



Algebraicity of higher Green functions at a CM point

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To Don Zagier on the occasion of his 70th birthday

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Abstract

In this paper, we investigate the algebraic nature of the value of a higher Green function on an orthogonal Shimura variety at a single CM point. This is motivated by a conjecture of Gross and Zagier in the setting of higher Green functions on the product of two modular curves. In the process, we will study analogue of harmonic Maass forms in the setting of Hilbert modular forms, and obtain results concerning the arithmetic of their holomorphic part Fourier coefficients. As a consequence, we answer a question of Zagier in his 1986 ICM proceeding.

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1 Introduction

Let $j(z)$ be the modular j -invariant on the modular curve $X_0(1) := \text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ with \mathbb{H} the upper-half plane. Its values at CM points are algebraic integers called singular moduli. They play an important role in the explicit construction of class fields of imaginary quadratic fields.

1.1 Conjecture and results

The function $G_1(z_1, z_2) := 2 \log |j(z_1) - j(z_2)|$ is the automorphic Green function on $X_0(1) \times X_0(1)$, and the limiting member of a family of automorphic functions

$$G_s(z_1, z_2) := -2 \sum_{\gamma \in \Gamma} Q_{s-1} \left(1 + \frac{|z_1 - \gamma z_2|^2}{2\Im(z_1)\Im(\gamma z_2)} \right), \Re(s) > 1, \tag{1.1}$$

$$Q_{s-1}(t) := \int_0^\infty (t + \sqrt{t^2 - 1} \cosh(u))^{-s} du,$$

that are eigenfunctions with respect to the Laplacians in z_1 and z_2 . For integral parameters $s = r + 1 \in \mathbb{N}$, these functions are called higher Green functions, and played an important role in calculating arithmetic intersections of Heegner cycles on Kuga-Sato varieties [39]. Given a weakly holomorphic modular form $f = \sum_{m \gg -\infty} c(m)q^m \in M_{-2r}^1$ on $X_0(1)$, one can associate to it a higher Green function

$$G_{r+1, f}(z_1, z_2) := \sum_{m \in \mathbb{N}} c(-m)m^r G_{r+1}^m(z_1, z_2),$$

$$G_s^m(z_1, z_2) = G_s(z_1, z_2) | T_m \tag{1.2}$$

$$= -2 \sum_{\gamma \in M_2(\mathbb{Z}), \det(\gamma)=m} Q_{s-1} \left(1 + \frac{|z_1 - \gamma z_2|^2}{2\Im(z_1)\Im(\gamma z_2)} \right).$$

Although the theory of complex multiplication does not directly apply, the values of $G_{r+1, f}$ at CM points on $X_0(1)^2$ should be algebraic in nature, as in the case of

the automorphic Green function. More precisely, these values should be algebraic multiples of logarithm of algebraic numbers. This was conjectured in [24, Conjecture (4.4)] when z_1, z_2 have the same discriminant, and mentioned as a question in [22, section V.1] for the general case (see also [30] and [34]). In this paper, we prove the following result, which in particular solves problem (ii) raised by Don Zagier at the end of his 1986 ICM proceeding [38].

Theorem 1.1 *Let $r \in \mathbb{N}$ and $f \in M_{-2r}^1$ with integral Fourier coefficients. Suppose d_1, d_2 are negative discriminants, such that one of them is fundamental when r is odd. For any CM point z_i with discriminant d_i , there exist $\kappa \in \mathbb{N}$ depending on d_1, d_2, r and f , and $\alpha = \alpha(z_1, z_2) \in H$ such that*

$$(d_1 d_2)^{r/2} G_{r+1, f}(z_1, z_2) = \frac{1}{\kappa} \log |\alpha|, \quad (1.3)$$

where $H = H_1 H_2$ with H_i the ring class field extension of $E_i := \mathbb{Q}(\sqrt{d_i})$ associated to z_i . Furthermore, we have

$$\alpha(z_1^\sigma, z_2^\sigma) = \sigma(\alpha(z_1, z_2)) \quad (1.4)$$

for any $\sigma \in \text{Gal}(H/E)$, where $E = E_1 E_2$.

Remark 1.2 The group $\text{Gal}(H/E)$ can be embedded as a subgroup of $\text{Gal}(H_1/E_1) \times \text{Gal}(H_2/E_2)$, which then acts on the CM point (z_1, z_2) .

There has been a lot of previous works concerning this question. The first such result is due to Gross, Kohlen and Zagier [22], where r is even¹ and one considers average of the whole $\text{Gal}(H/E)$ -orbit of (z_1, z_2) . In that case, the value is a rational multiple of the logarithm of a rational number. When $E_1 = E_2$, this conjecture follows from the work of Zhang [39], under the assumption of the non-degeneracy of certain height pairing of Heegner cycles on Kuga-Sato varieties. In [34], Viazovska gave an analytic proof without this assumption. When $E_1 \neq E_2$, Mellit [30] gave a strategy to systematically verify this conjecture with one of the points fixed, and carried it out for $z_1 = i$. In [28], we considered the average over the whole $\text{Gal}(H/E)$ -orbit with r odd, and were able to show that $\alpha \in \mathbb{Q}(\sqrt{d_1 d_2})$ and give an explicit factorization of the ideal it generates in the spirit of the seminal work of Gross and Zagier on singular moduli [23]. Very recently, Bruinier, Ehlen and Yang made significant progress and proved algebraicity result in the sense of Theorem 1.1 by averaging over the Galois orbit of one of the two CM points with fundamental discriminant [11]. We have now removed this averaging in Theorem 1.1 to obtain an algebraicity result at an individual CM point.

It is important to mention that one can replace $\text{SL}_2(\mathbb{Z})$ with a congruence subgroup $\Gamma_0(N)$, define higher Green functions on $X_0(N)^2$ analogously, and ask the same question. This was in fact the setting that [22] and [39] were in. By viewing $X_0(N)^2$ as the Shimura variety for the \mathbb{Q} -split group $\text{O}(2, 2)$, it is natural to generalize the

¹For odd r , they also obtain certain result, which turns out to be trivial in the case of level 1.

setting to an arbitrary orthogonal Shimura variety, and ask the question about the algebraic nature of the CM-values of higher Green functions on such varieties. This framework, which was adopted in [11], will be the one we work in.

To be more precise, let V be a rational quadratic space of signature $(n, 2)$, and

$$X_K(\mathbb{C}) = X_{V,K}(\mathbb{C}) = H(\mathbb{Q}) \backslash \mathbb{D}_V \times H(\hat{\mathbb{Z}}) / K \tag{1.5}$$

the Shimura variety associated to the algebraic group $H = H_V = \text{GSpin}_V$ and an open compact subgroup $K \subset H(\hat{\mathbb{Z}})$ (see Sect. 2.3 for details). Given an even, integral lattice $L \subset V$ such that K fixes $\hat{L} = L \otimes \hat{\mathbb{Z}}$ and acts trivially on the finite abelian group \hat{L}' / \hat{L} , one can associate a higher Green function $\Phi_L^r(z, h, f)$ on X_K to each weakly holomorphic modular form $f \in M_{-n/2-2r, \hat{\rho}_L}^1$ and $r \in \mathbb{N}$ (see Equation (3.29)). It has logarithmic singularity along special divisors on X_K .

For a totally real field F of degree d , a quadratic CM extension E/F becomes a binary F -quadratic space W with respect to a quadratic form $\alpha \text{Nm}_{E/F}$ for some $\alpha \in F^\times$. Suppose W has signature $((0, 2), (2, 0), \dots, (2, 0))$ with respect to the real embeddings $\sigma_1, \dots, \sigma_d$ of F and there is an isometric embedding $W_{\mathbb{Q}} := \text{Res}_{F/\mathbb{Q}} W \hookrightarrow V$. This not only implies

$$n + 2 \geq 2d, \tag{1.6}$$

but also gives CM points $Z(W_{\mathbb{Q}}) \subset X_K$ (see Equation (2.32)). This 0-cycle is defined over F , and each individual point $(z_0, h) \in Z(W_{\mathbb{Q}})$ is defined over certain abelian extension of E . We will prove the following result concerning the algebraic nature of Φ_L^r at CM points in $Z(W_{\mathbb{Q}})$.

Theorem 1.3 *In the setting above, suppose f has integral Fourier coefficients. Then there exist algebraic numbers $\lambda_j \in F$ and $\alpha_j \in E^{\text{ab}}$ for $1 \leq j \leq d$ such that*

$$\Phi_L^r(z_0, h_1, f) - \Phi_L^r(z_0, h_2, f) = \sum_{j=1}^d \lambda_j \log \left| \frac{\sigma_{h_1}(\alpha_j)}{\sigma_{h_2}(\alpha_j)} \right| \tag{1.7}$$

for any $(z_0, h_i) \in Z(W_{\mathbb{Q}})$. Here $\sigma_h \in \text{Gal}(E^{\text{ab}}/E)$ is the element associated to $h \in E^\times \backslash \hat{E}^\times$ via class field theory. In particular when $F = \mathbb{Q}(\sqrt{D})$ is real quadratic and $n + 2 = 2d = 4$, we can take $\lambda_1 = D^{r/2} / \kappa$ for some $\kappa \in \mathbb{N}$ and $\lambda_2 = 0$.

Remark 1.4 When $(V, Q) = (M_2(\mathbb{Q}), N \cdot \det)$, the Shimura variety $X_K(\mathbb{C})$ becomes $X_0(N)^2$ for suitable K [37, Sect. 3.1]. In that case, for CM points z_j with discriminant d_i , the CM point (z_1, z_2) is in $Z(W_{\mathbb{Q}})$ with W certain $F = \mathbb{Q}(\sqrt{d_1 d_2})$ -quadratic space. These are called ‘‘big CM points’’, resp. ‘‘small CM points’’, when F is real quadratic, resp. $F = \mathbb{Q}$.

Remark 1.5 Theorem 1.3 applies even when $Z(W_{\mathbb{Q}})$ intersects the singularity of Φ_L^r . In that case, the function $\Phi^r(z, 1, f) - \Phi^r(z, h, f)$ in z can be continued to a real-analytic function in the neighborhood of the singularity, and its value at $z = z_0$ defines the quantity on the left hand side of (1.7).

Previous results concerning any linear combinations of CM values of higher Green function either assume $n \leq 2$ or $d = 1$. Theorem 1.3 is the first result where the cases with $n \geq 3$ and $d \geq 2$ are addressed. In such cases, there is no known results even when one averages the higher Green function over all the CM points in all Galois conjugates of $Z(W_{\mathbb{Q}})$, unlike for Green functions studied in [13, 15]. Together with Theorem 1.1, Theorem 1.3 naturally leads one to expect the following.

Conjecture 1.6 *In the setting of Theorem 1.3, suppose f has integral Fourier coefficients and the singularity of $\Phi_L^r(z, h, f)$ does not intersect $Z(W_{\mathbb{Q}})$. Then there exists $\lambda_j \in F$ and $\alpha_j \in E^{\text{ab}}$ for $1 \leq j \leq d$ such that*

$$\Phi_L^r(z_0, h, f) = \sum_{j=1}^d \lambda_j \log |\sigma_h(\alpha_j)| \tag{1.8}$$

for all $(z_0, h) \in Z(W_{\mathbb{Q}})$.

When F is real quadratic, i.e. $d = 2$, we can confirm it in the following case.

Theorem 1.7 *Conjecture 1.6 holds when F is a real quadratic field, r is even, $n = 4$, and $Z(W_{\mathbb{Q}})$ is defined over \mathbb{Q} , in which case we can take $\lambda_1 \in \mathbb{Q}$ and $\lambda_2 = 0$.*

Remark 1.8 When E/\mathbb{Q} is Galois, there are many instances when $Z(W_{\mathbb{Q}})$ is defined over \mathbb{Q} (see e.g. Lemma 3.4 in [15]). In particular, the CM points on $X_0(1)^2$ satisfy this condition (see Example 2.7). Therefore, the case for even r in Theorem 1.1 follows from Theorem 1.7.

Remark 1.9 In a recent joint work [8], we have proved Conjecture 1.6 when E/\mathbb{Q} is biquadratic.

1.2 General proof strategy

When $F = \mathbb{Q}$, Conjecture 1.6 follows from Theorem 5.5 in [11]. Here we give a sketch of its proof, which is analytic in nature. First, one expresses $\Phi_L^r(z, h, f)$ as an integral of f against a suitable theta kernel $R_{\tau}^r \Theta_L(\tau, z, h)$, where R_{τ} is the raising operator (see (2.5)). Then a CM point $(z_0, h) \in Z(W_{\mathbb{Q}})$ leads to a rational splitting of V since $F = \mathbb{Q}$. Suppose it leads to an integral splitting of L into $L = \tilde{L} \oplus N$ with \tilde{L} and N definite lattices of signature $(n, 0)$ and $(0, 2)$ respectively. Then the theta kernel becomes²

$$\Theta_L(\tau, z_0, h) = \theta_{\tilde{L}}(\tau) \theta_N(\tau), \tag{1.9}$$

Note that θ_N is non-holomorphic and has weight -1 . One can then construct a preimage $\hat{\theta}_N$ of θ_N under the lowering operator L_{τ} . It is a harmonic Maass form of weight 1. The notion of harmonic Maass form was introduced in the seminal work

²For simplicity, we omit the detail about the modular forms being vector-valued.

of Bruinier and Funke [12], and studied around the same time by Zwegers in the context of modular completion of Ramanujan’s mock theta functions [40].

Applying the Rankin-Cohen operator (see (2.6)) to $\theta_{\tilde{L}}$ and θ_N then gives us a preimage of $R_{\tau}^r(\theta_{\tilde{L}}(\tau)\theta_N(\tau))$ under the L_{τ} . Putting these together and applying Stokes’ theorem gives us

$$\begin{aligned} \Phi_L^r(z_0, h, f) &= \int_{X_0(1)} f(\tau)R_{\tau}^r(\theta_{\tilde{L}}(\tau)\theta_N(\tau))d\mu(\tau) \\ &= \int_{X_0(1)} f(\tau)L_{\tau}\mathcal{C}_{(n/2,1),r}(\theta_{\tilde{L}}(\tau), \tilde{\theta}_N(\tau))d\mu(\tau) \\ &= \{f(\tau), \mathcal{C}_{(n/2,1),r}(\theta_{\tilde{L}}(\tau), \tilde{\theta}_N^+(\tau))\} = \text{CT}(\tilde{f}(\tau) \cdot \tilde{\theta}_N^+(\tau)), \end{aligned}$$

where $\{, \}$ is a pairing of formal Fourier series (see (4.13)) and CT denotes the constant term of a Fourier series. The function $\tilde{\theta}_N^+$ is the holomorphic part of $\tilde{\theta}_N$, and the modular form \tilde{f} in the last expression is weakly holomorphic with weight -1 and rational Fourier coefficients.

The harmonic Maass form $\tilde{\theta}_N$ of weight 1 was studied in [19, 20, 35]. It was shown that the term $\text{CT}(\tilde{f}(\tau) \cdot \tilde{\theta}_N^+(\tau))$ is the logarithm of an algebraic number. To see this, let P_1, P_2 be positive definite, unimodular lattices such that $\theta_{P_1} - \theta_{P_2}$ is holomorphic on \mathbb{H} . One can rewind the process above (with $r = 0$) and write

$$\begin{aligned} \text{CT}(\tilde{f}(\tau) \cdot \tilde{\theta}_N^+(\tau)) &= \int_{X_0(1)} \tilde{f}(\tau)\theta_N(\tau)d\mu(\tau) \\ &= \int_{X_0(1)} \hat{f}(\tau)(\theta_{P_1}(\tau) - \theta_{P_2}(\tau))\theta_N(\tau)d\mu(\tau) \\ &= \Phi_{L_1}(z_1, \hat{f}) - \Phi_{L_2}(z_2, \hat{f}), \end{aligned}$$

where $\hat{f} = \tilde{f} \cdot (\theta_{P_1} - \theta_{P_2})^{-1}$ is weakly holomorphic and $L_i = P_i \oplus N$. The functions $\Phi_{L_i}(z, \hat{f})$ are the regularized Borcherds lifts of \hat{f} and are logarithms of rational functions on Shimura varieties associated to L_i . Their values at CM points z_i are logarithms of algebraic numbers by the theory of complex multiplication. This finishes the sketch of the proof. The process of multiplying and dividing by $\theta_{P_1} - \theta_{P_2}$ a manifestation of the embedding trick (see [6, Sect. 8]).

The partial averaging result in [11, Theorem 1.2] used the coincidence that the average of $G_{r+1, f}(z_1, z)$ over the Galois orbit of z_1 is a higher Green function in z on the modular curve, i.e. $n + 2 = 3$. This is a rather special phenomenon that only happens when E/\mathbb{Q} is biquadratic. By (1.6), one is reduced to the case of $d = 1$ in Conjecture 1.6.

For $d \geq 2$, the lattice L splits as $\tilde{L} \oplus \text{Res}_{F/\mathbb{Q}}N$ with $N \subset W$ an \mathcal{O} -lattice of signature $((0, 2), (2, 0) \dots, (2, 0))$, and the analogue of (1.9) is

$$\Theta_L(\tau, z_0, h) = \theta_{\tilde{L}}(\tau)\theta_N^{\Delta}(\tau), \tag{1.10}$$

where $\theta_N^{\Delta}(\tau)$ is the diagonal restriction of the Hilbert theta function $\theta_N(\tau_1, \dots, \tau_d)$ of weight $(-1, 1 \dots, 1)$ associated to N . When one tries to execute the above strategy to

construct a preimage of θ_N^Δ under the lowering operator, it is necessary to work with Hilbert modular forms, and there are some serious obstacles.

- The analogous $\hat{\theta}_N$ should be a Hilbert modular form that satisfies suitable properties similar to those of harmonic Maass forms in the setting of elliptic modular forms. However, there is no suitable extension of the notion of harmonic Maass forms to higher rank groups.
- As harmonic Maass forms have singularities at the cusps, one would expect the same for the analogous $\hat{\theta}_N$. However, Koecher’s principle would imply that such $\hat{\theta}_N$ could not have singularity only at the cusps, but in the interior of the Hilbert modular variety as well. This also holds for its diagonal restriction and complicates the application of Stokes’ theorem.
- To extract information about the Fourier coefficient of $\hat{\theta}_N$, one needs the generalization of Borcherds’ lift over totally real fields. In a large part, this has been accomplished in [10] by considering regularized theta lifts of Whittaker forms. However, as the Shimura varieties appeared loc. cit. are compact, there is no Fourier expansion and one has limited information about the rationality of the lift. Furthermore, it seems hopeless to directly generalize the embedding trick in [6] to totally real fields.

Instead of studying the value at an individual CM point, one can average over CM points in $Z(W_{\mathbb{Q}})$, and those in $Z(W(j)_{\mathbb{Q}})$ for $2 \leq j \leq d$, where each $W(j)$ is a neighboring F -quadratic space of W (see Sect. 2.3). Then the rational quadratic spaces $\text{Res}_{F/\mathbb{Q}} W(j)$ are all isomorphic and

$$\sum_{1 \leq j \leq d} \sum_{(z,h) \in Z(W(j)_{\mathbb{Q}})} \Theta_L(\tau, z, h) = \theta_L(\tau) \otimes \sum_{1 \leq j \leq d} E_{N(j)}^\Delta(\tau),$$

$$E_{N(j)}(\tau_1, \dots, \tau_d) := \sum_{(z_j, h(j)) \in Z(W(j)_{\mathbb{Q}})} \theta_{N(j)}((\tau_1, \dots, \tau_d), h(j)),$$

with $N(j) \subset W(j)$ suitable lattices. The Hilbert modular form $\theta_{N(j_0)}$ is holomorphic in τ_j for $j \neq j_0$ and has weight $(1, \dots, 1, -1, 1, \dots, 1)$ with -1 at the j_0 -th place. One can now explicitly construct an incoherent Hilbert Eisenstein series \mathcal{E}_N of parallel weight 1 that maps to $E_{N(j)}$ under the lowering operator in τ_j for all $1 \leq j \leq d$. For $d = 2$, this is the real-analytic Eisenstein series that appeared in the seminal works of Gross and Zagier on singular moduli and the Gross-Zagier formula [23, 24]. It also appeared in [15], and has been combined with the regularized theta lifting of Borcherds to give fruitful generalizations of [23, 24] in [13, 16].

