



Effective Mass of the Polaron: A Lower Bound

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Abstract: We show that the effective mass of the Fröhlich Polaron is bounded below by $c\alpha^{2/5}$ for some constant $c > 0$ and for all coupling constants α . The proof uses the point process representation of the path measure of the Fröhlich Polaron.

1. Introduction and Results

The Polaron models the slow movement of an electron in a polar crystal. The electron drags a cloud of polarization along and thus appears heavier, it has an “effective mass” larger than its mass without the interaction. In the Fröhlich model of the Polaron, the Hamiltonian describing the interaction of the electron with the lattice vibrations is the operator given by

$$H_\alpha = \frac{1}{2}p^2 \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{N} + \frac{\sqrt{\alpha}}{\sqrt{2\pi}} \int_{\mathbb{R}^3} \frac{1}{|k|} (e^{ik \cdot x} a_k + e^{-ik \cdot x} a_k^*) dk$$

that acts on $L^2(\mathbb{R}^3) \otimes \mathcal{F}$ (where \mathcal{F} is the bosonic Fock space over $L^2(\mathbb{R}^3)$). Here x and p are the position and momentum operators of the electron, $\mathbf{N} \equiv \int_{\mathbb{R}^3} a_k^* a_k dk$ is the number operator, the creation and annihilation operators a_k^* and a_k satisfy the canonical commutation relations $[a_k^*, a_{k'}] = \delta(k - k')$ and $\alpha > 0$ is the coupling constant. The Hamiltonian commutes with the total momentum operator $p \otimes \mathbb{1} + \mathbb{1} \otimes P_f$, where $P_f \equiv \int_{\mathbb{R}^3} k a_k^* a_k dk$ is the momentum operator of the field, and has a fiber decomposition in terms of the Hamiltonians

$$H_\alpha(P) = \frac{1}{2}(P - P_f)^2 + \mathbf{N} + \frac{\sqrt{\alpha}}{\sqrt{2\pi}} \int_{\mathbb{R}^3} \frac{1}{|k|} (a_k + a_k^*) dk$$

at fixed total momentum P (which act on \mathcal{F}). Of particular interest is the energy-momentum relation $E_\alpha(P) := \inf \text{spec}(H_\alpha(P))$. It is known that E_α is rotationally

symmetric and has a global minimum at $P = 0$, which is believed to be unique (DS20). The effective mass $m_{\text{eff}}(\alpha)$ is defined as the inverse curvature at $P = 0$ i.e.

$$E_\alpha(P) - E_\alpha(0) = \frac{1}{2m_{\text{eff}}(\alpha)} |P|^2 + o(|P|^2).$$

While the asymptotics of the ground state energy $E_\alpha(0)$ in the strong coupling limit $\alpha \rightarrow \infty$ are known, the asymptotics of the effective mass $m_{\text{eff}}(\alpha)$ still remain an important open problem. For the ground state energy, Donsker and Varadhan (DV83) gave a probabilistic proof (using the path integral formulation) of Pekars conjecture (Pek49) that states

$$\lim_{\alpha \rightarrow \infty} \frac{E_\alpha(0)}{\alpha^2} = \min_{\psi \in H^1, \|\psi\|_{L^2}=1} \left[\|\nabla \psi\|_{L^2}^2 - \sqrt{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} dx dy \frac{|\psi(x)\psi(y)|^2}{|x-y|} \right]. \quad (1.1)$$

There also exists a functional analytic proof by Lieb and Thomas (LT97), that additionally contains explicit error bounds. Formal computations by Landau and Pekar (LP48) suggest that

$$\lim_{\alpha \rightarrow \infty} \frac{m_{\text{eff}}(\alpha)}{\alpha^4} = \frac{16\sqrt{2}\pi}{3} \int_{\mathbb{R}^3} dx |\psi(x)|^4, \quad \psi \text{ minimizer of (1.1),} \quad (1.2)$$

and Feynman (Fey55) obtains the same α^4 -asymptotics (but not the prefactor) by completely different methods. Rigorous results on the effective mass have been obtained only much later and they are significantly weaker: Lieb and Seiringer (LS20) show that $\lim_{\alpha \rightarrow \infty} m_{\text{eff}}(\alpha) = \infty$ by estimating

$$\frac{1}{2m_{\text{eff}}(\alpha)} \leq \lim_{P \rightarrow 0} \frac{1}{|P|^2} (\langle \phi_{\alpha,P}, H_\alpha(P) \phi_{\alpha,P} \rangle - E_\alpha(0))$$

with suitably chosen trial states $\phi_{\alpha,P}$. For models with stronger regularity assumptions (excluding the Fröhlich Polaron), quantitative estimates on the effective mass were recently obtained in (MS21). Our goal is to show that there exists some constant $c > 0$ such that for all $\alpha > 0$

$$m_{\text{eff}}(\alpha) \geq c\alpha^{2/5} \quad (1.3)$$

and thereby giving a first quantitative growth bound for the effective mass of the Fröhlich Polaron.

We give a short introduction into the probabilistic representation of the effective mass. We refer the reader to (DS20) for details in this representation, and to (Moe06) for a review covering functional analytic properties of the Polaron. An application of the Feynman-Kac formula to the semigroup $(e^{-T H_\alpha})_{T \geq 0}$ leads to the path measure

$$\mathbb{P}_{\alpha,T}(\mathbf{dx}) = \frac{1}{Z_{\alpha,T}} \exp \left(\frac{\alpha}{2} \int_0^T \int_0^T \frac{e^{-|t-s|}}{|\mathbf{x}_t - \mathbf{x}_s|} ds dt \right) \mathcal{W}(\mathbf{dx}) \quad (1.4)$$

in finite volume $T > 0$, where \mathcal{W} is the distribution of three dimensional Brownian motion and the partition function $Z_{\alpha,T}$ is a normalization constant. In particular, the partition function can be expressed as the matrix element $Z_{\alpha,T} = \langle \Omega, e^{-T H_\alpha(0)} \Omega \rangle$, where Ω is the Fock vacuum. This leads to $E_\alpha(0) = -\lim_{T \rightarrow \infty} \log(Z_{\alpha,T})/T$, which was applied (up to boundary conditions) in the aforementioned proof of Pekars conjecture by Donsker and Varadhan. Spohn conjectured (Spo87) that the path measure (in infinite

volume) converges under diffusive rescaling to Brownian motion with some diffusion constant $\sigma^2(\alpha)$, and showed that the effective mass then has the representation

$$m_{\text{eff}}(\alpha) = (\sigma^2(\alpha))^{-1}.$$

The existence of the infinite volume measure and the validity of the central limit theorem were shown by Mukherjee and Varadhan (MV19) for a restricted range of coupling parameters. The proof relies on a representation of the measures (1.4) as a mixture of Gaussian measures, the mixing measure being the distribution of a point process on $\{(s, t) \in \mathbb{R}^2 : s < t\} \times (0, \infty)$ which has a renewal structure. This approach was extended in (BP21) to a broader class of path measures and a functional central limit theorem, and, for the Fröhlich Polaron, to all coupling parameters (relying on known spectral properties of $H_\alpha(0)$). We also refer the reader to (MV21), (BMPV22), where proofs are given that do not need the spectral properties of $H_\alpha(0)$. We use this point process representation and derive a “variational like” formula for the effective mass (see Proposition 3.2). We obtain the estimate (1.3) by minimizing over sub-processes of the full point process.

