Products of Quasi-Banach Lattices

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Lizenz

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1 Introduction

One of the reasons for this thesis may be found in the following mail correspondence:

MM **Mitrea, Marius**
RE: Question on Wolff's reiteration theorem
À : Moritz Egert

Dear Moritz,

You are raising a valid issue: I am not aware of a proof of Wolff's reiteration theorem in general, save for the special setting you mentioned. Such a result would be quite useful, but it appears that a proof is lacking. If you come across one, please let me know…

By the way, congrats on the monograph – nice work!

Best regards,
Marius

Figure 1.1: We'll let him know.

In order to understand the context, we give a bit of background:

In Analysis, solving PDEs with given initial or boundary data is often at the core of the matter, where we might encounter the situation that a PDE can be solved for data in two distinct classes, whereas data classes "in between" are hard to approach directly. Interpolation is a mighty tool that can give all intermediate results "for free".

One way to make this precise is the complex interpolation method from Kalton and Mitrea (see [KM98]). We will however only consider a special case, which is very close to the method introduced by Calderón (see [Cal64]), that we just roughly sketch here: Let X_0, X_1 be Banach spaces and let $\mathcal{F}$ denote the space of all bounded functions $F : \{z \in \mathbb{C} \,|\, \mathrm{Re}(z) \in (0,1)\} \to X_0 + X_1$ s.t.

F can locally be expanded as a power series in $X_0 + X_1$ while also being continuously extendable to $\{z \in \mathbb{C} \,|\, \mathrm{Re}(z) \in [0,1]\}$ with the traces $F|_{j+\mathrm{i}\mathbb{R}}$ mapping continuously into X_j. For $\theta \in [0,1]$, the *complex interpolation space* $[X_0, X_1]_\theta$ is then defined via $[X_0, X_1]_\theta \coloneqq \{F(\theta) \,|\, F \in \mathcal{F}\}$. Of course, there are norms involved that turn these spaces into Banach spaces, which we leave for later on.
Very broadly speaking, the advantage of this rather abstract approach is the rich theory that comes with it, because the interpolation space $[X_0, X_1]_\theta$ in a way depends holomorphically on X_0, X_1, which is the best dependence one can hope for in any situation. Consequently, results like the maximum principle, the Schwarz lemma and the Cauchy integral formula are among the tools that are available to get information on $[X_0, X_1]_\theta$ from X_0 and X_1.
Classical examples for interpolation spaces are given by Lebesgue and Sobolev spaces. Let $1/p = (1-\theta)/p_0 + \theta/p_1$ and $k \in \mathbb{N}$. In [Lun18, Example 2.11] it was shown that when $1 \leq p_0, p_1 \leq \infty$ holds, we obtain

$$[\mathrm{L}^{p_0}(\mathbb{R}^n), \mathrm{L}^{p_1}(\mathbb{R}^n)]_\theta = \mathrm{L}^p(\mathbb{R}^n). \tag{1.1}$$

The Sobolev case was handled for example in [BL76, Theorem 6.4.5. (7)] and states that if $1 < p_0, p_1 < \infty$, we obtain

$$[\mathrm{W}^{k,p_0}(\mathbb{R}^n), \mathrm{W}^{k,p_1}(\mathbb{R}^n)]_\theta = \mathrm{W}^{k,p}(\mathbb{R}^n). \tag{1.2}$$

Both identifications continue to hold if the Lebesgue and Sobolev spaces are replaced by their respective duals, which is a consequence of [Cal64, Theorem 12.2].

We illustrate why complex interpolation is a powerful tool.

- Coming back to PDEs, we assume that the given data from a Banach space X can be understood as coming from an interpolation space $[X_0, X_1]_\theta$ for some $\theta \in [0,1]$ induced by two other Banach spaces X_0, X_1, for which the PDE can be solved. If $T_i : X_i \to Y_i$ denotes the solution operator for an endpoint space, where Y_i is an appropriate Banach space that contains the solutions for the respective data from X_i, and the T_i agree on $X_0 \cap X_1$ as a sort of compatibility condition, we interpolate by setting $T : X_0 + X_1 \to Y_0 + Y_1, x_0 + x_1 \mapsto T_0(x_0) + T_1(x_1)$ and then obtaining a solution operator $T|_{[X_0,X_1]_\theta} : [X_0, X_1]_\theta \to [Y_0, Y_1]_\theta$ by restriction to

$[X_0, X_1]_\theta = X$ ([Cal64, Theorem 4]). This procedure is used frequently, for example in [AE23].

- Sneiberg's extrapolation Theorem [Sne74] states that when $T_i : X_i \to Y_i$ are bounded operators and $T_\theta := T|_{[X_0,X_1]_\theta}$ is defined as in the above item, where $\theta \in (0,1)$, then the set of all θ s.t. T_θ is an isomorphism, is open in $(0,1)$.

  In order to illustrate how Sneiberg extrapolation can be used, we take a look at the following situation: Let $p \in (1,\infty)$, $A \in \mathrm{L}^\infty(\mathbb{R}^n, \mathbb{R}^{n\times n})$ be strongly elliptic, $f \in \left(\mathrm{W}^{1,p}(\mathbb{R}^n)\right)^*$ and consider the elliptic equation

$$-\mathrm{div}(A\nabla u) = f \qquad \text{on } \mathbb{R}^n, \tag{1.3}$$

  where we are interested in weak solutions to the above problem. Let also $1 < p_0 < p_1 < \infty$ and define

$$\mathcal{L}_p : \mathrm{W}^{1,p}(\Omega) \to \left(\mathrm{W}^{1,p'}(\Omega)\right)^*, \qquad u \mapsto \int_\Omega \langle A\nabla u, \nabla \cdot \rangle \, \mathrm{d}x$$

  and

$$\mathcal{L} : \mathrm{W}^{1,p_0}(\mathbb{R}^n) + \mathrm{W}^{1,p_1}(\mathbb{R}^n) \to \left(\mathrm{W}^{1,p_0'}(\mathbb{R}^n)\right)^* + \left(\mathrm{W}^{1,p_1'}(\mathbb{R}^n)\right)^*,$$
$$f + g \mapsto \mathcal{L}_{p_0}(f) + \mathcal{L}_{p_1}(g).$$

  Then $\mathcal{L}_p$ is bounded for every p simply by Hölder's inequality and the restriction $\mathcal{L}|_{[\mathrm{W}_0^{1,p_0}(\mathbb{R}^n), \mathrm{W}_0^{1,p_1}(\mathbb{R}^n)]_\theta}$ clearly just reproduces $\mathcal{L}_p$ when $1/p = (1-\theta)/p_0 + \theta/p_1$ (using (1.2) itself and with duality).

  Even though we didn't interpolate a solution operator $(\mathrm{W}^{1,p'}(\mathbb{R}^n))^* \to \mathrm{W}^{1,p}(\mathbb{R}^n)$ but a sort of "data operator" $\mathrm{W}^{1,p}(\mathbb{R}^n) \to (\mathrm{W}^{1,p'}(\mathbb{R}^n))^*$, this example is still instructive: Using the usual arguments involving Lax-Milgram and the Riesz representation theorem, we can show that $\mathcal{L}_2$ is an isomorphism. Let additionally $1 < p_0 < 2, p < p_1 < \infty$ and $\theta_2, \theta_p \in (0,1)$ be s.t. $1/2 = (1-\theta_2)/p_0 + \theta_2/p_1$ and $1/p = (1-\theta_p)/p_0 + \theta_p/p_1$. In order to further match the language of Sneibers' extrapolation theorem, we set $T_0 := \mathcal{L}_{p_0}$ and $T_1 := \mathcal{L}_{p_1}$, so that $T_{\theta_2} = \mathcal{L}_2$. The same theorem now states that there is $\varepsilon > 0$ s.t. $T_{\theta_p} = \mathcal{L}_p$ is an isomorphism as well when $|\theta_2 - \theta_p| < \varepsilon$.

This is remarkable in itself but also useful in the following situation: Let $n = 2$ and p be sufficiently close to 2 (i.e. $|\theta_2 - \theta_p| < \varepsilon$) and $f \in \left(\mathrm{W}^{1,p'}(\mathbb{R}^n)\right)^*$. Then (1.3) has a unique solution $u \in \mathrm{W}^{1,p}(\mathbb{R}^n)$, which without any further prior assumptions is also Hölder continuous due to Morrey's embedding (see [Eva10, p. 266, Theorem 4]).

- The Wolff reiteration theorem [Wol82, Theorem 2] states that if four Banach spaces X_0, X_1, X_2, X_3 are given, where $X_1 = [X_0, X_2]_\lambda$ and $X_2 = [X_1, X_3]_\mu$ holds, we then have $X_1 = [X_0, X_3]_\theta$ and $X_2 = [X_0, X_3]_\eta$ for appropriate parameters $\theta, \eta, \lambda, \mu \in [0, 1]$. The picture to have in mind is the following:

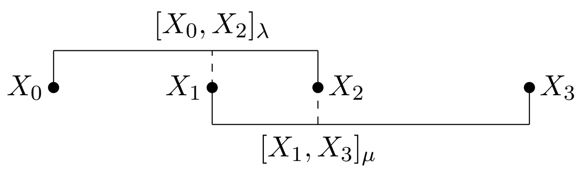


Figure 1.2: The dashed lines indicate the interpolation spaces from the requirements. If they are given, the interpolation ranges from λ and μ can be "glued" together so that X_1 and X_2 may be obtained from interpolation of X_0, and X_3 along θ and η.
This image is slightly altered, the original may be found in [BE19, Proposition 3.10].

As a motivation, let's say that in (1.1) $[\mathrm{L}^{p_0}, \mathrm{L}^{p_3}]_\theta = \mathrm{L}^p$ (we omit $\mathbb{R}^n$ for this paragraph) holds true when $1 \leq p_0, p_3 < \infty$. Due to duality (use [Cal64, § 12.2]), we can show the identification when $1 < p_0, p_3 \leq \infty$, so that the case $p_0 = 1, p_3 = \infty$ remains. Wolff reiteration now brings these two scales together as follows: If we want to show that $[\mathrm{L}^1, \mathrm{L}^\infty]_\theta = \mathrm{L}^{p_1}$ for some $\theta \in (0, 1)$ and the established identification should extend to this case, $p_1 = 1/(1-\theta)$ is uniquely determined. In order to actually prove this, for the specific choice of $p_1 = 1/(1-\theta)$ we introduce a "dummy space" L^{p_2} s.t. $[\mathrm{L}^1, \mathrm{L}^{p_2}]_\lambda = \mathrm{L}^{p_1}$, $[\mathrm{L}^{p_1}, \mathrm{L}^\infty]_\mu = \mathrm{L}^{p_2}$ holds for appropriate choices of $\lambda, \mu \in (0, 1)$ and $p_2 \in (p_1, \infty)$, using that in these ranges interpolation spaces can already be calculated. Now the requirements for Wolff reiteration are met and we can confirm that $[\mathrm{L}^1, \mathrm{L}^\infty]_\theta = \mathrm{L}^{1/(1-\theta)}$.

Actually obtaining explicit descriptions of interpolation spaces is hard work, where the disadvantage of the complex method comes forth: Due to the abstract nature of $X_0 + X_1$-valued analytic functions, interpolation spaces cannot easily be calculated. To overcome this, we require a bit more structure from the involved Banach spaces.

We now assume that the interpolated spaces consist of (equivalence classes) of functions, i.e. if $(\Omega, \mathcal{A}, \mu)$ is a σ-finite measure space, the elements of the X_i are from the space of $\mathcal{A}$-measurable and $\mathbb{C}$-valued functions, which we call $\mathrm{L}_0(\Omega)$, with the additional properties that for $f \in \mathrm{L}_0(\Omega)$ and $g \in X_i$ with $|f| \leq |g|$ it holds that $f \in X$ with $\| f \| \leq \| g \|$ and that there is $f \in X$ with $f > 0$ a.e. We will call such a Banach space a *Banach function space*. Given X_0, X_1 and $\theta \in (0,1)$, the *Calderón product* (see [Cal64, § 13.5]) is defined as

$$X_0^{1-\theta} X_1^{\theta} \coloneqq \{f \in \mathrm{L}_0(\Omega) \mid |f| \leq |f_0|^{1-\theta} |f_1|^{\theta}, \ f_i \in X_i\}.$$

Again, we may define a norm on this space but leave this for later.

Before we can state a result, which identifies interpolation spaces via Calderón products, we need to make the situation a bit more difficult by weakening the concept of a Banach space. Notice that when $0 < p < 1$, $\| \cdot \|_p$ isn't a norm on L^p, as it doesn't satisfy the triangle inequality anymore, because Young's inequality only implies

$$\| f + g \|_p \leq 2^{1/p-1} \left(\| f \|_p + \| g \|_p \right), \qquad f, g \in \mathrm{L}^p.$$

Maps such as $\| \cdot \|_p$ when $0 < p < 1$, that produce constants when using the triangle inequality, are called *quasi-norms*. If a vector space that is equipped with a quasi-norm, is complete with respect to it, it is called a *quasi-Banach space*. Of course, there are other quasi-Banach spaces that frequently pop up as data for PDEs, such as Hardy spaces or really any space that allows the correct parameter to come from the interval $(0, 1)$.
At first, this additional constant doesn't seem to make much of a difference, but the consequences are quite detrimental. We give just a few examples that illustrate the differing behaviour:

(i) In the case of a quasi-normed space X that is not a normed space, estimating multiple times via the quasi-triangle inequality comes at a cost, i.e. we will see that when $x_0, \ldots, x_N \in X$, estimating is basically as bad as

$$\left\| \sum_{n=0}^{N} x_n \right\|_X \leq \sum_{n=0}^{N} C^n \, \| x_n \|_X .$$

(ii) In general, the unit ball in quasi-normed spaces is not convex anymore. This is easily seen by drawing doodles in two dimensions when $\mathbb{R}^2$ is endowed with $\| (x, y) \|_p := (|x|^p + |y|^p)^{1/p}$, $(x, y) \in \mathbb{R}^2$ for $0 < p < 1$. But since we can fit an euclidean ball inside a $\| \cdot \|_p$-ball, the induced topology is at least locally convex.

(iii) However, this fails for general quasi-norms. When $0 < p < 1$, $\| \cdot \|_p$ induces a topology on $\mathrm{L}^p([0,1])$ via the metric $d(f, g) := \| f - g \|_p^p$, $f, g \in \mathrm{L}^p([0,1])$, which is not locally convex (see [Con, Example 2.19]).

(iv) The Hahn-Banach theorem is a useful tool in order to do meaningful duality theory, which in turn reveals much about the examined vector space. However, any quasi-normed space, for which the Hahn-Banach theorem holds true, needs to be locally convex (see [KPR84, Theorem 4.8]).

(v) Depending on the viewpoint, this is of no issue to $\mathrm{L}^p([0,1])$ when $0 < p < 1$, since in this case $(\mathrm{L}^p([0,1]))^* = \{0\}$ doesn't give much functionals to work with in the first place (see [Con, Theorem 2.21]).

(vi) No extensive integration theory is available to functions that are valued in quasi-normed spaces, as there is no way to show a triangle inequality for step functions, because due to (i) every term in a simple function produces a constant, that needs to be accounted for and only allows for very few functions to be integrable, if a (canonical) approach as for Bochner or Riemann integrals would be followed. For a more detailed explanation, see [AB22].

(vii) Finally, the maximum principle doesn't hold for all quasi-Banach spaces (see [Ale81, Proposition 4.2]) and no Cauchy integral formula is available (or at least none with as many useful properties as the one in the Banach case, see [BC90, p. 2]), which makes the usage of analytic functions much more difficult. Quasi-Banach spaces that satisfy the maximum principle are called A-*convex*.

Thus, replacing the word "Banach" with "quasi-Banach" each time it occurred in this introduction so far makes for a more general concept, which is why we retrospectively do exactly this. Due to (vi) and (vii) this comes with its own problems. We will deal with the most important basics of quasi-Banach spaces in Chapter 2 and use the developed theory to show that even in the quasi-Banach setting complex interpolation (Chapter 5) as well as Calderón products (Chapter 6) are meaningful.

Even under these adverse conditions, Kalton and Mitrea managed to identify interpolation spaces as Calderón products:

Theorem 1.1. *Let Ω be a complete, separable, and metric space and μ a σ-finite Borel measure on Ω. Let X_0, X_1 be separable quasi-Banach function spaces over $(\Omega, \mathcal{B}(\Omega), \mu)$ that are A-convex. Then X_0+X_1 is A-convex and the spaces $[X_0, X_1]_\theta$ and $X_0^{1-\theta}X_1^\theta$ agree up to equivalence of quasi-norms.*

Notice the separability assumption for both spaces. In practice this means that as soon as any involved space comes with an index that is ∞, it is likely not clear whether the above characterization should continue to hold. This causes problems at several points in the literature:

- In [Ame18] a class of Banach function spaces T^p, $p \in (0, \infty]$, which is separable for $p < \infty$ and quasi-Banach for $p \in (0,1)$, was considered, where the goal was to show that interpolation similarly to the L^p scale is possible. The case that $0 < p_0 < p_1 < \infty$ could be handled, as well as the case $1 < p_0 < p_1 \leq \infty$ because these spaces happen to satisfy a duality relation such as $(\mathrm{T}^p)^* = \mathrm{T}^{p'}$ when $1 \leq p < \infty$. In order to obtain the desired interpolation behaviour, it remains to treat the case that $0 < p_0 \leq 1 < p_1 = \infty$, which was thought to be handled via Wolff reiteration, overseeing that it is not available in the quasi-Banach setting when only one endpoint space is separable, which is to what the mail correspondence refers to.
- Similarly, in [AA18] Calderón products for certain quasi-Banach function spaces were calculated, where it is again not clear due to the quasi-Banach setting and separability of only one endpoint space, whether the

resulting products could be identified with interpolation spaces. However, results, which are only available for interpolation spaces but not Calderón products, were used. Most notably, Sneiberg's extrapolation theorem in the quasi-Banach setting due to Kalton and Mitrea.

- In [Hua16, §4] it was argued that one separability condition in Theorem 1.1 may be dropped if we require that both function spaces satisfy the so called *Fatou property* (which is what one would expect from it with regard to integration theory, but more on that later), but the corresponding source for this argument does not mention this claim. As it turns out, the Fatou property won't be needed when generalizing Theorem 1.1, but it gives a nice corollary, see Corollary 7.10.
- Furthermore, a remark in [KMM07, §7] suggests that the argument is rather straightforward in the case of sequence spaces by making use of the fact that dominated convergence (generalized to this setting by just replacing the 1-norm with an abstract quasi-norm on a function space) holds true for $X_0^{1-\theta}X_1^\theta$ if it holds for either X_0 or X_1. The idea points in the right direction, as we will use this fact as well, even without needing to restrict to sequence spaces. Whether dominated convergence holds true in a function space is characterized by separability, which will be the topic of Chapter 3.1.

Because the task of showing a general version of Wolff reiteration is too daunting, we leave it as an exercise for the reader and settle for the less involved path, by showing that the identification of interpolation spaces via Calderón products can be relaxed to only one separability assumption (which is done in Chapter 7) and that a Wolff reiteration theorem is indeed available (see Chapter 8) for the first gap that we identified above. Due to the identification itself, the other three gaps will also be closed. Additionally, we want to use this opportunity to elaborate on the proof of Theorem 1.1 in quite some detail, as it is rather succinctly formulated at times. The more general version reads as follows:

Theorem 1.2. *Let Ω be a separable metric space and μ a σ-finite Borel measure on Ω. Let $\theta \in (0,1)$ and let X_0, X_1 be p-convex quasi-Banach function spaces over $(\Omega, \mathcal{B}(\Omega), \mu)$, one of which is separable. Then $X_0 + X_1$ is p-convex and the spaces $[X_0, X_1]_\theta$ and $X_0^{1-\theta}X_1^\theta$ are separable and agree up to equivalence of quasi-norms.*

A short discussion of the changes is in order:

- The only property that separates the new Ω from the old one is completeness. Neither the topological nor the measure theoretic notion of completeness are needed, and a nice explanation as to why the latter isn't needed can be found in [LN24, Remark 2.1 (iii)].
- We swapped out A-convexity with the equivalent (at least in the case of functions spaces) notion of *p-convexity*, which states that for a quasi-Banach function space X there exists $p > 0$ s.t. for any $f_1, \ldots, f_n \in X$ in we have

$$\left\| \left(\sum_{i=0}^{n} |f_i|^p \right)^{1/p} \right\| \lesssim \left(\sum_{i=0}^{n} \| f_i \|^p \right)^{1/p}.$$

- And of course the relaxed conditions on separability of X_0, X_1. That the interpolation space is separable as well is a little extra that is an easy consequence from the correspondence of separability and dominated convergence, not a new insight and just added for completeness.

The role of p-convexity will be discussed in Chapter 3.3. In addition to the already mentioned theory, we will also need to elaborate on subharmonic functions and how their values are controlled by their boundary behaviour (Chapter 4.2) and introduce a generalization of the $(\mathrm{L}^p)^* \cong \mathrm{L}^{p'}$ correspondence for general function spaces (Chapter 3.2).

Lastly, we fix a bit of notation:

- Starting controversial, we assume $0 \in \mathbb{N}$.
- (Quasi-)norms will generally be written as $\| \cdot \|$, where any index is omitted if it is clear from the context to which space the (quasi-)norm refers to. If there is an index, it should again be clear from the context to which space the (quasi-)norm refers in this case.
- When estimating, we will sometimes write $\lesssim$ if the estimate holds true up to a constant that does not depend on any important parameters, where importance will depend on the context. The constant that is hidden beneath $\lesssim$ may vary from line to line without explicitly mentioning it.

- Balls of radius $\varepsilon > 0$ around a given point x are denoted as $B_\varepsilon(x)$.
- For a set A, its indicator function is denoted as χ_A.
- For real-valued functions (or numbers) f_n and f we write $f_n \nearrow f$ when $f_n \leq f$ and $(f_n)_n$ is increasing. Analogously, we define $f_n \searrow f$.
- For sets we denote with $\Omega_n \nearrow \Omega$ that Ω, Ω_n are measurable with $\cup_n \Omega_n = \Omega$, $\Omega_n \subseteq \Omega_{n+1}$ for all $n \in \mathbb{N}$.
- For a real valued function f we write $f_+ := \max(0, f)$ and $f_- := \min(0, f)$.
- We write $A \Subset B$ if A is compactly contained in B, i.e. $A \subseteq \overline{A} \subseteq B$.
- The interior of a set A will be denoted as A°.
- We denote the unit circle in $\mathbb{C}$ as $\mathbb{D} := B_1(0) \subseteq \mathbb{C}$ and the open strip between $\{\mathrm{Re} = 0\}, \{\mathrm{Re} = 1\}$ as $S := \{z \in \mathbb{C} \,|\, \mathrm{Re}(z) \in (0, 1)\}$.
- For $f : \Omega \to \mathbb{C}$ and $M \in \mathbb{C} \cup \{\infty\}$ we denote $[f = M] := \{\omega \in \Omega \,|\, f(\omega) = M\}$ and employ the same notation when f is real valued, $M \in \overline{\mathbb{R}}$ and $" = "$ is possibly exchanged with $" \leq ", " \geq ", " < "$, or $" > "$.
- In order to not pollute expressions with too many characteristic functions when dividing by a $\mathbb{C}$-valued function f we omit expressions like $\chi_{[f \neq 0]}$ and trust that the reader convinces themselves, that everything is in order.
- Similarly, when dealing with expressions like $\{\omega \in \Omega \,|\, f(\omega) = 0\}$ when f denotes an equivalence class of functions agreeing almost everywhere, we trust the reader to trust us when we say that problems with representatives are not an issue, unless explicitly discussed.
- At one point, we will need to distinguish between equivalence classes and representatives which we do by denoting equivalence classes with brackets, i.e. $[f]$ is an equivalence class while f is a representative.

2 Quasi-Banach spaces

In this chapter, we properly introduce quasi-Banach spaces and collect some properties that easily and more or less directly carry over from the Banach case. Easily only after investing some effort into the proof of the upcoming Aoki-Rolewicz-Theorem, which is a fundamental tool when dealing with quasi-Banach spaces.

Definition 2.1. Let X be a vector space over $\mathbb{C}$, $x, y \in X$ and $\lambda \in \mathbb{C}$. A map $\| \cdot \| : X \to \mathbb{R}$ is called a *quasi-norm*, if the following conditions are met:

(a) *Definiteness*: $x = 0$ if and only if $\| x \| = 0$.

(b) *Homogeneity*: $\| \lambda x \| = |\lambda| \, \| x \|$.

(c) *Quasi-triangle inequality*: There exists $C \geq 1$ such that: $\| x + y \| \leq C(\| x \| + \| y \|)$.

If X is endowed with a quasi-norm $\| \cdot \|$, we call X a *quasi-normed space*.

Furthermore, two quasi-norms $\| \cdot \|_1 , \| \cdot \|_2$ on a quasi-normd space X are said to be *equivalent*, if there exists $C_\sim \geq 1$ s.t. $\frac{1}{C_\sim} \| \cdot \|_2 \leq \| \cdot \|_1 \leq C_\sim \| \cdot \|_2$.

Convention 1. Every vector space in this thesis will have $\mathbb{C}$ as its groundfield.

We now want to define a topology on quasi-normed spaces. It seems inviting to just proceed as with normed spaces, where norms induce metrics which in turn induce topologies. But if X is a quasi-normed space and we were to directly do the same by saying that $O \subseteq X$ is open, if for all $x \in O$ there is $\varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq O$, then due to the quasi-triangle inequality it is not guaranteed that simple sets such as balls are actually open. More precisely, let $\varepsilon > 0, x \in X$ and

$y \in B_\varepsilon(x)$. Then for $w \in B_s(y)$, where we leave $s > 0$ open for now, we obtain by reproducing the arguments for a metric space

$$\| x - w \| \leq C(\| x - y \| + \| y - w \|) < C(\| x - y \| + s) \overset{!}{<} \varepsilon.$$

If this was to hold true, then $s < \frac{\varepsilon}{C} - \| x - y \|$, where the right-hand side isn't guaranteed to be positive, so that we could only conclude that $B_{\frac{\varepsilon}{C}}(x) \subseteq (B_\varepsilon(x))^\circ$. Without any further knowledge, this approach leads nowhere.
We use another observation instead: In the case of L^p-spaces, $\| \cdot \|_p$ can just be raised to the power of p in order so satisfy the requirements of a metric at the cost of homogeneity. This procedure works for all quasi-Banach spaces.

Theorem 2.2 (Aoki-Rolewicz)**.** *Let X be a quasi-normed space and C be a constant for which the quasi-triangle inequality holds true. Then there exists an equivalent quasi-norm $\| \cdot \|_\sim$ on X and $r \in (0, 1]$ s.t. $C = 2^{1/r-1}$ and*

$$\| x + y \|_\sim^r \leq \| x \|_\sim^r + \| y \|_\sim^r .$$

holds true for all $x, y \in X$. The above estimate continues to hold true if r is replaced by any $s \in (0, r]$. Also, $\| \cdot \|_\sim$ is continuous.

Proof. The proof is a retelling of [KPR84, Theorem 1.3].

Step 1: We show that $\| \cdot \|$ is already somewhat r-subadditive , i.e. whenever $x_0, \ldots, x_n \in X$ and $r \in (0, 1]$ s.t. $2C = 2^{1/r}$ we have

$$\left\| \sum_{i=0}^{n} x_i \right\|^r \leq 4 \sum_{i=0}^{n} \| x_i \|^r . \tag{2.1}$$

W.l.o.g we may assume that $\| x_i \| \geq \| x_{i+1} \|$ whenever $0 \leq i \leq n - 1$.

Let $r \in (0, 1]$ s.t. $2C = 2^{1/r}$. As a bit of preparation, we notice that when $x, y \in X$ the quasi-triangle inequality yields that

$$\| x + y \| \leq 2C \max\{\| x \| , \| y \|\} = 2^{1/r} \max\{\| x \| , \| y \|\}. \tag{2.2}$$

Using induction, we show that

$$\left\| \sum_{i=0}^{n} x_i \right\| \le \max_{0\le i\le n} 2^{(i+1)/r} \| x_i \| . \tag{2.3}$$

The claim is surely true when $n = 0$. Now assume that the claim holds true for some $n \ge 1$ and let $x_0, \dots, x_n \in X$, then

$$\begin{aligned}
\left\| \sum_{i=0}^{n+1} x_i \right\| &\le 2^{1/r} \max \left\{ \| x_0 \| , \left\| \sum_{i=0}^{n+1} x_i \right\| \right\} && (2.2)\\
&\le 2^{1/r} \max \left\{ \| x_0 \| , \max_{1\le i\le n+1} 2^{i/r} \| x_i \| \right\} && \text{(induction hypothesis)}\\
&= \max_{0\le i\le n+1} 2^{(i+1)/r} \| x_i \|
\end{aligned}$$

Finally, the key idea is to define an auxiliary function $H : X \to \mathbb{R}$ via

$$\begin{aligned}
&H(0) = 0,\\
&H(x) = 2^{m/r}, \qquad \text{if } 2^{(m-1)/r} < \| x \| \le 2^{m/r} \text{ for some } m \in \mathbb{Z},
\end{aligned}$$

which satisfies

$$\| x \| \le H(x) \le 2^{1/r} \| x \| , \qquad x \in X.$$

Notice that defining H not only easens notation, but also gives us an advantage in the sense that we can improve the quasi-triangle inequality

$$\| x_j + x_{j+1} \|^r \le 2^{1-r}(\| x_j \|^r + \| x_{j+1} \|^r) \le 2(\| x_j \|^r + \| x_{j+1} \|^r)$$

to something sharper that doesn't produce any constants, i.e.

$$\| x_j + x_{j+1} \|^r \le H(x_j + x_{j+1})^r \le H(x_j)^r + H(x_{j+1})^r \le 2(\| x_j \|^r + \| x_{j+1} \|^r),$$

as we will see in a bit.

We now proceed by showing the inequality

$$\left\| \sum_{i=0}^{n} x_i \right\|^r \le 2 \sum_{i=0}^{n} H(x_i)^r$$

by induction, from which (2.1) directly follows. The case when $n = 0$ is clear, so assume that the claim holds true for some $n \geq 0$ and let $x_0, \ldots, x_{n+1} \in X$.

If the values of the $H(x_i)$ are pairwise distinct, we don't actually need to do any induction. We have

$$
\begin{aligned}
\left\| \sum_{i=0}^{n+1} x_i \right\|^r &\leq \max_{0 \leq i \leq n+1} 2^{i+1} \| x_i \|^r && (2.3) \\
&\leq \max_{0 \leq i \leq n+1} 2^{i+1} H(x_i)^r \\
&\leq 2H(x_0)^r && (\text{since } 2^i H(x_i)^r \leq H(x_0)^r) \\
&\leq 2 \sum_{i=0}^{n+1} H(x_i)^r.
\end{aligned}
$$

Otherwise there exists $0 \leq j \leq n$ with $H(x_j) = H(x_{j+1})$ and $m' \in \mathbb{Z}$ with

$$
2^{(m'-1)/r} < \| x_{j+1} \| \leq \| x_j \| \leq 2^{m'/r},
$$

which gives

$$
\| x_j + x_{j+1} \| \leq 2^{1/r} \max\{\| x_j \|, \| x_{j+1} \|\} \leq 2^{(m'+1)/r}
$$

and thus

$$
H(x_j + x_{j+1})^r \leq 2^{m'+1} = 2^{m'} + 2^{m'} = H(x_j)^r + H(x_{j+1})^r.
$$

The claim follows now, beacause the set $\{x_i \mid 1 \leq i \leq n+1, i \neq j, j+1\} \cup \{x_j + x_{j+1}\}$ consists of n elements and we can conclude that

$$
\begin{aligned}
\left\| \sum_{i=0}^{n+1} x_i \right\|^r &\leq 2 \left(\sum_{i=0, i \neq j, j+1}^{n+1} H(x_i)^r + H(x_j + x_{j+1})^r \right) && (\text{induction hypothesis}) \\
&\leq 2 \sum_{i=0}^{n} H(x_i)^r.
\end{aligned}
$$

Step 2: In order to clean up the factor of 4 when estimating in (2.1), we use the right-hand side to define

$$\| x \|_\sim := \inf \left\{ \left(\sum_{i=0}^{n} \| x_i \|^r \right)^{1/r} \;\middle|\; \sum_{i=0}^{n} x_i = x \right\}, \qquad x \in X,$$

in order to put the factor into the equivalence of quasi-norms.

