Geometry and Topology of Bipolar Minimal Surfaces in the 5-Sphere

Vom Fachbereich Mathematik
der Technischen Universität Darmstadt
zur Erlangung des Grades
Doctor rerum naturalium
(Dr. rer. nat.)
genehmigte Dissertation

von
Melanie Rothe

1. Referentin:	Prof. Dr. Elena Mäder-Baumdicker
2. Referent:	Prof. Dr. Francisco Torralbo
3. Referent:	Prof. Dr. Guofang Wang
Tag der Einreichung:	08.07.2024
Tag der mündlichen Prüfung:	13.09.2024

Darmstadt 2024

Rothe, Melanie: Geometry and Topology of Bipolar Minimal Surfaces in the 5-Sphere
Darmstadt, Technische Universität Darmstadt,
Jahr der Veröffentlichung der Dissertation auf TUprints: 2024
URN: urn:nbn:de:tuda-tuprints-281645
Tag der mündlichen Prüfung: 13.09.2024

"*Of course it is happening inside your head, Harry, but why on earth should that mean that it is not real?*"

– J. K. Rowling, *Harry Potter and the Deathly Hallows*

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Acknowledgments

On the way to this thesis, I was supported by many inspiring people, which I would like to thank at this point.

First, I am grateful to my PhD supervisor, Elena Mäder-Baumdicker, for her unwavering support and new perspectives in mathematical discussions, her guidance through each phase of my PhD journey and for enabling me to participate in all the interesting events to share my research findings. Our collaborative relationship, characterized by honesty and mutual respect, will always be remembered.

At the same time, I thank Francisco Torralbo, my supervisor at the University of Granada, for being open to discuss all my mathematical questions at the time I arrived in Granada, for all the subsequent meetings where he kept me on right path to the results in the third chapter of this thesis and, in general, for preparing me to be a geometer.

Special thanks also go to Karsten Große-Brauckmann, who provided supervision during the early stages of my PhD and, since then, has remained a reliable advisor on both geometric and life questions.

Moreover, I would like to thank the geometry group at the TU Darmstadt for all the pleasant times including our lunch breaks and team events. Especially, I thank the PhD students in the group for making all our conference trips so enjoyable (even when flights were canceled).

I am also thankful to the geometry group in Granada, especially to Joaquín Pérez for inviting me to the IMAG. In this regard, special thanks moreover go to Antonio

Ros and Magdalena Rodríguez for their interest in my work. I am particularly grateful to the PhD students for their warm welcome and for making my time in Granada so much fun (especially when playing Pasapalabra every Thursday).

Overall, I am immensely grateful to the DAAD for awarding me one of the grants under the *Forschungsstipendien für Doktorandinnen und Doktoranden, 2022*, which enabled my valuable experiences in Granada and the subsequent work.

Furthermore, I would like to express my gratitude to Stefan Teufel, my master thesis supervisor in Tübingen, for encouraging me to pursue a PhD in geometry and for inspiring me from the very first lecture I attended during my studies.

In addition, I thank Prof. Dr. Guofang Wang for being a referee of this thesis.

Above all, I am thankful to my family – my husband Nicolai, my parents Simone and Walter, my stepfather Uli, my stepmother Edith, my brother Timo – for their unconditional support at all times. Especially, I thank Nicolai who was always open to hear about the latest problems from my PhD projects and to share his helpful perspectives. Thank you so much for standing by me when sometimes *smallest possible domains* seemed hardly possible.

Abstract

In the theory of closed minimal surfaces in the n-dimensional sphere $\mathbb{S}^n$, geometric and topological properties are closely intertwined. A classical question is whether there exist (primarily embedded) examples of every topological type – an issue, that particularly touches several other geometric variational problems. The current state of the art provides a rich theory and long list of examples for closed minimal surfaces in $\mathbb{S}^3$. However, knowledge about representatives in the individual topological classes and higher codimensions remains sparse. To this end, the main focus of this thesis lies on a specific class of minimal surfaces in $\mathbb{S}^5$, so-called bipolar surfaces, which arise from minimally immersed surfaces in $\mathbb{S}^3$.

On the one hand, we will topologically classify the bipolar minimal surfaces induced by two families among the prominent closed minimal surfaces in $\mathbb{S}^3$ that were constructed by H. Blaine Lawson in 1970. In that context, a notable phenomenon is that, regarding topology and embeddedness, bipolar surfaces can differ significantly from the original surfaces in $\mathbb{S}^3$.

On the other hand, we will consider bipolar surfaces as part of a more general class of minimal surfaces in $\mathbb{S}^5$. First, this leads to a deeper understanding of their geometric data. Finally, this will in fact enable us to prove that, under certain conditions, locally any immersed surface of the aforementioned class is congruent to a bipolar surface.

Zusammenfassung

In der Theorie geschlossener Minimalflächen in der n-dimensionalen Sphäre $\mathbb{S}^n$ sind geometrische und topologische Eigenschaften eng miteinander verwoben. Eine klassische Fragestellung ist, ob es (in erster Linie eingebettete) Beispiele für jeden topologischen Typ gibt – eine Frage, die insbesondere eine Vielzahl anderer geometrischer Variationsprobleme tangiert. Der gegenwärtige Stand beinhaltet eine umfassende Theorie und eine lange Liste von Beispielen für geschlossene Minimalflächen in $\mathbb{S}^3$. Das Wissen über Repräsentanten in den einzelnen topologischen Klassen und höheren Kodimensionen bleibt jedoch spärlich. Daher liegt der Schwerpunkt dieser Arbeit auf einer speziellen Klasse von Minimalflächen in $\mathbb{S}^5$, den sogenannten bipolaren Flächen, welche von Minimalflächen in $\mathbb{S}^3$ induziert werden.

Zum einen werden wir jene bipolaren Minimalflächen topologisch klassifizieren, die durch zwei Familien unter den weit bekannten, geschlossenen Minimalflächen in $\mathbb{S}^3$ erzeugt werden, die 1970 von H. Blaine Lawson konstruiert wurden. Ein bemerkenswertes Phänomen in diesem Zusammenhang ist, dass sich bipolare Flächen in Bezug auf Topologie und Eingebettetheit erheblich von den ursprünglichen Flächen in $\mathbb{S}^3$ unterscheiden können.

Andererseits werden wir bipolare Flächen als Teil einer allgemeineren Klasse von Minimalflächen in $\mathbb{S}^5$ betrachten. Dies führt zunächst zu einem tieferen Verständnis ihrer geometrischen Daten. Zuguterletzt werden wir dadurch in der Lage sein, zu beweisen, dass unter bestimmten Bedingungen lokal jede immersierte Fläche der oben genannten Klasse kongruent zu einer bipolaren Fläche ist.

Introduction

Our Topic

Minimal surface theory is a classical yet dynamic field that intersects with various areas of research. As critical points of the area functional, minimal surfaces are locally area-minimizing and geometrically characterized by a globally vanishing mean curvature vector. Generally, this condition on the curvature can be seen as dictated by a nonlinear elliptic partial differential equation. In this sense, minimal surfaces constitute exceptional, two-dimensional immersed submanifolds of an ambient Riemannian manifold of dimension ≥ 3. Of particular interest are the embedded examples.

The exploration of minimal surfaces can offer profound insights into the geometric properties of their ambient spaces. Especially compelling is the study of minimal surfaces in manifolds of constant curvature, such as Euclidean space $\mathbb{R}^n$, the sphere $\mathbb{S}^n$ or hyperbolic space $\mathbb{H}^n$. In this thesis, the primary focus lies on minimally immersed surfaces in $\mathbb{S}^5$.

Unlike Euclidean space $\mathbb{R}^n$ as an ambient manifold, the n-dimensional sphere $\mathbb{S}^n$ allows for closed minimal surfaces. Consequently, there is significant interest in their topological classification and its implications for their geometry. In this context, wide-ranging results are known for $\mathbb{S}^3$, including an infinite list of examples, e.g.,

- the geodesic 2-sphere and the Clifford torus as the earliest examples,
- Lawson's three infinite families of minimal surfaces (1970, cf. [37]),
- the surfaces of Karcher, Pinkall and Sterling (1988, cf. [31]),

- the surfaces of Kapouleas and Yang (2010, cf. [30]) and
- the two infinite families of Choe and Soret (2013, cf. [13]).

For a comprehensive survey, we refer to [7]. It must, however, be added that regarding each topological class, the aforementioned list is sparse. Additionally, compared to the findings for codimension 1, the settings of higher codimension are less explored. Nonetheless, these settings are also worth exploring due to many open questions about minimal surfaces in $\mathbb{S}^n$ that originate from various other classical geometric variational problems. This omnipresence is consistently leveraged towards finding corresponding answers.

In this regard, we start by providing an overview of key motivations.

Motivation: The Willmore Problem

First, we would like to mention the occurrence of minimal surfaces in $\mathbb{S}^n$ within the search for minimizers of the *Willmore functional*.

The absence of closed minimal surfaces in $\mathbb{R}^n$ leads to the question about the best shape a closed immersed surface $f\colon \Sigma \to \mathbb{R}^n$ with fixed topology can take in the ambient space so that it is not "unnecessarily" curved. Concerning that issue, a natural, conformally invariant functional to minimize is given by the *Willmore functional* (also called *Willmore energy*)

$$W[f] := \int_\Sigma |H_f|^2 \,\mathrm{d}\mu_f \,,$$

where H_f denotes the mean curvature vector of an immersion f and μ_f its induced density on Σ. Note that $W[f]$ is also well-defined for immersions of non-orientable 2-manifolds. The Willmore functional is named after Thomas Willmore, who made significant contributions to its analysis during the 1960s. Apart from the purely geometric question above, it has several applications, for example in general relativity (in the Hawking mass), elasticity theory or in cell biology (see for example [19, 20, 48, 50]).

