.

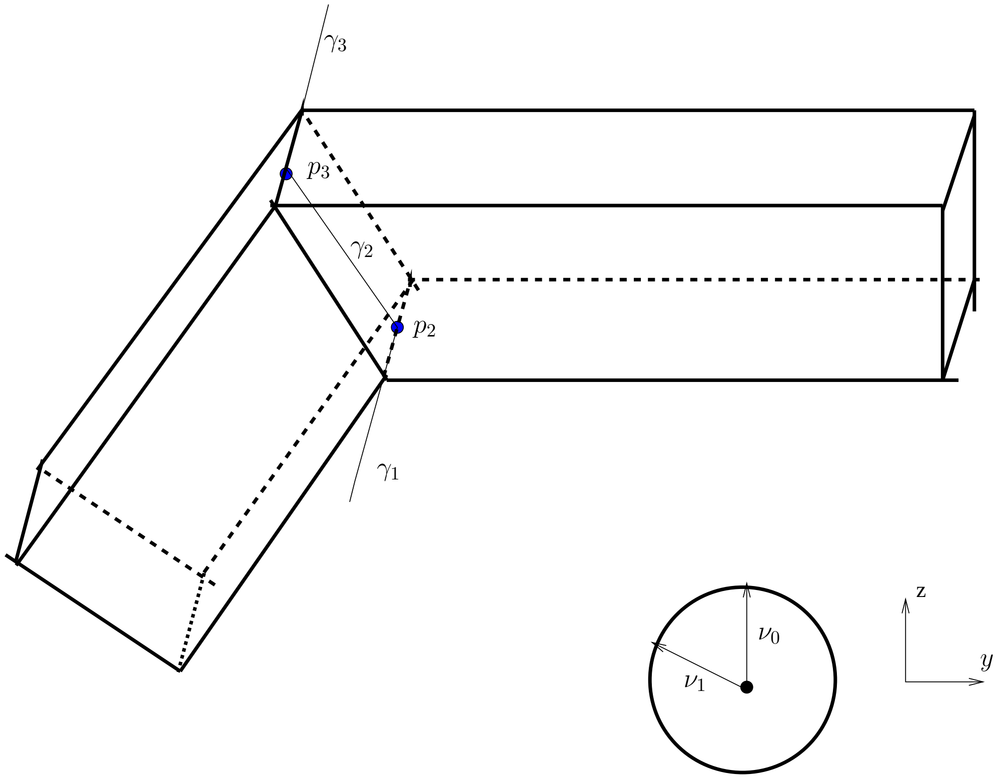


Figure 8.4: Wedge as barrier

Remark 50. In [Tom09] F. Tomi showed the existence of Plateau's solution to a certain class of properly embedded unbounded curves, which is valid in particular for the boundary Γ_φ we treat in our work.

8.5 Bifurcating Helicoids

Theorem 51. *(Main Theorem 2.) There is a one-parameter family M_φ of properly immersed minimal surface in $\mathbb{R}^3$, where $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ with the following properties:*
[1.] Each surface M_φ is singly periodic.
[2.] For $\varphi = 0$ the minimal surface M_φ is a standard helicoid.
[3.] For each surface it is valid: $M_\varphi = M_{-\varphi}$.
[4.] For $|\varphi| \to \frac{\pi}{2}$ the minimal surface M_φ tends to a plane with multiplicity.
[5.] The fundamental patch of each surface M_φ has an asymptotic normal vector $(0, 0, 1)$.

Proof. (1) **Periodicity.** Since each reflection is an isometry in $\mathbb{R}^3$, after one Schwarz reflection across γ_2 we get a minimal surface bounded by two parallel straight lines containing γ_1, γ_3 respectively: Let $M^*_{k,\varphi}$, S^*_φ denote the mirror image of $M_{k,\varphi}$, S_φ(Notaion44) by reflection across γ_2 as shown in the Figure (8.4). The surface $M^*_{k,\varphi}$ is contained in a wedge of a slab S^*_φ, which makes an angle $\pi - 2\varphi$ with S_φ and the two wedges meet at the strip determined by the two straight lines containing γ_1 and γ_3 respectively. These two parallel straight lines are orthogonal to γ_2, therefore by successive reflections across these two straight lines infinitely many times we get a complete minimal surface with period in the γ_2 direction. With the regularity results in Sect. 8.4 we can exclude the branch points after extending the minimal surface by Schwarz reflections.