We observe that whenever $x = \sum_{i=0}^{n} x_i$ for some $x_i \in X$, we have by (2.1) that

$$\| x \| = \left\| \sum_{i=0}^{n} x_i \right\| \leq \left(4 \sum_{i=0}^{n} \| x_i \|^r \right)^{1/r},$$

i.e.

$$4^{-1/r} \| x \| \leq \| x \|_\sim \leq \| x \| .$$

Through this, definiteness and the quasi-triangle inequality are clear for $\| \cdot \|_\sim$. Homogeneity is evident, so that $\| \cdot \|_\sim$ is indeed a quasi-norm that is equivalent to $\| \cdot \|$. We still need to verify the r-subadditivity. Let $x, y \in X$, then

$$\begin{aligned}
\| x + y \|_\sim^r &= \inf \left\{ \sum_{i=0}^{k} \| z_i \|^r \;\middle|\; \sum_{i=0}^{k} z_i = x + y \right\} \\
&\leq \inf \left\{ \sum_{i=0}^{n} \| x_i \|^r + \sum_{i=0}^{m} \| y_i \|^r \;\middle|\; \sum_{i=0}^{n} x_i = x, \sum_{i=0}^{m} y_i = y \right\} \\
&= \inf \left\{ \sum_{i=0}^{n} \| x_i \|^r \;\middle|\; \sum_{i=0}^{n} x_i = x \right\} + \inf \left\{ \sum_{i=0}^{m} \| y_i \|^r \;\middle|\; \sum_{i=0}^{m} y_i = y \right\} \\
&= \| x \|_\sim^r + \| y \|_\sim^r .
\end{aligned}$$

<u>Step 3:</u> That $\| \cdot \|_\sim$ is still s-subadditive when $s \in (0, r)$ (i.e. $s/r < 1$) is a simple consequence of Young's inequality:

$$\| x + y \|_\sim^s = (\| x + y \|_\sim^r)^{s/r} \leq (\| x \|_\sim^r + \| y \|_\sim^r)^{s/r} \leq \| x \|_\sim^s + \| y \|_\sim^s .$$

<u>Step 4:</u> Continuity of $\| \cdot \|_\sim$ is easily seen as for all $x, y \in X$ we have

$$| \| x \|_\sim^r - \| y \|_\sim^r | \leq \| x - y \|_\sim^r$$

similarly to the normed case. □

A thorough discussion with Sebastian Bechtel now allows us to define topologies on quasi-Banach spaces as we initially tried to do.

Proposition 2.3. *Let X be a quasi-normed space and $\|\cdot\|_\sim$ be an equivalent quasi-norm from the Aoki-Rolewicz-Theorem 2.2.*

(i) *$d(f,g) := \| f - g \|_\sim^r$, $f, g \in X$ defines a metric on X, inducing a topology*

$$\tau := \left\{ O \subseteq X \mid \textit{for all } x \in O \textit{ exists } \varepsilon_x > 0 \textit{ s.t. } B_{\varepsilon_x}^{\|\cdot\|_\sim^r}(x) \subseteq O \right\}$$

where $B_\varepsilon^{\|\cdot\|_\sim^r}(x) := \{ y \in X \mid \| x - y \|_\sim^r < \varepsilon \}$ for $\varepsilon > 0,\ x \in X$.

(ii) *The topology*

$$\tau' := \left\{ O \subseteq X \mid \textit{for all } x \in O \textit{ exists } \varepsilon_x > 0 \textit{ s.t. } B_{\varepsilon_x}^{\|\cdot\|}(x) \subseteq O \right\}$$

is the same as τ with $B_\varepsilon^{\|\cdot\|}(x)$ being defined analogously.

(iii) *Both τ and τ' don't depend on the specific choice of $\|\cdot\|_\sim$.*

Proof.

(i) Clear.

(ii) Let $D \geq 1$ be the constant from the equivalence of $\|\cdot\|$ and $\|\cdot\|_\sim$. Then we have $B_\varepsilon^{\|\cdot\|_\sim^r}(x) = B_{\varepsilon^{1/r}}^{\|\cdot\|_\sim}(x)$ for all $\varepsilon > 0$ and $x \in X$ and

$$B_{\frac{1}{D}\varepsilon^{1/r}}^{\|\cdot\|}(x) \subseteq B_{\varepsilon^{1/r}}^{\|\cdot\|_\sim}(x) \subseteq B_{D\varepsilon^{1/r}}^{\|\cdot\|}(x).$$

(iii) Also clear, because all norms that are equivalent to $\|\cdot\|$ are equivalent to each other as well. □

Convention 2. Let X be a quasi-normed space.

(i) In view of the last Proposition, from here on out we fix "the" equivalent quasi-norm from the Aoki-Rolewicz-Theorem 2.2 as the one that corresponds to

$$C := \inf\{K \geq 1 \mid K \text{ is a constant for which the quasi-triangle inequality holds true}\}$$

in order to avoid any ambiguity.

(ii) $\| \cdot \|_\sim$ will always refer to the quasi-norm given by the Aoki-Rolwicz-Theorem 2.2 without mentioning it explicitly every time. It should be clear from the context to which space it refers to.

(iii) The constant that comes from passing to $\| \cdot \|_\sim$ will only mostly be denoted with $C_\sim$, as we sometimes might just absorb it into $\lesssim$ without any explicit mention or write $C^\sim$ in order to have space for an index in the hopes that it doesn't obscure notation too much.

(iv) X will always be endowed with the topology from the above Proposition 2.3.

Remark 2.4. In general, it is not clear whether a quasi-norm is continuous, because the quasi-triangle inequality does not guarantee a "reverse quasi-triangle inequality". Because $\| \cdot \|_\sim$ is continuous and we don't care which of $\| f_n - f \|_\sim^r \to 0$ or $\| f_n - f \| \to 0$ holds true, it will often be of benefit for us to pass to the equivalent quasi-norm and raise its power in order to be able to use a triangle inequality.

In this way, all relevant notions from metric spaces are available to us in the usual way. We can in particular make

Definition 2.5. A quasi-normed space X is called a *quasi-Banach space*, if it is complete.

We end this chapter by showing that some properties directly carry over from the Banach case to the quasi-Banach case, while others need to be adjusted a bit, for which we need the following

Lemma 2.6.

(i) If X is a quasi-normed space and $x_0, \ldots, x_n \in X$, we have

$$\left\| \sum_{i=0}^{n} x_n \right\| \leq \sum_{i=0}^{n} C^n \| x_i \| ,$$

where $C \geq 1$ is a constant, for which the quasi-triangle inequality holds true.

(ii) Let $a_0, \dots, a_n \geq 0$ and $p \geq 1$. Then

$$\left(\sum_{i=0}^{n} a_i\right)^p \leq \sum_{i=0}^{n} \left(2^{p-1}\right)^{i+1} a_i^p.$$

Proof.

(i) This follows after showing the slightly sharper estimate

$$\left\| \sum_{i=0}^{n} x_n \right\| \leq \sum_{i=0}^{n-1} C^{i+1} \| x_i \| + C^n \| x_n \|$$

by induction over n.

(ii) This works similarly to (i) □

Lemma 2.7. *Let X be a quasi-normed space and let C be a constant for which the quasi-triangle inequality holds true.*

(i) X is a quasi-Banach space if and only if for every sequence $(x_n)_n \subseteq X$ with $\sum_{n=0}^{\infty} C^n \| x_n \| < \infty$ its corresponding series $\sum_{n=0}^{\infty} x_n$ converges in X.

(ii) Let $(x_n)_n \subseteq X$ be such that $\sum_{n=0}^{\infty} x_n$ converges in X. Then

$$\left\| \sum_{n=0}^{\infty} x_n \right\| \leq C_{\sim}^2 \left(\sum_{n=0}^{\infty} \| x_n \|^r \right)^{1/r} \leq C_{\sim}^2 \sum_{n=0}^{\infty} C^n \| x_n \| .$$

(iii) Let X_0, X_1 be quasi-Banach spaces that continuously embed into a Hausdorff topological vector space Z. Then:

(a) $X_0 + X_1$ becomes a quasi-Banach space if it is endowed with $\| x \| := \inf\{\| x_0 \|_{X_0} + \| x_1 \|_{X_1} \mid x = x_0 + x_1, x_i \in X_i\}$, $x \in X_0 + X_1$.

(b) $X_0 \cap X_1$ becomes a quasi-Banach space if it is endowed with $\| x \| := \max\{\| x \|_{X_0}, \| x \|_{X_1}\}$, $x \in X_0 \cap X_1$.

Proof.

(i) We generalize the ideas given in [Car00, Theorem 7.12] to the quasi-Banach case.

"$\Rightarrow$": Let $(x_n)_n \subseteq X$ with $\sum_{n=0}^{\infty} C^n \| x_n \| < \infty$ and set $s_N := \sum_{n=0}^{N} x_n$. Then for $N \geq M$ we have by the previous Lemma 2.6 (i)

$$\| s_N - s_M \| = \left\| \sum_{n=M+1}^{N} x_n \right\| \leq \sum_{n=M+1}^{N} C^n \| x_n \| ,$$

i.e. $(s_N)_N \subseteq X$ is a Cauchy sequence and thus converges in X.

"$\Leftarrow$": Let $(x_n)_n \subseteq X$ be a Cauchy sequence. As it suffices to find a convergent subsequence, we choose a subsequence that we again denote as $(x_n)_n$ with the property that for all $n \in \mathbb{N}$ we have $\| x_n - x_{n+1} \| < \frac{1}{(2C)^n}$. Then

$$\sum_{n=0}^{\infty} C^n \| x_n - x_{n+1} \| \leq \sum_{n=0}^{\infty} \frac{1}{2^n} < \infty,$$

i.e. $\sum_{n=0}^{\infty} x_n - x_{n+1}$ converges to a limit $x \in X$. Now $(x_n)_n$ converges to $x_0 - x$, since

$$x_N = x_0 - \sum_{n=0}^{N-1} x_n - x_{n+1}.$$

(ii) Let $(x_n)_n \subseteq X$ such that $\sum_{n=0}^{\infty} x_n$ converges in X. By passing to $\| \cdot \|_\sim$, which is continuous, we have $\| \sum_{n=0}^{\infty} x_n \|_\sim^r \leq \lim_{N \to \infty} \sum_{n=0}^{N} \| x_n \|_\sim^r$ (this limit exists in $[0, \infty]$). Using equivalence of norms leaves us with $\lesssim$ instead of $\leq$. More precisely, we calculate

$$\begin{aligned} \left\| \sum_{n=0}^{\infty} x_n \right\| &\leq \left(\left(C_\sim \left\| \sum_{n=0}^{\infty} x_n \right\|_\sim \right)^r \right)^{1/r} \\ &\leq C_\sim \left(\sum_{n=0}^{\infty} \| x_n \|_\sim^r \right)^{1/r} \\ &\leq C_\sim \left(\sum_{n=0}^{\infty} (C_\sim \| x_n \|)^r \right)^{1/r} \end{aligned}$$

$$\leq C_{\sim}^2 \sum_{n=0}^{\infty} \left(2^{1/r-1}\right)^n \| x_n \| \qquad \text{(2.6 (ii))}$$

$$= C_{\sim}^2 \sum_{n=0}^{\infty} C^n \| x_n \| .$$

(iii) The Banach case is dicussed in [Lun18, p. xi]. The quasi-Banach case works analogously, because all arguments only depend on the definiteness of quasi-norms. □

3 Quasi-Banach function spaces

As indicated in the introduction, quasi-Banach spaces and the generality of complex interpolation don't come with enough structure in order to further determine interpolation spaces. By implementing the order of $\mathbb{R}$ into the description of quasi-Banach spaces, they start to resemble Lebesgue spaces and many notions and results from integration theory carry over in an appropriate manner.

Convention 3. From here on out, we denote with Ω a separable metric space with a σ-finite measure μ on its Borel σ-algebra $\mathcal{B}(\Omega)$. We will abbreviate $(\Omega, \mathcal{B}(\Omega), \mu)$ with just Ω.

Definition 3.1. Let

$$\mathcal{L}_0(\Omega) := \{f : \Omega \to \mathbb{C} \mid f \text{ is } \mathcal{B}(\Omega)\text{-}\mathcal{B}(\mathbb{C})\text{-measurable}\}.$$

Then

$$\mathrm{L}_0(\Omega) := \mathcal{L}_0(\Omega)/\sim,$$

defines the space of *measurable functions on* Ω, where the equivalence relation $\sim$ is defined as $f \sim g :\iff f = g$ μ-almost everywhere when $f, g \in \mathcal{L}_0(\Omega)$.

Convention 4. In the following, we only write "a.e." for μ-a.e. and "measurable" for $\mathcal{B}(\Omega)$-$\mathcal{B}(\mathbb{C})$-measurable. If these expressions are modified, it should be clear from the context to what they refer to.
Of course, other notions that are written down in a pointwise sense are meant in a pointwise a.e. sense whenever equivalence classes of functions are involved.

Definition 3.2. Let $X \subseteq \mathrm{L}_0(\Omega)$ be a quasi-Banach space with the following properties:

(i) *Lattice property*: If $f \in \mathrm{L}_0(\Omega)$ and $g \in X$ with $|f| \leq |g|$, then $f \in X$ and $\| f \| \leq \| g \|$.

(ii) *Weak order unit*: There exists $f \in X$ with $f > 0$ a.e.

Then X is called a *quasi-Banach function space over* Ω. If $X \subseteq \mathrm{L}_0(\Omega)$ is a Banach space with the above properties, we call X a *Banach function space over* Ω.

Remark 3.3.

(i) A nice discussion as to why the definition as stated above is the "correct" one in order to do meaningful function space theory without being too restrictive can be found in [LN24]. Thanks goes to Sebastian Bechtel for pointing it out.

(ii) With this in mind, we should mention that requiring Ω to be a separable metric space is tailored to our needs, which will be be apparent when characterizing separability of quasi-Banach function spaces in Theorem 3.8.

But there are alternatives: In [LN24, Proposiiton 3.13] it was shown that the upcoming Theorem 3.8 continues to hold true if separability of a metric space is replaced with a purely measure theoretic definition of separability.

Convention 5. Because we will mostly deal with abstract function spaces and abstract Ω's anyway, we drop the "over Ω" part when talking about quasi-Banach function spaces. If a concrete example is examined, it will be clear from the example what Ω should be.

We collect some simple properties of function spaces.

Lemma 3.4. *Let X be a quasi-Banach function space.*

(i) If $f \in X$, then $|f| \in X$ with $\| f \| = \| |f| \|$ and $|f| < \infty$ a.e.

(ii) The sequence of sets $(\Omega_n)_n$ in the σ-finiteness of Ω can be chosen such that $\| \chi_{\Omega_n} \| < \infty$ for all $n \in \mathbb{N}$.

Proof.

(i) Since $|\,|f|\,| \leq |f|$, the lattice property gives us that $|f| \in X$ with $\|\,|f|\,\| \leq \|\,f\,\|$. Similarly, we have $|f| \leq |\,|f|\,|$, so that $\|\,f\,\| \leq \|\,|f|\,\|$.
For the second part, we first remark that the set $[|f| = \infty] = \cap_{M\in\mathbb{N}}[|f| > M]$ is measurable. Because $\chi_{[|f|=\infty]} \leq \frac{1}{n}|f|$, the lattice property implies $\chi_{[|f|=\infty]} \in X$ with

$$\left\|\,\chi_{[|f|=\infty]}\,\right\| = \frac{1}{n}\left\|\,n\chi_{[|f|=\infty]}\,\right\| \leq \frac{1}{n}\,\|\,|f|\,\|$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$, we obtain $\left\|\,\chi_{[|f|=\infty]}\,\right\| = 0$ and by definiteness of $\|\cdot\|$ also $\chi_{[|f|=\infty]} = 0$ and thus $\mu([|f| = \infty]) = 0$ as well.

(ii) Let $f \in X$ be a weak order unit. Denote $E_m := \{x \in \Omega \,|\, f(x) > \frac{1}{m+1}\}$ for $m \in \mathbb{N}$. Then

$$\Omega = \cup_{n=0}^{\infty}\Omega_n = \cup_{(n,m)\in\mathbb{N}^2}(\Omega_n \cap E_m),$$

where for all $n, m \in \mathbb{N}$ we have

$$\mu(\Omega_n \cap E_m) \leq \mu(\Omega_n) < \infty$$

and

$$\|\,\chi_{\Omega_n\cap E_m}\,\| = (m+1)\left\|\,\frac{1}{m+1}\chi_{\Omega_n\cap E_m}\,\right\| \leq (m+1)\,\|\,f\,\| < \infty.$$

Choosing our favourite enumeration of $\mathbb{N}^2$ and taking increasing unions now yields the claim. $\square$

Remark 3.5. In view of the Aoki-Rolewicz-Theorem 2.2 it would be nice if the r-subadditive equivalent quasi-norms on X would satisfy the lattice property as well. This is indeed true: Let $f, g \in \mathrm{L}_0(\Omega)$, $g \in X$ and $|f| \leq |g|$. Then $f \in X$ by X's lattice property w.r.t. $\|\cdot\|$ and any decomposition $g = \sum_{n=0}^{N} g_n$ where $g_n \in X$ yields a decomposition of f by $f = \sum_{n=0}^{N} \frac{f}{g} g_n$, since $\left|\frac{f}{g} g_n\right| \leq |g_n|$ and so $\frac{f}{g} g_n \in X$ with $\left\|\,\frac{f}{g} g_n\,\right\| \leq \|\,g_n\,\|$, so that $\|\,f\,\|_\sim \leq \|\,g\,\|_\sim$.

While it might seem reasonable to assume for simplicity that all quasi-Banach function spaces are from now on equipped with their corresponding quasi-norm from the Aoki-Rolewicz-Theorem, we won't do that here, since it increases the work we need to put in when working with explicit quasi-norms. For example, if X_0, X_1 are r-subadditive quasi-Banach spaces that embed into some topological vector space Z, it is not clear whether the induced quasi-norm on $X_0 + X_1$ is r-subadditive for the same value r, so that passing to the equivalent quasi-norm is still needed and would just complicate the arguments.

3.1 Order continuity

The question whether the lattice property can be related to monotonicity arises naturally and looks promising at first, as $x_n \searrow 0$ implies that $(\| x_n \|)_n$ already converges.

Definition 3.6. A quasi-norm on a quasi-Banach function space X is said to be *order continuous*, if for every sequence $(x_n)_n \in X$ with $x_n \searrow 0$ it holds that $\| x_n \| \searrow 0$.

$\mathrm{L}^p(\mathbb{R}^n)$ for $p \in [1, \infty)$ is a simple example for a Banach lattice with order continous norm due to the dominated convergence theorem. In fact, order continuity and dominated convergence are equivalent, so that the concept isn't new.

Proposition 3.7. *Let X be a quasi-Banach function space. Then X is order continous if and only if for every sequence $(f_n)_n \subseteq X$ with $|f_n| \le |g|$ and $f_n \to f$ a.e. for $f, g \in X$ implies $f_n \to f$ in X.*

Proof. We follow the proof of [BS88, Proposition 3.6].

$\underline{"\Rightarrow"}$: Let X be order continuous and $(f_n)_n \subseteq X$ with $|f_n| \le |g|$ and $f_n \to f$ a.e. for $f, g \in X$. We define

$$h_n := \sup_{k \ge n} |f_k - f| .$$

We have $h_n \in X$ as $|h_n| \le 2\,|g|$ and because of the pointwise a.e. convergence of the f_n we have $h_n \searrow 0$. Order continuity now yields that $0 \le \| f_n - f \| \le$

$\| h_n \| \searrow 0$.

$\underline{" \Leftarrow "}$: If X satisfies dominated convergence and we have for $(f_n)_n \subseteq X$ that $|f_n| \searrow 0$, we also have $|f_n| \leq |f_1|$ with $f_n \to 0$ a.e. Thus $\| f_n \| \to 0$. $\square$

$\mathrm{L}^\infty(\mathbb{R}^n)$ has various sequences violating order continuity at hand. The fact that $\mathrm{L}^p(\mathbb{R}^n)$ when $1 \leq p < \infty$ is separable, while $\mathrm{L}^\infty(\mathbb{R}^n)$ is not, is also the characterizing property.

Theorem 3.8. *Let X be a quasi-Banach function space. X is separable if and only if X is order-continuous.*

We will need two preparatory lemmata.

Lemma 3.9. *If $(f_n)_n \subseteq X$ is real-valued, monotonely increasing and there is $f \in X$ s.t. $f_n \to f$ in X, then $f = \sup_n f_n$.*

Proof. The proof is taken from [AB85, Theorem 11.2].

Let $\mathrm{Re}(X)_+ := \{h \in X \mid \mathrm{Im}(h) = \mathrm{Re}(h)_- = 0\}$. We want to show that this set is closed in X, so further let $(h_n)_n \subseteq \mathrm{Re}(X)_+$ with $h_n \to h$ in X. By the lattice property, we have $|h_n - h| \geq |\mathrm{Im}(h_n - h)|$ and $|h_n - h| \geq |\mathrm{Re}(h_n - h)| \geq |\mathrm{Re}(h_n)_- - \mathrm{Re}(h)_-|$, which directly translates to the corresponding estimates for $\| \cdot \|$. Thus $\mathrm{Im}(h) = \mathrm{Re}(h)_- = 0$ and $h \in \mathrm{Re}(X)_+$.

By assumption, we already obtain $f_n \leq f$. Now let $f_n \leq g$. Then $0 \leq g - f_n \to g - f$ and due to the closedness of $\mathrm{Re}(X)_+$, we get that $g \geq f$, i.e. $f = \sup_n f_n$. $\square$

To easen notation, we make the following definition.

Definition 3.10. Let X be a quasi-Banach function space and $(f_n)_n \subseteq X$ be a real valued sequence.

(i) $(f_n)_n$ is said to be *order bounded*, if there exist real-valued $g, h \in X$ with $g \leq f_n \leq h$ a.e. We then write $(f_n)_n \subseteq [g, h]$.

(ii) $(f_n)_n$ is said to be *disjoint*, if the f_n have pairwise disjoint supports, i.e. if $\min(f_n, f_m) = 0$ for $n \neq m$.

The following lemma makes a connection between order bounded and disjoint sequences. Its proof is rather technical and the construction uses (although very cleverly) only that quasi-Banach function spaces are partially ordered and doesn't contribute much to the understanding of function spaces. Thus, we omit it here and refer to [AB85, Theorem 12.11] for details.

Lemma 3.11. *Let X be a quasi-Banach function space and $(f_n)_n \subseteq [0, f]$ increasing. Then for every $k \in \mathbb{N}$ there exist disjoint sequences $(h_n^{(0)})_n, \ldots, (h_n^{(k)})_n \subseteq [0, f]$ with*

$$\sum_{j=0}^{k} h_n^{(j)} \leq f_{n+1} - f_n \leq \sum_{j=0}^{k} h_n^{(j)} + \frac{2}{k+3} f$$

a.e. for every $n \in \mathbb{N}$.

Proof of Theorem 3.8. $\underline{" \Rightarrow "}$: This direction is a culmination of results leading up to [AB85, Corollary 14.5] in the case that X is a Riesz space (a partially ordered Banach space, which has the lattice property and guarantees the existence of suprema and infima for finite subsets), so that we will only need to generalize the ideas to the quasi-normed case.

We argue by contraposition, so assume that X is not order continous.

Step 1: We construct an order bounded, monotonely increasing sequence that is not a Cauchy sequence.

By our assumption, there is (after possibly passing to a subsequence) $(f_n)_n \subseteq X$ and $\varepsilon > 0$ with $f_n \searrow 0$ but $\| f_n \| \geq \varepsilon$ for all $n \in \mathbb{N}$. Then the sequence $(f_0 - f_n)_n \subseteq [0, f_0]$ is increasing and not a Cauchy sequence. Assuming otherwise, it also would have a limit $g \in X$. Due to Lemma 3.9 we must have $g = \sup_n (f_0 - f_n) = f_0$, which yields $\| f_n \| = \| (f_0 - f_n) - f_0 \| = \| (f_0 - f_n) - g \| \to 0$, a contradiction. We write $g_n := f_0 - f_n$.

Step 2: Because $(g_n)_n$ is not a Cauchy sequence and $g_{n+1} - g_n$ can be approximated via disjoint sequences by Lemma 3.11, the disjoint sequences won't be Cauchy sequences as well.

Assume for a contradiction that every disjoint sequence $(h_n)_n \subseteq [0, f_0]$ is converging to 0 in X. After again possibly passing to a subsequence, we may assume that there exists some $\delta > 0$ s.t. $\| g_{n+1} - g_n \|_\sim^r \geq \delta$ for all $n \in \mathbb{N}$. Picking $k \in \mathbb{N}$ large enough s.t. $\left(\frac{2}{k+3} \| f_1 \|_\sim\right)^r < \frac{\delta}{2}$, Lemma 3.11 guarantees the existence of disjoint sequences $(h_n^{(0)})_n, \ldots, (h_n^{(k)})_n \subseteq [0, f_0]$ s.t.

$$\| g_{n+1} - g_n \|_\sim^r \leq \sum_{j=0}^{k} \| h_n^j \|_\sim^r + \left(\frac{2}{k+3} \| f_1 \|_\sim\right)^r.$$

For n large enough, this gives $\| g_{n+1} - g_n \|_\sim^r < \delta$, because we assumed that every disjoint sequence $(h_n)_n \subseteq [0, f_0]$ is converging to 0. A contradiction, thus there has to be some disjoint sequence $(h_n)_n \subseteq [0, f_0]$, which doesn't converge to 0.

Step 3: Using the disjointness of $(h_n)_n$, we can now show that X is not separable.

There is $\eta > 0$ s.t. after possibly passing to a subsequence we have that $\| h_n \| \geq \eta$ for all $n \in \mathbb{N}$. For every $A \in \mathcal{P}(\mathbb{N})$, we can define $h_A := \sum_{n \in A} h_n$. The series is well-defined by the disjointness of $(h_n)_n$ and $|h_A| \leq f_0$ gives that $h_A \in X$. The family $(h_A)_{A \in \mathcal{P}(\mathbb{N})}$ is uncountable, as $\mathcal{P}(\mathbb{N})$ is uncountable. It is also discrete in X, since for $A, B \in \mathcal{P}(\mathbb{N})$ with $A \neq B$ there is w.l.o.g. $n \in A \setminus B$ with $|h_A - h_B| \geq |h_n|$, which gives $\| h_A - h_B \| \geq \| h_n \| \geq \eta$. As X contains an uncountable discrete subset, it can't be separable.

"$\Leftarrow$": We follow the ideas for the separability of L^p-spaces as presented in [Str20, Theorem 3.2.10].

Step 1: We argue why we can simplify to characteristic functions of measurbale sets with finite quasi-norm and measure.

Let $f \in X$. Since $f = (\mathrm{Re} f)_+ - (\mathrm{Re} f)_- + i[(\mathrm{Im} f)_+ - (\mathrm{Im} f)_-]$ and using the quasi-triangle inequality three times doesn't hurt us, we can assume that f is real-valued and $f \geq 0$. Since we assume μ to be σ-finite, the order continuity allows us to approximate f by an increasing sequence w.r.t. sets that exhaust Ω, so we may assume that $\mu(\Omega), \| \chi_\Omega \| < \infty$ by Lemma 3.4 (ii). Lastly, since measurable functions can be approximated by non-negative incrasing simple functions, order continuity allows us to reduce to the case of simple functions and because these are finite linear combinations of characteristic functions, we can restrict ourselves to those.

Step 2: We make an educated guess for the countable dense subset of X, which is related to the separability of Ω.

Recall that in metric spaces, subsets of separable sets are also separable, so we didn't lose this assumption in the last step. Because Ω is a separable metric space, it is second-countable, i.e. there exists a countable family $\mathcal{U}$ of open subsets of Ω s.t. every open set in Ω can be written as a union of sets in $\mathcal{U}$. Set $\mathcal{U}' := \{\cap_{i=0}^{n} O_i \,|\, n \in \mathbb{N}, O_i \in \mathcal{U}\}$. Notice that $\mathcal{U}'$ is stable under finite intersections and that the Borel σ-algebra $\mathcal{B}$ of Ω is generated by $\mathcal{U}'$. Finally, we set

$$\mathcal{S} := \left\{ \sum_{i=0}^{n} a_m \chi_{A_m} \,|\, a_m \in \mathbb{Q} + \mathrm{i}\mathbb{Q},\ A_m \in \mathcal{U}' \right\}.$$

Obviously $\mathcal{S}$ is a countable subset of X. It remains to show that every characteristic function is the limit of a sequence in $\mathcal{S}$, i.e.

$$\mathcal{B}(\Omega) = \mathcal{A} := \{A \subseteq \Omega \,|\, A \text{ measurable and } \chi_A \in \overline{\mathcal{S}}\}.$$

Step 3: The inclusion $\mathcal{B}(\Omega) \supseteq \mathcal{A}$ is clear. For $\mathcal{B}(\Omega) \subseteq \mathcal{A}$, we show that $\mathcal{A}$ is a Dynkin system that is stable under finite intersections. It is then also a σ-algebra with $\mathcal{U}' \subseteq \mathcal{A}$ (immediate from the definitions), thus containing a generator of $\mathcal{B}$ and the claim follows.

$\mathcal{A}$ is a Dynkin system, as it satisfies the following properties:

(a) $\Omega \in \mathcal{A}$, since $\chi_\Omega \in \mathcal{S} \subseteq \overline{\mathcal{S}}$ due to $\Omega = \cap_{i\in\emptyset} O_i \in \mathcal{U}'$.

(b) If $A, B \in \mathcal{A}$ with $A \subseteq B$, then $\chi_{B\setminus A} = \chi_B - \chi_A \in \overline{\mathcal{S}}$ and thus $B \setminus A \in \mathcal{A}$.

(c) Let $(A_n)_n \subseteq \mathcal{A}$ be an increasing sequence. Then $\cup_n A_n \subseteq \Omega$, which implies that $\chi_{\cup_n A_n} \in X$ due to the lattice property (recall that we can assume that $\|\chi_\Omega\| < \infty$). Thus $\chi_{\cup_n A_n} - \chi_{A_n} \in X$ for all $n \in \mathbb{N}$, where $|\chi_{\cup_n A_n} - \chi_{A_n}| \searrow 0$, so that order continuity yields $\|\chi_{\cup_n A_n} - \chi_{A_n}\| \searrow 0$. Since $\overline{\mathcal{S}}$ is closed in X, we get $\cup_n A_n \in \mathcal{A}$.

Finally, $\mathcal{A}$ is closed under finite intersections, since for $A, B \in \mathcal{A}$ we have $\chi_{A\cap B} = \chi_A \cdot \chi_B$ and the product of the approximating sequences for χ_A, χ_B will approximate $\chi_A \cdot \chi_B$ after a single use of the quasi-triangle inequality.

Thus $\mathcal{S}$ is a countable dense subset of X, making X separable. $\square$

3.2 The associate space

Later on, we will need to test whether a real-valued function $f \in X$ is non-negative when X is a Banach function space. This can be realized easily in the case of Lebesgue spaces:

Lemma 3.12. *Let $p \in [1, \infty]$ and $f \in \mathrm{L}^p(\Omega)$ be real-valued. Then $f \geq 0$ if and only if for all measurable $A \subseteq \Omega$ with $\mu(A) < \infty$ it holds that $\int_\Omega f \chi_A \,\mathrm{d}\mu \geq 0$.*

Proof. $\underline{\text{"} \Rightarrow \text{"}}$: Immediate.

$\underline{\text{"} \Leftarrow \text{"}}$: Because Ω is σ-finite, there is $(\Omega_n)_n$ s.t. $\Omega_n \nearrow \Omega$ and $\mu(\Omega_n) < \infty$. Due to σ-subadditivity it thus suffices to show that for every $n, m \in \mathbb{N}$ we have

$$\mu\Bigg(\underbrace{\left[f < -\frac{1}{m+1}\right] \cap \Omega_n}_{=: M_{n,m}}\Bigg) = 0.$$

Notice that $M_{n,m}$ is measurable with $\mu(M_{n,m}) < \infty$. Using our assumption, we can estimate

$$0 \leq \int_\Omega f \chi_{M_{n,m}} \,\mathrm{d}\mu \leq -\frac{1}{m+1}\, \mu(M_{n,m}) \leq 0$$

to conclude $\mu(M_{n,m}) = 0$. $\square$

The proof of the above lemma was so easy, because the measure of a measurable set and the integral of its characteristic function are the same, which is not guaranteed in general quasi-Banach function spaces. Another point of view on the above result is that non-negativity may be checked by functionals, as $\chi_A \in \mathrm{L}^{p'}(\Omega) \hookrightarrow (\mathrm{L}^p(\Omega))^*$ for all $p \in [1, \infty]$ when $\mu(A) < \infty$. In order to check non-negativity in our more general function space setting as well, we need to proceed very carefully, as it is not obvious why any χ_A with $\mu(A) < \infty$ should induce a functional on any abstract Banach function space. We use the above proof to make the following definition:

Definition 3.13. Let X be a quasi-Banach function space. The *associate space* X' of X is defined as

$$X' := \left\{ f \in \mathrm{L}_0(\Omega) \;\middle|\; \| f \|' := \sup_{g \in X, \| g \| \leq 1} \left| \int_\Omega fg \,\mathrm{d}\mu \right| < \infty \right\}.$$

and is endowed with $\| \cdot \|'$.