The advantage of \mathcal{E}_N is that its Fourier coefficients can be computed explicitly, and shown to be logarithms of rational numbers. They furthermore can be interpreted as arithmetic intersection numbers. On the other hand, it provides limited information about the arithmetic of higher Green function at a single CM point, as the differential operator in the strategy for $d = 1$ does not readily generalize except in the case $n = 2d = 4$ and r even (see the discussion at the end of Sect. 5 in [11]). The higher Green functions studied by Gross, Kohnen and Zagier in [22] happen to be in this single case.

1.3 Ideas

We now describe some ideas and observations that help to overcome the obstacles mentioned in the previous section:

- For any holomorphic Hilbert cusp form $g(\tau)$ and $\ell \in \mathbb{N}$ sufficiently large, the product $g(\tau)^\ell \theta_N(\tau)$ has a modular preimage under the lowering operator in τ_1 with no singularity in \mathbb{H}^d . Furthermore, this preimage is harmonic in τ_1 and holomorphic in τ_2, \dots, τ_d .
- The generalization of Borcherds' lift in [10] differs from the logarithm of an F -rational function by a locally constant function, which can be canceled out when considering differences of linear combinations of CM values.
- The embedding trick only needs to work along the diagonal of \mathbb{H}^d , and one can apply the Siegel-Weil formula to replace the difficult task of finding suitable positive definite \mathcal{O} -lattice P to the simpler one of analyzing Eisenstein series.

The first idea is inspired by Zwegers' work [40], where the product of a mock theta function and a classical theta function is completed to become a real-analytic modular form without singularity in \mathbb{H} . Such products are also called "mixed mock-modular forms" in [18] and are natural objects to consider. Since the differential operators in τ_1, \dots, τ_d are all independent, this idea can be applied in the setting of Hilbert modular forms. The existence of the modular preimage will be proved using complex geometry (see Sect. 4.1), as done in the elliptic case in [12]. The parameter ℓ serves to ensure certain cohomology group vanishes (see Theorem 4.1). The Rankin-Cohen differential operator can also be generalized to be applied on such functions (see the differential operator $\mathcal{D}_{\kappa,r}$ in (2.13)).

The second idea is a compromise so that one can still use the generalization of Borcherds' lift in [10] to deduce algebraicity results. Considering differences is quite effective in removing the so-called "normalizing constant" in the regularized theta lift (see Theorem 1.1 in [9]), as different linear combinations could give rise to the same normalizing constant. Furthermore, considering the difference turns out to simplify many other situations as well. For example, it is enough to construct a preimage of $g(\tau)^\ell (\theta_{N_1}(\tau) - \theta_{N_2}(\tau))$ with $g(\tau)$ a holomorphic Hilbert cusp form. This is accomplished in Theorem 4.3, using the ampleness of twists of determinant of the Hodge bundle on toroidal compactifications of Hilbert modular varieties, which is contained in Theorem 4.1 and a result of independent interest.³ Also, one does not need to worry so much about the singularity of Φ_L^r (see Remark 1.5 and Lemma 4.5). The linear combination we take will come from multiplying this preimage with an Eisenstein series $E_{\tilde{p}}$. This leads to the crucial algebraicity result in Theorem 4.10, which is of independent interest.

For the embedding trick, the last idea reduces the problem of dividing by a Hilbert cusp form g , which is constructed from theta series, to dividing by its diagonal restriction g^Δ , which is an elliptic modular form. Using the Siegel-Weil formula, we can relate g to Hilbert Eisenstein series. By varying the weight, we will show that for any finite set of points in \mathbb{H}^d , there is a Hilbert Eisenstein series that does not vanish

³We thank the referee for a helpful suggestion that led to this result.

on this set (see Lemma 3.2). This observation has its root in the classical work [31] of Rankin and Swinnerton-Dyer on zeros of elliptic Eisenstein series, and leads to the “partition of unity” result in Proposition 3.4.

By putting these ideas together, we are able to overcome the obstacles and prove Theorem 1.3. When it is specialized to the case in Theorem 1.1, we can combine this result about differences with the result about partial averages in [11] to complete the proof.

1.4 Outlook and organization

To prove Conjecture 1.6, one needs algebraicity results concerning sums of CM points, in addition to the “difference result” in Theorem 1.3. For real quadratic F , we have worked out such a “sum result” when E/\mathbb{Q} is biquadratic in [8], which has led to a proof of Conjecture 1.6 in this case. When $d \geq 3$, one can try to relate the (in)coherent Eisenstein series to Eisenstein series on $O(2, 1)$ over F , and realize them as suitable theta lifts from SL_2 over F . We plan to pursue this idea in a future work.

The paper is organized as follows. In Sects. 2 and 3, we setup notations and collect various preliminary notions from the literature. Results such as Lemma 3.2 and Proposition 3.4 seem to be new, and form a crucial trick in the proof of Theorem 1.3. In Sect. 4, we construct certain real-analytic Hilbert modular form in Theorem 4.3 and prove algebraicity result about linear combinations of their Fourier coefficients in Theorem 4.10. Putting these together, we give the proofs of Theorems 1.1, 1.3 and 1.7 in Sect. 5.

2 Preliminary

Fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Throughout the paper, F will be a totally real field of degree d with ring of integers \mathcal{O} , different \mathfrak{d} and discriminant D . For $1 \leq j \leq d$ and $m \in F$, denote $\sigma_j : F \hookrightarrow \mathbb{R}$ the real embeddings of F and $m_j := \sigma_j(m)$. We write $m \gg 0$ if $m \in F$ is totally positive, i.e. $m_j > 0$ for all $1 \leq j \leq d$. For a number field E with ring of integers \mathcal{O}_E , let \mathbb{A}_E and $\hat{E} := E \otimes \hat{\mathbb{Z}}$ be the adèles and finite adèles respectively. The subgroup $\hat{\mathcal{O}}_E := \mathcal{O}_E \otimes \hat{\mathbb{Z}} \subset \hat{E}$ is open and compact.

Given $\tau = (\tau_1, \dots, \tau_d) \in \mathbb{H}^d$, we write $v = (v_1, \dots, v_d) := \mathfrak{I}(\tau) \in (\mathbb{R}_{>0})^d$. For a function f on \mathbb{H}^d , we will write f^Δ for its diagonal restriction to $\mathbb{H} \subset \mathbb{H}^d$. For $\alpha \in \mathbb{C}$, denote

$$e(\alpha) := e^{2\pi i \alpha}.$$

For $x = (x_j)_{1 \leq j \leq d}, y = (y_j)_{1 \leq j \leq d} \in \mathbb{C}^d$, we denote

$$xy = (x_j y_j)_{1 \leq j \leq d} \in \mathbb{C}^d, \text{tr}(x) := \sum_{1 \leq j \leq d} x_j, \text{Nm}(x) := \prod_{1 \leq j \leq d} x_j.$$

For a semigroup G and a G -graded ring $R = \bigoplus_{i \in G} R_i$, we use

$$R^{G_0} := \bigoplus_{i \in G_0} R_i \subset R$$

for a sub-semigroup $G_0 \subset G$. Also, we denote \mathbb{N} the positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

2.1 Modular forms

For a congruence subgroup $\Gamma \subset \text{SL}_2(\mathcal{O})$, a finite dimensional, unitary representation ρ of Γ on a finite dimensional hermitian space $(\mathcal{V}, \langle \cdot, \cdot \rangle)$, and weight $\kappa = (k_1, \dots, k_d) \in \mathbb{Z}^d$, let $\mathcal{A}_{\kappa, \rho}(\Gamma)$ denote the \mathbb{C} -vector space of vector-valued, real-analytic functions on \mathbb{H}^d invariant with respect to ρ on Γ of weight κ , and bounded near the cusps of $\Gamma \backslash \mathbb{H}^d$. It contains the subspaces $S_{\kappa, \rho}(\Gamma) \subset M_{\kappa, \rho}(\Gamma)$ of cuspidal and holomorphic Hilbert modular forms. We also write $\vec{k} := (k, \dots, k)$.

For any $f \in \mathcal{A}_{\kappa, \rho}(\Gamma)$, the function on \mathbb{H}^d

$$\|f(\tau)\|_{\text{Pet}}^2 := \langle f(\tau), f(\tau) \rangle \text{Nm}(v^\kappa) \tag{2.1}$$

is Γ -invariant. Given $f, g \in \mathcal{A}_{\kappa, \rho}(\Gamma)$ such that at least one of them has exponential decay near the cusps, we can define their Petersson inner product

$$(f, g)_{\text{Pet}} := \frac{1}{\sqrt{D}} \int_{\Gamma \backslash \mathbb{H}^d} \langle f(\tau), g(\tau) \rangle \text{Nm}(v^\kappa) d\mu(\tau), \tag{2.2}$$

where $d\mu(\tau) := d\mu(\tau_1) \dots d\mu(\tau_d)$ is the invariant measure on \mathbb{H}^d (see Equation (4.21) in [10]). For $\kappa = (k_1, \dots, k_d)$, denote the following related weights

$$\tilde{\kappa} := (-k_1, k_2, \dots, k_d), \hat{\kappa} := (2 - k_1, k_2, \dots, k_d). \tag{2.3}$$

We omit Γ , resp. ρ , from the notation when F is fixed and $\Gamma = \Gamma_F := \text{SL}_2(\mathcal{O})$, resp. it is trivial. When $F = \mathbb{Q}$, we will use the superscript ! to indicate modular forms with singularities at the cusps.

When ρ is trivial, it is known that $M_\kappa(\Gamma_F) = M_\kappa(\Gamma_F, \mathbb{Q}) \otimes \mathbb{C}$, where $M_\kappa(\Gamma_F, \mathbb{Q})$ is the subspace of modular forms with rational Fourier coefficients. This is also the case for $M_{\kappa, \rho}$ when ρ is a Weil representation defined below (see [29], [10, Sect. 7]).

For later purposes, we will be interested in the (\mathbb{N}^d) -graded ring

$$\mathcal{M}_F := \bigoplus_{\kappa \in \mathbb{N}^d} M_\kappa(\Gamma_F, \mathbb{Q}). \tag{2.4}$$

2.2 Differential operators

For $k \in \mathbb{Z}$, we have the usual raising, lowering and hyperbolic Laplacian operators on \mathbb{H}

$$\begin{aligned} R_{\tau, k} &:= 2i\partial_\tau + \frac{k}{v}, \tilde{R}_{\tau, k} := (4\pi)^{-1}R_{\tau, k}, L_{\tau, k} := -2iv^2\partial_{\bar{\tau}}, \\ \Delta_{\tau, k} &:= -R_{\tau, k-2}L_{\tau, k} = -L_{\tau, k+2}R_{\tau, k} - k \\ &= -v^2 \left(\partial_u^2 + \partial_v^2 \right) + ikv(\partial_u + i\partial_v). \end{aligned} \tag{2.5}$$

They change the weight by 2, -2 and 0 respectively. For $\kappa = (k_1, k_2) \in \mathbb{Q}^2$ and $r \in \mathbb{N}_0$, we can define the Rankin-Cohen operator on a real-analytic function $f(\tau_1, \tau_2) \in \mathbb{H}^2$ by

$$\begin{aligned} \mathcal{C}_{\kappa,r}(f)(\tau) &:= (2\pi i)^{-r} \sum_{s=0}^r (-1)^s \binom{k_1+r-1}{s} \binom{k_2+r-1}{r-s} \left(\partial_{\tau_1}^{(r-s)} \partial_{\tau_2}^{(s)} f \right) (\tau, \tau), \\ &= \sum_{s=0}^r (-1)^{r-s} \binom{k_1+r-1}{s} \binom{k_2+r-1}{r-s} \left(\tilde{R}_{\tau_1, k_1}^{r-s} \tilde{R}_{\tau_2, k_2}^s f \right) (\tau, \tau), \end{aligned} \tag{2.6}$$

where $\binom{m}{n} := \frac{m(m-1)(m-2)\dots(m-n+1)}{n!}$ is the binomial coefficient. The equality in the second line can be proved by considering the generating series constructed from the differential operators ∂_τ and R_τ . The details are contained in Sect. 5.2 of [14], in particular Propositions 18 and 19. The first expression shows that the operator preserves holomorphicity. When $\kappa \in \mathbb{Z}^2$, the second expression shows that it preserves modularity in the sense that

$$\mathcal{C}_{\kappa,r}(f) |_{k_1+k_2+2r} \gamma = \mathcal{C}_{\kappa,r}(f |_\kappa (\gamma, \gamma)) \tag{2.7}$$

for any $\gamma \in \text{SL}_2(\mathbb{R})$. The same result holds in the metaplectic setting when $\kappa \in \frac{1}{2}\mathbb{Z}^2$.

Example 2.1 Suppose $\kappa = (1, 1)$ and $f(\tau_1, \tau_2) = \mathbf{e}(\alpha_1 \tau_1 + \alpha_2 \tau_2)$ for $\alpha \in \mathbb{Q}(\sqrt{D})$ with $\text{tr}(\alpha) \neq 0$. Then

$$\begin{aligned} \mathbf{e}(-\text{tr}(\alpha)\tau) \mathcal{C}_{\kappa,r}(f)(\tau) &= \sum_{s=0}^r \binom{r}{s}^2 \alpha_1^{r-s} (-\alpha_2)^s \\ &= \text{tr}(\alpha)^r 2^{-r} \sum_{s=0}^r \binom{r}{s}^2 \left(\frac{2\alpha_1}{\alpha_1 + \alpha_2} \right)^{r-s} \left(\frac{-2\alpha_2}{\alpha_1 + \alpha_2} \right)^s \\ &= \text{tr}(\alpha)^r 2^{-r} \sum_{s=0}^r \binom{r}{s}^2 (x+1)^{r-s} (x-1)^s = \text{tr}(\alpha)^r P_r(x) \end{aligned}$$

with $x = \frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2} = \frac{\sqrt{D}\text{tr}(\alpha/\sqrt{D})}{\text{tr}(\alpha)}$ and $P_r(X)$ the r -th Legendre polynomial, which has parity $(-1)^r$. The last equality is a consequence of Rodrigues' formula (see (8.6.18) in [1]). This example will be used in the proof of Theorem 1.3.

When $f(\tau_1, \tau_2) = f_1(\tau_1) f_2(\tau_2)$ with f_i modular forms of weight $k_i \in \frac{1}{2}\mathbb{Z}$, the function $[f_1, f_2]_r := \mathcal{C}_{\kappa,r}(f)$ is the usual Rankin-Cohen bracket of f_1 and f_2 [14, Sect. 5.2]. If f_1 is harmonic of weight k_1 and f_2 is holomorphic of weight k_2 , then we have

$$L_\tau [f_1, f_2]_r = \binom{k_1+r-1}{r} L_\tau f_1 \tilde{R}_{k_2}^r f_2 = \binom{k_1+r-1}{r} \tilde{R}_{k_1+k_2-2}^r L_\tau (f_1 f_2) \tag{2.8}$$

for any $r \in \mathbb{N}_0$.

Lemma 2.2 For real-analytic functions $f, g : \mathbb{H} \rightarrow \mathbb{C}$, rational numbers k, ℓ and an integer $r \geq 0$, if $k + \ell \notin \{-2r + 2, -2r + 3, \dots, 0\}$,⁴ then there exists $c_{r,a,j} \in \mathbb{Q}$ such that

$$\tilde{R}_k^a f \tilde{R}_\ell^{r-a} g = \sum_{j=0}^r c_{r,a,j} \tilde{R}_{k+\ell+2j}^{r-j} [f, g]_j \tag{2.9}$$

for all $0 \leq a \leq r$.

Proof This is done by induction on r . The base case of $r = 0$ is trivial. For the inductive step to prove the case $r + 1$, we have $k + \ell \notin \{-2r, -2r + 1, \dots, 0\}$. Denote $x_a := \tilde{R}^a f \tilde{R}^{r+1-a} g$ for $0 \leq a \leq r + 1$. Applying \tilde{R} to (2.9) shows that $x_a + x_{a+1}$ is a rational linear combination of $\tilde{R}^{r+1-j} [f, g]_j$'s over $0 \leq j \leq r$ for any $0 \leq a \leq r$. From definition, we also have

$$\sum_{a=0}^{r+1} c_a x_a = [f, g]_{r+1}, \quad c_a := (-1)^a \binom{k+r}{r+1-a} \binom{\ell+r}{a}.$$

Therefore, the right hand side below is a rational linear combination of $\tilde{R}^{r+1-j} [f, g]_j$ for $0 \leq j \leq r + 1$ and it suffices to show that the square matrix on the left is invertible

$$\begin{pmatrix} A \\ c_0 & c_1 & \dots & c_r & c_{r+1} \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ \vdots \\ x_{r+1} \end{pmatrix} = \begin{pmatrix} x_0 + x_1 \\ \vdots \\ x_r + x_{r+1} \\ [f, g]_{r+1} \end{pmatrix}, \quad A := \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 \\ & & \dots & & & \\ 0 & 0 & \dots & 0 & 1 & 1 \end{pmatrix}.$$

The right kernel of A is spanned by the vector $((-1)^a)_{0 \leq a \leq r+1}$. On the other hand

$$\sum_{a=0}^{r+1} c_a \cdot (-1)^a = \sum_{a=0}^{r+1} \binom{k+r}{r+1-a} \binom{\ell+r}{a} = \binom{k+\ell+2r}{r+1},$$

which is zero precisely when $k + \ell \in \{-2r, -2r + 1, \dots, -r\}$. This is not possible by the condition imposed on $k + \ell$. Therefore the matrix $\begin{pmatrix} A \\ c_0 \dots c_{r+1} \end{pmatrix}$ is invertible. \square

Now we will extend the Rankin-Cohen operator to functions on \mathbb{H}^d for any $d \geq 2$ by first restricting it to \mathbb{H}^2 , before applying the usual Rankin-Cohen operator. This can be expressed as a linear combination of the generalized Rankin-Cohen operators studied in [26]. For $f : \mathbb{H}^d \rightarrow \mathbb{C}$, $\kappa = (k_1, \dots, k_d) \in \mathbb{Z}^d$, denote $f^{\Delta,1}(\tau', \tau_1) := f(\tau_1, \tau', \dots, \tau')$, $\kappa(1) := (\text{tr}(\kappa) - k_1, k_1) \in \mathbb{Z}^2$ and define

$$\mathcal{C}_{\kappa,r}^1(f)(\tau) := \mathcal{C}_{\kappa(1),r}(f^{\Delta,1})(\tau). \tag{2.10}$$

It is easy to check that $(f \mid_\kappa (\gamma, \dots, \gamma))^{\Delta,1} = f^{\Delta,1} \mid_{\kappa(1)} (\gamma, \gamma)$ and

$$\mathcal{C}_{\kappa,r}^1(f) \mid_{\text{tr}(\kappa)+2r} \gamma = \mathcal{C}_{\kappa,r}^1(f \mid_\kappa (\gamma, \dots, \gamma)) \tag{2.11}$$

⁴We take this set to be empty for $r = 0$.

for all $\gamma \in \text{SL}_2(\mathbb{R})$. Suppose $f(\tau) = q_1^{\alpha_1} \dots q_d^{\alpha_d}$ with $q_j := \mathbf{e}(\tau_j)$, $\alpha_j \in \mathbb{C}$, then

$$C_{\kappa,r}^1(f) = \sum_{s=0}^r (-1)^s \binom{k_1 + r - 1}{s} \binom{\text{tr}(\kappa) - k_1 + r - 1}{r - s} \alpha_1^{r-s} (\alpha - \alpha_1)^s q^\alpha, \tag{2.12}$$

where $\alpha := \alpha_1 + \dots + \alpha_d$. Analogous definitions also make sense when the index 1 above is replaced by any $j \in \{1, \dots, d\}$.

Finally for $f, g \in \mathbb{H}^d \rightarrow \mathbb{C}$ real-analytic and $\kappa \in \mathbb{Z}^d$, we define

$$\begin{aligned} \mathcal{D}_{\kappa,r}(f, g) &:= (g^{r+1})^\Delta C_{\kappa,r}^1(f/g) \\ &= (2\pi i)^{-r} \left(\sum_{\substack{e=(e_1, \dots, e_d) \in \mathbb{N}_0^d \\ \text{tr}(e)=r}} a_e g^{r+1} \partial_{\tau_1}^{e_1} \dots \partial_{\tau_d}^{e_d} (f/g) \right)^\Delta \end{aligned} \tag{2.13}$$

with $a_e \in \mathbb{Z}$ explicit constants given by

$$a_e := (-1)^{e_2 + \dots + e_d} \binom{k_1 + r - 1}{e_2 + \dots + e_d} \binom{k_2 + \dots + k_d + r - 1}{e_1} \frac{s!}{e_2! \dots e_d!}.$$

From the definition, one sees that $\mathcal{D}_{\kappa,r}(f, g)$ is real-analytic on \mathbb{H} and satisfies

$$\mathcal{D}_{\kappa,r}(f|_{\kappa+\lambda}(\gamma, \dots, \gamma), g|_\lambda(\gamma, \dots, \gamma)) = \mathcal{D}_{\kappa,r}(f, g)|_{\text{tr}(\kappa+(r+1)\lambda)+2r} \gamma \tag{2.14}$$

for $\kappa, \lambda \in \mathbb{Z}^d$ and $\gamma \in \text{SL}_2(\mathbb{R})$. The upshot of this operator is the following result.