(2) For $\varphi = 0$ the Plateau solution to each truncated curve $\Gamma_{n,\varphi}$ obtained in Proposition 45 is a helicoid piece. Thus the Plateau solution to Γ_φ is a fundamental piece of the standard helicoid.

(3) From the construction of mean convex sets in Notation 44 we can see that changing the sign of φ only carries the Plateau solution from a left-handed one to a right-handed one.

(4) As $\varphi \to \frac{\pi}{2}$ the wedge slab S_φ degenerates to the half xy -plane and the limiting boundary curve Γ_φ tends to be completely included in the xy -plane. The Plateau solution converges to a piece of xy- plane. Extending the Plateau solution (also a piece of plane) by Schwarz reflection across its boundary gives a plane with multiplicity.

(5) Lemma 47. □

8.6 Families of CMC-1 Surfaces in $\mathbb{H}^3$

As we mentioned in the introduction, differently scaled helicoids in $\mathbb{R}^3$ lead to non-congruent catenoid cousins in $\mathbb{H}^3$. We call the conjugate curve of the helicoid's axis the *waist circle* of the catenoid cousin. It is a geodesic and a line of curvature. We define the necksize of a catenoid cousin to be the circumference of the waist circle.

Corollary 52. *For every $m \in \mathbb{N}$ with $m \geq 2$ there is a one-parameter family $M^*_{m,\varphi}$ of CMC-1 surfaces in $\mathbb{H}^3$, where $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$, with the following properties:*

(1) [Bifurcation] The surface $M^*_{m,0}$ is a catenoid cousin with necksize $n = 2(m-1)\pi$.

(2) [Limit case] Fixing the symmetry of the surfaces $(M^*_{m,\varphi})$ as $\varphi \to \frac{\pi}{2}$, each surface $M^*_{m,\varphi}$ converges to a horosphere.

Proof. (1) By Def.38 the catenoid cousins are the conjugate CMC-1 surface in $\mathbb{H}^3$ to helicoids. Let us consider the immersed catenoid cousins (i.e. with self-intersection). All these surfaces are surfaces of revolution [Bry87]. In Poincaré ball model of hyperbolic geometry we may assume that the geometry center of these surfaces is the midpoint of the unit ball. Consider a catenoid cousin with necksize radius $\sinh r > 0$. Let $\theta = \frac{\pi}{m}$, where $m \in \mathbb{N} \setminus \{1\}$. We take a fundamental piece of the catenoid cousin, its boundary is a non-proper triangle in $\mathbb{H}^3$ with two vertices on the waist circle in the symmetry plane and one vertex N at the asymptotic boundary $\partial\mathbb{H} = \mathbb{S}^2$ (see Fig.8.5).

The length l_0 of the geodesic γ_2^* satisfies

$$l_0 = \sinh r \cdot \theta. \tag{8.5}$$

Let the normal curvature along the arc γ_2^* be κ^*. By Equation 3.5 we thus have

$$\kappa^* = \tau - 1.$$

The tilting angle of the normal along a symmetric arc of a CMC-1 surface in $\mathbb{H}^3$ is θ and we have $\theta = \int_{\gamma_2^*} \kappa^*$. Let ω be the angle between the conormals at the end points of the conjugate arc in $\mathbb{R}^3$. We have $\omega = \int_{\gamma_2} \tau$. From the construction we know the angle $\omega = \pi$. As already mentioned in Equation 3.7 we have an interesting relation

$$\theta = \omega - l_0.$$

Together with Equation (8.5) we get

$$\theta + \sinh(r)\theta \quad = \quad \omega$$

which gives

$$\frac{\pi}{m}(1 + \sinh r) = \pi \Rightarrow \frac{1 + \sinh r}{m} = 1 \Rightarrow \sinh r = m - 1 \tag{8.6}$$

and therefore

$$l_0 = \theta \cdot \sinh r = \frac{m-1}{m}\pi.$$

After $2m$ reflections across the infinite arcs successively the CMC-1 surface $M_{m,\varphi}^*$ is closed. That means the necksize is

$$n = 2(m-1)\pi.$$

(2) Fixing the symmetry of CMC-1 surface means in $\mathbb{R}^3$ fixing the period axis. As $\varphi \to \frac{\pi}{2}$ the length of the period axis tends to infinity. At the limit bifurcating helicoids degenerate to planes with multiplicity. Their CMC-1 cousins are horospheres which are congruent under scaling. Every horosphere touches tangentially the asymptotic boundary $\partial\mathbb{H} = \mathbb{S}^2$ at exactly one point. $\square$

