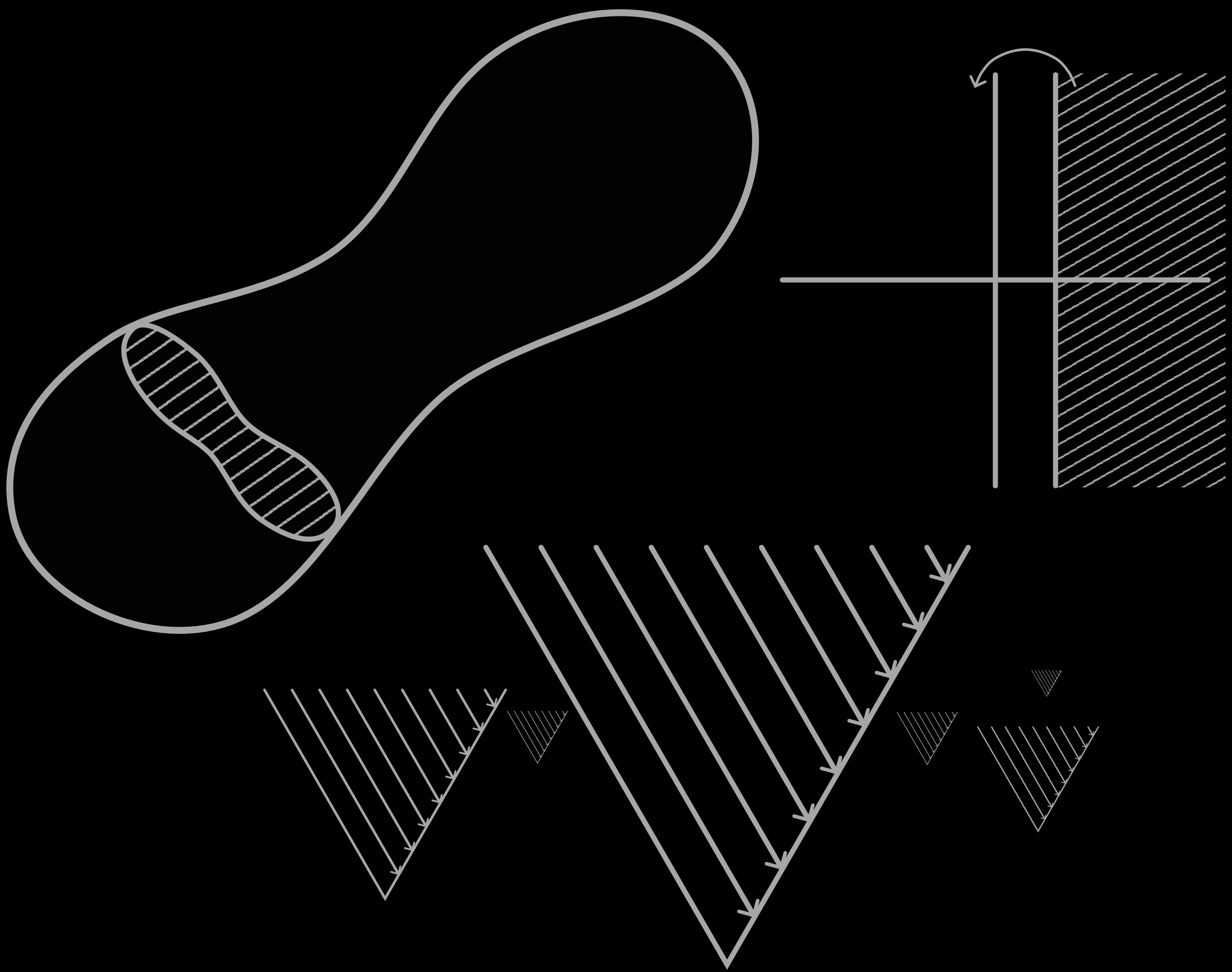
Period integrals of Kudla-Millson lifts and L-functions associated to vector valued modular forms



Ingmar Metzler

Cycling towards injectivity

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Period integrals of Kudla-Millson lifts and L-functions associated to vector valued modular forms

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For my parents and my beloved wife.

Abstract

The present thesis establishes a new converse theorem for Borcherds products. Moreover, the injectivity of the Kudla–Millson theta lift is demonstrated in the $\mathrm{O}(n, 2)$ case in greater generality than is currently available in the literature. Both results are derived under the assumption of a single hyperbolic split of the base lattice. Additionally, symmetric square type L-series associated to elliptic vector valued modular forms are examined and special values of these series are linked to cycle integrals of Kudla–Millson liftings.

For vector valued elliptic modular forms, asymptotic bounds for their Fourier coefficients are tailored to the specific setting and the structure of their index set with respect to Hecke actions is analysed. Subsequently, symmetric square type L-series are associated to these forms which are realised via a Rankin–Selberg integral and meromorphic continuation is concluded. Further, formulae for the operation of Hecke algebras are extended and proven to imply product expansions of the aforementioned L-series.

In the orthogonal setting, the construction of special divisors is reviewed, before integrals of Kudla–Millson liftings over these divisors are explicitly related to special values of the aforementioned L-series by means of the Siegel–Weil formula. The relations that emerge in this context are then exploited in order to derive the injectivity of the Kudla–Millson lift and, by means of a duality statement, a converse theorem for Borcherds products.

Zusammenfassung

In der vorliegenden Arbeit wird ein neuer Umkehrsatz für Borcherdsprodukte bewiesen. Außerdem wird die Injektivität des Kudla–Millsons lifts im $\mathrm{O}(n,2)$ Fall in größerer Allgemeinheit nachgewiesen als die aktuelle Literaturlage belegt. Beide Resultate werden unter der Annahme eines einzigen hyperbolischen Splits des zu Grunde liegenden Gitters gezeigt. Zusätzlich werden L-Reihen vom „symmetric square“ Typ, welche zu vektorwertigen Modulformen assoziiert werden, untersucht und spezielle Werte derselben mit Zykelintegralen von Kudla–Millson lifts in Verbindung gebracht.

Für vektorwertige Modulformen werden asymptotische Schranken ihrer Fourierkoeffizienten an den Rahmen der Arbeit angepasst und die Struktur ihrer Indizes im Hinblick auf die Wirkung von Hecke Algebren untersucht. Nachfolgend werden diesen Formen die oben benannten L-Reihen zugeordnet und als Rankin–Selberg Integrale realisiert, woraus eine meromorphe Fortsetzung gefolgert wird. Außerdem gelingt es, Formeln für die Wirkung von Hecke Algebren zu erweitern und daraus Produktentwicklungen der zuvor konstruierten L-Reihen zu berechnen.

Im ortogonalen Fall wird die Konstruktion spezieller Divisoren dargelegt, um anschließend Integrale von Kudla–Millson Lifts darüber unter zu Hilfenahme der Siegel–Weil Formel in Termen spezieller L-Werte auszudrücken. Die sich hieraus ergebenden Relationen werden ausgenutzt, um die Injektivität des Kudla–Millson Lifts zu zeigen und, mit Hilfe eines Dualitätsresultats, einen Umkehrsatz für Borcherdsprodukte zu folgern.

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Viele Menschen aus meinem Umfeld haben durch ihre fortwährende Unterstützung zu dieser Dissertation beigetragen. Ich möchte Euch allen gerne dafür danken!

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Contents

0 Introduction

A converse theorem for Borcherds products

In 1998, Richard Borcherds was awarded the fields medal in recognition of his pioneering contributions to the field of automorphic forms and mathematical physics [OR]. He had introduced vertex operator algebras as well as a multiplicative lift of automorphic forms and employed these tools to prove the Conway–Norton moonshine conjecture. The constructed lift is nowadays referred to as *Borcherds lift* and maps vector valued elliptic modular forms to orthogonal automorphic forms with infinite product expansions. These are also referred to as *Borcherds products* which provide a diverse array of examples of orthogonal modular forms, a fact that was instrumental in the jury's decision to bestow the fields medal upon Borcherds. In this thesis we prove a converse theorem for Borcherds products, demonstrating that all orthogonal modular forms that may a priori be targeted by the Borcherds lift are, in fact, in its range up to a nonzero constant factor.

To be more explicit, let (L, q) be an even lattice of signature $(m^+, 2)$ and $\mathbb{D}$ be the associated Grassmannian which carries a natural complex structure. The natural arithmetic subgroup to consider is the *discriminant kernel* $\Gamma(L) \leq \mathrm{O}(L)$ which is a finite index subgroup stabilising the discriminant group of L. On the elliptic side, let $\mathcal{L} = L'/L$ and inspect holomorphic modular functions $f : \mathbb{H} \to \mathbb{C}[\mathcal{L}]$ of weight $k \in \mathbb{Z}/2$, transforming with the dual Weil representation $\overline{\rho}_L$ which may possess poles at the cusp. This space is referred to as the space of *weakly holomorphic modular forms* and denoted by $\mathcal{M}^!_{L^-,k}$. Let $f \in \mathcal{M}^!_{L^-,k}$ and recall that such a function has a Fourier expansion

$$f(\tau) = \sum_{\mu \in L'/L} \sum_{m \in \mathbb{Z} - \mathrm{q}(\mu)} a(\mu, m) e^{2\pi i \tau m} \mathfrak{e}_\mu \tag{0.1}$$

where $\mathfrak{e}_\mu$ denotes the standard basis vector of $\mathbb{C}[\mathcal{L}]$ being 1 in the μ component and 0 elsewhere. Assume f has integral principal part, meaning that $a(\mu, m) \in \mathbb{Z}$ for all $m < 0$. Then the Borcherds lift associates to f a meromorphic modular form $\Psi(z, f)$ on $\mathbb{D}$ for $\Gamma(L)$ with unitary multiplier system of finite order such that

a) the weight of $\Psi(\,\cdot\,, f)$ is given by $a(0,0)/2$,

b) the divisor of $\Psi(\,\cdot\,, f)$ is equal to

$$\frac{1}{2}\sum_{\mu\in L'/L}\sum_{n>0} a(\mu,-n)Z(\mu,n), \tag{0.2}$$

where

$$Z(\mu,n) = \sum_{\substack{\lambda\in L+\mu\\ \mathrm{q}(\lambda)=n}} \{z\in\mathbb{D} \mid z\perp\lambda\}$$

is the *Heegner divisor* of discriminant (μ,n) (cf. Definition 4.70),

c) the target form $\Psi(\,\cdot\,, f)$ has explicit infinite product expansions.

As previously stated, this lift gives rise to a variety of new orthogonal modular forms and also to the monster denominator formula, which is utilised to prove the moonshine conjecture. In [Bor98, Pr. 16.10] Borcherds himself poses the question whether his lift may be reversed. That is to say, given a meromorphic modular form F for $\Gamma(L)$ whose zeros and poles are supported on special divisors $Z(\mu,n)$, is there a form $f\in\mathcal{M}^!_{L^-,k}$ such that $\Psi(z,f)$ equals F up to a constant factor? In this thesis we provide the following answer (cf. Theorem 7.19).

Theorem 0.1. *Assume that L splits a hyperbolic plane and $m^+>3$. Then every meromorphic modular form F with respect to $\Gamma(L)$ whose divisor is a linear combination of special divisors $Z(\mu,n)$ is (up to a nonzero constant factor) the Borcherds lift $\Psi(z,f)$ of a weakly holomorphic modular form $f\in\mathcal{M}^!_{L^-,1-m^+/2}$.*

The case of $m^+=3$ is also addressed in Section 7.3. More specifically, the same result holds for Witt rank 1 of the lattice L and a weaker version for Witt rank 2. The statement above improves upon the previously strongest result [Bru14, Thm. 1.2], in that only one hyperbolic split is required and no transition to a sublattice is needed. In order to prove Theorem 0.1, a method developed by Bruinier [Bru02] is utilised to characterise the above converse statement in terms of the injectivity of another theta lift, the *Kudla–Millson* lift.

The Kudla–Millson lift

In the 1980s Kudla and Millson [KM86] introduced special Schwartz forms φ_{KM} on the symmetric spaces attached to the classical groups $\mathrm{O}(m^+,m^-)$, $\mathrm{U}(m^+,m^-)$, and

$\mathrm{Sp}(m^+, m^-)$, taking values in closed differential forms. Their principal objective was to investigate cohomology classes of special cycles by means of a theta correspondence, generalising the celebrated work of Hierzebruch and Zagier [HZ76] on Hilbert modular surfaces. More precisely, in the orthogonal case, Kudla and Millson symmetrised the Schwartz forms φ_{KM} over a base lattice L and let the Weil representation of the symplectic group act on φ_{KM} in order to obtain a kernel (cf. Subsection 7.2.1 for details)

$$\Theta_{\mathrm{KM}}(\tau, z), \quad \tau \in \mathbb{H}, \quad z \in \mathbb{D}$$

in a symplectic and an orthogonal variable, which transforms automorphically in both variables. These may then be employed as an integral kernel to shift automorphic objects from the symplectic to the orthogonal setting or vice versa. Such an association is referred to as a *theta lift*. The authors continued their work in [KM87] in great generality and we recommend [KM90] and [BF04] as an introductory source. The applications of the Kudla–Millson theta correspondence range from the study of the cohomology of orthogonal and unitary Shimura varieties [KM86] and Arakelov theory of Shimura varieties [Kud04a] over specific counting problems [EGS23], constructing mock modular forms and higher dimensional error functions [FK17] to proving a converse theorem for Borcherds products [Bru02] [Bru14].

In this thesis, we consider the Kudla–Millson lift on the space of vector valued elliptic modular forms associated to the Weil representation ρ_L of an even lattice L of signature $(m^+, 2)$. To be more explicit, let $k = \mathrm{rank}(L)/2$ and $\mathcal{S}_{L,k}$ denote the space of cusp forms for the Weil representation ρ_L. Then

$$\Lambda_{\mathrm{KM}} : \mathcal{S}_{L,k} \to \mathcal{H}^{(1,1)}(Y_L), \quad f \mapsto \int_{\mathrm{Mp}_2(\mathbb{Z})\backslash\mathbb{H}} \langle f, \Theta_{\mathrm{KM}} \rangle \,\mathrm{Im}^k \;\mathrm{d}\mu$$

defines a linear map to the square integrable harmonic differential forms of Hodge type (1,1) on the variety $Y_L = \Gamma(L)\backslash\mathbb{D}$ and is referred to as the *Kudla–Millson lift*. Here μ denotes the hyperbolic measure on the upper half-plane $\mathbb{H}$ on which $\mathrm{Mp}_2(\mathbb{Z})$ acts via Möbius transforms. The question of its injectivity already arose in [KM90] and may be used to compute the rational Picard number of the underlying Shimura variety [Ber+16], as well as to derive properties of cones generated by special cycles [BM19] [Zuf22]. However, the main application we have in mind is the converse Theorem 0.1 for Borcherds products. There have been multiple results on the injectivity of the Kudla–Millson lift over the past two decades presented in [Bru02, Thm. 5.12 p. 139], [BF10, Cor. 4.11 p. 37], [Bru14, Thm. 5.3 p. 331], [Zuf24, Thm. 6.1 p. 22], and [Ste23, Cor. 7.8 p. 25]. These results are based on three fundamentally different methods which are sketched in Section 7.2. While there has been a recent success in unifying the results above by Zuffetti and the author [MZ23, Thm. 6.2 p. 24] by generalising Zuffetti's method, this has not shed any new light on the converse theorem for Borcherds products. The most potent result[1] for this application is Theorem 7.10 by Bruinier which provides injectivity under the assumption that the lattice L splits a hyperbolic plane as well as a scaled hyperbolic plane.

In the context of this thesis, we present a new approach to proving the injectivity of the Kudla–Millson lift by means of computing cycle integrals on the orthogonal variety. This approach eliminates the need for a scaled hyperbolic split and leads to the following result (cf. Theorem 7.16).

Theorem 0.2. *Let (L, q) be an even lattice of signature $(m^+, 2)$ with $m^+ > 3$. Assume that L splits a hyperbolic plane. Then the Kudla–Millson lift $\Lambda_{\mathrm{KM}} : \mathcal{S}_{L,1+m^+/2} \to \mathcal{H}^{(1,1)}(Y_L)$ is injective.*

The case of $m^+ = 3$ is also discussed in Subsection 7.3.2 and proven for the lattice L having Witt rank 1. It should be stressed that in the case of no hyperbolic split, Bruinier constructed a subspace of the kernel and demonstrated that it is in general non-trivial even in the simple instance of lattices of prime level [Bru14, Sec. 6.1].
The general proof strategy is to compute special cycle integrals of Kudla–Millson liftings $\Lambda_{\mathrm{KM}}(f)$ of cusp forms $f \in \mathcal{S}_{L,k}$. For suitably chosen cycles, these may be expressed in terms of special values of certain L-series $L(f, s)$ associated to the form f. If $\Lambda_{\mathrm{KM}}(f)$ vanishes, its cycle integral must vanish as well, yielding roots of the aforementioned L-series. As a consequence, a relation is established between the Fourier coefficients of f appearing in the construction of $L(f, s)$. This ultimately permits the deduction of their vanishing, which renders the initial form to vanish as well. Since the Kudla–Millson lift is linear, injectivity is inferred. Nonetheless, the appearing cycle integrals are interesting in their own right and we proceed with describing their computation below.

Cycle integrals

We describe how certain cycle integrals of Kudla–Millson liftings may be expressed in terms of special values of L-series. In order to construct these special cycles, note that for

[1] This result is, in fact, identical to [MZ23, Thm. 5.1 (i) p. 19] in this particular signature.

$\ell \in L$ of positive norm, there is a subvariety of the Grassmannian of the same type

$$\mathbb{D}_\ell = \{x \in \mathbb{D} \mid x \subseteq \ell^\perp\} \subset \mathbb{D}.$$

When considering the stabiliser Γ_ℓ of ℓ in $\Gamma(L)$,

$$\mathcal{Z}(\ell) := \Gamma_\ell \backslash \mathbb{D}_\ell \to \Gamma(L)\backslash \mathbb{D} = Y_L$$

defines a (in general relative) cycle of Y_L. Completing $\Lambda_{\mathrm{KM}}(f)$ with an adequate power of a Kähler form Ω allows for integration over $\mathcal{Z}(\ell)$:

$$\int_{\mathcal{Z}(\ell)} \Lambda_{\mathrm{KM}}(f) \wedge \Omega^{m^+-2}.$$

For the sake of simplicity, assume throughout the introduction that $L = \mathbb{Z}\ell \oplus K$ splits for some lattice K of signature $(m^+ - 1, 2)$. Then the theta series $\Theta_{\mathrm{KM}} = \Theta_{\mathbb{Z}\ell} \otimes \Theta_{\mathrm{KM},K}$ splits as a tensor product on $\mathbb{D}_\ell$ which, neglecting convergence issues, allows for isolating the integral over the divisor $\mathcal{Z}(\ell)$ as follows.

$$\begin{aligned} \int_{\mathcal{Z}(\ell)} \Lambda_{\mathrm{KM}}(f) \wedge \Omega^{m^+-2} &= \int_{Z(\ell)} \int_{\mathrm{Mp}_2(\mathbb{Z})\backslash\mathbb{H}} \langle f, \Theta_{\mathrm{KM}} \rangle \operatorname{Im}^k \, \mathrm{d}\mu \wedge \Omega^{m^+-2} \\ &= \int_{\mathrm{Mp}_2(\mathbb{Z})\backslash\mathbb{H}} \sum_{\substack{\lambda \in (\mathbb{Z}\ell)'/(\mathbb{Z}\ell) \\ \delta \in K'/K}} f_{\lambda\oplus\delta} \cdot \overline{\theta_{\mathbb{Z}\ell,\lambda}} \int_{\mathcal{Z}(\ell)} \overline{\theta_{\mathrm{KM},K,\delta}} \wedge \Omega^{m^+-2} \operatorname{Im}^k \, \mathrm{d}\mu. \end{aligned} \tag{0.3}$$

Here, subscripts in λ and δ indicate components of the corresponding vector valued modular forms. The inner integral $\int_{\mathcal{Z}(\ell)} \overline{\theta_{\mathrm{KM},K,\delta}} \wedge \Omega^{m^+-2}$ may be expressed in terms of an adelic standard theta integral which is suitable for an application of the Siegel–Weil formula[2]. Effectively, it is replaced by an Eisenstein series. Retranslating this to the classical setting reveals the expression as a scalar product

$$\int_{\mathcal{Z}(\ell)} \Lambda_{\mathrm{KM}}(f) \wedge \Omega^{m^+-2} \doteq \int_{\mathrm{Mp}_2(\mathbb{Z})\backslash\mathbb{H}} \left\langle f, \Theta_{\mathbb{Z}\ell} \otimes E_{K,k-1/2} \right\rangle \operatorname{Im}^k \, \mathrm{d}\mu. \tag{0.4}$$

Here, $E_{K,k-1/2}$ denotes an Eisenstein series of weight $k - 1/2$ associated to the lattice K. Note that $E_{K,k-1/2}$ is a Poincaré series, technically offering the possibility of unfolding. Selecting a primitive element $\ell_0 \in (\mathbb{Z}\ell)'$ and performing a subsequent calculation yields

[2] Compare Subsection 5.4.1 for details about its application.

the following special L-value (cf. Theorem 7.14).

Theorem 0.3. *Let (L, q) be an even lattice of signature $(m^+, 2)$, set $k = 1 + m^+/2$, let $f \in \mathcal{S}_{L,k}$ and select some primitive $\ell_0 \in L'$ of positive norm. If $m^+ > 3$, then*

$$\int_{\mathcal{Z}(\ell_0)} \Lambda_{\mathrm{KM}}(f) \wedge \Omega^{m^+-2} = C \cdot \sum_{n \in \mathbb{N}} \frac{a(n\overline{\ell_0}, n^2\,\mathrm{q}(\ell_0))}{n^{m^+}},$$

where $a(\lambda, n)$ denote the Fourier coefficients of the initial form $f \in \mathcal{S}_{L,k}$ (cf. (0.1)) and C is an explicit constant.

Before analysing the L-series presented above, it is imperative to elucidate the simplifications made previously and delineate an approach to the general case. Firstly, for general $\ell \in L$ such that the lattice L does not split the lattice $\mathbb{Z}\ell$, a splitting sublattice $M = \mathbb{Z}\ell \oplus K \leq L$ is considered. Subsequently, by applying lifting operators between vector valued modular forms to the respective discriminant forms, we may replace f by another form $\uparrow f$ to the discriminant form of the sublattice M, thus resulting in the theta function splitting once more as a tensor product. In order to relate the inner integral in (0.3) over the special cycle $\mathcal{Z}(\ell)$ to a standard theta integral and apply the Siegel–Weil formula, it is necessary to pass to a covering of the cycle and rewrite it in terms of the GSpin setting (compare Subsection 7.3.1 for details). Another obstacle is that the integral resulting from unfolding (0.4) is not convergent which is circumvented by passing to an analytic continuation and verifying that the function agrees with the resulting L-series on a right half-plane, thereby extending the identity to the edge of convergence. Ultimately, the coefficients of the initial form f must be recovered from the expression involving $\uparrow f$.

Symmetric square L-series

The L-series appearing in Theorem 0.3 are of independent interest and are analogous to symmetric square L-series in the scalar valued case. The latter have already been investigated by Shimura playing a key role in establishing the *Shimura lift* [Shi73] and led to further investigations in [Shi75]. When extending the construction of these L-series in Shimura's work to the vector valued setting, a straightforward approach is to associate a certain L-series to every component function of $f \in \mathcal{M}_{L,k}$, meaning for fixed $\mu \in L'/L$ in (0.1). However, upon closer examination it becomes evident that their natural analogues in the setting of vector valued modular forms intertwine the components. For the sake of simplicity, we present only a special instance of the series discussed in the main body in

Subsection 6.4.1. Namely, for a holomorphic vector valued modular form $f \in \mathcal{M}_{L,k}$ with Fourier expansion as in (0.1) and an anisotropic vector $\ell \in L'$, we define

$$L_\ell(f,s) = \sum_{n\in\mathbb{N}} \frac{a(n\bar{\ell}, n^2\, \mathrm{q}(\ell))}{n^s} \tag{0.5}$$

for $s \in \mathbb{C}$. In Subsection 3.3.4 we gather bounds of Fourier coefficients from existing literature and adjust these for our intended purpose. This is done in order to determine a right half-plane on which $L_\ell(f,s)$ converges absolutely with respect to the parameter s, thereby defining a holomorphic function. This is always the case for $\mathrm{Re}(s) > k + 1/2$ and we refer to Corollary 6.82 for sharper bounds. In Subsection 6.4.3 we proceed by realising the L-series in (0.5) as a Rankin–Selberg convolution and prove the convergence and analyticity of its unfolded counterpart in a certain region (cf. Proposition 6.97). Furthermore, we conclude that it admits meromorphic continuation to the whole complex plane and note that a symmetry property is inherited from an Eisenstein series. In fact, the main body of the thesis treats much more general L-series than the instance presented in (0.5).

A key feature of the L-series investigated by Shimura is that these admit infinite product expansions provided they are induced by Hecke eigenforms. This property may be replicated in the vector valued setting. In [BS08], Bruinier and Stein develop a theory of Hecke operators for vector valued modular forms, presenting a construction that is suitable for concrete computations. Additionally, the action of a variety of Hecke operators on Fourier coefficients of vector valued modular forms is also already presented in [BS08, Prop. 4.3 p. 258]. To be more precise, for a prime p not dividing the level of L and an eigenform $f \in \mathcal{M}_{L,k}$ for the Hecke operator $\mathcal{T}(p^2)$ with eigenvalue σ_p, there exists a constant C such that

$$\sigma_p a(\lambda, n) = p^{2k-2} a(\lambda/p, n/p^2) + C \cdot a(\lambda, n) + a(p\lambda, p^2 n). \tag{0.6}$$

Here, $a(\lambda, n)$ denote the Fourier coefficients of f as in (0.1). The recursion relation (0.6) is suitable for eliminating the prime p from the index n in (0.5). However, the case of primes p dividing the level of the lattice L is not contained in the source. Therefore, we derive a recursion formula analogous to (0.6) for primes dividing the level of the lattice L in Proposition 6.71 based on [Ste15]. However, in the general case, the recursion formula has ambiguities due to the phenomenon of torsion in the discriminant form. For maximal lattices, though, these ambiguities are resolved, resulting in a perfect recursion for certain forms. In this case, we derive an infinite product expansion for L-series as in (0.5). Simplifying, compared to the main body of the thesis, results in the following statement (cf. Corollary 6.92).

Theorem 0.4. *Assume L to be maximal with $2 \nmid \operatorname{lev}(L)$ and select a simultaneous eigenform $f \in \mathcal{S}_{L,k}$ of all $\mathcal{T}(p^2)$ with eigenvalue σ_p that is invariant under $\mathrm{O}(L'/L)$. Write*

$$f(\tau) = \sum_{\mu \in \mathcal{L}} \sum_{m \in \mathbb{Z} + \mathrm{q}(\mu)} a(\mu, m) e^{2\pi i \tau m} \mathfrak{e}_\mu$$

for the Fourier expansion of f. For an element $\ell \in L'$ such that $\operatorname{lev}(L) \cdot \mathrm{q}(\ell)$ is square free writing $(\lambda, t) \coloneqq (\bar{\ell}, \mathrm{q}(\ell)) \in \mathcal{L} \times \mathbb{Q}^\times$ yields the following product expansion

$$\begin{aligned}
&\sum_{n \in \mathbb{N}} \frac{a(n\lambda, n^2 t)}{n^s} \\
&= a(\lambda, t) \cdot \prod_{p \nmid \operatorname{lev}(L)} \frac{1 + \delta_{p \nmid t} \frac{G_{\mathcal{L}}(1)}{G_{\mathcal{L}}(p)} \cdot p^{k-1-s}}{1 - \left(\frac{\sigma_p}{p^{k-1}} - (1 - p^{-1}) \frac{G_{\mathcal{L}}(1)}{G_{\mathcal{L}}(p)} \right) \cdot p^{k-1-s} + p^{2(k-1-s)}} \\
&\cdot \prod_{p \mid \operatorname{lev}(L)} \frac{1 + K_{L,p} \begin{cases} \delta_{\lambda_p \neq 0}(1 - p^{-1}) + \delta_{\lambda_p = 0} \cdot \delta_{p \nmid t}, & 2 \mid R_p \\ -\delta_{\lambda_p = 0} \cdot p^{-1/2} \left(\frac{-t}{p} \right) \epsilon_p, & 2 \nmid R_p \end{cases} \cdot p^{k-1-s} + C(\lambda_p) \cdot p^{2(k-1-s)}}{1 - \left(\frac{\sigma_p}{p^{k-1}} - \delta_{2 \mid R_p} (1 - p^{-1}) K_{L,p} \right) p^{k-1-s} + p^{2(k-1-s)}}.
\end{aligned}$$

Here, $G_{\mathcal{L}}$ and $K_{L,p}$ are certain Gauss sums, λ_p denotes the projection of $\lambda \in \mathcal{L}$ to its p-component in $\mathcal{L}_p$, $C(\lambda_p)$ is a specified integer vanishing if $\lambda_p = 0$, $\chi_{\mathcal{L}}$ is a quartic character and ϵ_p equals 1 or i. Further, the quantity R_p is given in Definition 1.38.

In this context, we prove absolute convergence of the infinite product expansions as well as non-vanishing of the rational factors in p^{-s} within a specified range.

Injectivity of the Kudla–Millson lift

Turning to the question of injectivity, we recall that assuming $f \in \mathcal{S}_{L,k}$ lying in the kernel of the Kudla–Millson lift renders the special L-value in Theorem 0.3 to vanish. More explicitly, for any primitive $\ell_0 \in L'$ of positive norm, the L-series in (0.5) admits the root $L_{\ell_0}(f, m^+) = 0$. In conclusion, we have established relations stating that certain weighted sums of Fourier coefficients of any form f lying in the kernel of Λ_{KM} must equal zero. This will be exploited to prove that the form f had to vanish in the first place, thereby implying the triviality of the kernel of Λ_{KM}.

In the case of the lattice L splitting a hyperbolic plane, this is possible with an elegant trick. The key idea is to represent $L_{\ell_0}(f, s)$ as a recombination of subseries of the same type through an inclusion–exclusion argument. The assumption regarding the lattice L then guarantees that a substantial amount of these subseries vanish at m^+, effectively thinning out the original series $L_{\ell_0}(f, m^+)$. By exploiting absolute convergence of the L-series it will be concluded that the first coefficient of the series in (0.5) had to vanish, which is exactly the Fourier coefficient $a(\overline{\ell_0}, \mathrm{q}(\ell_0))$ of the form f. However, the set of Fourier coefficients representable in this fashion is, again due to the hyperbolic split, exhaustive, implying the vanishing of the initial form f. The details of this reasoning are presented in Subsection 7.3.2 and result in Theorem 0.2 above. Finally, an application of [Bru14, Thm. 4.2 p. 330] yields Theorem 0.1.

Part I

Preliminaries

In this part, the basics of the theory are provided before turning the attention to the theory of automorphic forms which lie at the heart of this thesis, as well as, the subject of modern number theory as a whole.

1 Fundamentals

For the sake of efficiency, the following content is not provided in chronological order. However, the majority of fundamental concepts that are not typically included in a regular Master's programme, although essential for comprehending this thesis, are addressed in the subsequent sections. Part of the following content has already been discussed in my Master's thesis and is restated here for the sake of convenience.

1.1 Quadratic spaces and lattices

The following content is primarily found in [Ehl03, Sec. 1.1], [Cas78, Sec. 7], [Bos08], [Kne02], [Lam73], [Lam05], and [Joh98]. Let R be a unitary, commutative, associative ring and M be a finitely generated R module and $\mathrm{Fr}(R)$ the fraction field if R is integral.

Definition 1.1. A *quadratic form* is a map $\mathrm{q} : M \to R$ such that

(i) $\mathrm{q}(rx) = r^2\,\mathrm{q}(x)$ for all $r \in R$ and $x \in M$,

(ii) $M \times M \ni (x, y) \mapsto \mathrm{b}(x, y) \coloneqq \mathrm{q}(x + y) - \mathrm{q}(x) - \mathrm{q}(y)$ is R bilinear.

Further, if R is integral, we allow $\mathrm{q} : M \to K$ for a field $K/\mathrm{Fr}(R)$.

a) The form b is called *associated bilinear form* (to q).

b) The pair (M, q) is called *quadratic module* over R. In case $R = F$ is a field, it is called *quadratic space* and in this case, M is denoted by V.

c) For a basis (m_i) of a free module M, the matrix $S = (\mathrm{b}(m_i, m_j))_{i,j}$ is called a *Gram matrix* and $\det(M) \coloneqq \det(\mathrm{q}) \coloneqq \det(S) \in R/(R^\times)^2$ the *determinant* of M or b or q.

d) If $\det(\mathrm{q}) \in R^\times$ (or $\in K^\times$), the form q or module (M, q) is called *regular*.

Example 1.2. Let F be a field of characteristic different from 2 and $V = F^n$, then

$$V \ni x \mapsto x^T \operatorname{diag}(a_1, \dots, a_n) x \in F \tag{1.1}$$

defines a quadratic form, where $a_i \in F$.

There is a notion of morphisms and a natural process of constructing new quadratic modules from known modules.

Definition 1.3. Let (M_1, q_1), (M_2, q_2) be quadratic modules over R mapping to the same ring.

a) A *morphism* $\sigma : (M_1, \mathrm{q}_1) \to (M_2, \mathrm{q}_2)$ of quadratic modules is an R module homomorphism $\sigma : M_1 \to M_2$ translating quadratic forms $\mathrm{q}_1 = \mathrm{q}_2 \circ \sigma$. An injective/bijective morphism is called *isometry/isomorphism*.

b) Define $(M_1, \mathrm{q}_1) \oplus (M_2, \mathrm{q}_2)$ as the quadratic module (M, q) with $M := M_1 \oplus M_2$ as R modules and $\mathrm{q} : (m_1, m_2) \mapsto \mathrm{q}_1(m_1) + \mathrm{q}_2(m_2)$.

c) For $r \in R$, we refer to $(M_1, r \cdot \mathrm{q}_1)$ as the *scaled lattice* $M_1(r)$.

Example 1.4. For $y \in (M, \mathrm{q})$ with $\mathrm{q}(y) \in R^\times$ the *reflection* at y is defined as the map $\tau_y : m \mapsto m - \mathrm{b}(m, y)\,\mathrm{q}(y)^{-1} y$ and determines an isometry. It is involutive, maps y to $-y$, and any $m \in M$ with $\mathrm{b}(m, y) = 0$ to itself.

If in Example 1.2 an element a_j was 0 for some j, the form would be considered on a smaller space where $a_i \in F^\times$ for all indices i. In fact, any quadratic space is isometrically isomorphic to a form as in (1.1). As a consequence, regularity may be assumed in general. Further, multiplying a_i with a nonzero square does not alter the isomorphism type of the space (V, q), so that a_i are regarded as elements of $\mathbb{F}^\times/(\mathbb{F}^\times)^2$. Over rings like $\mathbb{Z}$ the situation is more delicate (cf. Remark 1.13).

Example 1.5. In case of $F = \mathbb{R}$ and $V = \mathbb{R}^m$, every regular form q is isomorphic to

$$x \mapsto x^T \operatorname{diag}(\underbrace{1, \dots, 1}_{r \text{ times}}, -1, \dots, -1)x,$$

where the tuple $(r, m - r) =: (m^+, m^-)$ is called *signature* or *type* of (V, q) and is a complete invariant. We shall denote (V, q) by $\mathbb{R}^{(m^+, m^-)}$.

Definition 1.6. Let (M, q) be a quadratic module.

a) For a subset $N \subseteq M$ we refer to $N^\perp := \{x \in M \mid \mathrm{b}(x, y) = 0 \text{ for all } y \in N\}$ as the *orthogonal complement* of N.
b) Further, (M, q) is called *non-degenerate*, if $M^\perp = \{0\}$.
c) An element $0 \neq x \in M$ is called *isotropic*, if $\mathrm{q}(x) = 0$. We refer to (M, q) as being *isotropic* if it possesses an isotropic element and *anisotropic*, otherwise.

d) If (V, q) is a quadratic space, the dimension of a maximal totally isotropic subspace is called *Witt index* or *Witt rank* and is an invariant of (V, q).

Example 1.7. The standard isotropic regular module has Gram matrix $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. It is denoted by H and called *hyperbolic plane*. If e_1, e_2 denotes the standard basis, we have $e_i^\perp = \langle e_i \rangle$, i.e. e_i is isotropic, and $(e_1 + e_2)^\perp = \langle e_1 - e_2 \rangle$, rendering $H \simeq \left(\begin{smallmatrix}2&0\\0&-2\end{smallmatrix}\right)$ provided $2 \in R^\times$. Note that the Witt index of H is 1.

Remark 1.8. Any regular quadratic space splits into an orthogonal sum of an anisotropic space and hyperbolic planes, a fact known as *Witt's decomposition theorem* (cf. [Lam05, I.4 Thm. 4.1 p. 12]) rendering the Witt index r well defined as the number of hyperbolic planes. Also compare Theorem 4.6.

Definition 1.9. Let (V, q) be a quadratic space over F with basis $(v_i)_{i=1}^m$ and R be a unital subring of F.

a) The R module $L = \sum_{i=1}^m Rv_i$ together with q is called *quadratic R lattice*, $(v_i)_{i=1}^m$ is called *lattice basis* of L and m the *rank* of L. Equivalently, an R lattice L may be defined as an R submodule of (V, q) such that $L \otimes_R F \simeq V$.
b) Further, define the *discriminant* $\mathrm{disc}(L) := (-1)^{\frac{m^2-m}{2}} \det(L)$ of L.

Note that the determinant of a quadratic lattice is independent of the chosen basis if viewed as an element of $F^\times/(F^\times)^2 \cup \{0\}$.

Example 1.10. Prominent examples for lattices in this thesis are $\mathbb{Z}$ lattices in quadratic spaces over $\mathbb{Q}$ or $\mathbb{Z}_p$ lattices in quadratic spaces over $\mathbb{Q}_p$. In fact, every $\mathbb{Z}$ lattice in a quadratic $\mathbb{Q}$ vector space V lifts to a $\mathbb{Z}_p$ lattice L_p in a $\mathbb{Q}_p$ vector space V_p, by extending the bilinear form b onto the tensor product $V_p = \mathbb{Q}_p \otimes_{\mathbb{Q}} V$ and setting $L_p = \mathbb{Z}_p \otimes_{\mathbb{Z}} L$. A fundamental concrete example for $2 < p < \infty$ is the lattice $\mathbb{Z}_p^n$ with form (here, $r \in \mathbb{Z}_p^\times$)

$$\mathrm{q}_p^{n,r} : x \mapsto x^T \mathrm{diag}(1, \ldots, 1, r)x. \tag{1.2}$$

Proposition 1.11. *For $2 < p < \infty$ every non-degenerate quadratic $\mathbb{Z}_p$ lattice (L, q) is isomorphic to an orthogonal sum of scaled forms as in (1.2). More precisely, let $\dim(V) = m$, then there are $m_i \in \mathbb{N}$ for $1 \le i \le n$ such that $\sum_{i=1}^n m_i = m$, $r_i \in \mathbb{Z}_p^\times$ and a strictly increasing sequence of integers ν_i, such that*

$$(L, \mathrm{q}) \simeq \bigoplus_{i=1}^m \left(\mathbb{Z}_p^{n_i}, p^{\nu_i} \cdot \mathrm{q}_p^{n_i, r_i}\right).$$

Further, two of the above orthogonal sums are isomorphic to each other precisely when the associated ensembles of invariants $m_i, \nu_i, \epsilon_i := \left(\frac{r_i}{p}\right)$ *are identical.*

Corollary 1.12. *Assume* $p \neq 2$. *Every* $\mathbb{Z}_p$ *lattice has an orthogonal basis and there is a cancellation property for* $\mathbb{Z}_p$ *lattices.*

Remark 1.13. In fact, the case $p = 2$ is more complicated as the representation is not unique! However, these forms are required in Chapter 6 so that we refer to [Cas78, 8.4 p. 117] as well as [Joh98, 7.5 p. 881] and at least note that the fundamental building blocks of regular 2-adic quadratic lattices are, up to scaling with powers of 2, given by the Gram matrices

$$(r) \text{ with } r \in \{1, 3, 5, 7\}, \qquad H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \tag{1.3}$$

Further, there is a cancellation property with respect to hyperbolic planes, i.e. for L_1, L_2 quadratic $\mathbb{Z}_2$ lattices with $L_1 \oplus H \simeq L_2 \oplus H$, we have $L_1 \simeq L_2$ (also cf. [OMe73, 93:14 p. 256]). This remains true if H is replaced by a scaled hyperbolic plane.

Definition 1.14. The constituents $\left(\mathbb{Z}_p^{n_i}, p^{\nu_i} \cdot \mathrm{q}_p^{n_i,r}\right)$ in Proposition 1.11 are referred to as *Jordan components* and are also denoted $(p^{\nu_i})^{\epsilon_i n_i}$, where $\epsilon_i = (\frac{r_i}{p})$.

In case $p = 2$, the characterisation is more complicated, incorporating the *oddity* and two different *types* of forms (cf. [Joh98, 7.4 p. 380]).

After having learnt about characterisations of $\mathbb{Z}_p$ lattices and realising that a $\mathbb{Z}$ lattice L lifts to local lattices $L_p := L \otimes \mathbb{Z}_p$, it is natural to utilise local data in order to describe the global situation. Before continuing, note that with the correct identifications

$$L = \bigcap_{p<\infty} L_p.$$

Definition 1.15. On the set of non-degenerate $\mathbb{Z}$ lattices we introduce an equivalence relation by associating a lattice L with the collection of the isometry classes of $[L \otimes \mathbb{R}, (L_p)_{p<\infty}]$. These collections are called *genera* (singular: *genus*).

In fact, the local isomorphisms do not characterise $\mathbb{Z}$ lattices up to $\mathbb{Z}$ isomorphy; the relation is strictly coarser. Nonetheless, if the Gram matrix is indefinite and the rank is greater than 2, the significant case for this thesis, these equivalent classes are close. In fact, they coincide for $|\det(L)| < 128$ and divergences obey strict conditions (cf. [Joh98, 15.7 p. 378 and 15.9]). Also, two $\mathbb{Z}$ lattices L_1, L_2 have the same genus, if, and only if, $L_1 \oplus H \simeq L_2 \oplus H$.

Example 1.16. The following $\mathbb{Z}$ lattices have the same genus (note that $41 \in (\mathbb{Z}_2^\times)^2$ as well as $2 \in (\mathbb{Z}_{41}^\times)^2$) and are not isomorphic over $\mathbb{Z}$ (cf. [Cas78, p. 129]):

$$\begin{pmatrix} 2 & 0 \\ 0 & 164 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 4 & 0 \\ 0 & 82 \end{pmatrix}.$$

The following definition plays a major role for one of the main theorems of the present thesis (cf. Theorem 7.16).

Definition 1.17. A lattice (L, q) is said to *split* another lattice M, if there is a third lattice K, such that $L \simeq K \oplus M$ is an orthogonal decomposition. In case M is a hyperbolic plane, we also speak of a *hyperbolic split*.

Before introducing dual lattices and discriminant forms the primary consideration of lattices is concluded by sketching the Witt ring associated to a field F. To this end, another construction principle for lattices besides direct sums is required.

Definition 1.18. For two quadratic modules (M_i, q_i) over the same ring their tensor product $(M_1 \otimes M_2, \mathrm{q}_1 \otimes \mathrm{q}_2)$ may be defined canonically by forming the tensor product of the associated bilinear forms, so that $\mathrm{q}_1 \otimes \mathrm{q}_2(m_1 \otimes m_2) = \mathrm{q}_1(m_1)\, \mathrm{q}_2(m_2)$.

The set $M(F)$ of all isometry classes of finite dimensional quadratic spaces over a field F represents a commutative semiring with the orthogonal sum and the tensor product as concatenations. By Witt's cancellation Theorem, the additive monoid is a cancellation monoid. Consequently, there is a natural abelian group $\mathrm{Groth}(M(F))$ containing $M(F)$ as a submonoid, called *Grothendieck group* of $M(F)$. It consists of pairs $(x, y) \in M(F)^2$ modulo the equivalence $(x, y) \equiv (x', y') \iff x \oplus y' = x' \oplus y$, so that, for instance, $(0, y)$ is the additive inverse of $(y, 0) \in M(F)$.[1] The tensor product extends naturally onto this structure, equipping $\mathrm{Groth}(M(F))$ with the structure of a commutative ring.

Remark 1.19. For a field F, the quotient of the ring $(\mathrm{Groth}(M(F)), \otimes)$ and the ideal generated by hyperbolic planes is called *Witt ring* of F and denoted $W(F)$. The elements of the Witt ring are in bijection to the isometry classes of anisotropic quadratic spaces [Lam05, II.1 Prop. 1.4 p. 29].

[1] Recall the construction of rational numbers from the ring of integers.

Remark 1.20. Note that the discriminant is well defined on the Witt ring (cf. Definition 1.9 b)). Further, the signature (m^+, m^-) of a quadratic $\mathbb{R}$ space descends to an invariant on $W(\mathbb{R})$ when it is modified to $\mathrm{sig}(V) = m^+ - m^-$. For a $\mathbb{Z}$ lattice (L, q) we will also write $\mathrm{sig}(L) \coloneqq \mathrm{sig}(L \otimes_{\mathbb{Z}} \mathbb{R})$.

Dual lattices and discriminant forms

In the following, we assume the lattice L to be non-degenerate.

Definition 1.21. For a non-degenerate R lattice (L, q), define its *dual lattice*

$$L' \coloneqq \{x \in L \otimes Q(R) \mid \mathrm{b}(x, y) \in R \text{ for all } y \in L\}.$$

The lattice L is called

(i) *integral*, if $L \subseteq L'$ or equivalently $\mathrm{b}(L, L) \subseteq R$ or $S_{i,j} \in R$ for all indices,

(ii) *even*, if $q(L) \subseteq R$,

(iii) *unimodular*, if $L = L'$.

Note that every lattice which is unimodular or even is automatically integral. Further, in case of $2 < p < \infty$ we find $2 \in \mathbb{Z}_p^\times$, so that the notions of being even and integral coincide for $\mathbb{Z}_p$ lattices.

Example 1.22. a) The $\mathbb{Z}_p$ lattice $p^\nu q_p^{n,r}$ from Example 1.10 is even; also compare Table 1.1 for L'.

b) The hyperbolic plane H from Example 1.7 is unimodular over all rings R.

c) For R with $2, 3 \in R^\times$ consider (R^2, q) with Gram matrix A_2 from Remark 1.13, meaning $\mathrm{q}(x) = x_1^2 + x_1x_2 + x_2^2$. Clearly, the vectors $\ell_1 = \frac{1}{3} \cdot \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right), \ell_2 = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ are part of the dual lattice and since $|\langle \ell_1, \ell_2\rangle / L| = 3 = \det(A_2)$ which equals $|L'/L|$ by Remark 1.30, these represent a basis of L'.

Remark 1.23. a) For a non-degenerate R lattice L and $c \in R^\times$ we find $(cL)' = c^{-1}L'$.

b) If L is a non-degenerate R lattice with Gram matrix S, then $L' = S^{-1}L$.

Before turning to discriminant forms, we require some technical notions for reference.

Definition 1.24. Let (L, q) be an R lattice.

a) For $0 \neq l_1, l_2 \in L \otimes_R \mathrm{Fr}(R)$ and $z \in R \setminus R^\times$ such that $z \cdot l_2 = l_1$, we say that l_2 *divides* l_1 and write $l_2 \mid l_1$. If, in addition, $l_2 \in L$, we say z *divides* l_1, written $z \mid l_1$ and set $l_1/z \coloneqq l_2$ which might not be unambiguous!

b) The element $l_1 \neq 0$ is said to be *primitive* in L, if $l/z \notin L$ for all $z \in R \setminus R^\times$.

c) An even L is called *maximal* if there is no even R lattice in $L \otimes_R Q(R)$ properly containing L.

Remark 1.25. Let R be a principal ideal domain.

a) Any R lattice is contained in a maximal lattice.

b) If L is maximal and $L \otimes_R Q(R)$ splits a hyperbolic plane, so does $L = M \oplus H$ where M is automatically a maximal R lattice.

c) In case $R = \mathbb{Z}_p$ ($p < \infty$) and $\mathrm{rk}(L) \geq 5$ the space $L \otimes \mathbb{Q}_p$ splits a hyperbolic plane. By the theorem of Hasse–Minkowski every indefinite quadratic $\mathbb{Q}$ space of dimension at least 5 splits off a hyperbolic plane. This classical result is attributed to Meyer.

The theorem of Meyer implies the following result about the Witt rank over $\mathbb{Q}$.

Remark 1.26. Let (L, q) be an even $\mathbb{Z}$-lattice of signature (m^+, m^-). If $|\mathrm{sig}(L)| \geq 3$, then the Witt rank r of $L \otimes \mathbb{Q}$ is maximal, i.e. $r = \min\{m^+, m^-\}$.

Finally, discriminant forms may be derived from lattices.

Remark 1.27. If an R lattice L is integral, the bilinear form induces a form

$$\overline{\mathrm{b}} : L'/L \times L'/L \to F/R.$$

If a lattice L is even, the quadratic form induces a form

$$\overline{\mathrm{q}} : L'/L \to F/R.$$

The latter is called *discriminant form*.

We state some apparent observations before introducing more parameters.

Example 1.28. a) Since the hyperbolic plane H is unimodular, we find $H'/H = \{0\}$ and $\overline{\mathrm{q}} = 0$.

b) Let (L, q) be an even $\mathbb{Z}_p$ lattice and $d \in \mathbb{Z}_p$. Then $L(d) = L \leq L' \leq L(d)'$, so that there is a natural map $L'/L \hookrightarrow L(d)'/L(d)$.

Definition 1.29. Let (L, q) be an even $\mathbb{Z}$ lattice.

a) The smallest number $N \in \mathbb{N}$, such that $N\,\mathrm{q}(\ell) \in \mathbb{Z}$ for all arguments $\ell \in L'$ is called the *level* of L.

	L	L'	D	N
$p > 2$	$(\mathbb{Z}_p^n, p^\nu\, \mathrm{q}_p^{n,r})$	$\big(p^{-\nu} \cdot \mathbb{Z}_p^n, p^\nu\, \mathrm{q}_p^{n,r}\big)$	$((\mathbb{Z}/p^\nu\mathbb{Z})^n, p^{-\nu}\, \overline{\mathrm{q}}_p^{n,r})$	p^ν
$p = 2$	$(\mathbb{Z}_p^n, p^\nu\, \mathrm{q}_p^{n,r})$	$\big(p^{-(\nu+1)} \cdot \mathbb{Z}_p^n, p^\nu\, \mathrm{q}_p^{n,r}\big)$	$\big((\mathbb{Z}/p^{\nu+1}\mathbb{Z})^n, p^{-(\nu+2)}\, \mathrm{q}_p^{n,r}\big)$	$p^{\nu+2}$
	$(\mathbb{Z}_p^2, p^\nu H)$	$\big(\mathbb{Z}_p^2/p^\nu, p^\nu H\big)$	$\big((\mathbb{Z}/p^\nu\mathbb{Z})^2, p^{-\nu} H\big)$	p^ν
	$(\mathbb{Z}_p^2, p^\nu A_2)$	$(p^{-\nu}\mathbb{Z}_p^2, p^\nu A_2)$	$\big((\mathbb{Z}/p^\nu\mathbb{Z})^2, p^{-\nu} A_2\big)$	p^ν

Table 1.1: Examples of even lattices, their discriminant forms, and levels. Here, p is a prime, $n, \nu \in \mathbb{N}_0$, the form $\mathrm{q}_p^{n,r}$ is given in Example 1.10. In addition, note that H and A_2 (cf. Remark 1.13) denote the Gram matrix, meaning a factor of $\frac{1}{2}$ has to be taken into account when moving on to the associated quadratic form.

b) The *level* of $\ell \in L$ is the number $N_\ell \in \mathbb{N}$, such that $\mathrm{b}(\ell, L) = N_\ell \mathbb{Z}$.
c) A prime number p is referred to as being *bad* and as *good* otherwise. We also speak of these primes as *places*.

Remark 1.30. (cf. addendum) Let L be an even non-degenerate $\mathbb{Z}$ lattice of rank m.

a) We find $\det(L)L' \leq L$ as well as $NL' \leq L$.
b) We have $|L'/L| = |\det(L)| = \mathrm{vol}(L)^2$, in particular, L'/L is finite.
c) Further, the following division relations are true: $\mathrm{lev}(L) \mid 2\det(L) \mid 2\,\mathrm{lev}(L)^m$ and $\mathrm{lev}(L) \mid \det(L)^2$. In case of odd level N or even rank m, we find $\mathrm{lev}(L) \mid \det(L)$.
d) The level $\mathrm{lev}(L)$ and the determinant $\det(L)$ have the same prime divisors.

Further, if the rank of the lattice L is odd, then its level is divisible by 4. Equipped with these statements, discriminant groups may be defined and analysed locally.

Definition 1.31. In case of an even $\mathbb{Z}$ lattice L, the group L'/L is called *discriminant group*. Its *level* is the level of L.

Remark 1.32. In fact, any finite abelian group $\mathcal{L}$ with a non-degenerate quadratic form to $\mathbb{Q}/\mathbb{Z}$ arises as a quotient L'/L of a $\mathbb{Z}$ lattice L (cf. [Wal63, 4 p. 294]). Further, for any even $\mathbb{Z}$ lattice (L, q) we have *Milgram's formula* [MH73, App. 4]

$$\sum_{\lambda \in L'/L} \exp(2\pi i \cdot \overline{\mathrm{q}}(\lambda)) = \sqrt{|L'/L|} \cdot \exp(2\pi i \cdot \mathrm{sig}(L)/8). \tag{1.4}$$

As a consequence, the signature $\mathrm{sig}(\mathcal{L}) \coloneqq \mathrm{sig}(L) \mod 8$ is well defined.

Example 1.33. For a $\mathbb{Z}$ lattice, the group $\mathcal{L} \coloneqq L'/L$ factors orthogonally into maximal p subgroups

$$L'/L \simeq \bigoplus_{p<\infty} \mathcal{L}_p.$$

By the classification of finitely generated abelian groups $L'/L \simeq \prod_{p<\infty} \mathcal{L}_p$ is immediate. Note that if $x \in \mathcal{L}_p$, $y \in \mathcal{L}_{p'}$, then $\deg(x)\,\overline{\mathrm{b}}(x,y) \equiv 0$, so that the denominator of $\overline{\mathrm{b}}(x,y)$ must be a divisor of $\deg(x)$. But the same holds for y, implying $\overline{\mathrm{b}}(x,y) \equiv 0$. Further, we have $\mathcal{L}_p = L'_p/L_p$ with $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$, so that considerations of discriminant forms are reduced to local objects, signifying the relevance of Definition 1.15 and Proposition 1.11. In particular, Table 1.1 is essentially exhaustive.

Remark 1.34. An even lattice L is maximal if, and only if, L'/L is anisotropic. In particular, unimodular lattices are maximal. Further, by Example 1.33, the $\mathbb{Z}$ lattice L is maximal if, and only if, L_p is maximal for all primes p.

Definition 1.35. Let $\mathcal{L}$ be a discriminant group and $n \in \mathbb{N}$. Then ${}^n\mathcal{L}$ denotes the elements which are n-th powers in $\mathcal{L}$ and ${}_n\mathcal{L}$ denotes the n-torsion of $\mathcal{L}$.

Remark 1.36. For a discriminant group $\mathcal{L}$ the subgroup ${}^n\mathcal{L}$ is the orthogonal complement of ${}_n\mathcal{L}$ and the following sequence is exact

$$0 \longrightarrow {}_n\mathcal{L} \xrightarrow{\ \iota\ } \mathcal{L} \xrightarrow{\ \cdot n\ } {}^n\mathcal{L} \longrightarrow 0.$$

Example 1.37. Consider $p^\nu q_p^{n,r}$ (cf. Example 1.10) and $k \in \mathbb{N}$ with $\nu_p \coloneqq \nu_p(k)$. Then

$${}_k\mathcal{L} = \left\{ z \in \mathbb{Z}_p \;\middle|\; p^{\nu-\nu_p+2\delta_{p=2}} \mid z \right\}^n .$$

The following technical definition is required in Section 6.4.2 and in particular in Proposition 6.71, where (1.6) is needed.

Definition 1.38. Let (L, q) be an integral $\mathbb{Z}$ lattice. By [Ste15, Lemma 4.3] for any $n \in \mathbb{N}$ and odd prime p, there is an orthogonal decomposition of L/p^nL into $\mathbb{Z}/p^n\mathbb{Z}$ submodules

$$\left(\bigoplus_i L_i \right) \oplus \left(\bigoplus_j M_j \right) \oplus N,$$

where $L_i = (\mathbb{Z}/p^n\mathbb{Z})v_i$ is one dimensional with $\mathrm{b}(v_i, v_i) \in (\mathbb{Z}/p^n\mathbb{Z})^\times$, $M_j = (\mathbb{Z}/p^n\mathbb{Z})v_j$ with $\mathrm{b}(v_j, v_j) \in p^k(\mathbb{Z}/p^n\mathbb{Z})^\times$ where $1 \le k \le n-1$, and $\mathrm{b}(N, N) \subseteq p^n\mathbb{Z}$. We may assume the M_j being sorted with respect to increasing valuation k of $\mathrm{q}(v_j)$. Define

$$n_k = \#\{1 \leq i \leq m : \nu_p(\mathrm{b}(v_i, v_i)) = k\} \text{ and } R_p^n := \sum_{k=0}^{n-1}(n-k)n_k. \tag{1.5}$$

The most prominent case in this thesis is the one of $n = 1$, in which case we write $R_p := R_p^1$ and note that

$$R_p = n_0 = \#\{1 \leq i \leq m : \nu_p(\mathrm{b}(v_i, v_i)) = 0\} \tag{1.6}$$

equals the number of basis elements of L/pL such that the p-valuation of their norm is zero.

1.2 p-adic numbers and the adele ring

Number theorists are in general interested in the rational numbers $\mathbb{Q}$ or even primarily the discrete case of $\mathbb{Z}$. Sustainably, analytical approaches have been instrumental in developing sophisticated tools to simplify or even completely solve such discrete algebraic questions. The following section presents tools from Topology and functional Analysis (more precisely integration theory) that have the potential to facilitate solutions of purely algebraic questions, while also offering an intrinsic mathematical appeal.

Definition 1.39. For a field F, a mapping $|\cdot| : F \to [0, \infty]$ is called *absolute value function* (abs), if

a) $|\cdot|$ is definite, that is $|a| = 0 \implies a = 0$.

b) $|\cdot|$ satisfies the triangle inequality, i.e. for all $a, b \in F$: $|a + b| \leq |a| + |b|$.

c) $|\cdot|$ is multiplicative, meaning for all $a, b \in F$ we have $|ab| = |a| \cdot |b|$.

Further, such a function is said to be *non-Archimedean* if the following stricter version of b) is true:

$$|a + b| \leq \max\{|a|, |b|\}.$$

The inequality above is referred to as the *strong triangle inequality.*

Example 1.40. The most prominent examples are the standard *abs* on $\mathbb{Q}$ as well as the so called p-adic absolute value functions for a prime number p. The latter measure the appearance of p in the prime factor decomposition of a rational number. More precisely, let

$p \in \mathbb{N}$ be a prime number and $0 \neq q \in \mathbb{Q}$. Furthermore, let $a, b, k \in \mathbb{Z}$ with $\gcd(a, b) = 1$, such that $\gcd(a, b, p) = 1$ and $q = p^k \frac{a}{b}$. Then setting $|\, q\,|_p := p^{-k}$ gives rise to an *abs* on $\mathbb{Q}$ which is called *p-adic* absolute value function.
As expected, positive powers of p-adic *abs* are again *abs*, the same holds true for powers to $0 < \alpha < 1$ of the standard absolute value function on $\mathbb{Q}$. Clearly, these classes of *abs* generate the same topology on $\mathbb{Q}$, therefore two *abs* related this way are called *equivalent*. Another absolute value function is the trivial one, mapping every element to 1.

Proposition 1.41 (Ostrowski,[BS, 1.4.2 Thm. 3 p. 37])**.** *The list of absolute value functions on $\mathbb{Q}$ presented in Example, i.e. the standard abs, the class of p-adic abs, and the trivial abs 1.40 is complete up to equivalence.*

The Theorem of Ostrowski reveals that the definition of absolute value functions is eminently restrictive and that it is interconnected with the arithmetic structure of the field $\mathbb{Q}$. In fact, one can prove that any locally compact field with non-discrete topology is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p(t)$. The latter two classes are exactly the so called *local fields*, which might be characterised by being complete with respect to a discrete valuation (absolute value), and possess a finite residue class field.

Definition 1.42. The completion of $\mathbb{Q}$ with respect to a p-adic *abs* is called *field of p-adic numbers* and denoted $\mathbb{Q}_p$. It is referred to as the *local field* at that *place* p, in contrast to the *global* field $\mathbb{Q}$. The function $|\cdot|_p$ extends uniquely to $\mathbb{Q}_p$ and the fact that it is non-Archimedean results in the unit ball (which is open as well as compact)

$$\mathbb{Z}_p := \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\} = \overline{\mathbb{Z}}^{|\cdot|_p}$$

being a ring. It is called ring of *p-adic integers*, is a principal ideal domain and its unit group consists exactly of all elements with absolute value 1. As a consequence, $\mathbb{Q}_p$ is a locally compact, totally disconnected group. According to convention, define $\mathbb{Q}_\infty := \mathbb{R}$ and $\mathbb{Z}_\infty := \mathbb{Z}$ and write '$p < \infty$'for the set of all primes.

Remark 1.43. Note that since $\mathbb{Z}_p$ is compact for $p < \infty$, as well as open, the standard notation for the closure of balls is inadequate. For a fixed p, the field $\mathbb{Q}_p$ may be identified with the set of all Laurent series $\sum_{k=k_0}^{\infty} c_k p^k$ for integer coefficients $0 \leq c_k < p$ and an integer k_0. Then $\mathbb{Z}_p$ consists of the subring of power series.

It becomes apparent that the p-adic numbers encode arithmetic information about the rationals in an analytic fashion. Hence, a desirable tool would be an algebraic and topological structure containing all p-adic numbers $\mathbb{Q}_p$. In order to capitalise on functional analytical tools, a locally compact ring R seems to be preferable. However, in this case R would be a locally compact $\mathbb{Q}$ vector space and by a well known result from André Weil it had to be finite dimensional, conflicting with the requested properties. In order to obtain a structure which is at least additively locally compact, the inductive limit of the topological groups

$$\prod_{p\in S}\mathbb{Q}_p\times\prod_{p\notin S}\mathbb{Z}_p$$

with product topology is taken, where S is a finite set of primes. The resulting structure $\mathbb{A}_f$ is called *ring of finite adeles* and is a locally compact additive group with a topologically incompatible multiplication. Further, it contains the ring $\hat{\mathbb{Z}} := \prod_{p<\infty}\mathbb{Z}_p \subseteq \mathbb{A}_f = \hat{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathbb{Q}$.

Definition 1.44. The topological product

$$\mathbb{A} := \mathbb{A}_f\times\mathbb{R}$$

is called *Adele ring*. It is, again, a locally compact abelian group and its Haar measure μ is obtained by taking the product measure of the Haar measures of $\mathbb{Q}_p$ (the same holds for $\mathbb{A}_f$ where we assume the normalisation $\mu_f(\hat{\mathbb{Z}}) = 1$). The group of multiplicatively invertible elements $\mathbb{A}^\times$ of $\mathbb{A}$ is called *Idele group* and becomes a locally compact group with the inductive limit topology with respect to $\mathbb{Z}_p^\times$ as open compact subgroups in the components. Its Haar measure $\mu^\times$ is again the product of the Haar measures of $\mathbb{Q}_p^\times$.[2]

Example 1.45. For $s\in\mathbb{C}$ with $\mathrm{Re}(s) > 1$ we have that [Dei10, 5.3 Prop. 5.3.4 p. 128]

$$\zeta(s) = \int_{\hat{\mathbb{Z}}\cap\mathbb{A}^\times}\prod_{p<\infty}|x_p|_p^s\ \mathrm{d}\mu_f^\times.$$

The integral expression above contains, at a glance, an infinite product. In fact, the product is effectively finite for every argument and this type of function is usually considered when integrating adelically. The above identity was generalised by John Tate in his thesis [Tat16, [1]]. More precisely, for a Schwartz–Bruhat function f on $\mathbb{A}^\times$ (cf. Definition 1.53), the *Zeta integral*

$$\zeta(f,s) := \int_{\mathbb{A}^\times} f(x)\cdot\prod_{p\leq\infty}|x_p|_p^s\ \mathrm{d}\mu^\times$$

[2]These are given by their respective scaled additive measure μ such that $\mu^\times(t) = \frac{p}{p-1}\frac{\mu(t)}{|t|_p}$, where $\frac{\infty}{\infty} = 1$.

converges locally uniformly for $\mathrm{Re}(s) > 1$, has a meromorphic extension to $\mathbb{C}$ with known poles and fulfils the functional equation

$$\zeta(f, s) = \zeta(\hat{f}, 1 - s).$$

The construction presented above may be generalised by intertwining the integrand with finite characters. Further references are the elaborate [Kud04b], the comfortably accessible [Dei10, 6 p. 139–156], the concise [Bum98, 3.1 pp. 254–277], as well as Tate's original thesis [Tat16, [1] pp. 1–44].

The function

$$|\cdot| : \mathbb{A}^\times \to \mathbb{R}_{>0}, \qquad (x_p)_p \mapsto \prod_{p \le \infty} |x_p|_p$$

used above as an integral kernel defines a homomorphism whose kernel we denote by $\mathbb{A}_1$. It plays a significant role in classifying finite characters on $\mathbb{A}^\times/\mathbb{Q}^\times$.

Another important feature of adelic formulations is the so called *Hasse principle* or *local-global principle*, named after Helmut Hasse. The core concept is that global properties of $\mathbb{Q}$ correspond to an ensemble of properties at all places $p \le \infty$. A concrete example for this is the Theorem of Hasse-Minkowski (cf. [Has23] or [Has24]) or the fact that $\mathbb{Q}^\times < \mathbb{A}_1$.

1.3 Dual groups and Fourier analysis

A convenient setting for this theory is that of *locally compact abelian* groups (lca). Details are treated in depth in [Rud90, 1–2 pp. 1–58], which may serve as a comprehensive source for acquiring the theory. The examples of particular interest for this thesis are discussed in [Dei10, 5.4 pp. 129-138 and 6.3 pp. 146–152] and [Bum98, 3.1 pp. 254–262], featuring p-adic numbers and the adele ring.

Definition 1.46. A topological group $(G, \cdot)$ is called *locally compact* if it is Hausdorff and every element of G possesses a compact neighbourhood.
A positive measure μ on the borel σ-Algebra of G is called a *Haar measure*, if it satisfies the following conditions:

a) μ is locally finite, meaning for any compact set $K \subseteq G$ we have $\mu(K) < \infty$.

b) μ is regular.

c) μ is left invariant, hence for any subset $H \subseteq G$ and $g \in G$ the identity $\mu(gH) = \mu(H)$ holds.

By a theorem of Haar [Els11, Thm. 3.12 p. 362], any locally compact group possesses a non-trivial Haar measure μ which is unique up to a constant factor. As a consequence, for any $g \in G$ the measure $\mu_g(A) := \mu(Ag)$ must coincide with $\delta_G(g) \cdot \mu$ for an appropriately chosen factor $\delta_G(g) \in \mathbb{R}^\times_{>0}$. Moreover, δ_G defines a continuous homomorphism which is called *modular quasi character* of G (cf. [DE14, 1.4 Thm. 1.4.1 p. 14]).

Definition 1.47. A *character* of a lca G is a continuous homomorphism $\chi : G \to \mathbb{T}$. The group of characters G^* with the compact open topology is again a lca and called (*Pontryagin*) *dual group* .[3]

Example 1.48. The prototype of a character is the exponential function. In case of $(\mathbb{R}, +)$, define

$$x \mapsto \exp(2\pi i \cdot a \cdot x) =: e_\infty(a, x),$$

for $a \in \mathbb{R}$. In case of $(\mathbb{Q}_p, +)$ for any $p < \infty$ and $a \in \mathbb{Q}_p$ define

$$x \mapsto \exp(-2\pi i \cdot a \cdot x) =: e_p(a, x).$$

Note that by continuity, the mapping has to be constant on some neighbourhood $p^k\mathbb{Z}_p$ of 0. Hence, it is well defined by multiplicativity. In fact, this list of characters for $\mathbb{Q}_p$ is faithful and exhaustive. In particular, $\mathbb{Q}_p$ is isomorphic to its dual group via $a \mapsto (x \mapsto e_p(a, x))$.

The following theorem, initially proven by Pontryagin in a more restrictive setting and later proven in general by Egbert van Kampen and André Weil, justifies the name *dual group*.

Theorem 1.49 (Pontryagin)**.** *The bidual group of G is canonically isomorphic to G, in formulae $(G^*)^* \simeq G$.*

As a consequence, a lca G is compact if, and only if, G^* is discrete.
With the notion of a dual group, Fourier transformations may be defined. Recall that we write $L^1(G, \mu)$ for the space of integrable $\mathbb{C}$-valued functions on (G, μ).

Definition 1.50. Let $f \in L^1(G, \mu)$, then the mapping

$$G^* \ni \chi \mapsto \int_G f(g) \cdot \chi^{-1}(g) \, \mathrm{d}\mu(g)$$

is called *Fourier transform* of f and denoted by $\hat{f}$ or $\mathcal{F}(f)$.

[3] So characters are unitary, one dimensional representations of the group G, whereas for the notion of quasi characters unitaricity is not demanded.

Before illustrating this definition by specifying the main application within this thesis, a well established Lemma is stated.

Remark 1.51. Let $H \leq G$ be a compact subgroup and $\psi \in \mathrm{Hom}(G, \mathbb{C}^\times) \cap L^\infty_{\mathrm{loc}}(G)$. Then

$$\mathcal{F}(\mathbb{1}_H \psi)(\chi) = \begin{cases} \mu(H), & \text{if } \psi|_H = \chi|_H, \\ 0, & \text{else.} \end{cases}$$

Proof: For arbitrary $h \in H$ the following calculation immediately yields the result

$$\mathcal{F}(\mathbb{1}_H \psi)(\chi) = \psi(h)\chi^{-1}(h) \int_H \psi(g)\chi^{-1}(g)\, \mathrm{d}\mu(g) = \left(\psi\chi^{-1}\right)(h) \cdot \mathcal{F}(\mathbb{1}_H \psi)(\chi).$$

□

In case of $G = \mathbb{Q}_p$, the transform may be identified with a function on $\mathbb{Q}_p$ (cf. Example 1.48), resulting in the classical Fourier transformation in case of $p = \infty$ if the measure μ is scaled properly. In case of a non-degenerate quadratic space (V_p, q) over $\mathbb{Q}_p$, the bilinear form b provides an isomorphism to the linear dual of V_p, which induces an isomorphism of Pontryagin duals via

$$V_p \ni a \mapsto \Big(x \mapsto \psi_p(\mathrm{b}(a,x)) =: \psi_p^{\mathrm{b}}(a,x)\Big) \in V_p^*, \tag{1.7}$$

for a non-trivial character ψ_p of $\mathbb{Q}_p$. For convenience, we fix the choice $\psi_p(x) := e_p(1,x)$ (cf. Example 1.48). These isomorphisms give rise to a Fourier transformation on $L^1(V_p)$

$$\mathcal{F} := \mathcal{F}_{\psi_p^{\mathrm{b}}} : L^1(V_p) \ni f \mapsto \left(a \mapsto \int_{V_p} f(x) \cdot \psi_p^{\mathrm{b}}(a,x)^{-1}\, \mathrm{d}\mu(x)\right). \tag{1.8}$$

In fact, there is, analogously to the Schwartz space in the real setting, a subspace $\mathcal{S}(V_p) < L^1(V_p)$ on which $\mathcal{F}_{\psi_p^{\mathrm{b}}}$ defines an automorphism. Moreover, the space $\mathcal{S}(V_p)$ is characterised by this property[4] (cf. [Osb75, Thm. 1 p. 42]).

Example 1.52. Let $f \in L^1(V_p, \mu_p)$ and b be a non-degenerate bilinear form on V_p, ψ_p be a non-trivial character of $\mathbb{Q}_p$ and $c \in V_p$. Then we find the following calculation rules.

a) $\mathcal{F}[f(\,\cdot\, + c)](a) = \psi_p^{\mathrm{b}}(c,a) \cdot \mathcal{F}[f](a)$

[4]The classic approach from Bruhat is via an inductive limit involving Lie subquotient groups and hence requires extensive background knowledge in differential geometry, while the characterisation of Osborne is elementarily accessible. In addition, a key feature of the space and its maximality with respect to this feature is apparent.

b) $\mathcal{F}[\psi(c,\,\cdot\,)f](a) = \mathcal{F}[f](a-c)$

c) Let $p < \infty$ and $(L_p, \mathrm{q}) \subset (V_p, \mathrm{q})$ be an even lattice. Then

$$\mathcal{F}[\mathbb{1}_{L_p}] = \mu_p(L_p) \cdot \mathbb{1}_{L_p'} \tag{1.9}$$

by Remark 1.51 with $\psi = 1$ and $H = L_p$.

d) Let $p < \infty$, $(L_p, \mathrm{q}) \subset (V_p, \mathrm{q})$ be an even lattice, and $\lambda \in \mathbb{Q}_p^\times$. Then

$$\mathcal{F}[f(\lambda\,\cdot\,)](a) = \frac{1}{|\lambda|_p^m}\mathcal{F}[f](a/\lambda).$$

This follows from the fact that $A \mapsto \mu_p(\lambda A)$ defines a Haar measure as well, meaning it must agree with μ_p modulo a non-trivial factor. Evaluation on L_p in case $p < \infty$ yields the desired result.

Definition 1.53. Let $p < \infty$. For a $\mathbb{Q}_p$ vector space V_p, a *Schwartz–Bruhat* function is a locally constant function with compact support. These functions constitute a vector space denoted by $\mathcal{S}(V_p)$.

Clearly, for $l \in V_p$ and $\lambda \in \mathbb{Q}_p$, the function $\mathbb{1}_{l+\lambda L_p}$ is a Schwartz–Bruhat function. On the other hand, by decomposing the support of $f \in \mathcal{S}(V_p)$, we may represent it as a linear combination of such functions.

Remark 1.54. Let $p < \infty$, $(L_p, \mathrm{q}) \subset (V_p, \mathrm{q}, \mu_p)$ be an even lattice, $l \in V_p$, and $\lambda \in \mathbb{Q}_p^\times$. Then the observation $\mathbb{1}_{l+\lambda L_p}(x) = \mathbb{1}_{\lambda L_p}(x-l)$, as well as $\mathbb{1}_{\lambda L_p}(x) = \mathbb{1}_{L_p}(x/\lambda)$, imply in conjunction with Example 1.52 that

$$\begin{aligned}
\mathcal{F}[\mathcal{F}[\mathbb{1}_{l+\lambda L_p}]](a) &= \mathcal{F}[\psi_p^{\mathrm{b}}(-l,\,\cdot\,)\mathcal{F}[\mathbb{1}_{\lambda L_p}]](a) \\
&= \mathcal{F}[|\lambda|_p^m \mathcal{F}[\mathbb{1}_{L_p}](\lambda\,\cdot\,)](a+l) \\
&= \mathcal{F}[\mathcal{F}[\mathbb{1}_{L_p}]]\left(\frac{a+l}{\lambda}\right) \\
&= \mu_p(L_p)^2 |L_p'/L_p| \cdot \mathbb{1}_{L_p}\left(\frac{a+l}{\lambda}\right) \\
&= \mu_p(L_p)^2 |L_p'/L_p| \cdot \mathbb{1}_{-l+\lambda L_p}(a) \\
&= \mu_p(L_p)^2 |L_p'/L_p| \cdot \mathbb{1}_{l+\lambda L_p}(-a)\,.
\end{aligned}$$

In particular, the Fourier transform defines an automorphism of $\mathcal{S}(V_p)$. It is apparent, that the usual choice of Haar measure, in the following denoted by μ_p^{b}, has normalisation

$$\mu_p^{\mathrm{b}}(L_p) := \sqrt{|L_p'/L_p|}^{-1} = \sqrt{|\det(S)|_p} \tag{1.10}$$

as this results in

$$\mathcal{F}^2_{\psi_p^{\mathrm{b}}}[f](a) = f(-a) \tag{1.11}$$

for $f \in \mathcal{S}(V_p)$, rendering the Fourier transformation idempotent. As a consequence, *Parseval's formula* [Els11, 3.14 p. 199] holds as well as the Theorem of Plancherel, extending the result to L^2. With this normalisation the measure μ_p^{b} is called *self dual* and the product measure on $V_{\mathbb{A}}$ assigns the value 1 to the compact space $V_{\mathbb{A}}/V_{\mathbb{Q}}$ (cf. [Bum98, Prop. 3.1.3 p. 261]).

Schwartz–Bruhat functions may also be defined for the adele ring. Let V be a $\mathbb{Q}$ vector space of dimension m and $V_p = V \otimes_{\mathbb{Q}} \mathbb{Q}_p$ such as $V_{\mathbb{A}} = V \otimes_{\mathbb{Q}} \mathbb{A}$. Then an element $f \in \mathcal{S}(V_{\mathbb{A}})$ might be directly defined as a linear combination of products of the form

$$V_{\mathbb{A}} \ni a \mapsto \prod_{p \leq \infty} f_p(a_p),$$

where $f_p \in \mathcal{S}(V_p)$ and $f_p = \mathbb{1}_{\mathbb{Z}_p^m}$ for almost all $p < \infty$. Note that this space agrees with the inductive limit of finite tensor products of the local Schwartz–Bruhat spaces $\mathcal{S}(V_p)$[5]. Further, the space $\mathcal{S}(V_{\mathbb{A}})$ is generated by products as above when restricting possible choices of f_p at finite places to characteristic functions of elements of local discriminant groups associated to even quadratic lattices in V (cf. [KY10, p. 2286]).

For the final part of this section we return to characters, discussing the adelic case.

Remark 1.55. The instances of Example 1.48 may be combined to an adelic character. In fact, the mapping

$$\mathbb{A} \ni a \mapsto \left[x \mapsto \prod_{p \leq \infty} e_p(a_p, x_p) \right] \in \mathbb{A}^*$$

defines an isomorphism $\mathbb{A} \simeq \mathbb{A}^*$. Write $e(a,x) := \prod_{p\leq\infty} e_p(a,x)$ and note that these characters are trivial on $\mathbb{Q}$ if, and only if, $a \in \mathbb{Q}$, so that $(\mathbb{A}/\mathbb{Q})^* \simeq \mathbb{Q}$ (equipped with the discrete topology). Moreover, ψ will denote the *standard character* $\psi(x) = e(1,x)$, with

[5]For inducing the inclusions vectors of each $\mathcal{S}(V_p)$ have to be fixed which are taken to be $\chi_{\mathbb{Z}_p}$ for $p < \infty$. For further information consider [Bum98, p. 300-301].

components ψ_p, if not specified further.

Characterising the characters of the multiplicative group $\mathbb{A}^\times$, however, is more subtle and will not be discussed extensively.

Remark 1.56. Characters of $\mathbb{A}^\times/\mathbb{Q}^\times$ are called *Hecke characters* and decompose into a finite character and a quasi character of the form $a \mapsto |\cdot|^s$ for some $s \in \mathbb{C}$. Finite Hecke characters are in bijection to primitive Dirichlet characters (cf. [Bum98, Prop. 3.1.2 p. 259], [Dei10, Lem. 6.3.2 p. 147]), a link whose application becomes relevant in Section 5.3 and which is treated in detail in [Opi18, 1.10 p. 20]. If χ is a Dirichlet character, then χ_p will denote the associated local character on $\mathbb{Q}_p^\times$.

Further, quadratic Dirichlet characters are representable by the Hilbert symbol, which is the subject of the following section.

Remark 1.57. Remark 1.55 may be upgrade to characters of $V_\mathbb{A}$ by sending

$$V_\mathbb{A} \ni a \mapsto \left[x \mapsto \prod_{p \leq \infty} e_p^{\mathrm{b}}(a_p, x_p) \right] \in V_\mathbb{A}^*.$$

As a consequence, Fourier analysis as above may be carried out in the adelic setting, where computations are essentially reduced to treating finitely many places. For convenience, write $e_f^{\mathrm{b}}(a,x) := \prod_{p<\infty} e_p^{\mathrm{b}}(a_p, x_p)$ as well as $e^{\mathrm{b}}(a,x) := \prod_{p\leq\infty} e_p^{\mathrm{b}}(a_p, x_p)$

1.4 Hilbert symbol

The Hilbert symbol is a number theoretic bilinear form and induces characters which are essential for understanding the Weil representation which itself is of critical significance for the space of cusp forms appearing in the main theorem of this thesis.
The curious reader may find a compact and, concerning the framework of this thesis, comprehensive investigation of Hilbert symbols in [Ser12a, I.III p. 19–26] by Serre. Based on this reference, the following section provides the bare minimum of information required to grasp the conceptional use throughout this thesis.

Definition 1.58. Let F be a field. The *Hilbert symbol* $\mathcal{H}$ of $a, b \in F^\times$ is defined as

$$\mathcal{H}(a,b) = \begin{cases} 1, & \text{if } z^2 - ax^2 - by^2 = 0 \text{ has a non-trivial solution } (x,y,z) \in F^3, \\ -1, & \text{otherwise.} \end{cases}$$

This association is naturally viewed as a mapping $F^\times/(F^\times)^2 \times F^\times/(F^\times)^2 \to \{\pm 1\}$.

Remark 1.59. Let $a, a', b \in F^\times$ and $F_b = F(\sqrt{b})$. Then $\mathcal{H}(a,b) = 1$ holds if, and only if, a belongs to the group $N_{F_b^\times}$ of norms of $F_b^\times$.

i) $\mathcal{H}(a,-a) = \mathcal{H}(a, 1-a) = 1$,

ii) $\mathcal{H}(a,b) = 1 \implies \mathcal{H}(aa', b) = \mathcal{H}(a', b)$,

iii) $\mathcal{H}(a,b) = \mathcal{H}(a,-ab) = \mathcal{H}(a,(1-a)b)$, if $1 - a \neq 0$ in the last expression.

The Hilbert symbol may be evaluated in terms of the Legendre symbol. In order to to perform this reduction, a characterisation of $F^\times/(F^\times)^2$ is advantageous, which is provided in [Ser12a] in case of $F = \mathbb{Q}_p$. As a result, the following formulae are obtained.

Proposition 1.60. *If $F = \mathbb{Q}_p$ and $a, b \in \mathbb{Q}_p$ are written as $p^\alpha u, p^\beta v$ with $|u|_p = |v|_p = 1$, the Hilbert symbol of a, b admits the following representation:*

$$\mathcal{H}(a,b) = (-1)^{\alpha\beta\varepsilon(p)} \cdot \left(\frac{u}{p}\right)^\beta \left(\frac{v}{p}\right)^\alpha \qquad \text{if } p \neq 2,$$
$$\mathcal{H}(a,b) = (-1)^{\varepsilon(u)\varepsilon(v)+\alpha\omega(v)+\beta\omega(u)} \qquad \text{if } p = 2,$$

where $\left(\frac{u}{p}\right) := \left(\frac{\overline{u}}{p}\right) \equiv \overline{u}^{\frac{p-1}{2}} \mod p$ *is defined by the common* Legendre symbol *of $\overline{u}$ in $\mathbb{F}_p$ and $\overline{u}$ denotes the image of u under the canonical mapping $\mathbb{Z}_p^\times \to (\mathbb{Z}/p\mathbb{Z})^\times$. Furthermore, the notation* $\varepsilon(u) := \frac{u-1}{2} \mod 2$ *as well as* $\omega(u) := \frac{u^2-1}{8} \mod 2$ *has been used.*

Once the above formulae are established, they may be utilised to derive the following theorem.

Theorem 1.61. *For any $p \leq \infty$, the Hilbert symbol $\mathcal{H}_p$ defines a non-degenerated bilinear mapping on the $\mathbb{F}_2$ vector space $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$.*

The following examples of finite Hecke characters are central for the Weil representation of $\mathrm{SL}_2(\mathbb{A})$ and it is recommended to compare these with [Dei10, Lemma 6.3.2 p. 147].

Example 1.62. Let $b \in \mathbb{Q}^\times$, then for every $p \leq \infty$, $\chi_p^b : \mathbb{Q}_p^\times \to \{-1, 1\}, \quad a \mapsto \mathcal{H}_p(a,b)$ defines a character. Furthermore, χ_p^b is trivial on $\mathbb{Z}_p^\times$ for almost all p and the product over all $\chi_p^b(a_p)$ equals 1 for $(a_p)_{p\leq\infty} \in \mathbb{Q}^\times \subset \mathbb{A}^\times$, giving rise to a character

$$\chi^b : \mathbb{A}^\times/\mathbb{Q}^\times \to \{-1,1\}, \qquad (a_p)_{p\leq\infty} \mapsto \prod_{p\leq\infty} \chi_p^b(a_p).$$

For a quadratic $\mathbb{Q}$ vector space (V, q) of dimension m, the associated character is denoted by χ_V and equals χ^b with $b = (-1)^{\frac{m(m-1)}{2}} \det(V)$. Note that the mapping $(V, \mathrm{q}) \mapsto \chi_V$ is well defined on the underlying additive group of the Witt ring $W(\mathbb{Q})$ (cf. Section 1.1). Similarly, define χ_L for a quadratic $\mathbb{Z}$-lattice L and $\chi_L = \mathcal{H}_p(\,\cdot\,, \mathrm{disc}(L))$ for a local lattice.

In fact, all quadratic Dirichlet characters admit a representation via the Hilbert symbol as above. Each quadratic character, given by a fundamental discriminant $D \equiv 0, 1 \mod 4$ via the Kronecker symbol $m \mapsto \left(\frac{D}{m}\right)$, corresponds to the Hilbert symbol $\mathcal{H}_{\mathbb{A}}(\,\cdot\,, D)$.[6] Here,

$$\mathbb{A}^2 \ni (a, b) \mapsto \mathcal{H}_{\mathbb{A}}(a, b) := \prod_{p \le \infty} \mathcal{H}_p(a_p, b_p)$$

denotes the product of the local Hilbert symbols. Further, these characters are related to quadratic field extensions (cf. [Zag13]), which are all of the form $\mathbb{Q}(\sqrt{D})/\mathbb{Q}$ and will be discussed in the subsequent section.

1.5 Quadratic field extensions

In the following, elementary statements about quadratic field extensions of $\mathbb{Q}$ and $\mathbb{Q}_p$ are presented. The reader will find adequate references in [Neu06], [Hel90] as well as [Zag13] and a brief overview in [Fre90, Appendix A].
In the following, let R be a principal ideal domain and $Q(R)$ its field of fractions. In later sections the case $R = \mathbb{Z}_p$ for $p \le \infty$ will be primarily considered which is the reason for restricting the class of admissable rings R. If the field in question is $\mathbb{Q}_p$ for $p < \infty$, we associate the choice $R = \mathbb{Z}_p$, in case of $\mathbb{Q}$, the ring $R = \mathbb{Z}$ is considered.

Definition 1.63. Let $F/Q(R)$ be a field extension, then the roots of monic polynomials from $R[X]$ lying in F form a ring which is denoted $\mathcal{O}_F$ and referred to as *ring of integers* of F. Its elements are referred to as being *integral*.

Example 1.64. As a principal ideal domain, R is a unique factorisation domain and we have $\mathcal{O}_{Q(R)} = R$, so that there are no integral elements strictly above R in $Q(R)$. In particular, $\mathcal{O}_{\mathbb{Q}} = \mathbb{Z}$ and $\mathcal{O}_{\mathbb{Q}_p} = \mathbb{Z}_p$ for $p < \infty$.

Lemma 1.65. *If $F/Q(R)$ is algebraic, then every $\mathcal{O}_F$ module M in F is a free R module of rank $[F : Q(R)]$ In particular, M admits an R basis.*

[6] This relation is established in detail in [Opi18, 1.10 p. 20].

Example 1.66. Let $0, 1 \neq d \in \mathbb{Z}$ be square free and define

$$b = \begin{cases} \sqrt{d}, & \text{if } d \equiv 2,3 \mod 4, \\ (1+\sqrt{d})/2, & \text{if } d \equiv 1 \mod 4. \end{cases}$$

Then, $\mathcal{O}_F = \mathbb{Z} + b\mathbb{Z}$ for $F = \mathbb{Q}(\sqrt{d})$.

Definition 1.67. a) Let $\mathrm{tr}_{F/Q(R)}(x)$ and $\mathrm{N}_{F/Q(R)}(x)$ denote the *trace* and *norm* of an element $x \in F$ for a finite extension, given by the trace and determinant, respectively, of the associated linear Mapping $M_x : F \to F, \quad y \mapsto xy$.

b) The so called *discriminant* of (an R basis (α_i) of) $\mathcal{O}_F$ for separable $F/Q(R)$ is given by $D = \det(\mathrm{tr}_{F/Q(R)}(\alpha_i\alpha_j)_{i,j}) \in R/(R^\times)^2$. The discriminant $D_{F/Q(R)}$ of $F/Q(R)$ is the ideal generated by all discriminants of bases.[7]

Example 1.68. For $\mathbb{Q}(\sqrt{d})$ as above, a computation yields

$$D = \begin{cases} 4d, & \text{if } d \equiv 2,3 \mod 4, \\ d, & \text{if } d \equiv 1 \mod 4, \end{cases} \tag{1.12}$$

so that $F = \mathbb{Q}(\sqrt{d}) = \mathbb{Q}(\sqrt{D}) =: F_D$ and $\mathcal{O}_F = \mathbb{Z} + \frac{D+\sqrt{D}}{2}\mathbb{Z}$ for all choices of d. Discriminants of the above form correspond to determinants of binary even quadratic lattices over $\mathbb{Q}$.
Further, the norm $\mathrm{q} = \mathrm{N}_{F_D/\mathbb{Q}}$ restricted to $\mathcal{O}_{F_D}$ induces the structure of a quadratic lattice with Gram matrix (with respect to the basis $(1, \frac{D+\sqrt{D}}{2})$)

$$\begin{pmatrix} 2 & D \\ D & \frac{D^2-D}{2} \end{pmatrix},$$

so that $(F \otimes_{\mathbb{Q}} \mathbb{R}, \mathrm{q})$ is of signature

$$(1,1), \text{ if } D > 0,$$
$$(2,0), \text{ if } D < 0.$$

Further, note that $\mathrm{disc}(\mathcal{O}_F, \mathrm{q}) = (-1)(-D) = D$. In particular, if $D = \pm p$ is prime, we find that $L'/L \simeq (\mathbb{Z}/p\mathbb{Z}, rx^2/p)$ for some $p \nmid r$.

[7] In fact, this is a principle ideal, so that one basis suffices. As a consequence, the ideal D and generating element D will not be distinguished.

Definition 1.69. A finitely generated $\mathcal{O}_F$ module $\mathfrak{a}$ is called *fractional ideal.* The collection of such ideals forms an abelian group[8] $\mathscr{J}_{\mathcal{O}_F} = J_F$ with inversion defined by $\mathfrak{a}^{-1} = \{x \in F \mid x\mathfrak{a} \subseteq \mathcal{O}_F\}$ and identity element $\mathcal{O}_F$. The quotient of J_F and the principal fractional ideals $(a) = a\mathcal{O}_F$ for all $a \in F$ is called *ideal class group* and denoted Cl_F, so that the following sequence is exact:

$$1 \longrightarrow \mathcal{O}_F^* \longrightarrow F^* \longrightarrow J_F \longrightarrow Cl_F \longrightarrow 1.$$

For $R = \mathbb{Z}$ and an algebraic number field $F/\mathbb{Q}$, the number $|Cl_F|$ is finite and called *class number* of $F/\mathbb{Q}$.

The function $\mathrm{N}_{F_D/\mathbb{Q}}$ defines a quadratic form on F_D, so that every fractional ideal $\mathfrak{a}$ of F_D is naturally equipped with the structure of a quadratic lattice. For an ideal $0 \neq \mathfrak{a} \triangleleft \mathcal{O}_F$, the mapping $\mathfrak{a} \to [\mathcal{O}_F : \mathfrak{a}]$ extends to a homomorphism $\mathfrak{N} : J_F \to \mathbb{R}_+^\times$ which is called *absolute norm.*

Example 1.70. In fact, $\mathrm{q} = \mathrm{N}_{F_D/\mathbb{Q}} / \mathfrak{N}(\mathfrak{a}) : \mathfrak{a} \to \mathbb{Z}$ defines an integral (even) quadratic form on the fractional ideal $\mathfrak{a}$. This is true since $\mathrm{N}_{F_D/\mathbb{Q}}(a) = \mathfrak{N}(a\mathcal{O}_F)$ by definition, meaning that $\mathfrak{N}(\mathfrak{a}) \mid \mathrm{N}_{F_D/\mathbb{Q}}(a)$ for all $a \in \mathfrak{a}$. As a consequence, each fractional ideal is naturally an even $\mathbb{Z}$ lattice.

As mentioned above, fractional ideals admit a unique factorisation into prime ideals of $\mathcal{O}_F$ with exponents in $\mathbb{Z}$. However, a prime ideal $\mathfrak{p} \triangleleft R$ might not stay prime when passing to $\mathcal{O}_F$ (meaning to $\mathfrak{p} \equiv \mathfrak{p}\mathcal{O}_F$), so that there are prime ideals $\mathfrak{q}_i \triangleleft \mathcal{O}_F$, such that

$$\mathfrak{p} = \prod_{i=1}^{r} \mathfrak{q}_i^{e_i} \tag{1.13}$$

for adequate $e_i \in \mathbb{N}_0$ in a unique fashion. The exponents e_i are called *ramification index* and the degree of the field extension

$$f_i := [\mathcal{O}_F/\mathfrak{q}_i : R/\mathfrak{p}]$$

is called *inertia degree* of $\mathfrak{q}_i$ over $\mathfrak{p}$.

[8] Note that every element of this group admits a unique factorisation into prime ideals with exponents in $\mathbb{Z}$.

Proposition 1.71. *If $F/Q(R)$ is separable, then the following* fundamental identity *holds*

$$[F : Q(R)] = \sum_{i=1}^{r} e_i f_i.$$

There are the following notions which are advantageous for investigating ramification.

Definition 1.72. Let $\mathfrak{p}$ factor as in (1.13).

a) The prime ideal $\mathfrak{p}$ is said to *split completely* or *totally split* in F if $r = [F : Q(R)]$. On the other hand, $\mathfrak{p}$ is said to be *nonsplit* or *indecomposed* if $r = 1$.

b) Further, the prime ideal $\mathfrak{q}_i$ in the decomposition of $\mathfrak{p}$ is said to be *unramified* over $\mathcal{O}_F$ if $e_i = 1$ and $(\mathcal{O}_F/\mathfrak{q}_i)/(R/\mathfrak{p})$ is separable, otherwise it is called *ramified*. If, additionally, $f_i = 1$ for all i, then $\mathfrak{p}$ is called *totally ramified*.

c) Further, the ideal $\mathfrak{p}$ is called *unramified*, if all $\mathfrak{q}_i$ are unramified, and the extension $F/Q(R)$ is called *unramified*, if all of its prime ideals are unramified in F.

In fact, ramification barely occurs. For a given extension $F/Q(R)$, only finitely many prime ideals are ramified – these are exactly the prime divisors of the so called *different* (ideal). The *different ideal* of $F/Q(R)$ is given by the dual lattice of $\mathcal{O}_F$ with respect to the bilinear form $(x, y) \mapsto \operatorname{tr}_{F/Q(R)}(x, y) := \operatorname{tr}_{F/Q(R)}(xy)$ and is denoted by $\mathfrak{d}_F$.

Remark 1.73. Let $\mathfrak{a}$ be a fractional ideal of $F/Q(R)$ as in Example 1.70. Then the dual ideal can be computed by means of the different ideal via

$$\mathfrak{a}' = \mathfrak{d}_F^{-1}\mathfrak{a}.$$

Example 1.74. In case of $F_D/\mathbb{Q}$, we find $\mathfrak{d}_{F_D} = (\sqrt{D})$, so that $\mathfrak{a}' = (\sqrt{D})^{-1}\mathfrak{a}$. Further, for a prime ideal (identified with the number) $p \in \mathbb{Z}$, the occurrence of a case described in Definition 1.72 may be identified by means of the associated character:

$$\begin{aligned} p \text{ splits in } F_D &\iff & \left(\frac{D}{p}\right) &= 1, \\ p \text{ remains prime in } F_D &\iff & \left(\frac{D}{p}\right) &= -1, \\ p \text{ ramifies in } F_D &\iff & \left(\frac{D}{p}\right) &= 0. \end{aligned}$$

In order to investigate the global situation, local considerations prove to be useful. This is why the following definition for non-Archimedean fields is established.

Definition 1.75. A finite field extension of local fields F/K (with $K = Q(R)$) is called *unramified*, if

$$[F : K] = [\mathcal{O}_F/\mathfrak{m}_F : R/\mathfrak{m}_R],$$

where $\mathfrak{m}$ refers to the maximal ideal of the respective ring of integers.

Note that by Proposition 1.71, this coincides with Definition 1.72.

Valuations and global to local relations

This section is interconnected with Section 1.5, as the rings considered in elementary algebraic number theory are exactly these for which valuation rings can be naturally and consistently be associated. In Subsection 1.5 a prime factorisation in $\mathscr{I}_R$ for special types of rings R was introduced. Rings in which such a factorisation is possible are called *Dedekind domains* and these form a frame predestined to study discrete valuation rings (comparable to $\mathbb{Z}_p$). The reader further interested in the material will find the content provided in this thesis as well as additional relations in [Neu06] and [CF10, Chap. 1 p. 1].

Definition 1.76. A *discrete valuation ring* D is a principal ideal domain that is local, i.e. possesses exactly one maximal ideal $\mathfrak{m}_D$.

Given such a ring D, the maximal ideal $\mathfrak{m}_D$ is generated by an element π, which is uniquely determined up to a unit and called the *uniformiser* of D. Hence, every element $d \in D$ may be written as $d = \varepsilon \cdot \pi^n$ with a unit $\varepsilon \in D$ and $n \in \mathbb{N}_0$. The number n is denoted by $\nu_\mathfrak{p}(d)$, where we write $\mathfrak{p} = \mathfrak{m}_D$, and is called the *valuation* of d. The mapping extends uniquely to the quotient field of D and behaves in such a fashion, that $d \mapsto \rho^{\nu_\mathfrak{p}(d)}$ for $0 < \rho < 1$ defines a non-Archimedean absolute value function $|\cdot|_\mathfrak{p}$ on $Q(D)$.[9]

Definition 1.77. Let R be an integral domain, but not a field. Then, R is called *Dedekind domain* if every proper ideal factors into primes. Equivalently, it may be characterised by its localisations at every prime ideal $\mathfrak{p}$ being discrete valuation rings.

Example 1.78. Let $R = \mathbb{Z}$, then every prime ideal $\mathfrak{p}$ is represented by a prime number p. The localisation at $\mathfrak{p}$ identifies with

$$\mathbb{Z}_\mathfrak{p} = \left\{ \frac{x}{y} \mid x \in \mathbb{Z}, y \notin \mathfrak{p} \right\},$$

[9] In fact, every non-trivial absolute value function $|\cdot|$ on $Q(D)$ with $|D| \leq 1$ must be of this form.

meaning any number in $\mathbb{Q}$, whose nominator is not divided by p. The associated absolute value function is given by $z \mapsto |p|^{-\nu_p(z)}$ in accordance with Example 1.40. We see that $\mathbb{Z}_\mathfrak{p}$ is the unit ball with respect to this abs and that in fact $\mathbb{Q}_p$ is the completion of $Q(\mathbb{Z}_\mathfrak{p})$ along the abs. The same topological relation holds for $\mathbb{Z}_p$ and $\mathbb{Z}_\mathfrak{p}$.

Therefrom, the prime ideals $\mathfrak{p}$ of a Dedekind domain R give rise to complete non-Archimedean fields $\overline{Q(R)}^{|\cdot|_\mathfrak{p}} =: Q(R)_\mathfrak{p}$, similar to $\mathbb{Q}_p$ over $\mathbb{Q} = Q(\mathbb{Z})$ with ring of integers $\overline{R_\mathfrak{p}} = \{x \in Q(R)_\mathfrak{p} \mid |x|_\mathfrak{p} \leq 1\}$ and maximal ideal $\mathfrak{m}_{\overline{R_\mathfrak{p}}} = \{x \in Q(R)_\mathfrak{p} \mid |x|_\mathfrak{p} < 1\}$. These fields might be considered in order to characterise the field $Q(R)$ or compute invariants of $Q(R)$. Usually, $Q(R)$ is chosen to be a global field, in our setting a finite extension of $\mathbb{Q}$ and the prime ideals $\mathfrak{p}$ of $\mathcal{O}_{Q(R)}$ are called *places* of $Q(R)$ or $\mathcal{O}_{Q(R)}$, while the fields $Q(R)_\mathfrak{p}$ are referred to as *local fields*[10] associated to $Q(R)$.
In many cases the investigation of global properties might be reduced to the computation of local objects. A few supplementing examples will be presented in the following. For instance,

$$\mathscr{I}_R = \bigoplus_{\mathfrak{p} \text{ prime}} \mathscr{I}_{R_\mathfrak{p}} = \bigoplus_{\mathfrak{p} \text{ prime}} \mathscr{I}_{\overline{R_\mathfrak{p}}}, \tag{1.14}$$

so that global fractional ideals are completely described in terms of local ideals.[11] This should not be surprising, as $\mathscr{I}_{\overline{R_\mathfrak{p}}}$ is generated by one element, exactly the prime ideal $\mathfrak{p}$ and the left hand side of (1.14) equals the free group in exactly these prime ideals. In fact, the local objects are almost always trivial, so that only finitely many computations have to be carried out. For instance, consider two R modules M, N in a $Q(R)$ vector space V, which span V over $Q(R)$. Then for almost all $\mathfrak{p}$, we find $R_\mathfrak{p}M = R_\mathfrak{p}N$. Further, $M = \bigcap_{\mathfrak{p} \text{ prime}} R_\mathfrak{p}M$.

Example 1.79. Let L be a $\mathbb{Z}$ lattice in $\mathbb{Q}^n$, then $\mathbb{Z}_pL = \mathbb{Z}_p^n$ for almost all p and

$$\bigcap_{p<\infty} \mathbb{Z}_pL = L.$$

Next, two invariants of classical algebraic number theory are related to the local case, as they are useful for our proceedings. For two free[12] R modules M, N in a finite dimensional

[10] This notion might be defined rigorously and abstractly.
[11] Note that this implies that the class number is locally finite as well.
[12] In fact, requiring the modules to be free is not necessary, as by equation (1.14) the index can be defined locally and gives rise to one and only one ideal in $\mathscr{I}_R$.

$Q(R)$ vector space V, there is a linear isomorphism $\sigma : M \to N$ that extends to V. Define

$$[M : N]_R := R \det(\sigma)$$

as a principal ideal, the *module index* of N in M. It agrees with the group index, if $R = \mathbb{Z}$ and $M \supseteq N$.

Definition 1.80. The *discriminant* of an R module spanning a finite dimensional non-degenerate quadratic $Q(R)$ vector space V is defined to be

$$D(M) := D(M/R) := [M' : M]_R,$$

where M' is the dual module with respect to the bilinear form associated to V (cf. Section 1.1).

Note that $D(R_\mathfrak{p} M/R_\mathfrak{p}) = D(M/R)R_\mathfrak{p}$ as well as $D(\overline{R}M/\overline{R}) = D(M/R)\overline{R}$, and that in case of M being free,

$$D(M) = (\det(S))$$

where S is the Gram matrix to the trace on M. This, in fact, accords with Definition 1.67. Similarly to the above, the following relation may be established for duals:

$$(M\overline{R})' = M'\overline{R}.$$

Last, the discriminant and different ideal behave well when splitting considerations to local field extensions.

Remark 1.81. For a prime ideal $\mathfrak{p}$ in R and prime ideals $\mathfrak{B} \mid \mathfrak{p}$ in $\mathcal{O}_F$, we find

$$\mathfrak{d}(\mathcal{O}_F/R)\overline{\mathcal{O}_{F,\mathfrak{B}}} = \mathfrak{d}(\overline{\mathcal{O}_{F,\mathfrak{B}}}/R_\mathfrak{p}) \qquad \text{and} \qquad D(\mathcal{O}_F/R)\overline{R_\mathfrak{p}} = \prod_{\mathfrak{B}\mid\mathfrak{p}} D(\overline{\mathcal{O}_{F,\mathfrak{B}}}/\overline{R_\mathfrak{p}}).$$

Remark 1.82. The different ideal and discriminant give rise to a class of easily accessible lattices and are important constructions. Note that a lattice L in[13] $F = \mathbb{Q}_p(\sqrt{D})/\mathbb{Q}_p$ is distinct from its dual $\mathfrak{d}_F L$ (cf. Remark 1.73) if, and only if, $\mathfrak{d}_F$ is not trivial, which happens if, and only if, there is a prime ideal $\mathfrak{B}$ in $\mathcal{O}_F$ that divides $\mathfrak{d}_F$. This case occurs only if $p \mid D$, as $\mathfrak{d}_F = (\sqrt{D})$ (cf. Example 1.74) and (p) is the only prime ideal in $\mathcal{O}_{\mathbb{Q}_p}$.

[13] We explicitely state that D is assumed to be the discriminant of the extension, as the statement fails in case of $p = 2$, otherwise.

We conclude: the lattice $(\mathfrak{a}, N/N(\mathfrak{a}))$ in $\mathbb{Q}_p(\sqrt{D})/\mathbb{Q}_p$ is not unimodular if, and only if, $\mathbb{Q}_p(\sqrt{D})/\mathbb{Q}_p$ is ramified which is equivalent to $p \mid D$.

2 Classical modular forms

Modular forms are at the origin of the present thesis. Some notions and theorems of the classical theory are necessary for the later investigations in Sections 3.3 and 7.3. These are collected in a compact fashion in Section 2.2 of this chapter.

Apart from that, the author decided to devote a whole section to the motivation of classical modular forms, in order to build up an intuition for readers who are not familiar with a conceptual approach to modular forms. The path followed can be found in [DS05], [GBL10, III.59 p. 250], and [FB06, V p. 255], whereas introductory material offering different perspectives is, for instance, found in e.g. [Dei10], [KK98], [Iwa95], [Bum98], [Hec26], and [Bor97]. The curious reader will find elaborate material in [BC79], which is beyond the scope of this thesis, as well as a brief inspiring introduction to the general theory in [Bor65].

2.1 Introduction

Familiarity with at least classical modular forms[1] was an advantage in reading this thesis. Due to the sheer complexity[2] of the theory and the broad mathematical requirements to grasp its content, its introduction is often not realised conceptually. Nevertheless, a rough idea as well as fundamental concepts are discussed in this overview, which is based on a talk given by Solomon Friedberg at ICERM in 2013 [Fri13].

Based on mathematical experience, functions on a topological group G which are invariant under the action of a discrete (compact) subgroup Γ (or K) are particularly interesting. In a concrete setting, additional conditions may be imposed on these functions and the action of Γ frequently involves characters of G.

[1] These associated with $\mathrm{SL}_2(\mathbb{Z})$ on $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \simeq \mathbb{H}$ which are also referred to as *elliptic modular forms*.

[2] Cf. Robert Langlands on automorphic forms: 'It is a deeper subject than I appreciated and, I begin to suspect, deeper than anyone yet appreciates. To see it whole is certainly a daunting, for the moment even impossible, task.' according to [Fri13]

Example 2.1. Let $G = \mathbb{R}$ and $\Gamma = \mathbb{Z} < \mathbb{R}$, then a $\mathbb{Z}$ invariant function f is called *periodic* and gives rise to a Fourier expansion under reasonable circumstances (cf. [Con94, I.5 Thm. 5.11 p. 22]). In this sense, $\mathbb{Z}$ acts on $\mathbb{R}$ as a group if automorphism and f is an *automorphic function* (invariant under the action) with respect to Γ on G. In particular f may be regarded as a function on the quotient $\mathbb{R}/\mathbb{Z} \sim \mathbb{S}^1$ and by considering functions on $\mathbb{S}^1$ all automorphic functions on $\mathbb{R}$ with respect to $\mathbb{Z}$ are induced.

Remark 2.2. a) In case of $G = \mathrm{SL}_2(\mathbb{R})$ and a congruence subgroup Γ the theory of classical modular forms is obtained. Recall that by the Iwasawa decomposition $\mathbb{H} \simeq \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$, where the latter group is compact. A function f on G to $\mathbb{C}$ which is right invariant under $K = \mathrm{SO}_2(\mathbb{R})$ and invariant under Γ via the operation of the *Petersson operator* is called *classical modular form* if it fulfils certain analytic properties.[3] Usually, holomorphy or meromorphy is assumed. By imposing weaker conditions, for instance, real smoothness and satisfying certain differential equations, involving the hyperbolic Laplace operator, Maas forms are obtained (cf. [Bum98, 1.9 pp. 103-118]). This opens up the field to functional analysis, particularly operator theory, and representation theory. The latter was incorporated into the theory of automorphic forms by Gelfand and Fomin 1952 (cf. [GF52][4], while a reference for the former application is found in [Iwa95].

b) In the adelic setting, similar functions may be considered. For instance, $G = \mathbb{A}$ may be chosen along with functions f which are invariant under the operation of $\Gamma = \mathbb{Q}$ (Note that $\mathbb{Q} < \mathbb{A}$ is discrete and the quotient is compact). Further, the group $(\mathbb{A}/\mathbb{Q})^*$ may be identified with $\mathbb{Q}$ (in fact, with discrete topology - cf Remark 1.55), so that f admits a Fourier series analogously to the case $\mathbb{R}/\mathbb{Z}$. If G is set to be $\mathrm{GL}_1(\mathbb{A}) = \mathbb{A}^\times$ and $\Gamma = \mathrm{GL}_1(\mathbb{Q}) = \mathbb{Q}^\times$, automorphic forms are multiplicative functions on G, which are invariant under Γ and called *Hecke* characters (cf. Section 1.3).

c) Similarly, adelisations of rational groups may be considered, for instance, $\mathrm{SL}_2(\mathbb{A})$ (cf. [Bor91]). In this setting certain functions which transform with a character under a compact subgroup K from the right and are invariant under $\mathrm{SL}_2(\mathbb{Q})$ from the left may be related to classical modular forms (cf [Bum98, 3.6 p. 341] and the concrete example in Section 5.2.1), providing a broader context for classical modular forms.