Lemma 2.3 *For $\kappa = (k_1, \dots, k_d) \in \mathbb{Z}$, let $f : \mathbb{H}^d \rightarrow \mathbb{C}$ be a real-analytic function that is harmonic in τ_1 of weight k_1 and holomorphic in τ_2, \dots, τ_d . For any holomorphic function $g : \mathbb{H}^d \rightarrow \mathbb{C}$, we have*

$$L_\tau \mathcal{D}_{\kappa,r}(f, g) = \binom{k_1 + r - 1}{r} (g^\Delta)^{r+1} \tilde{R}_{\text{tr}(\kappa)-2}^r ((L_{\tau_1} f)/g)^\Delta \tag{2.15}$$

for all $r \in \mathbb{N}_0$.

Proof This follows directly from the definition and equation (2.8). □

We can also componentwisely apply $\mathcal{D}_{\kappa,r}$ when f is vector-valued, in which case we also write $\mathcal{D}_{\kappa,r}(f, g)$, and the result above holds as well.

2.3 Quadratic space and Shimura variety

Let V be a finite dimensional F -vector space of dimension $n + 2 \geq 0$ with a non-degenerate quadratic form Q . For our purpose, n is even when $d \geq 2$, i.e. $F \neq \mathbb{Q}$. Denote $V_{\sigma_j} := V \otimes_{F, \sigma_j} \mathbb{R}$ for $1 \leq j \leq d$, which is an \mathbb{R} -quadratic space of signature (p_j, q_j) , and $V(\mathbb{R}) = V \otimes_{\mathbb{Q}} \mathbb{R} = \oplus_i V_{\sigma_i}$ is an \mathbb{R} -quadratic space of signature (p, q)

with $p = \sum_j p_j, q = \sum_j q_j$. We say that V is totally positive if $V(\mathbb{R})$ is positive definite. The symmetric domain \mathbb{D} associated to $\text{SO}(V(\mathbb{R}))$ is realized as the Grassmannian of q -dimensional negative definite oriented subspaces of $V(\mathbb{R})$. It consists of 2 components unless $q = 0$, in which case it is a point.

Let GSpin_V be the general spin group of V . We will be interested in the \mathbb{Q} -algebraic group

$$H = H_V := \text{Res}_{F/\mathbb{Q}} \text{GSpin}_V, \tag{2.16}$$

which fits into the exact sequence

$$1 \rightarrow Z \rightarrow H \rightarrow \text{Res}_{F/\mathbb{Q}} \text{SO}(V) \rightarrow 1 \tag{2.17}$$

with $Z(\mathbb{Q}) \cong F^\times$. Denote $\nu : C(V) \rightarrow F^\times$ the spinor norm on the Clifford algebra $C(V)$ of V , which induces a surjection $\nu : H \rightarrow T := \text{Res}_{F/\mathbb{Q}} \mathbb{G}_m$ of algebraic groups.

Example 2.4 More generally, the group GSpin can be defined for a quadratic module M over a commutative ring R . For a nice example, we consider the hyperbolic plane, where $M = R^2$ is a free R -module with quadratic form $Q((a, b)) = ab$. Furthermore denote e_1, e_2 the images of $(1, 0), (0, 1) \in M$ in the Clifford algebra $C(M)$, and $e_0 := e_1 e_2, e_3 := e_2 e_1 \in C(M)$. Then we have $e_0 + e_3 = 1$ in $C(M)$ and an R -algebra isomorphism

$$C(M) = \bigoplus_{i=0}^3 R \cdot e_i \cong M_2(R), (a_0, a_1, a_2, a_3) \mapsto \begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix}.$$

The even Clifford algebra $C^0(M)$ corresponds precisely to the diagonal matrices in $M_2(R)$. The group GSpin then consists of invertible diagonal matrices, and the spinor norm ν is just the determinant.

For the rest of this subsection, suppose V has signature

$$((n, 2), (n + 2, 0), \dots, (n + 2, 0)), 2 \mid n. \tag{2.18}$$

Then the hermitian symmetric space associated to H can be realized as the Grassmannian $\mathbb{D} = \mathbb{D}_V = \mathbb{D}^+ \sqcup \mathbb{D}^-$ of oriented negative-definite 2-planes of V_{σ_1} . If we denote $V_{\mathbb{C}} := V \otimes_{F, \sigma_1} \mathbb{C}$ and extend the quadratic form \mathbb{C} -bilinearly to $V_{\mathbb{C}}$, then we can identify \mathbb{D} with the quadric

$$\mathcal{H} := \{[Z] \in \mathbb{P}(V_{\mathbb{C}}) : (Z, Z) = 0, (Z, \bar{Z}) < 0\}, \tag{2.19}$$

in the projective space $\mathbb{P}(V_{\mathbb{C}})$ by sending $[Z = X + iY]$ to the oriented 2-plane spanned by the ordered basis $\{X, Y\} \subset V_{\sigma_1}$. This endows \mathbb{D} with a complex structure. We can furthermore identify the tube domain

$$\mathcal{H} := \{z \in V_0 \otimes_{\mathbb{R}} \mathbb{C} : Q(\Im(z)) < 0\},$$

where $V_0 := V_{\sigma_1} \cap a^\perp \cap b^\perp$ for isotropic vectors $a, b \in V_{\sigma_1}$ with $(a, b) = 1$, with \mathcal{H} by sending z to the class of

$$w(z) := z + a - Q(z)b$$

in $\mathbb{P}(V_{\mathbb{C}})$. For $\gamma \in H(\mathbb{R})$, we have the automorphy factor $j(\gamma, z) = (\gamma w(z), b)$ from

$$\gamma w(z) = j(\gamma, z)w(\gamma z).$$

For $z \in \mathbb{D}^\pm$, denote $\bar{z} \in \mathbb{D}^\mp$ the 2-plane with the opposite orientation. The subgroup of $H(\mathbb{R})$ fixing \mathbb{D}^+ is the subgroup $H(\mathbb{R})_+$ consisting of elements with totally positive spinor norm. For a compact open $K \subset H(\hat{\mathbb{Q}})$, the \mathbb{C} -points of the Shimura variety associated to H

$$X_{V,K} = X_K := H(\mathbb{Q}) \backslash \mathbb{D} \times H(\hat{\mathbb{Q}}) / K \tag{2.20}$$

is a complex quasi-projective variety of dimension n , and has a canonical model over $\sigma_1(F)$ [32]. When V is anisotropic over F , the variety X_K is projective.

Example 2.5 For $F = \mathbb{Q}$ and $(V, Q) = (M_2(\mathbb{Q}), \det)$, we have

$$\mathbb{H}^2 \cup (\mathbb{H}^-)^2 \cong \mathbb{D}, \quad (z_1, z_2) \mapsto \mathbb{R}\Re Z + \mathbb{R}\Im Z,$$

where the line spanned by $Z = Z(z_1, z_2) := \begin{pmatrix} z_1 & -z_1 z_2 \\ 1 & -z_2 \end{pmatrix} \in V(\mathbb{C})$ is in the quadric \mathcal{H} defined in (2.19). For a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, there exists compact open $K_\Gamma \subset H(\hat{\mathbb{Q}})$ such that the connected component of the Shimura variety X_{V,K_Γ} can be identified with the product of modular curves $X_\Gamma \times X_\Gamma$. See Sect. 3.1 in [37] for more details.

A meromorphic modular form on X_K of weight $w \in \mathbb{Z}$ is a collection of meromorphic functions $\Psi(\cdot, h) : \mathcal{H} \rightarrow \mathbb{C}$ for each $h \in H(\hat{\mathbb{Q}})$ satisfying

$$\begin{aligned} \Psi(z, hk) &= \Psi(z, h) \text{ for all } k \in K, \\ \Psi(\gamma z, \gamma h) &= j(\gamma, z)^w \Psi(z, h), \text{ for all } \gamma \in H(\mathbb{Q}) \end{aligned} \tag{2.21}$$

and are meromorphic at the boundary.⁵ For such a meromorphic modular form, we also denote

$$\|\Psi(z, h)\|_{\mathrm{Pet}} := |\Psi(z, h)| \cdot |y|^w, \tag{2.22}$$

which is a real-analytic function on X_K (see Sect. 2 of [10]).

To describe the connected components of X_K , we write

$$H(\hat{\mathbb{Q}}) = \coprod_j H(\mathbb{Q})_+ h_j K,$$

⁵The boundary behavior is relevant for us when $d = 1$ as V will otherwise be anisotropic.

where $H(\mathbb{Q})_+ = H(\mathbb{Q}) \cap H(\mathbb{R})_+$. Then for $n > 0$, we have $X_K \cong \coprod_j \Gamma_{h_j} \backslash \mathbb{D}^+$ and (1.8) of [25] gives us

$$\pi_0(X_K) \cong H(\mathbb{Q})_+ \backslash H(\hat{\mathbb{Q}}) / K \cong F_+^\times \backslash \hat{F}^\times / \nu(K) \cong \text{Gal}(E_K / F), \tag{2.23}$$

where E_K / F is a finite Galois extension that the connected component $Y_K := \Gamma_1 \backslash \mathbb{D}^+$ is defined over. Furthermore, for $\sigma \in \text{Gal}(E_K / F)$ associated to $\nu(h_j^{-1})$, we have $Y_K^\sigma \cong Y_{h_j K h_j^{-1}}$ over E_K and

$$X_K \cong \coprod_j Y_{h_j K h_j^{-1}} \cong \coprod_{\sigma \in \text{Gal}(E_K / F)} Y_K^\sigma.$$

When restricted to the center Z in (2.17), the map ν above is simply the square map and its image consists of square elements in $\text{Gal}(E_K / F)$.

When $n = 0$, the domain \mathbb{D} has two points and the group GSpin_V can be identified with E_W^\times for a totally imaginary, quadratic extension E_W over F , the norm from E_W to F is simply the spinor norm, and

$$X_K \cong \mathbb{D} \times \mathbb{A}_{E_W}^\times / E_W^\times (E_W)_\infty^\times K. \tag{2.24}$$

For $1 \leq j \leq d$, there is a unique F -quadratic space $V(j)$ with signature

$$\text{sig}(V(j)) = ((n + 2, 0), \dots, (n + 2, 0), (n, 2), (n + 2, 0), \dots, (n + 2, 0))$$

and isomorphic to V at all finite places. They are neighboring quadratic spaces of an admissible incoherent quadratic space (\mathbb{V}, Q) over \hat{F} (see [9, Sect. 7]). One can carry out the construction before (2.20) to define $X_{V(j), K}$, which is the \mathbb{C} -points of a Shimura variety defined over $\sigma_j(F)$. There is a quasi-projective variety \mathbb{X}_K defined over F such that the base change to $\sigma_j(F)$ is $X_{V(j), K}$, and the union of $X_{V(j), K}$ over all j is the \mathbb{C} -points of \mathbb{X}_K considered as a scheme over \mathbb{Q} (see Lemma 7.1 of [9]).

2.4 Unimodular lattice

An \mathcal{O} -lattice $L \subset V$, i.e. a finitely generated \mathcal{O} -module satisfying $L \otimes F = V$, is called even, resp. integral, if $Q(L)$ is in \mathfrak{d}^{-1} , resp. \mathcal{O} . For an even \mathcal{O} -lattice (L, Q) , the quadratic form $Q_{\mathbb{Z}}(x) := \text{tr}_{F/\mathbb{Q}} Q(x)$ is \mathbb{Z} -valued, and we denote

$$L' := \{y \in V : (y, L) \subset \mathfrak{d}^{-1}\}, \tag{2.25}$$

which is the dual of the \mathbb{Z} -lattice L with respect to $Q_{\mathbb{Z}}$. For $\mu \in L' / L$ and $m \in F$, we write

$$L_{m, \mu} := \{\lambda \in L + \mu : Q(\lambda) = m\} \subset L'. \tag{2.26}$$

which is empty if $m \notin \mathfrak{d}^{-1} + Q(\mu)$.

Also, we denote

$$S_L := \bigoplus_{\mu \in \hat{L}' / \hat{L}} \mathbb{C} \phi_\mu \subset \mathcal{S}(V(\hat{F})), \quad \phi_\mu := \text{char}(\hat{L} + \mu) \tag{2.27}$$

the subspace of Schwartz functions with support on $\hat{L}' := L' \otimes \hat{\mathcal{O}}$ and constant on $\hat{L} := L \otimes \hat{\mathcal{O}}$. Note that $\hat{L}'/\hat{L} = L'/L$ is a finite abelian group, and we write $\hat{L}_{m,\mu} := L_{m,\mu} \otimes \hat{\mathcal{O}}$. For any $h \in H(\hat{\mathbb{Q}})$, the lattice⁶

$$L_h := V \cap h^{-1} \cdot \hat{L} \subset V \tag{2.28}$$

satisfies $h^{-1} \cdot \hat{L} = \hat{L}_h$ and is in the same genus as L . Using the left action of h^{-1} , we identify

$$L'/L = \hat{L}'/\hat{L} \cong \hat{L}'_h/\hat{L}_h = L'_h/L_h. \tag{2.29}$$

The linear isomorphism $\iota_h : S_L \rightarrow S_{L_h}$, which sends ϕ_μ to $\phi_{h^{-1}\mu}$, then identifies ρ_L with ρ_{L_h} .

We say that a lattice L is \mathbb{Z} -unimodular if $L' = L$. Then the set

$$\mathcal{U}_F := \{(L, Q) : L \text{ is an even } \mathbb{Z}\text{-unimodular } \mathcal{O}\text{-lattice}\} \tag{2.30}$$

is a commutative monoid with respect to \oplus . Let $\mathcal{U}_F^+ \subset \mathcal{U}_F$ denote the semigroup consisting of totally positive, non-trivial lattices. This set is non-empty by the following result.

Proposition 2.6 *For any totally real field F , the semigroup \mathcal{U}_F^+ is non-trivial.*

Proof For an integral \mathcal{O} -lattice $L \subset V$, let $\mathfrak{d}(L) \subset \mathcal{O}$ be the discriminant ideal of L (see [17]). Then $\mathfrak{d}(L) = \mathcal{O}$ if and only if $L^\# = L$, where $L^\# := \{y \in V : (y, L) \subset \mathcal{O}\}$ is the \mathcal{O} -dual of L . In this case, L is called unimodular. Satz 1 in [17] gives a necessary and sufficient condition for the existence of definite unimodular \mathcal{O} -lattices, which is easily seen to be satisfied when $V = W^{\oplus 4}$ with $\dim_F(W)$ is divisible by 2. Furthermore, for any one of the 2^d possible signatures for definite spaces, there is a space V having this signature and containing a unimodular \mathcal{O} -lattice. So for any $\alpha \in F^\times$, there is a definite space V such that it becomes totally positive definite after scaling its quadratic form by α .

It is a well-known result of Hecke (see the last Theorem in [36]) that the class of \mathfrak{d} in $\text{Cl}(F)$ is a square. So we can write $\mathfrak{d}^{-1} = \mathfrak{a}^2(\delta)$ with $\mathfrak{a} \subset \mathcal{O}$ and $\delta \in F$. Let (L, Q) be a non-trivial, integral unimodular \mathcal{O} -lattice such that δQ is totally positive definite. Then $(\mathfrak{a}L, \delta Q)$ is an even \mathcal{O} -lattice and

$$\delta(\lambda, \mathfrak{a}L) \subset \mathfrak{d}^{-1} = \mathfrak{a}^2(\delta) \Leftrightarrow (\lambda, \mathfrak{a}^{-1}L) \subset \mathcal{O}$$

for all $\lambda \in (\mathfrak{a}L)'$. So $\lambda \in \mathfrak{a}L^\# = \mathfrak{a}L$ and $(\mathfrak{a}L, \delta Q)$ is non-trivial, \mathbb{Z} -unimodular and totally positive definite. □

2.5 Special cycles

Now suppose V decomposes as $W \oplus U$ such that U is totally positive subspace of dimension r . Then the Grassmannian \mathbb{D}_U of U consists of one point z_U , and \mathbb{D}_W can

⁶The inverse in this definition makes the action of h a right action.

be realized as an analytic submanifold of \mathbb{D} via

$$z_W \mapsto (z_W, z_U) \in \mathbb{D}_V.$$

Similarly, the algebraic group $H_W := \text{Res}_{F/\mathbb{Q}}\text{GSpin}_W$, resp. $H_U := \text{Res}_{F/\mathbb{Q}}\text{GSpin}_U$, is isomorphic to the pointwise stabilizer of U , resp. W , in H_V , which induces $H_W \times H_U \hookrightarrow H_V$ and we write⁷ $(h_W, h_U) \in H_V$ for $h_W \in H_W, h_U \in H_U$. Then for $h \in H_V(\hat{\mathbb{Q}})$, the image of the natural map

$$H_W(\mathbb{Q}) \backslash \mathbb{D}_W \times H_W(\hat{\mathbb{Q}}) / H_W(\hat{\mathbb{Q}}) \cap hKh^{-1}, (z, h_1) \mapsto (z, h_1 h)$$

defines a codimension- r cycle on X_K , denoted by $Z(W, h)$. A word of caution about the notation: in [25], the items \mathbb{D}_W, H_W and $Z(W, h)$ were defined with W replaced by U . We decide to change the notation here as U will be varying later and it is important to keep track of W .

When $r = n$, the set $\mathbb{D}_W = \{z_W^\pm\}$ consists of two elements and points of $Z(W, h)$ are called (small) CM points. For a subfield $F_0 \subset F$, we can consider the F_0 -quadratic space

$$W_{F_0} := \text{Res}_{F/F_0} W. \tag{2.31}$$

For any F_0 -quadratic space $V_0 = W_{F_0} \oplus U_0$ with U_0 totally positive, the image of the above homomorphism $H_{W_{F_0}} \hookrightarrow H_{V_0}$ is a torus denoted by $T := T_W \subset H_{V_0}$. For any open compact $K \subset H_{V_0}(\hat{\mathbb{Q}})$, the torus T gives rise to the CM 0-cycle $Z(W_{F_0}, h)$ on $X_{V_0, K}$ defined over F . Its \mathbb{C} -points are given by

$$Z(W_{F_0}, h)(\mathbb{C}) = T(\mathbb{Q}) \backslash \{z_W^\pm\} \times T(\hat{\mathbb{Q}}) / K_T^h \rightarrow X_{V_0, K}, [z_W^\pm, t] \mapsto [z_W^\pm, th], \tag{2.32}$$

where $K_T^h := hKh^{-1} \cap T(\hat{\mathbb{Q}})$. These were called ‘‘big CM points’’ in [13] when $F_0 = \mathbb{Q}$ and U_0 is trivial. We omit h from the notation when it is trivial.

To obtain a 0-cycle defined over \mathbb{Q} , one considers the 0-cycle

$$Z(\mathbb{W}) := \sum_{1 \leq j \leq d, \tau_j = \sigma_j \circ \sigma_1^{-1}: \mathbb{R} \rightarrow \mathbb{R}} \tau_j(Z(W_{\mathbb{Q}})), \tag{2.33}$$

where \mathbb{W} is the admissible incoherent quadratic space with neighbors $W(j)$. Note that $Z(\mathbb{W})$ is $Z(W)$ in Equation (2.13) of [13]. The 0-cycles $\tau_j(Z(W))$ can be constructed as above with W replaced by $W(j)$ for $1 \leq j \leq d$ (see [13, Lemma 2.2]).

Example 2.7 We follow the discussions in [37, Sect. 3] and [27, Sect. 3.2] to realize the CM points appearing in Theorem 1.1 as big CM points. Let $d_1, d_2 < 0$ be discriminants such that $F = \mathbb{Q}(\sqrt{D})$ is a real quadratic field, where $D := d_1 d_2 > 0$. We label the real embeddings $\sigma_j : F \rightarrow \mathbb{R}$ such that $\sigma_1(\sqrt{D}) = \sqrt{D}$ and $\sigma_2(\sqrt{D}) = -\sqrt{D}$. Then $E := E_1 E_2$ is a CM extension of F and becomes an F quadratic space W of signature $((0, 2), (2, 0))$ with respect to the quadratic form $Q(\mu) := -\frac{N\mu\bar{\mu}}{\sqrt{D}}$. For

⁷We will sometimes view H_U, H_W as subgroups of H_V to lighten the notation.