Willmore showed that round spheres, as the most symmetric surfaces, are the only surfaces in $\mathbb{R}^3$ attaining the minimal energy of 4π. By [35], the same holds for higher codimension. However, little is known about minimizers for higher genera (both orientable and non-orientable) due to the challenging nature of handling the Willmore functional with variational methods, given its conformal invariance. Regarding potential candidates for minimizers, the relation to minimal surfaces in $\mathbb{S}^n$ comes into play. Given the inverse stereographic projection $P\colon \mathbb{R}^n \to \mathbb{S}^n \setminus \{-e_{n+1}\}$, the Willmore energy of a closed, immersed surface $f\colon \Sigma \to \mathbb{R}^n$ satisfies

$$W[f] = \int_\Sigma \left(|H_{P\circ f}|^2 + 1\right) \mathrm{d}\mu_{P\circ f}\,, \tag{1}$$

where $H_{P\circ f}$ denotes the mean curvature vector in $\mathbb{S}^n$ (see [71]). This implies that minimal surfaces in $\mathbb{S}^n$ stereographically project onto Willmore surfaces in $\mathbb{R}^n$, i.e., are critical for the functional W. In this process, their area is mapped to the Willmore energy of the stereographic projection.

The link to minimal surfaces in $\mathbb{S}^n$ was crucial for the work of Fernando Marques and André Neves in [44] from 2014, where they showed that stereographic projections of the Clifford torus in $\mathbb{S}^3$ (and conformal transformations thereof) have the minimal Willmore energy of $2\pi^2$ among orientable surfaces of genus $g \geq 1$ in $\mathbb{R}^3$. Thereby, they particularly proved the Willmore conjecture from the 1960s. The interested reader is referred to [45] for a comprehensive discussion.

Besides the sphere and the Clifford torus, only one more setting is clarified: Among surfaces $f : \mathbb{R}P^2 \to \mathbb{R}^n$, $n \geq 4$, we have $W[f] \geq 6\pi$ by the result of Peter Li and Shing-Tung Yau in [42], where equality holds only for the stereographic projections of the Veronese embedding in $\mathbb{S}^4$. Interestingly, as $6\pi < 2\pi^2$, one might be led to expect that the options of non-orientability and a higher codimension may also promote low Willmore energies.

For higher genera, there exist at least well-founded conjectures about explicit minimizers. On the one hand, it was conjectured by Rob Kusner (initially in [34]) that the stereographic projections of the Lawson surfaces $\xi_{g,1}$, which are minimally embedded in $\mathbb{S}^3$, are the minimizers for $g \geq 2$ in $\mathbb{R}^n$. This is supported by numerical simulations in [25]. On the other hand, Elena Mäder-Baumdicker and Jonas Hirsch conjectured the stereographic projections of the bipolar Lawson surface $\widetilde{\tau}_{3,1}$, which is minimally embedded in $\mathbb{S}^4$, to be the unique minimizers among immersed Klein bottles in $\mathbb{R}^n$, $n \geq 4$. This expectation is motivated by their finding in [23] that these are the unique minimizers in their conformal class.

Besides the explicit minimizers, there are various, non-explicit results. For instance, the results in [67, 3, 66] lead to the fact that the infimum β_g^n of the Willmore energy for immersed, orientable surfaces of genus $g \geq 2$ in $\mathbb{R}^n$ is attained by a smooth embedding, and $\beta_g^n < 8\pi$. By [8, 23], the same holds regarding the infimum of the Willmore energy among all immersed Klein bottles in $\mathbb{R}^n$.

Another important aspect calling for higher codimension is an observation made in conformal geometry. Within this framework, the Li-Yau inequality introduced in [42] serves as a tool for estimating the Willmore energy and hence, by (1), also the area of minimal surfaces in $\mathbb{S}^n$. Explicitly, [42] shows that for an immersed surface $f\colon \Sigma \to \mathbb{R}^n$, we have

$$W[f] \geq 4\pi \cdot \left|f^{-1}\big(\{x\}\big)\right| \qquad \text{for all } x \in \mathbb{R}^n \tag{2}$$

and thus, $W[f] \geq 8\pi$ for surfaces with self-intersections. Hence, such immersed surfaces are excluded as useful competitor surfaces within the search for embedded minimizers. Also note that embedded, non-orientable surfaces in $\mathbb{R}^n$ can only be realized if $n \geq 4$. Consequently, the setting of codimension 1 is obstructive for a search of embedded, non-orientable minimizers.

Motivation: Laplacian Eigenvalues

Closed minimal surfaces in $\mathbb{S}^n$ are moreover highly relevant in the geometric optimization of Laplacian eigenvalues on closed, two-dimensional manifolds. We proceed with a brief overview of this classical problem in spectral geometry.

Consider a closed, two-dimensional manifold Σ equipped with a Riemannian metric g. In terms of local coordinates $\left(x^1, x^2\right)$ on Σ, the *Laplace-Beltrami operator* on

(Σ, g) reads as

$$\Delta_g f = -\frac{1}{\sqrt{|g|}} \sum_{i,j=1}^{2} \partial_i \Big(\sqrt{|g|} g^{ij} \partial_j f \Big) , \qquad f \in C^\infty(\Sigma) ,$$

where we note that in the context of consideration, the sign is absorbed in the definition of Δ_g. The spectrum of Δ_g discrete and non-negative (with the previous convention). Its eigenvalues

$$0 = \lambda_0(\Sigma, g) < \lambda_1(\Sigma, g) \leq \lambda_2(\Sigma, g) \leq \ldots \leq \lambda_i(\Sigma, g) \leq \ldots$$

have finite multiplicities and tend to infinity. Under a scaling $g \mapsto tg$ for $t > 0$, they transform according to

$$\lambda_i(\Sigma, tg) = \frac{\lambda_i(\Sigma, g)}{t} .$$

The latter motivates to ask about Riemannian metrics on Σ attaining the supremum of the the *normalized i-th eigenvalue functional*

$$\Lambda_i(\Sigma, g) := \lambda_i(\Sigma, g) \cdot \mathrm{area}(\Sigma, g) ,$$

of the Laplace-Beltrami operator, which is invariant under scalings.

The search for such *maximal metrics* turned out to be quite difficult: Only few explicit maximal metrics have been found until now. As shown by Joseph Hersch [22], the standard metric on $\mathbb{S}^2 \subseteq \mathbb{R}^2$ is the only maximal metric for $\Lambda_1\big(\mathbb{S}^2, g\big)$. In [42], Peter Li and Shing-Tung Yau proved that the analog holds for the standard metric on $\mathbb{R}P^2$. For the torus $\mathbb{T}^2$, the only maximal metric, found by Nikolai Nadirashvili in [55], is the flat, equilateral metric. Moreover, the unique maximal metric on the Klein bottle $\mathbb{K}$ is given by the metric on the bipolar Lawson surface $\widetilde{\tau}_{3,1}$, which is a minimal surface in $\mathbb{S}^4$. This result is attributed to the works of Dmitry Jakobsen, Nikolai Nadirashvili and Iosif Polterovich in [26] together with the result of Ahmad El Soufi, Hector Giacomini and Mustapha Jazar in [15].

A notable property of the functional $\Lambda_i(\Sigma, g)$ is that the supremum is not necessarily attained by smooth metrics, in contrast to the Willmore functional. As shown in [56] and [64], the supremum of $\Lambda_2\big(\mathbb{S}^2, g\big)$ can be realized by a degenerated surface, namely the union of two spheres, both equipped with the standard metric, which share a single point. Moreover, it was recently shown [58] that for orientable genus $g = 2$, $\Lambda_1(\Sigma, g)$ is maximized by the Bolza surface that has six conical singularities. For further (implicit) results on the existence of maximal metrics under certain conditions, we refer to [57], [63], and [65] for the orientable case, and to [46] for the non-orientable case.

An important approach to maximal metrics has been the study of extremal points. As $\Lambda_i(\Sigma, g)$ is not differentiable but only continuous, a Riemannian metric g is called *extremal* if any analytic deformation g_t of g with $g_0 = g$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_i(\Sigma, g_t)\Big|_{t=0^-} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\Lambda_i(\Sigma, g_t)\Big|_{t=0^+} \leq 0\,.$$

Regarding extremal metrics for $\Lambda_1\big(\mathbb{T}^2, g\big)$ and $\Lambda_1(\mathbb{K}, g)$, considerable progress was made in the past years (for a comprehensive survey see [61]). For instance, the results in [26, 15] led to the unique maximal metric for the Klein bottle being realized as $\widetilde{\tau}_{3,1}$. Moreover, the metric on the Clifford torus was distinguished in [16] to be the only extremal metric for $\Lambda_1\big(\mathbb{T}^2, g\big)$ besides the equilateral metric mentioned above. Apart from that, many extremal metrics for higher, concretely specified eigenvalues on $\mathbb{T}^2$ and $\mathbb{K}$ were discovered by the metrics on the Lawson surfaces $\tau_{m,k}$ and the Otsuki tori $O_{\frac{p}{q}}$, both families of minimally immersed surfaces in $\mathbb{S}^3$, and in addition on their bipolar minimal surfaces in $\mathbb{S}^4$ (see [32, 33, 36, 60, 62]).

Now, extremal metrics are closely associated with minimal surfaces in spheres and the reasoning behind this connection is the following. According to Tsunero Takahashi's theorem in [68], an isometric immersion $\psi\colon \Sigma \to \mathbb{R}^{n+1}$ is minimal in $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ if and only if

$$\Delta_g \psi = 2\psi\,,$$

that is, the components of ψ are eigenfunctions of the Laplace-Beltrami operator with eigenvalue 2 (note once again the sign convention used here). Together with the result in [16], this implies that any minimal surface in a sphere induces an extremal metric for some $\Lambda_i(\Sigma, g)$.