Without additional effort, we can allow for a bit more generality and look at the probability measures $\mathbb{P}_{\alpha,T}$ on $C([0, \infty), \mathbb{R}^d)$ defined by

$$\mathbb{P}_{\alpha,T}(\mathbf{dx}) = \frac{1}{Z_{\alpha,T}} \exp\left(\frac{\alpha}{2} \int_0^T \int_0^T g(t-s)v(\mathbf{x}_{s,t}) \, ds dt\right) \mathcal{W}(\mathbf{dx}) \tag{1.5}$$

where $d \geq 2$, $v(x) = 1/|x|^\gamma$ with $\gamma \in [1, 2)$ and $g : [0, \infty) \rightarrow (0, \infty)$ is a probability density with finite first moment satisfying $\sup_{t \geq 0} (1+t)g(t) < \infty$ and $\mathbf{x}_{s,t} := \mathbf{x}_t - \mathbf{x}_s$ for $\mathbf{x} \in C([0, \infty), \mathbb{R}^d)$ and $s, t \geq 0$. By the results in (BMPV22), these conditions are sufficient for the existence of an infinite volume measure and the validity of a functional central limit theorem in infinite volume. By the proof of the central limit theorem given in (MV19)¹ the respective diffusion constant $\sigma^2(\alpha)$ can also be obtained by taking the “diagonal limit”, that is

$$\sigma^2(\alpha) = \lim_{T \rightarrow \infty} \frac{1}{dT} \mathbb{E}_{\alpha,T}[|X_{0,T}|^2].$$

Here $X_t(\mathbf{x}) := \mathbf{x}_t$ for any $\mathbf{x} \in C([0, \infty), \mathbb{R}^d)$ and $t \geq 0$, that is we denote by X the random variable that is given by the identity on $C([0, \infty), \mathbb{R}^d)$.

Theorem 1.1. *There exists a constant $C > 0$ (depending on γ , g and d) such that*

$$\sigma^2(\alpha) \leq C\alpha^{-2/(2+d)}$$

for all $\alpha > 0$. Consequently, there exists a constant $c > 0$ such that the effective mass of the Fröhlich Polaron satisfies $m_{\text{eff}}(\alpha) \geq c\alpha^{2/5}$ for all $\alpha > 0$.

As we will point out in Remark 2.1, the results obtained in (BMPV22) imply for the Fröhlich Polaron with minor additional effort the existence of some $\tilde{c} > 0$ such that $\sigma^2(\alpha) \geq e^{-\tilde{c}\alpha^2}$ for reasonably large α .

¹ It is generalizable to the measures above, see Remark 4.14 in (BP21).

2. Point Process Representation

By using the identity $\frac{1}{|x|} = \sqrt{\frac{2}{\pi}} \int_0^\infty du e^{-u^2|x|^2/2}$ and expanding the exponential into a series, Mukherjee and Varadhan (MV19) represented the path measure of the Fröhlich Polaron as a mixture of Gaussian measures $\mathbf{P}_{\xi,u}$, the mixing measure $\widehat{\Theta}_{\alpha,T}$ being the distribution of a suitable point process on $\{(s, t) \in \mathbb{R}^2 : s < t\} \times [0, \infty)$. In the following, we give an introduction to the point process representation in the form given in (BP21). We will additionally use the invariance of (1.5) under replacing v by $v_\varepsilon(x) := v(x) + \varepsilon$ for some $\varepsilon \geq 0$. While this transformation does not change the measure $\mathbb{P}_{\alpha,T}$, we will obtain different point process representations depending on the choice of ε .

Let $\Gamma_{\alpha,T}$ be the distribution of a Poisson point process on $\Delta := \{(s, t) \in \mathbb{R}^2 : s < t\}$ with intensity measure

$$\mu_{\alpha,T}(dsdt) := \alpha g(t - s) \mathbb{1}_{\{0 \leq s < t \leq T\}} dsdt.$$

By expanding the exponential into a series and interchanging the order of summation and integration, we obtain for any $A \in \mathcal{B}(C([0, \infty), \mathbb{R}^d))$ and any $\varepsilon \geq 0$

$$\begin{aligned} \mathbb{P}_{\alpha,T}(A) &= \frac{1}{Z_{\alpha,T}^\varepsilon} \int_A \mathcal{W}(d\mathbf{x}) \exp\left(\alpha \iint_{0 \leq s < t \leq T} dsdt g(t - s) v_\varepsilon(\mathbf{x}_{s,t})\right) \\ &= \frac{1}{Z_{\alpha,T}^\varepsilon} \sum_{n=0}^\infty \frac{1}{n!} \int_{\Delta^n} \mu_{\alpha,T}^{\otimes n}(ds_1 dt_1, \dots, ds_n dt_n) \int_A \mathcal{W}(d\mathbf{x}) \prod_{i=1}^n v_\varepsilon(\mathbf{x}_{s_i, t_i}) \\ &= \frac{e^{c_{\alpha,T}}}{Z_{\alpha,T}^\varepsilon} \int_{\mathbf{N}_f(\Delta)} \Gamma_{\alpha,T}(d\xi) \int_A \mathcal{W}(d\mathbf{x}) \prod_{(s,t) \in \text{supp}(\xi)} v_\varepsilon(\mathbf{x}_{s,t}) \end{aligned} \tag{2.1}$$

where $\mathbf{N}_f(\Delta)$ is the set of finite integer valued measures on Δ . We will view $\xi = \sum_{i=1}^n \delta_{(s_i, t_i)} \in \mathbf{N}_f(\Delta)$ as a collection of intervals $\{(s_i, t_i) : 1 \leq i \leq n\}$.

The measure $\Gamma_{\alpha,T}$ can be interpreted in the following way: Consider a $M/G/\infty$ -queue started empty at time 0 where the arrival process $\sum_{n=0}^\infty \delta_{s_n}$ is a homogeneous Poisson process with rate α , and the service times $(\tau_n)_n$ are iid with density g and are independent of the arrival process. Consider the process $\eta = \sum_{n=1}^\infty \delta_{(s_n, t_n)}$ where $t_n := s_n + \tau_n$ is for $n \in \mathbb{N}$ the departure of customer n . Then the process of all customers in η that arrive and depart before T has distribution $\Gamma_{\alpha,T}$.

For $\xi \in \mathbf{N}_f(\Delta)$ we define

$$F_\varepsilon(\xi) := \mathbb{E}_{\mathcal{W}} \left[\prod_{(s,t) \in \text{supp}(\xi)} v_\varepsilon(X_{s,t}) \right]$$

and the perturbed point process $\widehat{\Gamma}_{\alpha,T}^\varepsilon$ by

$$\widehat{\Gamma}_{\alpha,T}^\varepsilon(d\xi) := \frac{e^{c_{\alpha,T}}}{Z_{\alpha,T}^\varepsilon} F_\varepsilon(\xi) \Gamma_{\alpha,T}(d\xi).$$

Then Eq. (2.1) yields the representation $\mathbb{P}_{\alpha,T}(\cdot) = \int \widehat{\Gamma}_{\alpha,T}^\varepsilon(d\xi) \mathbf{P}_\xi^\varepsilon(\cdot)$ of $\mathbb{P}_{\alpha,T}$ as a mixture of the perturbed path measures

$$\mathbf{P}_\xi^\varepsilon(d\mathbf{x}) := \frac{1}{F_\varepsilon(\xi)} \prod_{(s,t) \in \text{supp}(\xi)} v_\varepsilon(\mathbf{x}_{s,t}) \mathcal{W}(d\mathbf{x}).$$

The function F_ε depends in an intricate manner on the configuration ξ , depending on number, length and relative position of the intervals. Nevertheless, the perturbed measure $\widehat{\Gamma}_{\alpha,T}^\varepsilon$ can still be interpreted as a queuing process by identifying $(s, t) \in \text{supp}(\xi)$ with a service interval $[s, t]$ as indicated above. Under the given assumptions on g and v , $\widehat{\Gamma}_{\alpha,T}^\varepsilon$ can then be expressed in terms of an iid sequence of “clusters” of overlapping intervals, separated by exponentially distributed dormant periods not covered by any interval. The processes of increments can be drawn independently along these dormant periods and clusters (according to the kernel $(\xi, A) \mapsto \mathbf{P}_\xi^\varepsilon(A)$). This yields the existence of the infinite volume limit as well as the functional central limit theorem (BP21), (BMPV22).