$i = 1, 2$, let $z_i = \frac{-b_i + \sqrt{d_i}}{2a_i}$ be a CM point of discriminant d_i . Denote H_i/E_i the ring class field corresponding to z_i , $H = H_1 H_2$ and

$$G_i := \text{Gal}(H_i/E_i) \subset \tilde{G}_i := \text{Gal}(H_i/\mathbb{Q}). \tag{2.34}$$

The group $\text{Gal}(H/\mathbb{Q})$ embeds into $\tilde{G}_1 \times \tilde{G}_2$ via restriction, under which the image of $\text{Gal}(H/E)$ is a subgroup of $G_1 \times G_2$. If $(V, \mathcal{Q}) = (M_2(\mathbb{Q}), \det)$, then the map⁸

$$V \rightarrow W_{\mathbb{Q}}, \gamma \mapsto (\gamma, Z(z_1, z_2))$$

is an isometry with $N = a_1 a_2$. The CM 0-cycle $Z(W_{\mathbb{Q}})$ defined in (2.32) is given by

$$\begin{aligned} Z(W) &= \sum_{\sigma \in \text{Gal}(H/E)} (z_1, z_2)^\sigma + (-\bar{z}_1, -\bar{z}_2)^\sigma \\ &= \sum_{\substack{(\sigma_1, \sigma_2) \in G_1 \times G_2 \\ \sigma_1|_{H_0} = \sigma_2|_{H_0}}} (z_1^{\sigma_1}, z_2^{\sigma_2}) + (-\bar{z}_1^{\sigma_1}, -\bar{z}_2^{\sigma_2}), \end{aligned}$$

where $H_0 = H_1 \cap H_2$. Note that $H_0 = \mathbb{Q}$ when d_1, d_2 are co-prime. On the other hand, we have

$$Z(W(2)_{\mathbb{Q}}) = \sum_{\sigma \in \text{Gal}(H/E)} (z_1, -\bar{z}_2)^\sigma + (-\bar{z}_1, z_2)^\sigma.$$

Lemma 3.2 in [27] tells us that H_0/\mathbb{Q} is abelian. Its proof even implies that every element in $\text{Gal}(H_0/\mathbb{Q})$ has order dividing 2. From these, we then know that the element $\sigma_2 \in \text{Gal}(H_2/E_2)$ satisfying $z_2^{\sigma_2} = -\bar{z}_2$ is a square and hence trivial when restricted to H_0 . Therefore, $Z(W_{\mathbb{Q}}) = Z(W(2)_{\mathbb{Q}})$, and $Z(W_{\mathbb{Q}})$ is already defined over \mathbb{Q} .

On the other extreme, when $r = 1$, we have $W = (F x_0)^\perp$ for some $x_0 \in F$ with $\mathcal{Q}(x_0) = m \gg 0$, and the cycle $Z(W, h)$ is a divisor. We define a weighted divisor by the finite sum

$$Z(m, \phi) := \sum_{H_W(\hat{\mathbb{Q}}) \setminus H(\hat{\mathbb{Q}})/K} \phi(h^{-1} x_0) Z(W, h) \tag{2.35}$$

for any $\phi \in \mathcal{S}(V(\hat{F}))^K$. We also write

$$Z(m, \mu) := Z(m, \phi_\mu) \tag{2.36}$$

for $\mu \in L'/L$ and $L \subset V$ an even \mathcal{O}_F -lattice.

2.6 A helpful lemma

In this section, we record a result that will be helpful in studying zeros of definite theta functions.

⁸Here (\cdot, \cdot) is the bilinear pairing on $M_2(\mathbb{C})$ induced by the determinant.

Lemma 2.8 *Let $\theta_1, \dots, \theta_N \in \mathbb{R}$ be \mathbb{Q} -linearly independent irrational numbers, $M \in \mathbb{N}$ and b an integer with $0 \leq b \leq M - 1$. For any $\alpha_1, \dots, \alpha_N \in \mathbb{R}$, there exists an infinite subsequence $\{n_i : i \in \mathbb{N}\} \subset \mathbb{N}$ such that*

$$\lim_{i \rightarrow \infty} \mathbf{e}((Mn_i + b)\theta_j) = \mathbf{e}(\alpha_j)$$

for all $1 \leq j \leq N$.

Proof By replacing θ_j with $M\theta_j$ and α_j with $\alpha_j - b\theta_j$, we can suppose that $M = 1$ and $b = 0$. We first prove the case $\alpha_j = 0$ for all $1 \leq j \leq N$ by constructing the sequence $\{n_i\}$ inductively. For any $\epsilon > 0$, by Kronecker’s approximation theorem [3, Theorem 7.10], there exists $n, h_j \in \mathbb{Z}$ such that

$$|n\theta_j - h_j - \epsilon/3| < \epsilon/3$$

for all $1 \leq j \leq N$, which is equivalent to

$$0 < n\theta_j - h_j < 2\epsilon/3.$$

By replacing n, h_j with $-n, -h_j$, we can ensure that $n \geq 1$ for $\epsilon < 3/2$, while $|n\theta_j - h_j| < \epsilon$ still holds. Denote $n(\epsilon) := n$ and $n_i := \max\{n(1/i') : 1 \leq i' \leq i\}$. Then

$$\lim_{i \rightarrow \infty} \mathbf{e}(n_i\theta_j) = 1, \quad 1 \leq j \leq N,$$

and the sequence $\{n_i\}$ is infinite since θ_j ’s are irrational.

In the general case, we can first use Kronecker’s approximation theorem to produce a sequence $\{n'_i\} \subset \mathbb{Z}$ such that $\lim_{i \rightarrow \infty} \mathbf{e}(n'_i\theta_j) = \mathbf{e}(\alpha_j)$ for all $1 \leq j \leq N$. For each i , we can find $i' > i$ such that $n''_i := n'_i + n_{i'}$ forms an increasing sequence in \mathbb{N} , where $\{n_i\}$ is the sequence we have constructed in the case all α_j ’s are 0. Then the new sequence $\{n''_i\}$ satisfies the condition of the lemma. \square

Lemma 2.9 *Suppose $\alpha_i, c_i \in \mathbb{C}$ for $i \in \mathbb{N}$ satisfy the condition that the series defining*

$$\phi(s) := \sum_{i \in \mathbb{N}} \alpha_i c_i^s$$

converges absolutely for $s = s_0 \in \mathbb{R}$ and equals to 0 for all but finitely many $s \in \mathbb{Z}_{>s_0}$. Then we have

$$\sum_{i \in \mathbb{N}, c_i=c} \alpha_i = 0 \tag{2.37}$$

for any $c \in \mathbb{C}^\times$. In other words, $\phi(s)$ is identically zero.

Remark 2.10 If $\alpha_i > 0$ for all $i \in \mathbb{N}$, then $c_i = 0$ for all $i \in \mathbb{N}$. This strengthens Lemma 5.6₁ in [21].

Proof Without loss of generality, we take $s_0 = 0$. After rearranging and scaling all the c_i 's, we can suppose that $|c_i| \geq |c_{i+1}|$ for all $i \in \mathbb{N}$, and $1 := |c_1| = |c_m| > |c_{m+1}|$. Denote

$$\phi_1(s) := \sum_{i=1}^m \alpha_i c_i^s.$$

Using induction, it is then enough to prove the lemma for $\phi_1(s)$.

The condition $\phi(s) = 0$ for all but finitely many $s \in \mathbb{N}$ implies that

$$\lim_{s \rightarrow \infty} |\phi_1(s)| = \lim_{s \rightarrow \infty} |\phi(s) - \phi_1(s)| = 0.$$

For $1 \leq i \leq m$ with $c_i \neq 0$, we can now write

$$c_i = \zeta^{r_{i,0}} \mathbf{e} \left(\sum_{j=1}^N r_{i,j} \theta_j \right)$$

with $\zeta = \mathbf{e}(1/M)$ for some $M \in \mathbb{N}$, $(r_{i,j})_{0 \leq j \leq N} \in \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}^N$ and $\theta_j \in \mathbb{R}$ such that $1, \theta_1, \dots, \theta_N$ are \mathbb{Q} -linearly independent. Then we have $c_i = c_{i'}$ if and only if $r_{i,j} = r_{i',j}$ for all $0 \leq j \leq N$. For any integer $0 \leq b \leq M - 1$, we have

$$\phi_1(s) = f_b(\mathbf{e}(s\theta_1), \dots, \mathbf{e}(s\theta_N)), \quad s \in M\mathbb{N} + b,$$

where $f_b(z_1, \dots, z_N) := \sum_{i=1}^m \alpha_i (\zeta^b)^{r_{i,0}} \prod_{j=1}^N z_j^{r_{i,j}} \in \mathbb{C}[z_1, z_1^{-1}, \dots, z_N, z_N^{-1}]$.

For any $\beta \in (\mathbb{R}/\mathbb{Z})^N$, Lemma 2.8 implies that there exists an infinite subsequence $\{n_k : k \in \mathbb{N}\} \subset \mathbb{N}$ such that

$$\lim_{k \rightarrow \infty} (\mathbf{e}((Mn_k + b)\theta_1), \dots, \mathbf{e}((Mn_k + b)\theta_N)) = \beta.$$

Since f_b is continuous, we then have

$$f_b(\beta) = \lim_{k \rightarrow \infty} f_b(\mathbf{e}((Mn_k + b)\theta_1), \dots, \mathbf{e}((Mn_k + b)\theta_N)) = \lim_{s \rightarrow \infty} \phi_1(s) = 0.$$

This implies that the polynomial f_b is identically zero, or equivalently

$$\sum_{1 \leq i \leq m, (r_{i,j})_{1 \leq j \leq N} = r} \alpha_i (\zeta^{r_{i,0}})^b = 0$$

for all $r \in \mathbb{Z}^N$ and $b \in \mathbb{Z}/M\mathbb{Z}$. This then implies

$$\sum_{1 \leq i \leq m, (r_{i,j})_{0 \leq j \leq N} = r'} \alpha_i = 0$$

for all $r' \in \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}^N$ and $\phi_1(s)$ is identically 0. So the lemma holds for ϕ_1 , and hence also for ϕ by induction. □

As an immediate consequence, we have the following result.

Corollary 2.11 *Let $\{c_i : i \in \mathbb{N}\}, \{b_j : j \in \mathbb{N}\}$ be subsets of \mathbb{C} , and*

$$I := \{i \in \mathbb{N} : c_i \neq 0\}, \quad J := \{j \in \mathbb{N} : b_j \neq 0\}.$$

Suppose there is $s_0 \in \mathbb{R}$ such that the series

$$f(s) := \sum_{i \in \mathbb{N}} c_i^s, \quad g(s) := \sum_{j \in \mathbb{N}} b_j^s$$

converge absolutely for $s = s_0$ and $f(s) = g(s)$ for all but finitely many $s \in \mathbb{Z}_{>s_0}$, then there is a bijection $\sigma : I \rightarrow J$ such that $c_i = b_{\sigma(i)}$ for all $i \in I$. In particular, I is finite if and only if J is finite, in which case they have the same size.

3 Functions

3.1 Weil representation and theta function

Let $G := \text{Res}_{F/\mathbb{Q}}(\text{SL}_2)$ and

$$\Gamma := \text{SL}_2(\mathcal{O}_F) \subset G(\mathbb{R}) \subset G(\mathbb{A}).$$

Also denote $\Gamma_f := \text{SL}_2(\hat{\mathcal{O}}) \subset G(\hat{\mathbb{Q}})$.

For an F -quadratic space (V, Q) of even dimension, let $\omega = \omega_\psi$ be the Weil representation of $G(\mathbb{A})$ on the space of Schwartz functions $\mathcal{S}(V(\mathbb{A}_F)) = \mathcal{S}(V(\hat{F})) \otimes \bigotimes_{1 \leq j \leq d} \mathcal{S}(V_{\sigma_j})$, where ψ is the standard additive character on $F \backslash \mathbb{A}_F$.

At the infinite local place, suppose (W, Q_W) is an \mathbb{R} -quadratic space of signature (p, q) with $2 \mid (p + q)$. For a point w in the symmetric space \mathbb{D}_W associated to $\text{SO}(W)$, we obtain an orthogonal decomposition $W = w^\perp \oplus w$ and a Schwartz function

$$\phi^{p,q}(w, \lambda) := \exp(-2\pi(Q_W(\lambda_{w^\perp}) + Q_W(\lambda_w))) \tag{3.1}$$

in $\mathcal{S}(W)$, which is acted on by $\text{SL}_2(\mathbb{R})$ via the Weil representation ω_W to produce

$$\phi^W(\tau, w, \lambda) := v^{q/2} \mathbf{e}(Q(\lambda_{w^\perp})\tau + Q(\lambda_w)\bar{\tau}). \tag{3.2}$$

Note that

$$\phi^W(\tau, w, \lambda) = \phi^W(\tau, \bar{w}, \lambda) \tag{3.3}$$

as it is independent of the orientation of w . If $q = 0$, then the expressions in (3.1) and (3.2) are independent of w and we omit them from the notation. In addition, we will also be interested in the following ‘singular Schwartz function’ when $q \geq 1$

$$\begin{aligned} &\phi^{W,*}(\tau, w, \lambda) \\ &:= \mathbf{e}(Q(\lambda)\tau) \begin{cases} -(-4\pi Q(\lambda_w))^{1-q/2} \Gamma(q/2 - 1, -4\pi Q(\lambda_w)v), & \lambda \notin w^\perp, \\ \text{CT}_{s=0} \frac{v^{s+q/2-1}}{s+q/2-1}, & \lambda \in w^\perp. \end{cases} \end{aligned} \tag{3.4}$$

Here $\Gamma(s, x) := \int_x^\infty t^{s-1} e^{-t} dt$ is the incomplete Gamma function. Direct calculations yield

$$L_\tau \phi^{W,*}(\tau, w, \lambda) = \phi^W(\tau, w, \lambda). \tag{3.5}$$

for all $w \in \mathbb{D}_W$ and $\lambda \in W$.

To describe the finite local place, let $L \subset V$ be an even lattice and $S_L \subset \mathcal{S}(V(\hat{F}))$ the subspace as in (2.27). Then Γ acts on the space S_L via ω , whose complex conjugate we denote by ρ_L . Its explicit values on the generators of Γ can be found in Sect. 3.2 of [10]. Furthermore, it is unitary with respect to the hermitian pairing on S_L

$$\langle \phi, \psi \rangle := \sum_{\mu \in L'/L} \phi(\mu) \overline{\psi(\mu)}, \quad \phi, \psi \in S_L. \tag{3.6}$$

More generally, for lattices L, M and $\phi \in S_{M \oplus L}, \psi \in S_M$, we define $\langle \phi, \psi \rangle_L \in S_L$ by

$$\langle \phi, \psi \rangle_L(\mu) := \sum_{v \in M'/M} \phi((v, \mu)) \overline{\psi(v)}, \quad \mu \in L'/L. \tag{3.7}$$

These pairings are then naturally defined for functions valued in S_L .

For the rest of the section, suppose V_{σ_j} is positive definite whenever $j \geq 2$. Then $\mathbb{D}_{V_{\sigma_j}}$ is \mathbb{D} for $j = 1$ and a point otherwise. For $\tau = (\tau_j) \in \mathbb{H}^d, z \in \mathbb{D}, h \in H(\hat{\mathbb{Q}})$ and $\phi_f \in \mathcal{S}(V(\hat{F}))$, we define the Siegel theta function

$$\begin{aligned} \Theta(\tau, z, h; \phi_f) &:= \sum_{\lambda \in V(F)} \phi_f(h^{-1}\lambda) \phi_\infty(\tau, z, \lambda), \\ \phi_\infty(\tau, z, \lambda) &:= \phi^{V_{\sigma_1}}(\tau_1, z, \lambda) \bigotimes_{2 \leq j \leq d} \phi^{V_{\sigma_j}}(\tau_j, \lambda). \end{aligned} \tag{3.8}$$

Note that $\Theta(\tau, \bar{z}, h; \phi_f) = \Theta(\tau, z, h; \phi_f)$ by (3.3). In the variable τ , this is a Hilbert modular form of weight

$$\tilde{\kappa} = \tilde{\kappa}(V) := ((p_1 - q_1)/2, n/2 + 1, \dots, n/2 + 1) \in \mathbb{Z}^d. \tag{3.9}$$

For any even lattice $L \subset V$, we also denote the associated theta function by

$$\Theta_L(\tau, z, h) := \sum_{\mu \in L'/L} \Theta(\tau, z, h; \phi_\mu) \phi_\mu, \tag{3.10}$$

which is valued in S_L . When $h = 1$, we omit it from the notation. The definition of Θ_L implies that

$$\Theta_L(\tau, z) = \Theta_{L_h}(\tau, z, h), \tag{3.11}$$

where L_h is defined in (2.28) and $L'/L \cong L'_h/L_h$ via (2.29).

If V_{σ_1} is definite, then $\Theta_L(\tau, z, h)$ is independent of z and we write

$$\theta_L(\tau, h) := \Theta_L(\tau, z, h). \tag{3.12}$$

In particular when $L = P$ is positive definite and \mathbb{Z} -unimodular, this is a scalar-valued, holomorphic Hilbert modular form on Γ_F of parallel weight $n/2 + 1$. We denote the graded subring

$$\mathcal{M}_F^\theta := \text{Span}\{\theta_P(\tau) : P \in \mathcal{U}_F^+\} \subset \mathcal{M}_F. \tag{3.13}$$

For future convenience, we also define

$$\begin{aligned} \Theta^*(\tau, z, h; \phi_f) &:= \sum_{\lambda \in V(F)} \phi_f(h^{-1}\lambda) \phi^{V_{\sigma_1},*}(\tau_1, z, \lambda) \otimes_{2 \leq j \leq d} \phi^{V_{\sigma_j}}(\tau_j, \lambda), \\ \Theta_L^*(\tau, z, h) &:= \sum_{\mu \in L'/L} \Theta^*(\tau, z, h; \phi_\mu) \phi_\mu. \end{aligned} \tag{3.14}$$

The sums converge absolutely since the singular Schwartz function decays as a Schwartz function and $z^\perp \cap L'$ is contained in a positive definite lattice. For fixed (z, h) , it defines a real-analytic function in τ , which satisfies the analogue of (3.11) as well as

$$L_{\tau_1} \Theta_L^*(\tau, z, h) = \Theta_L(\tau, z, h) \tag{3.15}$$

by (3.5). When V_{σ_1} is negative definite, we have $q_1 = n + 2$ and the function $\theta_L^*(\tau, h) := \Theta_L^*(\tau, z, h)$ can be written explicitly as

$$\begin{aligned} \theta_L^*(\tau, h) &= - \sum_{\lambda \in h(L') \cap V, \lambda \neq 0} \mathbf{e}(\text{tr}(Q(\lambda)\tau)) (4\pi |Q(\lambda)_1|)^{-n/2} \Gamma(n/2, 4\pi |Q(\lambda)_1| v_1) \phi_\lambda \\ &\quad + \text{CT}_{s=0} \frac{v_1^{s+n/2}}{s+n/2} \phi_0. \end{aligned} \tag{3.16}$$

Although θ_L^* is not modular in τ , difference of such functions will become modular after adding suitable holomorphic functions (see Theorem 4.3).

Remark 3.1 When $\dim_F V$ is odd, all the constructions above still hold by working with metaplectic covers. As this is not needed for most of the applications, we refrain from introducing more notations and refer the readers to [10].

3.2 Eisenstein series and Siegel-Weil formula

Let $B \subset \text{SL}_2$ the standard Borel subgroup, and $I(s, \chi) = \otimes_v I(s, \chi_v) := \text{Ind}_{B(\mathbb{A}_F)}^{\text{SL}_2(\mathbb{A}_F)} |\chi| \cdot | \cdot |^s$ the induced representation of $G(\mathbb{A}_\mathbb{Q})$ for $\chi = \chi_V$ the quadratic

Hecke character associated to an F -quadratic space (V, Q) with signature $((p_1, q_1), (n + 2, 0), \dots, (n + 2, 0))$. For a standard section $\Phi \in I(s, \chi)$, the Eisenstein series

$$E(g, s, \Phi) := \sum_{\gamma \in B(F) \backslash \mathrm{SL}_2(F)} \Phi(\gamma g, s) \tag{3.17}$$

converges absolutely for $\Re(s) \gg 0$ and has meromorphic continuation to $s \in \mathbb{C}$. When $s_0 := n/2$, the map $\lambda : \mathcal{S}(V(\mathbb{A}_F)) \rightarrow I(s_0, \chi)$ defined by

$$\lambda(\phi)(g) = (\omega(g)\phi)(0) \tag{3.18}$$

is $\mathrm{SL}_2(\mathbb{A}_F)$ -equivariant, and $\lambda(\phi)$ can be extended uniquely to all s and produce a standard sections in $I(s, \chi)$.