Remark 3.14.

(i) There are variations for naming and defining associate spaces. Some authors also denote them as "Köthe duals" or might define the norm via absolute values inside the integral, i.e.

$$\| f \|' := \sup_{g \in X, \| g \| \leq 1} \int_\Omega |fg| \,\mathrm{d}\mu < \infty,$$

which still yields the same spaces, see [Zaa67, Ch. 69, Theorem 1].

(ii) Since the associate space is essentially a space of functionals induced via integrals, we obtain $X' \subseteq X^*$ with continuous inclusion.

(iii) Every associate space of a quasi-Banach function space X turns out to satisfy a triangle inequality just by being defined via integrals, which clashes with the quasi-triangle inequality of X. Thus, associate spaces sometimes turn out to be spaces of functionals that might not be too exciting: In the case of $X = \mathrm{L}^p(\Omega)$ when $p \in (0, 1)$ we obtain $\mathrm{L}^p(\Omega)' = \{0\}$ by the previous item and (v) from the introduction. But there are cases with interesting associate spaces, see [LN24, Theorem 3.3] for a characterization for when the associate space is a Banach function space.

In order to reproduce Lemma 3.12 for general Banach function spaces (no quasi-prefix here, which will become more apparent in a bit), we need its associate space to consist of interesting functions. At least on the side of function spaces themselves, this can be guaranteed by weak order units.

Proposition 3.15. *Let $X \subseteq \mathrm{L}_0(\Omega)$ be a quasi-Banach space that satisfies the lattice property. Then the following are equivalent:*

(i) *For every increasing sequence of measurable sets $\Omega_n \nearrow \Omega$ there exists another increasing sequence of measurable sets $\widetilde{\Omega_n} \nearrow \Omega$ s.t. $\widetilde{\Omega_n} \subseteq \Omega_n$ and $\|\chi_{\widetilde{\Omega_n}}\| < \infty$ for all $n \in \mathbb{N}$.*

(ii) *X contains a weak order unit.*

(iii) *For every measurable $A \subseteq \Omega$ s.t. $\mu(A) > 0$ there is another measurable $E \subseteq A$ s.t. $\mu(E) > 0$ and $\| \chi_E \| < \infty$.*

Proof. Throughout this proof, let $(\Omega_n)_n$ denote the sequence from Ω's σ-finiteness.

"(i) $\Rightarrow$ (ii)": Using (i) for the sequence of Ω's σ-finiteness yields an increasing sequence $(\widetilde{\Omega_n})_n$ with $\left\| \chi_{\widetilde{\Omega_n}} \right\|_\sim < \infty$. Thus the functions $f_n := \sum_{i=0}^n 2^{-i} \frac{\chi_{\widetilde{\Omega_n}}}{\left\| \chi_{\widetilde{\Omega_n}} \right\|_\sim}$ belong to X and letting $M, N \in \mathbb{N}$ with $M \geq N$ we obtain by using Aoki-Rolewicz that

$$\| f_M - f_N \|_\sim^r = \left\| \sum_{i=N+1}^{M} 2^{-i} \frac{\chi_{\Omega_n}}{\| \chi_{\Omega_n} \|_\sim} \right\|_\sim^r \leq \sum_{i=N+1}^{M} (2^r)^{-i}.$$

Thus, $(f_n)_n$ is a Cauchy sequence in X with limit $f \in X$. That $f > 0$ a.e. follows from $(f_n)_n$ being monotonously increasing and $f_n > 0$ a.e. on $\widetilde{\Omega_n}$, which translates to $f > 0$ a.e. by getting ahead of our schedule and using that convergence in X implies pointwise a.e. convergence of a subsequence (Corollary 5.19 (i)).

"(ii) $\Rightarrow$ (iii)": Because Ω is σ-finite, there is $n \in \mathbb{N}$ s.t. $\mu(\Omega_n \cap A) > 0$ due to A having positive measure. Using that the Ω_n can be chosen s.t. their characteristic functions have finite quasi-norm by Lemma 3.4 (ii), we put $E := \Omega_n \cap A$ and obtain $\| \chi_E \| \leq \| \chi_{\Omega_n} \| < \infty$.

"(i) $\iff$ (iii)": See [Zaa67, Ch. 67, Theorem 4] for a proof in the Banach case, which remains valid in the quasi-Banach setting, because the quasi-triangle inequality is only used on finite sums. □

Definition 3.16. Every quasi-Banach space $X \subseteq \mathrm{L}_0(\Omega)$ that satisfies the lattice property and either one of the statements in Proposition 3.15 is called *saturated*.

In view of Proposition 3.15, the following is clear and already makes the previous definition somewhat meaningful:

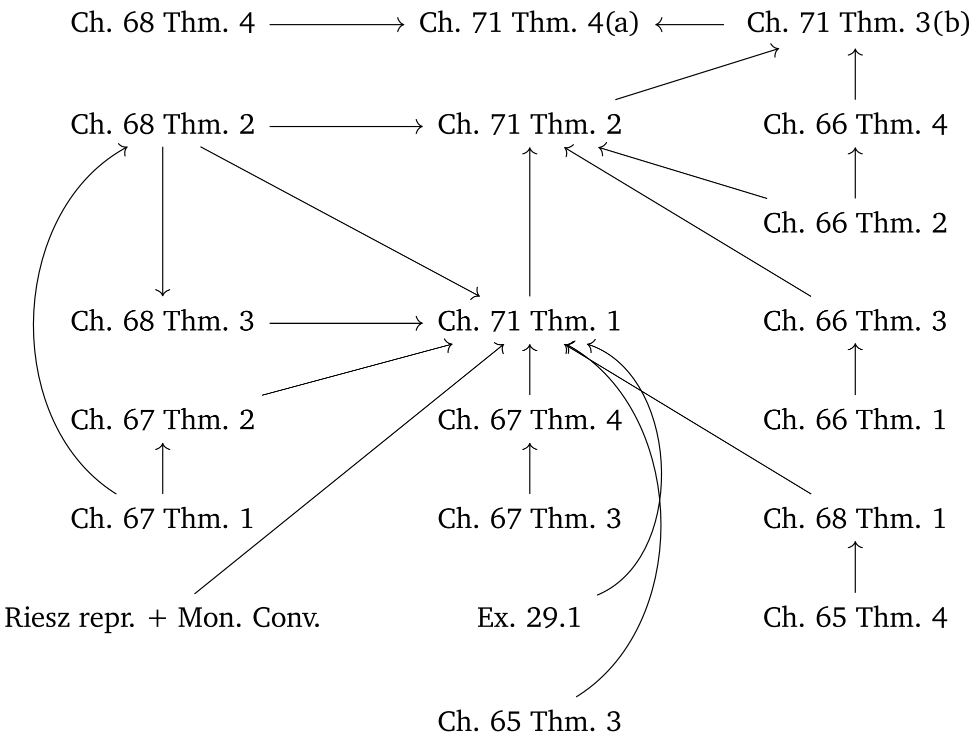
Proposition 3.17. *Every quasi-Banach function space is saturated.*

It remains to investigate whether the same holds true for associate spaces, which only in the Banach case turns out to always be correct.

Theorem 3.18. *Let X be a Banach function space. Then X' is a saturated Banach function space.*

Proof. That X' is a normed space, which is also complete, is treated in the opening remarks of [Zaa67, Ch. 69].

The proof for the saturatedness can be understood with a Bachelor student's background, but is too extensive to be written down here as well. As a compromise, we give minimal a list of implications, which can be found in [Zaa67], that need to be accounted for:



Our result is a consequence of Ch. 71 Thm. 4(a), where the most important argument is Ch. 71 Thm. 1, which makes use of a Hahn-Banach separation argument for Hilbert spaces for a point and $B_1(0) \cap \mathrm{L}^2(\Omega)$ where $B_1(0) \subseteq X$, which only works in locally convex spaces, where the unit ball is guaranteed to be convex. □

We can now show that testing non-negativity with functionals works for Banach function spaces by retelling the proof for the L^p case from Lemma 3.12 with slight modifications.

Corollary 3.19. *Let X be a Banach function space and $f \in \mathrm{L}_0(\Omega)$ be real-valued. Then $f \geq 0$ if and only if for all measurable $A \subseteq \Omega$ with $\| \chi_A \|' < \infty$ it holds that $\int_\Omega f\chi_A \,\mathrm{d}\mu \geq 0$.*

Proof. $\underline{\text{"} \Rightarrow \text{"}}$: Immediate.

$\underline{\text{"} \Leftarrow \text{"}}$: Theorem 3.18 used on the constant sequence $\Omega_n := \Omega$ yields the existence of a sequence $(\widetilde{\Omega_n})_n$ with $\widetilde{\Omega_n} \nearrow \Omega$ and $\left\| \chi_{\widetilde{\Omega_n}} \right\|' < \infty$ for all $n \in \mathbb{N}$. By μ's σ-subadditivity it thus suffices to show that for every $n, m \in \mathbb{N}$ we have

$$\mu\Bigg(\underbrace{\left[f < -\frac{1}{m+1}\right] \cap \widetilde{\Omega_n}}_{=:\widetilde{M_{n,m}}}\Bigg) = 0.$$

Notice that $\widetilde{M_{n,m}}$ is measurable with $\widetilde{M_{n,m}} \subseteq \widetilde{\Omega_n}$ and thus $\left\| \chi_{\widetilde{M_{n,m}}} \right\|' < \infty$, because X' satisfies the lattice property. Using our assumption, we can estimate

$$0 \leq \int_\Omega f\chi_{\widetilde{M_{n,m}}} \,\mathrm{d}\mu \leq -\frac{1}{m}\,\mu(\widetilde{M_{n,m}}) \leq 0$$

to conclude $\mu(\widetilde{M_{n,m}}) = 0$. □

Remark 3.20. The last Corollary 3.19 allows us to test estimates on functions ($f \leq g$ if and only if $0 \leq g - f$) with functionals, i.e. expressions that don't get arbitrarily large as

$$\left|\int_\Omega f\chi_A \,\mathrm{d}\mu\right| = \left|\int_\Omega \frac{f}{\| f \|}\chi_A \,\mathrm{d}\mu\right| \| f \| \leq \| \chi_A \|' \| f \| < \infty.$$

In order to achieve this, we needed to go the extra mile by proving this more general version of Lemma 3.12, because repeating its proof in general Banach function spaces leaves us with no control of $\| \chi_{M_{n,m}} \|'$, which would get us in trouble later on.

3.3 p-convexity

Even though there is no general integration theory for quasi-Banach spaces, some quasi-Banach function spaces have a way to partially recover a Riemann integral by means of

Definition 3.21. Let X be a quasi-Banach function space. X is said to be *p-convex*, if there exist $p \in (0,1]$ and a constant $C \geq 1$ s.t. for any $f_1, \ldots, f_n \in X$ we have

$$\left\| \left(\sum_{i=0}^{n} |f_i|^p \right)^{1/p} \right\| \leq C \left(\sum_{i=0}^{n} \| f_i \|^p \right)^{1/p}.$$

Definition 3.22. Let X be a p-convex quasi-Banach function space. Then $X^p := \{ f \in \mathrm{L}_0(\Omega) \mid |f|^{1/p} \in X \}$ is called *p-convexification of X*.

The above set is a vector space due to X's lattice property. We also want to endow it with a topology by showing

Proposition 3.23. *The p-convexification X^p of a quasi-Banach function space X can be normed s.t. it becomes a Banach function space.*

Proof. For $f \in X^p$ we define

$$\| f \|_{X^p} := \left\| |f|^{1/p} \right\|_X^p.$$

If $f \in X$ is a weak order unit, then $f^{1/p} \in X^p$ is a weak order unit as well and $\| \cdot \|_{X^p}$ satisfies the lattice property as well, because $\| \cdot \|_X$ does so and raising to powers is monotone. Furthermore, this map satisfies definiteness and homogeneity and a special kind of quasi-triangle inequality. For this, let $f, g \in X^p$, then

$$\| f + g \|_{X^p} = \left\| |f + g|^{1/p} \right\|_X^p$$

$$
\begin{aligned}
&\le \left\| \left[\left(|f|^{1/p} \right)^p + \left(|g|^{1/p} \right)^p \right]^{1/p} \right\|_X^p \\
&\le C \left\| |f|^{1/p} \right\|_X^p + \left\| |g|^{1/p} \right\|_X^p \\
&= C(\| f \|_{X^p} + \| g \|_{X^p}),
\end{aligned}
$$

where $C \ge 1$ is the constant from the p-convexity of X. Notice that since C is uniform w.r.t. the amount of vectors being estimated on the left-hand side (recall this fact from the definition of p-convexity, Definition 3.21), this is not the most general form of a quasi-triangle inequality and we can get rid of C by renorming $\| \cdot \|_{X^p}$ similarly to Step 2 in the proof of the Aoki-Rolewicz Theorem 2.2. This norm will be the one with which X^p will be endowed, but since it is equivalent to $\| \cdot \|_{X^p}$, it is sufficient to show completeness w.r.t. $\| \cdot \|_{X^p}$.

For the remaining completeness, we want to use the characterization of completeness in quasi-Banach spaces via series from Proposition 2.7 (i) where C is larger than the constant form X's p-convexity and the one from X's quasi-triangle inequality. Since we may take any such C that is convenient for us, we will do so by further requiring that $C \ge 2$. Let $(f_n)_n \subseteq X^p$ with $\sum_{n=0}^{\infty} C^n \| f_n \|_{X^p} < \infty$. Then for $N \ge M$ we obtain

$$
\begin{aligned}
&\le \frac{1}{C} \left\| \sum_{n=M}^{N} (C^n |f_n|)^{1/p} \right\|_X^p \\
&\le \frac{1}{C} \left\| \left(\sum_{n=M}^{N} ((C^n |f_n|)^{1/p})^p \right)^{1/p} \right\|_X^p && \text{(Young's inequality)} \\
&\le \sum_{n=M}^{N} \left\| (C^n |f_n|)^{1/p} \right\|_X^p && \text{(p-convexity)} \\
&= \sum_{n=M}^{N} C^n \| f_n \|_{X^p} .
\end{aligned}
$$

Thus, $\sum_{n=0}^{\infty} (C^n |f_n|)^{1/p}$ converges in X. Due to

$$
\left| \sum_{n=0}^{\infty} f_n \right|^{1/p} \overset{2.6\,\text{(ii)}}{\le} \sum_{n=0}^{\infty} \left(2^{1/p-1} \right)^n |f_n|^{1/p} \le \sum_{n=0}^{\infty} (C^n |f_n|)^{1/p}
$$

and the lattice property of X we also obtain $|\sum_{n=0}^{\infty} f_n|^{1/p} \in X$ and thus

$\sum_{n=0}^{\infty} f_n \in X^p$. This gives a reasonable candidate for the limit of $\left(\sum_{n=}^{N} f_n\right)_N$ in X^p, which after reusing the previous pointwise estimates and the lattice property of X turns out to be the correct one:

$$\begin{aligned}
&\left\| \sum_{n=0}^{N} f_n - \sum_{n=0}^{\infty} f_n \right\|_{X^p} \\
=&\left\| \left| \sum_{n=N+1}^{\infty} f_n \right|^{1/p} \right\|_X^p \\
\leq &\left\| \sum_{n=N+1}^{\infty} \left(\left(2^{1/p-1}\right)^{n+1} |f_n| \right)^{1/p} \right\|_X^p \\
\leq C &\left\| \sum_{n=N+1}^{\infty} \left(C^n |f_n|\right)^{1/p} \right\|_X^p \overset{N\to\infty}{\longrightarrow} 0.
\end{aligned}$$

□

The upshot is that a continuous and bounded function $F : \mathbb{R} \to X$ can't be integrated when X is a p-convex quasi-Banach function space, but $|F|^p : \mathbb{R} \to X^p$ can be, which might be enough to carry over arguments from the Banach to the quasi-Banach setting.

It turns out that p-convexity has much more meaning to it, because p-convex quasi-Banach function spaces are exactly those for which the maximum principle holds true. This is non-trivial, because, as mentioned in the introduction, in general not every quasi-Banach space satisfies the maximum principle.

Definition 3.24. Let X be a quasi-Banach space. X is said to be *A-convex* (or *analytically convex*), if there exists a constant $M \geq 1$ s.t. for any polynomial $P : \mathbb{D} \to X$ we have $\| P(0) \| \leq M \max_{z\in\partial\mathbb{D}} \| P(z) \|$.

Theorem 3.25. *Let X be a quasi-Banach function space. Then the following are equivalent:*

(i) X is A-convex.

(ii) X is p-convex for some $p > 0$.

(iii) There exists $p > 0$ s.t. X is r-convex for all $r \in (0, p)$.

Proof. Since the proof of this result goes way beyond the scope of this work, we won't open this can of worms and just refer to [Kal86b, Theorem 4.4] and [Kal84, Theorem 2.2]. □

However, in our case A-convexity is only a remnant of the original formulation of Theorem 1.1 as we won't need A-convexity explicitly and will thus state the main result with p-convexity.

4 Subharmonic functions

Let $U \subseteq \mathbb{R}^n$ be open. It is a well known fact that $u \in \mathrm{C}^2(U)$ is harmonic, i.e. $-\Delta u = 0$ if and only if u is representable via mean value formulas, i.e. for every $B(x,r) \subseteq U$ with $B(x,r) \subseteq \overline{B(x,r)} \subseteq U$ we have

$$u(x) = \fint_{\partial B(x,r)} u(y) \,\mathrm{d}S = \fint_{B(x,r)} u(y) \,\mathrm{d}y.$$

From there, we can define sub- und superharmonic functions by replacing equalities with appropriate inequalites and prove similar characterizations, i.e.

$$\begin{aligned} u \text{ is} \begin{Bmatrix} \text{superharmonic} \\ \text{harmonic} \\ \text{subharmonic} \end{Bmatrix} \quad &\iff \quad -\Delta u \begin{Bmatrix} \geq \\ = \\ \leq \end{Bmatrix} 0 \\ &\iff \quad u(x) \begin{Bmatrix} \geq \\ = \\ \leq \end{Bmatrix} \fint_{\partial B(x,r)} u(y) \,\mathrm{d}S. \end{aligned}$$

The expressions in the last equivalence make sense, if u is for example locally bounded and don't require any differentiability, which makes for a more general concept. In any case, the values of a sub-, super- or just harmonic function are in some way dependent on their boundary behaviour.

With view on the topics up to now, this concept seemingly comes out of nowhere and it will only become apparent when proving the main result Theorem 1.2 why we need this digression to take place.

4.1 Basic properties of subharmonic functions

First, we specify what exactly we mean by a subharmonic function.

Definition 4.1. Let $U \subseteq \mathbb{C}$ be open. A function $u : U \to \mathbb{R}$ is said to be *subharmonic*, if whenever $B_r(z) \Subset U$, then

$$u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + r\mathrm{e}^{\mathrm{i}t}) \, \mathrm{d}t.$$

The following characterization of subharmonicity will be very important for us and further sheds light on where the "sub"-syllable may come from.

Theorem 4.2. *Let $u : U \to \mathbb{C}$ be contnuous. Then the following are equivalent:*

(i) u is subharmonic.

(ii) For every compact subset $K \subseteq U$ and every $h \in C(K)$, which is harmonic in K° and satisfies $u \leq h$ on ∂K, it holds true that $u \leq h$ in K.

For its proof, we will need the Poisson kernel. We recall its formula and properties.

Definition 4.3. (i) The *Poisson kernel* $P : \mathbb{D} \times \partial\mathbb{D} \to \mathbb{R}$ is defined by

$$P(z, \zeta) := \operatorname{Re}\left(\frac{\zeta + z}{\zeta - z}\right) = \frac{1 - |z|^2}{|\zeta - z|^2}, \qquad z \in \mathbb{D}, \zeta \in \partial\mathbb{D}.$$

(ii) For $B_\rho(w) \subseteq \mathbb{C}$ and Lebesgue-integrable $\phi : \partial B_\rho(w) \to \mathbb{R}$ the *Poisson integral* $P_{B_\rho(w),\phi} : B_\rho(w) \to \mathbb{R}$ is defined as

$$P_{B_\rho(w),\phi}(z) := \frac{1}{2\pi} \int_0^{2\pi} P\left(\frac{z - w}{\rho}, \mathrm{e}^{\mathrm{i}\theta}\right) \phi(w + \rho\mathrm{e}^{\mathrm{i}\theta}) \, \mathrm{d}\theta, \qquad z \in B_\rho(w).$$

More explicitly, rewriting $B_\rho(w) \ni z = w + r\mathrm{e}^{\mathrm{i}t}$ for $r \in [0, \rho)$ and $t \in [0, 2\pi)$ we have

$$P_{B_\rho(w),\phi}(w + r\mathrm{e}^{\mathrm{i}t}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\rho^2 - r^2}{\rho^2 - 2\rho r \cos(\theta - t) + r^2} \phi(w + \rho\mathrm{e}^{\mathrm{i}\theta}) \, \mathrm{d}\theta.$$

This gives us many harmonic functions to work with. A proof for the following theorem can be found in [Ran95, Theorem 1.2.4].

Theorem 4.4. *With the notation in Definition 4.3 we have*

(i) $P_{B_\rho(w),\phi}$ *is harmonic on* $B_\rho(w)$.

(ii) If ϕ *is continuous at* $\zeta_0 \in \partial B_\rho(w)$, *then* $\lim_{z\to\zeta_0} P_{B_\rho(w),\phi}(z) = \phi(\zeta_0)$.

Proof of Theorem 4.2. "$\Rightarrow$": We present the proof of [Rud87, Theorem 17.4].

Assume for a contradiction that there is a compact $K \subseteq U$ and a harmonic function $h \in C(K)$ s.t. $u \leq h$ on ∂K and there is $z \in K^\circ$ with $u(z) - h(z) > 0$. We abbreviate $\widetilde{u} := u - h$. As $\widetilde{u}$ is continuous, it attains its maximum $m > 0$ on K, where the set $M := \{z \in K \mid \widetilde{u}(z) = m\}$ is a non-empty compact subset of K°, as $\widetilde{u} \leq 0$ on ∂K. Let $z_0 \in \partial M$. Then there exists $r > 0$ s.t. $\overline{B_r(z_0)} \subseteq K^\circ$ and there is $z' \in \overline{B_r(z_0)} \cap (K^\circ \setminus M)$. Again due to $\widetilde{u}$'s continuity, there is an open arc $V \subseteq \partial B_{|z_0-z'|}(z_0)$ s.t. $z' \in V$ and $\widetilde{u} \leq \frac{\widetilde{u}(z')+m}{2}$ on V. Thus

$$\begin{aligned}\widetilde{u}(z_0) = m &= \frac{1}{2\pi}\int_{\{t\in[0,2\pi]\,|\,|z_0-z'|\mathrm{e}^{\mathrm{i}t}\in V\}} m\,\mathrm{d}t + \frac{1}{2\pi}\int_{\{t\in[0,2\pi]\,|\,|z_0-z'|\mathrm{e}^{\mathrm{i}t}\notin V\}} m\,\mathrm{d}t \\ &> \frac{1}{2\pi}\int_0^{2\pi} \widetilde{u}(z_0 + |z_0 - z'|\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t\end{aligned}$$

and $\widetilde{u}$ is not subharmonic in K°. But since u is subharmonic and h is so as well due to the mean value formula for harmonic functions and sums of subharmonic functions are again subharmonic, this is a contradiction.

"$\Leftarrow$": Assuming for a contradiction that u is not subharmonic, there is $z_0 \in U$ and $r > 0$ s.t. $B_r(z_0) \Subset U$ and $u(z_0) > \fint_0^{2\pi} u(z_0 + r\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t$. As u is continuous, it is bounded on $\partial B_r(z_0)$ and thus Lebesgue-integrable on $\partial B_r(z_0)$. By Theorem 4.4 (i), the function $h := P_{B_r(z_0),u|_{\partial B_r(z_0)}}$ is harmonic in $B_r(z_0)$ and $u = h$ on $\partial B_r(z_0)$. By assumption, we have $u(z_0) \leq h(z_0) = \fint_0^{2\pi} u(z_0 + r\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t$, a contradiction. $\square$

Corollary 4.5. *Let* $U, V \subseteq \mathbb{C}$ *open,* u *be subharmonic on* U *and* $\varphi : V \to U$ *be biholomorphic. Then* $u \circ \varphi$ *is subharmonic on* V.

Proof. Let $K \subseteq V$ be compact, $h \in \mathrm{C}(K)$ and h harmonic in K° with $u \circ \varphi \leq h$ on ∂K. As φ is a homeomorphism, we have that $\varphi(K) \subseteq U$ is compact and $\partial\varphi(K) = \varphi(\partial K)$. Thus every $z \in \partial\varphi(K)$ may be rewritten as $z = \varphi(w)$ for exactly one $w \in \partial K \subseteq V$ and so

$$u(z) = u(\varphi(w)) \leq h(w) = h(\varphi^{-1}(z)).$$

Because h is harmonic, it can on locally simply connected domains be written as the real part of a holomorphic function f, so that locally $h \circ \varphi^{-1} = \mathrm{Re}(f) \circ \varphi^{-1} = \mathrm{Re}(f \circ \varphi^{-1})$. Since $f \circ \varphi^{-1}$ is again holomorphic, its real part is harmonic and thus $h \circ \varphi^{-1}$ is locally harmonic, i.e. harmonic. As u is subharmonic, the "if"-part of Theorem 4.2 yields that $u \leq h \circ \varphi^{-1}$ on $\varphi(K)$ resp. $u \circ \varphi \leq h$ on K. As $K \subseteq V$ is arbitrary, the "only if"-part of Theorem 4.2 proves the subharmonicity of $u \circ \varphi$. $\square$

4.2 An important class of subharmonic functions

Still, this rather abstract characterization doesn't give us a good idea of what a subharmonic function might look like while there are in fact many interesting subharmonic functions due to complex analysis.

Theorem 4.6. *Let $U \subseteq \mathbb{C}$ open, $u : U \to \mathbb{C}$ holomorphic. Then $|u|^p$ is subharmonic for all $p \in (0, \infty)$.*

To give a rough idea of how the proof works, we assume that u has no zeros in U so that the mean-value formula holds for $\ln(|u|)$ as it is harmonic in U, which can be seen by locally rewriting $\ln(|u|)$ as the real part of the holomorphic function $\ln(u) = \ln|u| + \mathrm{i}\arg(u)$, with the implicit convention that the branch cut is taken appropriately according to the values that u might take locally, which is fine because harmonicity is a local property. Or by brutally applying the Laplacian and invoking the Cauchy-Riemann equations a couple of times, which definitely wasn't the first thing that the author of these notes tried to do...

In any case, let $B_r(z) \Subset U$. Using Jensen's inequality, we can estimate

$$\begin{aligned}
|u(z)|^p &= \mathrm{e}^{p\ln(|u(z)|)} \\
&= \mathrm{e}^{p\frac{1}{2\pi}\int_0^{2\pi}\ln(|u(z+r\mathrm{e}^{\mathrm{i}t})|)\,\mathrm{d}t} \\
&\leq \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{p\ln(|u(z+r\mathrm{e}^{\mathrm{i}t})|)}\,\mathrm{d}t \\
&= \frac{1}{2\pi}\int_0^{2\pi} |u(z+r\mathrm{e}^{\mathrm{i}t})|^p\,\mathrm{d}t.
\end{aligned} \tag{4.1}$$

The challenge lies in the case where u might have zeros in U. In this case, $\ln(|u|)$ will not be harmonic but still satisfies a sub-mean value formula, which in conjunction with the monotonicity of the exponential function shows the claim as done above.

We will use another one of Jensen's doings, namely Jensen's formula, which is a generelization for the mean-value formula of $\ln(|u|)$ and accounts for how much the mean-value formula fails by introducing an error term that depends on the zeros of u.

First, we need

Lemma 4.7. $\int_0^{2\pi} \ln\left|1-\mathrm{e}^{\mathrm{i}t}\right|\,\mathrm{d}t = 0$.

Proof. We want to calculate the above integral by interpreting the integrand as the real part of an appropriate holomorphic function on $M := \{z \in \mathbb{C} \mid \mathrm{Re}(z) < 1\}$ in order to use Cauchy's integral formula along paths in M that approximate $\partial B_1(0)$ but cut out around $z = 1$. The proof stems from [Rud87, Lemma 15.17].

Let $\ln$ be the principal branch of the logarithm. It will serve us well to determine the properties of the real and imaginary parts of $\ln(1 - \cdot\,)$. Because $\mathrm{e}^{\ln(1-z)} = 1 - z$ for all $z \in M$, we already have

$$|1-z| = \left|\mathrm{e}^{\ln(1-z)}\right| = \mathrm{e}^{\mathrm{Re}(\ln(1-z))} \quad \text{i.e.} \quad \mathrm{Re}(\ln(1-z)) = \ln(|1-z|).$$

Due to $\mathrm{Re}(1-z) > 0$ on M we also obtain

$$\begin{aligned}
0 < \mathrm{Re}(\mathrm{e}^{\ln(1-z)}) &= \mathrm{e}^{\mathrm{Re}(\ln(1-z))}\mathrm{Re}(\mathrm{e}^{\mathrm{i}\mathrm{Im}(\ln(1-z))}) \\
&= \mathrm{e}^{\mathrm{Re}(\ln(1-z))}\cos(\mathrm{Im}(\ln(1-z))).
\end{aligned}$$

As $0 < \mathrm{e}^{\mathrm{Re}(h)}$ and $\ln(1-0) = 0$, we are left with $|\mathrm{Im}(\ln(1-z))| < \pi/2$.
For $\delta > 0$, we define $\Gamma_\delta : [\delta, 2\pi - \delta] \to \mathbb{C}, t \mapsto \mathrm{e}^{\mathrm{i}t}$ and γ_δ to be the circular arc inside M that connects $\mathrm{e}^{i\delta}$ with $e^{-\mathrm{i}\delta}$. Using Cauchy's integral theorem for the function $z \mapsto \frac{\ln(1-z)}{z}$, $z \in M$, which is holomorphic, because $\ln(1 - \cdot\,)$ is complex differentiable in $z = 0$, we obtain

$$\begin{aligned}\int_0^{2\pi} \ln\left|1-\mathrm{e}^{\mathrm{i}t}\right| \,\mathrm{d}t &= \lim_{\delta\to 0} \int_\delta^{2\pi-\delta} \ln\left|1-\mathrm{e}^{\mathrm{i}t}\right| \,\mathrm{d}t \\ &= \lim_{\delta\to 0} \mathrm{Re}\left[\int_{\Gamma_\delta} \frac{\ln(1-z)}{z}\,\mathrm{d}z\right] \\ &= \lim_{\delta\to 0} \mathrm{Re}\left[\int_{\gamma_\delta} \frac{\ln(1-z)}{z}\,\mathrm{d}z\right].\end{aligned}$$

All that's left is to estimate the integral over γ_δ. By restricting to $\delta > 0$ small enough s.t. $\ln(|1-z|) \geq \pi/2$ on γ_δ, we obtain

$$\begin{aligned}\left|\int_{\gamma_\delta} \frac{\ln(1-z)}{z}\,\mathrm{d}z\right| &\leq \int_{\gamma_\delta} \frac{|\ln(1-z)|}{|z|}\,\mathrm{d}z \\ &\leq \int_{\gamma_\delta} \frac{\sqrt{\ln(|1-z|)^2 + \left(\frac{\pi}{2}\right)^2}}{1-|1-\mathrm{e}^{\mathrm{i}\delta}|}\,\mathrm{d}z \\ &\leq \int_{\gamma_\delta} \frac{\ln(|1-z|)}{1-|1-\mathrm{e}^{\mathrm{i}\delta}|}\,\mathrm{d}z \\ &\leq \pi\delta \frac{\ln(\left|1-\mathrm{e}^{\mathrm{i}\delta}\right|)}{1-|1-\mathrm{e}^{\mathrm{i}\delta}|} \\ &= \pi\delta \frac{\ln(\left|1-\mathrm{e}^{\mathrm{i}\delta}\right|) - \ln(|\mathrm{i}\delta|) + \ln(|\mathrm{i}\delta|)}{1-|1-\mathrm{e}^{\mathrm{i}\delta}|} \\ &= \pi\delta \frac{\ln\left(\left|\frac{1-\mathrm{e}^{\mathrm{i}\delta}}{\delta}\right|\right)}{1-|1-\mathrm{e}^{\mathrm{i}\delta}|} + \frac{\pi\delta\ln(\delta)}{1-|1-\mathrm{e}^{\mathrm{i}\delta}|} \xrightarrow{\delta\to 0} 0. \qquad \square\end{aligned}$$

Proposition 4.8 (Jensen's formula)**.** *Let $r \in (0,1)$, f be holomorphic on $\mathbb{D}$, $f(0) \neq 0$ and $\alpha_0, \ldots, \alpha_N$ be the zeros of f in $\overline{B_r(0)}$, where N is the sum of all multiplicities of all distinct zeros of f in $\overline{B_r(0)}$. Then*

$$\ln(|f(0)|) = \frac{1}{2\pi}\int_0^{2\pi} \ln(|f(r\mathrm{e}^{\mathrm{i}t})|)\,\mathrm{d}t + \sum_{n=0}^{N} \ln\left(\frac{|\alpha_n|}{r}\right).$$

Proof. This proof is taken from [Rud87, Theorem 15.18].