In order to make the connection to the representation used in (MV19), we notice that for all $x \in \mathbb{R}^d \setminus \{0\}$

$$v_\varepsilon(x) = 1/|x|^\gamma + \varepsilon = \int_{[0,\infty)} v_\varepsilon(du) e^{-u^2|x|^2/2}$$

where

$$v_\varepsilon(du) = \frac{2^{(2-\gamma)/2}}{\Gamma(\gamma/2)} u^{\gamma-1} du + \varepsilon \delta_0(du).$$

It will turn out to be useful to make the atoms of out point process distinguishable. We will abuse notation and identify probability measures on $\mathbf{N}_f(\Delta)$ with symmetric probability measures on $\Delta^\cup := \bigcup_{n=0}^\infty \Delta^n$. With Eq. (2.1), we obtain

$$\mathbb{P}_{\alpha,T}(A) = \frac{e^{c_{\alpha,T}}}{Z_{\alpha,T}^\varepsilon} \int_{\Delta^\cup} \Gamma_{\alpha,T}(d\xi) \int_{[0,\infty)^{N(\xi)}} v_\varepsilon^{\otimes N(\xi)}(du) \int_A \mathcal{W}(d\mathbf{x}) e^{-\sum_{i=1}^{N(\xi)} u_i^2 |\mathbf{x}_{s_i,t_i}|^2/2}$$

where $N := \sum_{n=0}^\infty n \cdot \mathbb{1}_{\Delta^n}$. We define for $\xi = ((s_i, t_i))_{1 \leq i \leq N(\xi)} \in \Delta^\cup$ and $u \in [0, \infty)^{N(\xi)}$ the Gaussian measures

$$\mathbf{P}_{\xi,u}(d\mathbf{x}) := \frac{1}{\phi(\xi, u)} e^{-\sum_{i=1}^{N(\xi)} u_i^2 |\mathbf{x}_{s_i,t_i}|^2/2} \mathcal{W}(d\mathbf{x}) \quad (2.2)$$

where $\phi(\xi, u) := \mathbb{E}_{\mathcal{W}}[e^{-\sum_{i=1}^{N(\xi)} u_i^2 |\mathbf{x}_{s_i,t_i}|^2/2}]$ and, for $\xi \in \Delta^\cup$ with² $F_\varepsilon(\xi) < \infty$ the measures

$$\kappa_\varepsilon(\xi, du) := \frac{\phi(\xi, u)}{F_\varepsilon(\xi)} v_\varepsilon^{\otimes N(\xi)}(du)$$

and obtain

$$\mathbb{P}_{\alpha,T}(\cdot) = \int_{\Delta^\cup} \widehat{\Gamma}_{\alpha,T}^\varepsilon(d\xi) \int_{[0,\infty)^{N(\xi)}} \kappa_\varepsilon(\xi, du) \mathbf{P}_{\xi,u}(\cdot). \quad (2.3)$$

We will view $(s, t, u) \in \Delta \times [0, \infty)$ as an interval $[s, t]$ equipped with the mark u . In contrast to the representation in (MV19), we draw a sample of the mixing measure $\widehat{\Theta}_{\alpha,T}$ (which is the distribution of a point process on $\Delta \times [0, \infty)$) in two steps: First, draw a sample ξ according to $\widehat{\Gamma}_{\alpha,T}^0$ and then draw marks u according to the kernel $\kappa_0(\xi, \cdot)$. This added flexibility will be useful later. If we define

$$\sigma_T^2(\xi, u) := \frac{1}{dT} \mathbb{E}_{\mathbf{P}_{\xi,u}}[|X_{0,T}|^2]$$

² By finiteness of the partition function, we have $F_\varepsilon(\xi) < \infty$ for $\Gamma_{\alpha,T}$ almost all ξ .

we get with the previous considerations

$$\frac{1}{dT} \mathbb{E}_{\alpha,T} [|X_{0,T}|^2] = \int_{\Delta \cup [0, \infty)^{N(\xi)}} \widehat{\Gamma}_{\alpha,T}^\varepsilon(d\xi) \int_{[0, \infty)^{N(\xi)}} \kappa_\varepsilon(\xi, du) \sigma_T^2(\xi, u). \tag{2.4}$$

Taking the limit $T \rightarrow \infty$ in (2.4) then yields the diffusion constant i.e. the inverse of the effective mass.

One can also express the perturbed measure $\widehat{\Gamma}_{\alpha,T}^\varepsilon$ in terms of the path measures $\mathbb{P}_{\alpha,T}$ (a similar calculation was already made in (MV21, (4.18))). The Laplace transform of $\widehat{\Gamma}_{\alpha,T}^\varepsilon$ evaluated at the measurable function $f : \Delta \rightarrow [0, \infty)$ is

$$\begin{aligned} \int_{N_f(\Delta)} \widehat{\Gamma}_{\alpha,T}^\varepsilon(d\xi) e^{-\int_\Delta f d\xi} &= \frac{1}{Z_{\alpha,T}^\varepsilon} \sum_{n=0}^\infty \frac{1}{n!} \int_{\Delta^n} \mu_{\alpha,T}^{\otimes n}(dsdt) \mathbb{E}_{\mathcal{W}} \left[\prod_{i=1}^n v_\varepsilon(X_{s_i,t_i}) e^{-f(s_i,t_i)} \right] \\ &= \frac{1}{Z_{\alpha,T}^\varepsilon} \mathbb{E}_{\mathcal{W}} \left[\sum_{n=0}^\infty \frac{1}{n!} \left(\int_\Delta \mu_{\alpha,T}(dsdt) v_\varepsilon(X_{s,t}) e^{-f(s,t)} \right)^n \right] \\ &= \frac{1}{Z_{\alpha,T}^\varepsilon} \mathbb{E}_{\mathcal{W}} \left[\exp \left(\int_\Delta \mu_{\alpha,T}(dsdt) v_\varepsilon(X_{s,t}) e^{-f(s,t)} \right) \right] \\ &= \mathbb{E}_{\alpha,T} \left[\exp \left(\int_\Delta \mu_{\alpha,T}(dsdt) v_\varepsilon(X_{s,t}) (e^{-f(s,t)} - 1) \right) \right]. \end{aligned}$$

That is, $\widehat{\Gamma}_{\alpha,T}^\varepsilon$ is the distribution of a Cox process³ with driving measure $\mu_{\alpha,T}(dsdt) v_\varepsilon(X_{s,t})$ with $(X_t)_t \sim \mathbb{P}_{\alpha,T}$.

The introduction of v_ε with $\varepsilon > 0$, which in the context of (1.5) is a simple shift of energy, thus creates a component of $\widehat{\Gamma}_{\alpha,T}^\varepsilon$ that is quite easy to understand. Indeed, $\widehat{\Gamma}_{\alpha,T}^\varepsilon$ is the distribution of the sum $\eta_{\alpha,T} + \eta_{\alpha,T}^\varepsilon$ where $\eta_{\alpha,T} \sim \widehat{\Gamma}_{\alpha,T}^0$ and where the Poisson point process $\eta_{\alpha,T}^\varepsilon \sim \Gamma_{\varepsilon,\alpha,T}$ is independent of $\eta_{\alpha,T}$. $\eta_{\alpha,T}^\varepsilon$ leads to additional intervals, but on the other hand, for given ξ , under the measure $\kappa_\varepsilon(\xi, \cdot)$ now u has components that are equal to zero with positive probability. Intervals of ξ corresponding to these components will be invisible to the measure $\mathbf{P}_{\xi,u}$ as can be seen directly from its definition (2.2). These two effects balance since $\mathbb{P}_{\alpha,T}$ is independent of ε . However, given a realization ξ of $\eta_{\alpha,T} + \eta_{\alpha,T}^\varepsilon$ it is not the case that $\kappa_\varepsilon(\xi, \cdot)$ just produces zero components of u precisely for those intervals coming from $\eta_{\alpha,T}^\varepsilon$; on the contrary, $\kappa_\varepsilon(\xi, \cdot)$ does not depend on whether any given interval in ξ was produced by $\eta_{\alpha,T}$ or by $\eta_{\alpha,T}^\varepsilon$. The proof of Theorem 1.1 now uses the following strategy:

- (1) We show in Lemma 3.1 that $u \mapsto \sigma_T^2(\xi, u)$ is decreasing along each coordinate axis. This in particular implies that $\sigma_T^2(\xi, u)$ increases if intervals are deleted from ξ , which corresponds to setting the suitable components of u to zero. While not necessary⁴ for the proof of Theorem 1.1, we also give a characterization of $\sigma_T^2(\xi, u)$ in terms of a L^2 -distance which emphasizes the variational type flavor of our approach and might be useful for further refining the method.