The infinite part $\otimes_{1 \leq j \leq d} I(s, \chi_{\sigma_j})$ is generated by functions $\Phi_\infty^\kappa := \prod_j \Phi_\mathbb{R}^{\kappa_j}$ with $\kappa = (\kappa_j) \in \mathbb{Z}^d$ satisfying $\kappa_j \equiv n/2 + 1 \pmod{2}$, where

$$\Phi_\mathbb{R}([k_\theta, 1]) = e^{i\theta/2}, \quad -\pi < \theta \leq \pi$$

is the image of $\phi_\infty^{q+2, q}$ defined in (3.1) under λ for any $q \in \mathbb{N}$ [9, Lemma 4.1]. For $\phi_f \in \mathcal{S}(V(\hat{F}))$ and $\tilde{\kappa} = \tilde{\kappa}(V)$ as in (3.9), the Eisenstein series

$$E(\tau, \phi_f, \tilde{\kappa}) := v^{-\tilde{\kappa}/2} E(g_\tau, s_0, \lambda(\phi_f) \otimes \Phi_\infty^{\tilde{\kappa}}) \tag{3.19}$$

is a Hilbert modular form of weight $\tilde{\kappa}$. For any even lattice $L \subset V$, we have

$$E_L(\tau, \tilde{\kappa}) := \sum_{\mu \in L'/L} E(\tau, \phi_\mu, \tilde{\kappa}) \phi_\mu \in \mathcal{A}_{\tilde{\kappa}, \rho_L}. \tag{3.20}$$

Suppose $q = 2$. For any compact open $K \subset H(\hat{\mathbb{Q}})$ stabilizing \hat{L} and acting trivially on \hat{L}'/\hat{L} , we have the Siegel-Weil formula [9, Lemma 4.3]

$$E_L(\tau, \tilde{\kappa}) = \frac{1}{\mathrm{vol}(X_K)} \int_{X_K} \Theta_L(\tau, z, h) \Omega^n, \quad n \geq 1, \tag{3.21}$$

where Ω is the Kähler form on X_K normalized as in [10] and $\mathrm{vol}(X_K) = \int_{X_K} \Omega^n$. For $n = 0$, the Siegel-Weil formula yields

$$E_L(\tau, \tilde{\kappa}) = \frac{2}{\mathrm{vol}(X_K)} \sum_{[z, h] \in X_K} \Theta_L(\tau, z, h). \tag{3.22}$$

When $q = 0$, we have $\tilde{\kappa} = \overrightarrow{s_0 + 1}$ and the lattice $P = L$ is totally positive definite. The classical Siegel-Weil formula yields

$$\begin{aligned} E_P(\tau) &:= E_P(\tau, \tilde{\kappa}) = c_K^{-1} \sum_{h \in H(\mathbb{Q}) \backslash H(\hat{\mathbb{Q}})/K} \theta_P(\tau, h), \\ c_K &= \sum_{h \in H(\mathbb{Q}) \backslash H(\hat{\mathbb{Q}})/K} 1 \in \mathbb{N}. \end{aligned} \tag{3.23}$$

Though $\theta_{P^\ell} = \theta_P^\ell$, the Eisenstein series E_{P^ℓ} is almost never the same as E_P^ℓ !

Furthermore suppose that $P \in \mathcal{U}_F^+$ with rank 2τ . Then $E_P(\tau)$ is in \mathcal{M}_F^θ and coincides with the Hecke Eisenstein series for F of parallel weight $\tau = n/2 + 1$. It can be written as (see Sect. 2 in [5])

$$E_P(\tau) = \sum_{\beta \in \mathbb{P}^1(F)} c(\beta, \tau)^{-\tau}, \quad c(\beta, \tau) := \begin{cases} A_\beta \cdot \text{Nm}(\tau + \beta) & \beta \in F, \\ 1 & \beta = \infty, \end{cases} \tag{3.24}$$

for certain non-zero integers A_β depending only on β . Note that since $P^\ell \in \mathcal{U}_F^+$ for any $\ell \in \mathbb{N}$, we have $E_{P^\ell}(\tau) \in \mathcal{M}_F^\theta$. Applying Lemma 2.9, we can deduce the following results about zeros of theta functions.

Lemma 3.2 *Let $P \in \mathcal{U}_F^+$ and $S \subset \mathbb{H}^d$ a finite set of points. Then there exists $\ell \geq 1$ such that E_{P^ℓ} does not vanish on S .*

Proof Since $\text{SL}_2(\mathbb{R})^d$ acts transitively on \mathbb{H}^d , we can write $S = \{\tau_0, g_1\tau_0, \dots, g_N\tau_0\}$ with $\tau_0 \in \mathbb{H}^d$ and $g_i \in \text{SL}_2(\mathbb{R})^d$, where we set g_0 to be the identity. Assume that for every $\ell \geq 1$, there exists $0 \leq i \leq N$ such that $E_{P^\ell}(g_i\tau_0) = 0$. Then the function

$$f_\ell(\tau) := \prod_{i=0}^N E_{P^\ell}(g_i\tau)$$

vanishes at $\tau = \tau_0$ for all $\ell \geq 1$. Using the expression (3.24) we can write

$$f_\ell(\tau) = \sum_{\beta_0, \dots, \beta_N \in \mathbb{P}^1(F)} \left(\prod_{i=0}^N c(\beta_i, g_i\tau) \right)^{-\tau\ell}.$$

By Remark 2.10, we have $\prod_{i=0}^N c(\beta_i, g_i\tau_0) = 0$ for all $\beta_1, \dots, \beta_N \in \mathbb{P}^1(F)$, which is clearly a contradiction since $\prod_{i=0}^N c(\infty, g_i\tau_0) = 1$. □

Lemma 3.3 *For any $\tau_0 \in \mathbb{H}^d$, there exist $P_1, P_2 \in \mathcal{U}_F^+$ in the same genus such that $\theta_{P_1}(\tau_0) \neq \theta_{P_2}(\tau_0)$.*

Proof Assume otherwise. Then for any $P \in \mathcal{U}_F^+$, the function $\theta_P(\tau, h)$ takes the same value at τ_0 for all $h \in H(\hat{\mathbb{Q}})$ and we have

$$E_P(\tau_0) = c_K \sum_{h \in H(\mathbb{Q}) \backslash H(\hat{\mathbb{Q}})/K} \theta_P(\tau_0, h) = \theta_P(\tau_0) c_K \sum_{h \in H(\mathbb{Q}) \backslash H(\hat{\mathbb{Q}})/K} 1 = \theta_P(\tau_0).$$

Therefore $E_{P^\ell}(\tau_0) = \theta_{P^\ell}(\tau_0) = \theta_P(\tau_0)^\ell = E_P(\tau_0)^\ell$ for all $\ell \geq 1$ and $P \in \mathcal{U}_F^+$. Using expression (3.24), we obtain

$$\sum_{\beta \in \mathbb{P}^1(F)} c(\beta, \tau_0)^{-\tau\ell} = E_P(\tau_0)^\ell,$$

which contradicts Corollary 2.11 since $c(\beta, \tau_0) \neq 0$ for all $\beta \in \mathbb{P}^1(F)$. □

Proposition 3.4 (Partition of Unity) *Let F be a totally real field. Then there exist $m \in \mathbb{N}$, $\tilde{P}_i, P_{1,i}, P_{2,i} \in \mathcal{U}_F^+$ with ranks $2\tilde{\tau}_i$ and $2\tau_i$ for $1 \leq i \leq m$ such that $P_{1,i}$ and $P_{2,i}$ are in the same genus for all i and there is no $\tau_0 \in \mathbb{H}$ such that $(E_{\tilde{P}_i}(\theta_{P_{1,i}} - \theta_{P_{2,i}}))^\Delta(\tau_0) = 0$ for all i .*

Furthermore, for any $d_1, \dots, d_m, e_1, \dots, e_m \in \mathbb{N}$, there exists elliptic modular forms $g_i \in M_{-(\tilde{\tau}_i d_i + \tau_i e_i)d}^1$ with rational Fourier coefficients such

$$\sum_{i=1}^m g_i (E_{\tilde{P}_i}^{d_i} \cdot (\theta_{P_{1,i}} - \theta_{P_{2,i}})^{e_i})^\Delta = 1. \tag{3.25}$$

Proof Start with a point $\tau_0 \in \mathbb{H}$, we can find $P_1, P_2 \in \mathcal{U}_F^+$ in the same genus satisfying $(\theta_{P_1} - \theta_{P_2})^\Delta(\tau_0) \neq 0$ by Lemma 3.3. If $(\theta_{P_1} - \theta_{P_2})^\Delta$ has no zero on \mathbb{H} , then we can apply Lemma 3.2 to take $\tilde{P}_1, \tilde{P}_2 \in \mathcal{U}_F^+$ such that $E_{\tilde{P}_1}^\Delta$ and $E_{\tilde{P}_2}^\Delta$ do not have common zero on \mathbb{H} . The forms $E_{\tilde{P}_1} \cdot (\theta_{P_1} - \theta_{P_2})$ and $E_{\tilde{P}_2} \cdot (\theta_{P_1} - \theta_{P_2})$ satisfy the first claim.

Otherwise, let $\tau_2, \dots, \tau_m \in \mathbb{H}$ be the zeros of $(\theta_{P_1} - \theta_{P_2})^\Delta$ in a fundamental domain \mathcal{F} of $SL_2(\mathbb{Z}) \backslash \mathbb{H}$. By Lemma 3.3, there exists $P_{1,i}$ and $P_{2,i}$ such that $\theta_{P_{1,i}}^\Delta(\tau_i) \neq \theta_{P_{2,i}}^\Delta(\tau_i)$ for all $2 \leq i \leq m$. Let $S \subset \mathcal{F}$ be the finite set of the zeros of $\prod_{1 \leq i \leq m} (\theta_{P_{1,i}}^\Delta - \theta_{P_{2,i}}^\Delta)$. Applying Lemma 3.2, we can find $\tilde{P} \in \mathcal{U}_F^+$ such that $E_{\tilde{P}}$ does not vanish on S . Let $S' \subset \mathcal{F}$ be the finite set of the zeros of $E_{\tilde{P}}^\Delta \prod_{1 \leq i \leq m} (\theta_{P_{1,i}}^\Delta - \theta_{P_{2,i}}^\Delta)$. We can apply Lemma 3.2 to find $E_{\tilde{P}_i}$ for $2 \leq i \leq m$ such that they do not vanish on S' . Now the forms $E_{\tilde{P}} \cdot (\theta_{P_1} - \theta_{P_2})$ and $E_{\tilde{P}_i} \cdot (\theta_{P_{1,i}} - \theta_{P_{2,i}})$ with $2 \leq i \leq m$ satisfy the first claim.

To prove the second claim, we can write

$$\Delta^{-(\tilde{\tau}_i d_i + \tau_i e_i)d} (E_{\tilde{P}_i}^{12d_i} (\theta_{P_{1,i}} - \theta_{P_{2,i}})^{12e_i})^\Delta = A_i(j) \in M_0^1 = \mathbb{Q}[j]$$

with $j = j(\tau)$ the j -invariant and $A_i(x) \in \mathbb{Q}[x]$. The first claim implies $\gcd(A_1, \dots, A_m) = 1$, i.e. there exists $B_1, \dots, B_m \in \mathbb{Q}[x]$ such that

$$\sum_{i=1}^m B_i(x) A_i(x) = 1.$$

Setting $g_i := B_i(j) \Delta^{-(\tilde{\tau}_i d_i + \tau_i e_i)d} (E_{\tilde{P}_i}^{11d_i} (\theta_{P_{1,i}} - \theta_{P_{2,i}})^{11e_i})^\Delta$ proves the second claim. □

3.3 Higher Green function

We follow [11] to recall higher Green function on the Shimura variety X_K for $F = \mathbb{Q}$. Let V/\mathbb{Q} be a quadratic space of signature $(n, 2)$ and $L \subset V$ an even lattice with K an open compact stabilizing \hat{L} . Also, we denote

$$\sigma_0 := \frac{n}{4} - \frac{1}{2}. \tag{3.26}$$

For $\mu \in L'/L$ and $m \in \mathbb{Z} + Q(\mu)$, the automorphic Green function is defined by

$$\Phi_{m,\mu}(z, h, s) := 2 \frac{\Gamma(s + \sigma_0)}{\Gamma(2s)} \sum_{\lambda \in h(L_{m,\mu})} \left(\frac{m}{Q(\lambda_{z^\perp})} \right)^{s+\sigma_0} F\left(s + \sigma_0, s - \sigma_0, 2s; \frac{m}{Q(\lambda_{z^\perp})}\right), \tag{3.27}$$

where $F(a, b, c; z)$ is the Gauss hypergeometric function [1, Chap. 15]. The sum converges normally on $X_K \setminus Z(m, \mu)$ for $s > \sigma_0 + 1$ and defines an eigenfunction of the Laplacian Δ on \mathbb{D} , normalized as in [7], with eigenvalue $\frac{1}{2}(s - \sigma_0 - 1)(s + \sigma_0)$. Furthermore, it has a meromorphic continuation to $s \in \mathbb{C}$ with a simple pole at $s = \sigma_0 + 1$, whose constant term is denoted by $\Phi_{m,\mu}(z, h, \sigma_0 + 1)$ and the regularized theta lift of Hejhal-Poincaré series of index (m, μ) [7].

At $s = \sigma_0 + 1 + r$ with $r \in \mathbb{N}$, the function $\Phi_{m,\mu}(z, h, s)$ is called a *higher Green function*. For the unimodular lattice $L = M_2(\mathbb{Z})$ and $z = Z(z_1, z_2)$ as in Example 2.5, we have

$$\Phi_m(z, 1, r + 1) = \frac{2(-1)^r}{\Gamma(2r + 2)} G_{r+1}^m(z_1, z_2), \tag{3.28}$$

where G_s^m is defined in (1.2). For a harmonic Maass form $f = \sum_{m,\mu} c(m, \mu)q^{-m} \times \phi_\mu + O(1) \in H_{k-2r, \bar{\rho}_L}$ with $k := -2\sigma_0$, define

$$\Phi_L^r(z, h, f) := r! \sum_{m>0, \mu \in L'/L} c(m, \mu)m^r \Phi_{m,\mu}(z, h, \sigma_0 + 1 + r) \tag{3.29}$$

to be the associated higher Green function. Following from the work of Borcherds [6] and generalization by Bruinier [7] (also see [11, 34]), the function Φ_L^r has the following integral representation

$$\begin{aligned} \Phi_L^r(z, h, f) &= \lim_{T \rightarrow \infty} \int_{\mathcal{F}_T} \langle \tilde{R}_\tau^r f(\tau), \overline{\Theta_L(\tau, z, h)} \rangle d\mu(\tau) \\ &= (-1)^r \lim_{T \rightarrow \infty} \int_{\mathcal{F}_T} \langle f(\tau), \overline{\tilde{R}_\tau^r \Theta_L(\tau, z, h)} \rangle d\mu(\tau), \end{aligned} \tag{3.30}$$

where \mathcal{F}_T is the truncated fundamental domain of $\Gamma_{\mathbb{Q}} \backslash \mathbb{H}$ at height $T > 1$.

4 Real-analytic Hilbert modular forms and algebraicity of pairing

In this section, we will prove the existence of certain real-analytic Hilbert modular forms by generalizing the proof of Theorem 3.7 in [12], and give some results concerning their Fourier coefficients. The notations $F, D, \mathcal{O}, \mathfrak{d}, d$ are the same as in Sect. 2.

4.1 Certain real-analytic Hilbert modular forms

Let $\rho = \rho_L$ be a Weil representation, $\kappa = (k_1, \dots, k_d) \in \mathbb{Z}^d$, $\Gamma \subset \Gamma_F$ a congruence subgroup, and $X = X(\Gamma) = \Gamma \backslash \mathbb{H}^d$ be the open Hilbert modular variety. By adding finitely many cusps to X , we obtain the Baily-Borel compactification X^{BB} . It can also be constructed as the Proj of the ring of holomorphic modular forms on X , and is a normal, Noetherian scheme over \mathbb{C} . When Γ is neat, we fix a smooth toroidal compactification \tilde{X} of X . It is a compact complex manifold, and a desingularization of X^{BB} . We also have the natural map $\pi : \tilde{X} \rightarrow X^{\text{BB}}$ and let E be the boundary divisor on \tilde{X} . Suppose \tilde{X} is associated to a projective Γ -admissible decomposition (see [4, section II.2]).

Denote \mathcal{O} and $\mathcal{E}^{p,q}$ the sheaf of holomorphic functions and smooth differential forms of type (p, q) on \tilde{X} respectively, and take the subsheaf $\mathcal{E}' := \ker(\mathcal{E}^{0,1} \xrightarrow{\bar{\partial}} \mathcal{E}^{0,2})$. Then the Dolbeault resolution of \mathcal{O} gives us the short exact sequence

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{E}^{0,0} \xrightarrow{\bar{\partial}} \mathcal{E}' \rightarrow 0. \tag{4.1}$$

For a Cartier divisor D and quasi-coherent sheaf \mathcal{F} on \tilde{X} , we write $\mathcal{F}(D)$ for the corresponding twisting sheaf. Also, let $\mathcal{L}_{\kappa,\rho}$ be the sheaf of modular forms of weight κ and representation ρ on X . It extends to X^{BB} and \tilde{X} by Koecher’s principle, and we use $\mathcal{L}_{\kappa,\rho}$ and $\tilde{\mathcal{L}}_{\kappa,\rho}$ to denote these extensions. In particular,

$$H^0(\tilde{X}, \mathcal{E}^{0,0} \otimes \tilde{\mathcal{L}}_{\kappa,\rho}) \supset \mathcal{A}_{\kappa,\rho}(\Gamma). \tag{4.2}$$

Note that $\pi_* \tilde{\mathcal{L}}_{\kappa,\rho} = \mathcal{L}_{\kappa,\rho}$, $\pi^* \mathcal{L}_{\kappa,\rho} = \tilde{\mathcal{L}}_{\kappa,\rho}$. When ρ is trivial and $\kappa = (1, \dots, 1)$, $\mathcal{L} = \mathcal{L}_{\kappa,\rho}$ is the determinant of the Hodge bundle and ample on X^{BB} . However, the extension $\tilde{\mathcal{L}}$ is trivial at the fiber of a cusp, and in general not ample on \tilde{X} . Nevertheless, we can use it along with twisting by E to prove the following result.

Theorem 4.1 *In the notations above, for any $N \in \mathbb{N}$, there exists $n_0, k \in \mathbb{N}_{>N}$ such that the following map*

$$H^0(\tilde{X}, \mathcal{E}^{0,0} \otimes \tilde{\mathcal{L}}_{\kappa,\rho} \otimes \tilde{\mathcal{L}}_{\bar{k}}(-E)^{\otimes n}) \xrightarrow{\bar{\partial} \otimes 1 \otimes 1} H^0(\tilde{X}, \mathcal{E}' \otimes \tilde{\mathcal{L}}_{\kappa,\rho} \otimes \tilde{\mathcal{L}}_{\bar{k}}(-E)^{\otimes n}) \tag{4.3}$$

is surjective for all $n \geq n_0$.

Proof For simplicity, suppose X^{BB} has only one cusp x . By Theorem 2.2 in Chapter IV of [4], \tilde{X} is the normalization of the blowing-up of X^{BB} at certain coherent sheaf \mathcal{I} of ideals concentrated at x . As X^{BB} is Noetherian, so is \tilde{X} and π is quasi-compact. We claim that $\tilde{\mathcal{L}}_{\bar{k}}(-E) = \mathcal{O}_{\tilde{X}}(-E) \otimes \pi^* \mathcal{L}^{\otimes k}$ is ample on \tilde{X} for some $k > N$.