[3] In fact, this classical setting allows for a colourful geometric interpretation, discussed over the course of the subsequent sections.

[4] An English translation of that source is available by the American Mathematical Society.

2.1.1 A geometric approach

The viewpoints on automorphic forms are manifold. However, the classical case admits a certain concrete geometric interpretation which shall be discussed over the course of the subsequent sections. In short, classical automorphic forms (a generalised notion of modular forms) are global meromorphic differentials of the compactified coarse moduli space of complex elliptic curves. The following sections are devoted to fill these words with meaning for the reader unfamiliar with the aforementioned concepts.

Elliptic curves

At the origin of the geometrical approach lie *elliptic curves*, objects which are of a particular interest for number theorists. In fact, the ancient Greeks had already investigated equations such as

$$aX + bY = c \qquad a, b, c \in \mathbb{N}$$

and asked for its solutions for X and Y. In a modern context, the coefficients would be chosen from $\mathbb{Z}$ and it is well known, that the equation always has infinitely many solutions over $\mathbb{Q}$ and admits an integer solution, if, and only if, $\gcd(a, b) \mid c$. All of these solutions are computable via the Euclidean Algorithm.
An immediate problem arising from the above is to determine the solution of equations of the form

$$aX^n + bY^n = c$$

for natural numbers n which is exactly *Fermat's last Theorem* for $a, b, c = 1$. Questions of the type presented above are the subject of *Diophantine Geometry.* In fact, the case of a quadratic polynomial is solved by the Theorem of *Hasse–Minkowsky*, which relates solvability over $\mathbb{Q}$ to solvability over all p-adic numbers $\mathbb{Q}_p$. In this case there are infinitely many rational solutions. If, however, the degree of an irreducible polynomial is greater than 4, the number of rational solutions is finite, which was proven by Faltings[5] (1983) and conjectured by Mordell (1922).
At the verge of these cases lies the class of elliptic curves which correspond to cubic polynomials. They can have infinitely many rational solutions or finitely many, manifesting a bridge between both cases. This feature alone suffices to render them interesting for

[5] In fact, he received the fields medal for his proof of the conjecture in 1986.

number theorists.
In the following, we assume the characteristic of the field F to be different from 2 and 3.[6]

Definition 2.3. An *elliptic curve* over a field F, written E/F is given by a cubic equation

$$Y^2 = X^3 + aX + b \tag{2.1}$$

where $a, b \in F$ and the associated *discriminant* $\Delta := -2^4(4a^3 + 27b^2)$ does not vanish. Equation (2.1) is called *affine Weierstraß* equation and the parameter $j := -1728\frac{(4a)^3}{\Delta}$ is called *j-invariant*.

More geometrically, an elliptic curve is defined to be a projective, non-singular curve of genus 1 with a distinguished point. This results in the curve E having a cubic Weierstraß equation, which may look far more complicated than the above but may be transformed into that form via rational coordinate transformations, provided $\mathrm{char}(F) \neq 2, 3$. The condition on the discriminant translates to the curve being non-singular and the j-invariant parametrises elliptic curves over a field. The distinguished point may be chosen to be ∞ in the projective space. Compare Figure 2.1.1 for illustrations.

Nonetheless, there is more to elliptic curves than points in a plane – an algebraic structure! In fact, the Theorem of Riemann–Roch induces a natural addition on an elliptic curve, implying that every such curve can be understood as an abelian variety. A theorem of Mordell–Weil states that for any number field F the elliptic curve E/F as a group bears the structure of a finitely generated abelian group. The rank of it's free component is called *rank* of the elliptic curve and determines whether there are infinitely many solutions or not. In fact, the rank of an elliptic curve is the subject of the famous conjecture of Birch and Swinnerton-Dyer. Their conjecture relates the Laurent coefficients at a critical point of an L-series associated to the elliptic curve to different parameters of the curve. In particular, the pole order at the critical point is conjectured to equal the rank of the curve.
The aim of the following sections is to construct the coarse moduli space of complex elliptic curves with the aim of realising classical automorphic forms as meromorphic differentials on that space.

[6]This assumption guarantees that the Weierstraß equation can then be chosen to be much simpler.

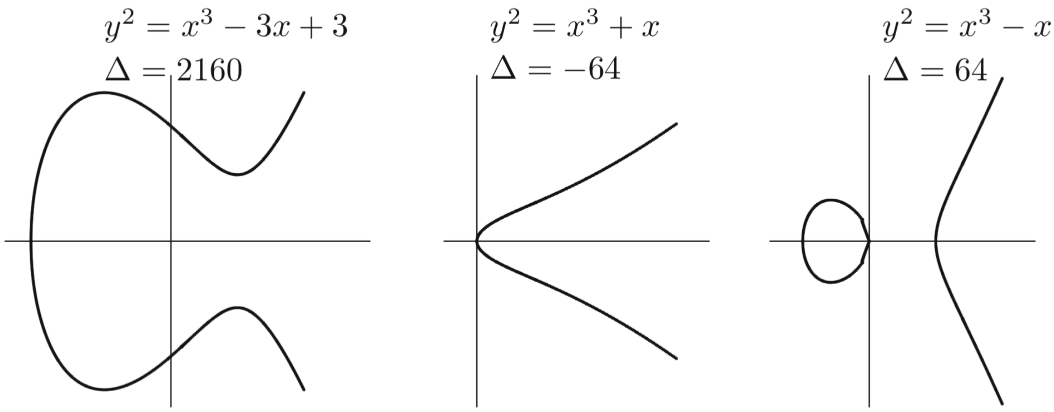


Figure 3.1: Three elliptic curves

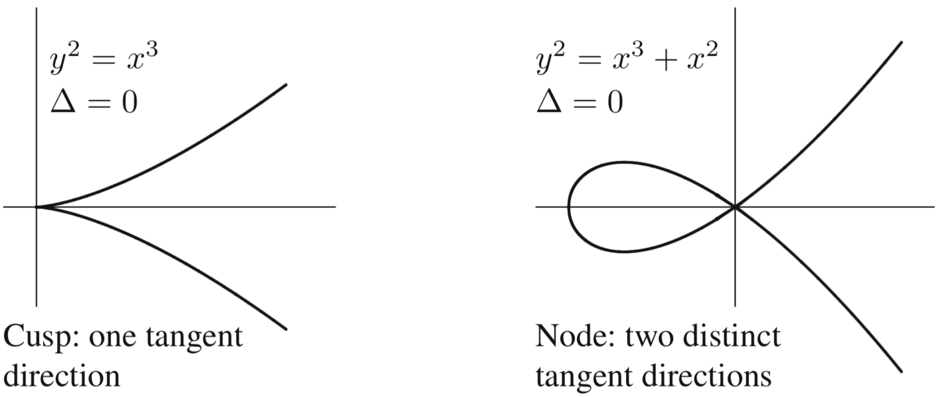


Figure 3.2: Two singular cubic curves.

Figure 2.1.1: Illustration of elliptic curves over $\mathbb{R}$ (cf. [Sil13, III.1 p. 43]). Indeed, cubic curves are smooth if, and only if, $\Delta \neq 0$. Also note that the last equation transforms to $y^2 = x^3 - 3x + 2$ under $x \mapsto (x-1)/3, y \mapsto y/3$, so it can be transformed to match Definition 2.3.

The coarse moduli space of complex elliptic curves

In order to construct the desired moduli space, a concrete realisation of complex elliptic curves is advantageous. Hence, these are related to tori in the following by the well known *Weierstraß* $\wp$ function.
Let $\Lambda < \mathbb{C}$ be a lattice of rank 2. Similar to the examples above, Λ-automorphic functions $f : \mathbb{C} \to \overline{\mathbb{C}}$ may be considered, meaning functions that fulfil

$$\forall \lambda \in \Lambda, z \in \mathbb{C} : f(z + \lambda) = f(z).$$

A naive attempt in order to construct such a function would be the summation over simple poles at the lattice points, which fails due to divergence. The next natural candidate is a sum over poles of order 2 which may be realised as a convergent series with the following correction terms:

$$\wp_\Lambda : \mathbb{C} \to \overline{\mathbb{C}}, \qquad z \mapsto \frac{1}{z^2} + \sum_{0 \neq \lambda \in \Lambda} \left(\frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2} \right).$$

This function is called *Weierstraß p-function* and represents a meromorphic Λ-automorphic function on $\mathbb{C}$.[7] In fact, $\wp_\Lambda$ may be used to classify elliptic functions. First, note that every even elliptic function can be written as a rational function in $\wp_\Lambda$ (cf. [FB06, Thm. 3.2 p. 275]). But the differential of $\wp_\Lambda$ is odd, so that the field of elliptic functions must be a quadratic extension of $\mathbb{C}(\wp_\Lambda)$. A direct computation confirms that the associated minimal polynomial is given by

$$(\wp'_\Lambda)^2 = 4 \cdot \wp_\Lambda^3 - g_2 \cdot \wp_\Lambda - g_3, \tag{2.2}$$

where

$$g_2 = 60 \sum_{\lambda \in \Lambda \setminus \{0\}} \lambda^{-4}, \qquad g_3 = 140 \sum_{\lambda \in \Lambda \setminus \{0\}} \lambda^{-6}.$$

These coefficients are called *Eisenstein series* and represent modular forms[8]. Nonetheless, equation (2.2) and (2.1) are of the same form, so that it appears to be not too far off to seek relating the associated solution sets. To be precise, the map to projective space

$$\mathbb{C}/\Lambda \ni z \mapsto [\wp_\Lambda(z), \wp'_\Lambda(z), 1] \in \mathbb{P}^2(\mathbb{C})$$

[7] Such a function, meaning an Λ-automorphic function on $\mathbb{C}$, is called *elliptic function*. The name is derived from the fact that these functions were discovered to occur as inverses to so called elliptic integrals – integrals which parametrise the arc length of an ellipse.

[8] However, this is not evident at the moment.

is a natural candidate, which maps 0 to infinity. In fact the map is bijective, in the sense that it relates elements of the Torus $\mathbb{C}/\Lambda$ to elements in $\mathbb{P}^2(\mathbb{C})$ which fulfil equation (2.2) for independent variables $(\wp_\Lambda, \wp'_\Lambda)$. The associated elliptic curve is denoted $E_\Lambda/\mathbb{C}$. Exporting the group structure from $\mathbb{C}/\Lambda$ yields exactly the addition induced by the Theorem of Riemann–Roch, so that the above mapping is, in fact, a biholomorphic homomorphism $\mathbb{C}/\Lambda \simeq E_\Lambda/\mathbb{C}$. Two of these tori $\mathbb{C}/\Lambda$ and $\mathbb{C}/\Lambda'$ can be shown to be biholomorphically equivalent[9] if, and only if, $\Lambda' = \alpha\Lambda$ for some $\alpha \in \mathbb{C}^\times$, meaning they arise from each other by homogeneous stretching and rotation.

After complex elliptic curves have been discussed briefly, the construction of the associated moduli space will be carried out. The approach below comprises the topological realisation followed by an outline of equipping it with a complex structure as well as its compactification.

In order to parametrise a space of elliptic curves, considering bases of lattices appears to be an advantageous starting point. A basis of a lattice Λ is given by a pair of complex non-collinear numbers (z_1, z_2). Since scaling does not change the class of the quotient $\mathbb{C}/\Lambda$ we are interested in, considering $\tau := \frac{z_1}{z_2}$ suffices. Further, note that changing the orientation of the basis does not give rise to another lattice, so that $\tau \in \mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ may be assumed, effectively fixing an orientation. In summary, the space $\mathcal{B}$ of unordered bases of lattices modulo scaling by $\mathbb{C}^\times$ may be viewed as the upper half plane $\mathbb{H}$:

$$\mathcal{B}/\mathbb{C}^\times \simeq \mathbb{H}.$$

However, this is not a one to one description of elliptic curves, since there is in general more than one basis for a lattice. In fact, a suitable $\mathbb{Z}$ linear combination of basis elements (z_1, z_2) represents a basis of the same lattice again. Such a base change can be identified with a matrix M with coefficients in $\mathbb{Z}$ and the most primitive examples of these are translation and rotation:

$$T(z_1, z_2) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)(z_1, z_2) = (z_1 + z_2, z_2), \qquad S(z_1, z_2) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)(z_1, z_2) = (-z_2, z_1).$$

Abusing the information of reversibility of this base change

$$M \in \mathrm{GL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z}) \,\dot{\cup}\, \mathrm{SL}_2(\mathbb{Z}) \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$$

[9] In fact, such a biholomorphism is a group isomophism if, and only if, it preserves 0, as the biholomorphisms are identified to be of the form $x \mapsto \alpha x + \beta$ by lifting them to the covering spaces $\mathbb{C}$, which only allow for affine linear transformations as biholomorphisms.

is derived. Since orientation had already been fixed, only base changes from $\mathrm{SL}_2(\mathbb{Z}) =: \Gamma(1)$ need to be considered which form a complete set of representatives. Note that the action of $\mathrm{SL}_2(\mathbb{Z})$ and $\mathbb{C}^\times$ commute on $\mathcal{B}$, so that the *Modular curve*

$$\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{B}/\mathbb{C}^\times \simeq \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} =: Y(\Gamma(1))$$

is well defined. In order to conclude that $\gamma \in \Gamma(1)$ operates via *Möbius transformations*[10] on $\mathbb{H}$ note that

$$\gamma\begin{pmatrix}\tau\\1\end{pmatrix} = \begin{pmatrix}a\tau+b\\c\tau+d\end{pmatrix} \simeq \begin{pmatrix}\frac{a\tau+b}{c\tau+d}\\1\end{pmatrix} \in \mathbb{H}.$$

In order to picture the geometry which will be induced on this quotient set, a *fundamental domain* which is a connected full set of representatives in $\mathbb{H}$ will be considered.[11] Such a domain for $Y(\Gamma(1))$ is given by

$$\mathcal{F}_{\Gamma(1)} = \left\{\tau \in \mathbb{H} \mid |\operatorname{Re}(\tau)| \leq \tfrac{1}{2}, |\tau| \geq 1\right\} \tag{2.3}$$

and is illustrated in Figure 2.1.2.

In order to glue the fundamental domain properly to a geometric representation of $Y(\Gamma(1))$, ambiguities have to be investigated. While the interior of $\mathcal{F}_{\Gamma(1)}$ does not contain multiple representatives of one class, its boundary is full of redundance which is illustrated in Figure 2.1.3. Utilising that $\mathcal{F}_{\Gamma(1)}$ is a fundamental domain for $\Gamma(1)$, the identity

$$\mathrm{SL}_2(\mathbb{Z}) = \langle S, T\rangle$$

may be derived rendering the examples of transformations presented essentially exhaustive.

The next step would be the introduction of a complex structure for the topological space $Y(\Gamma(1))$ as well as its compactification.
In order to endow $Y(\Gamma(1))$ with the desired structure, charts have to be constructed. Around most points of the quotient, this is trivial, since the natural projection from $\mathbb{H}$ to $Y(\Gamma(1))$ is one to one in a sufficiently small neighbourhood. The only exceptions to this are the image of i and ρ. This phenomenon occurs due to the fact that these points in $\mathbb{H}$ have a non-trivial stabiliser in $\mathrm{SL}_2(\mathbb{Z})$, rendering the projection not injective on any neighbourhood. To address this problem, the *Cayley* transform is applied to transfer the setting to the open unit disc, where it is solved via unwrapping (details are treated in

[10] In fact, this computation and the above motivation clarify the sudden appearance of Möbius transformations in the vast majority of textbooks on this subject.
[11] In fact, ambiguities on null sets are neglected.

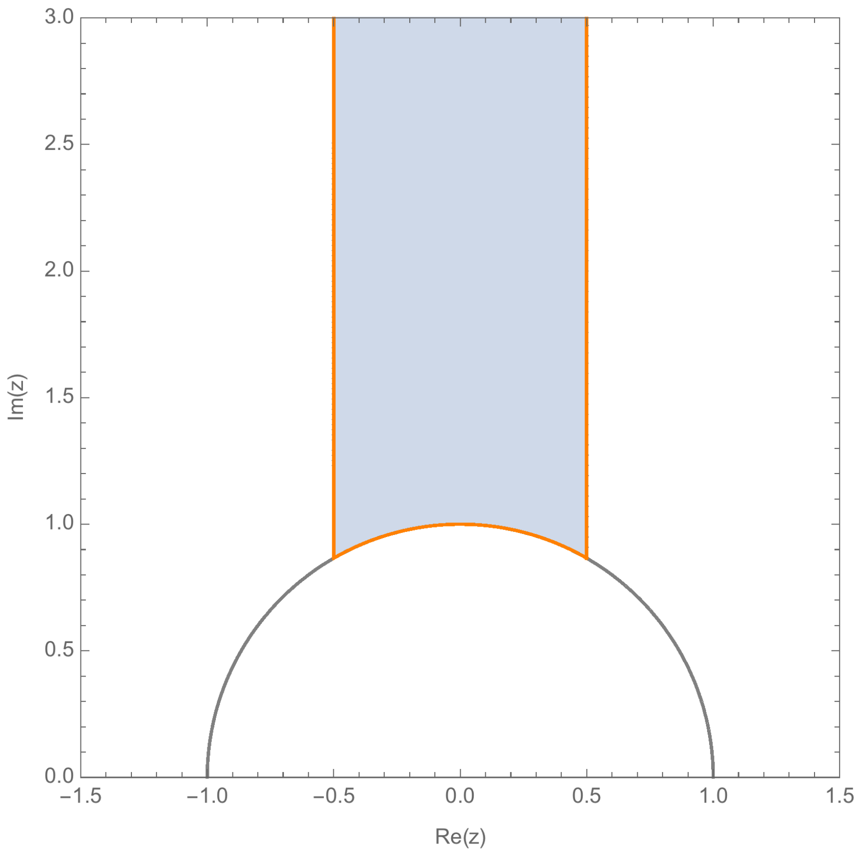


Figure 2.1.2: Illustration of $\mathcal{F}_{\Gamma(1)}$ (blue). Its border is marked in orange in addition to the upper half of the unit circle (grey) for the purpose of orientation. The corners match $\rho := \exp(2\pi i/6)$ and ρ^2.

[DS05, 2.2 p. 50]).

Next, the curve $Y(\Gamma(1))$ will be compactified, offering the application of advanced tools from algebraic geometry to study the resulting object. With this intention, the following chart, compatible with the above, is considered. Note that

$$\mathbb{H} \ni \tau \mapsto e^{2\pi i \tau} \in \dot{B}_1(0)$$

induces a biholomorphic map from $\mathbb{H}/\langle T \rangle$ (a vertical strip) to the open unit disc without zero $\dot{B}_1(0)$. Restricting this map to all elements $\tau \in \mathbb{H}$ with $\operatorname{Im}(\tau) > c$ for a fixed number $c > 1$ then provides a biholomorphic map from a vertical strip to a smaller open disc without its centre.[12] The situation is illustrated in Figure 2.1.4. The consequential step is

[12] As a consequence, every periodic (in real direction) function that is holomorphic on an upper half plane

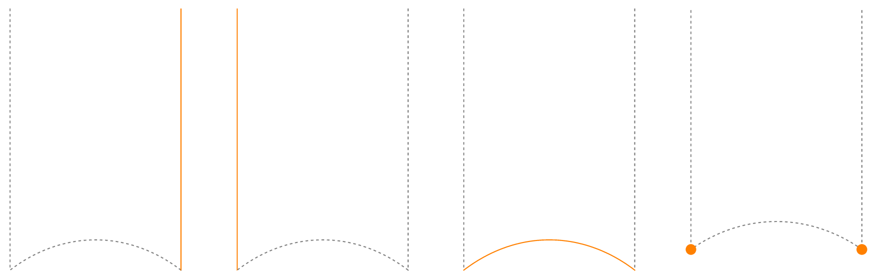


Figure 2.1.3: Illustration of ambiguities in $\mathcal{F}_{\Gamma(1)}$ together with the transformations they arise from. The orange lines correspond to the intersection of $\mathcal{F}_{\Gamma(1)}$ and its transform.

to formally add an element '∞' to $Y(\Gamma(1))$ and declare the geometry around it by the coordinate map ψ. This process yields a compactification of the curve $Y(\Gamma(1))$ which is denoted by $X(\Gamma(1))$ and referred to as the *Baily–Borel compactification*.

has a Fourier series. This is seen by pulling the function to the punctured unit disc, where it has a Laurent series around 0. Resubstituting the variable for $e^{2\pi i\tau}$ yields the desired Fourier series on an upper halfplane, which is truncated to the left. From this point of view, being holomorphic in ∞ is meaningful and translates to all Fourier coefficients being zero for negative indices.

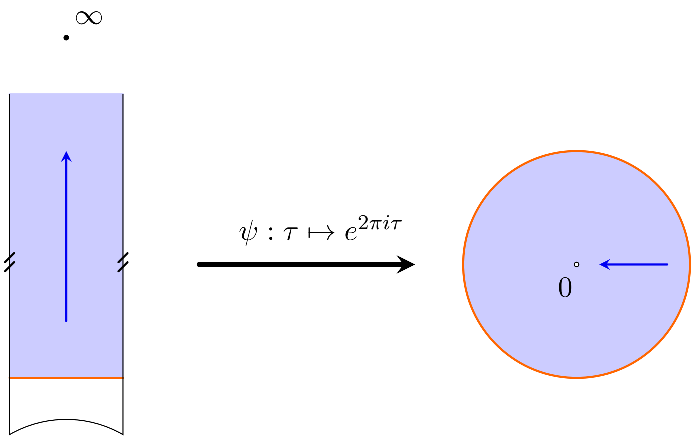


Figure 2.1.4: Illustration of the construction of a chart at '∞'. The blue arrow indicates the geometrical correspondence, while the orange lines are mapped to each other.

Classical modular forms as differentials

Previously, the space $X(\Gamma(1))$ has been constructed as a compactification of an important geometric description of elliptic curves. Hence, it appears natural to investigate functions on this space. Note that due to compactness, all holomorphic functions on $X(\Gamma(1))$ are trivial, leading to meromorphic functions as the next potential class to examine. However, *Klein*'s j-invariant can be proven to be a biholomorphism of $X(\Gamma(1))$ and the Riemann sphere, mapping ∞ to ∞, so that j parametrises elliptic curves over $\mathbb{C}$. As a consequence, meromorphic functions on $X(\Gamma(1))$ accord with rational functions in j, resulting in an immense space with a very simple structure. Fortunately, there is a natural class of function like objects attached to $X(\Gamma(1))$, which yields finite dimensional spaces of utmost significance in number theory. Instead of recognising functions only, meromorphic differentials might be considered.

The sheaf of meromorphic differentials of degree n on $\mathbb{C}$ is given by associating to each open set $V \subseteq \mathbb{C}$ objects

$$f(q)(\mathrm{d}q)^n,$$

where f is a meromorphic function on V. These are identified as sections through tensor powers of the cotangent space. The above definition translates instantly to manifolds and we shall be interested in global sections, leading to a compatibility criterion for local descriptions with respect to coordinate transformations. Let g be such a transformation, then we have

$$g^*(f(q)(\mathrm{d}q)^n) = f(g(q))g'(q)^n(\mathrm{d}q)^n,$$

where g^* denotes the pullback along g. However, it is not evident how to construct differentials on $X(\Gamma(1))$ unless a different perspective is chosen. In fact, a differential might be pulled back along the natural projection $\pi : \mathbb{H} \to X(\Gamma(1))$. The resulting differential $f(\tau)(\mathrm{d}\tau)^n$ must then be invariant under pullbacks from $\Gamma(1)$, so that

$$\begin{aligned} f(\tau)(d\tau)^n &= \gamma^* \left(f(\tau)(d\tau)^n\right) \\ &= f(\gamma\tau)(\gamma(\tau)')^n(d\tau)^n \\ &= f(\gamma\tau)\left(\frac{a(c\tau+d)-c(a\tau+b)}{(c\tau+d)^2}\right)^n (d\tau)^n \\ &= f(\gamma\tau)\left(\frac{ad-bc}{(c\tau+d)^2}\right)^n (d\tau)^n \\ &= \frac{f(\gamma\tau)}{(c\tau+d)^{2n}}(d\tau)^n \end{aligned}$$

has to be true for all $\gamma \in \Gamma(1)$. We introduce $j(\gamma, \tau) := (c\tau + d)$, the so called *factor of automorphy* and observe that j defines a cocycle:

$$j(\gamma \cdot \gamma', \tau) = j(\gamma, \gamma'\tau) j(\gamma', \tau). \tag{2.4}$$

Further, having the transformation property of $f(\mathrm{d}\tau)^n$ in mind, we define the *Petersson operator* of weight $k \in \mathbb{Z}$ given by

$$(f|_k\gamma)(\tau) = j(\gamma, \tau)^{-k} f(\gamma\tau). \tag{2.5}$$

This defines a right operation of $\Gamma(1)$ on the space of meromorphic (holomorphic) functions on $\mathbb{H}$. In addition to being invariant under pullbacks from $\Gamma(1)$, the differentials $f(\mathrm{d}\tau)^n$ have to be meromorphic in ∞, which translates to their Fourier expansion having only finitely many coefficients of negative index.
Combining the information above results in a bijective correspondence of meromorphic differentials of degree $k/2$ (for $k \in 2\mathbb{N}_0$) on $X(\Gamma(1))$ and the following objects.

Definition 2.4. An *automorphic form* of weight $k \in \mathbb{Z}$ on $\mathbb{H}$ for $\Gamma(1)$ is a meromorphic function f on $\mathbb{H}$, that fulfils the following properties.

i) f is invariant with respect to the Petersson operator $|_k$ for all $\gamma \in \Gamma(1)$.

ii) f is meromorphic in ∞.

This alternative description via functions on $\mathbb{H}$ is a convenient tool for constructing such differentials on $X(\Gamma(1))$. The resulting spaces will be infinite dimensional, but the alternative definition suggests the following modification. Instead of demanding f to be meromorphic, this property might be replaced by being holomorphic[13], resulting in the definition of classical modular forms which span finite dimensional spaces $\mathcal{M}_k(\Gamma(1))$ for each k:[14]

$$\dim(\mathcal{M}_k)_{k\in\mathbb{N}} = (0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 2, \ldots).$$

Further, instead of considering $\Gamma(1)$ as a group of automorphisms on $\mathbb{H}$, other discrete groups may be studied. A fruitful setting is given by *Fuchsian* groups Γ (cf. [Iwa95,

[13]This condition translates to orders of vanishing in the cusps and fix points of $\Gamma(1)$ for differentials, so it is not immeditely intuitive from a geometrical viewpoint.

[14]In fact, by i), every holomorphic modular form is 0 for even $k \in \mathbb{N}$. Upper bounds for the other cases up to 12 are found by means of the residue theorem of complex Analysis and are met by Eisenstein series and the Ramanujan τ-function Δ.

Chap. 2 p. 39]), which act discontinuously, rendering the quotient $\Gamma\backslash\mathbb{H}$ a T_2 space, so that it may be endowed with the structure of a manifold. Popular choices for groups are special subgroups of $\Gamma(1)$, introduced in the following.
Let $\Gamma(N)$ denote the kernel of the natural projection

$$\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}).$$

The index of $\Gamma(N)$ in $\Gamma(1)$ is finite, so that a set of representatives of $\Gamma(N)\backslash\mathbb{H}$ is constructed by gluing finitely many translates of $\mathcal{F}_{\Gamma(1)}$. A group Γ lying in between $\Gamma(N) \leq \Gamma \leq \Gamma(1)$ is called a *congruence subgroups* . These play a major role in the classical theory of modular forms. The reason why restricting the theory to these groups is that the associated spaces of modular forms offer a rich operator theory, called *Hecke* theory (cf. Section 6.2). This theory proves valuable when modular forms are associated to L-series or when elementary geometry[15] is considered (cf. [Bum98, 1.4 p. 41]). We shall have a glance at common examples of congruence subgroups: the group

$$\Gamma_0(N) := \left\{ \gamma \in \Gamma(1) \middle| \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, c \equiv 0 \mod N \right\}$$

for some $N \in \mathbb{N}$ occurs naturally, when considering elliptic curves with torsion data and we clearly find $\Gamma(N) \leq \Gamma_0(N)$. Further, $\Gamma_1(N)$ is defined to be the kernel of

$$\Gamma_0(N) \ni \gamma \mapsto d \in (\mathbb{Z}/N\mathbb{Z})^\times$$

and $\Gamma(N)$ equals the kernel of the homomorphism

$$\Gamma_1(N) \ni \gamma \mapsto b \in \mathbb{Z}/N\mathbb{Z}.$$

Both projections are surjective, so that the number of cosets may be computed elementarily. In summary,

$$\Gamma(N) \trianglelefteq \Gamma_1(N) \trianglelefteq \Gamma_0(N) \leq \mathrm{SL}_2(\mathbb{Z}).$$

At this point, the setting of a group Γ acting discontinuously on the upper half plane $\mathbb{H}$ might appear to be too specific unless the universal cover property of $\mathbb{H}$ is taken into account which clarifies the broadness of this choice. In fact, any compact Riemann

[15] There is a scalar product on the space of cusp forms, the Petersson product, which is compatible with Hecke operators.

surface M of genus strictly greater than 1 has the upper half plane $\mathbb{H}$ as its universal cover and its fundamental group $\pi_1(M)$ may be realised as a group Γ that acts discontinuously on $\mathbb{H}$ (cf. [Bum98, 3.7 p. 349]).

2.1.2 Constructions and generalisations

Before constructing classes of examples of modular forms and presenting more general notions than in the proceedings above, there are a few elementary facts to be collected about the action of $\mathrm{GL}_2^+(\mathbb{R})$ on $\mathbb{H}$. Recall that the group $\mathrm{GL}_2(\mathbb{R})$ acts on $\mathbb{C}\setminus\mathbb{R}$ via Möbius transformations. Namely, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})$ acts on $\tau \in \mathbb{C}\setminus\mathbb{R}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}.$$

This family of biholomorphic mappings of $\mathbb{C}\setminus\mathbb{R}$ is exhaustive. We also recall the factor of automorphy $j(\gamma, \tau) = c\tau + d$ and note that by direct computation

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau &= \frac{a\tau + b}{c\tau + d} \\ &= \frac{[a(u+iv)+b][c(u-iv)+d]}{(cu+d)^2 + (cv)^2} \\ &= \frac{ac(u^2+v^2) + bd + ad(u+iv) + bc(u-iv)}{(cu+d)^2 + (cv)^2} \\ &= \frac{ac(u^2+v^2) + bd + (ad+bc)u + iv(ad-bc)}{(cu+d)^2 + (cv)^2} \\ &= \frac{ac|\tau|^2 + bd + (ad+bc)u}{|j(\gamma,\tau)|^2} + i\det(\gamma)\frac{v}{|j(\gamma,\tau)|^2}. \end{aligned}$$

As a consequence, we find

$$\mathrm{Im}(\gamma\tau) = \det(\gamma)\frac{\mathrm{Im}(\tau)}{|j(\gamma,\tau)|^2}.$$

Eisenstein series

The investigation above portrays an elegant and aesthetic abstract approach to modular forms, but the existence of such functions has not been verified yet. This is where Eisenstein series come into play, as the first examples and most elementarily accessible class of modular forms.

Since a modular form has to be invariant with respect to the Petersson operator, symmetrising seems to be an obvious approach. For a function $f : \mathbb{H} \to \mathbb{C}$, the sum

$$\sum_{\gamma\in\Gamma(1)} f|_k\gamma$$

is a natural candidate, which fulfils, at least formally, the invariance with respect to the operator $|_k$. The next step is to choose the function f, such that the sum converges and is holomorphic. This, however, is hopeless for a holomorphic function f and it appears to be reasonable to restrict the summation. For simplicity[16], f shall be assumed to be 1-periodic, resulting in the group $\Gamma(1)_\infty = \langle \pm 1, T\rangle$ not changing the summand, so that

$$\sum_{\gamma\in\Gamma(1)_\infty\backslash\Gamma(1)} f|_k\gamma$$

would be an appropriate choice for an invariant function. If f is, for instance, chosen to be constant, the sum converges locally uniformly for $k > 2$ and defines a holomorphic function. In this case, the resulting function

$$E_k := \sum_{\gamma\in\Gamma(1)_\infty\backslash\Gamma(1)} 1|_k\gamma \tag{2.6}$$

is called *Eisenstein series* of weight k and the following computation yields that it is even holomorphic at ∞, with value 1:

$$\begin{aligned} 2\zeta(k)E_k &= 2\zeta(k)\cdot\sum_{\gamma\in\Gamma_\infty(1)\backslash\Gamma(1)} j(\gamma,\tau)^{-k} \\ &= \zeta(k)\cdot\sum_{\substack{c,d\in\mathbb{Z}\\(c,d)=1}}\frac{1}{(c\tau+d)^k} \\ &= \sum_{\substack{c,d\in\mathbb{Z}\\(c,d)\neq 0}}\frac{1}{(c\tau+d)^k} \\ &= \sum_{\lambda\in\Lambda_\tau\backslash\{0\}}\frac{1}{\lambda^k}. \end{aligned} \tag{$\star$}$$

The first equality is valid, since

$$\Gamma(1)_\infty\backslash\Gamma(1) \to \left\{(c,d)\in\mathbb{Z}^2 \mid (c,d)=1\right\}, \qquad (\bar{c},\bar{d}) \mapsto (c,d)$$

[16] A more general construction is known by the term *Poincaré series* (cf. Subsection 3.3.2).

defines a bijection and the representation via ($\star$) is the one commonly used in introductory texts on the topic. This sum also immediately yields that E_k is holomorphic in ∞, by letting $\text{Im}(\tau) \to \infty$. An adept computation involving the partial fraction expansion of the cotangent function[17] yields the Fourier expansion of the Eisenstein series (cf. [DS05, p. 5]):

$$E_k = 1 - \frac{2k}{B_k} \sum_{n>0} \sigma_{k-1}(n) \cdot e^{2\pi i n\tau}, \tag{2.7}$$

where B_k is the k-th Bernoulli–number and

$$\sigma_s(n) = \sum_{d|n} d^s \tag{2.8}$$

denotes the so called *divisor function* of degree $s \in \mathbb{C}$. The latter is a classical, multiplicative number theoretic function, meaning that the coefficients of E_k bear number theoretic information.

The Fourier expansion (2.7) together with the fact that the spaces of modular forms are finite dimensional may be utilised to derive unimagined identities between different sequences of Fourier coefficients. For instance, $E_4^2 = E_8$ has to hold, since the space $\mathcal{M}_8$ is 1 dimensional. As a result,

$$\sigma_7(n) = 120 \cdot \sum_{k=0}^{n} \sigma_3(n-k)\sigma_3(k)$$

is derived immediately.

Eisenstein series are not only the first examples of and most easily accessible modular forms, but to some extent *all* classical modular forms. To verify this claim note that the discriminant, technically defined above in Definition 2.3, can be written as

$$\Delta = \frac{E_4^3 - E_6^2}{1728}$$

and does not vanish on $\mathbb{H}$, by definition. It vanishes in ∞ and, as a consequence, defines a linear bijection

$$f \mapsto \Delta \cdot f, \qquad \mathcal{M}_k \to \mathcal{S}_{k+12} < \mathcal{M}_{k+12},$$

[17] In fact, this perspective offers another point for constructing modular forms from rational functions in the variable q – cf. [Fra20].

where $\mathcal{S}_k$ denotes the subspace of functions in $\mathcal{M}_k$ which vanish in ∞ and is called the space of *cusp forms*. Clearly, $\mathcal{M}_k = \mathbb{C}E_k \oplus \mathcal{S}_k$, so that all modular forms may be written as homogenous polynomials in E_4 and E_6 due to the known dimension of $\mathcal{M}_k$ up to the choice $k = 12$.
This closes the brief overview about a geometrical approach to classical modular forms as well as the particular case of Eisenstein series.

Non-holomorphic Eisenstein series

Examining Equation (2.6) results in realising that the series is convergent, if and only if, $k > 2$. Nonetheless, a similar function exists for $k = 2$ and even lower values of k, but these are not holomorphic. In order to generate such functions, a technique, named after Hecke, is commonly used.[18] The general approach discussed in Section 5.3 resembles this technique which is reason enough to devote a subsection to its introduction. The reader may find additional information in [DS05, Sec. 4.10 p. 147] or [Iwa95, Sec. 3.2 p. 61].

Definition 2.5. For $s \in \mathbb{C}, k \in \mathbb{Z}$ and $\tau \in \mathbb{H}$, let

$$E_k(\tau, s) := \sum_{\gamma \in \Gamma(1)_\infty \backslash \Gamma(1)} \operatorname{Im}(\tau)^s|_k \gamma$$

be the *augmented*[19] *Eisenstein series* of weight k.

In case of $\operatorname{Re}(s) > 1 - \frac{k}{2}$, this series converges absolutely in a uniform fashion on compact subsets, so that it is analytic on the half plane $\left\{s \in \mathbb{C} \mid \operatorname{Re}(s) > 1 - \frac{k}{2}\right\}$. Note that $\lim_{s \to 0} E_k(\tau, s) = E_k(\tau)$, in case of $k > 2$, so that this family of functions reproduces the classical Eisenstein series presented above. In case of $k \leq 2$, the series is not convergent for $s \to 0$, so that the naive definition of Eisenstein series given in Section 2.1.2 fails and a more advanced approach is required. This is exactly what the above series $E_k(\tau, s)$ are capable of, due to their remarkable analytic properties being reflected in the next proposition.

Proposition 2.6. *With $E_k(\tau, s)$ associate the complete Eisenstein series*

$$G_k^*(\tau, s) := \pi^{-s}\Gamma(k/2 + s)\zeta(2s)E_k(s - k/2).$$

[18] Compare, for instance, 'Darstellung von Klassenzahlen als Perioden von Integralen 3. Gattung aus dem Gebiet der elliptischen Modulfunktionen' found in [Hec83, p. 412].
[19] The series is augmented by the parameter s.

This function has analytic continuation to the whole plane of complex numbers in s, except for the case when $k = 0$. In that case, the function has simple poles at $s = 0, 1$. In any case it is invariant under the transformation $s \mapsto 1 - s$.

This result suggests using $E_k(\tau, 0)$ as a generalisation of classical Eisenstein series, which will then be non-holomorphic in case of $k = 2$. In fact, the Fourier expansion of that function looks like expected, but with the addition of a non-holomorphic correction term:

$$E_2(\tau) = -\frac{3}{\pi} \cdot \mathrm{Im}(\tau)^{-1} + 1 - 24 \sum_{n>0} \sigma_1(n) q^n.$$

More generally, the Fourier expansion of the automorphic function $G_0^*(\tau, s) = \zeta^*(2s) E_0(\tau, s)$ is given by (cf. [Bum98, proof of Thm. 1.6.1 p. 66])

$$\begin{aligned} G_0^*(\tau, s) &= \zeta^*(2s) y^s + \zeta^*(2(1-s)) y^{1-s} \\ &+ \sum_{\substack{r=-\infty,\\ r\neq 0}}^{\infty} 2|r|^{s-1/2} \sigma_{1-2s}(|r|) \sqrt{y} K_{s-1/2}(2\pi |r| y) e^{2\pi i r x}, \end{aligned} \tag{2.9}$$

where

$$K_s(y) = \frac{1}{2} \int_0^\infty e^{-y(t+1/t)/2} t^s \frac{\mathrm{d}t}{t}$$

is the *K Bessel function* and σ_s is the sum divisor function given in (2.8). The known applications of G_0^* are manifold (compare for instance the Rankin–Selberg methon [Bum98, 1.6 p. 65]). Note that Proposition 2.6 is derived from the Fourier expansion (2.9) for $k = 0$.

Define the *Maas differential operator*[20]

$$R_k := 2i\,\partial_z + \frac{k}{y}.$$

A simple calculation confirms that it satisfies the relations

$$(R_k f)|_{k+2}\gamma = R_k(f|_k\gamma) \quad \text{and} \quad R_k y^s = (s+k) y^{s-1},$$

[20] In fact, the original operator looks slightly different, as it had been designed for Maas wave forms. Minor modifications were necessary to make it work as desired in the present setting.

for smooth functions f and $\gamma \in \mathrm{SL}_2(\mathbb{R})$. As a consequence, we find

$$\begin{aligned} R_k G_k^*(s) &= \left(s + k - \frac{k}{2}\right) \cdot \Gamma\left(s + \frac{k}{2}\right) \pi^{-s} \zeta(2s) \cdot E_{k+2}\left(s - \frac{k}{2} - 1\right) \\ &= \Gamma\left(s + \frac{k+2}{2}\right) \pi^{-s} \zeta(2s) \cdot E_{k+2}\left(s - \frac{k+2}{2}\right) \\ &= G_{k+2}^*(s) \end{aligned}$$

and, in particular, Proposition 2.6 for $k > 0$. Further, the Fourier expansion of G_k^* follows from that of G_0^* by iterative application of differential operators, ultimately delivering that of E_k.

In fact, Proposition 2.6 might also be proven by means of theta functions (cf. [DS05, 4.10 p. 147 as.]), which are to be introduced in the next section before briefly relating them to Eisenstein series via a theorem of Siegel.

Theta series

Theta series are special modular forms related to quadratic lattices. Hence, let (L, q) be a quadratic $\mathbb{Z}$ lattice of rank $m \in \mathbb{N}$ and associate the theta function

$$\theta_L : \mathbb{H} \to \mathbb{C}, \qquad \tau \mapsto \sum_{x \in L} e^{2\pi i q(x)\tau}$$

with it.[21] The sum above converges locally uniformly for positive definite q, defines a holomorphic function in τ and is 1-periodic, if L is even. The transformation with respect to the inversion S is computed via Poisson summation which yields

$$\theta_{L'}(S\tau) = \left(\frac{\tau}{i}\right)^{m/2} \det(\mathrm{q})\, \theta_L(\tau).$$

Here the root is defined by using the main branch of the logarithm. Strictly speaking, this is not the transformation property of a classical modular form for $\Gamma(1)$. However, it resembles the desired property and for the special choices $\det(\mathrm{q}) = 1$ (equivalently $L = L'$), as well as, $m \in 8\mathbb{N}$ it is a classical modular form. If, on the other hand, L is at least even, θ_L still represents a modular form for a subgroup of $\Gamma(1)$. In fact, [Ebe12, 3.1 Thm. 3.2 p. 87] states the following.

Theorem 2.7. *For an even lattice (L, q) of rank $m \in 2\mathbb{N}$ and level $N \in \mathbb{N}$, the function*

[21] In case the quadratic lattice is encoded in the Gram matrix S, we shall denote the theta series by θ_S.

θ_L is a modular form of weight $\frac{m}{2}$ for $\Gamma_0(N)$ with character[22] $\left(\frac{\mathrm{disc}(L)}{\cdot}\right)$, *meaning that for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, we have*

$$\theta_L(\gamma\tau) = \left(\frac{\mathrm{disc}(L)}{d}\right) j(\gamma, \tau)^{m/2} \theta_L(\tau)$$

and that the form is holomorphic at the cusps.

Example 2.8. An as famous as immediate application of theta series is that to the problem of representation numbers of quadratic forms. Let (L, q) be an even positive $\mathbb{Z}$ lattice of rank m, then

$$\theta_L(\tau) = \sum_{k \geq 0} r_L(k) e^{2\pi i k \tau},$$

where $r_L(k) = \#\{x \in L \mid \mathrm{q}(x) = k\}$ is the *representation number* of n by (L, q). In fact, in case of $8 \mid m$ and $\det(\mathrm{q}) = 1$, θ_L is a classical modular form of weight $m/2$ as mentioned above, allowing for exact formulae for r_L in terms of coefficients of Eisenstein series in case $m \leq 16$. For higher ranks $m \geq 24$ estimates with bounded error are obtained at least:

$$r_L(k) = \begin{cases} -\frac{m}{B_{d/2}} \sigma_{m/2-1}(k), & \text{if} \quad m = 8, 16, \\ -\frac{m}{B_{d/2}} \sigma_{m/2-1}(k) + \mathcal{O}(k^{m/4}), & \text{else.} \end{cases}$$

The bound is derived from a general, elementary bound for coefficients of modular forms, named after Hecke (cf. Prop. 2.24), whereas the exact formulae for the low dimensional cases are due to the space of modular forms being one dimensional in this instance.

Another application of theta functions would be the proof of the functional equation of the Riemann Zeta function ζ, as the completed version

$$\zeta^*(s) = \zeta(s)\Gamma(s/2)\pi^{-s/2}$$

is essentially the Mellin transform of $(\theta_{\mathbb{Z}, x^2/2} - 1)$. As a consequence, the transformation law for that theta function may be abused, to represent ζ^* as an integral that is symmetric in s and $1 - s$, yielding[23] the following symmetry property

$$\zeta^*(s) = \zeta^*(1 - s).$$

[22]Note that the character is trivial, when the respective group is restricted to $\Gamma_1(N)$.
[23]A detailed computation is found in [Dei10, Thm. 6.1.2 p. 140]

Theorem of Siegel

The (arithmetic) Theorem of Siegel established a formula for weighted representation numbers of special quadratic lattices (cf. [Sie51a], [Sie51b]). As a consequence, an analytic result, namely realising Eisenstein series as a weighted sum of theta series, is obtained.[24] It is this version which we shall refer to as Siegel's Theorem over the course of this thesis. The reader may consult [Fre13, V.1.3 p. 261] or [KK98] for information exceeding the scope of the following brief overview.
As seen in the preceeding section, theta series correspond to positive definite lattices. In fact, representation numbers of the latter may be used for properly combining theta series in order to construct the full range of classical modular forms. In order to substantiate this claim, let $S \in M_n(\mathbb{Z})$ be symmetric positive definite and $G \in \mathbb{Z}^{n\times m}$, then $G^T SG$ is again symmetric positive definite. In fact, the number of matrices G, such that $G^T SG$ coincide with a fixed choice of matrix is finite. More specifically, for a symmetric matrix $T \in M_m(\mathbb{Z})$,

$$\#(S,T) := \# \left\{ G \in M_{n,m}(\mathbb{Z}) \mid G^T SG = T \right\}$$

is finite. It is called *representation number* of T by S. Further, on the set of symmetric positive definite matrices $\mathrm{Pos}_n(\mathbb{Z})$, the operation

$$S.G = G^T SG$$

with $G \in \mathrm{GL}_n(\mathbb{Z})$ defines a right action, which preserves evenness as well as unimodularity. This information suffices to state the analytic version of Siegel's Theorem.

Theorem 2.9 (Siegel, Witt)**.** *Let $S_1, \dots, S_h$ be representatives of the classes of positive definite even unimodular quadratic $\mathbb{Z}$-lattices of rank m and $k = \frac{m}{2}$. Then*

$$\sum_{j=1}^{h} \frac{\#(S_j, S_j)^{-1}}{\sum_{k=1}^{h} \#(S_k, S_k)^{-1}} \cdot \theta_{S_j} = E_k.$$

Altogether, the above theorem reveals that Eisenstein series may be obtained by a suitable combination of theta series, presenting them as an alternative foundation of the theory of classical modular forms.

The statement of Theorem 2.9 is briefly revisited in Section 5.4.1 where the *Siegel–Weil formula*, an intricate generalisation, is presented. In a phrase, it relates an integral

[24] The stated version was developed by Witt and can be proven with elementary Hecke theory.

(resembling the weighted sum) of a theta distribution[25] to a special value (in the parameter s - cf. Section 2.1.2) of an Eisenstein series.

[25] These are adelic distributions corresponding to classical theta functions in special cases (cf. Subsection 5.2.1).

2.2 Elliptic modular forms

This section is designed to provide a brief recollection of the classical theory and will serve as a point of reference in subsequent sections.

Definition 2.10. a) Define the *upper/lower half plane* as

$$\mathbb{H} := \{\tau \in \mathbb{C} \mid \operatorname{Im}(\tau) > 0\}, \qquad \mathbb{H}^- := \{\tau \in \mathbb{C} \mid \operatorname{Im}(\tau) < 0\}.$$

b) Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})$. Then the associated *Möbius transform* is given by

$$\mathbb{H} \cup \mathbb{H}^- \ni \tau \mapsto \gamma\tau := \frac{a\tau + b}{c\tau + d} \in \mathbb{H} \cup \mathbb{H}^-.$$

c) For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})$ and $\tau \in \mathbb{C} \setminus \mathbb{R}$ define the *factor of automorphy* as

$$j(\gamma, \tau) := (c\tau + d). \tag{2.10}$$

The following relations shed some light on these definitions.

Remark 2.11. a) For $\gamma, \gamma' \in \mathrm{GL}_2(\mathbb{R})$ and $\tau \in \mathbb{C} \setminus \mathbb{R}$ we find the *cocycle relation*

$$j(\gamma\gamma', \tau) = j(\gamma, \gamma'\tau) \cdot j(\gamma', \tau). \tag{2.11}$$

b) For $\tau \in \mathbb{C} \setminus \mathbb{R}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})$, we write $\tau = u + iv$ with $u, v \in \mathbb{R}$ and find

$$\begin{aligned}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau &= \frac{a\tau + b}{c\tau + d} \\ &= \frac{ac|\tau|^2 + bd + (ad + bc)u}{|j(\gamma, \tau)|^2} + i \det(\gamma) \frac{v}{|j(\gamma, \tau)|^2}.\end{aligned}$$

In particular, we have

$$\operatorname{Im}(\gamma\tau) = \frac{\operatorname{Im}(\tau)}{|j(\gamma, \tau)|^2}. \tag{2.12}$$

As a consequence, the subgroup $\mathrm{GL}_2(\mathbb{R})^+ < \mathrm{GL}_2(\mathbb{R})$ of matrices with positive determinant acts on $\mathbb{H}$ and $\mathrm{GL}_2(\mathbb{R})/\mathrm{GL}_2^+(\mathbb{R}) \simeq \langle \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \rangle$ has order 2. The centre $\mathbb{R}\mathcal{I}_2 < \mathrm{GL}_2(\mathbb{R})^+$ acts trivially and its quotient acts transitively on $\mathbb{H}$ (cf. (3.22)).

We consider the following arithmetic subgroup of $\mathrm{GL}_2^+(\mathbb{R})$.

Definition 2.12. The group $\Gamma(1) := \mathrm{SL}_2(\mathbb{Z}) \leq \mathrm{GL}_2^+(\mathbb{R})$ is called the *modular group*. Its special elements

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

are referred to as *translation* and *inversion*.

Definition 2.13. Let $k \in \mathbb{Z}$ and $f : \mathbb{H} \to \mathbb{C}$ be a holomorphic function. Then f is called *modular form* of weight k if it satisfies the following two conditions:

a) For all $\gamma \in \Gamma(1)$ it transforms with $f(\gamma\tau) = j(\gamma, \tau)^k f(\tau)$.

b) The function f is holomorphic at ∞.

The space of such functions is denoted $\mathcal{M}_k(\Gamma(1))$.

In order to give the last condition meaning, note that f is invariant under the translation matrix T, meaning it is periodic. As a consequence, it possesses a Fourier expansion

$$f(\tau) = \sum_{n\in\mathbb{Z}} a(n) \exp(2\pi i\tau n) \tag{2.13}$$

for all $\tau \in \mathbb{H}$. Here, $a(n) \in \mathbb{C}$ are complex numbers[26] and the series converges (and hence absolutely – cf. Lemma A.19) on the upper half plane. In the context of Fourier series, write

$$q := e(\tau) := \exp(2\pi i\tau). \tag{2.14}$$

With this notation, the expansion of f becomes

$$f(\tau) = \sum_{n\in\mathbb{Z}} a(n)e(n\tau) = \sum_{n\in\mathbb{Z}} a(n)q^n. \tag{2.15}$$

Definition 2.14. With the notation as above, we say that f is *holomorphic/meromorphic/vanishes* at ∞, if the power series in (2.15) has that property in the variable q in 0.

We define an operator leaving the space of modular forms invariant.

Definition 2.15. a) For a function $f : \mathbb{H} \to \mathbb{C}$, $k \in \mathbb{Z}$, and $\gamma \in \mathrm{GL}_2^+(\mathbb{R})$ define the *Petersson (slash) operator* of weight k as

$$|_k : \mathbb{C}^{\mathbb{H}} \to \mathbb{C}^{\mathbb{H}}, \quad f|_k\gamma(\tau) := \det(\gamma)^{k/2} j(\gamma, \tau)^{-k} f(\gamma\tau). \tag{2.16}$$

[26] For a note on the $\mathrm{Im}(\tau)$ independence of $a(n)$, compare A.2.

b) On $\mathbb{H}$ write $\tau = u + iv$ and define the *hyperbolic measure*

$$\mu := \frac{\mathrm{d}u\,\mathrm{d}v}{v^2}. \tag{2.17}$$

c) For $\delta > 0$, the set

$$\mathcal{S}(\delta) := \{\tau \in \mathbb{H} \mid \mathrm{Im}(\tau) > \delta, |\mathrm{Re}(\tau)| < 1/\delta\} \tag{2.18}$$

is called the *Siegel domain* associated to δ.

The Petersson slash operator defines a right action of $\mathrm{GL}_2^+(\mathbb{R})$ on $\mathbb{C}^{\mathbb{H}}$ preserving continuitiy, real smoothness and analyticity. Letting a matrix $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$ act preserves having a Fourier expansion and it preserves properties at ∞ of elements of $\mathcal{M}_k(\Gamma(1))$. Further, the measure μ is invariant under pullback via Möbius transformations from $\mathrm{GL}_2^+(\mathbb{R})$. Finally, Siegel domains have finite measure and may be used to prove that $\Gamma(1)$ acts properly discontinuously on $\mathbb{H}$, implying that the quotient $Y(\Gamma(1)) := \Gamma(1)\backslash\mathbb{H}$ carries a T_2 topology (cf. [Miy06, 1.5 p. 17]). A visualisation of a set of representatives in $\mathbb{H}$ of $Y(\Gamma(1))$ is presented in Figure 2.1.2.
The following series represents a canonical choice of a modular form.

Definition 2.16. Let $\Gamma_\infty := \langle T \rangle$ and consider for $k \geq 3$ the *Eisenstein series*

$$E_k(\tau) := \frac{1}{2} \cdot \sum_{\gamma \in \Gamma_\infty \backslash \Gamma(1)} 1|_k \gamma(\tau). \tag{2.19}$$

The series E_k converges normally on $\mathbb{H}$ and defines a holomorphic function. Further, it vanishes for $2 \nmid k$ and in fact, by the invariance condition, any $f \in \mathcal{M}_k(\Gamma(1))$ has to vanish due to the transformation under $-\mathcal{I}_2$ for uneven k. A comparison with the partial fraction expansion of the cotangent (cf. [DS05, p. 5]) delivers the following Fourier expansion:

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n>0} \sigma_{k-1}(n) q^n, \tag{2.20}$$

where B_k denotes the k-th Bernoulli number and

$$\sigma_s(n) = \sum_{d|n} d^s \tag{2.21}$$

denotes the *divisor function* of degree $s \in \mathbb{C}$ (with $\mathrm{Re}(s) > 0$).

Another example of automorphic forms are theta functions which are revisited in the more general setting of vector valued modular forms in Definition 3.29.

Definition 2.17. For (L, q) a positive definite even $\mathbb{Z}$ lattice of rank m, define

$$\theta_L : \mathbb{H} \to \mathbb{C}, \qquad \tau \mapsto \sum_{x \in L} e(\mathrm{q}(x)\tau) = \sum_{n \in \mathbb{N}} r_L(n) q^n. \tag{2.22}$$

For unimodular lattices, these are modular forms for $\Gamma(1)$ of weight $m/2$. However, instead of merely considering the arithmetic subgroup $\Gamma(1)$ as a group of transformations, a wider family of groups due to Hecke may be considered.

Definition 2.18. Let $N \in \mathbb{N}$. Then the kernel of the natural (surjective) projection

$$\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$$

is denoted by $\Gamma(N)$ and called the *principal congruence subgroup* of *level* N. Further, a subgroup $\Gamma \leq \Gamma(1)$ is called *congruence subgroup*, if there is some $\Gamma(N) \leq \Gamma$.

Remark 2.19. The following two classes of examples represent congruence subgroups:

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \mod N \right\}, \tag{2.23}$$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \,\middle|\, a \equiv d \equiv 1 \mod N \right\}. \tag{2.24}$$

We find for $N > 1$ that

$$\Gamma(N) \triangleleft \Gamma_1(N) \triangleleft \Gamma_0(N) < \Gamma(1),$$

where the natural kernel morphisms are given by

$$\Gamma_0(N) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d \in (\mathbb{Z}/N\mathbb{Z})^\times, \qquad \Gamma_1(N) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto b \in (\mathbb{Z}/N\mathbb{Z}).$$

In particular, any congruence subgroup has finite index in $\Gamma(1)$.

The action of $\mathrm{GL}_2^+(\mathbb{R})$ on $\mathbb{H}$ may be extended to the closure $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{P}^1(\mathbb{R})$ in a natural way. The modular group $\Gamma(1)$ then acts transitively on $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$.

Definition 2.20. For a congruence subgroup Γ, the equivalence classes $\Gamma\backslash\mathbb{P}^1(\mathbb{Q})$ are referred to as *cusps*. For a meromorphic function $f : \mathbb{H} \to \mathbb{C}$ with Fourier expansion at

Notation	Behaviour on $\mathbb{H}$	Behaviour at cusps	Name
$\mathcal{S}_k(\Gamma,\chi)$	holomorphic	vanishings	cusp forms
$\mathcal{M}_k(\Gamma,\chi)$	holomorphic	holomorphic	(holomorphic) modular forms
$\mathcal{M}^!_k(\Gamma,\chi)$	holomorphic	meromorphic	weakly holomorphic modular forms
$\mathcal{A}_k(\Gamma,\chi)$	meromorphic	meromorphic	meromorphic modular forms
$\mathcal{R}_k(\Gamma,\chi)$	real analytic	no condition	real analytic modular forms
$\mathcal{C}^m_k(\Gamma,\chi)$	$\mathcal{C}^m$	no condition	$\mathcal{C}^m$ modular functions

Table 2.1: Table of different types of modular forms. All of the forms f are assumed to transform with respect to Γ like in Definition 2.21, meaning for all $\gamma\in\Gamma$ they satisfy $f(\gamma\tau)=\chi(\gamma)j(\gamma,\tau)^kf(\tau)$.

∞ and a representative c of a cusp, we select $\gamma\in\Gamma(1)$ with $\gamma\infty=c$. Then the Fourier expansion of $f|_k\gamma$ at ∞ is referred to as the *Fourier expansion* of f at c (for $|_k$). It is not unique, but its properties like vanishings or being holomorphic are well defined.

The *modular curve* associated to some congruence subgroup Γ is denoted by $Y(\Gamma):=\Gamma\backslash\mathbb{H}$. It is a Riemannian manifold. By the theory of *Baily–Borel* , it may be compactified by adding its cusps to it, yielding the compactified modular curve $X(\Gamma):=\Gamma\backslash\mathbb{H}^*$, where $\mathbb{H}^*=\mathbb{H}\cup\mathbb{P}^1(\mathbb{Q})$. As a consequence, $Y(\Gamma)$ is a quasi projective variety.

Definition 2.21. Let $\Gamma\leq\Gamma(1)$ be a congruence subgroup, $k\in\mathbb{Z}$, and $\chi:\Gamma\to\mathbb{T}$ be a character. Then a holomorphic function $f:\mathbb{H}\to\mathbb{C}$ is called a *modular form* for Γ of weight k with character χ, if it satisfies the following two properties.

a) For all $\gamma\in\Gamma$ it transforms with $f(\gamma\tau)=\chi(\gamma)j(\gamma,\tau)^kf(\tau)$.

b) The function f is holomorphic at ∞.

The space of such functions is denoted by $\mathcal{M}_k(\Gamma,\chi)$.

The conditions on such functions f may be relaxed or tightened. Table 2.1 presents an overview of typical variants of Definition 2.21. The spaces $\mathcal{M}_k(\Gamma,\chi)$ are finite dimensional and if $\Gamma(N)\leq\Gamma$, then $T^N\in\Gamma$, so that f is invariant under a possibly higher power of T^N and hence possesses a Fourier expansion. Examples of such more general modular forms may be constructed again as Eisenstein series or, more generally, as Poincaré series (cf. [Miy06, 2.6 p. 61]). Further, theta functions represent modular forms with character.

Theorem 2.22 ([Ebe12, 3.1 Thm. 3.2 p. 87])**.** *For an even lattice (L,q) of rank $m\in2\mathbb{N}$ and level $N\in\mathbb{N}$, the function θ_L is a modular form of weight $\frac{m}{2}$ for $\Gamma_0(N)$ with character*[27]

[27]Note that the character vanishes, when the respective group is restricted to $\Gamma_1(N)$.

$\left(\frac{\mathrm{disc}(L)}{\cdot}\right)$, *meaning that for all* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, *we have*

$$\theta_L(\gamma\tau) = \left(\frac{\mathrm{disc}(L)}{a}\right) j(\gamma,\tau)^{m/2}\theta_L(\tau)$$

and that the form is holomorphic at the cusps.

Apart from these examples, modular forms with characters appear naturally, when considering these without character as a byproduct of representation theory of finite groups (cf. Remark 6.8).

Remark 2.23. There is a natural pairing serving as a scalar product on the space of cusp forms. For $f \in \mathcal{M}_k^!(\Gamma)$ and $g \in \mathcal{M}_k^!(\Gamma')$ congruence modular forms, so that $\Gamma(N) \leq \Gamma \cap \Gamma'$ for some $N \in \mathbb{N}$ and $f \cdot g \in \mathcal{S}_{2k}(\Gamma(N))$, set

$$\langle f, g\rangle := \frac{1}{[\Gamma(1):\Gamma(N)]} \cdot \int_{\Gamma(N)\backslash\mathbb{H}} f \cdot \overline{g} \cdot \mathrm{Im}^k \, \mathrm{d}\mu. \tag{2.25}$$

Then, the expression is absolutely convergent (cf. Example 3.52) and independent of the choice of N. It is referred to as the *Petersson product* of f and g.

In Section 6.4, L-functions, another significant number theoretic construct, will be associated to modular forms. For their convergence, asymptotic bounds on Fourier coefficients of modular forms are required. The following is a naive bound that already suffices for a vast array of cases.

Proposition 2.24 ([KK98, p. 200])**.** *For* $f \in \mathcal{S}_k(\Gamma_0(N), \chi)$ *and* $\gamma \in \Gamma(1)$*. Denote the Fourier expansion of* $f|_k\gamma$ *by* $\sum_{n_0 \leq n \in \mathbb{Z}/N} a(n)e(n\tau)$*, then we have that*

$$a(n) = \mathcal{O}(n^{k/2}), \qquad n \to \infty.$$

While the advantage of the proposition above is that its proof is relatively accessible, there are more effective bounds which become relevant in boundary cases.

Theorem 2.25 ([Ran39b, Thm. 2, p. 358])**.** *Let* $f \in \mathcal{S}_k(\Gamma(N))$ *with Fourier expansion* $\sum_{n_0 \leq n \in \mathbb{Z}/N} a(n)e(n\tau)$*. Then*

$$a_n = \mathcal{O}(n^{k/2-1/5}), \qquad n \to \infty.$$

The strongest possible bound is the following (cf. [Miy06, Thm. 4.5.17 p. 150]).

Theorem 2.26 (Ramanujan-Petersson-Deligne)**.** *For* $f \in S_k(\Gamma_0(N), \chi)$ *with Fourier expansion* $\sum_{n_0 \leq n \in \mathbb{Z}/N} a(n)e(n\tau)$*, we have for* $\gcd(n, N) = 1$ *and any* $\varepsilon > 0$ *that*

$$a(n) = \mathcal{O}(n^{k/2-1/2+\varepsilon}), \qquad n \to \infty.$$

Note that for primes dividing the level, Atkin–Lehner theory may be utilised to deduce similar bounds (cf. [Kna92, Thm. 9.27 p. 289]) and also 3.81. Additional bounds are presented in Section 3.3.4.

Part II

Automorphic forms

3 Symplectic theory

In Section 2.2, an introduction to the theory of classical modular forms associated with the group SL_2 has been presented, followed by a couple of generalisations. A further avenue for exploration would be groups of higher rank, such as SL_3. However, it turns out that the proper generalisation in this context is the symplectic setting. This will be briefly discussed in the following section, after which the focus will return to the essential subcase for this thesis, namely the elliptic setting, which will include the Weil representation and vector valued modular forms. The chapter concludes with a collection of asymptotic bounds for Fourier coefficients of vector valued modular forms which will be used to derive convergence results of L–series.

3.1 Symplectic modular forms

We provide a brief overview of the symplectic notions of modular forms – called *Siegel modular forms*. For a slightly more elaborate but concise description of the following, we refer to [Fre91, Chap. 1], while the German source [Fre83] provides a more comprehensive picture.

As in case of the orthogonal group, symplectic groups may be defined as the invariance group of a bilinear form. For that purpose, let $g \in \mathbb{N}$ and $\mathcal{I}_g$ denote the identity matrix of rank g. Then

$$J = \begin{pmatrix} 0 & \mathcal{I}_g \\ -\mathcal{I}_g & 0 \end{pmatrix} \in M_{2g}$$

denotes the standard *alternating matrix*.

Definition 3.1. Let R be a commutative ring with unit. Then the *symplectic group* of *genus* $g \in \mathbb{N}$ over R is defined to be

$$\mathrm{Sp}_g(R) \coloneqq \left\{ M \in M_{2g}(R) \mid M^T J M = J \right\}. \tag{3.1}$$

Example 3.2. In case $g = 1$, the conditions reduce to the case of the special linear group

$$\mathrm{Sp}_1(R) = \mathrm{SL}_2(R).$$

As in case of SL_2, there is also a variety of symplectic matrices that may be presented more explicitly in the case of a general ring R. Let $S \in M_g(R)$ be symmetric, i.e. $S^T = S$ and $U \in \mathrm{GL}_g(R)$. Then the following are symplectic matrices.

$$N(S) := \begin{pmatrix} \mathcal{I}_g & S \\ 0 & \mathcal{I}_g \end{pmatrix}, \tag{3.2}$$

$$M(U) := \begin{pmatrix} U^T & 0 \\ 0 & U^{-1} \end{pmatrix}, \tag{3.3}$$

$$J = \begin{pmatrix} 0 & \mathcal{I}_n \\ -\mathcal{I}_n & 0 \end{pmatrix}. \tag{3.4}$$

In case $R = \mathbb{Z}$ or $R = F$ a field, the group $\mathrm{Sp}_g(R)$ is generated by matrices as in (3.2) and (3.4). Further, the unitary group $U(g)$ of rank g embeds into $\mathrm{Sp}_g(\mathbb{R})$ via

$$U(g) \hookrightarrow \mathrm{Sp}_g(\mathbb{R}), \qquad U \mapsto \begin{pmatrix} \mathrm{Re}(U) & \mathrm{Im}(U) \\ -\mathrm{Im}(U) & \mathrm{Re}(U) \end{pmatrix}. \tag{3.5}$$

This is a maximal compact subgroup and it agrees with $\mathrm{SO}_2(\mathbb{R})$ in the instance of $\mathrm{Sp}_1(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$.

As in case of the special linear group of rank 2, there is a theory of congruence subgroups and of multiplier systems which are described in [Fre91]. In order to define modular forms in the symplectic setting, an analogue of the upper half plane has to be constructed.

Definition 3.3. Let $g \in \mathbb{N}$. Then the *Siegel upper half space* of *degree g* is defined to be

$$\mathbb{H}_g := \{Z = X + iY \in M_{2g}(\mathbb{C}) \mid Z^T = Z,\, Y > 0\}.$$

Here, X, Y denote the real, respectively imaginary part, of Z and the condition $Y > 0$ means that Y is a positive definite matrix, i.e. as a Gram Matrix of a real quadratic vector space.

The Siegel upper half space is convex and as such connected and carries a natural complex structure. Analogous to the SL_2 case, there is an action of $\mathrm{Sp}_g(\mathbb{R})$ on the upper half space $\mathbb{H}_g$. Namely, let

$$Z \in \mathbb{H}_g, \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_g(\mathbb{R}).$$

Then $CZ + D$ is invertible and the matrix

$$MZ := (AZ + B)(CZ + D)^{-1} \tag{3.6}$$

lies in $\mathbb{H}_g$, again. This association defines an action of $\mathrm{Sp}_g(\mathbb{R})$ on $\mathbb{H}_g$ via biholomorphic maps and all biholomorphic maps on the upper half space are of this form. For instance, the action of the matrices in (3.2), (3.3), and (3.4) looks like the following.

$$N(S)Z = Z + S, \qquad M(U)Z = U^T ZU, \qquad JZ = -Z^{-1}.$$

Note that to the map

$$\mathbb{H}_g \ni Z \mapsto \det(CZ + D),$$

there is a holomorphic square root. We fix one of these roots and denote it by

$$Z \mapsto \det(CZ + D)^{1/2}.$$

Definition 3.4. Let $r \in \mathbb{Z}$, V be a finite dimensional $\mathbb{C}$ vector space, and

$$f : \mathbb{H}_g \to V$$

be holomorphic. Then f is called holomorphic *Siegel modular form* of *weight* r if it satisfied the following transformation property for all $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_g(\mathbb{Z})$

$$f(MZ) = \det(CZ + D)^r f(Z) \tag{3.7}$$

and for any $Z_0 \in \mathbb{H}_g$, the expression $f(Z)$ is uniformly bounded for $\mathrm{Im}(Z) > \mathrm{Im}(Z_0)$.

We note that such a function is invariant under the operation of matrices $N(S)$ with $S \in M_g(\mathbb{Z})$ symmetric which form a $\mathbb{Z}$ lattice with bilinearform induced by the trace. Hence, it possesses a Fourier expansion of the form

$$f(Z) = \sum_{S \text{ rational},\, S = S^T} a(S)\, e\left(\mathrm{tr}(SZ)\right). \tag{3.8}$$

The condition on a modular form f being universally bounded on domains with $\mathrm{Im}(Z) \geq \mathrm{Im}(Z_0)$ for any fixed $Z_0 \in \mathbb{H}_g$ translates to the Fourier coefficients $a(S)$ vanishing unless $S \geq 0$ is semi positive definite.

In case of genus greater than 1, there are harsher restrictions on the Fourier coefficients given by the so called Koecher principle.

Theorem 3.5 (Koecher principle)**.** *Let $f : \mathbb{H}_g \to \mathbb{C}$ be a holomorphic function with the properties that*

a) $f(Z + S) = f(Z)$ for all $S \in M_g(\mathbb{Z})$ with $S^T = S$,

b) $f(U^T ZU)$ for all $U \in \mathrm{SL}_g(\mathbb{Z})$.

Then the condition $g > 1$ implies that f has a Fourier expansion of the form

$$f(Z) = \sum_{\substack{S=S^T,\\ S\geq 0 \text{ even}}} a(S)\, e\left(\mathrm{tr}(SZ)\right).$$

In particular, for any $Z_0 \in \mathbb{H}_g$ the expression $f(Z)$ is uniformly bounded for $\mathrm{Im}(Z) \geq \mathrm{Im}(Z_0)$*.*

There is also a rich theory of Siegel modular forms reproducing a variety of results from the classical setting of genus $g = 1$ and we refer to [Fre83] for details before turning towards the essential case for this thesis being $\mathrm{Sp}_1 = \mathrm{SL}_2$ and its Weil representation.

3.2 Weil representation

Vector valued modular forms for the Weil representation of SL_2 play a significant role within the scope of this thesis (cf. Section 3.3). However, when forms of half integral weight are taken into account, it becomes necessary to consider a twofold cover, the *metaplectic group* Mp_2, of SL_2. Such a cover exists and an abstract construction is sketched in the addendum (cf. Section B.3). However, these details are not required for the purpose of the thesis and we will be presenting a briefer more ad hoc approach in the following.

As mentioned in the chapter above, the symplectic group over a field is generated by the elements $N(S)$ and J (cf. (3.2) and (3.4)). In case of genus $g = 1$ and a field F we follow [Lan98, XI.2 pp. 209-214] and write $a \in F^\times$ and $b \in F$. Then we find the following relations

$$J^2 = M(-1), \tag{3.9}$$

$$M(a)N(b)M(a^{-1}) = N(ba^2), \tag{3.10}$$

$$M(a) = JN(a^{-1})JN(a)JN(a^{-1}). \tag{3.11}$$

We note that

$$F \ni b \mapsto N(b) \in \mathrm{SL}_2(F), \qquad F^\times \ni a \mapsto M(a) \in \mathrm{SL}_2(F)$$

are embeddings. These matrices may be used to represent any element in $\mathrm{SL}_2(F)$.

Lemma 3.6. *Every element of* $\mathrm{SL}_2(F)$ *has a unique representation of the form*

$$N(b)M(a) \qquad \text{or} \qquad N(b_1)M(a)JN(b_2)$$

where $a \in F^\times$ *and* $b, b_1, b_2 \in F$*.*

In fact, we have the following stronger assertion.

Proposition 3.7. *Let* F *be a field. The group* $\mathrm{SL}_2(F)$ *is, up to isomorphy, the free group generated by an injective additive homomorphism* N *from* F *and an element* J *such that* M *defined by* (3.11) *is an injective multiplicative homomorphism from* $F^\times$ *modulo the relations* (3.9) *and* (3.10)*.*

This reveals that in order to construct a representation of $\mathrm{SL}_2(F)$, solely the action of $N(b)$ and J have to be established and the relations stated in Proposition 3.7 have to be verified. Also from a practical point of view, relation (3.11) alongside Lemma 3.6 suggest computing the action of $M(a)$ in order to manipulate emerging expressions conveniently. In the following, we will describe the respective action of matrices.

The setting of this paragraph is mainly a brief overview of necessary notions from [KRY06, 8.5 pp. 320-342] – the source also covers the case of higher genus. We will restrict to the case of a prime $p > 2$ and refer to the source for the case of $p = 2$ which is similar in nature. Let ψ_p denote the standard additive character of $\mathbb{Q}_p$ meaning the one from Example 1.48 with the choice $a = 1$ and V_p a vector space of dimension m.[1] Recall that to a Schwartz–Bruhat form $\phi \in \mathcal{S}(V_p)$, the Fourier transform described in (1.8) will be denoted $\hat{\phi}$. Then, for $a \in \mathbb{Q}_p^\times$ and $b \in \mathbb{Q}_p$ a projective representation of $\mathrm{SL}_2(\mathbb{Q}_p)$ on $\mathcal{S}(V_p)$,

[1] Selecting different characters only requires minor modifications.

is given by the following operator

$$ {}^{\mathrm{pr}}r_p\left[\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\right] : \phi \mapsto \psi(b\,\mathrm{q}(x))\cdot\phi(x), \tag{3.12}$$

$$ {}^{\mathrm{pr}}r_p\left[\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right] : \phi \mapsto \hat{\phi}(x), \tag{3.13}$$

$$ {}^{\mathrm{pr}}r_p\left[\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}\right] : \phi \mapsto |a|_p^{m/2}\phi(ax). \tag{3.14}$$

In fact, it satisfies the relation ${}^{\mathrm{pr}}r_p(g_1){}^{\mathrm{pr}}r_p(g_2) = c_L(g_1,g_2){}^{\mathrm{pr}}r_p(g_1g_2)$ for the Leray cocycle c_L. That cocycle is given by

$$c_L(g_1,g_2) = \gamma(\psi\circ q(g_1,g_2)),$$

where γ is the Weil index of a character of second degree and the associated quadratic form is given by the Leray invariant of the isotropic subspaces (Y, g_1, Y, Yg_2^{-1}) (cf. [LV80, 1.6 p. 47 as.] for an extensive discussion or [Rao93, Introduction and Thm. 4] for a brief reference). The cocycle amounts to a parametrisation of the *metaplectic extension* $\overline{\mathrm{SL}}_2(\mathbb{Q}_p)$ of $\mathrm{SL}_2(\mathbb{Q}_p)$ satisfying

$$1\to\mathbb{T}\to\overline{\mathrm{SL}}_2(\mathbb{Q}_p)\to\mathrm{SL}_2(\mathbb{Q}_p)\to 1 \tag{3.15}$$

via $\mathrm{SL}_2(\mathbb{Q}_p)\times\mathbb{T}$. Namely,

$$(g,z)\mapsto[g,z]_L, \qquad \text{where} \qquad [g_1,z_1]_L\cdot[g_2,z_2]_L = [g_1g_2, c_L(g_1,g_2)z_1z_2].$$

For later global use this cocycle has to be modified, as it is not trivial on compact open subgroups.[2] The Weil representation in Leray coordinates is given by

$$\omega_{V_p}([g,z]_L)\phi(x) = \chi_V(x(g))\left(z\gamma\left(\psi_{p,\frac{1}{2}}\right)\right)^{\#}\gamma\left(\psi_{p,\frac{1}{2}}\circ V\right)^{-1}{}^{\mathrm{pr}}r_p(g)\,\phi(x) \tag{3.16}$$

where

$$\# = \begin{cases}1 & \text{if } m \text{ is odd,}\\ 0 & \text{if } m \text{ is even,}\end{cases} \qquad\qquad \psi_{p,a}(x) \coloneqq e_p(a,x),$$

[2]This issue is fixed explicitly in [KRY06, (8.5.11)], resulting in a complex cocycle that is trivial on compact open subgroups as well as the standard parabolic subgroup.

and

$$\chi_V(x(g)) = \mathcal{H}_p\left(x(g), \operatorname{disc}(q)\right), \quad \text{where} \quad x\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = \begin{cases} c, & \text{if } c \neq 0, \\ d, & \text{else.} \end{cases} \tag{3.17}$$

Recall that $\mathcal{H}_p$ denotes the Hilbert symbol on $\mathbb{Q}_p$ described in Section 1.4. Note also that χ_V, as well as, γ are well defined on isometry classes of quadratic spaces (V, q) and the latter even defines a character of the Witt Ring (cf. Definition 1.19) mapping to the 8-th roots of unity.

Example 3.8. In case of $\dim(V_p) = m$ is even, the representation factors through SL_2.[3] In that case, the concrete operations are given by

$$\omega_{V_p}\left[\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right] : \phi \mapsto \psi(b\,\mathrm{q}(x)) \cdot \phi(x),$$
$$\omega_{V_p}\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] : \phi \mapsto \gamma(V_p) \cdot \hat{\phi}(x),$$
$$\omega_{V_p}\left[\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right] : \phi \mapsto \chi_p(a)|a|_p^{m/2}\phi(ax),$$

where $\gamma(V_p)$ is the *local splitting index* defined in [Kud94, Thm. 3.1].

This closes the brief overview of the local case and we turn towards sketching an adelisation of the construction.

Adelisation

The concepts above, namely the constructed group and their representations may be transferred to the adelic setting. For instance, define

$$\mathrm{SL}_2(\mathbb{A}) := \prod_{p \leq \infty}{}' \mathrm{SL}_2(\mathbb{Q}_p)$$

[3] In fact, that statement is an if, and only if, meaning the associated representation is ordinary on SL_2 in that case exactly.

to be the restricted product with respect to $\mathrm{SL}_2(\mathbb{Z}_p)$.[4] Similarly to the local case, there is an extension

$$1 \longrightarrow \mathbb{T} \longrightarrow \overline{\mathrm{SL}}_2(\mathbb{A}) \longrightarrow \mathrm{SL}_2(\mathbb{A}) \longrightarrow 1. \tag{3.18}$$

The Weil representation is given as a direct product. In order to prove that this definition is meaningful, the reader verifies that $\mathrm{SL}_2(\mathbb{Z}_p)$ operates trivially on almost all components of a Schwartz Bruhat function from $\mathcal{S}(V_{\mathbb{A}})$ (cf. [Yos84, Lemma 2.1 p. 190]). So that

$$\omega_{\mathbb{A}}([g_{\mathbb{A}}, z]_L) = \bigotimes_{p\leq\infty} \omega_{V_p}([g_p, z_p]_L) \tag{3.19}$$

might be realised as a *restricted tensor product* (cf. [Bum98, pp 300-303]).

Note that this definition is given in Leray coordinates, as were the above representations, even though the group itself is more conveniently parametrised by the cocycle from [KRY06, (8.5.11)], so that it is trivial on a suitable compact open subgroup.
There is a product formula for the Weil indices

$$1 = \prod_{p\leq\infty} \gamma(\psi \circ q_p) = 1,$$

which is also valid for the modified local Weil indices $\gamma(V)$, used in [KY10] to describe Fourier coefficients of adelic Eisenstein series in terms of Whittaker functions (also cf. [Rao93]).

We continue by stating some properties of SL_2 and its metaplectic extension based on [KY10, 1 p. 2278]. We restate the following notation

$$M(F_p) := \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} =: M(a) \;\middle|\; a \in F_p^{\times} \right\}, \tag{3.20}$$

$$N(F_p) := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} =: N(b) \;\middle|\; b \in F_p \right\}. \tag{3.21}$$

Recall that $\mathrm{SL}_2(\mathbb{R})$ admits the so called *Iwasawa* decomposition. Namely, that $\mathrm{SO}_2(\mathbb{R}) <$

[4] Since the product does not depend on changes to finitely many components, we select an arbitrary open subgroup at the Archemedian place.

$\mathrm{SL}_2(\mathbb{R})$ is a maximal compact subgroup, such that $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \simeq \mathbb{H}$. Here,

$$k_\vartheta = \begin{pmatrix} \cos(\vartheta) & \sin(\vartheta) \\ -\sin(\vartheta) & \cos(\vartheta) \end{pmatrix} \in K_\infty := \mathrm{SO}_2(\mathbb{R})$$

constitutes a parametrisation of K_∞ via $\vartheta \in \mathbb{T}$.[5] Consequently, we find for $\tau = u + iv \in \mathbb{H}$ and $g_\tau := N(u)M(\sqrt{v}) \in P_\infty$ that

$$\begin{aligned} g_\tau k_\vartheta &= [N(u)M(\sqrt{v})]k_\vartheta \\ &= \left[\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right) \cdot \begin{pmatrix} \sqrt{v} & 0 \\ 0 & 1/\sqrt{v} \end{pmatrix}\right] \cdot \begin{pmatrix} \cos(\vartheta) & \sin(\vartheta) \\ -\sin(\vartheta) & \cos(\vartheta) \end{pmatrix} \\ &\mapsto g_\tau k_\vartheta i = \tau = u + iv \in \mathbb{H} \end{aligned} \tag{3.22}$$

defines a bijection $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \simeq \mathbb{H}$. As a consequence, the *Iwasawa decomposition*

$$\mathrm{SL}_2(\mathbb{R}) = P_\infty K_\infty \tag{3.23}$$

follows, where $P_\infty := P(\mathbb{R}) = N(\mathbb{R})M(\mathbb{R}) < \mathrm{SL}_2(\mathbb{R})$ denotes the standard Borel group of upper triangular matrices and the representation is unique.[6] This decomposition will be presented in more detail in the prerequisites of Chapter 5.
Similar to the above, define $K_p := \mathrm{SL}_2(\mathbb{Z}_p)$ as well as $K_f := \mathrm{SL}_2(\hat{\mathbb{Z}}) = \prod_{p<\infty} K_p$. Then, for the maximal compact subgroup $K_\mathbb{A} := K_\infty K_f < \mathrm{SL}_2(\mathbb{A})$, an analogous decomposition $\mathrm{SL}_2(\mathbb{A}) = P_\mathbb{A} K_\mathbb{A}$ is acquired. Such a decomposition is used to compute the action of SL_2, globally or locally, on the principal series representation, discussed in Section 5.3.1.

Recall that $\mathrm{SL}_2(\mathbb{A})$ is described in terms of $\overline{\mathrm{SL}}_2(\mathbb{A})$ and $\mathbb{T}$ by means of a cocycle in accordance with (3.18) that is given by local Leray cocycles. In the following, we assume these to be modified as in [KRY06, (8.5.11)] in order to yield a cocycle that is trivial on K_p which is advantageous for global use. To the group $P(\mathbb{A})$ there is a splitting homomorphism $P(\mathbb{A}) \to \overline{\mathrm{SL}}_2(\mathbb{A})$, so that when $P'_\mathbb{A}$ denotes the inverse image, we have

$$P(\mathbb{A}) \times \mathbb{T} \simeq P'_\mathbb{A}.$$

Similarly, the group K_∞ come with a corresponding group in K'_∞ in $\overline{\mathrm{SL}}_2(\mathbb{A})$. However, for K_f, passing to a subgroup is necessary, due to the place $p = 2$. If that component is

[5] Note that other authors oftentimes choose $\vartheta \mapsto -\vartheta$, even though this results in a negative direction of rotation. However, as a consequence, the standard character does not have to be conjugated in the transformation formula of Θ functions (cf. (5.12)).
[6] The group P is a parabolic subgroup which manifests itself in the notation.

replaced with $K_0(4)_2 = \overline{\Gamma_0(4)}$, which is the completion of $\Gamma_0(4)$ in $K_2 = \mathrm{SL}_2(\mathbb{Z}_2)$, then

$$K_0(4) = K_0(4)_2 \times \prod_{2<p<\infty} K_p$$

is an open compact subgroup of $\mathrm{SL}_2(\mathbb{A}_f)$ and there is a splitting morphism $K_0(4) \to \overline{\mathrm{SL}}_2(\mathbb{A})$.
Further, the group $\mathrm{SL}_2(\mathbb{A})$ might be exhausted by simple components – a feature known as *strong approximation*, after Kneser [Kne65]. Namely, for any compact open subgroup $K_{\mathbb{A}_f}$, multiplication by $G_{\mathbb{R}}$ and $\mathrm{SL}_2(\mathbb{Q})$ suffices to reconstruct the whole group $\mathrm{SL}_2(\mathbb{A})$:

$$\mathrm{SL}_2(\mathbb{A}) = \mathrm{SL}_2(\mathbb{Q})\,(\mathrm{SL}_2(\mathbb{R}) \times K_f)\,,$$

where $\mathrm{SL}_2(\mathbb{Q})$ is to be understood as lying diagonally in $\mathrm{SL}_2(\mathbb{A})$. If K_0 is chosen to be an open compact subgroup in $K_0(4)$ and identified with its image in $\overline{\mathrm{SL}}_2(\mathbb{A})$ under the splitting morphism, the same result holds for the metaplectic group:

$$\overline{\mathrm{SL}}_2(\mathbb{A}) = \overline{\mathrm{SL}}_2(\mathbb{Q})\,\overline{\mathrm{SL}}_2(\mathbb{R})K_0. \tag{3.24}$$

3.2.1 Classical Weil representation and relation

In the extension $\overline{\mathrm{SL}}_2(\mathbb{Q}_p)$ there is a twofold cover $\mathrm{Mp}_2(\mathbb{Q}_p)$ included [Kud94] satisfying the analogue of (3.15), namely

$$1 \to \{\pm 1\} \to \mathrm{Mp}_2(\mathbb{Q}_p) \to \mathrm{SL}_2(\mathbb{Q}_p) \to 1. \tag{3.25}$$

Two fold covers of $\mathrm{SL}_2(\mathbb{Q}_p)$ are unique up to isomorphism and are relevant for considering half integral weight modular forms. In addition to generalising the weight to the half integral case, vector valued modular forms, like in Section 3.3, may be considered. For these, the most prominent choice of representation is the Weil representation we have encountered above.

In fact, there is also a classical notion of Weil representations on $\mathrm{Mp}_2(\mathbb{Z})$, which may be taken to be the subgroup of $\mathrm{Mp}_2(\mathbb{R})$ projecting to $\mathrm{SL}_2(\mathbb{Z})$, dating back to Schoenberg in [Sch39]. We shall understand such a representation adelically before verifying that it, in fact, coincides with the classical definition found in the literature (cf. [Str13] which builds upon [Sch09]). The classical notion is related to discriminant forms (cf. Remark 1.27) which are without exception representable by quotients of even $\mathbb{Z}$ lattices. Let (L, q) be such an even non-degenerate lattice. From that object we obtain in a natural way a quadratic space (V_p, q) by taking a tensor product with $\mathbb{Q}_p$. The sections above explain how a representation of $\mathrm{Mp}_2(\mathbb{A})$ on $\mathcal{S}_{\mathbb{A}}(V_{\mathbb{A}})$ is induced by this – the associated Weil representation $\omega_{\mathbb{A}}$. We follow [Kud03] and [BY09] for sketching the extraction of a discrete representation on a finite dimensional space from $\omega_{\mathbb{A}}$. Let $K' < \mathrm{Mp}_2(\mathbb{A})$ be the preimage of $\mathrm{SL}_2(\hat{\mathbb{Z}}) < \mathrm{SL}_2(\mathbb{A}_f)$ under the natural projection to $\mathrm{SL}_2(\mathbb{A})$. Recall that there is a canonical splitting morphism of $\mathrm{SL}_2(\mathbb{Q})$ into $\mathrm{Mp}_2(\mathbb{A})$. Then

$$\mathrm{Mp}_2(\mathbb{A}) = \mathrm{SL}_2(\mathbb{Q})\,\mathrm{Mp}_2(\mathbb{R})K'. \tag{3.26}$$

Note that $\mathrm{SL}_2(\mathbb{Z}) \simeq \mathrm{Mp}_2(\mathbb{Q}) \cap \mathrm{Mp}_2(\mathbb{R})K'$ and that for any $\gamma \in \mathrm{Mp}_2(\mathbb{Z})$ there are unique elements $\gamma_0 \in \mathrm{SL}_2(\mathbb{Z})$ and $\gamma' \in K'$ such that

$$\gamma_0 = \gamma\gamma'.$$

The association $\mathrm{Mp}_2(\mathbb{Z}) \ni \gamma \mapsto \gamma' \in K'$ defines a homomorphism and the idea is to construct a finite dimensional subspace of $\mathcal{S}(V_{\mathbb{A}_f})$ that arises naturally from L and is invariant under the action of $\mathrm{Mp}_2(\mathbb{Z})$ via ω_f. Then we had a finite dimensional representation of $\mathrm{Mp}_2(\mathbb{Z})$.
Note that for $\hat{L} = L \otimes \hat{\mathbb{Z}}$ we have that $\mathbb{1}_{\mu+\hat{L}} \in \mathcal{S}(V_{\mathbb{A}_f})$ with $\mu \in L'/L$. Recall Example 1.33 to verify that this notation is meaningful and the denoted functions generate a space $\mathcal{S}_{L'/L} < \mathcal{S}(V_{\mathbb{A}_f})$ of complex dimension $|L'/L|$. Next, we verify that this space is invariant under $\omega_{\mathbb{A}_f}|_{\mathrm{Mp}_2(\mathbb{Z})}$. For that purpose, it is sufficient to consider the representation locally. Further, (3.16) reveals that the action of the representation ω_p only differs by an element of $\mathbb{T}$ from the action of ${}^{\mathrm{pr}}r_p$. But the latter defined a projective representation of $\mathrm{SL}_2(\mathbb{Q}_p)$ and hence it suffices to consider the action of elements of $\mathrm{SL}_2(\mathbb{Z})$ via ${}^{\mathrm{pr}}r_p$. The group $\mathrm{SL}_2(\mathbb{Z})$ is, however, generated by the two matrices T representing translation by 1 and S representing an inversion at i (cf. Definition 2.12).
By (3.12), we find for $l \in L_p'$ that

$${}^{\mathrm{pr}}r_p(T)\mathbb{1}_{l+L_p}(l+x) = e_p(q(l+x))\mathbb{1}_{l+L_p}(l+x) = e_p(q(l))\mathbb{1}_{l+L_p}(l+x) \doteq \mathbb{1}_{l+L_p}(l+x)$$

For the action of S we remark that by (3.13) it suffices to consider only the operation of the Fourier transformation. By assuming a self dual measure, Remark 1.54 then yields in the notation of Remark 1.57 that

$$\mathcal{F}[\mathbb{1}_{l+\hat{L}}(x)](y) = e_f^{\mathrm{b}}(-l, y) \cdot \mathbb{1}_{\hat{L}'}(y)$$

which is again in $\mathcal{S}_{L'/L}$, as $e_f^{\mathrm{b}}(l, \cdot\,)$ descends to a character on L'/L. Hence, we obtain a representation

$$\rho_L : \mathrm{Mp}_2(\mathbb{Z}) \ni \gamma \mapsto \omega^*_{\mathbb{A}_f}(\gamma') : \mathcal{S}_{L'/L} \to \mathcal{S}_{L'/L}, \tag{3.27}$$

where ω^* denotes the dual representation. Recall that the normalisation of the local measure

$$\mu_p^{\mathrm{b}}(L_p) = \sqrt{|L_p'/L_p|}^{-1} = \sqrt{|\det(S)|_p}$$

implies $\hat{\mu}^{\mathrm{b}}(\hat{L}) = \sqrt{|L'/L|}^{-1} = \sqrt{|S|_\infty}^{-1}$ for the measure of the finite adele module, where S denotes the Gram matrix of L. Of course, there is a scalar product on the space $\mathcal{S}_{L'/L}$ that will be normalised to $\langle \mathbb{1}_{\hat{L}}, \mathbb{1}_{\hat{L}} \rangle = 1$ for reasons that will become clear later on.
Note that $\mathcal{S}_{L'/L}$ in fact carries the structure of a commutative $\mathbb{C}$ algebra, by considering convolution as a product. We have

$$\begin{aligned}
[\mathbb{1}_{l+L_p} * \mathbb{1}_{l'+L_p}](y) &= \int_{\mathbb{Q}_p} \mathbb{1}_{l+L_p}(x)\mathbb{1}_{l'+L_p}(y-x)\,\mathrm{d}\mu_p(x) \\
&= \int_{L_p} \mathbb{1}_L(y-(l+l')-x)\,\mathrm{d}\mu_p(x) \\
&= \mathbb{1}_{l+l'+L_p}(y)
\end{aligned}$$

meaning that the convolution product coincides with the one induced by the group structure of the discriminant form L'/L. Hence, we have proven the following lemma.

Lemma 3.9. *For an even non-degenerate lattice L the space $\mathcal{S}_{L'/L}$ equipped with the convolution product is isometrically isomorphic to the group algebra $\mathbb{C}[L'/L]$*

Further, we conclude the existence of a natural representation on the group algebra $\mathbb{C}[L'/L]$, based on the discussion succeeding (3.27).

Corollary 3.10. *For an even $\mathbb{Z}$ lattice there is a natural representation of $\mathrm{Mp}_2(\mathbb{Z})$ on the group algebra $\mathbb{C}[L'/L]$, which we denote ρ_L and call* (integral) Weil representation*.*

We would like to describe this representation more explicitly. For that purpose, the group $\mathrm{Mp}_2(\mathbb{Z})$ has to be examined more closely. We begin by studying the twofold metaplectic cover of $\mathrm{GL}_2^+(\mathbb{R})$ which we will denote by $\overline{\mathrm{GL}_2^+}(\mathbb{R})$. In order to provide a convenient model for this group, we require the *factor of automorphy* j that has been given in Definition 2.10. Equipped with that notion, $\overline{\mathrm{GL}_2^+}(\mathbb{R})$ may be realised as the group of pairs (γ, ϕ), where $\gamma \in \mathrm{GL}_2^+(\mathbb{R})$ and ϕ denotes a holomorphic square root of $j(\gamma, \cdot\,)$ on $\mathbb{H}$. Multiplication is given by

$$(\gamma, \phi) \cdot (\gamma', \phi') = (\gamma\gamma', \phi(\gamma' \cdot)\phi'). \tag{3.28}$$

Note that $\overline{\mathrm{GL}_2^+}(\mathbb{R})$ inherits an action on $\mathbb{H}$ from $\mathrm{GL}_2(\mathbb{R})^+$ via pullback along the covering map. Namely, for $(\gamma, \phi) \in \overline{\mathrm{GL}_2^+}$ with $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\tau \in \mathbb{H}$, set

$$(\gamma, \phi)\tau := \gamma\tau = \frac{a\tau + b}{c\tau + d}. \tag{3.29}$$

Also, the preimage of SO_2 in $\overline{\mathrm{GL}_2^+}(\mathbb{R})$ will be denoted $\overline{\mathrm{SO}}_2$ and defines a double cover. Recall that the preimage of $\mathrm{SL}_2(\mathbb{Z})$ in $\overline{\mathrm{GL}_2^+}(\mathbb{R})$ is denoted by $\mathrm{Mp}_2(\mathbb{Z})$. It is then generated by the two elements

$$\overline{T} := \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right), \tag{3.30}$$

$$\overline{S} := \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right) \tag{3.31}$$

which project to the two generators T and S of $\mathrm{SL}_2(\mathbb{Z})$ (cf. Definition 2.12). The element

$$\overline{Z} := \overline{S}^2 = (\overline{ST})^3 = \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, i \right) \tag{3.32}$$

generates the centre of $\mathrm{Mp}_2(\mathbb{Z})$.

Remark 3.11. To ease the notation, an element $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ will be associated to $(\gamma, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$ for the standard choice of square root ϕ of the factor of automorphy $j(\gamma, \tau)$. We will also write $\tilde{\gamma} = (\gamma, \phi)$. Note that this association does not define a section of groups as it is not a homomorphism, but only in terms of covering maps!
Similarly, we embed Γ_∞ into $\mathrm{Mp}_2(\mathbb{Z})$ by sending $\gamma \mapsto (\gamma, 1)$. This, however, is a section in the category of groups.

Comparing the transformation property of theta functions in the adelic setting with the transformation rule of theta functions in the classical setting (cf. [Bor98, Thm. 4.1 p. 505]) yields the identification of ρ_L as the classical Weil representation [BY09, p. 639]. The concrete formulae are given as follows.

Remark 3.12. Let (L, q) be an even non-degenerate lattice and $\mathcal{L} = (L'/L, \overline{\mathrm{q}})$ be the associated discriminant form. It suffices to describe the action of ρ_L on the generators $\overline{T}$ and $\overline{S}$. These operations are given by

$$\rho_L(\overline{T})\mathfrak{e}_\lambda = e(\overline{\mathrm{q}}(\lambda)) \cdot \mathfrak{e}_\lambda, \tag{3.33}$$

$$\rho_L(\overline{S})\mathfrak{e}_\lambda = \frac{e(-\operatorname{sig}(\mathcal{L})/8)}{\sqrt{|\mathcal{L}|}} \cdot \sum_{\mu \in \mathcal{L}} e(-\beta(\lambda, \mu))\mathfrak{e}_\mu. \tag{3.34}$$

From the remark above it is apparent that the representation ρ_L only depends on the discriminant form $\mathcal{L}$.

Definition 3.13. Let $(\mathcal{L}, \overline{\mathrm{q}})$ be a discriminant form. Then $\rho_{\mathcal{L}} := \rho_L$ is a unitary representation of $\mathrm{Mp}_2(\mathbb{Z})$ on $\mathbb{C}[\mathcal{L}]$ that is fully determined by (3.33) and (3.34).

We require some more insights into this representation for later investigations and will, in the following, recollect a number of useful facts based on [BS08].

Remark 3.14. a) Note that the generator of the centre $\overline{Z}$ operates for $\lambda \in \mathcal{L}$ as

$$\rho_{\mathcal{L}}(\overline{Z})\mathfrak{e}_\lambda = e(-\operatorname{sig}(\mathcal{L})/4) \cdot \mathfrak{e}_{-\lambda}. \tag{3.35}$$

As a consequence, the Weil representation $\rho_{\mathcal{L}}$ factors through $\mathrm{Mp}_2(\mathbb{Z})/\langle \overline{Z}^2 \rangle \simeq \mathrm{SL}_2(\mathbb{Z})$, if, and only if, $\operatorname{sig}(\mathcal{L}) \equiv 0 \mod 2$.

b) Another useful action which is computable is the one of $\overline{U} = \overline{ST}^{-1}\overline{S}^{-1}$:

$$\rho_{\mathcal{L}}(\overline{U}^m)\mathfrak{e}_\lambda = \frac{1}{|D|} \sum_{\mu, \nu \in D} e\left(-mq(\mu) + \beta(\mu, \lambda - \nu)\right) \mathfrak{e}_\nu \tag{3.36}$$

found in [BS08, 2 Lem. 2.3 p. 254].

c) Further, following [BS08, 2 Lem. 2.1 p. 253] define for integers $a, d \in \mathbb{Z}$ fulfilling $\gcd(a, N) = 1 = \gcd(d, N)$ such that $a \cdot d \equiv 1 \mod N$ the element

$$R_d := \overline{ST}^d \overline{S}^{-1} \overline{T}^a \overline{ST}^d.$$

It has the form $R_d = (\gamma, \phi)$ with $\gamma \equiv \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \mod N$. Then

$$\rho_{\mathcal{L}}(R_d)\mathrm{e}_\lambda = \frac{G_{\mathcal{L}}(d)}{G_{\mathcal{L}}(1)} \cdot \mathrm{e}_{d\lambda}, \tag{3.37}$$

where $G_{\mathcal{L}}(d) = \sum_{\nu \in \mathcal{L}} e(d\,\overline{\mathrm{q}}(\nu))$ are Gauss sums (cf. Definition A.16).

Remark 3.15. Let $(L, \mathrm{q}) = (L_1, \mathrm{q}_1) \oplus (L_2, \mathrm{q}_2)$ be a direct composition into even lattices. Recall that $\mathbb{C}[\mathcal{L}] = \mathbb{C}[\mathcal{L}_1] \otimes \mathbb{C}[\mathcal{L}_2]$. Then $\rho_L = \rho_{L_1} \otimes \rho_{L_2}$.

Remark 3.16. In case of even rank it had been noted above that the representation $\rho_{\mathcal{L}}$ factors through $\mathrm{SL}_2(\mathbb{Z})$. Further, writing $N := \mathrm{lev}(\mathcal{L})$ we find that $\rho_{\mathcal{L}}$ is trivial on $\Gamma(N)$ for even rank, so that it factors through the finite group

$$\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) \simeq \Gamma(1)/\Gamma(N).$$

In case of odd rank, the oddity formula [Joh98, Chap. 15 p. 383 (30)] implies $4 \mid N$, in particular, $\mathcal{L}$ contains 2-adic Jordan components. In this case, there is a well known section (see also [Shi73, p. 447])

$$s : \Gamma_0(4) \to \overline{\Gamma}_0(4), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \epsilon_d^{-1} \left(\frac{c}{d}\right) \sqrt{j(\gamma, \tau)} \right) \tag{3.38}$$

with $\epsilon_d = 1$ or i, depending on whether $d \equiv 1$ or $3 \mod 4$. Here, $\overline{\Gamma_0}(4)$ denotes the preimage of $\Gamma_0(4)$ under the projection $\overline{\mathrm{SL}_2}(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z})$.[7] The argument at the end of [Bor00, Thm. 5.4 p. 330] yields that $\rho_{\mathcal{L}}$ is trivial on $s(\Gamma(N))$ and factors through the central extension of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ by $\{\pm 1\}$ given by

$$\mathrm{Mp}_2(\mathbb{Z})/s(\Gamma(N)).$$

Finally, we require the following piece of notation for the computation of Fourier expansions of non-holomorphic Eisenstein series.

Definition 3.17. For $\lambda, \mu \in \mathcal{L}$ and $(\gamma, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$, we introduce the following notation for the λ, μ-*coefficient*:

$$\rho_{\lambda,\mu}(\gamma, \phi) := \langle \rho_L(\gamma, \phi)\mathfrak{e}_\lambda, \mathfrak{e}_\mu \rangle. \tag{3.39}$$

[7] Note that only the section on $\Gamma(N)$ maps to Mp_2 and $\Gamma_0(N)$ is mapped to a central extension by $\mathbb{C}^\times$.

3.3 Vector valued modular forms

The theory of scalar valued modular forms is rich and attempting to generalise it in different directions is apparent. Recall that the compellingly simple construction of Jacobi's theta function

$$\sum_{n\in\mathbb{Z}} q^n$$

yields a function that is not modular in the standard sense, but almost. In addition, classical functions like the Dedekind eta function are almost modular in the sense that they transform with a multiplier, which is correcting signs. Also recall that Theorem 2.22 yields a whole class of theta functions that transform almost like a modular form - up to a character. Characters are unital one dimensional representations and cocycles appear when projecting representations from coverings to the underlying groups. So it appears natural to generalise the concept of modular forms to include a (possibly projective) representation. However, the step from allowing a character to allowing a multidimensional representation may not be immediately intelligible and we will recall an example. Consider the generalised theta function (compare Definition 2.17) for an even positive definite lattice (L, q)

$$\theta_{\lambda+L} : \mathbb{H} \to \mathbb{C}, \qquad \sum_{x\in\lambda+L} q^{\mathrm{q}(x)},$$

with $\lambda \in L'/L$ and $q = \exp(2\pi i\tau)$. Note that then

$$\theta_{\lambda+L}|_{n/2}S$$

is in fact a linear combination of theta functions of this very form $\theta_{\mu+L}$ for $\mu \in L'/L$. (cf. [Ebe12, 3.1 p. 81 (T2)]). To rephrase this phenomenon: there is a linear action intertwining the different theta functions $\theta_{\lambda+L}$ associated to an even lattice L. In fact, their action is compatible in the above sense.

Theorem 3.18 ([Bor98, Thm. 4.1 p. 505])**.** *For an even lattice* (L, q)*, the theta series* $(\theta_{\lambda+L})_{\lambda\in L'/L}$ *is a vector valued modular form for* $\mathrm{Mp}_2(\mathbb{Z})$ *that transforms under the Weil representation associated to the discriminant form* $(L'/L, \overline{\mathrm{q}})$*.*

This means that the linear intertwining of the individual components may be described by a complex representation, yielding a new perspective on the topic at hand. Indeed, the theory of vector valued modular forms is suited to unify all of the above phenomena in a single framework and we shall, in the following, review the elementary parts of its theory. The first published record of a call for the development of the theory of vector valued modular forms the author is aware of is from Selberg [Sel65]. The reader may find a source for vector valued modular forms in a quite general setting in [Gan14]. The subcase that is required for this thesis is the one of vector valued modular forms for the Weil representation ρ_L of $\mathrm{Mp}_2(\mathbb{Z})$ and it will be discussed in the following. However, note that not all naturally occurring vector valued modular forms are associated to these types of representations – compare for instance [Zhu96].

3.3.1 Discrete Weil representation

For the following, compare [Str13] and [BS08].

One of the most significant examples of representations for considering vector valued modular forms is the discrete Weil representation introduced in Subsection 3.2.1.

Recall that for a discriminant form $(\mathcal{L}, \overline{\mathrm{q}})$ there is a unitary representation $\rho_{\mathcal{L}} : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{Aut}(\mathbb{C}[\mathcal{L}])$ introduced in (3.27) and described in Remark 3.12. As before, set $e(x) = \exp(2\pi i x)$ and recall that the representation is determined by the actions of the two generators $\overline{T} = \left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1\right)$, and $\overline{S} = \left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau}\right)$ of $\mathrm{Mp}_2(\mathbb{Z})$ given by

$$\begin{aligned} \rho_{\mathcal{L}}(\overline{T})\mathfrak{e}_\lambda &= e(\overline{\mathrm{q}}(\lambda)) \cdot \mathfrak{e}_\lambda, \\ \rho_{\mathcal{L}}(\overline{S})\mathfrak{e}_\lambda &= \frac{e(-\operatorname{sig}(\mathcal{L})/8)}{\sqrt{|\mathcal{L}|}} \cdot \sum_{\mu \in \mathcal{L}} e(-\beta(\lambda, \mu))\mathfrak{e}_\mu. \end{aligned}$$

After this recap, we may introduce vector valued modular forms.

Definition 3.19. A function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ is called *vector valued modular function* of weight $k \in \frac{1}{2}\mathbb{Z}$ for $\Gamma \leq \mathrm{Mp}_2(\mathbb{Z})$ if for all $\gamma = (M, \phi) \in \Gamma$

$$f(\gamma\tau) = \phi(\tau)^{2k} \cdot [\rho_L(\gamma) f](\tau).$$

It carries the predicate *holomorphic*, *weakly holomorphic*, *meromorphic*, or *real analytic*, if it satisfies the usual conditions on $\Gamma \backslash \mathbb{H}$ respectively. The associated spaces are denoted by $\mathcal{M}_{L,k}, \mathcal{M}^!_{L,k}, \mathcal{A}_{L,k}, \mathcal{R}_{L,k}$ in accordance with Table 2.1.

By Remark 3.16, we recognise that every component function of such a vector valued modular form is a scalar valued modular form for $\Gamma(N)$ for some $N \in \mathbb{N}$. However, following [Bor98, Exa. 2.2 p. 500] we realise that any scalar valued modular form may be induced to a vector valued modular form to $\mathrm{Mp}_2(\mathbb{Z})$ for some Weil representation. Hence, we we will stick with the case of vector valued modular forms of level 1; meaning for the full group $\mathrm{Mp}_2(\mathbb{Z})$.

Remark 3.20. In case of $2k \not\equiv \mathrm{sig}(\mathcal{L}) \mod 2$, we find $\mathcal{M}_{L,k} = \{0\} = \mathcal{R}_{L,k}$.

Proof: In fact, applying $|_{\mathcal{L}}\overline{Z}$ to a function f twice yields

$$e(\mathrm{sig}(\mathcal{L})/2 - k) \cdot f.$$

Now if f was a modular form for $\mathrm{Mp}_2(\mathbb{Z})$, i.e. invariant under that action, we concluded that either $f = 0$ or $e(k - \mathrm{sig}(\mathcal{L})/2) = 1$ which is equivalent to $2k \equiv \mathrm{sig}(\mathcal{L}) \mod 2$. □

Remark 3.21. If $f \in \mathcal{M}^!_{L,k}$ we find that

$$f|_{L,k}\overline{T} = \sum_{\lambda \in L'/L} e(-\overline{\mathrm{q}}(\lambda)) f_\lambda(\tau+1)\mathfrak{e}_\lambda.$$

By comparing components we infer that $e(-\overline{\mathrm{q}}(\lambda)(\tau+1))f_\lambda(\tau+1) = e(-\overline{\mathrm{q}}(\lambda)(\tau))f_\lambda(\tau)$, i.e. $e(-\overline{\mathrm{q}}(\lambda)(\,\cdot\,))f_\lambda(\,\cdot\,)$ is 1 periodic and holomorphic in a neighbourhood of infinity, meaning it possesses a Fourier expansion. As a consequence, we find

$$f(\tau) = \sum_{\lambda \in L'/L} \sum_{n \in \overline{\mathrm{q}}(\lambda)+\mathbb{Z}} a(\lambda, n) e(n\tau)\mathfrak{e}_\lambda \tag{3.40}$$

for appropriate numbers $a(\lambda, n) \in \mathbb{C}$. In particular, the set of admissible indices (cf. Definition 3.67)

$$\mathscr{I} = \{(\lambda, n) \in \mathcal{L} \times \mathbb{Q} \mid n \equiv \overline{\mathrm{q}}(\lambda)\}$$

interpreted as a subset of the dual space $(\mathcal{M}^!_{k,L})'$ is separating.