Let the zeros of f be enumerated s.t. $\alpha_0, \dots, \alpha_m \in B_r(0)$ and $\alpha_{m+1}, \dots, \alpha_N \in \partial B_r(0)$ if possible. If just one or none of the previous cases applies, the corresponding products resp. sums won't have a contribution, which does not change the outcome of the proof. We want to make f into a function with no zeros inside $\overline{B_r(0)}$ by removing its zeros and thus define

$$g(z) := f(z) \prod_{n=0}^{m} \frac{r^2 - \overline{\alpha_n} z}{r(\alpha_n - z)} \prod_{n=m+1}^{N} \frac{\alpha_n}{\alpha_n - z}, \qquad z \in O$$

where $O \subseteq \mathbb{D}$ is some open neighbourhood of $\overline{B_r(0)}$ that does not contain any zeros of f other than $\alpha_0, \dots, \alpha_N$. Notice that when $|z| = r$, we have that

$$\left| \frac{r^2 - \overline{\alpha_n} z}{r(\alpha_n - z)} \right| = \left| \frac{z\,\overline{z} - \overline{\alpha_n} z}{r(\alpha_n - z)} \right| = \left| \frac{z(\overline{\alpha_n} - \overline{z})}{r(\alpha_n - z)} \right| = 1.$$

g is holomorphic in O, because f is complex differentiable in each of its zeros, and g has no zeros, because the multiplicity of each zero is accountend for, so that $\ln(|g|)$ is harmonic in O, which gives us

$$\ln(|g(0)|) = \frac{1}{2\pi} \int_0^{2\pi} \ln(\left|g(r\mathrm{e}^{\mathrm{i}t})\right|) \,\mathrm{d}t.$$

By plugging in the definition of g and rewriting $\alpha_n = r\mathrm{e}^{\mathrm{i}t_n}$ for $m+1 \leq n \leq N$, we obtain

$$\begin{aligned} &\ln(|f(0)|) + \sum_{n=0}^{m} \ln\left(\frac{r}{|\alpha_n|}\right) \\ =&\frac{1}{2\pi} \int_0^{2\pi} \ln(\left|f(r\mathrm{e}^{\mathrm{i}t})\right|) - \sum_{n=m+1}^{N} \ln\left(\left|1 - \mathrm{e}^{i(t-t_n)}\right|\right) \,\mathrm{d}t. \end{aligned}$$

By Lemma 4.7, the second term inside the integral can be omitted, so that after rearranging the claim is shown. □

Corollary 4.9. *If f is holomorphic on $\mathbb{D}$, then for all $r \in (0,1)$ it holds that*

$$\ln\left(|f(0)|\right) \leq \frac{1}{2\pi} \int_0^{2\pi} \ln\left(\left|f(r\mathrm{e}^{\mathrm{i}t})\right|\right) \,\mathrm{d}t.$$

Proof. If $f(0) = 0$, the left-hand side is $-\infty$ and the estimate trivially holds true. If $f(0) \neq 0$, let $\alpha_0, \dots, \alpha_N$ be the zeros of f in $\overline{B_r(0)}$, where N is the sum of all multiplicities of all distinct zeros of f in $\overline{B_r(0)}$. Then $\frac{|\alpha_n|}{r} \leq 1$ and so $\ln\left(\frac{|\alpha_n|}{r}\right) \leq 0$, which yields the desired estimate by using Jensen's formula (Proposition 4.8) as follows:

$$\begin{aligned} \ln(|f(0)|) &= \frac{1}{2\pi}\int_0^{2\pi} \ln(|f(re^{\mathrm{i}t})|)\,\mathrm{d}t + \sum_{n=0}^{N} \ln\left(\frac{|\alpha_n|}{r}\right) \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} \ln(|f(re^{\mathrm{i}t})|)\,\mathrm{d}t. \end{aligned}$$ □

Proof of Theorem 4.6. Due to the previous Corollary 4.9, we can proceed as in the motivational calculation in (4.1) to show the claim. □

4.3 Controlling subharmonic functions through their boundary behaviour

In this section we take a look at subharmonic functions with boundary values on $\mathbb{D}$ and S and show that values in the interior are not only controlled by balls inside the domain but also by the boundary values on $\partial\mathbb{D}$ and ∂S.

Proposition 4.10. *Let u be continuous and subharmonic in $\mathbb{D}$ s.t. u extends to a bounded function on $\overline{\mathbb{D}}$ which is continuous along radial limits, i.e. $u(re^{\mathrm{i}\theta}) \xrightarrow{r\to 1} u(e^{\mathrm{i}\theta})$ for almost every $\theta \in [0, 2\pi]$. Then*

$$u(z) \leq \fint_{\partial\mathbb{D}} P(z,\zeta)u(\zeta)\,\mathrm{d}S(\zeta)$$

for all $z \in \mathbb{D}$, where P is the Poisson kernel.

Proof. Fix $z \in \mathbb{D}$. Then for every $r \in (\frac{1+|z|}{2}, 1)$ we have $z \in B_r(0)$. Theorem 4.4 asserts that the Poisson integral $h_r := P_{B_r(0), u|_{B_r(0)}}$ is harmonic on $B_r(0)$ with continuous boundary data $h_r = u$ on $\partial B_r(0)$. Using the equivalent characterization of subharmonicity from Theorem 4.2 for the compact set $\overline{B_r(0)} \subseteq \mathbb{D}$, we have

$$u(z) \le h_r(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2 - |z|^2}{r^2 - 2r\,|z| \cos(\theta - t) + |z|^2} u(r\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta$$

where $z = |z|\,\mathrm{e}^{\mathrm{i}t}$. Because $r \in (\frac{1+|z|}{2}, 1)$, the integrand satisfies the estimate

$$\begin{aligned}\left| \frac{r^2 - |z|^2}{r^2 - 2r\,|z| \cos(\theta - t) + |z|^2} u(r\mathrm{e}^{\mathrm{i}\theta}) \right| &\le \frac{1 - |z|^2}{(r - |z|)^2} \left| u(r\mathrm{e}^{\mathrm{i}\theta}) \right| \\ &\le \frac{1}{\left(\frac{1+|z|}{2} - |z|\right)^2} \| \, u \, \|_\infty \\ &\le \frac{4}{(1 - |z|)^2} \| \, u \, \|_\infty ,\end{aligned}$$

which is uniform in r as z is fixed. Furthermore, by assumption the integrand converges for almost every θ to $P(z, \mathrm{e}^{\mathrm{i}\,\cdot\,})u(\mathrm{e}^{\mathrm{i}\,\cdot\,})$ when $r \to 1$. Thus, dominated convergence yields the claim. □

Because the composition of a subharmonic function with a biholomorphic function is again subharmonic (see Corollary 4.5), it seems reasonable that a similar result should hold true on S. For this, we need

Lemma 4.11. *The map*

$$C : S \to \mathbb{D}, \qquad z \mapsto \frac{\mathrm{e}^{\pi \mathrm{i} z} - \mathrm{i}}{\mathrm{e}^{\pi \mathrm{i} z} + \mathrm{i}}$$

is biholomorphic and extends to a homeomorphism on $\overline{S}$ with $C(\partial S) = \partial\mathbb{D} \setminus \{-1, 1\}$.

Proof. That C maps S to $\mathbb{D}$ biholomorphically is pointed out in [Lun18, Lemma 2.9]. That the natural extension is still injective (and a homeomorphism) is due to the fact that $z \mapsto \frac{z-\mathrm{i}}{z+i}$ is a Möbius transform, which is biholomorphic on $\mathbb{C}$, and that $\mathrm{e}^{\pi\mathrm{i}\,\cdot}$ fails to be injective for $z, z' \in \mathbb{C}$ if and only if $z - z' \in 2\mathbb{Z}$, which is not possible on $\overline{S}$. By the same reasoning, $\mathrm{e}^{\pi\mathrm{i}\,\cdot}$ is biholomorphic on a neighbourhood of $\overline{S}$ and thus C extends to a homeomorphism onto its image. It only remains to calculate said image, which won't be the whole of $\overline{\mathbb{D}}$.

Let $t \in \mathbb{R}$. First, we observe that

$$C(\mathrm{i}t) = \frac{\mathrm{e}^{-\pi t} - \mathrm{i}}{\mathrm{e}^{-\pi t} + \mathrm{i}}$$

and

$$C(1 + \mathrm{i}t) = \frac{\mathrm{e}^{\pi \mathrm{i}(1+\mathrm{i}t)} - \mathrm{i}}{\mathrm{e}^{\pi \mathrm{i}(1+\mathrm{i}t)} + \mathrm{i}} = \frac{-\mathrm{e}^{-\pi t} - \mathrm{i}}{-\mathrm{e}^{-\pi t} + \mathrm{i}} = C(\mathrm{i}t)^{-1}.$$

Now,

$$|C(\mathrm{i}t)| = \left| \frac{\mathrm{e}^{-\pi t} - \mathrm{i}}{\mathrm{e}^{-\pi t} + \mathrm{i}} \right| = \left| \frac{\mathrm{e}^{-\pi t} - \mathrm{i}}{\overline{\mathrm{e}^{-\pi t} - \mathrm{i}}} \right| = 1$$

and so $|C(1 + \mathrm{i}t)| = 1$ as well. Thus $C(1 + \mathrm{i}t) = 1/(\mathrm{Re}(C(\mathrm{i}t)) + \mathrm{Im}(C(\mathrm{i}t))) = \mathrm{Re}(C(\mathrm{i}t)) - \mathrm{i}\mathrm{Im}(C(\mathrm{i}t))$ and it suffices to only check the properties of $C(\mathrm{i}t)$. Further writing this out gives

$$C(\mathrm{i}t) = \frac{\mathrm{e}^{-2\pi t} - 1}{\mathrm{e}^{-2\pi t} + 1} - \mathrm{i}\frac{2\mathrm{e}^{-\pi t}}{\mathrm{e}^{-2\pi t} + 1},$$

so that $\mathrm{Im}(C(\mathrm{i}t)) > 0$. It follows that $\mathrm{Im}(C(1 + \mathrm{i}t)) < 0$ and we already obtain $C(\partial S) \subseteq \partial \mathbb{D} \setminus \{-1, 1\}$.

To see that $C(\mathrm{i}\mathbb{R}) = [\mathrm{Im}(z) > 0] \cap \overline{\mathbb{D}}$, it suffices to check that

$$\arg(C(\mathrm{i}t)) = \begin{cases} \arctan\left(\frac{\mathrm{Im}(C(\mathrm{i}t))}{\mathrm{Re}(C(\mathrm{i}t))}\right), & \text{if } \mathrm{Re}(C(\mathrm{i}t)) > 0 \\ \arctan\left(\frac{\mathrm{Im}(C(\mathrm{i}t))}{\mathrm{Re}(C(\mathrm{i}t))}\right) + \pi, & \text{if } \mathrm{Re}(C(\mathrm{i}t)) < 0 \\ \frac{\pi}{2}, & \text{if } \mathrm{Re}(C(\mathrm{i}t)) = 0. \end{cases}$$

takes on every value in $(0, \pi)$. Notice that $\arg(C(\mathrm{i} \cdot))$ is continuous. Since $t \to \infty$ implies $\arg(C(\mathrm{i}t)) \to 0$ while $t \to -\infty$ implies $\arg(C(\mathrm{i}t)) \to \pi$, the intermediate value theorem implies that every value $(0, \pi)$ indeed does get taken. Similarly, we conclude $C(1 + \mathrm{i}\mathbb{R}) = [\mathrm{Im}(z) < 0] \cap \overline{\mathbb{D}}$ and the claim is shown. $\square$

Corollary 4.12. *Let $u : S \to \mathbb{R}$ be continuous and subharmonic s.t. u extends to a bounded and continuous function on $\overline{S}$. Then there exists a positive function $P' : S \times \partial S \to \mathbb{R}$ that is integrable on ∂S for every $z \in S$ s.t.*

$$u(z) \leq \int_{\partial S} P'(z, \zeta) u(\zeta) \,\mathrm{d}S(\zeta).$$

In this case we can also write

$$u(z) \leq \int_{\mathbb{R}} P_0(z, \mathrm{i}t) u(\mathrm{i}t) \, \mathrm{d}t + \int_{\mathbb{R}} P_1(z, 1 + \mathrm{i}t) u(1 + \mathrm{i}t) \, \mathrm{d}t.$$

where $P_0 := P'|_{S \times \mathrm{i}\mathbb{R}}$ *and* $P_1 := P'|_{S \times (1+\mathrm{i})\mathbb{R}}$ *satisfy* $\int_{\mathbb{R}} P_0(z,t) \, \mathrm{d}t = 1 - \mathrm{Re}(z)$ *and* $\int_{\mathbb{R}} P_1(z,t) \, \mathrm{d}t = \mathrm{Re}(z)$ *for every* $z \in S$*.*

Proof. Let C be as in Lemma 4.11, which shows that C induces homeomorphisms from $\partial\mathbb{D} \cap \{z \in \mathbb{C} \,|\, \mathrm{Re}(z) > 0\}$ to $\mathrm{i}\mathbb{R}$ and from $\partial\mathbb{D} \cap \{z \in \mathbb{C} \,|\, \mathrm{Re}(z) < 0\}$ to $(1+\mathrm{i})\mathbb{R}$. Furthermore, the composition $u \circ C^{-1}$ is subharmonic on $\mathbb{D}$ by Corollary 4.5 and extends to a bounded and continuous function on $\mathbb{D} \cup \partial\mathbb{D} \setminus \{-1, 1\}$ due to u's properties. By choosing any values for the extension to $\{-1, 1\}$, the requirements of Theorem 4.10 are satisfied and we can estimate

$$\begin{aligned} u(z) &= u(C^{-1}(C(z))) \\ &\leq \int_{\partial\mathbb{D}} P(C(z), \zeta) \, u(C^{-1}(\zeta)) \, \mathrm{d}S(\zeta) \\ &= \int_{\partial S} \frac{1}{2\pi} P(C(z), C(\zeta)) \, |C'(\zeta)| \, u(\zeta) \, \mathrm{d}S(\zeta) \qquad \text{(Transformation rule)}. \end{aligned}$$

Thus we obtain the first part with $P'(z, \zeta) := 1/(2\pi) P(C(z), C(\zeta)) \, |C'(\zeta)|$, $(z, \zeta) \in S \times \partial S$.

We now embark on our journey to calculate P' explicitly. Let $\zeta = \mathrm{i}t \in \mathrm{i}\mathbb{R}$, $z = x + \mathrm{i}y \in S$, then

$$\begin{aligned} P'(z,\zeta) &= \frac{1}{2\pi} \frac{1 - \left| \frac{e^{\mathrm{i}\pi z} - \mathrm{i}}{e^{\mathrm{i}\pi z} + \mathrm{i}} \right|^2}{\left| \frac{e^{-\pi t} - \mathrm{i}}{e^{-\pi t} + \mathrm{i}} - \frac{e^{\mathrm{i}\pi z} - \mathrm{i}}{e^{\mathrm{i}\pi z} + \mathrm{i}} \right|^2} \left| \frac{2\pi \mathrm{i} \, e^{-\pi t}}{(\mathrm{e}^{-\pi t} + \mathrm{i})^2} \right| \\ &= \frac{\left|\mathrm{i} + \mathrm{e}^{\mathrm{i}\pi z}\right|^2 - \left|\mathrm{i} - \mathrm{e}^{\mathrm{i}\pi z}\right|^2}{\left|(e^{-\pi t} - \mathrm{i})\,(e^{\mathrm{i}\pi z} + \mathrm{i}) - (e^{\mathrm{i}\pi z} - \mathrm{i})\,(e^{-\pi t} + \mathrm{i})\right|^2} \mathrm{e}^{-\pi t} \\ &= \frac{1 + 2\mathrm{Im}(e^{\mathrm{i}\pi z}) + (\mathrm{e}^{-\pi y})^2 - \left[1 - 2\mathrm{Im}(e^{\mathrm{i}\pi z}) + (\mathrm{e}^{-\pi y})^2\right]}{\left|\mathrm{i}\,[2\mathrm{e}^{-\pi t} - 2\mathrm{e}^{i\pi z}]\right|^2} \mathrm{e}^{-\pi t} \\ &= \frac{4\mathrm{e}^{-\pi y} \sin(\pi x)}{4\left[(e^{-\pi t})^2 - 2\mathrm{e}^{-\pi t}\mathrm{Re}(e^{\mathrm{i}\pi z}) + (\mathrm{e}^{-\pi y})^2\right]} \mathrm{e}^{-\pi t} \\ &= \frac{\mathrm{e}^{-\pi(y+t)} \sin(\pi x)}{(\mathrm{e}^{-\pi y})^2 \left[\left(e^{\pi(y-t)}\right)^2 - 2\mathrm{e}^{\pi(y-t)} \cos(\pi x) + \cos(\pi x)^2 + \sin(\pi x)^2\right]} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\mathrm{e}^{\pi(y-t)}\sin(\pi x)}{\sin(\pi x)^2 + \left[e^{\pi(y-t)} - \cos(\pi x)\right]^2} \\
&=: P_0(z, \mathrm{i}t).
\end{aligned}
$$

If $\zeta = 1 + \mathrm{i}t \in 1 + \mathrm{i}\mathbb{R}$, then $e^{\mathrm{i}\pi\zeta} = -\mathrm{e}^{-\pi t}$ and making the proper replacements in the above calculation yields

$$
\begin{aligned}
P'(z, \zeta) &= \frac{\mathrm{e}^{\pi(y-t)}\sin(\pi x)}{\sin(\pi x)^2 + \left[e^{\pi(y-t)} + \cos(\pi x)\right]^2} \\
&=: P_1(z, 1 + \mathrm{i}t).
\end{aligned}
$$

To see that $\int_{\mathbb{R}} P_0(z, \mathrm{i}t)\,\mathrm{d}t = 1 - \mathrm{Re}(z)$ for every $z \in S$, we first observe that $\int_{\mathbb{R}} P_0(z, t)\,\mathrm{d}t$ doesn't depend on y. We may thus assume that $y = 0$ and can directly compute the integral:

$$
\begin{aligned}
\int_{\mathbb{R}} P_0(z, \mathrm{i}t)\,\mathrm{d}t &= \frac{1}{\pi}\left[\arctan\left(\frac{\cos(\pi x) - \mathrm{e}^{-\pi t}}{\sin(\pi x)}\right)\right]_{-\infty}^{\infty} \\
&= \frac{1}{\pi}\left[\arctan\left(\frac{1}{\tan(\pi x)}\right) + \frac{\pi}{2}\right] \\
&= \frac{1}{\pi}\left[\arctan\left(\tan\left(\frac{\pi}{2} - \pi x\right)\right) + \frac{\pi}{2}\right] \\
&= \frac{1}{\pi}\left[\frac{\pi}{2} - \pi x + \frac{\pi}{2}\right] && (\text{when } x \in (0, 1)) \\
&= 1 - x.
\end{aligned}
$$

The evaluation of the second integral works similarly. □

5 Complex interpolation

Calderón's interpolation method finds its motivation in a classical result from interpolation theory.

Theorem 5.1 (Riesz-Thorin). *Let $(M, \mathcal{A}, \eta)$ and $(N, \mathcal{B}, \zeta)$ be measure spaces and denote with $\mathcal{M}(M, \mathcal{A}, \eta)$ the measurable (similarly for $\mathcal{M}(N, \mathcal{B}, \zeta)$) and with $\mathcal{S}(M, \mathcal{A}, \eta)$ the step functions on M. Let*

$$T : \mathcal{S}(M, \mathcal{A}, \eta) \to \mathcal{M}(N, \mathcal{B}, \zeta)$$

be a linear operator that satisfies

$$\| Tf \|_{q_0} \leq M_0 \| f \|_{p_0}, \qquad \| Tf \|_{q_1} \leq M_1 \| f \|_{p_1}$$

for all $f \in \mathcal{S}(M, \mathcal{A}, \eta)$, where $1 \leq p_0 < p_1 \leq \infty$, $1 \leq q_0, q_1 \leq \infty$. For $t \in (0, 1)$ set

$$\frac{1}{p_t} := \frac{1-t}{p_0} + \frac{t}{p_1}, \qquad \frac{1}{q_t} := \frac{1-t}{q_0} + \frac{t}{q_1}.$$

Then for every $t \in [0, 1]$, T is uniquely extendable to an operator

$$T : \mathrm{L}^{p_t}(M, \mathcal{A}, \eta) \to \mathrm{L}^{q_t}(N, \mathcal{B}, \zeta)$$

with

$$\| T \|_{q_t, p_t} \leq M_0^{1-t} M_1^t.$$

The proof uses duality of L^p-spaces in order to rewrite certain norms as integrals, that can be interpreted as evaluations of holomorphic functions, for which complex analysis is used.

5.1 Interpolation of quasi-Banach spaces

With this in mind, we can view Calderón's version of interpolation as the proof of the above theorem, made into a definition, which we now make precise.

Definition 5.2. Let X_0, X_1 be quasi-Banach spaces.

(i) *Interpolation pair*: The pair (X_0, X_1) is called an interpolation pair, if there exists a Hausdorff topological vector space Z s.t. X_0, X_1 continuously embed into Z.

(ii) *Analytic functions*: A function $F : S \to X_0 + X_1$ is called analytic, if F can be expressed locally as a power series, i.e. for every $z_0 \in S$ there is $(f_n)_n \subseteq X$ and $r > 0$ s.t. $F(z) = \sum_{n=0}^{\infty} f_n(z - z_0)^n$ holds in $X_0 + X_1$ for every $z \in B_r(z_0)$.

(iii) *Admissible functions*: Define the space of admissible functions $\mathcal{F}$ as the space of analytic functions $F : S \to X_0 + X_1$ which are bounded and extend continuously up to $\overline{S}$ s.t. the traces $t \mapsto F(j + \mathrm{i}t)$ are bounded continuous functions into X_j, $j = 0, 1$.

(iv) *Interpolation space*: Endow $\mathcal{F}$ with

$$\| F \|_{\mathcal{F}} := \max \left\{ \sup_{t\in\mathbb{R}} \| F(\mathrm{i}t) \|_{X_0}, \sup_{t\in\mathbb{R}} \| F(1 + \mathrm{i}t) \|_{X_1}, \sup_{z\in S} \| F(z) \|_{X_0+X_1} \right\}$$

where $F \in \mathcal{F}$. Let $\theta \in [0, 1]$. The interpolation space $[X_0, X_1]_\theta \subseteq X_0 + X_1$ is then defined as

$$[X_0, X_1]_\theta := \{F(\theta) \mid F \in \mathcal{F}\}$$

and endowed with

$$\| f \|_\theta := \inf\{\| F \|_{\mathcal{F}} \mid f = F(\theta), F \in \mathcal{F}\}, \qquad f \in [X_0, X_1]_\theta .$$

A few remarks are in order.

Remark 5.3.

(i) Some authors also require the following density assumption in Defintion 5.2 (i):

The intersection $X_0 \cap X_1$ is dense in X_i, $i = 0, 1$.

This requirement excludes pathological but also interesting examples for interpolation spaces (since in general, $[X_0, X_1]_j$ when $j = 0, 1$ does not need to reproduce X_j) and is useful for duality results, see for example [Cal64, Theorem 9.5, 12.1]. Since this type of density assumption won't play a major role (in fact, it will only come up in the Wolff reiteration theorem) in this thesis, we leave it out.

(ii) Notice that we didn't require absolute convergence of the series in Definition 5.2 (ii).

(iii) The usage analytic functions opens up the machinery that is complex analysis, at least in the normed case. We already mentioned that in the quasi-Banach setting, integration theory and results like the maximum principle are in general not feasible. We will manage, but need to be much more careful.

(iv) None of the conditions in Defintion 5.2 (iii) should be omitted as then only trivial interpolation spaces are produced or the whole construction does not work. Let $f = f_0 + f_1 \in X_0 + X_1$ be arbitrary.

(a) If boundedness is omitted, there is no obvious way to quasi-norm $\mathcal{F}$.

(b) If the trace condition is omitted and $\|\cdot\|_{\mathcal{F}}$ is changed appropriately, constant functions are admissible functions, so that $F := f$ realizes f via $F(\theta) = f$ and $[X_0, X_1]_\theta = X_0 + X_1$ for every $\theta \in [0, 1]$.

(c) If continuous extension is omitted, in the case of $\theta \in (0, 1)$ we can choose

$$F(z) := \begin{cases} f_0, & \text{if } \mathrm{Re}(z) = 0 \\ f_1, & \text{if } \mathrm{Re}(z) = 1 \\ f, & \text{otherwise.} \end{cases}$$

to realize f via an admissible function and obtain $[X_0, X_1]_\theta = X_0 + X_1$. In the case of $\theta = 0$, we choose

$$F(z) := \begin{cases} f_0, & \text{if } \mathrm{Re}(z) = 0 \\ 0, & \text{otherwise.} \end{cases}$$

to obtain $[X_0, X_1]_0 = X_0$. Similarly, we obtain $[X_0, X_1]_1 = X_1$.

(d) Lastly, if analyticity is omitted, for $\theta \in (0,1)$ the map $z \mapsto F(z) := \frac{1-\mathrm{Re}(z)}{1-\theta} f_0 + \frac{\mathrm{Re}(z)}{\theta} f_1$ is admissible, as it is bounded and for every $t \in \mathbb{R}$ we have

$$F(\mathrm{i}t) = \frac{1}{1-\theta} f_0 \in X_0$$
$$F(1+\mathrm{i}t) = \frac{1}{\theta} f_1 \in X_1.$$

Again, F clearly realizes f so that $[X_0, X_1]_\theta = X_0 + X_1$.
If $\theta = 0, 1$, we omit the summand in F that is not well-defined and get $[X_0, X_1]_0 = X_0, [X_0, X_1]_1 = X_1$.

In order to do anlysis in interpolation spaces, we at least need to show that they are complete, for which we need some basic properties of quasi-Banach valued analytic functions.

Theorem 5.4. *Let X be a quasi-Banach space and $U \subseteq \mathbb{C}$ be open. Then the following are equivalent:*

(i) $f : U \to X$ is analytic.

(ii) For every $z \in U$ there exists $r > 0$ s.t. $B_r(z) \Subset U$, a Banach space X_z, a linear and continuous operator $T_z : X_z \to X$ and an analytic function $f_z : B_r(z) \to X_z$ s.t. $f|_{B_r(z)} = T_z \circ f_z$.

(iii) For every $z \in U$ and $r > 0$ s.t. $B_r(z) \Subset U$ there exists a Banach space X_z, a linear and continuous operator $T_z : X_z \to X$ and an analytic function $f_z : B_r(z) \to X_z$ s.t. $f|_{B_r(z)} = T_z \circ f_z$.

Proof. "(i)⇒(iii)": This is shown in [Tur76, Theorem 9.3.2].

"(iii)⇒(ii)": This is clear.

"(ii)$\Rightarrow$(i)": If f satisfies (ii), due to the continuity of T_z the series representation of f_z directly carries over to f, making f analytic. $\square$

Corollary 5.5. *Let X be a quasi-Banach function space, $U \subseteq \mathbb{C}$ open and $f : U \to X$ analytic.*

(i) If $V \subseteq \mathbb{C}$ is open and $\varphi : V \to U$ is holomorphic, then $f \circ \varphi : V \to \mathbb{C}$ is analytic as well.

(ii) Let $z \in U$. Then the power series representation of f conicides with f in the largest open disc around z that is sill contained in U.

Proof.

(i) Let $z \in V$. Then the characterization of analytic functions from Theorem 5.4 (ii) applies and for $\varphi(z) \in U$ there exists $r > 0$ s.t. $B_r(\varphi(z)) \Subset U$, a Banach space $X_{\varphi(z)}$, a linear and continuous operator $T_{\varphi(z)} : X_{\varphi(z)} \to X$ and an analytic function $f_{\varphi(z)} : B_r(\varphi(z)) \to X_{\varphi(z)}$ s.t. $f|_{B_r(\varphi(z))} = T_{\varphi(z)} \circ f_{\varphi(z)}$. By continuity of φ we may pick some $B_{r'}(z) \subseteq \varphi^{-1}(B_r(\varphi(z)))$. Then for every $a \in B_{r'}(z)$ it holds that

$$(f \circ \varphi)(a) = f(\varphi(a)) = (T_{\varphi(z)} \circ f_{\varphi(z)})(\varphi(a)) = (T_{\varphi(z)} \circ (f_{\varphi(z)} \circ \varphi))(a).$$

The function $f_z := f_{\varphi(z)} \circ \varphi$ is analytic on $B_{r'}(z)$ because in the Banach setting, analyticity is equivalent to complex differentiability [HJ14, Theorem 160], which is easily seen here by the chain rule. Again by Theorem 5.4, $f \circ \varphi$ is thus analytic.

(ii) This is a direct consequence of the arbitrary choice of $r > 0$ in the factorization theorem (Theorem 5.4 (iii)). $\square$

Obviously, $\|\cdot\|_{\mathcal{F}}$ is well defined, as admissible functions are bounded. Its quasi-norm properties are immediate, where $\max\{C_0, C_1, \max\{C_0, C_1\}\} = \max\{C_0, C_1\}$ is a possible constant for the quasi-triangle inequality. Except for analyticity, we could copy the proof of $\mathrm{C}([0,1])$'s completeness (which makes use of the fact that $X_0 + X_1$ is complete, see Lemma 2.7 (iii) (a)).

For analyticity, we would use Morera's theorem which is not available in the case of quasi-Banach spaces, so that we need to bring out the big guns, whose proofs are out of the scope of this work.

That $\mathcal{F}$ is complete is a consequence of [Kal86a, Theorem 6.3], but we intend to also give another seemingly elementary proof that doesn't rely on integration theory and uses more involved tools, wherever elementary approaches would fail. In addition to the above already established properties of analytic functions, we will mainly need a weaker version of the maximum principle that holds true in all quasi-Banach spaces.

Theorem 5.6. *Let X be a quasi-Banach space. Then for every $s \in (0,1)$ there exists $C = C(s,X) \geq 1$ s.t. whenever $F : \overline{\mathbb{D}} \to X$ is a uniform limit of functions $F_n : \overline{\mathbb{D}} \to X$, which are continuous in $\overline{\mathbb{D}}$ and analytic on $\mathbb{D}$, it holds that*

$$\| F(0) \| \leq C \sup_{s \leq |z| \leq 1} \| F(z) \| .$$

Proof. The hard part is done in [Kal86c, Theorem 5.2], where it is shown that for every $s \in (0,1)$ there exists $C(s,X) \geq 1$ s.t. whenever $F : \overline{\mathbb{D}} \to X$ is continuous in $\overline{\mathbb{D}}$ and analytic on $\mathbb{D}$ it holds that

$$\| F(0) \| \leq C(s,X) \sup_{s \leq |z| \leq 1} \| F(z) \| .$$

Of course, this continues to hold true for the equivalent quasi-norm from the Aoki-Rolewicz theorem with the constant $C_\sim^2 C(s,X)$.

Now for $n \in \mathbb{N}$ let F_n be X-valued, analytic on $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$ and $F_n \to F$ uniformly on $\overline{\mathbb{D}}$. Using the previous case, we obtain

$$\begin{aligned} \| F(0) \|_\sim^r &\leq \| F(0) - F_n(0) \|_\sim^r + \| F_n(0) \|_\sim^r \\ &\leq \| F(0) - F_n(0) \|_\sim^r + \left(C_\sim^2 C(s,X) \sup_{s \leq |z| \leq 1} \| F_n(z) \|_\sim \right)^r . \end{aligned}$$

By continuity of $\| \cdot \|_\sim$ and uniform convergence of the F_n we obtain

$$\| F(0) \|_\sim \leq C_\sim^2 C(s,X) \sup_{s \leq |z| \leq 1} \| F(z) \|_\sim .$$

Passing back to $\|\cdot\|$ we obtain the claim with the constant $C := C_\sim^4 C(s, X)$. $\square$

This can be improved a bit.