A prominent conjecture in this context is due to Shing-Tung Yau (see [73]) stating that the smallest eigenvalue λ_1 for a closed, embbedded minimal surface in $\mathbb{S}^3$ is 2 (or more generally, n for closed, embedded minimal hypersurfaces in $\mathbb{S}^{n+1}$). This conjecture is still open, but strongly supported by the result of Jaigyoung Choe and Marc Soret in [12]. In this article, they prove that Yau's conjecture is true for a certain class of symmetric, closed, embbedded minimal surfaces in $\mathbb{S}^3$ that includes Lawson's ξ-family from [37] as well as the surfaces of Hermann Karcher, Ulrich Pinkall and Ivan Sterling from [31].

Motivation: Minimal Surfaces in $\mathbb{S}^2 \times \mathbb{S}^2$

Besides their appearance in the aforementioned problems, minimal surfaces in $\mathbb{S}^3$ occur in the theory of minimal Lagrangian surfaces in $\mathbb{S}^2 \times \mathbb{S}^2$, inducing an important class of examples.

Four-dimensional manifolds are particularly interesting in differential geometry due to their rich and complex structure, allowing for unique phenomena not present in any other dimension (see for example [39] or [49]). The theory of minimal surfaces in such ambient spaces is particularly well-understood for Einstein-Kähler surfaces (see [9, 11, 17, 51, 70] among others) and is based on the interaction of immersed surfaces with the given Kähler structure. In this context, the Einstein-Kähler surface $\mathbb{S}^2 \times \mathbb{S}^2$ is, besides the complex projective plane $\mathbb{C}P^2$, the only other compact Hermitian symmetric space of complex dimension 2.

A comprehensive geometric characterization of Lagrangian minimal surfaces in $\mathbb{S}^2 \times \mathbb{S}^2$, for which the pullback of the Kähler form vanishes equally, is presented in [11]. In this article, Ildefonso Castro and Francisco Urbano discuss the *so-called* Gauss maps of minimal surfaces in $\mathbb{S}^3$ as an important source of examples. They prove that, under a certain condition, locally any minimal Lagrangian surface arises as the Gauss map of a minimal surface in $\mathbb{S}^3$. In addition, they characterize the images of the Gauss map of geodesic two-spheres, the Clifford torus and the Lawson surface $\tau_{3,1}$ in $\mathbb{S}^3$ as the unique closed, minimal Lagrangian surfaces in $\mathbb{S}^2 \times \mathbb{S}^2$ admitting Hamiltonian stability. In [70], Francisco Torralbo and Francisco Urbano work in a comparable setting to geometrically characterize general minimal surfaces in $\mathbb{S}^2 \times \mathbb{S}^2$ (not only the Lagrangian ones). Specifically, they provide a construction method for minimal surfaces in $\mathbb{S}^2 \times \mathbb{S}^2$ given a certain type of pair of conformally equivalent minimal surfaces in $\mathbb{S}^3$. The latter in particular generalizes the Gauss map. Similarly as in [11], they demonstrate, under certain conditions, that locally any minimally immersed surface in $\mathbb{S}^2 \times \mathbb{S}^2$ occurs in that way, i.e., comes from two minimal surfaces in $\mathbb{S}^3$.

Results of this Thesis

The present thesis comprises two main projects, which are summarized below.

Before we begin, note that the constructions and results of the first project, presented in Chapter 2, can be found in the preprint [54] (except for Section 2.3.4). Furthermore, we remark that the second project, treated in Chapter 3, resulted from a research stay in 2022 supported by the DAAD (within a grant among the *Forschungsstipendien für Doktorandinnen und Doktoranden, 2022*) at the department of mathematics (IMAG) of the University of Granada and was supervised by Prof. Dr. Francisco Torralbo.

Chapter 1: Preliminaries

In **Chapter 1** of this thesis, we provide the derivation of bipolar minimal surfaces as in [37], including all the necessary preliminaries.

Encountering open questions about minimal surfaces in $\mathbb{S}^n$, one strategy is to construct new minimal surfaces from a given minimal immersion. Pioneering examples in that context came from H. Blaine Lawson in [37]: First, given an oriented, minimally immersed surface

$$\psi \colon \Sigma \to \mathbb{S}^3\,,$$

where Σ possibly arises as an oriented double cover of a non-orientable manifold, he showed that its Gauss map

$$\psi^*\colon \Sigma \to \mathbb{S}^3\,,$$

that is, a smooth unit normal field tangent to $\mathbb{S}^3 \subseteq \mathbb{R}^4$ (shifted to the origin), also yields a minimally immersed surface in $\mathbb{S}^3$, which generally may have isolated singularities (i.e., $\operatorname{rank}\big(\mathrm{d}\psi^*\big) < 2$ at isolated points). Lawson called this generalized surface the *polar variety* of the surface immersed by ψ. Second, given the above pair of minimal surfaces in $\mathbb{S}^3$, he furthermore introduced a minimally immersed surface in $\mathbb{S}^5$ (viewed as an embedded submanifold of $\mathbb{R}^6$), the so-called bipolar surface

$$\widetilde{\psi}\colon \Sigma \to \mathbb{S}^5\,, \quad \widetilde{\psi} := \psi \wedge \psi^*\,.$$

This definition is based on the identification $\mathbb{R}^6 \cong \Lambda^2\mathbb{R}^4$, where $\Lambda^2\mathbb{R}^4$ is the 2-fold exterior product of $\mathbb{R}^4$, the linear space of bivectors. Formally, the components of $\widetilde{\psi}$ are obtained from the anti-symmetric expressions

$$\psi^i\big(\psi^*\big)^j - \psi^j\big(\psi^*\big)^i\,, \qquad i,\, j \in \{1,2,3,4\}, \quad i < j\,,$$

in the components of ψ and ψ^*. Remarkably, the metric $\widetilde{g}$ induced by $\widetilde{\psi}$ on Σ, is conformally equivalent to the metric g induced by ψ. More precisely,

$$\widetilde{g} = (2 - K)\, g\,, \tag{3}$$

where K denotes the Gaussian curvature of (Σ, g).

Chapter 2: Lawson's Bipolar Minimal Surfaces in $\mathbb{S}^5$

Bipolar minimal surfaces are still actively discussed in minimal surface theory (see for example [6] or [52]). They have furthermore experienced a notable breakthrough in the context of extremal metrics for Laplacian eigenvalues – which has also drawn considerable attention to their potential for the Willmore problem. The first project we present in **Chapter 2** of this thesis, is located in that context.

In this project, we study the bipolar minimal surfaces arising from the families of closed minimal surfaces in $\mathbb{S}^3$ introduced by H. Blaine Lawson in 1970. In [37], he presented a construction method for complete minimal surfaces in $\mathbb{S}^3$ based on the successive application of the Schwarz reflection principle to an initial, embedded minimal disk $f\colon \Sigma \to \mathbb{S}^3$. More precisely, this initial piece of surface is the unique embedded Plateau solution for a boundary given by a certain type of geodesic polygon $\Gamma = f(\partial\Delta)$ in $\mathbb{S}^3$. In these terms, the resulting complete surface $\mathcal{M}_\Gamma \subseteq \mathbb{S}^3$ can be written as

$$\mathcal{M}_\Gamma = \bigcup_{g\in G} (g \circ f)(\Delta)\,,$$

where $G \subseteq \mathrm{SO}(4)$ denotes the group generated by the geodesic reflections across the arcs of Γ. By additionally applying the above method to explicit polygons, Lawson constructed his well-known, infinite families of closed minimal surfaces $(\xi_{m,k})$, $(\tau_{m,k})$ and $(\eta_{m,k})$ in $\mathbb{S}^3$. After geodesic 2-spheres and the Clifford torus, these were the first examples of closed minimal surfaces of higher genus. Specifically, they also provided embedded examples for every orientable genus $g \geq 2$.

The initial motivation for our project was a notable phenomenon observed by Hugues Lapointe in [36] for the bipolar Lawson surfaces $\widetilde{\tau}_{m,k}$. He showed that various properties of the bipolar surface can crucially differ from the original surface in $\mathbb{S}^3$. First, this concerns the topology: For example, it is known (Theorem 1.3.1 in [36]) that if $mk \equiv 3 \bmod 4$, then $\tau_{m,k}$ is a torus in $\mathbb{S}^3$, but $\widetilde{\tau}_{m,k}$ is a Klein bottle in $\mathbb{S}^5$. In more detail, suppose that $\tau_{m,k}$ is immersed by $\psi : \Sigma \to \mathbb{S}^3$ on a smallest possible closed domain Σ (here a torus) distinguished by covering the surface only one time. Then, $\widetilde{\psi} : \Sigma \to \mathbb{S}^5$ is already well-defined on a closed quotient of Σ (here a Klein bottle), which is occurs as the base of a suitable version of Σ as a covering space. We note that in such a case the actual area of the bipolar surface is only half of the area measured by the immersion $\widetilde{\psi} : \Sigma \to \mathbb{S}^5$. A further point of comparison is the embeddedness: The example of the surface $\widetilde{\tau}_{3,1}$, as detected by Jonas Hirsch and Elena Mäder-Baumdicker in [23], demonstrates that the bipolar surface can be embedded, even if the original surface was not.

The findings of Lapointe prompted the question whether there exist closed, minimally immersed surfaces in $\mathbb{S}^5$ which have higher genus and arise as bipolar surfaces.

Particular interest was directed to the possibility of non-orientable, embedded examples. With this in mind, we analyzed the other two families of bipolar Lawson surfaces $(\widetilde{\xi}_{m,k})$ and $(\widetilde{\eta}_{m,k})$, regarding their topology, embeddedness and area. Note that by the *topology* we refer to the topological class of a smallest possible closed domain of the immersion $\widetilde{\psi}$ which is, perhaps, realized by a quotient covered by Σ. We remark that the results on the bipolar τ-family heavily rely on the knowledge of explicit parametrizations, whereas such parametrizations are not known for the ξ- and η-family.