Remark 3.22. Let $f \in \mathcal{M}^!_{L,k}$ with Fourier expansion as in (3.40). Applying $|_{\mathcal{L}}\overline{Z}$ to f yields

$$f_\lambda = e[(\mathrm{sig}(\mathcal{L}) - 2k)/4] \cdot f_{-\lambda} \tag{3.41}$$

for any $\lambda \in \mathcal{L}$. In particular, assuming $\lambda = -\lambda$ which may only happen if λ is trivial in all p components of $\mathcal{L}$ for $p > 2$, we conclude that f_λ vanishes, unless $\mathrm{sig}(\mathcal{L}) \equiv 2k \mod 4$. On the level of Fourier coefficients (3.41) means that these fulfil the following symmetry property

$$a(-\lambda, n) = e[(\mathrm{sig}(\mathcal{L}) - 2k)/4] \cdot a(\lambda, n). \tag{3.42}$$

In fact, applying $|_{\mathcal{L}}\overline{Z}$ (cf. Example 3.14 a)) yields

$$\sum_{\lambda \in L'/L} \sum_{n \in \overline{q}(\lambda)+\mathbb{Z}} a(\lambda, n)e(n\tau)\mathfrak{e}_\lambda = e(\mathrm{sig}(\mathcal{L})/4 - k/2) \sum_{\lambda \in L'/L} \sum_{n \in \overline{q}(\lambda)+\mathbb{Z}} a(\lambda, n)e(n\tau)\mathfrak{e}_{-\lambda}.$$

While properties of vector valued modular forms have already been discussed, we have yet to verify their existence. Hence, we will have a look at the following trivial example, before turning towards more interesting construction principles.

Example 3.23. Let L be unimodular, i.e. $\mathcal{L} = L'/L = \{0\}$. Then $\mathbb{C}[\mathcal{L}] \simeq \mathbb{C}$ and necessarily $\mathrm{rk}(L) = m \equiv 0 \mod 8$, so that the discrete Weil representation ρ_L acts trivially by Remark 3.12. This implies that the notion of a vector valued modular form for the Weil representation coincides with the classical notion of a scalar valued modular form in that case. Also note that by Remark 3.20 the weight k must be even in that context.

3.3.2 Some constructions

There are various construction principles for vector valued modular forms. We will present a selection of these principles and begin by making direct observations. We will then introduce concrete classes of modular forms and subsequently move on to discuss more advanced construction principles before touching on some structure results.

Combining known forms

Clearly, we may abstractly induce forms from subrepresentations, but this has already been cast aside as a special technical case when explaining that we stick to forms for the full group $\mathrm{Mp}_2(\mathbb{Z})$. Instead, we recall ideas from the classical setting where we have encountered two accessible construction principles: products and symmetrisation. The latter has been used to construct Eisenstein series, which will also be revisited below, while the further offered a trivial construction of new forms and will be stated first.

Remark 3.24. Consider finitely many discriminant forms $(\mathcal{L}_i, \overline{q}_i)$ for $1 \leq i \leq m$ and for $k_i \in \mathbb{Z}/2$ modular forms $f_i \in \mathcal{M}_{\mathcal{L}_i,k_i}$ with Fourier expansion

$$f_i(\tau) = \sum_{\lambda_i \in \mathcal{L}_i} \sum_{n_i \in \overline{q}_i(\lambda_i)+\mathbb{Z}} c_i(\lambda_i, n_i)e(n_i\tau)\mathfrak{e}_{\lambda_i}.$$

For $\mathcal{L} = \bigoplus_i \mathcal{L}_i$ we use the notation $\underline{\lambda} = (\lambda_i)_i \in \bigoplus_i \mathcal{L}_i = \mathcal{L}$ as well as $\underline{n} = (n_i)_i$ to define the function

$$\bigotimes_i f_i : \mathbb{H} \ni \tau \mapsto \sum_{\underline{\lambda}\in\mathcal{L}} \sum_{\substack{\underline{n}\in\mathbb{Q}^m,\\ \forall i: n_i\in\overline{\mathrm{q}}_i(\lambda_i)+\mathbb{Z}}} c(\underline{\lambda},\underline{n}) e(\underline{n}\tau)\cdot \mathrm{e}_{\underline{\lambda}} \in \mathbb{C}[\mathcal{L}]$$

where $c(\underline{\lambda},\underline{n}) = \prod_i c_i(\lambda_i, n_i)$, $e(\underline{n}\tau) = e(\sum_i n_i\tau)$, and $\mathfrak{e}_{\underline{\lambda}} = \bigotimes_i \mathfrak{e}_{\lambda_i}$. Then for $k = \sum_i k_i$ we have that

$$f_1 \otimes \ldots \otimes f_m \coloneqq \bigotimes_i f_i \in \mathcal{M}_{\mathcal{L},k}.$$

The above remark offers trivial means to construct new forms from already known ones, but so far we have not encountered a single true vector valued modular form. Hence, selecting an even lattice L of level $\mathrm{lev}(L) \mid N$ and a modular form for $\Gamma(N)$, we may inject this form into a component of $\mathbb{C}[L'/L]$. This yields a vector valued modular form for $\Gamma(N)$ which may be symmetrised to a vector valued modular form for the full group $\mathrm{SL}_2(\mathbb{Z})$. Such a construction has been given by Scheithauer in [Sch15, Thm. 3.1 p. 7] along others and we will present a special case of the more delicate variant for $\Gamma_0(N)$.
Note that the Weil representation acts in fact via a character $\chi_{\mathcal{L}}$ under $\Gamma_0(\mathrm{lev}(L))$ on the zero component of $\mathbb{C}[L'/L]$. This character is given by (cf. [Sch15, p. 7] for details)

$$\chi_{\mathcal{L}}\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right) = \left(\frac{a}{|\mathcal{L}|}\right) e[(a-1)\,\mathrm{oddity}(\mathcal{L})/8].$$

Proposition 3.25. *Let L be an even lattice, $N \in \mathbb{N}$ with $\mathrm{lev}(L) \mid N$ and $\chi_{\mathcal{L}}$ be the character of $\mathrm{SL}_2(\mathbb{Z})$ presented above. Given a modular form $f \in \mathcal{M}_k(\Gamma_0(N), \chi_{\mathcal{L}})$, we find*

$$F(\tau) \coloneqq \sum_{\gamma\in\Gamma_0(N)\backslash\Gamma(1)} f|_k\gamma\rho_L(\gamma)^{-1}\mathfrak{e}_0 \in \mathcal{M}_{L,k}.$$

Remarkably, a significant portion of vector valued modular forms are of this type. Namely, if $\mathrm{lev}(L)$ is assumed to be square free and we choose $F \in \mathcal{M}_{L,k}$ that is invariant under the action of $\mathrm{O}(L'/L)$, then F may be represented as such a lift of a $\Gamma_0(\mathrm{lev}(L))$ scalar valued modular form (cf. [Sch15, Cor. 5.5 p. 24]).

Siegel theta series

We will briefly review Siegel theta series, based on [Bor98]. In this subsection let (L, q) be an even lattice of signature (m^+, m^-) and rank $m = m^+ + m^-$. Further, let $\mathcal{P} : \mathbb{R}^m \to \mathbb{C}$ be a polynomial. We identify $\mathbb{R}^m$ with the underlying space of the quadratic

space $\mathbb{R}^{(m^+,m^-)}$ (cf. Example 1.5). Then we say that $\mathcal{P}$ has *degree* (κ^+,κ^-), if $\mathcal{P}$ has degree κ^+ in the first m^+ variables and degree κ^- in the last m^- variables.
Denote by $\sigma : L\otimes_{\mathbb{Z}}\mathbb{R}\to\mathbb{R}^{(m^+,m^-)}$ an isometry and set $z:=\sigma^{-1}(\mathbb{R}^{0,m^-})$. Then $z^\perp=\sigma^{-1}(\mathbb{R}^{m^+,0})$ and $V=z^\perp\oplus z$ with $z\in\mathbb{D}$. For any element $v\in V_{\mathbb{R}}=L\otimes\mathbb{R}$ introduce the decomposition $v=v_{z^\perp}+v_z\in z^\perp\oplus z$. With this notation, we may introduce Siegel theta functions.

Definition 3.26. Let $\lambda\in L'/L$ and $\mathcal{P}$ be homogeneous of degree (κ^+,κ^-), $\tau=u+iv\in\mathbb{H}$ and σ, z be as above as well as $\alpha,\beta\in V$. Define

$$\begin{aligned}\theta_{L,\lambda}(\tau,\alpha,\beta,\sigma,\mathcal{P}) := v^{\frac{m^-}{2}+\kappa^-}\sum_{\mu\in L+\lambda}\exp\left(-\frac{\Delta}{8\pi y}\right)(\mathcal{P})(\sigma(\mu+\beta))\\ \times e\left(\tau\,\mathrm{q}[(\mu+\beta)_{z^\perp}]+\overline{\tau}\,\mathrm{q}[(\mu+\beta)_z]-\mathrm{b}(\mu+\beta/2,\alpha)\right),\end{aligned} \tag{3.43}$$

where Δ is the *Laplacian*.

Remark 3.27. Note that the prefactor $v^{\frac{m^-}{2}+\kappa^-}$ has been introduced in order to completely shift the transformation to the holomorphic variable as in [MZ23].

Remark 3.28. In the particularly relevant case of $\alpha=0=\beta$ these elements are ommitted from the notation and the function may be written in more plane terms. To this end, note that $\mathrm{q}(v)=\mathrm{q}(v_{z^\perp})+\mathrm{q}(v_z)$ so that we write $\mathrm{q}_z(v):=\mathrm{q}(v_z)$ and call q_z^+ given by $\mathrm{q}_z^+(v):=\mathrm{q}(v)-2\,\mathrm{q}_z(v)=\mathrm{q}(v_{z^\perp})-\mathrm{q}(v_z)$ the *standard majorant* of q with respect to $z\in\mathbb{D}$. With this notation, we find for $\tau=u+iv\in\mathbb{H}$ that

$$\theta_{L,\lambda}(\tau,\sigma,\mathcal{P})=v^{\frac{m^-}{2}+\kappa^-}\sum_{\mu\in L+\lambda}\exp\left(-\frac{\Delta}{8\pi y}\right)(\mathcal{P})(\sigma(\mu))e\left(u\,\mathrm{q}(\mu)+iv\,\mathrm{q}_z^+(\mu)\right). \tag{3.44}$$

We have defined such a Siegel theta function for every $\lambda\in\mathcal{L}=L'/L$ and consider these to carry local information. Piecing them together creates a complete theta function.

Definition 3.29. Let $\mathcal{P}$ be homogeneous of degree (κ^+,κ^-), $\tau=u+iv\in\mathbb{H}$ and σ, z be as above and $\alpha,\beta\in V$. Define the *Siegel* theta function of L to be

$$\Theta_L(\tau,\alpha,\beta,\sigma,\mathcal{P}) := \sum_{\lambda\in\mathcal{L}}\theta_{L,\lambda}(\tau,\alpha,\beta,\sigma,\mathcal{P})\mathfrak{e}_\lambda. \tag{3.45}$$

In case $\alpha=\beta=0$, these are omitted from the notation. The same holds in case $\mathcal{P}=1$.

Borcherds has proven the following transformation property for Siegel theta functions.

Theorem 3.30 ([Bor98, Thm. 4.1 p. 505])**.** *For $\mathcal{P}$ a homogeneous polynomial of degree (κ^+, κ^-) and every $\gamma = \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \phi\right) \in \mathrm{Mp}_2(\mathbb{Z})$, we find*

$$\Theta_L(\gamma\tau, a\alpha + b\beta, c\alpha + d\beta, \sigma, \mathcal{P}) = \phi^{m^+ + 2\kappa^+ - (m^- + 2\kappa^-)} \rho_L(\gamma)\Theta_L(\tau, \alpha, \beta, \sigma, \mathcal{P}). \tag{3.46}$$

As usual, the proof of the transformation property is reduced to the cases of the two generators $\overline{T}$ and $\overline{S}$ of $\mathrm{Mp}_2(\mathbb{Z})$. The first identity is a triviality, while the second is achieved with the typical tool of Poisson summation.

In Section 5.2, theta functions are introduced in adelic fashion, depending on a choice of Schwartz function which is why we introduce the following semiclassical perspective.

Theta functions for general Schwartz forms

The Siegel theta functions introduced above already represent a fairly general framework for theta functions which suffices for most applications. However, the kernel symmetrised in that setting has a very particular form and while technically being adequate for the applications within this thesis, formulae appearing in the later parts of this thesis are much more amenable in a different notation, arising from a more conceptual approach to theta functions. In order to approach another construction perspective recall the classical setting for theta functions (cf. Definition 2.17) which were constructed by building a family of Schwartz functions $\varphi(\tau) \in \mathcal{S}(V_\mathbb{R})$ from the form q and symmetrising $\varphi(\tau)$ over the lattice L. These series were clearly periodic and constructed in such a fashion that Poisson summation yielded a transformation property for the action of $S \in \mathrm{SL}_2(\mathbb{Z})$.
However, in case of an indefinite quadratic form, the choice $V_\mathbb{R} \ni x \mapsto \exp(2\pi i\, \mathrm{q}(x)\tau)$ does not define a Schwartz function, for it is not rapidly decreasing. Hence, we will have to select a different variant of that prototype in order to guarantee convergence. In addition, it is unclear how to construct a whole family $\varphi(\tau) \in \mathcal{S}(V_\mathbb{R})$ of Schwartz functions. In order to shed light on a possible solution we have to take another perspective on the classical situation.
We note that in the classical setting for a positive definite quadratic space (V, q) of even signature and

$$\varphi : V_\mathbb{R} \to \mathbb{C}, \qquad v \mapsto \exp(-2\pi\, \mathrm{q}(v))$$

we find for $\tau = u + iv \in \mathbb{H}$, $g_\tau = N(u)M(\sqrt{v})$ as in (3.22) and ω_∞ the Weil representation of Example 3.8 that

$$\omega_\infty(g_\tau)\varphi(x) = e^{2\pi i\, \mathrm{q}(v)u} v^{m/4} \varphi(\sqrt{v}x) = v^{m/4} \exp(2\pi i\, \mathrm{q}(x)\tau). \tag{3.47}$$

With that new perspective, we may define theta functions associated to Schwartz forms. First, recall that there is a section $P_\infty \to \mathrm{Mp}_2(\mathbb{R})$ of the standard Borel subgroup, so that g_τ may directly be interpreted as an element in $\mathrm{Mp}_2(\mathbb{R})$.

Definition 3.31. Let $\varphi \in \mathcal{S}(V_\mathbb{R})$ and $\lambda \in \mathcal{L} = L'/L$. Then

$$\theta_{L,\lambda}(\tau; \varphi) = v^{-m/4} \cdot \sum_{x \in L + \lambda} [\omega_\infty(g_\tau)\varphi](x)$$

is called *theta function* associated to φ and the coset $\lambda + L$.

We note that we may let $\mathrm{O}(V_\mathbb{R})$ act on φ via the left regular representation, i.e.

$$h : \varphi \mapsto \varphi(h^{-1} \cdot) \in \mathcal{S}(V_\mathbb{R}).$$

In that sense, $\theta_{L,\lambda}$ may be understood as a function on $\mathbb{H} \times \mathrm{O}(V_\mathbb{R})$. Combining the above theta functions for different cosets of the dual lattice yields the following theta function.

Definition 3.32. Let $\varphi \in \mathcal{S}(V_\mathbb{R})$. Then

$$\Theta_L(\tau, h; \varphi) = v^{-m/4} \cdot \sum_{x \in L'} [\omega_\infty(g_\tau)\varphi](h^{-1}x)\mathfrak{e}_{x+L}$$

is called *theta function* associated to φ. In case the Schwartz form φ is invariant under the action of a maximal compact $K \subseteq \mathrm{O}(V_\mathbb{R})$, we find that the dependence on h factors through $\mathbb{D}$ and we write $\Theta_L(\tau, z; \varphi)$ for $z \in \mathbb{D}$. Also, for fixed z, we omit this variable.

A prominent example is the classical case based on (3.47) given that $(V_\mathbb{R}, \mathrm{q})$ is positive definite. However, in case of an indefinite quadratic form, the naively chosen function $V_\mathbb{R} \ni x \mapsto \exp(-2\pi\, \mathrm{q}(x))$ does not define a Schwartz function, for it is not rapidly decreasing. Recall that we had for an element $z \in \mathbb{D}$ defined the standard majorant

$$\mathrm{q}_z^+(v) = \mathrm{q}(v_{z^\perp}) - \mathrm{q}(v_z)$$

which is positive definite. This enables us to associated theta functions to indefinite quadratic spaces by introducing a parameter $z \in \mathbb{D}$.

Example 3.33. With $\mathrm{q}_z^+ : V_\mathbb{R} \to \mathbb{R}$ as above, $\varphi(x) = e(i\,\mathrm{q}_z^+(x))$, and $\tau = u + iv \in \mathbb{H}$ we find

$$\Theta_L(\tau, z; \varphi) = \sum_{x \in L'} e\left(u\,\mathrm{q}(x) + iv\,\mathrm{q}_z^+(x)\right) \mathfrak{e}_{x+L}.$$

We will revisit this construction later in Section 5.2, where it will be given adellically. Before finishing the current subsection we note that theta functions do factor, provided the associated Schwartz forms do.

Remark 3.34. Let the lattice $L = L_1 \oplus L_2$ split and assume $\varphi_\infty \in \mathcal{S}(V_\infty)$ splits as a tensor product $\varphi_\infty = \varphi'_\infty \otimes \varphi''_\infty \in \mathcal{S}(V_{21\infty}) \otimes \mathcal{S}(V_{2,\infty})$. Then, if the theta function $\Theta(\tau; \varphi_\infty)$ is absolutely convergent, it also decomposes as a tensor product

$$\Theta_L(\tau; \varphi_\infty) = \Theta_{L_1}(\tau; \varphi'_\infty) \otimes \Theta_{L_2}(\tau; \varphi''_\infty). \tag{3.48}$$

Proof: A straight forward calculation yields the desired result

$$\begin{aligned}
&\Theta_L(\tau; \varphi_\infty) \\
&= v^{-\frac{m}{4}} \cdot \sum_{(\lambda_1,\lambda_2) \in L_1'/L_1 \oplus L_2'/L_2} \; \sum_{l_1 \in L_1+\lambda_1, l_2 \in L_\ell + \lambda_2} \left(\omega_\infty(g_\tau)\varphi_\infty\left((l_1, l_2)\right)\right) \mathrm{e}_{\lambda_1+\lambda_2} \\
&= \sum_{\substack{\lambda_1 \in L_1'/L_1 \\ \lambda_2 \in L_2'/L_2}} v^{-\frac{1}{4}} \sum_{l_1 \in L_1+\lambda_1} \left(\omega_\infty(g_\tau)\varphi'_\infty\right)(l_1) \cdot v^{-\frac{m-1}{4}} \sum_{l_2 \in L_2+\lambda_2} \left(\omega_\infty(g_\tau)\varphi''_\infty\right)(l_2) \mathrm{e}_{\lambda_1+\lambda_2} \\
&= \sum_{\substack{\lambda_1 \in L_1'/L_1 \\ \lambda_2 \in L_2'/L_2}} \theta_{L_1,\lambda_1}(\tau; \varphi'_\infty) \cdot \theta_{L_2,\lambda_2}(\tau; \varphi''_\infty) \mathrm{e}_{\lambda_1+\lambda_2} \\
&= \Theta_{L_1}(\tau; \varphi'_\infty) \otimes \Theta_{L_2}(\tau; \varphi''_\infty).
\end{aligned}$$

□

Lifts between discriminant forms

Assuming we have an even lattice L and a sublattice $M \leq L$, it is natural to ask, whether modular forms for one of these lattices may be used to construct modular forms for the other. In fact, there is a rather natural construction for relating functions in the spaces $\mathbb{C}[M'/M]^\mathbb{H}$ and $\mathbb{C}[L'/L]^\mathbb{H}$. However, its following discussion is technical and the reader only interested in references may consult (3.49) and (3.51) as well as Proposition 3.36.

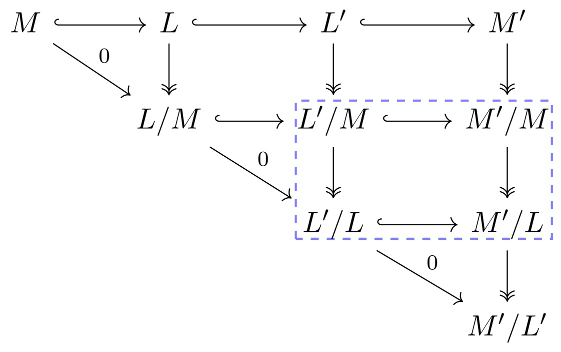


Figure 3.3.1: Visualisation of natural projections and injections between quotients related to $M \leq L$. Diagonal arrows correspond to trivial mappings. The significant rectangle for our construction is framed in blue. Passing to the generated group algebras, the diagram remains the same.

To this end, let

$$M \leq L \leq L' \leq M'$$

be a chain of lattices of finite index, so that $L/M \leq L'/M \leq M'/M$ – compare Figure 3.3.1.

Recall the construction of the coordinate mappings $M'/M \ni \mu \mapsto (f \mapsto f_\mu)$. An element $f \in \mathcal{A}_{M'/M,k}$ is determined by its values under all of these mappings. Hence, it suffices to construct from such an f images for the respecting mappings $L'/L \ni \nu \mapsto g_\nu$. We may reduce to the case $f(\tau) \in \mathbb{C}^{M'/M}$ by fixing τ.
Clearly, an element in $\mathbb{C}^{M'/L}$ has a natural reduction to an element in $\mathbb{C}^{L'/L}$, by pulling back along the natural injection $\iota : L'/L \to M'/L$. Hence, constructing from $f(\tau) \in \mathbb{C}^{M'/M}$ a natural element in $\mathbb{C}^{M'/L}$ suffices. If there was a natural section s to the projection $M'/M \to M'/L$, applying the pullback of it to $f(\tau)$ yielded the desired result. However, the projection $\pi : M'/M \to M'/L$ might not be injective, in fact its kernel is L/M, so that it represents a $|L/M|$ fold cover of M'/L on which the kernel L/M acts on the fibres. Hence, for an arbitrary section s_0 to π we obtain for every $\nu \in L/M$ another section $s_\nu : \mu \mapsto s_0(\mu) + \nu$. Their images cover M'/M disjointly so that the mapping

$$s^* : \sum_{\nu \in \ker(\pi)} (s_\nu)^* : \mathbb{C}^{M'/M} \to \mathbb{C}^{M'/L}$$

is well defined and, in particular, independent of any choice of ν and s_0. As a consequence, there is the natural map

$$\iota^* \circ s^* : \mathbb{C}^{M'/M} \to \mathbb{C}^{M'/L} \to \mathbb{C}^{L'/L}.$$

In conclusion, there is an operator

$$\downarrow_L^M : \mathbb{C}[M'/M]^{\mathbb{H}} \to \mathbb{C}[L'/L]^{\mathbb{H}}, \quad g \mapsto \downarrow_L^M g, \tag{3.49}$$

with

$$(\downarrow_L^M g)_{\overline{\mu}} = \sum_{\nu \in L/M} g_{\mu+\nu} \tag{3.50}$$

where $\mu := s_0 \circ \iota(\overline{\mu})$ is an arbitrary but fixed lift of some $\overline{\mu} \in L'/L$ to M'/M.

Remark 3.35. The operator $\downarrow_L^M$ translates the theta function associated with M to a theta function of L. In the notation of Definition 3.32 this means

$$\begin{aligned}\downarrow_L^M (\Theta_M(\tau, z; \varphi)) &= \downarrow_L^M \left(\sum_{\lambda \in M'/M} \sum_{x \in M+\lambda} (\omega_\infty(g_\tau)\varphi)(x, z) \mathrm{e}_\lambda \right) \\ &= \sum_{\lambda \in L'/L} \sum_{x \in L+\lambda} (\omega_\infty(g_\tau)\varphi)(x, z) \mathrm{e}_\lambda \\ &= \Theta_L(\tau, z; \varphi).\end{aligned}$$

The operator $\downarrow_L^M$ clearly preserves smoothness and polynomial or exponential growth conditions so that it preserves the property of being a modular (cusp) form, if only it translated the transformation property. This will become apparent later, so that we continue to ask for its adjoint operator with respect to the Petersson scalar product.

Let $f \in \mathcal{A}_{L,k}, g \in \mathcal{A}_{M,k}$ such that their Petersson scalar product is defined and compute

$$\begin{aligned}\langle f, \downarrow_L^M g \rangle_L &= \sum_{\overline{\mu} \in L'/L} f_{\overline{\mu}} \cdot \overline{\downarrow_L^M g_{\overline{\mu}}} \\ &= \sum_{\overline{\mu} \in L'/L} f_{\overline{\mu}} \cdot \overline{\sum_{\alpha \in L/M} g_{\mu+\alpha}} \\ &= \sum_{\overline{\mu} \in L'/L} f_{\overline{\mu}} \cdot \overline{\sum_{\substack{\alpha \in L'/M \\ \overline{\alpha} = \overline{\mu}}} g_{\alpha}}\end{aligned}$$

$$= \sum_{\mu \in L'/M} f_{\overline{\mu}} \cdot \overline{g_\mu}$$
$$= \sum_{\mu \in M'/M} \left(\uparrow_L^M f\right)_\mu \cdot \overline{g_\mu}$$
$$= \langle \uparrow_L^M f, g \rangle_M$$

where $\uparrow_L^M f$ denotes the naive embedding from $f \in \mathbb{C}^{L'/L}$ to $\mathbb{C}^{M'/M}$ given by $f_{\overline{\mu}}$, if $\mu \in L'/M$ and 0, otherwise. Explicitly,

$$\left(\uparrow_L^M f\right)_\mu = \begin{cases} f_{\overline{\mu}}, & \text{if } \mu \in L'/M, \\ 0, & \text{otherwise,} \end{cases} \tag{3.51}$$

which defines the lift

$$\uparrow_L^M \colon \mathcal{A}_{L,k} \to \mathcal{A}_{M,k}, \quad g \mapsto g_M. \tag{3.52}$$

Note that these lifts are not inverse to each other, as $\downarrow_L^M \circ \uparrow_L^M = |L/M| \cdot \mathrm{id}$ and $\uparrow_L^M \circ \downarrow_L^M$ equals $|L/M| \cdot \mathrm{id}$ only on the in general proper subspace given by the image of $\uparrow_L^M$, whose elements are supported on L'/M. In fact, the latter gives a complete description of the image of $\uparrow_L^M$, in particular, any element $f \in \mathcal{M}_{M,k}$ supported on L'/M satisfies $f_\mu = f_{\mu+\mu'}$ for $\mu' \in L/M$ (cf. [Bru14, Prop. 3.3]).

In fact, there is an underlying more conceptual construction of such operators which is also the origin of the naming convention in this thesis [Sch15, Sec. 4] and [Bru14, Sec. 3]. In fact, for a discriminant group D and an isotropic subgroup H, the group $D' := H^\perp/H$ defines a discriminant group of the same signature. Then there are operators

$$\uparrow_{D'}^D \colon M_{k,D'} \to M_{k,D}, \qquad g \mapsto \sum_{\mu \in H^\perp} g_{\mu+H} \mathfrak{e}_\mu,$$
$$\downarrow_{D'}^D \colon M_{k,D} \to M_{k,D'}, \qquad f \mapsto \sum_{\mu \in H^\perp} f_\mu \mathfrak{e}_{\mu+H}.$$

The relation to the operators constructed in this thesis is explained in [Sch15, p. 16] and is the following: The subgroup $H := L/M \leq M'/M =: D$ is isotropic and we find $H^\perp = L'/M$, yielding the natural isomorphism $D' = H^\perp/H \simeq L'/L$. The formulae are then identical to these above and the respective results of Scheithauer and Bruinier yield that the operators do in fact transfer the transformation properties with respect to the

respective Weil representations. That is, the operators map modular forms to modular forms.
Note that the more abstract description is in fact not more general, but equivalent, for if $H \leq D$, its preimage $\pi^{-1}(H) \subseteq M'$ under the projection $\pi : M' \to D$ is an even sublattice. In fact, if $H := L/M$, we obtain $\pi^{-1}(H) = L$.

In conclusion we find the following proposition.

Proposition 3.36. *For $M \leq L$ even non-degenerate lattices and $k \in \mathbb{Z}/2$, there are linear operators*

$$\downarrow_L^M : \mathcal{A}_{M,k} \to \mathcal{A}_{L,k}, \qquad \uparrow_L^M : \mathcal{A}_{L,k} \to \mathcal{A}_{M,k}$$

fulfilling the following properties.

a) The operators map modular (cusp) forms to modular (cusp) forms.

b) The lift $\downarrow_L^M$ is compatible with theta functions, i.e. $\downarrow_L^M (\Theta_M) = \Theta_L$.

c) The operators are adjoint to each other on the space of cusp forms:

$$\langle \cdot , \downarrow_L^M (\cdot) \rangle_L = \langle \uparrow_L^M (\cdot), \cdot \rangle_M .$$

This identity also holds, if one of the cusp forms is replaced by a modular form.

d) The image $B := \operatorname{im}(\uparrow_L^M)$ is the subspace consisting of functions that are supported on L'/M. We find

$$\begin{aligned} \downarrow_L^M \circ \uparrow_L^M &= |L/M| \cdot \mathrm{id}, \\ \uparrow_L^M \circ \downarrow_L^M \big|_B &= |L/M| \cdot \mathrm{id}_B . \end{aligned}$$

Note that the latter identity is an abuse of notation, since $\uparrow_L^M$ had to formally be endowed with a restriction of its target to B.

Eisenstein series

The following exposition is based upon [Kie21, 2.5 pp. 30-36]. The statements are essential for our investigations, more specifically for deriving meromorphic continuation of the symmetric square type L-functions arising from the Rankin–Selberg type integrals in Subsection 6.4.3. In the following, let (L, q) be a non-degenerate lattice and $k \in \mathbb{Z}/2$ be a half integral number.

Definition 3.37. Let $\lambda \in \mathcal{L}$ be isotropic and $k \in \mathbb{Z}/2$. Define the *Eisenstein series*

$$E_{L,\lambda,k}(\tau, s) := \frac{1}{2} \sum_{\gamma \in \overline{\Gamma_\infty} \backslash \mathrm{Mp}_2(\mathbb{Z})} (\mathrm{Im}(\tau)^s \mathfrak{e}_\lambda) \,|_{L,k} \gamma. \tag{3.53}$$

By construction this function is $\mathrm{Mp}_2(\mathbb{Z})$ modular of weight k.

Lemma 3.38. *The Eisenstein series $E_{L,\lambda,k}$ converges normally on $\mathbb{H}$ for $\mathrm{Re}(s) > 1 - \frac{k}{2}$, is real analytic in τ and an eigenfunction of the hyperbolic Laplace operator of weight k with Eigenvalue $s(s + k - 1)$.*

Remark 3.39. Recall that the Weil representation ρ_L acts trivially on $\Gamma(\mathrm{lev}(L))$, so that the components of $E_{L,\lambda,k}(\tau, s)$ are scalar-valued Eisenstein series for $\Gamma(\mathrm{lev}(L))$. These, however, have meromorphic continuation in s.

Bruinier and Kühn compute the Fourier expansion of these Eisenstein series. Note that they do so with respect to the dual Weil representation, which is why the following is a slight reformulation of their statement, taking that difference into account.

Proposition 3.40 ([BK03, Prop. 3.1 p. 1695])**.** *For $\mathrm{sig}(\mathcal{L}) \equiv 2k \mod 2$, the Eisenstein series $E_{L,\lambda,k}$ has the Fourier expansion*

$$E_{L,\lambda,k}(\tau, s) = \sum_{\mu \in L'/L} \sum_{n \in \mathbb{Z} + \overline{\mathrm{q}}(\mu)} c_\lambda(\mu, n, s, v) \mathrm{e}(nu) \mathfrak{e}_\gamma \tag{3.54}$$

where the coefficients $c_\lambda(\mu, n, s, v)$ are given by

$$\begin{cases} (\delta_{\lambda,\mu} + i^{\mathrm{sig}(\mathcal{L})+2k} \delta_{-\lambda,\mu}) v^s + 2\pi v^{1-k-s} \frac{\Gamma(k+2s-1)}{\Gamma(k+s)\Gamma(s)} \cdot \sum_{x \in \mathbb{Z}\backslash\{0\}} |2c|^{1-k-2s} H_c(\beta, 0, \mu, 0), & \text{if } n = 0, \\ \frac{2^k \pi^{s+k-1}}{\Gamma(s+k)} \mathcal{W}_s(4\pi n v) \cdot \sum_{c \in \mathbb{Z}\backslash\{0\}} |c|^{1-k-2s} H_c(\lambda, 0, \mu, n), & \text{if } n > 0, \\ \frac{2^k \pi^{s+k-1}}{\Gamma(s)} \mathcal{W}_s(4\pi n v) \cdot \sum_{c \in \mathbb{Z}\backslash\{0\}} |c|^{1-k-2s} H_c(\lambda, 0, \mu, n), & \text{if } n < 0. \end{cases}$$

Here, H_c denotes the generalised Kloostermann sum

$$H_c(\lambda, m, \mu, n) = \frac{e^{-\pi i \,\mathrm{sgn}(c) k/2}}{|c|} \sum_{\substack{0 \not\equiv d \mod c \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1)_\infty \backslash \Gamma(1) / \Gamma(1)_\infty}} \overline{\rho}_{\mu,\lambda} \widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \mathrm{e}\left(\frac{am + nd}{c}\right), \tag{3.55}$$

the matrix coefficients $\rho_{\mu,\lambda}$ are as in Definition 3.17, and the Whittaker functions $\mathcal{W}_s$ are given in Definition A.20.

For the notation $\tilde{\gamma} \in \mathrm{Mp}_2(\mathbb{Z})$ for an element $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, compare Remark 3.11.

Proof: In [BK03] only the case $\mathrm{sig}(\mathcal{L}) + 2k \equiv 0 \mod 4$ is considered. In order to be certain that, apart from the obvious change to the constant coefficient, no additional term has to be altered, we compute the Fourier coefficients by consulting [Bru02, 1.2.3 p. 23] . Begin with

$$c_\lambda(\mu, n, s, v) = \frac{1}{2}\int_0^1 \Big\langle \sum_{(\gamma,\phi)\in\widetilde{\Gamma(1)_\infty}\setminus \mathrm{Mp}_2(\mathbb{Z})} \mathrm{Im}(\tau)^s \mathfrak{e}_\lambda|_k\gamma, \mathfrak{e}_\mu(n\overline{\tau})\Big\rangle.$$

Note that there are 4 non-trivial representatives (γ, ϕ) with $\gamma_{2,1} = 0$, namely $\overline{Z}^0, \overline{Z}^1, \overline{Z}^2, \overline{Z}^3$. The remaining matrices are represented by the double coset $\Gamma(1)_\infty\backslash\Gamma(1)/\Gamma(1)_\infty$. Recall that by Remark 3.20 we may reduce to the case $\mathrm{sig}(\mathcal{L}) \equiv 2k \mod 2$, so that $\overline{Z}$ acts trivially and we find

$$\begin{aligned} c_\lambda(\mu, n, s, v) &= \int_0^1 \langle \mathfrak{e}_\lambda, \mathfrak{e}_\mu(n\overline{\tau})\rangle \, \mathrm{d}u \cdot v^s + \int_0^1 \langle \mathfrak{e}_\lambda|_{L,k}\overline{Z}, \mathfrak{e}_\mu(n\overline{\tau})\rangle \, \mathrm{d}u \cdot v^s \\ &+ \sum_{\substack{\gamma\in\Gamma(1)_\infty\backslash\Gamma(1)\\ \gamma_{2,1}\neq 0}} \int_0^1 \frac{\mathrm{Im}(\tau)^s}{|j(\gamma,\tau)|^{2s}} j(\gamma,\tau)^{-k} \big\langle \rho(\tilde{\gamma})^{-1}\mathfrak{e}_\lambda, \mathfrak{e}_\mu(n\overline{\tau})\big\rangle \, \mathrm{d}u \end{aligned}$$

Define $\epsilon_k := i^{\mathrm{sig}(\mathcal{L})+2k}$ and recall that by Remark 3.22 we have

$$\mathfrak{e}_\lambda|_{L,k}\overline{Z} = \epsilon_k \mathfrak{e}_{-\lambda}.$$

Further, we find that

$$\big\langle \rho(\tilde{\gamma})^{-1}\mathfrak{e}_\lambda(m\gamma\tau), \mathfrak{e}_\mu(n\overline{\tau})\big\rangle = \overline{\rho_{\mu,\lambda}(\tilde{\gamma})}\mathrm{e}(m\gamma\tau)\mathrm{e}(-n\tau).$$

Note that by unitaricity, the expression $\overline{\rho_{\mu,\lambda}(\tilde{\gamma})}$ does not depend on altering $\tilde{\gamma}$ from the right by an element of $\widetilde{\Gamma(1)}_\infty$. With these computations, we obtain

$$\begin{aligned} c_\lambda(\mu, n, s, v) &= \delta_{0,n}\left(\delta_{\lambda,\mu} + \epsilon_k\delta_{-\lambda,\mu}\right) v^s \\ &+ \sum_{\gamma\in\Gamma(1)_\infty\backslash\Gamma(1),\gamma_{2,1}\neq 0} \int_0^1 \frac{\mathrm{Im}(\tau)^s}{|j(\gamma,\tau)|^{2s}} j(\gamma,\tau)^{-k}\overline{\rho_{\mu,\lambda}(\tilde{\gamma})}\mathrm{e}(-n\tau)\, \mathrm{d}u \\ &= \delta_{0,n}\left(\delta_{\lambda,\mu} + \epsilon_k\delta_{-\lambda,\mu}\right) v^s \end{aligned}$$

$$+ \sum_{\substack{\gamma\in\Gamma(1)_\infty\backslash\Gamma(1)/\Gamma(1)_\infty \\ \gamma_{2,1}\neq 0}} \overline{\rho_{\mu,\lambda}(\tilde{\gamma})} v^s \exp(2\pi v) \int_{-\infty}^{\infty} \frac{\mathrm{e}(-nu)}{|j(\gamma,\tau)|^{2s}} j(\gamma,\tau)^{-k} \,\mathrm{d}u.$$

Now writing $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$, we have

$$\sqrt{j(\gamma,\tau)} = \operatorname{sgn}(c)\sqrt{c}\sqrt{\tau + d/c},$$

so that we find for the appearing integral

$$\begin{aligned} &v^s \exp(2\pi v) \int_{-\infty}^{\infty} \frac{\mathrm{e}(-nx)}{|j(\gamma,\tau)|^{2s}} j(\gamma,\tau)^{-k} \,\mathrm{d}u \\ =&v^s \exp(2\pi v) |c|^{-k-2s} \operatorname{sgn}(c)^k \int_{-\infty}^{\infty} \frac{\mathrm{e}(-nu)}{(\tau + d/c)^{k+s}\,(\overline{\tau} + d/c)^s} \,\mathrm{d}u \\ =&v^s \exp(2\pi v) |c|^{-k-2s} \operatorname{sgn}(c)^k e\left(\frac{nd}{c}\right) \int_{-\infty}^{\infty} \frac{\mathrm{e}(-nu)}{\tau^{k+s}\overline{\tau}^s} \,\mathrm{d}u \\ =&v^s \exp(2\pi v) |c|^{-k-2s} \operatorname{sgn}(c)^k e\left(\frac{nd}{c}\right) \int_{-\infty}^{\infty} \frac{\mathrm{e}(-nu)}{\tau^{k+s}\overline{\tau}^s} \,\mathrm{d}u. \end{aligned}$$

The integral

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}(-nu)}{\tau^{k+s}\overline{\tau}^s} \,\mathrm{d}u = \int_{-\infty}^{\infty} \frac{\mathrm{e}(-nu)}{(v - iu)^{k+s}(v+iu)^s} \,\mathrm{d}u$$

is then rewritten by Bruinier and Kühn in terms of Whittaker functions as

$$2^k \pi^{s+k} i^{-k} |n|^{s+k-1} y^{-s} \mathcal{W}_s(4\pi n y) \begin{cases} \Gamma(k+s)^{-1}, & n > 0, \\ \Gamma(s)^{-1}, & n < 0, \end{cases}$$

and

$$2^{2-k-2s} \pi i^{-k} \frac{\Gamma(k+2s-1)}{\Gamma(k+s)\Gamma(s)} y^{1-k-2s}, \qquad \text{in case of } n = 0.$$

Finally, we find for the sum

$$\begin{aligned} &\sum_{\substack{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in\Gamma(1)_\infty\backslash\Gamma(1)/\Gamma(1)_\infty \\ c\neq 0}} |c|^{-k-2s} \operatorname{sgn}(c)^k \overline{\rho}_{\mu,\lambda}(\tilde{\gamma}) e\left(\frac{nd}{c}\right) \\ =&\sum_{c\neq 0} |c|^{-k-2s} \operatorname{sgn}(c)^k \sum_{\substack{0\neq d \mod c \\ \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in\Gamma(1)_\infty\backslash\Gamma(1)/\Gamma(1)_\infty}} \overline{\rho}_{\mu,\lambda}(\tilde{\gamma}) e\left(\frac{nd}{c}\right) \\ =&\sum_{c\neq 0} |c|^{1-k-2s} H_c(\lambda, 0, \mu, n). \end{aligned}$$

□

Remark 3.41. The Fourier expansion of a slight generalisation, with non-trivial periodic term, of the above is computed in [Völ18, Prop 3.6.7 p. 58]. In that case, however, the series is not an eigenfunction of the hyperbolic Laplacian anymore, but suffers from a shift in the parameter s.

We also introduce the following notation for simplifying statements further below.

$$c_\lambda(\mu, n, s, v) = \begin{cases} (\delta_{\lambda,\mu} + i^{\mathrm{sig}(\mathcal{L})+2k}\delta_{-\lambda,\mu})v^s + c_\lambda(\mu, 0, s)v^{1-k-s}, & n = 0, \\ c_\lambda(\mu, n, s)\mathcal{W}_s(4\pi n v), & n \neq 0. \end{cases} \tag{3.56}$$

Proposition 3.42. *Let $k > 2$, then $E_{L,\lambda,k}(\tau, 0)$ defines a vector valued holomorphic modular form with respect to ρ_L and $\mathrm{Mp}_2(\mathbb{Z})$, in short $E_{L,\lambda,k}(\tau, 0) \in \mathcal{M}_{L,k}$. Its Fourier expansion is of the form*

$$E_{L,\lambda,k}(\tau) = \mathfrak{e}_\lambda + (-1)^k \mathfrak{e}_\lambda + \sum_{\gamma \in L'/L} \sum_{0 < n \in \mathbb{Z} + \overline{\mathrm{q}}(\gamma)} c_\lambda(\gamma, n)\mathfrak{e}_\lambda(n\tau),$$

where we set $c_\lambda(\gamma, n) := c_\lambda(\gamma, n, 0)$.

According to [Hej83, p. 372] we have the following transformation property.

Proposition 3.43. *For an isotropic element $\lambda \in \mathcal{L}$, we find the following functional equation*

$$E_{L,\lambda,k}(\tau, s) = \frac{1}{2} \sum_{\mu \in \mathrm{Iso}(L'/L)} c_\lambda(\mu, 0, s) E_{L,\mu,k}(\tau, 1 - k - s). \tag{3.57}$$

From that transformation, a symmetry property for a special value of the Eisenstein series is derived which may naively play a role in investigating the values of special L-functions in Subsection 6.4.3 – also compare Remark 6.99 on that matter.

Corollary 3.44. *Imposing the additional condition $\lambda = -\lambda$ on an isotropic element $\lambda \in \mathcal{L}$ and assuming $k > 2$ for the purpose of convergence, we obtain for the choice $s = s_0 = 0$ that*

$$E_{L,\lambda,k}(\tau, s_0) = \frac{2 - \delta_{\lambda,0}}{2}\left(1 + i^{\mathrm{sig}(\mathcal{L})+2k}\right) \cdot E_{L,\lambda,k}(\tau, 1 - k - s_0). \tag{3.58}$$

From the Fourier expansion of $E_{L,\lambda,k}$ and the properties of the Whittaker function $\mathcal{W}_s$ (cf. [Erd81, 6.9 p. 264], [Olv+10, Sec. 13.14], and [DAR84, Sec. 13 p. 189]) we derive an asymptotic bound for the Eisenstein series.

Corollary 3.45. *We find for* $\tau = u + iv \in \mathbb{H}$ *and* $\sigma = \max\{\mathrm{Re}(s), \mathrm{Re}(1-k-s)\}$ *that*

$$E_{L,\lambda,k}(\tau, s) = \mathcal{O}(v^{\sigma}), \qquad v \to \infty. \tag{3.59}$$

In fact, the constant required to bound may be chosen locally uniformly in s*.*

Proof: Indeed, the constant term of the expansion gives rise to the growth $\sim v^{\sigma}$, while the other terms may be locally uniformly bounded in s. Note that the coefficients $\rho_{\mu,\lambda}$ are universally bounded by 1, since ρ is unitary, effectively bounding the Kloostermann sum (3.55) in the Fourier expansion of $E_{L,\lambda,k}$. This gives rise to a universal bound of the infinite series in the coefficients in (3.54) on right half planes by comparing it to the Riemann ζ function. The exponential terms may be uniformly bounded in s on vertical strips, as may the Γ factors in the Numerator. Further, the Beta function is entire and as such continuous, yielding a locally uniformly bound in s. Finally, Lemma A.21 yields the desired bound on the remaining Whittaker functions to justify an absolute bound on the series of all other coefficients by bounding it by a geometric series. □

Remark 3.46. This implies the same bound for the scalar valued Eisenstein series by selecting the lattice L to be unimodular.

Parabolic Poincaré series

The construction of Eisenstein series above may be significantly generalised. Instead of sticking with $\mathfrak{e}_\lambda$ for some $\lambda \in \mathcal{L}$ with $\overline{\mathrm{q}}(\lambda) = 0$ isotropic for a kernel, we may select a general element $\lambda \in \mathcal{L}$ and adjust for the additional prefactor that came with the transformation under $\overline{T}$ by cancelling it with a proper choice of the following form

$$\mathfrak{e}_\lambda(m\tau) := e(m\tau) \cdot \mathfrak{e}_\lambda.$$

Now for $m \in \mathbb{Z} + \overline{\mathrm{q}}(\lambda)$, we find that $\mathfrak{e}_\lambda(m\tau)|_{L,k}\overline{T} = \mathfrak{e}_\lambda(m\tau)$ for any weight k. The generalisation goes further. In fact, instead of symmetrising $\mathrm{Im}(\tau)^s$ we may consider much more general functions. To this end, let $\psi : \mathbb{R}_{>0} \to \mathbb{C}$ be piecewise infinitely differentiable and for reasons of convergence, assume there is an exponent $\alpha \in \mathbb{R}_{\geq 0}$ such that $\psi(v) = \mathcal{O}(v^{\alpha})$ for $v \to 0$. Then we find

$$\psi(v)\mathfrak{e}_\lambda(m\tau)|_{L,k}\overline{T} = \psi(v)\mathfrak{e}_\lambda(m\tau),$$

so that the following expression is well defined.

Definition 3.47. Let $\lambda \in L'/L$ and $m \in \overline{\mathrm{q}}(\lambda) + \mathbb{Z}$. Further, let $\psi \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_{>0})$. Then

$$P_{\lambda,m,\psi,k}(\tau) \coloneqq \frac{1}{2} \sum_{(\gamma,\phi) \in \overline{\Gamma_\infty} \backslash \mathrm{Mp}_2(\mathbb{Z})} \psi(v)\mathfrak{e}_\lambda(m\tau)|_{L,k}(\gamma,\phi) \tag{3.60}$$

is called the *(parabolic) Poincaré series* of weight k with parameters λ, m and ψ.

Clearly, ψ may be replaced by a family ψ_s of functions with the analogous property depending on a complex parameter s and we write $P_{\lambda,m,\psi,k}(\tau, s)$ for the Poincaré series in that case. This way, we may reproduce the Eisenstein series from the section above. The Fourier series of such parabolic Poincaré series for special choices of ψ may be computed. Compare [Bru02, Thm. 1.4 p. 19] for the choice $\psi = 1$ and for different choices of ψ which play a role in the field of Maass forms consult [Völ18, p. 54].

Remark 3.48. Assume there is $\alpha \in \mathbb{R}$ such that $\psi(v) = \mathcal{O}(v^\alpha)$ for $v \to 0$ and that the weight fulfils $k \geq 2(1-\alpha)$. Then $P_{\lambda,m,\psi,k}(\tau)$ may be proven to be locally uniformly convergent, by comparing it to the classical Eisenstein series $E_k(\tau,\alpha)$. Hence, if ψ is (locally) continuous or holomorphic, so is $P_{\lambda,m,\psi,k}(\tau)$.

Remark 3.49. Note that $\psi\mathfrak{e}_\lambda(m\tau)|_{L,k}\overline{Z} = \psi\mathfrak{e}_{-\lambda}(m\tau)$, so that the kernel, and as such also the Poincaré series, are invariant under the action of $\overline{Z}^2$.

Petersson scalar product

As in the scalar valued case, the space of cusp forms is equipped with a scalar product. This subsection serves two purposes: firstly, it introduces this product, and secondly, it realises pairings of cusp forms against more general automorphic forms which are required for constructing a class of L-series as Rankin–Selberg type integrals in Subsection 6.4.3.

Definition 3.50. Let $f \in \mathcal{S}_{L,k}$ and $g \in \mathcal{M}_{L,k}$. Define the *Petersson product* of f and g to equal

$$\int_{\mathcal{F}_{\Gamma(1)}} \langle f, g \rangle \cdot \mathrm{Im}^k \,\mathrm{d}\mu, \tag{3.61}$$

where $\mathrm{d}\mu$ is the standard invariant hyperbolic measure on $\mathbb{H}$ as in Definition 2.15 b) and $\mathcal{F}_{\Gamma(1)}$ denotes a fundamental domain for $\Gamma(1)$ (cf. (2.3)).

We will need to verify that this expression is convergent in order to obtain a well defined notion. In fact, we require a more general convergence statement for a later application in the context of the Rankin–Selberg method and will state it here. Note first, that an integral of the form

$$\int_{\mathcal{F}_{\Gamma(1)}} f \, \mathrm{d}\mu$$

is absolutely convergent, as long as $f \in L^\infty(\mathcal{F}_{\Gamma(1)})$. In fact, it is easy to verify that the volume of $\mathcal{F}_{\Gamma(1)}$ is finite, by computing the volume of an encompassing Siegel domain.

Definition 3.51. A function $f : \mathbb{R} \to \mathbb{C}$ is called *rapidly decreasing* towards ∞, if for all $n \in \mathbb{N}$ we have

$$f(x) = \mathcal{O}(x^{-n}), \qquad x \to \infty. \tag{3.62}$$

A family of functions f_i is called rapidly decreasing, if all f_i are. It is called *uniformly rapidly decreasing*, if the constants in (3.62) may be chosen independently of the index i.

Example 3.52.

a) Let $f : \mathbb{R} \to \mathbb{C}$ be rapidly decreasing towards ∞ and $p : \mathbb{R} \to \mathbb{C}$ be of at most polynomial growth. Then the product $f \cdot p$ is rapidly decreasing.

b) Let $f : \mathbb{H} \to \mathbb{C}$ have a convergent Fourier expansion of the form

$$f(\tau) = \sum_{0 < n \in \mathbb{Z}/N} a(n)e(n\tau)$$

for some natural number $N \in \mathbb{N}$. Then for $\tau = u + iv \in \mathbb{H}$, the family $f(u + iv)$ is uniformly rapidly decreasing for $v \to \infty$.

c) Let f, g be as in Definition 3.50. Then $\langle f, g \rangle$ is uniformly (in $\mathrm{Re}(\tau)$) rapidly decreasing towards ∞.

d) Let L be a non-degenerate quadratic lattice that splits $L = L_1 \oplus L_2$. Given $f \in \mathcal{S}_{L,k}$, $g \in \mathcal{M}_{L_1,k_1}$ and an Eisenstein series E_{L_2,λ,k_2} as in Definition 3.37 the product

$$\langle f, g \otimes E_{L_2,\lambda,k_2}(\,\cdot\,, s) \rangle \tag{3.63}$$

is uniformly rapidly decreasing towards ∞ for all fixed s for which E_{L_2,λ,k_2} does not have a pole. Further, the prefactor for the bound can be chosen locally uniformly in the parameter s (cf. Corollary 3.45).

In fact, part b) is verified by the following computation. Note first, that the Fourier series must then be absolutely convergent on $\mathbb{H}$ by a standard argument for power series (cf. Lemma A.19). In fact, we find that for any $\delta > 0$ the series is absolutely bounded on the upper half plane $\{\tau \mid \mathrm{Im}(\tau) > \delta\}$ by

$$\sum_{0<n\in\mathbb{Z}/N} |a(n)| \exp(-2\pi n\delta). \tag{3.64}$$

Consequently, we infer

$$\begin{aligned} |f(\tau)| \leq & \sum_{0<n\in\mathbb{Z}/N} |a(n)| \exp(-2\pi n v) \\ = & \exp(-\pi v/N) \sum_{0<n\in\mathbb{Z}/N} |a(n)| \exp(-\pi n v) \exp(-\pi(n-1/N)v) \\ \leq & \exp(-\pi v/N) \sum_{0<n\in\mathbb{Z}/N} |a(n)| \exp(-\pi n\delta). \end{aligned}$$

Recall that the series on the right hand side equals a finite value, since the Fourier expansion of f is absolutely convergent at $i\delta/2$. Consequently, we find $f(u+iv) = \mathcal{O}(\exp(-\pi v/N))$ for $v \to \infty$ uniformly in u.

In order to verify that (3.63) is rapidly decreasing we restrict to the case $\mathrm{Re}(s) > 1 - k/2$ first and note that

$$\begin{aligned} & |\langle f, g \otimes \sum_{\gamma\in\overline{\Gamma_\infty}\backslash \mathrm{Mp}_2(\mathbb{Z})} (\mathrm{Im}(\tau)^s \mathfrak{e}_\lambda) \,|_{L,k}\gamma\rangle| \\ \leq & \left| \sum_{\gamma\in\overline{\Gamma_\infty}\backslash \mathrm{Mp}_2(\mathbb{Z})} \mathrm{Im}(\tau)^s|_k\gamma \right| \cdot |\langle f, g \otimes \rho_{\mathcal{L}_2}(\gamma)^{-1}\mathfrak{e}_\lambda\rangle|. \end{aligned}$$

Observe that

$$|\langle f, g \otimes \rho_{\mathcal{L}_2}(\gamma)^{-1}\mathfrak{e}_\lambda\rangle| \leq \sum_{\mu\in\mathcal{L}_2} |\langle \mathfrak{e}_\lambda, \rho_{\mathcal{L}_2}(\gamma)\mathfrak{e}_\mu\rangle| \cdot |\langle f, g \otimes \mathfrak{e}_\mu\rangle|.$$

Since $\rho_{\mathcal{L}_2}$ is unitary, we find that the prefactor in the sum is universally bounded by 1. Further, $\langle f, g \otimes \mathfrak{e}_\mu\rangle$ possesses a Fourier expansion as in part b), so that it remains to recall that

$$\sum_{\gamma\in\overline{\Gamma_\infty}\backslash \mathrm{Mp}_2(\mathbb{Z})} \mathrm{Im}(\tau)^s|_k\gamma = E_k(\tau, s)$$

is by Corollary 3.45 locally in s uniformly bounded in u by a single polynomial for $v \to \infty$. The concrete polynomial is given by $v^{\max\{\mathrm{Re}(s), 1-k-\mathrm{Re}(s)\}}$.

Deriving the statement for more general parameters $s \in \mathbb{C}$ is more technical. The idea is

to use the Fourier expansion of $E_{L,\lambda,k}(\tau,s)$ presented in Theorem 3.40 and note that it has meromorphic continuation (this is derived from the scalar valued case for the coordinate functions). Then, the L-series appearing in the Fourier expansion has meromorphic continuation and is independent of v, so that it may be locally uniformly bounded as long as it has no pole. Further, in order to bound the Whittaker function $\mathcal{W}_s$, we refer to Definition A.20 to recall its definition and to [Olv+10, (13.19.3)][8] in order to verify that the asymptotic behaviour $\mathcal{W}_s(v) \sim e^{-1/2v}$ declared in [Olv+10, (13.14.21)] is true for a locally uniform bound in s. Consequently, only the constant term of the Fourier expansion plays a role which may again be asymptotically bounded by $v^{\max\{\mathrm{Re}(s),1-k-\mathrm{Re}(s)\}}$.
Note that part c) is a subcase of part b) and part a) is immediate.

By part c) and a) of the above Example, we conclude the following Corollary implying that the Petersson product in Definition 3.50 is well defined.

Corollary 3.53. *Assume f, g to be chosen as in Definition 3.50 and let $p : \mathbb{H} \to \mathbb{C}$ be a function of at most polynomial growth in $\mathrm{Im}(\tau) \to \infty$. Then the integral*

$$\int_{\mathcal{F}_{\Gamma(1)}} \langle f, g\rangle \cdot p \,\mathrm{d}\mu \tag{3.65}$$

converges absolutely.

Further, we obtain by part d) of Example 3.52 that the following Rankin–Selberg type integrals yield holomorphic functions.

Corollary 3.54. *Let L be a non-degenerate quadratic lattice that splits $L = L_1 \oplus L_2$. Given $f \in \mathcal{S}_{\mathcal{L},k}$, $g \in \mathcal{M}_{\mathcal{L}_1,k_1}$ and an Eisenstein series E_{L_2,λ,k_2} as in Definition 3.37, the integral*

$$\int_{\mathcal{F}_{\Gamma(1)}} \langle f, g \otimes E_{L_2,\lambda,k_2}(\,\cdot\,,s)\rangle \cdot v^k \,\mathrm{d}\mu \tag{3.66}$$

converges absolutely for $\mathrm{Re}(s) > 1 - k/2$. In fact, if $E_{L_2,\lambda,k}$ has no pole at some $s \in \mathbb{C}$ the convergence is locally uniformly, so that the integral in (3.66) defines a holomorphic function in s.

The case of higher genus

Many of the ideas that have been presented above for the SL_2 case may be reproduced in case of genus $g > 1$. There is, for instance, also a metaplectic extension which at the

[8] The reader may also consult [Olv+10, (5.2.4)] for the definition of the appearing *Pochhammer* symbol.

infinite place may also be realised by pairs

$$\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \phi(Z)\right),$$

where ϕ is a holomorphic square root of $(CZ+D)$. Its integral points $\mathrm{Mp}_{2g}(\mathbb{Z})$ act via the discrete Weil representation on $\mathbb{C}[\mathcal{L}^g]$, where $\mathcal{L} = L'/L$ is the discriminant form as before. For $\underline{\lambda}, \underline{\mu} \in \mathcal{L}^g$ we understand $\mathrm{b}(\underline{\mu}, \underline{\lambda}) = \sum_{j=1}^{g} \mathrm{b}(\mu_j, \lambda_j)$. Then the action of ρ_L is given by the following formulae.

a) For $U \in \mathrm{GL}_g(\mathbb{Z})$ we find

$$\rho_L\left(M(U), \sqrt{\det(U)}\right)\mathfrak{e}_{\underline{\lambda}} = \sqrt{\det(U)}^{m^- - m^+}\mathfrak{e}_{U^{-1}\underline{\lambda}}.$$

b) For $S \in M_g(\mathbb{Z})$ symmetric we have

$$\rho_L\left(N(S), 1\right)\mathfrak{e}_{\underline{\lambda}} = e[\mathrm{tr}(\overline{\mathrm{q}}(\lambda)S)]\mathfrak{e}_{\underline{\lambda}}.$$

c) For J the standard alternating matrix we find

$$\rho_L\left(-J, \sqrt{\det(Z)}\right)\mathfrak{e}_{\underline{\lambda}} = \frac{e[-g\frac{(m^+-m^-)}{8}]}{|L'/L|^{g/2}} \sum_{\underline{\mu}\in\mathcal{L}^g} e[-\mathrm{b}(\underline{\mu}, \underline{\lambda})]\mathfrak{e}_{\underline{\mu}}.$$

We note that a holomorphic function

$$f : \mathbb{H}_g \to \mathbb{C}[\mathcal{L}^g], f(Z) = \sum_{\underline{\lambda}\in\mathcal{L}^g} f_{\underline{\lambda}}\mathfrak{e}_{\underline{\lambda}}$$

transforming like a modular form for the Weil representation ρ_L fulfils

$$f|_{L,k}(N(S), 1) = \sum_{\underline{\lambda}\in\mathcal{L}^g} e[-\mathrm{tr}(\overline{\mathrm{q}}(\underline{\lambda})S)]f_{\underline{\lambda}}\mathfrak{e}_{\underline{\lambda}},$$

meaning $e[-\mathrm{tr}(\overline{\mathrm{q}}(\underline{\lambda})(Z+S))]f_{\underline{\lambda}}(Z) = e[-\mathrm{tr}(\overline{\mathrm{q}}(\underline{\lambda})Z)]f_{\underline{\lambda}}(Z)$, i.e. it is periodic and hence possesses a Fourier expansion just as in the case of a trivial representation (cf. (3.8)).

For additional details, the reader may consult [KRY06] and [Zha09].

3.3.3 Indices of Fourier coefficients

In this subsection we lay the foundation for investigations of Fourier coefficients in the context of examining the injectivity of the Kudla–Millson lift in Section 7.3. In case the base lattice in this section does not split a hyperbolic plane, partial orders will play a key role whose relevance arises from the action of Hecke algebras on Fourier coefficients of the vector valued modular forms involved. In fact, these partial orders are induced by the action of monoids. Consequently, we begin our investigation with introducing monoidal preorders before defining crucial notions for the investigation of indices of Fourier coefficients of automorphic forms.

Monoid preorders

Definition 3.55. Let $(M, \cdot)$ be a monoid and S be a set.

a) We say that $(M, \cdot)$ *acts* on S, written $M \circlearrowright S$, if there is a monoid-morphism $\sigma : (M, \cdot) \to \mathrm{Aut}(S)$. For $m \in M$ and $s \in S$ write $m \cdot s := \sigma(m)(s)$.

b) For an action $M \circlearrowright S$ define a preorder on S via

$$s_1 \preceq s_2 \iff \exists m \in M : s_2 = m \cdot s_1.$$

The following examples are meant to build some intuition and also lay the foundation for the key instance we have in mind.

Example 3.56. a) For the monoid $(\mathbb{N}_0, +)$ and the set $\mathbb{N}_0$, where the action is given by addition, we find that $\preceq$ equals the usual order on $\mathbb{N}_0$.

b) Let $(\mathbb{N}_0, +)$ be the monoid and $S = \mathbb{Q}^\times$. Pick a prime p and a natural number $k \in \mathbb{N}$ and let $\mathbb{N}_0$ act via $q \mapsto n \cdot q := p^{nk} \cdot q$. Then $q_1 \preceq q_2$ encodes that q_2 is a multiple of q_1 by a power of p^k.

c) Let $(\mathbb{N}, \cdot)$ be the monoid and $S = \mathbb{Q}^\times$. Then $\mathbb{N} \circlearrowright \mathbb{Q}^\times$ via multiplication and the relation $\preceq$ detects natural multiples.

d) As a refinement of the above, let $P \subseteq \mathbb{P}$ be a set of primes and for every $p \in P$ select $k_p \in \mathbb{Z}$. Construct the monoid $\bigoplus_{p \in P}(\mathbb{N}_0, +)$ and $S = \mathbb{Q}^\times$ with action

$$q \mapsto (n_p)_p \cdot q := \left(\prod_{p \in P} p^{n_p k_p} \right) \cdot q$$

where multiplication on the right denotes multiplication in $\mathbb{Q}^\times$. Note that the choice $P = \{p\}$ is b), the choice $P = \mathbb{P}$ and $k_p = 1$ for all $p \in \mathbb{P}$ equals part c). Unique factorisation domains may be treated similarly.

The notion of a preorder is too weak for the application in later stages of this thesis. However, the desired property of antisymmetry may be characterised in terms of the monoid action.

Lemma 3.57. *For the preorder $\preceq$ induced by $(M, \cdot) \circlearrowright S$ the following are equivalent:*

a) The preorder $\preceq$ is a partial order (i.e. antisymmetric).

b) For all $m_1, m_2 \in M$ and all $s \in S$ we find that $m_2 m_1 \cdot s = s$ already implies $m_1 \cdot s = s$.

Proof: Assume there are $m_1, m_2 \in M$ and some $s \in S$ such that $m_2 m_1 \cdot s = s$, in other words $s \preceq m_1 \cdot s \preceq m_2 m_1 \cdot s = s$. If we assume antisymmetry, we obtain $m_1 \cdot s = s$, settling the first implication.
On the other hand, assuming b) and given $s_1 \preceq s_2$ and $s_2 \preceq s_1$, we spell that out as there are $m_1, m_2 \in M$ such that $s_2 = m_1 s_1$ and $s_1 = m_2 s_2 = m_2 m_1 s_1$. By assumption we find $s_2 = m_1 s_1 = s_1$. □

Example 3.58. All subexamples of Example 3.56 define partial orders since there is no unit other than the neutral element and no other element acts with a fixed point.

This prompts us to define the following notion.

Definition 3.59. A monoid action $M \circlearrowright S$ is called *faithful* if only the neutral element $e \in M$ acts as the identity on S. It is further called *free* if $e \in M$ is the only element fixing a point in S.

Remark 3.60. A monoid action $M \circlearrowright S$ that is free induces a partial order. Further, every free action is faithful.

Clearly, we would like to apply the notion of monoidal preorders to discriminant forms and consider the following example as a prototype.

Example 3.61. Let (L, q) be an even lattice of level $N \in \mathbb{N}$. With the notation of Example 3.56 part d), we select Q to be a subset of the prime divisors of N, choose $S = \oplus_{q \in Q} \mathcal{L}_q = \oplus_{q \in Q} L'_q / L_q$, and declare a similar action

$$\lambda \mapsto (n_p)_p \cdot \lambda := \left(\prod_{p \in P} p^{n_p k_p} \right) \cdot \lambda.$$

We remark that the associated preorder is almost never antisymmetric. In fact, let $2 \neq q \in Q$ and assume there is $p \in P$ with $p \neq q$. Then, multiplication by p acts as an automorphism on $\mathcal{L}_q$ and there must be some number r_p such that multiplication by p^{r_p} acts as the identity on $\mathcal{L}_q$. In consequence, select some non-trivial $\lambda \in \mathcal{L}_q \leq S$ and verify that

$$(p^{r_p-1} \cdot p^1) \cdot \lambda = \lambda \qquad \wedge \qquad p^1 \cdot \lambda \neq \lambda. \tag{3.67}$$

However, (3.67) contradicts part b) of Lemma 3.57. In case Q holds only one element and $P \subseteq Q$, the preorder will be antisymmetric.

When considering indices of Fourier coefficients of vector valued automorphic forms, however, not only the associated discriminant form $\mathcal{L}$ plays a role but pairs taken from $\mathcal{L} \times \mathbb{Q}$. Hence, it is natural to ask how to combine actions onto products of sets.

Definition 3.62. Let $(M_i, \cdot)$ be monoids and (S_i) be sets with an M_i monoid action σ_i.

a) Then we call

$$\sigma : \prod_i (M_i, \cdot) \to \operatorname{Aut}\left(\prod_i S_i\right), \qquad (m_i) \cdot (s_i) \coloneqq (\sigma_i(m_i)s_i) \in \prod_i S_i$$

the *product action* of the σ_i.

b) In case there is a monoid $(M, \cdot)$ that embeds into each $(M_i, \cdot)$, the induced action via a diagonal embedding into the product

$$\sigma : M \to \operatorname{Aut}\left(\prod_i S_i\right), \qquad m \cdot (s_i) \coloneqq (\sigma_i(m)s_i) \in \prod_i S_i$$

is called the *diagonal action* of $(M, \cdot)$ via the family (σ_i).

Remark 3.63. If in the above definition for a diagonal action one of the σ_i is faithful, so is the product of the family (σ_i). Then, however, it induces a partial order.

The following is the prime example for the application in this thesis.

Example 3.64. Let $M = (\mathbb{N}, \cdot)$ and (L, q) be an even lattice. Then $\mathbb{N}$ acts naturally on $\mathcal{L} = L'/L$ via multiplication and on $\mathbb{Q}^\times$ via multiplication with squares. The resulting diagonal action of the product is given by letting $n \in \mathbb{N}$ act by

$$n \cdot (\lambda, q) = (n\lambda, n^2 q) \in \mathcal{L} \times \mathbb{Q}^\times. \tag{3.68}$$

Refine the above by selecting some $N \in \mathbb{N}$ and replacing M by the submonoid of $(\mathbb{N}, \cdot)$ generated by prime numbers that are (co)prime to N. The action defines a partial order.

Remark 3.65. Example 3.64 may be realised in the following fashion. Select $P \subseteq \mathbb{P}$ to be the collection of primes coprime to N, consider Example 3.56 d) with $k_p = 2$ and Example 3.61 with $k_p = 1$ and Q representing all prime divisors of $\operatorname{lev}(L)$. Then taking the diagonal action of $\bigoplus_{p \in P}(\mathbb{N}_0, +)$ via the actions described above yields the desired result. The advantage of decomposing along primes becomes apparent in Subsection 6.4.2.

The following definition subsumes partial orders generated by the cases of Example 3.64 that are crucial for investigating the action of Hecke operators on Fourier coefficients of vector valued modular forms for the Weil representation.

Definition 3.66. Let $N \in \mathbb{N}$ and define $\preceq_N$ ($\preceq^N$) to be the partial order on $\mathcal{L} \times \mathbb{Q}^\times$ induced by the monoid action from Example 3.64 with the monoid $M \subseteq (\mathbb{N}, \cdot)$ being generated by primes dividing (being coprime to) N.
Explicitly, this means that for $s_0, s_f \in S = \mathcal{L} \times \mathbb{Q}^\times$ and some prime $p \in \mathbb{P}$ we define

$$s_0 \preceq_p s_f \quad :\iff \quad \text{there is some } k \in \mathbb{N} \text{ such that} \quad p^k \cdot s_0 = s_f,$$

$$s_0 \preceq_N s_f \quad :\iff \quad \begin{array}{l} \text{there is a finite set of primes } \{p_i \in \mathbb{P} : 0 \le i \le k,\, p_i \mid N\} \\ \text{and corresponding elements } s_i \in S \text{ such that} \\ s_0 \preceq_{p_0} s_1 \preceq_{p_1} \cdots \preceq_{p_k} s_k = s_f, \end{array}$$

$$s_0 \preceq^N s_f \quad :\iff \quad \begin{array}{l} \text{there is a finite set of primes } \{p_i \in \mathbb{P} : 0 \le i \le k,\, p_i \nmid N\} \\ \text{and corresponding elements } s_i \in S \text{ such that} \\ s_0 \preceq_{p_0} s_1 \preceq_{p_1} \cdots \preceq_{p_k} s_k = s_f. \end{array}$$

The corresponding strict partial orders are denoted $\prec_N$ ($\prec^N$).

Indices of Fourier coefficients

First, we will have a look at indices of Fourier coefficients of vector valued modular forms associated to an even quadratic lattice (L, q) of signature (m^+, m^-). Recall that the possible indices lie in the set $\mathcal{L} \times \mathbb{Q}$, where $\mathcal{L} = L'/L$ denotes the discriminant form.

Definition 3.67. For an even quadratic lattice (L, q) of signature (m^+, m^-), define the set of *admissible* indices with respect to (L, q) as

$$\mathscr{I} := \{(\lambda, n) \in \mathcal{L} \times \mathbb{Q} \mid \overline{\mathrm{q}}(\lambda) \equiv n\}.$$

Further, define the set of *representable* indices with respect to (L, q) as

$$\mathscr{R} := \left\{(\lambda, n) \in \mathcal{L} \times \mathbb{Q} \;\middle|\; \exists \ell \in L' : (\lambda, n) = (\bar{\ell}, \mathrm{q}(\ell))\right\}.$$

Here, $\bar{\ell}$ denotes the natural projection of ℓ to $\mathcal{L}$. Define the subsets

$$\mathscr{I}^\times := \mathscr{I} \cap \mathcal{L} \times \mathbb{Q}^\times \subseteq \mathscr{I}, \qquad \mathscr{R}^\times := \mathscr{R} \cap \mathscr{I}^\times \subseteq \mathscr{R},$$

with non-vanishing second component. The subsets with positive/negative second component are denoted $\mathscr{I}^\pm, \mathscr{R}^\pm$. Further, we require the *primitively representable* pairs

$$\mathscr{R}_1 := \left\{(\lambda, n) \in \mathcal{L} \times \mathbb{Q} \;\middle|\; \exists \ell \in L' \text{ primitive } : (\lambda, n) = (\bar{\ell}, \mathrm{q}(\ell))\right\}.$$

With these notions, we may formulate the following key lemma for prooving the injectivity of the Kudla–Millson lift under the assumption of a hyperbolic split.

Lemma 3.68. *In case $L \simeq K \oplus H$ splits a hyperbolic plane we find $\mathscr{R}_1 = \mathscr{R} = \mathscr{I}$, i.e. any possible index (λ, n) may be written as $(\overline{\ell_0}, \mathrm{q}(\ell_0))$ for some primitive element $\ell_0 \in L'$.*

Proof: Let $(\lambda, n) \in \mathscr{I}$, choose a representative $\ell \in L'$ with $\bar{\ell} = \lambda$, meaning $\mathrm{q}(\ell) \equiv n \mod \mathbb{Z}$, and note that the class $\bar{\ell} \in \mathcal{L}$ is invariant under translating ℓ by elements in H. Since $L = K \oplus H$ splits a hyperbolic plane $H = \langle e_1, e_2 \rangle_{\mathbb{Z}}$, where e_1, e_2 denote the standard generators with $\mathrm{q}(e_1) = \mathrm{q}(e_2) = 1$ and $\mathrm{b}(e_1, e_2) = 1$, we find

$$\mathrm{q}(\ell) = \mathrm{q}(\ell|_K) + \mathrm{q}(\ell|_H)$$

and for all $m_1, m_2 \in \mathbb{Z}$ that

$$\mathrm{q}(m_1 e_1 + m_2 e_2) = m_1 m_2$$

so that the primitive vector

$$\ell_\lambda := \ell|_K \oplus [(n - \mathrm{q}(\ell|_K))e_1 + e_2]$$

fulfils the desired property $(\overline{\ell_\lambda}, \mathrm{q}(\ell_\lambda)) = (\lambda, n)$. In particular, $(\lambda, n) \in \mathscr{R}_1 \subseteq \mathscr{R} \subseteq \mathscr{I}$. $\square$

Remark 3.69. The action of Example 3.64 on $\mathcal{L} \times \mathbb{Q}^\times$ and the corresponding partial orders from Definition 3.66 restrict to $\mathscr{I}^\times$ and $\mathscr{R}^\times$. They may, in the same spirit be extended to $\mathcal{L} \times \mathbb{Q}$ and then restricted to $\mathscr{I}$ and $\mathscr{R}$, if antisymmetry is given up (also cf. Example 3.61).

The monoid action of Example 3.64 on $\mathcal{L} \times \mathbb{Q}$ may be partly reversed for good primes.

Definition 3.70. For a pair $(\lambda, n) \in \mathcal{L} \times \mathbb{Q}$ and $k \in \mathbb{N}$ with $\gcd(k, \operatorname{lev}(L)) = 1$ define

$$(\lambda, n)/k := (\lambda/k, n/k^2).$$

Note that in the notation above and for natural numbers $k_1, k_2 \in \mathbb{N}$ satisfying $\gcd(k_1, \operatorname{lev}(L)) = \gcd(k_2, \operatorname{lev}(L)) = 1$ we find that

$$[(\lambda, n)/k_1]/k_2 = (\lambda, n)/(k_1 k_2) = [(\lambda, n)/k_2]/k_1.$$

Next assume we have an element $(\lambda, n) \in \mathscr{I}^\times$. Then, necessarily $n \in \mathbb{Z}/\operatorname{lev}(L)$ so that for $k \in \mathbb{N}$ with $\gcd(k, \operatorname{lev}(L)) = 1$, we conclude $(\lambda, n)/k \notin \mathscr{I}^\times$, unless $n/k^2 \in \mathbb{Z}/\operatorname{lev}(L)$. So there must be elements in $\mathscr{I}$ which are minimal with respect to division.

Definition 3.71. Let $N \in \mathbb{N}$.

a) The pair $(\lambda, n) \in \mathscr{I}^\times$ (respectively contained in $\mathscr{R}^\times$) is called N-*primitive*, if there is no $(\mu, m) \in \mathscr{I}^\times$ (respectively contained in $\mathscr{R}^\times$) with

$$(\mu, m) \prec^N (\lambda, n).$$

b) A number $n \in \mathbb{Z}/N$ is called N-*square free* (also N-$\boxslash$), if for any prime $p \nmid N$ we find $n/p^2 \notin \mathbb{Z}/N$.

In case of good primes, N-primitive indices may be classified by their second component as follows.

Lemma 3.72. *For $N \in \mathbb{N}$ with $\operatorname{lev}(L) \mid N$ and $p \in \mathbb{P}$ the following are true.*

a) A pair $(\lambda, n) \in \mathscr{R}^\times$ is N-primitive if, and only if, n is N–square free.

b) If $(\lambda, n) \in \mathscr{I}^\times$ with $p \nmid N$, $p^2 \mid n$, and (λ, n) only differs (additively in the second component) from an element in $\mathscr{R}$ by $(0, p^2 \cdot z) \in \mathcal{L} \times p^2\mathbb{Z}$, we find that it cannot be N-primitive.

Proof: Clearly, if n is N-square free, there is nothing to show.
For the other direction write $(\lambda, n) = (\lambda, \mathrm{q}(\ell_\lambda))$ with $\ell_\lambda \in L'$ projecting to λ. Assume

there was some prime $p \nmid N$ such that $p^2 \mid \mathrm{num}(\mathrm{q}(\ell_\lambda))$ (which is equivalent to $p^2 \mid \mathrm{q}(\ell_\lambda)$, since $p \nmid \mathrm{lev}(L)$ and $\mathrm{den}(\mathrm{q}(\ell_\lambda)) \mid \mathrm{lev}(L)$).
We have $\mu := \lambda/p \iff p\mu = \lambda$, so that there must be some $\ell_\mu \in L'$ projecting to μ. This means $\ell_\lambda = p\ell_\mu + l$ for some $l \in L$. Hence,

$$\mathrm{q}(\ell_\lambda) = p^2\,\mathrm{q}(\ell_\mu) + p\,\mathrm{b}(\ell_\mu, l) + \mathrm{q}(l) \tag{$*$}$$

but since $p^2 \mid \mathrm{q}(\ell_\lambda)$, so does $p^2 \mid \mathrm{q}(\ell_\lambda) - p^2\,\mathrm{q}(\ell_\mu) \in \mathbb{Z}$, meaning $p\,\mathrm{b}(\ell_\mu, l) + \mathrm{q}(l) = p^2 \cdot r$ for some $r \in \mathbb{Z}$. Hence, the numerator on the right side of $(*)$ is divisible by p^2 so that

$$\mathrm{q}(\ell_\lambda)/p^2 = \mathrm{q}(\ell_\mu) + r.$$

This, however, means nothing else than

$$\mathrm{q}(\ell_\lambda)/p^2 \equiv \overline{\mathrm{q}}(\mu) = \overline{\mathrm{q}}(\lambda/p)$$

and hence, $(\lambda/p, \mathrm{q}(\ell_\lambda)/p^2)$ was admissible, so that (λ, n) could not be primitive.
Now, in case $n = \mathrm{q}(\ell_\lambda) + p^2 \cdot z$ with $z \in \mathbb{Z}$, we find $p^2 \mid \mathrm{q}(\ell_\lambda)$ and the same argument as above works. □

Lemma 3.73. *Assume there are $\lambda, \sigma, \mu \in \mathcal{L}$ and $m, n \in \mathbb{Z}$ with $\gcd(n, m) = 1$ as well as $n\lambda = \sigma = m\mu$. Then there is $\delta \in \mathcal{L}$ such that $m\delta = \lambda$ and $n\delta = \mu$. In other words*

$$\begin{array}{ccc} & \sigma & \\ {\scriptstyle \cdot n}\nearrow & & \nwarrow{\scriptstyle \cdot m} \\ \lambda & & \mu \end{array} \wedge \gcd(n,m)=1 \implies \exists \delta\in\mathcal{L}: \begin{array}{ccc} & \sigma & \\ {\scriptstyle \cdot n}\nearrow & & \nwarrow{\scriptstyle \cdot m} \\ \lambda & & \mu \\ {\scriptstyle \cdot m}\nwarrow & & \nearrow{\scriptstyle \cdot n} \\ & \delta & \end{array}.$$

Proof: For the notation compare Definition 1.35. We begin with the left diagram and note that $\sigma \in {}^{n}\mathcal{L} \cap {}^{m}\mathcal{L}$. However, multiplication by m acts as an automorphism on $\mathcal{L}_p$ for all primes p with $p \nmid m$, so that $\mu \in {}^{n}\mathcal{L}$. In other words, there is $\delta_\mu \in \mathcal{L}$ with $n\delta_\mu = \mu$. Now by assumption $nm\delta_\mu = mn\delta_\mu = m\mu = \sigma = n\lambda$, meaning $\lambda - m\delta_\mu \in {}_{n}\mathcal{L}$. However, by symmetry of the diagram, we also find $\lambda \in {}^{m}\mathcal{L}$, so that $\lambda - m\delta_\mu \in {}_{n}\mathcal{L} \cap {}^{m}\mathcal{L}$. This means, there is an m-th root of $\lambda - m\delta_\mu$ which we will call δ' and which lies in ${}_{n}\mathcal{L}$, since multiplication by m acts as an automorphism on this submodule. In total, we find for

$\delta := \delta_\mu + \delta'$ that

$$n\delta = n\delta_\mu + n\delta' = n\delta_\mu = \mu, \qquad \text{and}$$
$$m\delta = m\delta_\mu + m\delta' = m\delta + \lambda - m\delta = \lambda.$$

□

We will have to explain the purpose of these investigations which originate from an approach to proving the injectivity of the Kudla–Millson theta lift.

For a lattice of signature $(m^+, 2)$ with level N and a modular form $f \in \mathcal{M}_{L,k}$ we find that $\mathscr{I}$ is the set of possible indices of Fourier coefficients of f. If f is further a cusp form, these lie in $\mathscr{I}^+$ which decomposes into maximal cones with respect to the partial order $\prec^N$. For an index $i \in \mathscr{I}$ we speak of the value of i on f when referring to the Fourier coefficient with index i.
By Proposition 6.54 we find that for a prime $p \nmid N$ the Hecke operator $\mathcal{T}(p^2)$ relates the values of indices on f which are linked by $\prec_p$. If f is an eigenform, the value of some index in $\mathscr{I}$ may actually be reduced to the value of a minimal index with respect to $\prec_p$, in many cases. In fact, the same is true for the partial order $\prec^N$.
The role of the representable indices $\mathscr{R}$ is the following. Recall that a major goal of this thesis is to prove the injectivity of the Kudla–Millson lift. For that purpose we consider cycle integrals of liftings. In Subsection 7.3.1 we learn that coefficients with indices in $\mathscr{R}$ are exactly these that appear in descriptions of the aforementioned cycle integrals (cf. Theorem 7.14). The same is true for different descriptions of the Fourier expansion of Kudla–Millson liftings appearing in the literature (cf. [Bru14], [Zuf24], and [MZ23]). Hence, the set $\mathscr{R}$ contains the indices that play a role for the Kudla–Millson lift.
The purpose of the primitively representable indices $\mathscr{R}_1$ is then the following. The values of these indices represent the first terms of certain L-series associated to f which are introduced in Subsection 6.4.1. These series are shown to admit roots if the form f is annihilated by the Kudla–Millson lift. Now if these indices happen to be minimal with respect to $\prec^N$, their value on f can be shown to vanish if f is an eigenform for a certain Hecke algebra. Then, however, all indices related to it by $\prec^N$ have to vanish on f. In many cases, this suffices to prove that f has to vanish, if its Kudla–Millson lift does. By employing a Hecke equivariance result for theta lifts [Yos85], the assumption of an eigenform may be eliminated, implying the injectivity of the Kudla–Millson theta lift.

However, for now, results obtained by this procedure are less general than Theorem 7.16.

3.3.4 Bounds on Fourier coefficients

Recall that a vector valued modular form $f \in \mathcal{M}_{L,k}$ has a Fourier expansion of the form

$$f(\tau) = \sum_{\lambda \in \mathcal{L}} \sum_{n \in \overline{\mathfrak{q}}(\lambda)+\mathbb{Z}} a(\lambda, n) \cdot q^n \mathfrak{e}_\lambda.$$

Typically these coefficients encode arithmetic information, in the case of theta series representation numbers of lattices, for instance. This alone justifies the desire for asymptotic bounds of these coefficients, as $n \to \infty$. In our case, in Section 6.4, we are associating L-series to vector valued modular forms and exploit that the asymptotic behaviour of the coefficients $a(\lambda, n)$ determines the range of convergence of these series.
In the following, we will collect information on this asymptotic behaviour. In fact, there is an abundance of bounds for scalar valued modular forms available and one could hope to lift these results to the vector valued case. Clearly, the Weil representation intertwines the different components of f, so directly applying bounds for scalar valued modular forms for $\mathrm{Mp}_2(\mathbb{Z})$ cannot work and we will have to do further investigations. Along these lines, we attempt to understand the behaviour of the discrete Weil representation ρ_L better with respect to subgroups of $\mathrm{Mp}_2(\mathbb{Z})$ and state the following result.

Proposition 3.74 ([Bor00, Thm. 5.4]). *Suppose that $\mathcal{L}$ is a discriminant form of level dividing N. If b and c are divisible by N, then $\gamma = \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \sqrt{c\tau + d}\right) \in \mathrm{Mp}_2(\mathbb{Z})$ acts on the Weil representation by*

$$\rho_L(g)\mathfrak{e}_\lambda = \chi_{\mathcal{L}}(g)\mathfrak{e}_{d\lambda}$$

where $\chi_{\mathcal{L}}$ denotes the character of $\overline{\Gamma}_0(N)$ defined in [Bor00, Thm. 5.4].

However, in case of $4 \nmid N$, the character $\chi_{\mathcal{L}}$ is trivial on the preimage of $\Gamma(N)$ in $\mathrm{Mp}_2(\mathbb{Z})$. In the other case, meaning $4 \mid N$, the section $s(\Gamma(N))$ in $\mathrm{Mp}_2(\mathbb{Z})$ is contained in the kernel of the character (cf. [Bor00, Lem. 5.3 p. 329]). Hence, the component functions f_λ of a vector valued modular form are actually contained in $\mathcal{M}_k(\Gamma(\mathrm{lev}(L)))$.

As a consequence, for the purpose of bounding the coefficients of the vector valued modular form f, it suffices to bound the asymptotic behaviour of scalar valued modular forms for $\Gamma(N)$. There is the following standard bound available in the literature (cf. [Shi71, Lem. 3.62 p. 90], [Shi73, p. 447]).

Lemma 3.75. *Let $k \in \mathbb{Z}/2$ and $N \in \mathbb{N}$ with $4 \mid N$ if k is not integral. Suppose that*

$f \in \mathcal{S}_k(\Gamma(N))$ has Fourier expansion

$$f(\tau) = \sum_{n \in \mathbb{Z}} a(n) q^{n/N}.$$

Then

$$a(n) \in \mathcal{O}(n^{k/2}) \text{ for } n \to \infty.$$

Also compare [JLL18] deriving that if the component functions of a vector valued modular form satisfy this bound, then it is a cusp form. Combining this bound with the bounds on coefficients of Eisenstein series yields the following result.

Corollary 3.76. *Let $f \in \mathcal{M}_k(\Gamma(N))$ with Fourier expansion as above. Then*

$$a(n) \in \mathcal{O}(n^{k-1}).$$

In many cases there are sharper bounds on the asymptotic behaviour known, for instance, the bound $\mathcal{O}(n^{k/2-1/5})$ by Rankin in the even case (cf. Theorem 2.25). However, even sharper bounds are to be found in the literature for forms to larger congruence subgroups. Consequently, the idea is to reduce the cases relevant to us to the setting of these bigger congruence subgroups.

In fact, further analysis of the situation yields that the component functions f_λ for $\lambda \in \mathcal{L}$ may indeed be described by scalar valued modular forms for $\Gamma_1(\mathrm{lev}(L))$ with character. Recall that $\Gamma(\mathrm{lev}(L)) \triangleleft \Gamma_1(\mathrm{lev}(L))$ with abelian quotient, which remains true under the section s so that by Remark 6.8, we may reduce to the case of a modular form for $\Gamma_1(\mathrm{lev}(L))$ with character.

In a subsequent step, this information may be utilised to reduce to the even simpler case of Γ_0 modular forms. However, First note that a scalar valued modular form f that has been transferred by the Petersson operator still possesses a Fourier expansion. Hence, it is meaningful to speak of a Fourier expansion of forms that have been altered by such a procedure. The proof idea of the following lemma is found in [DS05, 1.2.11 p. 24].

Lemma 3.77. *Let Γ be a congruence subgroup, $k \in \mathbb{Z}/2$, $f \in \mathcal{M}_k(\Gamma)$, and select $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$. Then $f|_k\gamma$ has a Fourier expansion and its constant term vanishes if f is a cusp form.*

Proof: Assuming $\gamma \in \mathrm{Mp}_2(\mathbb{Z})$, $f|_k\gamma$ must have a Fourier expansion. In fact, recall that there is $N \in \mathbb{N}$ such that $\Gamma(N) \leq \Gamma$. However, $s(\Gamma(N)) \trianglelefteq \mathrm{Mp}_2(\mathbb{Z})$ is normal. Hence, $f|_k\gamma$

is N periodic and holomorphic at ∞, so that it possesses a Fourier expansion.
In the general case, γ may be written as $\gamma = \alpha\gamma m a'$ where $\alpha \in \mathrm{Mp}_2(\mathbb{Z})$ and $\gamma' = r \cdot \left(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right), \sqrt{d}\right)$ with $r \in \mathbb{Q}^+$ and $\gcd(a,b,d) = 1$. Then also $f|_k\gamma$ must have a Fourier expansion. □

With this tool at hand, we may reduce to the case of a modular form for $\Gamma_1(N)$ without character which, in turn, reduce to modular forms for $\Gamma_0(N)$ with character factoring through $\Gamma_1(N)$.

Lemma 3.78. *Let $N \in \mathbb{N}$, $k \in \mathbb{Z}/2$, $\psi : \Gamma_1(N) \to \mathbb{T}$ be a character that is trivial on $\Gamma(N)$, and*

$$f = \sum_{n \in \mathbb{N}/N} a(n) q^n \in \mathcal{S}_k(\Gamma_1(N), \psi).$$

Assume that there is some $\alpha \in \mathbb{R}$ such that for all characters $\chi : \Gamma_0(N) \to \mathbb{T}$ that are trivial on $\Gamma_1(N)$ and any choice $g = \sum_{n \in \mathbb{N}} b(n) q^n \in \mathcal{S}_k(\Gamma_0(N^2), \chi)$ we have $b(n) \in \mathcal{O}(n^\alpha)$. Then also

$$a(n) \in \mathcal{O}(n^\alpha).$$

Remark 3.79. The statement above, meaning the transferal of bounds, remains true if the space of cusp forms $\mathcal{S}$ is replaced by holomorphic modular forms $\mathcal{M}$, weakly holomorphic modular forms $\mathcal{M}^!$ or meromorphic modular forms $\mathcal{A}$.

Proof: Assume f to be as above and $k \notin \mathbb{Z}$ implying $4 \mid N$ (the case of $k \in \mathbb{Z}$ is similar but easier to prove). We will prove that $f(N\cdot)$ is a modular form without character for $\Gamma_1(N^2)$. Recall that there is a section $s : \Gamma_0(N) \to \overline{\mathrm{SL}}_2(\mathbb{Z})$ given by (3.38) which is implicitly used to define modular forms of half integral weight and may be utilised to pull back some computations to $\Gamma_0(N)$. Let $\gamma = \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \phi(\tau)\right) \in \overline{\Gamma}_1(N^2)$ and compute

$$\begin{aligned}
&\left(\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}, 1\right) \cdot \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \phi(\tau)\right) \cdot \left(\begin{pmatrix} 1/N & 0 \\ 0 & 1 \end{pmatrix}, 1\right) \\
= &\left(\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}, 1\right) \cdot \left(\begin{pmatrix} a/N & b \\ c/N & d \end{pmatrix}, \phi(\tau/N)\right) \\
= &\left(\begin{pmatrix} a & Nb \\ c/N & d \end{pmatrix}, \phi(\tau/N)\right) \in \overline{\Gamma}_1(N).
\end{aligned}$$

Now, without loss of generality, we may pass to the case of N being a square and $3 \mid \nu_2(N)$. Under this assumption, we infer from (3.38) that if $\gamma \in s(\Gamma_1(N^2))$ then also the conjugate above is contained in $s(\Gamma_1(N))$. In fact, it even lies in $s(\Gamma(N))$. Next, define $M_N := \left(\left(\begin{smallmatrix} N & 0 \\ 0 & 1 \end{smallmatrix}\right), 1\right) \in \overline{\mathrm{GL}_2^+}(\mathbb{Q})$ and conclude that for any $\gamma \in s(\Gamma_1(N^2))$ we find that

$$\begin{aligned} f(M_N\gamma\tau) &= f(M_N\gamma M_N^{-1}M_N\tau) \\ &= \psi(M_N\gamma M_N^{-1})f(M_N\tau)\phi(\tau)^k. \end{aligned}$$

Recall that by assumption ψ was trivial on $s(\Gamma(N))$. As a result, we have that

$$f(N\tau) = \sum_{n\in\mathbb{N}/N} a(n)q^{Nn} \in \mathcal{S}_k(\Gamma_1(N^2)).$$

Recall that the space $\mathcal{S}_k(\Gamma_1(N^2))$ may be represented as a direct sum of spaces of modular forms for $\Gamma_0(N^2)$ with character (cf. Remark 6.8 and Example 6.9) so that the coefficients fulfil $a(n) \in \mathcal{O}(n^\alpha)$ by assumption. □

As a consequence, it suffices to cite asymptotic bounds of Fourier coefficients for modular forms for $\Gamma_0(N)$ with nebentypus. There are some sharper bounds available in this setting. For integral weight modular forms, there is the Deligne bound, stated in Thm. 2.26, providing the best possible bound. However, it had been formulated only for natural numbers n being coprime to the level N of the modular form. Due to the structure of the Hecke algebra (cf. Proposition 6.41) it will then suffice to only prove a similar bound for coefficients at primes that divide the level. In fact, by Atkin–Lehner theory, it suffices to do so for newforms. The following is due to Ogg and Li and part of [Li74, Thm. 3 p. 295].

Proposition 3.80. *Let $k \in \mathbb{Z}_{>0}$ and $f \in \mathcal{S}_k^{\mathrm{new}}(\Gamma_0(N), \chi)$ with Fourier expansion*

$$f(\tau) = \sum_{n=1}^{\infty} a(n)e(n\tau)$$

be a normalised form that is an eigenform for the Hecke algebra $\mathscr{H}_N$. Then for $p \mid N$ we find the following

i) If χ is not induced by a character modulo N/p, then $|a(p)| = p^{k/2-1/2}$.

ii) If χ is a character modulo N/p and $p^2 \nmid N$, then $a(p)^2 = \chi(p)p^{k-1}$.

iii) If χ is a character modulo N/p and $p^2 \mid N$, then $a(p) = 0$.

As a Corollary we obtain the following bound on Fourier coefficients.

Theorem 3.81. *Let $k \in \mathbb{Z}_{>0}$ and $f \in \mathcal{S}_k(\Gamma_0(N), \chi)$ with Fourier expansion*

$$f(\tau) = \sum_{n=1}^{\infty} a(n)e(n\tau).$$

Then

$$a(n) \in \mathcal{O}_\varepsilon(n^{k/2-1/2+\varepsilon}).$$

However, for half integral weight forms, the situation is more delicate. Nevertheless, there are the following two bounds available in the literature. The first has been proven by Blomer and Harcos and is derived from [BH08, Cor. 2 p. 55].

Theorem 3.82. *Let $k \in (\mathbb{Z}/2) \setminus \mathbb{Z}$ with $k \geq 5/2$, $N \in \mathbb{N}$, and χ be a character modulo $4N$. Let*

$$f(\tau) = \sum_{n=1}^{\infty} a(n)e(n\tau)$$

be in $\mathcal{S}_k(\Gamma_0(4N), \chi)$. Then for $t \in \mathbb{N}$ and $n \nmid (2N)^\infty$ we find

$$a(tn) \in \mathcal{O}_\varepsilon(n^{k/2-5/16+\varepsilon}).$$

Note that the above theorem is also true in case of $k = 3/2$, if the choice of cusp form is reduced to the orthogonal complement of theta functions.
The second bound goes back to Iwaniec [Iwa87] and Duke [Duk88] in a special case and has been generalised by Waibel [Wai18, Thm. 1 p. 186].

Theorem 3.83. *Let $k \in (\mathbb{Z}/2) \setminus \mathbb{Z}$ with $k \geq 5/2$, $4 \mid N \in \mathbb{N}$, and χ be a character modulo $4N$. Let*

$$f(\tau) = \sum_{n=1}^{\infty} a(n)e(n\tau)$$

be in $\mathcal{S}_k(\Gamma_0(4N), \chi)$. Then it holds for indices $n = tv^2w^2$ with t squarefree, $v \mid N^\infty$, and $\gcd(w, N) = 1$ that

$$a(n) \in \mathcal{O}_\varepsilon(n^{k/2-1/2+\varepsilon}v^{1/2}).$$

Note that the above implies $a(n) \in \mathcal{O}_\varepsilon(n^{k/2-1/4+\varepsilon})$, which is the Weil bound.

In conclusion we have the following bounds on Fourier coefficients of vector valued modular forms.

Corollary 3.84. *Let* $2 \leq k \in \mathbb{Z}/2$ *and* $f \in \mathcal{S}_{L,k}(\mathrm{Mp}_2(\mathbb{Z}))$ *with Fourier expansion*

$$f(\tau) = \sum_{\lambda \in \mathcal{L}} \sum_{n \in \overline{\mathrm{q}}(\lambda)+\mathbb{Z}} a(\lambda, n) \cdot e(n\tau) \otimes \mathfrak{e}_\lambda.$$

Assume $n \in \mathbb{Q}$ *is in a progression with an upper bound on the number of bad primes appearing in the prime number decomposition of* n*. Then for all* $\lambda \in \mathcal{L}$

$$a(\lambda, n) \in \begin{cases} \mathcal{O}_\varepsilon(n^{k/2-1/2+\varepsilon}), & 2 \mid \mathrm{rank}(L), \\ \mathcal{O}_\varepsilon(n^{k/2-5/16+\varepsilon}), & 2 \nmid \mathrm{rank}(L). \end{cases}$$

For general n *and* $t \in \mathbb{Q}$*, the following bound is true*

$$a(\lambda, tn^2) \in \begin{cases} \mathcal{O}_\varepsilon(n^{k-1+\varepsilon}), & 2 \mid \mathrm{rank}(L), \\ \mathcal{O}_\varepsilon(n^{k-1/2+\varepsilon}), & 2 \nmid \mathrm{rank}(L). \end{cases}$$

In later applications, certain progressions of indices of Fourier coefficients of the vector valued form f are considered. Hence, we conclude with statements about how the bounds above transfer to setting we have in mind.

Corollary 3.85. *Let* $f \in \mathcal{M}_{k,L}(\mathrm{Mp}_2(\mathbb{Z}))$ *be a modular form with Fourier expansion*

$$f(\tau) = \sum_{\lambda \in \mathcal{L}} \sum_{n \in \mathbb{Z}+\lambda^2/2} a(\lambda, n) \cdot e(n\tau) \otimes \mathfrak{e}_\lambda.$$

Assume that there is $\sigma \in \mathbb{R}$ *such that for all* $\lambda \in \mathcal{L}$ *we find* $a(\lambda, n) \in \mathcal{O}(n^{k/2-\sigma})$*. Select* $d \in \mathbb{N}$ *and fix a pair* (λ, n)*. Then*

$$a(m\lambda, m^d \cdot n) \in \mathcal{O}(m^{d(k/2-\sigma)}). \tag{3.69}$$

Proof: We find

$$|a(m\lambda, m^d n)| \leq \sum_{\mu \in \mathcal{L}} |a(\mu, m^d n)| \in \mathcal{O}(m^{d(k/2-\sigma)}).$$

□

Remark 3.86. Clearly, the above corollary is also valid for progressions $\mathfrak{s} : \mathcal{L} \times \mathbb{Q} \to \mathcal{L} \times \mathbb{Q}$ such that the projection to the second component of $\mathfrak{s}$ is in $\mathcal{O}(m^d)$ for some $d > 0$ which is independent of $\lambda \in \mathcal{L}$. Formalising this statement, however, interfered with readability.

The section will be concluded with a collection of bounds that are required for proving convergence of L-series in Section 6.4.1.

Corollary 3.87. *Let* $2 \leq k \in \mathbb{Z}/2$ *and* $f \in \mathcal{S}_{L,k}(\mathrm{Mp}_2(\mathbb{Z}))$ *with Fourier expansion*

$$f(\tau) = \sum_{\lambda \in \mathcal{L}} \sum_{n \in \overline{\mathrm{q}}(\lambda)+\mathbb{Z}} a(\lambda, n) \cdot e(n\tau)\mathfrak{e}_\lambda.$$

Select $t \in \mathbb{Q}$*, then the following bound is true for all* $\lambda \in \mathcal{L}$

$$a(\lambda, tn^2) \in \begin{cases} \mathcal{O}_\varepsilon(n^{k-1+\varepsilon}), & 2 \mid \mathrm{rank}(L), \\ \mathcal{O}_\varepsilon(n^{k-1/2+\varepsilon}), & 2 \nmid \mathrm{rank}(L). \end{cases}$$

If in addition $n \in \mathbb{Q}$ *is assumed to be coprime to* $\mathrm{lev}(L)$ *we obtain*

$$a(\lambda, tn^2) \in \begin{cases} \mathcal{O}_\varepsilon(n^{k-1+\varepsilon}), & 2 \mid \mathrm{rank}(L), \\ \mathcal{O}_\varepsilon(n^{k-5/8+\varepsilon}), & 2 \nmid \mathrm{rank}(L). \end{cases}$$

In order to recollect the different bounds relevant to our setting, we introduce a parameter $\sigma \in \mathbb{R}$ associated to some $f \in \mathcal{M}_{L,k}(\mathrm{Mp}_2(\mathbb{Z}))$ describing the improvement of the asymptotic growth of its Fourier coefficients compared to the Hecke bound, i.e. $\sigma \in \mathbb{R}$ is chosen such that in the case of a progression which is linear in n we have

$$a(\lambda, n) \in \mathcal{O}_\varepsilon(n^{k/2-\sigma+\varepsilon}). \tag{3.70}$$

In case $f \notin \mathcal{S}_{L,k}$ and sufficiently large weight, we select the obvious bound $\sigma = 1 - k/2$. In case of $f \in \mathcal{S}_{L,k}$, Table 3.1 provides an overview of possible choices, also for a variant of progression in the index. In other cases, for theta functions of low weight, for instance, the parameter σ has to be determined separately.

σ	$n \in \mathbb{N}$ $a(\lambda, n)$	$n \in \mathbb{N}$ $a(\lambda, tn^2)$	$n \nmid \mathrm{lev}(L)^\infty$ $a(\lambda, tn^2)$
$2 \mid \mathrm{rk}(L)$	1/2	1/2	1/2
$2 \nmid \mathrm{rk}(L)$	0	1/4	5/16

Table 3.1: To a cusp form $f \in \mathcal{S}_{L,k}$ of weight $k \geq 2$ we associate the parameter $\sigma \in \mathbb{R}$ above, determined by the asymptotic growth of its Fourier coefficients $a(\lambda, n)$ relative to the Hecke bound, i.e. $a(\lambda, n) \in \mathcal{O}_\varepsilon(n^{k/2-\sigma+\varepsilon})$. Its direct application is determining the range of convergence of L-series associated to f (cf. Subsection 6.4.1). The number $t \in \mathbb{Q}$ appearing above allows for a shift in the index in order to consider the variant $a(\lambda, tn^2) \in \mathcal{O}_\varepsilon(n^{k-2\sigma+\varepsilon})$.

4 Orthogonal theory

The preceding chapter focussed on introducing the symplectic setting for modular forms. This chapter presents a similar theory in the orthogonal setting. First, properties of orthogonal groups and the construction of the spin group are reviewed. Subsequently, models of modular varieties in this context, the standard arithmetic subgroups and modular forms are introduced. Finally, a brief review of special divisors which play a significant role for cycle integrals in Section 7.3 is provided.

4.1 Group properties

In this section we summarise the theory of orthogonal and Spin groups in the context of our investigations.

4.1.1 Orthogonal groups

Definition 4.1. Let (M, q) be a free non-degenerate quadratic R module. Then the associated *orthogonal group* is defined as

$$\mathrm{O}(M) \coloneqq \{\gamma \in \mathrm{Aut}(M) \mid \mathrm{b}(\gamma x, \gamma y) = \mathrm{b}(x, y) \text{ for all } x, y \in M\}\,.$$

Here, b denotes the associated bilinear form. Further, the subgroup of elements with determinant 1 is denoted $\mathrm{SO}(M)$. Further, if $M = V$ is an $\mathbb{R}$ vector space of signature (m^+, m^-), we write $\mathrm{O}(m^+, m^-) \coloneqq \mathrm{O}(V)$.

One of the most significant examples of orthogonal transformations are reflections.

Example 4.2. Let (V, q) be a quadratic space over K. Then every $u \in V$ with non-trivial norm $\mathrm{q}(u)$ induces a *reflection* at the hyperplane $u^\perp$ via $\tau_u : v \mapsto v - \mathrm{b}(v, u)\,\mathrm{q}(u)^{-1}u$. In fact, we have $\tau_u \in \mathrm{O}(V)$. The name reflection derives from the properties $\tau_u(u) = -u$ and $\tau_u(v) = v$ for $v \in u^\perp$. In particular, $\tau_u^2 = \mathrm{id}$ and we find $\det(\tau_u) = -1$, so that $\tau_u \notin \mathrm{SO}(V)$.

We will present more examples, which are discussed in more detail in [Fis10, 5.5 p. 303].

Example 4.3. Let V be a real vector space of dimension 2 with standard scalar product as bilinear form, so that $x \mapsto \mathrm{q}(x) = \frac{1}{2}\langle x, x\rangle$ defines the quadratic form.

a) The map from the unit circle

$$\mathbb{T} \to \mathrm{SO}(V), \quad \vartheta \mapsto k_\vartheta := \begin{pmatrix} \cos(\vartheta) & -\sin(\vartheta) \\ \sin(\vartheta) & \cos(\vartheta) \end{pmatrix} \tag{4.1}$$

defines an injective group homomorphism from the complex unit circle. The associated map may be visualised as a rotation by the angle ϑ.

b) We find that for any $\vartheta \in \mathbb{T}$ the map represented by the matrix

$$r_\vartheta := \begin{pmatrix} \cos(\vartheta) & \sin(\vartheta) \\ \sin(\vartheta) & -\cos(\vartheta) \end{pmatrix}$$

is also orthogonal. In fact, it is elementary to verify that any element in $\mathrm{O}(2,0)$ is represented by either k_ϑ or r_ϑ for a suitable choice of $\vartheta \in \mathbb{T}$, yielding a homeomorphism $\mathrm{O}(2,0) \simeq \mathbb{T} \sqcup \mathbb{T}$, if $\mathrm{O}(2,0)$ is equipped with the relative topology.

Remark 4.4. In case of $M = V$ a K vector space, the group $\mathrm{O}(M)$ may be defined as the stabiliser of q of the natural right action $\mathrm{GL}(V) \ni g \mapsto q \circ g$ on quadratic forms on M, as long as the characteristic of K is different from 2.

Remark 4.5. If (M, q) is a free non-degenerate quadratic R module and $R \to S$ is a map of rings, we have a natural induced map $\mathrm{O}(M) \to \mathrm{O}(M \otimes S)$. In particular, for a $\mathbb{Z}$ lattice $\mathrm{O}(L) \hookrightarrow \mathrm{O}(L \otimes S)$ for $S \in \{\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{R}\}$.

The following two theorems are essential for our investigations and central to the orthogonal theory. The extension result is due to Witt, while the generation result is due to Cartan–Dieudonné.

Theorem 4.6 ([Cas78, Thm. 4.1])**.** *Let (V, q) be a quadratic space over a field of characteristic different from 2 with regular subspaces V_1, V_2. Suppose there is an isomorphism $\sigma : (V_1, \mathrm{q}|_{V_1}) \to (V_2, \mathrm{q}|_{V_2})$, then σ extends to an automorphism of (V, q).*

Theorem 4.7 ([Gro02, Thm. 6.6])**.** *Let (V, q) be a quadratic K vector space with $\mathrm{char}(K) \neq 2$ of dimension m. Then every element in $\mathrm{O}(V)$ is the product of at most m reflections.*

Remark 4.8 ([Fis10, 5.5.5, 5.5.6, p. 307])**.** In fact, the examples presented above provide a complete list of orthogonal elements for definite quadratic $\mathbb{R}$ spaces. Let (V, q) be such a space of dimension m, and $\gamma \in \mathrm{O}(V)$, then there are $r^+, r^-, r \in \mathbb{N}_0$ with $r^+ + r^{-1} + 2r = m$, and $\vartheta_i \in (0, \pi) \cup (\pi, 2\pi)$ such that γ is represented by a matrix

$$\mathcal{I}_{r^+} \oplus (-1) \cdot \mathcal{I}_{r^-} \oplus \bigoplus_{i=1}^{r} k_{\vartheta_i}$$

with respect to a suitable orthogonal basis of V. Further, in case of a complex quadratic space, all Gram matrices are diagonal.

Recall that $\mathrm{GL}_n(\mathbb{R}) \simeq \mathrm{Aut}(V)$ carries the trace topology of $\mathbb{R}^{n \times n}$.