Since normalization is a finite morphism in this case (see Lemma 33.27.1 in [33, Tag 0BXX]), it preserves ampleness and we can suppose that \tilde{X} is the blowing-up of X^{BB} . By the discussion and Lemma 31.32.4(3) in [33, Tag 01OF] and Lemma 29.38.2 in [33, Tag 01VL], we know that $\mathcal{O}_{\tilde{X}}(-E) = \mathcal{O}_{\tilde{X}}(1)$ is π -relatively ample. Since \mathcal{L} is ample on X^{BB} , Lemma 29.37.7 in [33, Tag 01VG] proves the claim.

By considering the long exact sequence in cohomology associated to (4.1), we see that the surjectivity of (4.3) is equivalent to the vanishing of $H^1(\tilde{X}, \tilde{\mathcal{L}}_{\kappa, \rho} \otimes \tilde{\mathcal{L}}_{\tilde{\kappa}}(-E)^{\otimes n})$ for all n sufficiently large. This follows from standard vanishing result for cohomology (e.g. Lemma 30.17.1 in [33, Tag 01XO]), which finishes the proof. \square

Proposition 4.2 *For $\kappa \in \mathbb{Z}^d$, let $\tilde{\kappa}, \hat{\kappa} \in \mathbb{Z}^d$ be as in (2.3). Suppose $f \in \mathcal{A}_{\tilde{\kappa}, \rho}(\Gamma)$ is holomorphic in τ_j for $2 \leq j \leq d$. Given any $g \in S_{\kappa'}(\Gamma)$ of parallel weight, there exists $\ell_0 \in \mathbb{N}$ and functions $\hat{G}_\ell \in \mathcal{A}_{\hat{\kappa} + \ell\kappa', \rho}(\Gamma)$ for all $\ell \geq \ell_0$ such that they are holomorphic in τ_j for $2 \leq j \leq d$,*

$$L_{\tau_1} \hat{G}_\ell = g^\ell \cdot f, \tag{4.4}$$

and $\hat{G}_{\ell'} = g^{\ell' - \ell} \hat{G}_\ell$ for all $\ell' \geq \ell \geq \ell_0$.

Proof Suppose Γ is neat and g is non-zero. Let $\kappa' = \vec{k'}$ with $k' \in \mathbb{N}$ and fix some $N > k'$. Now choose $n_0, k > N$ as in Theorem 4.1 and set $\ell_0 = n_0 k$. Given $f \in \mathcal{A}_{\tilde{\kappa}, \rho}(\Gamma)$, the differential form $v_1^{-2} g(\tau)^{\ell_0} f(\tau) d\bar{\tau}_1$ is in the kernel of $\bar{\partial} \otimes 1 \otimes 1$ since f is holomorphic in τ_2, \dots, τ_d . Furthermore, it is orders of vanishing at the cusps are at least ℓ_0 since f is bounded near the cusps. Therefore, it is a global section of $\mathcal{E}' \otimes \tilde{\mathcal{L}}_{\tilde{\kappa}, \rho} \otimes \tilde{\mathcal{L}}_{\kappa'}(-E)^{\otimes \ell_0}$. Note that

$$\tilde{\mathcal{L}}_{\kappa'}(-E)^{\otimes \ell_0} = \tilde{\mathcal{L}}_{\vec{k'k}}(-kE)^{\otimes n_0} \subset \tilde{\mathcal{L}}_{\vec{k'k}}(-k'E)^{\otimes n_0} = \tilde{\mathcal{L}}_{\vec{k}}(-E)^{\otimes n_0 k'}$$

since $k' < N < k$ and $\mathcal{O}_{\tilde{X}}(-kE)$ is a subsheaf of $\mathcal{O}_{\tilde{X}}(-k'E)$. By Theorem 4.1, there exists $\hat{G}_{\ell_0} \in H^0(\tilde{X}, \mathcal{E}^{0,0} \otimes \tilde{\mathcal{L}}_{\hat{\kappa}, \rho} \otimes \tilde{\mathcal{L}}_{\vec{k}}(-E)^{\otimes n_0 k'})$ such that $\bar{\partial}(-2i\hat{G}_{\ell_0}) = v_1^{-2} g^{\ell_0} f d\bar{\tau}_1$. As f, g are real-analytic, so is \hat{G}_{ℓ_0} . So for any $\ell \geq \ell_0$, the real-analytic modular form $\hat{G}_\ell := g^{\ell - \ell_0} \hat{G}_{\ell_0} \in \mathcal{A}_{\hat{\kappa} + \ell\kappa', \rho}$ is holomorphic in τ_2, \dots, τ_d and satisfies

$$L_{\tau_1} \hat{G}_\ell(\tau) = -2i v_1^2 \partial_{\bar{\tau}_1} \hat{G}_\ell(\tau) = -2i g(\tau)^{\ell - \ell_0} v_1^2 \partial_{\bar{\tau}_1} \hat{G}_{\ell_0}(\tau) = g(\tau)^\ell f(\tau).$$

From the construction, the last condition is also satisfied.

Finally for any congruence subgroup $\Gamma \subset \Gamma_F$, there exists a neat, normal subgroup $\Gamma' \subset \Gamma$ of finite index. Averaging the function $\hat{G}_\ell \in \mathcal{A}_{\hat{\kappa} + \ell\kappa', \rho}(\Gamma')$ constructed above over Γ/Γ' then gives the desired function in level Γ . \square

Now, we will apply this result to the case when f is the special value of a theta kernel and g is the holomorphic theta function for a positive definite lattice.

Theorem 4.3 *Let W be an F -quadratic space of dimension 2 with signature as in (2.18), and $P_1, P_2 \in \mathcal{U}_F^+$ positive definite, \mathbb{Z} -unimodular \mathcal{O} -lattices of ranks 2τ and in the same genus. For an \mathcal{O} -lattice $N \subset W$, there exists $\ell_0 \in \mathbb{N}$ and $\hat{\delta}(\tau) = \hat{\delta}(\tau; N, h, P_1, P_2, \ell) \in \mathcal{A}_{1+\ell\tau, \rho_N}^+$ for all $\ell \geq \ell_0$ and $h \in H_W(\hat{\mathbb{Q}})$ having the following properties.*

1. *It is holomorphic in τ_2, \dots, τ_d and has exponential decay near the cusps.*

2. It satisfies

$$L_{\tau_1} \hat{\delta}(\tau) = (\theta_{P_2}(\tau) - \theta_{P_2}(\tau))^\ell (\theta_N(\tau) - \theta_N(\tau, h)), \tag{4.5}$$

for all $\tau \in \mathbb{H}^d$, $h \in H_W(\hat{\mathbb{Q}})$ and $\ell \geq \ell_0$.

3. We can write

$$\hat{\delta}(\tau) = \hat{\delta}^+(\tau) + (\theta_{P_1}(\tau) - \theta_{P_2}(\tau))^\ell (\theta_N^*(\tau) - \theta_N^*(\tau, h)), \tag{4.6}$$

where $\theta_N^*(\tau, h)$ is defined in (3.16), and $\hat{\delta}^+$ is holomorphic in τ and Γ_∞ -invariant with respect to ρ_N .

4. Given the Fourier expansions

$$\hat{\delta}(\tau) = \sum_{m, \mu} \hat{a}_{m, \mu}(v_1) \mathbf{e}(\text{tr}(m\tau)) \phi_\mu, \quad \hat{\delta}^+(\tau, h) = \sum_{m, \mu} \hat{a}_{m, \mu}^+ \mathbf{e}(\text{tr}(m\tau)) \phi_\mu,$$

we have $\hat{a}_{m, \mu}^+ = 0$ unless $m \gg 0$, and

$$\lim_{v_1 \rightarrow \infty} \hat{a}_{m, \mu}(v_1) = \hat{a}_{m, \mu}^+ \tag{4.7}$$

for all $m \in F$ and $\mu \in L'/L$.

5. For any $r_1, \dots, r_d \in \mathbb{N}_0$, we have

$$\lim_{|v| \rightarrow \infty} \partial_{\tau_1}^{r_1} \dots \partial_{\tau_d}^{r_d} (\hat{\delta} - \hat{\delta}^+) = 0. \tag{4.8}$$

Remark 4.4 Up to holomorphic cusp forms of parallel weight $\nu\ell + 1$, the holomorphic part $\hat{\delta}^+$ is uniquely determined by the conditions above. We will show later in Theorem 4.10 that certain rational linear combinations of the Fourier coefficients of $\hat{\delta}^+$ are logarithms of algebraic numbers.

Proof By the Siegel-Weil formula, $\theta_N(\tau)$ and $\theta_N(\tau, h)$ have the same constant terms at all cusps, and their difference decays rapidly towards all cusps (see e.g. Proposition 5.1 in [9]). By the same reason, $\theta_{P_1} - \theta_{P_2}$ is a holomorphic cusp form, and we can apply Proposition 4.2 above to $f(\tau) = \theta_N(\tau) - \theta_N(\tau, h) \in \mathcal{A}_{\kappa, \rho}$ and $g = \theta_{P_1} - \theta_{P_2} \in S_{\vec{\nu}}$. Note that ℓ_0 a priori depends on h . Since $H_W(\mathbb{Q}) \backslash H_W(\hat{\mathbb{Q}}) / K$ is finite for any open compact $K \subset H_W(\hat{\mathbb{Q}})$, there are only finitely many possible h that give rise to different $\theta_N(\tau) - \theta_N(\tau, h)$, with possibly different ℓ_0 's. By taking the maximum removes its dependence on h .

Notice that $\hat{\delta}(\tau) - (\theta_{P_1}(\tau) - \theta_{P_2}(\tau))^\ell (\theta_N^*(\tau) - \theta_N^*(\tau, h))$ is annihilated by L_{τ_1} by (3.15) and (4.5). Therefore, it is holomorphic in τ_1, \dots, τ_d . This proves part (3). Equation (3.16) implies that $\theta_N^*(\tau) - \theta_N^*(\tau, h)$ decays exponentially as $v_1 \rightarrow \infty$. Therefore, the same holds for $\hat{\delta}$ and the holomorphic part $\hat{\delta}^+$, whose Fourier coefficients are then supported only on totally positive indices. Equations (4.7) and (4.8) now follows directly from (3.16). \square

4.2 Whittaker forms

Suppose $d \geq 2$ for this section. We follow [9, 10] to recall Whittaker forms and their regularized theta lifts. For an even \mathcal{O} -lattice (L, Q) with signature as in (2.18) and $n > 2$, denote

$$s_0 := n/2, \tag{4.9}$$

$\tilde{\kappa} = \tilde{\kappa}(V) = (s_0 - 1, s_0 + 1, \dots, s_0 + 1)$ as in (3.9) and $\kappa = (1 - s_0, s_0 + 1, \dots, s_0 + 1) \in \mathbb{Z}^d$. Given $\mu \in L'/L$ and totally positive $m \in \mathfrak{d}^{-1} + Q(\mu)$, the function

$$f_{m,\mu}(\tau) := \frac{\text{Nm}(4\pi m)^{s_0}}{(4\pi m_1)^{s_0} \Gamma(s_0)^d} (\Gamma(s_0) - \Gamma(s_0, 4\pi m_1 v_1)) \times e^{4\pi m_1 v_1} \mathbf{e}(-\text{tr}(m\bar{\tau})) \phi_\mu, \tag{4.10}$$

is called a harmonic Whittaker form of weight κ in the sense of [10]. The space generated by such forms is denoted by $H_{\kappa, \bar{\rho}_L}$. Given $f = \sum_{\mu \in L'/L, m \gg 0} c(m, \mu) f_{m,\mu} \in H_{\kappa, \bar{\rho}_L}$, the Fourier polynomial

$$\mathcal{P}(f) := \sum_{\mu \in L'/L} \sum_{\substack{m \in \mathfrak{d}^{-1} + Q(\mu) \\ m \gg 0}} c(m, \mu) q^{-m} \phi_\mu \tag{4.11}$$

is called its principal part. It is invariant under $\Gamma_\infty := \left\{ \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \in \text{SL}_2(\mathcal{O}) \right\}$ with respect to $\bar{\rho}_L$. Conversely, given any polynomial of the above form, there is a unique harmonic Whittaker form $f_{\mathcal{P}} \in H_{\kappa, \bar{\rho}_L}$ with this principal part. Note that such polynomial only depends on the finite quadratic modular L'/L . We say that f or $\mathcal{P}(f)$ is *rational* if the polynomial $\mathcal{P}(f)$ has rational coefficients.

Let $\hat{\kappa}$ be the dual weight of κ as in (2.3). There is a natural surjection $\xi = \xi_{\hat{\kappa}}^{(1)} : H_{\kappa, \bar{\rho}_L} \rightarrow S_{\hat{\kappa}, \rho_L}$

$$\xi(f) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_F} v_1^{-s_0-1} \overline{L_{\tau_1} f(\tau)} |_{\kappa, \rho_L} \gamma, \tag{4.12}$$

where the sum above converges absolutely as $n > 2$. This induces a bilinear pairing between $g = \sum_{n, \nu} b(n, \nu) q^n \phi_\nu \in M_{\hat{\kappa}, \rho_L}$ and $f \in H_{\kappa, \bar{\rho}_L}$ given by

$$\begin{aligned} \{g, f\} &= \{g, \mathcal{P}(f)\} := (g, \xi(f))_{\text{Pet}} = \text{CT}((\mathcal{P}(f), g)) \\ &= \sum_{\mu \in L'/L, m \gg 0} c(m, \mu) b(m, \mu). \end{aligned} \tag{4.13}$$

A harmonic Whittaker form f is called *weakly holomorphic* if $\xi(f)$ vanishes identically, i.e. $\{g, f\} = 0$ for all $g \in S_{\hat{\kappa}, \rho_L}$. We use $M_{\kappa, \rho_L}^! \subset H_{\kappa, \rho_L}$ to denote the subspace of such forms. Using the last expression in (4.13), we can extend $\{, \}$ to formal Fourier series in the parameter $\mathbf{e}(\text{tr}(mu))$. For a subfield $\mathcal{F} \subset \mathbb{C}$, let $M_{\kappa, \bar{\rho}_L}^!(\mathcal{F})$

denote the subspace of $M_{\kappa, \bar{\rho}_L}^1$ with Fourier coefficients in \mathcal{F} . Then the fact that $M_{\tilde{\kappa}, \rho_L}(\mathcal{F}) = M_{\tilde{\kappa}, \rho_L}(\mathbb{Q}) \otimes \mathcal{F}$ implies that

$$M_{\kappa, \bar{\rho}_L}^1(\mathcal{F}) = M_{\tilde{\kappa}, \bar{\rho}_L}^1(\mathbb{Q}) \otimes \mathcal{F}. \tag{4.14}$$

4.3 Regularized theta lifts

For each $f = f_{\mathcal{P}} = \sum_{m, \mu} c(m, \mu) f_{m, \mu} \in H_{\kappa, \bar{\rho}_L}$, Bruinier computed its regularized theta lift in [10] and constructed an Arakelov Green function $\Phi(z, h, f)$ for the divisor

$$Z(f) = Z(\mathcal{P}) := \sum_{\mu \in L'/L, m \gg 0} c(m, \mu) Z(m, \phi_{\mu}) \tag{4.15}$$

on the Shimura variety X_K . When $s_0 = n/2 > 1$, Corollary 5.3 of [9] expressed this as

$$\begin{aligned} \Phi_L(z, h, f) &= \frac{1}{\sqrt{D}} \int_{\Gamma_{\infty} \backslash \mathbb{H}^d}^{\text{reg}} \langle f(\tau), \overline{\Theta_L(\tau, z, h) - E_L(\tau, \tilde{\kappa})} \rangle (v_2 \dots v_d)^{s_0+1} d\mu(\tau) \\ &\quad + B(f)(\Gamma'(1) + s_0^{-1}), \end{aligned} \tag{4.16}$$

where $\Theta_L \in \mathcal{A}_{\tilde{\kappa}, \rho_L}$ with the weight $\tilde{\kappa}$ given (2.3), $B(f) = B(\mathcal{P}) := \{E_L(\tau, \tilde{\kappa}), \mathcal{P}\} \in \mathbb{C}$ and

$$\int_{\Gamma_{\infty} \backslash \mathbb{H}^d}^{\text{reg}} G(\tau) d\mu(\tau) := \int_{v \in (\mathbb{R}_{>0})^d} \left(\int_{u \in \mathcal{O} \backslash \mathbb{R}^d} G(\tau) du \right) \frac{dv}{\text{Nm}(v)^2} \tag{4.17}$$

for any Γ_{∞} -invariant function G on \mathbb{H}^d such that the integral converges.

The integral in (4.16) converges for $s_0 = n/2 > 1$. (see Proposition 5.2 of [9]). Furthermore, Equation (3.11) implies that

$$\Phi_L(z, h(h')^{-1}, f) = \Phi_{L_{h'}}(z, h, \iota_{h'} \circ f) \tag{4.18}$$

for any $h, h' \in H(\hat{\mathbb{Q}})$, where the isomorphism $\iota_{h'} : S_L \rightarrow S_{L_{h'}}$ is defined in Sect. 2.4.

By Theorem 5.14 of [10], for any $(z_0, h_0) \in \mathbb{D} \times H(\hat{\mathbb{Q}})$, the function

$$\Phi_L(z, h, f) + \sum_{m, \mu} c(m, \mu) \sum_{\lambda \in h_0(L_{m, \mu}) \cap z_0^{\perp}} \log |Q(\lambda_z)| \tag{4.19}$$

is real-analytic in a neighborhood of (z_0, h_0) . By inspecting its proof, we have the following consequence.

Lemma 4.5 *Suppose $n > 2$ and $h_0, h'_0 \in H(\hat{\mathbb{Q}})$ satisfy⁹*

$$h_0|_{V \cap z_0^{\perp}} = h'_0|_{V \cap z_0^{\perp}}. \tag{4.20}$$

⁹Via the diagonal embedding, we tacitly view elements in $V(\mathbb{Q})$ as in $V(\hat{\mathbb{Q}})$. The set of z_0 in \mathbb{D} such that $V \cap z_0^{\perp}$ is non-trivial has measure 0.

Then $\Phi_L(z, h_0, f) - \Phi_L(z, h'_0, f)$ is real-analytic in an open neighborhood of $z_0 \in \mathbb{D}$, where it is given by

$$\begin{aligned} & \Phi_L(z, h_0, f) - \Phi_L(z, h'_0, f) \\ &= \frac{1}{\sqrt{D}} \int_{\Gamma_\infty \setminus \mathbb{H}^d}^{\text{reg}} \langle f(\tau), \overline{\Theta_L(\tau, z, h_0) - \Theta_L(\tau, z, h'_0)} \rangle (v_2 \dots v_d)^{s_0+1} d\mu(\tau) \end{aligned} \tag{4.21}$$

In particular, the integral above converges at $z = z_0$.

Proof Without loss of generality, we can suppose $f = f_{m, \mu}$. From (4.20), we have

$$h_0(L_{m, \mu}) \cap z_0^\perp = h_0(L_{m, \mu} \cap z_0^\perp) = h'_0(L_{m, \mu} \cap z_0^\perp) = h'_0(L_{m, \mu}) \cap z_0^\perp.$$

So the singularities of $\Phi_L(z, h, f)$ and $\Phi_L(z, h', f)$ agree for (z, h) and (z, h') in open neighborhoods $\mathcal{N}_1 \times \mathcal{N}_2$ and $\mathcal{N}'_1 \times \mathcal{N}'_2$ of (z_0, h_0) and (z_0, h'_0) respectively. Then $\Phi_L(z, h_0, f) - \Phi_L(z, h'_0, f)$ is real-analytic in the open neighborhood $\mathcal{N} := \mathcal{N}_1 \cap \mathcal{N}_2 \subset \mathbb{D}$ of z_0 , which could be made to satisfy $h_0(L_{m, \mu}) \cap z^\perp = \{0\}$ for all $z \in \mathcal{N} \setminus \{z_0\}$.

Then convergence of the integral at $z \notin Z(m, \mu)$ follows directly from Proposition 5.2 of [9]. For such z , we can apply the unfolding calculation in the proof of Theorem 5.3 of [10] while evaluating at $s = s_0$. Then the following integrals are identically equal for any $z \in \mathcal{N} \setminus \{z_0\}$

$$\begin{aligned} & \int_{u \in \mathcal{O} \setminus \mathbb{R}^d} \langle f_{m, \mu}(\tau), \overline{\Theta_L(\tau, z, h_0) - \Theta_L(\tau, z, h'_0)} \rangle (v_2 \dots v_d)^{s_0+1} du \\ &= \int_{u \in \mathcal{O} \setminus \mathbb{R}^d} \langle f_{m, \mu}(\tau), \overline{\Theta'_L(\tau, z, h_0) - \Theta'_L(\tau, z, h'_0)} \rangle (v_2 \dots v_d)^{s_0+1} du, \end{aligned}$$

where $\Theta'_L(\tau, z, h) := \Theta_L(\tau, z, h) - \sum_{\lambda \in h(L_{m, \mu}) \cap z^\perp} \phi_\mu(h^{-1}\lambda)\phi_\infty(\tau, z, \lambda)\phi_\mu$. From the second expression, we see that the integral in (4.21) also converges for $z = z_0$. \square

To evaluate the regularized theta integral, one can apply Stokes' theorem when certain primitive exist. The following lemma distilled from Theorems 6.3 and 7.2 in [9] will be helpful for this purpose.