Our approach starts from defining an immersion of a bipolar surface in the special case of an underlying Schwarz reflection process in $\mathbb{S}^3$, as presented in **Section 2.1**. To this end, we first translate Lawson's construction procedure from [37] of a closed minimal surface in $\mathbb{S}^3$ into a corresponding immersion $\psi\colon S \to \mathbb{S}^3$ on a specific smallest possible closed domain $S := G \times \Delta/\sim$. In principle, this is done by gluing the preimages $\{g\} \times \Delta$ of the minimal disks $g \circ f$ for $g \in G$. Then, in terms of the group G (which is finite in the closed case), we can determine the topological class of S. This characterization in particular allows to define a Gauss map $\psi^*\colon S \to \mathbb{S}^3$ in the case when S is orientable. If S is non-orientable, we define a Gauss map $\psi^*\colon \overline{S} \to \mathbb{S}^3$ on the orientable double cover $\overline{S}$, which is similarly obtained as S itself. Finally, given the maps ψ and ψ^*, we can define the bipolar immersion $\widetilde{\psi} = \psi \wedge \psi^*$ on S, or on $\overline{S}$ if required. Recalling the results of Lapointe, this immersion is perhaps not yet defined on a smallest possible domain.

In **Section 2.2**, we use the above framework to take up the question about how often a smallest possible domain for the bipolar surface is covered by S or $\overline{S}$. Our basic idea about such a scenario is that different G-copies of the initial piece of surface in $\mathbb{S}^3$ are mapped to the same pieces in the bipolar surface $\widetilde{\mathcal{M}}_\Gamma$. With this in mind, we identify the following, exemplary condition for the occurrence of a double cover. In that case, the behavior of the orientation can be tracked in terms of purely algebraic properties of the group G, always leading to an orientable quotient. Note that when S is orientable, we define the *parity* $\sigma(g) \in \mathbb{Z}_2$ of $g \in G$ as follows: If

$g = r_{i_1} \circ \ldots \circ r_{i_k}$ for $i_1, \ldots, i_k \in \{1, \ldots N\}$ and $k \in \mathbb{N}$,

$$\sigma(g) := \begin{cases} 0 & \\ & \text{if } k \text{ is} \begin{array}{c} \text{even} \\ \text{odd} \end{array} \\ 1 & \end{cases} .$$

Theorem 1.

(i) *If S is orientable and $-\mathrm{id}_{\mathbb{R}^4} \in G$ with $\sigma(-\mathrm{id}_{\mathbb{R}^4}) = 0$, then the action*

$$\langle -\mathrm{id}_{\mathbb{R}^4} \rangle \times S \to S, \; \big(h, [(g,p)]\big) \mapsto [(hg, p)]$$

leaves the bipolar immersion $\widetilde{\psi} \colon S \to \mathbb{S}^5$ invariant and induces a smooth covering map of degree 2 on S such that the corresponding quotient $S/\langle -\mathrm{id}_{\mathbb{R}^4} \rangle$ is orientable. In particular, we have

$$\mathrm{area}\Big(\widetilde{\mathcal{M}_\Gamma}\Big) \leq \mathrm{area}(\mathcal{M}_\Gamma) - \pi \chi(S) \, .$$

(ii) *If S is non-orientable and $-\mathrm{id}_{\mathbb{R}^4} \in G$, then the action*

$$\langle -\mathrm{id}_{\mathbb{R}^4} \rangle \times \overline{S} \to \overline{S}, \; \big(h, [(s, g,p)]\big) \mapsto [(s, hg, p)]$$

leaves the bipolar immersion $\widetilde{\psi} \colon \overline{S} \to \mathbb{S}^5$ invariant and induces a smooth covering map of degree 2 on $\overline{S}$ such that the corresponding quotient $\overline{S}/\langle -\mathrm{id}_{\mathbb{R}^4} \rangle$ is orientable. In particular, we have

$$\mathrm{area}\Big(\widetilde{\mathcal{M}_\Gamma}\Big) \leq 2 \, \mathrm{area}(\mathcal{M}_\Gamma) - 2\pi \chi(S) \, .$$

Concerning the surfaces $\xi_{m,k}$, $\eta_{m,k} \subseteq \mathbb{S}^3$, we find in **Section 2.3** that the above method is, in fact, applicable and already provides a full characterization of the topology of the corresponding bipolar surfaces $\widetilde{\xi}_{m,k}$, $\widetilde{\eta}_{m,k} \subseteq \mathbb{S}^5$. To see the latter, it turns out that analyzing certain vertex points of the initial piece of surface in $\mathbb{S}^3$ is sufficient here. More precisely, we find that at such vertex points, a higher multiplicity in the bipolar surface only arises from their group copies in $\mathbb{S}^3$ (and particularly not from any interior points). Thereby, we finally show that in each case we can choose a bipolar image point of multiplicity $\mu > 1$ with μ transversally intersecting tangent planes – ruling out any further covers. As a last step, we also determine area bounds, based on the area formula

$$\operatorname{area}\left(\widetilde{\psi}\right) = 2\operatorname{area}(\psi) - 2\pi\chi(\Sigma)\,,$$

which is, for closed domains, an immediate consequence of (3) together with the Gauss-Bonnet theorem. Then, lower bounds are derived from the detected covers combined with the area estimates of Rob Kusner (cf. [34]) and upper bounds from the computed multiplicities plugged into the Li-Yau inequality (2).

After briefly introducing the specific setup of Lawson's construction method for his families in **Section 2.3.1**, we finally arrive at our main theorems of the project in **Section 2.3.2** and **Section 2.3.3**. Note that, in these sections, we do not include the bipolar surfaces of the Clifford torus $\xi_{1,1}$ and the Klein bottle $\eta_{1,1}$ for technical reasons. However, as mentioned above, these particular surfaces were already treated in [36] as they coincide with $\tau_{1,1}$ and $\tau_{2,1}$. A description of the $\widetilde{\tau}$-family from our viewpoint is given in **Section 2.3.4**.

At first, we have the following characterization for the $\widetilde{\xi}$-family. Note that, for a nicer presentation of formulas, we shifted the indices. Moreover, we remark that the ξ-family in $\mathbb{S}^3$ consists of orientable surfaces of Euler characteristic

$$\chi(\xi_{m-1,k-1}) = 2\big(1-(m-1)(k-1)\big)\,.$$

Theorem 2. *Let m, $k \in \mathbb{Z}_{\geq 2}$ such that $m > 2$ or $k > 2$. Then, the bipolar surface $\widetilde{\xi}_{m-1,k-1} \subseteq \mathbb{S}^5$ is orientable. Moreover,*

(i) *if both m and k are even, we have*

$$\chi\left(\widetilde{\xi}_{m-1,k-1}\right) = 1-(m-1)(k-1)\,,$$

$$2\pi\max\{m,k\} \leq \operatorname{area}\left(\widetilde{\xi}_{m-1,k-1}\right) < 2\pi(mk+k-m)\,;$$

(ii) *if m or k is odd, we have*

$$\chi\Big(\widetilde{\xi}_{m-1,k-1}\Big) = 2 - 2(m-1)(k-1)\,,$$

$$4\pi\max\{m,k\} \leq \operatorname{area}\Big(\widetilde{\xi}_{m-1,k-1}\Big) < 4\pi(mk+k-m)\,.$$

Analogously, we prove the following theorem for the family $(\widetilde{\eta}_{m,k})$. Note that the surface $\eta_{m-1,k-1} \subseteq \mathbb{S}^3$ is non-orientable and

$$\chi(\eta_{m-1,k-1}) = 1 - (m-1)(k-1)$$

when k is even. Otherwise, $\eta_{m-1,k-1}$ is orientable and

$$\chi(\eta_{m-1,k-1}) = 2\big(1-(m-1)(k-1)\big)\,.$$

Theorem 3. *Let m, $k \in \mathbb{Z}_{\geq 2}$ such that $m > 2$ or $k > 2$. Then, the bipolar surface $\widetilde{\eta}_{m-1,k-1} \subseteq \mathbb{S}^5$ is orientable. Moreover,*

(i) *if both m and k are even, we have*

$$\chi\Big(\widetilde{\eta}_{m-1,k-1}\Big) = 1 - (m-1)(k-1)\,,$$

$$2\pi\max\{m,k\} \leq \operatorname{area}\Big(\widetilde{\eta}_{m-1,k-1}\Big) < 2\pi(3mk-3k-m)\,;$$

(ii) *if m or k is odd, we have*

$$\chi\Big(\widetilde{\eta}_{m-1,k-1}\Big) = 2\big(1-(m-1)(k-1)\big)\,,$$

$$4\pi\max\{m,k\} \leq \operatorname{area}\Big(\widetilde{\eta}_{m-1,k-1}\Big) < 4\pi(3mk-3k-m)\,.$$

Since, within the proofs of the above theorems, we furthermore detect transversally intersecting tangent planes in both cases, we moreover find the following.

Corollary 4. *Let m, $k \in \mathbb{Z}_{\geq 2}$ such that $m > 2$ or $k > 2$. Then, the bipolar surfaces $\widetilde{\xi}_{m-1,k-1} \subseteq \mathbb{S}^5$ and $\widetilde{\eta}_{m-1,k-1} \subseteq \mathbb{S}^5$ are not embedded.*

In the light of our initial question about embedded, closed minimal surfaces of higher genera in $\mathbb{S}^5$, as competitor surfaces for the Willmore problem, the above theorems suggest the need for alternative approaches. However, the mechanism developed for our analysis could be promising in future explorations of bipolar surfaces of minimal surfaces in $\mathbb{S}^3$ obtained from a reflection process (as for example given in [13], [31] or, quite recently, [5]).