Corollary 5.7. *In the situation of Theorem 5.6, it also holds true that for every $s \in (0,1)$ there exists $r' \in (0,1)$ and a constant $C = C(s, X) \geq 1$ s.t.*

$$\sup_{|z|\leq s} \| f(z) \| \leq C \sup_{r' \leq |z| \leq 1} \| f(z) \| .$$

Proof. Let $z \in \overline{B_s(0)}$. We recall from complex analysis that

$$\varphi_z(v) := \frac{z - v}{1 - \overline{z}v}, \qquad v \in \overline{\mathbb{D}}.$$

is biholomorphic on $\mathbb{D}$ with $\varphi_z(0) = z$ and $\varphi_z(\partial\mathbb{D}) = \partial\mathbb{D}$. Let $r \in (s, 1)$. We obtain

$$\begin{aligned} \| f(z) \| &= \| f(\varphi_z(0)) \| \\ &\leq C \sup_{r \leq |v| \leq 1} \| f(\varphi_z(v)) \| \qquad \text{(Corollary 5.5 (i) and Theorem 5.6)} \\ &= C \sup_{w \in \varphi_z(\{r \leq |v| \leq 1\})} \| f(w) \| \end{aligned}$$

and the claim follows, if we can show that there is $r' \in (0,1)$ s.t. for all $z \in \overline{B_s(0)}$ it holds that $\varphi_z(\{r \leq |v| \leq 1\}) \subseteq \{r' \leq |v| \leq 1\}$, because then we can further estimate with

$$\leq C \sup_{r' \leq |w| \leq 1} \| f(w) \|$$

and taking the supremum on the left-hand side yields the desired estimate, since r' doesn't depend on $z \in \overline{B_s(0)}$ but only on s.

To this end, we will show that $r' := \frac{r-s}{1+s}$ is an appropriate choice. Let $\rho \leq |v| \leq 1$. Then $|\varphi_z(v)| \leq 1$, because $\varphi_z(\overline{\mathbb{D}}) = \overline{\mathbb{D}}$ and

$$|\varphi_z(v)| \geq \frac{|r - s|}{1 + |z|\,|v|} \geq \frac{r - s}{1 + s} = r'.$$

Thus $\varphi_z(\{r \leq |v| \leq 1\}) \subseteq \{r' \leq |v| \leq 1\}$ indeed holds true for all $z \in \overline{B_s(0)}$. $\square$

We will also need that the power series of analytic functions converge absolutely and thus locally uniformly in their respective disc of convergence.

Lemma 5.8. *Let X be a quasi-Banach function space, $\rho > 0$ and $\sum_{n=0}^{\infty} f_n z^n$ be a power series with coefficients in X that converges for all $z \in B_\rho(0)$. Then the power series $\sum_{n=0}^{\infty} |f_n| \, |z|^n$ converges for all $z \in B_\rho(0)$ as well.*

Proof. Let $z \in B_\rho(0)$. The sequence $(\sum_{n=0}^{N} f_n z^n)_N$ is a Cauchy sequence, so it follows that $\| f_n z^n \|_\sim \overset{n\to\infty}{\longrightarrow} 0$. In particular, there is $n_0 \in \mathbb{N}$ s.t. for all $n \geq n_0$ it follows that $\| f_n \|_\sim |z|^n < 1$, which is equivalent to $\| f_n \|_\sim^{1/n} < 1/|z|$. Thus $\limsup_{n\to\infty} \| f_n \|_\sim^{1/n} \leq 1/|z|$, where the left-hand side is now independent of the specific choice of z. Using that $\| \cdot \|_\sim$ satisfies the lattice property (Remark 3.5) and thus doesn't depend on any absolute values on the inside (Lemma 3.4 (i)) and passing to the limit $|z| \to \rho$ yields $\limsup_{n\to\infty} \| \, |f_n| \, \|_\sim^{1/n} = \limsup_{n\to\infty} \| f_n \|_\sim^{1/n} \leq 1/\rho$.
We proceed by showing that for every $z \in B_\rho(0)$ the sequence $(\sum_{n=0}^{N} |f_n| \, |z|^n)_N$ is a Cauchy sequence in X. For $m, n \in \mathbb{N}$, $m \geq n$ we estimate

$$\left\| \sum_{k=n}^{m} |f_k| \, |z|^k \right\|_\sim \leq \left(\sum_{k=n}^{m} (\| \, |f_k| \, \|_\sim |z|^k)^r \right)^{1/r} . \tag{5.1}$$

It holds that

$$\limsup_{k\to\infty} \left((\| \, |f_k| \, \|_\sim |z|^k)^r \right)^{1/k} = \left(\limsup_{k\to\infty} \| \, |f_k| \, \|_\sim^{1/k} |z| \right)^r \leq \left(\frac{1}{\rho} |z| \right)^r < 1,$$

so that by the root test, the expression inside the large brackets on the right-hand side of (5.1) represents a converging series, making the left-hand side converge in X as well. $\square$

Remark 5.9. In particular, Lemma 5.8 shows that convergent power series with coefficients in a quasi-Banach space X converge absolutely inside their radius of convergence, as the RHS in (5.1) was shown to represent a converging series.

Lemma 5.10. *In the situation of Lemma 5.8, both power series converge locally uniformly inside $B_r(0)$.*

Proof. It suffices to only consider balls $B_s(0) \Subset B_r(0)$ with $s \in (0, r)$. Because $\sum_{n=0}^{\infty} f_n z^n$ converges if $|z| = s$, Lemma 5.8 implies that $\sum_{n=0}^{\infty} |f_n| \, |z|^n$ converges as well and the lattice property shows for all $z \in \overline{B_s(0)}$ and $m \in \mathbb{N}$ that

$$\left\| \sum_{n=m}^{\infty} f_n z^n \right\| = \left\| \left| \sum_{n=m}^{\infty} f_n z^n \right| \right\| \leq \left\| \sum_{n=m}^{\infty} |f_n| \, |z|^n \right\| \leq \left\| \sum_{n=m}^{\infty} |f_n| \, s^n \right\|.$$

Taking the supremum on the left-hand side in the above estimate before letting $m \to \infty$ yields the claim. □

Theorem 5.11. *$\mathcal{F}$ is a quasi-Banach space.*

Proof. Even though the arguments leading up to [BC90, Theorem 3.1] are used in a slightly different context (showing that holomorphic functions (= approximable by analytic functions of finite rank) are again analytic), they are applicable to this situation as well.

As already mentioned, it only remains to show that limits in $\mathcal{F}$ preserve analyticity. To this end, let $F_n \to F$ in $\mathcal{F}$, F_n analytic and $z_0 \in S$. By Corollary 5.5 (ii) we can pick some $r_0 > 0$ s.t. $\overline{B_{r_0}(z_0)} \subseteq S$ and s.t. all F_n have a series expansion that agrees with F_n in $\overline{B_{r_0}(z_0)}$, i.e.

$$F_n(z) = \sum_{k=0}^{\infty} f_k^{(n)} (z - z_0)^k, \qquad z \in \overline{B_{r_0}(z_0)}$$

for appropriate $f_k^{(n)} \in X$. Furthermore, $F_n \to F$ uniformly on $\overline{B_{r_0}(z_0)}$ w.r.t. $\| \cdot \|_{X_0 + X_1}$. To easen notation, from this point on we abbreviate $X := X_0 + X_1$ and write $\| \cdot \|_{\infty}$ for the sup-norm on $\overline{B_{r_0}(z_0)}$. We also pass to $\| \cdot \|_{\sim}$, which for readability we only denote as $\| \cdot \|$ as this would otherwise clash with the ∞-index. Of course, Corollary 5.7 continues to hold true for $\| \cdot \|_{\sim}$.

Step 1: We construct a candidate for the series expansion of F. For this, we show by induction that for every $k \in \mathbb{N}$ the sequence $(f_k^{(n)})_n$ converges in X.

Because uniform convergence implies pointwise convergence, we already obtain $f_0^{(n)} = F_n(0) \to F(0)$ in X.

Now let $m \geq 1$ and assume that $f_k := \lim_{n\to\infty} f_k^{(n)}$ exists for all $0 \leq k \leq m-1$. For $n \in \mathbb{N}$ we define

$$h_{n,m}(z) := \sum_{k=m}^{\infty} f_k^{(n)}(z-z_0)^{k-m}, \qquad z \in \overline{B_{r_0}(z_0)}.$$

Then

$$h_{n,m}(z) = \begin{cases} f_m^{(n)}, & z = z_0, \\ \frac{F_n(z)}{(z-z_0)^m} - \sum_{k=0}^{m-1} \frac{f_k^{(n)}}{(z-z_0)^{m-k}}, & z \neq z_0. \end{cases}$$

Let $\widetilde{r} \in (0, r_0)$. Because of Lemma 5.10, $h_{n,m}$ is a uniform limit of X-valued polynomials, which are trivially analytic. Thus, the weaker version of the maximum principle (Theorem 5.6, which is applicable after translation and dilation of the unit circle by Corollary 5.5 (i)) holds true for $h_{n,m} - h_{n',m}$ whenever $n, n' \in \mathbb{N}$, $\widetilde{r} \in (0, r_0)$ and we can estimate as follows:

$$\begin{aligned} &\left\| f_m^{(n)} - f_m^{(n')} \right\|^r \\ &= \| h_{n,m}(z_0) - h_{n',m}(z_0) \|^r \\ &\lesssim \sup_{\widetilde{r} \leq |z-z_0| \leq r_0} \| h_{n,m}(z) - h_{n',m}(z) \|^r \\ &\leq \sup_{\widetilde{r} \leq |z-z_0| \leq r_0} \frac{\| F_n(z) - F_{n'}(z) \|^r}{\widetilde{r}^m} + \sum_{k=0}^{m-1} \frac{\left\| f_k^{(n)} - f_k^{(n')} \right\|^r}{\widetilde{r}^{m-k}} \overset{n,n'\to\infty}{\longrightarrow} 0. \end{aligned}$$

By completeness of X, the limit of $(f_m^{(n)})_n$, which we will call $f_m := \lim_{n\to\infty} f_m^{(n)}$, exists. Our candidate thus reads as follows: For a value of $R > 0$, that is yet to be determined, we would like to have

$$F(z) \overset{!}{=} \sum_{k=0}^{\infty} f_k (z-z_0)^k, \qquad z \in B_R(z_0).$$

Step 2: As an auxilliary step, we show that $h_{n,m}$ converges uniformly on $\overline{B_{r_0}(z_0)}$ to some function h_m, in order to be able to apply the weaker version of the maximum principle to $h_{n,m} - h_m$.

For $m \in \mathbb{N}$ we define a function

$$h_m(z) := \begin{cases} f_m, & z = z_0, \\ \frac{F(z)}{(z-z_0)^m} - \sum_{k=0}^{m-1} \frac{f_k}{(z-z_0)^{m-k}}, & z \in \overline{B_{r_0}(z_0)} \setminus \{z_0\}. \end{cases}$$

By the previous step, we already have $h_{n,m} \overset{n\to\infty}{\longrightarrow} h_m$ pointwise on $\overline{B_{r_0}(z_0)}$ and that $h_{n,m}$ is a Cauchy sequence w.r.t uniform convergence on any annulus $\{z \in \overline{B_{r_0}(z_0)} \,|\, \tilde{r} \leq |z| \leq r_0\}$, $\tilde{r} \in (0, r_0)$. For $\overline{B_{\tilde{r}}(z_0)}$ we can use the improved version of the weaker maximum principle Corollary 5.7 to also show that $h_{n,m}$ is a Cauchy sequence w.r.t uniform convergence on $\overline{B_{\tilde{r}}(z_0)}$, showing that $h_{n,m}$ is a Cauchy sequence w.r.t uniform convergence on $\overline{B_{r_0}(z_0)}$, whose limit can be identified as h_m due to pointwise convergence.

Step 3: We need better control of $\left\| f_m^{(n)} - f_m \right\|$ in order to later on be able to estimate the difference of $\sum_{k=0}^{\infty} f_k (z-z_0)^k$ and $\sum_{k=0}^{\infty} f_k^{(n)} (z-z_0)^k$ component wise. To this end, we show that $\left\| f_m^{(n)} - f_m \right\| \leq (1+D)^m \| F_n - F \|_{\infty}$, where $D > 0$ does not depend on m.

By the previous step, the weaker version of the maximum principle (Theorem 5.6) is applicable to $h_{n,m} - h_m$, so that for all $n \in \mathbb{N}, m \geq 1$ we obtain

$$\begin{aligned}
&\left\| \left(\frac{r_0}{2}\right)^m (f_m^{(n)} - f_m) \right\|^r \\
=&\left\| \left(\frac{r_0}{2}\right)^m (h_{n,m}(z_0) - h_m(z_0)) \right\|^r \\
\leq& C^r \sup_{r_0/2 \leq |z-z_0| \leq r_0} \left\| \left(\frac{r_0}{2}\right)^m (h_{n,m}(z) - h_m(z)) \right\|^r \\
=& C^r \sup_{r_0/2 \leq |z-z_0| \leq r_0} \left\| \left(\frac{r_0}{2}\right)^m \left(\frac{F_n(z) - F(z)}{(z-z_0)^m} + \sum_{k=0}^{m-1} \frac{f_k^{(n)} - f_k}{(z-z_0)^{m-k}} \right) \right\|^r \\
\leq& C^r \left[\| F_n - F \|_{\infty}^r + \sum_{k=0}^{m-1} \left\| \left(\frac{r_0}{2}\right)^k (f_k^{(n)} - f_k) \right\|^r \right],
\end{aligned} \tag{5.2}$$

where $C \geq 1$ is from Theorem 5.6.

We can now show by induction that

$$\| F_n - F \|_{\infty}^r + \sum_{k=0}^{m} \left\| \left(\frac{r_0}{2}\right)^k (f_k^{(n)} - f_k) \right\|^r \leq (1 + C^r)^m \| F_n - F \|_{\infty}^r .$$

for all $m, n \in \mathbb{N}$. For now, fix $n \in \mathbb{N}$.

When $m = 0$, we use $f_0 := \lim_{n\to\infty} f_0^{(n)}$ to estimate

$$\| F_n - F \|_\infty^r + \left\| f_0^{(n)} - f_0 \right\|^r \le 2 \| F_n - F \|_\infty^r \le (1 + C^r) \| F_n - F \|_\infty^r .$$

When $m \ge 1$, we use (5.2) to estimate

$$\begin{aligned} &\| F_n - F \|_\infty^r + \sum_{k=0}^{m} \left\| \left(\frac{r_0}{2}\right)^k (f_k^{(n)} - f_k) \right\|^r \\ &\le \| F_n - F \|_\infty^r + \sum_{k=0}^{m-1} \left\| \left(\frac{r_0}{2}\right)^k (f_k^{(n)} - f_k) \right\|^r \\ &\qquad + C^r \left(\| F_n - F \|_\infty^r + \sum_{k=0}^{m-1} \left\| \left(\frac{r_0}{2}\right)^k (f_k^{(n)} - f_k) \right\|^r \right) \end{aligned}$$

and conclude using the induction hypothesis

$$\begin{aligned} &\le (1 + C^r)^{m-1} \| F_n - F \|_\infty^r + C^r (1 + C^r)^{m-1} \| F_n - F \|_\infty^r \\ &= (1 + C^r)^m \| F_n - F \|_\infty^r . \end{aligned}$$

Thus

$$\begin{aligned} \left\| f_m^{(n)} - f_m \right\| &\le C \left(\frac{2}{r_0}\right)^m \left(\| F_n - F \|_\infty^r + \sum_{k=0}^{m-1} \left\| \left(\frac{r_0}{2}\right)^k (f_k^{(n)} - f_k) \right\|^r \right)^{1/r} \\ &\le \left(\frac{2}{r_0} (1 + C^r)^{1/r} \right)^m \| F_n - F \|_\infty , \end{aligned}$$

which concludes this step with $D := \frac{2}{r_0}(1 + C^r)^{1/r}$.

Step 4: We can finally show that $\sum_{k=0}^\infty f_k (z - z_0)^k$ defines a power series that agrees with $F(z)$ for all z in a certain neighbourhood of z_0.

Set $R := \min\{r_0, 1/(2D)\}$ and let $z \in B_R(z_0)$. Then $\sum_{k=0}^\infty f_k (z - z_0)^k$ converges and agrees with $F(z)$, because

$$\left\| \sum_{k=0}^{\infty} f_k (z - z_0)^k - F(z) \right\|^r$$

$$
\begin{aligned}
&\leq \left\| \sum_{k=0}^{\infty} (f_k - f_k^{(n)})(z-z_0)^k \right\|^r + \left\| \sum_{k=0}^{\infty} f_k^{(n)}(z-z_0)^k - F(z) \right\|^r \\
&\leq \sum_{k=0}^{\infty} \left\| f_k - f_k^{(n)} \right\|^r |z-z_0|^{kr} + \| F_n(z) - F(z) \|^r && (R \leq r_0) \\
&\leq \sum_{k=0}^{\infty} D^{kr} \| F_n - F \|_\infty^r |z-z_0|^{kr} + \| F_n(z) - F(z) \|^r && (\text{Step 3}) \\
&\leq \| F_n - F \|_\infty^r \sum_{k=0}^{\infty} \left(\frac{1}{2^r} \right)^k + \| F_n(z) - F(z) \|^r && (R \leq 1/(2D)) \\
&\overset{n\to\infty}{\longrightarrow} 0.
\end{aligned}
$$

Thus, F is analytic. □

We can finally show that the notion of interpolation spaces is a meaningful one.

Proposition 5.12. *Let X_0, X_1 be quasi-Banach spaces and (X_0, X_1) an interpolation pair. Then:*

(i) $[X_0, X_1]_\theta$ *is a quasi-Banach space.*

(ii) The embeddings

$$
X_0 \cap X_1 \hookrightarrow [X_0, X_1]_\theta \hookrightarrow X_0 + X_1
$$

are continuous.

Proof.

(i) A very obvious approach to quasi-norm $[X_0, X_1]_\theta$ is to endow it with the subspace topology of $X_0 + X_1$. But then the quasi-norm doesn't remember information about $\mathcal{F}$ and closedness of $[X_0, X_1]_\theta$ in $X_0 + X_1$ and thus completeness of $[X_0, X_1]_\theta$ isn't obvious.

By choosing the correct quasi-norm as done in Defintion 5.2 (iv), completeness is guaranteed. To see this, we elaborate on [Cal64, Theorem 3]:

The map $\varphi : \mathcal{F} \to X_0 + X_1, F \mapsto F(\theta)$ admits the commutative diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & {[X_0, X_1]_\theta = \varphi(\mathcal{F})} \\ {\mathcal{F}/\mathrm{Ker}(\varphi)} \arrow[ur] & \end{tikzcd},$$

i.e. there is a bijection between $[X_0, X_1]_\theta$ and $\mathcal{F}/\mathrm{Ker}(\varphi)$, where the latter is already a quasi-Banach space due to $\mathrm{Ker}(\varphi)$ being closed in $\mathcal{F}$ as φ is bounded. For $f \in [X_0, X_1]_\theta$ there is exactly one $\overline{F} \in \mathcal{F}/\mathrm{Ker}(\varphi)$ with $\overline{F}(\theta) = f$, for which we observe that

$$\begin{aligned} \| \overline{F} \| = \mathrm{dist}(F, \mathrm{Ker}(\varphi)) &= \inf\{\| F - G \|_{\mathcal{F}} \mid G \in \mathrm{Ker}(\varphi)\} \\ &= \inf\{\| H \|_{\mathcal{F}} \mid H \in F + \mathrm{Ker}(\varphi)\} \\ &= \inf\{\| H \|_{\mathcal{F}} \mid H(\theta) = f,\ H \in \mathcal{F}\} \\ &= \| f \|_\theta , \end{aligned}$$

i.e. $[X_0, X_1]_\theta$ and $\mathcal{F}/\mathrm{Ker}(\varphi)$ have the same quasi-norms, making the bijection into an isometric isomorphism and $[X_0, X_1]_\theta$ into a quasi-Banach space as well.

(ii) Let $x \in X_0 \cap X_1$. Then $F(z) := x$, $z \in S$ is constant and trivially admissible. Thus $x = F(\theta) \in [X_0, X_1]_\theta$ and

$$\begin{aligned} \| x \|_\theta \leq \| F \|_{\mathcal{F}} &= \max\{\| x \|_0 , \| x \|_{X_0+X_1} , \| x \|_1\} \\ &\leq \max\{\| x \|_0 , \| x \|_1\} = \| x \|_{X_0 \cap X_1} . \end{aligned}$$

Now let $x \in [X_0, X_1]_\theta$. Then there is $F \in \mathcal{F}$ s.t. $F(\theta) = x$. Since F is $X_0 + X_1$-valued, $x \in X_0 + X_1$ and

$$\| x \|_{X_0+X_1} = \| F(\theta) \|_{X_0+X_1} \leq \sup_{z \in S} \| F(z) \|_{X_0+X_1} \leq \| F \|_{\mathcal{F}} .$$

Taking the infimum over all $F \in \mathcal{F}$ with $F(\theta) = x$ yields $\| x \|_{X_0+X_1} \leq \| x \|_\theta$. $\square$

Remark 5.13. In the beginning of the introduction, we spoke of "intermediate results", where intermediate is to be understood w.r.t. $X_0 \cap X_1$ and $X_0 + X_1$, see Proposition 5.12 (ii).

5.2 Interpolation of quasi-Banach function spaces

In order to calculate interpolation spaces of quasi-Banach function spaces, we need to address the Hausdorff topological vector space that is needed in order to define the sum of two quasi-Banach function spaces. With view to Definition 3.1, $\mathrm{L}_0(\Omega)$ seems like a natural candidate for all interpolation pairs consisting of quasi-Banach function spaces. We need a bit of preparation.

Proposition 5.14.

(i) There exists a strictly positive function $f \in \mathrm{L}^1(\Omega) \cap \mathrm{L}^\infty(\Omega)$.

(ii) The map

$$\mathrm{L}_0(\Omega) \times \mathrm{L}_0(\Omega) \to \mathbb{R}, \qquad (x, y) \mapsto \int_\Omega \frac{|x-y|}{1+|x-y|}\, f \,\mathrm{d}\mu$$

defines a metric on $\mathrm{L}_0(\Omega)$. *In particular,* $\mathrm{L}_0(\Omega)$ *endowed with the above metric becomes a Hausdorff topological vector space.*

(iii) Let $x_n, x \in \mathrm{L}_0(\Omega)$ *and* $x_n \to x$ *in measure on every set of finite measure. Then* $x_n \to x$ *in* $\mathrm{L}_0(\Omega)$.

Proof.

(i) By Ω's σ-finiteness there is an increasing sequence $\Omega_n \nearrow \Omega$ with $\mu(\Omega_n) < \infty$ for all $n \in \mathbb{N}$, where we may assume that $\inf_n \mu(\Omega_n) > 0$. Letting $q \in (0, 1)$, we can define

$$f := \sum_{n=0}^{\infty} \frac{q^n}{\mu(\Omega_n)} \chi_{\Omega_n}.$$

Then $f > 0$ and $f \in \mathrm{L}^\infty(\Omega)$ by construction and by monotone convergence we obtain

$$\| f \|_{\mathrm{L}^1(\Omega)} = \lim_{N\to\infty} \int_\Omega \sum_{n=1}^{N} \frac{q^n}{|\Omega_n|} \chi_{\Omega_n} \,\mathrm{d}\mu = \lim_{N\to\infty} \sum_{n=1}^{N} q^n = \frac{1}{1-q} - 1 < \infty.$$

(ii) Elementary.

(iii) Let $x_n \to x$ in measure on sets of finite measure, $\varepsilon > 0$ and $\Omega_m \nearrow \Omega$ with $\mu(\Omega_m) < \infty$ for all $m \in \mathbb{N}$. Then

$$
\begin{aligned}
&d(x_n, x) \\
&= \int_{\Omega_m} \frac{|x_n - x|}{1 + |x_n - x|} f \, \mathrm{d}\mu + \int_{\Omega \setminus \Omega_m} \frac{|x_n - x|}{1 + |x_n - x|} f \, \mathrm{d}\mu \\
&= \int_{[|x_n - x| > \varepsilon] \cap \Omega_m} \frac{|x_n - x|}{1 + |x_n - x|} f \, \mathrm{d}\mu + \int_{[|x_n - x| \leq \varepsilon] \cap \Omega_m} \frac{|x_n - x|}{1 + |x_n - x|} f \, \mathrm{d}\mu + \\
&\qquad \int_{\Omega \setminus \Omega_m} \frac{|x_n - x|}{1 + |x_n - x|} f \, \mathrm{d}\mu \\
&\leq \| f \|_{\mathrm{L}^\infty(\Omega)} \, \mu([|x_n - x| > \varepsilon] \cap \Omega_m) + \varepsilon \, \| f \|_{\mathrm{L}^1(\Omega)} + \int_{\Omega \setminus \Omega_m} f \, \mathrm{d}\mu
\end{aligned}
$$

First letting $m \to \infty$ and then $n \to \infty$ and $\varepsilon \to 0$ yields the claim. □

Remark 5.15. Our assumption that Ω is σ-finite can't be relaxed. For example $\mathrm{L}^1(\mathbb{R}, \mathcal{P}(\mathbb{R}), \zeta)$, where ζ denotes the counting measure, does not admit a strictly positive integrable function.

This candidate indeed does the trick for every quasi-Banach function space. To show this, we will need the following two results.

Theorem 5.16 (Egorov's theorem)**.** *Let $(N, \mathcal{A}, \nu)$ be a finite measure space and let $(f_n)_n$ be a sequence of measurable functions that converge to some f a.e. Then for every $\varepsilon > 0$ there is a set $A \in \mathcal{A}$ with $\nu(N \setminus A) < \varepsilon$ and $f_n \to f$ uniformly on A.*

Proof. See [Kal21, Lemma 1.38]. □

Proposition 5.17. *Let X be a quasi-Banach function space and $(f_n)_n \subseteq X$, $f \in X$. If $f_n \to f$ in X, then $f_n \to f$ in measure on sets of finite measure.*

Proof. The proof is already done in the Banach case in [Cal64, Theorem 13.2]. We will just generalize this result to the quasi-Banach case by adding powers of r at appropriate places. W.l.o.g., we may assume that $f = 0$.

We argue by contradiction, so further assume that $f_n \to 0$ in X but $(f_n)_n$ doesn't converge to 0 in measure on sets of finite measure. Then there exists $\varepsilon > 0$, $E \subseteq \Omega$ measurable with $\mu(E) \in (0, \infty)$, $\delta > 0$ and a subsequence that we again denote as $(f_n)_n$ s.t. for all $n \in \mathbb{N}$ we have

$$\mu([|f_n| > \varepsilon] \cap E) > \delta.$$

Denote $E_n := [|f_n| > \varepsilon] \cap E$. To easen notation, we may now assume $E = \Omega$.

Intuitively, the idea is the following: If $X = \mathrm{L}^p$ for some $p \in (0, \infty]$, we could immediately conclude that

$$\| f_n \|_{\mathrm{L}^p(E)} \geq \| \varepsilon \chi_{E_n} \|_{\mathrm{L}^p(E)} \geq \varepsilon \delta^{1/p} > 0$$

and obtain a contradiction. In our more general setting, we try to follow the same idea but need to be more careful when making connections between pointwise estimates and abstract quasi-norms when using the lattice property. We do this in the following manner: For a given $\omega \in E$ we will count appropriately for how many functions $|f_n(\omega)| > \varepsilon$ occurs. If this happens infinitely often on a set of positive measure, which we can understand as $\mu(\cap_{n\in\mathbb{N}}[|f_n| > \varepsilon]) > 0$, then $f_n \to 0$ in X will also fail.

By our assumption, we have $\varepsilon\chi_{E_n} \leq |f_n|$ on E and thus $\| \chi_{E_n} \|_{\sim}^{r} \to 0$ by the lattice property. We select a subsequence that we again denote $(\chi_{E_n})_n$ s.t. $\sum_{n=0}^{\infty} \| \chi_{E_n} \|_{\sim}^{r} < \infty$.
For $N \in \mathbb{N}$ define $S_N := \sum_{n=0}^{N} \chi_{E_n}$. Then $S_N \leq S_{N+1}$ a.e. and $S : E \to [0, \infty], \omega \mapsto S(\omega) := \lim_{N\to\infty} S_N(\omega)$ is well-defined, $S_N \leq S$ a.e. and the set $[S = \infty] = \cap_{M\in\mathbb{N}} \cup_{N\in\mathbb{N}} [S_N > M]$ is measurable. Assume that S is finite almost everywhere. As then $\cap_{M\geq 0}[S > M] = \emptyset$ and $[S > M] \supseteq [S > M + 1]$ for all $M \in \mathbb{N}$, continuity of μ gives the existence of an $M \in \mathbb{N}$ s.t. $\mu([S > M]) < \frac{\delta}{2}$. We have

$$\delta < \mu(E_n) = \underbrace{\mu(E_n \cap [S > M])}_{<\frac{\delta}{2}} + \mu(E_n \cap [S \leq M]),$$

thus $\mu(E_n \cap [S \leq M]) \geq \frac{\delta}{2}$. This already provides us with a first contradiction, as for all $N \in \mathbb{N}$ we now would have

$$\begin{aligned}(N+1)\frac{\delta}{2} &\leq \sum_{n=0}^{N} \mu(E_n \cap [S \leq M]) \\ &= \sum_{n=0}^{N} \int_{[S\leq M]} \chi_{E_n} \,\mathrm{d}\mu \\ &= \int_{[S\leq M]} S_N \,\mathrm{d}\mu \\ &\leq M\mu(E) < \infty.\end{aligned}$$

Consequently, $S = \infty$ on a set of positive measure and by Egorov's theorem (used on the sequence $\frac{1}{S_N}$) we even have $S_N \to \infty$ uniformly on a subset $D \subseteq E$ with $\mu(D) \in (0, E]$. Given $m \in \mathbb{N}$, we have $m \leq S_N$ on D for N large enough with

$$0 < \| \chi_D \|_{\sim} \leq \frac{1}{m} \| S_N \|_{\sim} \leq \frac{1}{m} \left(\sum_{n=0}^{N} \| \chi_{E_n} \|_{\sim}^{r} \right)^{1/r} \leq \frac{1}{m} \left(\sum_{n=0}^{\infty} \| \chi_{E_n} \|_{\sim}^{r} \right)^{1/r}.$$

This again gives a contradiction, as now letting $m \to \infty$, we would have $\| \chi_D \|_X = 0$ and thus $\chi_D = 0$, i.e. $\mu(D) = 0$. $\square$

Theorem 5.18. *Let $X \subseteq \mathrm{L}_0(\Omega)$ be a quasi-Banach function space. Then the embedding $X \hookrightarrow \mathrm{L}_0(\Omega)$ is continuous.*

Proof. This is a direct consequence of the fact that convergence in X implies convergence in measure on sets of finite measure (Proposition 5.17) and that the latter implies convergence in $\mathrm{L}_0(\Omega)$ (Proposition 5.14 (iii)). $\square$

We can now show that limits in quasi-Banach functions spaces are determined by pointwise limits and say a bit more about abstract limits of Cauchy sequences in quasi-Banach function spaces.

Corollary 5.19. *Let $X \subseteq \mathrm{L}_0(\Omega)$ be a quasi-Banach function space.*

(i) *If $f_n \to f$ in X, there is a subsequence $(f_{n_k})_k$ s.t. $f_{n_k} \to f$ a.e. Furthermore, if there is $g \in \mathrm{L}_0(\Omega)$ s.t. $f_n \to g$ a.e., then $f = g$.*

(ii) Let $(f_n)_n \subseteq X$ s.t. $\sum_{n=0}^{\infty} C^n \parallel f_n \parallel_X < \infty$, where C is a constant for which the quasi-triangle inequality holds true. Then $\sum_{n=0}^{\infty} f_n$ converges absolutely pointwise a.e. and is the limit of $(\sum_{n=0}^{N} f_n)_N$ in X.

Proof.

(i) This is because convergence in X implies convergence in measure on sets of finite measure (Proposition 5.17) and because the latter implies (up to a subsequence) convergence a.e. (see [Kle20, Corollary 6.13])

(ii) This is done in the second part of [Cal64, Theorem 13.2] in the Banach case, which we generalize.