³ I.e. $\widehat{\Gamma}_{\alpha,T}^\varepsilon$ is the distribution of a point process that is conditionally on $(X_t)_t \sim \mathbb{P}_{\alpha,T}$ a Poisson point process with intensity measure $\mu_{\alpha,T}(dsdt) v_\varepsilon(X_{s,t})$.

⁴ Lemma 3.3 later in the text can also be shown directly by using independence of Brownian increments.

- (2) The dependence of the distribution $\kappa_\varepsilon(\xi, \cdot)$ on the whole configuration ξ prevents us from simply deleting intervals from ξ in order to obtain an estimate on the variance in (2.4). To get around this, we fix some $C(\alpha)$ (to be determined later), and show that $\kappa_\varepsilon(\xi, \cdot)$ stochastically dominates a kernel that assigns the marks 0 and $C(\alpha)$ independently to each interval in ξ with suitable probabilities. This is done in Sect. 4.
- (3) The two previous items mean that we obtain an upper bound on the value of (2.4) when we first replace κ_ε by our new kernel and then delete intervals from ξ . On the other hand, for interval configurations ξ that consist only of non-overlapping intervals, $\sigma_T^2(\xi, u)$ can be computed explicitly, see Lemma 3.3. By deleting a sufficient number of intervals (including all those coming from $\eta_{\alpha, T}$), and by optimizing over the parameter C , we obtain our result.

Remark 2.1. For the Fröhlich Polaron, i.e. for $d = 3$, $\gamma = 1$ and $g(t) = e^{-t}$ for all $t \geq 0$, the point process representation implies the existence of some constant $\tilde{c} > 0$ such that $\sigma^2(\alpha) \geq e^{-\tilde{c}\alpha^2}$ for all reasonably large $\alpha > 0$. Given $\xi \in \mathbf{N}_f(\Delta)$ and $r \in \mathbb{R}$ let $N_r(\xi) := \xi(\{(s, t) : s \leq r \leq t\})$ be the number of intervals that contain r . We call t dormant under ξ , if $N_t(\xi) = 0$, else we call t active under ξ . Under ξ , the interval $[0, T]$ can be decomposed into alternating dormant and active periods. Let

$$\mathcal{D}_T(\xi) := \lambda(\{t \in [0, T] : N_t(\xi) = 0\})$$

(where λ is the Lebesgue measure) be the sum of lengths of all dormant periods in $[0, T]$. One can easily convince oneself that $\sigma_T^2(\xi, u) \geq \frac{1}{T} \mathcal{D}_T(\xi)$ for all $u \in [0, \infty)^n$ (this follows from Proposition 3.2 later in the text as well). By applying Jensens inequality to the measure $\frac{1}{T} dt \mathbb{P}_{\alpha, T}(\mathbf{dx})$ and using Brownian scaling in order to interpret the resulting expression in terms of the derivative of $\psi(\alpha) := \lim_{T \rightarrow \infty} \log(Z_{\alpha, T})/T = -E_\alpha(0)$, it was shown in (BMPV22) that

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \int_0^T \widehat{\Gamma}_{\alpha, T}^0(N_t = 0) dt = \liminf_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{Z_{\alpha, t} Z_{\alpha, T-t}}{Z_{\alpha, T}} dt \geq e^{-\frac{3}{2}\alpha\psi'(\alpha)}.$$

By Fubini's Theorem

$$\int_0^T \widehat{\Gamma}_{\alpha, T}^0(N_t = 0) dt = \mathbb{E}_{\widehat{\Gamma}_{\alpha, T}^0}[\mathcal{D}_T]$$

and thus $\sigma^2(\alpha) \geq e^{-\frac{3}{2}\alpha\psi'(\alpha)}$ by (2.4). By convexity of ψ and since $\lim_{\alpha \rightarrow \infty} \psi(\alpha)/\alpha^2 \in (0, \infty)$ we have

$$\psi'(\alpha) \leq \frac{\psi(2\alpha) - \psi(\alpha)}{\alpha} \in \mathcal{O}(\alpha)$$

and hence there exists some $\tilde{c} > 0$, $\alpha_0 > 0$ such that $\sigma^2(\alpha) \geq e^{-\tilde{c}\alpha^2}$ for all $\alpha \geq \alpha_0$.

3. Properties of $\sigma_T^2(\xi, u)$

We derive a few properties of $\sigma_T^2(\xi, u)$. First, we will show that $\sigma_T^2(\xi, \cdot)$ is decreasing with respect to the partial order on $[0, \infty)^{N(\xi)}$ defined by $u \leq \tilde{u}$ iff $u_i \leq \tilde{u}_i$ for all $1 \leq i \leq N(\xi)$. We will then express $\sigma_T^2(\xi, u)$ in terms of a L^2 -distance and calculate $\sigma_T^2(\xi, u)$ in case that ξ consists of pairwise disjoint intervals.

Lemma 3.1. *For fixed $\xi \in \Delta^\cup$, the function $\sigma_T^2(\xi, \cdot)$ is decreasing on $[0, \infty)^{N(\xi)}$.*

Proof. Let $\xi = ((s_i, t_i))_{1 \leq i \leq N(\xi)} \in \Delta^\cup$. Then

$$\begin{aligned} \frac{\partial \sigma_T^2(\xi, u)}{\partial u_i} &= \frac{\partial}{\partial u_i} \frac{\mathbb{E}_{\mathcal{W}}[|X_{0,T}|^2 e^{-\sum_{j=1}^{N(\xi)} u_j^2 |X_{s_j, t_j}|^2 / 2}]}{\mathbb{E}_{\mathcal{W}}[e^{-\sum_{j=1}^{N(\xi)} u_j^2 |X_{s_j, t_j}|^2 / 2}]} \\ &= -u_i \operatorname{Cov}_{\mathbf{P}_{\xi, u}}(|X_{0,T}|^2, |X_{s_i, t_i}|^2). \end{aligned}$$

By Isserlis' theorem (and as the coordinate processes X^1, \dots, X^d are iid under the centered Gaussian measure $\mathbf{P}_{\xi, u}$)

$$\begin{aligned} \operatorname{Cov}_{\mathbf{P}_{\xi, u}}(|X_{0,T}|^2, |X_{s_i, t_i}|^2) &= d \cdot \operatorname{Cov}_{\mathbf{P}_{\xi, u}}((X_{0,T}^1)^2, (X_{s_i, t_i}^1)^2) \\ &= 2d \cdot \mathbb{E}_{\mathbf{P}_{\xi, u}}[X_{0,T}^1 X_{s_i, t_i}^1]^2 \geq 0. \end{aligned}$$

□

In particular, $\sigma_T^2(\xi, u)$ is increasing under deletion of a marked interval (s_i, t_i, u_i) (set $u_i = 0$).