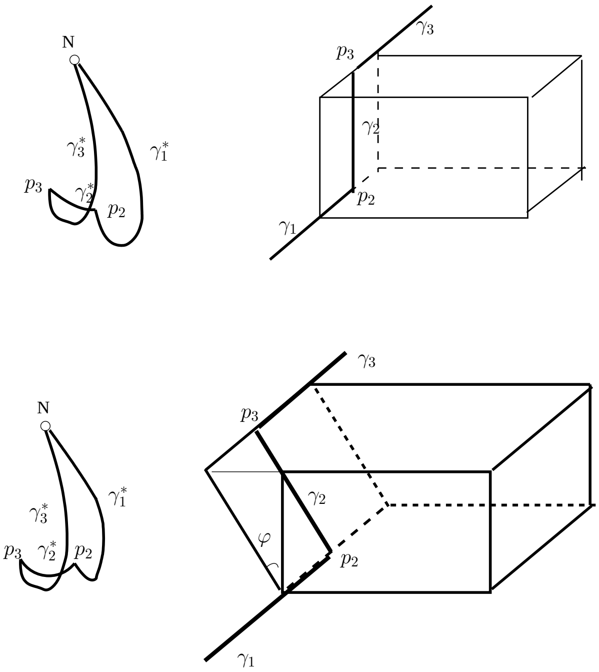


Figure 8.5: Boundaries of a helicoid piece and its catenoid cousins. The upper pair is the boundary of a fundamental patch of a catenoid cousin with necksize 2π (i.e. $m = 3$) and its conjugate boundary in $\mathbb{R}^3$. The lower pair is the boundary of a bifurcating helicoid (on the right) and its conjugate boundary in $\mathbb{H}^3$ (on the left).

8.6.1 Some final remarks

Remark 53. The genus zero CMC-1 surfaces in $\mathbb{H}^3$ are classified in [UY93](Theorem 6.2) and [RUY03](Theorem 2.1): A complete conformal CMC-1 immersion with finite total curvature and two regular ends is either a finite cover of a catenoid cousin, or a finite cover of a warped catenoid cousin with the data

$$g = \frac{\delta^2 - l^2}{4l} z^l + b, \quad w = z^{-l-1} dz$$

where $l, \delta \in \mathbb{Z}^+$, $l \neq \delta$, and $b \geq 0$.
If $b = 0$, the surface (g, ω) is just a $\delta-$fold cover of a catenoid cousin. If $b > 0$, the surface f has the symmetry group of f is $D_l \times \mathbb{Z}^2$. Thus the warped catenoid cousins are bifurcating surfaces from catenoid cousins.

In [ET01] Sa Earp and Toubiana studied symmetry properties of CMC-1 surfaces in $\mathbb{H}^3$. In [ET04] they showed: Every non-totally umbilic conformal CMC-1 immersion $X \colon U \to \mathbb{H}^3$ of a simply connected domain $U \subset \mathbb{C}$ into the half-space model of the hyperbolic three-space gives rise to two meromorphic data (h, T) defined on U that completely describe X. Using the meromorphic data (h, T) they give the families of warped catenoid cousins independently of Rossman-Umahara-Yamada.

Remark 54. This gives us reason to conjecture that the CMC-1 families in $\mathbb{H}^3$ we have obtained by using conjugate surface construction with bifurcating helicoids are the warped catenoids cousins. As we know the rate of convergence of the bifurcating minimal surfaces is linear, it remains unknown if the rate of convergence is invariant under Lawson correspondence. For that we need to know more about the estimation of the coefficients in the Gauss and Codazzi equations in $\mathbb{H}^3$.

Remark 55. We believe that the one-parameter families of minimal surfaces in Theorem 51 are continuous in φ. To show this, we could use the maximum principle for minimal surfaces at infinity to show that each Plateau solution M_φ is unique to the given boundary Γ_φ.

Appendix

In this appendix we show the auxiliary calculations of Sect. 4.1.