Lemma 4.9. *Let* (V, q) *be a regular real quadratic space. Then* $\mathrm{O}(V)$ *is closed. Further, it is compact, if, and only if,* V *is definite.*

Proof: Recall that S denotes the Gram matrix of q. Consider the map $M_n(\mathbb{R}) \ni g \mapsto g^T \mathrm{S}\, g$, wich is continuous as it is polynomial in the coordinates of $M_n(\mathbb{R})$. Then $\mathrm{O}(V)$ is given by the preimage of S and as such closed.
Recall further that $\sqrt{\mathrm{q}(\,\cdot\,)}$ defines a norm on V if, and only if, q is definite. This norm induces an operator norm on the operator algebra $\mathscr{B}(V) \simeq M_n(\mathbb{R})$ whose unit ball contains $\mathrm{O}(V)$. Since $M_n(\mathbb{R})$ is finite dimensional, all of its norms are equivalent and $\mathrm{O}(V)$ is compact as it is closed and totally bounded.
Assume $(V, \mathrm{q}) \simeq H$ is a hyperbolic plane. Further, we find for any $a \in \mathbb{R}^\times$ that

$$\mathrm{diag}(a, a^{-1})^T H \,\mathrm{diag}(a, a^{-1}) = H,$$

meaning $s_a := \mathrm{diag}(a, a^{-1}) \in \mathrm{O}(H)$. However, in the standard operator topology we have $\|s_a\| = \max\{|a|, |a^{-1}|\}$, meaning $\mathrm{O}(H)$ is unbounded. Now if (V, q) was not a hyperbolic plane but indefinite, we may assume a hyperbolic split, rendering $\mathcal{I}_{m-2} \oplus s_a \in \mathrm{O}(V)$ with $\|\mathcal{I}_{m-2} \oplus s_a\| = \|s_a\|$, settling the general case. □

Before turning to constructing the general spin group, which will be an extension of the special orthogonal group SO, we will state some topological properties of SO and O in the following remark which is not trivial to proove.

Remark 4.10. Let (V, q) be a real quadratic space of signature (m^+, m^-).

a) If (V, q) is definite, $\mathrm{O}(V)$ is compact and has two connected components, one of which is given by $\mathrm{SO}(V)$. Then $\mathrm{SO}(V)$ is the identity component and is even pathconnected, as may be proven by means of Remark 4.8.

b) If (V, q) is indefinite, $\mathrm{O}(V)$ is non-compact and has four connected components. The components correspond to whether the elements preserve the orientation on the maximal (positive and negative) definite subspaces. $\mathrm{SO}(V)$ then has two connected components, corresponding to elements (not) changing the orientation on both maximal positive and negative definite subspaces. Further,

$$\begin{aligned} \mathrm{O}(m^+) \times \mathrm{O}(m^-) &< \mathrm{O}(m^+, m^-), \\ \mathrm{SO}(m^+, m^-) \cap (\mathrm{O}(m^+) \times \mathrm{O}(m^-)) &< \mathrm{SO}(m^+, m^-), \\ \mathrm{SO}(m^+) \times \mathrm{SO}(m^-) &< \mathrm{SO}^+(m^+, m^-) \end{aligned}$$

are maximal compact subgroups, where SO^+ denotes the identity component. We find that

$$\mathrm{O}(m^+, m^-)/\,\mathrm{O}(m^+) \times \mathrm{O}(m^-) \simeq \mathrm{SO}^+(m^+, m^-)/\,\mathrm{SO}(m^+) \times \mathrm{SO}(m^-).$$

4.1.2 Spin groups

In the following section the Clifford algebra is constructed in order to derive the general Spin group from it. This group functions as a cover of the special orthogonal group and is a valid choice for defining Shimura varieties in the orthogonal setting, since these embed naturally into Siegel varieties and are hence defined over $\mathbb{Q}$, making the transition to a number field superfluous. For basic facts about the tensor product and tensor algebra we refer to [GW20] and [Lan02, XVI-7 p. 632]. The primary source for the introduction of the Clifford algebra and the Spin group is [Bru+08, 2.2 p. 129] which may be supplemented with [Kit99].

Recall the definition of tensor products.

Definition 4.11. Let R be a ring and M, N be R modules. A *tensor product* of M, N is an R module $M \otimes_R N$ together with an R bilinear map $\tau : M \times N \to M \otimes_R N$, denoted $(m, n) \mapsto m \otimes n$, which is universal for the following diagram.

$$\begin{array}{ccc} M \otimes_R N & \xrightarrow{\exists!\beta} & P \\ \tau \uparrow & \nearrow_{\alpha} & \\ M \times N & & \end{array}$$

This means for every R module and R bilinear map $\alpha : M \times N \to P$ there exists exactly one linear map $\beta : M \otimes_R N \to P$, such that $\alpha = \beta \circ \tau$.

Remark 4.12. i) The pair $(M \otimes_R N, \tau)$ is unique up to unique isomorphism.

ii) If $(m_i), (n_j)$ are generating sets of M, N, respectively, then the *elementary tensors* $(m_i \otimes n_j)$ form a generating set of $M \otimes_R N$, in particular, the tensor product is a finitely generated R module, if M, N are.

Example 4.13. Given a ring homomorphism $f : R \to S$, the R module M induces an S module via tensoring. Namely, $M \otimes_R S$, where S is regarded as an R module via f carries the structure of an S module, since S itself is an S module. More precisely, the multiplication in S is associative, so that for $r \in R$, s_1, $s_2 \in S$ we find that the corresponding actions commute $f(r) \cdot (s_1 \cdot s_2) = (f(r) \cdot s_1) \cdot s_2$, so that S inherits an S right action and is rendered an (R, S) bimodule. Then, however, $M \otimes_R S$ carries the structure of an S (right) module. This is called *extension of scalars*.
A common realisation is what we have encountered in Section 1.1 in the context of quadratic modules, where $R \hookrightarrow S$ injects and the name extension of scalars becomes quite graphical. If, for instance, $R = K$ is a field, and F/K its algebraic closure and (b_i) denotes a basis of the vector space V, then $V(F) \coloneqq V \otimes_K F$ is a vector space over F with basis (b_i) again. This process is then called *complexification* of the K vector space V.

The tensor product as an operation on modules has remarkable properties, constituting the structure of the Witt ring.

Remark 4.14. a) The modules $M \otimes_R N$ and $N \otimes_R M$ are isomorphic via $m \otimes n \mapsto n \otimes m$, i.e. the tensor product is a commutative operation.

b) The modules $(M_1 \otimes_R M_2) \otimes_R M_3$ and $M_1 \otimes_R (M_2 \otimes_R M_3)$ are isomorphic via $(m_1 \otimes m_2) \otimes m_3 \mapsto m_1 \otimes (m_2 \otimes m_3)$, i.e. the tensor product is an associative operation. This is why we write $m_1 \otimes m_2 \otimes m_3$ for $(m_1 \otimes m_2) \otimes m_3$ and $\bigotimes_{i \in I} M_i$ for the tensor product of a finite family $(M_i)_{i \in I}$ of R modules.

c) The map $R \otimes_R M \to M$, $r \otimes m \mapsto rm$ defines an R linear isomorphism, i.e. the module R functions as a unit element with respect to the tensor product.

d) For a family of R modules M_i and an R modules N, the map $(m_i)_i \otimes n \mapsto (m_i \otimes n)_i$ defines an isomorphism $\left(\bigoplus_i M_i\right) \otimes_R N \simeq \bigoplus_i (M_i \otimes N)$, i.e. the tensor product commutes with direct sums or in short *distributivity.*

For an R module M, we seek an R algebra, that canonically includes R, as well as M. The tensor product appears to possess the correct properties to be beneficial in that quest. Let M be an R module and $l \in \mathbb{N}_0$. We write $M^{\otimes 0}$ for R and $M^{\otimes l+1} := M^{\otimes l} \otimes_R M$. Then from the associativity of the tensor product we obtain for any l_1, $l_2 \in \mathbb{N}_0$ a bilinear associative map

$$M^{\otimes l_1} \otimes M^{\otimes l_2} \to M^{\otimes (l_1+l_2)}, \quad (m_1, m_2) \mapsto m_1 \otimes m_2.$$

This declares a ring structure on the direct sum $\bigoplus_{l=0}^{\infty} M^{\otimes l}(M)$ rendering it an algebra.

Definition 4.15. Let M be an R module. The *tensor algebra* $\mathcal{T}(M)$ of M is defined as

$$\mathcal{T}(M) := \bigoplus_{l=0}^{\infty} M^{\otimes l}. \tag{4.2}$$

The product of two elements m, $n \in \mathcal{T}(M)$ is denoted $m \otimes n$. For $l \in \mathbb{N}_0$, we refer to $M^{\otimes l}$ in $\mathcal{T}(M)$ as the tensors of *degree* l and denote the summand by $\mathcal{T}^l(M)$, rendering $\mathcal{T}(M)$ a graded asscociative algebra with canonical injection $\iota : M \to \mathcal{T}(M)$.

Remark 4.16. Note that for $l \in \mathbb{N}_0$ the association $M \mapsto \mathcal{T}^l(M)$ is functorial. In fact, if $(M_i), (N_i)$ are finite families of R modules and $f_i : M_i \to N_i$ are R linear maps. Then we may concatenate $\prod_i f_i$ with the tensor map $\tau : \prod_i N_i \to \bigotimes_i N_i'$ to obtain a multilinear map $\prod_i M_i \to \bigotimes_i N_i$ that factors uniquely through $\bigotimes_i M_i$ and gives by the universal property of $\bigotimes_i M_i$ rise to a linear map

$$\bigotimes_i f_i : \bigotimes_i M_i \to \bigotimes_i N_i. \tag{4.3}$$

This shows that $\mathcal{T}^l : M \mapsto M^{\otimes l}, f \mapsto f^{\otimes l} := \bigotimes_{i=1}^{l} f$ is functorial in the category of R modules.

Remark 4.17. For an R module map $f : M \to N$ the family of associated maps $(\mathcal{T}^l(f))$ induces a map $\mathcal{T}(M) \to \mathcal{T}(N)$ which we denote by $\mathcal{T}(f)$. In fact, for a finite family of elements $m_i \in M$ we find

$$\mathcal{T}(f)(\otimes_i m_i) = \otimes_i f(m_i),$$

so that we obtain an algebra morphism and hence end up with a functorial association $\mathcal{T} : M \mapsto \mathcal{T}(M), \quad f \mapsto \mathcal{T}(f)$ of R modules to graded R algebras.

Theorem 4.18. *The tensor algebra satifies the following universal property: For any linear map $f : M \to A$ for an associative R algebra A, there is a unique linear algebra morphism $\rho : \mathcal{T}(M) \to A$ such that the following diagramm commutes.*

$$\begin{array}{ccc} \mathcal{T}(M) & \xrightarrow{\exists!\rho} & A \\ \uparrow{\scriptstyle\iota} & \nearrow{\scriptstyle f} & \\ M & & \end{array}$$

Example 4.19. The universal property of $\mathcal{T}(M)$ implies that the mapping defined by $J : M \to M \subset \mathcal{T}(M), \quad m \mapsto -m$ lifts to a map $\mathcal{T}(M) \to \mathcal{T}(M)$. It is called the *canonical automorphism* of $\mathcal{T}(M)$.

There are several algebras derived from the tensor algebra. Recall that morphisms from $\mathcal{T}(M)$ to commutative algebras factor through the quotient by the commutator ideal $\langle [m,n] \rangle$.

Definition 4.20. Let M be an R module.

a) Let $\mathcal{C} := \langle [\mathcal{T}(M), \mathcal{T}(M)] \rangle$ denote the commutator ideal of $\mathcal{T}(M)$. Then the *symmetric algebra* of M is defined to be the quotient $\mathrm{Sym}(M) := \mathcal{T}(M)/\mathcal{C}$. The product is denoted by ‚·'. The image of $\mathcal{T}^l(M)$ in $\mathrm{Sym}(M)$ is denoted by $\mathrm{Sym}^l(M)$ and called the l-th *symmetric power* of M.

b) Let $\mathcal{I} = \langle \{ m \otimes n + n \otimes m \mid m, n \in \mathcal{T}(M) \} \rangle$ denote the anticommutator ideal of $\mathcal{T}(M)$. Then the *exterior algebra* of M is defined to be the quotient $\bigwedge(M) := \mathcal{T}(M)/\mathcal{I}$. The product is denoted by ‚$\wedge$'. The image of $\mathcal{T}^l(M)$ in $\bigwedge(M)$ is denoted by $\bigwedge^l(M)$ and called the l-th *exterior power* of M.

The following universal property is directly inherited from the one of the tensor algebra.

Remark 4.21. By construction, $M^l \to \mathcal{T}^l(M) \to \mathrm{Sym}^r(M)$ is universal for multilinear symmetric maps.

Example 4.22. Let V be an F vector space of dimension n. Then $\bigwedge^n(V)$ has dimension one and we find by functoriality of $\bigwedge^l$ that for any endomorphism $f : V \to V$, we have an associated linear map $\bigwedge^n(f) : \bigwedge^n(V) \to \bigwedge^n(V) \simeq F$. Clearly, $\mathrm{Hom}_{\mathrm{VS}}(F,F) \simeq F$ canonically, so that $\bigwedge^n(f)$ may be interpreted as an element in F. It is called the *determinant* of f.

Remark 4.23. Let $n \in \mathbb{N}_0$ and $\mathfrak{S}_n$ denote the n-th symmetric group, i.e. the automorphism group of the set of n elements. Then there is a canonical injection $\iota_n : \mathfrak{S}_n \hookrightarrow \mathfrak{S}_{n+1}$ by letting an element $\sigma \in \mathfrak{S}$ act on the first n elements of $\{1, \dots, n+1\}$ and fixing the last element. This gives rise to a filtered inductive system and the colimit along this system is called the *general finite symmetric group* and denoted $\mathfrak{S}_f$.
It is clear that $\mathfrak{S}_n$ acts on $\mathcal{T}^n(M)$ via $\sigma(\otimes_{1\leq i\leq n} m_i) \coloneqq \otimes_i m_{\sigma(i)}$. This action extends to a linear action of $\mathfrak{S}_f$ on $\mathcal{T}(M)$ and gives rise to the ideal

$$\mathcal{C} \coloneqq \langle \{m - \sigma(m) \mid m \in \mathcal{T}(M), \sigma \in \mathfrak{S}_f\} \rangle.$$

The quotient $\mathcal{T}(M)/\mathcal{C}$ is isomorphic to the *symmetric algebra* $\mathrm{Sym}(M)$.

Definition 4.24. Assume (M, q) is a quadratic R module mapping to R. Consider the ideal $\mathcal{I} \coloneqq \langle \{m \otimes m - \mathrm{q}(m) \mid m \in M\} \rangle \trianglelefteq \mathcal{T}(M)$. Then the quotient $\mathcal{C}_M \coloneqq \mathcal{T}(M)/\mathcal{I}$ is called the *Clifford algebra*.

Remark 4.25. a) Write $m \cdot n \coloneqq m \otimes n \in \mathcal{C}_M$. Then

$$m \cdot m = \mathrm{q}(m), \qquad m \cdot n + n \cdot m = \mathrm{b}(m, n). \tag{4.4}$$

In particular, we have $m \cdot n = -n \cdot m$, if, and only if, m and n are orthogonal to each other.

b) Both R as a ring and M as a module are canonically injected into $\mathcal{C}_M$.

c) If M is free and of finite rank with basis $b_1, \dots, b_n$ then the family

$$b_{i_1} \cdot \dots \cdot b_{i_r} \qquad (1 \leq r \leq n, 1 \leq i_1, < \dots < i_r \leq n)$$

forms a basis of $\mathcal{C}_M$, rendering it a free R module of rank 2^n.

Example 4.26. If $\mathrm{char}(R) \neq 2$ and $\mathrm{q} \equiv 0$, we find $\mathcal{C}_M \simeq \bigwedge(M)$.

Remark 4.27. The Clifford algebra satifies the following universal property: For any linear map $f : M \to A$ for an associative unital R algebra A, such that $f(m)^2 = \mathrm{q}(m) \cdot 1 \in A$ for all $m \in M$, there is a unique linear algebra morphism $\rho : \mathcal{C}_M \to A$ such that the following diagramm commutes.

$$\begin{array}{ccc} \mathcal{C}_M & \xrightarrow{\exists ! \rho} & A \\ {\scriptstyle \iota}\uparrow & \nearrow_{f} & \\ M & & \end{array}$$

Note that, as a consequence, any isometry $M_1 \to M_2$ lifts to an R algebra morphism $\mathcal{C}_{M_1} \to \mathcal{C}_{M_2}$ so that the Clifford construction yields a functor from quadratic R modules with isometries as morphisms to the category of unital associative R algebras.

From now on, we will consider the case of free quadratic R modules (M, q). There are two canonical automorphisms on $\mathcal{C}_M$ which will become important for constructing and working with the Spin group (cf. Subsection 5.4.3).

Definition 4.28. Let (M, q) be a free quadratic R module.

a) Multiplication by -1 induces an isometry on M. By the universal property of $\mathcal{C}_M$ it lifts to a map $J : \mathcal{C}_M \to \mathcal{C}_M$ called the *canonical automorphism*.

b) Define the *canonical involution* by

$$.^T : \mathcal{C}_M \to \mathcal{C}_M, \qquad m_1 \cdot \dots \cdot m_k \mapsto m_k \cdot \dots \cdot m_1. \tag{4.5}$$

c) Define the *Clifford norm* on $\mathcal{C}_M$ by

$$\mathrm{N} : \mathcal{C}_M \to \mathcal{C}_M, \qquad m \mapsto m^T \cdot m. \tag{4.6}$$

Example 4.29. a) For $m \in M \leq \mathcal{C}_M$ we find $m^T = m$ and hence $\mathrm{N}(m) = \mathrm{q}(m)$.

We are ultimately interested in finite dimensional quadratic vector spaces over fields of characteristic 0, so assume in the following that $2 \in R^\times$ and that M has the finite orthogonal basis $(b_i)_{i=1}^n$.

Definition 4.30. Write $\mathcal{C}_M^0$ for the subalgebra generated by products of an even number of basis elements b_i and call it the *even* Clifford algebra. On the contrary, the submodule $\mathcal{C}_M^1$ is defined as the submodule generated by products of an odd number of basis elements.

Remark 4.31. a) The decomposition $\mathcal{C}_M = \mathcal{C}_M^0 \oplus \mathcal{C}_M^1$ as a modules defines a $\mathbb{Z}/2\mathbb{Z}$ grading on $\mathcal{C}_M$. In case $-1 \neq 1 \in R$, note that $\mathcal{C}_M^0 = (\mathcal{C}_M)^J$ equals the fix space of the canonical automorphism J.

b) Let $\delta := b_1 \cdot \dots \cdot b_n \in \mathcal{C}_M$. Then

$$\delta b_i = (-1)^{n-1} b_i \delta, \qquad \delta^2 = (-1)^{(n-1)n/2} 2^{-n} \det(\mathrm{q}) \in R/(R^\times)^2. \tag{4.7}$$

Further, this special element determines the centre of the Clifford algebra

$$Z(\mathcal{C}_M) = \begin{cases} R & \text{if } n \text{ is even,} \\ R + R\delta & \text{if } n \text{ is odd,} \end{cases}$$

$$Z(\mathcal{C}_M^0) = \begin{cases} R + R\delta & \text{if } n \text{ is even,} \\ R & \text{if } n \text{ is odd.} \end{cases}$$

A sizeable range of examples is found in [Bru+08, p. 132] that is further extended by [Kit99, 1.5 p. 24]

Now that the Clifford algebra has been introduced we turn towards the main application of this construction within the scope of our quest: the construction of the Spin group. Note that for every invertible element $x \in \mathcal{C}_M^\times$ we obtain a linear map

$$v_x : M \to M, \quad m \mapsto xmJ(x)^{-1}.$$

Definition 4.32. Define the *Clifford group* of M as

$$\mathfrak{G}_M := \{x \in \mathcal{C}_M^\times \mid v_x(M) = M\}.$$

The induced representation $v : \mathfrak{G}_M \to \mathrm{Aut}_R(M)$ is referred to as the *vector representation* of $\mathfrak{G}_M$.

Example 4.33. Note that $x \in \mathfrak{G}_M \cap V$ gives rise to the reflection τ_x at the hyperplane $x^\perp$ that had already appeared in Example 4.2. In fact, with $w = rx + sy$ for $y \in x^\perp$, $r, s \in R$, we find

$$v_x(w) = xwJ(x)^{-1} = -rxxx^{-1} - sxyx^{-1} = -rx + syxx^{-1} = \tau_x(w).$$

Lemma 4.34. *Let* $-1 \neq 1 \in R$

a) We find $\ker(v) = R^\times$*.*

b) The mapping $\mathrm{N} : \mathfrak{G}_M \to R^\times$ *defines a homomorphism.*

c) Moreover, $v : \mathfrak{G}_M \to \mathrm{Aut}(M)$ *only targets the subgroup* $\mathrm{O}(M) \subseteq \mathrm{Aut}(M)$*.*

Proof: It suffices to verify that $\ker(v) \subseteq R^\times$. Let $x \in \ker(v)$, then $x = x_0 + x_1 \in \mathcal{C}_M^0 \oplus \mathcal{C}_M^1$ and we have $xmJ(x)^{-1}$ for all $m \in M$. This means

$$x_0 m = m x_0,$$
$$x_1 m = -m x_1.$$

However, these m generate $\mathcal{C}_M$, so that we find $x_0 \in Z(\mathcal{C}_M) \cap \mathcal{C}_M^0 = R$. Also, the second condition can be tested with basis elements $m = b_i$ of M. Then we find $b_i x_1 = x_1 b_i$ and

hence by (4.7) $x_1 = 0$.
b) For $m \in M$ and $x \in \mathfrak{G}_M$, we have $v_x(m) = xmJ(x)^{-1} \in M$ and hence the identity $v_x(m) = -J(v_x(m))^T$. This implies

$$xmJ(x)^{-1} = v_x(m) = -J(v_x(m))^T = (x^T)^{-1}mJ(x^T)$$

which in turn provides

$$m = (x^T x)mJ(x^T x)^{-1} = N(x)mJ(N(x))^{-1} = v_{N(x)}(m).$$

However, the kernel of v had been identified to equal $R^\times$ and hence $N(x) \in R^\times$.
c) Recall that for $x \in \mathfrak{G}_M, m \in M$ we have $v_x(m) \in M$, so that

$$\mathrm{q}(v_x(m)) = N(v_x(m)) = J(x^{-1})^T m^T x^T xmJ(x^{-1}) = \mathrm{q}(m),$$

meaning $v_x \in \mathrm{O}(M)$. □

With these preparations we may define a cover of the special orthogonal group that is regularly used to define orthogonal Shimura varieties, the general *Spin* group. The advantage of using it in order to define varieties in place of the orthogonal group is that the corresponding varieties embed into Siegel varieties and are hence defined over $\mathbb{Q}$, meaning it is not necessary to pass to an extension of $\mathbb{Q}$.

Definition 4.35. The *general Spin* group and *Spin* group of M are defined as

$$\mathrm{GSpin}_M := \mathfrak{G}_M \cap \mathcal{C}_M^0, \tag{4.8}$$

$$\mathrm{Spin}_M := \mathrm{GSpin}_M \cap \ker(\mathrm{N}). \tag{4.9}$$

Remark 4.36. In case $R = K$ is a field of characteristic unequal to 2, we write $V = M$ and note that the orthogonal group is generated by reflections (cf. Theorem 4.7), so that the special orthogonal group is the subgroup of elements given by an even number of reflections. Consequently, we obtain the following exact sequences

$$1 \longrightarrow K^\times \longrightarrow \mathfrak{G}_V \longrightarrow \mathrm{O}(V) \longrightarrow 1,$$

$$1 \longrightarrow K^\times \longrightarrow \mathrm{GSpin}_V \longrightarrow \mathrm{SO}(V) \longrightarrow 1.$$

Remark 4.37. The above exact sequence implies $\mathfrak{G}_V/K^\times \simeq \mathrm{O}(V)$, so by Lemma 4.34, the Clifford norm induces a homomorphism $\mathrm{O}(V) \to K^\times/(K^\times)^2$ which we denote by $\mathcal{N}$ and call the *Spinor norm*.

Example 4.38. We find for an anisotropic vector $x \in V$ and the corresponding reflection τ_x that $\mathcal{N}(\tau_x) \equiv \mathrm{N}(x) \equiv \mathrm{q}(x) \mod (K^\times)^2$.

Remark 4.39. We obtain an exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}_V \xrightarrow{\;v\;} \mathrm{SO}(V) \xrightarrow{\;\mathcal{N}\;} K^\times/(K^\times)^2.$$

Note that if L is indefinite, then $\mathrm{Spin}_{V_\mathbb{R}}$ is connected (cf. [PRR93, Prop. 7.6 p. 407]).

We end this subsection by describing Spin_{L_p} in more concrete terms, following [Att15, p. 20]. Let L be an R lattice of rank m with $R \in \{\mathbb{Z}, \mathbb{Z}_p\}$ and basis $(b_i)_i$. Then the Clifford algebra $\mathcal{C}_L$ has the basis

$$\{b_1^{e_1} \cdot \ldots \cdot b_m^{e_m}, \mid e_i \in \{0,1\}\}$$

as an R module. The involution J identifies with the restriction from $\mathcal{C}_V$ and the $\mathbb{Z}/2\mathbb{Z}$ grading on it descends to $\mathcal{C}_L$. Hence, it is possible to define the spin groups as in Definition 4.35, meaning in explicit terms

$$\mathrm{GSpin}_L := \{x \in \mathcal{C}_L^0 \mid x \cdot J(x) \in R^\times, xLJ(x)^{-1} = L\}, \tag{4.10}$$

$$\mathrm{Spin}_L := \{x \in \mathcal{C}_L^0 \mid x \cdot J(x) = 1, xLJ(x)^{-1} = L\}. \tag{4.11}$$

Eichler transforms

We have already seen some examples of orthogonal transformations constructed via the vector representation of the Clifford group. However, there is another significant class that plays a key role for the existence of Fourier expansions of orthogonal modular forms – Eichler transforms.

Definition 4.40. For $x \in L$ and $y \in V_\mathbb{R}$ with $\mathrm{q}(x) = \mathrm{b}(x,y) = 0$ the vector representation of $1 + xy \in \mathrm{Spin}(V_\mathbb{R})$ is called the *Eichler transform* of x and y and is denoted by $E(x,y)$.

Remark 4.41. In explicit terms, we find for $v \in V_\mathbb{R}$ the following formula

$$E(x,y)(v) = v - \mathrm{b}(v,x)y + \mathrm{b}(v,y)x - \mathrm{q}(y)\,\mathrm{b}(v,x)x. \tag{4.12}$$

In fact, a computation reveals that

$$v_{1+xy}(w) = (1+xy)wJ(1+xy)^{-1}$$

$$= (1 + xy)w(1 - xy)$$
$$= w - wxy + xyw - xywxy.$$

Note that $wxy = (\mathrm{b}(w,x) - xw)y = \mathrm{b}(w,x)y - x(\mathrm{b}(w,y) - yw)$, notice that $[x,y] = 0$ and $xx = \mathrm{q}(x) = 0$ to learn that

$$\begin{aligned} wxy &= (\mathrm{b}(w,x) - xw)y = \mathrm{b}(w,x)y - x(\mathrm{b}(w,y) + yw), \\ xyw &= (\mathrm{b}(x,y) - yx)w, \\ xywxy &= x\,\mathrm{b}(w,x)\,\mathrm{q}(y). \end{aligned}$$

This yields the desired result.

Remark 4.42. If $x, y \in L$, then $E(x,y)$ restricts to an isometry of L. Additionally, $E(x,y)$ also preserves $v \in L'/L$ in this case.

Further, Eichler transforms have the following properties.

Lemma 4.43. *Let x, y be as above.*

a) We find $E(x,y)(x) = x$.

b) We find $E(x,y)(y) = y + 2\,\mathrm{q}(y)x$.

c) For $y_1, y_2 \in x^{\perp}$ we find $E(x,y_1) \circ E(x,y_2) = E(x, y_1 + y_2)$.

4.2 Modular varieties

4.2.1 Models

A fundamental reference for the following is [Bru02, p. 78], while [Bru+08, 2.4 p. 136] provides a brief and accessible overview.

This section will explore various ways in which orthogonal spaces that serve as domains for automorphic forms can be realised. As previously stated, (V, q) denotes a regular rational quadratic space of signature (m^+, m^-). Let $G = \mathrm{O}(m^+, m^-)$ denote the orthogonal group of the real quadratic vector space $V_{\mathbb{R}} = V \otimes \mathbb{R}$. A canonical candidate for a domain to study modular forms on is the symmetric domain G/K, where K is a maximal compact subgroup. The resulting symmetric space is hermitian, if, and only if, either m^+ or m^- is equal to 2. As a consequence, we shall restrict to assuming $m^- = 2$ from Subsection 4.2.2 onwards.

The Grassmannian

We begin with a model providing a conveniently simple descritpion, the Grassmannian. It is a special case of a Flag manifold.

Definition 4.44. Let $V_\mathbb{R}$ be a real quadratic space of signature $(m^+, 2)$. Then its Grassmannian is defined to be

$$\mathbb{D}(V_\mathbb{R}) \coloneqq \{z \subseteq V_\mathbb{R} \mid \dim(z) = 2 \text{ and } \mathrm{q}\,|_z < 0\}\,. \tag{4.13}$$

We would like to realise $\mathbb{D}$ as a quotient G/K for some maximal compact $K \leq G$. Note that by Witt's extension theorem (Theorem 4.6), the group $\mathrm{O}(V_\mathbb{R})$ acts transitively on $\mathbb{D}$ and that any real orthogonal group associated to a definite space is compact (cf. Lemma 4.9). So fix $z_0 \in \mathbb{D}$, then $K \coloneqq \mathrm{O}(z_0) \times \mathrm{O}(z_0^\perp) \hookrightarrow \mathrm{O}(V_\mathbb{R}) = G$ is included in the stabiliser of z_0. However, an element of $\mathrm{O}(V_\mathbb{R})$ leaving z_0 invariant, must also leave $z_0^\perp$ invariant, so that K is, in fact, identified to equal the stabiliser of z_0. Additionally, the group K is maximally compact among subgroups of G. Further, $G/K \simeq \mathbb{D}$, so that the Grassmannian might be equipped with a symmetric structure, inhertited from the quotient. Nevertheless, the clear advantage of the model is the apparent description, while the models below provide different additional insights.

The projective Model

Another model is induced through projective space and called the *projective model.* It offers the advantage of an immediate complex structure. Consider $V_\mathbb{C} = \mathbb{C} \otimes_\mathbb{Q} V$, which induces a projective variety

$$P(V_\mathbb{C}) = (V_\mathbb{C} \setminus \{0\})/\mathbb{C}^\times.$$

Equipping this space with the final topology of the natural projection results in a compact space. Further, it naturally carries the structure of a complex analytic manifold. In fact, it is the standard example of such a space. We consider the closed subvariety

$$\mathscr{N} \coloneqq \{[Z] \in P(V_\mathbb{C}) \mid \mathrm{b}(Z, Z) = 0\}\,,$$

which, as a consequence, bears an analytic structure. It is called *zero quadric.* Note that elements $Z \in V_\mathbb{C}$ may be decomposed into $X + iY$ for $X, Y \in V_\mathbb{R}$, where the elements X, Y are referred to as the *real* and *imaginary part* of the vector z. As a consequence, we find

$$\mathrm{b}(X+iY, X \pm iY) = \mathrm{b}(X,X) \mp \mathrm{b}(Y,Y) + i\left(\mathrm{b}(X,Y) \pm \mathrm{b}(X,Y)\right),$$

yielding $\mathrm{b}(Z,\overline{Z}) = \mathrm{b}(X,X) + \mathrm{b}(Y,Y)$ and that $\mathrm{b}(Z,Z) = 0$ is equivalent to $\mathrm{b}(X,X) = \mathrm{b}(Y,Y)$ and $\mathrm{b}(X,Y) = 0$. We consider the open subset of the zero quadratic

$$\mathcal{K} := \left\{[Z] \in P(V_{\mathbb{C}}) \mid \mathrm{b}(Z,Z) = 0, \mathrm{b}(Z,\overline{Z}) < 0\right\}$$

which inherits the complex structure of $\mathscr{N}$. Note that the orthogonal group $\mathrm{O}(V_{\mathbb{R}})$ acts, by Witt's extension theorem (cf. Theorem 4.6), transitively on $\mathcal{K}$.

In the following, we reduce to the case $m^- = 2$. Then the space $\mathcal{K}$ may be explicitly related to the Grassmannian.

Proposition 4.45. *The assignment $\mathcal{K} \ni [Z] \mapsto \mathbb{R}X + \mathbb{R}Y \in \mathbb{D}$ defines a continuous and open two to one map.*

Proof: In fact, the real part X and imaginary part Y of a representative Z of $[Z] \in \mathcal{K}$ fulfil $\mathrm{q}(X) = \mathrm{q}(Y) < 0$ and $\mathrm{b}(X,Y) = 0$, as we have computed above. As a consequence, they form an orthogonal normed basis of an element of $\mathbb{D}$. Note that the pair $(X,-Y)$ associated to $\overline{Z}$ gives rise to the same element in $\mathbb{D}$, so that the mapping is at least two to one, where the ambiguity of the cover is given by the orientation of the basis that is considered. If there was another preimage, it had to arise from (X,Y) via an orthogonal transformation. If the transformation was special, it reduced to multiplying Z with an element of $\mathbb{T}$, acting as the identity on $\mathscr{N}$, if it was not special, it reduced to $(X,Y) \mapsto (X,-Y)$.
Checking the topological conditions is left to the reader and plays no major role in the following exposition. □

In fact, the space $\mathcal{K}$ decomposes into two connected components which agree with the sheets of the twofold cover above. We fix one of these sheets and denote it by $\mathcal{K}^+$, the other by $\mathcal{K}^-$. Then the elements of $\mathrm{O}(V_{\mathbb{R}})$ preserving $\mathcal{K}^+$ form a subgroup $\mathrm{O}^+(V_{\mathbb{R}}) < \mathrm{O}(V_{\mathbb{R}})$. This may be characterised as follows: recall that the Spinor norm $\mathcal{N} : \mathrm{O}(V_{\mathbb{R}}) \to \{\pm 1\}$ defines a homomorphism (cf. Remark 4.37) so that its product with the determinant map $\mathcal{N} \cdot \det$ represents a homormophism, as well. The kernel of this map equals $\mathrm{O}^+(V_{\mathbb{R}})$, which is open and closed in $\mathrm{O}(V_{\mathbb{R}})$. Its complement $\mathrm{O}(V_{\mathbb{R}}) \setminus \mathrm{O}^+(V_{\mathbb{R}})$ consists of these orthogonal transformations that interchange $\mathcal{K}^+$ and $\mathcal{K}^-$.

The projective model provides a natural complex structure that may be exported to the Grassmannian $\mathbb{D}$ via Proposition 4.45. In the next section, we identify $\mathcal{K}$ with a tube domain.

The tube domain model

The last model that is to be discussed is the tube domain model, reminiscent of the upper half plane. In the following, we restrict to the hermitian case of $m^+ = 2$. We choose a nonzero isotropic vector $e_1 \in V$ and some $e_2 \in V$ with $\mathrm{b}(e_1, e_2) = 1$.[1] Next, we define the quadratic space $W = e_1^\perp \cap e_2^\perp$ which is of type $(n-1, 1)$ (meaning it is Lorentzian), so that

$$V = W \oplus \mathbb{Q}e_1 \oplus \mathbb{Q}e_2. \tag{4.14}$$

We write $Z = (z, a, b) \in V_\mathbb{C}$ for an element in the associated complex space given by $z + ae_1 + be_2$ with $z \in W_\mathbb{C}$.

Assume, we wanted to construct an element $[Z] \in \mathcal{K}$ from some $z \in W_\mathbb{C}$. We recall that the conditions on such a representative Z for inducing an element in the zero quadric $\mathscr{N}$ imply that for the decomposition $Z = X + iY$ with $X, Y \in V_\mathbb{R}$ we must have that X, Y have equal negative weight and span a 2 dimensional subspace of $V_\mathbb{R}$. Hence, one of these must have an e_2-component, meaning $\mathrm{b}(Z, e_1) \neq 0$. This prompts us to set $(z, a, 1)$ as a candidate for Z, with a complex number $a \in \mathbb{C}$ which has yet to be determined. If we now require $[z + ae_1 + e_2] \in \mathscr{N}$, we obtain by definition

$$\begin{aligned} 0 = {} & \mathrm{b}((z, a, 1), (z, a, 1)) \\ & = 2\,\mathrm{q}(z) + 2a + 2\,\mathrm{q}(e_2), \end{aligned}$$

yielding $a = -\,\mathrm{q}(z) - \mathrm{q}(e_2)$. We realise that e_2 could without loss of generality have been assumed to fulfil $\mathrm{q}(e_2) = 0$, by replacing it with $e_2 - \mathrm{q}(e_2)e_1$, since $\mathrm{q}(e_2 - \mathrm{q}(e_2)e_1) = 0$ and $\mathrm{b}(e_1, e_2 - \mathrm{q}(e_2)e_1) = \mathrm{b}(e_1, e_2)$. If we write $z = x + iy$ with $x, y \in W_\mathbb{R}$, again, we find for the real and imaginary part

$$(z, -\,\mathrm{q}(z) - \mathrm{q}(e_2), 1) = (x, \mathrm{q}(y) - \mathrm{q}(x) - \mathrm{q}(e_2), 1) + i(y, \mathrm{b}(x, y), 0).$$

[1] Should the rational space V be not isotropic, we may perform the construction in the real setting.

Now the second condition to lie in $\mathcal{K}$ reads

$$\begin{aligned} 0 > \; & \mathrm{b}((z, -\mathrm{q}(z) - \mathrm{q}(e_2), 1), (\overline{z}, -\overline{\mathrm{q}(z)} - \mathrm{q}(e_2), 1)) \\ & = 2(\mathrm{q}(x) + \mathrm{q}(y)) + 2(-\mathrm{Re}(\mathrm{q}(z)) - \mathrm{q}(e_2)) + 2\,\mathrm{q}(e_2) \\ & = 4\,\mathrm{q}(y). \end{aligned} \tag{4.15}$$

Meaning lying in $\mathcal{K}$ is equivalen to $\mathrm{q}(y) < 0$. As a consequence, we restrict to the subspace

$$\mathcal{H}^{\pm}_{m^+} := \{z = x + iy \in W_{\mathbb{C}} \mid \mathrm{q}(y) < 0\}, \tag{4.16}$$

which is an analogy of the union of the upper and lower halfplane. We see in particular, that the constructed association is injective, as $(z, -\mathrm{q}(z) - \mathrm{q}(e_2), 1)$ is determined by z. In order to recall that given $[Z] \in \mathcal{K}$, we had seen that Z must have a nonzero e_2 component, so that we may without loss of generality pick a representative $Z = (z, a, 1)$ for some $a \in \mathbb{C}$. Above we had seen that this already implies $a = -\mathrm{q}(z) - \mathrm{q}(e_2)$, which results in a unique element $z \in \mathcal{H}^{\pm}$. In conclusion, we have proven the bijectivity result of the following proposition.

Proposition 4.46. *The map*

$$\mathcal{H}^{\pm}_{m^+} \ni z \mapsto [(z, -\mathrm{q}(z) - \mathrm{q}(e_2), 1)] \in \mathcal{K}$$

defines a biholomorphic association.

Definition 4.47. Define $\mathcal{H}^{+}_{m^+}$, to be the preimage of $\mathcal{K}^+$ under the above isomorphism as the *generalised upper half plane* or *tube domain model*.

To emphasise the analogy to the upper half plane model in the elliptic case, we set

$$\mathcal{C} := \{y \in W_{\mathbb{R}} \mid \mathrm{q}(y) < 0\}$$

and note that it possesses two connected components, each of which forms a cone. One component is given by

$$\mathcal{C}^+ := \mathcal{H}^{+}_{m^+} \cap i\mathcal{C},$$

so that

$$\mathcal{H}^{+}_{m^+} = W_{\mathbb{R}} + i\mathcal{C}^+. \tag{4.17}$$

In the following the superscript $+$ of the tube domain model will frequently be omitted.

The action of $\mathrm{O}^+(V_\mathbb{R})$ on $\mathcal{K}^+$ may be pulled back to an action on the tube domain $\mathcal{H}_{m^+}$.

4.2.2 Modular groups

Similarly to the classic case of elliptic modular forms, the base space will be constructed as a quotient of G/K by an arithmetic subgroup of $\mathrm{O}(V_\mathbb{R})$ and compactified. In the following, we consider lattices which comprise a central class of objects in this thesis as a foundation for the construction. Let (L, q) be an even $\mathbb{Z}$ lattice of signature (m^+, m^-). Recall that $V = \mathbb{Q} \otimes_\mathbb{Z} L$ is the associated rational quadratic space.

Note that $\mathrm{O}(L)$ also acts on L'. In fact, let $\ell \in V$, $\gamma \in \mathrm{O}(L)$ and (b_i) be a basis of L. Then (γb_i) also constitutes a basis of L. Hence, we may test $\gamma\ell$ against (γb_i) for integrality. Writing ℓ in the basis (b_i) yields that $\gamma\ell \in L'$ if, and only if, $\ell \in L'$. As a consequence, we obtain a natural projection

$$\mathrm{O}(L) \to \mathrm{Aut}(L'/L). \tag{4.18}$$

Of course, the same is true for any subgroup of $\mathrm{O}(L)$, in particular the following definition is meaningful.

Definition 4.48. Let $H \leq \mathrm{O}(L)$ be a finite index subgroup. Then the kernel of the natural map obtained from restricting (4.18)

$$H \hookrightarrow \mathrm{O}(L) \to \mathrm{Aut}(L'/L) \tag{4.19}$$

is called the *discriminant kernel* subgroup of H. It is denoted H_L. In case of the choice $H = \mathrm{O}^+(L)$ or $\mathrm{SO}^+(L)$, also write $\Gamma(L) = H_L$ or $\Gamma_L = H_L$.

Typically, the group H is chosen to be $\mathrm{O}^+(L) := \mathrm{O}(L) \cap \mathrm{O}^+(V)$, the subgroup which preserves orientation or $\mathrm{SO}^+(L) = \mathrm{SO}(L) \cap \mathrm{O}^+(V)$. These cases will be assumed if not specified further.
Note that the group $\Gamma(L)$ has finite index in $\mathrm{O}(L)$. Next, recall that for any $d \in \mathbb{Z} \setminus \{0\}$ the lattice $L(d)$ was defined as $(L, d \cdot \mathrm{q})$, so that $(L(d))' = (1/d)L'$. The orthogonal group $\mathrm{O}(L(d))$ agrees with $\mathrm{O}(L)$ if embedded canonically into $\mathrm{GL}(V_\mathbb{R})$.[2] Further, we find the embedding $L'/L \hookrightarrow L(d)'/L(d) = L(d)'/L$. Clearly, any element $\sigma \in \mathrm{Aut}(L(d)'/L)$ induced from $\mathrm{O}(L)$ restricts to an element $\sigma' \in \mathrm{Aut}(L(d_2)'/L)$ for $d_2 \mid d$. As a special case, any element in the kernel of $\mathrm{O}(L) \to \mathrm{Aut}(L(d)'/L)$ also lies in the kernel of $\mathrm{O}(L) \to$

[2] To do so, realise $L(d)$ as $(\sqrt{d} \cdot L, \mathrm{q})$.

$\mathrm{Aut}(L(d_2)'/L)$, meaning there is the following inclusion

$$\Gamma(L(d)) \leq \Gamma(L(d_2)) \leq \Gamma(L) \leq \mathrm{O}^+(L) \leq \mathrm{O}(L) \tag{4.20}$$

of subgroups of finite index.

Example 4.49. Consider the case of a hyperbolic plane H presented in Example 1.7. In the following, for a nonzero integer d, the discriminant kernel $\Gamma(H(d))$ is computed. Assume we had an orthogonal transformation $\gamma \in \mathrm{O}(H)$, described in coordinates as $\gamma = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$. Then we find

$$\gamma^T H \gamma = \begin{pmatrix} 2\,b_1 b_3 & b_2 b_3 + b_1 b_4 \\ b_2 b_3 + b_1 b_4 & 2\,b_2 b_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

From this equation we infer that in every column of the matrix representation of γ, an entry must vanish. Further, a whole row cannot vanish, as the side diagonal entries were contradicted (and bijectivity of γ). So either $b_1 = b_4 = 0$ or $b_2 = b_3 = 0$. Then the side diagonals read $b_1 b_4 = 1$ or $b_2 b_3 = 1$. Since the entries will have to be integers, this may be reformulated as $b_1 = b_4 = \pm 1$ and $b_2 = b_3 = \pm 1$. So the only orthogonal matrices are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

This group is clearly generated by the matrices $-\mathcal{I}, H$.
Recall that H was unimodular, so that $L(d)' = \mathbb{Z}/d \cdot e_1 + \mathbb{Z}/d \cdot e_2$ and $L(d)'/L(d) = L(d)'/L \simeq \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$ where the latter carries the quadratic form $(x, y) \mapsto xy/d \in \mathbb{Q}/\mathbb{Z}$. Hence, for any $d \in \mathbb{Z}$, the automorphisms induced by $\mathrm{O}(L(d)) = \mathrm{O}(L)$ in $\mathrm{Aut}(L(d)'/L)$ are generated by the images of

$$\begin{aligned} -\mathcal{I} :& (x, y) \mapsto (-x, -y) \in \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}, \\ H :& (x, y) \mapsto (y, x) \in \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}. \end{aligned}$$

This results in the following discriminant kernel:

$$\Gamma(H(d)) = \begin{cases} \{\pm\mathcal{I}, \pm H\}, & d = \pm 1, \\ \{\pm\mathcal{I}\}, & d = \pm 2, \\ \{\mathcal{I}\}, & |d| > 2. \end{cases} \tag{4.21}$$

Remark 4.50. The discriminant kernel is, other than the orthogonal group, functorial in the lattice. Explicitly, if there are two lattices $M \leq L$, then $\Gamma(M) \leq \Gamma(L)$. For the identification of elements in $\mathrm{O}(L)$ and $\mathrm{O}(M)$, these are embedded into $\mathrm{O}(L \otimes_{\mathbb{Z}} \mathbb{Q})$.

Proof: Denote $V = L \otimes_{\mathbb{Z}} \mathbb{Q} = M \otimes_{\mathbb{Z}} \mathbb{Q}$ and let $\gamma \in \Gamma(M) \subset \mathrm{O}(V)$. Then γ acts trivially on M'/M. In particular, it acts trivially on the subgroup $L/M \leq M'/M$, meaning it must act on L, so that $\gamma \in \mathrm{O}(L)$. However, the trivial action on M'/M is preserved under pushforward via $M'/M \to M'/L$. Conclusively, the element γ lies in $\mathrm{O}(L)$ and acts trivially on M'/L and hence on the subgroup $L'/L \leq M'/L$. Intersection with $\mathrm{O}^+(V_{\mathbb{R}})$ does not interfere with any of the presented arguments, implying $\Gamma(M) \leq \Gamma(L)$. □

Note that $\Gamma(M)$ in the above Remark has finite index in $\Gamma(L)$ which follows from the fact that it must contain $\Gamma(L(d))$ for some $d \in \mathbb{N}$, again by Remark 4.50. In analogue to the elliptic case, this prompts us to define the notion of a congruence subgroup.

Definition 4.51. Let L be as above and $d \in \mathbb{N}$ be a natural number. Then $\Gamma(L(d))$ is called its d-th *principal congruence* subgroup. Further, a subgroup $\Gamma \leq \mathrm{O}^+(L)$ is called a *congruence subgroup* with respect to L, if there is some $d \in \mathbb{N}$ such that $\Gamma(L(d)) \leq \Gamma$.

Note that all of these groups have finite index in $\mathrm{O}(L)$ and that the notion of congruence subgroups depends solely on the associated rational quadratic space $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$.

Remark 4.52. For the hermitian case of $m^- = 2$, there is a theorem stating that the opposite is also true, provided $m^+ \geq 3$. We will see below that this theorem does not hold in case of $m^+ = 1$.

In fact, there are some accidental isomorphisms that realise symplectic cases via orthogonal groups. Most notably the elliptic case may be realised via the $\mathrm{O}(1,2)$ setting.

Example 4.53. Assume V has signature $(1,2)$, then the associated real space has quadratic form $x_1 x_2 - x_0^2$ and we may realise it as

$$V_{\mathbb{R}} = \left\{ \begin{pmatrix} x_1 & x_0 \\ x_0 & x_2 \end{pmatrix} \,\middle|\, x_i \in \mathbb{R} \right\},$$

so that $\mathrm{q} = \det$. Then for $g \in \mathrm{GL}_2(\mathbb{R})$ the transformation $V \ni v \mapsto gvg^T$ is orthogonal as long as $\det(g) = \pm 1$ is presumed. The corresponding natural association defines an isogeny with kernel $\pm\mathcal{I}$ which descends to an isogeny

$$\mathrm{SL}_2(\mathbb{R}) \to \mathrm{O}^+(V_{\mathbb{R}}).$$

The image has index 2 and adding the reflection $v \mapsto -v$ to the image generates the whole group. The association

$$\mathbb{P}^1(\mathbb{C}) \ni \tau \mapsto \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & \text{if } \tau = \infty, \\ \begin{pmatrix} \tau^2 & \tau \\ \tau & 1 \end{pmatrix}, & \text{else,} \end{cases}$$

defines a biholomorphic map onto the zero quadric $\mathscr{N} \subset \mathbb{P}(V_{\mathbb{C}})$. The induced action of $\mathrm{PSL}_2(\mathbb{R})$ is given by Möbius transformations. In fact, let $\gamma \in \mathrm{SL}_2(\mathbb{R})$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \tau^2 & \tau \\ \tau & 1 \end{pmatrix} \cdot \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} (b+a\tau)^2 & (b+a\tau)(d+c\tau) \\ (b+a\tau)(d+c\tau) & (d+c\tau)^2 \end{pmatrix}$$
$$\simeq \begin{pmatrix} (\gamma\tau)^2 & \gamma\tau \\ \gamma\tau & 1 \end{pmatrix}.$$

The space $\mathcal{K}$ in that model identifies with the upper and lower half plane, where $\mathcal{K}^+$ represents exactly the upper (or lower) half plane. In fact, the condition to lie in $\mathcal{K}$, namely $\mathrm{b}(\tau, \overline{\tau}) < 0$, reads

$$0 > \left(\tau, \tau^2, 1\right) \begin{pmatrix} -2 & & \\ & & 1 \\ & 1 & \end{pmatrix} \begin{pmatrix} \overline{\tau} \\ \overline{\tau^2} \\ 1 \end{pmatrix} = -2|\tau|^2 + \tau^2 + \overline{\tau}^2 = -4y^2 \iff y \neq 0.$$

The discriminant kernel of L may also be realised over GSpin_L. In fact, we have the following theorem.

Theorem 4.54 ([Att15, Cor 2.5.3])**.** *Let (L, q) be a non-degenerate R-lattice. Then the image of $\mathrm{GSpin}(L)$ in $\mathrm{SO}(L)$ under the natural map equals $\Gamma(L)$, which is the discriminant kernel of $\mathrm{SO}^+(L)$, where $\mathrm{SO}^+(L)$ denotes the intersection of $\mathrm{SO}(L)$ with $\mathrm{SO}^+(L \otimes \mathbb{R})$ if $R = \mathbb{Z}$ or $\mathrm{SO}^+(L \otimes \mathbb{Q}_p)$ if $R = \mathbb{Z}_p$.*

We will often consider the following variety in analogue to the classical case.

Definition 4.55. Let $\Gamma \leq \Gamma(L)$ be a subgroup of finite index. Then denote $\Gamma\backslash\mathbb{D}$ by Y_Γ. In case of $\Gamma = \Gamma(L)$, we also write $Y_L = Y_{\Gamma(L)}$.

4.2.3 Boundary components

Additional background material on the subject is found in the base source [BF01] for this subsection and [Kie21].

Throughout this section we shall assume $m^- = 2$, so that L is a lattice of signature $(m^+, 2)$. Boundary components of G/K are represented by isotropic spaces and the projective model is relatively accessible for the purpose of describing boundary components. Hence, we consider the boundary of $\mathcal{K}^+$ in the zero quadric $\mathscr{N}$.

In fact, the boundary points are described as follows. If a point in $[Z] \in \mathscr{N}$ should be a boundary point of $\mathcal{K}$, its representative $Z = X + iY$ has to fulfil $\mathrm{q}(X) = \mathrm{q}(Y) = 0$ and $\mathrm{b}(X, Y) = 0$. This might happen, if the vectors X and Y are collinear or not, giving rise to a one dimensional or respectively two dimensional isotropic subspace. In fact, X and Y are collinear if, and only if, $[X + iY]$ may be represented by a real vector (namely either X or Y). Otherwise, there is no real representation. These points are then collected into components.

Definition 4.56. We have the following classification of boundary points.

a) If the space spanned by X, Y is one dimensional, we call $[Z]$ a *special boundary point*.

b) If the space spanned by X, Y is two dimensional, we call $[Z]$ a *generic boundary point*.

c) A set consisting of one special boundary point is called zero dimensional *boundary component*.

d) For an isotropic plane $I \subset V_{\mathbb{R}}$, the collection of generic boundary points which may be represented by elements in $I \otimes_{\mathbb{R}} \mathbb{C}$ is called one dimensional *boundary component*.

The reason for bundling these points is due to the following structural result about one dimensional boundary components.

Remark 4.57. There is a bijective correspondence between boundary components and isotropic subspaces of $V_{\mathbb{R}}$. Moreover, one dimensional boundary components are isomorphic to upper half planes $\mathbb{H}$.

In fact, the image of $I \otimes_{\mathbb{R}} \mathbb{C}$ in $\mathscr{N}$ for a two dimensional isotropic space $I \subset V(\mathbb{R})$ may be identified with $\mathbb{H} \cup \mathbb{R} \cup \{\infty\}$ as in the classical case.

Definition 4.58. A boundary component is called *rational*, if the corresponding isotropic space I is defined over $\mathbb{Q}$, meaning for $[Z] \in \mathscr{N}$ that there is a representative $Z = X + iY$, such that $X, Y \in V_{\mathbb{Q}} \subset V_{\mathbb{R}}$.

Example 4.59. Assume $V_{\mathbb{Q}}$ is anisotropic and $\Gamma < \mathrm{O}^+(V)$ is commensurable with $\mathrm{O}(L)$. Then the rational boundary of Y_Γ is empty and $X_\Gamma = Y_\Gamma$ is compact.

4.2.4 Modular forms

The following content is essentially contained in [Bru02, 3.3 p. 84]. Another reference providing a brief overview is [Kie22].

Similar to the elliptic case, we will describe modular forms, which may be interpreted as global sections of a hermitian line bundle over the modular variety $Y(\Gamma)$, as functions on the cover G/K of that space with a certain transformation property. In order to state this transformation property analogously to the classical case, we are required to introduce the factor of automorphy in the orthogonal setting.

Recall that there is a natural action of $\mathrm{O}^+(V_{\mathbb{R}})$ on the projective model $\mathcal{K}$ of G/K, which is transferred by the biholomorphic association discussed above to the tube domain model $\mathcal{H}_{m^+}$. We use the notation of the preceding subsections and select for $z \in \mathcal{H}_{m^+}$ a specific representative of its corresponding element $[Z] \in \mathcal{K}^+$ following Proposition 4.46. In fact, selecting $V_{\mathbb{C}} \ni \tilde{z} = z - (\mathrm{q}(z) + \mathrm{q}(e_2))e_1 + e_2$ gives rise to an embedding of $\mathcal{H}_{m^+}$ into the preimage of $\mathcal{K}^+$ in $V_{\mathbb{C}}$. The latter is exactly the cone over the chosen embedding and will be denoted $\tilde{\mathcal{K}}^+$.

Starting from the perspective of the cone, it is relatively straightforward how to define modular functions in analogy to the classical case:

Definition 4.60. A function $F : \tilde{\mathcal{K}}^+ \to \mathbb{C}$ is called *modular* of weight $r \in \mathbb{Z}$ with respect to a congruence subgroup $\Gamma \leq \Gamma(L)$ if it satisfies the following two properties.

a) For all $t \in \mathbb{C}^\times$ the function is homogeneous of degree $-r$, meaning $F(tZ) = t^{-r}F(Z)$.

b) For all $\gamma \in \Gamma(L)$ the function is invariant under pullback: $F(\gamma Z) = F(Z)$.

From that definition, it is immediate how to generalise it to modular functions with a character. In fact, let $\chi : \Gamma \to \mathbb{T}$ ba a character, then b) may be generalised to read $F(\gamma Z) = \chi(\gamma)F(Z)$. In order to translate that action to the function on the embedded tube domain $\mathcal{H}_{m^+}$, we need to introduce the factor of automorphy for which we will have to consider the action of $\mathrm{O}^+(V_{\mathbb{R}})$ that is transferred from the canonical action on $\mathcal{K}$. We find for $\gamma \in \mathrm{O}^+(V_\mathbb{R})$ and $z \in \mathcal{H}_{m^+}$ with corresponding lift $\tilde{z} \in \tilde{\mathcal{K}}^+$ (cf. Proposition 4.46) that[3]

$$\gamma(\tilde{z}) = z' - ae_1 + b(e_2 - \mathrm{q}(e_2)e_1)$$

for some $a, b \in \mathbb{R}$ and another $z' \in \mathcal{H}_{m^+}$. By definition, we obtain via normalising that the preimage of $[\gamma(\tilde{z})]$ in $\mathcal{H}_{m^+}$ equals

$$\gamma(z) = b^{-1}z' \qquad \text{where} \qquad b = \mathrm{b}(\gamma(\tilde{z}), e_1).$$

If we repeat the procedure by letting another $\gamma_2 \in \mathrm{O}^+(V_\mathbb{R})$ act on z', we find that the new prefactor b' (in place of b) fulfils a cocycle relation. This prompts us to define.

Definition 4.61. Let $\gamma \in \mathrm{O}^+(V_\mathbb{R})$ and $z \in \mathcal{H}_{m^+}$ corresponding to $\tilde{z}$ as above. Then the *factor of automorphy* is defined to be

$$j(\gamma, z) := \mathrm{b}(\gamma(\tilde{z}), e_1).$$

Lemma 4.62. *The factor of automorphy fulfils*

$$j(\gamma, z)\widetilde{(\gamma z)} = \gamma(\tilde{z}) \tag{4.22}$$

as well as the cocycle relation

$$j(\gamma_1\gamma_2, z) = j(\gamma_1, \gamma_2 z)j(\gamma_2, z). \tag{4.23}$$

In addition, the imaginary part of an element $z \in \mathcal{H}_{m^+}$ transforms with the factor of automorphy in analogy to the elliptic case presented in (2.12).

Remark 4.63. For $\gamma \in \mathrm{O}^+(V_\mathbb{R})$ and $z \in \mathcal{H}_{m^+}$, we have the following transformation property

$$\mathrm{q}(\mathrm{Im}(\gamma z)) = \frac{\mathrm{q}(\mathrm{Im}(z))}{|j(\gamma, z)|^2} \tag{4.24}$$

Proof: By (4.15) reading $4\,\mathrm{q}(y) = \mathrm{b}(\tilde{z}, \overline{\tilde{z}})$ and (4.22) we obtain

$$\begin{aligned}\mathrm{q}(\mathrm{Im}(\gamma z)) &= \frac{1}{4}\left(\frac{\gamma(\tilde{z})}{\mathrm{b}(\gamma\tilde{z}, e_1)}, \frac{\gamma(\overline{\tilde{z}})}{\mathrm{b}(\gamma\overline{\tilde{z}}, e_1)}\right) \\ &= \frac{1}{4|\mathrm{b}(\gamma\tilde{z}, e_1)|^2}\left(\gamma\tilde{z}, \gamma\overline{\tilde{z}}\right)\end{aligned}$$

[3] Recall that we may replace e_2 by $e_2 - \mathrm{q}(e_2)e_1$ to guarantee $\mathrm{q}(e_2) = 0$.

$$= \frac{\mathrm{q}(\mathrm{Im}(z))}{|\mathrm{b}(\gamma\tilde{z}, e_1)|^2}.$$

□

In the light of Lemma 4.62, we may translate Definition 4.60 to the tube domain model. To be explicit, let

$$F_{\mathcal{H}}(z) : \mathcal{H}_{m^+} \to \mathbb{C}, \qquad z \mapsto F(\tilde{z}) = F(z - (\mathrm{q}(z) + \mathrm{q}(e_2))e_1 + e_2).$$

Then $F_{\mathcal{H}}$ satisfies the following transformation property for $\gamma \in \Gamma$ (cf. (4.22))

$$\begin{aligned} F_{\mathcal{H}}(\gamma z) &= F(\widetilde{\gamma z}) \\ &= F(j(\gamma, z)^{-1}\gamma\tilde{z}) \\ &= j(\gamma, z)^r \chi(\gamma) F(\tilde{z}) \\ &= j(\gamma, z)^r \chi(\gamma) F_{\mathcal{H}}(z) \end{aligned} \tag{4.25}$$

and we see that there is a bijective correspondence between such functions.

Technically, (4.25) already yields the notion of a modular function on $\mathcal{H}_{m^+}$. However, as we will consider the Borcherds lift in Section 7.1, we require a more general notion of modular forms than these; namely to characters and have to introduce multiplier systems. To this end, let $\ln(j(\gamma, z))$ denote a fixed logarithm of $j(\gamma, z)$. For a rational number $r \in \mathbb{Q}$, set

$$j(\gamma, z)^r := \exp\left(r \cdot \ln(j(\gamma, z))\right).$$

Then there is a map $w_r : \mathrm{O}^+(V_{\mathbb{R}}) \times \mathrm{O}^+(V_{\mathbb{R}}) \to \mathbb{T}$ restricting to the roots of unity of order bounded by $\mathrm{den}(r)$ such that

$$j(\gamma_1\gamma_2, Z)^r = w_r(\gamma_1, \gamma_2) \cdot j(\gamma_1, \gamma_2 z)^r j(\gamma_2, z)^r$$

which only depends on $r \mod \mathbb{Z}$. With this preparation, we may replace the notion of a character with a more general notion, satisfying the same transformation property to cancel the above factor when transforming.

Definition 4.64. Let $\Gamma \leq \mathrm{O}^+(V_{\mathbb{R}})$ and $r \in \mathbb{Q}$. A multiplier system of weight r for Γ is a map

$$\chi : \Gamma \to \mathbb{T}, \qquad \text{s.t.} \qquad \chi(\gamma_1\gamma_2) = w_r(\gamma_1, \gamma_2)^{-1} \cdot \chi(\gamma_1)\chi(\gamma_2).$$

With the notion of a multiplier system, modular forms on the tube domain $\mathcal{H}_{m^+}$ may be defined. For simplicity we assume $m^+ \geq 3$ first, in order to simplify to the following definition.

Definition 4.65. Assume $m^+ \geq 3$. Let $\Gamma \leq \Gamma(L)$ be a subgroup of finite index and χ be a multiplier system for Γ of weight $r \in \mathbb{Q}$. A holomorphic function F on $\mathcal{H}_{m^+}$ is called a *holomorphic modular function* of weight r and multiplier system χ with respect to Γ, if it fulfils

$$F(\gamma z) = \chi(\gamma) j(\gamma, z)^r F(z) \tag{4.26}$$

for all $\gamma \in \Gamma$. Write $\mathcal{M}_r(\Gamma, \chi)$ for the space of such functions.

Recall that in the classical case, there has been a growth condition towards the boundary. In case of a holomorphic modular function this condition is superfluous by the Köcher principle (cf. [Bru02, Prop. 4.15 p. 109]) in case of $m^+ \geq 3$. If $m^+ = 2$, however, the situation is more convoluted and we would like to briefly touch on how to proceed. Assuming $m^+ = 2$ and that there is no isotropic element in V, the quotient $\Gamma \backslash \mathbb{D}$ is compact and we may stick with Definition 4.60 for integral weight $r \in \mathbb{Z}$ and impose holomorphicity or meromorphicity.[4] In case there is an isotropic element, we are required to speak about vanishing at the cusps. For this instance, we sketch how to derive a Fourier expansion of such functions.[5]

Recall for $x, y \in L$ the Eichler transform $E(x, y)$ of Definition 4.40. Then by Remark 4.42, we find that the Eichler transform $E(x, y) \in \operatorname{Aut}(L)$ is contained in $\Gamma(L)$. In fact, for $d \in \mathbb{N}$ and $y \in d \cdot L$, we even find $E(x, y) \in \Gamma(L(d))$. These transformations may be used to associate a Fourier expansion to a modular function. For that purpose, recall the construction of the tube domain model, in particular of the subspace $W_{\mathbb{R}} < V_{\mathbb{R}}$ whose rational points are described in (4.14). Instead of constructing W directly as a vector space, we may, as long as $V_{\mathbb{Q}}$ is isotropic, select a primitive isotropic vector $e_1 \in L$ and choose $e_2 \in L'$ such that $\mathrm{b}(e_1, e_2) = 1$. Then the lattice $M := L \cap e_1^{\perp} \cap e_2^{\perp}$ is a Lorentzian, i.e. it has signature $(m^+ - 1, 1)$. If we set $W_{\mathbb{R}} := K \otimes \mathbb{R}$, we obtain the same construction as for the tube domain model. The difference is that this time we have a lattice $M \subset L$ as a base of the construction.
Select $z \in \mathcal{H}_{m^+}$ with associated element $\tilde{z} = z + ae_1 + e_2 \in V_{\mathbb{C}}$.

[4] In case of non-integral weight $r \in \mathbb{Q} \setminus \mathbb{Z}$ one has to consider covers of $\mathcal{K}^+$, based on the denominator or r.
[5] This is also valid in case of $m^+ \geq 3$, as there will always be isotropic vectors by Meyer's theorem.

Next, notice that for $l \in M$ we find that $E(e_1, l)$ acts as follows (cf. (4.12))

$$\begin{aligned} E(e_1, l)(\tilde{z}) &= \tilde{z} - \mathrm{b}(\tilde{z}, e_1)l + \mathrm{b}(\tilde{z}, l)e_1 - \mathrm{q}(l)\,\mathrm{b}(\tilde{z}, e_1)e_1 \\ &= \tilde{z} - l + \mathrm{b}(z, l)e_1 - \mathrm{q}(l)e_1 \\ &= (z - l) + (a + \mathrm{b}(z, l) - \mathrm{q}(l))e_1 + e_2. \end{aligned}$$

Hence, the induced action of $E(e_1, l)$ on $\mathcal{H}_{m^+}$ is given by $z \mapsto z - l$ and we obtain for the corresponding factor of automorphy that $j(E(e_1, l), z) = \mathrm{b}(E(e_1, l)\tilde{z}, e_1) = 1$. As a consequence, every holomorphic function $F : \mathcal{H}_{m^+} \to \mathbb{C}$ transforming with respect to $\Gamma(L(d))$ as in (4.26) for some $d \in \mathbb{N}$ possesses a Fourier expansion

$$F(z) = \sum_{\lambda \in \rho + K'/d} c(\lambda) e(\mathrm{b}(\lambda, z)) \tag{4.27}$$

for some $\rho \in M \otimes \mathbb{Q} = W_{\mathbb{Q}}$ which is unique modulo K'/d.

With this preparation, one may define modular forms also in case $m^+ \ngeq 3$.

Definition 4.66. Let V be isotropic. Let $\Gamma \leq \Gamma(L)$ be a subgroup of finite index and χ be a multiplier system for Γ of weight $r \in \mathbb{Q}$. A holomorphic function F on $\mathcal{H}_{m^+}$ is called a *holomorphic modular function* of weight r and multiplier system χ with respect to Γ, if it fulfils

$$F(\gamma z) = \chi(\gamma) j(\gamma, z)^r F(z) \tag{4.28}$$

for all $\gamma \in \Gamma$ and for the associated Fourier expansion (4.27) the coefficient $c(\lambda)$ vanishes, unless their index λ lies in $\mathcal{C}^+$. We write $\mathcal{M}_r(\Gamma, \chi)$ for the space of such functions.

As already alluded to, the second condition stated above about coefficients vanishing unless being contained in the cone manifesting the imaginary part of the tube domain model is automatic in case of $m^+ \geq 3$ by the Köcher Principle [Bru02, Prop 4.15 p. 109]. With the notion of holomorphic modular forms, the meromorphic version may be defined in a straightforward fashion.

Definition 4.67. Let V be isotropic. An orthogonal *meromorphic modular form* with respect to the group $\Gamma \leq \Gamma(L)$ of finite index of weight $r \in \mathbb{Q}$ and multiplier system χ is a function F on $\mathcal{H}_{m^+}$ with transformation property

$$F(\gamma z) = \chi(\gamma) j(\gamma, z)^r F(z)$$

for all $\gamma \in \Gamma$ that may be written as a quotient $F = F_1/F_2$ with $F_1 \in \mathcal{M}_s(\Gamma, \chi_1)$ and $F_2 \in \mathcal{M}_t(\Gamma, \chi_2)$ for suitable parametres $s, t \in \mathbb{Q}$ and multiplier systems χ_1, χ_2.

Analogously to the classical case presented in Chapter 2, there is a natural class of modular forms that may be constructed to prove the existence of such forms (cf. Subsection 2.1.2). To this end, we define, in analogy to the classical case, the following operator.

Definition 4.68. Let f be a function on $\mathcal{H}_{m^+}$ and $k \in \mathbb{Z}$. Then for $\gamma \in \Gamma(L)$

$$f|_k\gamma(z) := j(\gamma, z)^{-k} f(\gamma z)$$

defines a function on $\mathcal{H}_{m^+}$ and the induced right operation of $\Gamma(L)$ preserves holomorphicity and meromorphicity, respectively. The operator $|_k$ is referred to as *Petersson operator* of weight k or *slash operator*.

As in the elliptic setting, the Petersson operator leaves the space of modular forms invariant.

An example of orthogonal modular forms has yet to be constructed! Recall Eisenstein series in the classical setting introduced in Definition 2.16. These are constructed to be the simplest possible instance of a $\Gamma(1)$ invariant function, by symmetrising the constant function with respect to the Petersson operator. The same procedure will be mimicked in the orthogonal setting.

Definition 4.69. Let $k \in \mathbb{Z}$, $z \in L$ be a cusp, i.e. a primitive isotropic vector, and $\Gamma \leq \Gamma(L)$ be a subgroup of finite index. Then

$$E_{k,z}(Z) := \sum_{\sigma \in \Gamma} 1|_k\sigma$$

as a function on $\mathcal{K}^+$ is called *Eisenstein series.*

The series converges for $k > l + 2$ by comparison to an Epstein zeta function. As in the elliptic setting non-holomorphic Eisenstein series may be defined. We will not require these in the following, but the curious reader may consult [Kie22, 7 p. 2867]. Further, non trivial examples of modular forms may be constructed via the Borcherds lift, relating weakly holomorphic vector valued elliptic modular forms to meromorphic orthogonal modular forms (cf. Section 7.1).

4.2.5 Special divisors

In algebraic geometry divisors are a common tool for investigating the structure of varieties. A feature of the orthogonal setting is that the associated Shimura varieties have an abundance of divisors being induced by sub-varieties of the same type in all codimensions. These give rise to so called *special cycles* which will play a key role in proving the injectivity of the Kudla–Millson lift in Section 7.3. In the following, we describe the construction of these codimension 1 special cycles.

Recall that we assume the rational quadratic space (V, q) to have signature $(m^+, 2)$ and select some $v \in V$ with $\mathrm{q}(v) > 0$. Then the subspace $V_v := v^\perp$ has signature $(m^+ - 1, 2)$ and the complement in the projective model $\mathcal{K}^+$ denoted by

$$\mathcal{Z}_v := \left\{ [z] \in \mathcal{K}^+ \mid (z, v) = 0 \right\} \tag{4.29}$$

is an analytic divisor on $\mathcal{K}^+$, which is exactly the hermitian symmetric domain corresponding to (V_v, q). This means

$$\mathcal{Z}_v \simeq \mathbb{D}_v := \{ U \subset V_v(\mathbb{R}) \mid \dim(U) = 2 \text{ and } \mathrm{q}\,|_U < 0 \}. \tag{4.30}$$

Here, $\mathbb{D}_v$ can be realised as the Grassmannian attached to the stabiliser $\mathrm{O}(V_\mathbb{R})_v$ of v in $\mathrm{O}(V_\mathbb{R})$. It has become apparent that the divisor $\mathcal{Z}_v$ coincides with a subsymmetric domain of the same type and these divisors are referred to as *special divisors*. Its corresponding description in the tube domain model looks as follows. Write $v = v_W + ae_2 + be_1$ as above (cf. (4.14)) so that

$$\mathcal{Z}_v \simeq \left\{ z \in \mathcal{H}^+ \mid a\,\mathrm{q}(z) - \mathrm{b}(z, v_W) - a\,\mathrm{q}(e_2) - b = 0 \right\} \subset \mathcal{H}^+.$$

This description gives rise to the term *rational quadratic divisor*. Further, the following combination of these divisors appears naturally in many applications (cf. Theorem 7.3).

Definition 4.70. Select $\lambda \in L'/L$ as well as a rational number $n > 0$. Then

$$Z(\lambda, n) := \sum_{\substack{v \in \lambda + L \\ \mathrm{q}(v) = n}} \mathcal{Z}_v \tag{4.31}$$

defines an analytic divisor on $\mathcal{K}^+$ which is called *Heegner divisor* of discriminant (λ, n).

Recall that arithmetic subgroups $\Gamma \leq \mathrm{O}(L)$ have been considered which give rise to modular varieties $Y_\Gamma = \Gamma\backslash\mathbb{D}$. If the arithmetic group Γ lies in the discriminant kernel, i.e. acts trivially on the discriminant group L'/L, then the Heegner divisor descends to an algebraic divisor on Y_Γ, which is also denoted by $Z(\lambda, n)$. This may be verified by Chow's lemma.

Alternatively, the Heegner divisors might be symmetrised over L'/L to yield an algebraic divisor on Y_Γ, namely

$$Z(m) \coloneqq \frac{1}{2} \sum_{\lambda \in L'/L} Z(\lambda, m). \tag{4.32}$$

We also require the following more primitive divisor which is directly induced by (4.29). Let $\Gamma_v \coloneqq \Gamma \cap \mathrm{O}(V_\mathbb{R})_v$ be the intersection with the stabiliser of v. Then we find that

$$\mathcal{Z}(v) \coloneqq \Gamma_v\backslash\mathbb{D}_v \to Y_\Gamma \tag{4.33}$$

defines a (in general relative) cycle. These cycles play a fundamental role in the proof of the injectivity of the Kudla–Millson lift in Section 7.3.1.

The existence of such a family of algebraic divisors is distinctive for orthogonal and unitary groups.

5 Automorphic forms on SL_2

In the preceding sections, the concept of modular forms has been introduced in classical terms, both in the orthogonal and elliptic setting. In particular, Eisenstein series and theta functions have been discussed, both playing a central role in the theory of automorphic forms. Furthermore, there is a rather representation theoretic perspective on modular forms involving the adelic points of the respective group. An introduction to this perspective is presented in [Dei10], while a more thorough discussion can be found in [Bum98] and [Gel+90]. We restrict ourselves to describing the analogous versions of classical theta functions and Eisenstein series on $\mathrm{SL}_2(\mathbb{A})$ in an ad hoc fashion and encourage the reader to consult the literature for additional background information.

The following sections introduce theta distributions 5.2 and Eisenstein series 5.3 associated to Schwartz–Bruhat functions are introduced before relating both to each other by means of the Siegel–Weil formula (5.4.1).

5.1 Preliminaries

Before introducing examples of automorphic forms, we will have to state some facts about the group $\mathrm{SL}_2(\mathbb{R})$ based on [Dei10, Chap. 3 p. 81]. Recall the following three embeddings into $\mathrm{SL}_2(\mathbb{R})$:

$$m : (\mathbb{R}^\times, \cdot\,) \ni a \mapsto m(a) := \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \tag{5.1}$$

$$n : (\mathbb{R}, +) \ni b \mapsto n(b) := \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \tag{5.2}$$

$$k : \mathbb{T} \ni \vartheta \mapsto k(\vartheta) := \begin{pmatrix} \cos(\vartheta) & \sin(\vartheta) \\ -\sin(\vartheta) & \cos(\vartheta) \end{pmatrix}. \tag{5.3}$$

The image of m equals the subgroup of diagonal matrices in $\mathrm{SL}_2(\mathbb{R})$. We will denote it by M and the subgroup $m(\mathbb{R}_{>0})$ by M^0. Further, the image of n will be denoted by N and the one of k, which equals $\mathrm{SO}_2(\mathbb{R})$, will also be denoted by K_∞.

Example 5.1. a) For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ we consider the action on $i \in \mathbb{H}$ and compute

$$\gamma i = \frac{ai+b}{ci+d} = \frac{ac+bd+i(ad-bc)}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + i\frac{1}{c^2+d^2}. \tag{5.4}$$

b) For $a \in \mathbb{R}^\times$ and $\tau \in \mathbb{H}$ we find

$$m(a)\tau = a^2\tau.$$

c) For $b \in \mathbb{R}$ and $\tau \in \mathbb{H}$ we find

$$n(b)\tau = \tau + b.$$

d) Combining the last two instances, we obtain for $u + iv = \tau \in \mathbb{H}$ that

$$n(u)m(\sqrt{v})i = n(u)iv = u + iv = \tau. \tag{5.5}$$

The example above prompts the following definition.

Definition 5.2. Let $u + iv = \tau \in \mathbb{H}$ and define the associated matrix $g_\tau \coloneqq n(u)m(\sqrt{v}) \in \mathrm{SL}_2(\mathbb{R})$, acting as $g_\tau i = \tau$.

With these notions, the *Iwasawa* decomposition of the group $\mathrm{SL}_2(\mathbb{R})$ may be constructed. Consider the stabiliser of $i \in \mathbb{H}$ with respect to the action $\mathrm{SL}_2(\mathbb{R}) \circlearrowright \mathbb{H}$ via Möbius transformations. A swift computation yields that $\mathrm{Stab}(i) = \mathrm{SO}_2(\mathbb{R})$. Recall that by Example 5.1 the action of $\mathrm{SL}_2(\mathbb{R})$ is transitive and we obtain the desired decomposition.

Proposition 5.3 (Iwasawa decomposition)**.** *Let M^0, N, and $K_\infty = \mathrm{Stab}(i) = \mathrm{SO}_2(\mathbb{R})$ be as above. Then*

$$N \times M^0 \times K_\infty \to \mathrm{SL}_2(\mathbb{R}), \qquad (n, m, k) \mapsto nmk$$

defines a homeomorphism.

Proof: In fact, we find that associating the following matrices to $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ defines a topological inverse mapping.

$$m = \begin{pmatrix} \sqrt{c^2+d^2}^{-1} & 0 \\ 0 & \sqrt{c^2+d^2} \end{pmatrix},$$

$$n = \begin{pmatrix} 1 & \frac{ac+bd}{c^2+d^2} \\ 0 & 1 \end{pmatrix},$$

$$k = \frac{1}{\sqrt{c^2+d^2}}\begin{pmatrix} d & -c \\ c & d \end{pmatrix}.$$

For constructing these matrices, we have employed (5.4) and (5.5). □

We will also require the interplay of the factor of automorphy with the special matrices appearing in the Iwasawa decomposition.

Example 5.4. Let $a \in \mathbb{R}^\times$, $b \in \mathbb{R}$, $\vartheta \in \mathbb{T}$, and $\tau \in \mathbb{C} \setminus \mathbb{R}$. Then we find

$$j(m(a), \tau) = a^{-1}, \tag{5.6}$$

$$j(n(b), \tau) = 1, \tag{5.7}$$

$$j(k_\vartheta, i) = e^{-i\vartheta}. \tag{5.8}$$

Further, combining these with the cocycle relation (cf. Remark 2.11), we obtain

$$\begin{aligned} j(n(b)m(a)k_\vartheta, i) &= j(n(b)m(a), i)j(k_\vartheta, i) \\ &= j(n(b), m(a)i)j(m(a), i)j(k_\vartheta, i) \\ &= a^{-1}e^{-i\vartheta}. \end{aligned}$$

For the purpose of explicitly constructing Eisenstein series in Section 5.3, we are required to express the factor of automorphy of the constitute k of an element $\gamma \in \mathrm{SL}_2(\mathbb{R})$ in Proposition 5.3 in terms of entries of γ and refer to [BY09, p. 641] on that matter.

Remark 5.5. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and $\tau \in \mathbb{H}$. Then $\gamma\tau = \gamma g_\tau i$, where the Iwasawa decomposition guarantees $\gamma g_\tau = g_{\gamma\tau} k_\vartheta = n(\beta)m(\alpha)k_\vartheta$ for suitable β, α and ϑ. In fact, we find

$$e^{i\vartheta} = \overline{j(k_\vartheta, i)} = \frac{j(\gamma, \overline{\tau})}{|j(\gamma, \tau)|}, \tag{5.9}$$

$$\alpha = \frac{\sqrt{v}}{|j(\gamma, \tau)|} = \sqrt{\mathrm{Im}(\gamma\tau)}. \tag{5.10}$$

Proof: Recall that the first equation of (5.9) has already been given in Example 5.4. For the second, we begin by calculating

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} g_\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \sqrt{v} & u/\sqrt{v} \\ 0 & 1/\sqrt{v} \end{pmatrix} = \begin{pmatrix} \sqrt{v}a & (ua+b)/\sqrt{v} \\ \sqrt{v}c & (uc+d)/\sqrt{v} \end{pmatrix} \tag{5.11}$$

yielding by the Iwasawa decomposition (cf. proof of Prop. 5.3) as an associated matrix

$$k_\vartheta = \frac{1}{\sqrt{(cv)^2 + (uc+d)^2}} \begin{pmatrix} (uc+d) & -cv \\ cv & (uc+d) \end{pmatrix}.$$

With this expression, we obtain by inserting the definition of the factor of automorphy and by writing $\tau = u + iv$ that

$$\overline{j(k_\vartheta, i)} = \frac{-cvi + (uc+d)}{\sqrt{(cv)^2 + (uc+d)^2}} = \frac{c(u-iv)+d}{\sqrt{[c(u+iv)+d][c(u-iv)+d]}} = \frac{j(\gamma,\overline{\tau})}{|j(\gamma,\tau)|}.$$

We may also read the parameter α off by employing (5.11) deriving that

$$a = \sqrt{vc^2 + (uc+d)^2/v}^{-1} = \frac{\sqrt{v}}{|j(\gamma,\tau)|} = \sqrt{\mathrm{Im}(\gamma\tau)}$$

which finishes the proof. □

After these prerequisites, we may turn towards concrete examples of automorphic forms.

5.2 Theta distributions

The following section introduces the first example of automorphic forms on $\mathrm{Mp}_2(\mathbb{A})$ in the form of theta distributions. Their construction requires less effort than the construction of Eisenstein series, which is the reason for favouring their introduction for the first section. Further information is provided in [Kud03]. Let (V, q) be a quadratic $\mathbb{Q}$ vector space, $V_\mathbb{A}$ its adelisation, and let ω denote the Weil representation of $\mathrm{Mp}_2(\mathbb{A})$ on $\mathcal{S}(V_\mathbb{A})$ as in (3.19).

Definition 5.6. Define the *theta distribution*, also known as *theta kernel*,

$$\theta(g,h;\varphi) = \sum_{x \in V(\mathbb{Q})} (\omega(g)\varphi)(h^{-1}x)$$

which is a tempered distribution in $\varphi \in \mathcal{S}(V_\mathbb{A})$, where $g \in \mathrm{Mp}_2(\mathbb{A})$ and $h \in \mathrm{O}(V_\mathbb{A})$. Here, the latter denote the adelic points of the orthogonal group $\mathrm{O}(V)$.

Poisson summation yields that the theta kernel is left invariant under $\mathrm{Mp}_2(\mathbb{Q})$ as well as $\mathrm{O}(V)$. Furthermore, it is only slowly increasing on the quotient $\mathrm{Mp}_2(\mathbb{Q})\backslash \mathrm{Mp}_2(\mathbb{A})$ and $\mathrm{O}(V)\backslash \mathrm{O}(V_\mathbb{A})$.

We would like to compare the context of the definition above to previous notations of theta functions.

Remark 5.7. Comparing Definition 5.6 with Definition 3.31 we find the following identity. Select $\lambda \in L'/L$ and $\varphi_f = \mathbb{1}_{\lambda+L}$ as well as $\varphi_\infty \in \mathcal{S}(V_\infty)$ and $h_\infty \in \mathrm{O}(V_\mathbb{R})$ as well as $h_f = \mathrm{id} \in \mathrm{O}(V_{\mathbb{A}_f})$. Then we find

$$\theta_{L,\lambda}(\tau, h_\infty; \varphi_\infty) = \mathrm{Im}(\tau)^{-m/4} \cdot \theta(g_\tau, h_\infty h_f; \varphi_\infty \otimes \varphi_f).$$

Theta lifts and as such theta functions are central objects in the present thesis and we will present a link to Eisenstein series, the Siegel–Weil formula (cf. Section 5.4.1), which is also crucial for the main objective of this thesis of proving injectivity and surjectivity results of theta lifts. We note that in many parts of the thesis, we use classical notation, also for theta functions which is why we explicitly outline the realisation of a classical theta function by a theta distributions as above.

5.2.1 The classic setting

Theta distributions may be viewed as generalisations of classical theta series (cf. Definition 2.17). In this context, let (L, q) be a classical $\mathbb{Z}$ lattice with positive definite quadratic form and even rank m. Then $V := \mathbb{Q} \otimes_\mathbb{Z} L$ is a vector space with quadratic form q and the association

$$V_\infty \ni x \mapsto \exp(-2\pi\, \mathrm{q}(x))$$

defines a Schwartz–Bruhat function on $V_\infty := V \otimes_\mathbb{Q} \mathbb{R}$.

Recall that the Weil representation factors through SL_2 in the even case and we may restrict to working with the SL_2 setting. Further, we select $N \in \mathbb{N}$ and introduce the abbreviation $K_0(N)_p = \overline{\Gamma_0(N)} \leq \mathrm{SL}_2(\mathbb{Z}_p)$ to note that

$$\Gamma_0(N) \setminus \mathrm{SL}_2(\mathbb{R}) \simeq \mathrm{SL}_2(\mathbb{Q}) \setminus \mathrm{SL}_2(\mathbb{A}) / \prod_{p<\infty} K_0(N)_p$$

via the association

$$\alpha : \Gamma_0(N)x \mapsto \mathrm{SL}_2(\mathbb{Q})(x, 1) \prod_{p<\infty} K_0(N)_p.$$

As a Schwartz form choose $\varphi = e_\infty(i\,\mathrm{q},\,\cdot\,) \cdot \prod_{p<\infty} \mathbb{1}_{L_p}$, where e_∞ is the standard character at the archimedean place from Example 1.48 and let $g_\tau k_\vartheta = g_\infty \in \mathrm{SL}_2(\mathbb{R})$ be the element corresponding to $u + iv = \tau \in \mathbb{H}$ with respect to the Iwasawa decomposition as in

Definition 5.2:

$$g_\tau k_\vartheta = \left[\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right) \cdot \begin{pmatrix} \sqrt{v} & 0 \\ 0 & 1/\sqrt{v} \end{pmatrix}\right] \cdot \begin{pmatrix} \cos(\vartheta) & \sin(\vartheta) \\ -\sin(\vartheta) & \cos(\vartheta) \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

With this notation, we obtain

$$\theta(g_\tau k_\vartheta, 1; \varphi) = e^{-i\vartheta m/2} \cdot \sqrt{v}^{m/2} \cdot \theta_L(\tau). \tag{5.12}$$

Given $\tau \in \mathbb{H}$, define $\psi : \tau \mapsto \theta(g_\tau k_\vartheta, 1; \varphi) j(g_\tau k_\vartheta, i)^{m/2} = \theta_L(\tau)$. Then, for arbitrary $\gamma \in \Gamma_0(N)$, where N is the level of L we obtain:

$$\begin{aligned} \psi(\gamma.\tau) &= \theta((\gamma g_\tau k_\vartheta, 1), 1; \varphi) \cdot j(\gamma g_\tau k_\vartheta, i)^{m/2} \\ &= \theta(\gamma(g_\tau k_\vartheta, \gamma^{-1}), 1; \varphi) \cdot j(\gamma g_\tau k_\vartheta, i)^{m/2} \\ &= \theta((g_\tau k_\vartheta, 1), 1; \varphi)\chi_V(\gamma^{-1}) \cdot j(\gamma g_\tau k_\vartheta, i)^{m/2} \\ &= \theta_L(\tau) \cdot \chi_V(\gamma^{-1}) \cdot j(\gamma, \tau)^{m/2} \\ &= \psi(\tau) \cdot \left(\frac{\mathrm{disc}(L)}{\gamma_{2,2}}\right) \cdot j(\gamma, \tau)^{m/2}, \end{aligned}$$

where the isolation of the character χ_V from Example 1.62, which is meant to act on the lower right entry of the argument, is performed locally. For that purpose the local matrices are decomposed in order to apply the formulae of Example 3.8 and we refer to [Rao93] to relate the local Weil index appearing to the values of the character χ_V. The final step to express this character in terms of the Kronecker symbol is performed in [Opi18, Cor. 1.10.2 p. 22]. Consequently, the function $\psi : \mathbb{H} \to \mathbb{C}$ not only equals the theta series θ_L on $\mathbb{H}$ but the calculation also proves that it transforms with a character under $\Gamma_0(N)$. Additionally, holomorphy may be verified directly by a trivial estimate of the sum, so that θ_L is identified as a holomorphic modular form (cf. Theorem 2.22).

Corollary 5.8. *The function $\theta_L : \mathbb{H} \to \mathbb{C}$ is a modular form of weight $\frac{m}{2}$ with character $\left(\frac{\mathrm{disc}(L)}{\cdot}\right)$ with respect to the group $\Gamma_0(\mathrm{lev}(L))$.*

5.3 Eisenstein series

This section is devoted to the introduction of Eisenstein series on SL_2 which play a pivotal role in applying the Siegel–Weil formula for proving the central result of this thesis (cf. Theorem 7.16). At first, principal series representations, which are indispensable

for constructing Eisenstein series, are briefly discussed, before defining Eisenstein series associated to standard sections. Subsequently, Eisenstein series are ascribed to Schwartz–Bruhat functions which lays the foundation for forging a bridge to theta kernel from the previous section by means of the Siegel–Weil formula.
This entire section relies upon [KY10], in which Kudla and Yang provide a comprehensive setting in a compact fashion that is adequate for the investigation of Eisenstein series on $\overline{\mathrm{SL}}_2(\mathbb{A})$. Readers interested in further details may wish to consult the more elaborate [Bum98], which also addresses the Lie theory upon which individual results are built.

We fix a character ψ of $\mathbb{A}/\mathbb{Q}$ which we recall are identifiable with rational numbers. For explicit computations ψ is assumed to be the standard character as in Remark 1.55 with $a = 1$, even though the theory works for arbitrary non-trivial characters, with proper normalisation of the Haar measures involved. In general, χ will denote a finite character on $\mathbb{A}^\times/\mathbb{Q}^\times$, if it is not specified further.

5.3.1 Principal series representations

The principal series representations constitute an essential component of adelic Eisenstein series. They are vital for investigating unitary representations of SL_2, as they comprise two of essentially four infinitesimal isomorphism classes of representations of $\mathrm{SL}_2(\mathbb{R})$ – besides the *(mock) discrete series* and the *complementary series* [Lan98, VI.6 Thm. 8 p. 123]. The reason for the appearance of these representations in the construction of Eisenstein series is described in detail in [Bum98]. The case of SL_2 is discussed in [Dei10, Sec. 7.1 p. 157], a more sophisticated compact approach is given by [Gel74] or [Kud96], whereas [Bum98, p. 213, Sec. 4.5 pp. 469-489] presents the content extensively, featuring a significant representation theoretic discussion. Supplementing material is also found in [Kud94], [KR92], as well as [KRY99, pp. 353-454].

The principal series representations of SL_2 are induced by characters on the Borel group of upper triangular matrices. As in the case of finite groups, induced representations can be constructed as tensor products or explicitly by right action of the comprising group. However, in the infinite case involving topology is indispensable so that working only on dense subspaces with properties which are apparently advantageous for computations is convenient. This leads to demanding analytical and symmetry features of investigated functions.

The following construction is nearly identical in the $\mathrm{SL}_2(\mathbb{R})$ case. Let G be a totally disconnected locally compact group with a closed subgroup H which possesses a smooth representation (π_H, V). Consider the space of smooth functions $f : G \to V$ which satisfy

$$f(hg) = \delta_G(h)^{-1/2}\delta_H(h)^{1/2}\pi_H(h)f(g)$$

with modular quasi characters δ_G, δ_H of the respective groups.[1] Then, the representation of G *induced* by π_H (of H) on the space of functions defined above is denoted $\pi_G = \mathrm{Ind}_H^G(\pi_H)$ and given by right translation of G:

$$\pi_G(g)(f)(x) = f(xg).$$

The essential case for this thesis is the choice $G = \mathrm{SL}_2(F_p)$ with upper triangular matrices H as a maximal compact subgroup. When the representation π_H of H is chosen to be a multiplicative quasi character $\chi\left(\begin{smallmatrix} a & b \\ 0 & a^{-1}\end{smallmatrix}\right) := \chi(a)$, $\mathrm{Ind}_H^G(\chi)$ consists of functions $f : G \to \mathbb{C}$, such that

$$f(n(b)m(a)g) = \chi(a)|a|_p \cdot f(g),$$

where $n(b) = \left(\begin{smallmatrix} 1 & b \\ 0 & 1\end{smallmatrix}\right)$ and $m(a) = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1}\end{smallmatrix}\right)$ as usual. These are called *principal series representations* in case of irreducibility and an element of them is determined by the values it attains on K_p.[2] The choice of particular interest is that of a Hecke quasi character, which naturally decomposes into a family of local finite characters χ_p on $\mathbb{Q}_p^\times$ and quasi characters $|\cdot|_p^s$ (cf. Remark 1.56). Hence, the functions in the associated induced representation transform in the following fashion

$$f(n(b)m(a)g) = \chi_p(a)|a|_p^{s+1} \cdot f(g). \tag{5.13}$$

The collection of spaces $\mathrm{Ind}_H^G(\chi|\cdot|_p^s)$ form vector bundles over $\mathbb{C}$ (parametrised by s) denoted by $I_p(s,\chi)$ which admit holomorphic sections $\Phi_p(s)$ – at least for $\mathrm{Re}(s) > 1$. In particular, there is a so called *spherical section* Φ_p^0, attaining the value 1 on the neutral element of SL_2 and being right K_p invariant.

For a family of Hecke quasi characters $\chi \cdot |\cdot|^s$ parametrised by $s \in \mathbb{C}$, the local representations described above (for all $p \leq \infty$) may be combined via *restricted tensor products* (cf. [Bum98, p. 300 as.]). For this purpose the spherical section is fixed in each component. This results in a collection of adelic representations, denoted by $I(s,\chi)$ and also called *principal series representations* associated to the Hecke character. A similar

[1]Note that the factors are chosen in such a fashion that induction preserves unitaricity (cf. [Bum98, Thm. 2.6.1 p. 224]).
[2]This is true by strong approximation (cf. [Kne65]).

construction can be done in case of $\overline{\mathrm{SL}}_2(\mathbb{A})$ (cf. [KY10]) and the resulting principal series representation will also be denoted $I(s,\chi)$. In case these factor through $\mathrm{SL}_2(\mathbb{A})$, they are called *even*, if not, they are called *odd*. Again, there are holomorphic sections $\Phi(s)$, which are utilised as kernels for symmetrisation resulting in non-holomorphic Eisenstein series which are to be discussed in the following section. To be more explicit, a section $\Phi(s)$ from $I(s,\chi)$ is a smooth function on $G'_{\mathbb{A}}$ such that

$$\Phi(p'g', s) = \chi(a)\cdot|a|^{s+1}\Phi(g', s)\begin{cases}1, & \text{in the even case,}\\ z, & \text{in the odd case,}\end{cases} \tag{5.14}$$

where $p' = [n(b)m(a), z] \in P'_{\mathbb{A}}$ is in the standard Borel subgroup (cf. Subsection 3.2) and the same normalisation as in [KRY06, p. 287] is chosen. A section $\Phi(s)$ is called *standard* if its restriction to the maximal compact subgroup $K_{\mathbb{A}} < \mathrm{SL}_2(\mathbb{A})$ ($K'_{\mathbb{A}} < \overline{\mathrm{SL}}_2$, respectively) is independent of s and *factorisable* if $\Phi = \otimes_p \Phi_p$ is a primitive tensor. Further, every section $\Phi(g,s) \in I(s,\chi)$ is determined by its values on $K'_{\mathbb{A}}$ which is verified via the Iwasawa decomposition in (3.23).

The simplest example of a section at a finite place is the normalised standard section.

Example 5.9. Let $2 < p < \infty$ and. Then the section $\Phi_p^0 \in I_p(s,\chi)$ determined by

$$\Phi_p^0(k, s) = 1$$

for $k \in K_p = \mathrm{SL}_2(\mathbb{Z}_p)$ is a standard section. It is called the *spherical section* (since it is right invariant under K_p). To make sense of the above, recall that in case $2 < p$, there is a section of $K_p = \mathrm{SL}_2(\mathbb{Z}_p) \hookrightarrow \overline{\mathrm{SL}}_2(\mathbb{Q}_p)$.

Recall that the characters of the double cover $\overline{\mathrm{SO}}_2(\mathbb{R})$ sitting in K'_∞ are of the following form (cf. [BF04, Sec. 2]).

Remark 5.10. For every $l \in \mathbb{Z}/2$ there is a character of $\overline{\mathrm{SO}}_2(\mathbb{R})$ denoted by ν_l with values

$$\nu_l\left(k_\vartheta, \pm\sqrt{j(k_\vartheta,\tau)}\right) \mapsto \pm\sqrt{j(k_\vartheta, i)}^{-1} = \pm e^{il\vartheta}.$$

With this family of characters, one may define concrete elements of the principal series representation at the infinite place.

Example 5.11. Assume χ is a quadratic character. For $l \in \mathbb{Z}/2$ imposing the conditions

$$\Phi^l_\infty(gk, s) = \nu_l(k)\Phi^l_\infty(g, s), \quad \Phi^l_\infty(1, s) = 1$$

on a candidate Φ^l_∞ for a section through $I_\infty(s,\chi)$ almost works. In fact, the parameter l has to satisfy a parity condition to comply with the action of $-\mathcal{I}$ when letting it act via χ, for otherwise the section Φ^l_∞ is forced to vanish. The concrete condition is stated in (5.18). We refer to Φ^l_∞ as the *normalised eigenfunction* of weight l and note that these functions span the K_∞ finite functions in $I_\infty(s,\chi)$.

For $\tau = u + iv \in \mathbb{H}$ we recall the following matrices from the Iwasawa decomposition appearing in Proposition 5.3. Let $g_\tau = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)\begin{pmatrix} \sqrt{v} & 0 \\ 0 & 1/\sqrt{v} \end{pmatrix}$ and $k_\vartheta = \begin{pmatrix} \cos(\vartheta) & \sin(\vartheta) \\ -\sin(\vartheta) & \cos(\vartheta) \end{pmatrix}$ for some radiant ϑ. Inserting these, we find

$$\Phi^l_\infty(g_\tau k_\vartheta, s) = \chi_\infty\left(\sqrt{\mathrm{Im}(\tau)}\right)\mathrm{Im}(\tau)^{(s+1)/2}\nu_l(k_\vartheta)\Phi^l_\infty(1,s).$$

In particular, for $\gamma \in \mathrm{Mp}_2(\mathbb{Z})$ we find $\gamma g_\tau = g_{\gamma\tau}k_\vartheta$, resulting by Remark 5.5 in

$$\begin{aligned}\Phi^l_\infty(g_{\gamma\tau}k_\vartheta, s) &= \chi_\infty(\sqrt{\mathrm{Im}(\gamma\tau)})\,\mathrm{Im}(\gamma\tau)^{(s+1)/2}\nu_l(k_\vartheta)\Phi^l_\infty(1,s)\\ &= \mathrm{Im}(\gamma\tau)^{(s+1)/2}\left(\frac{\overline{j(\gamma,\tau)}}{|j(\gamma,\tau)|}\right)^l\Phi^l_\infty(1,s)\\ &= \mathrm{Im}(\tau)^{l/2}\frac{\mathrm{Im}(\gamma\tau)^{(s+1-l)/2}}{j(\gamma,\tau)^l}.\end{aligned}$$

There is an intertwining map which may be used to construct elements of the principal series representations by means of the Weil representation. In fact, when considering the action of the Weil representation in Example 3.8, for instance, and comparing it to (5.14), the following statement is inferred.

Lemma 5.12. *Let* (V,q) *be a quadratic* $\mathbb{Q}$ *space of dimension* m *with character* χ_V *as in Example 1.62. Further, let* $p \le \infty$, ω_p *be the associated Weil representation as in* (3.16) *and set* $s_0 \coloneqq \frac{m}{2} - 1$. *Then*

$$\lambda_p : \mathcal{S}(V_p) \to I_p(s_0,\chi), \qquad \lambda_p(\varphi)(g') \coloneqq \omega_p(g')\varphi(0) \tag{5.15}$$

is a well defined map. In fact, it is almost always surjective, except for three special cases that are described in [KY10, Sec. 4 p. 2286].

Clearly, to any of the sections arising from the above intertwining operator (5.15), there is an associated canonical standard section. This method reproduces the already known examples from above but also yields new examples.

Example 5.13. a) Let V_∞ have signature (m^+,m^-). Then for any element z of the associated Grassmannian, there is an associated positive standard majorant q_z^+ of q by decomposing orthogonally along z and flipping the sign of the negative definite part (cf. Remark 3.28). Setting

$$\varphi_\infty(x,z) \coloneqq e^{-2\pi\,\mathrm{q}_z^+(x)} \in \mathcal{S}(V_\infty) \tag{5.16}$$

associates the standard section from Example 5.11 of weight $l = \frac{m^+ - m^-}{2}$ via the intertwining operator of Lemma 5.12 in the archimedean case.

b) Let L_p be unimodular, then the standard section associated to $\lambda_{V_p}(\mathbb{1}_L)$ equals the spherical section $\Phi_p^0(s)$ from Example 5.9.

c) Next, assume L_p is simply even, let $\mu_p \in L_p'/L_p$ and write $\varphi_{\mu_p} \coloneqq \mathbb{1}_{\mu_p+L_p}$. Then the action of ω_p on φ_{μ_p} is described in Section 3.2, where the Fourier transform of φ_{μ_p} is computed explicitly in Remark 1.54. This gives rise to a standard section in $I_p(\chi, s)$.

d) Further, we find an operator for the finite adelic part

$$\lambda_f : \mathcal{S}(V_{\mathbb{A}_f}) \to I_f(s_0,\chi) = \otimes'_{p<\infty} I_p(s_0,\chi), \qquad \lambda_f(\varphi)(g_f') = \omega_f(g_f')\varphi(0).$$

For $\mu \in L'/L \simeq \hat{L}'/\hat{L}$ there is the associated element $\varphi_\mu \coloneqq \mathbb{1}_{\mu+L_f} = \otimes_{p<\infty}\varphi_{\mu_p} \in \mathcal{S}(V_{\mathbb{A}_f})$ for which we find $\lambda_f(\varphi_\mu) \in I_f(s_0,\chi)$. The associated standard section will be denoted by Φ_μ and plays a major role in the construction of Eisenstein series below. Clearly, we may perform the same procedure for a Schwartz–Bruhat function $\varphi \in \mathcal{S}(V_{\mathbb{A}})$ and associate to it a standard section $\Phi \in I(s,\chi)$.

With the tools described and examples constructed above, we may turn towards Eisenstein series.

5.3.2 Eisenstein series

The following exposition is based on [KY10, Sec. 2] and [BY09, Sec. 2]. We begin by defining Eisenstein series in analogy to the classical case by symmetrising sections of the principal series representation.

Definition 5.14. For a standard section $\Phi(g,s) \in I(s,\chi)$ define the associated *Eisenstein series*

$$\mathcal{E}(g,s;\Phi) \coloneqq \sum_{\gamma \in P(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{Q})} \Phi(\gamma g, s). \tag{5.17}$$

Note that this series converges for $\mathrm{Re}(s) > 1$, is, by definition, left invariant under $\mathrm{SL}_2(\mathbb{Q})$, and has a meromorphic continuation to the whole complex plane in s due to Langlands [Lan06].[3]

We are interested in concrete instances of the above series and restrict, from now on, to a quadratic character χ which may be realised by a square free integer $d \in \mathbb{Z}$ via

$$\chi(x) = \begin{cases} \mathcal{H}_{\mathbb{A}}(x, d), & \text{even case,} \\ \mathcal{H}_{\mathbb{A}}(x, 2d), & \text{odd case.} \end{cases}$$

Here, $\mathcal{H}_{\mathbb{A}}$ denotes the Hilbert symbol discussed in Section 1.4 and the even or odd case refers to the principal series representation factoring through SL_2 or not.[4] Recall that in order for the section Φ_∞^l of Example 5.11 to be non-trivial and hence part of $I_\infty(s, \chi)$, the following parity condition has to be satisfied

$$\begin{cases} (-1)^l = \mathrm{sig}(d), & \text{even case,} \\ l \equiv \mathrm{sig}(d)/2 \mod 2, & \text{odd case.} \end{cases} \tag{5.18}$$

In this context, we are able to represent classical Eisenstein series.

Example 5.15. For χ being the trivial character and g_τ as a above we find by Example 5.11 and Example 5.9 with the definition $\Phi_f^0 = \otimes_p \Phi_p^0$ that

$$\mathcal{E}(g_\tau, s; \Phi_\infty^l \otimes \Phi_f^0) = \mathrm{Im}(\tau)^{l/2} \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \frac{\mathrm{Im}(\gamma\tau)^{(s+1-l)/2}}{j(\gamma, \tau)^l}.$$

The above example demonstrates how classical Eisenstein series may be realised. In fact, this motivates considering the Eisenstein series in (5.17) as functions on the upper half plane. Recall that any factorisable finite standard section $\Phi_f(g, s) \in I_f(\chi, s)$ is invariant under some open subgroup K_0 of $K_0(4)$. Hence, by strong approximation the following Eisenstein series, whose definition is motivated by the example above, determines the series $\mathcal{E}(g', s; \Phi_\infty^l \otimes \Phi_f)$ completely.

Definition 5.16. For Φ_∞^l from Example 5.11 transforming with ν_l under $\overline{K_\infty}$ and a finite

[3] Compare 6 Lemma 6.1 p. 91 for instance. However, that treatment is written in a completely different notation and with a different perspective than the current document (a more analytic point of view), rendering it not particularly helpful on a glance if only our setting is known.

[4] This will agree with the signature being even or odd in our later setting.

standard section $\Phi_f \in I_f(\chi, s)$, define

$$\mathcal{E}(\tau, s; \Phi_\infty^l \otimes \Phi_f) := \mathrm{Im}(\tau)^{-l/2} \cdot \mathcal{E}(g_\tau, s; \Phi_\infty^l \otimes \Phi_f). \tag{5.19}$$

The series $\mathcal{E}(\tau, s; \Phi_\infty^l \otimes \Phi_f)$ then defines a non-holomorphic modular form of weight l and we have seen how to realise classical scalar valued Eisenstein series. However, in this thesis, vector valued modular forms play a major role and we seek means to describe these also in the adelic setting.
In the spirit of Subsection 3.2.1, we will realise non-holomorphic vector valued Eisenstein series for the discrete Weil representation (cf. Definition 3.37) in the present setting. To this end, let (V, q) be the rational quadratic space associated to an even $\mathbb{Z}$-lattice (L, q) and select the quadratic character $\chi = \chi_V = \mathcal{H}_{\mathbb{A}}(\,\cdot\,, \mathrm{disc}(V))$ described in Example 1.62. Recall that in Example 5.13 d), for $\mu \in L'/L$ the finite adelic Schwartz form $\varphi_\mu := \mathbb{1}_{\mu+\hat{L}} := \otimes_{p<\infty} \mathbb{1}_{\mu+L_p}$ has been associated to a standard section Φ_μ via the intertwining operator.

Definition 5.17. With the notation as above, we define the following Eisenstein series

$$\mathcal{E}_{\hat{L},l}(\tau, s) := \sum_{\mu \in L'/L} \mathcal{E}\left(\tau, s; \Phi_\infty^l \otimes \Phi_\mu\right) \varphi_\mu. \tag{5.20}$$

The shape of this series not only resembles that of the classical non-holomorphic vector valued Eisenstein series presented in Definition 3.37 but may be directly translated to that setting [BY09].

Proposition 5.18. *With the notation as above we find the following identity*

$$\mathcal{E}_{\hat{L},l}(\tau, s) = E_{L,0,l}\left(\tau, \tfrac{s+1-l}{2}\right). \tag{5.21}$$

For this equality, the natural identification $\mathcal{S}_{L'/L} \simeq \mathbb{C}[L'/L]$ from Lemma 3.9 is used.

Proof: We follow [BY09, Sec. 2 pp. 637-642] for part of the computation. We begin our calculation by inserting the explicit term for the section Φ_∞^l that has been calculated in Example 5.11.

$$\begin{aligned}\mathcal{E}_{\hat{L},l}(\tau, s) &= \mathrm{Im}(\tau)^{-l/2} \sum_{\mu \in \mathcal{L}} \mathcal{E}(g_\tau, s; \Phi_\infty^l \otimes \Phi_\mu)\varphi_\mu \\ &= \frac{1}{2} \sum_{\mu \in \mathcal{L}} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma(1)} \frac{\mathrm{Im}(\gamma\tau)^{(s+1-l)/2}}{j(\gamma, \tau)^l} \Phi_\mu(\gamma)\varphi_\mu.\end{aligned}$$

Now, recall that Φ_μ was induced by $\lambda_f(\varphi_\mu) \in I_f(s_0, \chi)$ via the finite intertwining operator introduced in Example 5.13. We then find via the correspondence of $\mathcal{S}_{L'/L}$ and $\mathbb{C}[L'/L]$ constructed before Lemma 3.9 that for $\gamma \in \mathrm{Mp}_2(\mathbb{Z})$ the following identity is true.

$$\begin{aligned}
\lambda_f(\varphi_\mu)(\gamma) &= (\omega_f(\gamma)\varphi_\mu)(0) \\
&= \langle \omega_f(\gamma)\varphi_\mu, \varphi_0 \rangle \\
&= \langle \varphi_\mu, \omega_f(\gamma)^{-1}\varphi_0 \rangle \\
&= \overline{\langle \omega_f(\gamma^{-1})\varphi_0, \varphi_\mu \rangle} \\
&= \langle \rho_L(\gamma^{-1})\varphi_0, \varphi_\mu \rangle \\
&= \langle \rho_L(\gamma^{-1})\mathfrak{e}_0, \mathfrak{e}_\mu \rangle \\
&= \langle \mathfrak{e}_0, \rho_L(\gamma)\mathfrak{e}_\mu \rangle.
\end{aligned}$$

Here, we have used the correspondence (3.27) between the classical Weil representation and the dual of the adelic version on $\mathcal{S}_{L'/L}$. Inserting this relation in the above expression for the Eisenstein series and using the isomorphism $\mathcal{S}_{L'/L} \simeq \mathbb{C}[L'/L]$ we obtain

$$\begin{aligned}
&\mathcal{E}_{\hat{L},l}(\tau, s) \\
&\simeq \frac{1}{2} \sum_{\gamma \in \overline{\Gamma_\infty} \backslash \mathrm{Mp}_2(\mathbb{Z})} \frac{\mathrm{Im}(\gamma\tau)^{(s+1-l)/2}}{j(\gamma,\tau)^l} \sum_{\nu \in \mathcal{L}} \langle \rho_L(\gamma^{-1})\mathfrak{e}_0, \mathfrak{e}_\mu \rangle \mathfrak{e}_\mu \\
&= \frac{1}{2} \sum_{\gamma \in \overline{\Gamma_\infty} \backslash \mathrm{Mp}_2(\mathbb{Z})} \left[\mathrm{Im}(\tau)^{(s+1-l)/2}\right]\Big|_l \cdot \rho_L(\gamma^{-1})\mathfrak{e}_0 \\
&= \frac{1}{2} \sum_{\gamma \in \overline{\Gamma_\infty} \backslash \mathrm{Mp}_2(\mathbb{Z})} \left[\mathrm{Im}(\tau)^{(s+1-l)/2}\mathfrak{e}_0\right]\Big|_{L,l}\gamma \\
&= E_{L,0,l}(\tau, \tfrac{s+1-l}{2}).
\end{aligned}$$

This completes the proof. □

It is possible to compute the Fourier expansion of $\mathcal{E}(\tau, s; \Phi)$ for factorisable sections in terms of local Whittaker functions as explained in [KY10, Thm. 2.4 p. 2282]. The computations in concrete cases involve solving Gauss sums for the appearing local integrals and a sizable amount of case distinctions. This has been carried out in the special case of lattices associated to quadratic field extensions in a Master's thesis [Met19].