As already used in (MV19), the normalization constant $\phi(\xi, u)$ can also be expressed in terms of a determinant of the covariance matrix of a suitable Gaussian vector. To see this, let $(B_t)_t \geq 0$ be an one dimensional Brownian motion and $(Z_n)_n$ be an iid sequence of $\mathcal{N}(0, 1)$ distributed random variables, independent of $(B_t)_t \geq 0$. Then, by Lemma 6.1 of the appendix, we have

$$\phi(\xi, u) = \frac{1}{\det(C(\xi, u))^{d/2}}$$

where $C(\xi, u)$ is the covariance matrix of the Gaussian vector

$$(u_1 B_{s_1, t_1} + Z_1, \dots, u_{N(\xi)} B_{s_{N(\xi)}, t_{N(\xi)}} + Z_{N(\xi)}).$$

Proposition 3.2. *The diffusion constant has the representation*

$$\sigma^2(\alpha) = \lim_{T \rightarrow \infty} \int_{\Delta^\cup} \widehat{\Gamma}_{\alpha, T}^\varepsilon(d\xi) \int_{[0, \infty)^{N(\xi)}} \kappa_\varepsilon(\xi, du) \sigma_T^2(\xi, u)$$

where for $\xi = ((s_i, t_i))_{1 \leq i \leq N(\xi)} \in \Delta^\cup$ and $u \in [0, \infty)^{N(\xi)}$

$$\sigma_T^2(\xi, u) = \frac{1}{T} \operatorname{dist}_{L^2} \left(B_{0,T}, \operatorname{span}\{u_i B_{s_i, t_i} + Z_i : 1 \leq i \leq N(\xi)\} \right)^2. \quad (3.1)$$

Proof. It is left to show Equality (3.1). We have

$$T \sigma_T^2(\xi, u) = -\frac{2/d}{\phi(\xi, u)} \frac{\partial}{\partial v} \mathbb{E}_{\mathcal{W}} \left[e^{-\frac{1}{2}v|X_{0,T}|^2 - \sum_{i=1}^{N(\xi)} \frac{1}{2}u_i^2 |X_{s_i, t_i}|^2} \right] \Big|_{v=0}. \quad (3.2)$$

For $v, w \in \mathbb{R}$ let $C_T(\xi, u, v, w) \in \mathbb{R}^{(N(\xi)+1) \times (N(\xi)+1)}$ be the covariance matrix of the Gaussian vector $(u_1 B_{s_1, t_1} + Z_1, \dots, u_{N(\xi)} B_{s_{N(\xi)}, t_{N(\xi)}} + Z_{N(\xi)}, \sqrt{v} B_{0,T} + w Z_{N(\xi)+1})$. Then, with Lemma 6.1, Eq. (3.2) becomes

$$\begin{aligned} T \sigma_T^2(\xi, u) &= -\frac{2}{d} \det(C(\xi, u))^{d/2} \frac{\partial}{\partial v} \frac{1}{\det(C_T(\xi, u, v, 1))^{d/2}} \Big|_{v=0} \\ &= \frac{1}{\det(C(\xi, u))} \frac{\partial}{\partial v} \det(C_T(\xi, u, v, 1)) \Big|_{v=0} \end{aligned}$$

since $\det(C_T(\xi, u, 0, 1)) = \det(C(\xi, u))$. By using that the determinant is multilinear in the rows and columns, we have

$$\det(C_T(\xi, u, v, 1)) = \det(C(\xi, u)) + v \det(C_T(\xi, u, 1, 0))$$

and hence

$$\frac{\partial}{\partial v} \det(C_T(\xi, u, v, 1)) \Big|_{v=0} = \det(C_T(\xi, u, 1, 0)).$$

Finally, by Lemma 6.2

$$\frac{\det(C_T(\xi, u, 1, 0))}{\det(C(\xi, u))} = \text{dist}_{L^2} \left(B_{0,T}, \text{span}\{u_i B_{s_i, t_i} + Z_i : 1 \leq i \leq N(\xi)\} \right)^2$$

which concludes the proof.

Lemma 3.3. *Let $\xi = ((s_i, t_i))_{1 \leq i \leq N(\xi)} \in \Delta^\cup$ with $[s_i, t_i] \subseteq [0, T]$ for all $1 \leq i \leq N(\xi)$ and $[s_i, t_i] \cap [s_j, t_j] = \emptyset$ for $i \neq j$. Then, for $u \in [0, \infty)^{N(\xi)}$*

$$\sigma_T^2(\xi, u) = 1 - \frac{1}{T} \sum_{i=1}^{N(\xi)} \tau_i + \frac{1}{T} \sum_{i=1}^{N(\xi)} \frac{\tau_i}{\tau_i u_i^2 + 1}$$

where $\tau_i := t_i - s_i$ for $1 \leq i \leq N(\xi)$.

Proof. We project $B_{0,T}$ onto $\text{span}\{u_i B_{s_i, t_i} + Z_i : 1 \leq i \leq N(\xi)\}$ and obtain

$$\sum_{i=1}^{N(\xi)} \mathbb{E}[B_{0,T} \cdot (u_i B_{s_i, t_i} + Z_i)] \frac{u_i B_{s_i, t_i} + Z_i}{\mathbb{E}[(u_i B_{s_i, t_i} + Z_i)^2]} = \sum_{i=1}^{N(\xi)} \frac{\tau_i u_i}{u_i^2 \tau_i + 1} (u_i B_{s_i, t_i} + Z_i).$$

Hence

$$\begin{aligned} T \sigma_T^2(\xi, u) &= \text{dist} \left(B_{0,T}, \text{span}\{u_i B_{s_i, t_i} + Z_i : 1 \leq i \leq N(\xi)\} \right)^2 \\ &= \mathbb{E} \left[\left(B_{0,T} - \sum_{i=1}^{N(\xi)} B_{s_i, t_i} + \sum_{i=1}^{N(\xi)} \frac{1}{u_i^2 \tau_i + 1} B_{s_i, t_i} - \frac{u_i \tau_i}{u_i^2 \tau_i + 1} Z_i \right)^2 \right] \\ &= \mathbb{E} \left[\left(B_{0,T} - \sum_{i=1}^{N(\xi)} B_{s_i, t_i} \right)^2 \right] + \sum_{i=1}^{N(\xi)} \frac{\tau_i}{(u_i^2 \tau_i + 1)^2} + \frac{u_i^2 \tau_i^2}{(u_i^2 \tau_i + 1)^2} \\ &= T - \sum_{i=1}^{N(\xi)} \tau_i + \sum_{i=1}^{N(\xi)} \frac{\tau_i}{\tau_i u_i^2 + 1}. \end{aligned}$$

□

In the situation above, $1 - \frac{1}{T} \sum_{i=1}^{N(\xi)} \tau_i = \frac{1}{T} \lambda([0, T] \setminus \bigcup_{i=1}^{N(\xi)} [s_i, t_i])$, where λ denotes the Lebesgue measure. Assume that $\tau_i \geq 1$ and $u_i \geq C$ for all $1 \leq i \leq N(\xi)$ where $C > 0$. Then

$$\frac{1}{T} \sum_{i=1}^{N(\xi)} \frac{\tau_i}{\tau_i u_i^2 + 1} \leq \frac{1}{C^2 + 1} \frac{1}{T} \sum_{i=1}^{N(\xi)} \tau_i \leq \frac{1}{C^2 + 1}. \quad (3.3)$$

For a general configuration (ξ, u) , the previous considerations allow us to obtain estimates on $\sigma_T^2(\xi, u)$ by considering subconfigurations (ξ', u') of pairwise disjoint marked intervals with intervals lengths exceeding 1 and marks exceeding C . In combination with an application of renewal theory, this implies the following technical Lemma:

Lemma 3.4. *Let $f : [0, \infty) \rightarrow [0, \infty)$ be measurable with $\int_0^\infty (1+t)f(t) dt < \infty$ and $f(t) = 0$ for almost all $t \in [0, 1]$. For $T > 0$ let Ξ_T be the law of a Poisson point process with intensity measure $f(t-s)\mathbb{1}_{\{0 < s < t < T\}} ds dt$ and let $C > 0$. Then*

$$\limsup_{T \rightarrow \infty} \int_{\Delta^U} \Xi_T(d\xi) \int_{[0, \infty)^{N(\xi)}} \delta_C^{\otimes N(\xi)}(du) \sigma_T^2(\xi, u) \leq \frac{1}{1 + \int_0^\infty tf(t) dt} + \frac{1}{C^2 + 1}.$$