A.0 Some Topological Definitions

Definition 56. A *fiber bundle* consists of the data (E, B, Π, F), where E, B and F are topological spaces and $\Pi\colon E \to B$ is a continuous surjection satisfying a *local triviality condition*: for each point $b \in B$ there is a neighborhood U and a homeomorphism $h\colon \Pi^{-1}(U) \to U \times F$, such that $\Pi(x) = pr_1 \circ h(x)$ for each $x \in \Pi^{-1}(U)$, where $pr_1\colon U \times F \to U$ is the projection to the first component.

The space B is called the base space of the bundle, E the total space, and F the fiber. The map Π is called the projection map.

Every fiber bundle $\Pi\colon E \to B$ is an open map, since projections of products are open maps.

Definition 57. A *principal G-bundle* is a fiber bundle $\Pi\colon E \to B$ together with a continuous right action $E \times G \to E$ by a topological group G such that G preserves the fibers of E and acts freely and transitively on them.

Lemma 58. *Any fibre bundle over a contractible space is trivial. A principal bundle has a global section if and only if it is trivial.*

Definition 59. Given two spaces X and Y, we say they are *homotopy equivalent* or of the same homotopy type if there exist continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $g \circ f$ is homotopic to the identity map id_X and $f \circ g$ is homotopic to id_Y.

A.1 The Calculation of Equation 4.5

Set $v_a = \langle N, e_a \rangle_{\mathbb{R}^{m+1}}$ we write the differential of x and h as 1-forms

$$dx = \sum_a \omega_a e_a, \quad \text{and} \quad dh = \sum_a h^2 v_a \omega_a$$

Since by definition

$$h(x) = \frac{1}{1 - \langle x, N \rangle}$$

the differential of h is

$$\begin{aligned} dh &= \frac{\langle dx, N \rangle}{(1 - \langle x, N \rangle)^2} \\ &= h^2 \langle dx, N \rangle \\ &= h^2 \langle \sum_a \omega_a e_a, N \rangle \\ &= h^2 \sum_a \omega_a v_a. \end{aligned}$$

A.2 The Calculation of Equation 4.7

For sake of simplicity we write $d\pi$ for the evaluated differential form $d\pi_p(X)$ at the point $p \in S$ and a tangential vector $X \in T_pS$. We want to show that Equation 4.7 is valid:

$$\langle d\pi, d\pi \rangle_{\mathbb{R}^{m+1}} = \sum_a h^2 \omega_a^2$$

Evaluating $d\pi$ by Equation 4.6 we have

$$\begin{aligned}
\langle d\pi, d\pi\rangle_{\mathbb{R}^{m+1}} &= \langle\sum_a h^2 v_a\omega_a(x-N) + \sum_b h\omega_b e_b, \sum_a h^2 v_a\omega_a(x-N) + \sum_b h\omega_b e_b\rangle \\
&= \langle\sum_a h^2 v_a\omega_a(x-N), \sum_a h^2 v_a\omega_a(x-N)\rangle \\
&\quad +2\langle\sum_a h^2 v_a\omega_a(x-N), \sum_b h\omega_b e_b\rangle + \langle\sum_b h\omega_b e_b, \sum_b h\omega_b e_b\rangle \\
&= h^4\sum_a v_a\omega_a\sum_b v_b\omega_b\langle x-N, x-N\rangle + 2\sum_a h^3 v_a\omega_a\langle(x-N), \sum_b \omega_b e_b\rangle + \sum_a h^2\omega_a^2 \\
&= h^4\sum_a v_a\omega_a\sum_b v_b\omega_b\frac{2}{h} + 2\sum_a h^3 v_a\omega_a\sum_b \omega_b\langle(x-N), e_b\rangle + \sum_a h^2\omega_a^2 \\
&= 2h^3\sum_a v_a\omega_a\sum_b v_b\omega_b + 2\sum_a h^3 v_a\omega_a\sum_b \omega_b\langle -N, e_b\rangle + \sum_a h^2\omega_a^2 \\
&= 2h^3\sum_a v_a\omega_a\sum_b v_b\omega_b - 2\sum_a h^3 v_a\omega_a\sum_b \omega_b v_b + \sum_a h^2\omega_a^2 \\
&= \sum_a h^2\omega_a^2
\end{aligned}$$