Chapter 3: Geometry of Bipolar Minimal Surfaces

In **Chapter 3**, we develop a framework which provides a more geometric perspective on bipolar minimal surfaces. Inspiration for this came from the results of Castro and Urbano on Gauss maps as Lagrangian minimal surfaces in $\mathbb{S}^2 \times \mathbb{S}^2$ presented in [11]. For our purposes, we partly adapted their description to minimally immersed surfaces in $\mathbb{S}^5 \subseteq \mathbb{R}^6 \cong \Lambda^2\mathbb{R}^4$. This starts with **Section 3.1**, where we demonstrate how the Hodge isomorphism, here seen as a linear map $*\colon \mathbb{R}^6 \to \mathbb{R}^6$, allows for a certain embedding of $\mathbb{S}^2(1/\sqrt{2}) \times \mathbb{S}^2(1/\sqrt{2}) \subseteq \mathbb{S}^5$, namely,

$$\mathcal{M} := \left\{ p \in \mathbb{S}^5 : \langle p, *p \rangle = 0 \right\}.$$

In this sense, the two almost complex structures $J^+ = (J_0, J_0)$ and $J^- = (J_0, -J_0)$ on $\mathbb{S}^2(1/\sqrt{2}) \times \mathbb{S}^2(1/\sqrt{2})$ become available on $\mathcal{M}$, where J_0 denotes the standard almost complex on $\mathbb{S}^2(1/\sqrt{2})$.

In **Section 3.2**, we proceed with an investigation of the class of oriented, minimally immersed surfaces in $\mathbb{S}^5$, whose image lies in $\mathcal{M}$. For an immersed surface $\Phi\colon \Sigma \to \mathbb{S}^5$ in that class, we have $\langle \Phi, *\Phi \rangle = 0$ at each point, implying that $\eta := *\Phi$ is a unit normal field along Φ. More generally, the corresponding normal bundle $N\Sigma$ of the immersed surface splits according to

$$N\Sigma = \mathcal{N} \oplus \mathbb{R}\eta\,,$$

where $\mathcal{N}$ is tangent to the submanifold $\mathcal{M}$. On this base, we first find that the considered class of minimal surfaces in $\mathbb{S}^5$ coincides with the Lagrangian minimal surfaces in $\mathcal{M}$, with respect to J^+ or to J^-. In our framework, this reads as a compatibility of the tangent and normal bundle of the immersed surface, i.e., we have

$$\mathcal{N} \oplus \{0\} = J^+(T\Sigma) \quad \text{or} \quad \mathcal{N} \oplus \{0\} = J^-(T\Sigma)\,. \tag{4}$$

Under these circumstances, we will see that a powerful tool assigned to an immersion Φ, which was also extensively studied in [11], is given by the function C, defined by extending the local expression

$$C = \frac{1}{2}\langle JE_1, *E_2\rangle\,, \qquad J \in \left\{J^+, J^-\right\},$$

with respect to a positively oriented local orthonormal frame (E_1, E_2) on Σ. More precisely, we find that C seems to dictate the exterior geometry of the immersed surface by entering the description of the second fundamental form of Φ.

Finally, in **Section 3.3**, we arrive at the link to bipolar minimal surfaces by observing that the latter are in fact part of the class we inspected, where always $J = J^-$ in (4). In this regard, a specification of the preceding findings yields a full resolution of the fundamental geometric data of bipolar minimal surfaces. Compared with Lawson's initial discussion in [37], the theorem below additionally includes the extrinsic data.

Theorem 5. *Let $\widetilde{\psi} = \psi \wedge \psi^* \colon \Sigma \to \mathbb{S}^5$ be the bipolar surface of an oriented, minimally immersed surface $\psi\colon \Sigma \to \mathbb{S}^3$ with induced metric g, Levi-Civita connection ∇ and shape operator A with respect to the unit normal field ν associated to ψ^*. Then, the fundamental data of $\widetilde{\psi}$ are given as follows:*

(i) *The induced metric reads as*

$$\widetilde{g} = \frac{2}{1+2C}\, g\,.$$

(ii) *The shape operator with respect to the normal field $\eta = *\widetilde{\psi}$ is given by*

$$\widetilde{A}_\eta = (1+2C)\, R_{\frac{\pi}{2}} \circ A\,,$$

where $R_{\frac{\pi}{2}}(p)$ denotes the rotation by $\frac{\pi}{2}$ on $T_p\Sigma$.

(iii) *Regarding the normal subbundle $\mathcal{N} = J^-(T\Sigma)$, the components of a shape operator $\widetilde{A}_{J^-Z}$ for $Z \in \mathfrak{X}(T\Sigma)$, are given by*

$$\left\langle \widetilde{A}_{J^-Z}(X), Y \right\rangle = -\big(\nabla\langle\sigma,\nu\rangle\big)(Z;X,Y) \qquad \text{for } X,\, Y \in \mathfrak{X}(\Sigma)\,.$$

In this context, furthermore notice that (4) allows to see the normal bundle of the bipolar surface as the sum of the tangent and normal bundle of the corresponding immersed surface in $\mathbb{S}^3$.

Afterwards, we finally arrive at the main theorem of the section, which demonstrates that locally, any immersed minimal surface in $\mathbb{S}^5$, whose image lies in $\mathcal{M}$, is in fact congruent to a bipolar surface of a minimally immersed surface in $\mathbb{S}^3$.

Theorem 6. *Let Σ be an oriented, simply connected, two-dimensional manifold and let $\Phi\colon \Sigma \to \mathbb{S}^5$ be a minimal immersion with $\langle\Phi, *\Phi\rangle = 0$. Moreover, suppose that $C(p) \neq -\frac{1}{2}$ for all $p \in \Sigma$ or $C(p) \neq \frac{1}{2}$ for all $p \in \Sigma$. Then, there exists a minimal immersion $\psi\colon \Sigma \to \mathbb{S}^3$ such that up to an isometry of $\mathbb{S}^5$*

$$\Phi = \widetilde{\psi}\,.$$

This theorem resembles Theorem 4.4 in [11] but with a condition that is relaxed compared to Castro and Urbano's requirement of $C(p)^2 < \frac{1}{4}$ for all $p \in \Sigma$. More precisely, while in [11] neither of $-\frac{1}{2}$ or $\frac{1}{2}$ can be attained by C, in our version one of them is allowed. Now, this relaxed version has the benefit that it holds for all bipolar surfaces and is optimal since the values $\pm\frac{1}{2}$ are typically attained by bipolar surfaces and their images under the antipodal map on $\mathbb{S}^5$.

Concerning the proof of the above theorem, we remark that our strategy is different from Castro and Urbano. Their approach is based on a specific holomorphic quadratic differential on Σ and finally uses the sinh-Gordon equation to show the existence of a suitable immersion $\psi\colon \Sigma \to \mathbb{S}^3$. Our method to access ψ starts from defining geometric data (g, A) that satisfy the Gauss and Codazzi equations in $\mathbb{S}^3$. More precisely, the latter relies on proving certain partial differential equations to be satisfied by C. By finally deducing that the fundamental data of $\widetilde{\psi}$ (which we know from Theorem 5) and the given immersion Φ are the same, we arrive at the above result.

We finish our studies with an interpretation of the aforementioned construction method from [70] in **Section 3.4**. In particular, we will demonstrate that a suitable pair of minimal surfaces in $\mathbb{S}^3$ is given by two Lawson surfaces $\widetilde{\tau}_{m,k}$ and $\widetilde{\tau}_{\hat{m},\hat{k}}$ whenever $mk = \hat{m}\hat{k}$.

Outline of the Thesis

We conclude this introductory part with a brief overview of the contents in this thesis.

In **Chapter 1**, we present the derivation of bipolar minimal surfaces as introduced in [37], including a preparation on isometric immersions, the minimal surface equation in $\mathbb{S}^n$ and particularly $\mathbb{S}^3$ and in addition exterior products of vector spaces.

Chapter 2 provides the characterization of the bipolar surfaces $\widetilde{\xi}_{m,k}$ and $\widetilde{\eta}_{m,k}$ regarding their topological type and embeddedness, based on the construction of corresponding immersions on a smallest possible domain. We also apply our methods to the surfaces $\widetilde{\tau}_{m,k}$ and compare it with the original classification in [36].

In **Chapter 3**, we derive that bipolar surfaces belong to a class of minimal surfaces in $\mathbb{S}^5$, which lie in a specific embedding of $\mathbb{S}^2(1/\sqrt{2}) \times \mathbb{S}^2(1/\sqrt{2})$. On that base, we specify their fundamental geometric data and prove that, under a certain condition, locally any minimal surface in that particular class is congruent to a bipolar surface.

Chapter 1

Preliminaries

1.1 Basics on Isometric Immersions

In this section, we will examine the framework of isometrically immersed submanifolds, with the aim of establishing the notation used in subsequent sections and revisiting the classical results necessary for our discussion. For a more comprehensive exploration of this theory, we refer the reader to [14], [38] and [41].

If not stated differently, we always consider smooth manifolds without boundary in this thesis. For a smooth manifold M, we denote the set of smooth vector fields on M by $\mathfrak{X}(M)$. Given a smooth vector bundle $E \to M$, we denote the set of smooth sections of this bundle by $\Gamma(E)$. If M is additionally equipped with a Riemannian metric, we will conventionally use the induced bundle metric and connection on the tensor bundles over M.

A differentiable immersion $\psi\colon M \to \overline{M}$ of an m-dimensional manifold M into a Riemannian manifold $\big(\overline{M}, \langle\cdot,\cdot\rangle\big)$ of dimension $m+k$, $k \geq 1$, induces a Riemannian metric $g := \psi^*\langle\cdot,\cdot\rangle$ on M. In particular, $\psi\colon (M,g) \to \big(\overline{M}, \langle\cdot,\cdot\rangle\big)$ becomes an isometric immersion. Identifying M with its image under ψ, the tangent space to $\overline{M}$ at $p \in M$ splits into

$$T_p\overline{M} = T_pM \oplus N_pM\,,$$

where $N_pM := (T_pM)^\perp$ is called the normal space at p. In this sense, any $v \in T_p\overline{M}$

has a unique decomposition $v = v^T + v^N$ into its tangential and normal components, smoothly depending on p. Note that, whenever required, we consider a local vector field $X \in \mathfrak{X}(M)$ or a local section $\nu \in \Gamma(NM)$ by a corresponding extension to a local, tangential or normal vector field on $\overline{M}$.