5.4 Comparing theta and Eisenstein series

Theta and Eisenstein series represent a sizable share of the theory of automorphic forms. In fact, in many cases it is known that the span of theta series encompasses the space of cusp forms (cf. [Mül24, Thm. 6.7]) which is then complemented by Eisenstein series. Further, it is worth noting the existence of a celebrated formalism for relating theta series to Eisenstein series, called the *Siegel–Weil* formula, which is the subject of the subsequent subsection. Afterwards, the Siegel–Weil formula is applied to link a certain geometric integral of the Kudla–Millson Schwartz form to Eisenstein series, as outlined by Kudla [Kud03, Sec. 4.3 pp. 328-335]. Recall that L is an even lattice of signature $(m^+, 2)$ and the dimension of $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$ is denoted by m.

5.4.1 Siegel–Weil formula

In the previous section, Eisenstein series have been associated to Schwartz–Bruhat functions. The subject of this subsection is the so called *Siegel–Weil* formula which establishes a relation between such Eisenstein series (5.17) evaluated at a special point s_0 and theta integrals as described in (5.22). It is an adelic extension of the classic result Theorem 2.9 presented in Section 2.1.2 and has first been proven by André Weil [Wei65]. The curious reader may find further information in [KR94] where the result is notably extended. Recall that the classic result mentioned above reads

$$\sum_i \frac{\#(S_i, S_i)^{-1}}{\sum_j \#(S_j, S_j)^{-1}} \theta_{S_i} = E_{m/2},$$

where S_i are representatives of classes of even unimodular lattices of rank m. In other words, a weighted sum, or integral for an appropriate measure, over theta series equals an Eisenstein series of weight $m/2$. Weil performed the step of reinterpreting the sum as an integral and transformed the above formula into the adelic context [Wei65].

In order to state the Siegel–Weil formula, we have to introduce the aforementioned theta integral. Recall the definition of adelic theta distributions as in Definition 5.6. Namely, for a Schwartz–Bruhat function $\varphi \in \mathcal{S}(V_{\mathbb{A}})$ we set

$$\theta(g, h; \varphi) = \sum_{x \in V_{\mathbb{Q}}} (\omega(g)\varphi)(h^{-1}x)$$

as a function in $g \in \mathrm{Mp}_2(\mathbb{A})$ and $h \in \mathrm{O}(V_{\mathbb{A}})$. Averaging over the argument h delivers the following standard theta integral.

Definition 5.19. For a Schwartz–Bruhat function $\varphi \in \mathcal{S}(V_{\mathbb{A}})$ define the following theta integral.

$$I(g;\varphi) := \int_{\mathrm{O}(V_{\mathbb{Q}})\backslash \mathrm{O}(V_{\mathbb{A}})} \theta(g', h; \varphi)\, \mathrm{d}h, \tag{5.22}$$

where $\mathrm{d}h$ is the standard invariant normalised measure on the quotient such that $\mathrm{vol}(\mathrm{O}(V_{\mathbb{Q}}) \setminus \mathrm{O}(V_{\mathbb{A}})) = 1$.

By Weil's convergence criterion, the integral $I(g;\varphi)$ above is absolutely convergent in case the rational space V is anisotropic or $\dim(V) = m > 2 + r$, where r denotes the Witt rank of V. In case this criterion is not matched, the integral may still be regularised as in [KR94, p. 41] and numerous results may be transferred to this setting.

With these preparations we require only one additional piece of notation to state the Siegel–Weil formula. Let $\varphi \in \mathcal{S}(V_{\mathbb{A}})$ be a Schwartz–Bruhat form and $\Phi \in I(s,\chi)$ its associated standard section through the principal series representation arising from the intertwining operator as described in Example 5.13 d) (also compare Lemma 5.12).

Definition 5.20. With the notation as above, write

$$\mathcal{E}(g', s; \varphi) := \mathcal{E}(g', s; \Phi), \qquad \mathcal{E}(\tau, s; \varphi) := \mathcal{E}(\tau, s; \Phi).$$

Theorem 5.21 (Siegel–Weil formula, [Kud03, Thm. 4.1 p. 36])**.** *Assume $m > 2+r$ or that V is anisotropic. Then the theta integral $I(g';\varphi)$ for $\varphi \in \mathcal{S}(V_{\mathbb{A}})$ is absolutely convergent. Further, $\mathcal{E}(g', s; \varphi)$ is holomorphic at $s_0 = \frac{m}{2} - 1$ and the following identity holds:*

$$\mathcal{E}(g', s_0; \varphi) = \kappa \cdot I(g'; \varphi),$$

where $\kappa = 2$, if $m \leq 2$, and $\kappa = 1$ otherwise.

In particular, the Eisenstein series associated to a lattice as it has been discussed in the previous section bears information about representation numbers of the quadratic lattice in its Fourier expansion at the critical point $s_0 = \frac{m}{2} - 1$. This is remarkable since the Eisenstein series is essentially build from local data, while the theta integral involves global arithmetic. Hence, theta integrals associated to different quadratic spaces may be related to each other, via the Eisenstein series, once the local data agrees.

Definition 5.22. Let V_p, V'_p be two quadratic $\mathbb{Q}_p$ spaces of dimension m. Suppose these spaces induce the same character $\chi_{V_p} = \chi_{V'_p}$ (cf. Example 1.62). Two functions $\varphi_p \in \mathcal{S}(V_p)$ and $\varphi'_p \in \mathcal{S}(V'_p)$ are called *matching*, if they give rise to the same section via the intertwining operator $\lambda_p(\varphi_p) = \lambda'_p(\varphi'_p)$.

In fact, in case of $m > 4$ and p non-Archimedian, every $\varphi_p \in \mathcal{S}(V_p)$ has a matching function $\varphi'_p \in \mathcal{S}(V'_p)$, since the local principal series $I_p(s_0, \chi_p)$ is irreducible.

In order to compare theta integrals as suggested above, it is hence sufficient to find a non-trivial match at the archimendian place. One solution to that problem is given by the Kudla–Millson Schwartz function which is introduced in the following section.

5.4.2 Kudla–Millson Schwartz form

Let (V, q) be a quadratic $\mathbb{Q}$ vector space of signature $(m^+, 2)$ and dimension m. Recall that for a complex manifold D, the space of smooth forms of Hodge type $(1,1)$ is denoted by $\mathscr{A}^{(1,1)}(D)$. Kudla and Millson have constructed a form $\varphi_{\mathrm{KM}} \in \mathcal{S}(V_{\mathbb{R}}) \otimes \mathscr{A}^{(1,1)}(\tilde{\mathbb{D}})$, where $\tilde{\mathbb{D}}$ denotes the oriented Grassmannian associated to the base lattice L, with the following properties.

i) For all $h \in \mathrm{O}(V_{\mathbb{R}})$
$$h^*\varphi_{\mathrm{KM}}(h^{-1}x) = \varphi_{\mathrm{KM}}(x). \tag{5.23}$$

ii) The form φ_{KM} has weight $m/2$ for K'_∞ for the Weil representation ω_∞, meaning
$$\omega_\infty(k')\varphi_{\mathrm{KM}} = \nu_{m/2}(k')\varphi_{\mathrm{KM}}, \tag{5.24}$$
where the character ν_l is presented in Remark 5.10.

iii) The form φ_{KM} is closed, i.e.
$$\mathrm{d}\varphi_{\mathrm{KM}} = 0 \tag{5.25}$$
for the exterior differential d on $\tilde{\mathbb{D}}$.

This form *matches* the Gaussian on a positive definite space of the same dimension in the sense that it gives rise to the same section through the principal series representation. To specify the meaning of this statement, we define the following form.

Definition 5.23. Define the following closed $\mathrm{O}(V_\infty)$ invariant $(1,1)$ form on $\tilde{\mathbb{D}}$:
$$\Omega := \varphi_{\mathrm{KM}}(0). \tag{5.26}$$

An explicit description in local coordinates on the tube domain model of that form is presented in [Kud03, Prop. 4.11 p. 330]. It is given by

$$\Omega = -\frac{1}{2\pi i}\left[-\,\mathrm{b}(y,y)^{-2}\,\mathrm{b}(y,\,\mathrm{d}z)\wedge\mathrm{b}(y,\,\mathrm{d}\overline{z}) + \mathrm{b}(y,y)^{-1}\tfrac{1}{2}\,\mathrm{b}(\,\mathrm{d}z,\,\mathrm{d}\overline{z})\right] \tag{5.27}$$

in coordinates on the tube domain model $\mathcal{H}^{\pm}_{m^+}$ (cf. (4.16)). Its negative $-\Omega$ is an invariant *Kähler form* on $\tilde{\mathbb{D}}$ and descends to a Kähler form on the variety X_K given in (5.30).

Note that we may write

$$\varphi_{\mathrm{KM}}(x)\wedge\Omega^{m-1} = \tilde{\varphi}_{\mathrm{KM}}(x)\Omega^{m}$$

for a function $\tilde{\varphi}_{\mathrm{KM}} \in \mathcal{S}(V_{\mathbb{R}})\otimes A^{(0,0)}(\tilde{\mathbb{D}})$. The $\mathrm{O}(V_\infty)$ invariance of Ω then implies that

$$\tilde{\varphi}_{\mathrm{KM}}(hx,hz) = \tilde{\varphi}_{\mathrm{KM}}(x,z)$$

for any $h\in\mathrm{O}(V_\infty)$. Further, for $k \coloneqq \frac{m}{2}$ we find

$$\lambda_\infty(\varphi_{\mathrm{KM}}(\,\cdot\,,z)) = \Phi^k(s_0)\Omega, \quad \lambda_\infty(\tilde{\varphi}_{\mathrm{KM}}(\,\cdot\,,z)) = \Phi^k(s_0) \tag{5.28}$$

for any $z\in\tilde{\mathbb{D}}$, where λ_∞ denotes the intertwining operator described in Lemma 5.12 and Φ^k_∞ denotes the standard section in $I_\infty(s,\chi_V)$ at the infinite place presented in Example 5.11. This means, the function $\tilde{\varphi}_{\mathrm{KM}}$ will be associated by λ_∞ to the same section as the standard Gaussian φ_0 on a space V' of signature $(m,0)$ if the local characters agree, i.e. they are *matching*. If further, we have two matching functions at the finite places $\varphi_f\in\mathcal{S}(V_f)$ and $\varphi'_f\in\mathcal{S}(V'_f)$, the Siegel–Weil formula (Theorem 5.21) yields

$$I(g';\tilde{\varphi}_{\mathrm{KM}}\otimes\varphi_f) = E(g',s_0;\tilde{\varphi}_{\mathrm{KM}}\otimes\varphi_f) = E(g',s_0;\varphi_0\otimes\varphi'_f) = I(g';\varphi_0\otimes\varphi'_f). \tag{5.29}$$

Before turning towards an essential application of the Siegel–Weil formula within the scope of this thesis, we remark that a more constructive approach to the Kudla–Millson Schwartz form will be sketched in Subsection 7.2.1 while an even more explicit representation is found in [MZ23].

5.4.3 A geometric integral

The following discussion evolves around [Kud03, Prop. 4.17 p. 332]. Before stating the main proposition, understanding the structure of the variety in terms of the considered reductive group is instructive.

Lemma 5.24. *Let G be a reductive group, $K_f \subset G(\mathbb{A}_f)$ be a compact open subgroup and $K_\infty \subset G(\mathbb{R})$ be a maximal compact subgroup. Assume $G(\mathbb{A}) = \bigsqcup_j G(\mathbb{Q})G^+(\mathbb{R})h_jK_f$ for suitable $h_j \in G(\mathbb{A}_f)$*[5] *and set $\Gamma_j \coloneqq G(\mathbb{Q}) \cap (G^+(\mathbb{R})h_jK_fh_j^{-1})$. Then by abuse of notation we understand $\Gamma_j \leq G^+(\mathbb{R})$ and find*

$$G(\mathbb{Q})\backslash(G(\mathbb{R}) \times G(\mathbb{A}_f))/(K_\infty \times K_f) = \bigsqcup_j \Gamma_j\backslash(G^+(\mathbb{R})/K_\infty).$$

Further, the following map is bijective and equivariant

$$\bigsqcup_j \Gamma_j\backslash G^+(\mathbb{R}) \to G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \quad \Gamma_jx \mapsto G(\mathbb{Q})(x,h_j)K_f.$$

Proof: By abuse of notation, we understand $\Gamma_j \subset G(\mathbb{Q}_p)$ for all p. First, we assume $x = \gamma_jy \in G^+(\mathbb{R})$ with $\gamma_j \in \Gamma_j$. Then recall that $\Gamma_j \subset G(\mathbb{Q}) \cap h_jK_fh_j^{-1}$, implying $(x,h_j) \equiv (\gamma_jy,h_j) \equiv (y,\gamma_j^{-1}h_j) \equiv (y,h_j)$, so that the map is well defined.
Next, we verify that the mapping is injective: Assume there are two $x,y \in \bigsqcup_j \Gamma_j\backslash G^+(\mathbb{R})$ with $(x,h_i) \equiv (y,h_j)$. Clearly, $i = j$, for otherwise the decomposition of $G(\mathbb{A})$ provided by the h_j is not disjoint. So assume $(x,h_j) \equiv (y,h_j)$. Then there are $\gamma \in G(\mathbb{Q})$ and $k \in K_f$ with $(x,h_j) = (\gamma y, \gamma h_jk)$. Hence, $h_j = \gamma h_jk \iff h_jk^{-1}h_j^{-1} = \gamma \in G(\mathbb{Q})$, meaning $\gamma = h_jk^{-1}h_j^{-1} \in \Gamma_j$. As a consequence, we find $x = \gamma y$, meaning the map is injective.
For surjectivity let there be $(x_\infty,x_f) \in G(\mathbb{A})$. By assumption, we know that there is $\gamma \in G(\mathbb{Q})$ with $\gamma^{-1}x_f \in h_jK_f$ for some j and $\gamma^{-1}x_\infty \in G^+(\mathbb{R})$. Then necessarily $(x_\infty,x_f) = \gamma(\gamma^{-1}x_\infty,1)\gamma^{-1}x_f$, i.e. $\Gamma_j\gamma^{-1}x_\infty \in \Gamma_jG^+(\mathbb{R})$ is a preimage. □

In our context, the considered group will be denoted by H and represents the GSpin group associated to the lattice (L,q) of signature $(m^+,2)$. Denote by $\tilde{\mathbb{D}}$ the oriented Grassmannian of L and choose a compact open subgroup $K_f \subset H(\mathbb{A}_f)$. Then we consider the variety

$$X_K \coloneqq H(\mathbb{Q})\backslash(\tilde{\mathbb{D}} \times H(\mathbb{A}_f))/K_f \tag{5.30}$$

[5] Compare [Kne65] as a reference for the method of strong approximation or [PRR93, Sec. 7.4 p. 427].

and have that

$$X_K = \bigsqcup_{j=1} \Gamma_j \backslash \mathbb{D}$$

as in the proposition above, where $\mathbb{D}$ is a connected component of $\tilde{\mathbb{D}}$. We write $Y_j \coloneqq \Gamma_j\backslash\mathbb{D}$ and note that there are $|\hat{\mathbb{Z}}^\times : \mathrm{N}(K_f)|$ of these components. Here, N denotes the Clifford norm given in Definition 4.28. With this preparation we may state the following modified version of [Kud03, Prop. 4.17 p. 332].

Proposition 5.25. *We find for a compact subgroup $K_f \leq H(\mathbb{A}_f)$ with $K_f \cap Z(\mathbb{A}) \simeq \hat{\mathbb{Z}}^\times$ under the isomorphism $Z(\mathbb{A}) \simeq \mathbb{A}^\times$ that*

$$(-1)^{m^+}\tfrac{1}{4}\,\mathrm{vol}(K_f)\cdot \int_{X_K} \theta(g';\varphi_{\mathrm{KM}}\otimes\varphi_f)\wedge\Omega^{m^+-1} = I(g';\tilde{\varphi}_{\mathrm{KM}}\otimes\varphi_f).$$

Further, the integral on the right hand side over each component Y_j of X_K is identical. Hence, for any component Y_j the following identity holds

$$(-1)^{m^+}\tfrac{1}{4}\,\mathrm{vol}(K_f)\cdot|\hat{\mathbb{Z}}^\times : \mathrm{N}(K_f)|\cdot \int_{Y_j} \theta(g';\varphi_{\mathrm{KM}}\otimes\varphi_f)\wedge\Omega^{m^+-1} = I(g';\tilde{\varphi}_{\mathrm{KM}}\otimes\varphi_f).$$

We would like to apply this result in the more classical setting of varieties defined at the infinite place over O as in [Bor98] and recall that for $H = \mathrm{GSpin}$ we find

$$1 \to \mathbb{Q}^\times \to H(\mathbb{Q}) \to \mathrm{SO}(V) \to 1,$$

as well as

$$H(\hat{\mathbb{Z}}) \to \Gamma'(\hat{L}) \to 1,$$

where $\Gamma'(\hat{L})$ denotes the discriminant kernel in $\mathrm{SO}(\hat{L})$. For $\Gamma \leq \mathrm{SO}(L)$, we may consider the closure $\overline{\Gamma} < \mathrm{SO}(V_{\mathbb{A}_f})$ and further select a cover $K_f < H(\mathbb{A}_f)$, containing $\hat{\mathbb{Z}}^\times$. In that case, we have that the Shimura variety X_K of the pair (H, K_f) is the same as the variety associated to $(\mathrm{SO}, \overline{\Gamma})$ (cf. [BY21, p. 1669]):

$$\mathrm{SO}(V)\backslash\mathbb{D}\times\mathrm{SO}(V_{\mathbb{A}_f})/\overline{\Gamma} \simeq H(\mathbb{Q})\backslash\mathbb{D}\times H(\mathbb{A}_f)/K_f.$$

In the special case of $K_f = \mathrm{GSpin}_{\hat{L}}$, we find that for the choice of $h_j = e$ being equal to the neutral element that $Y_j \simeq Y'(L) = \Gamma'(L)\backslash\mathbb{D}$, where $\Gamma'(L)$ is the discriminant kernel in SO. In fact, if we specialise further, to the case of L splitting a hyperbolic plane, we find that the index $|\hat{\mathbb{Z}}^\times : \mathrm{N}(K_f)|$ equals 1 so that there is only one connected component identifying with $Y'(L) = \Gamma'(L)\backslash\mathbb{D}$. All in all, we obtain the following corollary.

Corollary 5.26. *Let (L, q) be an even quadratic lattice of signature $(m^+, 2)$ and select $K_f = \mathrm{GSpin}_{\hat{L}}$. Further, write $Y_L = \Gamma(L)\backslash\mathbb{D}$ for the variety associated to the discriminant kernel $\Gamma(L)$ in $\mathrm{O}(L)$ and select an element $\varphi_f \in \mathcal{S}(V_{\mathbb{A}_f})$ that is invariant under $\Gamma(L)$. Then*

$$\int_{Y_L} \theta(g'; \varphi_{\mathrm{KM}} \otimes \varphi_f) \wedge \Omega^{m^+-1} = (-1)^{m^+} \frac{4c_K}{|\hat{\mathbb{Z}}^\times : \mathrm{N}(K_f)|} \cdot I(g', \tilde{\varphi}_{\mathrm{KM}} \otimes \varphi_f), \tag{5.31}$$

where $c_K = 1$ or $1/2$, depending on whether $\Gamma(L) \cap \mathrm{SO}(V)\backslash\mathbb{D} \to \Gamma(L)\backslash\mathbb{D}$ is bijective or two to one.

Proof: If considered in the special orthogonal setting, meaning integrating over $\Gamma'(L)\backslash\mathbb{D}$ instead of $\Gamma(L)\backslash\mathbb{D}$ on the left hand side of (5.31), the statement follows immediately from Proposition 5.25. The only task remaining is hence the translation to the orthogonal setting. By assumption, the integrand is invariant under pullback by elements from $\Gamma(L)$. In fact, the form Ω is invariant under $\mathrm{O}(V_\infty)$ and φ_{KM} fulfils (5.23). Further, φ_f was chosen to be invariant under $\Gamma(L)$ as well, so that we may translate the domain of integration by $\gamma \in \Gamma'(L)\backslash\Gamma(L)$ and combine a fundamental domain for $\Gamma'(L)\backslash\mathbb{D}$ from translates of a domain for $\Gamma(L)\backslash\mathbb{D}$. We obtain that the integral in question in (5.31) equals the integral over $\Gamma'(L)\backslash\mathbb{D}$ divided by the multiplicity of the cover $\Gamma'(L)\backslash\mathbb{D} \to \Gamma(L)\backslash\mathbb{D}$. Finally, we employ the determinant map and the fundamental theorem on group homomorphism or apply Lemma A.8 with $G = \mathrm{O}(V)$, $H = \mathrm{SO}(V)$ and $K = \Gamma(L)$ to find

$$[\Gamma(L) : \Gamma'(L)] \leq [\mathrm{O}(V) : \mathrm{SO}(V)] = 2,$$

meaning that the considered cover represents at most a double-cover. □

Part III

Intertwining worlds

6 L-series

The main purpose of this chapter is to introduce and investigate symmetric square type L-series associated to vector valued modular forms (cf. Definition 6.77). After a brief overview of classical L-functions in 6.1, classical Hecke theory for $\Gamma_1(N)$ modular forms is investigated and applied to derive properties of associated L-series. Afterwards, Hecke theory for vector valued modular forms is briefly reviewed based on [BS08] and properties of Hecke algebras arising in that context are verified. In the last section, L-series are associated to vector valued modular forms. We prove their convergence and realise them as a Rankin–Selberg integral to derive meromorphic continuation. In special cases, product expansions for these L-series are proven.

6.1 Introduction

The concept of an L-function is loosely speaking a generalisation of the Riemann ζ-function

$$\zeta : \{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1\} \to \mathbb{C}, \qquad s \mapsto \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

This famous function attributed to Riemann encodes deep number theoretic insights – the distribution of prime numbers among the naturals and it is equipped with remarkable analytic properties:

- An elementary bound, involving geometric sums, yields absolute convergence of the series for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. In this range, it converges normally and defines a holomorphic function without roots.
- For these $s \in \mathbb{C}$, the series has an alternative representation as a so called *Euler product* over the finite primes:

$$\zeta(s) = \prod_{p<\infty} \left(1 - p^{-s}\right)^{-1}. \tag{6.1}$$

- The Riemann zeta function ζ has a meromorphic continuation to the whole complex plane $\mathbb{C}$ with a single pole in $s = 1$ with residue 1.
- The so called *completed Riemann zeta function* $\Lambda(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$ has a point symmetry around 1/2:

$$\Lambda(s) = \Lambda(1 - s). \tag{6.2}$$

There is no universally accepted definition of L-functions. However, inspired by the properties of the Riemann ζ–function and later applications, the following description delimits a rough notion of what to expect of an L-function.

An L-function should be a holomorphic function L on a right half plane $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > \sigma\}$ for some $\sigma \in \mathbb{R}$ such that the following properties are fulfilled.

a) There are rational functions Q_p for any prime p, such that

$$L(s) = \prod_{p<\infty} Q_p(p^{-s})$$

on some right half plane.

b) There exist rational functions r_e and $r_1, \ldots, r_m$, signs $\pm_1, \ldots, \pm_m$, and a constant $C \in \mathbb{C}$, such that

$$\Lambda(s) := C \cdot e^{r_e(s)} \cdot \prod_{l=1}^{m} \Gamma(r_l(s))^{\pm_l} \cdot L(s)$$

admits meromorphic continuation to $\mathbb{C}$ with only finitely many poles all lying along the real line.

c) There is a functional equation. Explicitly, there are $\epsilon \in \mathbb{T}$ and $k \in \mathbb{R}$ such that

$$\Lambda(s) = \epsilon\Lambda(1 - s + k).$$

An L-series should then be a series of the form

$$L(a, s) := \sum_{n\in\mathbb{N}} \frac{a(n)}{n^s}$$

for a sequence $a : \mathbb{N} \to \mathbb{C}$ with the potential to represent an L-function in its range of convergence.

In the following, we will have a brief look at some elementary notions and the classical example of a Dirichlet L-series associated to a Dirichlet character.

Definition 6.1. A function $a : \mathbb{N} \to \mathbb{C}$ is called *weakly multplicative*, if $a(m \cdot n) = a(m) \cdot a(n)$ for $m, n \in \mathbb{N}$ with $\gcd(m, n) = 1$.

Example 6.2. i) Evidently, any multiplicative function $\chi : \mathbb{N} \to \mathbb{C}$ is weakly multiplicative.

ii) The sum divisor function $\sigma_s(n) = \sum_{d|n} d^s$ for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$ is weakly multiplicative.

iii) if ν_p denotes the p-adic evaluation of an integer, then the function $\mathbb{N} \ni n \mapsto \lambda(n) = (-1)^{\sum_{p<\infty} \nu_p(n)}$ is weakly multiplicative.

iv) The *Möbius function*

$$n \mapsto \mu(n) := \begin{cases} 0, & \text{if } n \text{ contains a square,} \\ (-1)^{\#\{p \mid \nu_p(n) > 0\}}, & \text{else.} \end{cases} \tag{6.3}$$

In compact and uncommon notation, we may write $\mu(n) = \delta_{\nexists \square | n} \cdot \lambda(n)$.

Example 6.3. a) If $a, b : \mathbb{N} \to \mathbb{C}$ are weakly multiplicative, so is their product $a \cdot b$, and a^s for $s \in \mathbb{C}$ for $\mathrm{Re}(s) > 0$. In particular, $\mathbb{N} \ni n \mapsto \frac{a(n)}{n^s}$ is multiplicative.

b) If $a, b : \mathbb{N} \to \mathbb{C}$ are weakly multiplicative, so is the convolution

$$[a * b](n) = \sum_{d|n} a(d) b(n/d).$$

This product naturally arises as the coefficient sequence of $L(a, s) \cdot L(b, s) = L(a * b, s)$. It is commutative, admits a neutral element and an inverse to a, if, and only if, $a(1) \neq 0$.

Remark 6.4. If a is weakly multiplicative and $\sum_n a(n) \neq 0$ is absolutely convergent then

$$\sum_{n=1}^{\infty} a(n) = \prod_p \left(\sum_{l=0}^{\infty} a\left(p^l\right) \right).$$

If a is in fact multiplicative,

$$\sum_{n=1}^{\infty} a(n) = \prod_{p<\infty} \frac{1}{1-a(p)}.$$

Example 6.5. For a Dirichlet character χ the L-function $L(\chi, s) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$ converges absolutely for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ by comparing it to the Riemann zeta function $\zeta(s)$ and admits the following *Euler product*

$$L(\chi, s) = \prod_{p<\infty} \frac{1}{1-\chi(p)p^{-s}}.$$

In particular, for $\chi = \chi_0$ the trivial character on $(\mathbb{Z}/N\mathbb{Z})^\times$ we have

$$L(s, \chi_0) = \prod_{p|N} (1-p^{-s}) \cdot \zeta(s).$$

Also, the series $L(\chi, s)$ converge conditionally in case the character χ is non-trivial[1] and never vanishes at $s = 1$.

Should the reader be interested in obtaining a better impression of Dirichlet series he may consider the compactly presented discussion in [Zag13, pp 1-15]. Further, L-series may also be associated to scalar valued modular forms. In fact, if $f \in \mathcal{M}_k(\Gamma_0(N), \chi)$ has Fourier expansion

$$f(\tau) = \sum_{n\in\mathbb{N}_0} a(n)q^n$$

we may associate to it the L-series

$$L(f, s) := \sum_{n\in\mathbb{N}} \frac{a(n)}{n^s}. \tag{6.4}$$

This series converges on a right half plane due to asymptotic bounds on the Fourier coefficients $a(n)$ (cf. Corollary 3.76) and possesses a meromorphic continuation. There is even a converse theorem for such L-functions due to André Weil presented in [Bum98, Thm. 1.5.1 p. 60]. Further, by developing a theory of Hecke operators, which will be partially carried out in the following section, product expansions of such L-series may be constructed for certain classes of modular forms (cf. Theorem 6.45).

[1]IN case of trivial character, it has a pole at $s = 1$.

6.2 Classical Hecke theory

In this section, we explore classical Hecke theory for $\Gamma_1(N)$ modular forms. In fact, the theory is reduced to the case $\Gamma_0(N)$ with character. Hecke operators are endomorphisms of modular forms and allow for proving that L-series associated to modular forms have not only meromorphic continuation, but also admit product expansions. For the content of this section, compare [Bum98, 1.4 pp. 41-53] and [Kna92, IX.6 p. 273].

We return to elliptic modular forms for a moment. These form, for fixed weight and congruence group, a finite dimensional vector space. In order to comprehend the structure of objects considering morphisms on them is a fundamental approach. Since these are finite dimensional vector spaces the ring of endomorphisms is, of course, represented by square matrices with entries in the ground field. This, however, does not provide any meaningful insight.

Hence, the aim is to naturally construct special algebras of operators on these spaces. Recall the Petersson slash operator of weight $k \in \mathbb{Z}$ for $\gamma \in \mathrm{GL}_2(\mathbb{R})^+$:

$$|_k\gamma : \mathbb{C}^{\mathbb{H}} \to \mathbb{C}^{\mathbb{H}}, \qquad f \mapsto \Big(z \mapsto f|_k\gamma(\tau) := \det(\gamma)^{k/2} j(\gamma,\tau)^{-k} f(\gamma\tau)\Big).$$

This defines a right action of $\mathrm{GL}_2(\mathbb{R})^+$ that preserves smoothness conditions.[2] The subspace of $\mathcal{C}^{\mathrm{hol}}(\mathbb{H},\mathbb{C})$ on which a subgroup $\Gamma < \mathrm{GL}_2(\mathbb{Q})^+$ that is commensurable to $\Gamma(1)$ operates as the identity comprises the space of weakly holomorphic modular forms of weight k to Γ.

These operators form the basis of constructing the desired endomorphism ring. Observe that for $g \in \mathrm{GL}_2(\mathbb{Q})^+$ and $f \in \mathcal{M}_k(\Gamma)$ we find

$$f|_k g \in \mathcal{M}_k(g^{-1}\Gamma g).$$

The basic idea is to construct an appropriate linear combination of operators $|_k g$ in order to cancel terms to receive a form that is invariant under Γ, again. The first obvious candidate of an index set would be $g\Gamma$ in order to achieve invariance by summation. However, this does obviously not converge in general so means of forcing convergence are to be sought. A naive idea for diminishing the index set of the sum would be utilising the invariance of f to quotient out Γ from the left. However, Γ then was required to operate on the index set $g\Gamma$ so the set, indeed, had to be $\Gamma g\Gamma$. This, on the other hand, appears

[2] Note that the centraliser $\mathbb{R}\cdot\mathcal{I}$ operates trivially if k is even and $\mathbb{R}^+\cdot\mathcal{I}$ if k is odd causing spaces of automorphic forms to vanish if k is odd and $-\mathcal{I} \in \Gamma$.

to be a promising candidate. Now, a reasonable condition to force convergence is that the quotient $\Gamma\backslash\Gamma g\Gamma$ is finite. There is a wide range of groups for which this is possible and uniformly computable: the so called *Hecke* or *congruence* subgroups (cf. Definition 2.18).

Remark 6.6. Let $\Gamma \leq \Gamma(1)$ be a congruence subgroup and $g \in \mathrm{GL}_2(\mathbb{Q})^+$. Then there is $M \in \mathbb{N}$ such that $\Gamma(M) \leq g^{-1}\Gamma g$. In particular, $\Gamma\backslash\Gamma g\Gamma$ is finite.

Proof: Let $\Gamma(N) \leq \Gamma$ and choose N_1, N_2 such that $N_1 g$ and $N_2 g^{-1}$ have integer entries. Define $M = NN_1N_2$ and write $\gamma = \mathcal{I} + M\gamma'$ for $\gamma \in \Gamma(M)$ where γ' has integer entries. Clearly, $g\gamma g^{-1} = \mathcal{I} + N(N_1 g)\gamma'(N_2 g^{-1}) \in \Gamma(N) \leq \Gamma$.
Further, $|\Gamma\backslash\Gamma g\Gamma| = [\Gamma : g^{-1}\Gamma g \cap \Gamma]$. The intersection, however, must contain a congruence group which has finite index in $\Gamma(1)$ and hence in Γ. □

This statement not only yields the desired finiteness result of the symmetrised sum discussed above, directly yielding the operators that had been advertised, but also that the property of being a congruence modular form is preserved by the action of $\mathrm{GL}_2(\mathbb{Q})^+$. In fact, every space of congruence modular forms can be injected into a space of modular forms for $\Gamma_1(M)$ for an adequate $M \in \mathbb{N}$ which is beneficial for explicit computations and stated below. Note that this operation preserves the property of vanishing at cusps (cf. Lemma 3.77).

Remark 6.7. Let $\Gamma \leq \Gamma(1)$ be a congruence subgroup. Then there is $g \in \mathrm{GL}_2(\mathbb{Q})^+$ and $M \in \mathbb{N}$ such that $\Gamma_1(M) \leq g^{-1}\Gamma g$.

Proof: Let $\Gamma(N) \leq \Gamma$ and $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q})^+$ with entries in $\mathbb{Z}$ such that $N \mid a^2$ and $N \mid c^2$. To ensure the former condition, multiply with the least common multiple of the denominators. Then for any $\mathcal{I} + \gamma \in \Gamma_1(N\det(g))$ there is $h \in \mathbb{N}$ such that

$$\begin{aligned} g(\mathcal{I}+\gamma)g^{-1} &\equiv \mathcal{I} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 0 & h \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \mod N \\ &\equiv \mathcal{I} + h \cdot \begin{pmatrix} -ac & a^2 \\ -c^2 & ac \end{pmatrix} \mod N \\ &\equiv \mathcal{I} \mod N. \end{aligned}$$

Consequently, $g\Gamma_1(N\det(g))g^{-1} \leq \Gamma(N) \leq \Gamma$. □

In particular, for any congruence modular form f of weight k there is $g \in \mathrm{GL}_2(\mathbb{Q})^+$ and a natural number N such that $f|_k g \in \mathcal{M}_k(\Gamma_1(N))$ ($\iff f \in \mathcal{M}_k(g\Gamma_1(N)g^{-1})$). In fact, the reduction may be taken further by harnessing a decomposition of $\Gamma_1(N)$ modular forms into $\Gamma_0(N)$ modular forms with Dirichlet character.

Remark 6.8 ([Miy06, Lemma 4.3.1 p. 114])**.** Let Γ be commensurable to $\Gamma(1)$ and assume there is $\Gamma' \trianglelefteq \Gamma$ such that Γ/Γ' is finite abelian. Then for $R_k \in \left\{\mathcal{A}_k, \mathcal{M}^!_k, \mathcal{M}_k, \mathcal{S}_k, \mathcal{S}_k^\perp \leq \mathcal{M}_k\right\}$

$$R_k(\Gamma') = \bigoplus_{\chi \in (\Gamma/\Gamma')^*} R_k(\Gamma, \chi).$$

Proof: The group Γ acts on $R_k(\Gamma')$ via $\gamma \mapsto (f \mapsto f|_k\gamma)$, descending to a representation of Γ/Γ' which decomposes by Example B.13 into irreducible representations, meaning characters as the quotient is abelian. □

Example 6.9. For $N \in \mathbb{N}$: $\Gamma_1(N) \trianglelefteq \Gamma_0(N)$ and $\Gamma_0(N)/\Gamma_1(N) = (\mathbb{Z}/N\mathbb{Z})^\times$ by

$$\Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^\times, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \overline{a}.$$

As a consequence, we have for instance

$$\mathcal{M}_k(\Gamma_1(N)) = \bigoplus_{\substack{\chi \text{ Dirichlet} \\ \text{mod } N}} \mathcal{M}_k(\Gamma_0(N), \chi). \tag{6.5}$$

This sum is exploited to justify only explicitly constructing the theory of Hecke operators for $\mathcal{M}_k(\Gamma_0(N), \chi)$. To do so, the Petersson slash operator will be altered for $\Gamma_0(N)$ to include characters:

$$|_{k,\chi} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : f \mapsto \left(z \mapsto f|_{k,\chi} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(z) = \overline{\chi(a)} \det(\gamma)^{k/2} j(\gamma, z)^{-k} f(\gamma z)\right).$$

However, in order to treat modular forms with characters, the modified Petersson slash operator has to be extended (analogously to the extension of the regular one to $\mathrm{GL}_2(\mathbb{Q})^+$). Hence, an extension of $\Gamma_0(N) < \mathrm{GL}_2(\mathbb{Q})^+$ on which the Dirichlet character χ mod N operates naturally has to be constructed. For a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ evaluating a character $\chi \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \chi(a)$ clearly fails for general rationals a but may be defined with ease in case that the denominator of a is coprime to N.
To this end let S_N be the prime divisors of N and $\mathbb{Z}_{S_N^C}$ denote the localisation along all other primes (rendering them invertible). For $r/s \in \mathbb{Z}_{S_N^C}$ write $\chi(r/s) := \chi(r)/\chi(s)$, so χ is applicable to elements of $\mathrm{GL}_2(\mathbb{Z}_{S_N^C})^+$. However, it fails to be a character of this group

unless restricting to the subgroup of matrices with left lower entry being in $N\mathbb{Z}_{S_N^C}$ (an analogue of $\Gamma_0(N)$).

Definition 6.10. Define the following subgroup of $\mathrm{GL}_2^+(\mathbb{Q})$

$$G_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_{S_N^C}) \;\middle|\; c \in N\mathbb{Z}_{S_N^C} \right\} \cap \mathrm{GL}_2^+(\mathbb{Q}).$$

Recall that by Remark 6.6 there are finitely many representatives $g_i \in \mathrm{GL}_2(\mathbb{Q})^+$ giving rise to the following decomposition into right cosets

$$T_g := \Gamma_0(N) g \Gamma_0(N) = \bigcup_i \Gamma_0(N) g_i. \tag{6.6}$$

Hence, the following is an immediate definition for a natural operator on spaces of modular forms.

Definition 6.11. Let $g \in G_0(N)$, $k \in \mathbb{Z}$ and χ be a Dirichlet character for $\Gamma_0(N)$ and assume the decomposition in (6.6). Then the associated *Hecke operator* is defined to be

$$|_{k,\chi} T_g : \mathcal{C}_k^\alpha(\Gamma_0(N), \chi) \to \mathcal{C}_k^\alpha(\Gamma_0(N), \chi), \quad f \mapsto \sum_i f|_{k,\chi} g_i. \tag{6.7}$$

Remark 6.12. The operator $|_{k,\chi} T_g$ above descends to an endomorphism of the space of holomorphic modular forms $\mathcal{M}_k(\Gamma_0(N), \chi)$ and of cusp forms $\mathcal{S}_k(\Gamma_0(N), \chi)$.

Now that there is a relatively natural family of operators on a wide spectrum of modular forms, the structure of their algebra has to be investigated. Ultimately, this will yield product expansions of L-functions associated to cusp forms (cf. Proposition 6.42).

For $f \in \mathcal{C}_k^\infty(\Gamma_0(N), \chi)$ and $\alpha, \beta \in G_0(N)$ consider the expression

$$f|_{k,\chi} T_\alpha T_\beta = \sum_{i,j} f|_{k,\chi} \alpha_i \beta_j.$$

Since f is invariant with respect to $|_{k,\chi}\Gamma_0(N)$, $\alpha_i\beta_j$ may be pooled in families $\Gamma_0(N)\alpha_i\beta_j$. However, choosing other α_i, which amounts to multiplying them from the left with $\gamma_i \in \Gamma_0(N)$, does not change the family $\Gamma_0(N)\alpha_i\beta_j$ and multiplying the collection α_i from the right side by a fixed element $\gamma \in \Gamma_0(N)$ results in permuting the cosets in (6.6) yielding a family $\gamma_i \in \Gamma_0(N)$ such that $\{\alpha_i\}$ is replaced by $\{\gamma_i\alpha_i\}$. As a consequence, the number of elements $\alpha_i\beta_j$ occuring in $\Gamma_0(N)\alpha_i\beta_j$ is independent of the concrete choices for

α_i, β_j and may be denoted $n(\alpha, \beta; \sigma)$ for some $\sigma \in \Gamma_0(N)\alpha_i\beta_j$. As a second consequence, it is independent under multiplying σ with an element $\gamma \in \Gamma_0(N)$ from the right, yielding

$$\sum_{i,j} f|_{k,\chi}\alpha_i\beta_j = \sum_{\sigma \in \Gamma_0(N)\backslash G_0(N)} n(\alpha, \beta; \sigma) f|_{k,\chi}\sigma = \sum_{\sigma \in \Gamma_0(N)\backslash G_0(N)/\Gamma_0(N)} n(\alpha, \beta; \sigma) f|_{k,\chi} T_\sigma.$$

This inspires defining a multiplication on the objects T_α by the above formula. To be precise, let $\mathscr{H}_0(N)$ denote the free abelian group over $\{T_\alpha \mid \alpha \in \Gamma_0(N)\backslash G_0(N)/\Gamma_0(N)\}$ and define a multiplication via

$$T_\alpha T_\beta := \sum_{\sigma \in \Gamma_0(N)\backslash G_0(N)/\Gamma_0(N)} n(\alpha, \beta; \sigma) \cdot T_\sigma. \tag{6.8}$$

This is immediately recognised to be associative by explicitly writing the triple product out yielding a sum of the form

$$T_\alpha T_\beta T_\gamma = \sum_{\sigma \in \Gamma_0(N)\backslash G_0(N)/\Gamma_0(N)} n(\alpha, \beta, \gamma; \sigma) \cdot T_\sigma.$$

Definition 6.13. The ring $\mathscr{H}_0(N)$ is called *abstract Hecke algebra* for $\Gamma_0(N)$.

Remark 6.14. Usually the ring $\mathscr{H}_0(N)$ is tensorised with $\mathbb{Q}$ (or $\mathbb{R}$) over $\mathbb{Z}$ and becomes an algebra - there will be no distinction in notation.

In order to characterise the ring $\mathscr{H}_0(N)$, the different symbols T_α are classified, first. This is a multi step process and may be inferred from the base case $N = 1$. In that instance, the elementary divisor theorem yields the following.

Remark 6.15. The set $\Gamma(1)\backslash \mathrm{GL}_2(\mathbb{Q})^+/\Gamma(1)$ is faithfully represented by

$$\begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}$$

where $D_1, D_2 \in \mathbb{Q}_{>0}$ such that $D_1/D_2 \in \mathbb{N}$. In particular, $\Gamma(1)\alpha\Gamma(1) = \Gamma(1)\alpha^T\Gamma(1)$.

Proof: The proof is completely contained in [Bum98, 1.4.2 p. 44]. □

Remark 6.16. The set $\Gamma(1)\left(\begin{smallmatrix} D_1 & 0 \\ 0 & D_2 \end{smallmatrix}\right)\Gamma(1)$, with D_1, D_2 as above, equals

$$\left\{D_2 \cdot M \mid M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{Z}), \gcd(a, b, c, d) = 1, \det(M) = D_1/D_2\right\}.$$

Proof: First,

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \cdot \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} = \begin{pmatrix} \alpha D_1 \alpha' + \beta D_2 \gamma' & \alpha D_1 \beta' + \beta D_2 \delta' \\ \gamma D_1 \alpha' + \delta D_2 \gamma' & \gamma D_1 \beta' + \delta D_2 \delta' \end{pmatrix}.$$

Assume $D_2 = 1$ by factoring out. Obviously, a common divisor had to divide D_1, since the determinant is D_1. However, it cannot divide D_1 since then w.l.o.g. β must be divisible by it, preventing δ to be. But then γ' and δ' had to, leading to a contradiction.
On the other hand, by Remark 6.15 any matrix in $\mathrm{GL}_2(\mathbb{Q})^+$ is in such a double coset and D_1, D_2 are determined by the greatest common divisor of the entries and the determinant $D_1 D_2$ (paired with the condition $D_1/D_2 \in \mathbb{N}$). □

With these statements as a foundation we may prove the general case. Recall that we write $\mathbb{Z}_{S_N^C}$ for the localisation of $\mathbb{Z}$ along all primes not dividing N. Explicitly, let $R \coloneqq \{z \in \mathbb{Z} \mid \gcd(z, N) = 1\}$, then $\mathbb{Z}_{S_N^C} = R^{-1}\mathbb{Z}$.

Remark 6.17. $\Gamma_0(N)\backslash G_0(N)/\Gamma_0(N)$ is faithfully represented by matrices

$$\begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}$$

with $D_1, D_2 \in \mathbb{Z}_{S_N^C}^\times \cap \mathbb{Q}^+$ such that $D_1/D_2 \in \mathbb{N}$. Consequently, for all $\alpha \in G_0(N)$, we have $\Gamma_0(N)\alpha\Gamma_0(N) = \Gamma_0(N)\alpha^T\Gamma_0(N)$.

Proof: Let $\alpha \in G_0(N)$ and select (by virtue of Remark 6.15) $\gamma_1, \gamma_2 \in \Gamma(1)$ such that $\alpha = \gamma_1 \delta \gamma_2$ with $\delta = \mathrm{diag}(D_1, D_2)$ such that $D_2 \mid D_1$ and $D_1/D_2 \in \mathbb{N}$.
By Remark 6.16 we find that N cannot have a common divisor with D_1 and D_2, meaning $\gcd(D_1/D_2, N) = 1$. Further, we find for an arbitrary element $\gamma_D = \left(\begin{smallmatrix} a & b \\ c \cdot D_1/D_2 & d \end{smallmatrix}\right) \in \Gamma_0(D_1/D_2)$ that

$$\begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c\frac{D_1}{D_2} & d \end{pmatrix} = \begin{pmatrix} aD_1 & bD_1 \\ cD_1 & dD_2 \end{pmatrix} = \begin{pmatrix} a & b\frac{D_1}{D_2} \\ c & d \end{pmatrix} \cdot \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}.$$

Hence, $\gamma_2 = \left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right)$ may be altered by an element γ_D like above from the left, if γ_1 is altered, respectively. The left lower entry of the resulting matrix $\gamma_D\gamma_2$ is $c\frac{D_1}{D_2}a_2 + dc_2$ and we endeavour to select c, d such that this number is congruent to zero modulo N. Let $g \coloneqq \gcd\left(\frac{D_1}{D_2}a_2, c_2\right)$. The Euclidean algorithm yields a solution (cN, dN) such that $g\left(c\frac{D_1}{D_2 g}a_2 + dc_2/g\right) \cdot N = gN$. By passing to the solution $(c', d') \coloneqq (cN \pm c_2, dN \mp \frac{D_1}{D_2}a_2)$

we obtain a tuple for which $\gcd(c', d') = 1$, remaining valid for the pair $\left(\frac{D_1}{D_2}c', d'\right)$.[3] As a consequence, the Lemma of Bézout implies the existence of the desired matrix $\gamma_D \in \Gamma_0(D_1/D_2)$, such that $\gamma_D\gamma_2 \in \Gamma_0(N)$. Hence, w.l.o.g. $\gamma_2 \in \Gamma_0(N)$ implying $\gamma_1 = \alpha\gamma_2^{-1}\delta^{-1} \in G_0(N) \cap \Gamma(1) \leq \Gamma_0(N)$. □

This characterisation of the symbols appearing in the Hecke algebra is, in fact, more useful than it might initially appear to be. Indeed, it may be used to prove that the algebra $\mathscr{H}_0(N)$ is commutative. First, however, the following preparatory assertion is stated.

Remark 6.18. The representatives g_i in Equation (6.6) may be chosen in such a fashion that

$$\Gamma_0(N)g\Gamma_0(N) = \bigcup \Gamma_0(N)g_i = \bigcup g_i\Gamma_0(N).$$

Proof: By Remark 6.17

$$\bigcup \Gamma_0(N)g_i = \Gamma_0(N)\alpha\Gamma_0(N) = \Gamma_0(N)\alpha^t\Gamma_0(N) = \bigcup g_i^t\Gamma_0(N)$$

and the matrices g_i, g_i^t generate the same double coset. Hence, $g_i = \gamma g_i^t\gamma'$ for $\gamma, \gamma' \in \Gamma_0(N)$, so $\gamma^{-1}g_i = g_i^t\gamma' \in \Gamma_0(N)g_i \cap g_i^t\Gamma_0(N)$ are the desired representatives. □

Proposition 6.19. *The algebra $\mathscr{H}_0(N)$ is commutative.*

Proof: The proof is, neglecting minor modifications, carried out in [Bum98, p. 45]. □

This algebra is not only remarkably well behaved, but the extent of naturality of the action on the spaces of modular forms is reflected in its compatibility with the Petersson product. For that insight recall the following expression for the Petersson product from Remark 2.23 where we write $\tau = u + iv \in \mathbb{H}$

$$\langle f, g\rangle = \frac{1}{[\Gamma(1):\Gamma(N)]}\int_{\Gamma(N)\backslash\mathbb{H}} f(\tau)\overline{g(\tau)}v^k\,\frac{\mathrm{d}x\,\mathrm{d}y}{v^2}$$

defining a scalar product on $\mathcal{S}_k(\Gamma_0(N), \chi)$.

Proposition 6.20. *The algebra $\mathscr{H}_0(N)$ acts on $\mathcal{S}_k(\Gamma_0(N), \chi)$ as a commutative algebra of normal operators. As such, it admits an orthogonal basis of simultaneous eigenforms of $\mathcal{S}_k(\Gamma_0(N), \chi)$. In case $\chi = 1$, the operators are self adjoint.*

[3] Note that if there was a divisor, it had to be a divisor of D_1/D_2, but then it also had to divide dN. However, D_1/D_2 has neither a common divisor with N nor d.

The proof is split into the following two lemmas.

Lemma 6.21. *For $f, g \in \mathcal{S}_k(\Gamma_0(N), \chi)$ and $\alpha \in G_0(N)$ we have*

$$\langle f|\alpha, g\rangle = \langle f, g|\alpha^{-1}\rangle. \tag{6.9}$$

Proof: Recall that by Remark 6.6 there is $\Gamma' \coloneqq \Gamma(N') \leq \alpha^{-1}\Gamma(N)\alpha$. Since the measure is Moebius invariant the transformation $\tau \mapsto \alpha^{-1}\tau$ yields

$$\begin{aligned}
&\langle f|\alpha, g\rangle \\
&\overset{(*)}{=} \frac{1}{[\Gamma(1):\Gamma']}\int_{\Gamma'\backslash\mathbb{H}} f|_{k,\chi}\alpha(\alpha^{-1}\tau)\cdot\overline{g}(\alpha^{-1}\tau)\left(\frac{\det(\alpha^{-1})}{j(\alpha^{-1},\tau)\overline{j(\alpha^{-1},\tau)}}\right)^k y^k\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \\
&= \frac{1}{[\Gamma(1):\Gamma']}\int_{\Gamma'\backslash\mathbb{H}} \frac{\overline{\chi(a)}\det(\alpha)^{k/2}}{j(\alpha,\alpha^{-1}\tau)^k}f(\tau)\cdot\overline{j(\alpha^{-1},\tau)^{-k}g(\alpha^{-1}\tau)}\left(\frac{\det(\alpha^{-1})}{j(\alpha^{-1},\tau)}\right)^k y^k\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \\
&= \frac{1}{[\Gamma(1):\Gamma']}\int_{\Gamma'\backslash\mathbb{H}} \frac{\overline{\chi(a)}}{\chi(d/\det(\alpha))}f(\tau)\cdot\overline{g|_{k,\chi}\alpha^{-1}}(\tau)y^k\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \\
&= \langle f, g|\alpha^{-1}\rangle.
\end{aligned}$$

In the last step $\alpha \in G_0(N)$ has been used, so that $\det(\alpha) = ad \mod N\mathbb{Z}_{S_N^C}$. However, then

$$\frac{\overline{\chi(a)}}{\chi(d/\det(\alpha))} = \chi(\det(\alpha)/ad) = \chi(ad-bc)/\chi(ad) = 1.$$

In $(*)$ the group Γ' is replaced by $\alpha\Gamma'\alpha^{-1} \leq \Gamma(N) \leq \Gamma(1)$. By the invariance of the measure (used in $\downarrow$)

$$\mathrm{vol}(\mathcal{F}_{\Gamma(1)})\cdot[\Gamma(1):\alpha\Gamma'\alpha^{-1}] = \mathrm{vol}(\alpha\Gamma'\alpha^{-1}) \overset{\downarrow}{=} \mathrm{vol}(\Gamma') = \mathrm{vol}(\mathcal{F}_{\Gamma(1)})\cdot[\Gamma(1):\Gamma'],$$

so that descending to a Hecke subgroup suffices implying that this inconvenience could be ignored. □

Lemma 6.22. *The adjoint of $|_{k,\chi}T_\alpha$ with $\alpha = \mathrm{diag}(D_1, D_2)$ as in Remark 6.17 with respect to the Petersson scalar product is given by*

$$(|_{k,\chi}T_\alpha)^* = M_{\chi(D_1D_2)}|_{k,\chi}T_\alpha.$$

Here, M_z for $z \in \mathbb{C}$ denotes the multiplication operator by z.

Proof: Clearly, Equation (6.9) states that the expression is invariant under translating α from the left or right by an element in $\Gamma_0(N)$ implying that each element $\beta \in \Gamma_0(N)\backslash\Gamma_0(N)\alpha\Gamma_0(N)$ may be replaced by α under the scalar product. By Remark 6.6 the number of these is finite, say $n_\alpha \in \mathbb{N}$. Then, however, the statement is trivial for $\mathcal{S}_k(\Gamma(1))$, as in that case $\det(\alpha)\alpha^{-1} = S\alpha S^{-1}$ and scalar matrices operate trivially yielding

$$\langle f|_k T_\alpha, g\rangle = n_\alpha \langle f|_k\alpha, g\rangle = n_\alpha\langle f, g|_k\alpha\rangle = \langle f, g|_k T_\alpha\rangle.$$

In case of $\Gamma_0(N)$ with $N > 1$, however, the computation is more subtle. First note that scalar matrices act by multiplication with a character and, hence, transform nearly trivially:

$$(|_{k,\chi}T_{D_2\mathcal{I}})^* = \left(M_{\overline{\chi(D_2)}}\right)^* = M_{\chi(D_2)} = |_{k,\chi}T_{D_2^{-1}\mathcal{I}}.$$

As a consequence, only T_β with $\beta = \beta(D_1) = \left(\begin{smallmatrix} D_1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $D_1 \in \mathbb{N}$ remain to be considered. Select $m_1, m_2, \in \mathbb{Z}$ such that

$$D_1^2 m_1 - N m_2 = 1$$

and notice that

$$\begin{pmatrix} D_1 m_1 & m_2 \\ N & D_1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & D_1 \end{pmatrix} \cdot \begin{pmatrix} m_1 & m_2 \\ N & D_1^2 \end{pmatrix}^{-1} = \begin{pmatrix} D_1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Employing that computation immediately yields the desired adjoint, where $n_{\beta(D_1)} \in \mathbb{N}$ is chosen suitably:

$$\begin{aligned} \langle f|_{k,\chi}T_{\beta(D_1)}, g\rangle &= n_{\beta(D_1)}\langle f|_{k,\chi}\beta(D_1), g\rangle \\ &= n_{\beta(D_1)}\langle f, g|_{k,\chi}(D_1^{-1}\mathcal{I})\beta(D_1)\rangle \\ &= \langle f, \chi(D_1)\cdot g|_{k,\chi}T_{\beta(D_1)}\rangle. \end{aligned}$$

Combining the computations above yields for $\alpha = \operatorname{diag}(D_1, D_2)$

$$(|_{k,\chi}T_\alpha)^* = \left(|_{k,\chi}T_{D_2\mathcal{I}}T_{\beta(D_1/D_2)}\right)^* = M_{\chi(D_1)}|_{k,\chi}T_{\beta(D_1/D_2)} = M_{\chi(D_1D_2)}|_{k,\chi}T_\alpha,$$

meaning the operator $|_{k,\chi}T_\alpha$ is in general normal. Further, it is self adjoint, if, and only if, $\chi(D_1D_2) = 1$. Note that the proof relied upon $\gcd(D_1, N) = 1$. □

We find the following application to the eigenvalues of Hecke operators.

Example 6.23. Note that the lemma above implies for the Hecke eigenvalue σ of T_α

$$\overline{\sigma} = \chi(D_1 D_2) \cdot \sigma. \tag{6.10}$$

This yields, in particular, for the angle ϑ of σ in polar coordinates that $2\vartheta(\sigma) \equiv -\vartheta(\chi(D_1D_2))$, implying

$$\sigma \in \mathbb{R} \cdot \zeta_{2\,\mathrm{ord}(\chi)} \subseteq \mathbb{R} \cdot \zeta_{2N}. \tag{6.11}$$

In particular, in case of $\chi = 1$, the value of σ must be real. If χ, on the other hand, is quadratic, we find that σ must be real or purely imaginary, depending on whether $\chi(D_1D_2)$ equals 1 or -1.

Proposition 6.20 has the significant application of yielding Euler products. In order to relate these operators directly to the coefficients of L-functions associated to modular forms, the operators are clustered in an appropriate fashion.
Recall that $q\mathcal{I} \in (\mathbb{Z}_{S_N^C})^\times_{>0} \cdot \mathcal{I} < G_0(N)$ acts trivially via the Petersson slash operator as multiplication with $\overline{\chi}(q)$, so that any Hecke operator $|_{k,\chi} T_\alpha$ is essentially represented by a matrix α with integer entries. Further, let $n \in \mathbb{N}$ with $\gcd(n, N) = 1$ and denote for the moment[4] by $T(n)$ the operator arising from summing over all T_α where $\alpha \in G_0(N)$ is of the form $\mathrm{diag}(D_1, D_2)$, such that $D_2 \mid D_1$ (cf. Remark 6.17) and $D_1, D_2 \in \mathbb{N}$, with the limitation $\det(\alpha) = D_1 \cdot D_2 = n$.
By Definition 6.11

$$f|_{k,\chi} T(n) = \sum_i f|_{k,\chi} \beta_i \tag{6.12}$$

for suitable choices of $\beta_i \in M_2(\mathbb{Z})$. In fact, if $M(n, N)$ denotes the subset of $M_2(\mathbb{Z}) \cap G_0(N)$ consisting of matrices with determinant n we have that

$$M(n, N) = \bigcup_i \Gamma_0(N) \beta_i. \tag{6.13}$$

Definition 6.24. The $\mathbb{Z}$ algebra generated by T_α with $\alpha \in G_0(N) \cap M_2(\mathbb{Z})$ is denoted $\mathscr{H}_{0,\mathbb{Z}}(N)$.

In order to carry out explicit computations, the following decomposition is useful.

Lemma 6.25. *For $D_1, D_2 \in \mathbb{N}$ as in Remark 6.17 there is the following disjoint decompo-*

[4] Note that this operator is renormalised in Definition 6.27!

sition

$$\Gamma_0(N)\begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}\Gamma_0(N) = \dot{\bigcup}_{\substack{a,d\in\mathbb{N},\, ad=D_1D_2 \\ b \mod d \\ \gcd(a,b,d)=D_2}} \Gamma_0(N)\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}. \tag{6.14}$$

In particular,

$$M(n,N) = \dot{\bigcup}_{\substack{a,d\in\mathbb{N},\, ad=n \\ b \mod d}} \Gamma_0(N)\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}.$$

Proof: Only (6.14) requires proof. The inclusion '$\supseteq$' is obvious by Remark 6.17.
Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be an element of the left set. Then $(\gamma,\delta) = (c,-a)\,/\gcd(a,c)$ has gcd 1 and $\gcd(a,c,N) = 1$, so that $N \mid c$ carries over to $c/\gcd(a,c)$. Hence, the pair (γ,δ) gives rise to a matrix $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma_0(N)$, such that the product $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \cdot g$ has lower left entry $(ca - ac)/\gcd(a,c) = 0$, yielding w.l.o.g. $c = 0$. As a consequence, $ad = n$ and possible multiplication with $-\mathcal{I}$ forces $a, d > 0$, while $\gcd(a,b,d) = D_2$ is deduced from Remark 6.16. Finally, multiplication with a suitable element from $\Gamma_0(N)_\infty = \Gamma(1)_\infty$ yields $0 \le b < d$.
Lastly, it remains to be verified that the decomposition is disjoint. Assume there were representatives $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right), \left(\begin{smallmatrix} a' & b' \\ 0 & d' \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma_0(N)$ such that:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix}. \tag{6.15}$$

Solving yields

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix}\begin{pmatrix} d & -b \\ 0 & a \end{pmatrix} \cdot \frac{1}{n}$$

from which we infer that $\gamma = 0$ and $\delta > 0$. However, the matrix is an element of $\Gamma_0(N) \le \Gamma(1)$, so $\alpha = \delta = 1$. Inserting this result in (6.15) yields $b + d\beta = b'$ or, alternatively, $b \equiv b' \mod d$. □

As an immediate application we find (cf. [Dei10, Prop. 2.5.4 p. 40]) the following.

Lemma 6.26. *For $m, n \in \mathbb{N}$ coprime to N with $\gcd(m,n) = 1$ we have*

$$T(mn) = T(m)T(n).$$

Recall that the operator $T(m)$ is (up to now) empty in case $(m,N) > 1$.

Proof: Lemma 6.25 yields sets of representatives R_m, R_n for $T(m)$ and $T(n)$. Their product $T(m)T(n)$ will, as set of representatives, admit the product set R_mR_n. That the following mapping is well defined is clear by the definition of $M(n,N)$ and Lemma 6.25.

$$R_m \times R_n \to \Gamma_0(N)_\infty \backslash R_{mn}, \qquad (A,B) \mapsto \Gamma_\infty \backslash AB.$$

For injectivity: assume there were (the relation $\simeq$ in the following means that the matrices are identical module left operation of $\Gamma_\infty = \langle T \rangle$, where T denotes the translation matrix)

$$\begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} \cdot \begin{pmatrix} a_1' & b_1' \\ 0 & d_1' \end{pmatrix} = \begin{pmatrix} a_1a_1' & a_1b_1' + b_1d_1' \\ 0 & d_1d_1' \end{pmatrix}$$
$$\simeq \begin{pmatrix} a_2a_2' & a_2b_2' + b_2d_2' \\ 0 & d_2d_2' \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix} \cdot \begin{pmatrix} a_2' & b_2' \\ 0 & d_2' \end{pmatrix}.$$

Since the diagonal entries are identical and $\gcd(m,n) = 1$ we deduce that indices do not matter for a and d. Hence, the identity of the right upper entries becomes

$$ab_1' + b_1d' \equiv ab_2' + b_2d' \mod dd'.$$

Reduction modulo d' yields

$$ab_1' \equiv ab_2' \mod d'.$$

Again, by $\gcd(m,n) = 1$ we infer $\gcd(a,d') = 1$, yielding $b_1' \equiv b_2' \mod d'$. By assumption, however, this means $b_1' = b_2'$, implying $b_1d' \equiv b_2d' \mod dd'$ which reduces to $b_1d' \equiv b_2d' \mod d$. Once more, $\gcd(d,d') = 1$ yields $b_1 = b_2$ by choice of the set of representatives. The surjectivity is implied once Lemma 6.25 is taken into consideration to deduce that $|R_m| = \sigma_1(m)$ which is evidently weakly multiplicative. □

Before employing the above results, the operator $T(n)$ is renormalised.

Definition 6.27. For $n \in \mathbb{N}$ with $\gcd(n,N) = 1$ define ${}^kT(n) := n^{k/2-1} \cdot T(n)$, and, more generally, ${}^kT_g = \det(g)^{k/2-1} \cdot T_g$, so that

$$|_{k,\chi}{}^kT(n) := n^{k/2-1} \cdot \sum_{\beta \in \Gamma_0(N) \backslash M(n,N)} |_{k,\chi}\beta.$$

With these modifications in place L-functions associated to modular forms (cf. (6.4)) may be investigated. First, the operation of the operators on Fourier expansions of modular forms is examined before exploiting the result to extract (incomplete) product expansions for associated L-series.

Example 6.28. If $\sum_{m=0}^{\infty} A(m)q^m = f \in \mathcal{M}_k(\Gamma_0(N), \chi)$ and $n \in \mathbb{N}$ with $\gcd(n, N) = 1$ the Fourier coefficients of $f|_{k,\chi}{}^kT(n) = \sum_{m=0}^{\infty} B(m)q^m$ are given by

$$B(m) = \sum_{a|\gcd(m,n)} \overline{\chi(a)} a^{k-1} A\left(\frac{mn}{a^2}\right).$$

This is verified by the direct computation, where in the following $b(d)$ runs through a set of representatives mod d.

$$\begin{aligned} f|_{k,\chi}{}^kT(n)(z) &= n^{k/2-1} \cdot \sum_{ad=n} \sum_{b\,(d)} \left(\frac{a}{d}\right)^{k/2} \overline{\chi(a)} f\left(\frac{az+b}{d}\right) \\ &= \sum_{ad=n} \overline{\chi(a)} \frac{a^k}{n} \sum_{m=0}^{\infty} A(m) e\left(\frac{amz}{d}\right) \cdot \sum_{b\,(d)} e\left(\frac{mb}{d}\right) \\ &= \sum_{m=0}^{\infty} \sum_{\substack{ad=n \\ d|m}} \overline{\chi(a)} \frac{a^k}{n} A(m) \cdot e\left(\frac{amz}{d}\right) \cdot d \\ &= \sum_{m=0}^{\infty} \sum_{\substack{ad=n \\ a|m}} \overline{\chi(a)} a^{k-1} A\left(\frac{md}{a}\right) \cdot e\left(mz\right) \\ &= \sum_{m=0}^{\infty} \sum_{a|\gcd(m,n)} \overline{\chi(a)} a^{k-1} A\left(\frac{mn}{a^2}\right) \cdot e\left(mz\right). \end{aligned}$$

With this example at hand, we may prove properties of eigenforms to Hecke operators.

Definition 6.29. Let $N \in \mathbb{N}$ and $\chi : \Gamma_0(N) \to \mathbb{T}$ be a Dirichlet Character modulo N. A form $f \in \mathcal{S}_k(\Gamma_0(N), \chi)$ is referred to as a *Hecke eigenform*, if it is an eigenvector to $T(n)$ for all $n \in \mathbb{N}$ with $\gcd(n, N) = 1$.

Hecke eigenforms have remarkable properties derived from the properties of the Hecke algebra.

Proposition 6.30 ([Bum98, Prop 1.4.5 p. 48])**.** *Assume $f \in \mathcal{S}_k(\Gamma_0(N), \chi)$ is a Hecke eigenform with Fourier coefficients $a(m)$ and eigenvalues $\lambda(n)$ for ${}^kT(n)$. Then*

i) In case $N = 1$ we have $a(1) \neq 0$.

ii) If $a(1) = 1$, then $\lambda(n) = a(n)$ for all n with $\gcd(n, N) = 1$.

iii) If $a(1) = 1$ the association $n \mapsto a(n)$ for $\gcd(n, N) = 1$ is weakly multiplicative.

Proof: By Example 6.28 we find for $n \in \mathbb{N}$ with $\gcd(n, N) = 1$ that

$$\lambda(n)a(m) = \sum_{d|\gcd(m,n)} \overline{\chi(d)} d^{k-1} a\left(\frac{mn}{d^2}\right).$$

Assuming $\gcd(m, n) = 1$ we infer

$$\lambda(n)a(m) = a(mn).$$

Setting $m = 1$ provides $\lambda(n)a(1) = a(n)$. This implies ii) immediately and iii) after comparison with Lemma 6.26. In case of i) $a(1) = 0$ renders $f = 0$, contradicting the assumption of f being an eigenform. □

Hecke eigenforms for which the first Fourier coefficient equals 1 are referred to as *normalised*. Their relevance is demonstrated by Proposition 6.31.

Proposition 6.31. *If $f \in \mathcal{S}_k(\Gamma_0(N), \chi)$ is a normalised eigenform the associated L-function has a partial product expansion:*

$$L(f, s) = \sum_{m=1}^{\infty} \frac{a(m)}{m^s} = \left(\sum_{\substack{m \in \mathbb{N} \\ p|m \implies p|N}} \frac{a(m)}{m^s} \right) \cdot \prod_{p \nmid N} \left(1 - a(p)p^{-s} + \overline{\chi(p)} p^{k-1-2s}\right)^{-1}.$$

Before proving this result, another beneficial application of Lemma 6.25 shedding light on the structure of $\mathscr{H}_{0,\mathbb{Z}}(N)$ is presented.

Lemma 6.32. *For $r \in \mathbb{N}$ and $k, p \nmid N$ as above we find*

$$T(p^{r+1}) = T(p)T(p^r) - p \cdot T_{p \cdot \mathcal{I}} \cdot T(p^{r-1}).$$

The identity remains valid if the symbol T is replaced by ${}^kT$.

Proof: The proof is presented in Section A.3 based on the idea found in [Dei10, Prop 2.5.4 p 40]. □

Example 6.33. Employing Definition 6.11 and 6.27 in Lemma 6.32 yields for k, χ, p as above and $r \in \mathbb{N}$

$$|_{k,\chi} {}^kT(p)\, {}^kT(p^r) = |_{k,\chi} {}^kT\left(p^{r+1}\right) + p^{k-1}\overline{\chi(p)} \cdot |_{k,\chi} {}^kT\left(p^{r-1}\right). \tag{6.16}$$

Proof of Prop 6.31: The weak multiplicativity of the coefficients for arguments coprime to N yields

$$\sum_{m\in\mathbb{N}} a(m)m^{-s} = \left(\sum_{\substack{M\in\mathbb{N} \\ p|m \implies p|N}} \frac{a(m)}{m^s} \right) \cdot \prod_{p\nmid N} \left(\sum_{l=0}^{\infty} a(p^l)p^{-ls} \right).$$

For $p \nmid N$ Equation 6.16 and $\lambda(p) = a(p)$ yield for $r \in \mathbb{N}$

$$a(p^{r+1}) - a(p)a(p^r) + \overline{\chi(p)}p^{k-1}a(p^{r-1}) = 0$$

translating to

$$\left(1 - a(p)X + \overline{\chi(p)}p^{k-1}X^2\right) \cdot \left(\sum_{r=0}^{\infty} a(p^r)X^r \right) = 1$$

as an identity of formal power series. Selecting $X = p^{-s}$ yields the desired result. □

Proposition 6.34. *The algebra $\mathscr{H}_{0,\mathbb{Z}}(N)$ has the following properties:*

a) It is commutative.

b) It is generated (over $\mathbb{Z}$) by $T_{\mathcal{I}}$ and $T(p)$ for primes p with $p \nmid N$.

c) It acts normally on $\mathcal{S}_k(\Gamma_0(N), \chi)$.

d) $\mathscr{H}_0(N)$ and $\mathscr{H}_{0,\mathbb{Z}}(N)$ generate the same algebra over $\mathbb{Q}$.

Proof: For a) and c) the properties are inherited from $\mathscr{H}_0(N)$ (cf. Proposition 6.19, 6.20). Part b): Lemma 6.26 and Lemma 6.32 imply immediately that every element in $\mathscr{H}_0(N)$ is generated by the operators mentioned in b).
Part d) is realised as follows: by Lemma 6.26 in case of $r = 1$, the operator $T_{p\mathcal{I}}$ is reconstructable from $T(p)$, but then $T_{m\mathcal{I}}$ for $m \in N$ with $\gcd(m, N) = 1$ is. By the discussion before Definition 6.24 operators in $\mathscr{H}_0(N)$ are decomposed into elements from $\mathscr{H}_{0,\mathbb{Z}}(N)$ and $T_{q\mathcal{I}}$ with $q = \frac{q_1}{q_2} \in \mathbb{Z}^{\times}_{S_N^C}$. The latter act as multiplication with $\chi(q_1)/\chi(q_2)$, where χ is a Dirichlet character mod N. However, there is $m \in \mathbb{N}$ with $\gcd(m, N) = 1$ and $\chi(m)\chi(q_2) = 1$, meaning $T_{q\mathcal{I}}$ acts as $T_{m\mathcal{I}}$. □

Evidently, a full product expansion of L-series associated to a Hecke eigenform instead of the partial expansion of Proposition 6.31 is desirable, though cannot be expected in general, as the following example testifies.

Example 6.35. The modular form $\Delta(\tau) + \Delta(6\tau) = f \in \mathcal{S}_{12}(\Gamma_0(6), 1)$ is an eigenfunction of all Hecke operators T_g with $g \in G_0(6)$, since $\Delta \in \mathcal{S}_{12}$ which is one dimensional. The associated L-function $L(f, s)$, however, has no Euler product, since its Fourier coefficients $C(m)$ are not multiplicative. Indeed, $C(2)$ and $C(3)$ equal the Fourier coefficient $C_\Delta(2), C_\Delta(3)$ of Δ while $C(6)$ equals $C_\Delta(6) + C_\Delta(1) \neq C_\Delta(6) = C_\Delta(2) \cdot C_\Delta(3)$.

However, in order to hope for a full product expansion, Hecke operators for primes dividing the level have to be constructed and investigated.

6.2.1 Dividing the level

We fix a level $N \in \mathbb{N}$. There is, in fact, a theory of Hecke operators $T(p)$ for $\Gamma_0(N)$ in case $p \mid N$. However, these are, in general, not normal and hence there is no basis of Hecke eigenforms for $\mathcal{S}_k(\Gamma_0(N), \chi)$ which have the strong properties stated above (cf. Example 6.35).

In order to extend the definition of Hecke operators $T(n)$ to $n \in \mathbb{N}$ with $\gcd(n, N) > 1$ note that the collection of matrices $M(n, N)$ (cf. (6.13)) was empty, so it has to be redefined. In fact, instead of a congruence subgroup Γ acting on a suitable subgroup of $\mathrm{GL}_2(\mathbb{Q})^+$ the condition may be attenuated to semigroups (cf. [Shi71] or [Miy06, 4.5 p. 131]). This had implicitly been done before, when replacing $G_0(N)$ with $M_2(\mathbb{Z}) \cap G_0(N)$ in the investigation prior to this subsection. In the case at hand, define

$$\Delta_0(N) := \left\{ \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \mod N, (a, N) = 1 \right\} \cap \mathrm{GL}_2(\mathbb{Q})^+$$

to extend $G_0(N) \cap M_2(\mathbb{Z}) \subseteq \Delta_0(N)$. Note that the conditions on a, c were chosen suitably to extend χ to the semigroup. The semigroup $\Delta_0(N)$ decomposes as before into the sets

$$M(n, N) := \left\{ \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \;\middle|\; \det(\alpha) = n, c \equiv 0 \mod N, (a, N) = 1 \right\}.$$

In case of $\gcd(n, N) = 1$, this agrees with the definition previously settled for, so it is, in fact, an extension. There is a geometric reason for the condition on a, emerging from symmetrising over superlattices of index n with a natural torsion condition (cf. [Kna92, IX.6 p 275]). Note that the condition on a stays intact, when such a matrix is multiplied from the left or right by an element of $\Gamma_0(N)$. The proof of Lemma 6.25 stays intact as well and we find (cf. [Kna92, p. 277])

$$M(n,N) = \dot{\bigcup}_{\substack{a,d\in\mathbb{N},\, ad=n\\ (a,N)=1\\ b \mod d}} \Gamma_0(N)\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}. \tag{6.17}$$

Now, if we define

$$f|_{k,\chi}\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \overline{\chi(a)}\det(\alpha)^{k/2} j(\alpha,\tau)^{-k} f(\alpha\tau),$$

we obtain an extension of the action used above on $\Gamma_0(N)$ that yields the same Hecke operators in case $\gcd(n,N)=1$. We attempt to reproduce the results of the case treated above and note that for the following [Miy06, p. 134] is helpful.

Remark 6.36. If $\sum_{m=1}^{\infty} A(m)q^m = f \in \mathcal{M}_k(\Gamma_0(N),\chi)$ and $n\in\mathbb{N}$, the Fourier coefficients of $f|_{k,\chi}{}^k T(n) = \sum_{m=0}^{\infty} B(m)q^m$ are given by

$$B(m) = \sum_{a|\gcd(m,n)} \overline{\chi(a)} a^{k-1} A\left(\frac{mn}{a^2}\right).$$

Proof: The computation is identical to that of Example 6.28:

$$\begin{aligned}
f|_{k,\chi}{}^k T(n)(z) &= n^{k/2-1}\cdot \sum_{\substack{ad=n\\ (a,N)=1}} \sum_{b\ (d)} \left(\frac{a}{d}\right)^{k/2} \overline{\chi(a)} f\left(\frac{az+b}{d}\right)\\
&= \sum_{\substack{ad=n\\ (a,N)=1}} \overline{\chi(a)}\frac{a^k}{n}\sum_{m=0}^{\infty} A(m) e\left(\frac{amz}{d}\right)\cdot \sum_{b\ (d)} e\left(\frac{mb}{d}\right)\\
&= \sum_{m=0}^{\infty} \sum_{\substack{ad=n\\ (a,N)=1\\ d|m}} \overline{\chi(a)}\frac{a^k}{n} A(m)\cdot e\left(\frac{amz}{d}\right)\cdot d\\
&= \sum_{m=0}^{\infty} \sum_{\substack{ad=n\\ (a,N)=1\\ a|m}} \overline{\chi(a)} a^{k-1} A\left(\frac{md}{a}\right)\cdot e\left(mz\right)\\
&= \sum_{m=0}^{\infty} \sum_{\substack{a|\gcd(m,n)\\ (a,N)=1}} \overline{\chi(a)} a^{k-1} A\left(\frac{mn}{a^2}\right)\cdot e\left(mz\right).
\end{aligned}$$

Note that $\chi(a)=0$ if $(a,N)>1$, so the requirement may be eliminated from the sum. □

Definition 6.37. The algebra of operators generated by T_α for all $\alpha \in \bigcup_{n\in\mathbb{N}} M(n,N)$ is denoted $\overline{\mathscr{H}_{0,\mathbb{Z}}}(N)$. It contains the algebra $\mathscr{H}_{0,\mathbb{Z}}(N)$ from the previous section.

Lemma 6.38. *For coprime $m, n \in \mathbb{N}$ the Hecke operators are multiplicative:*

$$T(m)T(n) = T(mn).$$

Proof: The first part of the proof is identical to the proof of Lemma 6.26. Only for identifying the map to fulfil surjectivity counting has to be reconsidered. Note that

$$|R_n| = \sum_{(a,N)=1, a|n} \frac{n}{a} = n \cdot \sum_{a|n/\gcd(n,N)} a^{-1} = \gcd(n,N) \cdot \sigma_1\left(\frac{n}{\gcd(n,N)}\right).$$

However, $n \mapsto \gcd(n,N)$ is clearly weakly multiplicative, so that $n \mapsto |R_n|$ is, yielding bijectivity from injectivity. □

Together with the Lemma below the structure of $T(m)$ where m only has prime divisors appearing in N is recognised to be trivial.

Lemma 6.39. *If $p \mid N$ the following reduction is valid $T(p^r) = T(p)^r$. It is evidently also true for ${}^kT$ in place of T.*

Proof: The proof is reduced to the identity $T(p^{r+1}) = T(p)T(p^r)$. The decomposition (6.17) yields the following sets of representatives for $T(p)$ and $T(p)^r$

$$R_p = \left\{ \begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix} \middle| \, b \mod p \right\}, \qquad R_{p^r} = \left\{ \begin{pmatrix} 1 & b_r \\ 0 & p^r \end{pmatrix} \middle| \, b_r \mod p^r \right\}.$$

The product of representatives is, evidently,

$$\begin{pmatrix} 1 & b_r + p^r b \\ 0 & p^{r+1}. \end{pmatrix}$$

However, this yields exactly the set of representatives $R_{p^{r+1}}$. □

This yields a complete picture of the structure of the algebra $\overline{\mathscr{H}_{0,\mathbb{Z}}}(N)$.

Corollary 6.40. *The algebra $\overline{\mathscr{H}_{0,\mathbb{Z}}}(N)$ is commutative and generated (over $\mathbb{Z}$) by the operators $T_{q\mathcal{I}}$ for primes $q \nmid N$ and $T(p)$ for all primes p. It extends the action of $\mathscr{H}_{0,\mathbb{Z}}(N)$.*

Proof: This is an immediate consequence of Proposition 6.34, Lemma 6.38, and Lemma 6.39. □

There is also an analogous result to Lemma 6.22 for bad primes that is more complicated and we refer to [Miy06, Thm. 4.5.4 p 136] The analogue of Proposition 6.30 is also true without the condition $(n, N) = 1$.

Proposition 6.41. *Assume $f \in \mathcal{S}_k(\Gamma_0(N), \chi)$ is an eigenform for all ${}^kT(n)$ with Fourier coefficients $a(m)$ and eigenvalues $\lambda(n)$ for ${}^kT(n)$. Then w.l.o.g. $a(1) = 1$ and*

i) $\lambda(n) = a(n)$ for all $n \in N$,

ii) the association $n \mapsto a(n)$ is weakly multiplicative.

iii) For $p \mid N$ and $r \in \mathbb{N}$ we find necessarily $a(p^r) = a(p)^r$.

iv) For $p \nmid N$ and $r \in \mathbb{N}$ we find $a(p^{r+1}) = a(p)a(p^r) - p^{k-1}\chi(p)a(p^{r-1})$.

Proof: Except for iii) the proof ist identical to the proof of 6.30. Part iii), however, is a direct consequence of Lemma 6.39. □

Further, a full product expansion of simultaneous eigenforms (cf. Prop 6.31) is obtained:

Proposition 6.42. *If $f \in \mathcal{S}_k(\Gamma_0(N), \chi)$ is a simultaneous eigenform of ${}^kT(n)$ for all $n \in \mathbb{N}$ with Fourier coefficients $a(n)$, $a(1) = 1$ may be assumed and the associated L-function has the product expansion:*

$$L(f, s) = \sum_{m=1}^{\infty} \frac{a(m)}{m^s} = \prod_{p|N} \left(1 - a(p)p^{-s}\right)^{-1} \cdot \prod_{p\nmid N} \left(1 - a(p)p^{-s} + \overline{\chi(p)}p^{k-1-2s}\right)^{-1}.$$

Proof of Prop 6.42: From Proposition 6.31 follows, that only

$$\left(\sum_{p|m \implies p|N} \frac{a(m)}{m^s} \right)$$

has to be factored. For $p \mid N$ Part iii) of Proposition 6.41 translates to

$$(1 - a(p)X) \cdot \left(\sum_{r=0}^{\infty} a(p^r)X^r \right) = 1$$

as an identity of formal power series. Selecting $X = p^{-s}$ yields the desired result. □

Clearly, the question remains whether such simultaneous eigenfunctions exist. Example 6.35 already indicated a weakness in the construction: there exist Hecke eigenforms in

$\mathcal{S}_k(\Gamma_0(N), \chi)$ arising from cusp forms of different level or weight which may be effortlessly combined to violate multiplicative conditions. However, there is still a decomposition into eigenspaces.

Theorem 6.43 (Hecke-Petersson)**.** *The space $\mathcal{S}_k(\Gamma_0(N))$ decomposes into a direct sum of eigenforms for $T(p)$ for all primes p.*

The above theorem also provides a *multiplicity one* statement for forms with non-vanishing first coefficient, as the Fourier coefficients are by Proposition 6.41 determined by the Hecke eigenvalues of $T(p)$.
In fact, the result may even be improved. If the modular forms induced by lower weight and level are eliminated from consideration by restricting to their orthogonal complement, the theory of $T(p)$ for primes not dividing the level N works as desired (cf. [AL70]). The remaining subspace of forms which do not arise in such a fashion is called space of *newforms* $\mathcal{S}_k^{\text{new}}(\Gamma_0(N), \chi)$. It is stable under the operation of the Hecke algebra, admits an orthogonal basis of Hecke eigenforms and, if f is such an eigenform, its first Fourier coefficient is automatically different from zero. We restrict to the case of trivial character for the following statement.

Theorem 6.44 (strong multiplicity one)**.** *The space $\mathcal{S}_k^{\text{new}}(\Gamma_0(N))$ admits a basis of normalised eigenforms for the operators $T(p)$ for $p \nmid N$. Further, two normalised eigenforms $f \in \mathcal{S}_k^{\text{new}}(\Gamma_0(N))$ and $g \in \mathcal{S}_k(\Gamma_0(N'))$ having the same eigenvalues for all but finitely many $T(p)$ result in $N = N'$ and $f = g$. Additionally, in this case f is an eigenfunction for all $T(p)$.*

Compare [AL70, Thm 2 p 144 and Thm 4 p 150] for the origin of the theorem, [Miy71, Thm 2 p 185][5] for a representation theoretic perspective.[6] The name *multiplicity one* arises from the fact that it is described in adelic terms as only admitting every smooth irreducible admissible representation once. This is true for GL_n (cf. [Jac70, Prop 11.1.1 p. 354] for the case $n = 2$ and [Sha74] for the generalisation to $n > 2$) and SL_2. The part *strong* referes to the fact that these cuspidal automorphic representations are determined up to isomorphism locally and if they differ they must differ at infinitely many places, compare [Cas73, Thm. 2 p. 307] and [JS81].

The L-function associated to a newform does not only admit a product expansion but carries additional analytic properties (cf. [DS05, pp 200-205]).

[5] note that normalised eigenforms are referred to as *primitive* in this source
[6] For translating the representation theoretic perspective [Cas73, 3. p. 308] might be considered.

Theorem 6.45. *Let $f \in \mathcal{S}_k^{\mathrm{new}}(\Gamma_0(N), \chi)$ be a normalised Hecke eigenform and $L(f,s)$ denote its L-series. Then*

a) The series $L(f,s)$ admits analytic continuation to all $s \in \mathbb{C}$.

b) There is a product expansion

$$L(f,s) = \prod_{p<\infty} \left(1 - a(p)p^{-s} + \overline{\chi}(p)p^{k-1-2s}\right)^{-1}.$$

c) Assume f is an eigenfunction for the Fricke involution with eigenvalue ϵ. The completed *L-function $L^*(f,s) := \left(\frac{2\pi}{\sqrt{N}}\right)^{-s} \Gamma(s)L(f,s)$ satisfies the functional equation*

$$L^*(f,s) = \epsilon \cdot i^k \cdot L^*(f, k-s).$$

That the form f is assumed in the last part to be an eigenform for the Fricke involution is no remarkable restriction. The Fricke involution is a special case of an Atkin–Lehner involution, commutes with the Hecke operators for good primes and is self adjoint [Kna92, Prop 9.19 p.281], so that these are simultaneously diagonisable. We will briefly review Atkin–Lehner involutions as these are helpful for computing product expansions of L-series associated to vector valued modular forms arising as a symmetrisation of scalar valued forms for $\Gamma_0(N)$.

Atkin–Lehner involution

Atkin–Lehner involutions that already appeared briefly above, are of particular interest for investigating L-series associated to lifts of scalar valued modular forms in $\mathcal{S}_k(\Gamma_0(N), \chi)$ to the vector valued case. For the following, compare [AL70].

Definition 6.46. Let $e \mid N$ be a divisor of N, such that $\gcd(e, N/e) = 1$. Such a number is called a *Hall divisor* of N and this fact is denoted $e \,||\, N$.
For any Hall divisor $e \,||\, N$ select a matrix

$$\gamma_e = \begin{pmatrix} ae & b \\ cN & de \end{pmatrix} \in M_2(\mathbb{Z})$$

with determinant e.[7] The induced operator is called an *Atkin–Lehner involution*:

$$\mathcal{M}_k(\Gamma_0(N),\chi) \to \mathcal{M}_k(\Gamma_0(N),\chi), \qquad f \mapsto f|_k\gamma_e.$$

Remark 6.47. a) Atkin Lehner involutions on $\mathcal{M}_k(\Gamma_0(N),\chi)$ are well defined.

b) For all $e \,||\, N$ the matrices γ_e are normalising $\Gamma_0(N)$.

c) Different choices γ_e, γ_e' are equivalent mod $\Gamma_0(N)$.

d) For $e,h \,||\, N$ we have $\gamma_e \cdot \gamma_h = \gamma_{\mathrm{lcm}(e,h)}$.

Proof: Note that for two different choices γ_e, γ_e' of matrices and $\gamma \in \Gamma_0(N)$

$$\gamma_e\gamma\gamma_e' \in e^2 \cdot \Gamma_0(N).$$

Additionally, $\gamma_e'' := e\gamma_e'^{-1}$ is an alternative choice, meaning $\gamma_e\gamma\gamma_e'^{-1} \in \Gamma_0(N)$ $(\star)$. However, for an element $\delta \in \Gamma_0(N)$, select $\gamma = \gamma_e^{-1}\delta\gamma_e' \in \Gamma_0(N)$ and find that $\gamma_e\Gamma_0(N)\gamma_e^{-1} = \Gamma_0(N)$, i.e. b).
For c), $(\star)$ implies $\gamma_e\gamma = \gamma'\gamma_e'$ for some $\gamma' \in \Gamma_0(N)$. By b), a modified $\gamma' \in \Gamma_0(N)$ yields $\gamma_e\gamma = \gamma_e'\gamma'$, meaning $\gamma_e \equiv \gamma_e' \mod \Gamma_0(N)$.
Lastly, for a), $f \in \mathcal{M}_k(\Gamma_0(N),\chi)$ together with b) (select $\delta \in \Gamma_0(N)$ s.t. $\delta\gamma_e = \gamma_e\gamma$) and the computation for c) imply $f|_k\gamma_e = f|_k\delta\gamma_e = f|_k\gamma_e\gamma = f|_k\gamma_e'\gamma' = f|_k\gamma_e'$. The preservation of vanishing conditions at cusps, however, is immediate. □

6.3 Vector valued Hecke theory

So far we have had a look at Hecke operators acting on scalar valued modular forms in $\mathcal{S}_k(\Gamma_0(N),\chi)$. However, in Section 3.3 we have introduced vector valued modular forms that are a collection of scalar valued forms interacting under transformation of the full modular group to guarantee automorphic behaviour. In this setting, L-series may also be defined and one might hope to reproduce some of the results in the classical case. For the purpose of deriving product expansions, specifically, an analogous theory to Hecke operators in the elliptic case has to be developed which has been carried out by [BS08].

[7] Of course, this condition already implies that such a divisor e had to be a Hall divisor.

Recall that a vector valued modular form $f \in \mathcal{M}_{\mathcal{L},k}(\mathrm{Mp}_2(\mathbb{Z}))$ is associated to a discriminant form $\mathcal{L}$ and has a Fourier expansion

$$f(\tau) = \sum_{\lambda \in \mathcal{L}} \sum_{n \in \overline{\mathrm{q}}(\lambda)+\mathbb{Z}} a(\lambda, n) \cdot e(n\tau)\mathfrak{e}_\lambda.$$

We fix a $\mathbb{Z}$ lattice L with $\mathcal{L} = L'/L$. In order to construct Hecke operators in the classical case, the action of the Petersson slash operator on a supergroup of $\Gamma_0(N)$ was utilised. This action already exists and may be reused for acting on the component functions. The task that remains, however, is to extend the action of the Weil representation $\rho_\mathcal{L}$ on $\mathbb{C}[\mathcal{L}]$ to a supergroup of $\mathrm{Mp}_2(\mathbb{Z})$.
Assume $\mathrm{sig}(\mathcal{L})$ is even so that we are in the case of $\mathrm{SL}_2(\mathbb{Z})$ modular forms and write $N \coloneqq \mathrm{lev}(L)$. Then the Weil representation is trivial on $\Gamma(N)$, meaning it factors through

$$\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) \simeq \mathrm{SL}_2(\mathbb{Z})/\Gamma(N).$$

Hence, a naive approach would be an extension of the Weil representation to $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$. This, however, fails. Bruinier and Stein note that with the definitions $S(N) \coloneqq \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$, $G(N) \coloneqq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$, and $U(N) \coloneqq (\mathbb{Z}/N\mathbb{Z})^\times$, the following sequence is exact and splitting:

$$1 \to S(N) \to G(N) \to U(N) \to 1.$$

Hence, $G(N)$ is a semidirect product of $S(N)$ and $U(N)$. Then they construct a group $Q(N)$, such that there is a natural exact sequence

$$1 \to S(N) \to Q(N) \to U(N) \to 1$$

and, in fact, $Q(N) \simeq S(N) \times U(N)$. Hence, the authors choose a unitary representation of $U(N)$ that commutes with $\rho_\mathcal{L}$ on $S(N)$ and have thus extended the representation. Their choice is consistent with the classical theory, in case the base lattice is unimodular and we will denote the extended representation by $\rho_\mathcal{L}$, again. They then construct a group $\mathcal{Q}(N)$ (cf. [BS08, (4.2) p. 257]) containing $\mathrm{Mp}_2(\mathbb{Z})$ and covering $Q(N)$. This setting is finally suitable to reproduce the ideas of the scalar valued case.

The full picture is presented in [BS08, Sec. 3]. In the end, the authors prove that in the case we are interested in, the coset decomposition of the appearing double cosets are equivalent to the standard decomposition. We will briefly sketch the action of these operators after introducing notation and proving identities for their abstract counterparts.

Note that in Lemma 6.32 and 6.39 relations of Hecke operators in the abstract Hecke algebra have been proven. We recall that the subalgebra $\mathscr{H}_{0,\mathbb{Z}}(N)$ generated by the operators $T(n)$ for $n \in \mathbb{N}$ (cf. paragraph before (6.12)) was of particular interest for the theory of L-series and we would like to replicate part of these results in the vector valued setting. The operators considered by [BS08], however, correspond not to $T(p^2)$, but to so called *primitive* operators, serving as simpler building blocks of the abstract Hecke algebra. This justifies the new notation $\mathcal{T}(p^2)$ in contrast to $T(p^2)$ from the previous section.

Definition 6.48. For $n \in \mathbb{N}$ with $\gcd(n, N) = 1$, define the matrix

$$g(n) := \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$$

and in the notation of (6.8) set the *primitive* Hecke operator associated to n to be

$$\mathcal{T}(n) := T_{g(n)}.$$

Note that these operators may be used to decompose the previously defined operators $T(n)$ appearing in (6.12).

Remark 6.49. Let $n \in \mathbb{N}$ and p a prime not dividing $\mathrm{lev}(L)$, then Lemma 6.25 implies

$$T(p^n) = \sum_{l=0}^{\lfloor n/2 \rfloor} T_{p^l \cdot \begin{pmatrix} p^{n-2l} & 0 \\ 0 & 1 \end{pmatrix}} = \sum_{l=0}^{\lfloor n/2 \rfloor} T_{p^l \mathcal{I}} \cdot \mathcal{T}(p^{n-2l}).$$

Hence, for $n \in \mathbb{N}$ with $\gcd(n, \mathrm{lev}(L)) = 1$ to understand the action of the family of operators $T(n)$, it suffices to know how $\mathcal{T}(n)$ and $T_{n\mathcal{I}}$ act.

There is the following analog to Lemma 6.32.

Lemma 6.50. *We find for $r \in \mathbb{N}$ and a prime p not dividing N that*

$$\mathcal{T}(p^{r+1}) = \mathcal{T}(p)\mathcal{T}(p^r) - p \cdot T_{p\mathcal{I}} \cdot \mathcal{T}(p^{r-1}) - \delta_{r=1} \cdot T_{p\mathcal{I}}.$$

Proof: See Section A.3 for details. □

Note that this is consistent with [Miy06, Lem. 4.5.7 (1) p. 140], where $T(p,p) = T_{p\mathcal{I}}$ and $T(1, p^e) = \mathcal{T}(p^e)$ in our notation. Note also that the number N in Miyake may be assumed to equal 1 in the current subsection. By utilising the statement above, we may derive a relation for square indices which may prove useful later.

Lemma 6.51. *For $2 \leq r \in \mathbb{N}$ and a prime p not dividing N, the following relation is true:*

$$\mathcal{T}(p^{r+2}) = \left[\mathcal{T}(p^2) + (1-p) \cdot T_{p\mathcal{I}}\right] \mathcal{T}(p^r) - (1 + \delta_{r=2} p^{-1}) \left(p \cdot T_{p\mathcal{I}}\right)^2 \mathcal{T}(p^{r-2}).$$

Proof: We apply Lemma 6.50 repeatedly to obtain the following:

$$\begin{aligned}
\mathcal{T}(p^{r+2}) &= \mathcal{T}(p)\mathcal{T}(p^{r+1}) - p \cdot T_{p\mathcal{I}}\mathcal{T}(p^r) \\
&= \mathcal{T}(p) \left[\mathcal{T}(p)\mathcal{T}(p^r) - p \cdot T_{p\mathcal{I}}\mathcal{T}(p^{r-1})\right] - p \cdot T_{p\mathcal{I}}\mathcal{T}(p^r) \\
&= \left[\mathcal{T}(p^2) + (p+1) \cdot T_{p\mathcal{I}}\right] \mathcal{T}(p^r) \\
&\quad - p \cdot T_{p\mathcal{I}} \left[\mathcal{T}(p^r) + (p + \delta_{r=2}) \cdot T_{p\mathcal{I}}\mathcal{T}(p^{r-2}) + \mathcal{T}(p^r)\right] \\
&= \left[\mathcal{T}(p^2) + (1-p) \cdot T_{p\mathcal{I}}\right] \mathcal{T}(p^r) - (1 + \delta_{r=2} p^{-1}) \left(p \cdot T_{p\mathcal{I}}\right)^2 \mathcal{T}(p^{r-2}).
\end{aligned}$$

□

In order to define an action of the abstract Hecke operators, also in the vector valued case, it is necessary to declare how the representatives in the double cosets of the operators $\mathcal{T}(m)$ act on the group algebra $\mathbb{C}[\mathcal{L}]$. For $m \in \mathbb{N}$ with $\gcd(m, \mathrm{lev}(L))$, the approach presented in [BS08] and sketched above yields the following action for any $\lambda \in \mathcal{L}$:

$$\mathfrak{e}_\lambda|_{\mathcal{L}} g(m^2) = \mathfrak{e}_{m\lambda}. \tag{6.18}$$

It also translates to the odd case, where the metaplectic cover is required and we set for $m \in \mathbb{N}$ with $\gcd(m, \mathrm{lev}(L)) = 1$ analogously to the even case

$$g(m^2) := \left(\begin{pmatrix} m^2 & 0 \\ 0 & 1 \end{pmatrix}, 1 \right) \in \mathrm{Mp}_2^+(\mathbb{R}). \tag{6.19}$$

With this notation and the operation on $\mathbb{C}[\mathcal{L}]$ obtained from linearising (6.18), we may define the action of the Hecke operator $\mathcal{T}(m^2)$.

Definition 6.52. Let $m \in \mathbb{N}$ with $\gcd(m, \mathrm{lev}(L)) = 1$ and let $g(m^2) \in \mathrm{Mp}_2^+(\mathbb{R})$ be as above. Further, let

$$\mathrm{Mp}_2(\mathbb{Z}) \cdot g(m^2) \cdot \mathrm{Mp}_2(\mathbb{Z}) = \bigsqcup_i \mathrm{Mp}_2(\mathbb{Z}) \cdot h_i$$

be a right coset decomposition. Then we declare for $f \in \mathcal{M}_{\mathcal{L},k}$ the following operation

$$f|_{L,k}\mathcal{T}(m^2) := \det(g(m^2))^{k/2-1} \sum_i \sum_{\mu \in \mathcal{L}} (f|_k \delta_i) \otimes (\mathfrak{e}_\mu|_{\mathcal{L}} h_i)\,.$$

Note that unlike the scalar valued case, the factor $\det(g(m))^{k/2-1}$ has directly been incorporated into the action instead of writing ${}^k\mathcal{T}$ for a separate, rescaled operator. Since this is the only case required in the following, we decided to lighten the notation. We also note that $|_{L,k}\mathcal{T}(m^2)$ equals exactly the primitive operator $|_{k,A}T^*(m^2)$ in [BS08, (4.30)].

We turn towards describing how these Hecke operators manipulate the Fourier expansion of a modular form.

Operation on Fourier coefficients

First, we fix some notation for the following sections.

Notation 6.53. To lighten the notation write $\overline{\Gamma(1)}$ for $\mathrm{Mp}_2(\mathbb{Z})$. We fix an even $\mathbb{Z}$ lattice (L, q) of rank m and level $\mathrm{lev}(L)$. Its discriminant form is denoted $\mathcal{L} := L'/L$ and for $n \in \mathbb{N}$ with $\gcd(n, \mathrm{lev}(L)) = 1$, we write $|_{L,k}\mathcal{T}(n^2)$ for the Hecke operator represented by $\overline{\Gamma(1)}g(n^2)\overline{\Gamma(1)}$ which is denoted by $|_{k,\mathcal{L}}T(n^2)^*$ in [BS08] and represents the primitive Hecke operator associated to n^2 in the existing literature. Also, we resort to eliminating the metaplectic group from the notation of the space of modular forms, writing $\mathcal{M}_{L,k}$ or $\mathcal{S}_{L,k}$ for the space of cusp forms, respectively.

Recall that for $k \in \mathbb{Z}/2$ a vector valued modular form $f \in \mathcal{M}_{L,k}$ has a Fourier expansion by Remark 3.21:

$$f(\tau) = \sum_{\lambda \in L'/L} \sum_{n \in \overline{\mathrm{q}}(\lambda)+\mathbb{Z}} a(\lambda, n) e(n\tau)\mathfrak{e}_\lambda. \tag{6.20}$$

For once, understanding how the above Hecke operators manipulate the Fourier expansion of a modular form provides insights into the structure of the operators and arithmetic of the expansion. On the other hand, it has an immediate application to a family of L-series associated to modular forms. For that purpose, Bruinier and Stein have investigated the action of Hecke operators on the Fourier coefficients of such a form.

Proposition 6.54 ([BS08, Prop. 4.3 p. 258])**.** *Let L have even signature and p be a prime coprime to* $\mathrm{lev}(L)$*. Let $f \in \mathcal{M}_{L,k}$ and denote the Fourier expansion as in* (6.20)*. Then*

$$f|_{L,k}\mathcal{T}(p^2) = \sum_{\lambda \in \mathcal{L}} \sum_{n \in \mathbb{Z}+\overline{\mathrm{q}}(\lambda)} b(\lambda, n) q^n \otimes \mathfrak{e}_\lambda,$$

where

$$b(\lambda,n) = p^{2k-2}a(\lambda/p, n/p^2) + \frac{G_{\mathcal{L}}(1)}{G_{\mathcal{L}}(p)}p^{k-2}(p\delta_{p|n} - 1)a(\lambda,n) + a(p\lambda, p^2n) \tag{6.21}$$

and

$$\delta_{p|n} = \begin{cases} 1, & p \mid n, \\ 0, & p \nmid n. \end{cases}$$

Moreover, we understand that $a(\lambda/p, n/p^2) = 0$ *if* $p^2 \nmid n$.

Here, for $d \in \mathbb{N}$, the Gauss sum $G_{\mathcal{L}}(d) = \sum \lambda \in \mathcal{L} e(d\,\overline{\mathrm{q}}(\lambda))$ is further described in Definition A.16 and below. Bruinier and Stein have also proven the case of odd signature.

Theorem 6.55 ([BS08, Thm 4.10 p 263])**.** *Let L have odd signature and p be a prime coprime to* $\operatorname{lev}(L)$*. Let $f \in \mathcal{M}_{L,k}$ and denote the Fourier expansion as in* (6.20)*. Then*

$$f|_{L,k}\mathcal{T}(p^2) = \sum_{\lambda \in \mathcal{L}} \sum_{n \in \mathbb{Z} + \overline{\mathrm{q}}(\lambda)} b(\lambda,n) q^n \otimes \mathfrak{e}_\lambda,$$

where

$$b(\lambda,n) = p^{2k-2}a(\lambda/p, n/p^2) + \epsilon_p^{\mathrm{sig}(\mathcal{L}) + \left(\frac{-1}{|\mathcal{L}|}\right)} \left(\frac{p}{|\mathcal{L}|2^{\mathrm{sig}(\mathcal{L})}}\right)\left(\frac{-n}{p}\right) p^{k-3/2}a(\lambda,n) + a(p\lambda, p^2n) \tag{6.22}$$

and for an odd integer d we set

$$\epsilon_d = \begin{cases} 1, & d \equiv 1 \mod 4, \\ i, & d \equiv -1 \mod 4. \end{cases}$$

Moreover, we understand that $a(\lambda/p, n/p^2) = 0$ *if* $p^2 \nmid n$.

Based upon this operation on the Fourier coefficients of vector valued modular forms $f \in \mathcal{M}_{L,k}$, product expansions of L-series associated to f may be established, provided that the form f is an eigenform of the respective Hecke operators. This approach is pursued in Subsection 6.4.2.

6.3.1 Dividing the level

Next, we will have to turn to the bad places, following Bruinier and Stein. In case $p \mid N$, the matrix $g(p^2) = \left(\begin{smallmatrix} p^2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ does not belong to $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$. However, the action of the case $p \nmid N$ might be mimicked by defining for $\lambda \in \mathcal{L}$ the operation

$$\mathfrak{e}_\lambda|_{\mathcal{L}} g(p^2) \coloneqq \mathfrak{e}_{p\lambda}. \tag{6.23}$$

This coincides with the action of $g(p^2)$ in case of $p \nmid \mathrm{lev}(L)$ constructed in [BS08, Sec. 4] and extends to $g(m^2)$ for $m \in \mathbb{N}$. Since the goal is to define associated Hecke operators, products $\gamma g(m^2)\gamma' \in \overline{\Gamma(1)} g(m^2) \overline{\Gamma(1)}$ have to be considered, where we understand $\overline{\Gamma(1)} = \mathrm{Mp}_2(\mathbb{Z})$. It is clear how to define their action, however, consistency remains to be verified. In the following we frequently drop the index $\mathcal{L}$ from $|_{\mathcal{L}}$ to improve readability.

Proposition 6.56 ([BS08, Prop. 5.1])**.** *For fixed $\delta(m^2) = \gamma g(m^2)\gamma' \in \overline{\Gamma(1)} g(m^2) \overline{\Gamma(1)}$ different choices of γ, γ' result in the same action*

$$\mathfrak{e}_\lambda|\gamma|g(m^2)|\gamma'.$$

Proof: Assume a different set of choices, such that $\gamma g(m^2)\gamma' = \gamma_1 g(m^2)\gamma_1'$. It is sufficient to assume $\gamma' = \gamma_1 = 1$, meaning $\delta(m^2) = \gamma g(m^2) = g(m^2)\gamma_1'$ so that

$$\mathfrak{e}_\lambda|\gamma g(m^2) = \mathfrak{e}_\lambda|g(m^2)\gamma_1'. \tag{6.24}$$

Write $\gamma = \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \pm\sqrt{c\tau + d}\right)$, then necessarily $\gamma_1' = \left(\left(\begin{smallmatrix} a & b/m^2 \\ m^2 c & d \end{smallmatrix}\right), \pm\sqrt{m^2 c\tau + d}\right)$. As a consequence, $b \equiv 0 \mod m^2$, meaning $\gamma \in \overline{\Gamma}^0(m^2)$ and $\gamma' \in \overline{\Gamma}_0(m^2)$.
However, $\overline{\Gamma}^0(m^2)$ is generated by $\overline{\Gamma}^0(m^2) \cap \overline{\Gamma}_0^0(N), T^{m^2}$ and U.[8] In the first case, the identity follows directly from a well known theorem by Borcherds, stating that matrices in $\overline{\Gamma}_0^0(N)$ act by multiplication with a character in d and multiplication of the index of $\mathfrak{e}_\lambda$ with d. For T^{m^2} the representation acts by multiplication, trivialising the computation. The delicate case is that of $U = \left(\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right), \sqrt{\tau + 1}\right)$. The operation of U is found in Lemma 2.3 of [BS08], so that the left hand side of (6.24) reads

$$\mathfrak{e}_\lambda|U|g(m^2) = \frac{1}{|\mathcal{L}|} \sum_{\mu,\nu \in \mathcal{L}} e\left(q(\mu) + \beta(\mu, \lambda - \nu)\right) \mathfrak{e}_{m\nu}.$$

[8]Note that γ may be manipulated by multiplication with T^{lm^2} for suitable $l \in \mathbb{Z}$ such that b is divisible by N, since $(a, d) = 1$ so that $N \mid (l_1 a + l_2 d + b/m^2)$. However, the same argument applies to altering c with U^l for $l \in \mathbb{Z}$ adequately chosen.