Proof. In case that $f = 0$ a.e. the statement is trivial, so assume $\int_0^\infty f(t) dt > 0$. Let

$$\beta := \int_0^\infty f(r) dr, \quad \rho(t) := \frac{f(t)}{\int_0^\infty f(r) dr}$$

for $t \geq 0$. We consider the Poisson point process $\eta = \sum_{n=1}^\infty \delta_{(s_n, t_n)}$ of a $M/G/\infty$ -queue where the arrival process $\sum_{n=1}^\infty \delta_{s_n}$ is a homogenous Poisson point process with intensity β , the service times $(\tau_n)_n$ are iid, have density ρ and are independent of the arrival process and $t_n := s_n + \tau_n$ is for $n \in \mathbb{N}$ the departure of customer n . Then the restriction of η to the process of all customers that arrive and depart before T has distribution Ξ_T . We inductively define $i_0 := 0, t_0 := 0$ and

$$i_{n+1} := \inf\{j > i_n : s_j > t_{i_n}\}$$

for $n \geq 0$, i.e. customer i_{n+1} is for $n \geq 1$ the first customer that arrives after the departure of customer i_n . The waiting times $(s_{i_n} - t_{i_{n-1}})_{n \geq 1}$ are iid $\text{Exp}(\beta)$ distributed and are independent of the iid service times $(t_{i_n} - s_{i_n})_{n \geq 1}$ which have density ρ . Notice that $(i_n)_{n \geq 0}$ defines a renewal process. Let $N_T := \inf\{n : t_{i_n} \geq T\}$. By the considerations after Lemma 3.3, we have

$$\int_{\Delta^U} \Xi_T(d\xi) \int_{[0, \infty)^{N(\xi)}} \delta_C^{\otimes N(\xi)}(du) \sigma_T^2(\xi, u) \leq \mathbb{E}\left[\frac{B_T}{T} + \frac{1}{T} \sum_{n=1}^{N_T-1} s_{i_n} - t_{i_{n-1}}\right] + \frac{1}{C^2 + 1}$$

where $t_0 := 0$ and $B_T := T - t_{i_{N_T-1}}$ is the backward recurrence time of the renewal process $(i_n)_n$ at time T . By renewal theory, $(B_T)_T \geq 0$ converges in distribution as $T \rightarrow \infty$. Let $(T_k)_k$ be a sequence in $[0, \infty)$ that converges to infinity. Then $B_{T_k}/T_k \rightarrow 0$ in probability as $k \rightarrow \infty$. Hence, there exists a subsequence $(T_{k_j})_j$ such that $B_{T_{k_j}}/T_{k_j} \rightarrow 0$ almost surely as $j \rightarrow \infty$. By dominated convergence, $\mathbb{E}[B_{T_{k_j}}/T_{k_j}] \rightarrow 0$ as $j \rightarrow \infty$. As $(T_k)_k$ was arbitrary, we have $\mathbb{E}[B_T/T] \rightarrow 0$ as $T \rightarrow \infty$. By renewal theory

$$N_T/T \rightarrow \frac{1}{\mathbb{E}[t_{i_1}]} = \frac{1}{\beta^{-1} + \int_0^\infty r\rho(r) dr} \tag{3.4}$$

almost surely as $T \rightarrow \infty$ and by the law of large numbers we thus have

$$\frac{1}{T} \sum_{n=1}^{N_T-1} s_{i_n} - t_{i_{n-1}} \rightarrow \frac{\beta^{-1}}{\beta^{-1} + \int_0^\infty r\rho(r) dr} = \frac{1}{1 + \int_0^\infty rf(r) dr}$$

almost surely as $T \rightarrow \infty$. By the dominated convergence theorem, we have convergence in L^1 as well and the statement follows.

4. Estimation of the Kernel κ_ε

While the component $\eta_{\alpha,T}^\varepsilon \sim \Gamma_{\varepsilon\alpha,T}$ is far easier to understand than the whole process, the distribution of marks of $\eta_{\alpha,T}^\varepsilon$ under κ_ε still depends on the whole process. We get rid of this problem by estimating κ_ε by a suitable product kernel that marks the intervals independent of each other. For $t, C > 0$ we define

$$h_\varepsilon(t) := \mathbb{E}_{\gamma\mathcal{V}}[v_\varepsilon(X_t)] = \int_{[0,\infty)} v_\varepsilon(du) \frac{1}{(1+u^2t)^{d/2}}$$

and

$$p_\varepsilon(C) := \frac{2^{-\gamma/2}}{(1+4C^2)^{d/2}} (1 - \varepsilon/h_\varepsilon(2)).$$

Notice that there exists some constant $c_{\gamma,d}$ such that $h_\varepsilon(t) = c_{\gamma,d}t^{-\gamma/2} + \varepsilon$ for all $t > 0$. In particular, $\varepsilon/h_\varepsilon(2) < 1$ and

$$\mathcal{P}_{\varepsilon,C}(t, \cdot) := \begin{cases} \delta_0 & \text{if } t > 2 \\ p_\varepsilon(C)\delta_C + (1 - p_\varepsilon(C))\delta_0 & \text{if } t \leq 2 \end{cases}$$

defines for $t > 0$ a probability measure. Fix $\xi = ((s_i, t_i))_{1 \leq i \leq N(\xi)} \in \Delta^\cup$ with $F_\varepsilon(\xi) < \infty$. We will show that $\kappa_\varepsilon(\xi, \cdot)$ stochastically dominates

$$\tilde{\kappa}_{\varepsilon,C}(\xi, \cdot) := \bigotimes_{i=1}^{N(\xi)} \mathcal{P}_{\varepsilon,C}(t_i - s_i, \cdot)$$

that is

$$\int_{[0,\infty)^{N(\xi)}} \kappa_\varepsilon(\xi, du) f(u) \leq \int_{[0,\infty)^{N(\xi)}} \tilde{\kappa}_{\varepsilon,C}(\xi, du) f(u)$$

for any decreasing function $f : [0, \infty)^{N(\xi)} \rightarrow \mathbb{R}$. By Proposition 3.2 and the monotonicity of $\sigma_T^2(\xi, \cdot)$, we then obtain

$$\sigma^2(\alpha) \leq \limsup_{T \rightarrow \infty} \int_{\Delta^\cup} \widehat{\Gamma}_{\alpha,T}^\varepsilon(d\xi) \int_{\{0,C\}^{N(\xi)}} \tilde{\kappa}_{\varepsilon,C}(\xi, du) \sigma_T^2(\xi, u).$$

Since $\sigma_T^2(\xi, u)$ is increasing under deletion of marked intervals we can further estimate

$$\sigma^2(\alpha) \leq \limsup_{T \rightarrow \infty} \int_{\Delta^\cup} \Gamma_{\varepsilon\alpha,T}(d\xi) \int_{\{0,C\}^{N(\xi)}} \tilde{\kappa}_{\varepsilon,C}(\xi, du) \sigma_T^2(\xi, u). \tag{4.1}$$

Notice that intervals with zero mark do not contribute to $\sigma_T^2(\xi, u)$. Estimate (4.1) in combination with Lemma 3.4 will yield Theorem 1.1.