By the characterization of completeness in quasi-Banach spaces through series (Lemma 2.7 (i)), the sequence $(\sum_{n=0}^{N} f_n)_N$ converges in X and also in measure on sets of finite measure (Proposition 5.17) to some $f \in X$. We want to show that $\sum_{n=0}^{\infty} f_n$ exists a.e. by showing that $\sum_{n=0}^{\infty} |f_n| < \infty$ a.e., because then $\sum_{n=0}^{N} f_n \to \sum_{n=0}^{\infty} f_n$ a.e. and by (i) we can conclude that $f = \sum_{n=0}^{\infty} f_n$.

By monotonicity we know that $\sum_{n=0}^{\infty} |f_n|$ exists with values in $[0, \infty]$. Because $\| \cdot \|$ doesn't depend on absolute values (Lemma 3.4 (i)) we have $\sum_{n=0}^{\infty} C^n \| |f_n| \| < \infty$ as well and thus there is $g \in X$ s.t. $\sum_{n=0}^{N} |f_n| \to g$ in X. By (i), we have $\sum_{n=0}^{\infty} |f_n| = g$, which is finite a.e. (Lemma 3.4 (i)). Thus $\sum_{n=0}^{\infty} |f_n| < \infty$ a.e. and so $\sum_{n=0}^{\infty} f_n$ exists a.e. $\square$

Because any quasi-Banach function space continuously embeds into the Hausdorff topological vector space $L_0(\Omega)$ (Theorem 5.18), any pair of quasi-Banach function spaces (X_0, X_1) is eligible for complex interpolation. It remains to check whether the arising interoplation space respects the function space properties of the interpolated ones.

Theorem 5.20. *Given an interpolation pair (X_0, X_1) consisting of quasi-Banach function spaces, $X_0 + X_1$, $[X_0, X_1]_\theta$ and $X_0 \cap X_1$ are quasi-Banach function spaces.*

Proof. In all cases it only remains to check whether the lattice property is satisfied and a weak order unit is contained.

<u>$X_0 + X_1$:</u>

(i) This step is taken from [Cal64, Theorem 13.5]. Let $f \in \mathrm{L}_0(\Omega), g \in X_0+X_1$ with $|f| \leq |g|$. For every decomposition $g = g_0 + g_1$ with appropriate $g_i \in X_i$ we have

$$f = \frac{f}{g} g = \frac{f g_0}{g} + \frac{f g_1}{g}$$

with $\left| \frac{f g_i}{g} \right| \leq |g_i|$. Thus $\frac{f g_i}{g} \in X_i$ with $\left\| \frac{f g_i}{g} \right\|_{X_i} \leq \| g_i \|_{X_i}$ due to the lattice properties of the X_i. This gives $f \in X_0 + X_1$ and

$$\| f \|_{X_0+X_1} \leq \left\| \frac{f g_0}{g} \right\|_{X_0} + \left\| \frac{f g_1}{g} \right\|_{X_1} \leq \| g_0 \|_{X_0} + \| g_1 \|_{X_1} .$$

Taking the infimum over all decompositions $g = g_0+g_1$ gives $\| f \|_{X_0+X_1} \leq \| g \|_{X_0+X_1}$.

(ii) Taking a positive $f_0 \in X_0$, the function $f_0 \in X_0 + X_1$ clearly is positve as well.

$[X_0, X_1]_\theta$:

(i) Let $f \in \mathrm{L}_0(\Omega), g \in [X_0, X_1]_\theta$ with $|f| \leq |g|$. Then for every $G \in \mathcal{F}$ with $G(\theta) = g$ we define $F := \frac{f}{g} G$. F is admissible, as $\left| \frac{f}{g} \right| \leq 1$ so that all properties of G carry over to F through the involved lattice properties. We also have $\| F(z) \|_{X_0+X_1} \leq \| G(z) \|_{X_0+X_1}$ for all $z \in S$ and $\| F(j + \mathrm{i}t) \|_{X_j} \leq \| G(j + \mathrm{i}t) \|_{X_j}$ for all $j = 0, 1$ and $t \in \mathbb{R}$, which gives $\| F \|_{\mathcal{F}} \leq \| G \|_{\mathcal{F}}$. Since this is true for every $G \in \mathcal{F}$ with $G(\theta) = g$, first taking the infimum on the left-hand side and the on the right-hand one we get $\| f \|_\theta \leq \| g \|_\theta$ as well.

(ii) For positive $f_i \in X_i$ we define $F(z) := \min(f_0, f_1), z \in S$. By the lattice properties of the X_i, the traces of the constant extension to $\overline{S}$ actually lie in the X_i. All other properties of admissibility are immediate as F is constant. Lastly, we have $F(\theta) > 0$ as well.

$X_0 \cap X_1$:

(i) Let $f \in \mathrm{L}_0(\Omega), g \in X_0 \cap X_1$ with $|f| \leq |g|$. Then $f \in X_i$ with $\| f \|_i \leq \| g \|_i$ and so it is immediate from the definition of $X_0 \cap X_1$'s quasi-norm that $\| f \|_{X_0 \cap X_1} \leq \| g \|_{X_0 \cap X_1}$.

(ii) For positive $f_i \in X_i$ we again define $f := \min(f_0, f_1) > 0$, which clearly is in $X_0 \cap X_1$.

$\square$

6 Calderón products

Before proving Theorem 7.1 it now only remains to introduce Calderón products. We give them meaning by showing that they are quasi-Banach function spaces and show some basic properties.

Throughout this chapter, let X_0, X_1 be quasi-Banach function spaces and $\theta \in (0,1)$.

Definition 6.1. The *Calderón product* $X_0^{1-\theta} X_1^{\theta}$ *of* X_0 *and* X_1 is defined as the set of all $f \in \mathrm{L}_0(\Omega)$ s.t.

$$\| f \|_{X_0^{1-\theta} X_1^{\theta}} := \inf\{ \| f_0 \|_0^{1-\theta} \| f_1 \|_1^{\theta} \mid |f| \leq |f_0|^{1-\theta} |f_1|^{\theta},\ f_0 \in X_0, f_1 \in X_1 \}$$

is finite.

Remark 6.2. Notice that now we require $\theta \in (0,1)$, which is due to the fact that for the boundary cases $\theta = 0, 1$ the Calderón product always reproduces X_0 resp. X_1.

Theorem 6.3. $X_0^{1-\theta} X_1^{\theta}$ *is a quasi-Banach function space.*

Proof. The proofs of steps 2 and 3 are based on [Cal64, Theorem 13.5]. We abbreviate $X := X_0^{1-\theta} X_1^{\theta}$ and denote the constants from the quasi-triangle inequality of the X_i as C_i.

Step 1: We start by showing the function space properties, which will be useful for showing completeness later on.

(a) Let $f \in \mathrm{L}_0(\Omega)$ and $g \in X$ with $|f| \leq |g|$. Then

$$\begin{aligned}&\{\| g_0 \|_0^{1-\theta} \| g_1 \|_1^{\theta} \mid |g| \leq |g_0|^{1-\theta} |g_1|^{\theta}\} \\ \subseteq&\{\| f_0 \|_0^{1-\theta} \| f_1 \|_1^{\theta} \mid |f| \leq |f_0|^{1-\theta} |f_1|^{\theta}\},\end{aligned}$$

so that $\| f \|_X \leq \| g \|_X < \infty$ and consequently $f \in X$ as well.

(b) If $f_i \in X_i$ are positive, $f := f_0^{1-\theta} f_1^{\theta}$ is positive as well and is contained in X due to $\| f \|_X \leq \| f_0 \|_0^{1-\theta} \| f_1 \|_1^{\theta} < \infty$.

Step 2: Now we show the quasi-norm properties of $\| \cdot \|_X$, which will also play a role for the completeness of X.

(a) Definiteness:
$\| 0 \|_X = 0$ is obvious.

Let $f \in X$ with $\| f \|_X = 0$. Then for every $n \in \mathbb{N}$ there are $g_n \in X_0, h_n \in X_1$ with

$$\| g_n \|_0^{1-\theta} \| h_n \|_1^{\theta} \leq \frac{1}{n}$$

and

$$|f| \leq |g_n|^{1-\theta} |h_n|^{\theta} = \Bigg(\underbrace{|g_n| \frac{\| h_n \|_1^{\theta}}{\| g_n \|_0^{\theta}}}_{=:\widetilde{g_n}}\Bigg)^{1-\theta} \Bigg(\underbrace{|h_n| \frac{\| g_n \|_0^{1-\theta}}{\| h_n \|_1^{1-\theta}}}_{=:\widetilde{h_n}}\Bigg)^{\theta}$$

where we may assume that $\| g_n \|_0 , \| h_n \|_1 \neq 0$, as otherwise we can use the definiteness of the quasi-norms on the X_i to conclude g_n or $h_n = 0$ for some n and thus $|f| = 0$. In particular,

$$\| \widetilde{g_n} \|_0 = \| g_n \|_0^{1-\theta} \| h_n \|_1^{\theta} \leq \frac{1}{n}$$

and similarly

$$\|\widetilde{h_n}\|_1 = \| g_n \|_0^{1-\theta} \| h_n \|_1^{\theta} \leq \frac{1}{n}.$$

Thus $\widetilde{g_n} \to 0$, and $\widetilde{h_n} \to 0$ in both X_0 and X_1 and after passing to appropriate subsequences, which we again denote as $(\widetilde{g_n})_n, (\widetilde{h_n})_n$, we obtain $\widetilde{g_n} \to 0$, and $\widetilde{h_n} \to 0$ a.e. (see Corollary 5.19 (i)). Since $|f| \leq |g_n|^{1-\theta} |h_n|^{\theta} \overset{n\to\infty}{\longrightarrow} 0$ a.e., we get $|f| = 0$.

(b) Homogeneity:
Evident.

(c) Quasi-triangle inequality:
Let $f, g \in X$, $\varepsilon > 0$. Then there exist $f_i, g_i \in X_i$ s.t.

$$\| f_0 \|_0^{1-\theta} \| f_1 \|_1^{\theta} \leq \| f \|_X + \frac{\varepsilon}{2}, \qquad |f| \leq |f_0|^{1-\theta} |f_1|^{\theta},$$
$$\| g_0 \|_0^{1-\theta} \| g_1 \|_1^{\theta} \leq \| g \|_X + \frac{\varepsilon}{2}, \qquad |g| \leq |g_0|^{1-\theta} |g_1|^{\theta}.$$

We may again assume that the involved norms are non-zero, as otherwise f or $g = 0$ and the claim becomes trivial in this case. Using Hölder, we obtain

$$\begin{aligned}
|f+g| &\leq |f| + |g| \\
&\leq |f_0|^{1-\theta} |f_1|^{\theta} + |g_0|^{1-\theta} |g_1|^{\theta} \\
&= \Big(\underbrace{\frac{\| f_1 \|_1^{\theta}}{\| f_0 \|_0^{\theta}} |f_0|}_{=:\widetilde{f_0}}\Big)^{1-\theta} \Big(\underbrace{\frac{\| f_0 \|_0^{1-\theta}}{\| f_1 \|_1^{1-\theta}} |f_1|}_{=:\widetilde{f_1}}\Big)^{\theta} + \\
&\qquad \Big(\underbrace{\frac{\| g_1 \|_1^{\theta}}{\| g_0 \|_0^{\theta}} |g_0|}_{=:\widetilde{g_0}}\Big)^{1-\theta} \Big(\underbrace{\frac{\| g_0 \|_0^{1-\theta}}{\| g_1 \|_1^{1-\theta}} |g_1|}_{=:\widetilde{g_1}}\Big)^{\theta} \\
&\leq \left(\widetilde{f_0} + \widetilde{g_0}\right)^{1-\theta} \left(\widetilde{f_1} + \widetilde{g_1}\right)^{\theta}.
\end{aligned}$$

Thus

$$\begin{aligned}
\| f+g \|_X &\leq \left\| \widetilde{f_0} + \widetilde{g_0} \right\|_0^{1-\theta} \left\| \widetilde{f_1} + \widetilde{g_1} \right\|_1^{\theta} \\
&\leq C_0^{1-\theta} \left(\| f_0 \|_0^{1-\theta} \| f_1 \|_1^{\theta} + \| g_0 \|_0^{1-\theta} \| g_1 \|_1^{\theta} \right)^{1-\theta} \times \\
&\qquad C_1^{\theta} \left(\| f_0 \|_0^{1-\theta} \| f_1 \|_1^{\theta} + \| g_0 \|_0^{1-\theta} \| g_1 \|_1^{\theta} \right)^{\theta} \\
&= C_0^{1-\theta} C_1^{\theta} \left(\| f_0 \|_0^{1-\theta} \| f_1 \|_1^{\theta} + \| g_0 \|_0^{1-\theta} \| g_1 \|_1^{\theta} \right) \\
&\leq C_0^{1-\theta} C_1^{\theta} \left(\| f \|_X + \| g \|_X + \varepsilon \right).
\end{aligned}$$

As this is true for all $\varepsilon > 0$, the quasi-triangle inequality is shown.

Additionally, we also showed that X is a vector space in the first place, because the validity of the quasi-triangle inequality now implies for $f, g \in X$ that $\| f + g \|_X \leq C_0^{1-\theta} C_1^\theta (\| f \|_X + \| g \|_X) < \infty$. The other requirements of a vector space are immediate.

Step 3: To tackle completeness, we use the characterization of completeness via series (Lemma 2.7 (i)) for $C := \max\{C_0, C_1\} \geq C_0^{1-\theta} C_1^\theta$, which is eligible by step 2.

Let $(f_n)_n \in X$ with $\sum_{n=0}^\infty C^n \| f_n \| < \infty$ and $\varepsilon > 0$. Then there exist $f_{n,i} \in X_i$ with

$$\| f_{n,0} \|_0^{1-\theta} \| f_{n,1} \|_1^\theta \leq \| f_n \|_X + \frac{\varepsilon}{(2C)^n}, \qquad |f_n| \leq |f_{n,0}|^{1-\theta} |f_{n,1}|^\theta .$$

We may w.l.o.g. assume again that all norms are non-zero. Repeating the procedure from step 2 we get

$$\sum_{n=0}^\infty |f_n| \leq \ldots \leq \left(\sum_{n=0}^\infty \frac{\| f_{n,1} \|^\theta}{\| f_{n,0} \|^\theta} |f_{n,0}| \right)^{1-\theta} \left(\sum_{n=0}^\infty \frac{\| f_{n,0} \|^{1-\theta}}{\| f_{n,1} \|^{1-\theta}} |f_{n,1}| \right)^\theta .$$

By Corollary 5.19 (ii), the term in the first bracket defines a function in X_0 as X_0 is a quasi-Banach function space and

$$\begin{aligned} \sum_{n=0}^\infty C_0^n \left\| \frac{\| f_{n,1} \|^\theta}{\| f_{n,0} \|^\theta} f_{n,0} \right\|_0 &= \sum_{n=0}^\infty C_0^n \| f_{n,0} \|^{1-\theta} \| f_{n,1} \|^\theta \\ &\leq \sum_{n=0}^\infty C_0^n \left(\| f_n \|_X + \frac{\varepsilon}{(2C)^n} \right) \\ &\leq \sum_{n=0}^\infty C^n \| f_n \|_X + \varepsilon \end{aligned}$$

for all $\varepsilon > 0$, i.e.

$$\sum_{n=0}^\infty C_0^n \left\| \frac{\| f_{n,1} \|^\theta}{\| f_{n,0} \|^\theta} f_{n,0} \right\|_0 \leq \sum_{n=0}^\infty C^n \| f_n \|_X < \infty.$$

Similarly, the term in the second bracket defines a function in X_1 as well.

Thus $\sum_{n=0}^{\infty} |f_n| \in X$. In particular, we may define $f := \sum_{n=0}^{\infty} f_n \in \mathrm{L}_0(\Omega)$ for which we obtain $f \in X$ by the lattice property as $|f| \leq \sum_{n=0}^{\infty} |f_n|$. We conclude by showing that $(\sum_{n=0}^{N} f_n)_N$ converges to f, which is easily seen by using a version of the quasi-triangle inequality for series (Lemma 2.7 (ii)), as

$$\left\| f - \sum_{n=0}^{N} f_n \right\|_X = \left\| \sum_{n=N+1}^{\infty} f_n \right\|_X \lesssim \sum_{n=N+1}^{\infty} C^n \| f_n \|_X \overset{N\to\infty}{\longrightarrow} 0$$

by our assumption that $\sum_{n=0}^{\infty} C^n \| f_n \|_X < \infty$. □

Remark 6.4. If both X_0 and X_1 are Banach function spaces, step 2 may be skipped as step 3 works the same for $C = C_0 = C_1 = 1$, resulting in $X_0^{1-\theta} X_1^{\theta}$ being a Banach function space in this case as well.

We collect some further properties of Calderón products.

Proposition 6.5.

(i) The embeddings

$$X_0 \cap X_1 \hookrightarrow X_0^{1-\theta} X_1^{\theta} \hookrightarrow X_0 + X_1$$

are continuous.

(ii) $X_0^{1-\theta} X_1^{\theta}$ *is also quasi-normed via equality, i.e.*

$$\| f \|_{X_0^{1-\theta} X_1^{\theta}} = \inf\{\| f_0 \|_0^{1-\theta} \| f_1 \|_1^{\theta} \mid |f| = |f_0|^{1-\theta} |f_1|^{\theta},\, f_i \in X_i\}$$

for all $f \in X_0^{1-\theta} X_1^{\theta}$.

(iii) If X_0 *and* X_1 *are separable, then* $X_0^{1-\theta} X_1^{\theta}$ *is separable as well.*

(iv) If X_0 *or* X_1 *is order-continuous, then* $X_0^{1-\theta} X_1^{\theta}$ *is order-continuous as well.*

(v) In particular, if X_0 *or* X_1 *is separable, then* $X_0^{1-\theta} X_1^{\theta}$ *is separable as well.*

Proof.

(i) Let $f \in X_0 \cap X_1$. Then $|f| \leq |f|^{1-\theta} |f|^{\theta}$ and thus

$$\begin{aligned}
\| f \|_{X_0^{1-\theta} X_1^{\theta}} &\leq \| f \|_0^{1-\theta} \| f \|_1^{\theta} \\
&\leq \max\{\| f \|_0 , \| f \|_1\}^{1-\theta} \max\{\| f \|_0 , \| f \|_1\}^{\theta} \\
&= \| f \|_{X_0 \cap X_1} .
\end{aligned}$$

Now let $f \in X_0^{1-\theta} X_1^{\theta}$ and $|f| \leq |f_0|^{1-\theta} |f_1|^{\theta}$ be any decomposition. Then

$$\begin{aligned}
|f| &\leq \| f_0 \|_0^{1-\theta} \| f_1 \|_1^{\theta} \left(\frac{|f_0|}{\| f_0 \|_0} \right)^{1-\theta} \left(\frac{|f_1|}{\| f_1 \|_1} \right)^{\theta} \\
&\leq \| f_0 \|_0^{1-\theta} \| f_1 \|_1^{\theta} \left[(1-\theta) \frac{|f_0|}{\| f_0 \|_0} + \theta \frac{|f_1|}{\| f_1 \|_1} \right] \quad \text{(convexitiy of } \exp).
\end{aligned}$$

Thus

$$F_0 := \frac{1}{1-\theta} \left(\frac{|f|}{\| f_0 \|_0^{1-\theta} \| f_1 \|_1^{\theta}} - \theta \frac{|f_1|}{\| f_1 \|_1} \right) \leq \frac{|f_0|}{\| f_0 \|_0}$$

is in X_0 and satisfies $|f| = \| f_0 \|_0^{1-\theta} \| f_1 \|_1^{\theta} \left[(1-\theta) F_0 + \theta \frac{|f_1|}{\| f_1 \|_1} \right]$. Thus $|f| \in X_0 + X_1$ and so $f \in X_0 + X_1$ with

$$\begin{aligned}
\| f \|_{X_0 + X_1} &= \| |f| \|_{X_0 + X_1} \\
&\leq \| f_0 \|_0^{1-\theta} \| f_1 \|_1^{\theta} \left[(1-\theta) \| F_0 \|_0 + \theta \left\| \frac{|f_1|}{\| f_1 \|_1} \right\|_1 \right] \\
&\leq \| f_0 \|_0^{1-\theta} \| f_1 \|_1^{\theta} .
\end{aligned}$$

Taking the infimum on the right-hand side gives the continuous inclusion.

(ii) Since

$$\begin{aligned}
&\{\| f_0 \|_0^{1-\theta} \| f_1 \|_1^{\theta} \mid |f| = |f_0|^{1-\theta} |f_1|^{\theta} ,\ f_0 \in X_0, f_1 \in X_1\} \\
\subseteq &\{\| f_0 \|_0^{1-\theta} \| f_1 \|_1^{\theta} \mid |f| \leq |f_0|^{1-\theta} |f_1|^{\theta} ,\ f_0 \in X_0, f_1 \in X_1\}
\end{aligned}$$

we already have

$$\| f \|_{X_0^{1-\theta} X_1^{\theta}} \leq \inf\{\| f_0 \|_0^{1-\theta} \| f_1 \|_1^{\theta} \mid |f| = |f_0|^{1-\theta} |f_1|^{\theta} ,\ f_i \in X_i\}.$$

For the other direction, let $|f| \leq |f_0|^{1-\theta} |f_1|^{\theta}$. We can then decompose Ω as follows:

$$\Omega = \underbrace{\Big[|f_0| \geq |f|,\ |f_1| \geq |f|\Big]}_{=:A} \cup \underbrace{\Big[|f_0| \geq |f|,\ |f_1| < |f|\Big]}_{=:B}$$
$$\cup \underbrace{\Big[|f_0| < |f|,\ |f_1| \geq |f|\Big]}_{=:C} \cup \underbrace{\Big[|f_0| < |f|,\ |f_1| < |f|\Big]}_{=:D}.$$

Notice that $\mu(D) = 0$ as otherwise $|f| \leq |f_0|^{1-\theta} |f_1|^{\theta}$ a.e. can't hold.

The idea is now to decrease the $|f_i|$ whenever they are larger than $|f|$, because increasing isn't covered by the lattice property. Just picking $|f|$ on B for $|f_0|$ and on C for $|f_1|$ doesn't make $|f| \leq |f_0|^{1-\theta} |f_1|^{\theta}$ into an equality, so that lowering the $|f_i|$ needs to be done more carefully by rearranging $|f| \leq |f_0|^{1-\theta} |f_1|^{\theta}$ so that $|f_i|$ remains on the right-hand side.

We define

$$f_0' := |f|\chi_A + \left(\frac{|f|}{|f_1|^{\theta}}\right)^{1/(1-\theta)} \chi_B + |f_0|\,\chi_C,$$
$$f_1' := |f|\chi_A + |f_1|\,\chi_B + \left(\frac{|f|}{|f_0|^{1-\theta}}\right)^{1/\theta} \chi_C.$$

These functions satisfy $|f_i'| \leq |f_i|$ a.e. and $|f| = |f_0'|^{1-\theta} |f_1'|^{\theta}$ a.e. Using the lattice property, we see that $f_i' \in X_i$ with $\| f_0' \|_0^{1-\theta} \| f_1' \|_1^{\theta} \leq \| f_0 \|_0^{1-\theta} \| f_1 \|_1^{\theta}$ and obtain

$$\inf\{\| f_0 \|_0^{1-\theta} \| f_1 \|_1^{\theta} \mid |f| = |f_0|^{1-\theta} |f_1|^{\theta},\ f_i \in X_i\} \leq \| f \|_{X_0^{1-\theta} X_1^{\theta}}.$$

(iii) Let $f \in X_0^{1-\theta} X_1^{\theta}$ be non-negative. By (ii) we may assume that $f = |f_0|^{1-\theta} |f_1|^{\theta}$ for some $f_j \in X_j$. By separability of both spaces, there are sequences $(f_{n,j})_n$ from a countable dense subset $D_j \subseteq X_j$ with $f_{n,j} \to f_j$ in X_j. We then obtain

$$\left\| f - |f_{n,0}|^{1-\theta} |f_{n,1}|^{\theta} \right\|_{X_0^{1-\theta} X_1^{\theta}}$$
$$\leq \left\| |f_0|^{1-\theta} |f_1|^{\theta} - |f_{n,0}|^{1-\theta} |f_1|^{\theta} \right\|_{X_0^{1-\theta} X_1^{\theta}} +$$

$$\begin{aligned}
&\left\| \, |f_{n,0}|^{1-\theta} \, |f_1|^{\theta} - |f_{n,0}|^{1-\theta} \, |f_{n,1}|^{\theta} \, \right\|_{X_0^{1-\theta} X_1^{\theta}} \\
\leq &\, \| \, |f_0| - |f_{n,0}| \, \|_0^{1-\theta} \, \| \, |f_1| \, \|_1^{\theta} + \\
&\| \, |f_{n,0}| \, \|_0^{1-\theta} \, \| \, |f_1| - |f_{n,1}| \, \|_1^{\theta} \\
= &\, \| \, ||f_0| - |f_{n,0}|| \, \|_0^{1-\theta} \, \| \, f_1 \, \|_1^{\theta} + \\
&\| \, f_{n,0} \, \|_0^{1-\theta} \, \| \, ||f_1| - |f_{n,1}|| \, \|_1^{\theta} \\
\lesssim &\, \| \, |f_0 - f_{n,0}| \, \|_0^{1-\theta} + \\
&\| \, |f_1 - f_{n,1}| \, \|_1^{\theta} \\
= &\, \| \, f_0 - f_{n,0} \, \|_0^{1-\theta} + \\
&\| \, f_1 - f_{n,1} \, \|_1^{\theta} \\
&\overset{n \to \infty}{\longrightarrow} 0.
\end{aligned}$$

Arguing similarly for $\mathrm{Re}(f)_-$ and $\mathrm{Im}(f)_\pm$ yields that $X_0^{1-\theta} X_1^{\theta}$ is separable with countable and dense subset

$$\widetilde{D} := \left\{ \sum_{k=0}^{3} \left| f_0^{(k)} \right|^{1-\theta} \left| f_1^{(k)} \right|^{\theta} \mid f_j^{(k)} \in D_j, \, \overline{D_j} = X_j \right\}.$$

(iv) The following proof is taken from [CNS03, Theorem 1.29].

W.l.o.g. let X_1 be order-continuous and $(f_n)_n \subseteq X_0^{1-\theta} X_1^{\theta}$ with $|f_n| \searrow 0$. Since $\| \, f_n \, \|_{X_0^{1-\theta} X_1^{\theta}} \leq \| \, f_0 \, \|_{X_0^{1-\theta} X_1^{\theta}} < \infty$, there exist decompositions $|f_n| = |f_{0,n}|^{1-\theta} \, |f_{1,n}|^{\theta}$ with $f_{i,n} \in X_i$. We rewrite

$$|f_n| = |f_{0,0}|^{1-\theta} \left| \left| \frac{f_n}{|f_{0,0}|^{1-\theta}} \right|^{1/\theta} \right|^{\theta}$$

and hope to have recovered another decomposition of $|f_n|$ with more useful properties. Indeed, since $|f_n| \leq |f_0|$, we have

$$\begin{aligned}
|f_n| \leq |f_0| \quad &\overset{(*)}{\iff} \quad |f_{0,n}|^{1-\theta} \, |f_{1,n}|^{\theta} \leq |f_{0,0}|^{1-\theta} \, |f_{1,0}|^{\theta} \\
&\iff \quad \left(\left| \frac{f_{0,n}}{f_{0,0}} \right| \right)^{(1-\theta)/\theta} |f_{1,n}| \leq |f_{1,0}| \, ,
\end{aligned}$$

(notice that $(*)$ is only possible due to (ii), as estimates on $|f_n|$ wouldn't translate to another inequality as above) which yields

$$\left|\frac{f_n}{|f_{0,0}|^{1-\theta}}\right|^{1/\theta} = \left|\frac{|f_{0,n}|^{1-\theta}\,|f_{1,n}|^{\theta}}{|f_{0,0}|^{1-\theta}}\right|^{1/\theta} = \left(\left|\frac{f_{0,n}}{f_{0,0}}\right|\right)^{(1-\theta)/\theta} |f_{1,n}| \leq |f_{1,0}|\,,$$

i.e. $\left|\frac{f_n}{|f_{0,0}|^{1-\theta}}\right|^{1/\theta} \in X_1$. As $\left|\frac{f_n}{|f_{0,0}|^{1-\theta}}\right|^{1/\theta} \searrow 0$, it follows by the order-continuity of X_1 that $\left\|\left|\frac{f_n}{|f_{0,1}|^{1-\theta}}\right|^{1/\theta}\right\|_1 \searrow 0$. Using the definition of $\|\cdot\|_{X_0^{1-\theta}X_1^{\theta}}$, we get

$$\| f_n \|_{X_0^{1-\theta}X_1^{\theta}} \leq \| f_{0,1} \|_0^{1-\theta} \left\|\left|\frac{f_n}{|f_{0,1}|^{1-\theta}}\right|^{1/\theta}\right\|^{\theta} \xrightarrow{n\to\infty} 0,$$

i.e. $X_0^{1-\theta}X_1^{\theta}$ is order-continuous.

(v) This is a direct consequence of (iv) and the characterization of separability through order-continuity (Theorem 3.8). □

Remark 6.6. Proposition 6.5 (i) serves as a "sanity check" in the sense that Calderón products are intermediate as well.

Remark 6.7. In general, order-continuity only respects products but not sums. If X_0, X_1 are quasi-Banach spaces and only one of which is order-continuous, then X_0+X_1 doesn't need to be order-continuous. To see this, let $X_0 = \mathrm{L}^1(\mathbb{R}^n)$, $X_1 = \mathrm{L}^\infty(\mathbb{R}^n)$ and for $m \in \mathbb{N}$ consider $x_m := \chi_{\mathbb{R}^n\setminus B_m(0)} \in X_0 + X_1$. Then for any decompostion $x_m = x_m^0 + x_m^1$ with $x_m^i \in X_i$ we have

$$\begin{aligned}
&\| x_m^0 \|_{\mathrm{L}^1} + \| x_m^1 \|_{\mathrm{L}^\infty} \\
&= \lim_{r\to\infty} \int_{B_r(0)} |x_m^0| \,\mathrm{d}x + \fint_{B_r(0)} \| x_m^1 \|_{\mathrm{L}^\infty} \,\mathrm{d}x \\
&\geq \lim_{r\to\infty} \int_{B_r(0)} |x_m^0| \,\mathrm{d}x + \fint_{B_r(0)} |x_m^1| \,\mathrm{d}x \\
&\geq \lim_{r\to\infty} \fint_{B_r(0)} |x_m^0 + x_m^1| \,\mathrm{d}x && (|B_r(0)| \geq 1 \text{ for } r \text{ large}) \\
&= \lim_{r\to\infty} \fint_{B_r(0)} |x_m| \,\mathrm{d}x \\
&= \lim_{r\to\infty} \frac{r^n - m^n}{r^n} && (\text{when } r \geq m)
\end{aligned}$$

$$=1.$$

Thus $\| x_m \|_{\mathrm{L}^1+\mathrm{L}^\infty} \geq 1$ but $|x_m| \searrow 0$ pointwise a.e.

7 Identifying interpolation spaces with Calderón products

After establishing the general foundation for the proof of Theorem 1.2, we are finally ready to tackle its proof. For convenience, we restate the result here.

Theorem 7.1. *Let $\theta \in (0,1)$ and let X_0, X_1 be p-convex quasi-Banach function spaces over Ω, one of which is separable. Then $X_0 + X_1$ is p-convex and the spaces $[X_0, X_1]_\theta$ and $X_0^{1-\theta} X_1^\theta$ are separable and agree up to equivalence of quasi-norms.*

We split the proof up in three larger sections: First, we get the easy parts out of the way, meaning p-convexity and separability. We then devote one section for each inclusion $[X_0, X_1]_\theta \supseteq X_0^{1-\theta} X_1^\theta$ and $[X_0, X_1]_\theta \subseteq X_0^{1-\theta} X_1^\theta$, where further preliminaries specific to the respective proofs will be handled as well.

7.1 The easy part of Theorem 7.1

This will be a short one.

Proof of the easy part of Theorem 7.1. We show that $X_0 + X_1$ is p-convex and that the resulting interpolation space is separable (if the identification already holds). The following arguments can be found in [KM98, Theorem 3.4].

Since X_0 and X_1 are p-convex for some values $p_0, p_1 \in (0,1]$, we may assume by Theorem 3.25 that they are p-convex for some $p \in (0, \min\{p_0, p_1\}]$ and will show that $X_0 + X_1$ is p-convex for this specific p.