Using the decomposition from Remark 1.36 yields

$$= \frac{1}{|\mathcal{L}|} \sum_{\mu \in \mathcal{L}} \sum_{\substack{\nu' \in \mathcal{L}_m \\ \nu'' \in \mathcal{L}_m}} e\left(q(\mu) + \beta(\mu, \lambda - \nu'/m - \nu'')\right) \mathfrak{e}_{\nu'},$$

where ${}_m\mathcal{L}$ denotes the m-torsion of $\mathcal{L}$. By a simple character argument (cf. Remark 1.51)

$$= \frac{|\mathcal{L}_m|}{|\mathcal{L}|} \sum_{\mu \in \mathcal{L}^m} \sum_{\nu \in \mathcal{L}_m} e\left(q(\mu) + \beta(\mu, \lambda - \nu/m)\right) \mathfrak{e}_{\nu}$$
$$= \frac{1}{|\mathcal{L}|} \sum_{\mu \in \mathcal{L}} \sum_{\nu \in \mathcal{L}_m} e\left(q(m\mu) + \beta(m\mu, \lambda - \nu/m)\right) \mathfrak{e}_{\nu}.$$

Considering the right hand side of (6.24) results in

$$\mathfrak{e}_\lambda | g(m^2) | U^{m^2} = \frac{1}{|\mathcal{L}|} \sum_{\mu, \nu \in \mathcal{L}} e\left(m^2 q(\mu) + \beta(\mu, m\lambda - \nu)\right) \mathfrak{e}_{\nu}.$$

Shifting μ by an element $\mu' \in \mathcal{L}_m$ does not change the sum over μ meaning

$$= \frac{1}{|\mathcal{L}||\mathcal{L}_m|} \sum_{\mu, \nu \in \mathcal{L}} \sum_{\mu' \in \mathcal{L}_m} e\left(m^2 q(\mu) + \beta(\mu + \mu', m\lambda - \nu)\right) \mathfrak{e}_{\nu}.$$

However, the sum over μ' is, as a sum over a character, non-vanishing if, and only if, $\nu \in \mathcal{L}^m$, so that

$$= \frac{1}{|\mathcal{L}|} \sum_{\mu \in \mathcal{L}} \sum_{\nu \in \mathcal{L}^m} e\left(m^2 q(\mu) + \beta(\mu, m\lambda - \nu)\right) \mathfrak{e}_{\nu}$$
$$= \frac{1}{|\mathcal{L}|} \sum_{\mu \in \mathcal{L}} \sum_{\nu \in \mathcal{L}^m} e\left(q(m\mu) + \beta(m\mu, \lambda - \nu/m)\right) \mathfrak{e}_{\nu}.$$

□

As a consequence, the natural extension of the action is meaningful.

Definition 6.57. For $m \in \mathbb{N}$ and $\delta(m^2) = \gamma g(m^2)\gamma' \in \overline{\Gamma} g(m^2) \overline{\Gamma}$ set

$$\mathfrak{e}_\lambda | \delta(m^2) := \mathfrak{e}_\lambda | \gamma | g(m^2) | \gamma'.$$

Moreover this acts[9] weakly contravariantly in the sense of the following Proposition,

[9]Note that it is not derived from an actual mathematical group action like in the case $\gcd(m, \mathrm{lev}(L)) = 1$.

being essential for the Hecke theory developed by Bruinier and Stein.

Proposition 6.58 ([BS08, Prop. 5.4]). *For coprime m_1, $m_2 \in \mathbb{N}$ we find for elements $\alpha \in \overline{\Gamma(1)}g(m_1^2)\overline{\Gamma(1)}$ and $\beta \in \overline{\Gamma(1)}g(m_2^2)\overline{\Gamma(1)}$ that*

$$\mathfrak{e}_\lambda|\alpha|\beta = \mathfrak{e}_\lambda|(\alpha\beta).$$

Proof: Write $g_1 := g(m_1^2), g_2 := g(m_2^2)$ and $\alpha = \gamma g_1 \gamma', \beta = \delta g_2 \delta'$. By the elementary divisor Theorem we find $\alpha\beta = \epsilon g_1 g_2 \epsilon'$. In view of the definition of the action, the assertion reduces to the case $\gamma = \delta' = 1$. So far,

$$\alpha\beta = g_1\gamma'\delta g_2 = \epsilon g_1 g_2 \epsilon'. \tag{6.25}$$

First, assume $\epsilon' = 1$. Then $\alpha\beta = g_1\gamma'\delta g_2 = \epsilon g_1 g_2$ implies $g_1\gamma'\delta = \epsilon g_1 \in \overline{\Gamma(1)}g_1\overline{\Gamma(1)}$ yielding with the definition of the action preceding this Proposition

$$\begin{aligned} & \mathfrak{e}_\lambda \mid g_1\gamma'\delta &&= \mathfrak{e}_\lambda \mid \epsilon g_1 \\ \Longrightarrow\quad & \mathfrak{e}_\lambda \mid g_1\gamma'\delta| \; g_2 &&= \mathfrak{e}_\lambda \mid \epsilon g_1 \mid g_2. \end{aligned}$$

Clearly, $|g_1|g_2 = |g_1 g_2$, so that the right hand side's action reads $|\epsilon g_1 g_2 = |\alpha\beta$, while the left hand sides action is by definition and the slash operator defining an action of $\overline{\Gamma(1)}$ $|g_1\gamma'|\delta g_2 = |\alpha|\beta$.
Next, consider the case of general ϵ'. Rearranging (6.25) yields

$$\delta g_2 \epsilon'^{-1} = \gamma'^{-1} g_1^{-1} \epsilon g_1 g_2.$$

However, since the matrix component of the left hand side has integral entries, so does the right hand. Since $(m_1, m_2) = 1$ it may be inferred that $\tilde{\delta} := \gamma'^{-1} g_1^{-1} \epsilon g_1 \in \overline{\Gamma(1)}$. Introducing this notation results in

$$\delta g_2 = \tilde{\delta} g_2 \epsilon' \overset{(6.25)}{\Longleftrightarrow} (g_1\gamma')(\tilde{\delta} g_2) = \epsilon g_1 g_2.$$

However, the assertion has already been proven in case $\epsilon' = 1$, so that

$$\mathfrak{e}_\lambda \mid (g_1\gamma') \mid (\tilde{\delta} g_2) = \mathfrak{e}_\lambda \mid (\epsilon g_1 g_2).$$

Acting from the right with ϵ' renders this statement equivalent to the original. □

For $g(m^2) = \left(\left(\begin{smallmatrix} m^2 & 0 \\ 0 & 1 \end{smallmatrix}\right), 1\right) \in \mathrm{Mp}_2(\mathbb{Q})^+$, pick a disjoint left coset decomposition

$$\overline{\Gamma(1)}g(m^2)\overline{\Gamma(1)} = \bigcup_i \overline{\Gamma(1)}h_i.$$

Definition 6.59. For $g(m^2)$ and h_i as above define the action of $\mathcal{T}(m^2)$ on $f \in M_{\mathcal{L},k}$ by

$$f \mapsto f|_{L,k}\mathcal{T}(m^2) := \det(g(m^2))^{k/2-1} \cdot \sum_i \sum_{\lambda \in \mathcal{L}} f_\lambda|_k h_i \otimes \mathfrak{e}_\lambda|_{\mathcal{L}} h_i. \tag{6.26}$$

Before stating the main theorem of this section, a variant of Lemma 6.25 has to be established.

Lemma 6.60. *Let $m, n \in \mathbb{N}$ and R_m^* denote a set of representatives of $\mathcal{T}(m) = T_{\left(\begin{smallmatrix} m & 0 \\ 0 & 1 \end{smallmatrix}\right)}$. Then* $\gcd(m, n) = 1$ *implies*

$$R_m^* R_n^* = R_{mn}^*.$$

Proof: We begin with the case of $\Gamma(1)$. Equation (6.14) of Lemma 6.25 yields

$$R_m^* = \dot{\bigcup}_{\substack{a,d \in \mathbb{N},\, ad=m \\ b \mod d \\ \gcd(a,b,d)=1}} \Gamma(1) \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}. \tag{6.14}$$

The proof is heavily based upon that of Lemma 6.26, as translates of $R_\cdot^*$ form a partition of the set of representatives R_m appearing in its proof. First, we check that the following map is well defined:

$$R_m^* \times R_n^* \to \Gamma(1)_\infty \backslash R_{mn}^*, \qquad (A, B) \mapsto \Gamma(1)_\infty \backslash AB.$$

It is clear that

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \cdot \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} = \begin{pmatrix} aa' & ab' + bd' \\ 0 & dd' \end{pmatrix}$$

is in R_{mn}, however, note that by $\gcd(m, n) = 1$ we have

$$\gcd(aa', dd', ab' + bd') \mid \gcd(aa', dd') = \gcd(a, d) \cdot \gcd(a', d').$$

Assume $p \mid \gcd(aa', dd', ab' + bd')$ then

$$p \mid \gcd(a, d) \implies p \mid bd' \implies p \mid b \implies p \mid \gcd(a, b, d) \quad \bot,$$
$$p \mid \gcd(a', d') \implies p \mid ab' \implies p \mid b' \implies p \mid \gcd(a', b', d') \quad \bot.$$

So the right hand side is, in fact, contained in R^*_{mn}. The proof of Lemma 6.26 shows that the mapping is bijective, if the non-primitive version R_m is considered. However, note that by Lemma 6.25

$$R_m = \bigcup_{\substack{a,d\in\mathbb{N},\, ad=m \\ b \mod d}} \Gamma(1)\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \dot{\bigcup}_{d^2|m} (d\mathcal{I})\cdot R^*_{m/d^2},$$

so that it must, by $d\mathcal{I}\in Z(\mathrm{GL}_2(\mathbb{R}))$, be bijective on the level of R^*_m. In case of a square we find the identity

$$R_{m^2} = \bigcup_{\substack{a,d\in\mathbb{N},\, ad=m^2 \\ b \mod d}} \Gamma(1)\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \dot{\bigcup}_{d|m} (d\mathcal{I})\cdot R^*_{(m/d)^2}, \tag{6.27}$$

which is used to explicitly partition $T(m^2)$ essentially into primitive operators $\mathcal{T}(\ \cdot^2\ )$. Now in case of $\overline{\Gamma(1)}$, it suffices to consider only squares m, n. In that case, however, the computation is identical. □

We also require the action of another element. Recall that we had defined the metaplectic element $g(m^2) = \left(\left(\begin{smallmatrix} m^2 & 0 \\ 0 & 1\end{smallmatrix}\right), 1\right)$ in (6.19) and define the following associated element

$$g^*(m^2) := \left(\begin{pmatrix} 1 & 0 \\ 0 & m^2 \end{pmatrix}, m\right) = \overline{S}g(m^2)\overline{S}^{-1} \in \overline{\Gamma(1)}g(m^2)\overline{\Gamma(1)}. \tag{6.28}$$

Lemma 6.61. *For $\lambda\in\mathcal{L}$ and $m\in\mathbb{N}$ we find*

$$\mathfrak{e}_\lambda|_{\mathcal{L}}g^*(m^2) = \sum_{\substack{\sigma\in\mathcal{L} \\ m\sigma=\lambda}} \mathfrak{e}_\sigma \tag{6.29}$$

and conclude for the standard scalar product on $\mathbb{C}[\mathcal{L}]$ and v_1, $v_2\in\mathbb{C}[\mathcal{L}]$ that

$$\langle v_1|_{\mathcal{L}}g(m^2), v_2\rangle = \langle v_1, v_2|_{\mathcal{L}}g^*(m^2)\rangle, \tag{6.30}$$

meaning these operators are adjoint to each other: $\left(|_{\mathcal{L}}g(m^2)\right)^* = |_{\mathcal{L}}g^*(m^2)$.

Proof: By linearity it suffices to prove the identity on a basis of $\mathbb{C}[\mathcal{L}]$. Let $\lambda,\mu\in\mathcal{L}$ and consider the following computation:

$$\langle\mathfrak{e}_\lambda|g(m^2), \mathfrak{e}_\mu\rangle = \langle\mathfrak{e}_{m\lambda}, \mathfrak{e}_\mu\rangle$$

$$= \delta_{m\lambda=\mu}$$
$$= \langle e_\lambda, \sum_{\substack{\sigma\in\mathcal{L}\\ m\sigma=\mu}} \mathfrak{e}_\sigma\rangle.$$

Note that the right entry of the scalar product may be realised as $\mathfrak{e}_\mu|g^*(m^2)$. In fact, we find by (3.34), (3.35), and $\overline{S}^{-1} = \overline{S}^7 = Z^3\overline{S} = \left(\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right), -i\sqrt{\tau}\right)$ that

$$\begin{aligned}\mathfrak{e}_\mu|_{\mathcal{L}}\overline{S}g(m^2)\overline{S}^{-1} &= \frac{e(-\operatorname{sig}(\mathcal{L})7/8)}{\sqrt{\mathcal{L}}}\sum_{\nu\in\mathcal{L}} e(\mathrm{b}(\mu,\nu))\mathfrak{e}_{m\nu}|\overline{S}^{-1}\\ &= \frac{1}{|\mathcal{L}|}\cdot\sum_{\nu\in\mathcal{L}} e(\beta(\mu,\nu))\sum_{\sigma\in\mathcal{L}} e(-\mathrm{b}(m\nu,\sigma))\mathfrak{e}_\sigma\\ &= \frac{1}{|\mathcal{L}|}\cdot\sum_{\nu,\sigma\in\mathcal{L}} e(\mathrm{b}(\mu-m\sigma,\nu))\mathfrak{e}_\sigma\\ &= \sum_{\substack{\sigma\in\mathcal{L}\\ m\sigma=\mu}} \mathfrak{e}_\sigma,\end{aligned}$$

so that

$$g^*(m^2) = \det(g(m^2))g(m^2)^{-1} = \overline{S}g(m^2)\overline{S}^{-1}$$

acts adjointly to $g(m^2)$. □

As a consequence, we obtain the following adjointness on the space of modular forms.

Corollary 6.62. *Let $f, h \in \mathcal{S}_{L,k}(\overline{\Gamma(1)})$, $m \in \mathbb{N}$ and $g(m^2)$ as above. Then we find for every $\alpha \in \overline{\Gamma(1)}g(m^2)\overline{\Gamma(1)}$ that*

$$\langle f|_{L,k}\alpha, h\rangle = \langle f, h|_{L,k}\det(\alpha)\alpha^{-1}\rangle. \tag{6.31}$$

The expression does not depend on the choice of α in the double coset.

Proof: Recall that

$$\langle f|_{L,k}\alpha, h\rangle = \sum_{\lambda,\mu\in\mathcal{L}} \langle f_\lambda|_k\alpha, h_\mu\rangle\langle \mathfrak{e}_\lambda|\alpha, \mathfrak{e}_\mu\rangle.$$

Assume first that $\alpha \in \overline{\Gamma(1)} = \mathrm{Mp}_2(\mathbb{Z})$. Note that the scalar valued case of Corollary 6.62 as in Lemma 6.21 is also valid for $\Gamma(N)$ – the proof is identical but simpler, as there is no character. Recall that the component functions satisfy $f_\lambda, h_\mu \in \mathcal{S}_k(\Gamma(\mathrm{lev}(L)))$ and combine this with the unitaricity of the Weil representation for the desired result.

In the general case, by the contravariance of the slash operator (cf. Definition 6.57), and invariance of f and h under $\mathrm{Mp}_2(\mathbb{Z})$, we may reduce to the choice $\alpha = g(m^2)$. This, however is solved by the same argument as above combined with Lemma 6.61 in the components of the group algebra. □

Combining the results above leads to the following observation which is also stated in [BS08, Thm. 5.6 p. 269].

Theorem 6.63. *The operation of $\mathcal{T}(m^2)$ for arbitrary $m \in \mathbb{N}$ determines a linear, cusp form preserving action on the space of modular forms. Further, it is identified to be self adjoint with respect to the Petersson scalar product and the operators satisfy*

$$\mathcal{T}(m_1^2)\mathcal{T}(m_2^2) = \mathcal{T}(m_1^2 m_2^2)$$

for coprime elements $m_1, m_2 \in \mathbb{N}$.

We install the following piece of notation in order to state two corollaries.

Definition 6.64. For $M \in \mathbb{N}$ let $\mathcal{H}_M^2$ denote the Hecke algebra generated by all $\mathcal{T}(p^2)$ and $T_{p\mathcal{I}}$ for primes p with $\gcd(p, M) = 1$.

Corollary 6.65. *For any $M \in \mathbb{N}$, the algebra $\mathcal{H}_M^2$ acts as a commutative algebra of cusp form preserving linear operators.*

The Corollary follows from Theorem 6.63. In conjunction with Lemma 6.51 we conclude that for any prime p with $p \nmid M\,\mathrm{lev}(L)$, the algebra $\mathcal{H}_M^2$ contains $\mathcal{T}(p^{2r})$ for all $r \in \mathbb{N}$.

Corollary 6.65 immediately implies the existence of simultaneous eigenforms.

Corollary 6.66. *Let $M \in \mathbb{N}$ and assume $W \leq \mathcal{S}_{L,k}$ is a nonzero subspace that is invariant under the operation of $\mathcal{H}_M^2$. Then W has an orthonormal basis consisting of simultaneous eigenforms of $\mathcal{H}_M^2$.*

We turn towards proving Theorem 6.63.

Proof: The first assertion is immediate. For the proof of the second claim, let $f, h \in \mathcal{S}_{L,k}$ and apply Corollary 6.62 to see that there must be a natural number $n_m \in \mathbb{N}$ such that

$$\begin{aligned}\langle f|_{L,k}\mathcal{T}(m^2), h\rangle &= \det(g(m^2))^{k/2-1} \cdot n_m \cdot \langle f|_{L,k} g(m^2), h\rangle \\ &= \det(g(m^2))^{k/2-1} \cdot n_m \cdot \langle f, h|_{L,k}\overline{S} g(m^2)\overline{S}^{-1}\rangle \\ &= \det(g(m^2))^{k/2-1} \cdot n_m \cdot \langle f, h|_{L,k} g(m^2)\rangle\end{aligned}$$

$$= \langle f, h|_{L,k} T_{g(m^2)} \rangle.$$

The last assertion is a consequence of Lemma 6.60 and the fact that the contravariance of the slash operator extends to the definition for $p \mid \mathrm{lev}(L)$ in case of coprime numbers as in Proposition 6.58. The product $\mathcal{T}(m_1^2)\mathcal{T}(m_2^2)$ will, as set of representatives, admit the product set $R^*_{m_1^2} R^*_{m_2^2}$ which, by the stated lemma, equals $R^*_{(m_1m_2)^2}$ representing the operator $\mathcal{T}((m_1m_2)^2)$. □

Now that the existence of eigenforms is certain, a bound on the respective eigenvalues was advantageous. The following method for such bounds was discovered by Kohnen [Koh87] even though the bound itself had been known before [Wei84, Sec. 6].

Lemma 6.67. *Let $m \in \mathbb{N}$ and $f \in \mathcal{S}_{L,k}(\overline{\Gamma(1)})$ be a nonzero eigenform of $\mathcal{T}(m^2) = T_{g(m^2)}$ with eigenvalue σ_m, where $g(m^2) = \left(\left(\begin{smallmatrix} m^2 & 0 \\ 0 & 1 \end{smallmatrix} \right), 1 \right)$. Then*

$$|\sigma_m| < m^{k-2} |\overline{\Gamma(1)} \backslash \overline{\Gamma(1)} g(m^2) \overline{\Gamma(1)}| \cdot |{}_m\mathcal{L}|. \tag{6.32}$$

In the particular case where $m = p$ is a prime, we find

$$|\sigma_p| < p^{k-1} \cdot (p+1) \cdot |{}_p\mathcal{L}|. \tag{6.33}$$

Here, ${}_m\mathcal{L}$ for a natural number m denotes the m-torsion of $\mathcal{L}$.

Proof: Recall the definition of $g^*(m^2)$ from (6.28) and note that for $\lambda \in \mathcal{L} = L'/L$ we have by Lemma 6.61 that

$$\mathfrak{e}_\lambda |_{\mathcal{L}} g^*(m^2) |_{\mathcal{L}} g(m^2) = \sum_{\substack{\sigma \in \mathcal{L} \\ m\sigma = \lambda}} \mathfrak{e}_\sigma |_{\mathcal{L}} g(m^2) = \delta_{\lambda \in {}^m\mathcal{L}} \cdot |{}_m\mathcal{L}| \cdot \mathfrak{e}_\lambda.$$

First, note that Cauchy–Bunyakowsky–Schwarz implies in conjunction with Corollary 6.62 that

$$\| f|_{L,k} g^*(m^2) \|_2^2 \leq \| f|_{L,k} g^*(m^2) g(m^2) \|_2 \cdot \|f\|_2.$$

Further,

$$\begin{aligned} &\| f|_{L,k} g^*(m^2) g(m^2) \|_2^2 \\ &= \sum_{\lambda, \mu \in \mathcal{L}} \langle f_\lambda |_k g^*(m^2) g(m^2), f_\mu |_k g^*(m^2) g(m^2) \rangle \langle \mathfrak{e}_\lambda |_k g^*(m^2) g(m^2), \mathfrak{e}_\mu |_k g^*(m^2) g(m^2) \rangle \end{aligned}$$

$$= \sum_{\lambda,\mu\in {}^m\mathcal{L}} \langle f_\lambda, f_\mu\rangle |_m\mathcal{L}|^2 \langle \mathfrak{e}_\lambda, \mathfrak{e}_\mu\rangle$$
$$\leq |_m\mathcal{L}|^2 \|f\|_2^2.$$

In particular, in case of $\gcd(m, \mathrm{lev}(L))$, we have equality in the last line. Here, ${}_m\mathcal{L}$ and ${}^m\mathcal{L}$ denote the m-torsion and m-multiples in $\mathcal{L}$, respectively. Also compare Remark 1.36. Second,

$$\begin{aligned}\sigma_m \cdot \langle f, f\rangle &= \left\langle \mathcal{T}(m^2) f, f\right\rangle \\ &= m^{k-2} \sum_{h\in\overline{\Gamma(1)}\backslash\overline{\Gamma(1)}g(m^2)\overline{\Gamma(1)}} \langle f|_k h, f\rangle \\ &= m^{k-2} \cdot |\overline{\Gamma(1)}\backslash\overline{\Gamma(1)}g(m^2)\overline{\Gamma(1)}| \cdot \left\langle f|_k g(m^2), f\right\rangle,\end{aligned}$$

where we have used, in the last line that by Corollary 6.62, the value of the inner product does not depend on the representative in the double coset. Again, by Cauchy–Bunyakowsky–Schwarz and the above calculations, we infer

$$\begin{aligned}|\sigma_m| &\leq m^{k-2} \left|\overline{\Gamma(1)}\backslash\overline{\Gamma(1)}g(m^2)\overline{\Gamma(1)}\right| \frac{\|f|_{L,k}g(m^2)\|_2 \cdot \|f\|_2}{\|f\|_2^2} \\ &= m^{k-2} \cdot \left|\overline{\Gamma(1)}\backslash\overline{\Gamma(1)}g(m^2)\overline{\Gamma(1)}\right| |_m\mathcal{L}|.\end{aligned} \tag{6.34}$$

Considering the case $\gcd(m, \mathrm{lev}(L)) = 1$, we have equality, if, and only if, $f|_{L,k}g(m^2)$ and f are proportional to each other. Hence, assume there is some constant $C \in \mathbb{C}$ with

$$C \cdot \sum_{\lambda\in\mathcal{L}} \sum_{n\in\overline{\mathfrak{q}}(\lambda)+\mathbb{Z}} a(\lambda, n)\, e\,(\tau n) \otimes \mathfrak{e}_\lambda = \sum_{\lambda\in\mathcal{L}} \sum_{n\in\overline{\mathfrak{q}}(\lambda)+\mathbb{Z}} a(\lambda, n)\, e\left(\tau n m^2\right) \otimes \mathfrak{e}_{m\lambda}. \tag{6.35}$$

Here, we see that components $\lambda \in {}_m\mathcal{L}$ could not have contributed. Assume first, that $\gcd(m, \mathrm{lev}(L)) = 1$ in which case multiplication by m acts as an automorphism on $\mathcal{L}$. So there is a natural number $l \in \mathbb{N}$, such that m^l acts as the identity on $\mathcal{L}$. Now, unless $\sigma_m = 0$, we must have $C \neq 0$. In the latter case, assume $a(\lambda, n)$ was a nonzero coefficient and possibly replace the l above by a multiple, large enough, so that $m^l \nmid n$. Applying the operator $|_{L,k}g(m^2)$ in succession for l iterations, we obtain (6.35) but with (C, m) replaced by (C^l, m^l). Hence, we find by comparing coefficients that $a(\lambda, n)$ could only be different from zero if $m^l \mid n$ which contradicts our assumption. Hence, we must have strict inequality in equation (6.34).

In the case of $\gcd(m, \mathrm{lev}(L)) > 1$, we find that if the bound was not strict, there could not be any contribution from any p part $\mathcal{L}_p$ of $\mathcal{L}$, for primes $p \mid \gcd(m, \mathrm{lev}(L))$. Explicitly, this means that $f_\lambda = 0$ for any λ, such that the projection to $\mathcal{L}_p$ for p as above was different from 0. On the remaining components, the argument is the same as above.
Finally, for the choice $m = p$, there are $p(p+1)$ cosets in the class $\overline{\Gamma(1)} \backslash \overline{\Gamma(1)} g(m^2) \overline{\Gamma(1)}$ (cf. [Shi73, p. 451]), finishing the proof. □

We turn our attention towards the irregular case of bad primes for the action of Hecke operators on Fourier coefficients of vector valued modular forms, in analogy to Proposition 6.54. Such a formula has been given by [Ste15, Thm. 5.4 p. 246] for bad odd primes. However, it has been pointed out by [BCJ18] that the cited Theorem is flawed. The original author has submitted a correction [Ste21], representing the previously arising but flawed coefficients in a more abstract fashion in terms of certain representation numbers of the lattice that also appear in [BK01, p. 447] as coefficients of Eisenstein series.

These representation numbers, however, are not explicit enough for the application we have in mind. In particular, they are unfit to derive product expansions of L-series. Hence, the coefficients had to be calculated more explicitly based on the exposition [Ste15]. To this end we have to let $\mathcal{T}(p^2)$ explicitly act on $f \in \mathcal{M}_{L,k}$ where the operation is given by letting right cosets of

$$\overline{\Gamma(1)} g(p^2) \overline{\Gamma(1)}, \text{ where } g(p^2) = \left(\begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix}, 1 \right)$$

operate as in Definition 6.59. Recall that by [BS08, p. 263] or [Shi73, p. 451] we find

$$\overline{\Gamma(1)} g(p^2) \overline{\Gamma(1)} = \overline{\Gamma(1)} g(p^2) \cup \bigcup_{h(p)^*} \overline{\Gamma(1)} \beta_h \cup \bigcup_{b(p^2)} \overline{\Gamma(1)} \gamma_b, \tag{6.36}$$

where

$$\beta_h = \left(\begin{pmatrix} p & h \\ 0 & p \end{pmatrix}, \sqrt{p} \right), \qquad \gamma_b = \left(\begin{pmatrix} 1 & b \\ 0 & p^2 \end{pmatrix}, p \right). \tag{6.37}$$

The operation on the scalar component functions of the vector valued modular form via the Petersson slash operator is well known and understood. As a consequence, it suffices to compute the action of the elements in (6.37) on the discriminant group. The action of

γ_b is presented in [Ste15, Thm. 5.1 p. 243] in greater generality than required.

Proposition 6.68. *Let p be an odd prime, $b \in \mathbb{Z}/p^2\mathbb{Z}$, and $\lambda \in \mathcal{L}$. Then γ_b acts in the following fashion:*

$$\mathfrak{e}_\lambda|_{\mathcal{L}}\gamma_b = \sum_{\substack{\nu\in\mathcal{L} \\ p\nu=\lambda}} e(-b\,\overline{\mathrm{q}}(\nu))\mathfrak{e}_\nu. \tag{6.38}$$

The action of β_h, however, will be extracted from [BCJ18, Prop. 5.3 p. 26] where we select the special case of $l = a = 1$.

Proposition 6.69. *Let p be an odd prime and $h \in (\mathbb{Z}/p\mathbb{Z})^\times$, write $m = \operatorname{rank}(L)$, and select $\lambda \in \mathcal{L}$. Then*

$$\mathfrak{e}_\lambda|_{\mathcal{L}}\beta_h = p^{-m/2} \sum_{\substack{\delta\in\mathcal{L}(p) \\ p\delta=\lambda}} e(-h\,\overline{\mathrm{q}}_p(\delta))\mathfrak{e}_\lambda, \tag{6.39}$$

where $\overline{\mathrm{q}}_p$ denotes $\overline{p\,\mathrm{q}} : \mathcal{L}(p) = L(p)'/L(p) \to \mathbb{Q}/\mathbb{Z}$.

This will have to be rewritten for our purposes.

Lemma 6.70. *With the notation as above, we find*

$$\mathfrak{e}_\lambda|_{\mathcal{L}}\beta_h = p^{-m/2}\delta_{\lambda\in{}^p\mathcal{L}} \cdot e\left(-hp\,\mathrm{q}(\ell_{\lambda/p})\right) G_{L,p}(1,-h)\mathfrak{e}_\lambda,$$

where $G_{L,p}(n,h) = \sum_{v\in L/p^nL} e\left(\frac{h}{p^n}\,\mathrm{q}(v)\right)$ is the Gauss sum defined in Definition A.12, $\ell_{\lambda/p} \in L'$ is some element, such that $p \cdot \overline{\ell_{\lambda/p}} = \lambda$, ${}^p\mathcal{L}$ are the p-multiples in $\mathcal{L}$ and δ denotes Kronecker symbol.

Proof: Beginning with (6.39) we transform the sum by utilising the following description $\mathcal{L}(p) = p^{-1}L'/L \simeq L'/pL$:

$$p^{-m/2} \sum_{\substack{\delta\in\mathcal{L}(p) \\ p\delta=\lambda}} e(-h\,\overline{\mathrm{q}}_p(\delta))\mathfrak{e}_\lambda = p^{-m/2} \sum_{\substack{\mu\in L'/pL \\ \mu=\lambda \mod L}} e\left(-\frac{h}{p}\,\mathrm{q}(\mu)\right)\mathfrak{e}_\lambda,$$

where $\mathrm{q}(\mu)$ is do be understood as the quadratic form on L' of some preimage in L' of $\mu \in L'/pL$. In order for the sum above to not vanish, it is necessary that $\lambda \in {}^p\mathcal{L}$ (cf. [Bar03, Lemma 5.2.1]). Note that by the isomorphism $(L'/pL)/(L/pL) \simeq \mathcal{L}$, the solutions in μ of $\mu = \lambda \mod L$ are represented by $\lambda + L/pL$ so that the expression above equals

$$p^{-m/2}\delta_{\lambda\in{}^p\mathcal{L}} \cdot \sum_{v\in L/pL} e\left(-\frac{h}{p}\,\mathrm{q}(\ell_\lambda + v)\right)\mathfrak{e}_\lambda,$$

where ℓ_λ is some lift of λ to L'. In fact, the sum does not depend on the chosen representative, as the shift may be absorbed into v. We would like to separate the dependency on λ from the sum. To this end, let $\nu \in \mathcal{L}$ be a p-th root of λ, i.e. $p\cdot\nu = \lambda \in \mathcal{L}$ and denote a lift of ν to L' by ℓ_ν. Then

$$\delta_{\lambda\in{}^p\mathcal{L}} \sum_{v\in L/pL} e\left(-\tfrac{h}{p}\,\mathrm{q}(\ell_\lambda + v)\right) = \delta_{\lambda\in{}^p\mathcal{L}} \cdot e\left(-\tfrac{h}{p}\,\mathrm{q}(p\ell_\nu)\right) \cdot \sum_{v\in L/pL} e\left(-\tfrac{h}{p}\,\mathrm{q}(v)\right). \qquad (*)$$

In fact, since $\lambda \in {}^p\mathcal{L}$ there is $\nu \in \mathcal{L}$ such that $p\nu = \lambda$. Hence, there must be $\ell_\nu \in L'$ and $\delta \in L$ such that $p\ell_\nu + \delta = \ell_\lambda$. Nevertheless, the term $\delta \in L$, may be absorbed in v and will henceforth be ignored resulting in replacing $\mathrm{q}(\ell_\lambda + v)$ by $\mathrm{q}(p\ell_\nu + v)$. Then, however,

$$\mathrm{q}(p\ell_\nu + v) = \mathrm{q}(p\ell_\nu) + p\beta(\ell_\nu, v) + \mathrm{q}(v) \equiv \mathrm{q}(p\ell_\nu) + \mathrm{q}(v) \mod p.$$

This, in fact, means that the reduction above is valid.
Note that Stein appears to have made the same reduction, first. Then, however, he noted in [Ste21] that it was not valid because the factored expression in $(*)$ depended on the choice of representative. If, however, a p-th root ν of λ is considered, there is no ambiguity as the following computations testifies: Note that two chosen ν may differ by $\nu' \in {}_p\mathcal{L}$. As a consequence, ℓ_ν might be replaced with $\ell_\nu + \ell_{\nu'} + \nu_0$ with $\nu_0 \in L$. Then

$$\begin{aligned}
\mathrm{q}\left[p(\ell_\nu + \ell_{\nu'} + \nu_0)\right] &= \mathrm{q}\left[p(\ell_\nu + \ell_{\nu'})\right] + p^2\,\mathrm{b}(\ell_\nu + \ell_{\nu'}, \nu_0) + p^2\,\mathrm{q}(\nu_0) \\
&\equiv \mathrm{q}\left[p(\ell_\nu + \ell_{\nu'})\right] \mod p \\
&= \mathrm{q}(p\ell_\nu) + p\,\mathrm{b}(\ell_\nu, p\ell_{\nu'}) + \mathrm{q}(p\ell_{\nu'}) \\
&\equiv \mathrm{q}(p\ell_\nu) \mod p.
\end{aligned}$$

In order to verify the last line, note that $p\ell_{\nu'} \in L$, so that the term involving the bilinear form is, in fact, in $p\mathbb{Z}$ and we have $\mathrm{q}(p\ell_{\nu'}) \in \mathbb{Z}$. On the other hand we find $\mathrm{q}(p\ell_{\nu'}) = 2^{-1}p\,\mathrm{b}(\ell_{\nu'}, p\ell_{\nu'}) \in \frac{p}{2}\mathbb{Z} \cap \mathbb{Z} = p\mathbb{Z}$. □

We follow [Ste15] in order to compute the action of Hecke operators on the Fourier expansion of vector valued modular forms more explicitly. In fact, the following is a modified version of [Ste15, Thm. 5.4 p. 246].

Proposition 6.71. *Let p be an odd prime and $f \in \mathcal{M}_{L,k}$ with Fourier expansion*

$$f(\tau) = \sum_{\lambda \in \mathcal{L}} \sum_{n \in \overline{\mathrm{q}}(\lambda)+\mathbb{Z}} a(\lambda, n) \cdot e(n\tau)\mathfrak{e}_\lambda.$$

Then the Fourier coefficients of $f|_{L,k}\mathcal{T}(p^2)$ are given by

$$\begin{aligned} b(\lambda, n) = p^{2(k-1)} \delta_{\lambda \in {}^p\mathcal{L}} & \sum_{\substack{\lambda' \in {}_p\mathcal{L} \\ n - p^2\,\overline{\mathrm{q}}(\lambda/p+\lambda') \in p^2\mathbb{Z}}} a\left(\lambda/p + \lambda', \frac{n - p^2\,\overline{\mathrm{q}}(\lambda/p+\lambda')}{p^2} + \overline{\mathrm{q}}(\lambda/p+\lambda')\right) \\ & + p^{k-2}\delta_{\lambda \in {}^p\mathcal{L}} K_{L,p} g_p\left[1, {}_{-}\chi_p{}^R, n - \mathrm{q}(p \cdot \ell_{\lambda/p})\right] \, a(\lambda, n) \\ & + a(p\lambda, p^2 n). \end{aligned}$$

Here, $K_{L,p} = p^{-m/2} G_{L,p}(1,1)$, where $G_{L,p}(r,1) = \sum_{\nu \in L/p^rL} e\left(\frac{1}{p^r} q(\nu)\right)$ and $g_p[r,\chi,n] = \sum_{h \in (\mathbb{Z}/p^r\mathbb{Z})^*} \chi(h) e\left(\frac{hn}{p^r}\right)$, while $\ell_{\lambda/p} \in L'$ is any element such that $\overline{p\ell_{p/\lambda}} = \lambda \in \mathcal{L}$. Further, ${}_p\mathcal{L}$ is the p-torsion of $\mathcal{L}$, ${}^p\mathcal{L}$ are the p-multiples and $\delta_{\lambda \in {}^p\mathcal{L}} = 1$ if $\lambda \in {}^p\mathcal{L}$ and 0, otherwise. In addition, ${}_{-}\chi_p(h) = \left(\frac{-h}{p}\right)$ is the Legendre symbol and the exponent R has been presented in Definition 1.38 (1.6) as R_p.

Proof: Most of the proof is identical for higher powers of p^2, but we restrict to the operation of $\mathcal{T}(p^2)$ in order to lighten the notation. We have

$$f|_{k,L}\mathcal{T}(p^2) = p^{k-2} \sum_{\lambda \in \mathcal{L}} f_\lambda|_k g(p^2)\ \mathfrak{e}_\lambda|_L g(p^2) \tag{6.40}$$

$$+ p^{k-2} \sum_{h \in (\mathbb{Z}/p\mathbb{Z})^\times} \sum_{\lambda \in \mathcal{L}} f_\lambda|_k \beta_h\ \mathfrak{e}_\lambda|_L \beta_h \tag{6.41}$$

$$+ p^{k-2} \sum_{b \in \mathbb{Z}/p^2\mathbb{Z}} \sum_{\lambda \in \mathcal{L}} f_\lambda|_k \gamma_b\ \mathfrak{e}_\lambda|_L \gamma_b. \tag{6.42}$$

For the first part (6.40) we find

$$\begin{aligned} & p^{k-2} \sum_{\lambda \in \mathcal{L}} f_\lambda|_k g(p^2)\ \mathfrak{e}_\lambda|_{\mathcal{L}} g(p^2) \\ = {} & p^{2(k-1)} \sum_{\lambda \in \mathcal{L}} f_\lambda\left(p^2\tau\right)\ \mathfrak{e}_{p\lambda} \\ = {} & p^{2(k-1)} \sum_{\lambda \in \mathcal{L}} \sum_{n \in \mathbb{Z}+\overline{\mathrm{q}}(\lambda)} a(\lambda, n) \cdot e\left(p^2 n\tau\right)\ \mathfrak{e}_{p\lambda} \end{aligned}$$

$$= p^{2(k-1)} \sum_{\lambda \in {}^p\mathcal{L}} \sum_{\lambda' \in {}_p\mathcal{L}} \sum_{n \in p^2(\mathbb{Z}+\overline{\mathrm{q}}(\lambda/p+\lambda'))} a(\lambda/p+\lambda', n/p^2) \cdot e(n\tau) \;\mathfrak{e}_\lambda.$$

Here, λ/p is any fixed p-th root of λ in $\mathcal{L}$. Also compare Remark 1.36. The inner sum in the expression above may be rewritten as

$$\sum_{n-p^2\,\overline{\mathrm{q}}(\lambda/p+\lambda') \in p^2\mathbb{Z}} a\left(\lambda/p+\lambda', \frac{n-p^2\,\overline{\mathrm{q}}(\lambda/p+\lambda')}{p^2} + \overline{\mathrm{q}}(\lambda/p+\lambda')\right) \cdot e(n\tau) \;\mathfrak{e}_\lambda$$

in order to separate the integer part of the index from the rest.
Next, we evaluate the sum (6.41) involving the action of the elements β_h:

$$p^{k-2} \sum_{h \in (\mathbb{Z}/p\mathbb{Z})^\times} \sum_{\lambda \in \mathcal{L}} f_\lambda|_k\beta_h \;\mathfrak{e}_\lambda|_L\beta_h.$$

First, we utilise Lemma 6.70 to obtain

$$\begin{aligned} &p^{k-2} \sum_{h \in (\mathbb{Z}/p\mathbb{Z})^\times} \sum_{\lambda \in \mathcal{L}} f_\lambda|_k\beta_h \;\mathfrak{e}_\lambda|_L\beta_h \\ &= p^{2(k-1)} \sum_{h \in (\mathbb{Z}/p\mathbb{Z})^\times} \sum_{\lambda \in {}^p\mathcal{L}} p^{-k} f_\lambda\left(\frac{p\tau+h}{p}\right) \cdot p^{-\frac{m}{2}} e\left(-hp\,\mathrm{q}(\ell_{\lambda/p})\right) G_{L,p}(1,-h)\mathfrak{e}_\lambda. \qquad (*) \end{aligned}$$

Here, ℓ_μ for some $\mu \in \mathcal{L}$ is understood to be an arbitrary but fixed lift of μ to L' and λ/p is any fixed p-th root of λ in $\mathcal{L}$. In $(*)$, the dependency of the Gauss sum $G_{L,p}(1,-h)$ on h may be extracted. In fact, an application of Remark A.14 yields

$$G_{L,p}(1,-h) = \left(\frac{-h}{p}\right)^R \cdot G_{L,p}(1,1) \qquad (6.43)$$

with the following choice of R_p. The $(\mathbb{Z}/p\mathbb{Z})$ module L/pL with quadratic form to $\mathbb{Z}/p\mathbb{Z}$ decomposes into one dimensional submodules (cf. Remark A.14). Let R_p be the number of these one dimensional modules, such that their generator is non-isotropic, i.e. if v is a generator, then $\mathrm{q}(v) \notin p\mathbb{Z}$.
With these computations, and by writing ${}_{-}\chi_p(h)$ for the Legendre symbol appearing in (6.43), we find the following expression for $(*)$:

$$p^{k-2-m/2} G_{L,p}(1,1) \sum_{h \in (\mathbb{Z}/p\mathbb{Z})^\times} \sum_{\lambda \in {}^p\mathcal{L}} f_\lambda\left(\frac{p\tau+h}{p}\right) \cdot e\left(-hp\,\mathrm{q}(\ell_{\lambda/p})\right) {}_{-}\chi_p(h)^R \mathfrak{e}_\lambda$$

$$= p^{k-2-m/2} G_{L,p}(1,1) \sum_{h \in (\mathbb{Z}/p\mathbb{Z})^\times} \sum_{\lambda \in {}^p\mathcal{L}} \sum_{n \in \overline{\mathrm{q}}(\lambda)+\mathbb{Z}} a(\lambda, n)\, e(n\tau)\, e\left(\tfrac{h}{p}[n - \mathrm{q}(p \cdot \ell_{\lambda/p})]\right) {}_{-}\chi_p(h)^R \mathfrak{e}_\lambda.$$

Set $K_{L,p} := p^{-m/2} G_{L,p}(1,1)$ and compare Definition A.12 to recognise the appearing term above as the following Gauss sum:

$$g_p\left[1, {}_{-}\chi_p^R, n - \mathrm{q}(p \cdot \ell_{\lambda/p})\right] = \sum_{h \in (\mathbb{Z}/p\mathbb{Z})^\times} e\left(\tfrac{h}{p}[n - \mathrm{q}(p \cdot \ell_{\lambda/p})]\right) \chi_p(-h)^R$$

which yields for the term in $(*)$ to equal

$$p^{k-2} K_{L,p} \sum_{\lambda \in {}^p\mathcal{L}} \sum_{n \in \overline{\mathrm{q}}(\lambda)+\mathbb{Z}} a(\lambda, n) g_p\left[1, {}_{-}\chi_p^R, n - \mathrm{q}(p \cdot \ell_{\lambda/p})\right] e\left(n\tau\right) \mathfrak{e}_\lambda.$$

Last, the sum in (6.42) involving the action of γ_b is evaluated, using Proposition 6.68.

$$\begin{aligned} &p^{k-2} \sum_{b \in \mathbb{Z}/p^2\mathbb{Z}} \sum_{\lambda \in \mathcal{L}} f_\lambda|_k \gamma_b \; \mathfrak{e}_\lambda|_L \gamma_b \\ &= p^{-2} \sum_{b \in \mathbb{Z}/p^2\mathbb{Z}} \sum_{\lambda \in \mathcal{L}} f_\lambda\left(\frac{\tau + b}{p^2}\right) \cdot \sum_{\substack{\nu \in \mathcal{L} \\ p\nu = \lambda}} e\left(-b\, \overline{\mathrm{q}}(\nu)\right) \mathfrak{e}_\nu \\ &= p^{-2} \sum_{\lambda \in \mathcal{L}} \sum_{\substack{\nu \in \mathcal{L} \\ p\nu = \lambda}} \sum_{n \in \overline{\mathrm{q}}(\lambda)+\mathbb{Z}} a(\lambda, n) \sum_{b \in \mathbb{Z}/p^2\mathbb{Z}} e\left(b \frac{n - \mathrm{q}(p\ell_\nu)}{p^2}\right) e\left(\frac{n\tau}{p^2}\right) \mathfrak{e}_\nu. \qquad (*') \end{aligned}$$

Note that the inner sum is p^2, if $n - \mathrm{q}(p\ell_\nu) \in p^2\mathbb{Z}$, otherwise the geometric formula yields

$$\sum_{b \in \mathbb{Z}/p^2\mathbb{Z}} e\left(\frac{n - \mathrm{q}(p\ell_\nu)}{p^2}\right)^b = \frac{1 - e\left(n - \mathrm{q}(p\ell_\nu)\right)}{1 - e\left([n - \mathrm{q}(\ell_\nu)]/p^2\right)} = 0.$$

Hence,

$$\sum_{b \in \mathbb{Z}/p^2\mathbb{Z}} e\left(\frac{n - \mathrm{q}(p\ell_\nu)}{p^2}\right)^b = p^2 \delta_{n \equiv \mathrm{q}(p\ell_\nu) \mod p^2\mathbb{Z}}.$$

Further, $p\nu = \lambda$ may only be fulfilled if $\lambda \in {}^p\mathcal{L}$. The solutions to the equation are exactly the elements $\lambda/p + \delta$ where $\delta \in {}_p\mathcal{L}$ and λ/p is a fixed inverse image. However, by Remark 1.36, these are all elements in $\mathcal{L}$. In total, we obtain the following expression for

$(*')$:

$$\begin{aligned}
&\sum_{\lambda\in{}^p\mathcal{L}}\sum_{\substack{\nu\in\mathcal{L}\\ p\nu=\lambda}}\sum_{\substack{n\in\mathbb{Z}+\overline{\mathrm{q}}(\lambda)\\ n-\mathrm{q}(p\ell_\nu)\in p^2\mathbb{Z}}} a(\lambda,n)e\left(\frac{n\tau}{p^2}\right)\mathfrak{e}_\nu\\
&=\sum_{\mu\in\mathcal{L}}\sum_{n-p^2\,\mathrm{q}(\ell_\mu)\in p^2\mathbb{Z}} a(p\mu,n)e\left(\frac{n\tau}{p^2}\right)\mathfrak{e}_\mu\\
&=\sum_{\mu\in\mathcal{L}}\sum_{m\in\mathbb{Z}+\overline{\mathrm{q}}(\mu)} a(p\mu,p^2m)e\left(m\tau\right)\mathfrak{e}_\mu. \qquad \square
\end{aligned}$$

As a consequence of the above, we obtain the case of good primes (different from 2), meaning essentially Proposition 6.54.

Corollary 6.72 (Bruinier, Stein)**.** *Let L have even signature, p be an odd prime not dividing $\operatorname{lev}(L)$ and $f\in\mathcal{M}_{L,k}$ with Fourier coefficients $a(\lambda,n)$. Denote the Fourier coefficients of $f|_{L,k}\mathcal{T}(p^2)$ by $b(\lambda,n)$. Then we find*

$$b(\lambda,n)=p^{2k-2}a(\lambda/p,n/p^2)+\frac{G_{\mathcal{L}}(1)}{G_{\mathcal{L}}(p)}p^{k-2}(\delta_{p|n}p-1)a(\lambda,n)+a(p\lambda,p^2n). \tag{6.44}$$

Proof: In the present case, the formulae of Proposition 6.71 read

$$\begin{aligned}
b(\lambda,n)=\;&p^{2(k-1)}a\left(\lambda/p,\frac{n}{p^2}\right)\\
&+p^{k-2-m/2}G_{L,p}(1,1)g_p\left[1,{}_{-}\chi_p{}^R,n-q(p\cdot\lambda/p)\right]\,a\left(\lambda,n\right)\\
&+a(p\lambda,p^2n).
\end{aligned}$$

As a consequence, only the middle term has to be evaluated. So the sums g_p and $G_{L,p}$ are to be understood. Recall that Equation (A.5) in Remark A.18 yields the desired quotient of Gauss sums. Hence, only proving that the second Gauss sum g_p reduces to $(\delta_{p|n}p-1)$ is required. Note that $R=m$ is even in our case, so that $\chi_p^R=\chi_1$ is the trivial character. As a consequence, Remark A.15 implies

$$g_p[1,\chi_1,n-\mathrm{q}(p\cdot\ell_{\lambda/p})]=\delta_{p|n-\mathrm{q}(p\cdot\ell_{\lambda/p})}p-1.$$

However, the denominator of the individual fractions does not contain p, since $p\nmid\operatorname{lev}(L)$. Hence, investigating the numerator for divisibility by p suffices. However, $\mathrm{q}(p\cdot\ell_{\lambda/p})$ is clearly divisible by p, so that the question reduces to whether $p\mid n$ or not. $\square$

6.4 L-functions of modular forms

L-functions and modular forms are closely related analytical objects in the field of number theory. In fact, there is a classical theory of the association of L-functions to scalar valued modular forms, as well as a converse theorem[10]. Furthermore, the interconnection between these classes of objects has also been exploited to deliver the famous proof of Fermat's last theorem. We refer to [DS05] and [Sil13] for more on that matter and to [Bum98] for an accessible overview of the classical theory of L-functions and their relation to modular forms. In the following, our focus lies on associating symmetric square type L-series that have appeared in [Shi73] to vector valued modular forms for the discrete Weil representation which play a key role in the present thesis.

6.4.1 L-series of vector valued modular forms

In this subsection, convergence statements depend on a parameter $\sigma \in \mathbb{R}$. Its value is given by the improvement of the asymptotic growth of Fourier coefficients compared to the Hecke bound of cusp forms and a possible, not necessarily optimal, choice is found in Table 3.1 or Corollary 6.82.

In Section 6.1 we have briefly reviewed L-functions associated to classical modular forms and stated their main properties in Theorem 6.45. In the following we will define similar L-functions associated to vector valued modular forms $f \in \mathcal{M}_{L,k}$ for some lattice L. For this subsection, assume that the Fourier expansion of the modular form f is written as

$$f(\tau) = \sum_{\mu \in L'/L} \sum_{n \in \overline{\mathrm{q}}(\mu)+\mathbb{Z}} a(\mu, n) \cdot e(n\tau)\mathfrak{e}_\mu. \tag{6.45}$$

Given the structure of the operation of the Hecke algebra on Fourier coefficients, a definition resembling symmetric square L-functions comparable to these in [Shi73] is motivated. Recall that (L, q) was an even lattice of signature (m^+, m^-) and rank m. For simplicity, we will initially assume a splitting $L = L_1 \oplus L_2$ with $L_1 \otimes \mathbb{R}$ definite of rank m_1.

Definition 6.73. For $f \in \mathcal{M}_{L,k}$, $L = L_1 \oplus L_2$ an orthogonal splitting with definite $L_1 \otimes_{\mathbb{Z}} \mathbb{R}$, an isotropic element $\eta \in L_2'/L_2$, and $r \in \mathbb{Q}^\times$ we define the formal series

$$L_{L_1,\eta,r}(f, s) := \sum_{0 \neq l \in L_1'} \frac{a(\bar{l} + \eta, r\,\mathrm{q}(l))}{\mathrm{q}(l)^s}. \tag{6.46}$$

[10] Compare [Bum98, Thm 1.5.1 p 60], for instance, for a general Theorem due to André Weil.

The origin of this definition lies in the construction of such L-series via the Rankin–Selbgerg method as in Proposition 6.97.

Remark 6.74. Note that in case $\eta = 0$, the series in (6.46) has a symmetry property inherited from the symmetry of the Fourier coefficients of the modular form f. Recall Remark 3.22, in particular (3.42), reading

$$a(-\lambda, n) = e[(\mathrm{sig}(\mathcal{L}) - 2k)/4]a(\lambda, n)$$

which is satisfied for all $\lambda \in \mathcal{L} = L'/L$ and $n \in \overline{\mathrm{q}}(\lambda) + \mathbb{Z}$. In the light of Remark 3.20, we may assume $\mathrm{sig}(\mathcal{L}) - 2k \equiv 0, 2 \mod 4$, in any case, so that $e[(\mathrm{sig}(\mathcal{L}) - 2k)/4] = \pm 1$.

It is natural to ask for the region of convergence of these L-series for complex arguments $s \in \mathbb{C}$ which may be derived from asymptotic bounds on the Fourier coefficients appearing in their construction. The reader may compare Corollary 3.84 and Corollary 3.85 on that matter. We require the following technical result, before continuing.

Lemma 6.75. *Let $m \in \mathbb{N}$ be a natural number and $\alpha \in \mathbb{R}$. Then*

$$\sum_{0 \neq z \in \mathbb{Z}^m} \|z\|_\infty^{-\alpha}$$

converges if $\alpha > m$, where $\|\cdot\|_\infty$ denotes the ∞–norm.

Proof: For $n \in \mathbb{N}$, we will compute the size of the discrete sphere $\mathbb{S}_\infty^{m-1}(n)$ of radius n in $\mathbb{Z}^m$ with respect to $\|\cdot\|_\infty$. Using the abbreviation $[n] \coloneqq \{0, \ldots, n\}$ for $n \in \mathbb{N}$, we find

$$\begin{aligned}
\mathbb{S}_\infty^{m-1}(n) &= \|\cdot\|_\infty^{-1}(n) \\
&= \{(z_i) \in \mathbb{Z}^m \mid \max |z_i| = n\} \\
&= \|\cdot\|_\infty^{-1}([n]) \setminus \|\cdot\|_\infty^{-1}([n-1]).
\end{aligned}$$

Clearly, $\#\|\cdot\|_\infty^{-1}([n]) = (2n+1)^m$, so that we obtain

$$|\mathbb{S}_\infty^{m-1}(n)| = (2n+1)^m - (2n-1)^m.$$

Using the binomial formula yields

$$(2n \pm 1)^m = \sum_{k=0}^{m} \binom{m}{k} (\pm 1)^k (2n)^{m-k},$$

resulting in

$$\begin{aligned}|\mathbb{S}_\infty^{m-1}(n)| &= (2n+1)^m - (2n-1)^m \\ &= 2\sum_{2\nmid k=0}^{m} \binom{m}{k}(2n)^{m-k} \\ &\leq K \cdot n^{m-1}\end{aligned}$$

for some $K > 0$. With that bound, we obtain

$$\sum_{0\neq z\in\mathbb{Z}^m} \|z\|_\infty^{-\alpha} \leq K \sum_{n\in\mathbb{N}} \frac{1}{n^{1-m+\alpha}}$$

However, the right hand series equals $\zeta(1-m+\alpha)$ which converges exactly if $1-m+\alpha > 1$ or, equivalently, $\alpha > m$. □

With this tool, we may prove convergence of the investigated L-series.

Lemma 6.76. *Assume $L = L_1 \oplus L_2$ splits with definite $L_1 \otimes_\mathbb{Z} \mathbb{R}$ of rank m_1 and $2 \leq k \in \mathbb{Z}/2$. In case $f \in \mathcal{M}_{L,k}$, the series $L_{L_1,\eta,r}(f,s)$ converges normally for $\mathrm{Re}(s) > k - 1 + \frac{m_1}{2}$. If, in addition, $f \in \mathcal{S}_{L,k}$, we find that $L_{L_1,\eta,r}(f,s)$ converges normally for $\mathrm{Re}(s) > \frac{k+m_1}{2} - \sigma$ for $\sigma = 1/2$ in the even case and $\sigma = 1/4$ in the odd case.*

Proof: Recall that $\sqrt{|\mathrm{q}|} : L_1 \otimes_\mathbb{Z} \mathbb{R} \to \mathbb{R}_{\geq 0}$ defines a norm. Hence, it is equivalent to $\|\cdot\|_\infty$, meaning, in particular, there is a constant $0 < c$ such that for all $v \in L_1 \otimes_\mathbb{Z} \mathbb{R}$ we have

$$c \cdot \|v\|_\infty \leq \sqrt{|\mathrm{q}(v)|}.$$

Further recall that we have for any $\lambda \in \mathcal{L}$ a bound

$$|a(\lambda, rn^2)| \leq C \cdot n^{k-2\sigma}$$

for some $\sigma \in \mathbb{R}$ and some positive constant $C > 0$. For instance, we may always pick $\sigma = 1 - k/2$ (cf. Corollary 3.76) and select $\sigma = \frac{1}{5}$ for $f \in \mathcal{S}_{L,k}$ by Corollary 3.84. The best choices available in this thesis are found in Corollary 3.87. In particular, we find

$$\sum_{0\neq l\in L_1'} \frac{|a(\bar{l}+\eta, r\,\mathrm{q}(l))|}{|\mathrm{q}(l)|^{\mathrm{Re}(s)}} \leq \frac{C}{c^2} \sum_{0\neq l\in L_1'} \frac{\|l\|_\infty^{k-2\sigma}}{\|l\|_\infty^{2(\mathrm{Re}(s))}}$$

$$= \frac{C}{c^2} \sum_{0 \neq l \in \mathbb{Z}^{m_1}} \|l\|_\infty^{k-2(\sigma+\mathrm{Re}(s))}.$$

However, by Lemma 6.75 the latter series converges as long as

$$2(\sigma + \mathrm{Re}(s)) - k > m_1$$

holds, which is equivalent to $\mathrm{Re}(s) > \frac{k+m_1}{2} - \sigma$. Clearly, the bound is decreasing in $\mathrm{Re}(s)$, so that normal convergence on a right half plane follows immediately. □

Clearly, it is dissatisfying to rely on a splitting of the lattice L in order to associate the symmetric square L-function from Definition 6.73 to a modular form. In the following, we will erase that assumption. Let L_1 be a $\mathbb{Z}$-submodule of L such that $L_1 \otimes \mathbb{R}$ is definite. By choosing $L_2 := (L_1^\perp \cap L)$ we construct an even non-degenerate lattice $M := L_1 \oplus L_2 \leq L$ of finite index. Then by Proposition 3.36 there is a lifting operator mapping $f \in \mathcal{M}_{L,k}(\mathrm{Mp}_2(\mathbb{Z}))$ to $\uparrow_L^M f \in \mathcal{M}_{M,k}$ and for $\eta \in L_2'/L_2$ isotropic and $r \in \mathbb{Q}^\times$ we may declare the L-series $L_{L_1,\eta,k}(\uparrow_L^M f, s)$ as above. We abbreviate $\tilde{f} :=\uparrow_L^M f$ and denote its Fourier coefficients by $\tilde{a}(\lambda, n)$. Then

$$L_{L_1,\eta,r}(\tilde{f}, s) = \sum_{0 \neq l \in L_1'} \frac{\tilde{a}(\bar{l} + \eta, r\,\mathrm{q}(l))}{\mathrm{q}(l)^s}.$$

Note that if η is not induced by an element of L', then the coefficients $\tilde{a}(\bar{l}+\eta, r\,\mathrm{q}(l))$ vanish. Hence, we may assume η to be induced by an element of L'. Further, $\tilde{a}(\bar{l}+\eta, \mathrm{q}(l))$ vanishes unless $l \in L'$. But if also $l \in L'$, necessarily $\tilde{a}(\bar{l} + \eta, \mathrm{q}(l)) = a(\bar{l} + \eta, \mathrm{q}(l))$. Consequently, if we define $a(\bar{l} + \eta, \mathrm{q}(l))$ to vanish, unless $l \in L'$, we conclude

$$L_{L_1,\eta,r}(\tilde{f}, s) = \sum_{0 \neq l \in L_1'} \frac{a(\bar{l} + \eta, r\,\mathrm{q}(l))}{\mathrm{q}(l)^s}.$$

This prompts us to generalise Definition 6.73.

Definition 6.77. For $f \in \mathcal{M}_{L,k}$, a splitting sublattice $M = L_1 \oplus L_2 \leq L$ with definite $L_1 \otimes \mathbb{R}$, an isotropic element $\eta \in (L' \cap L_2')/L_2$, and $r \in \mathbb{Q}^\times$ define the formal series

$$L_{L_1,\eta,r}(f, s) := \sum_{0 \neq l \in L_1' \cap L'} \frac{a(\bar{l} + \eta, r\,\mathrm{q}(l))}{\mathrm{q}(l)^s}. \tag{6.47}$$

Remark 6.78. In the setting of Definition 6.77 we have

$$L_{L_1,\eta,r}(f,s) = L_{L_1,\eta,r}(\uparrow_L^M f, s),$$

where the right hand side may be interpreted as in Definition 6.73. As a consequence, the definition above is a proper extension to the case without splitting lattice L_1 and, in particular, Lemma 6.76 is applicable to obtain the same range of convergence.

In addition to these general series, we are also interested in the following specialisations which will appear as cycle integrals of Kudla–Millson liftings in Subsection 7.3.1.

Definition 6.79. Let (L, q) be an even lattice and

$$f = \sum_{\mu\in\mathcal{L}} \sum_{n\in\mathbb{Z}+\overline{\mathrm{q}}(\mu)} a(\mu,n)\cdot e(n\tau)\mathfrak{e}_\mu \in \mathcal{M}_{L,k}$$

be a modular form. For $\lambda \in \mathcal{L}$ and $r \in \mathbb{Q}^\times$ we define the associated *symmetric square* L-function

$$L_{(\lambda,r)}(f,s) \coloneqq \sum_{n\in\mathbb{N}} \frac{a(n\lambda, n^2 r)}{n^s} \tag{6.48}$$

and its specialisation induced by $\ell \in L'$ satisfying $\mathrm{q}(\ell) \neq 0$ with $\lambda \coloneqq \overline{\ell}$ and $r \coloneqq \mathrm{q}(\ell)$:

$$L_\ell(f,s) \coloneqq L_{(\overline{\ell},\mathrm{q}(\ell))}(f,s) = \sum_{n\in\mathbb{N}} \frac{a(n\overline{\ell}, n^2\,\mathrm{q}(\ell))}{n^s}. \tag{6.49}$$

The range of convergence of these special series is derived from Lemma 6.76.

Lemma 6.80. *For $f \in \mathcal{M}_{L,k}$, $k \geq 2$, $\lambda \in \mathcal{L} = L'/L$, and $r \in \mathbb{Q}^\times$ the series $L_{(\lambda,r)}(f,s)$ converges absolutely for* $\mathrm{Re}(s) > 2k-1$*. If, in addition, $f \in \mathcal{S}_{L,k}$, we find that $L_{(\lambda,r)}(f,s)$ converges absolutely for* $\mathrm{Re}(s) > k+1-2\sigma$ *for $\sigma = 1/2$ in the even case and $\sigma = 1/4$ in the odd case.*

Proof: Begin by choosing $\ell \in L'$ such that $\overline{\ell} = \lambda$ and recall that we have to assume $r = \mathrm{q}(\ell) \neq 0$. Note that $(\mathbb{Q}\ell \cap L, \mathrm{q}\,|_{\mathbb{Z}\ell})$ defines an even lattice, implying $\ell \in (\mathbb{Q}\ell \cap L, \ell)'$. Denote by ℓ_0 a primitive element of the dual $(\mathbb{Q}\ell \cap L)'$. Then

$$L_{\mathbb{Z}\ell,0,r} = \sum_{0\neq l\in(\mathbb{Z}\ell)'} \frac{a(\overline{l}, r\,\mathrm{q}(l))}{\mathrm{q}(l)^s}$$

$$= \sum_{0\neq n\in\mathbb{Z}} \frac{a(n\overline{\ell_0}, r\,\mathrm{q}(\ell_0)n^2)}{\mathrm{q}(\ell_0)^s n^{2s}}.$$

In particular, if $t \in \mathbb{Z}$ denotes the unique element with $t\ell_0 = \ell$ we have

$$\mathrm{q}(\ell)^{s/2} \cdot L_{\mathbb{Z}\ell,0,r/\mathrm{q}(\ell)}(s/2) = \sum_{0\neq n\in\mathbb{Z}} \frac{a(n\overline{\ell_0}, r(n/t)^2)}{(n/t)^s}.$$

Recall that by Lemma 6.76 this expression converges absolutely for $\mathrm{Re}(s)/2 > \frac{k+1}{2} - \sigma$ for $\sigma = 1 - k/2$ in general and some $1/4 \leq \sigma < 1/2$ in case f was a cusp form (cf. Corollary 3.87). However, by considering the subseries of indices $n \in t\mathbb{N}$ we obtain exactly $L_{(\lambda,r)}(f, s)$. Hence, the latter series converges absolutely for $\mathrm{Re}(s) > k + 1 - 2\sigma$. □

For the proof of Theorem 7.16 we will require subseries of the above L-functions that are of a special shape. A naive motivation for defining these is to isolate coefficients associated to good primes for which there is a product expansion due to a well behaved Hecke theory (cf. Corollary 6.87).

Definition 6.81. In the setting of Definition 6.79, meaning $f \in \mathcal{M}_{L,k}$ with Fourier coefficients $a(\mu, n)$ and $\lambda \in \mathcal{L} = L'/L$ as well as $r \in \mathbb{Q}^\times$, consider for some $N \in \mathbb{N}$ the subseries

$$L^N_{(\lambda,r)}(f, s) := \sum_{\substack{n\in\mathbb{N}\\ \gcd(n,N)=1}} \frac{a(n\lambda, n^2 r)}{n^s} \tag{6.50}$$

and its specialisation induced by $\ell \in L'$ satisfying $\mathrm{q}(\ell) \neq 0$ with $\lambda := \overline{\ell}$ and $r := \mathrm{q}(\ell)$:

$$L^N_\ell(f, s) := L^N_{(\overline{\ell},\mathrm{q}(\ell))}(f, s) = \sum_{\substack{n\in\mathbb{N}\\ \gcd(n,N)=1}} \frac{a(n\overline{\ell}, n^2\,\mathrm{q}(\ell))}{n^s}. \tag{6.51}$$

As subseries of $L_{(\lambda,r)}(f, s)$, these clearly converge absolutely, if the original series does, meaning Lemma 6.80 is applicable as well. Further, the concrete asymptotic bounds for Fourier coefficients of vector valued cusp forms in Section 3.3.4 that are subsumed in Corollary 3.87 provide information on how far absolute convergence may be pushed.

Corollary 6.82. *Let $2 \leq k \in \mathbb{Z}/2$, $\ell \in L'$ be non-isotropic and $f \in \mathcal{S}_{L,k}$ with Fourier expansion*

$$f(\tau) = \sum_{\lambda\in\mathcal{L}} \sum_{n\in\overline{\mathrm{q}}(\lambda)+\mathbb{Z}} a(\lambda, n) \cdot e(n\tau)\mathfrak{e}_\lambda.$$

Then for $N \in \mathbb{N}$ the series

$$L_\ell^N(f,s) = \sum_{\substack{n\in\mathbb{N}\\ \gcd(n,N)=1}} \frac{a(n\bar{\ell}, n^2\,\mathrm{q}(\ell))}{n^s}$$

converges absolutely for $\mathrm{Re}(s) > k+1-2\sigma$ *for some* $1/4 \le \sigma \le 1/2$*. The possible choices of* σ *are given as follows.*

σ	$N \in \mathbb{N}$	$\mathrm{lev}(L) \mid N^\infty$
$2 \mid \mathrm{rk}(L)$	$1/2$	$1/2$
$2 \nmid \mathrm{rk}(L)$	$1/4$	$5/16$

We turn to the application of Hecke theory to the L-functions discussed above.

6.4.2 Product expansions

In case $f \in \mathcal{S}_{L,k}$ is an eigenform of a family of Hecke operators $\mathcal{T}(m^2)$ as in Definition 6.59, the associated L-series considered in Definition 6.79 admits a partial product expansion. These products are induced by the action of Hecke operators on the Fourier expansion of f presented in Proposition 6.54 and Theorem 6.55. Recall that for $N \in \mathbb{N}$ the expression $\mathscr{H}_N^2$ denotes the Hecke algebra generated by $\mathcal{T}(p^2)$ for primes $p \nmid N$. By Corollary 6.66 these operators always admit simultaneous eigenforms unless $\mathcal{S}_{L,k}$ is trivial.

In special cases, even full product expansions of the associated L-series may be derived, though requiring restrictive assumptions on the lattice L and the form f as presented in Corollary 6.92. We begin by isolating a single factor associated to a prime number based on [Shi73, Cor. 1.8 p. 451].

Proposition 6.83. *Let p be a prime with $p \nmid \mathrm{lev}(L)$, $N \in \mathbb{N}$ fulfil $p \nmid N$ and assume $f \in \mathcal{S}_{L,k}$ is an eigenform of $\mathcal{T}(p^2)$ with eigenvalue σ_p. Then for $\lambda \in \mathcal{L}$ and $0 < t \in \mathbb{Z}+\overline{\mathrm{q}}(\lambda)$ such that $p^2 \nmid \mathrm{lev}(L)t$ we find that*

$$L_{(\lambda,t)}^N(f,s) = L_{(\lambda,t)}^{Np}(f,s) \cdot \begin{cases} \dfrac{1+\delta_{p\nmid t}\frac{G_{\mathcal{L}}(1)}{G_{\mathcal{L}}(p)}\cdot p^{k-1-s}}{1-\left(\frac{\sigma_p}{p^{k-1}}-(1-p^{-1})\frac{G_{\mathcal{L}}(1)}{G_{\mathcal{L}}(p)}\right)\cdot p^{k-1-s}+p^{2(k-1-s)}} & ,2 \mid \mathrm{rk}(L), \\[2ex] \dfrac{1-\chi_{\mathcal{L}}(p)\left(\frac{-t}{p}\right)p^{-1/2}\cdot p^{k-1-s}}{1-\frac{\sigma_p}{p^{k-1}}\cdot p^{k-1-s}+p^{2(k-1-s)}} & ,2 \nmid \mathrm{rk}(L). \end{cases}$$

Here, $\chi_{\mathcal{L}}(p) = \epsilon_p^{\mathrm{sig}(\mathcal{L})+\left(\frac{-1}{|\mathcal{L}|}\right)} \left(\frac{p}{|\mathcal{L}|2^{\mathrm{sig}(\mathcal{L})}}\right)$ *and* $\epsilon_p = 1$ *or* i *depending on wether* $p \equiv 1$ *or* -1 mod 4*. Given* $L^N_{(\lambda,t)}(f,\,\cdot\,)$ *is nonzero, the rational function in* p^{-s} *is well defined and does not vanish in the right half plane of absolute convergence of* $L^N_{(\lambda,t)}(f,\,\cdot\,)$ *determined by Corollary 6.82.*

Before proving the result, recall that for some integer d we set $G_{\mathcal{L}}(d) = \sum_{\lambda\in\mathcal{L}} e(d\,\overline{\mathrm{q}}(\lambda))$ in Definition A.16, preceding alternative descriptions of that Gauss sum.

Proof: Recall that the Fourier coefficients of $f \in \mathcal{S}_{L,k}$ are denoted $a(\lambda, n)$ as in (6.20). For $n \in \mathbb{N}$, following [Shi73, p. 452], define the formal power series

$$H_n(x) \coloneqq \sum_{m=0}^{\infty} a(p^m n\lambda, (p^m n)^2 t)x^m.$$

Our goal is to prove that there is a rational function $Q(x)$ such that

$$H_n(x) = a(n\lambda, n^2 t) \cdot Q(x). \tag{$*$}$$

Then,

$$\begin{aligned} L^N_{(\lambda,t)}(f,s) &= \sum_{\substack{n=1\\ \gcd(n,N)}}^{\infty} a(n\lambda, n^2 t)n^{-s} \\ &= \sum_{\substack{n=1\\ \gcd(n,Np)}}^{\infty} H_n(p^{-s})n^{-s} = L^{Np}_{(\lambda,t)}(f,s) \cdot Q(p^{-s}). \end{aligned}$$

We will compute $Q(x)$ in the case of even and odd rank separately.

The case of even rank: In this case, Proposition 6.54 yields

$$\sigma_p \cdot a(\lambda, n) = a(p\lambda, p^2 n) + \frac{G_{\mathcal{L}}(1)}{G_{\mathcal{L}}(p)} p^{k-2} \left(p\delta_{p|n} - 1\right) a(\lambda, tn) + p^{2(k-1)} a(\lambda/p, n/p^2)$$

resulting for $n, m \in \mathbb{N}$ with $p \nmid n$ and $p^2 \nmid \mathrm{lev}(L)t$ in

$$\begin{aligned} \sigma_p \cdot a(n\lambda, n^2 t) &= a(pn\lambda, p^2 n^2 t) + \frac{G_{\mathcal{L}}(1)}{G_{\mathcal{L}}(p)} p^{k-2} \left(p\delta_{p|t} - 1\right) a(n\lambda, n^2 t), \qquad \text{(I)} \\ \sigma_p \cdot a(p^m n\lambda, p^{2m} n^2 t) &= a(p^{m+1} n\lambda, p^{2(m+1)} n^2 t) \\ &\quad + (p-1)\frac{G_{\mathcal{L}}(1)}{G_{\mathcal{L}}(p)} p^{k-2} a(p^m n\lambda, p^{2m} n^2 t) \end{aligned}$$

$$+ p^{2(k-1)} a(p^{m-1} n\lambda, p^{2(m-1)} n^2 t). \tag{II}$$

Multiplying these equations by the formal variable x^{m+1} and summing them results in

$$\begin{aligned}\sigma_p x \cdot H_n(x) = H_n(x) - a(n\lambda, n^2 t) + (p-1)\frac{G_{\mathcal{L}}(1)}{G_{\mathcal{L}}(p)} p^{k-2} x H_n(x) \\ - \delta_{p \nmid t} \frac{G_{\mathcal{L}}(1)}{G_{\mathcal{L}}(p)} p^{k-1} a(n\lambda, n^2 t) x + p^{2(k-1)} x^2 H_n(x).\end{aligned}$$

A straightforward rearrangement yields

$$\begin{aligned}H_n(x) = a(n\lambda, tn^2) \cdot \left(1 + \delta_{p \nmid t} \frac{G_{\mathcal{L}}(1)}{G_{\mathcal{L}}(p)} p^{k-1} x\right) \\ \cdot \left[1 - \sigma_p x + (p-1)\frac{G_{\mathcal{L}}(1)}{G_{\mathcal{L}}(p)} p^{k-2} x + p^{2(k-1)} x^2\right]^{-1}.\end{aligned}$$

The case of odd rank: In this case, Theorem 6.55 yields

$$\sigma_p \cdot a(\lambda, n) = a(p\lambda, p^2 n) + \frac{\chi_{\mathcal{L}}(p)}{\sqrt{p}} \left(\frac{-n}{p}\right) p^{k-1} a(\lambda, n) + p^{2(k-1)} a(\lambda/p, n/p^2)$$

where we have abbreviated $\chi_{\mathcal{L}}(p) := \varepsilon_p^{\mathrm{sig}(A) + \left(\frac{-1}{|A|}\right)} \left(\frac{p}{|A| 2^{\mathrm{sig}(A)}}\right)$. This results for $p \nmid n$, $m \in \mathbb{N}$ and $p^2 \nmid \mathrm{lev}(L)t$ in

$$\sigma_p \cdot a(n\lambda, n^2 t) = a(pn\lambda, p^2 n^2 t) + \frac{\chi_{\mathcal{L}}(p)}{\sqrt{p}} \left(\frac{-t}{p}\right) p^{k-1} a(n\lambda, n^2 t), \tag{I}$$

$$\sigma_p \cdot a(p^m n\lambda, p^{2m} n^2 t) = a(p^{m+1} n\lambda, p^{2(m+1)} n^2 t) + p^{2(k-1)} a(p^{m-1} n\lambda, p^{2(m-1)} n^2 t). \tag{II}$$

Analogously to the situation of even rank we find

$$\begin{aligned}\sigma_p x \cdot H_n(x) = H_n(x) - a(n\lambda, n^2 t) + \frac{\chi_{\mathcal{L}}(p)}{\sqrt{p}} \left(\frac{-t}{p}\right) p^{k-1} a(n\lambda, n^2 t) x \\ + p^{2(k-1)} x^2 H_n(x)\end{aligned}$$

resulting in

$$H_n(x) = a(n\lambda, n^2 t) \left(1 - \frac{\chi_{\mathcal{L}}(p)}{\sqrt{p}} \left(\frac{-t}{p}\right) p^{k-1} x\right) \cdot \left[1 - \sigma_p x + p^{2(k-1)} x^2\right]^{-1}.$$

This settles the shape of the rational factor.

Next, we will investigate the behaviour of the numerator and denominator of the rational factor. Write

$$Q(x) = \frac{R(x,t)}{P(x)}$$

for the rational function in x. Now assume that $L^N_{(\lambda,t)}(f,s) \neq 0$, for otherwise there is no behaviour to consider. Then there is some $n \in \mathbb{N}$ with $\gcd(n,N) = 1$, such that $a(n\lambda, n^2t) \neq 0$. We fix this n. Recall that there exists a whole right half plane such that if $s \in \mathbb{C}$ is contained in it, the series $L^N_{(\lambda,t)}(f,s)$ converges absolutely. Then the proof above shows that we have for such s the identity

$$P(p^{-s}) \cdot H_n(p^{-s}) = a(n\lambda, n^2t) \cdot R(p^{-s}, t).$$

Observe that unless $R(p^{-s},t) = 0$, no factor in the above equation may vanish. In fact, in case of $p \mid t$, we find $R(p^{-s},t) = 1$ regardless of s. If $p \nmid t$, further analysis is required. In that case, all factors in the second term of $R(p^{-s},t)$ that are not a power of p have absolute value 1. Hence, we find that

$$R(p^{-s},t) \neq 0 \text{ for } \begin{cases} \mathrm{Re}(s) > k-1, & 2 \mid \mathrm{rk}(L), \\ \mathrm{Re}(s) > k-3/2 & 2 \nmid \mathrm{rk}(L). \end{cases}$$

These bounds are strictly better than the bounds of convergence for $L^N_{(\lambda,t)}(f,s)$ imposed on $\mathrm{Re}(s)$. Hence, in the right half plane of absolute convergence of $L^N_{(\lambda,t)}(f,s)$, the factor $Q(p^{-s})$ is well defined and different from zero. □

The result above may be utilised to construct infinite product expansions. However, before employing this result, a remarkable notion of convergence has to be introduced.

Definition 6.84. Let $(a_n)_{n\in\mathbb{N}} \in \mathbb{C}$ be a sequence of numbers. Then, the infinite product

$$\prod_{n\in\mathbb{N}} (a_n)$$

is said to be *absolutely convergent*, if the following series converges:

$$\sum_{n\in\mathbb{N}} |a_n - 1|.$$

The practical application of that notion is that an absolutely convergent product is zero, if, and only if, one of its factors equals zero. We refer the reader to [FB06, IV p. 200] for more on that matter and proceed by applying Proposition 6.83 in order to derive infinite product expansions.

Corollary 6.85. *Let $N \in \mathbb{N}$, S be a family of good primes p, and assume $f \in \mathcal{S}_{L,k}$ is a simultaneous eigenform of $\mathcal{T}(p^2)$ with eigenvalue σ_p for all $p \in S$. Denote the Fourier expansion of f by*

$$f = \sum_{\lambda \in \mathcal{L}} \sum_{n \in \overline{\mathrm{q}}(\lambda)+\mathbb{Z}} a(\lambda, n) e(n\tau) \mathfrak{e}_\mu$$

and select $\lambda \in \mathcal{L} = L'/L$, as well as $t \in \overline{\mathrm{q}}(\lambda) + \mathbb{Z}$ with $p^2 \nmid \mathrm{lev}(L)t$ for all $p \in S$. Then we find in the range of absolute convergence of $L^N_{(\lambda,t)}(f, s)$ presented in Corollary 6.82 that

$$L^N_{(\lambda,t)}(f,s) = \left(\sum_{\substack{n \in \mathbb{N} \\ \gcd(n,N)=1 \\ \forall p \in S: p \nmid n}} \frac{a(n\lambda, nt)}{n^s} \right) \cdot \prod_{\substack{p \in S \\ \gcd(p,N)=1}} \begin{cases} \dfrac{1+\delta_{p \nmid t} \frac{G_{\mathcal{L}}(1)}{G_{\mathcal{L}}(p)} \cdot p^{k-1-s}}{1-\left(\frac{\sigma_p}{p^{k-1}} - (1-p^{-1}) \frac{G_{\mathcal{L}}(1)}{G_{\mathcal{L}}(p)}\right) \cdot p^{k-1-s} + p^{2(k-1-s)}} & , 2 \mid \mathrm{rk}(L), \\ \dfrac{1-\chi_{\mathcal{L}}(p)\left(\frac{-t}{p}\right) p^{-1/2} \cdot p^{k-1-s}}{1-\frac{\sigma_p}{p^{k-1}} \cdot p^{k-1-s} + p^{2(k-1-s)}} & , 2 \nmid \mathrm{rk}(L). \end{cases}$$

Further, the possibly infinite product is absolutely convergent for $\mathrm{Re}(s) > k+1$.

Proof: The existence of the product is a consequence of Proposition 6.83 and a standard argument.

In order to prove absolute convergence of the product in the claimed range, it suffices to prove that its individual rational factors we denote by $Q_p(p^{-s}, t)$ converge to 1 for $p \to \infty$ and all choices of $s \in \mathbb{C}$ with $\mathrm{Re}(s) > k+1$ fast enough – compare Definition 6.84 on that matter. Introduce the following notation for the numerator and denominator

$$Q_p(p^{-s}, t) = \frac{R_p(p^{-s}, t)}{P_p(p^{-s})}.$$

Then we have

$$Q_p(p^{-s}, t) - 1 = \frac{R_p(p^{-s}, t) - P_p(p^{-s})}{P_p(p^{-s})}$$

and it suffices to investigate the convergence of the numerator and denominator for this expression, separately. The denominator will be shown to converge to 1. Then it suffices to prove that the numerator goes to 0 faster than linearly.[11] As before, we distinguish the

[11] Hence, the series over $Q_p(p^{-s}, t) - 1$ is absolutely convergent by comparison to the Riemann ζ function.

odd and even case, however, before specialising to one of these, recall that by Lemma 6.67 the following bound is true:

$$|\sigma_p| < p^{k-1}(p+1). \tag{6.52}$$

The Case of even signature: We begin by bounding

$$\begin{aligned}\left|P_p(p^{-s})-1\right| &\leq \left|\frac{\sigma_p}{p^{k-1}}-(1-p^{-1})\frac{G_{\mathcal{L}}(1)}{G_{\mathcal{L}}(p)}\right| p^{k-1-\mathrm{Re}(s)}+p^{2(k-1-\mathrm{Re}(s))}\\ &= \left(\frac{p+1}{p}+\frac{1-p^{-1}}{p}\right)\cdot p^{k-\mathrm{Re}(s)}+p^{2(k-1-\mathrm{Re}(s))}.\end{aligned}$$

We see that for $\mathrm{Re}(s)>k$, the expression converges to 0 for $p\to\infty$ as desired. Next, the numerator is investigated:

$$\begin{aligned}&R_p(p^{-s},t)-P_p(p^{-s})\\ &=\left[\frac{\sigma_p}{p^{k-1}}+\left(\delta_{p\nmid t}-(1-p^{-1})\right)\frac{G_{\mathcal{L}}(1)}{G_{\mathcal{L}}(p)}\right]\cdot p^{k-1-s}-p^{2(k-1-s)}.\end{aligned}$$

We refer to Remark A.18 for a proof of the fraction of Gauss sums to be of absolute value one in order to verify that the term in brackets may be absolutely bounded by

$$\left|\frac{\sigma_p}{p^{k-1}}+\left(\delta_{p\nmid t}-(1-p^{-1})\right)\frac{G_{\mathcal{L}}(1)}{G_{\mathcal{L}}(p)}\right|\leq(p+1)+(1-p^{-1/2})=\left(1+2p^{-1}-p^{-2}\right)\cdot p.$$

As a consequence, the difference $R_p(p^{-s},t)-P_p(p^{-s})$ goes to zero for $p\to\infty$ in case of $\mathrm{Re}(s)>k$. It goes to 0 faster than linearly for $\mathrm{Re}(s)>k+1$. This settles absolute convergence of the product in the even case.

The case of odd signature: We begin by bounding

$$\begin{aligned}|P_p(p^{-s})-1| &\leq \left|\frac{\sigma_p}{p^{k-1}}\right|\cdot p^{k-1-\mathrm{Re}(s)}+p^{2(k-1-\mathrm{Re}(s))}\\ &=\frac{p+1}{p}\cdot p^{k-\mathrm{Re}(s)}+p^{2(k-1-\mathrm{Re}(s))}.\end{aligned}$$

We see that for $\mathrm{Re}(s)>k$, the expression converges to 0 for $p\to\infty$. Next, the numerator is investigated:

$$R_p(p^{-s},t)-P_p(p^{-s})$$

$$= \left[\frac{\sigma_p}{p^{k-1}} - \chi_{\mathcal{L}}(p)\left(\frac{-t}{p}\right)p^{-1/2}\right] \cdot p^{k-1-s} - p^{2(k-1-s)}$$

The term in brackets may be bounded by

$$\left|\frac{\sigma_p}{p^{k-1}} - \chi_{\mathcal{L}}(p)\left(\frac{-t}{p}\right)p^{-1/2}\right| \leq (p+1) + \sqrt{p}^{-1} = [1 + p^{-1} + p^{-3/2}] \cdot p.$$

As a consequence, the difference $Q_p(p^{-s}, t) - P_p(p^{-s})$ goes to zero for $p \to \infty$ in case $\mathrm{Re}(s) > k$. It goes to 0 more than linearly for $\mathrm{Re}(s) > k+1$. This settles absolute convergence of the product in the odd case.
Recall that the the different ranges of absolute convergence of $L^N_{(\lambda,t)}(f, s)$ determined by Corollary 6.82 cover the condition $\mathrm{Re}(s) > k + 1/2$. □

The following special case conveys the essence of the above statement but is not general enough for applications that we have in mind.

Corollary 6.86. *Let $2 \leq k \in \mathbb{Z}/2$ and assume $f \in \mathcal{S}_{L,k}$ is a simultaneous eigenform of $\mathcal{T}(p^2)$ with eigenvalue σ_p for all primes $p \nmid \mathrm{lev}(L)$. Denote the Fourier expansion of f by*

$$f(\tau) = \sum_{\mu \in \mathcal{L}} \sum_{n \in \overline{\mathrm{q}}(\lambda)+\mathbb{Z}} a(\mu, n) \cdot e(n\tau)\mathfrak{e}_\mu$$

and select $\lambda \in \mathcal{L}$ as well as $t \in \overline{\mathrm{q}}(\lambda) + \mathbb{Z}$ with $p^2 \nmid \mathrm{lev}(L)t$ for all $p \nmid \mathrm{lev}(L)$. Then

$$L_{(\lambda,t)}(f, s) = \left(\sum_{\substack{n \in \mathbb{N} \\ n \mid \mathrm{lev}(L)^\infty}} \frac{a(n\lambda, nt)}{n^s}\right) \cdot \prod_{p \nmid \mathrm{lev}(L)} \begin{cases} \dfrac{1+\delta_{p\nmid t}\frac{G_{\mathcal{L}}(1)}{G_{\mathcal{L}}(p)}\cdot p^{k-1-s}}{1-\left(\frac{\sigma_p}{p^{k-1}}-(1-p^{-1})\frac{G_{\mathcal{L}}(1)}{G_{\mathcal{L}}(p)}\right)\cdot p^{k-1-s}+p^{2(k-1-s)}} & , 2 \mid \mathrm{rk}(L), \\ \dfrac{1-\chi_{\mathcal{L}}(p)\left(\frac{-t}{p}\right)p^{-1/2}\cdot p^{k-1-s}}{1-\frac{\sigma_p}{p^{k-1}}\cdot p^{k-1-s}+p^{2(k-1-s)}} & , 2 \nmid \mathrm{rk}(L) \end{cases}$$

is absolutely convergent as an infinite product for $\mathrm{Re}(s) > k+1$.

Corollary 6.87. *Select $2 \leq k \in \mathbb{Z}/2$ and $f \in \mathcal{S}_{L,k}$ with Fourier expansion*

$$f(\tau) = \sum_{\mu \in \mathcal{L}} \sum_{n \in \overline{\mathrm{q}}(\lambda)+\mathbb{Z}} a(\mu, n) \cdot e(n\tau)\mathfrak{e}_\mu$$

and let (λ, t) be an index of a Fourier coefficient of f such that for some $N \in \mathbb{N}$ with

$\mathrm{lev}(L) \mid N$ *we have* $p^2 \nmid \mathrm{lev}(L)t$ *for all* $p \nmid N$*. Assume* f *is a simultaneous eigenform for* $\mathcal{T}(p^2)$ *with eigenvalue* σ_p *for all primes* $p \nmid N$*. Then*

$$\begin{aligned} L^N_{(\lambda,t)}(f,s) &\overset{\mathrm{def}}{=} \sum_{\substack{n\in\mathbb{N},\\ \gcd(n,N)=1}} \frac{a(n\lambda, n^2t)}{n^s} \\ &= a(\lambda,t)\cdot\prod_{p\nmid N}\begin{cases} \frac{1+\delta_{p\nmid t}\frac{G_{\mathcal{L}}(1)}{G_{\mathcal{L}}(p)}\cdot p^{k-1-s}}{1-\left(\frac{\sigma_p}{p^{k-1}}-(1-p^{-1})\frac{G_{\mathcal{L}}(1)}{G_{\mathcal{L}}(p)}\right)\cdot p^{k-1-s}+p^{2(k-1-s)}} & ,2\mid \mathrm{rk}(L),\\ \frac{1-\chi_{\mathcal{L}}(p)\left(\frac{-t}{p}\right)p^{-1/2}\cdot p^{k-1-s}}{1-\frac{\sigma_p}{p^{k-1}}\cdot p^{k-1-s}+p^{2(k-1-s)}} & ,2\nmid \mathrm{rk}(L)\end{cases}\end{aligned}$$

converges absolutely for $\mathrm{Re}(s) > k+1$*. Here we have* $G_{\mathcal{L}}(d) = \sum_{\mu\in\mathcal{L}} e(d\,\overline{\mathrm{q}}(\mu))$ *and* $\chi_{\mathcal{L}}(p) = \epsilon_p^{\mathrm{sig}(\mathcal{L})+\left(\frac{-1}{|\mathcal{L}|}\right)}\left(\frac{p}{|\mathcal{L}|2^{\mathrm{sig}(\mathcal{L})}}\right)$ *with* $\epsilon_p = 1$ *or* i *depending on wether* $p \equiv 1$ *or* -1 mod 4.

Remark 6.88. Specialising to the case of a negative definite lattice of rank 1 in Corollary 6.87 and the choice $(\lambda,t) = (\overline{\ell_0}, \mathrm{q}(\ell_0))$ for some suitable $\ell_0 \in L'$, we find by [BS08, Rem. 4.11] that $\chi_{\mathcal{L}}(p) = \left(-\frac{\mathrm{q}(\ell_0)}{p}\right)$, implying

$$L^N_{\ell_0}(f,s) = a(\overline{\ell_0}, \mathrm{q}(\ell_0))\cdot\prod_{p\nmid N}\frac{1-\delta_{p\nmid \mathrm{q}(\ell_0)}p^{-1/2}\cdot p^{k-1-s}}{1-\frac{\sigma_p}{p^{k-1}}\cdot p^{k-1-s}+p^{2(k-1-s)}}.$$

After settling the case of primes not dividing the level $\mathrm{lev}(L)$, we turn to the more delicate case of a prime $p \mid \mathrm{lev}(L)$. Recall that in this instance the map on the discriminant form

$$\mathcal{L} \ni \lambda \to p\cdot\lambda \in \mathcal{L}$$

is not an automorphism, but has a non-trivial kernel denoted by ${}_p\mathcal{L}$. In this case, the recursion formula for Fourier coefficients of Hecke eigenforms obtained from Proposition 6.71 fans out in general, inhibiting the construction of a product expansion. However, this fan-out may be trivialised by imposing adequate constraints.

To this end, we recall that by Example 1.33 a discriminant form $\mathcal{L}$ decomposes orthogonally into maximal p subgroups

$$\mathcal{L} \simeq \bigoplus_{p<\infty}\mathcal{L}_p$$

and we refer to the component of $\lambda \in \mathcal{L}$ lying in $\mathcal{L}_p$ as λ_p. With this preparation we may state appropriate conditions for a factorisation at bad places.

Proposition 6.89. *Let $f \in \mathcal{S}_{L,k}$ with Fourier expansion*

$$f(\tau) = \sum_{\mu \in \mathcal{L}} \sum_{n \in \overline{\mathrm{q}}(\mu)+\mathbb{Z}} a(\mu, n) \cdot e(n\tau)\mathfrak{e}_\mu.$$

Assume $p \neq 2$ is a prime such that f is an eigenform of $\mathcal{T}(p^2)$ with eigenvalue σ_p and that there is no nonzero $\lambda' \in \mathcal{L}_p$ with $\overline{\mathrm{q}}(\lambda') = 0$. Select $\lambda \in \mathcal{L}$ and $t \in \overline{\mathrm{q}}(\lambda) + \mathbb{Z}$ such that the p-component $\lambda_p \in \mathcal{L}_p$ of λ is non-trivial or $p^2 \nmid \mathrm{lev}(L)t$. If $\lambda_p \neq 0$ assume further that f is invariant under $\mathrm{O}(\mathcal{L}_p)$. Then we find for a natural number $N \in \mathbb{N}$ with $p \nmid N$ that

$$L^N_{(\lambda,t)}(f,s) = L^{Np}_{(\lambda,t)}(f,s)$$
$$\cdot \frac{1 + K_{L,p}\begin{cases} \delta_{\lambda_p \neq 0}(1-p^{-1}) + \delta_{\lambda_p=0}\delta_{p\nmid t}, & 2 \mid R_p \\ -\delta_{\lambda_p=0}p^{-1/2}\left(\frac{-t}{p}\right)\epsilon_p, & 2 \nmid R_p \end{cases} \cdot p^{k-1-s} + C(\lambda_p)p^{2(k-1-s)}}{1 - \left(\frac{\sigma_p}{p^{k-1}} - \delta_{2|R_p}(1-p^{-1})K_{L,p}\right)p^{k-1-s} + p^{2(k-1-s)}}.$$

The rational factor in p^{k-1-s} extracted is well defined and nonzero for $\mathrm{Re}(s) > \mathrm{rk}(\mathcal{L}_p)/2 + k$. Here, $K_{L,p} = p^{-\mathrm{rk}(L)/2}\sum_{\nu \in L/pL} e\left(\frac{1}{p}\mathrm{q}(\nu)\right)$ and R_p has been given in Definition 1.38. Further, $\epsilon_p = 1, i$ depending on whether $p = 1, 3 \mod 4$ and the constant $C(\lambda_p) + 1$ equals the size of the orbit of $\lambda_p \in \mathcal{L}_p$ under $\mathrm{O}(\mathcal{L}_p)$. In particular, $C(\lambda_p) = 0$, if $\lambda_p = 0$.

Remark 6.90. Sharper bounds for the non-vanishing result are found in the proof in a variety of subcases.

Proof: The proof is analogous to the proof of Proposition 6.83 but more convoluted. Recall that the Fourier coefficients of $f \in \mathcal{S}_{L,k}$ in question are denoted by $a(\lambda, n)$ as in Equation (6.20). For $n \in \mathbb{N}$, following [Shi73, p. 452], define the formal power series

$$H_n(x) := \sum_{m=0}^{\infty} a(p^m n\lambda, (p^m n)^2 t)x^m.$$

Our goal is to prove that there is a rational function $Q(x)$ such that

$$H_n(x) = a(n\lambda, n^2 t) \cdot Q(x). \qquad (*)$$

Then,

$$\begin{aligned} L^N_{(\lambda,t)}(f,s) &= \sum_{\substack{n=1\\ \gcd(n,N)}}^{\infty} a(n\lambda, n^2t)n^{-s} \\ &= \sum_{\substack{n=1\\ \gcd(n,Np)}}^{\infty} H_n(p^{-s})n^{-s} \\ &= L^{Np}_{(\lambda,t)}(f,s)\cdot Q(p^{-s}). \end{aligned}$$

Recall that by Proposition 6.71, the operation of $\mathcal{T}(p^2)$ on the Fourier coefficients of f reads

$$\begin{aligned} \sigma_p a(\lambda,n) = p^{2(k-1)}\delta_{\lambda\in^p\mathcal{L}} \sum_{\substack{\lambda'\in_p\mathcal{L}\\ n-p^2\,\overline{\mathrm{q}}(\lambda/p+\lambda')\in p^2\mathbb{Z}}} a\left(\lambda/p+\lambda', \frac{n-p^2\,\overline{\mathrm{q}}(\lambda/p+\lambda')}{p^2}+\overline{\mathrm{q}}(\lambda/p+\lambda')\right) \\ + p^{k-2}\delta_{\lambda\in^p\mathcal{L}}K_{L,p}g_p\left[1, {}_{-}\chi_p{}^{R_p}, n-\mathrm{q}(p\cdot\ell_{\lambda/p})\right]\, a\left(\lambda,n\right) \\ + a(p\lambda,p^2n). \end{aligned}$$

This results for some $n,m\in\mathbb{N}$ with $p\nmid n$ and $p^2\nmid \mathrm{lev}(L)t$ or $\lambda_p\neq 0$ in[12]

$$\begin{aligned} \sigma_p a(n\lambda,n^2t) = p^{k-2}\delta_{\lambda\in^p\mathcal{L}}K_{L,p}g_p\left[1, {}_{-}\chi_p{}^{R_p}, n^2(t-\mathrm{q}(p\cdot\ell_{\lambda/p}))\right]\, a\left(n\lambda,n^2t\right) \\ + a(pn\lambda,(pn)^2t) \end{aligned} \tag{I}$$

and

$$\begin{aligned} \sigma_p a(p^mn\lambda,(p^mn)^2t) = p^{2(k-1)} \sum_{\substack{\lambda'\in_p\mathcal{L}\\ (p^mn)^2t-p^2\,\overline{\mathrm{q}}(p^mn\lambda/p+\lambda')\in p^2\mathbb{Z}}} a\left((p^mn\lambda)/p+\lambda', (p^mn)^2t/p^2\right) \\ + p^{k-2}K_{L,p}g_p\left[1, {}_{-}\chi_p{}^{R_p}, (p^mn)^2t-\mathrm{q}(p\cdot\ell_{(p^mn\lambda)/p}))\right] a\big(p^mn\lambda,(p^mn)^2t\big) \\ + a(p^{m+1}n\lambda,(p^{m+1}n)^2t). \end{aligned} \tag{II}$$

The Gauss sum g_p given in Definition A.12 appearing above will be made more explicit. We begin by understanding the sum appearing in (II) meaning for $m\in\mathbb{N}$ being different from zero. Note that for a fixed choice $\ell_\lambda\in L'$ projecting to $\lambda\in\mathcal{L}$ we have $t=\mathrm{q}(\ell_\lambda)+r$ for some integer $r\in\mathbb{Z}$. The expression g_p in (II) above, however, is independent of

[12]Note that the group $\mathcal{L}_p$ may only contain constituents of the form $\mathbb{Z}/p\mathbb{Z}$.

that residue r as it is multiplied by p^m. Further, recall that by Proposition 6.71, the choice of representative $\ell_{(p^m n\lambda)/p}$ does not matter, meaning we may choose $p^{m-1}n\ell_\lambda$. As a consequence, the Gauss sum in (II) collapses to

$$g_p\left[1, {}_{-}\chi_p{}^{R_p}, (p^m n)^2 t - \mathrm{q}(p \cdot \ell_{(p^m n\lambda)/p}))\right] = g_p\left[1, {}_{-}\chi_p{}^{R_p}, 0\right].$$

It should be noted that this Gauss is computed in Remark A.15. In fact, we find that

$$g_p\left[1, {}_{-}\chi_p{}^{R_p}, 0\right] = \begin{cases} 0, & \text{if } 2 \nmid R_p, \\ p-1, & \text{if } 2 \mid R_p. \end{cases}$$

Next, the case $m = 0$, corresponding to (I), is considered. Here, Remark A.15 implies

$$g_p\left[1, {}_{-}\chi_p{}^{R_p}, t - \mathrm{q}(p \cdot \ell_{\lambda/p})\right] = \begin{cases} \delta_{p \nmid t - \mathrm{q}(p\ell_{\lambda/p})} p^{1/2} \left(\frac{\mathrm{q}(p \cdot \ell_{\lambda/p}) - t}{p}\right) \epsilon_p, & \text{if } 2 \nmid R_p, \\ \delta_{p \mid t - \mathrm{q}(p\ell_{\lambda/p})} p - 1, & \text{if } 2 \mid R_p. \end{cases}$$

We deduce that the Gauss sum is indifferent to multiplication of the last argument by a square that is coprime to p. Multiplication by $(\mathrm{lev}(L)/p^{\nu_p(\mathrm{lev}(L))})^2$ yields that the number $(\mathrm{lev}(L)/p^{\nu_p(\mathrm{lev}(L))})^2 p^2\, \mathrm{q}(\ell_{\lambda/p})$ is an integer divisible by p.[13] Hence, $\delta_{p \nmid t - \mathrm{q}(p\ell_{\lambda/p})} = \delta_{p \nmid t}$ and this expression is already implicitly included in the Legendre symbol, yielding

$$g_p\left[1, {}_{-}\chi_p{}^{R_p}, t - \mathrm{q}(p \cdot \ell_{\lambda/p})\right] = \begin{cases} p^{1/2} \left(\frac{-t}{p}\right) \epsilon_p, & \text{if } 2 \nmid R_p, \\ \delta_{p \mid t} p - 1, & \text{if } 2 \mid R_p. \end{cases}$$

In the next step, we investigate the index of the sum in (II). Further analysis of the discriminant group yields that $\mathcal{L}_p = {}_p\mathcal{L}_p \simeq {}_p\mathcal{L}$ equals the p-torsion of the discriminant group $\mathcal{L}$. As a consequence, any p multiple $\mu \in {}^p\mathcal{L}$ has trivial p component μ_p, meaning multiplication by p acts as an automorphism on ${}^p\mathcal{L}$, yielding

$$\{p^m n\lambda/p + \lambda' \mid \lambda' \in {}_p\mathcal{L}\} = \{p^{m-1}n\lambda + \lambda' \mid \lambda' \in {}_p\mathcal{L}\}.$$

In addition, by Remark 1.36, ${}^p\mathcal{L}$ is the orthogonal complement of ${}_p\mathcal{L}$, so that for $\lambda' \in {}_p\mathcal{L}$

$$\overline{\mathrm{q}}(p^{m-1}n\lambda + \lambda') = \overline{\mathrm{q}}(p^{m-1}n\lambda) + \overline{\mathrm{q}}(\lambda')$$

[13] Recall that we have assumed $\mathcal{L}_p$ to be non-isotropic, implying $p^2 \nmid \mathrm{lev}(L)$.

in case of $m > 1$ or $\lambda_p = 0$. Recall that $t \in \overline{\mathrm{q}}(\lambda) + \mathbb{Z}$ so that $(p^m n)^2 t - p^2\,\mathrm{q}(p^{m-1} n \ell_\lambda) \in p^2\mathbb{Z}$. However, we have assumed that there is no isotropic element in $\mathcal{L}_p$, meaning we find that the index of the sum in (II) collapses to the case $\lambda' = 0 \in \mathcal{L}$.
Should we have $\lambda_p \neq 0$, only the case $m = 1$ remains. We may begin arguing as above by replacing λ by $\lambda - \lambda_p$ but note that there will be at least two choices of $\lambda' \in {}_p\mathcal{L}$ this time. Namely, $\lambda_p \neq 0$ and $-\lambda_p$. However, there might be even more and the number of these choices $\lambda' \in \mathcal{L}_p$ will be denoted by $C(\lambda_p) + 1$. In order to proceed with the calculation the different Fourier coefficients appearing in the sum in (II) have to be related.[14] This is where the assumption of invariance with respect to the action of the orthogonal group comes into play. Recall that for $\lambda_p \neq 0$, we find $\overline{\mathrm{q}}(\lambda_p) \neq 0$ by assumption on the lattice. This implies the local lattice L_p to be maximal (cf. Remark 1.34) which renders $\mathcal{L}_p$ an $\mathbb{F}_p$ vector space. Then, mapping λ_p to a different choice of root with the same norm defines an isometry of subspaces, which, by Witts extension theorem 4.6, extends to an isometry of $\mathcal{L}_p$, i.e. an element in $\mathrm{O}(\mathcal{L}_p)$. By assumption the Fourier coefficients of f are invariant with respect to this action so that we obtain $C(\lambda_p) + 1$ contributions from this particular coefficient and identify this number as the size of the orbit of λ_p under $\mathrm{O}(\mathcal{L}_p)$.
With these manipulations, (I) and (II) become

$$\begin{aligned}\sigma_p a(n\lambda, n^2 t) = p^{k-2}\delta_{\lambda\in{}^p\mathcal{L}} K_{L,p}\left[\delta_{2|R_p}(\delta_{p|t}p - 1) + \delta_{2\nmid R_p} p^{1/2}\left(\frac{-t}{p}\right)\epsilon_p\right]\, a\left(n\lambda, n^2 t\right) \\ + a(pn\lambda, (pn)^2 t)\end{aligned} \tag{I'}$$

and

$$\begin{aligned}\sigma_p a(p^m n\lambda, (p^m n)^2 t) = p^{2(k-1)}[\delta_{m>1} + \delta_{m=1}(C(\lambda_p)+1)]a\left(p^{m-1}n\lambda, \frac{(p^m n)^2 t}{p^2}\right) \\ + \delta_{2|R_p} p^{k-2} K_{L,p}\cdot(p-1)\; a\left(p^m n\lambda, (p^m n)^2 t\right) \\ + a(p^{m+1}n\lambda, (p^{m+1}n)^2 t).\end{aligned} \tag{II'}$$

We multiply the equations above by x^{m+1} and sum them to obtain

$$\begin{aligned}&\sigma_p \cdot H_n(x)x \\ &= p^{2(k-1)} H_n(x)x^2 + C(\lambda_p)p^{2(k-1)} a(n\lambda, n^2 t)x^2 \\ &\quad + \delta_{2|R_p} p^{k-2} K_{L,p}\left[(p-1)H_n(x)x - a(n\lambda, n^2 t)x\left\{(p-1) - \delta_{\lambda_p=0}(\delta_{p|t}p - 1)\right\}\right]\end{aligned}$$

[14] In fact, in the sum in (II), there are now indices appearing for different choices of λ' which were not contained in the initial series $L^N_{(\lambda,t)}(f,s)$.

$$+ \delta_{2\nmid R_p} p^{k-2} K_{L,p} \delta_{\lambda_p=0} p^{1/2} \left(\frac{-t}{p}\right) \epsilon_p a(n\lambda, n^2 t) x$$
$$+ H_n(x) - a(n\lambda, n^2 t).$$

This will have to be rearranged. Beforehand, note that the term in curly braces may be rewritten as

$$\begin{aligned} &(p-1) - \delta_{\lambda_p=0}(\delta_{p|t} p - 1) \\ &= (p-1) - \delta_{\lambda_p=0}[(p - 1 + (\delta_{p|t} p - p)] \\ &= \delta_{\lambda_p\neq 0}(p-1) - \delta_{\lambda_p=0} p(\delta_{p|t} - 1) \\ &= \delta_{\lambda_p\neq 0}(p-1) + \delta_{\lambda_p=0}\delta_{p\nmid t} p. \end{aligned}$$

With that transformation, we may rearrange the above equation for $H_n(x)$ to read

$$\begin{aligned} &H_n(x) - H_n(x) x \left(\sigma_p - \delta_{2|R_p}(p-1) p^{k-2} K_{L,p}\right) + H_n(x) x^2 p^{2(k-1)} \\ &= a(n\lambda, n^2 t) \times \left[1 + p^{k-2} K_{L,p} \begin{cases} \delta_{\lambda_p\neq 0}(p-1) + \delta_{\lambda_p=0}\delta_{p\nmid t} p, & 2 \mid R_p \\ -\delta_{\lambda_p=0} p^{1/2} \left(\frac{-t}{p}\right) \epsilon_p, & 2 \nmid R_p \end{cases} \cdot x - C(\lambda_p) p^{2(k-1)} x^2 \right] \end{aligned}$$

which yields the desired rational expression, once x is substituted with p^{-s}:

$$\begin{aligned} &H_n(p^{-s}) \\ &= a(n\lambda, n^2 t) \cdot \frac{1 + K_{L,p} \begin{cases} \delta_{\lambda_p\neq 0}(1 - p^{-1}) + \delta_{\lambda_p=0}\delta_{p\nmid t}, & 2 \mid R_p \\ -\delta_{\lambda_p=0} p^{-1/2} \left(\frac{-t}{p}\right) \epsilon_p, & 2 \nmid R_p \end{cases} \cdot p^{k-1-s} - C(\lambda_p) p^{2(k-1-s)}}{1 - \left(\sigma_p - \delta_{2|R_p}(1 - p^{-1}) p^{k-1} K_{L,p}\right) p^{-s} + p^{2(k-1-s)}}. \end{aligned}$$

However, it remains to check that the potential denominator is well defined. First, we may assume $L^N_{(\lambda,t)}(f,s) \neq 0$ for there is nothing to show otherwise. Then we may fix n such that $a(n\lambda, n^2 t) \neq 0$ and write

$$Q(p^{-s}) = \frac{R(p^{-s}, t)}{P(p^{-s})}$$

for the numerator and denominator of the rational expression above. In the right half plane of absolute convergence of $L^N_{(\lambda,t)}(f,s)$, also $H_n(p^{-s})$ converges absolutely, yielding

$$P(p^{-s}) H_n(p^{-s}) = a(n\lambda, n^2 t) \cdot R(p^{-s}, t)$$

in this right half plane. Note that in the above equation no factor vanishes, unless $R(p^{-s}, t)$ does, meaning it suffices to verify that $R(p^{-s}, t)$ does not vanish, to conclude that $P(p^{-s})$ does not vanish as well. To this end, we shall specify a right half plane, such that $|1 - R(p^{-s}, t)| < 1$. The first factor that needs to be tended to is $K_{L,p}$. By Remark A.18 we find

$$|K_{L,p}| = p^{-\operatorname{rk}(L)/2}|G_{L,p}(1,1)| = \sqrt{|{}_p\mathcal{L}|}.$$

Recall that in our case the local lattice L_p was assumed to be maximal, implying ${}_p\mathcal{L} \simeq \mathcal{L}_p$ which results in its size being equal to $p^{\operatorname{rk}(\mathcal{L}_p)}$. Also note that $C(\lambda_p) = 0$ in case $\lambda_p = 0$. If $\lambda_p \neq 0$, we necessarily find $p \mid \operatorname{lev}(L)$ and conclude that $C(\lambda_p) \leq |\mathcal{L}_p| - 2 = p^{\operatorname{rk}(\mathcal{L}_p)} - 2$. We will have a look at the event that $2 \mid R_p$ first which yields

$$|1 - R(p^{-s}, t)| \leq \begin{cases} \left(p^{\operatorname{rk}(\mathcal{L}_p)/2}(1 - p^{-1}) + [p^{\operatorname{rk}(\mathcal{L}_p)} - 2]p^{k-1-\operatorname{Re}(s)}\right) p^{k-1-\operatorname{Re}(s)}, & \lambda_p \neq 0, \\ p^{\operatorname{rk}(\mathcal{L}_p)/2+k-1-\operatorname{Re}(s)}, & \lambda_p = 0. \end{cases}$$

Next, the case $2 \nmid R_p$ is considered which gives

$$|1 - R(p^{-s}, t)| \leq \begin{cases} [p^{\operatorname{rk}(\mathcal{L}_p)} - 2]p^{2(k-1-\operatorname{Re}(s))}, & \lambda_p \neq 0, \\ p^{\operatorname{rk}(\mathcal{L}_p)/2+k-3/2-\operatorname{Re}(s)}, & \lambda_p = 0. \end{cases}$$

With the gathered information we may guarantee non-vanishing of the numerator for

$\operatorname{Re}(s) >$	$\lambda_p = 0$	$\lambda_p \neq 0$
$2 \mid R_p$	$\operatorname{rk}_p(\mathcal{L})/2 + k - 1$	$\operatorname{rk}_p(\mathcal{L})/2 + k$
$2 \nmid R_p$	$\operatorname{rk}_p(\mathcal{L})/2 + k - 1 - 1/2$	$\operatorname{rk}_p(\mathcal{L})/2 + k - 1$

For the case $2 \mid R_p$ and $\lambda_p \neq 0$, we chose a suboptimal bound by bounding $(1 - p^{-1}) \leq 1$ and $(1 - p^{-\operatorname{rk}(\mathcal{L}_p)}) \leq 1/3$. The exact choice may be computed by means of a quadratic equation. In detail, note that in this case the expression is strictly decreasing in $\operatorname{Re}(s)$, write $z = p^{k-1-\operatorname{Re}(s)}$ and $c_p = p^{\operatorname{rk}(\mathcal{L}_p)/2}\frac{1-p^{-1}}{p^{\operatorname{rk}(\mathcal{L}_p)}-2}$. Then we seek the even point of

$$z^2 + c_p z = 1.$$

Solving the quadratic equation results in only the positive solution being acceptable, meaning

$$p^{k-1-\mathrm{Re}(s)} = -\frac{c_p}{2} + \sqrt{\frac{c_p^2}{4} + 1},$$

yielding

$$\mathrm{Re}(s) = k - 1 - \ln_p(2) + \ln_p\left(\sqrt{p^{\mathrm{rk}(\mathcal{L}_p)}\left(\frac{1-p^{-1}}{p^{\mathrm{rk}(\mathcal{L}_p)}-2}\right)^2 + 4} - p^{\mathrm{rk}(\mathcal{L}_p)/2}\frac{1-p^{-1}}{p^{\mathrm{rk}(\mathcal{L}_p)}-2}\right).$$

□

Note that in case of a good prime $p \nmid \mathrm{lev}(L)$, the parameter R_p equals the rank and $\lambda_p = 0$ vanishes trivially as the associated local lattice is unimodular, so that we may recover Proposition 6.83 in case $p \neq 2$. For the reduction to that case, also compare Corollary 6.72 and Remark A.18 where the respective Gauss sums are explicitly compared to each other in the even case.