Let now $\overline{\nabla}$ be the Levi-Civita connection on $T\overline{M}$, ∇ be the Levi-Civita connection on TM and $\nabla^\perp$ be the normal connection on NM. We denote the associated curvatures by $\overline{R}$, R and $R^\perp$, respectively[1]. Moreover, let σ the second fundamental form of M and A_ν the shape operator corresponding to $\nu \in \Gamma(NM)$, which is defined by

$$\langle A_\nu(X), Y\rangle = \langle \sigma(X,Y), \nu\rangle \qquad \text{for all } X,\, Y \in \mathfrak{X}(M)\,.$$

The following proposition provides the essential characterization of the covariant derivative, the second fundamental form and the associated shape operators by their appearance in the derivatives of tangential and normal vector fields in the ambient space.

Proposition 1.1.1 (Gauss Formula and Weingarten Equation)**.** *An immersed submanifold $\psi\colon M \to \overline{M}$ satisfies the Gauss formula, that is,*

$$\overline{\nabla}_X Y = \nabla_X Y + \sigma(X,Y) \qquad \text{for all } X,\, Y \in \mathfrak{X}(M)\,. \tag{1.1}$$

Furthermore, we have

$$\overline{\nabla}_X \nu = -A_\nu(X) + \nabla^\perp_X \nu \qquad \text{for all } X \in \mathfrak{X}(M),\, \nu \in \Gamma(NM)\,. \tag{1.2}$$

In both equations above, the terms on the right-hand side precisely correspond to the tangential and normal components. The tangential part of the second equation,

$$\left(\overline{\nabla}_X \nu\right)^T = -A_\nu(X)\,, \tag{1.3}$$

is called the Weingarten equation.

[1] Note that for the curvature R on a Riemannian M with Riemannian connection ∇, we stick to the convention $R(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ for $X,\, Y,\, Z \in \mathfrak{X}(M)$.

Moreover, the following equations capture the relation between the intrinsic and extrinsic geometry of an immersed submanifold, i.e., the geometry of its tangent and normal bundle. Regarding the proof, we refer to [14].

Proposition 1.1.2 (Fundamental Equations)**.** *Let $\psi\colon M \to \overline{M}$ be an immersed submanifold, W, X, Y, $Z \in \mathfrak{X}(M)$ and ν, $\eta \in \Gamma(NM)$. Then, the following equations hold:*

(i) *the Gauss equation, i.e.,*

$$\left\langle \overline{R}(W,X)Y,Z \right\rangle = \left\langle R(W,X)Y,Z \right\rangle - \left\langle \sigma(W,Z),\sigma(X,Y) \right\rangle + \left\langle \sigma(W,Y),\sigma(X,Z) \right\rangle,$$

(ii) *the Codazzi equation, i.e.,*

$$\left\langle \overline{R}(X,Y)Z,\nu \right\rangle = \overline{\nabla}_X \langle \sigma(Y,Z),\nu \right\rangle - \overline{\nabla}_Y \langle \sigma(X,Z),\nu \right\rangle,$$

(iii) *the Ricci equation, i.e.,*

$$\left\langle \overline{R}(X,Y)\nu,\eta \right\rangle - \left\langle R^{\perp}(X,Y)\nu,\eta \right\rangle = -\left\langle [A_\nu,A_\eta]X,Y \right\rangle,$$

where $[A_\nu, A_\eta] := A_\nu \circ A_\eta - A_\eta \circ A_\nu$.

Remark 1.1.3. (i) Proposition 1.1.1 and Proposition 1.1.2 can be further interpreted (see [69] or [14]). In terms of local parametrizations, the Gauss formula and the Weingarten equation demonstrate that an immersion and a corresponding local orthonormal frame $N = (\nu_1, \dots \nu_k)$ for $N\Sigma$ satisfy a system of partial differential equations, fixed by the components of tensors g and σ (or the operators $A_{\nu_1}, \dots, A_{\nu_k}$, respectively). Given initial values $\psi(p)$, $\mathrm{d}\psi_p$ and $N(p)$, a solution of such system is unique. Moreover, in a homogeneous, isotropic space, two solutions for the same pair (g, A) are always congruent.

(ii) For simply connected domains and ambient spaces of constant sectional curvature, the Gauss, Codazzi and Ricci equations can be read as compatibility conditions on g and σ, required for the existence of a solution of the aforementioned system. Note that for codimension $k > 1$, this implicit definition of an immersed surface initially requires the definition of an appropriate vector bundle $E \to M$ (with fibers of dimension k) equipped with a bundle metric and compatible connection. If the fundamental equations are satisfied E, plays the role of the normal bundle of the immersed submanifold.

(iii) In the above sense, along with the fundamental equations, the forms g and σ are locally consistent with a smooth immersion in the ambient space and fully encode its geometry. Therefore, we call the pair (g, σ) (or locally, $(g, A_{\nu_1}, \dots, A_{\nu_k})$) *fundamental data* of an immersed surface.

Example 1.1.4. We specify the Gauss equation and the Codazzi equation for an immersion $\psi\colon \Sigma \to \overline{M}$ of a two-dimensional manifold Σ into a Riemannian manifold $\overline{M}$ of constant sectional curvature c. In this case, we have

$$\big\langle \overline{R}\big(\overline{W}, \overline{X}\big)\overline{Y}, \overline{Z}\big\rangle = c\Big(\big\langle \overline{X}, \overline{Y}\big\rangle\big\langle \overline{W}, \overline{Z}\big\rangle - \big\langle \overline{X}, \overline{Z}\big\rangle\big\langle \overline{W}, \overline{Y}\big\rangle\Big)$$

for all $\overline{W}$, $\overline{X}$, $\overline{Y}$, $\overline{Z} \in \mathfrak{X}\big(\overline{M}\big)$.

Let now (E_1, E_2) be a local orthonormal frame on Σ and $\nu \in \Gamma(N\Sigma)$. Then, the Gaussian curvature K of Σ reads as

$$K = \big\langle R(E_1, E_2)E_2, E_1\big\rangle\,.$$

Due to the symmetries of R and σ, the Gauss equation reduces to

$$K = c + \big\langle \sigma(E_1, E_1), \sigma(E_2, E_2)\big\rangle - \big|\sigma(E_1, E_2)\big|^2\,. \tag{1.4}$$

Moreover, given X, Y, $Z \in \mathfrak{X}(\Sigma)$ and $\nu \in \Gamma(N\Sigma)$, the Codazzi equation is

$$\overline{\nabla}_X\big\langle \sigma(Y, Z), \nu\big\rangle - \overline{\nabla}_Y\big\langle \sigma(X, Z), \nu\big\rangle = 0$$

or, more explicitly,

$$\begin{aligned} &X\Big(\big\langle\sigma(Y,Z),\nu\big\rangle\Big) - \big\langle\sigma\big(\nabla_X Y,Z\big),\nu\big\rangle - \big\langle\sigma\big(Y,\nabla_X Z\big),\nu\big\rangle - \big\langle\sigma(Y,Z),\nabla^\perp_X\nu\big\rangle \\ &\quad = Y\Big(\big\langle\sigma(X,Z),\nu\big\rangle\Big) - \big\langle\sigma\big(\nabla_Y X,Z\big),\nu\big\rangle - \big\langle\sigma\big(X,\hat{\nabla}_Y Z\big),\nu\big\rangle - \big\langle\sigma(X,Z),\nabla^\perp_Y\nu\big\rangle\,. \end{aligned} \tag{1.5}$$

Note that in codimension 1 (i.e., $k = 1$), a *unit* normal field $\nu \in \Gamma(N\Sigma)$ satisfies

$$0 = X\big(\langle\nu,\nu\rangle\big) = \big\langle\nabla^\perp_X\nu,\nu\big\rangle$$

and thus $\nabla^\perp\nu = 0$. In this case, (1.5) simplifies to

$$\begin{aligned} &X\Big(\big\langle\sigma(Y,Z),\nu\big\rangle\Big) - \big\langle\sigma\big(\nabla_X Y,Z\big),\nu\big\rangle - \big\langle\sigma\big(Y,\nabla_X Z\big),\nu\big\rangle \\ &\quad = Y\Big(\big\langle\sigma(X,Z),\nu\big\rangle\Big) - \big\langle\sigma\big(\nabla_Y X,Z\big),\nu\big\rangle - \big\langle\sigma\big(X,\hat{\nabla}_Y Z\big),\nu\big\rangle\,. \end{aligned} \tag{1.6}$$

We proceed with the definition of the following extrinsic, geometric invariant, which from now on plays a major role in our considerations.

Definition 1.1.5 (Mean Curvature Vector)**.** The *mean curvature vector* of an immersed submanifold $\psi\colon M \to \overline{M}$ is defined by

$$H := \operatorname{tr}_g(\sigma)\,.$$

In terms of a local, orthonormal frame $(E_1, \ldots, E_m)$ on M, we have

$$H = \sum_{i=1}^{m} \sigma(E_i, E_i)\,.$$

The mean curvature vector provides valuable information about the geometric characteristics of an immersed submanifold. For example, it is essential in minimal surface theory (which we will address soon) and the evolution of submanifolds under geometric flows (see for example [43]). Qualitatively, it can be understood as the

C^∞-gradient of the *volume functional* (or, the *area functional* if $m = 2$)

$$\mathrm{vol}(\psi) := \int_M \mathrm{d}\mu_g \,.$$

More precisely, given a smooth variation of ψ, i.e., a smooth family $\Psi\colon (-1,1)\times M \to \overline{M}$ of immersions such that $\Psi(0,\cdot) = \psi$ (and $\Psi(t,\cdot)|_{\partial M} = \psi|_{\partial M}$ for all $t \in$ (-1, 1) if M is a manifold with boundary), then

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{vol}\big(\Psi(t,\cdot)\big)\bigg|_{t=0} = -\int_M \langle H, E\rangle \,\mathrm{d}\mu_g \,,$$

where $E := \Psi_*\partial_t\big|_{t=0}$ with the canonical vector field ∂_t along $(-1,1)$ (see for example [38]). This means deformations of the submanifold in the direction of H provide the fastest decrease of the volume.