Lemma 4.6 *Suppose $n = 2s_0 > 2$ and $f = \sum_{m \gg 0, \mu \in L'/L} c(m, \mu) f_{m, \mu}$ is a harmonic Whittaker form in $H_{\kappa, \bar{\rho}_L}$. Let $\eta \in \mathcal{A}_{\hat{\kappa}, \rho_L}$ such that $\int_{\Gamma_\infty \setminus \mathbb{H}^d}^{\text{reg}} \langle f(\tau), \overline{\eta(\tau)} \rangle (v_2 \dots v_d)^{s_0+1} d\mu(\tau)$ converges. Suppose there exists*

$$\hat{\eta}(\tau) = \sum_{m \in F, \mu \in L'/L} \hat{b}_{m, \mu}(v_1) \mathbf{e}(\text{tr}(m\tau)) \phi_\mu \in \mathcal{A}_{\hat{\kappa}, \rho_L}$$

such that $L_{\tau_1} \hat{\eta}(\tau) = -2i v_1^2 \partial_{\bar{\tau}_1} \hat{\eta}(\tau) = \eta(\tau)$ and the Γ_F -invariant function $\|\hat{\eta}(\tau)\|_{\text{Pet}}$ is bounded on \mathbb{H}^d . Then

$$\begin{aligned} & \int_{\Gamma_\infty \backslash \mathbb{H}^d}^{\text{reg}} \langle f(\tau), \overline{\eta(\tau)} \rangle (v_2 \dots v_d)^{s_0+1} d\mu(\tau) \\ &= -(\hat{\eta}, \xi(f))_{\text{Pet}} + \lim_{T \rightarrow \infty} \sum_{m, \mu} c(m, \mu) \hat{b}_{m, \mu}(T). \end{aligned} \tag{4.22}$$

Remark 4.7 Since $\xi(f)$ is a cusp form, the integral defining the Petersson inner product exists, and so does the limit in T .

Proof We will give the proof for $f = f_{m, \mu}$. The general case can be proved the same way. Note that m is totally positive. By applying L_{τ_1} to the Fourier expansion of $\hat{\eta}$, we can write

$$\eta(\tau) = \sum_{m \in F, \mu \in L'/L} b_{m, \mu}(v_1) \mathbf{e}(\text{tr}(m\tau)) \phi_\mu, \quad \partial_{v_1} \hat{b}_{m, \mu}(v_1) = v_1^{-2} b_{m, \mu}(v_1).$$

Substituting this into the left hand side of (4.22) gives us

$$\begin{aligned} & \frac{1}{\sqrt{D}} \int_{\Gamma_\infty \backslash \mathbb{H}^d}^{\text{reg}} \langle f(\tau), \overline{\eta(\tau)} \rangle (v_2 \dots v_d)^{s_0+1} d\mu(\tau) \\ &= \lim_{T \rightarrow \infty} \int_{T^{-1}}^T b_{m, \mu}(v_1) \frac{\Gamma(s_0) - \Gamma(s_0, 4\pi m_1 v_1)}{\Gamma(s_0)} \frac{dv_1}{v_1^2}. \end{aligned}$$

Using integration by parts, we can rewrite the integral as

$$\begin{aligned} & \int_{T^{-1}}^T b_{m, \mu}(v_1) \frac{\Gamma(s_0) - \Gamma(s_0, 4\pi m_1 v_1)}{\Gamma(s_0)} \frac{dv_1}{v_1^2} \\ &= -\frac{(4\pi m_1)^{s_0}}{\Gamma(s_0)} \int_{T^{-1}}^T \hat{b}_{m, \mu}(v_1) v_1^{s_0-1} e^{-4\pi m_1 v_1} dv_1 \\ & \quad + \hat{b}_{m, \mu}(T) \frac{\Gamma(s_0) - \Gamma(s_0, 4\pi m_1 T)}{\Gamma(s_0)} - \hat{b}_{m, \mu}(T^{-1}) \frac{\Gamma(s_0) - \Gamma(s_0, 4\pi m_1 T^{-1})}{\Gamma(s_0)} \end{aligned}$$

As $T \rightarrow \infty$, the first term on the right hand side becomes $-(\hat{\eta}, \xi(f))_{\text{Pet}}$ after unfolding, and the second term becomes $\lim_{T \rightarrow \infty} \hat{b}_{m, \mu}(T)$. For the third term, the boundedness of $\langle \hat{\eta}, \hat{\eta} \rangle \text{Nm}(v)^{s_0+1}$ implies that

$$\hat{b}_{m, \mu}(t) = O(t^{(-s_0-1)/2}) \text{ as } t \rightarrow 0,$$

while $\Gamma(s_0) - \Gamma(s_0, 4\pi m_1 t) = O(t^{s_0})$ as $t \rightarrow 0$. Since $s_0 > 1$, the third term vanishes as $T \rightarrow \infty$. This finishes the proof. □

4.4 Algebraicity result

Fix a quadratic space W over F having signature as in (2.18) with $n = 0$. Let $N \subset W$ be an even \mathcal{O} -lattice, $K_N \subset H_W(\hat{\mathbb{Q}})$ an open compact fixing \hat{N} and acting trivially on $\hat{N}'/\hat{N} = S_N$. Now, let $\tilde{P}, P_1, P_2 \in \mathcal{U}_F^+$ be lattices such that P_1, P_2 are in the same genus. Denote their ranks by $2\tilde{\tau}$ and 2τ with $\tilde{\tau}, \tau \in \mathbb{N}$. Given integers $\ell \in \mathbb{N}$ and $0 \leq i \leq \ell$, we write

$$\begin{aligned} L_i &:= N \oplus \tilde{P} \oplus P_1^i \oplus P_2^{\ell-i}, \\ \tilde{U} &:= \tilde{P} \otimes F, \quad U := P_1^\ell \otimes F = P_2^\ell \otimes F, \\ V &:= L_i \otimes F = \tilde{U} \oplus U \oplus W. \end{aligned} \tag{4.23}$$

Note that $\rho_{L_i} = \rho_N$ since \tilde{P}, P_1, P_2 are \mathbb{Z} -unimodular. For $h_W \in H_W(\hat{\mathbb{Q}}) \subset H_V(\hat{\mathbb{Q}})$, we have

$$\Theta_{L_i}(\tau, z_0, h_W) = \theta_N(\tau, h_W) \theta_{\tilde{P}}(\tau) \theta_{P_1}(\tau)^i \theta_{P_2}(\tau)^{\ell-i} \in \mathcal{A}_{\tilde{\tau}+\tau\ell+(-1,1,\dots,1), \rho_N} \tag{4.24}$$

for a CM point $(z_0, h_W) \in Z(W, 1) \subset X_{V,K}$ defined over E_W^{ab} for a CM field E_W/F , and $K \subset H_V(\mathbb{A})$ an open compact fixing L_i for all $0 \leq i \leq \ell$. Note that z_0 depends only on the rational splitting $V = \tilde{U} \oplus U \oplus W$.

Theorem 4.3 shows that for all $\ell \in \mathbb{N}$ sufficiently large, there exists $\hat{\delta}(\tau; N, h_W, P_1, P_2, \ell) \in \mathcal{A}_{\tau\ell+1, \rho_N}$ for every $h_W \in H_W(\hat{\mathbb{Q}})$ such that

$$L_{\tau_1} \hat{\delta}(\tau) = (\theta_{P_1}(\tau) - \theta_{P_2}(\tau))^\ell (\theta_N(\tau) - \theta_N(\tau, h_W)) \tag{4.25}$$

and $L_{\tau_j} \hat{\delta} = 0$ for $2 \leq j \leq d$. Recall that $\hat{\delta}^+(\tau)$ is the holomorphic part of $\hat{\delta}(\tau)$ as in (4.6). We can now apply Lemmas 4.5 and 4.6 to the function

$$\hat{\eta}_{\tilde{P}}(\tau) := \theta_{\tilde{P}}(\tau) \cdot \hat{\delta}(\tau), \quad \hat{\eta}_{\tilde{P}}^+(\tau) := \theta_{\tilde{P}}(\tau) \cdot \hat{\delta}^+(\tau), \tag{4.26}$$

and express linear combinations of the Fourier coefficients of $\hat{\eta}_{\tilde{P}}^+$ in terms of special values of the regularized theta lift $\Phi_{L_i}(z, h, f)$ for any harmonic Whittaker form $f \in H_{\kappa, \tilde{\rho}_N}$.

Proposition 4.8 *Let $L_i, h_W, \hat{\delta}, \hat{\eta}_{\tilde{P}}$ be as above. Then*

$$\begin{aligned} &\sum_{0 \leq i \leq \ell} (-1)^{\ell-i} \binom{\ell}{i} (\Phi_{L_i}(z_0, 1, f) - \Phi_{L_i}(z_0, h_W, f)) \\ &= -(\hat{\eta}_{\tilde{P}}, \xi(f))_{\text{Pet}} + \{\hat{\eta}_{\tilde{P}}^+, f\} \end{aligned} \tag{4.27}$$

for any harmonic Whittaker form $f \in H_{\kappa, \tilde{\rho}_N}$ with $\kappa = \widetilde{\tilde{\kappa}(V)}$.

Proof It is easy to see that $h_0 = 1, h'_0 = h_W$ satisfy the condition (4.20). Since $\hat{\delta}$ has exponential decay near the cusps, the function $\|\hat{\eta}_{\tilde{P}}(\tau)\|_{\text{Pet}}$ is bounded on \mathbb{H}^d .

Furthermore, equation (4.25) and the holomorphicity of $\theta_{\tilde{P}}$ implies that

$$\begin{aligned} L_{\tau_1} \hat{\eta}_{\tilde{P}}(\tau) &= L_{\tau_1}(\theta_{\tilde{P}}(\tau) \hat{\delta}(\tau)) = \theta_{\tilde{P}}(\tau) L_{\tau_1}(\hat{\delta}(\tau)) \\ &= \theta_{\tilde{P}}(\tau)(\theta_{P_1}(\tau) - \theta_{P_2}(\tau))^\ell (\theta_N(\tau) - \theta_N(\tau, h_W)). \end{aligned}$$

If we denote $\hat{b}_{m,\mu}(v_1)$ and $\hat{b}_{m,\mu}^+$ the Fourier coefficients of $\hat{\eta}_{\tilde{P}}$ and $\hat{\eta}_{\tilde{P}}^+$ respectively, then we have

$$\lim_{v_1 \rightarrow \infty} \hat{b}_{m,\mu}(v_1) = \hat{b}_{m,\mu}^+$$

by Theorem 4.3. Therefore Lemmas 4.5 and 4.6 together with (4.24) imply

$$\begin{aligned} &\sum_{0 \leq i \leq \ell} (-1)^{\ell-i} \binom{\ell}{i} (\Phi_{L_i}(z_0, 1, f) - \Phi_{L_i}(z_0, h_W, f)) \\ &= \int_{\Gamma_\infty \backslash \mathbb{H}^d}^{\text{reg}} \sum_{0 \leq i \leq \ell} (-1)^{\ell-i} \binom{\ell}{i} \langle f(\tau), \overline{\Theta_{L_i}(\tau, z_0, 1) - \Theta_{L_i}(\tau, z_0, h_W)} \rangle \\ &\quad \times (v_2 \dots v_d)^{s_0+1} d\mu(\tau) \\ &= \int_{\Gamma_\infty \backslash \mathbb{H}^d}^{\text{reg}} \langle f(\tau), \overline{(\theta_N(\tau) - \theta_N(\tau, h_W)) \theta_{\tilde{P}}(\tau) (\theta_{P_1}(\tau) - \theta_{P_2}(\tau))^\ell} \rangle \\ &\quad \times (v_2 \dots v_d)^{s_0+1} d\mu(\tau) \\ &= -(\hat{\eta}_{\tilde{P}}, \xi(f))_{\text{Pet}} + \lim_{T \rightarrow \infty} \sum_{m,\mu} c(m, \mu) \hat{a}_{m,\mu}(T) = -(\hat{\eta}_{\tilde{P}}, \xi(f))_{\text{Pet}} + \{\hat{\eta}_{\tilde{P}}^+, f\}. \end{aligned}$$

This finishes the proof. □

Remark 4.9 If we replace \tilde{P} by $\tilde{P}_{\tilde{h}}$ with $\tilde{h} \in H_{\tilde{U}}(\hat{\mathbb{Q}})$, then $\tilde{P}_{\tilde{h}}$ is in the same genus as \tilde{P} and $N \oplus \tilde{P}_{\tilde{h}} \oplus P_1^i \oplus P_2^{\ell-i} = (\tilde{h}^{-1} L_i) \cap V = L_{i,\tilde{h}}$ with $\tilde{h} \in H_{\tilde{U}}(\hat{\mathbb{Q}}) \subset H_V(\hat{\mathbb{Q}})$. Furthermore, $\iota_{\tilde{h}} \circ f = f$ since \tilde{P} is \mathbb{Z} -unimodular. The proposition above and Equation (3.11) imply that the function $\hat{\eta}_{\tilde{P}_{\tilde{h}}}$ satisfies

$$\begin{aligned} &\sum_{0 \leq i \leq \ell} (-1)^{\ell-i} \binom{\ell}{i} (\Phi_{L_i}(z_0, (1, \tilde{h}^{-1}), f) - \Phi_{L_i}(z_0, (h_W, \tilde{h}^{-1}), f)) \\ &= -(\hat{\eta}_{\tilde{P}_{\tilde{h}}}, \xi(f))_{\text{Pet}} + \{\hat{\eta}_{\tilde{P}_{\tilde{h}}}^+, f\} \end{aligned} \tag{4.28}$$

for all $\tilde{h} \in H_{\tilde{U}}(\hat{\mathbb{Q}})$.

If $c(m, \mu) \in \mathbb{Z}$ for all m and μ , then Theorem 6.8 of [10] implies that there exists a meromorphic modular form $\Psi_L(z, h, f)$ on X_K with weight $-B(f)$ and a finite order multiplier system such that

$$-\log \|\Psi_L(z, h, f)\|_{\text{Pet}}^2 = \Phi_L(z, h, f). \tag{4.29}$$

Furthermore, the divisor of $\Psi_L(z, h, f)$ is the special cycle $Z(f)$, which is defined over F . Let $M \in \mathbb{N}$ be the order of the multiplier system. Then up to a locally constant function $C_L(z, h, f)$ on X_K , the form $\Psi_L(z, h, f)^M$ equals to a meromorphic modular form $R_L(z, h, f)$ of weight $-B(M \cdot f) \in \mathbb{Z}$ on X_K defined over $\sigma_1(F)$, and we can write

$$\Phi_L(z, h, f) = 2 \log |C_L(z, h, f)| - \frac{2}{M} \log \|R_L(z, h, f)\|_{\text{Pet}}. \tag{4.30}$$

Now we let $L = L_i$ for $0 \leq i \leq \ell$ as in Proposition 4.8. Since the spinor norm $v : H_{\tilde{U}} \rightarrow T$ is surjective, the subgroup $H_{\tilde{U}}(\hat{\mathbb{Q}}) \subset H_V(\hat{\mathbb{Q}})$ acts transitively on the connected components of X_K by (2.23). Therefore for any open compact $K_{\tilde{U}} \subset K \cap H_{\tilde{U}}(\hat{\mathbb{Q}})$, the quantity

$$\mathcal{C}_i(K_{\tilde{U}}, f) := \prod_{\tilde{h} \in H_{\tilde{U}}(\mathbb{Q})_+ \backslash H_{\tilde{U}}(\hat{\mathbb{Q}})/K_{\tilde{U}}} C_{L_i}(z, (h_W, \tilde{h}), f) \tag{4.31}$$

is independent of $h_W \in H_W(\hat{\mathbb{Q}})$ and $z \in \mathbb{D}$.

Now, we are ready to state and prove the main result of this section.

Theorem 4.10 *Let N, h, P_1, P_2, ℓ and $\hat{\delta}(\tau; N, h, P_1, P_2, \ell) \in \mathcal{A}_{\tau\ell+1, \rho_L}^{\rightarrow}$ be the same as in Theorem 4.3 and $\tilde{P}, L_i, V, \kappa$ the same as in Proposition 4.8. For any weakly holomorphic Whittaker form $f = f_{\mathcal{P}} \in M_{\kappa, \tilde{\rho}_N}^1$ with coefficients in a number field $\mathcal{F} \subset \mathbb{C}$, there exists $\lambda_1, \dots, \lambda_m \in \mathcal{F}$ and $\alpha_1, \dots, \alpha_m \in E_W^{\text{ab}}$ independent of h_W such that*

$$\{E_{\tilde{P}}(\tau)\hat{\delta}^+(\tau), \mathcal{P}\} = \sum_{i=1}^m \lambda_i \log |\alpha_i/\sigma(\alpha_i)|, \tag{4.32}$$

where $\sigma \in \text{Gal}(E_W^{\text{ab}}/E_W)$ is the element associated to $h_W \in H_W(\hat{\mathbb{Q}}) \cong \hat{E}_W^\times$ via class field theory.

Remark 4.11 We can choose λ_i, α_i above such that the index m is bounded by the degree of \mathcal{F}/\mathbb{Q} . Due to the order the multiplier system of the meromorphic modular form $\Psi(z, h, f)$ in (4.29), the denominator of λ_i depends on f even when it has integral coefficients

Proof By (4.14), we can write $f = \sum_{i=1}^m \lambda_i f_i$ with $f_i \in M_{\kappa, \tilde{\rho}_N}^1(\mathbb{Q})$ and suppose that $\mathcal{F} = \mathbb{Q}$. By replacing f with $M \cdot f$ for some $M \in \mathbb{N}$, we can suppose that f has integral Fourier coefficients and the modular form $\Psi_{L_i}(z, h, f)$ in (4.29) has trivial character for all $0 \leq i \leq \ell$. Let $R_L(z, h, f)$ and $C_L(z, h, f)$ be the same as in (4.30).

Let $K \subset H_V(\hat{\mathbb{Q}})$ be an open compact fixing \hat{L}_i , acting trivially on $\hat{L}'_i/\hat{L}_i = \hat{N}'/\hat{N}$, and $K \cap H_W(\hat{\mathbb{Q}})$ contains K_N . Denote $K_{\tilde{U}} := K \cap H_{\tilde{U}}(\hat{\mathbb{Q}})$ an open compact in $H_{\tilde{U}}(\mathbb{Q})$ and $C_i(K_{\tilde{U}}, f)$ the same as in (4.31). Fix $S_W \subset H_W(\hat{\mathbb{Q}})$ coset representatives of $H_W(\mathbb{Q}) \backslash H_W(\hat{\mathbb{Q}})/K_N$.