Let $f_0, \dots, f_n \in X_0 + X_1$. Then there are $f_{i,j} \in X_j$ s.t. $f_i = f_{i,0} + f_{i,1}$ and $\| f_{i,0} \|_0 + \| f_{i,1} \|_1 \leq 2 \| f_i \|_{X_0 + x_1}$ for all $0 \leq i \leq n$. We get

$$\begin{aligned}
&\left\| \left(\sum_{i=0}^n |f_i|^p \right)^{1/p} \right\|_{X_0+X_1} \\
&\leq \left\| \left(\sum_{i=0}^n (|f_{i,0}| + |f_{i,1}|)^p \right)^{1/p} \right\|_{X_0+X_1} \\
&\lesssim \left\| \left(\sum_{i=0}^n |f_{i,0}|^p \right)^{1/p} + \left(\sum_{i=0}^n |f_{i,1}|^p \right)^{1/p} \right\|_{X_0+X_1} && \text{(Minkowski)} \\
&\leq \left\| \left(\sum_{i=0}^n |f_{i,0}|^p \right)^{1/p} \right\|_0 + \left\| \left(\sum_{i=0}^n |f_{i,1}|^p \right)^{1/p} \right\|_1 \\
&\lesssim \left(\sum_{i=0}^n \| f_{i,0} \|_0^p \right)^{1/p} + \left(\sum_{i=0}^n \| f_{i,1} \|_1^p \right)^{1/p} && (p\text{-convexity of } X_i) \\
&\leq \left(\sum_{i=0}^n (\| f_{i,0} \|_0 + \| f_{i,1} \|_1)^p \right)^{1/p} + \\
&\qquad \left(\sum_{i=0}^n (\| f_{i,1} \|_1 + \| f_{i,0} \|_0)^p \right)^{1/p} \\
&\lesssim \left(\sum_{i=0}^n \| f_i \|_{X_0+X_1}^p \right)^{1/p} && \text{(Choice of } f_{i,0}, f_{i,1}.)
\end{aligned}$$

After having shown that $[X_0, X_1]_\theta$ and $X_0^{1-\theta} X_1^\theta$ agree up to equivalence of quasi-norms, separability of both spaces is clear by Proposition 6.5 (v). □

7.2 Proving $[X_0, X_1]_\theta \supseteq X_0^{1-\theta} X_1^\theta$

We start with a quick outline of the proof of this direction in the Banach case, where the differences between the Banach and quasi-Banach case are minimal.

In order to show that $[X_0, X_1]_\theta \supseteq X_0^{1-\theta} X_1^\theta$, we take $f \in X_0^{1-\theta} X_1^\theta$ with $|f| = |f_0|^{1-\theta} |f_1|^\theta$ and truncate via $f_n \coloneqq f \cdot \chi_{E_n}$ where $E_n \coloneqq [1/n \leq |f_0|, |f_1| \leq n]$. Since the non-zero values of $|f_n| = |\chi_{E_n} f_0|^{1-\theta} |\chi_{E_n} f_1|^\theta$ behave nicely, this gives rise to an admissible function by replacing θ with $z \in S$. At this point, it is easy to show that $f_n \in [X_0, X_1]_\theta$ with continuous inclusion. Order-continuity of $X_0^{1-\theta} X_1^\theta$ (recall Proposition 6.5 (iv)) now ensures that the class of truncated functions is dense in $X_0^{1-\theta} X_1^\theta$, so that the continuous inclusion may be extended by density.

And so, we only need to prepare an admissible function for this part of the proof of Theorem 7.1.

Proposition 7.2. *Let X_0, X_1 be quasi-Banach function spaces and $f_i \in X_i$ s.t. the absolute values of their non-zero values are contained in $[M^{-1}, M]$ for some $M > 1$. Then $z \mapsto F(z) := |f_0|^{1-z} |f_1|^z$, $z \in S$, is an admissible function.*

Proof. Because $|f_0|^{1-z} |f_1|^z = |f_0|^{1-z} |f_1|^z \chi_{\text{supp}(f_0) \cap \text{supp}(f_1)}$ for all $z \in S$, we may assume that f_0, f_1 have the same support, which in the following we will call $\text{supp}(F)$. It holds that $\chi_{\text{supp}(F)} \in X_0 \cap X_1$, because of $\chi_{\text{supp}(F)} \leq M |f_i|$ and the respective lattice properties of the X_i. We now go through the properties of admissible functions.

(i) Boundedness: By Young's inequality we have for all $z \in S$

$$\left| |f_0|^{1-z} |f_1|^z \right| = |f_0|^{\text{Re}(1-z)} |f_1|^{\text{Re}(z)} \leq \text{Re}(1-z) |f_0| + \text{Re}(z) |f_1| .$$

As $X_0 + X_1$ satisfies the lattice property, we obtain for all $z \in S$ that

$$\begin{aligned} \| F(z) \|_{X_0+X_1} &\leq \| \text{Re}(1-z) |f_0| + \text{Re}(z) |f_1| \|_{X_0+X_1} \\ &\leq \text{Re}(1-z) \| f_0 \|_0 + \text{Re}(z) \| f_1 \|_1 \\ &\leq \| f_0 \|_0 + \| f_1 \|_1 , \end{aligned} \tag{7.1}$$

which also gives the well-definedness of $S \to X_0 + X_1, z \mapsto |f_0|^{1-z} |f_0|^z$.

(ii) Traces bounded in X_i and continuity up to the boundary of $\overline{S}$: For the extension of F we choose the natural extension $F(z) := |f_0|^{1-z} |f_1|^z \chi_{\text{supp}_F}$ with the convention that $0^z = 0$ when $z \in \overline{S}$. The map $t \mapsto F(j + \text{i}t)$ is bounded into X_j, because we have $|F(j + \text{i}t)| \leq |f_j|$, so that the lattice properties of the X_j imply $F(j + \text{i}t) \in X_j$ and $\| F(j + \text{i}t) \|_j \leq M \left\| \chi_{\text{supp}(F)} \right\|$ for all $t \in \mathbb{R}$. To tackle continuity up to the boundary, we use the following: For each choice of real numbers $a, b > 0$ the map

$z \mapsto a^{1-z}b^z$ is holomorphic on $\mathbb{C}$, so that we can show a sort of mean-value estimate. Let $z, z' \in \mathbb{C}$ and denote the line segment connecting z and z' as $\overline{zz'}$. Then

$$\begin{aligned} \left| a^{1-z}b^z - a^{1-z'}b^{z'} \right| &= \left| \int_{\overline{zz'}} \ln(b/a) a^{1-\xi} b^{\xi} \, \mathrm{d}\xi \right| \\ &\leq |z - z'| \ln(b/a) \sup_{\xi \in \overline{zz'}} a^{1-\mathrm{Re}(\xi)} b^{\mathrm{Re}(\xi)}. \end{aligned}$$

When $z, z' \in \overline{S}$, this allows us to make the pointwise estimate

$$\begin{aligned} &\left| |f_0|^{1-z} |f_1|^z - |f_0|^{1-z'} |f_1|^{z'} \right| \\ &\leq |z - z'| \ln\left(\frac{|f_1|}{|f_0|} \right) \sup_{\xi \in \overline{zz'}} |f_0|^{1-\mathrm{Re}(\xi)} |f_1|^{\mathrm{Re}(\xi)} \chi_{\mathrm{supp}(F)} \\ &\leq |z - z'| \ln(M^2) M \chi_{\mathrm{supp}(F)}. \end{aligned}$$

Due to the lattice properties of $X_0,\ X_1,\ X_0 + X_1$ and $\chi_{\mathrm{supp}(F)}$ being contained in all those spaces, continuity of the traces $t \mapsto F(j + \mathrm{i}t)$ and the continuity of F up to $\overline{S}$ directly follow, because after estimating as done above, any convergence is separated into $|\cdot|$ and away from any involved quasi-norms.

(iii) Analyticity: To get an idea of how a power series expansion of F might look like, let $z_0 \in S$. Then for appropriate $z \in S$ we wish to use the known series expansion of $\exp$ and get something along the lines of

$$\begin{aligned} F(z) &= |f_0|^{1-z} |f_1|^z \\ &= |f_0| \, \mathrm{e}^{z_0 \ln\left(\left|\frac{f_1}{f_0}\right|\right)} \mathrm{e}^{(z-z_0)\ln\left(\left|\frac{f_1}{f_0}\right|\right)} \\ &= \sum_{n=0}^{\infty} \frac{|f_0| \, \mathrm{e}^{z_0 \ln\left(\left|\frac{f_1}{f_0}\right|\right)}}{n!} \left[\ln\left(\left| \frac{f_1}{f_0} \right| \right) \right]^n (z - z_0)^n. \end{aligned} \tag{7.2}$$

By comparing with the exponential series, the above series converges a.e. and agrees with $F(z)$ a.e. if $z \in B_r(z_0)$ where

$$r := \min\left\{ \frac{1}{2\ln(M)}, \mathrm{dist}(z_0, \partial S) \right\}.$$

We now show that this series converges in $X_0 + X_1$ and agrees with F

inside $B_r(z_0)$. First, note that the coefficients actually lie in $X_0 + X_1$, because the right-hand side in the following estimate

$$\left| \frac{|f_0| \, \mathrm{e}^{z_0 \ln\left(\left|\frac{f_1}{f_0}\right|\right)}}{n!} \left[\ln\left(\left|\frac{f_1}{f_0}\right|\right)\right]^n \right| \leq |f_0| \, \frac{2\ln(M)}{n!} \left[2\ln(M)\right]^n$$

is only a multiple of $|f_0|$.
Now let $m, n \in \mathbb{N}, m \geq n$. It holds for all such z as above that

$$\begin{aligned}
&\left\| \sum_{k=n}^{m} \frac{|f_0| \, \mathrm{e}^{z_0 \ln\left(\left|\frac{f_1}{f_0}\right|\right)}}{k!} \left[\ln\left(\left|\frac{f_1}{f_0}\right|\right)\right]^k (z - z_0)^k \right\|_{X_0+X_1} \\
&\leq \left\| \sum_{k=n}^{m} \frac{|f_0| \left[2\ln(M)\right]^{k+1}}{k!} |z - z_0|^k \right\|_{X_0+X_1} \\
&\leq 2\ln(M) \, \| \, f_0 \, \|_{X_0+X_1} \sum_{k=n}^{m} \frac{1}{k!},
\end{aligned}$$

so that the convergence of the exponential series implies the convergence of the series from (7.2) in $X_0 + X_1$. Because pointwise limits determine limits in $X_0 + X_1$ (Corollary 5.19 (i)), $F(z)$ is the limit of said series. Thus, F is analytic. $\square$

Remark 7.3. In the remark in [KMM07, p. 25] it is mentioned that the separability assumption in Theorem 7.1 is used in order to show admissibilty of functions classes as in the previous proposition. The same proposition shows that this is not needed.

Proof of "$\supseteq$" in Theorem 7.1. The proof comes together from many different places. For the inclusion $X_0^{1-\theta} X_1^{\theta} \subseteq [X_0, X_1]_\theta$ we work out Moritz Egert's handwritten notes and use ideas from [KPS82, Chapter 4, Theorem 1.14].

Step 1: We show that the inlcusion holds true for a special kind of function class, that will turn out to be a dense subset of $X_0^{1-\theta} X_1^{\theta}$.
For $\delta > 0$ set

$$\mathcal{D} := \{ f \in X_0^{1-\theta} X_1^{\theta} \, | \, \exists \, M > 1, \text{ s.t. } |f| = K \, \| \, f \, \|_{X_0^{1-\theta} X_1^{\theta}} \, |f_0|^{1-\theta} \, |f_1|^{\theta} \, ,$$

$$\| f_i \|_i \leq 1, \;\; M^{-1} \leq \chi_{[|f_i| \neq 0]} |f_i| \leq M \text{ a.e.}\}$$

where $K := (1+\delta)\left(C^{\sim}_{X_0^{1-\theta}X_1^{\theta}}\right)^2$. The specific value of $\delta > 0$ is not important (for this proof, but its following remark), so that we omit the dependence $K = K(\delta)$.

If $f \in \mathcal{D}$, Proposition 7.2 guarantees that the function

$$F(z) := \frac{f}{|f|} |f_0|^{1-z} |f_1|^z, \qquad z \in S$$

is admissible with its extension given by $z \mapsto \frac{f}{|f|} |f_0|^{1-z} |f_1|^z \chi_{\text{supp}(f_0)\cap\text{supp}(f_1)}$, $z \in \overline{S}$, so that its quasi-norm satisfies the following estimate:

$$\begin{aligned}
\| F \|_{\mathcal{F}} = \max \Bigg\{ & \sup_{t\in\mathbb{R}} \left\| \frac{f}{|f|} |f_0|^{1-\mathrm{i}t} |f_1|^{\mathrm{i}t} \chi_{\text{supp}(f_0)\cap\text{supp}(f_1)} \right\|_0 , \\
& \sup_{t\in\mathbb{R}} \left\| \frac{f}{|f|} |f_0|^{-\mathrm{i}t} |f_1|^{1+\mathrm{i}t} \chi_{\text{supp}(f_0)\cap\text{supp}(f_1)} \right\|_1 , \\
& \sup_{z\in S} \| F(z) \|_{X_0+X_1} \Bigg\} \\
\leq {} & \max \{ \| f_0 \|_0 , \| f_1 \|_1 , \| f_0 \|_0 + \| f_1 \|_1 \} \qquad \text{(by (7.1))} \\
\leq {} & 2.
\end{aligned}$$

As $F(\theta) = \frac{f}{K \| f \|_{X_0^{1-\theta}X_1^{\theta}}}$, we can now further estimate

$$\begin{aligned}
\| f \|_\theta &= K \| f \|_{X_0^{1-\theta}X_1^{\theta}} \left\| \frac{f}{K \| f \|_{X_0^{1-\theta}X_1^{\theta}}} \right\|_\theta \\
&\leq K \| f \|_{X_0^{1-\theta}X_1^{\theta}} \| F \|_{\mathcal{F}} \\
&\leq 2K \| f \|_{X_0^{1-\theta}X_1^{\theta}}
\end{aligned}$$

Step 2: We show that $\mathcal{D}$ is dense in $X_0^{1-\theta}X_1^{\theta}$ by using its order-continuity (guaranteed by Propsition 6.5 (iv)).

Let $f \in X_0^{1-\theta}X_1^{\theta}$. Then there exist $f_i \in X_i$ s.t.

$$|f| = |f_0|^{1-\theta} |f_1|^\theta, \quad \| f_0 \|_0^{1-\theta} \| f_1 \|_1^\theta \leq (1+\delta/2) \| f \|_{X_0^{1-\theta}X_1^{\theta}} .$$

For $n \in \mathbb{N}$ set $E_n := [1/n \leq |f_0|, |f_1| \leq n]$ and define $f_n := f\chi_{E_n}$. Then

$X_0^{1-\theta}X_1^\theta$'s lattice property implies $f_n \in X_0^{1-\theta}X_1^\theta$ while its order-continuity yields $f_n \to f$ in $X_0^{1-\theta}X_1^\theta$ due to $|f - f_n| \searrow 0$.
To show that $f_n \in \mathcal{D}$, notice that with $f_n \to f$ in $X_0^{1-\theta}X_1^\theta$ we also have $\| f_n \|_\sim \to \| f \|_\sim$, so that by continuity of $\| \cdot \|_\sim$ and $1 < \frac{1+\delta}{1+\delta/2}$ there exists $N \in \mathbb{N}$ s.t. $\| f \|_\sim \le \frac{1+\delta}{1+\delta/2} \| f_n \|_\sim$ for all $n \ge N$. We obtain for all those n that

$$
\begin{aligned}
|f_n| &= \| f_0\chi_{E_n} \|_0^{1-\theta} \| f_1\chi_{E_n} \|_1^\theta \underbrace{\left| \frac{f_0\chi_{E_n}}{\| f_0\chi_{E_n} \|_0} \right|}_{:=h_{0,n}}^{1-\theta} \underbrace{\left| \frac{f_1\chi_{E_n}}{\| f_1\chi_{E_n} \|_1} \right|}_{:=h_{1,n}}^{\theta} \\
&\le (1+\delta/2) \| f \|_{X_0^{1-\theta}X_1^\theta} |h_{0,n}|^{1-\theta} |h_{1,n}|^\theta \\
&\le (1+\delta/2) C^\sim_{X_0^{1-\theta}X_1^\theta} \| f \|_\sim |h_{0,n}|^{1-\theta} |h_{1,n}|^\theta \\
&\le (1+\delta) C^\sim_{X_0^{1-\theta}X_1^\theta} \| f_n \|_\sim |h_{0,n}|^{1-\theta} |h_{1,n}|^\theta \\
&\le (1+\delta) \left(C^\sim_{X_0^{1-\theta}X_1^\theta} \right)^2 \| f_n \|_{X_0^{1-\theta}X_1^\theta} |h_{0,n}|^{1-\theta} |h_{1,n}|^\theta .
\end{aligned}
$$

Replacing $h_{0,n}$ by $\widetilde{h_{0,n}} := \left| \frac{|f_n|}{K \| f_n \|_{X_0^{1-\theta}X_1^\theta} |h_{1,n}|^\theta} \right|^{1/(1-\theta)} \le |h_{0,n}|$ achieves equality and shows that $f_n \in \mathcal{D}$.

Step 3: Because it is not clear whether $\mathcal{D}$ is actually a linear subspace (because $f \in \mathcal{D}$ is determined through $|f|$, which is not linear in f), we can't directly make Cauchy sequences in $X_0^{1-\theta}X_1^\theta$ to Cauchy sequences in $[X_0, X_1]_\theta$ and extend the inclusion from $\mathcal{D}$ to the whole of $X_0^{1-\theta}X_1^\theta$ in this way. We will need to proceed more carefully.

Let $f \in X_0^{1-\theta}X_1^\theta$. Inductively, we will construct a sequence $(f_n)_n \subseteq \mathcal{D}$ s.t.

$$\left\| f - \sum_{j=0}^{n} f_j \right\|_\sim^r \le \rho^{n+1} \| f \|_\sim^r \tag{7.3}$$

$$\| f_n \|_\sim^r \le \rho^{n+1} \| f \|_\sim^r \tag{7.4}$$

holds for some $\rho \in (0,1)$.

For $n = 0$, let $c, \varepsilon > 0$ to be fixed in a bit. By the previous step there is $f_0 \in \mathcal{D}$ s.t. $\| cf - f_0 \|_\sim^r \le \varepsilon \| f \|_\sim^r$. This f_0 also satisfies

$$\| f - f_0 \|_\sim^r \le \| cf - f_0 \|_\sim^r + \| (1-c)f \|_\sim^r \le (\varepsilon + (1-c)^r) \| f \|_\sim^r$$

$$\| f_0 \|_\sim^r \le \| f_0 - cf \|_\sim^r + \| cf \|_\sim^r \qquad \le (\varepsilon + c^r) \| f \|_\sim^r .$$

Picking $c = 1/2$ and ε so that $\rho := \varepsilon + c^r < 1$ yields the claim for $n = 0$.

Assuming that for some $n \ge 0$ we already found $f_0, \dots, f_n$ with (7.3) and (7.4), we can repeat the arguments for when $n = 0$ by finding a $f_{n+1} \in \mathcal{D}$ with $\left\| c\left(f - \sum_{j=0}^n f_j\right) - f_{n+1} \right\|_\sim^r \le \varepsilon \left\| f - \sum_{j=0}^n f_j \right\|_\sim^r$. Then

$$\begin{aligned}
&\left\| f - \sum_{j=0}^{n+1} f_j \right\|_\sim^r \\
\le &\left\| c\left(f - \sum_{j=0}^{n} f_j \right) - f_{n+1} \right\|_\sim^r + \left\| (1-c)\left(f - \sum_{j=0}^{n} f_j \right) \right\|_\sim^r \\
\le &(\varepsilon + (1-c)^r) \left\| f - \sum_{j=0}^{n} f_j \right\|_\sim^r \\
\le &\rho^{n+2} \| f \|_\sim^r
\end{aligned}$$

and

$$\begin{aligned}
&\| f_{n+1} \|_\sim^r \\
\le &\left\| f_{n+1} - c\left(f - \sum_{j=0}^{n} f_j \right) \right\|_\sim^r + \left\| c\left(f - \sum_{j=0}^{n} f_j \right) \right\|_\sim^r \\
\le &(\varepsilon + c^r) \left\| f - \sum_{j=0}^{n} f_j \right\|_\sim^r \\
\le &\rho^{n+2} \| f \|_\sim^r .
\end{aligned}$$

Since $\rho \in (0,1)$, (7.3) implies that $\sum_{j=0}^n f_j \overset{n\to\infty}{\longrightarrow} f$ in $X_0^{1-\theta} X_1^\theta$. Since $f_n \in \mathcal{D}$, we also obtain (by the Aoki-Rolewicz-Theorem we may assume that the equivalent quasi-norms for $[X_0, X_1]_\theta$ and $X_0^{1-\theta} X_1^\theta$ are subadditive for the same value r)

$$\begin{aligned}
\| f_n \|_\theta^r &\le (2K)^r \| f_n \|_{X_0^{1-\theta} X_1^\theta}^r && (f_n \in \mathcal{D} \text{ and step 1}) \\
&\le (2K)^r (C^\sim_{X_0^{1-\theta} X_1^\theta})^r \| f_n \|_\sim^r
\end{aligned}$$

$$\leq (2K)^r (C^{\sim}_{X_0^{1-\theta} X_1^\theta})^r \rho^{n+1} \| f \|^r_{\sim} \qquad \text{(by (7.4))}$$
$$\leq (2K)^r \left(C^{\sim}_{X_0^{1-\theta} X_1^\theta} \right)^{2r} \rho^{n+1} \| f \|^r_{X_0^{1-\theta} X_1^\theta} .$$

Thus we estimate for $n, m \in \mathbb{N}$ with $n \geq m$

$$\begin{aligned}
&\left\| \sum_{j=m}^{n} f_j \right\|_\theta^r \\
\leq& \left(C^{\sim}_{[X_0,X_1]_\theta} \right)^{2r} \sum_{j=m}^{n} \| f_j \|^r_\theta \qquad \text{(Lemma 2.7 (ii))} \\
\leq& (2K)^r \left(C^{\sim}_{X_0^{1-\theta} X_1^\theta} C^{\sim}_{[X_0,X_1]_\theta} \right)^{2r} \sum_{j=m}^{n} \rho^{j+1} \| f \|^r_{X_0^{1-\theta} X_1^\theta}
\end{aligned}$$

to see that $(\sum_{j=0}^{n} f_j)_n \subseteq [X_0, X_1]_\theta$ is a Cauchy sequence. Its limit $\widetilde{f} \in [X_0, X_1]_\theta$ satisfies

$$\begin{aligned}
&\left\| \widetilde{f} \right\|_\theta \\
\leq& C_\theta \left(\left\| \widetilde{f} - \sum_{j=0}^{n} f_j \right\|_\theta + \left\| \sum_{j=0}^{n} f_j \right\|_\theta \right) \\
\leq& C_\theta \left(\left\| \widetilde{f} - \sum_{j=0}^{n} f_j \right\|_\theta + 2K \left(C^{\sim}_{X_0^{1-\theta} X_1^\theta} C^{\sim}_{[X_0,X_1]_\theta} \right)^2 \left(\sum_{j=0}^{n} \rho^{j+1} \right)^{1/r} \| f \|_{X_0^{1-\theta} X_1^\theta} \right)
\end{aligned}$$

and so letting $n \to \infty$

$$\left\| \widetilde{f} \right\|_\theta \leq 2K\, C_\theta \left(C^{\sim}_{X_0^{1-\theta} X_1^\theta} C^{\sim}_{[X_0,X_1]_\theta} \right)^2 \left(\frac{1}{1-\rho} - 1 \right)^{1/r} \| f \|_{X_0^{1-\theta} X_1^\theta} \qquad (7.5)$$

Since $\sum_{j=0}^{n} f_j \to f$ in $X_0^{1-\theta} X_1^\theta$, $\sum_{j=0}^{n} f_j \to \widetilde{f}$ in $[X_0, X_1]_\theta$ and both spaces continuously embed into the Hausdorff space $\mathrm{L}_0(\Omega)$ (Theorem 5.18), the limits can be identified and this inclusion is done. $\square$

Remark 7.4.

(i) We need the correct norming to appear in the definition of $\mathcal{D}$, because if we omitted $K \| f \|_{X_0^{1-\theta} X_1^\theta}$ and $\| f_i \|_i \leq 1$ and repeated the previous steps, we would have ended up with $\| f \|_\theta \lesssim \| f_0 \|^{1-\theta} \| f_1 \|^\theta$, where it is not clear whether taking the infimum over all f_i with $|f| = |f_0|^{1-\theta} |f_1|^{\theta}$ and $M^{-1} \leq \chi_{[|f_i| \neq 0]} |f_i| \leq M$ a.e. for some $M > 0$ in fact realizes $\| f \|_{X_0^{1-\theta} X_1^{\theta}}$ on the right-hand side.

We also need $K \geq 1$ in $K \| f \|_{X_0^{1-\theta} X_1^{\theta}}$, because for $|f| = |f_0|^{1-\theta} |f_1|^{\theta}$ it holds in general that

$$|f| \geq \| f \|_{X_0^{1-\theta} X_1^{\theta}} \left(\frac{|f_0|}{\| f_0 \|_0} \right)^{1-\theta} \left(\frac{|f_1|}{\| f_1 \|_1} \right)^{\theta},$$

so that we need to "elevate" the right-hand side to an equality in order to have a chance at estimating in the right direction.

(ii) In the previous proof we kept explicit track of all constants so that we can see how the isometry between $X_0^{1-\theta} X_1^{\theta}$ and $[X_0, X_1]_\theta$ gets lost when going from the Banach to the quasi-Banach case.

In $X_0^{1-\theta} X_1^{\theta} \subseteq [X_0, X_1]_\theta$ we ended up with (7.5). In the Banach case, there is no need to pass to equivalent and continuous norms and the triangle inequality doesn't come at a cost, so that we may assume that $C_\theta = C^{\sim}_{X_0^{1-\theta} X_1^{\theta}} = C^{\sim}_{[X_0, X_1]_\theta} = r = 1$. Because $\rho = \varepsilon + (1/2)^r$ for $\varepsilon > 0$ s.t. $\rho < 1$, we may pass to the limit $\varepsilon \to 0$ so that $\rho = 1/2$ eliminates another expression. The number 2 is a remnant of the very rough estimate from (7.1). Since the maximum modulus principle holds true for Banach space valued analytic functions defined on S (see [KMM07, p. 21]), we can replace this number by 1 as well because for $z \in \overline{S}$ it holds that $\| F(z) \|_{X_0+X_1} \leq \sup_{z \in \partial S} \| F(z) \|_{X_0+X_1} \leq \max\{\| f_0 \|_0 , \| f_1 \|_1\} \leq 1$ due to the norming condition in $\mathcal{D}$. All that remains now is $(1+\delta)$ that hides in K due to the definition of $\mathcal{D}$. But since all steps remain true independently of the specific choice of $\delta > 0$, we may pass to the limit $\delta \to 0$ as well and obtain $\| f \|_\theta \leq \| f \|_{X_0^{1-\theta} X_1^{\theta}}$ in (7.5).

7.3 Proving $[X_0, X_1]_\theta \subseteq X_0^{1-\theta} X_1^{\theta}$

Again, we start with an outline for the proof in the Banach case.

For $[X_0, X_1]_\theta \subseteq X_0^{1-\theta} X_1^{\theta}$, take $f = F(\theta) \in [X_0, X_1]_\theta$ for some $F \in \mathcal{F}$. In order to find a candidate for the decomposition, we note that [KPS82, p. 216] pointed out that the values of admissible functions on S are determined by their behaviour on ∂S, i.e.

$$F(z) = \int_{\mathbb{R}} F(\mathrm{i}t)P_0(z, \mathrm{i}t)\,\mathrm{d}t + \int_{\mathbb{R}} F(1+\mathrm{i}t)P_1(z, 1+\mathrm{i}t)\,\mathrm{d}t, \qquad z \in S. \tag{7.6}$$

By definition of Riemann integrals and admissibility of F, the above integrals already lie in X_0 resp. X_1. Assuming further that $X_0 + X_1$ is a function space, as indicated in [KPS82, p. 240] we can use some trickery involving logarithms and the convexity of exponential functions in order to estimate the sum with a product and arrive at

$$\begin{aligned} |f| &= |F(\theta)| \\ &\le \left(\frac{1}{1-\theta}\int_{\mathbb{R}} |F(\mathrm{i}t)|\, P_0(\theta, \mathrm{i}t)\,\mathrm{d}t\right)^{1-\theta} \times \\ &\qquad \left(\frac{1}{\theta}\int_{\mathbb{R}} |F(1+\mathrm{i}t)|\, P_1(\theta, 1+\mathrm{i}t)\,\mathrm{d}t\right)^{\theta} \\ &=: |f_0|^{1-\theta}\, |f_1|^{\theta} \end{aligned} \tag{7.7}$$

a.e. in Ω. Here, we can use integration theory for Banach-valued functions and the associated triangle inequality for integrals to deduce that

$$\| f_0 \|_0 \le \frac{1}{1-\theta}\int_{\mathbb{R}} \| F(\mathrm{i}t) \|_0\, P_0(\mathrm{i}t)\,\mathrm{d}t \le \sup_{t\in\mathbb{R}} \| F(\mathrm{i}t) \|_0 \le \| F \|_{\mathcal{F}}$$

and similarly for f_1, so that $\| f \|_{X_0^{1-\theta}X_1^{\theta}} \le \| F \|_{\mathcal{F}}^{1-\theta} \| F \|_{\mathcal{F}}^{\theta}$, i.e. $\| f \|_{X_0^{1-\theta}X_1^{\theta}} \le \| f \|_{\theta}$.

We see that the crucial part is the one where integration theory for Banach space valued functions was used. We will tackle this problem by forcing $F(\theta)$ to be in a Banach space using the p-convexity of $X_0 + X_1$, which is guaranteed by the easy part of Theorem 7.1. More precisely, if F is analytic and quasi-Banach valued, the map $z \mapsto |F(z)|^q$ is Banach valued for appropriate values of $q > 0$. Because of the absolute value, any analyticity gets lost and we would get stuck if the above map did not look awfully similar to Theorem 4.6, where a subharmonic function is obtained by putting $|f|^q$ for some holomorphic function f. Corollary 4.12 states, that the values of subharmonic functions are at least controlled by their behaviour on ∂S, which in our case represents the spaces with respect to which we want to obtain a decomposition and is enough to actually derive a

decomposition similar to (7.7).
The problem is with the fact that an admissible function F is not $\mathbb{C}$-valued, but only the function $F(z)$, to which F evaluates for some $z \in \overline{S}$. We need to somehow fix appropriate $\omega \in \Omega$ and view F as a hopefully holomorphic function $S \to \mathbb{C},\ z \mapsto (F(z))(\omega)$ in order to apply our established theory for subharmonic functions. This is why for the following proof, we make an explicit distinction between equivalence classes and representatives.

Lemma 7.5. *Let X_0 and X_1 be p-convex quasi-Banach function spaces, F be an admissible function, $z_0 \in S$, $q > 0$ and $r > 0$ s.t. $B_r(z_0) \Subset S$. Then there exist representatives $F(z_0) \in [F(z_0)]$, $F(z_0 + r\mathrm{e}^{\mathrm{i}t}) \in [F(z_0 + r\mathrm{e}^{\mathrm{i}t})]$ and a null set $N \subseteq \Omega$ s.t. for all $\omega \in \Omega \setminus N$ it holds that*

$$|(F(z_0))(\omega)|^q \leq \frac{1}{2\pi} \int_0^{2\pi} \left|(F(z_0 + r\mathrm{e}^{\mathrm{i}t}))(\omega)\right|^q \,\mathrm{d}t.$$

Furthermore, it holds that

$$\frac{1}{2\pi} \int_0^{2\pi} \left|(F(z_0 + r\mathrm{e}^{\mathrm{i}t}))(\,\cdot\,)\right|^q \,\mathrm{d}t = \left[\frac{1}{2\pi} \int_0^{2\pi} \left|F(z_0 + r\mathrm{e}^{\mathrm{i}t})\right|^q \,\mathrm{d}t\right],$$

a.e., where the right-hand side represents an $(X_0 + X_1)^q$-valued Riemann integral.