A.3 The Calculation of Equation 4.15

Here we want to show the following equation

$$\beta_{rij} = \frac{1}{h}b_{rij} - v_r\delta_{ij}.$$

We apply the Equation 4.13, 4.14 in Equation 4.12 and we have

$$\sum_i \beta_{rij}\theta_i = \sum_i b_{rij}\omega_i - v_r\theta_j$$

Remember the definition $\theta_i = h\omega_i$ therefore

$$\sum_i \beta_{rij}\theta_i = \sum_i \frac{b_{rij}}{h}\theta_i - v_r\theta_j = (\sum_i \frac{b_{rij}}{h} - v_r\delta_{ij})\theta_i$$

A.4 The Calculation of Equation 4.16

We want to show

$$\frac{1}{h}\langle f_r, Y\rangle = v_r.$$

A direct calculation shows:

$$\begin{aligned} f_r &= \frac{1}{h} d\pi(e_r) \\ &= \frac{1}{h}(h^2 v_r(y-A) + he_r) \\ &= hv_r(y-A) + e_r \end{aligned}$$

and

$$\begin{aligned} \frac{1}{h}\langle f_r, Y\rangle &= \frac{1}{h}\langle hv_r(y-A) + e_r, A + h(y-A)\rangle \qquad \text{(by (4.1))} \\ &= \frac{1}{h}(h^2 v_r|y-A|^2 + hv_r\langle y, A\rangle - hv_r + \langle e_r, A\rangle + h\langle e_r, y\rangle - h\langle e_r, A\rangle)) \\ &= \frac{1}{h}(2hv_r + hv_r(1-\frac{1}{h}) - hv_r + v_r + 0 - hv_r)\text{(by (4.3) (4.2))} \\ &= v_r \end{aligned}$$

A.5 The Hyberbolic 3-space

The Hyperboloid Model

The Lorentz space L^4 is $\mathbb{R}^4$ together with the standard symmetric bi-linear form of signature $(3,1)$:

$$\langle x, y\rangle_L = \sum_{i=1}^{3} x_i \cdot y_i - x_0 \cdot y_0$$

The set $\langle x, y\rangle_L = -1$ is geometrically a two sheeted hyperboloid in $\mathbb{R}^4$. We define the hyperbolic 3-space $\mathbb{H}^3$ to be the upper sheet of this hyperboloid:

$$\mathbb{H}^3 := \{x \in \mathbb{R}^4 \colon \langle x, y\rangle_L = -1 \text{ und } x_0 > 0\}.$$

Let x, y be unit tangent vectors in $\mathbb{H}^3$ and let $\nu(x,y)$ be the Lorentzian time-like angle between x and y. The hyperbolic distance between x and y is defined to be the real number

$$d_H(x,y) := \nu(x,y).$$

As $\langle x, y\rangle_L = |x|_L |y|_L \cosh \nu(x,y)$, we have the equation

$$\cosh d_H(x,y) = -\langle x, y\rangle_L.$$

The real function d_H is a Riemannian metric and with that $\mathbb{H}^3$ is a Riemannian manifold.

The Poincaré Disk Model

Using stereographic projection of the upper sheet of the hyperboloid to the unit disk in $\{x_0 = 0\}$ from the point $(-1, 0, 0, 0) \in \mathbb{R}^4$ yields the Poincaré disk model of $\mathbb{H}^3$. Thus the Poincaré Disk model is the 3- dimensional unit ball

$$B^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \colon x_1^2 + x_2^2 + x_3^2 < 1\}$$

with the metric

$$g_{ij} = \frac{4}{(1 - |x|^2)^2}\delta_{ij}, \quad i, j \in \{1, 2, 3\}.$$

This metric is conformal to the Euclidean metric and so angles are the same as the Euclidean angles. With this metric the unit ball is complete, simply-connected, and has constant sectional curvature -1. The asymptotic boundary $\partial\mathbb{H}^3$ is the unit 2-sphere $\mathbb{S}^2$. The geodesics in the Poincaré model are segments of Euclidean lines and circles that intersect $\partial\mathbb{H}^3$ orthogonally. The CMC hyperbolic planes with $H = 0$ are the intersections of B^3 with Euclidean spheres and planes which meet $\partial\mathbb{H}^3$ orthogonally. The horospheres are the Euclidean spheres, which are contained in B^3 and tangent to $\partial\mathbb{H}$ at one point.