Now, critical points of the volume (or area) functional are particularly distinguished. According to the previous considerations, they are geometrically characterized by a globally vanishing mean curvature vector.

Definition 1.1.6 (Minimal Submanifold)**.** An immersed submanifold $\psi\colon M \to \overline{M}$ is called *minimal* if $H \equiv 0$.

On M, the property $H \equiv 0$ is equivalent to an elliptic, in general non-linear system of partial differential equations, which depends on the ambient space $\overline{M}$. For $m = 2$, this is often referred to as the *minimal surface equation*.

Note that below, Δ_g denotes the *Laplace-Beltrami* operator of (M, g), which in terms of local coordinates $\big(x^1, x^2\big)$ on Σ reads as

$$\Delta_g f = \frac{1}{\sqrt{|g|}} \sum_{i,j=1}^{2} \partial_i\Big(\sqrt{|g|}g^{ij}\partial_j f\Big)\,, \qquad f \in C^\infty(\Sigma)\,,$$

using a sign convention different from the spectral geometric context in the introduction.

Proposition 1.1.7. *For an immersed submanifold $\psi\colon M \to \mathbb{R}^n$, we have*

$$H^{\mathbb{R}^n} = \Delta_g \psi\,.$$

Thus, in Euclidean space $\mathbb{R}^n$, an immersed submanifold $\psi\colon M \to \mathbb{R}^n$ is minimal if and only if

$$\Delta_g \psi = 0\,. \tag{1.7}$$

Proof. We denote the Euclidean connection by D and use that $D_X Y = XY\psi$ for all X, $Y \in \mathfrak{X}(M)$. Given a local orthonormal frame $(E_1, \ldots, E_m)$ on M, the assertion follows as

$$\begin{aligned} H^{\mathbb{R}^n} &= \sum_{i=1}^{m} (D_{E_i} E_i)^N \\ &= \sum_{i=1}^{m} (D_{E_i} E_i - \nabla_{E_i} E_i) \\ &= \sum_{i=1}^{m} \big(E_i E_i \psi - (\nabla_{E_i} E_i)\psi\big) \\ &= \operatorname{tr}_g(\nabla\nabla\psi) \\ &= \Delta_g \psi\,. \end{aligned}$$

□

Now, the following proposition can be helpful to characterize minimality if the ambient space is an embedded submanifold of Euclidean space $\mathbb{R}^n$.

Lemma 1.1.8 (cf. [38])**.** *Suppose that $\overline{M} \subseteq \mathbb{R}^n$ is an embedded submanifold equipped with the induced metric. Furthermore, let $\psi\colon M \to \overline{M} \subseteq \mathbb{R}^n$ be an immersed submanifold in $\overline{M}$. Then, we have*

$$H^{\overline{M}} = (\Delta_g \psi)^{\overline{T}}\,,$$

where $(\cdot)^{\overline{T}}$ is the tangential projection onto $T\overline{M}$.

Proof. This is a direct consequence of the fact that orthogonal projections commute.

More precisely, using a local, orthonormal frame $(E_1, \dots, E_m)$ on M and the Gauss formula (1.1), we have

$$H^{\overline{M}} = \sum_{i=1}^{m} \left(\overline{\nabla}_{E_i} E_i\right)^N = \sum_{i=1}^{m} \left((E_i E_i \psi)^{\overline{T}}\right)^N = \left(\sum_{i=1}^{m} (E_i E_i \psi)^N\right)^{\overline{T}} \overset{(1.7)}{=} (\Delta_g \psi)^{\overline{T}} . \quad \square$$

At this stage, we are able to specify minimality in the n-dimensional, Euclidean sphere $\mathbb{S}^n := \left\{x \in \mathbb{R}^{n+1} \colon |x| = 1\right\}$.

Theorem 1.1.9. *An immersed, m-dimensional submanifold $\psi \colon M \to \mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ is minimal in $\mathbb{S}^n$ if and only if*

$$\Delta_g \psi = -m\psi .$$

Proof. We have $T_x\mathbb{S}^n \cong x^\perp$ for all $x \in \mathbb{S}^n$. Consequently, by Lemma 1.1.8, the immersion $\psi \colon M \to \mathbb{S}^n$ is minimal if and only if there exists some $F \in C^\infty(M)$ such that

$$\Delta_g \psi = F\psi .$$

Hence, using that $|\psi|^2 \equiv 1$, we have

$$0 = \frac{1}{2}\Delta_g |\psi|^2 = \langle \psi, \Delta_g \psi \rangle + |\mathrm{d}\psi|^2 = F|\psi|^2 + |\mathrm{d}\psi|^2 = F + |\mathrm{d}\psi|^2 .$$

Therefore, given a local, orthonormal frame $(E_1, \dots, E_m)$ on M, we have

$$F = -|\mathrm{d}\psi|^2 = -\sum_{i=1}^{m} \left|\mathrm{d}\psi(E_i)\right|^2 = -\sum_{i=1}^{m} |E_i|^2 = -m$$

and the assertion follows. $\square$

Now, in what follows, we are focusing on minimal surfaces, i.e., the case when $m = 2$. A remarkable fact is that unlike Euclidean space $\mathbb{R}^n$, the n-sphere $\mathbb{S}^n$ allows for closed minimal surfaces. From this perspective, the minimal surface equation in $\mathbb{S}^n$ reveals an interplay between geometry and topology. In the context of this classical topic, we will explore the attributes and specific examples of closed minimal surfaces in $\mathbb{S}^n$ in this thesis. Note that due to the topological classification of closed, two-dimensional manifolds by genus (or Euler characteristic) and orientability, it is particularly interesting to examine existence and geometric properties of minimally immersed surfaces within each topological class.

We finish this part of our preparations with specifying the Gauss equation in $\mathbb{S}^n$ for minimally immersed surfaces.

Proposition 1.1.10. *For a minimally immersed surface $\psi\colon \Sigma \to \mathbb{S}^n$, the Gauss equation is given by*

$$K = 1 - \frac{1}{2}|\sigma|^2\,. \tag{1.8}$$

In particular, we have $K \leq 1$ for a minimal surface in $\mathbb{S}^n$.

Proof. This follows directly from (1.4) with $c = 1$, using that minimality implies

$$\sigma(E_1, E_1) = -\sigma(E_2, E_2)$$

in terms of any local orthonormal frame (E_1, E_2) on Σ. □

1.2 Characteristics of Minimal Surfaces in $\mathbb{S}^3$

At this point, we examine in more detail the minimal surface equation in $\mathbb{S}^3$ and focus on general conclusions regarding the geometric properties of solutions (for examples, we direct to Chapter 2). Mainly, this section revisits the results in [37].

From now on, let $\psi\colon \Sigma \to \mathbb{S}^3$ be an isometrically immersed surface as described in the previous section. Additionally, we assume without loss of generality that Σ is orientable (in the sense that any non-orientable surface can be considered by its orientable double cover). This setting always allows for an oriented, conformal atlas of isothermal charts for g (see for example [1]). Therefore, our convention below is to consider local coordinates $\left(x^1, x^2\right)$ on Σ such that the metric g is locally of the form

$$g = \lambda\, \delta_{\mathbb{R}^2}$$

with a smooth conformal factor λ or, in other words,

$$|\partial_1 \psi| = |\partial_2 \psi| = \sqrt{\lambda} \quad \text{and} \quad \langle \partial_1 \psi, \partial_2 \psi \rangle = 0\,.$$

Since an oriented, conformal atlas is the same as a complex atlas, we also view Σ as a Riemann surface with a local complex coordinate $z = x^1 + ix^2$ and accordingly define the local vector fields

$$\partial \psi := \frac{1}{2}(\partial_1 \psi - i\partial_2 \psi), \quad \overline{\partial}\psi := \frac{1}{2}(\partial_1 \psi + i\partial_2 \psi)\,,$$

which satisfy

$$\langle \partial\psi, \partial\psi \rangle = \langle \overline{\partial}\psi, \overline{\partial}\psi \rangle = 0 \quad \text{and} \quad \langle \partial\psi, \overline{\partial}\psi \rangle = \frac{\lambda}{2}\,. \tag{1.9}$$

Note that from this viewpoint, we call $\psi\colon \Sigma \to \mathbb{S}^3$ a *conformal immersion* (and we hide it if a statement does not require that perspective). In this setup, the Gaussian curvature K of (Σ, g) is given by

$$K = -\frac{2}{\lambda}\partial\overline{\partial}\log(\lambda)\,. \tag{1.10}$$

To describe the geometry of the immersed surface as in Section 1.1, we begin by selecting a unit normal field.

Definition 1.2.1. Let $\nu \in \Gamma(N\Sigma)$ be the unit normal vector field along the conformally immersed surface $\psi\colon \Sigma \to \mathbb{S}^3$ which is tangent to $\mathbb{S}^3 \subseteq \mathbb{R}^4$ and satisfies that

$$\left(\psi, \frac{1}{\sqrt{\lambda}}\partial_1\psi, \frac{1}{\sqrt{\lambda}}\partial_2\psi, \nu\right)$$

is a positively oriented, orthonormal basis of $\mathbb{R}^4$ at each point. In more explicit terms, this means that we can write ν in terms of the generalized cross product, i.e.,

$$\nu = \frac{1}{\lambda} \sum_{k=1}^{4} \det \Big(\psi, \partial_1 \psi, \partial_2 \psi, e_k \Big) e_k \,,$$

where e_k denotes the k-th vector in the standard basis of $\mathbb{R}^4$.