Lemma 4.1. *Let $C > 0$ and $\xi = ((s_i, t_i))_{1 \leq i \leq N(\xi)} \in \Delta^\cup$ with $F_\varepsilon(\xi) < \infty$. Then $\kappa_\varepsilon(\xi, \cdot)$ stochastically dominates $\tilde{\kappa}_{\varepsilon,C}(\xi, \cdot)$.*

Proof. Let $J(\xi) := \{1 \leq i \leq N(\xi) : t_i - s_i \leq 2\}$. The statement is trivial for $J(\xi) = \emptyset$, so assume $J(\xi) \neq \emptyset$. We define

$$\chi : [0, \infty)^{N(\xi)} \rightarrow \{0, C\}^{J(\xi)}, \quad \chi_i := C \cdot \mathbb{1}_{\{u_i \geq C\}}$$

for $i \in J(\xi)$. Let $\pi : [0, \infty)^{N(\xi)} \rightarrow [0, \infty)^{J(\xi)}$ be the restriction map $u \mapsto (u_i)_{i \in J(\xi)}$ and let $\phi : [0, \infty)^{N(\xi)} \rightarrow [0, \infty)^{N(\xi)}$ be defined by

$$\phi_i(u) := \begin{cases} u_i & \text{if } i \in J(\xi) \\ 0 & \text{else} \end{cases}$$

for $u \in [0, \infty)^{N(\xi)}$. Let $f : [0, \infty)^{N(\xi)} \rightarrow \mathbb{R}$ be decreasing and let $\tilde{f} : [0, \infty)^{J(\xi)} \rightarrow \mathbb{R}$ be such that $\tilde{f} \circ \pi = f \circ \phi$. Then

$$\begin{aligned} \int_{[0, \infty)^{N(\xi)}} \kappa_\varepsilon(\xi, du) f(u) &\leq \int_{[0, \infty)^{N(\xi)}} \kappa_\varepsilon(\xi, du) \tilde{f}(\chi(u)) \\ \int_{[0, \infty)^{N(\xi)}} \tilde{\kappa}_{\varepsilon, C}(\xi, du) f(u) &= \int_{[0, \infty)^{N(\xi)}} \tilde{\kappa}_{\varepsilon, C}(\xi, du) \tilde{f}(\chi(u)). \end{aligned}$$

Hence, it is sufficient to show that the distribution of χ under $\mu := \kappa_\varepsilon(\xi, \cdot)$ stochastically dominates the distribution of χ under $\tilde{\mu} := \tilde{\kappa}_{\varepsilon, C}(\xi, \cdot)$. In order to show this, it is sufficient (see e.g. (FV17, pp. 137–138)) to show that for any $\omega \in \{0, C\}^{J(\xi)}$ and any $i \in J(\xi)$

$$\mu(\chi_i = C | \chi_j = \omega_j \forall j \neq i) \geq p_\varepsilon(C) = \tilde{\mu}(\chi_i = C | \chi_j = \omega_j \forall j \neq i).$$

To simplify notation, we assume w.l.o.g. that $i = 1 \in J(\xi)$ and set $\tau_1 := t_1 - s_1$. Let $\omega \in \{0, C\}^{J(\xi)}$ be fixed and let $A \subseteq [0, \infty)^{N(\xi)-1}$ be such that $\{\chi_j = \omega_j \forall j \neq 1\} = [0, \infty) \times A$. For $u_1 \in [0, \infty)$ and $u = (u_2, \dots, u_{N(\xi)}) \in [0, \infty)^{N(\xi)-1}$ we have by Corollary 6.3 of the appendix and since $(C + u_1)^2/2 \leq C^2 + u_1^2$

$$\begin{aligned} \phi(\xi, (u_1 + C, u)) &= \mathbb{E}_{\mathcal{W}} \left[e^{-(C+u_1)^2 |X_{s_1, t_1}|^2/2} \prod_{i=2}^{N(\xi)} e^{-u_i^2 |X_{s_i, t_i}|^2/2} \right] \\ &\geq \mathbb{E}_{\mathcal{W}} \left[e^{-C^2 |X_{s_1, t_1}|^2} e^{-u_1^2 |X_{s_1, t_1}|^2} \prod_{i=2}^{N(\xi)} e^{-u_i^2 |X_{s_i, t_i}|^2/2} \right] \\ &\geq \frac{1}{(1 + 4C^2)^{d/2}} \phi(\xi, (\sqrt{2}u_1, u)) \end{aligned}$$

since $\tau_1 \leq 2$ as $1 \in J(\xi)$. We get (by using the fact that the density of $v_\varepsilon|_{(0, \infty)}$ with respect to the Lebesgue measure is increasing)

$$\begin{aligned} &\mu(\chi_1 = C, \chi_j = \omega_j \forall j \neq 1) \\ &= \frac{1}{F_\varepsilon(\xi)} \int_{(C, \infty)} v_\varepsilon(du_1) \int_A v_\varepsilon^{\otimes(N(\xi)-1)}(du) \phi(\xi, (u_1, u)) \\ &\geq \frac{1}{F_\varepsilon(\xi)} \int_{(0, \infty)} v_\varepsilon(du_1) \int_A v_\varepsilon^{\otimes(N(\xi)-1)}(du) \phi(\xi, (u_1 + C, u)) \\ &\geq \frac{1}{(1 + 4C^2)^{d/2}} \frac{1}{F_\varepsilon(\xi)} \int_{(0, \infty)} v_\varepsilon(du_1) \int_A v_\varepsilon^{\otimes(N(\xi)-1)}(du) \phi(\xi, (\sqrt{2}u_1, u)) \end{aligned}$$

$$\begin{aligned}
 &= \frac{2^{-\gamma/2}}{(1+4C^2)^{d/2}} \frac{1}{F_\varepsilon(\xi)} \int_{(0,\infty)} v_\varepsilon(\mathbf{d}u_1) \int_A v_\varepsilon^{\otimes(N(\xi)-1)}(\mathbf{d}u) \phi(\xi, (u_1, u)) \\
 &= \frac{2^{-\gamma/2}}{(1+4C^2)^{d/2}} \frac{1}{F_\varepsilon(\xi)} \left(\int_{[0,\infty)} v_\varepsilon(\mathbf{d}u_1) \int_A v_\varepsilon^{\otimes(N(\xi)-1)}(\mathbf{d}u) \phi(\xi, (u_1, u)) \right. \\
 &\quad \left. - \varepsilon \int_A v_\varepsilon^{\otimes(N(\xi)-1)}(\mathbf{d}u) \phi(\xi', u) \right)
 \end{aligned}$$

where $\xi' = ((s_2, t_2), \dots, (s_N(\xi), t_N(\xi)))$. By Corollary 6.3, for any $u_1 \in [0, \infty)$, $u \in [0, \infty)^{N(\xi)-1}$

$$\phi(\xi, (u_1, u)) \geq \frac{\phi(\xi', u)}{(1+u_1^2 \tau_1)^{d/2}} \geq \frac{\phi(\xi', u)}{(1+2u_1^2)^{d/2}}$$

which yields

$$\begin{aligned}
 &\int_{[0,\infty)} v_\varepsilon(\mathbf{d}u_1) \int_A v_\varepsilon^{\otimes(N(\xi)-1)}(\mathbf{d}u) \phi(\xi, (u_1, u)) \\
 &\geq \int_{[0,\infty)} \frac{v_\varepsilon(\mathbf{d}u_1)}{(1+2u_1^2)^{d/2}} \int_A v_\varepsilon^{\otimes(N(\xi)-1)}(\mathbf{d}u) \phi(\xi', u) \\
 &= h_\varepsilon(2) \int_A v_\varepsilon^{\otimes(N(\xi)-1)}(\mathbf{d}u) \phi(\xi', u).
 \end{aligned}$$

Hence, we get

$$\begin{aligned}
 &\mu(\chi_1 = C, \chi_j = \omega_j \forall j \neq 1) \\
 &\geq \frac{2^{-\gamma/2}}{(1+4C^2)^{d/2}} (1 - \varepsilon/h_\varepsilon(2)) \frac{1}{F_\varepsilon(\xi)} \int_{[0,\infty)} v_\varepsilon(\mathbf{d}u_1) \int_A v_\varepsilon^{\otimes(N(\xi)-1)}(\mathbf{d}u) \phi(\xi, (u_1, u)) \\
 &= p_\varepsilon(C) \mu(\chi_j = \omega_j \forall j \neq 1)
 \end{aligned}$$

which yields the claim. \square

5. Finishing the Proof of Theorem 1.1

Proof of Theorem 1.1. For $C > 0$ let $\zeta_{\alpha,T}^{\varepsilon,C}$ be a Poisson point process on $\Delta \times \{0, C\}$ with intensity measure

$$\alpha \varepsilon g(t-s) \mathbb{1}_{\{0 < s < t < T\}} \mathbf{d}s \mathbf{d}t \mathcal{P}_{\varepsilon,C}(t-s, \mathbf{d}u).$$