Since f is weakly holomorphic, the cusp form $\xi(f)$ vanishes identically. Combining the Siegel-Weil formula in (3.23), Proposition 4.8, Remark 4.9 and Equations (4.30), (4.31), we can then write

$$\begin{aligned} \{E_{\tilde{p}}(\tau)\hat{\delta}^+(\tau), \mathcal{P}\} &= c_{K_{\tilde{U}}}^{-1} \sum_{\tilde{h} \in H_{\tilde{U}}(\mathbb{Q}) \backslash H_{\tilde{U}}(\hat{\mathbb{Q}})/K_{\tilde{U}}} \{\theta_{\tilde{p}}(\tau, \tilde{h})\hat{\delta}^+(\tau), \mathcal{P}\} \\ &= c_{K_{\tilde{U}}}^{-1} \sum_{\substack{\tilde{h} \in H_{\tilde{U}}(\mathbb{Q}) \backslash H_{\tilde{U}}(\hat{\mathbb{Q}})/K_{\tilde{U}} \\ 0 \leq i \leq \ell}} (-1)^{\ell-i} \binom{\ell}{i} \left(\Phi_{L_i}(z_0, (1, \tilde{h}), f) - \Phi_{L_i}(z_0, (h_W, \tilde{h}), f) \right) \\ &= 2c_{K_{\tilde{U}}}^{-1} \sum_{0 \leq i \leq \ell} (-1)^{\ell-i} \binom{\ell}{i} \\ &\quad \times \left(\log \left| \frac{C_i(K_{\tilde{U}}, f)}{C_i(K_{\tilde{U}}, f)} \right| - \sum_{\tilde{h} \in H_{\tilde{U}}(\mathbb{Q}) \backslash H_{\tilde{U}}(\hat{\mathbb{Q}})/K_{\tilde{U}}} \log \left\| \frac{R_{L_i}(z_0, (1, \tilde{h}), f)}{R_{L_i}(z_0, (h_W, \tilde{h}), f)} \right\|_{\text{Pet}} \right) \\ &= -\frac{2}{c_{K_{\tilde{U}}} \cdot \#S_W} \sum_{\tilde{h} \in H_{\tilde{U}}(\mathbb{Q}) \backslash H_{\tilde{U}}(\hat{\mathbb{Q}})/K_{\tilde{U}}} \log |Q(z_0, (1, \tilde{h})) / Q(z_0, (h_W, \tilde{h}))|, \end{aligned}$$

where $Q(z, h)$ is a meromorphic function on $X_{K'}$ defined over $\sigma_1(F)$ given by

$$Q(z, h) := \prod_{0 \leq i \leq \ell} \prod_{h_0 \in S_W} \left(\frac{R_{L_i}(z, h, f)}{R_{L_i}(z, h(h_0, 1), f)} \right)^{(-1)^i \binom{\ell}{i}}$$

with $K' := K \cap_{h_0 \in S_W} h_0 K h_0^{-1}$. Since H_W is abelian, we have $K_N \times K_{\tilde{U}} \subset K'$. Therefore, the CM 0-cycle $Z(W)$ also lies on $X_{K'}$ and each CM point is defined over a number field $E_{K_N} \subset E_W^{\text{ab}}$. The function R_{L_i} is defined over $\sigma_1(F)$, as well as the natural map $X_{K'} \rightarrow X_K$ given by right multiplication with $h_0 \in H_W(\hat{\mathbb{Q}}) \subset H_V(\hat{\mathbb{Q}})$ (see [25, page 46]). Therefore, the modular function $Q(z, h)$ is also defined over $\sigma_1(F)$. Furthermore, it is non-zero at $(z, h) = (z_0, (h_W, 1))$ for all $h_W \in H_W(\hat{\mathbb{Q}})$. These values are algebraic numbers satisfying

$$\sigma(Q(z_0, (1, \tilde{h}))) = Q(z_0, (h_W, \tilde{h}))$$

as σ fixes the function $Q(z, h)$, which is defined over $\sigma_1(F)$, and acts on CM points in the CM 0-cycle $Z(W, \tilde{h})$ by Shimura’s reciprocity law (see Sects. 3.1 and 5.3 in [2] for the relevant case here). Setting $\alpha = \prod_{\tilde{h} \in H_{\tilde{U}}(\mathbb{Q}) \backslash H_{\tilde{U}}(\hat{\mathbb{Q}})/K_{\tilde{U}}} Q(z_0, (1, \tilde{h}))$ and $\lambda = -\frac{2}{c_{K_{\tilde{U}}} \cdot \#S_W}$ finishes the proof. □

5 Proofs of theorems

Now we are ready to prove the three theorems from the introduction.

5.1 Proof of Theorem 1.3

Let $N \subset W$ be a lattice such that $K \cap H_W(\hat{\mathbb{Z}})$ fixes \hat{N} and acts trivially on \hat{N}'/\hat{N} . By passing to sublattice, we can suppose that $L = \tilde{L} \oplus N_0$ with $N_0 = \text{Res}_{F/\mathbb{Q}}N$ and \tilde{L} positive definite of rank $2 + n - 2d$. Then for any $(z_0, h) \in Z(W_{\mathbb{Q}})$, we have $\Theta_L(\tau, z_0, h) = \theta_{\tilde{L}}(\tau) \otimes \theta_N^{\Delta}(\tau, h)$. By replacing N with $h_1 \hat{N} \cap V$ if necessary, we can suppose that $h_1 = 1$ and write $h = h_2$. We can apply Lemma 2.2 to $g = \theta_{\tilde{L}}$, $k = 1 - n/2 - 2r$, $\ell = 1 + n/2 - d$ as $k + \ell = -d + 2 - 2r \leq 1 - 2r$. This implies

$$\begin{aligned} \langle \tilde{R}_{\tau}^r f(\tau), \overline{\Theta_L(\tau, z_0, h)} \rangle &= \left\langle \tilde{R}_{\tau}^r f(\tau), \overline{\theta_{\tilde{L}}(\tau)}_{N_0}, \overline{\theta_N^{\Delta}(\tau, h)} \right\rangle \\ &= \sum_{j=0}^r c_{r,r,j} \langle \tilde{R}_{\tau}^{r-j} f_j(\tau), \overline{\theta_N^{\Delta}(\tau, h)} \rangle, \end{aligned}$$

where $c_{r,r,j} \in \mathbb{Q}$ and

$$f_j(\tau) := \mathcal{C}_{(k,\ell),j} \langle f(\tau_1), \overline{\theta_{\tilde{L}}(\tau_2)} \rangle_{N_0} \in M_{2-2r-d+2j, \bar{\rho}_{N_0}}^1$$

has rational Fourier coefficients. From the integral representation (3.30), we see that it suffices to prove the theorem with $f = f_0$ and $L = N_0$, in which case

$$\langle f(\tau), \overline{\tilde{R}_{\tau}^r \Theta_L(\tau, z_0, h)} \rangle = \langle f_0(\tau), \overline{\tilde{R}_{\tau}^r \theta_N^{\Delta}(\tau, h)} \rangle.$$

Now for $1 \leq i \leq m$, let $\tilde{P}_i, P_{1,i}, P_{2,i} \in \mathcal{U}_F^+$ be \mathcal{O} -lattices of ranks $2\tilde{\tau}_i, 2\tau_i$ as in Proposition 3.4. For any $\ell \in \mathbb{N}$, it gives us $g_i \in M_{-(\tilde{\tau}_i + \tau_i \ell_i)(r+1)d}^1$ such that

$$\begin{aligned} \langle f_0(\tau), \overline{\tilde{R}_{\tau}^r \theta_N^{\Delta}(\tau, h)} \rangle \\ = \langle f_0(\tau), \overline{\tilde{R}_{\tau}^r \theta_N^{\Delta}(\tau, h)} \rangle \sum_{i=1}^m g_i(\tau) (E_{\tilde{P}_i}^{\Delta}(\tau) (\theta_{P_{1,i}}^{\Delta}(\tau) - \theta_{P_{2,i}}^{\Delta}(\tau))^{\ell_i})^{r+1}, \end{aligned}$$

and it suffices to prove the theorem for each i . We drop the index i and have

$$\begin{aligned} \Phi_L^r(z_0, 1, f) - \Phi_L^r(z_0, h, f) \\ = \lim_{T \rightarrow \infty} \int_{\mathcal{F}_T} \langle f_0(\tau) g(\tau), \overline{(G^{\Delta}(\tau))^{r+1} \tilde{R}_{\tau}^r (\theta_N^{\Delta}(\tau) - \theta_N^{\Delta}(\tau, h))} \rangle d\mu(\tau). \end{aligned} \tag{5.1}$$

where we set $G := E_{\tilde{P}} \cdot (\theta_{P_1} - \theta_{P_2})^{\ell}$. Fix an $\ell \geq \ell_0$ as in Theorem 4.3 and we obtain a real-analytic Hilbert modular form $\hat{\delta}(\tau) = \hat{\delta}(\tau; N, h, P_1, P_2, \ell) \in \mathcal{A}_{1+\tau\ell, \rho_N}^{\rightarrow}$ with the property

$$L_{\tau_1} \hat{\eta}(\tau) = G(\tau) (\theta_N(\tau) - \theta_N(\tau, h)), \quad \hat{\eta} := E_{\tilde{P}} \cdot \hat{\delta} \in \mathcal{A}_{\hat{\kappa}, \rho_N}, \tag{5.2}$$

where $\kappa := (1 - \tilde{\tau} - \tau\ell, \tilde{\tau} + \tau\ell + 1, \dots, \tilde{\tau} + \tau\ell + 1)$ and $\hat{\kappa}$ is defined as in (2.3). We can now use the differential operator $\mathcal{D}_{1,r}^{\rightarrow}$ from (2.13) to define a real-analytic, elliptic

modular form

$$\hat{\eta}_r := \mathcal{D}_{\vec{1},r}(\hat{\eta}, G) \in \mathcal{A}_{k,\rho_{N_0}}, \quad k := d((1+r)(\tilde{\nu} + \nu\ell) + 1) + 2r. \tag{5.3}$$

By Lemma 2.3, this function satisfies

$$L_\tau \hat{\eta}_r = G^\Delta(\tau)^{r+1} \tilde{R}^r(L_{\tau_1} \hat{\eta} / G)^\Delta = G^\Delta(\tau)^{r+1} \tilde{R}^r(\theta_N^\Delta(\tau) - \theta_N^\Delta(\tau, h)).$$

We can now apply Stokes' theorem to the right hand side of (5.1) and part (5) of Theorem 4.3 to obtain

$$\begin{aligned} \Phi_L^r(z_0, 1, f) - \Phi_L^r(z_0, h, f) &= - \lim_{T \rightarrow \infty} \int_0^1 \langle f_0(u + iT)g(u + iT), \overline{\hat{\eta}_r(u + iT)} \rangle du \\ &= -\{f_0g, \hat{\eta}_r^+\}, \end{aligned}$$

where $\hat{\eta}_r^+ := \mathcal{D}_{\vec{1},r}(\hat{\eta}^+, G)$ is a formal Fourier series with coefficients being F -linear combinations of those of $\hat{\eta}^+$. Note there exists a harmonic Whittaker form $g_r \in H_{\kappa,\bar{\rho}_N}$ with Fourier coefficients in F such that

$$\{f_0g, \mathcal{D}_{\vec{1},r}(\delta, G)\} = \{g_r, \delta\}.$$

for any Fourier series $\delta(\tau) = \sum_{\mu \in S_N, m \in F} c(m, \mu) \mathbf{e}(\text{tr}(\mu\tau)) \phi_\mu$ that is Γ_∞ -invariant with respect to ρ_N . If $\delta \in S_{\tilde{\kappa},\rho_N}$, then $\mathcal{D}_{\vec{1},r}(\delta, G)$ is in $S_{k,\rho_{N_0}}$ and this pairing vanishes as $f_0g \in M_{2-k,\bar{\rho}_{N_0}}^1$. Therefore g_r is weakly holomorphic and (1.7) follows from

$$\Phi_L^r(z_0, 1, f) - \Phi_L^r(z_0, h, f) = -\{f_0g, \hat{\eta}_r^+\} = -\{g_r, \hat{\eta}^+\} = -\{g_r, E_{\vec{p}} \hat{\delta}^+\}$$

and Theorem 4.10.

When $d = 2 = n/2$, we already have $L = N_0$ and the reduction step in the first paragraph above is not necessary. The function $\hat{\eta}_r^+$ is simply $(G^\Delta)^{r+1} \mathcal{C}_{(1,1),r}(\hat{\eta}^+ / G)$, and the last claim follows from Example 2.1.

5.2 Proof of Theorem 1.7

As $Z(W_{\mathbb{Q}})$ is defined over \mathbb{Q} , i.e. $Z(W_{\mathbb{Q}}) = Z(W(2)_{\mathbb{Q}})$ on $X_{V,K}$, we can apply Theorem 5.10 in [11] to conclude that

$$\frac{2}{\text{deg}(Z(W_{\mathbb{Q}}))} \Phi_L^r(Z(W_{\mathbb{Q}}), f) = \text{CT}(\langle f, \mathcal{C}_{(1,1),r}(\mathcal{E}_L^+) \rangle),$$

where \mathcal{E}_L^+ is the holomorphic part of the derivative of an incoherent Eisenstein series, which is a real-analytic Hilbert modular form of weight $(1, 1)$. From Example 2.1, we see that the constant term of $\mathcal{C}_{(1,1),r}(\mathcal{E}_L^+)$ vanishes when $r \geq 1$. Furthermore, since f has rational Fourier coefficients, the term $\text{CT}(\langle f, \mathcal{C}_{(1,1),r}(\mathcal{E}_L^+) \rangle)$ is a rational linear combinations of the non-zero Fourier coefficients of \mathcal{E}_L^+ , which are rational multiple

of logarithms of integers by Proposition 4.6 in [13]. Therefore, we have

$$\sum_{(z_0, h) \in Z(W_{\mathbb{Q}})} \Phi_L^r(z_0, h, f) = \frac{1}{c'} \log |a| \tag{5.4}$$

for some $c' \in \mathbb{N}$ and $a \in \mathbb{Q}$.

We can now apply Theorem 1.3 to find $c \in \mathbb{N}$ and $\alpha \in E' \subset E_W^{\text{ab}}$ such that

$$\Phi_L^r(z_0, 1, f) - \Phi_L^r(z_0, h, f) = \frac{1}{c'c} \log \left| \frac{\alpha}{\sigma(\alpha)} \right|$$

for all $(z_0, h) \in Z(W_{\mathbb{Q}})$. Denote $N := |Z(W_{\mathbb{Q}})|/2 \in \mathbb{N}$ and $\beta := \prod_{\sigma \in \text{Gal}(E'/E_W)} \sigma(\alpha) \in E_W$. The system of equations above has the unique solution

$$\Phi_L^r(z_0, h, f) = \frac{1}{2Nc'c} \log |\sigma(\tilde{\alpha})|$$

with $\tilde{\alpha} = \alpha^{2N} a^c / \beta^2 \in E'$. This finishes the proof.

5.3 Proof of Theorem 1.1

When r is even, this follows from the discussion in Example 2.7 and Theorem 1.7. When r is odd and d_1 is fundamental, Theorem 7.13 of [11] gives the algebraicity analogous to (5.4) with the left hand side replaced by certain partial average. Using this and proceeding with the rest of the argument in the proof of Theorem 1.7 gives Theorem 1.1.

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References

1. Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. National Bureau of Standards Applied Mathematics Series, vol. 55. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington (1964)
2. Andreatta, F., Goren, E.Z., Howard, B., Pera, K.M.: Faltings heights of Abelian varieties with complex multiplication. *Ann. Math. (2)* **187**(2), 391–531 (2018)
3. Apostol, T.M.: Modular Functions and Dirichlet Series in Number Theory, 2nd edn. Graduate Texts in Mathematics, vol. 41. Springer, New York (1990)
4. Ash, A., Mumford, D., Rapoport, M., Tai, Y.-S.: Smooth Compactifications of Locally Symmetric Varieties, 2nd edn. Cambridge Mathematical Library. Cambridge University Press, Cambridge (2010). With the collaboration of Peter Scholze
5. Bogo, G., Li, Y.: Span of restriction of Hilbert theta functions. *Pure Appl. Math. Q.* **19**(1), 61–83 (2023)
6. Borcherds, R.E.: Automorphic forms with singularities on Grassmannians. *Invent. Math.* **132**(3), 491–562 (1998)
7. Bruinier, J.H.: Borcherds Products on $O(2, l)$ and Chern Classes of Heegner Divisors. Lecture Notes in Mathematics, vol. 1780. Springer, Berlin (2002)
8. Bruinier, J.H., Li, Y., Yang, T.: Deformations of theta integrals and a conjecture of Gross-Zagier (2022). [arXiv:2204.10604](https://arxiv.org/abs/2204.10604), submitted
9. Bruinier, J.H., Yang, T.: CM values of automorphic Green functions on orthogonal groups over totally real fields. In: Arithmetic Geometry and Automorphic Forms. *Adv. Lect. Math. (ALM)*, vol. 19, pp. 1–54. Int. Press, Somerville (2011)
10. Bruinier, J.H.: Regularized theta lifts for orthogonal groups over totally real fields. *J. Reine Angew. Math.* **672**, 177–222 (2012)
11. Bruinier, J.H., Ehlen, S., Yang, T.: CM values of higher automorphic Green functions for orthogonal groups. *Invent. Math.* **225**(3), 693–785 (2021)
12. Bruinier, J.H., Funke, J.: On two geometric theta lifts. *Duke Math. J.* **125**(1), 45–90 (2004)
13. Bruinier, J.H., Kudla, S.S., Yang, T.: Special values of Green functions at big CM points. *Int. Math. Res. Not.* **9**, 1917–1967 (2012)
14. Bruinier, J.H., van der Geer, G., Harder, G., Zagier, D.: The 1-2-3 of Modular Forms. Universitext. Springer, Berlin (2008). Lectures from the Summer School on Modular Forms and their Applications held in Nordfjordeid, June 2004, Edited by Kristian Ranestad
15. Bruinier, J.H., Yang, T.: CM-values of Hilbert modular functions. *Invent. Math.* **163**(2), 229–288 (2006)
16. Bruinier, J.H., Yang, T.: Faltings heights of CM cycles and derivatives of L -functions. *Invent. Math.* **177**(3), 631–681 (2009)
17. Chang, K.: Diskriminanten und Signaturen gerader quadratischer Formen. *Arch. Math. (Basel)* **21**, 59–65 (1970)
18. Dabholkar, A., Murthy, S., Zagier, D.: Quantum black holes, wall crossing, and mock modular forms (2012). [arXiv:1208.4074](https://arxiv.org/abs/1208.4074)
19. Duke, W., Li, Y.: Harmonic Maass forms of weight 1. *Duke Math. J.* **164**(1), 39–113 (2015)
20. Ehlen, S.: CM values of regularized theta lifts and harmonic weak Maassforms of weight 1. *Duke Math. J.* **166**(13), 2447–2519 (2017)
21. Freitag, E.: Hilbert Modular Forms. Springer, Berlin (1990)
22. Gross, B., Kohlen, W., Zagier, D.: Heegner points and derivatives of L -series. II. *Math. Ann.* **278**(1–4), 497–562 (1987)
23. Gross, B.H., Zagier, D.B.: On singular moduli. *J. Reine Angew. Math.* **355**, 191–220 (1985)
24. Gross, B.H., Zagier, D.B.: Heegner points and derivatives of L -series. *Invent. Math.* **84**(2), 225–320 (1986)
25. Kudla, S.S.: Algebraic cycles on Shimura varieties of orthogonal type. *Duke Math. J.* **86**(1), 39–78 (1997)
26. Lee, M.H.: Multilinear differential operators on modular forms. *Proc. Am. Math. Soc.* **132**(5), 1267–1277 (2004)
27. Li, Y.: Singular units and isogenies between CM elliptic curves. *Compos. Math.* **157**(5), 1022–1035 (2021)
28. Li, Y.: Average CM-values of higher Green’s function and factorization. *Am. J. Math.* **144**(5), 1241–1298 (2022)

29. McGraw, W.J.: The rationality of vector valued modular forms associated with the Weil representation. *Math. Ann.* **326**(1), 105–122 (2003)
30. Mellit, A.: Higher Green's functions for modular forms (2008). [arXiv:0804.3184](https://arxiv.org/abs/0804.3184)
31. Rankin, F.K.C., Swinnerton-Dyer, H.P.F.: On the zeros of Eisenstein series. *Bull. Lond. Math. Soc.* **2**, 169–170 (1970)
32. Shih, K.: Existence of certain canonical models. *Duke Math. J.* **45**(1), 63–66 (1978)
33. The Stacks project authors, The stacks project (2023). <https://stacks.math.columbia.edu>
34. Viazovska, M.: CM values of higher Green's functions (2011). [arXiv:1110.4654](https://arxiv.org/abs/1110.4654)
35. Viazovska, M.: Petersson inner products of weight-one modular forms. *J. Reine Angew. Math.* **749**, 133–159 (2019)
36. Weil, A.: *Basic Number Theory*, 3rd edn. Die Grundlehren der Mathematischen Wissenschaften, vol. 144. Springer, New York (1974)
37. Yang, T., Yin, H.: Difference of modular functions and their CM value factorization. *Trans. Am. Math. Soc.* **371**(5), 3451–3482 (2019)
38. Zagier, D.: *L-Series and the Green's functions of modular curves*. In: *Proceedings of the International Congress of Mathematicians, Vol. 1, 2*, Berkeley, Calif., 1986, pp. 689–698. Am. Math. Soc., Providence (1987)
39. Zhang, S.: Heights of Heegner cycles and derivatives of *L*-series. *Invent. Math.* **130**(1), 99–152 (1997)
40. Zwegers, S.P.: *Mock theta functions*. Proefschrift Universiteit Utrecht, Thesis (Ph.D.), Universiteit Utrecht (2002)

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