Proof. Since F is analytic, we know that $[F(z)] = \sum_{n=0}^{\infty} [f_n](z - z_0)^n$ for appropriate $[f_n] \in X$ and all $z \in B_{\mathrm{dist}(z_0, \partial S)}(z_0)$ (Corollary 5.5 (ii)). Choosing any $z' \in B_{\mathrm{dist}(z_0, \partial S)}(z_0)$ s.t. $r' := |z' - z_0| > r$, we in particular obtain that $\sum_{n=0}^{\infty} [|f_n|]\, |z' - z_0|^n$ converges in X (Lemma 5.8). Because we are looking at series of non-negative terms, passing to subsequences in Corollary 5.19 (i) is not necessary and we obtain after choosing some representatives $f_n \in [f_n]$ that $\sum_{n=0}^{\infty} |f_n|\, (r')^n$ converges on $\Omega \setminus N$, where $N \subseteq \Omega$ is an appropriate null set. In particular, we also obtain

$$\left|\sum_{n=0}^{\infty} f_n (z - z_0)^n\right| \leq \sum_{n=0}^{\infty} |f_n|\, |z - z_0|^n \leq \sum_{n=0}^{\infty} |f_n|\, (r')^n \tag{7.8}$$

on $\Omega \setminus N$ and all $z \in B_{r'}(z_0)$.

With this in mind, we suspect that the desired representatives will amount to be f_0 and $\sum_{n=0}^{\infty} f_n \left(r\mathrm{e}^{\mathrm{i}t}\right)^n$. The former is clear, since $f_0 \in [F(z_0)]$ already holds trivially by F's analyticity. For the latter, we obtain again by F's analyticity that $\left[\sum_{n=0}^{N} f_n(re^{\mathrm{i}t})^n\right] = \sum_{n=0}^{N} [f_n]\,(re^{\mathrm{i}t})^n \to [F(z_0 + re^{\mathrm{i}t})]$ in X and by (7.8) that $\sum_{n=0}^{N} f_n(re^{\mathrm{i}t})^n \to \sum_{n=0}^{\infty} f_n(re^{\mathrm{i}t})^n$ pointwise on $\Omega \setminus N$, so that the limits can be identified (Corollary 5.19 (i)) and we can conclude that indeed $\sum_{n=0}^{\infty} f_n(re^{\mathrm{i}t})^n \in [F(z_0 + re^{\mathrm{i}t})]$.

For fixed $\omega \in \Omega \setminus N$, we can now define the function

$$G_\omega : B_{r'}(z_0) \to \mathbb{C}, \qquad z \mapsto \sum_{n=0}^{\infty} f_n(\omega)(z - z_0)^n,$$

which is well-defined and holomorphic by (7.8). In particular, $|G_\omega|^q$ is subharmonic by Theorem 4.6.

This yields

$$\begin{aligned}
|f_0(\omega)|^q = |G_\omega(z_0)|^q &\le \frac{1}{2\pi}\int_0^{2\pi} \left|G_\omega(z_0 + re^{\mathrm{i}t})\right|^q \,\mathrm{d}t \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{n=0}^{\infty} f_n(\omega)\left(re^{\mathrm{i}t}\right)^n\right|^q \,\mathrm{d}t \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left|(F(z_0 + re^{\mathrm{i}t}))(\omega)\right|^q \,\mathrm{d}t.
\end{aligned}$$

Regarding the pointwise identification, we can argue as follows: Because $\frac{1}{2\pi}\int_0^{2\pi} \left|F(z_0 + re^{\mathrm{i}t})\right|^q \,\mathrm{d}t$ is defined as the limit of Riemann sums in $(X_0 + X_1)^p$ (which is guaranteed to be a Banach function space by Proposition 3.23), we have

$$\frac{1}{2\pi}\int_0^{2\pi} \left|F(z_0 + re^{\mathrm{i}t})\right|^q \,\mathrm{d}t = \lim_{n\to\infty} \sum_{j=0}^{N(n)} \left|F(z_0 + re^{\mathrm{i}t_j^{(n)}})\right|^q (x_{j+1}^{(n)} - x_j^{(n)})$$

for some appropriate sequences of tagged partitions of $[0, 2\pi]$. This continues to hold true pointwise a.e. for a subsequence of the above sequence of tagged partitions, so we may write a.e.

$$\begin{aligned}
&\left[\frac{1}{2\pi}\int_0^{2\pi} \left|F(z_0 + re^{\mathrm{i}t})\right|^q \,\mathrm{d}t\right](\omega) \\
&= \lim_{k\to\infty} \left[\sum_{j=0}^{N(n_k)} \left|F(z_0 + re^{\mathrm{i}t_j^{(n_k)}})\right|^q (x_{j+1}^{(n_k)} - x_j^{(n_k)})\right](\omega)
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{k\to\infty} \sum_{j=0}^{N(n_k)} \left| F(z_0 + r\mathrm{e}^{\mathrm{i}t_j^{(n_k)}})(\omega) \right|^q (x_{j+1}^{(n_k)} - x_j^{(n_k)}) \\
&= \lim_{k\to\infty} \sum_{j=0}^{N(n_k)} \left| \sum_{k=0}^{\infty} f_k(\omega) \left(r\mathrm{e}^{\mathrm{i}t_j^{(n_k)}} \right)^k \right|^q (x_{j+1}^{(n_k)} - x_j^{(n_k)}) \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=0}^{\infty} f_k(\omega) \left(r\mathrm{e}^{\mathrm{i}t} \right)^k \right|^q \mathrm{d}t \qquad (7.9) \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left| (F(z_0 + r\mathrm{e}^{\mathrm{i}t}))(\omega) \right|^q \mathrm{d}t.
\end{aligned}
$$

Notice that going to (7.9) only works, because we established $\frac{1}{2\pi}\int_0^{2\pi} \left|(F(z_0 + r\mathrm{e}^{\mathrm{i}t}))(\omega)\right|^q \mathrm{d}t$ beforehand as the Riemann integral of the holomorphic function G_ω, so that the limit for a subsequence of tagged partitions still yields the same Riemann integral. $\square$

Thus, the outline in the quasi-Banach case is as follows: Similar to the Banach case, where analytic functions on S are determined by their behaviour on ∂S, we want to control quasi-Banach valued admissible functions by their boundary values. The closest that we can get to integral expressions as in (7.7) is by using that $(X_0 + X_1)^q$ is a Banach space for all $q \in (0, p]$ due to Proposition 3.23, which is naturally provided by $X_0 + X_1$'s p-convexity, and pass from F to $|F|^q$. Mimicking (7.7), this could yield something along the lines of

$$
\begin{aligned}
|F(\theta)|^q \leq \left(\frac{1}{1-\theta} \int_{\mathbb{R}} |F(\mathrm{i}t)|^q \, P_0(\theta, \mathrm{i}t) \, \mathrm{d}t \right)^{1-\theta} \times \\
\left(\frac{1}{\theta} \int_{\mathbb{R}} |F(1+\mathrm{i}t)|^q \, P_1(\theta, 1+\mathrm{i}t) \, \mathrm{d}t \right)^{\theta}.
\end{aligned}
$$

Due to the previous Lemma 7.5, Corollary 4.12 would look like a promising first step in this direction, if $|F(\theta)|^q$ was $\mathbb{R}$-valued, which it isn't. But since $((X_0 + X_1)^q)'$ is saturated, estimates like the one above can be achieved by testing the above inequality against functionals as in Corollary 3.19. The induced map $z \mapsto \int_\Omega |F(z)|^q \, \chi_A \, \mathrm{d}x$ will be real-valued as well and is the correct function to use Corollary 4.12 on, since $|F(\theta)|^q$ is real-valued a.e. At some point in the proof, we will need to exchange integrals over Ω with ones over $\mathbb{R}$ when $|F|^q$ is involved. In order to circumvent Fubinis theorem, which would require measurability from $|F|^q$ w.r.t $\mathbb{R} \times \Omega$, we want to argue via continuity of the functionals given by Corollary 3.19. Handling the q-dependence correctly, a limiting process will yield an appropriate decomposition for $|F(\theta)|$.

Before diving into the proof, we get some technial lemmata out of the way.

Lemma 7.6. *Let X be a Banach function space and $F : \mathbb{R} \to X$ be a map.*

(i) If F is Riemann integrable and $\varphi \in X^*$, it holds that $\varphi\left(\int_{\mathbb{R}} F \,\mathrm{d}t\right) = \int_{\mathbb{R}} \varphi(F) \,\mathrm{d}t$.

(ii) If F is continuous and bounded, then $t \mapsto P_j(\theta, j + \mathrm{i}t)F(t)$ is Riemann integrable for $j = 0, 1$, $\theta \in (0, 1)$.

Proof.

(i) This follows immediately after rewriting $\int_{\mathbb{R}} F \,\mathrm{d}t$ via Riemann sums and using φ's continuity.

(ii) $P_j(\theta, j + \mathrm{i}\,\cdot\,)$ is continuous and Riemann integrable on $\mathbb{R}$ by Corollary 4.12. Thus $t \mapsto P_j(\theta, j + \mathrm{i}t)F(t)$ is Riemann integrable on compact subsets of $\mathbb{R}$ and if $x, x' \in \mathbb{R}$, we obtain its Riemann integrability as follows:

$$\left\| \int_x^{x'} P_j(\theta, j + \mathrm{i}t)F(t) \,\mathrm{d}t \right\| \leq \sup_{t\in\mathbb{R}} \| F(t) \| \int_x^{x'} P_j(\theta, j + \mathrm{i}t) \,\mathrm{d}t \overset{x,x'\to\pm\infty}{\longrightarrow} 0. \qquad \square$$

At some point, we would like to estimate expressions like $\int_{\mathbb{R}} P_j(\theta, j+\mathrm{i}t) \,|F(t)|^q \,\mathrm{d}t$ via Jensen's inequality, but because they only exist as vector-valued Riemann integrals, estimating pointwise a.e. is a fickle thing, for which we will need

Lemma 7.7. *Let $c : \mathbb{R} \to \mathbb{R}$ be convex and $x_0 \in \mathbb{R}$. Then there exists a function $l(x) = ax + b$, $a, b \in \mathbb{R}$ s.t. $l(x_0) = c(x_0)$ and $l \leq c$ on $\mathbb{R}$.*

Proof. By convexity of c, the expressions $\frac{c(x_0)-c(x_0-h)}{h}$ and $\frac{c(x_0+h)-c(x_0)}{h}$ are monotonely increasing resp. descreasing in h. Thus, the limits

$$a := \lim_{h\downarrow 0} \frac{c(x_0) - c(x_0 - h)}{h} \qquad \text{and} \qquad b := \lim_{h\downarrow 0} \frac{c(x_0 + h) - c(x_0)}{h}.$$

exist and satisfy $a \leq b$. Let $\Phi \in [a, b]$, then

$$c(x_0) - c(x) \leq \Phi(x_0 - x)$$

for all $x \in \mathbb{R}$ and thus the function $l(x) := \Phi(x - x_0) + c(x_0)$ has the desired properties. $\square$

Proof of $[X_0, X_1]_\theta \subseteq X_0^{1-\theta} X_1^\theta$. This direction originates from [KM98, Theorem 3.4].

Let $f \in [X_0, X_1]_\theta$ and F be an admissible function s.t. $F(\theta) = f$. Due to the easy part of this proof, we may assume that X_0, X_1 and $X_0 + X_1$ are p-convex for the same value $p > 0$.

Step 1: We start by taking a look at the mentioned functionals which should be composed with $|F|^q$ in order to arrive at something subharmonic.

Let $q \in (0, p]$. Because F is $X_0 + X_1$-valued, the function $|F|^q$ is $(X_0 + X_1)^q$-valued and thus Banach-valued (as seen in Proposition 3.23). Let $A \subseteq \Omega$ be measurable s.t. $\| \chi_A \|'_{(X_0+X_1)^q} < \infty$. We then define

$$\varphi_A : (X_0 + X_1)^q \to \mathbb{C}, \qquad G \mapsto \int_\Omega G \chi_A \, \mathrm{d}\omega.$$

This functional is easily seen to be bounded by the definition of $\| \cdot \|'_{(X_0+X_1)^q}$, as

$$\begin{aligned} \left| \int_\Omega G \chi_A \, \mathrm{d}\omega \right| &= \left| \int_\Omega \frac{G}{\| G \|_{(X_0+X_1)^q}} \chi_A \, \mathrm{d}\omega \right| \| G \|_{(X_0+X_1)^q} \\ &\leq \| \chi_A \|'_{(X_0+X_1)^q} \| G \|_{(X_0+X_1)^q} . \end{aligned}$$

In order to meet the conditions of Corollary 4.12, we need to verify subharmonicity and continuous and bounded extendability of $\varphi_A(|F(\,\cdot\,)|^q)$.

- Subharmonicity: Let $z_0 \in S$ and $r > 0$ s.t. $B_r(z_0) \Subset S$. We get

$$\begin{aligned} \varphi_A(|F(z_0)|^q) &\leq \varphi_A \left(\frac{1}{2\pi} \int_0^{2\pi} \left| F(z_0 + r\mathrm{e}^{\mathrm{i}t}) \right|^q \, \mathrm{d}t \right) && \text{(Lemma 7.5)} \\ &= \frac{1}{2\pi} \int_0^{2\pi} \varphi_A(\left| F(z_0 + r\mathrm{e}^{\mathrm{i}t}) \right|^q) \, \mathrm{d}t && \text{(Lemma 7.6 (i).)} \end{aligned}$$

- Continuous extendability: The natural extension of $\varphi_A(|F(\,\cdot\,)|^q)$ is the one that is given by the extension of F from S to $\overline{S}$. Let $z, z' \in \overline{S}$, then

$$\begin{aligned}
&\big\|\, |F(z)|^q - |F(z')|^q \,\big\|_{(X_0+X_1)^q} \\
&= \Big\|\, \big||F(z)|^q - |F(z')|^q\big|^{1/q} \,\Big\|^q_{(X_0+X_1)} \\
&\leq \big\|\, \big||F(z)| - |F(z')|\big| \,\big\|^q_{(X_0+X_1)} \qquad \text{(Hölder-continuity of } x \mapsto x^q) \\
&\leq \big\|\, |F(z) - F(z')| \,\big\|^q_{(X_0+X_1)} \\
&= \|\, F(z) - F(z') \,\|^q_{(X_0+X_1)}
\end{aligned}$$

  so that the continuity of $z \mapsto F(z)$ implies the continuity of $\varphi_A(|F(\,\cdot\,)|^q)$.

- Bounded extendability: For all $z \in \overline{S}$ we obtain by boundedness of φ_A that

$$\begin{aligned}
\varphi_A(|F(z)|) &\leq \|\, \chi_A \,\|'_{(X_0+X_1)^q} \,\|\, |F(z)|^q \,\|_{(X_0+X_1)^q} \\
&= \|\, \chi_A \,\|'_{(X_0+X_1)^q} \,\|\, |F(z)| \,\|^q_{X_0+X_1} \\
&\leq \|\, \chi_A \,\|'_{(X_0+X_1)^q} \,\|\, F \,\|^q_{\mathcal{F}}\,.
\end{aligned}$$

  (Here, the meaning of Remark 3.20 comes through.)

Step 2: We can now relate $|F|^q$ to its boundary values and reproduce an estimate similar to (7.7).

With the properties of $\varphi_A(|F(\,\cdot\,)|^q)$ being established, Corollary 4.12 yields

$$\varphi_A(|F(\theta)|^q) \leq \int_{\mathbb{R}} P_0(\theta, \mathrm{i}t)\, \varphi_A(|F(\mathrm{i}t)|^q)\, \mathrm{d}t + \int_{\mathbb{R}} P_1(\theta, 1+\mathrm{i}t)\, \varphi_A(|F(1+\mathrm{i}t)|^q)\, \mathrm{d}t.$$

By Lemma 7.6 (ii), we obtain $\int_{\mathbb{R}} P_j(\theta, j+\mathrm{i}t)\, |F(\mathrm{i}t)|^q \,\mathrm{d}t \in (X_j)^q$ and because $\varphi_A \in ((X_0+X_1)^q)^* \subseteq ((X_j)^q)^*$ for $j = 0, 1$, Lemma 7.6 (i) and (ii) ensure that

$$\begin{aligned}
\varphi_A(|F(\theta)|^q) &\leq \varphi_A\left(\int_{\mathbb{R}} P_0(\theta, \mathrm{i}t)\, |F(\mathrm{i}t)|^q \,\mathrm{d}t\right) + \\
&\qquad \varphi_A\left(\int_{\mathbb{R}} P_1(\theta, 1+\mathrm{i}t)\, |F(1+\mathrm{i}t)|^q \,\mathrm{d}t\right) \\
&= \varphi_A\bigg(\int_{\mathbb{R}} P_0(\theta, \mathrm{i}t)\, |F(\mathrm{i}t)|^q \,\mathrm{d}t +
\end{aligned}$$

$$\int_{\mathbb{R}} P_1(\theta, 1+\mathrm{i}t)\, |F(1+\mathrm{i}t)|^q \Big).$$

Testing with Corollary 3.19 yields

$$|F(\theta)|^q \leq \int_{\mathbb{R}} P_0(\theta, \mathrm{i}t)\, |F(\mathrm{i}t)|^q \,\mathrm{d}t + \int_{\mathbb{R}} P_1(\theta, 1+\mathrm{i}t)\, |F(1+\mathrm{i}t)|^q \,\mathrm{d}t. \tag{7.10}$$

Now the right-hand side needs to be turned into a product in order to make $F(\theta)$ compatible with the structure of $X_0^{1-\theta} X_1^{\theta}$. To this end, we define

$$f_0(q) := \left(\frac{1}{1-\theta} \int_{\mathbb{R}} P_0(\theta, \mathrm{i}t)\, |F(\mathrm{i}t)|^q \,\mathrm{d}t \right)^{1/q},$$
$$f_1(q) := \left(\frac{1}{\theta} \int_{\mathbb{R}} P_1(\theta, 1+\mathrm{i}t)\, |F(1+\mathrm{i}t)|^q \,\mathrm{d}t \right)^{1/q}.$$

We have $f_j(p) \in X_j^p$ by definition of Riemann integrals and if $C_j^{(p)}$ denotes the constant that comes from passing to the equivalent norm on X_j^p from the proof of Proposition 3.23, we obtain

$$\| f_j(p) \|_j = \left(\| f_j(p)^p \|_{(X_j)^p} \right)^{1/p} \leq \left(C_j^{(p)} \right)^{2/p} \| F \|_{\mathcal{F}}$$

by the validity of the triangle inequality for $(X_j)^p$-valued integrals. We now want to relate $f_j(q)$ to $f_j(p)$ in order to pull q out of the integral and observe that by Theorem 5.19 (i) we can choose subsubsequences (which we still denote w.r.t to n as otherwise denoting them w.r.t. n_{k_l} becomes a mess), s.t.

$$f_j(q)^q = \lim_n \sum_{j=0}^{N(n)} P_0(\theta, \mathrm{i}t_j^{(n)}) \left| F(\mathrm{i}t_j^{(n)}) \right|^q (x_{j+1}^{(n)} - x_j^{(n)})$$
$$f_j(p)^p = \lim_n \sum_{j=0}^{N(n)} P_0(\theta, \mathrm{i}t_j^{(n)}) \left| F(\mathrm{i}t_j^{(n)}) \right|^p (x_{j+1}^{(n)} - x_j^{(n)})$$

holds a.e. for the same choice of appropriate tagged partitions of $[-n, n]$ induced by $(x_j^{(n)})_{0 \leq j \leq N(n)}$ and $t_j^{(n)} \in [x_j^{(n)}, x_{j+1}^{(n)}]$. Notice that $\int_{\mathbb{R}} P_0(\theta, \mathrm{i}t)\,\mathrm{d}t$ may be rewritten as a limit of Riemann sums w.r.t to the above tagged partitions aswell. Because $c(x) := x^{p/q} \chi_{[x \geq 0]}$ is a convex function on $\mathbb{R}$, Lemma 7.7 guarantees that for almost every $\omega \in \Omega$ there is a function l_ω s.t. $l_\omega([f_j(q)^q](\omega)) = c_\omega([f_j(q)^q](\omega))$, $l_\omega(x) = a_\omega x + b_\omega$ and $l_\omega(x) \leq c(x)$ for all $x \in \mathbb{R}$. Notice that the latter part is uniform in ω. Then we obtain a.e. that

$$
\begin{aligned}
&\left(\frac{1}{1-\theta}\int_{\mathbb{R}} P_0(\theta,\mathrm{i}t)\,|F(\mathrm{i}t)|^q\,\mathrm{d}t\right)^{p/q}\\
&= c(f_j(q)^q)\\
&= l(f_j(q)^q)\\
&= a\,f_j(q)^q + b\\
&= a\,\frac{1}{1-\theta}\int_{\mathbb{R}} P_0(\theta,\mathrm{i}t)\,|F(\mathrm{i}t)|^q\,\mathrm{d}t + \frac{1}{1-\theta}\int_{\mathbb{R}} P_0(\theta,\mathrm{i}t)\,b\,\mathrm{d}t\\
&= \lim_n \sum_{j=0}^{N(n)} P_0(\theta,\mathrm{i}t_j^{(n)})\left[a\left|F(\mathrm{i}t_j^{(n)})\right|^q + b\right](x_{j+1}^{(n)} - x_j^{(n)}) \qquad (7.11)\\
&\le \lim_n \sum_{j=0}^{N(n)} P_0(\theta,\mathrm{i}t_j^{(n)})\left|F(\mathrm{i}t_j^{(n)})\right|^p (x_{j+1}^{(n)} - x_j^{(n)})\\
&= \frac{1}{1-\theta}\int_{\mathbb{R}} P_0(\theta,\mathrm{i}t)\,|F(\mathrm{i}t)|^p\,\mathrm{d}t\\
&= f_j(p)^p.
\end{aligned}
$$

Notice that rewriting w.r.t. Riemann sums in (7.11) is needed, as it is not clear beforehand whether $\int_{\mathbb{R}} P_0(\theta,\mathrm{i}t)\,|F(\mathrm{i}t)|^q\,\mathrm{d}t$ can be represented pointwise a.e. as a Riemann integral and should be compatible with $\int_{\mathbb{R}} P_0(\theta,\mathrm{i}t)\,b\,\mathrm{d}t$. Nonetheless, a similar estimate holds for $f_1(q)$ and $f_1(p)$, so that (7.10) turns into

$$|F(\theta)|^q \le (1-\theta)f_0(p)^q + \theta f_1(p)^q,$$

which after abbreviating $f_j \coloneqq f_j(p)$ is equivalent to

$$|F(\theta)| \le [(1-\theta){f_0}^q + \theta {f_1}^q]^{1/q} = \exp\left(\frac{\ln[(1-\theta){f_0}^q + \theta {f_1}^q]}{q}\right).$$

Using L'Hôpital's rule, we finally obtain

$$
\begin{aligned}
|F(\theta)| &\le \lim_{q\to 0}\exp\left(\frac{\ln[(1-\theta){f_0}^q + \theta {f_1}^q]}{q}\right)\\
&= \lim_{q\to 0}\exp\left(\frac{(1-\theta)\ln(f_0){f_0}^q + \theta\ln(f_1){f_1}^q}{(1-\theta){f_0}^q + \theta {f_1}^q}\right)\\
&= \exp\left((1-\theta)\ln(f_0) + \theta\ln(f_1)\right)\\
&= f_0^{1-\theta} f_1^{\theta}.
\end{aligned}
$$

This shows that $F(\theta) \in X_0^{1-\theta} X_1^\theta$ with

$$\| F(\theta) \|_{X_0^{1-\theta} X_1^\theta} \leq \| f_0 \|_0^{1-\theta} \| f_1 \|_1^\theta \leq \left(C_0^{(p)} \right)^{2/p} \left(C_1^{(p)} \right)^{2/p} \| F \|_{\mathcal{F}} . \qquad \square$$

Remark 7.8. Here, the isometry gets lost because of the usage of p-convexity when estimating $\| f_j \|_j$. In the Banach case, p-convexity isn't needed as every Banach function space is 1-convex and we may after arriving at (7.10) for $q = 1$ refer to the methods that lead to (7.7) to obtain $X_0^{1-\theta} X_1^\theta \supseteq [X_0, X_1]_\theta$ with $\| f \|_{X_0^{1-\theta} X_1^\theta} \leq \| f \|_\theta$.

7.4 Identification without any separability assumption

In the proof of $X_0^{1-\theta} X_1^\theta \subseteq [X_0, X_1]_\theta$, the crucial part is the one where order-continuity of $X_0^{1-\theta} X_1^\theta$ guarantees that pointwise a.e. converging cutoffs also converge in $X_0^{1-\theta} X_1^\theta$, which then guarantees that the continuous inclusion on the set of cutoffs can be extended. More generally, there should be some notion of closedness w.r.t. pointwise convergence. One possible requirement that pops up in the literature is

Definition 7.9. Let X be a quasi-Banach function space. X satisfies the *Fatou property*, if for all $f \in \mathrm{L}_0(\Omega)$ and sequences $(f_n)_n \subseteq X$ with $0 \leq f_n \nearrow f$ a.e. and $\sup_n \| f_n \| < \infty$ it follows that $f \in X$ with $\| f \| = \lim_{n \to \infty} \| f_n \|$.

This gives another version of Theorem 7.1 that does not need any separability assumption at the cost of the Fatou property.

Corollary 7.10. *Let $\theta \in (0, 1)$ and let X_0, X_1 be p-convex quasi-Banach function spaces over a σ-finite measure space Ω and assume that $[X_0, X_1]_\theta$ satisfies the Fatou property. Then $X_0 + X_1$ is p-convex and the spaces $[X_0, X_1]_\theta$ and $X_0^{1-\theta} X_1^\theta$ agree up to equivalence of quasi-norms.*

Proof. The proof of this version is basically the Corollary in [KPS82, p. 242].

Separability was only used in Step 2 of $[X_0, X_1]_\theta \supseteq X_0^{1-\theta} X_1^\theta$ in order to show that $\mathcal{D}$ is dense in $X_0^{1-\theta} X_1^\theta$. If $[X_0, X_1]_\theta$ satisfies the Fatou property, we can manage this inclusion with less.

Let $f \in X_0^{1-\theta} X_1^\theta$ and $\varepsilon > 0$. Then there exist $f_i \in X_i$ s.t.

$$|f| = |f_0|^{1-\theta} |f_1|^\theta, \quad \| f_0 \|_0^{1-\theta} \| f_1 \|_1^\theta \leq (1+\varepsilon) \| f \|_{X_0^{1-\theta} X_1^\theta}.$$

For $n \in \mathbb{N}$ set $E_n := [1/n \leq |f_0|, |f_1| \leq n]$ and define $f_n := f\chi_{E_n}$. The map

$$F : S \to X_0 + X_1, \qquad z \mapsto \left| \frac{f_0 \chi_{E_n}}{\| f_0 \chi_{E_n} \|_0} \right|^{1-z} \left| \frac{f_1 \chi_{E_n}}{\| f_1 \chi_{E_n} \|_1} \right|^z$$

is admissible by Proposition 7.2 and so $\| f_0 \chi_{E_n} \|_0^{1-\theta} \| f_1 \chi_{E_n} \|_1^\theta F(\theta) = |f_n| \in [X_0, X_1]_\theta$ with

$$\begin{aligned}
\| f_n \|_\theta &= \| f_0 \chi_{E_n} \|_0^{1-\theta} \| f_1 \chi_{E_n} \|_1^\theta \left\| \left| \frac{f_0 \chi_{E_n}}{\| f_0 \chi_{E_n} \|_0} \right|^{1-\theta} \left| \frac{f_1 \chi_{E_n}}{\| f_1 \chi_{E_n} \|_1} \right|^\theta \right\|_\theta \\
&\leq \| f_0 \chi_{E_n} \|_0^{1-\theta} \| f_1 \chi_{E_n} \|_1^\theta \| F \|_{\mathcal{F}} \\
&\leq 2 \| f_0 \|_0^{1-\theta} \| f_1 \|_1^\theta \\
&\leq 2(1+\varepsilon) \| f \|_{X_0^{1-\theta} X_1^\theta}.
\end{aligned}$$

Then we have $0 \leq |f_n| \nearrow |f|$, $|f| \in \mathrm{L}_0(\Omega)$ with $\sup_n \| |f_n| \|_\theta < \infty$. Now the Fatou property of $[X_0, X_1]_\theta$ implies that $f \in [X_0, X_1]_\theta$ and

$$\| f \|_\theta = \lim_{n \to \infty} \| f_n \|_\theta \leq 2(1+\varepsilon) \| f \|_{X_0^{1-\theta} X_1^\theta}.$$

The remainder of this proof now works as the remaining parts of the proof of Theorem 7.1. $\square$

Remark 7.11.

(i) We can weaken the assumptions on Ω as well, since the requirement that Ω is a separable metric space was only needed when proving the equivalence of separability and order-continuity in Theorem 3.8, which isn't needed in this version. We still need σ-finiteness, because the characterization of saturatedness in Proposition 3.15 needs it and $[X_0, X_1]_\theta \subseteq X_0^{1-\theta} X_1^\theta$ does not work without the latter.

(ii) We again end up without an isometry for this inclusion in the quasi-Banach case, because we didn't use the maximum principle and instead the bound from (7.1). In the Banach case, it then only remains to let $\varepsilon \to 0$ to retreive the isometry.

Of course, the value of the above theorem depends heavily on how difficult it is to establish a Fatou property for an interpolation space that is only determined through abstract analytic functions.

8 Wolff reiteration

By now, three of our identified gaps are closed and as a finishing touch, the Wolff reiteration theorem remains. For completeness, we start by stating its Banach space version.

Theorem 8.1 (Wolff reitaration)**.** *Let X_0, X_1, X_2, X_3 be Banach spaces that continuously embed into some topological vector space Z with $X_0 \cap X_3$ being dense in X_1 and X_2. Further, let $\theta, \eta, \lambda, \mu \in [0, 1]$ be parameters subject to the following conditions:*

(i) $0 < \theta < \eta < 1$.

(ii) $\theta = \lambda\eta$.

(iii) $\eta = (1-\mu)\theta + \mu$ *(or equivalently* $1 - \eta = (1-\mu)(1-\theta)$*).*

If $X_1 = [X_0, X_2]_\lambda$ *and* $X_2 = [X_1, X_3]_\mu$*, then* $X_1 = [X_0, X_3]_\theta$ *and* $X_2 = [X_0, X_3]_\eta$.

Proof. See [Wol82, Theorem 2] for a proof. □

Aside from the more general identification of Calderón products and interpolation spaces, all other hard work is already done as there is a Wolff reiteration theorem for quasi-Banach function spaces.

Theorem 8.2. *Let X_0, X_1, X_2, X_3 be quasi-Banach function spaces over Ω with $X_0 \cap X_3$ being dense in X_1 and X_2. Further, let $\theta, \eta, \lambda, \mu \in [0, 1]$ be parameters subject to the following conditions:*

(i) $0 < \theta < \eta < 1$.

(ii) $\theta = \lambda\eta$.

(iii) $\eta = (1-\mu)\theta + \mu$ (or equivalently $1-\eta = (1-\mu)(1-\theta)$).

If $X_1 = X_0^{1-\lambda} X_2^{\lambda}$ and $X_2 = X_1^{1-\mu} X_3^{\mu}$, then $X_1 = X_0^{1-\theta} X_3^{\theta}$ and $X_2 = X_0^{1-\eta} X_3^{\eta}$.

Proof. See [GM89, Theorem 4.13] for a proof. □

With this out of the way, we can prove a Wolff reiteration theorem for the complex interpolation method for quasi-Banach function spaces, when only one of the endpoint spaces is separable.

Theorem 8.3. *Let X_0, X_1, X_2, X_3 be p-convex quasi-Banach function spaces over Ω that satisfy the Fatou-property, where either X_0 or X_3 is separable with $X_0 \cap X_3$ being dense in X_1 and X_2. Further, let $\theta, \eta, \lambda, \mu \in [0,1]$ be parameters subject to the following conditions:*

(i) $0 < \theta < \eta < 1$.

(ii) $\theta = \lambda\eta$.

(iii) $\eta = (1-\mu)\theta + \mu$ (or equivalently $1-\eta = (1-\mu)(1-\theta)$).

If $X_1 = [X_0, X_2]_\lambda$ and $X_2 = [X_1, X_3]_\mu$, then up to equivalence of quasi-norms $X_1 = [X_0, X_3]_\theta$ and $X_2 = [X_0, X_3]_\eta$.

Proof. We only need to notice that in either case of X_0 or X_3 being separable, we may calculate X_1 and X_2 by Calderón products (Theorem 7.1):
If X_0 is separable, then X_1 is separable as well with $X_1 = [X_0, X_2]_\lambda = X_0^{1-\lambda} X_2^{\lambda}$. Since $X_2 = [X_1, X_3]_\mu$, X_1's now established separability yields X_2's separability and $X_2 = X_1^{1-\mu} X_3^{\mu}$. The same applies when X_3 is separable. Now Wolff reiteration for Calderón products (Theorem 8.2) applies and after using Theorem 7.1 again to convert Calderón products back to interpolation spaces, the claim is shown. □

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