The Upper Half-Space Model

Now we want an orientation preserving isometry from B to the upper-half space model $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \colon x_3 > 0\}$.
First do an inversion in the sphere of radius $\sqrt{2}$ centered at $(0, 0, 1)$:

$$i \colon B^3 \to \mathbb{R}^3, \quad p \mapsto \frac{2(p - (0, 0, 1))}{||p - (0, 0, 1)||^2} + (0, 0, 1).$$

This is however orientation reversing and takes B^3 to the lower half-space. Thus we compose this map with the reflection map

$$r \colon \mathbb{R}^3 \to \mathbb{R}^3, \quad (x_1, x_2, x_3) \mapsto (x_1, x_2, -x_3)$$

to obtain the orientation preserving isometry.

The boundary $\partial\mathbb{H}^3$ in this model is $\{x_3 = 0\} \cup \{\infty\}$.

The geodesics are semi-circles intersecting $\{x_3 = 0\}$ orthogonally, or rays parallel to the x_3-axis starting in the hyperplane $\{x_3 = 0\}$. The horoshperes are planes contained in $\mathbb{H}^3$ which are parallel to $\{x_3 = 0\}$ or spheres contained in $\mathbb{H}^3$ which is tangent to $\{x_3 = 0\}$ at one point. The conformal metric is given by

$$g_{ij} = \frac{4}{x_3^2}\delta_{ij}, \quad i, j \in \{1, 2, 3\}.$$

The Hermitian Model

We get the hermitian model by identifying L^4 with the hermitian symmetric 2×2 matrices by identifying (x_1, x_2, x_3, x_4) with

$$v = \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix}$$

Then $\mathbb{H}^3$ is the set of such matrices v with $\det v = 1$. The complex Lie group $SL(2, \mathbb{C})$ acts naturally on L^4 by

$$v \mapsto gvg^*$$

where we regard v as a 2×2 Hermitian symmetric matrix as above and $g^* = \overline{g}^t$. Since $\det v = -\langle v, v \rangle$ this action preserves the inner product and leaves $\mathbb{H}^3$ invariant. The isometry group of $\mathbb{H}^3$ is $PSL(2, \mathbb{C})$. The map

$$SL(2, \mathbb{C}) \to Herm(2), \quad F \mapsto FF^*$$

takes its values in $\mathbb{H}^3$.

The catenoid cousins

In his work [Bry87] Robert Bryant found a representation for CMC-1 surfaces in $\mathbb{H}^3$. This representation is similar to the Weierstrass representation for minimal surfaces in $\mathbb{R}^3$, in that it also produces surfaces from a meromorphic function g and a holomorphic 1-form ω on a Riemann surface. The representation $g = z^\mu$, $\omega = \frac{1-\mu^2}{4\mu} z^{-\mu-1} dz$ describes all CMC-1 surfaces of revolution, which are called *catenoid cousins*. These surfaces form a one parameter family, depending on a parameter $\mu \in (-\frac{1}{2}, 0) \cup (0, \infty)$. When μ is negative, the surface is embedded. When μ is positive, the surface has self-intersections. As μ converges to zero, the surfaces converge to two horopheres that are tangent at one point (Horospheres also have mean curvature 1). The catenoid cousins are the only CMC-1 surfaces of revolution in $\mathbb{H}^3$.

Lawson-correspondence in $\mathbb{H}^3$

Denote by $M^3(K)$ the simply connected space form of curvature K. As we have already discussed in early section, the Gauss and Codazzi equations 3.1 are the integrability conditions which guarantee the existence of an immersion $f \colon M^2 \to M^3(K)$ from a simply connected Riemannian surface M^2 in $M^3(K)$ with given first fundamental form g and shape operator S. Assume that f has constant mean curvature $H = \operatorname{tr}(S)$. For a constant $c \in \mathbb{R}$ we define

$$S^* = J \circ S - c \cdot id, \quad \overline{K} = K - 2c \operatorname{tr}(S) - c^2.$$

Set $g^* = g$. We can check that (g^*, S^*) satisfies the Gauss and Codazzi equations in $M^3(\overline{K})$. Thus there exists an immersion $f^*\colon M^2 \to M^3(\overline{K})$ with the metric g^* and shape operator S^*. The mean curvature H^* of f^* is

$$H^* = H + c$$

where H is the mean curvature of f.

Set $H = K = 0$ and $c = 1$. Then we have the correspondence between minimal surfaces in $\mathbb{R}^3$ and CMC 1 surfaces in $\mathbb{H}^3$.

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