As another corollary, we obtain a factorisation of L-series. Note that it suffices in light of Corollary 6.86 to reduce to the case of bad primes and compare Remark 1.34 for the appearing maximality condition.

Corollary 6.91. *Select $N \mid \mathrm{lev}(L)^\infty$ and a Hall divisor $N_0 \mid N$ with $2 \nmid N_0$ such that L_p is maximal for all primes $p \mid N_0$. Further, let $f \in \mathcal{S}_{L,k}$ be a simultaneous eigenform of all $\mathcal{T}(p^2)$ with eigenvalue σ_p for the primes $p \mid N_0$. Denote the Fourier expansion of f by*

$$f(\tau) = \sum_{\mu\in\mathcal{L}} \sum_{n\in\overline{\mathrm{q}}(\mu)+\mathbb{Z}} a(\mu, n) \cdot e(n\tau)\mathfrak{e}_\mu$$

and select an index (λ, t) such that for all primes $p \mid N_0$ we have $p^2 \nmid \mathrm{lev}(L)t$ or $\lambda_p = 0$. For the primes $p \mid N_0$ for which $\lambda_p \neq 0$, assume that f is invariant with respect to $\mathrm{O}(\mathcal{L}_p)$. Then

$$\sum_{\substack{n\in\mathbb{N}\\ n\mid N^\infty}} \frac{a(n\lambda, nt)}{n^s}$$

$$= \sum_{\substack{n\in\mathbb{N}\\ n\mid (N/N_0)^\infty}} \frac{a(n\lambda, nt)}{n^s}$$

$$\cdot \prod_{p \mid N_0} \frac{1 + K_{L,p} \begin{cases} \delta_{\lambda_p \neq 0}(1 - p^{-1}) + \delta_{\lambda_p = 0}\delta_{p \nmid t}, & 2 \mid R_p \\ -\delta_{\lambda_p = 0} p^{-1/2} \left(\frac{-t}{p}\right) \epsilon_p, & 2 \nmid R_p \end{cases} \cdot p^{k-1-s} + C(\lambda_p) p^{2(k-1-s)}}{1 - \left(\frac{\sigma_p}{p^{k-1}} - \delta_{2 \mid R_p}(1 - p^{-1}) K_{L,p}\right) p^{k-1-s} + p^{2(k-1-s)}}$$

for $\mathrm{Re}(s) > \max_{p \mid N_0}\{\mathrm{rk}(\mathcal{L}_p)\}/2 + k$ *in the notation of Proposition 6.89.*

Combining this factorisation with Corollary 6.86 yields the following product expansion.

Corollary 6.92. *Assume L to be maximal and select a simultaneous eigenform $f \in \mathcal{S}_{L,k}$ of all $\mathcal{T}(p^2)$ with eigenvalue σ_p. Write*

$$f(\tau) = \sum_{\mu \in \mathcal{L}} \sum_{n \in \overline{\mathrm{q}}(\mu)+\mathbb{Z}} a(\mu, n) \cdot e(n\tau)\mathfrak{e}_\mu$$

for its Fourier expansion and select an index (λ, t) such that for all primes p we have $p^2 \nmid \mathrm{lev}(L)t$ or $\lambda_p \neq 0$. Assume f to be invariant with respect to the action of $\mathrm{O}(\mathcal{L}_p)$ for these primes for which $0 \neq \lambda_p \in \mathcal{L}_p$. Then

$$\begin{aligned}
&L_{(\lambda,t)}(f,s)\\
&\stackrel{\text{def}}{=} \sum_{n\in\mathbb{N}} \frac{a(n\lambda, n^2 t)}{n^s}\\
&= \begin{cases} \sum_{n\in\mathbb{N}} \frac{a(2^n\lambda, 2^n t)}{2^{ns}}, & 2 \mid \mathrm{lev}(L), \\ a(\lambda, t), & 2 \nmid \mathrm{lev}(L) \end{cases}\\
&\cdot \prod_{p \nmid \mathrm{lev}(L)} \begin{cases} \frac{1+\delta_{p\nmid t}\frac{G_{\mathcal{L}}(1)}{G_{\mathcal{L}}(p)}\cdot p^{k-1-s}}{1-\left(\frac{\sigma_p}{p^{k-1}}-(1-p^{-1})\frac{G_{\mathcal{L}}(1)}{G_{\mathcal{L}}(p)}\right)\cdot p^{k-1-s}+p^{2(k-1-s)}} & ,2 \mid \mathrm{rk}(L), \\ \frac{1-\chi_{\mathcal{L}}(p)\left(\frac{-t}{p}\right)p^{-1/2}\cdot p^{k-1-s}}{1-\frac{\sigma_p}{p^{k-1}}\cdot p^{k-1-s}+p^{2(k-1-s)}} & ,2 \nmid \mathrm{rk}(L) \end{cases}\\
&\cdot \prod_{2 \neq p \mid \mathrm{lev}(L)} \frac{1 + K_{L,p}\begin{cases} \delta_{\lambda_p \neq 0}(1-p^{-1}) + \delta_{\lambda_p=0}\delta_{p\nmid t}, & 2 \mid R_p \\ -\delta_{\lambda_p=0}p^{-1/2}\left(\frac{-t}{p}\right)\epsilon_p, & 2 \nmid R_p \end{cases} \cdot p^{k-1-s} + C(\lambda_p)\cdot p^{2(k-1-s)}}{1-\left(\frac{\sigma_p}{p^{k-1}} - \delta_{2|R_p}(1-p^{-1})K_{L,p}\right)p^{k-1-s} + p^{2(k-1-s)}}.
\end{aligned}$$

The product converges absolutely for $\mathrm{Re}(s) > k+1$ as long as the factors in the finite product are well defined. This is definitely the case for $\mathrm{Re}(s) > \max\{\mathrm{rk}(\mathcal{L}_p)\}/2 + k$ and in this range all rational factors in p^{-s} are different from zero.
Here, $G_{\mathcal{L}}(d) = \sum_{\lambda\in\mathcal{L}} e(d\,\overline{\mathrm{q}}(\lambda))$ and $K_{L,p} = p^{-\mathrm{rk}(L)/2}\sum_{v\in L/pL} e(\mathrm{q}(v))$, while λ_p denotes the projection of $\lambda \in \mathcal{L}$ to the p-component $\mathcal{L}_p$. Further, R_p is given in Definition 1.38 and

$$\chi_{\mathcal{L}}(p) = \epsilon_p^{\mathrm{sig}(L)+\left(\frac{-1}{|\mathcal{L}|}\right)}\left(\frac{p}{|\mathcal{L}|2^{\mathrm{sig}(\mathcal{L})}}\right), \quad \text{with } \epsilon_p = \begin{cases} 1, & p \equiv 1 \mod 4, \\ i, & p \equiv 3 \mod 4. \end{cases}$$

6.4.3 Rankin–Selberg L-series

In this section, the L-series associated to vector valued modular forms in Subsection 6.4.1 are constructed via an integral pairing in Rankin–Selberg fashion. We begin with a brief introduction of the central tool for the method, the unfolding trick.

The unfolding trick

Let G be a group, (X, μ) be a measured space, such that $G \circlearrowleft X$ acts faithfully and invariantly with respect to the measure. Further, let $H \leq G$ be a subgroup.

Lemma 6.93. *Assume there is a measurable representation $\mathcal{F}_G$ of $G\backslash X$ and select a subgroup $H \leq G$ with countable $H\backslash G$. Then*

$$\mathcal{F}_H \coloneqq \bigcup_{\gamma \in H\backslash G} \gamma^{-1} \mathcal{F}_G$$

is a measurable representation of $H\backslash X$.

Proof: Let $x \in X$, then there is $g \in G$ with $gx \in \mathcal{F}_G$. Now there is some $\gamma \in H\backslash G$ and $h \in H$ with $\gamma h = g$. Apparently, $hx = \gamma^{-1}\gamma h x = \gamma^{-1} g x \in \gamma^{-1}\mathcal{F}_G$.
Assume we had $x_1, x_2 \in \mathcal{F}_H$ and $h \in H$ with $x_1 = hx_2$. Then there are $\gamma_1, \gamma_2 \in H\backslash G$ with $x_i \in \gamma_i^{-1}\mathcal{F}_G$, so that there are $y_i \in \mathcal{F}_G$ such that $\gamma_1^{-1} y_1 = x_1 = x_2 = h\gamma_2^{-1} y_2$, meaning $x_1 = \gamma_1 h \gamma_2^{-1} y_2 \in \mathcal{F}_G$. Hence, $\gamma_1 h \gamma_2^{-1} = e$, translating to $\gamma_1 h = \gamma_2 \in \gamma_1 H$, implying $\gamma_1 = \gamma_2$, and resulting in $h = e$. □

Lemma 6.94. *Assume there is a measurable representation of $G\backslash X$, a subgroup $H \leq G$ of at most countably infinite index and a function $f : (X, \mu) \to \mathbb{C}$ which is invariant under H and integrable on a measurable representation of $H\backslash X$. Then*

$$\int_{G\backslash X} \sum_{\gamma \in H\backslash G} \gamma^* f \, \mathrm{d}\mu = \int_{H\backslash X} f \, \mathrm{d}\mu. \tag{6.53}$$

Proof: We have

$$\int_{G\backslash X} \sum_{\gamma \in H\backslash G} \gamma^* f \, \mathrm{d}\mu = \sum_{\gamma \in H\backslash G} \int_{G\backslash X} \gamma^* f \, \mathrm{d}\mu = \sum_{\gamma \in H\backslash G} \int_{\gamma^{-1}(G\backslash X)} f \, \mathrm{d}\mu = \int_{H\backslash X} f \, \mathrm{d}\mu,$$

where convergence is verified from right to left. □

There are some straightforward generalisations to the above setting. First, one may assume that $G \circlearrowright X$ acts not faithfully, but that its isotropy group of an element $x \in X$ is normal in G and contained in H. However, even that is not necessary, but their are measure theoretic inconveniences arising when considering more general situations – compare [Neu29] for a note on that matter. Further, the last result is indifferent to null sets, meaning that conditions may be neglected on null-sets and also the notion of a fundamental domain – a set of representatives of the orbits – may be defined up to null sets.

Constructing L-series

The most prominent application of the unfolding trick that has been briefly reviewed in the section above is the choice of $X = \mathbb{H}$ the upper half plane, $G = \Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$ acting via Möbius transformations and as a subgroup $H = \Gamma_\infty$. Recall that a fundamental domain $\mathcal{F}_{\Gamma(1)}$ is, up to a null-set, given by

$$\mathcal{F}_{\Gamma(1)} = \{\tau \in \mathbb{H} \mid |\mathrm{Re}(\tau)| \leq 1/2, |\tau| \geq 1\}$$

(cf. Figure 2.1.2) and except for i, ρ, and ρ^2, the isotropy group is $\{\pm\mathcal{I}\} < \Gamma(1)$, which may be divided out. The considered measure is the hyperbolic measure on the upper half plane which is given by $\mathrm{d}\mu = \frac{\mathrm{d}u\,\mathrm{d}v}{v^2}$, where we represent $\tau = u + iv \in \mathbb{H}$ with $u = \mathrm{Re}(\tau), v = \mathrm{Im}(\tau)$ (cf.).

In this context, Rankin[15] [Ran39b] and Selberg have considered integrals of the form

$$\int_{\mathcal{F}_{\Gamma(1)}} \varphi \cdot G_0(\tau, s)\, \mathrm{d}\mu,$$

where $\varphi = f \cdot \overline{g} \cdot \mathrm{Im}^k$ was essentially a product of two modular forms $f, g \in \mathcal{M}_k(\Gamma(1))$ that was assumed to be a cusp form, so that φ was automorphic. Further, G_0 denotes a non-holomorphic Eisenstein series of weight 0. The resulting integral would unfold and define an L-series with analytic continuation and functional equation [Ran39b, Thm. 3 p. 360].

In the following, we will mimic that procedure in order to construct L-series as in (6.47) for cusp forms $f \in \mathcal{S}_{L,k}$. For that purpose, we will pair f against a non-holomorphic Eisenstein series that is twisted with a theta series. We will first prove that the unfolding trick is applicable in our context. However, we will do so in greater generality, as we have

[15] Note that Rankin has already considered the case of the arithmetic group $\Gamma(N)$.

future applications in mind. As a consequence, the following technical result is meant for the reader who is interested in details. Its Corollary 6.96 suffices for the proof of Theorem 7.16 which is the main application within the scope of the present thesis.

Recall that for $\eta \in \mathcal{L} = L'/L$ and suitable parameters m, ψ, k we have defined vector valued parabolic Poincaré series $P_{\eta,m,\psi,k}$ (cf. Definition 3.47 and its succeeding comments).

Proposition 6.95. *Let L be an even lattice that splits as $L = L_1 \oplus L_2$,*

- *$f \in \mathcal{S}_{L,k}$ with Fourier coefficients $a(\lambda, n) \in \mathcal{O}_\varepsilon(n^{\nu_f})$,*
- *$g \in \mathcal{M}_{L_1,k_1}$ with Fourier coefficients $b(\lambda, n) \in \mathcal{O}_\varepsilon(n^{\nu_g})$,*
- *and $P_{\eta,m,\psi,k_2} = \sum_{\gamma \in \overline{\Gamma_\infty} \backslash \mathrm{Mp}_2(\mathbb{Z})} \psi \mathfrak{e}_\eta(m\,\cdot)|_{L_2,k_2}\gamma \in \mathcal{A}_{L_2,k_2}$ be a parabolic Poincaré series in the notation of Definition 3.47 such that $k = k_1 + k_2$. Asssume that $\psi \in \mathcal{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{>0})$ and that there are constants $\alpha_0, \alpha_\infty \in \mathbb{R}$ such that $\psi(v) \in \mathcal{O}(v^{\alpha_0})$ for $v \to 0$ and $\psi(v) \in \mathcal{O}(v^{\alpha_\infty})$ for $v \to \infty$.*

Then

$$\int_{\mathcal{F}_{\Gamma(1)}} \langle f, g \otimes P_{\eta,m,\psi,k} \rangle \cdot \mathrm{Im}^k \,\mathrm{d}\mu = \int_{\mathcal{F}_{\Gamma_\infty}} \big\langle f, g \otimes \psi \mathfrak{e}_\eta(m\,\cdot) \big\rangle \cdot \mathrm{Im}^k \,\mathrm{d}\mu \tag{6.54}$$

provided $\alpha_0 > \max\{3 - k + \nu_f + \nu_g, 1 - k/2\}$.

Note that by assumption of f, g being modular forms we find by Theorem 2.25 and Corollary 3.85 that the parameters ν_f, ν_g are explicitly selectable such that $a(\lambda, n) = \mathcal{O}(n^{\nu_f})$ and $b(\mu, n) = \mathcal{O}(n^{\nu_g})$.

Proof: First, we assume absolute convergence of the integral on the right hand side of (6.54) and verify the result of the unfolding method. We use the abbreviations $H := \overline{\Gamma_\infty}$, $G = \mathrm{Mp}_2(\mathbb{Z})$, as well as $h_{\eta,m} = \psi \mathfrak{e}_\eta(m\,\cdot)$ and begin by computing

$$\begin{aligned}
\int_{\mathcal{F}_{\Gamma(1)}} \langle f, g \otimes P_{\eta,m,\psi,k} \rangle \cdot \mathrm{Im}^k \,\mathrm{d}\mu &= \int_{\mathcal{F}_{\Gamma(1)}} \left\langle f, g \otimes \sum_{\gamma \in H \backslash G} h_{\eta,m}|_{k_2}\gamma \right\rangle \cdot \mathrm{Im}^k \,\mathrm{d}\mu \\
&= \int_{\mathcal{F}_{\Gamma(1)}} \sum_{\gamma \in H \backslash G} \langle f, g \otimes h_{\eta,m}|_{k_2}\gamma \rangle \cdot \mathrm{Im}^k \,\mathrm{d}\mu \\
&= \int_{\mathcal{F}_{\Gamma(1)}} \sum_{\gamma \in H \backslash G} \langle f|_k\gamma, g|_{k_1}\gamma \otimes h_{\eta,m}|_{k_2}\gamma \rangle \cdot \mathrm{Im}^k \,\mathrm{d}\mu \\
&= \int_{\mathcal{F}_{\Gamma(1)}} \sum_{\gamma \in H \backslash G} \big\langle f|_k\gamma, (g \otimes h_{\eta,m})|_k\gamma \big\rangle \cdot \mathrm{Im}^k \,\mathrm{d}\mu
\end{aligned}$$

$$= \int_{\mathcal{F}_{\Gamma(1)}} \sum_{\gamma \in H\backslash G} \gamma^* \left(\langle f, g \otimes h_{\eta,m} \rangle \cdot \mathrm{Im}^k \right) \, \mathrm{d}\mu$$

$$= \int_{\mathcal{F}_{\Gamma_\infty}} \langle f, g \otimes h_{\eta,m} \rangle \cdot \mathrm{Im}^k \, \mathrm{d}\mu. \tag{6.55}$$

Subsequently, we have to prove absolute convergence of the unfolded integral. Note that

$$\begin{aligned} \langle f, g \otimes h_{\eta,m} \rangle &= \Big\langle \sum_{\substack{\lambda_1 \in L_1'/L_1 \\ \lambda_2 \in L_2'/L_2}} f_{\lambda_1+\lambda_2} \mathfrak{e}_{\lambda_1+\lambda_2}, \sum_{\lambda_1 \in L_1'/L_1} g_{\lambda_1} \psi \mathfrak{e}_{\lambda_1+\eta}(m\,\cdot\,) \Big\rangle \\ &= \sum_{\lambda_1 \in L_1'/L_1} f_{\lambda_1+\eta} \overline{g_{\lambda_1} \psi e(m\,\cdot\,)} \end{aligned} \tag{6.56}$$

where indices of f and g denote the respective component function. Assume we had $\psi(v) = 1$ and $m = 0$, then

$$\langle f, g \otimes h_{\eta,m} \rangle = \sum_{n \in \mathbb{Z}/\operatorname{lev}(L)} c(n) q^n$$

has a Fourier expansion that is convergent and hence absolutely convergent on $\mathbb{H}$ (cf. Lemma A.19). Returning to the case of general $h_{\eta,m}$, we restrict to considering the behaviour $v \to \infty$, first and note that it suffices to prove convergence of the integral on a strip $S \subseteq \{z \in \mathbb{C} \mid |\mathrm{Re}(z)| \leq 1/2\}$ with arbitrary positive minimum height such that $\psi(v)$ is bounded by some positive multiple of v^{α_∞}. Then, we may use the same trick as in Example 3.52 d) to verify that for general $\psi(v)$ we have the following bound on S

$$\begin{aligned} \left| \langle f, g \otimes h_{\eta,m} \rangle \right| &\leq |\psi(v)| \sum_{n \in \mathbb{N}/\operatorname{lev}(L)} |c(n)| e^{-2\pi v n} \\ &\leq |\psi(v)| e^{-2\pi v/(2\operatorname{lev}(L))} \sum_{n \in \mathbb{N}/\operatorname{lev}(L)} |c(n)| e^{2\pi i n (iv/2)}. \end{aligned}$$

The right sum however, is bounded by a constant, since the Fourier series is absolutely convergent on $\mathbb{H}$ and the integral over S of the last expression converges absolutely.

We turn to the case $v \to 0$ and will proceed with inserting the Fourier expansion of both modular forms f and g in (6.56) to proceed with the computation. Recall that there are Fourier coefficients, such that

$$f(\tau) = \sum_{\lambda \in L'/L} \sum_{0 < n \in \mathbb{Z} + \overline{\mathrm{q}}(\lambda)} a(\lambda, n) e(n\tau) \mathfrak{e}_\lambda,$$

$$g(\tau) = \sum_{\mu \in L_1'/L_1} \sum_{0 \le n \in \mathbb{Z} + \overline{\mathrm{q}}(\mu)} b(\mu, n) e(n\tau) \mathfrak{e}_\mu.$$

With that notation, we derive the following explicit expansion.

$$\begin{aligned} &\sum_{\lambda_1 \in L_1'/L_1} f_{\lambda_1+\eta} \overline{g_{\lambda_1}} \\ = &\sum_{\lambda_1 \in L_1'/L_1} \sum_{0 < n \in \mathbb{Z} + \overline{\mathrm{q}}(\lambda_1) + \overline{\mathrm{q}}(\eta)} \sum_{0 \le n_1 \in \mathbb{Z} + \overline{\mathrm{q}}(\lambda_1)} a(\lambda_1 + \eta, n) \overline{b(\lambda_1, n_1)} \cdot e(n\tau - n_1 \overline{\tau}). \end{aligned}$$

With this expansion, we shall prove absolute convergence of the unfolded integral. First, we note that it is sufficient to bound each of the $|L_1'/L_1|$ summands in λ_1 arising in (6.56) separately. Consequently, we fix λ_1 in such a fashion that the associated term dominates the others, when multiplied with a suitable constant $C_1 > 0$. Next, we rewrite

$$|e(n\tau - n_1\overline{\tau})| = \exp[-2\pi v(n + n_1)].$$

In the following, we may assume $\nu_f, \nu_g \ge 0$. Combining these observations with (6.56) yields that the scalar product isbounded by

$$|\langle f, g \otimes h_{\eta,m} \rangle| \le |L_1'/L_1| C_1 \sum_{\substack{0 < n \in \mathbb{Z} + \overline{\mathrm{q}}(\lambda_1) + \overline{\mathrm{q}}(\eta) \\ 0 \le n_1 \in \mathbb{Z} + \overline{\mathrm{q}}(\lambda_1)}} n^{\nu_f} n_1^{\nu_g} \cdot \exp[-2\pi v(n + n_1)] \cdot |\psi(v)|.$$

We write $N \coloneqq \mathrm{lev}(L)$ and note that a summation over more possible choices of n, n_1 potentially increases the sum.

$$\begin{aligned} |\langle f, g \otimes h_{\eta,m} \rangle| &\le |L_1'/L_1| C_1 \sum_{\substack{0 \ne n \in \mathbb{Z}/N \\ n_1 \in \mathbb{Z}/N}} |n|^{\nu_f} |n_1|^{\nu_g} \cdot \exp[-2\pi v(|n| + |n_1|)] \cdot |\psi(v)| \\ &= |L_1'/L_1| C_1 \sum_{\substack{0 \ne n \in \mathbb{Z} \\ n_1 \in \mathbb{Z}}} \frac{|n|^{\nu_f} |n_1|^{\nu_g}}{N^{(\nu_f + \nu_g)}} \cdot \exp[-2\pi v(|n| + |n_1|)/N] \cdot |\psi(v)|. \end{aligned}$$

We simplify the appearing series. First, we observe that we may bound the numerator

$$|n|^{\nu_f} |n_1|^{\nu_g} \le \max\{|n|, |n_1|\}^{(\nu_f + \nu_g)}.$$

Recall further that $|\psi(v)|$ was bounded for $v \to 0$ by v^{α_0}. Hence, we may bound $|\psi(v)| \le \frac{C_2}{|L_1'/L_1| C_1} v^{\alpha_0}$ close to the real line for a positive constant $C_2 > 0$ and without loss of generality we may assume that the bound for ψ is valid on $\mathcal{F}_{\Gamma_\infty}$. Consequently, we deduce

$$\begin{aligned}
&\int_{\mathcal{F}_{\Gamma_\infty}} |\langle f, g \otimes h_{\eta,m}\rangle| \cdot \operatorname{Im}^k \ \mathrm{d}\mu \\
&\leq C_2 \cdot \int_0^\infty \sum_{\substack{0 \neq n \in \mathbb{Z} \\ n_1 \in \mathbb{Z}}} \frac{\max\{|n|, |n_1|\}^{(\nu_f+\nu_g)}}{N^{\nu_f+\nu_g}} \cdot e^{-\frac{2\pi}{N}(|n|+|n_1|)\cdot v} v^{\alpha_0+k-2} \, \mathrm{d}v
\end{aligned}$$

substituting $v \mapsto v \cdot N/[2\pi(|n| + |n_1|)]$ yields

$$\begin{aligned}
&= C_2 \cdot \sum_{\substack{0 \neq n \in \mathbb{Z} \\ n_1 \in \mathbb{Z}}} \frac{\max\{|n|, |n_1|\}^{\nu_f+\nu_g}}{N^{(\nu_f+\nu_g)}[2\pi(|n| + |n_1|)/N]^{\alpha_0+k-1}} \cdot \Gamma(\alpha_0 + k - 1) \\
&= \frac{C_2}{N^{\nu_f+\nu_g}} \frac{\Gamma(\alpha_0 + k - 1)}{(2\pi/N)^{\alpha_0+k-1}} \cdot \sum_{\substack{0 \neq n \in \mathbb{Z} \\ n_1 \in \mathbb{Z}}} \frac{\max\{|n|, |n_1|\}^{\nu_f+\nu_g}}{(|n| + |n_1|)^{\alpha_0+k-1}}.
\end{aligned}$$

We are left with the task of bounding the series to the right. First, we increase it by summing over $0 \neq (n, n_1) \in \mathbb{Z}^2$ and write $x := (n, n_1)$. Then we note that $|n|+|n_1| = \|x\|_1$ and $\max\{|n|, |n_1|\} = \|x\|_\infty$. However, these norms are equivalent (cf. Remark A.7), implying that the series converges, if and only if,

$$\sum_{0 \neq x \in \mathbb{Z}^2} \frac{1}{\|x\|_\infty^{\alpha_0+k-1-(\nu_f+\nu_g)}}$$

converges. The convergence of this type of series, however, has already been established in Lemma 6.75 by computing the size of the discrete $\|\cdot\|_\infty$–sphere and comparing the result with the Riemann ζ-function. It converges for $\alpha_0 + k - 1 - (\nu_f + \nu_g) > 2$, meaning

$$\alpha_0 > 3 - k + \nu_f + \nu_g.$$

□

A special instance of Proposition 6.95 where the Poincaré series is given by an Eisenstein series is integral for the main goal of this thesis. This yields the following observation which is relevant for the computation of cycle integrals of the Kudla–Millson liftings in Section 7.3.1.

Corollary 6.96. *Let $f \in \mathcal{S}_{L,k}$, $L = L_1 \oplus L_2$ split, $g \in \mathcal{M}_{L_1,k_1}$, and $\eta \in L_2'/L_2$ be isotropic. Further, choose $E_{L_2,\eta,k_2}(\tau, s)$ be the Eisenstein series from Definition 3.37 such* *that* $k = k_1 + k_2$*. Then there is some* $s_0 \in \mathbb{R}$ *such that*

$$\int_{\mathcal{F}_{\Gamma(1)}} \langle f, g \otimes E_{L_2,\eta,k_2}(\,\cdot\,, s)\rangle \cdot \operatorname{Im}^k \, \mathrm{d}\mu = \int_{\mathcal{F}_{\Gamma_\infty}} \big\langle f, g \otimes \mathfrak{e}_\eta \big\rangle \cdot \operatorname{Im}^{\overline{s}+k} \, \mathrm{d}\mu \tag{6.57}$$

for all $s \in \mathbb{C}$ *with* $\operatorname{Re}(s) > s_0$ *and both sides are holomorphic in* $\overline{s}$*.*

In order to construct the L-series presented in (6.47), we will have to choose g to equal a theta series. Assume, we had $(L, \mathrm{q}) = (L_1, \mathrm{q}_1) \oplus (L_2, \mathrm{q}_2)$ with (L_1, q_1) positive definite of rank m_1 and associated theta function $\Theta_{L_1}(\tau) \coloneqq \Theta_{L_1}(\tau; \exp(-2\pi\, \mathrm{q}_1))$. Selecting $g = \Theta_{L_1}(\tau)$, we obtain for $f \in \mathcal{S}_{L,k}$ a cusp form and suitable $s \in \mathbb{C}$ that

$$\int_{\mathcal{F}_{\Gamma(1)}} \langle f, \Theta_{L_1} \otimes E_{L_2,\eta,k_2}(\,\cdot\,, s)\rangle \cdot \operatorname{Im}^k \, \mathrm{d}\mu = \int_{\mathcal{F}_{\Gamma_\infty}} \big\langle f, \Theta_{L_1} \otimes \mathfrak{e}_\eta \operatorname{Im}^s \big\rangle \cdot \operatorname{Im}^k \, \mathrm{d}\mu.$$

We compute the integrand before continuing:

$$\begin{aligned} \big\langle f, \Theta_{L_1}(\tau) \otimes \mathfrak{e}_\eta \big\rangle &= \Big\langle \sum_{\substack{\lambda_1 \in L_1'/L_1 \\ \lambda_2 \in L_2'/L_2}} f_{\lambda_1+\lambda_2} \mathfrak{e}_{\lambda_1+\lambda_2}, \sum_{\lambda_1 \in L_1'/L_1} \theta_{L_1,\lambda_1} e_{\lambda_1+\eta} \Big\rangle \\ &= \sum_{\lambda_1 \in L_1'/L_1} f_{\lambda_1+\eta} \overline{\theta_{L_1,\lambda_1}}. \end{aligned}$$

We proceed similarly to the proof of the technical Proposition 6.95. This time, however, we do not have to consider the absolute value of the integrand and a significant portion of the terms appearing will cancel, allowing for a relatively concrete computation. For that purpose, we assume $\operatorname{Re}(s)$ to be large enough to guarantee absolute convergence. To this end, recall that f has a representation as a Fourier series as well as the definition of Θ_{L_1} (compare for instance Definition 3.31 in conjunction with Example 3.33 in the positive definite case).

$$\begin{aligned} f(\tau) &= \sum_{\lambda \in L'/L} \sum_{0 < n \in \mathbb{Z} + \overline{\mathrm{q}}(\lambda)} a(\lambda, n) e(n\tau) \mathfrak{e}_\lambda, \\ \Theta_{L_1}(\tau) &= \sum_{\lambda_1 \in L_1'/L_1} \sum_{l \in L_1 + \lambda_1} e(\mathrm{q}(l)\tau) \mathfrak{e}_{\lambda_1}. \end{aligned}$$

We insert the Fourier expansion of both functions in order to obtain (recall that $\eta \in L_2'/L_2$

was isotropic)

$$\begin{aligned}&\sum_{\lambda_1\in L_1'/L_1} f_{\lambda_1+\eta}\overline{\theta_{L_1,\lambda_1}}\\&=\sum_{\lambda_1\in L_1'/L_1}\sum_{0<n\in\mathbb{Z}+\overline{\mathrm{q}}(\lambda_1)}\sum_{l\in L_1+\lambda_1} a(\lambda_1+\eta,n)\cdot e(n\tau)e\left[-\overline{\tau}\,\mathrm{q}(l)\right]\\&=\sum_{\lambda_1\in L_1'/L_1}\sum_{0<n\in\mathbb{Z}+\overline{\mathrm{q}}(\lambda_1)}\sum_{l\in L_1+\lambda_1} a(\lambda_1+\eta,n)\cdot e[iv(n+\mathrm{q}(l))]e\left[u(n-\mathrm{q}(l))\right].\end{aligned}$$

Clearly, $n-\mathrm{q}(l)\in\mathbb{Z}$, so that $\mathbb{R}\ni u\mapsto e\left[u(n-\mathrm{q}(l))\right]$ defines a character which descends to a character on $\mathbb{T}$. The unit circle $\mathbb{T}$, however, is compact, implying by Remark 1.51 the following identity:

$$\int_0^1 e\left[u(n-\mathrm{q}(l))\right]\,\mathrm{d}u=\delta_{n,\mathrm{q}(l)},$$

where the right hand side is to be understood as a Kronecker symbol. As a consequence, we conclude

$$\begin{aligned}&\int_{\mathcal{F}_{\Gamma_\infty}}\left\langle f,\Theta_{L_1}\otimes\mathfrak{e}_\eta\right\rangle\cdot\mathrm{Im}^{\overline{s}+k}\,\mathrm{d}\mu\\&=\int_0^\infty\sum_{\lambda_1\in L_1'/L_1}\sum_{0\neq l\in L_1+\lambda_1} a(\lambda_1+\eta,\mathrm{q}(l))\cdot e[iv2\,\mathrm{q}(l)]\cdot v^{\overline{s}+k-2}\,\mathrm{d}v\\&=\sum_{\lambda_1\in L_1'/L_1}\sum_{0\neq l\in L_1+\lambda_1} a(\lambda_1+\eta,\mathrm{q}(l))\cdot\int_0^\infty\exp[-4\pi\,\mathrm{q}(l)\cdot v]\cdot v^{\overline{s}+k-2}\,\mathrm{d}v.\end{aligned}$$

Applying the subtitution $v\mapsto v/4\pi\,\mathrm{q}(l)$ yields that this equals

$$\begin{aligned}&=\sum_{\lambda_1\in L_1'/L_1}\sum_{0\neq l\in L_1+\lambda_1}\frac{a(\lambda_1+\eta,\mathrm{q}(l))}{(4\pi\,\mathrm{q}(l))^{\overline{s}+k-1}}\cdot\int_0^\infty e^{-v}\cdot v^{\overline{s}+k-2}\,\mathrm{d}v\\&=\frac{\Gamma(\overline{s}+k-1)}{(4\pi)^{\overline{s}+k-1}}\sum_{\lambda_1\in L_1'/L_1}\sum_{0\neq l\in L_1+\lambda_1}\frac{a(\lambda_1+\eta,\mathrm{q}(l))}{\mathrm{q}(l)^{\overline{s}+k-1}},\end{aligned}$$

where instead of summing over all elements of all cosets of L_1' by L_1, we may as well sum over the whole lattice L_1' to obtain the following, simpler description

$$=\frac{\Gamma(\overline{s}+k-1)}{(4\pi)^{\overline{s}+k-1}}\sum_{0\neq l\in L_1'}\frac{a(\overline{l}+\eta,\mathrm{q}(l))}{\mathrm{q}(l)^{\overline{s}+k-1}}.$$

Comparing the last result with Definition 6.77, we notice that the series appearing is exactly $L_{L_1,\eta,1}(f,\overline{s}+k-1)$. Recall that there is $\sigma \in \mathbb{R}$ such that $a(\lambda, \mathrm{q}(l)) \in \mathcal{O}_\varepsilon(\mathrm{q}(l)^{k/2-\sigma+\varepsilon})$ – compare Table 3.1 for concrete choices of σ. Then a comparison with Lemma 6.76 yields that the last expression converges absolutely for $\mathrm{Re}(s) > \frac{m_1-k}{2} + 1 - \sigma$ with σ as in the Lemma and defines a holomorphic function in $\overline{s}$ in that right half plane. As a consequence, we have proven the following Proposition.

Proposition 6.97. *Let $f \in \mathcal{S}_{L,k}$, $k \geq 2$, $L = L_1 \oplus L_2$ with positive definite L_1 of rank m_1, $E_{L_2,\eta,k_2}(\,\cdot\,, s)$ be the Eisenstein series of Definition 3.37, and assume $k = m_1/2 + k_2$. Then we have the identity*

$$\int_{\mathcal{F}_{\Gamma(1)}} \langle f, \Theta_{L_1} \otimes E_{L_2,\eta,k_2}(\,\cdot\,, s)\rangle \cdot \mathrm{Im}^k \;\mathrm{d}\mu = \frac{\Gamma(\overline{s}+k-1)}{(4\pi)^{\overline{s}+k-1}} \cdot L_{L_1,\eta,1}(f,\overline{s}+k-1) \tag{6.58}$$

of holomorphic functions in $\overline{s}$ for $\mathrm{Re}(s) > \frac{m_1-k}{2} + 1 - \sigma$, where $\sigma = 1/2$ or $1/4$ depending on whether $2 \mid \mathrm{rk}(L)$ or not. Further, the left hand side gives rise to a meromorphic continuation of $L_{L_1,\eta,1}$.

There is a specialisation that plays a key role in the proof of Theorem 7.16.

Corollary 6.98. *Let (L,q) be even such that $L = \mathbb{Z}\ell \oplus L_2$ for some $\ell \in L$ of positive norm and $f \in \mathcal{S}_{L,k}$. Assume we have $k > 3 - 2\sigma$ with σ as above. Then*

$$\int_{\mathcal{F}_{\Gamma(1)}} \big\langle f, \Theta_{\mathbb{Z}\ell} \otimes E_{L_2,0,k-1/2}(\,\cdot\,,0)\big\rangle \cdot \mathrm{Im}^k \;\mathrm{d}\mu = \frac{2\Gamma(k-1)}{(4\pi\,\mathrm{q}(\ell_0))^{k-1}} \delta_{\mathrm{sig}(L)\equiv 2k\ (4)} \cdot L_{\ell_0}(f, 2k-2) \tag{6.59}$$

for a primitive $\ell_0 \in L'$ such that $\ell_0 \mid \ell \in L'$. Here, δ denotes a Kronecker symbol.

Proof: In Proposition 6.97, we select $L_1 = \mathbb{Z}\ell$ and find $m_1 = \mathrm{rk}(\mathbb{Z}\ell) = 1$. Further, the choice $s = 0$ necessitates $k > 2 + m_1 - 2\sigma = 3 - 2\sigma$ for reasons of convergence. Writing out (6.58) yields

$$\begin{aligned}\int_{\mathcal{F}_{\Gamma(1)}} \big\langle f, \Theta_{L_1} \otimes E_{L_2,0,k-1/2}(\,\cdot\,,0)\big\rangle \cdot \mathrm{Im}^k \;\mathrm{d}\mu &= \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{0 \neq l \in L_1'} \frac{a(\overline{l}, \mathrm{q}(l))}{\mathrm{q}(l)^{k-1}} \\ &= \frac{\Gamma(k-1)}{(4\pi\,\mathrm{q}(\ell_0))^{k-1}} \sum_{0 \neq n \in \mathbb{Z}} \frac{a(n\overline{\ell_0}, n^2\,\mathrm{q}(\ell_0))}{|n|^{2k-2}}.\end{aligned}$$

Recall that by Remark 3.20 there are no non-trivial modular forms in case $2 \nmid \mathrm{sig}(\mathcal{L}) + 2k$, anyway. Further, in case of $\mathrm{sig}(\mathcal{L}) \equiv 2k+2 \mod 4$ the series vanishes by the symmetry property of the coefficients under reflection of the index (cf. Remark 3.22). Hence,

$$\sum_{0 \neq n \in \mathbb{Z}} \frac{a(n\overline{\ell_0}, n^2\,\mathrm{q}(\ell_0))}{|n|^{2k-2}} = 2 \cdot \delta_{\mathrm{sig}(L) \equiv 2k\ (4)} \cdot \sum_{0 \neq n \in \mathbb{N}} \frac{a(n\overline{\ell_0}, n^2\,\mathrm{q}(\ell_0))}{n^{2(k-1)}}.$$

Finally, the series on the right hand side equals $L_{\ell_0}(f, 2(k-1))$ by Definition 6.79. □

The special value $L_{\ell_0}(f, 2(k-1))$ in (6.59) plays a key role in the context of computing cycle integrals of Kudla–Millson liftings in the subsequent chapter about theta lifts.

Remark 6.99. We recall that the Eisenstein series $E_{L_2,0,k-1/2}$ appearing in (6.59) has a symmetry property (cf. Corollary 3.44). This symmetry may naively be utilised to relate the special value to another value for which properties like non-vanishing might be derivable. However, upon closer investigation we find that the right hand side of (6.59) had, under the induced reflection property, to be evaluated at $s = 1 - k$, where neither the L-series nor the Γ–prefactor converge.

7 Automorphic lifts

In the preceding sections two distinct types of varieties have been discussed, namely symplectic and orthogonal varieties as well as their associated automorphic forms. It turns out that forms of one type may be utilised to construct forms of the other type by means of so called *lifts*. The lifts considered in this thesis are of a special type, referred to as *theta lifts*. These are given by integrating against a kernel that is constructed by symmetrising a smooth object with respect to representations of a dual reductive pair. A concrete description, tailored to our setting, is presented below.
The celebrated paper [BF04] of Bruinier and Funke serves as a fundamental source for the subsequent discussion and a lattice (L, q) of signature (m^+, m^-) is assumed to be fixed.

7.1 Borcherds Lift

The celebrated product expansion of the discriminant function

$$\Delta = q \prod_{n=1}^{\infty} (1 - q^n)^{24} \tag{7.1}$$

derived from the respective product of the Dedekind eta function is well established in the field of automorphic forms. It was this expression that prompted Richard Borcherds to pursue a more conceptual approach to product expansions of modular forms.
In 1995 Borcherds published a paper [Bor95] in which he presented a rigorous method for obtaining infinite product expansions of modular forms with character for $\mathrm{SL}_2(\mathbb{Z})$ by lifting weakly holomorphic modular forms of weight $1/2$ to the latter. The lift is multiplicative, transforming the sum of modular forms into a product of their images. Furthermore, the constant coefficient of the input form represents the weight of the resulting function [Bor95, Thm. 14.1 p. 204]. This theorem even yields information about the zeroes and poles of the target function as algebraic number theoretic data. As an example Borcherds retrieves identity (7.1) and product expansions for the Klein j-invariant as well as several

Eisenstein series. The following selection represents some of these examples.

Example 7.1. For $12 \cdot \theta(\tau) = 12 \sum_{n\in\mathbb{Z}} e^{2\pi i n^2 \tau} = \sum_{n\in\mathbb{Z}} c(n)q^n$ and the series $\sum_{n\in\mathbb{Z}} a(n)q^n$ as in [Bor95, p. 204] we obtain

$$\Delta(\tau) = q^1 \prod_{n>0} (1-q^n)^{c(n^2)} \quad \text{and} \quad j(\tau) = q^{-1} \prod_{n>0} (1-q^n)^{a(n^2)}.$$

A far more intricate example is the following construction.

Example 7.2. The modified Klein's j invariant $j(\tau) - 744 = \sum_{n\geq -1} c(n)q^n$ lifts to

$$j(\tau_1) - j(\tau_2) = q_1^{-1} \prod_{m>0, n\in\mathbb{Z}} (1 - q_1^m q_2^n)^{c(mn)}$$

known as the *monster denominator formula*. This may be utilised to endow the monster module with a multiplication, inducing a vertex operator algebra structure, which ultimately leads to solving the Moonshine conjecture.

In his renowned 1998 paper [Bor98] Borcherds significantly expanded on his initial approach, presenting a multiplicative lift [Bor98, Thm. 13.3 p. 544] that maps weakly holomorphic vector valued modular forms of weight $1 - m^+/2$ to modular forms for certain discrete automorphism groups in $\mathrm{O}_{m^+,2}(\mathbb{R})$, i.e. to the orthogonal setting. In the case of $m^+ = 1$, the association may be interpreted as a purely symplectic lift, due to the accidental isomorphism of $\mathrm{SO}_{1,2}(\mathbb{R})$ and $\mathrm{Sp}_1(\mathbb{R})$. For an excellent concise source on the fundamentals which are presented in the following, see [Bru14].

It should be noted that the basis of Borcherds' construction is a theta lift utilising a Siegel theta function as in Definition 3.29 (see also Remark 3.28)

$$\Theta_L(\tau, z) = v^{\frac{m^-}{2}} \sum_{\lambda \in L'} e\left[u\, \mathrm{q}(\lambda) + iv\, \mathrm{q}_z^+(\lambda)\right] \mathfrak{e}_{\lambda+L}$$

as an integral kernel which is paired with a weakly holomorphic vector valued modular form $f \in \mathcal{M}^!_{L,k}$, namely

$$\Phi(z, f) := \int^\bullet_{\mathrm{Mp}_2(\mathbb{Z})\backslash\mathbb{H}} \langle f(\tau), \Theta_L(\tau, z)\rangle \frac{\mathrm{d}u\, \mathrm{d}v}{v^2}. \tag{7.2}$$

Here, we have written $u + iv = \tau \in \mathbb{H}$ and $z \in \mathbb{D}$ is an element of the Grassmannian. Further, the dot accompanying the integral sign signifies a necessary regularisation (cf. [Bor98]). This pairing is then utilised to derive the multiplicative Borcherds lift, the main features of which are collected in the following theorem.[1]

Theorem 7.3 (Borcherds)**.** *Let $f \in \mathcal{M}^!_{L^-,1-m^+/2}$ be a weakly holomorphic modular form with Fourier coefficients $a(\lambda, n)$ as in (3.40). Assume that $a(\lambda, n)$ is integral for $n \leq 0$. Then there exists a meromorphic modular form $\Psi(\,\cdot\,, f)$ for the discriminant kernel $\Gamma(L)$ with unitary multiplier system of finite order, such that the following conditions are satisfied.*

a) The weight of $\Psi(\,\cdot\,, f)$ is given by $a(0,0)/2$.

b) The divisor of the target form $\Psi(\,\cdot\,, f)$ is given by the principal part of f via

$$\frac{1}{2} \sum_{\mu \in L'/L} \sum_{n>0} a(\mu, -n) Z(\mu, n), \tag{7.3}$$

where

$$Z(\mu, n) = \sum_{\substack{v \in \mu + L \\ \mathrm{q}(v) = n}} \{z \in \mathbb{D} \mid z \perp v\}$$

is the Heegner divisor *of discriminant (μ, n) (cf. Definition 4.70 plus the subsequent comment).*

c) The target form $\Psi(\,\cdot\,, f)$ has explicit infinite product expansions.

For further details regarding the product expansions that are constructed for different cusps, or background material on the construction itself, we refer the reader to [Bor98, Thm. 13.3 p. 544] and conclude with the following definition.

Definition 7.4. For a form $f \in \mathcal{M}^!_{L^-,1-m^+/2}$ as above, we refer to $\Psi(\,\cdot\,, f)$ as the *Borcherds lift* of the form f or, alternatively, as the *Borcherds product* of f.

7.2 Kudla–Millson lift

In the 1980s Kudla and Millson [KM86] introduced special Schwartz forms φ_{KM} on the symmetric spaces attached to the classical groups $\mathrm{O}(m^+, m^-)$, $\mathrm{U}(m^+, m^-)$, and $\mathrm{Sp}(m^+, m^-)$, taking values in closed differential forms. Their principal objective was to investigate cohomology classes of special cycles by means of a theta correspondence, generalising the celebrated work of Hirzebruch and Zagier [HZ76] on Hilbert modular

[1] We denote by L^- the lattice $(L, -\mathrm{q})$.

surfaces. More precisely, in the orthogonal case, Kudla and Millson symmetrised the Schwartz forms φ_{KM} over a base lattice L and let the Weil representation of the symplectic group act on φ_{KM} in order to obtain a kernel $\Theta_{\mathrm{KM}}(\tau, z)$ in a symplectic and an orthogonal variable, which transforms automorphically in both variables. These may then be employed as an integral kernel to lift automorphic objects from the symplectic to the orthogonal setting or vice versa. The authors continued their work in [KM87] in great generality and we recommend [KM90] as an introductory source and [BF04] for a perspective restricting to the elliptic case on the symplectic side.

In the following subsection, the theta function Θ_{KM}, as well as the associated lift will be described in a more explicit manner. We begin by reviewing the construction of the Schwartz form φ_{KM}.

7.2.1 The Kudla–Millson Schwartz form

In this section we will provide a brief overview of the construction of the Kudla–Millson Schwartz form within the context relevant to our study, while also establishing the necessary notation. It should be noted that some of its properties have already been stated in Subsection 5.4.2, as they were required in the context of the Siegel–Weil formula. This section is based on [BF04, Sec. 2, 4 p. 64], which serves as an excellent source of further background information and some of the content of the following is also found in [MZ23].

We begin by recalling the setting. Let (L, b) be an even $\mathbb{Z}$ lattice of signature (m^+, m^-) and rank m. It should be recalled that $V = L \otimes \mathbb{Q}$ is the associated rational quadratic space and that $V(\mathbb{R}) = V \otimes \mathbb{R}$ denotes the enclosing real space. An orthogonal basis (e_i) of $V(\mathbb{R})$ with $\mathrm{b}(e_i, e_i) = 1$ for $1 \leq i \leq m^+$ and $\mathrm{b}(e_i, e_i) = -1$ for $m^+ < i \leq m$ is selected, effectively choosing an isometry to the standard real quadratic vector space $\mathbb{R}^{(m^+, m^-)}$ of Example 1.5. Recall that the symmetric domain associated to L has a model as the Grassmannian

$$\mathbb{D} = \{z \subset V \mid \dim z = m^-,\ \mathrm{q}|_z < 0\},$$

which will be equipped with the base point z_0 spanned by $(e_i)_{m^+ < i \leq m}$. Further, we denote its stabiliser in $\mathrm{SO}^+(V(\mathbb{R}))$ by K and note that it is maximal compact. Additional details can be found in Subsection 4.2.1. Recall that for any $z \in \mathbb{D}$ we find $V(\mathbb{R}) = z \oplus z^{\perp}$ so that each element $v \in V(\mathbb{R})$ may be decomposed uniquely into $v = v_z + v_{z^{\perp}} \in z \oplus z^{\perp}$. Note that this decomposition gives rise to the positive definite standard majorant of q, namely

$$\mathrm{q}_z(v)^+ = \mathrm{q}(v_{z^\perp}) - \mathrm{q}(v_z).$$

Let $\mathfrak{g}$ denote the Lie algebra of $\mathrm{SO}^+(V(\mathbb{R}))$ and $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$ be its Cartan decomposition. Then $\mathfrak{p} \simeq \mathfrak{g}/\mathfrak{k}$ identifies with the tangent space at z_0 of $\mathbb{D}$. With respect to the basis above, we have

$$\mathfrak{p} \simeq \left\{ \begin{pmatrix} 0 & X \\ X^T & 0 \end{pmatrix} \,\middle|\, X \in \mathbb{R}^{m^+ \times m^-} \right\} \simeq \mathbb{R}^{m^+ \times m^-}. \tag{7.4}$$

We denote the m^--forms on $\mathbb{D}$ by $\mathscr{A}^{m^-}(\mathbb{D})$. Then

$$\left[\mathcal{S}(V(\mathbb{R})) \otimes \mathscr{A}^{m^-}(\mathbb{D})\right]^{\mathrm{SO}^+(V(\mathbb{R}))} \simeq \left[\mathcal{S}(V(\mathbb{R})) \otimes \bigwedge\nolimits^{m^-} \mathfrak{p}^*\right]^K,$$

where the isomorphism is given by evaluation at the base point and the form may be extended to the entire Grassmannian $\mathbb{D}$ by the action of $\mathrm{SO}(V(\mathbb{R}))$ via pullback. Therefore, we may represent φ_{KM} as an element in $\left[\mathcal{S}(V(\mathbb{R})) \otimes \bigwedge^{m^-} \mathfrak{p}^*\right]^K$. To this end, let $X_{\alpha,\beta} \in \mathbb{R}^{m^+ \times m^-}$ for $1 \leq \alpha \leq m^+$ and $1 \leq \beta \leq m^-$ denote the basis matrix with value one in the (α,β)-entry and zero elsewhere. The corresponding dual element will be denoted by $w_{\alpha,\beta}$ and multiplication with it by $A_{\alpha,\beta}$. Then, define the following differential operator where we identify the cotangent space of $\mathbb{D}$ at z_0 with $\mathfrak{p}^*$ under (7.4):

$$\mathscr{D}^{m^+,m^-} := \frac{1}{2^{m^-/2}} \prod_{\beta=1}^{m^-} \left[\sum_{\alpha=1}^{m^+} \left(x_\alpha - \frac{1}{2\pi} \frac{\partial}{\partial x_\alpha} \right) \otimes A_{\alpha,\beta} \right]. \tag{7.5}$$

This operator is applied to the standard Gaussian $\varphi_0 \otimes 1 \in \left[\mathcal{S}(V(\mathbb{R})) \otimes \bigwedge^0 \mathfrak{p}^*\right]^K$ in order to obtain the Kudla–Millson Schwartz form

$$\varphi_{\mathrm{KM}} = \mathscr{D}^{m^+,m^-} \varphi_0 \otimes 1 \tag{7.6}$$

which is closed. Based upon this construction, a more explicit description of the form is computed in [MZ23, Sec. 2.2] which invokes products of Hermite polynomials. It is noteworthy that $\Omega := \varphi_{\mathrm{KM}}(0, \cdot\,)$ is the Euler form of the symmetric space $\mathbb{D}$ and in the hermitian case of $m^- = 2$, its negative defines a Kähler form. In the latter case, a description in coordinates on the associated tube domain model has been provided in (5.27). Of particular interest for this thesis is the hermitian case of signature $(m^+, 2)$, for which the form φ_{KM} is, in fact, a $(1,1)$-form.

With the Kudla–Millson Schwartz form available, the associated theta kernel may be constructed as presented in Definition 3.32.

Definition 7.5. Let $\tau = u + iv \in \mathbb{H}$ and $z \in \mathbb{D}$. Then the *Kudla–Millson theta function* of the lattice L is given by

$$\Theta_{\mathrm{KM}}(\tau, z) := \Theta_L(\tau, z; \varphi_{\mathrm{KM}}) := v^{-m/4} \sum_{\lambda \in L'/L} \sum_{l \in \lambda + L} \left(\omega_\infty(g_\tau)\varphi_{\mathrm{KM}}\right)(l, z)\mathfrak{e}_\lambda. \tag{7.7}$$

It should be noted that this form is explicitly expressed in terms of Siegel–theta functions in [MZ23]. With the form Θ_{KM} at hand, we may construct associated lifts of automorphic forms.

7.2.2 The Kudla–Millson lift

As with the Borcherds lift, the theta kernel associated to the Kudla–Millson Schwartz form may be utilised as an integral kernel to construct a theta lift.

Definition 7.6. Let L be an even lattice of signature (m^+, m^-) and rank m. Set $k := m/2$, then for a cusp form $f \in \mathcal{S}_{L,k}(\mathrm{Mp}_2(\mathbb{Z}))$ and $\tau = u + iv \in \mathbb{H}$, the association

$$f \mapsto \Lambda_{\mathrm{KM}}(f) := \int_{\mathrm{Mp}_2(\mathbb{Z})\backslash\mathbb{H}} v^k \langle f(\tau), \Theta_L(\tau, z; \varphi_{\mathrm{KM}})\rangle \, \frac{\mathrm{d}u\,\mathrm{d}v}{v^2} \tag{7.8}$$

defines a map to the m^--forms $\mathscr{A}^{m^-}(\mathbb{D})$ and is referred to as *Kudla–Millson lift*.

The target form $\Lambda_{\mathrm{KM}}(f)$ is in general closed and inherits the invariance under the discriminant kernel subgroup $\Gamma(L)$ of L on the orthogonal side from the theta form Θ_{KM}. Consequently, it descends to a form on the quotient space $Y_L = \Gamma(L)\backslash\mathbb{D}$. In the hermitian case of $m^- = 2$, we obtain a map

$$\Lambda_{\mathrm{KM}} : \mathcal{S}_{L,1+m^+/2} \to \mathcal{H}^{(1,1)}(Y_L), \tag{7.9}$$

to the space of square integrable harmonic differential forms of Hodge type $(1,1)$ on the orthogonal modular variety associated to the discriminant kernel $\Gamma(L)$.

The applications of the Kudla–Millson theta correspondence range from the study of the cohomology of orthogonal and unitary Shimura varieties [KM86] and Arakelov theory of Shimura varieties [Kud04a] over specific counting problems [EGS23], constructing mock modular forms and higher dimensional error functions [FK17], to proving a converse theorem for Borcherds products [Bru02] [Bru14]. It is the latter application that we have in mind which is related to the injectivity of the Kudla–Millson lift. The question of its injectivity already arose in [KM90, p. 122] and may be used to compute the rational Picard number of the underlying Shimura variety [Ber+16], as well as to derive properties of cones generated by special cycles [BM19] [Zuf22]. Since the Kudla–Millson lift has been constructed, there have been a number of advances in prooving its injectivity and we would like to present these results. The first statement found in the literature, the author is aware of, is the following.

Theorem 7.7 ([Bru02, Thm. 5.12 p. 139])**.** *Let (L, q) be an even lattice of signature $(m^+, 2)$ such that it splits two hyperbolic planes $L \simeq K \oplus H \oplus H$. Then the Kudla–Millson lift $\Lambda_{\mathrm{KM}} : \mathcal{S}_{L,1+m^+/2} \to \mathcal{H}^{(1,1)}(Y_L)$ is injective.*

The proof is based upon the computation of the Fourier expansion of the target form, which reveals information about the Fourier coefficients of the initial form f and leads to the conclusion that it must vanish if its image does. The second hyperbolic split is required in order to guarantee that indices of Fourier coefficients are representable by the lattice in a certain fashion similar to Lemma 3.68.

There has been a second advance by the same author in collaboration with Jens Funke in [BF10]. In this instance, the authors develop a new strategy of proof, by utilising the doubling method to compute the L^2–norm of the Kudla–Millson lift by means of the Rallis inner product formula in order to conclude injectivity in the unimodular case for general signature. In addition, the authors consider a twist of the Kudla–Millson lift, denoted by $\Lambda_{\mathrm{KMF},l}$, which was introduced by Funke and Millson [FM12] with a parameter $l \in \mathbb{N}_0$ that allows for inputting forms of higher weight. The twisted lift $\Lambda_{\mathrm{KMF},l}$ maps to the space of $\widetilde{\mathrm{Sym}}^l(V)$ valued closed differential forms on Y_L. Here, $\widetilde{\mathrm{Sym}}^l(V)$ denotes the local system on $\mathbb{D}$ associated to the l-the symmetric power of V (cf. Definition 4.20). Details are provided in the primary source or in a compact fashion in [BF10] and we refer to the lift $\Lambda_{\mathrm{KMF},l}$ as the *Kudla–Millson–Funke* lift of degree $l \in \mathbb{N}_0$. The latter source also contains the following injectivity statement.

Theorem 7.8 ([BF10, Cor. 4.11 p. 37])**.** *Assume that $m > \max\{4, 3+r\}, m^+ > 1, m^- + l$ even, and that L is even unimodular.*
Then the theta lift $\Lambda_{\mathrm{KMF},l} : \mathcal{S}_{L,k+l} \to \mathscr{A}^{m^-}(Y_L, \widetilde{\mathrm{Sym}}^l(V))$ is injective.

This approach has recently been generalised by Stein to include the case of maximal lattices by solving intricate local integrals for bad primes in the same setting.

Theorem 7.9 ([Ste23, Cor. 7.8 p. 25])**.** *Let* $m > \max\{6, 2l-2, 3+r\}$ *and assume that* $m^- + l$ *as well as* $k = \frac{m}{2} + l$ *are even. If* L'/L *is assumed to be anisotropic, then* $\Lambda_{\mathrm{KMF},l} : \mathcal{S}_{L,k+l} \to \mathscr{A}^{m^-}(Y_L, \widetilde{\mathrm{Sym}}^l(V))$ *is injective.*

Nevertheless, the case with the most compelling application within this thesis remains the $\mathrm{O}(m^+, 2)$ setting since it relates to the Borcherds lift – a connection that is discussed in further detail below. In this case, the strongest result available in the literature is attributed to Bruinier and proven by refining his approach in [Bru02] through the development of a newform theory for vector valued modular forms in order to weaken the assumption of a second hyperbolic split.

Theorem 7.10 ([Bru14, Thm. 5.3 p. 331])**.** *Let* (L, q) *be an even lattice of signature* $(m^+, 2)$ *such that it splits a hyperbolic plane and a scaled hyperbolic plane* $L \simeq K \oplus H(N) \oplus H$ *for some* $N \in \mathbb{N}$*. Then the Kudla–Millson lift* $\Lambda_{\mathrm{KM}} : \mathcal{S}_{L,1+m^+/2} \to \mathcal{H}^{(1,1)}(Y_L)$ *is injective.*

It should also be noted that quite recently, Zuffetti and the author have extended this result to the case of general signature, also including the Funke–Millson twist.

Theorem 7.11 ([MZ23, Thm. 6.2 p. 24])**.** *Let* L *be an even indefinite lattice of signature* (m^+, m^-)*.*

i) If $L \simeq K \oplus H(N) \oplus H$ *for some even lattice* K *and some positive integer* N*, then the lift* $\Lambda_{\mathrm{KMF},l}$ *is injective.*

ii) Let $m^- = 1$*. If* $L \simeq M \oplus H$ *for some positive definite even lattice* M *and* $M \otimes \mathbb{Z}_p$ *splits off a hyperbolic plane for every prime* p*, then the lift* $\Lambda_{\mathrm{KMF},l}$ *is injective.*

iii) Let $m^+ = 1$*. Then the lift* $\Lambda_{\mathrm{KMF},l}$ *associated to* L *is identically zero.*

It is noteworthy that this result implies all of the above. The proof of Theorem 7.11 relies on the utilisation of Borcherds' method of expressing theta functions with respect to sublattices [Bor98, Sec. 5] and realising these as Poincaré series in order to apply the unfolding method. The application of this method yields a Fourier expansion of the lift in terms of the Fourier coefficients of the initial form. In this context, the first hyperbolic split is then employed to simplify this expansion allowing to extract vanishing results for the initial coefficients, while the additional split of a scaled hyperbolic plane is required to guarantee that the lattice represents a sufficient number of indices of Fourier coefficients of the initial form. In order to rely on the weaker assumption of a *scaled* hyperbolic split, the newform theory of [Bru14, Sec. 3] is invoked.

7.2.3 Linking lifts

An alternative approach to constructing the Kudla–Millson lift Λ_{KM} based on Borcherds' additive lift $\Phi(z, f)$, as presented in (7.2), has been described by Bruinier and Funke and will be outlined in the following paragraphs. In their celebrated paper [BF04], based upon this construction, the authors prove a duality statement between the Borcherds and Kudla–Millson lift. In this context, they formally introduce a new space of Maass wave forms that has been foreshadowed by [Bru02] and relate it to holomorphic modular forms via a differential operator.

We will briefly present the setting and highlight the aforementioned connection between the injectivity of the Kudla–Millson lift and the converse theorem for Borcherds products, as outlined in [Bru14, Sec. 4.1].

Definition 7.12. Let (L, q) be an even lattice. For $k \in \mathbb{Z}/2$ with $k \neq 1$ denote by $H_{L,k}$ the space of *weak Maass forms* of weight k with respect to the Weil representation ρ_L. This space consists of all real analytic functions $f : \mathbb{H} \to \mathbb{C}[L'/L]$, satisfying the following.

a) f transforms as a modular form: $f(\gamma\tau) = \phi^{2k}\rho_L(\gamma, \phi)f(\tau)$ for all $(\gamma, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$.

b) f grows at most exponentially towards the cusps.

c) f is annihilated by the hyperbolic Laplace operator of weight k given by

$$\Delta_k = -v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + ikv \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right).$$

Such a form has a Fourier expansion on the upper half plane, however, its coefficients are not invariant under imaginary shifts. It should also be noted that the hyperbolic Laplace operator may be expressed in terms of the *Maass raising* and *lowering* operator. To be explicit, the operators

$$R_k = 2i \frac{\partial}{\partial \tau} + kv^{-1}, \qquad L_k = -2iv^2 \frac{\partial}{\partial \overline{\tau}}$$

can be concatenated to construct $-\Delta_k = L_{k+2}R_k + k = R_{k-2}L_k$. Bruinier and Funke then craft an antilinear operator

$$\xi_k : H_{L,k} \to \mathcal{M}^!_{L^-,2-k}, \qquad f(\tau) \mapsto v^{k-2}\overline{L_k f(\tau)} = R_{-k} v^k \overline{f(\tau)} \tag{7.10}$$

with kernel $\mathcal{M}^!_{L,k}$. Recall that L^- denotes the lattice L with quadratic form rescaled by the factor -1. The crucial point is that the operator ξ_k is in fact surjective, resulting in the exact sequence

$$0 \to \mathcal{M}^!_{L,k} \to H_{L,k} \xrightarrow{\xi_k} \mathcal{M}^!_{L^-,2-k} \to 0. \tag{7.11}$$

If we restrict to the preimage of $\mathcal{S}_{L^-,2-k}$ which will be denoted by $H^+_{L,k}$, this yields the following exact sequence.

$$0 \to \mathcal{M}^!_{L,k} \to H^+_{L,k} \xrightarrow{\xi_k} \mathcal{S}_{L^-,2-k} \to 0.$$

In the following, fix $k = m/2$ and compare [BF04, Thm. 6.1 p. 77] to verify that the following equality is true for all $f \in H^+_{L^-,2-k}$:

$$\langle \Theta_L(\tau, z; \varphi_{\mathrm{KM}}), \xi_{2-k}(f)\rangle_{L,k} - a^+(0,0)\Omega = dd^c\Phi(z, f),$$

where Ω is the negative of a Kähler form considered in (5.23), $a^+(0,0)$ denotes the constant term of the Fourier expansion of f and the expression $\Phi(z, f)$ might require regularisation as in [Bru02]. Hence, for a cusp form $g \in \mathcal{S}_{L,k}$, select a weak harmonic Maass form $f \in H^+_{L^-,2-k}$ with vanishing constant Fourier coefficient such that $\xi_{2-k}(f) = g$, inducing a linear map

$$\Lambda(g, z) = dd^c\Phi(z, f) \tag{7.12}$$

that agrees with the Kudla–Millson lift on $\mathcal{S}_{L,k}$. Based on this construction a subspace of the kernel of the linear lift Λ_{KM} on $\mathcal{S}_{L,k}$ may be specified. Namely, let

$$N_{L^-,2-k} = \{f \in H_{L^-,2-k} \mid Z(f) = 0 \in \mathrm{Div}(Y_L) \otimes \mathbb{C}\},$$

where $Z(f)$ denotes the divisor of f on Y_L. Then by [Bru02, Thm. 4.23 p. 114], the space $\xi(N_{L^-,2-k})$ is contained in the kernel of Λ_{KM} and Bruinier defines $\mathcal{S}^+_{L,k}$ to be the orthogonal complement of that subspace in $\mathcal{S}_{L,k}$. Moreover, he outlines the characteristics of the spaces $N_{L^-,2-k}$ and $\mathcal{S}^+_{L,k}$, including their dependence on the lattice L and not merely on the discriminant form L'/L, that they are stable under $\mathrm{O}^+(L)$, and that $N_{L^-,2-k}$ vanishes, provided L splits a hyperbolic plane which implies in particular $\mathcal{S}^+_{L,k} = \mathcal{S}_{L,k}$. Further, the constant term in the Fourier expansion of an element in $N_{L^-,2-k}$ vanishes automatically.

The relationship (7.12) is then utilised to prove the following alternative characterisation of a converse theorem for Borcherds products. For that purpose, write Λ^+_{KM} for the restriction of Λ to $\mathcal{S}^+_{L,k}$.

Theorem 7.13 ([Bru14, Thm 4.2 p. 330]). *Suppose that $m^+ \geq 2$ and that m^+ is greater than the Witt rank of V. The following are equivalent:*

i) The map $\Lambda_{\mathrm{KM}}^+ : \mathcal{S}_{L,k}^+ \to \mathcal{H}^{(1,1)}(Y_L)$ is injective.

ii) Every meromorphic modular form F with respect to $\Gamma(L)$ whose divisor is a linear combination of special divisors as in (7.3) is (up to a nonzero constant factor) the Borcherds lift $\Psi(z, f)$ of a weakly holomorphic modular form $f \in \mathcal{M}^!_{L^-,2-k}$ with integral principal part.

The proof of the first implication is based on a weak converse theorem for Borcherds products of Bruinier, stating that there is a desired preimage $f \in H_{L^-,2-k}$ in the space of weak harmonic Maass forms such that $\Lambda_{\mathrm{KM}}(\xi(f)) = dd^c\Phi(z, f) = 0$. For the sake of convenience, assume that the lattice L splits a hyperbolic plane. Then, if Λ_{KM}^+ is injective, we must have $\xi(f) = 0$, i.e. f is in the kernel of the operator ξ. This kernel, however, agrees with the space $\mathcal{M}^!_{L^-,2-k}$ by (7.11), meaning f was, in fact, a weakly holomorphic modular form. In conclusion, a converse theorem for Borcherds products is derived as demanded in [Bor95, Problem 10] and [Bor98, Problem 16.10].

7.3 Injectivity of a theta lift

Let (L, q) be an even $\mathbb{Z}$ lattice of signature $(m^+, 2)$ and rank m and set $k := m/2$. In this section we will prove the injectivity of the Kudla–Millson lift introduced in Definition 7.6:

$$\Lambda_{\mathrm{KM}} : \mathcal{S}_{L,k} \to \mathcal{H}^{(1,1)}(Y_L).$$

First, note that the lift Λ_{KM} is linear, hence it suffices to prove that its kernel is trivial. As described in the introduction, the idea is for $f \in \mathcal{S}_{L,k}$ to integrate the form $\Lambda_{\mathrm{KM}}(f)$ over special cycles of Y_L. If it is assumed that f is annihilated by the Kudla–Millson lift, then these cycle integrals must vanish. The nature of the special cycles will allow us to extract information about the Fourier coefficients of the initial form f. Subsequently, we shall conclude that they vanish without exception, i.e. $f = 0$, proving the injectivity.
In the following, abbreviate $\mathcal{L} = L'/L$ and denote the Fourier expansion of f by

$$f(\tau) = \sum_{\lambda \in \mathcal{L}} \sum_{n \in \overline{\mathrm{q}}(\lambda) + \mathbb{Z}} a(\lambda, n) \cdot e(n\tau)\mathfrak{e}_\lambda. \tag{7.13}$$

7.3.1 Cycle integrals of Kudla–Millson lifts

The first major step is to prove that integrals of a target form $\Lambda_{\mathrm{KM}}(f)$ of the Kudla–Millson lift over certain divisors reduce to special L-values of symmetric square type L-functions of f. These divisors have been presented in Subsection 4.2.5 in (4.33) and we briefly recall their shape. Let $\ell \in L$ have positive norm, $\mathbb{D}$ denote the Grassmannian of L and $\Gamma(L)$ denote the discriminant kernel in $\mathrm{O}(L)$. If $\Gamma(L)_\ell$ denotes the stabiliser of ℓ in $\Gamma(L)$, then

$$\mathcal{Z}(\ell) := \Gamma(L)_\ell \backslash \mathbb{D}_\ell \to \Gamma(L) \backslash \mathbb{D} = Y_L$$

defines a (in general relative) cycle of Y_L, where we have set $\mathbb{D}_\ell = \{x \in \mathbb{D} \mid x \subseteq \ell^\perp\}$. With the application of Kudla's geometric example [Kud03, Sec. 4.3 p. 328] in mind, consider

$$\int_{\mathcal{Z}(\ell)} \int_{\mathcal{F}} \langle f(\tau), \Theta_L(\tau, z; \varphi_{\mathrm{KM}}) \rangle v^k \frac{\mathrm{d}u \, \mathrm{d}v}{v^2} \wedge \Omega^{m^+-2}, \tag{7.14}$$

where Ω is the negative of a Kähler form on Y_L as in Definition 5.23 and $\mathcal{F} = \mathrm{Mp}_2(\mathbb{Z}) \backslash \mathbb{H}$. In order to apply Kudla's observation [Kud03, Prop. 4.17 p. 332], several reduction steps are performed. To this end, recall that $L_\ell := L \cap \ell^\perp$ is a lattice of signature $(m^+ - 1, 2)$, such that $M := \mathbb{Z}\ell \oplus L_\ell \leq L$ defines a sublattice of finite index. If $\mathrm{O}(V)_\ell$ denotes the stabiliser of ℓ, we find

$$\Gamma(L_\ell) \simeq \Gamma(M) \cap \mathrm{O}(V)_\ell \leq \Gamma(L) \cap \mathrm{O}(V)_\ell = \Gamma(L)_\ell.$$

Here, we have used that the discriminant kernel is inclusion preserving (cf. Remark 4.50). Further, an application of Lemma A.8 with the choices $G = \Gamma(L), H = \Gamma(M)$, and $K = \Gamma(L)_\ell$ necessitates $[\Gamma(L)_\ell : \Gamma(L_\ell)] \leq [\Gamma(L) : \Gamma(M)]$. The latter index is certainly finite, and so is the multiplicity $C(\ell)$ of the covering

$$\mathcal{Y}(\ell) := \Gamma(L_\ell) \backslash \mathbb{D}_\ell \to \Gamma(L)_\ell \backslash \mathbb{D} = \mathcal{Z}(\ell).$$

Since the integrand in (7.14) is invariant under pullbacks from $\Gamma(L)$ (cf. Definition 5.23 and (5.23)), we may rewrite the integral as an integral over $\mathcal{Y}(\ell)$, provided we adjust for the prefactor $C(\ell)^{-1}$.

We will begin the computation in the following and note that convergence will be verified backwards at a later stage. The first aim is to separate part of the integrand that belongs to the cycle $\mathcal{Z}(\ell)$ in order to apply the Siegel–Weil formula.

In Subsection 3.3.2 a lift $\uparrow_L^M\colon \mathcal{S}_{L,k} \to \mathcal{S}_{M,k}$ has been described that behaves well with respect to the respective scalar products as well as theta functions. In fact, by Proposition 3.36 the operator $\uparrow_L^M$ fulfils the following relation:

$$\langle f(\tau), \Theta_L(\tau, z; \varphi_{\mathrm{KM}})\rangle = \langle \uparrow_L^M f(\tau), \Theta_M(\tau, z; \varphi_{\mathrm{KM}})\rangle.$$

Note that $\Theta_M(\tau, z; \varphi_{\mathrm{KM}})$ fulfils the conditions of Remark 3.34 on $\mathbb{D}_\ell$ (cf. [Fun02, Thm. 2.1 p. 294]). As a consequence, this theta function splits as a tensor product and by abbreviating $\tilde{f} :=\uparrow_L^M f$ and writing $\varphi_{\mathrm{KM},L_\ell}$ for the Kudla–Millson Schwartz form on $L_\ell \otimes \mathbb{R}$, we obtain the following identity on $\mathbb{D}_\ell$:

$$\langle f(\tau), \Theta_L(\tau, z; \varphi_{\mathrm{KM}})\rangle = \langle \tilde{f}(\tau), \Theta_{\mathbb{Z}\ell}(\tau, z; \varphi_{0,\ell}) \otimes \Theta_{L_\ell}(\tau, z; \varphi_{\mathrm{KM},L_\ell})\rangle,$$

where $\varphi_{0,\ell}(x) = \exp(-2\pi\, \mathrm{q}(x))$ is the standard Gaussian on the space $\mathbb{R}\ell$. Now, if we swap the order of integration in (7.14) under the assumption of absolute convergence, we may integrate over only a part of the scalar product. For that purpose, write $\lambda_1 \in (\mathbb{Z}\ell)'/\mathbb{Z}\ell$ as well as $\lambda_2 \in (L_\ell)'/L_\ell$ and find in the notation of Definition 3.31 that

$$\begin{aligned}&\int_{\mathcal{Y}(\ell)} \int_{\mathcal{F}} \langle f(\tau), \Theta_L(\tau, z; \varphi_{\mathrm{KM}})\rangle v^k \frac{\mathrm{d}u\, \mathrm{d}v}{v^2} \wedge \Omega^{m^+-2} \\ = &\int_{\mathcal{F}} \sum_{\lambda_1,\lambda_2} \tilde{f}_{\lambda_1\oplus\lambda_2}(\tau) \cdot \overline{\theta_{\mathbb{Z}\ell,\lambda_1}(\tau; \varphi_{0,\ell})} \int_{\mathcal{Y}(\ell)} \overline{\theta_{L_\ell,\lambda_2}(\tau, z; \varphi_{\mathrm{KM},L_\ell})} \wedge \Omega^{m^+-2} v^k \frac{\mathrm{d}u\, \mathrm{d}v}{v^2}.\end{aligned}$$

Recall that $\Theta_{\mathbb{Z}\ell}$ has no z-dependence, since the associated Grassmannian is trivial. The inner integral over $\mathcal{Y}(\ell)$, however, has already been explicitly related to a standard theta integral by means of Corollary 5.26 which in turn equals a special value of an Eisenstein series by the application of the Siegel–Weil formula (cf. Theorem 5.21).[2] In fact, we obtain under the assumption that $m^+ - 1$ is bigger than the Witt index of $V_\ell = L_\ell \otimes \mathbb{Q}$ that

$$\begin{aligned}&\int_{Z(\ell)} \int_{\mathcal{F}} \langle f(\tau), \Theta_L(\tau, z; \varphi_{\mathrm{KM}})\rangle v^k \frac{\mathrm{d}u\, \mathrm{d}v}{v^2} \wedge \Omega^{m^+-2} \\ = C_L(\ell) \cdot &\int_{\mathcal{F}} \sum_{\lambda_1,\lambda_2} \tilde{f}_{\lambda_1\oplus\lambda_2} \overline{\theta_{\mathbb{Z}\ell,\lambda_1}(\tau; \varphi_{0,\ell}) \mathcal{E}(\tau, s_0; \tilde{\varphi}_{\mathrm{KM},L_\ell} \otimes \varphi_{\lambda_2})} v^k \frac{\mathrm{d}u\, \mathrm{d}v}{v^2}\end{aligned} \tag{7.15}$$

for a nonzero explicit constant[3] $C_L(\ell)$. Here, the Eisenstein series is to be understood

[2] Compare Remark 5.7 as well as Definition 5.16 for translating Theorem 5.21 from functions on the group to functions in $\tau \in \mathbb{H}$. A factor of $\mathrm{Im}(\tau)^{-k/2}$ appears in the process on both sides.

[3] In fact, we find that $C_L(\ell)$ is given by the product of the prefactors on the right hand side of (5.31) in

adellically and has been presented in Definition 5.20. Further, it is holomorphic at the critical point $s_0 = k - 3/2$ while φ_{λ_2} denotes the finite Schwartz form represented by the coset of λ_2 (cf. Example 5.13 part d)). By (5.28), the intertwining operator maps $\tilde{\varphi}_{\mathrm{KM},L_\ell}$ to the weight $k - 1/2$ section $\Phi_\infty^{k-1/2}$ through the principal series representation. Hence, collecting terms for λ_1, λ_2 and utilising Definition 5.17, the inner sum in (7.15) may again be written as a scalar product by using the identification of $\mathbb{C}[\mathcal{L}]$ with $\mathcal{S}_\mathcal{L}$ presented in Lemma 3.9:

$$\sum_{\lambda_1,\lambda_2} \tilde{f}_{\lambda_1\oplus\lambda_2}\overline{\theta_{\mathbb{Z}\ell,\lambda_1}(\tau;\varphi_{0,\ell})\mathcal{E}(\tau,s_0;\tilde{\varphi}_{\mathrm{KM},L_\ell}\otimes\varphi_{\lambda_2})} = \langle \tilde{f}, \Theta_{\mathbb{Z}\ell}(\tau;\varphi_{0,\ell})\otimes\mathcal{E}_{\hat{L}_\ell,k-1/2}(\tau,s_0)\rangle.$$

Here, $\mathcal{E}_{\hat{L}_\ell,k-1/2}$ is an adelic version of the vector valued Eisenstein series (cf. Definition 5.17) which has been rewritten in classical terms in Proposition 5.18. In fact, (5.21) reads for $l \in \mathbb{Z}/2$, representing the weight,

$$\mathcal{E}_{\hat{L}_\ell,l}(\tau,s_0) = E_{L_\ell,0,l}\left(\tau,\tfrac{s_0+1-l}{2}\right), \tag{7.16}$$

where $E_{L_\ell,0,l}$ is described in Definition 3.37. As a consequence, selecting $l = k - 1/2$ yields a new expression for the initial integral in classical terms:

$$\begin{aligned}&\int_{\mathcal{Z}(\ell)}\int_{\mathcal{F}}\langle f(\tau),\Theta_L(\tau,z;\varphi_{\mathrm{KM}})\rangle v^k\frac{\mathrm{d}u\,\mathrm{d}v}{v^2}\wedge\Omega^{m^+-2}\\ &= C_L(\ell)\cdot\int_{\mathcal{F}}\langle\tilde{f}(\tau),\Theta_{\mathbb{Z}\ell}(\tau;\varphi_{0,\ell})\otimes E_{L_\ell,0,k-1/2}(\tau,0)\rangle v^k\frac{\mathrm{d}u\,\mathrm{d}v}{v^2}.\end{aligned} \tag{7.17}$$

Recall that by Example 3.52 part d) the integral in (7.17) converges absolutely as long as the Eisenstein series $E_{L_\ell,0,k-1/2}$ converges. This is definitely the case if $k > 5/2$ is assumed (cf. Lemma 3.38) which is equivalent to $m^+ > 3$, justifying all steps so far. Further, the expression in (7.17) is a Petersson scalar product type integral containing a parabolic Poincaré series as in Proposition 6.95. As such, it qualifies for unfolding, provided the resulting integral converges absolutely. By Proposition 6.95, it does so for $0 > 3 - k/2$, which is true in our case as long as $m^+ > 10$. This bound may be marginally improved, however, a bound below 7 appears to be unfeasible. Clearly, it is possible to continue the desired computation under the assumption $m^+ > 10$. Nevertheless, regularising the integral appears to be more attractive.

For that purpose, recall that by Definition 3.37 the series $E_{L_\ell,0,k-1/2}(\tau,0)$ is a special value

Corollary 5.26 with $C(\ell)^{-1}$.

of a series $E_{L_\ell,0,k-1/2}(\tau,s)$ depending on some parameter $s\in\mathbb{C}$ which may be proven to be normally convergent if s satisfies $\mathrm{Re}(s)>\frac{5}{4}-k/2=\frac{5-m}{4}$ (cf. Lemma 3.38). This implies the condition $m^+\geq 3$ to guarantee holomorphicity to the right of $s=0$. Now, instead of considering the Eisenstein series $E_{L_\ell,0,k-1/2}$ for $s=0$, we may investigate the integral in (7.17) for general arguments $s\in\mathbb{C}$ (also compare the analytic continuation of $E_{L_\ell,0,k-1/2}$ in s by Remark 3.39). Then, unfolding is possible for choices $s\in\mathbb{C}$ with sufficiently large real part as a byproduct of Proposition 6.95. This has been spelled out in Corollary 6.96 stating explicitly that for all s with sufficiently large real part we find

$$\int_{\mathcal{F}}\left\langle\tilde{f},\Theta_{\mathbb{Z}\ell}\otimes E_{L_\ell,0,k-1/2}(\,\cdot\,,s)\right\rangle\cdot\mathrm{Im}^k\,\frac{\mathrm{d}u\,\mathrm{d}v}{v^2}=\int_{\mathcal{F}_{\Gamma_\infty}}\left\langle\tilde{f},\Theta_{\mathbb{Z}\ell}\otimes\mathfrak{e}_0\right\rangle\cdot\mathrm{Im}^{\overline{s}+k}\,\frac{\mathrm{d}u\,\mathrm{d}v}{v^2}.$$

Here, $\mathcal{F}_{\Gamma_\infty}$ denotes a suitable representation of $\Gamma_\infty\backslash\mathbb{H}$. We may then proceed to manipulate the integral as it is carried out directly below Corollary 6.96 to obtain a representation as an L-series. This will yield the special case of Proposition 6.97 that is

$$\int_{\mathcal{F}}\left\langle\tilde{f},\Theta_{\mathbb{Z}\ell}\otimes E_{L_\ell,0,k-1/2}(\,\cdot\,,s)\right\rangle\cdot\mathrm{Im}^k\,\frac{\mathrm{d}u\,\mathrm{d}v}{v^2}=\frac{\Gamma(\overline{s}+k-1)}{(4\pi)^{\overline{s}+k-1}}\cdot\sum_{0\neq l\in(\mathbb{Z}\ell)'}\frac{\tilde{a}(\bar{l},\mathrm{q}(l))}{\mathrm{q}(l)^{\overline{s}+k-1}}.\quad(7.18)$$

Here, $\tilde{a}(\lambda,n)$ denote the Fourier coefficients of $\tilde{f}$ as in (7.13). The series on the right hand side identifies with $L_{\mathbb{Z}\ell,0,1}(\tilde{f},\overline{s}+k-1)$ given in Definition 6.73. For that naive definition, the lattice $\mathbb{Z}\ell$ was assumed to split from the initial lattice. However, the succeeding more thorough Definition 6.77 also allowed for considering splitting sublattices and Remark 6.78 concluded that

$$\begin{aligned}\sum_{0\neq l\in(\mathbb{Z}\ell)'}\frac{\tilde{a}(\bar{l},\mathrm{q}(l))}{\mathrm{q}(l)^{\overline{s}+k-1}}&=L_{\mathbb{Z}\ell,0,1}(\tilde{f},\overline{s}+k-1)\\&=L_{\mathbb{Z}\ell,0,1}(f,\overline{s}+k-1)\\&=\sum_{0\neq l\in(\mathbb{Z}\ell)'}\frac{a(\bar{l},\mathrm{q}(l))}{\mathrm{q}(l)^{\overline{s}+k-1}}.\end{aligned}$$

To ease the notation before continuing, we may select a primitive element $\ell_0\in(\mathbb{Z}\ell)'\cap L'$ and write

$$L_{\mathbb{Z}\ell,0,1}(f,\overline{s}+k-1)=\sum_{0\neq l\in(\mathbb{Z}\ell)'}\frac{a(\bar{l},\mathrm{q}(l))}{\mathrm{q}(l)^{\overline{s}+k-1}}=\sum_{0\neq n\in\mathbb{Z}}\frac{a(\overline{n\ell_0},n^2\,\mathrm{q}(\ell_0))}{\mathrm{q}(n\ell_0)^{\overline{s}+k-1}}.$$

Notice that the Fourier coefficients of cusp forms have a symmetry property enforced by the action of the generator $\overline{Z} \in \mathrm{Mp}_2(\mathbb{Z})$ of the centre (cf. Remark 6.74), yielding that the series vanishes unless $2k \equiv \mathrm{sig}(L) \mod 4$. In our case $2k = m^+ + 2 \equiv m^+ - 2 \mod 4$. The right side is, in fact, the signature, so that the negative and positive indices of the series contribute the same terms, yielding

$$\begin{aligned} L_{\mathbb{Z}\ell,0,1}(f, \overline{s} + k - 1) &= \sum_{0 \neq n \in \mathbb{Z}} \frac{a(n\overline{\ell_0}, n^2\, \mathrm{q}(\ell_0))}{\mathrm{q}(n\ell_0)^{\overline{s}+k-1}} \\ &= \frac{2}{\mathrm{q}(\ell_0)^{\overline{s}+k-1}} \cdot \sum_{n \in \mathbb{N}} \frac{a(n\overline{\ell_0}, n^2\, \mathrm{q}(\ell_0))}{n^{2(\overline{s}+k-1)}} \\ &= \frac{2}{\mathrm{q}(\ell_0)^{\overline{s}+k-1}} \cdot L_{\ell_0}(f, 2(\overline{s} + k - 1)), \end{aligned}$$

where we use the notation of Definition 6.79 in the last line. By utilising the asymptotic growth of the Fourier coefficients $a(\lambda, n)$, absolute convergence of $L_{\ell_0}(f, 2(\overline{s} + k - 1))$ may be proven in a right half plane. This is carried out in Lemma 6.80, proving that absolute convergence is guaranteed for $2 \cdot \mathrm{Re}(s) > 3 - k - 2\sigma$ with some positive σ depending on the asymptotic behaviour of the progression of Fourier coefficients. Concrete choices of σ have been derived in Corollary 3.87 based on bounds by [Ran39b], [BH08], and [Wai18] and collected in Table 3.1. The currently relevant cases are also contained in Lemma 6.80 and read

$$\sigma = \begin{cases} 1/2, & 2 \mid \mathrm{sig}(L), \\ 1/4, & 2 \nmid \mathrm{sig}(L). \end{cases}$$

If we were to insert the critical point $s = 0$, the conditions read

$$m^+ > 4(1 - \sigma) = \begin{cases} 2, & 2 \mid \mathrm{sig}(L), \\ 3, & 2 \nmid \mathrm{sig}(L). \end{cases} \tag{7.19}$$

Assuming (7.19), we terminate the manipulation of the right hand side of (7.18) and apply our findings to (7.17) in order to infer the following Theorem.

Theorem 7.14. *Let (L, q) be an even lattice of signature $(m^+, 2)$, set $k := 1 + \frac{m^+}{2}$, let $f \in \mathcal{S}_{L,k}$ and select some $\ell \in L$ of positive norm. If $m^+ > 3$, then*

$$\int_{\mathcal{Z}(\ell)} \Lambda_{\mathrm{KM}}(f) \wedge \Omega^{m^+-2} = 2C_L(\ell) \frac{\Gamma(k-1)}{(4\pi\, \mathrm{q}(\ell_0))^{k-1}} \cdot L_{\ell_0}(f, m^+), \tag{7.20}$$

where Λ_{KM} is the Kudla–Millson lift, Ω the negative of a Kähler form presented in

Definition 5.23, $\mathcal{Z}(\ell)$ is the special divisor in (4.33), $C_L(\ell)$ is a nonzero explicit constant, ℓ_0 is some primitive element in $\mathbb{Q}\ell \cap L'$, and $L_{\ell_0}(f, s)$ denotes the symmetric square type L-series from Definition 6.79.

As a consequence, we conclude that a family of special L-values vanishes, if the associated cusp form is annihilated by the Kudla–Millson lift.

Corollary 7.15. *In the notation of Theorem 7.14 we find that for any $f \in \ker(\Lambda_{\mathrm{KM}})$ and all primitive $\ell_0 \in L'$ of positive norm the following L-value vanishes*

$$L_{\ell_0}(f, m^+) \overset{\text{def}}{=} \sum_{n \in \mathbb{N}} \frac{a(n\overline{\ell_0}, n^2\, \mathrm{q}(\ell_0))}{n^{m^+}} = 0. \tag{7.21}$$

The corollary above will be employed to derive an injectivity result.

7.3.2 Injectivity for a hyperbolic split

The aim of this subsection is to prove the following theorem.

Theorem 7.16. *Let (L, q) be an even lattice of signature $(m^+, 2)$ with $m^+ > 3$. Assume that L splits a hyperbolic plane. Then the Kudla–Millson lift $\Lambda_{\mathrm{KM}} : \mathcal{S}_{L,1+m^+/2} \to \mathcal{H}^{(1,1)}(Y_L)$ is injective.*

We begin by recalling that the Kudla–Millson lift associates to an elliptic cusp form $f \in \mathcal{S}_{L,k}$ a form

$$\Lambda_{\mathrm{KM}}(f) = \int_{\mathrm{Mp}_2(\mathbb{Z})\backslash\mathbb{H}} \langle f(\tau), \Theta(\tau, z; \varphi_{\mathrm{KM}})\rangle \frac{\mathrm{d}u\, \mathrm{d}v}{v^2} \in \mathcal{H}^{(1,1)}(Y_L).$$

Before continuing, we will sketch the idea of the proof of Theorem 7.16. By Corollary 7.15, the vanishing of $\Lambda_{\mathrm{KM}}(f)$ for some $f \in \mathcal{S}_{L,k}$ implies the vanishing of special L-values associated to primitive positive $\ell_0 \in L'$. Briefly recall that primitivity derived from having to sum over all coefficients whose indices are generated by a rational line in V. Under the hypothesis of a hyperbolic split of L, we may eliminate the primitivity assumption on ℓ_0. With that advantage, the series $L_{\ell_0}(f, m^+)$ may be dissected into vanishing subseries, guaranteeing the vanishing of all of its coefficients.

There are two challenges to be addressed in order to prove the vanishing of the initial form f based on the vanishing of the special L-values in (7.21) which contain Fourier coefficients of f. The first is, that the only Fourier coefficients $a(\lambda, n)$ of f ever appearing with the chosen method are of the form $a(\overline{\ell}, \mathrm{q}(\ell))$ for some $\ell \in L'$ of positive norm; the second is that the relations involve infinitely many of these coefficients – the system of equations does not appear to be determined. While the second issue may be addressed by Hecke theory, the first is solely a property of the lattice L.

By assuming that L splits off a hyperbolic plane, we may eliminate the first problem. In fact, in this case, Lemma 3.68 states that any possible index (λ, n) of a Fourier coefficient of $f \in \mathcal{S}_{L,k}$ may be represented by some primitive (!) vector $\ell_0 \in L'$ in the sense that $(\lambda, n) = (\overline{\ell_0}, \mathrm{q}(\ell_0))$.[4] However, the L-series $L_\ell(f, s)$ depends solely on the pair $(\overline{\ell}, \mathrm{q}(\ell))$, so that in conjunction with Corollary 7.15 we immediately conclude the following assertion.

Corollary 7.17. *Assume (L, q) is an even $\mathbb{Z}$ lattice of signature $(m^+, 2)$ that splits a hyperbolic plane and set $k := 1 + m^+/2$. If $m^+ > 3$ and $f \in \mathcal{S}_{L,k}$ is annihilated by Λ_{KM}, then for any $\ell \in L'$ of positive norm the following special L-value vanishes:*

$$L_\ell(f, m^+) \stackrel{\text{def}}{=} \sum_{n \in \mathbb{N}} \frac{a(n\overline{\ell}, \mathrm{q}(n\ell))}{n^{m^+}} = 0.$$

As we will see, this also eliminates the second problem in proving the main theorem without resorting to Hecke theory by utilising an inclusion–exclusion trick.

Lemma 7.18. *Assume L splits a hyperbolic plane and $f \in \mathcal{S}_{L,k}$ for $m^+ > 3$ is annihilated by Λ_{KM}. Let $N \in \mathbb{N}$ be a natural number and $\ell \in L'$ be of positive norm. Then*

$$L_\ell^N(f, m^+) \stackrel{\text{def}}{=} \sum_{\substack{n \in \mathbb{N} \\ \gcd(n,N)=1}} \frac{a(n\overline{\ell}, \mathrm{q}(n\ell))}{n^{m^+}} = 0.$$

Proof: Note that it suffices to consider square free N. We will prove the assertion by induction on the number of distinct prime divisors of N. The case $N = 1$ is identical to Corollary 7.17. Assume that the statement is true for some $N \in \mathbb{N}$. Select a prime number $p \nmid N$ and note that by the assumption $m^+ > 3$ absolute convergence guarantees

$$\begin{aligned}\sum_{\substack{n \in \mathbb{N} \\ \gcd(n,pN)=1}} \frac{a(n\overline{\ell}, \mathrm{q}(n\ell))}{n^{m^+}} &= \sum_{\substack{n \in \mathbb{N} \\ \gcd(n,N)=1}} \frac{a(n\overline{\ell}, \mathrm{q}(n\ell))}{n^{m^+}} - \frac{1}{p^{m^+}} \sum_{\substack{n \in \mathbb{N} \\ \gcd(n,N)=1}} \frac{a(np\overline{\ell}, \mathrm{q}(np\ell))}{n^{m^+}} \\ &= L_\ell^N(f, m^+) - \frac{1}{p^{m^+}} L_{p\ell}^N(f, m^+).\end{aligned}$$

[4] Note that the proof ot the Lemma in question may be understood without learning about the notation of the respective subsection.

Now the right hand side vanishes by our induction hypothesis, completing the proof. □

The lemma above is the last ingredient for the proof of the main theorem.

Proof of Theorem 7.16: Assume there was a non-trivial form $f \in \ker(\Lambda_{\mathrm{KM}})$ with Fourier expansion

$$f(\tau) = \sum_{\lambda\in\mathcal{L}} \sum_{n\in\overline{\mathrm{q}}(\lambda)+\mathbb{Z}} a(\lambda, n) \cdot e(n\tau)\mathfrak{e}_\lambda.$$

Fix a pair (λ, n) for which $a(\lambda, n) \neq 0$. By Lemma 3.68, the hyperbolic split guarantees the existence of some $\ell \in L'$ of positive norm with $(\bar{\ell}, \mathrm{q}(\ell)) = (\lambda, n)$. Further, the assumption of $m^+ > 3$ implies that $L_\ell(f, m^+) = \sum_{d\in\mathbb{N}} a(d\bar{\ell}, d^2\,\mathrm{q}(\ell))/d^{m^+}$ converges absolutely. Hence, there must be some number $M \in \mathbb{N}$ such that

$$\sum_{\substack{d\in\mathbb{N}\\ d>M}}^{\infty} \frac{|a(d\bar{\ell}, d^2\,\mathrm{q}(\ell))|}{d^{m^+}} < |a(\bar{\ell}, \mathrm{q}(\ell))|.$$

Consequently, selecting $N := M!$ results in

$$L_\ell^N(f, m^+) = a(\bar{\ell}, \mathrm{q}(\ell)) + \sum_{\substack{1<d\in\mathbb{N}\\ \gcd(d,N)=1}} \frac{a(d\bar{\ell}, d^2\,\mathrm{q}(\ell))}{d^{m^+}} \neq 0. \tag{7.22}$$

However, f was assumed to be annihilated by Λ_{KM}, implying by Lemma 7.18 that the left hand side of (7.22) vanishes – a contradiction! As a consequence, the assumption of a non-trivial element in the kernel of Λ_{KM} must be incorrect i.e. the Kudla–Millson lift is injective. □

An application of Theorem 7.13 yields a converse theorem for Borcherds products (also compare Theorem 7.25).

Theorem 7.19. *Assume that $L \simeq K \oplus H$ for some lattice K of signature $(m^+ - 1, 1)$ with $m^+ > 3$. Then every meromorphic modular form F for $\Gamma(L)$ whose divisor is a linear combination of special divisors $Z(\mu, n)$ as in (4.31) is (up to a nonzero constant factor) the Borcherds lift $\Psi(z, f)$ of some weakly holomorphic modular form $f \in \mathcal{M}^!_{L^-,1-m^+/2}$.*

Remark 7.20. We note that the essential assumption of a hyperbolic split in Theorem 7.16 may not be omitted. This has been shown by [Bru14, Sec. 6.1 p. 333]. Also, the theorem cannot be proven for $m^+ = 1$ – cf. Theorem 7.11 iii). The case of $m^+ = 3$ is discussed in the following, while the case of $m^+ = 2$ remains out of reach for a single hyperbolic split with the current method.

The case of $m^+ = 3$

The case of $m^+ = 3$, which was previously excluded for the sake of convenience, may still be treated by the same procedure. A careful analysis of the proof reveals that the majority of arguments remains intact if vanishing of the L-value $L_\ell(f, m^+)$ is replaced by convergence $L_\ell(f, s) \to 0$ for $s \to m^+$. This approach is universally applicable for Witt rank 1 of the lattice L which is precisely the case that remains unproven. The crucial additional element for such an advancement is Theorem 3.82, guaranteeing that the subseries $L_\ell^N(f, m^+)$ in fact converges absolutely, provided $\mathrm{lev}(L) \mid N$.
Indeed, the sole issue with the case $m^+ = 3$ in the proof of Theorem 7.16 is convergence. More explicitly, in order to apply the Siegel–Weil formula to insert the Eisenstein series $\mathcal{E}$ in (7.15) and obtain holomorphicity at $s_0 = k - 1/2$, Weil's convergence criterion is applied. This criterion fails, unless $L \cap \ell^\perp$ has Witt rank at most 1 which is guaranteed if the initial lattice L is assumed to have Witt rank 1. Continuing the proof under this assumption, yields that (7.18) equals

$$2\frac{\Gamma(\overline{s} + k - 1)}{(\mathrm{q}(\ell_0)4\pi)^{\overline{s}+k-1}} \cdot L_{\ell_0}(f, 2(\overline{s} + k - 1))$$

for all $s \in \mathbb{C}$ in a right half plane, where the L-series converges absolutely and is holomorphic in $\overline{s}$. This is the case for $\mathrm{Re}(s) > 0$. However, the left hand side of (7.18) is holomorphic in $\overline{s}$ at $s = m^+$ by Weil's convergence criterion (cf. Theorem 5.21 and Corollary 3.54). Hence, the right hand side of (7.18) can be holomorphically continued to $s = m^+$. As a consequence, we obtain the following version of Corollary 7.15.

Corollary 7.21. *Let (L, q) be an even lattice of signature $(3, 2)$, set $k := \frac{5}{2}$, let $f \in \mathcal{S}_{L,k}$ and select some $\ell \in L$ of positive norm such that $L \cap \ell^\perp$ has Witt rank at most 1. Then*

$$\int_{\mathcal{Z}(\ell)} \Lambda_{\mathrm{KM}}(f) \wedge \Omega^1 = 2C_L(\ell)\frac{\Gamma(k-1)}{(4\pi\,\mathrm{q}(\ell_0))^{k-1}} \cdot \lim_{\substack{s \to m^+ \\ \mathrm{Re}(s) > m^+}} L_{\ell_0}(f, s), \tag{7.23}$$

for some primitive element $\ell_0 \in \mathbb{Q}\ell \cap L'$. For the rest of the notation, compare Theorem 7.14.

Corollary 7.22. *In the notation of Corollary 7.21, assume $f \in \ker(\Lambda_{\mathrm{KM}})$. Then*

$$\lim_{\substack{s \to m^+ \\ \mathrm{Re}(s) > m^+}} L_{\ell_0}(f, s) \overset{\mathrm{def}}{=} \lim_{\substack{s \to m^+ \\ \mathrm{Re}(s) > m^+}} \sum_{n \in \mathbb{N}} \frac{a(n\overline{\ell_0}, n^2\,\mathrm{q}(\ell_0))}{n^s} = 0, \tag{7.24}$$

where $a(\lambda, n)$ denote the Fourier coefficients of f as in (7.13).

The same line of reasoning that was used to prove Corollary 7.17 and Lemma 7.18 can be applied to prove their respective analogues which are subsumed in the following Corollary.

Corollary 7.23. *Assume the lattice L of signature $(3,2)$ has Witt rank 1, splits a hyperbolic plane and that some $f \in \mathcal{S}_{L,k}$ is annihilated by Λ_{KM}. Let $N \in \mathbb{N}$ be a natural number and $\ell \in L'$ be of positive norm. Then*

$$\lim_{\substack{s \to m^+ \\ \mathrm{Re}(s) > m^+}} L_\ell(f,s) = \lim_{\substack{s \to m^+ \\ \mathrm{Re}(s) > m^+}} L_\ell^N(f,s) = 0.$$

Proposition 7.24. *Let (L, q) be an even lattice of Witt rank 1 and signature $(3,2)$ such that L splits a hyperbolic plane. Then the lift $\Lambda_{\mathrm{KM}} : \mathcal{S}_{L,5/2} \to \mathcal{H}^{(1,1)}(Y_L)$ is injective.*

Proof: The proof is almost identical to the proof of Theorem 7.16, only differing by convergence issues at the critical point $s = 3$ of the L-series involved.
Assume there was a non-trivial form $f \in \ker(\Lambda_{\mathrm{KM}})$ with Fourier expansion

$$f(\tau) = \sum_{\lambda \in \mathcal{L}} \sum_{n \in \overline{\mathrm{q}}(\lambda) + \mathbb{Z}} a(\lambda, n) \cdot e(n\tau)\mathfrak{e}_\lambda.$$

Fix a pair (λ, n) for which $a(\lambda, n) \neq 0$. By Lemma 3.68, the hyperbolic split guarantees the existence of some $\ell \in L'$ of positive norm with $(\bar{\ell}, \mathrm{q}(\ell)) = (\lambda, n)$. Further, $m^+ = 3$ is chosen such that $L_\ell(f,s) = \sum_{d \in \mathbb{N}} a(d\bar{\ell}, d^2\, \mathrm{q}(\ell))/d^s$ converges absolutely to the right of $s = m^+$. However, for $N \in \mathbb{N}$ with $\mathrm{lev}(L) \mid N$, the series

$$L_\ell^N(f,s) = \sum_{\substack{d \in \mathbb{N} \\ \gcd(d,N)=1}} \frac{a(d\bar{\ell}, d^2\, \mathrm{q}(\ell))}{d^s}$$

converges absolutely at $s = m^+$ by Corollary 3.87. Hence, there must be some number $M \in \mathbb{N}$ such that

$$\sum_{\substack{M < d \in \mathbb{N} \\ \gcd(d,N)=1}}^{\infty} \frac{|a(d\bar{\ell}, d^2\, \mathrm{q}(\ell))|}{d^{m^+}} < |a(\bar{\ell}, \mathrm{q}(\ell))|.$$

Consequently, selecting $N := (M \cdot \mathrm{lev}(L))!$ results in

$$L_\ell^N(f, m^+) = a(\bar{\ell}, \mathrm{q}(\ell)) + \sum_{\substack{1 < d \in \mathbb{N} \\ \gcd(d,N)=1}} \frac{a(d\bar{\ell}, d^2\, \mathrm{q}(\ell))}{d^{m^+}} \neq 0. \tag{7.25}$$

However, the form f was assumed to be annihilated by Λ_{KM}, implying by Corollary 7.23 that

$$L_\ell^N(f, m^+) = \lim_{\substack{s \to m^+ \\ \mathrm{Re}(s) > m^+}} L_\ell(f, s) = 0,$$

contradicting (7.25). As a consequence, the assumption of a non-trivial element in the kernel of Λ_{KM} must be incorrect, i.e. the Kudla–Millson lift is injective. □

This result may be employed to augment the converse theorem for Borcherds products. Recall that Theorem 7.19 already settled the case of a hyperbolic split with $m^+ > 3$. For the case of $m^+ = 3$ and a hyperbolic split, we may find that the lattice L has Witt rank 1 or 2. The case of Witt rank 2 is treated in [Bru14, Thm. 1.2 p. 317] with a minor caveat, while the case of Witt rank 1 is a consequence of Proposition 7.24 in conjunction with Theorem 7.13.

Theorem 7.25. *Assume $m^+ > 2$ and that $L \simeq K \oplus H$ splits a hyperbolic plane. Then there is a sublattice $K_0 \leq K$ such that every meromorphic modular form F with respect to $\Gamma(L)$ whose divisor is a linear combination of special divisors $Z(\mu, n)$ as in (4.31) is (up to a nonzero constant factor) the Borcherds lift $\Psi(z, f)$ of a weakly holomorphic modular form $f \in \mathcal{M}^!_{K_0^-, 1-m^+/2}$.*

Remark 7.26. Note that the only case, where passing to a proper sublattice $K_0 < K$ in Theorem 7.25 might be necessary is the case of $m^+ = 3$ and L having Witt rank 2. We expect the method of proof employed in the current thesis to also work in this case by bypassing Weil's convergence criterion, rendering the transition to a sublattice K_0 superfluous in each instance.

Part IV

Appendix

A Supplementing proofs

A.1 Lattices

In this section, we provide proofs of statements from Section 1.1 and also provide a non-trivial example of diagonalising a Gram matrix shedding some light on the $\mathrm{O}(2,2)$ case. We assume the reader to be familiar with the notation of Section 1.1.

Remark A.1. Let (L, q) be an R lattice of rank m in the quadratic space $L \otimes_R F$. Then choosing a Gram matrix S corresponds to defining an isometry to the lattice $R^m \subseteq F^m$ with bilinear form

$$\mathrm{b} : R^m \times R^m \ni (x, y) \mapsto x^t S y \in F.$$

In the p-adic case $R = \mathbb{Z}_p$, we may for $p \neq 2, \infty$ also bring the quadratic form into a standard form according to Proposition 1.11 which then results in choosing a different basis. In case of $F = \mathbb{Q}_p$, the additive group $(F, +)$ carries a locally compact topology and hence possesses a Haar measure, which induces a measure on F^m and will be normalised on the unit cube ($\mathbb{Z}_p^m$ in case $p < \infty$). Hence, it is meaningful to assign to L a volume $\mathrm{vol}(L)$ by taking the measure of the $\mathbb{Z}_p$ cube generated by a basis of L. Note that for a $\mathbb{Z}$ lattice L we find the following product formula for the corresponding local lattices $\prod_{p \leq \infty} \mathrm{vol}(L_p) = 1$.

Remark A.2. (1.30) Let L be a non-degenerate $\mathbb{Z}$ lattice.

[back]

a) We find $\det(L)L' \leq L$ as well as $NL' \leq L$.
b) We have $|L'/L| = |\det(L)| = \mathrm{vol}(L)^2$, in particular, L'/L is finite.
c) Further, the following division relations are true $N \mid 2\det(L) \mid 2N^n$ and $N \mid \det(L)^2$. In case of odd level N or even rank m, we find $N \mid \det(L)$.
d) The level N and the determinant $\det(L)$ have the same prime divisors.