Then Estimate (4.1) becomes⁵

$$\sigma^2(\alpha) \leq \limsup_{T \rightarrow \infty} \mathbb{E}[\sigma_T^2(\zeta_{\alpha,T}^{\varepsilon,C})] \leq \limsup_{T \rightarrow \infty} \mathbb{E}[\sigma_T^2(\tilde{\zeta}_{\alpha,T}^{\varepsilon,C})] \quad (5.1)$$

where $\tilde{\zeta}_{\alpha,T}^{\varepsilon,C}$ is the restriction of $\zeta_{\alpha,T}^{\varepsilon,C}$ to $\{(s, t) : 1 \leq t-s \leq 2\} \times \{C\}$ and thus a Poisson point process with intensity measure

$$\alpha \varepsilon p_\varepsilon(C) g(t-s) \mathbb{1}_{\{1 \leq t-s \leq 2\}} \mathbb{1}_{\{0 < s < t < T\}} \mathbf{d}s \mathbf{d}t \delta_C(\mathbf{d}u).$$

⁵ Here $\sigma_T^2((s_1, t_1, u_1), \dots, (s_n, t_n, u_n)) := \sigma_T^2(((s_1, t_1), \dots, (s_n, t_n)), (u_1, \dots, u_n))$.

By Lemma 3.4 we have

$$\limsup_{T \rightarrow \infty} \mathbb{E}[\sigma_T^2(\tilde{\zeta}_{\alpha, T}^{\varepsilon, C})] \leq \frac{1}{1 + \alpha \varepsilon p_\varepsilon(C) \int_1^2 r g(r) dr} + \frac{1}{C^2 + 1}.$$

Choosing $C = \alpha^{1/(2+d)}$ yields the claim.

Remark 5.1. Replacing the arbitrarily chosen cut-off parameters 1 and 2 for the lifespan by general $A < B$ and optimizing over A, B, C, ε leads to

$$\sigma^2(\alpha) \leq \inf_{0 < A < B, 0 < C, 0 < \varepsilon} \left(1 + \frac{2^{-\gamma/2} \alpha}{(1 + 2BC^2)^{d/2}} \frac{\varepsilon h_0(B)}{\varepsilon + h_0(B)} \int_A^B r g(r) dr \right)^{-1} + (1 + AC^2)^{-1}$$

which leads to the same exponent $-2/(2 + d)$ for the decay in α . The non-dependency of this exponent on γ is presumably the result from the estimate $(C + r)^{\gamma-1} \geq r^{\gamma-1}$ that has been used in the proof of Lemma 4.1.

6. Appendix: Facts About Gaussian Measures

In the following, we collect a few facts about Gaussian measures (which were in a similar form already applied in (MV19)) in order to be self contained.

Lemma 6.1. *Let $X = (X_1, \dots, X_n)$ be a centered Gaussian vector with covariance matrix C . Then*

$$\mathbb{E} \left[\prod_{i=1}^n e^{-X_i^2/2} \right] = \frac{1}{\sqrt{\det(I_n + C)}}.$$

Proof. Write $C = O\Lambda O^T$ with $O \in \mathbb{R}^{n \times n}$ orthogonal and $\Lambda \in \mathbb{R}^{n \times n}$ diagonal with non-negative entries. Then $C = AA^T$ with $A := O\Lambda^{1/2}$. Let $Z = (Z_1, \dots, Z_n) \sim \mathcal{N}(0, 1)^{\otimes n}$. Then $AZ \stackrel{d}{=} X$ and thus

$$\begin{aligned} \mathbb{E} \left[\prod_{i=1}^n e^{-X_i^2/2} \right] &= \mathbb{E} \left[e^{-|X|^2/2} \right] = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-|Az|^2/2} e^{-|z|^2/2} dz \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\sum_{i=1}^n (1+\lambda_i)z_i^2/2} dz \\ &= \prod_{i=1}^n (1 + \lambda_i)^{-1/2}. \end{aligned}$$

Since $\prod_{i=1}^n (1 + \lambda_i) = \det(I + \Lambda) = \det(I + C)$ the statement follows. □

Lemma 6.2. *Let $X = (X_1, \dots, X_n)$ be a centered Gaussian vector with covariance matrix C . Then*

$$\det(C) = \prod_{k=1}^n \text{dist}_{L^2}(X_k, \text{span}\{X_1, \dots, X_{k-1}\})^2.$$

Proof. Write $X_n = Y_n + Z_n$ with $Y_n \in \text{span}\{X_1, \dots, X_{n-1}\}$ and $Z_n \perp \text{span}\{X_1, \dots, X_{n-1}\}$. Define $Y_i := X_i$ for $1 \leq i \leq n - 1$. Let C' and C'' be the covariance matrices of (Y_1, \dots, Y_{n-1}) and (Y_1, \dots, Y_n) respectively. Then, by the Leibniz formula for the determinant

$$\begin{aligned} \det(C) &= \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n \mathbb{E}[X_i X_{\sigma(i)}] \\ &= \mathbb{E}[X_n^2] \sum_{\sigma: \sigma(n)=n} \text{sgn}(\sigma) \prod_{i=1}^{n-1} \mathbb{E}[X_i X_{\sigma(i)}] + \sum_{\sigma: \sigma(n) \neq n} \text{sgn}(\sigma) \prod_{i=1}^n \mathbb{E}[X_i X_{\sigma(i)}] \\ &= (\mathbb{E}[Y_n^2] + \mathbb{E}[Z_n^2]) \sum_{\sigma: \sigma(n)=n} \text{sgn}(\sigma) \prod_{i=1}^{n-1} \mathbb{E}[Y_i Y_{\sigma(i)}] + \sum_{\sigma: \sigma(n) \neq n} \text{sgn}(\sigma) \prod_{i=1}^n \mathbb{E}[Y_i Y_{\sigma(i)}] \\ &= \mathbb{E}[Z_n^2] \det(C') + \det(C''). \end{aligned}$$

Since $Y_n \in \text{span}\{Y_1, \dots, Y_{n-1}\}$ we have $\det(C'') = 0$ and the statement follows inductively.

Corollary 6.3. *For any centered Gaussian vector $X = (X_1, \dots, X_n)$ we have*

$$\mathbb{E}\left[\prod_{i=1}^n e^{-X_i^2/2}\right] \geq \frac{1}{\sqrt{1 + \mathbb{E}[X_n^2]}} \mathbb{E}\left[\prod_{i=1}^{n-1} e^{-X_i^2/2}\right].$$

Proof. Let $Z = (Z_1, \dots, Z_n) \sim \mathcal{N}(0, 1)^{\otimes n}$ be independent of X . Then the covariance matrix of $Z + X$ is $I + C$, where C is the covariance matrix of X . By the previous two lemmas,

$$\mathbb{E}\left[\prod_{i=1}^n e^{-X_i^2/2}\right] = \frac{\mathbb{E}\left[\prod_{i=1}^{n-1} e^{-X_i^2/2}\right]}{\text{dist}_{L^2}(X_n + Z_n, \text{span}\{X_1 + Z_1, \dots, X_{n-1} + Z_{n-1}\})}.$$

The statement follows from

$$\begin{aligned} \text{dist}_{L^2}(X_n + Z_n, \text{span}\{X_1 + Z_1, \dots, X_{n-1} + Z_{n-1}\}) &\leq \mathbb{E}[(X_n + Z_n)^2]^{1/2} \\ &= \sqrt{1 + \mathbb{E}[X_n^2]}. \end{aligned}$$

□

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