Proof: a) Let S denote a Gram matrix of the lattice L. Then both inclusions follow from 1.23 once NS^{-1} is identified as an integral matrix. For a basis $(l_i)_i$ of L and respective dual basis $(l_i')_i$ we have $S^{-1} = (\mathrm{b}(l_i', l_j'))_{i,j}$ and verify

$$N\,\mathrm{b}(l_i', l_j') = N\left[q(l_i' + l_j') - q(l_i') - q(l_j')\right] \in \mathbb{Z}.$$

c) The latter implies $\det(S) \mid \det(S)\det(NS^{-1}) = N^m$. Since $2\det(L)S^{-1} \in M_m(\mathbb{Z})$ and by a) we have $\det(L)^2\,\mathrm{q}(L') \in M_m(\mathbb{Z})$, N must divide both by minimality.
Finally, in case of even lattice and rank m, $\det(S)S^{-1}$ represents an even lattice. To verify this, the Leibniz formula for a minor of S reading $\sum_{\sigma \in S_{m-1}} \mathrm{sgn}(\sigma) \cdot \prod_i s_{i,\sigma(i)}$ yields identical terms for permutations σ and σ^{-1}. In case $\sigma^2 = \mathrm{id}$, there is a fixed index n, as $m-1$ is odd, so that $s_{n\sigma(n)} = s_{nn} \in 2\mathbb{Z}$ is part of the product. As a consequence, all linear combinations of NS^{-1} and $\det(S)S^{-1}$ are even, yielding by definition of the level $N \mid \det(S)$.
b) Further, the elementary divisor Theorem for $L \leq L'$ and a straightforward calculation yield $\det(S) = |L'/L|^2 \det(S)^{-1}$. The statement about the volume is completely reduced to the case $\mathbb{Q}_p^n$. Then Remark 1.23 and the transformation formula for integrals yield the desired result. □

Remark A.3. Let (L, q) be an even $\mathbb{Z}$-lattice of signature (m^+, m^-). If $|\mathrm{sig}(L)| \geq 3$, then the Witt rank $r_{\mathbb{Q}}$ of L is maximal, i.e. $r = \min\{m^+, m^-\}$.

Proof: The definite case is clear. Assume $V_{\mathbb{Q}}$ is indefinite, i.e. $m^+, m^- > 0$. Note that $|\mathrm{sig}(L)| = |m^+ - m^-| = \max\{m^+, m^-\} - \min\{m^+, m^-\}$, so that

$$|\mathrm{sig}(L)| \geq 3 \iff \max\{m^+, m^-\} + \min\{m^+, m^-\} \geq 3 + 2\min\{m^+, m^-\}.$$

This translates to $\dim(V) \geq 3 + 2 \cdot r_{\mathbb{R}}$, with $r_{\mathbb{R}} = \min\{m^+, m^-\}$ being the Witt rank over $\mathbb{R}$. By applying the Theorem of Meyer (cf. Remark 1.25) inductively to an indefinite space fulfilling the last inequality, we see that it splits at least $r_{\mathbb{R}}$ hyperbolic planes. □

In the following, we present the diagonalisation of two important cases of lattices, one of which may play a key role for the injectivity of the Kudla–Millson lift in signature $(2, 2)$.

Example A.4. Let L be a non-degenerate quadratic $\mathbb{Z}$ lattice of rank 2 with Gram matrix S. Clearly, given $\ell \in V = L \otimes_{\mathbb{Z}} \mathbb{Q}$ with $\mathrm{q}(\ell) = 0$, it cannot be completed to an orthogonal basis, since the lattice was assumed to be non-degenerate. So assume ℓ fulfilled $\mathrm{q}(\ell) \neq 0$. Then $\ell^T S \neq 0$ and by non-degeneracy, this vector has 1 dimensional kernel generated by

$0 \neq v := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) S\ell$. Now ℓ, v represent an orthogonal basis of V. We find

$$v^T S = \ell^T S \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) S = \ell^T \det(S) \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = \det(S) \left[\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \ell\right]^T.$$

This results in

$$v^T S v = \det(S)\ell^T \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) S\ell = \det(S) 2\,\mathrm{q}(\ell),$$

yielding the following Gram matrix with respect to the basis $\mathcal{B} = (\ell, v)$

$$S_{\mathcal{B}} = 2\,\mathrm{q}(\ell) \begin{pmatrix} 1 & 0 \\ 0 & \det(S) \end{pmatrix}.$$

Example A.5. Assume L is a non-degenerate $\mathbb{Z}$ lattice of rank 4, such that $L = M \oplus N$ splits with $\mathrm{rk}(M) = 2$. Note that this renders M, N non-degenerate $\mathbb{Z}$ lattices of rank 2. We would like to complete $\ell \in V = L \otimes_{\mathbb{Z}} \mathbb{Q}$ to an orthogonal basis. The vector ℓ must clearly be non-isotropic and we write $\ell = (\ell_1, \ell_2, \ell_3, \ell_4)^T \in V$ as for the Gram matrix

$$S = S_M \oplus S_N = \begin{pmatrix} s_{11} & s_{12} & & \\ s_{12} & s_{22} & & \\ & & s_{33} & s_{34} \\ & & s_{34} & s_{44} \end{pmatrix}.$$

Without loss of generality, $\ell|_M$ is not isotropic, meaning $\mathrm{q}(\ell|_M) \neq 0$. If $\ell = \ell|_M \oplus 0$, meaning the restriction $\ell|_N = 0$ vanishes identically, the situation has been solved in Example A.4, by giving an OB for M and N individually.

So assume $\ell|_M, \ell|_N \neq 0$. Then we may, without loss of generality assume that $\ell_1, \ell_3 \neq 0$. Further, $v_1 = \begin{pmatrix} \ell_1 s_{12} + \ell_2 s_{22} & -\ell_1 s_{11} - \ell_2 s_{12} & 0 & 0 \end{pmatrix}^T \in M'$ is orthogonal to ℓ. In order to complete this to a basis, we have two cases to consider, namely whether $\ell|_N$ is isotropic or not.

1. Assume $\ell|_N$ is not isotropic. Then we may repeat the argument above to find $v_2 = \begin{pmatrix} 0 & 0 & \ell_3 s_{34} + \ell_4 s_{44} & -\ell_3 s_{33} - \ell_4 s_{34} \end{pmatrix}^T \in N'$ to be orthogonal to $\ell|_N$ and hence also to ℓ. Now in order to find the last vector of the orthogonal basis, we need to solve the following system of equations (we use the computation of Example A.4,

again)

$$\begin{pmatrix} \ell_2 & -\ell_1 & 0 & 0 \\ 0 & 0 & \ell_4 & -\ell_3 \\ \ell_1 s_{11} + \ell_2 s_{12} & \ell_1 s_{12} + \ell_2 s_{22} & \ell_3 s_{33} + \ell_4 s_{34} & \ell_3 s_{34} + \ell_4 s_{44} \end{pmatrix} w = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

For the last line, we want to eliminate the second and fourth entry. In order to complete the step, note that $\ell_1 s_{11} + \ell_2 s_{12} + (\ell_1 s_{12} + \ell_2 s_{22})/\ell_1 = 2\,\mathrm{q}(\ell|_M)/\ell_1$, so that we need to solve the following system:

$$\begin{pmatrix} \ell_2 & -\ell_1 & 0 & 0 \\ 0 & 0 & \ell_4 & -\ell_3 \\ \mathrm{q}(\ell|_M)/\ell_1 & 0 & \mathrm{q}(\ell|_N)/\ell_3 & 0 \end{pmatrix} w = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We choose $w_1 = \mathrm{q}(\ell|_N)\ell_1$, yielding by the first line $w_2 = \mathrm{q}(\ell|_N)\ell_2$. Now the third line yields $\mathrm{q}(\ell|_M)\,\mathrm{q}(\ell|_N) = -\,\mathrm{q}(\ell|_N)w_3/\ell_3$, which returns $w_3 = -\,\mathrm{q}(\ell|_M)\ell_3$. Combining this fact with the second line results in $w_4 = \mathrm{q}(\ell|_M)\ell_4$. This completes the construction of the orthogonal basis. We end the computation by determining the representing matrix. To this end, it suffices to reduce the computation to $w|_M = \mathrm{q}(\ell|_N)\ell|_M$. This yields $w|_M^T S_M w|_M = \mathrm{q}(\ell|_N)^2\,\mathrm{q}(\ell|_M)$, which may be combined to $w^T S w = (\mathrm{q}(\ell|_N)^2\,\mathrm{q}(\ell|_M) + \mathrm{q}(\ell|_M)^2\,\mathrm{q}(\ell|_N)) = \mathrm{q}(\ell|_M)\,\mathrm{q}(\ell|_N)\,\mathrm{q}(\ell)$. All in all, we obtain the following orthogonal basis with representing Gram matrix

$$\begin{aligned} \mathcal{B} &= \begin{pmatrix} \ell_1 & \ell_1 s_{12} + \ell_2 s_{22} & 0 & \mathrm{q}(\ell|_N)\ell_1 \\ \ell_2 & -\ell_1 s_{11} - \ell_2 s_{12} & 0 & \mathrm{q}(\ell|_N)\ell_2 \\ \ell_3 & 0 & \ell_3 s_{34} + \ell_4 s_{44} & -\,\mathrm{q}(\ell|_M)\ell_3 \\ \ell_4 & 0 & -\ell_3 s_{33} - \ell_4 s_{34} & -\,\mathrm{q}(\ell|_M)\ell_4 \end{pmatrix} \\ &= \left(\ell, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) S_M \ell|_M, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) S_N \ell|_N, \mathrm{q}(\ell|_N)\ell|_M - \mathrm{q}(\ell|_M)\ell|_N\right), \end{aligned}$$

$$S_{\mathcal{B}} = 2\begin{pmatrix} \mathrm{q}(\ell) & & & \\ & \det(S_M)\,\mathrm{q}(\ell|_M) & & \\ & & \det(S_N)\,\mathrm{q}(\ell|_N) & \\ & & & \mathrm{q}(\ell|_M)\,\mathrm{q}(\ell|_N)\,\mathrm{q}(\ell) \end{pmatrix}.$$

2. Assume $\ell|_N$ is isotropic. Then the space $\ell^\perp$ will clearly be isotropic as well, as it contains $\ell|_N$. However, we still have ℓ and v_1 as non-isotropic, orthogonal elements.

Also, N is isotropic, non-degenerate and 2 dimensional and as such isometric to a hyperbolic plane, meaning we may assume $N = H$. We may still assume $\ell_3 \neq 0$, forcing $\ell_4 = 0$. This leaves us with finding elements from the kernel of the following system that are different from $(0, 0, \ell_3, 0)$.

$$\begin{pmatrix} \ell_2 & -\ell_1 & 0 & 0 \\ 2\,\mathrm{q}(\ell|_M)/\ell_1 & 0 & 0 & \ell_3 \end{pmatrix} u = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We assume $u_4 \neq 0$, say $-2\,\mathrm{q}(\ell|_M)/\ell_3$. Then the last equation yields $2\,\mathrm{q}(\ell|_M)/\ell_1 \cdot u_1 = 2\,\mathrm{q}(\ell|_M)$, meaning $u_1 = \ell_1$. In addition, the first equation yields $\ell_1\ell_2 = \ell_1 u_2$, meaning $u_2 = \ell_2$. All in all, we find

$$v_2 \coloneqq u = \begin{pmatrix} \ell_1 & \ell_2 & u_3 & -2\,\mathrm{q}(\ell|_M)/\ell_3 \end{pmatrix}^T,$$

with u_3 undetermined, yet. We choose $u_3 = 0$. Clearly, we have $\ell|_M = u|_M$, so that for the orthogonality condition on M, we may reproduce the same computation as above and only have to compute $u|_N^T S_N = u|_N^T H = \begin{pmatrix} -2\,\mathrm{q}(\ell|_M)/\ell_3 & 0 \end{pmatrix}^T$. This yields as a condition for the last vector w

$$\begin{pmatrix} \ell_2 & -\ell_1 & 0 & 0 \\ 2\,\mathrm{q}(\ell|_M)/\ell_1 & 0 & 0 & \ell_3 \\ 2\,\mathrm{q}(\ell|_M)/\ell_1 & 0 & -2\,\mathrm{q}(\ell|_M)/\ell_3 & 0 \end{pmatrix} w = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We recall that assuming $w_4 \neq 0$ (as above for u will determine w_1 and vice versa). Further, w_1 determines w_2 by the first line. Hence, w is of the same form as u, but with $w_3 \neq u_3 = 0$. In fact, we find from the last line that $2\,\mathrm{q}(\ell|_M)/\ell_1 \cdot \ell_1\ell_3 = 2\,\mathrm{q}(\ell|_M) w_3$, meaning $w_3 = \ell_3$. We recollect the basis elements

$$\ell = \begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \\ \ell_4 \end{pmatrix}, v_1 = \begin{pmatrix} \ell_1 s_{12} + \ell_2 s_{22} \\ -\ell_1 s_{11} - \ell_2 s_{12} \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} \ell_1 \\ \ell_2 \\ 0 \\ -2\,\mathrm{q}(\ell|_M)/\ell_3 \end{pmatrix}, v_3 = \begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \\ -2\,\mathrm{q}(\ell|_M)/\ell_3 \end{pmatrix}.$$

Next, we need to compute the representing matrix for v_2, v_3, finding

$$v_2^T S_H v_2 = 2\,\mathrm{q}(\ell|_M), \qquad v_3^T S v_3 = 2\,\mathrm{q}(\ell|_M) - 4\,\mathrm{q}(\ell|_M) = -2\,\mathrm{q}(\ell|_M).$$

Now recall that we have assumed in this case that $\mathrm{q}(\ell|_N) = 0$, rendering $\mathrm{q}(\ell|_M) = \mathrm{q}(\ell)$. Further, we had assumed $L = M \oplus H$, meaning $\det(S) = \det(S_M) \cdot \det(S_H) = -\det(S_K)$. This ultimately leads to the following representing matrix for the orthogonal basis $\mathcal{B} := (\ell, v_1, v_2, v_3)$:

$$S_{\mathcal{B}} = 2\,\mathrm{q}(\ell) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\det(S) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

A.2 Miscellaneous

Norms

We briefly recall the following definition.

Definition A.6. Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and V be an $\mathbb{F}$ vector space. A map $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ is called a norm, if for $x, y \in V$ and $\lambda \in \mathbb{F}$ the following conditions are satisfied.

a) It is positive definite, i.e. $\|x\| = 0 \iff x = 0$.

b) It fulfils the *triangle inequality*, i.e. $\|x + y\| \leq \|x\| + \|y\|$.

c) It is *positive homogeneous* of *degree* one, i.e. $\|\lambda x\| = |\lambda| \cdot \|x\|$.

We call the pair $(V, \|\cdot\|)$ a *normed space*.

We also recall the following fact about normed spaces that is applied to prove convergence of certain L-series in Lemma 6.76 and Proposition 6.95.

Remark A.7. Let $(V_i, \|\cdot\|_i)$ for $i \in I$ an index set be normed spaces over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

a) For $c \in \mathbb{R}_{>0}$, the space $(V_i, c \cdot \|\cdot\|_i)$ is a normed space, again.

b) The direct sum

$$\bigoplus_i (V_i, \|\cdot\|_i) := \left(\oplus_i V_i, \sum_i \|\cdot\|_i \right)$$

is a normed space, again.

c) Let V_i be finite dimensional. Then any norm $\|\cdot\|$ is equivalent to $\|\cdot\|_i$, i.e. there are constants $C \geq c > 0$ such that

$$c\|\cdot\| \leq \|\cdot\|_i \leq C\|\cdot\|.$$

Groups and index transfers

We will briefly recall a property of groups that is required for a key reduction to domains of integration in the proof of the main theorem of this thesis.

Lemma A.8. *Let G be a group, $H, K \leq G$ be subgroups. Then we find*

$$[K : H \cap K] \leq [G : H].$$

Proof: First, note that $H \cap K \leq K$ is a subgroup. Assume, we establish a bijection between the left cosets of $H \cap K$ in K and the left cosets of H in HK. We will call the latter $[HK : H]$. Then since $HK \subseteq G$, we find

$$[K : H \cap K] = [HK : H] \leq [G : H].$$

The bijection is identical to the construction in the first isomorphism theorem of group theory [Bos18, Prop. 8 p. 18]. To be explicit, send

$$(H \cap K)k \mapsto Hk.$$

Assume there are two different $k_1, k_2 \in K$ such that $Hk_1 = Hk_2$. Then there is $h \in H$ such that $k_1 k_2^{-1} = h \in H \cap K$. In other words, $(H \cap K)k_2 = (H \cap K)k_1$, proving that the association is injective. However, surjectivity of the association is apparent, finishing the proof. □

Corollary A.9. *Let G be a group and, $H, K \leq G$ be subgroups. Then we have*

$$[G : H \cap K] \leq [G : H] \cdot [G : K].$$

Invariance of Fourier coefficients under imaginary shifts

It has been stated in (2.13) that periodic holomorphic functions on the upper half plane possess a Fourier expansion, i.e. let $f : \mathbb{H} \to \mathbb{C}$ be holomorphic with $f(\tau + 1) = f(\tau)$, then there is a Fourier expansion of f:

$$f(z) = \sum_{n\in\mathbb{Z}} a(n) \cdot e^{2\pi i n\tau}$$

with coefficients

$$a(n) = \int_{[0,1]} f(\tau) \cdot e^{-2\pi i n\tau} \, \mathrm{d}\lambda(\mathrm{Re}(\tau)).$$

From the real perspective on Fourier theory, however, it is unclear why the Fourier coefficients $a(n)$ should be independent of the chosen height $\mathrm{Im}(\tau)$. The following computation is based on [FB06, III.5, p. 150].

We investigate the derivative of $a(n)$ with respect to $\mathrm{Im}(\tau)$ and use the product integration rule to perform the following calculation:

$$\begin{aligned}\partial_{\mathrm{Im}(\tau)} a(n) &= \int_{[0,1]} \partial_{\mathrm{Im}(\tau)}(f(\tau)) \cdot e^{-2\pi i n\tau} \, \mathrm{d}\lambda(\mathrm{Re}(\tau)) + 2\pi n \int_{[0,1]} f(\tau) \cdot e^{-2\pi i n\tau} \, \mathrm{d}\lambda(\mathrm{Re}(\tau)) \\ &= f(\tau) e^{-2\pi i n\tau} \Big|_{\mathrm{Re}(\tau)=0}^{\mathrm{Re}(\tau)=1} = 0,\end{aligned}$$

where we have used that f is holomorphic, i.e. $\frac{\partial f}{\mathrm{Im}(\tau)} = i \frac{\partial f}{\partial \, \mathrm{Re}(\tau)}$. In consequence, $a(n)$ is constant with respect to $\mathrm{Im}(\tau)$.

A.3 Relations of Hecke operators

In this section, we present some proofs of lemmas in Subsection 6.2 and 6.3 that have been outsourced in order to improve readability.

Lemma A.10 (Lemma 6.32)**.** *For $r \in \mathbb{N}$ and $k, p \nmid N$ as above we find*

$$T(p^{r+1}) = T(p)T(p^r) - p \cdot T_{p \cdot \mathcal{I}} \cdot T(p^{r-1}).$$

The identity remains valid if the symbol T is replaced by ${}^kT$.

Proof of Lemma 6.32: Lemma 6.25 yields the following sets of representatives for $T(p)$

$$R_p = \left\{ \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right\} \bigcup \left\{ \begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix} \middle| \; b \mod p \right\}$$

and for $T(p^r)$

$$R_{p^r} = \left\{ \begin{pmatrix} p^{r_1} & b \\ 0 & p^{r_2} \end{pmatrix} \middle| \begin{array}{l} r_1 + r_2 = r, \\ b \mod p^{r_2} \end{array} \right\}.$$

Their product $R_p R_{p^r}$ will be a system of representatives for $T(p)T(p^r)$ and reads

$$\left\{ \begin{pmatrix} p^{r_1+1} & pb \\ 0 & p^{r_2} \end{pmatrix} \middle| \begin{array}{l} r_1 + r_2 = r, \\ b \mod p^{r_2} \end{array} \right\} \bigcup \left\{ \begin{pmatrix} p^{r_1} & b_r + b_1 p^{r_2} \\ 0 & p^{r_2+1} \end{pmatrix} \middle| \begin{array}{l} r_1 + r_2 = r, \\ b_1 \mod p, \\ b_r \mod p^{r_2} \end{array} \right\}.$$

The second set falls only short of a set of representatives of $T(p^{r+1})$ by the matrix $\operatorname{diag}(p^{r+1}, 1)$. Eliminating this element from the first set leaves

$$\left\{ p \cdot \begin{pmatrix} p^{r_1} & b \\ 0 & p^{r_2-1} \end{pmatrix} \middle| \begin{array}{l} r_1 + r_2 = r, \\ r_2 > 0, \\ b \mod p^{r_2} \end{array} \right\} = \left\{ p \cdot \begin{pmatrix} p^{r_1} & b \\ 0 & p^{r_2} \end{pmatrix} \middle| \begin{array}{l} r_1 + r_2 = r - 1, \\ b \mod p^{r_2+1} \end{array} \right\}.$$

The associated Hecke element is given by

$$p \cdot T_{p \cdot \mathcal{I}} \cdot T(p^{r-1}) = \frac{p}{(p^{r+1})^{k/2-1}} {}^k T_{p \cdot \mathcal{I}} \cdot {}^k T(p^{r-1}).$$

Combining the information above, we obtain in light of Definition 6.13 that

$$T(p)T(p^r) = T(p^{r+1}) + p \cdot T_{p \cdot \mathcal{I}} \cdot T(p^{r-1}).$$

□

Lemma A.11 (Lemma 6.50)**.** *We find for $r \in \mathbb{N}$ and a prime p not dividing N that*

$$\mathcal{T}(p^{r+1}) = \mathcal{T}(p)\mathcal{T}(p^r) - p \cdot T_{p\mathcal{I}} \cdot \mathcal{T}(p^{r-1}) - \delta_{r=1} \cdot T_{p\mathcal{I}}.$$

Proof of Lemma 6.50: The classes $\Gamma(1)\backslash\Gamma(1) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \Gamma(1)$ are represented by

$$R_p^* = \left\{ \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix} \middle| b \mod p \right\}.$$

For p^r the set of representatives reads as follows:

$$R^*_{p^r} = \left\{ \begin{pmatrix} p^{r_1} & b_r \\ 0 & p^{r_2} \end{pmatrix} \;\middle|\; \begin{array}{l} r_1 + r_2 = r \\ b_r \mod p^{r_2} \\ 0 < r_2 < r \implies p \nmid b_r \end{array} \right\}.$$

Their product $R^*_p R^*_{p^r}$ has the following decomposition:

$$\left\{ \begin{pmatrix} p^{r_1+1} & b_r p \\ 0 & p^{r_2} \end{pmatrix} \;\middle|\; \begin{array}{l} r_1 + r_2 = r \\ b_r \mod p^{r_2} \\ 0 < r_2 < r \implies p \nmid b_r \end{array} \right\} \tag{I}$$

$$\bigcup \left\{ \begin{pmatrix} p^{r_1} & b_r + b_1 p^{r_2} \\ 0 & p^{r_2+1} \end{pmatrix} \;\middle|\; \begin{array}{l} r_1 + r_2 = r \\ b_r \mod p^{r_2} \\ b_1 \mod p \\ 0 < r_2 < r \implies p \nmid b_r \end{array} \right\}. \tag{II}$$

The second set's conditions may be reformulated as

$$\left\{ \begin{pmatrix} p^{r_1} & b_{r+1} \\ 0 & p^{r_2} \end{pmatrix} \;\middle|\; \begin{array}{l} 0 < r_2 \\ r_1 + r_2 = r + 1 \\ b_{r+1} \mod p^{r_2} \\ 1 < r_2 < r + 1 \implies p \nmid b_{r+1} \end{array} \right\}$$

$$= \left\{ \begin{pmatrix} p^{r_1} & b_{r+1} \\ 0 & p^{r_2} \end{pmatrix} \;\middle|\; \begin{array}{l} 0 < r_2 \\ r_1 + r_2 = r + 1 \\ b_{r+1} \mod p^{r_2} \\ 0 < r_2 < r + 1 \implies p \nmid b_{r+1} \end{array} \right\} \tag{II.1}$$

$$\bigcup \left\{ p \begin{pmatrix} p^{r-1} & 0 \\ 0 & 1 \end{pmatrix} \right\}. \tag{II.2}$$

Clearly, (II.1) falls short of a set of representatives for $R^*_{p^{r+1}}$ by $\operatorname{diag}(p^{r+1}, 1)$ only which is taken from (I) (the case of $r_2 = 0$), leaving in place of (I)

$$\left\{ p \begin{pmatrix} p^{r_1} & b_{r-1} \\ 0 & p^{r_2} \end{pmatrix} \;\middle|\; \begin{array}{l} r_1 + r_2 = r - 1 \\ b_{r-1} \mod p^{r_2+1} \\ r_2 < r - 1 \implies p \nmid b_{r-1} \end{array} \right\}$$

$$= \left\{ p \begin{pmatrix} p^{r-1} & b_{r-1} \\ 0 & 1 \end{pmatrix} \;\middle|\; \begin{array}{l} b_{r-1} \mod p \\ 1 < r \implies p \nmid b_{r-1} \end{array} \right\}$$

$$\bigcup \left\{ p \begin{pmatrix} p^{r_1} & b_{r-1} \\ 0 & p^{r_2} \end{pmatrix} \;\middle|\; \begin{array}{l} 0 < r_2 \\ r_1 + r_2 = r - 1 \\ b_{r-1} \mod p^{r_2+1} \\ 0 < r_2 < r - 1 \implies p \nmid b_{r-1} \end{array} \right\}.$$

We modify the elements via multiplication with suitable elements from $\Gamma(1)$ to obtain

$$\Gamma(1) \simeq \bigsqcup_{i=1}^{p-1+\delta_{r=1}} \left\{ p \begin{pmatrix} p^{r-1} & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$\bigcup \bigsqcup_{i=1}^{p} \left\{ p \begin{pmatrix} p^{r_1} & b_{r-1} \\ 0 & p^{r_2} \end{pmatrix} \;\middle|\; \begin{array}{l} 0 < r_2 \\ r_1 + r_2 = r - 1 \\ b_{r-1} \mod p^{r_2} \\ 0 < r_2 < r - 1 \implies p \nmid b_{r-1} \end{array} \right\}.$$

Adding the representative from (II.2) yields

$$= \bigsqcup_{i=1}^{p+\delta_{r=1}} \left\{ p \begin{pmatrix} p^{r-1} & 0 \\ 0 & 1 \end{pmatrix} \right\} \bigcup \bigsqcup_{i=1}^{p} \left\{ p \begin{pmatrix} p^{r_1} & b_{r-1} \\ 0 & p^{r_2} \end{pmatrix} \;\middle|\; \begin{array}{l} 0 < r_2 \\ r_1 + r_2 = r - 1 \\ b_{r-1} \mod p^{r_2} \\ 0 < r_2 < r - 1 \implies p \nmid b_{r-1} \end{array} \right\}$$

$$= \delta_{r=1} \left\{ p \begin{pmatrix} p^{r-1} & 0 \\ 0 & 1 \end{pmatrix} \right\} \bigcup \bigsqcup_{i=1}^{p} \left\{ p \begin{pmatrix} p^{r_1} & b_{r-1} \\ 0 & p^{r_2} \end{pmatrix} \;\middle|\; \begin{array}{l} r_1 + r_2 = r - 1 \\ b_{r-1} \mod p^{r_2} \\ 0 < r_2 < r - 1 \implies p \nmid b_{r-1} \end{array} \right\}.$$

This represents the operator

$$\delta_{r=1} \cdot T_{p\mathcal{I}} + p \cdot T_{p\mathcal{I}} \cdot \mathcal{T}(p^{r-1}).$$

In total we find

$$\mathcal{T}(p)\mathcal{T}(p^r) = \mathcal{T}(p^{r+1}) + p \cdot T_{p\mathcal{I}} \cdot \mathcal{T}(p^{r-1}) + \delta_{r=1} \cdot T_{p\mathcal{I}}.$$

□

A.4 Gauss sums

This is a complementary section containing elementary computations for different Gauss sums appearing throughout the thesis. The presented sums and a significant portion of the computations are already found in [Ste15] but had to be recomputed in order to detect a computational error in the expression for Fourier coefficients in Theorem 5.4 of this source.

Definition A.12. Let (L, q) be a quadratic $\mathbb{Z}$ lattice of rank $m \in \mathbb{N}$, p a prime, $h \in \mathbb{Z}$ with $\gcd(h, p) = 1$ and $n, l \in \mathbb{N}$ with $l \geq n$, plus χ be a Dirichlet Character modulo n. Define

$$G_{L,p}(n, h) := \sum_{v \in L/p^n L} e\left(\frac{h}{p^n}\,\mathrm{q}(v)\right), \tag{A.1}$$

$$g_p(n, l, h) := \sum_{k \in \mathbb{Z}/p^l\mathbb{Z}} e\left(\frac{h}{p^n} k^2\right), \tag{A.2}$$

$$g_p\left[n, \chi, h\right] := \sum_{k \in \mathbb{Z}/p^n\mathbb{Z}} \chi(k) e\left(\frac{hk}{p^n}\right). \tag{A.3}$$

Remark A.13. For odd primes p, we have the following identity:

$$g_p(n, l, h) = \left(\frac{h}{p^n}\right) \epsilon(p^n) p^{l-n/2},$$

where

$$\epsilon(k) = \begin{cases} 1, & \text{if } k \equiv 1 \mod 4, \\ i, & \text{if } k \equiv 3 \mod 4. \end{cases}$$

For $p = 2$, we find

$$g_p(n, l, h) = \left(\frac{p^n}{h}\right) (1 + i^h) p^{l-n/2}.$$

Proof: We find

$$\begin{aligned} g_p(n, l, h) &= \sum_{k_1 \in \mathbb{Z}/p^{l-n}\mathbb{Z}} \sum_{k_2 \in \mathbb{Z}/p^n\mathbb{Z}} e\left(\frac{h}{p^n}(k_1 p^n + k_2)^2\right) \\ &= p^{l-n} \sum_{k_2 \in \mathbb{Z}/p^n\mathbb{Z}} e\left(\frac{h}{p^n} k_2^2\right). \end{aligned}$$

The sum on the right is equal to

$$\sqrt{p^n} \cdot \begin{cases} \left(\frac{h}{p^n}\right) \epsilon(p^n), & \text{if } p > 2, \\ \left(\frac{p^n}{h}\right) (1+i^h), & \text{if } p = 2, \end{cases}$$

by [BEW98, 1.5 Thm. 1.5.2 p. 26] and [BEW98, 1.5 Thm. 1.5.4 p. 27]. □

Remark A.14. For $p > 2$ we find that the following reduction is true

$$G_{L,p}(n,h) = \left(\frac{h}{p}\right)^{R^n} \underbrace{p^{nR_0} \cdot \prod_{k=0}^{n-1} \left(\epsilon(p^{n-k})\sqrt{p^{n+k}}\right)^{n_k} \cdot \prod_{i=1}^{n_k} \left(\frac{\mathrm{q}(v_i) \| \mathrm{q}(v_i) \|_p}{p^n \| \mathrm{q}(v_i)\|_p}\right)}_{=G_{L,p}(n,1)}$$

for suitable but explicit numbers $R^n, R_0 \in \mathbb{N}_0$.

Proof: By [Ste15, Lemma 4.3] there is an orthogonal decomposition of L/p^nL into $\mathbb{Z}/p^n\mathbb{Z}$ submodules

$$\left(\bigoplus_i L_i\right) \oplus \left(\bigoplus_j M_j\right) \oplus N,$$

where $L_i = (\mathbb{Z}/p^n\mathbb{Z})v_i$ is one dimensional with $\mathrm{b}(v_i, v_i) \in (\mathbb{Z}/p^n\mathbb{Z})^\times$, $M_j = (\mathbb{Z}/p^n\mathbb{Z})v_j$ where $\mathrm{b}(v_j, v_j) \in p^k(\mathbb{Z}/p^n\mathbb{Z})^\times$ with $1 \le k \le n-1$ and $\mathrm{b}(N,N) \subseteq p^n\mathbb{Z}$. We may assume the M_j being sorted with respect to increasing valuation k of $\mathrm{q}(v_j)$. Recall that we set

$$n_k = \#\{1 \le i \le m : \nu_p(\mathrm{b}(v_i, v_i)) = k\}, \quad R := \sum_{k=0}^{n-1} (n-k) n_k$$

in Definition 1.38. Here $\{v_i\}$ denotes, as suggested before, a basis of L as a $\mathbb{Z}/p^n\mathbb{Z}$ module. With these definitions, we find

$$\begin{aligned} G_{L,p}(n,h) &= \sum_{(x_j)_j \in (\mathbb{Z}/p^n\mathbb{Z})^m} e\left(\sum_i \frac{h\,\mathrm{q}(v_j)}{p^n} x_i^2\right) \\ &= \prod_{i, \mathrm{q}(v_i) \neq 0} \sum_{x_i \in \mathbb{Z}/p^n\mathbb{Z}} e\left(\frac{h\,\mathrm{q}(v_i) \| \mathrm{q}(v_i)\|_p}{p^n \| \mathrm{q}(v_i)\|_p} x_i^2\right) \cdot \prod_{i, \mathrm{q}(v_i) = 0} p^n \\ &\overset{\star}{=} \prod_{k=0}^{n-1} \left(\frac{h}{p^{n-k}}\right)^{n_k} \cdot \prod_{\substack{i \\ 0 \le \nu_p(\mathrm{q}(v_i)) < n}} \sum_{x_i \in \mathbb{Z}/p^n\mathbb{Z}} e\left(\frac{\mathrm{q}(v_i) \| \mathrm{q}(v_i)\|_p}{p^n \| \mathrm{q}(v_i)\|_p} x_i^2\right) \cdot \prod_{\substack{i \\ \mathrm{q}(v_i) \in p^n\mathbb{Z}}} p^n \end{aligned}$$

$$= \left(\frac{h}{p}\right)^R \cdot G_{L,p}(n,1).$$

For $\star$, Remark A.13 has been used to obtain for $0 \le \nu_p(\mathrm{q}(v_i)) < n$ and fixed i

$$\begin{aligned}
& g_p(n-\nu_p(\mathrm{q}(v_i)), n, h\,\mathrm{q}(v_i)\|\,\mathrm{q}(v_i)\|_p) \\
= & \sum_{x_i \in \mathbb{Z}/p^n\mathbb{Z}} e\left(\frac{h\,\mathrm{q}(v_i)\|\,\mathrm{q}(v_i)\|_p}{p^n\|\,\mathrm{q}(v_i)\|_p} x_i^2\right) \\
= & \left(\frac{h}{p^n\|\,\mathrm{q}(v_i)\|_p}\right)\left(\frac{\mathrm{q}(v_i)\|\,\mathrm{q}(v_i)\|_p}{p^n\|\,\mathrm{q}(v_i)\|_p}\right)\epsilon(p^{n-\nu_p(\mathrm{q}(v_i))})p^{(n+\|\,\mathrm{q}(v_i)\|_p)/2} \\
= & \left(\frac{h}{p}\right)^{n-\nu_p(\mathrm{q}(v_i))} \cdot g_p(n-\nu_p(\mathrm{q}(v_i)), n, \mathrm{q}(v_i)\|\,\mathrm{q}(v_i)\|_p).
\end{aligned}$$

This computation, however, yields directly

$$\begin{aligned}
G_{L,p}(n,1) &= p^{nR_0} \cdot \prod_{\substack{i \\ 0 \le \nu_p(\mathrm{q}(v_i)) < n}} \left(\frac{\mathrm{q}(v_i)\|\,\mathrm{q}(v_i)\|_p}{p^n\|\,\mathrm{q}(v_i)\|_p}\right)\epsilon(p^{n-\nu_p(\mathrm{q}(v_i))})p^{(n+\|\,\mathrm{q}(v_i)\|_p)/2} \\
&= p^{nR_0} \cdot \prod_{k=0}^{n-1}\left(\epsilon(p^{n-k})\sqrt{p^{n+k}}\right)^{n_k} \cdot \prod_{i=1}^{n_k}\left(\frac{\mathrm{q}(v_i)\|\,\mathrm{q}(v_i)\|_p}{p^n\|\,\mathrm{q}(v_i)\|_p}\right)
\end{aligned}$$

for $R_0 = \{i \mid q(v_i) \in p^n\mathbb{Z}\} = m - \sum_{k=0}^{n-1} n_k$ Recall that the elements v_i have been assumed to be sorted with respect to increasing valuation of $\mathrm{q}(v_i)$ □

Remark A.15. For $\chi = \chi_1$ the trivial character on $(\mathbb{Z}/p^n\mathbb{Z})^\times$ or $\chi = \chi_p = \left(\frac{\cdot}{p}\right)$ the Legendre symbol and $p \neq 2$, we find

$$g_p[n, \chi_1, h] = \begin{cases} p^{n-1}(p-1), & \text{if } \nu_p(h) \ge n, \\ -p^{n-1}, & \text{if } \nu_p(h) = n-1, \\ 0, & \text{if } \nu_p(h) < n-1. \end{cases}$$

$$g_p[n, \chi_p, h] = \begin{cases} p^{n-1/2}\left(\frac{h/p^{n-1}}{p}\right)\epsilon_p, & \text{if } \nu_p(h) = n-1, \\ 0, & \text{if } \nu_p(h) \neq n-1. \end{cases}$$

Note that $\chi_1(0) = \chi_1(p) = 0$ and $\epsilon_p = 1, i$ depending on $p \equiv 1, 3 \mod 4$.

Proof: If χ is induced by a character mod p, we find

$$\begin{aligned} g_p[n,\chi,h] &= \sum_{k\in\mathbb{Z}/p^n\mathbb{Z}} \chi(k)e\left(\frac{kh}{p^n}\right) \\ &= \sum_{k_1\in\mathbb{Z}/p\mathbb{Z}} \chi(k_1) \sum_{k_2\in\mathbb{Z}/p^{n-1}\mathbb{Z}} e\left(\frac{(k_1+pk_2)h}{p^n}\right) \\ &= \sum_{k_1\in\mathbb{Z}/p\mathbb{Z}} \chi(k_1)e\left(\frac{k_1h}{p^n}\right) \sum_{k_2\in\mathbb{Z}/p^{n-1}\mathbb{Z}} e\left(\frac{k_2h}{p^{n-1}}\right) \\ &= \delta_{p^{n-1}|h} \cdot p^{n-1} \cdot \sum_{k_1\in\mathbb{Z}/p\mathbb{Z}} \chi(k_1)e\left(\frac{k_1h}{p^n}\right). \end{aligned}$$

Now, if $\chi=\chi_1$ is the trivial Dirichlet character (mod p),

$$\sum_{k_1\in\mathbb{Z}/p\mathbb{Z}} \chi_1(k_1)e\left(\frac{k_1h}{p^n}\right) = \begin{cases} p-1, & \text{if } \nu_p(h)\geq n, \\ -1, & \text{if } \nu_p(h)=n-1. \end{cases}$$

In addition, if $\chi_p=\left(\frac{\cdot}{p}\right)$, we have

$$\sum_{k_1\in\mathbb{Z}/p\mathbb{Z}} \chi_p(k_1)e\left(\frac{k_1h}{p^n}\right) = \begin{cases} 0, & \text{if } \nu_p(h)\geq n, \\ \left(\frac{h/p^{n-1}}{p}\right)\epsilon(p)\sqrt{p}, & \text{if } \nu_p(h)=n-1. \end{cases}$$

The last assertion follows from a well known Theorem by Gauss [IK04, 3.5 Thm. 3.3 p. 49], who required years finding an adequate solution to the problem but finally came out with multiple different successful approaches. □

The following Gauss sum is required in Proposition 6.54 and in the subsequent results building upon it.

Definition A.16. Let $\mathcal{L}$ be a discriminant form. For $d\in\mathbb{N}$ define

$$G_{\mathcal{L}}(d) = \sum_{\lambda\in\mathcal{L}} e\left(d\,\overline{\mathrm{q}}(\lambda)\right). \tag{A.4}$$

In [BS08], the authors compare [McG03, Lem. 4.6 p. 115] to [Bor00, Thm. 5.4 p. 329] in order to conclude the following statement.

Remark A.17. For $\gcd(d, \operatorname{lev}(\mathcal{L})) = 1$ the association

$$(\mathbb{Z}/\operatorname{lev}(\mathcal{L})\mathbb{Z})^\times \ni d \mapsto \frac{G_\mathcal{L}(1)}{G_\mathcal{L}(d)}$$

defines a character.

For the following remark that is required in the proof of Corollary 6.72, recall that for a natural number $d \in \mathbb{N}$ and a discriminant form $\mathcal{L}$, the symbol ${}_d\mathcal{L}$ denoted the d-torsion of $\mathcal{L}$.

Remark A.18. Let $\mathcal{L}$ be the discriminant form of a lattice L of rank m and assume $\gcd(d, \operatorname{lev}(\mathcal{L})) = 1$. Then, by [Bar03, Thm. 5.2.2] we find

$$G_\mathcal{L}(d) = \sqrt{|\mathcal{L}|}\sqrt{i}^{\operatorname{sgn}(\mathcal{L})} \frac{1}{d^{m/2}} \sum_{\nu \in L/dL} e\left(-\frac{1}{d}\operatorname{q}(\nu)\right).$$

In particular, the case $d = p^n$ yields

$$G_\mathcal{L}(p^n) = \sqrt{|\mathcal{L}|}\sqrt{i}^{\operatorname{sgn}(\mathcal{L})} \frac{1}{p^{nm/2}} \cdot G_{L,p}(n,1).$$

As a consequence,

$$\frac{G_\mathcal{L}(1)}{G_\mathcal{L}(p)} = p^{m/2} \frac{1}{\overline{G_{L,p}(1,1)}} = \frac{p^{-m/2}}{|{}_p\mathcal{L}|} G_{L,p}(1,1). \tag{A.5}$$

Proof: Only the last equation has to be proven. By [Sch09, Prop. 3.8] we find the identity $|G_\mathcal{L}(d)| = \sqrt{|{}_d\mathcal{L}||\mathcal{L}|}$. As a consequence,

$$|G_{L,p}(n,1)| = \sqrt{|p^{mn}||{}_p\mathcal{L}|}.$$

This immediately implies the result. □

Do note that explicit formulae for a more general version of $G_\mathcal{L}$ are also contained in [Sch09, Thm. 3.9 p. 11].

A.5 Whittaker functions and asymptotics

In these complementary notes, we explore convergence and boundedness properties of special functions which are required for Rankin–Selberg integrals in Subsection 6.4.3.

Lemma A.19. *Let $\sum_{n\in\mathbb{N}} a_n z^n$ with $a_n \in \mathbb{C}$ be a power series that is convergent at a point $0 \neq z \in \mathbb{C}$. Then it is absolutely and locally uniformly convergent on the interior of the disc $B_{|z|}(0)$ of radius $|z|$.*

Proof: Recall that by the convergence of the series, it must form a Cauchy sequence and as such, there is some $N \in \mathbb{N}$ such that $|a_n z^n| < 1$ for all $n \geq N$. Now let $z_2 \in \mathbb{C}$ fulfil $|z_2| < |z|$. Then

$$\sum_{n=N}^{\infty} |a_n z_2^n| = \sum_{n=N}^{\infty} |a_n z^n||\tfrac{z_2}{z}|^n \leq \sum_{n=N}^{\infty} |\tfrac{z_2}{z}|^n < \infty.$$

In fact, the last term is strictly decreasing, when decreasing $|z_2|$. As a consequence, we obtain uniform absolute convergence on the interior of the disc $B_{|z|}(0)$. □

The following special functions are required for the Fourier expansion of the non-holomorphic Eisenstein series in Proposition 3.40. We follow [DAR84, Sec. 13 p. 189] and [Olv+10, Chap. 13 p. 321] as a reference for the following special functions.

Definition A.20. 1. For $a, b \in \mathbb{C}$ with $\mathrm{Re}(a) > 0$ and $z \in \mathbb{R}$ define the integral representation of the *confluent hypergeometric function*

$$U(a, b, z) := \Gamma(a)^{-1} \int_0^{\infty} e^{-zt} t^{a-1} (1+t)^{b-a-1} \, \mathrm{d}t.$$

2. Define the *Whittaker function*

$$\mathcal{W}_{\kappa,\mu}(z) := e^{-z/2} z^{1/2+\mu} U(1/2 + \mu - \kappa, 1 + 2\mu, z).$$

3. The following special version plays a role for the Fourier expansion of Eisenstein series in Proposition 3.40. For $v \in \mathbb{R}^{\times}$ and $k \in \mathbb{Z}/2$ representing the weight in the respective section, set

$$\mathcal{W}_s(v) := |v|^{-k/2} \mathcal{W}_{\mathrm{sgn}(v)k/2,(1-k)/2-s}(|v|).$$

Lemma A.21. *Given $\delta > 0$, the modified Whittaker function $\mathcal{W}_s(v)$ is bounded for any $v \in \mathbb{R} \setminus B_\delta(0)$ and any $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1 - k$ by*

$$C_s \cdot e^{-\frac{|v|}{2}(1-\varepsilon)},$$

with some arbitrarily small $\varepsilon > 0$ and a constant $C_s > 0$ that may be chosen locally *uniformly in s.*

Proof: We write out the modified Whittaker function $\mathcal{W}_s$ explicitly.

$$\begin{aligned}
&\mathcal{W}_s(v)\\
&= |v|^{-\frac{k}{2}} \cdot e^{-\frac{|v|}{2}} |v|^{\frac{1}{2}+\frac{1-k}{2}-s} U\left(\frac{1}{2}+\frac{1-k}{2}-s-\operatorname{sgn}(v)\frac{k}{2}, 1+2\frac{1-k}{2}-2s, |v|\right)\\
&= e^{-\frac{|v|}{2}} |v|^{1-\frac{k}{2}-s} \cdot U\left(1-\frac{k}{2}-s-\operatorname{sgn}(v)\frac{k}{2}, 2-k-2s, |v|\right)\\
&= e^{-\frac{|v|}{2}} |v|^{1-s-\frac{k}{2}} \cdot U\left(1-s-k\delta_{v>0}, 2(1-s)-k, |v|\right).
\end{aligned}$$

Now, by the differential equation of the confluent hypergeometric functions, we may replace $U(a, b, |v|)$ by a finite sum of such functions with $a, b \gg 0$ with prefactors that are polynomial in a, b, and $|v|$ (cf. [Olv+10, 13.3.14 p. 325]). Hence, it suffices to bound the above term for $s \in \mathbb{C}$ with $\operatorname{Re}(s) \ll 0$. In that case, we find

$$\begin{aligned}
&\mathcal{W}_s(v)\\
&= \frac{e^{-\frac{|v|}{2}} |v|^{1-\frac{k}{2}-s}}{\Gamma(1-s-k\delta_{v>0})} \cdot \int_0^\infty e^{-|v|t} t^{-s-k\delta_{v>0}} (1+t)^{2(1-s)-k-(1-s-k\delta_{v>0})-1} \,\mathrm{d}t\\
&= \frac{e^{-\frac{|v|}{2}} |v|^{1-\frac{k}{2}-s}}{\Gamma(1-s-k\delta_{v>0})} \cdot \int_0^\infty e^{-|v|t} t^{-s-k\delta_{v>0}} (1+t)^{-s-k\delta_{v<0}} \,\mathrm{d}t.
\end{aligned}$$

It is clear, that the integral is bounded for $|v|$ above a fixed threshold and that the bound is uniform on vertical strips in s. Recall that the Beta functions is entire, in particular continuous. Hence, the whole expression is bounded by

$$C_s e^{-\frac{|v|}{2}(1-\varepsilon)}$$

for $\varepsilon > 0$ arbitrarily small and some C_s that may be chosen as a constant on a sufficiently small neighbourhood. □

B Further background material

In this chapter, we present some more general background material that is less subject to direct referencing in the main body of the thesis. It is presented in a less formal manner and thought of as additional notes and not part of the main content.

B.1 Some categorical notions

Restricted products

The construction of $\mathbb{A}$ as a locally compact Hausdorff ring is the paradigm of a natural extension of locally compact structures in this category. However, it may be generalised and is based on the observation that a product of locally compact topological spaces is again locally compact with respect to the product topology if and only if almost all of the factors are compact. We refer to [Dei10, 5.1 p. 121] for more details on the subject than we have presented below.

Definition B.1. Let I be an index set and X_i be locally compact spaces as well as $K_i \subset X_i$ compact and open. Then

$$\prod_{i \in I}{}^{K_i} X_i = \bigcup_{\substack{E \subset I \\ \text{finite}}} \prod_{i \in E} X_i \times \prod_{i \in I \setminus E} K_i$$

together with the topology generated by the sets

$$\prod_{i \in E} U_i \text{ where } U_i \subset X_i \text{ is open and almost all } U_i \text{ are equal to} K_i \tag{B.1}$$

is called *restricted product* of the family $(X_i)_{i \in I}$ with respect to $(K_i)_{i \in I}$.

From the above commentary it is immediate that the individual products which are being united are locally compact. However, the restricted product is a locally compact space itself. Note that since every K_i is open, the sets presented in B.1 form a basis of a topology, meaning they are closed under intersection. The following remark states some elementary properties of $\prod_{i\in I}{}^{K_i} X_i$.

Remark B.2. a) For any partition of $I = A \dot\cup B$, we have

$$\prod_{i\in I}{}^{K_i} X_i \simeq \left(\prod_{i\in A}{}^{K_i} X_i\right) \times \left(\prod_{i\in B}{}^{K_i} X_i\right).$$

b) The canonical injection

$$\iota : \prod_{i\in I}{}^{K_i} X_i \hookrightarrow \prod_{i\in I} X_i$$

is continuous, but it is only homeomorphic onto its image, if $K_i = X_i$ for almost all $i \in I$.

c) If every space X_i is locally compact, $\prod_{i\in I}{}^{K_i} X_i$ is as well.

Both products in a) are equal as sets. The topological equivalence follows immediately by definition of the restricted product topology and the product topology itself.
In case b) the continuity of ι is a consequence of a), by splitting the product according to every open set $\prod_{i\in E} U_i \times \prod_{i\in I\setminus E} X_i$ in $\prod_{i\in I} X_i$ and considering both factors seperately. The homeomorphicity follows from the fact, that $\prod_{i\in I} K_i$ is open in $\prod_{i\in I} X_i$ if and only if $X_i = K_i$ for almost all I.
For c) let $(x)_{i\in I} \in \prod_{i\in I}{}^{K_i} X_i$, then there is a finite subset $E \subset I$, such that $x_i \in K_i$ for $i \in I \setminus E$. Choose compact neighbourhoods U_i of x_i for $i \in E$, then $U := \prod_{i\in E} U_i \times \prod_{i\in I\setminus E} K_i$ is, by use of a) and b), a compact neighbourhood of x.

Projective limits

Both, $\mathbb{Z}_p$ and $\mathbb{Z}_p^\times$ can be viewed as projective limits of families of finite algebraic structures, inheriting many of their properties. Hence, it might be sufficient to prove certain properties for the families of finite groups they arise from to lift these properties to the adelic structure. We will define projective limits and give $\mathbb{Z}_p$ and $\mathbb{Z}_p^\times$ as concrete realisations. Compare again [Dei10, 4.5 p. 113] for additional details.

Definition B.3. A partially ordered set $(I, \leq)$ is called *directed*, if to $a, b \in I$ there exists a common upper bound c, e.g. $a \leq c$ and $b \leq c$.

Projective limits will be defined for rings explicitly, implying the same structure for groups.

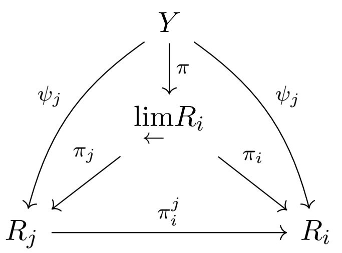
Definition B.4. A *projective system* of rings is a family $(R_i)_{i\in I}$ of rings R_i where $(I,\leq)$ is a partially ordered directed set together with homomorphisms

$$\pi_i^j : R_j \to R_i$$

for every pair $i,j\in I$ with $i\leq j$, such that

$$\pi_i^i = \mathrm{id}_{R_i} \quad \text{and} \quad \pi_i^j = \pi_i^k \circ \pi_k^j \quad \text{for } i\leq k\leq j\in I.$$

The last property is called *compatibility* of the homomorphisms. The *projective limit* of the projective system $((R_i)_{i\in I}, (\pi_i^j)_{i\leq j\in I})$ is the subring of all $(a_i)_{i\in I}\in\prod_{i\in I} A_i$ with $a_i = \pi_i^j(a_j)$ for all $i\leq j\in I$ and is denoted by $\varprojlim R_i$. It comes with mappings $\pi_i : \varprojlim R_i \to R_i$ such that for any Y with mappings $\psi_i : Y\to R_i$ fulfilling the same properties, there is exactly one $\pi : Y \to \varprojlim R_i$ such that the following diagram commutes.



The following example illustrates the concept and provides an idea of why the structure is called *projective* limit.

Example B.5.

- Let $(I,\leq) = (\mathbb{N}_0,\leq)$, p be a prime number and $R_i := \mathbb{Z}/p^{i+1}\mathbb{Z}$ as well as

$$\pi_i^j : R_j \to R_i, \quad x + p^{j+1}\mathbb{Z} \mapsto x + p^{i+1}\mathbb{Z}$$

  be the canonical projections. Then $((R_i)_{i\in I}, (\pi_i^j)_{i\leq j\in I})$ is a projective system and its limit is isomorphic to $\mathbb{Z}_p = \left\{\sum_{i=0}^{\infty} b_i p^i \mid 0\leq b_i < p-1 \text{ and } b_i\in\mathbb{N}\right\}$. In fact the mapping

$$\phi : \mathbb{Z}_p \to \varprojlim R_i, \quad \sum_{i=0}^{\infty} b_k p^k \mapsto \left(\sum_{k=0}^{i} b_k p^k \mod p^{i+1}\right)_{i\in I}$$

  is a topological ring isomorphism (note that both spaces are compact Hausdorff spaces).

- This example is used in [Ser12a, 3.1 p 15] to derive the results presented in 1.4 nd we stick to the notation of the source, denoting $U := \mathbb{Z}_p^\times$ and $U_n := 1 + p^n\mathbb{Z}_p$ for

$n \geq 1$ (here a projective system of groups is considered). Then $U \simeq \lim\limits_{\leftarrow} U/U_n$ with respect to the canonical projections:

$$\pi_i^j : U/U_j \to U/U_i, \quad x \cdot (1 + p^j \mathbb{Z}_p) \mapsto x \cdot (1 + p^i \mathbb{Z}_p).$$

Note that $U/U_n \simeq (\mathbb{Z}/p^n\mathbb{Z})^\times$, since

$$\mathbb{Z}_p^\times = \left\{ \sum_{i=0}^{\infty} b_i p^i \mid 0 \leq b_i < p - 1 \text{ and } b_i \in \mathbb{N} \text{ and } b_0 \neq 0 \right\}.$$

Then

$$\psi : \mathbb{Z}_p^\times \to \lim_{\leftarrow} U/U_i, \quad \sum_{i=0}^{\infty} b_k p^k \mapsto \left(\sum_{k=0}^{i} b_k p^k \mod p^{i+1} \right)_{i \in I}$$

is the desired isomorphism, which was directly derivable from example one with $\psi := \phi|_{\mathbb{Z}_p^\times}$.

B.2 Representation theory in characteristic 0

Groups, as such, appear throughout all sciences. In nearly every subfield ob mathematics the author is aware of but also in information technologies, physics (symmetry groups, atomic spectra, ...), biology, chemistry, and more . This is, in fact, due to the general fashion of their notion, alternatively expressed: their poor structure. In order to investigate particular representatives, a different perspective is advantageous: a more geometric approach that results in an apparent enrichment of structure (in fact it is more of a revelation of structure). The author speaks of realising groups as symmetry groups of geometric objects, by embedding them into GL_n. This is the fundamental idea of finite representation theory and it yields, in the case of complex finite dimensional representations, a self-contained, accessible edifice of ideas. There are, however, significant obstacles when representations in infinite dimensional spaces are considered, which prevent purely algebraic approaches from finding success. In this section we discuss, in a brief fashion, representation theoretic aspects of relevance for our subsequent investigations. They are based on [Ser12b], [Ber07], [Bum98], and [Con94].

Definition B.6. A *representation* ρ of a (topological) group G [or ring] is a homomorphism $\rho : G \to \mathrm{Aut}(V)$ for some complex vector space. Two representations ρ_1, ρ_2 of G are isomorphic, if there is a G-equivariant automorphism F of V, meaning for all $g \in G$:

$F\rho_1(g) = \rho_2(g)F$. Write $\dim \rho := \dim V$. In case a representation of a Banach $(*)$ algebra or a C^* algebra is considered, it will map to $\mathscr{B}(\mathcal{H})$, the bounded linear operators on a Hilbert space $\mathcal{H}$. A representation is called *unitary* if $\rho(G)$ lies in the unitary group of $\mathcal{H}$.

Remark B.7. The notion of a G-equivariant mapping is particularly restrictive in this setting. In fact, there is no non-trivial G-equivariant mapping between two non-isomorphic representations and in the case of identical representations these are given by homotheties. This fact is known as *Schur*'s lemma.

Example B.8. a) If G is a topological group, then the representations of G into $\mathrm{GL}_1(\mathbb{C})$ are *quasi characters*. Those which are unitary comprise the class of *characters* (cf. Definition 1.47). This indicates how restrictive the requirement of unitaricity is.

b) Let G be compact implying the existence of a right Haar measure μ_R. Further, let $\rho : G \to \mathscr{B}(\mathcal{H})$ be a representation of G. The symmetrised scalar product $\int_G \langle \rho(g)\,\cdot\,, \rho(g)\,\cdot\,\rangle \,\mathrm{d}\mu_R(g)$ renders ρ unitary. In particular, all representations of finite groups may be assumed to be unitary.

c) Let $G = \mathrm{SO}(2)$, then $\mathbb{T} \simeq \mathrm{SO}(2)$ via $\theta \mapsto \left(\begin{smallmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{smallmatrix}\right) = k_\theta$. In this sense, the unitary representations of $\mathrm{SO}(2)$ are exactly the characters of $\mathbb{T}$, which are given by mapping $\theta \mapsto e(k\theta)$ for $k \in \mathbb{Z}$.

d) The Weil representation (3.16) is an example of a unitary representation of $\mathrm{Mp}_2(\mathbb{Q}_p)$.

e) Let F be a field. Then $\mathrm{GL}_n(F)$ has a representation on the vector space of $n \times n$ matrices $M_n(F)$ via

$$r(M)(X) := M \cdot X \cdot M^T.$$

In case $n = 2$, the matrix $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ is an eigenvector of $r(M)$ for all $M \in \mathrm{GL}_n$ with Eigenvalue $\det(M)$. This matrix plays a major role in the theory of modular forms and also this thesis (cf. (3.13), for instance).

f) Let G be a toplogical (Lie) group [possessing a Haar measure], then the *right regular representation* ρ_R maps into the automorphism group of $C(G)$ $(C^\infty(G))$ $[L^2(G)]$ and is given by

$$\rho_R : g \mapsto [f \mapsto (h \mapsto f(hg))]\,,$$

in short $[\rho_R(g)f](h) = f(hg)$. Analogously, the *left regular representation* is given by $[\rho_L(g)f](h) = f\left(g^{-1}h\right)$. [These are unitary, provided a right, respectively left Haar measure is chosen.] This construction generalises readily to an action $G \circlearrowright X$

for proper X instead of $G \circlearrowleft G$ by multiplication, yielding a representation on $L^2(X)$ for instance.

In fact, the left regular representation is the ultimate example for finite groups as it carries their entire representation theoretic information. In this sense it is the most important example of a representation of finite groups.

Remark B.9. Let $H \leq G$ be groups and $\sigma : H \to \mathrm{Aut}(V)$ be a representation. The *induced representation* of σ to G is given by

$$\mathrm{Ind}_H^G(\sigma) = K[G] \otimes_{K[H]} V.$$

As in most theories objects may be used to construct new ones in the category at hand and in this spirit there is the obvious notion of finite direct sums and tensor products of representations. However, there is another naive construction becoming relevant later.

Example B.10. Let ρ_G, ρ_H be representations of (topological) groups [rings] G, H into $\mathrm{Aut}(V)$, $\mathrm{Aut}(W)$, and $\varphi : V \to W$ be an isomorphism. Then ρ_H is pulled back along φ via pointwise conjugation to a representation ρ'_H into $\mathrm{Aut}(V)$. If ρ_G, ρ'_H commute pointwise, there is an obvious representation

$$\rho_{G\times H} : G \times H \to \mathrm{Aut}(V), \quad (g, h) \mapsto \left[v \mapsto (\rho_G(g) \circ \rho'_H(h))(v)\right]$$

called the *product representation* of ρ_G, ρ'_H.

As usual the basic building blocks of the theory are to be investigated; in this case the constituents of direct sums of representations which are exactly constructed as expected.

Definition B.11. A *representation* $\rho : G \to \mathrm{Aut}(\mathcal{H})$ is called *irreducible* if there is no proper subspace $0 \neq \mathcal{K} < \mathcal{H}$ that is invariant under ρ, meaning $\rho(G)\mathcal{K} \subseteq \mathcal{K}$. Otherwise, it is called *reducible*.

The reason for the cautious fashion of the definition is that the theory does not behave particularly well for non-unitary infinite dimensional representations. A closed subspace $\mathcal{K}$ is obviously complementable such that $\mathcal{K} \oplus \mathcal{K}^\perp = \mathcal{H}$, but the invariance of $\mathcal{K}$ under ρ carries only in general over if G is unitary (cf. Example B.8 b) to recall that any representation of a finite group is essentially unitary).

Example B.12. Let $G = \mathrm{SO}(2)$ and consider the trivial representation given by the action of G on $\mathbb{C}^2$. Then let $U := \left(\begin{smallmatrix} 1 & i \\ i & 1 \end{smallmatrix}\right)$ and note that $Uk(\theta)U^{-1} = \left(\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{smallmatrix}\right)$. Meaning that conjugation with U yields an isomorphism to an obvious direct sum of characters of $\mathrm{SO}(2)$.

The next instance depicts the classical theory of finite groups.

Remark B.13. Let G be finite. Then any finite dimensional representation ρ of G decomposes into a direct sum of irreducible representations. Further, let $(\rho_i)_{i\in I}$ be a complete set of isomorphically distinct irreducible representations of G. For ρ as above, define the *character* $\chi_\rho(g) \coloneqq \mathrm{tr}(\rho(g))$ of ρ. It is invariant on conjugacy classes of G. Such functions are called *class functions* and they form a vector space $Cl(G)$ whose dimension equals the number of conjugacy classes of G. It is equipped with the scalar product

$$\langle \psi, \phi \rangle = |G|^{-1} \sum_{g\in G} \psi(g)\overline{\phi}(g).$$

Crucially, $\langle \chi_{\rho_i}, \chi_\rho \rangle$ equals the number of irreducible constituents of ρ isomorphic to ρ_i, consequently χ_ρ characterises ρ completely and we find $\langle \chi_\rho, \chi_\rho \rangle = 1$ if, and only if, ρ is irreducible. Computing the character of ρ_L results in

$$\rho_L \simeq \bigoplus_{i\in I} \bigoplus_{k=1}^{\dim \rho_i} \rho_i$$

causing $\sum_{i\in I} \dim \rho_i \rho_i(g) = \delta_{g,1}$ which in turn implies $\sum_{i\in I} (\dim \rho_i)^2 = |G|$. Further, applying this information to the regular representation yields that $(\rho_i)_i$ is an ortho normal basis of $Cl(G)$ meaning $|I|$ equals the number of conjugacy classes of G. As a corollary one obtains that G is abelian, if, and only if, all ρ_i are one dimensional. So far everything was also applicable in characteristic different from 0.
If, in addition, characteristic 0 is imposed, another harsh constraint is forced upon the numbers $\dim \rho_i$: these divide $|G|$.

B.3 Symplectic and metaplectic groups

The term *metaplectic group* describes a covering group of the symplectic group SL_2, which, in general, will be a central extension[1] of SL_2 by $\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{T}$. In our case, however, an extension by $\mathbb{T}$ is considered, as it splits over subgroups for which the common $\mathbb{Z}/2\mathbb{Z}$ extension does not (cf. [Kud94]). It will be referred to as *the metaplectic group* as if there were only one and denoted $\overline{\mathrm{SL}}_2$. A concrete description is provided by [KRY06, 8.5 pp. 320-336], which accords with the choice in [KY10]. The extension by $\mathbb{Z}/2\mathbb{Z}$, usually

[1] Cf. [Tao10] for a brief overview of group extensions.

referred to as the metaplectic group, is derivable from $\overline{\mathrm{SL}}_2$ as the latter determines a cocycle $\overline{c} \in H^2(\mathrm{SL}_2, \mathbb{T})$. Since this homology group contains only one non-trivial element of order two, the cover is essentially unique and coordinates are chosen by picking a concrete representative for $\overline{c}$, which shall be denoted by c and equal the one from [KRY06, 8.5 p. 320]. If chosen, $\overline{\mathrm{SL}}_2$ is realised as the set of pairs $(\gamma, \zeta) \in \mathrm{SL}_2 \times \mathbb{T}$ with multiplication $(\gamma, \zeta) \cdot (\gamma', \zeta') = (\gamma\gamma', c(\gamma, \gamma')\zeta\zeta')$. Formally, the information above, supplemented by a few remarks about subgroups and splits mentioned in Section 3.2 in the adelic case, suffices in order to deal with the metaplectic group as far as the major aim of this thesis is concerned. However, the author sought to supplement the thesis by additional details concerning the construction of metaplectic groups.
Throughout this section, the mapping $x \mapsto 2x$ is usually assumed to be an isomorphism.

Abstract symplectic groups

There is a conceptional approach to the metaplectic group motivated by quantum mechanics which shall be discussed in the following paragraphs. It was introduced by André Weil [Wei64] in general in order to describe the theory of theta functions in group theoretic terms (cf. [Gel74]). The concrete case, sufficient for our investigation, is sketched concisely in [Gel76, 2.3 p. 28] or, more crisply but less conceptually in [Kud96], and the basis of its construction shall be presented in greater detail, based on [Wei64]. Starting with locally compact abelian groups G, H isomorphic to their Pontryagin duals G^*, H^*, respectively, a completion of $\mathbb{Q}$ with respect to an absolute value function for instance, every morphism $\alpha : G \to H$ induces a dual morphism α^* via pullback:

$$\alpha^* : H^* \to G^*, \qquad h^* \mapsto h^* \circ \alpha.$$

The association $\alpha \mapsto \alpha^*$ defines a contravariant, isomorphism preserving functor, which also conserves the modulus of the morphisms, meaning the scaling factor for pushforward measures. Furthermore, we have $(\alpha^*)^* = \alpha$ when the canonical isomorphy $G \simeq (G^*)^*$ is taken into account. Note that product group $G \times G^*$ is isomorphic to its dual $G^* \times G$ via

$$\begin{pmatrix} x \\ x^* \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ x^* \end{pmatrix}.$$

Similarly, endomorphisms σ of $G \times G^*$ may be written in matrixform

$$\begin{pmatrix} x \\ x^* \end{pmatrix} \mapsto \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ x^* \end{pmatrix}, \quad \text{with dual morphism } \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix} = \sigma^*, \tag{B.2}$$

where α, β, γ and δ denote morphisms mapping appropriately between G or G^*, respectively. The pairing

$$F : (G \times G^*) \times (G \times G^*) \to \mathbb{T}, \quad [(x, x^*), (y, y^*)] \mapsto (x, x^*) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y \\ y^* \end{pmatrix} = y^*(x) \tag{B.3}$$

will become relevant later, whereas for now we are interested in the pairing

$$\kappa : (G \times G^*) \times (G \times G^*) \ni [\omega_1, \omega_1] \mapsto F(\omega_1, \omega_2)F(\omega_2, \omega_1)^{-1} = y^*x/x^*y \in \mathbb{T}, \tag{B.4}$$

which has the *symplectic group* as its invariance group.

Definition B.14. The *symplectic group* $\mathrm{Sp}(G)$ of a locally compact abelian group G is defined to be the group of automorphisms of $G \times G^*$ which fix the pairing κ. Hence, with the notation as above an automorphism σ is in $\mathrm{Sp}(G)$ if $\kappa(\sigma\omega_1, \sigma\omega_2) = \kappa(\omega_1, \omega_2)$ holds for all $\omega_1, \omega_2 \in G \times G^*$.

By direct computation this condition for σ may be expressed equivalently as

$$\sigma^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Example B.15. In the concrete case of $G = \mathbb{Q}_p$ the group $\mathbb{Q}_p$ may be identified with its Pontryagin dual via the exponential mapping and the entries of the endomorphism σ of $G \times G^* \simeq \mathbb{Q}_p^2$ are interpreted as elements of $\mathbb{Q}_p$. Consequently, the invariance condition for a symplectic automorphism σ reduces to

$$1 = (\alpha\delta - \beta\gamma) = \det(\sigma).$$

In particular, this testifies that the symplectic group is exactly the special linear group $\mathrm{SL}_2(\mathbb{Q}_p)$ if G is chosen to be the additive group of a p-adic field. This closes the paragraph about realising SL_2 via a symplectic pairing.

The Heisenberg group

The subsequent sections cover the construction of the metaplectic group, following [Wei64]. He viewed the symplectic group as a group of unitary operators on $L^2(G)$ and utilised tools from the theory of C^* Algebras to prove the existence of a group extension, which he called *metaplectic group*. As a first step, the Heisenberg group of G is constructed on which $\mathrm{Sp}(G)$ acts.
For $\omega = (u, u^*) \in G \times G^*$ define the unitary operator

$$U(\omega) : L^2(G) \to L^2(G), \quad f(x) \mapsto f(x+u) \cdot u^*(x).$$

Moreover, we find

$$U(\omega_1)U(\omega_2) = F(\omega_1, \omega_2) \cdot U(\omega_1 + \omega_2). \tag{B.5}$$

Consequently, the mapping

$$G \times G^* \times \mathbb{T} \ni (\omega, t) \mapsto U(\omega)t \tag{B.6}$$

defines a bijection onto an operator group which can be used to pullback the group law, obtaining

$$(\omega_1, t_1) \cdot (\omega_2, t_2) = (\omega_1\omega_2, F(\omega_1, \omega_2)t_1t_2). \tag{B.7}$$

This law makes $A(G) := G \times G^* \times \mathbb{T}$ into a locally compact group with the ordinary product topology, which, exported to the corresponding operator group, equals the strong operator topology. The group $A(G)$ equipped with multiplication given by (B.7) is called *Heisenberg group* of G, for a reason which will become clear after the following historical example.

Example B.16 (history, quantum mechanics). In 1925 Werner Heisenberg suggested solving the harmonic oscillator, a physical system, with operators P, Q on which he imposed a commutator relation $[Q, P] = i \cdot \mathrm{id}$. Nearly simultaneously, Erwin Schrödinger suggested solving the very same system with a partial differential equation. Both resulted in the same physical solution, leaving the community puzzled before these seemingly different approaches.
John Von Neumann showed not only that both descriptions were mathematically equivalent,

but that Heisenberg's abstract condition determines the mathematical system up to unitary equivalence [Neu31] if irreducibility is imposed and that all representations are direct sums of this essentially unique irreducible one. His proof relied upon a one to one relation between self adjoint operators on Hilbert spaces and unitary strongly continuous 1-parameter operator groups suggested by Herrmann Weyl and published by Marshall Harvey Stone [Sto30]. Transferring the operators P, Q to the corresponding unitary operators translates the commutator relation essentially to (B.5), called the *Weyl relation*. Roughly 20 years later, 1949, George Whitelaw Mackey published a theorem analogous to that of von Neumann, but for locally compact abelian (seperable) groups [Mac49], so that the unitary irreducible representations of the Heisenberg group in the above abstract setting were understood.[2]

The centre of $A(G)$ is evidently $\mathbb{T}$, which is closed as it is the kernel of the continuous homomorphism $\pi_1 : A(G) \to G \times G^*, \quad (\omega, t) \mapsto \omega$ and therefore the diagram induces a topological group isomorphism

$$\begin{array}{ccc} A(G) & \xrightarrow{\pi_1} & G \times G^* \\ \downarrow & \nearrow & \\ A(G)/\mathbb{T} & & \end{array}$$

Recall that the concern of this section is to construct a covering group for the symplectic group $\mathrm{Sp}(G)$ which is a group of automorphisms of $G \times G^*$.

The symplectic group as a group of automorphisms of $A(G)$

The next step is to consider the group of automorphisms $B(G)$ of $A(G)$ and to prove that it contains a copy of $\mathrm{Sp}(G)$. This realises the symplectic group as a group of automorphisms of operators and ultimately leads to the existence of a projective representation of $\mathrm{Sp}(G)$, called *Weil representation*. Elements of $B(G)$ which operate trivially on $\mathbb{T}$ form a subgroup, denoted $B_0(G)$. An automorphism $s \in B_0(G)$ evidently induces an automorphism σ on $G \times G^*$ and may be represented as

$$s(\omega, t) = (\sigma\omega, f(\omega)t),$$

[2]Elaborate historical notes discussing the development of unitary representation theory from this context may be found in [Mac76, Appendix p. 209 a.s.].

where $f : G \times G^* \to \mathbb{T}$ is a continuous mapping. The condition that an automorphism of this form defines an element of $B(G)$ is equivalent to

$$\forall\, \omega_1, \omega_2 \in G \times G^* : \quad f(\omega_1 + \omega_2)\, f(\omega_1)^{-1}\, f(\omega_2)^{-1} = F(\sigma\omega_1, \sigma\omega_2)\, F(\omega_1, \omega_2)^{-1}, \qquad \text{(B.8)}$$

implying that $B_0(G)$ may be identified with the group of pairs (σ, f) satisfying (B.8), with multiplication given by

$$(\sigma, f)\,(\sigma', f') = (\sigma\sigma', f''),$$

where $f''(\omega) = f(\omega)f'(\sigma\omega)$. Since the left hand side of equation (B.8) is symmetric in ω_1, ω_2 the automorphism σ has to be a symplectic element, leaving κ, defined in (B.4), invariant. This suggests seeking a decomposition of $B_0(G)$ by $\mathrm{Sp}(G)$ and in fact there is a splitting exact sequence:

$$1 \longrightarrow (G \times G^*)^* \longrightarrow B_0(G) \longrightarrow \mathrm{Sp}(G) \longrightarrow 1. \qquad \text{(B.9)}$$

Note that the kernel of the natural projection $B_0(G) \to \mathrm{Sp}(G)$ is exactly $(G \times G^*)^*$, according to condition (B.8). As a result, it suffices to construct the splitting morphism for (B.9), which is done by computation. For the sake of lucidity the application of characters is denoted with duality brackets. Henceforth, $f \in (G \times G^*)^*$ may be written as $f(g, g^*) = \langle g, a^* \rangle \cdot \langle a, g^* \rangle$ with suitable $a \in G, a^* \in G^*$. Before constructing the splitting morphism for (B.9) explicitly, a general construction concerning generators of characters of second degree has to be carried out. For that sake let f denote a character of *second degree*[3] of G, that is a function $f : G \to \mathbb{T}$, such that

$$G \times G \ni (g, g') \mapsto f(g + g')f(g)^{-1}f(g')^{-1} \in \mathbb{T} \qquad \text{(B.10)}$$

is a bilinear pairing. These functions f form a group with pointwise multiplication denoted $X^2(G)$. Furthermore, the associated pairing (B.10) may be represented by a morphism $\rho : G \to G^*$:

$$f(g + g')f(g)^{-1}f(g')^{-1} = \langle g, \rho g' \rangle.$$

Since the lefthand side is obviously symmetric, ρ has to be as well, namely $\rho = \rho^*$ and the mapping $\pi : f \mapsto \rho$ defines a homomorphism onto the additive group of symmetric morphisms $\mathrm{Hom}_s(G, G^*)$ and has kernel G^*. If doubling defines an automorphism of G,

[3] The meaning of characters of second degree for bilinear pairings is similar to the one of quadratic forms for bilinear forms in linear Algebra.

namely $G \ni g \mapsto 2g \in G$ is bijective[4] with inverse 2^{-1}, then the association

$$\iota : \mathrm{Hom}_s(G, G^*) \ni \rho \mapsto (g \mapsto \langle g, \rho 2^{-1} g\rangle) \in X^2(G) \tag{B.11}$$

defines a section of π, making the following exact sequence split:

$$1 \longrightarrow G^* \longrightarrow X^2(G) \underset{\iota}{\overset{\pi}{\rightleftarrows}} \mathrm{Hom}_s(G, G^*) \longrightarrow 1. \tag{B.12}$$

Ultimately, the split of (B.9) may be dealt with. For this purpose define the function

$$f'(g, g^*) = f(g, g^*)\langle \gamma g^*, -\beta g\rangle,$$

where $\sigma \in \mathrm{Sp}(G)$ was written in matrixform as in (B.2). Then, equation (B.8) reads

$$f'(g_1 + g_2, g_1^* + g_2^*) = f'(g_1, g_1^*)f'(g_2, g_2^*) \cdot \langle g_1, \beta^*\alpha g_2\rangle \cdot \langle \delta^*\gamma g_1^*, g_2^*\rangle$$

for f' and the definitions $h(g) = f'(g, 0), h^*(g^*) = f'(0, g^*)$ lead to

$$h(g_1 + g_2) = h(g_1)h(g_2) \cdot \langle g_1, \beta^*\alpha g_2\rangle,$$
$$h^*(g_1^* + g_2^*) = h(g_1^*)h(g_2^*) \cdot \langle \delta^*\gamma g_1^*, g_2^*\rangle.$$

Since σ is assumend to be symplectic, the morphisms $\beta^*\alpha, \delta^*\gamma$ are symmetric, hence the above relation describes the transformation property of a character of *second degree*. Thereupon, f may be written as a product of two characters of second degree and a bilinear form:

$$f(g, g^*) = h(g)h^*(g^*) \cdot \langle \gamma g^*, \beta g\rangle.$$

In view of (B.12), especially (B.11), the function f is effortlessly expressed in terms of σ as

$$f_\sigma(g, g^*) = \langle g, \beta^*\alpha 2^{-1} g\rangle \cdot \langle \delta^*\gamma 2^{-1} g^*, g^*\rangle \cdot \langle \gamma g^*, \beta g\rangle.$$

The association $\sigma \mapsto f_\sigma$ defines a right split of (B.9), which proves that $B_0(G)$ contains a copy of the symplectic group $\mathrm{Sp}(G)$, as claimed. Next, these are related to unitary operators on $L^2(G)$.

[4]This requirement is in fact equivalent to $G \times G^*$ having only one equivalence class of irreducible σ-representations [Mac64].

A theorem of Segal

The following theorem, initially from Segal [Seg63, Thm. 2 p. 39], and reformulated by Weil [Wei64, 10. Thm. 1 p. 157] is not proven here, but its implications relevant for our discourse are discussed.

Weil shows that every automorphism of $A(G)$ which fixes its centre is given by an inner automorphism of the unitary group of $L^2(G)$. This ensures an abstract argument for the existence of the Weil representation (cf. section B.3).

Theorem B.17. *Let $\overline{B}_0(G)$ denote the normaliser of $A(G)$ in the group of automorphisms of $L^2(G)$. Then every element of $B_0(G)$ is a restriction of an inner automorphism of the unitary group where the latter may be chosen from $\overline{B}_0(G)$. Furthermore, this natural projection $\pi_0 : \overline{B}_0(G) \to B_0(G)$ has kernel $\mathbb{T}$, producing the exact sequence:*

$$1 \longrightarrow \mathbb{T} \longrightarrow \overline{B}_0(G) \xrightarrow{\pi_0} B_0(G) \longrightarrow 1. \tag{B.13}$$

The theorem above provides an operator theoretic characterisation of the automorphism group $B_0(G)$ of $A(G)$ which contains a copy of $\mathrm{Sp}(G)$. In fact, it plays a central role in constructing Weil's abstract metaplectic group Mp which is discussed briefly in the subsequent section.

The metaplectic extension

What has been done above may be partly transferred to the situation where $G = V$ is a finite dimensional vector space over a local field F. In this case the theory may be linearised, replacing the Pontryagin dual G^* with the algebraic dual vector space V'. In particular, equation (B.4) reads

$$\beta : (V\times V') \times (V\times V') \ni [\omega_1, \omega_1] \mapsto F(\omega_1, \omega_2) - F(\omega_2, \omega_1) = (x, x') \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix} \in F, \tag{B.14}$$

with F analogous to (B.3). Similarly, the product $V \times V' \times F$ becomes a group with the law

$$(\omega_1, t_1) \cdot (\omega_2, t_2) = (\omega_1 + \omega_2, F(\omega_1, \omega_2) + t_1 + t_2), \tag{B.15}$$

denoted $H(V)$. In case of $V = F$, the associated symplectic group is exactly $\mathrm{SL}_2(F)$, analogous to example B.15. Inspired by the abstract treatment above, automorphisms of $H(V)$ which are composed of an automorphism σ of $V \times V'$ and a quadratic form f on $V \times V'$ and operate via

$$(\omega, t) \mapsto (\sigma\omega, f(\omega) + t)$$

are considered. In order to define an automorphism of $H(V)$ via this association, it is necessary and sufficient to satisfy

$$\forall \omega_1, \omega_2 \in V \times V' : \quad f(\omega_1 + \omega_2) - f(\omega_1) - f(\omega_2) = F(\sigma\omega_1, \sigma\omega_2) - F(\omega_1, \omega_2). \quad \text{(B.16)}$$

Nearly identically to the multiplicative case, these automorphisms comprise a group with group law

$$(\sigma, f)(\sigma', f') = (\sigma\sigma', f'')$$

where $f''(\omega) = f(\omega) + f'(\sigma\omega)$. This group is denoted by $\mathrm{Ps}(V)$ and named *pseudo symplectic* group of V. Again, by symmetry of the left hand side of (B.16) σ has to be symplectic, where on the other hand, every symplectic element σ determines a quadratic form via (B.16). As a consequence, $(\sigma, f) \to \sigma$ defines an isomorphism of $\mathrm{Ps}(V)$ and $\mathrm{Sp}(V)$. In addition, if $F = \mathbb{Q}_p$ and χ denotes a non-trivial character of $\mathbb{Q}_p$, then the relations above may be transformed to the multiplicative case by applying χ to both sides of the equations. This may be utilised to pull fragments of the multiplicative theory back to the additive case. Explicitly, the mapping

$$H(V) \to A(G), \quad (\omega, t) \mapsto (\omega, \chi(t))$$

defines a homomorphism as well as

$$\mu : \mathrm{Ps}(V) \to B_0(G), \quad (\sigma, f) \mapsto (\sigma, \chi \circ f).$$

The latter is injective, if $\mathrm{char}(F) \neq 2$.[5] Note that the group was denoted by G, instead of V, since it is considered to be a locally compact topological group with Pontryagin dual, rather than a linear vector space. This difference in notation might appear captious but it is meant to remind the reader which point of view led to the associated results.

[5] In fact, the mapping μ is surjective if, and only if, $F = \mathbb{Q}_p$ for some $p \leq \infty$ (cf. [Rao93, p. 350]). Rao's $\mathrm{Ps}(H, F)$ is essentially our $\mathrm{Ps}(V)$ (where the second component is the induced multiplicative function) and $\mathrm{Ps}(H)$ in his notation is identical with our $B_0(G)$.

Definition B.18. Define the *abstract metapletic group* $\mathrm{Mp}(V)$ to be the subgroup of $\mathrm{Ps}(V) \times \overline{B}_0(V)$, consisting of those elements $(s, \overline{s})$, such that $\mu(s) = \pi_0(\overline{s})$ (cf. Figure B.3.1).

The natural projection onto $\mathrm{Ps}(V)$ will be denoted by π. Its surjectivity is a consequence of π_0's (cf. Theorem B.17). The kernel of π consists exactly of $\mathbb{T} < \overline{B}_0(V)$, so that the upper horizontal sequence in the commutative Diagram B.3.1 is exact. Consequently, $\mathrm{Mp}(V)$ is a central extension of $\mathrm{Sp}(V)$.
Furthermore, the injectivity of μ implies that the projection of $\mathrm{Mp}(V)$ into $\overline{B}_0(G)$ must be injective as well. Hence, $\mathrm{Mp}(V)$ admits a faithful unitary continuous representation on $L^2(G)$, where the automorphism group is equipped with the strong operator topology. In fact, it even restricts to a representation on the subspace of Schwartz Bruhat functions $S(V)$, since $B_0(G)$ operates on $S(V)$.
After the introduction of the metaplectic group, Weil constructs the adelic pendant to Mp and proves [Wei64, IV p. 194] that the cover is never trivial and always reduces to a topological central extension

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \overline{\mathrm{Sp}}(V) \longrightarrow \mathrm{Sp}(V) \longrightarrow 1,$$

since the cohomology class it determines in $H^2(G, \mathbb{T})$ is of order 2. If $V = F$, which is the case we are interested in, this double cover is unique (cf. [Gel76, 2 Prop. 2.3 p. 14]).[6]
However, working with the extension $\mathrm{Mp}(V)$ is more convenient for the adelic setting due to its splitting over certain subgroups for which the $\mathbb{Z}_2$ extension does not (cf. [Kud94]). In the process of Weils proceedings, a class of representations named after him is constructed which will be discussed in the following section.

The Weil representation

The most concise conceptional approaches to the relevant setting of the Weil representation the author is aware of is found in [Kud96] and the slightly less dense [Pra93]. An extended description is found in [LV80] and a slightly more explicit approach in [Rao93] (note that Rao's notation is slightly different, for instance, he writes $Ps(G)$ for $B_0(G)$.). The abstract reason for the existence of a representation of $\mathrm{Sp}(V)$ is a theorem by von Neumann [Neu31] which was later generalised by [Mac49] (cf. Remark B.16). This essentially means that any irreducible unitary representation of the Heisenberg group $A(G)$ on $L^2(G)$ is unique

[6] Further, [Wed16, 2.6 p. 34] or [Hat02, 1.3 p. 54] may be considered for a conceptual approach to coverings.

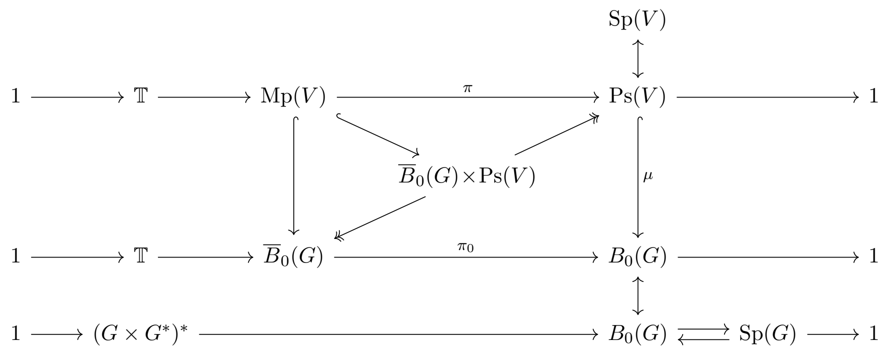


Figure B.3.1: Cf. (B.13) from Theorem B.17 for the horizontal sequence in the middle. The bottom sequence is in accord with the right split of (B.9) discussed in Section B.3. By injectivity of μ, the projection of $\mathrm{Mp}(V)$ into $\overline{B}_0(G)$ must be injective as well.

up to unitary equivalence[7]. In fact, if (B.6) defines a unitary representation, then for any element $\sigma \in \mathrm{Sp}(G) = \mathrm{Sp}(V)$

$$A(G) = G \times G^* \times \mathbb{T} \ni (\omega, t) \mapsto U(\sigma\omega)t \tag{B.17}$$

defines another one. Consequently,

$$U(\sigma\omega) = \Gamma(\sigma)U(\omega)\Gamma(\sigma)^{-1}$$

for some unitary operator $\Gamma(\sigma)$, which is determined up to a scalar. As a result, the mapping $\sigma \mapsto \Gamma(\sigma)$ gives rise to a projective representation[8] of the symplectic group $\mathrm{Sp}(V)$, meaning a continuous homomorphism into $\mathcal{U}(L^2(G))/\mathbb{T}$. This projective representation may be lifted to a central extension of $\mathrm{Sp}(V)$, which may be taken as a double cover. As mentioned above there is only one double cover of $\mathrm{Sp}(V)$, resulting in a representation of $\overline{\mathrm{Sp}}(V)$.

However, the above discussion is not very instructive, especially if computations are required. For this thesis only Weil representations of SL_2, more accurately its metaplectic cover $\overline{\mathrm{SL}}_2$, are required. Weil representations for this setting are given more explicitely

[7]The proof features elementary fragments of the theory of C^*-Algebras [Seg63, p. 39, 41], which happen to be of no further use in supporting a deeper understanding of the ultimate goal of this thesis. As a consequence, it will be disregarded.

[8]Details about projective representations are found in [Mac58]. If the reader is interested in the historic approach a far more elementary and less sophisticated source is [Sch04]. Note that the multipliers of projective representations are also called *Schur multiplier*.

in terms of integral and multiplication operators in Subsection B.3. Note that the term *local Weil representation* denotes a representation of the metaplectic extension $\overline{\mathrm{SL}}_2(F_p)$ of $\mathrm{SL}_2(F_p)$ into the space of automorphisms of a Schwartz Bruhat space $S(V_p)$ for a finite vector space V_p over F_p, while the *global* (adelic) representation emerges from their combination. The representation factors through SL_2 if, and only if, the space V_p is even dimensional, rendering it an ordinary representation of SL_2 in that case.

B.4 Figures

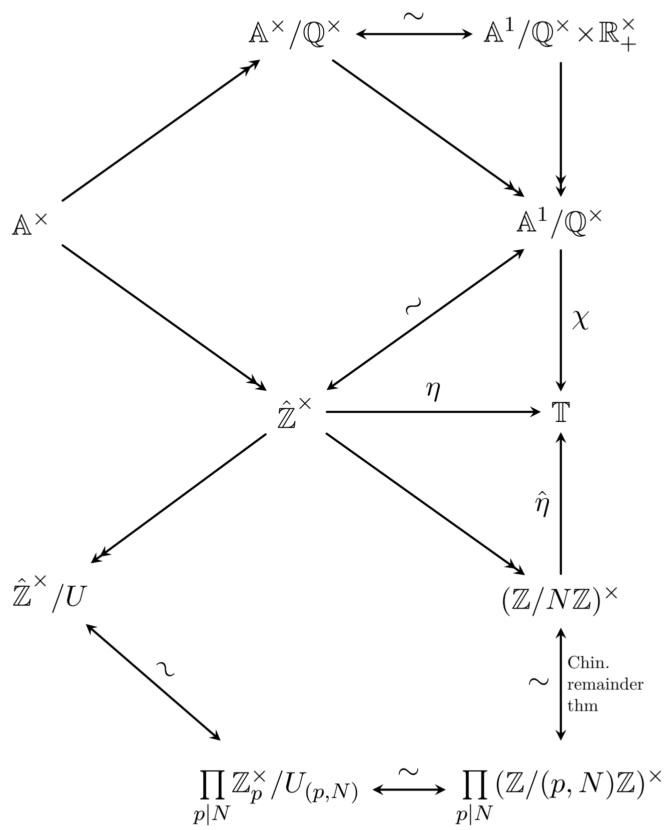


Figure B.4.1: Illustration of the relation between finite Hecke characters and primitive Dirichlet characters in form of a commutative diagram. Double tipped arrows mark surjective projections, where the ones from $\mathbb{A}^\times \to \hat{\mathbb{Z}}^\times \to (\mathbb{Z}/N\mathbb{Z})^\times$ are induced by taking the corresponding long paths. However, both of them are canonical and using the Chinese remainder theorem $\hat{\mathbb{Z}}^\times \simeq \varprojlim (\mathbb{Z}/N\mathbb{Z})^\times$ may be verified. An in depth description is found in [Opi18, Sec. 1.10 p. 20].

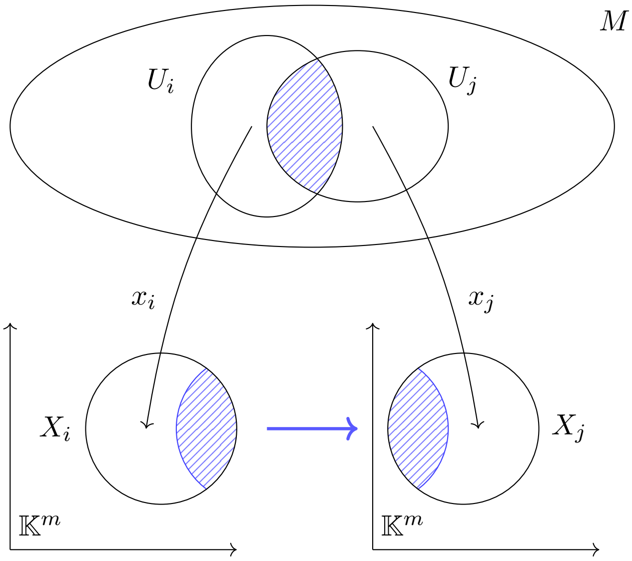


Figure B.4.2: Visualisation of charts for a manifold M of dimension m over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and their respective transition map (represented by a blue arrow). Here, U_i, U_j are neighbourhoods on M, X_i, X_j neighbourhoods in $\mathbb{K}^m$ and x_i, x_j denote the local coordinates. The illustration is based on [Tho15].

Index

Glossary

Notation	Description	Page List
R	A commutative, unitary, and associative ring.	13
q	A quadratic form.	13
b	The bilinear form associated to a quadratic form.	13
F	A field. If the characteristic is not mentioned, it will never equal 2.	13
V	A vector space. If the associated field is not mentioned, it will be a $\mathbb{Q}$ vector space.	13
S	The Gram matrix of q.	13
$\det(M)$	The determinant of a free quadratic Module (M, q), also $\det(\mathrm{q}) = \det(S) \in R/(R^\times)^2$ for the Gram matrix S.	13
$N^\perp$	The orthogonal complement of a subset N of a quadratic module (M, q).	14
H	The hyperbolic plane, meaning an unimodular $\mathbb{Z}$ lattice of rank 2 and signature $(1, 1)$.	15
L	A lattice (usually accompanied by a quadratic form q).	15
V_p	The tensor product $V \otimes_{\mathbb{Q}} \mathbb{Q}_p$ usually accompanied by a quadratic form q.	15
L_p	The tensor product $L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ usually accompanied by a quadratic form q.	15
$\mathrm{q}_1 \otimes \mathrm{q}_2$	The tensor product of two quadratic forms.	17
$\mathrm{Groth}(M)$	The Grothendieck group of a cancellation monoid M.	17
$W(F)$	The Witt ring of a field F.	17
L'	The dual lattice of L.	18
$\mathbb{Q}_p$	The field of p-adic numbers.	23
$\mathbb{Z}_p$	The ring of p-adic integers.	23

Notation	Description	Page List
$\mathbb{A}_f$	The finite Adele ring - containing all $\mathbb{Q}_p$ for finite primes $p < \infty$.	24
$\hat{\mathbb{Z}}$	The product of all $\mathbb{Z}_p$ for $p < \infty$. It is an open and compact subring of $\mathbb{A}_f$.	24
$\mathbb{A}$	The Adele ring - containing all $\mathbb{Q}_p$.	24
$\zeta(s)$	The Riemann Zeta function.	24, 61
$\lvert \cdot \rvert$	The absolute value on $\mathbb{A}^\times$.	25
$\mathbb{A}_1$	The kernel of the absolute value on $\mathbb{A}^\times$.	25
G	A locally compact abelian group with elements g.	25
μ	The Haar measure of G.	25
δ_G	The modular quasi character of G, where $\delta_G(g)\mu(A) = \mu(Ag)$.	26, 168
χ	A character of G.	26, 327
$\mathbb{T}$	The group of unitary complex numbers.	26, 319, 323
G^*	The dual group of a locally compact abelian group G.	26
e_∞	Bicharacter for $\mathbb{Q}_\infty$.	26
e_p	Bicharacter for $\mathbb{Q}_p$.	26
$\hat{f}$	The Fourier transform of a function $f \in L^1(G)$.	26
$\psi^{\mathrm{b}}(a, x)$	The concatenation of a character ψ and a bilinearform b.	27
$\mathcal{S}(S)$	The space of Schwartz Bruhat functions on an appropriate topological space S.	28, 79
χ_p	The character associated to the place p.	30
$\left(\frac{v}{p}\right)$	The Legendre symbol of $v \in \mathbb{Z}$.	31
$\mathbb{F}_p$	The finite field with p elements.	31
χ^b	The finite Hecke character associated with $b \in \mathbb{Q}$.	31
χ_L	The character associated to a quadratic lattice (L, q).	32
$Q(R)$	The quotient field of an integral domain R.	32
$\mathcal{O}_F$	The ring of integers of a field $F/Q(R)$.	32
$\mathrm{tr}_{F/K}$	The trace of an element $x \in F$ over the extension $F/Q(R)$.	33
$\mathrm{N}_{F/K}$	The norm of an element $x \in F$ over the extension $F/Q(R)$.	33

Notation	**Description**	**Page List**
M_x	The linear Mapping $M_x : F \to F, y \mapsto xy$ associated to $x \in F$.	33
$\mathfrak{a}$	A fractional ideal of $F/Q(R)$.	34
J_F	The group of fractional ideals $F/Q(R)$.	34
Cl_F	The ideal class group of $F/Q(R)$.	34
Γ	A discrete subgroup of a topological group G – usually with additional properties to ensure that the quotient is equipped with an appropriate structure (cf. Fuchsian subgroup of SL_2).	41
K	A compact subgroup of a topological group G. In an adelic setting K will usually assumed to be open.	41
SL_2	The special linear group of 2×2 matrices.	42, 319
SO_2	The special orthogonal group of 2×2 matrices.	42, 82
E/F	An elliptic curve over the field F.	44
$\wp$	The Weierstraß $\wp$-function.	46
$\mathbb{H}$	The upper half plane consisting of all complex numbers c with $\mathrm{Im}(c) > 0$.	47
$\Gamma(1)$	The modular group $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$.	48
$Y(\Gamma)$	The modular curve $Y(\Gamma) = \Gamma\backslash\mathbb{H}$ for an automorphism group Γ of $\mathbb{H}$.	48
$\mathcal{F}_\Gamma$	A fundamental domain for the automorphism group Γ of $\mathbb{H}$.	48
∞	The element added to the modular curve $Y(\Gamma(1))$ for compactification. It is also referred to as the *cusp* ∞ or just the cusp.	50
$X(\Gamma)$	The compactified modular curve for Γ.	50
$j(\gamma, \tau)$	The factor of automorphy, relying on $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$.	53
$\vert_k$	Petersson operator of weight $k \in \mathbb{Z}$, where $f\vert_k\gamma = j(\gamma, \tau)^{-k} f(\gamma\tau)$. It defines a right operation of $\mathrm{SL}_2(\mathbb{Z})$ on meromorphic (holomorphic) functions on $\mathbb{H}$.	53
$\mathcal{M}_k(\Gamma)$	The Weierstraß $\wp$-function.	53
$\Gamma(N)$	The kernel of the natural projection $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ for a natural number N.	54

Notation	**Description**	**Page List**
$\Gamma_0(N)$	A special congruence subgroup.	54
$\Gamma_1(N)$	A special congruence subgroup.	54
$\pi_1(M)$	The fundamental group of a Riemann surface M.	55
E_k	The classical normalised Eisenstein series of weight $k \in 2\mathbb{N}$.	56
$\sigma_s(n)$	The divisor function of the number n with parameter $s \in \mathbb{C}$.	57
$E_k(\tau, s)$	The normalised (non-holomorphic) augmented Eisenstein series of weight $k \in 2\mathbb{N}$.	58
$G_k^*(\tau, s)$	The complete augmented Eisenstein series of weight $k \in 2\mathbb{N}$.	58
K_s	The K Bessel function.	59
R_k	A Maass differential operator.	59
θ_L	The theta function associated to the quadratic lattice L.	60
$r_L(n)$	Representation number of the natural number n by the quadratic lattice L.	61
ζ^*	The completed Riemann Zeta function.	61
$\#(S, T)$	The representation number of the Gram matrix T by S.	62, 175
$\mathrm{Pos}_n(\mathbb{Z})$	The set of symmetric positive definite $n \times n$-matrices with coefficients in $\mathbb{Z}$.	62
${}^{\mathrm{pr}}r_p$	The projective representation on SL_2, inducing the Weil representation.	80
c_L	The Leray cocycle.	80
$\gamma(\eta)$	The Weil index of the character η of second degree.	80
$q(g_1, g_2)$	The Leray invariant associated to the Leray triple of isotropic subspaces determined by g_1, g_2.	80
$[g, z]_L$	The Leray bracket, parametrising $\overline{\mathrm{SL}}_2$.	80
ω_V	The Weil representation of $\overline{\mathrm{SL}}_2$ into the space $\mathcal{S}(V_{\mathbb{A}})$.	80
$\psi_{p,a}$	The mapping $x \mapsto e_p(a, x)$.	80
$M(F_p)$	The matrix group of elements of the form $m(a)$ for $a \in F_p^\times$.	82
$N(F_p)$	The matrix group of elements of the form $n(b)$ for $b \in F_p$.	82

Notation	Description	Page List
k_ϑ	Elements of $\mathrm{SO}_2(\mathbb{R})$ parametrised by an angle $\vartheta \in \mathbb{T}$.	83, 166
g_τ	The matrix $n(x)m(\sqrt{y})$ representing the number $\tau = x + iy \in \mathbb{H}$	83, 166
P_∞	The real matrix group of 2×2 upper triangular matrices.	83
K_p	The group $\mathrm{SL}_2(\mathbb{Z}_p)$ for $p < \infty$.	83
K_f	The group $\mathrm{SL}_2(\hat{\mathbb{Z}})$.	83
$K_\mathbb{A}$	The adelic group $K_\infty \times K_f$.	83
$P_\mathbb{A}$	The adelic Borel group of upper triangular matrices in $G_\mathbb{A}$.	83
$P'_\mathbb{A}$	The preimage under the splitting morphism of $P_\mathbb{A}$ in $G'_\mathbb{A}$.	83
K'_∞	The preimage under the splitting morphism of K_∞ in $G'_\mathbb{A}$.	83
$K_0(4)_2$	The completion of $\Gamma_0(4)$ in K_2.	84
$K_0(4)$	The group $K_0(4)_2 \times \prod_{2<p<\infty} K_p$.	84
K_0	A subgroup of $K_0(4)$, which is open and compact in K_f – it is identified with its preimage in $G'_\mathbb{A}$.	84
Mp_2	A subgroup of $\overline{\mathrm{SL}}$, representing a central extension of SL_2 by $\{\pm 1\}$.	84
$\mathcal{S}_{L'/L}$	The subspace of $\mathcal{S}(V_f)$ generated by indicator functions of cosets of L' by L.	85
ρ_L	The discrete Weil representation on $\mathrm{Mp}_2(\mathbb{Z})$ to $\mathbb{C}[L'/L]$.	86
$\overline{\mathrm{GL}_2^+}$	The metaplectic cover of GL_2^+.	87
$\overline{T}$	The element $\left(\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right), 1\right) \in \mathrm{Mp}_2(\mathbb{Z})$.	87
$\overline{S}$	The element $\left(\left(\begin{smallmatrix}0&-1\\0&1\end{smallmatrix}\right), \sqrt{\tau}\right) \in \mathrm{Mp}_2(\mathbb{Z})$.	87
$\overline{Z}$	The element $\left(\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right), i\right) \in \mathrm{Mp}_2(\mathbb{Z})$.	87
$\overline{U}$	The element $\overline{ST}^{-1}\overline{S}^{-1} \in \mathrm{Mp}_2(\mathbb{Z})$.	88
R_d	The element $\overline{ST}^d\overline{S}^{-1}\overline{T}^a\overline{ST}^d \in \mathrm{Mp}_2(\mathbb{Z})$ for suitable elements $a, b \in \mathbb{Z}$.	88
$\chi_\mathcal{L}$	A special character describing the action of $\Gamma_0(\mathrm{lev}(\mathcal{L}))$ under the discrete Weil representation.	94
$\mathrm{O}(M)$	The orthogonal group of a quadratic module (M, q).	129
$\mathbb{T}$	The unit circle in $\mathbb{C}$.	130

Notation	**Description**	**Page List**
$\mathcal{T}(M)$	The tensor algebra of the module M.	134
$\mathcal{C}_M$	The Clifford algebra of the quadratic module M.	136
N	The Clifford norm.	137
$\mathfrak{G}_M$	The Clifford group of the quadratic module M.	138
GSpin_M	The general Spin group of the quadratic module M.	139
Spin_M	The Spin group of the quadratic module M.	139
$\mathbb{D}(V)$	The Grassmannian of the quadratic vector space V.	142
$\mathcal{K}$	The projective model of the Grassmannian.	143
$\Gamma(L)$	The discriminant kernel of the lattice L.	146
$\mathcal{Z}_v$	The special divisor of $\mathbb{D}$ induced by some $v \in \mathbb{D}$.	157
θ	The theta distribution associated to $h \in \mathrm{O}_{\mathbb{A}}(V)$.	164
θ_L	The theta series associated with the quadratic lattice L.	166
H	A closed subgroup of the topological group G.	167
π_H	A smooth representation of the closed subgroup H.	168
$\mathrm{Ind}_H^G(\pi_H)$	The induced representation of G by π_H of G.	168
$I_p(s,\chi)$	The local principal series representation associated to s, χ.	168
$\Phi_p(g,s)$	Holomorphic sections through the bundles of local principal series representations $I_p(s,\chi)$.	168
$\Phi_p^0(g,s)$	The spherical section, characterised by being right K_p invariant and normalised.	168
$I(s,\chi)$	The global principal series representation associated to s, χ.	168
$\Phi(g,s)$	Holomorphic sections through the bundles of principal series representations $I(s,\chi)$.	169
$I(g;\varphi)$	The theta integral of the theta distribution associated with a Schwartz Bruhat form φ.	176
$\vert_k$	The Petersson slash operator of weight $k \in \mathbb{Z}$.	189
$T(n)$	The standard Hecke operator of level $n \in \mathbb{N}$.	198
$g(n)$	The matrix $\left(\begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Q})$ for some $n \in \mathbb{N}$.	212
$\mathcal{T}(n)$	The primitive Hecke operator of level $n \in \mathbb{N}$, directly associated to $g(n)$.	212
σ_n	The eigenvalue of a modular form f under the primitive Hecke operator $\mathcal{T}(n)$.	223

Notation	**Description**	**Page List**
$L_\ell(f,s)$	The L-series $L_\ell(f,s) = \sum_{n\in\mathbb{N}} a(n\bar{\ell}, n^2\,\mathrm{q}(\ell))/n^s$ associated to a modular form $f \in \mathcal{M}_{L,k}$ with Fourier coefficients $a(\lambda,m)$ and an isotropic element $\ell \in L'$.	236
$\Phi(z,f)$	Borcherds' regularised additive theta lift.	268
$\Psi(z,f)$	Borcherds' regularised multiplicative lift.	269
φ_{KM}	The Kudla–Millson Schwartz form.	271
Θ_{KM}	The Kudla–Millson theta form.	272
Λ_{KM}	The Kudla–Millson theta lift.	272
$\overline{\mathrm{SL}}_2$	The metaplectic extension of SL_2.	319
α^*	The dual morphism to $\alpha : G \to H$, given as a pullback.	320
F	A bilinear pairing inducing – the multiplier of the Heisenberg group.	321
κ	The symplectic pairing of $G \times G^*$.	321
$\mathrm{Sp}(G)$	The symplectic group of $G \times G^*$.	321
$L^2(G)$	The complex Hilbert space of square integrable functions on (G,μ).	322
$U(\omega)$	The unitary operator induced by $\omega \in G \times G^*$.	322
$A(G)$	The Heisenberg group of G.	322
$[Q,P]$	The commutator $QP(PQ)^{-1}$ of Q, P.	322
$B(G)$	The group of automorphisms of $A(G)$.	323
$B_0(G)$	The group of automorphisms of $A(G)$ fixing its centre.	323
$\langle\,\cdot\,,\,\cdot\,\rangle$	Duality brackets, taking an element of $G \times G^*$ and merging the components.	324
$X^2(G)$	The group of characters of second degree for G.	324
ρ	The morphism associated to a bilinear paring.	324
$\mathrm{Hom}_s(G,G^*)$	The subgroup of symmetric morphisms of $\mathrm{Hom}(G,G^*)$.	324
f_σ	The image of σ under the section $\mathrm{Sp}(G) \to B_0(G)$, splitting the Sequence (B.9).	325
$\overline{B}_0(G)$	The normaliser of $A(G)$ in $\mathrm{Aut}(L^2(G))$.	326
π_0	The projection of $\overline{B}_0(G)$ onto $B_0(G)$.	326
$H(V)$	The Heisenberg group of the Vector space V.	327
$\mathrm{Ps}(V)$	The pseudo symplectic group of a vector space V.	327

Notation	**Description**	**Page List**
Mp	The abstract metaplectic group.	328
$\mathbb{N}$	The natural numbers, excluding the zero.	
$\mathbb{N}_0$	The natural numbers, including the zero.	
$\mathbb{Z}$	The integers.	
$\mathbb{Q}$	The rational numbers.	
$\mathbb{R}$	The real numbers.	
$\mathbb{C}$	The complex numbers.	
$\gcd(a, b)$	The greatest common divisor of $a, b \in \mathbb{Z}$.	
$\mathrm{Im}(\tau)$	The imaginary part of $\tau \in \mathbb{H}$.	
$\mathrm{Re}(s)$	The real part of $s \in \mathbb{C}$.	
$\langle \cdot, \cdot \rangle$	A scalar product.	

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Erklärungen

Ich versichere, dass die elektronische Version dieser Dissertation mit der vorliegenden schriftlichen Version übereinstimmt.

Ich versichere, noch nicht zu einem vorherigen Zeitpunkt eine Promotion versucht zu haben.

Die Dissertation ist von mir mit einem Verzeichnis aller benutzten Quellen versehen. Ich erkläre, dass ich die Arbeit – abgesehen von den in ihr ausdrücklich genannten Hilfen – selbstständig verfasst habe.

Des Weiteren versichere ich, die "Grundsätze zur Sicherung guter wissenschaftlicher Praxis an der Technischen Universität Darmstadt" und die "Leitlinien zum Umgang mit digitalen Forschungsdaten an der TU Darmstadt" in den jeweils aktuellen Versionen bei der Verfassung der vorliegenden Dissertation beachtet zu haben.

Ich versichere, dass die Arbeit bisher noch nicht zu Prüfungszwecken gedient hat.

29.08.2024, Darmstadt

Unterschrift