As a next step, we specify the second fundamenal form σ and the scalar-valued second fundamental form $\beta := \langle \sigma, \nu \rangle$ of the conformally immersed surface $\psi \colon \Sigma \to \mathbb{S}^3$. We denote their components with respect to the chosen coordinates by σ_{ij} and β_{ij}, i, $j \in \{1, 2\}$. To this end, first note that the Gauss formula (1.1) yields

$$\partial_i \partial_j \psi + \lambda \delta_{ij} \psi = \nabla_{\partial_i} \partial_j + \sigma_{ij} \,, \tag{1.11}$$

where

$$\nabla_{\partial_i} \partial_j = (\partial_i \partial_j \psi)^T = \frac{1}{\lambda} \sum_{k=1}^{2} \langle \partial_i \partial_j \psi, \partial_k \psi \rangle \partial_k \psi \,.$$

Hence, we have

$$\sigma_{ij} = \partial_i \partial_j \psi + \lambda \delta_{ij} \psi - \frac{1}{\lambda} \sum_{k=1}^{2} \langle \partial_i \partial_j \psi, \partial_k \psi \rangle \partial_k \psi$$

and

$$\begin{aligned} \beta_{ij} &= \langle \sigma_{ij}, \nu \rangle \\ &= \langle \partial_i \partial_j \psi, \nu \rangle \\ &= \frac{1}{\lambda} \det \big(\psi, \partial_1 \psi, \partial_2 \psi, \partial_i \partial_j \psi \big) \\ &= \frac{2}{i\lambda} \det \big(\psi, \partial \psi, \overline{\partial} \psi, \partial_i \partial_j \psi \big) \,. \end{aligned} \tag{1.12}$$

We continue with the assumption that $\psi \colon \Sigma \to \mathbb{S}^3$ is additionally minimal. Since locally

$$\Delta_g = \frac{4}{\lambda}\partial\overline{\partial}\,,$$

the minimal surface equation from Theorem 1.1.9 is given by

$$\partial\overline{\partial}\psi = -\frac{\lambda}{2}\psi\,. \tag{1.13}$$

Remark 1.2.2. As presented in Lemma 1.1 in [37], this shows that any conformal, minimal immersion $\psi\colon \Sigma \to \mathbb{S}^3$ is a real analytic mapping.

Furthermore, as $\beta_{11} = -\beta_{22}$ by minimality and in particular

$$|\sigma|^2 = \frac{1}{\lambda^2}\sum_{i,j=1}^{2}|\sigma_{ij}|^2 = \frac{1}{\lambda^2}\sum_{i,j=1}^{2}\beta_{ij}^2 = \frac{2}{\lambda^2}\left(\beta_{11}^2 + \beta_{12}^2\right),$$

we conclude from (1.8) that the Gauss equation is

$$\lambda^2(1-K) = \beta_{11}^2 + \beta_{12}^2\,. \tag{1.14}$$

Finally, note that in the considered setting the Weingarten equation (1.3) reads as

$$\begin{aligned} \partial_1\nu &= -\frac{\beta_{11}}{\lambda}\partial_1\psi - \frac{\beta_{12}}{\lambda}\partial_2\psi\,,\\ \partial_2\nu &= -\frac{\beta_{12}}{\lambda}\partial_1\psi + \frac{\beta_{11}}{\lambda}\partial_2\psi\,. \end{aligned} \tag{1.15}$$

Now, as demonstrated by the following lemma, regarding Σ as a Riemann surface not only offers technical advantage but also allows for a deeper characterization of minimal surfaces in $\mathbb{S}^3$.

Proposition 1.2.3 (cf. [37], Lemma 1.2)**.** *If $\psi\colon \Sigma \to \mathbb{S}^3$ is a conformal, minimal immersion, then*

$$\varphi := \frac{1}{2}(\beta_{11} - i\beta_{12}) = -\langle\partial\psi, \partial\nu\rangle$$

defines a holomorphic quadratic differential $\varphi(z)\,\mathrm{d}z^2$ on Σ, the so-called Hopf diffe-

rential.

Proof. Due to the minimal surface equation (1.13) we have

$$\partial_1^2\psi = 4\partial\overline{\partial}\psi - \partial_2^2\psi = -2\lambda\psi - \partial_2^2\psi\,,$$

and therefore

$$\partial_1^2\psi - i\partial_1\partial_2\psi = \frac{1}{2}\big(\partial_1^2\psi - \partial_2^2\psi - 2i\partial_1\partial_2\psi\big) - \lambda\psi = 2\partial^2\psi - \lambda\psi\,.$$

Consequently,

$$\begin{aligned}\varphi &= \frac{1}{i\lambda}\det\Big(\psi, \partial\psi, \overline{\partial}\psi, \big(\partial_1^2\psi - i\partial_1\partial_2\psi\big)\Big)\\ &= \frac{2}{i\lambda}\det\Big(\psi, \partial\psi, \overline{\partial}\psi, \partial^2\psi\Big)\,. \end{aligned} \tag{1.16}$$

Now, by a straightforward computation we obtain

$$\big\langle\partial\psi, \overline{\partial}\psi\big\rangle = \frac{\lambda}{2} \tag{1.17}$$

and

$$\big\langle\partial^i\psi, \partial^j\psi\big\rangle = \big\langle\overline{\partial}^i\psi, \overline{\partial}^j\psi\big\rangle = 0 \quad \text{for all } i,\, j \text{ with } 1 \le i+j \le 3\,. \tag{1.18}$$

Thereby, we compute

$$\varphi^2 = -\frac{4}{\lambda^2}\det\begin{pmatrix} \langle\psi,\psi\rangle & \langle\psi,\partial\psi\rangle & \langle\psi,\overline{\partial}\psi\rangle & \langle\psi,\partial^2\psi\rangle \\ \langle\partial\psi,\psi\rangle & \langle\partial\psi,\partial\psi\rangle & \langle\partial\psi,\overline{\partial}\psi\rangle & \langle\partial\psi,\partial^2\psi\rangle \\ \langle\overline{\partial}\psi,\psi\rangle & \langle\overline{\partial}\psi,\partial\psi\rangle & \langle\overline{\partial}\psi,\overline{\partial}\psi\rangle & \langle\overline{\partial}\psi,\partial^2\psi\rangle \\ \langle\partial^2\psi,\psi\rangle & \langle\partial^2\psi,\partial\psi\rangle & \langle\partial^2\psi,\overline{\partial}\psi\rangle & \langle\partial^2\psi,\partial^2\psi\rangle \end{pmatrix} = \big\langle\partial^2\psi, \partial^2\psi\big\rangle\,.$$

Employing the minimal surface equation (1.13) and (1.18), this yields

$$\begin{aligned}\overline{\partial}\big(\varphi^2\big) &= 2\big\langle\partial\big(\partial\overline{\partial}\psi\big), \partial^2\psi\big\rangle\\ &= -\big\langle\partial(\lambda\psi), \partial^2\psi\big\rangle\end{aligned}$$

$$
\begin{aligned}
&= -(\partial\lambda)\big\langle \psi, \partial^2\psi \big\rangle - \lambda\big\langle \partial\psi, \partial^2\psi \big\rangle \\
&= 0\,,
\end{aligned}
$$

i.e., φ^2 is holomorphic and, by the continuity of φ, so is φ. In particular, $\varphi(z)\,\mathrm{d}z^2$ defines a holomorphic quadratic differential on Σ. □

Now, the Gauss equation immediately connects the Hopf differential with the intrinsic geometry of minimaly immersed surfaces in $\mathbb{S}^3$.

Corollary 1.2.4 (cf. [37], Lemma 1.4)**.** *For a conformal, minimal immersion* $\psi\colon \Sigma \to \mathbb{S}^3$ *we have*

$$
|\varphi|^2 = \frac{\lambda^2}{4}(1 - K)\,. \tag{1.19}
$$

Hence, the points where $K = 1$ *are precisely the isolated zeros of the Hopf differential.*

To continue, the existence of the Hopf differential is particularly useful to characterize *closed* minimally immersed surfaces in $\mathbb{S}^3$. In contrast to holomorphic functions on closed Riemann surfaces (which are necessarily constant due to the maximum principle), holomorphic quadratic differentials provide a rich structure to study global properties. For an introduction to this theory, especially covering the detailed proofs of the statements below, we refer to [27].

Proposition 1.2.5 (cf. [37], Proposition 1.5, and originally, [2])**.** *Let* $\psi\colon \Sigma \to \mathbb{S}^3$ *be a minimal immersion of a two-dimensional, closed and oriented manifold* Σ *of genus* g*.*

(i) *If* $g = 0$*, then the immersed surface is a geodesic 2-sphere in* $\mathbb{S}^3$*.*

(ii) *If* $g = 1$*, then the conformal parametrization* ψ *can be chosen such that* φ *is constant and non-vanishing.*

(iii) *If* $g > 1$*, there exist points such that* $K = 1$*.*

Proof. Recall that we can view Σ as a Riemann surface. If genus $g = 0$ (i.e., Σ is homeomorphic to $\mathbb{S}^2$), any holomorphic quadratic differential, in particular the

Hopf differential, must vanish. In that case, (1.2.4) implies that $K = 0$ globally and therefore assertion (i) follows. In the case that $g \geq 1$, any holomorphic quadratic differential on Σ has $4g - 4$ zeros. Consequently, if $g = 1$ (i.e., Σ is topologically a torus), the Hopf differential must be a non-vanishing holomorphic quadratic differential. So, by a change of complex coordinates it can be chosen to be constant. Furthermore, if $g > 1$ and hence $4g - 4 > 0$, (iii) is a direct consequence of Corollary 1.2.4. □

Based on the fact that closed, two-dimensional manifolds are topologically classified by their genus and (non-)orientability, it is natural to ask about minimal immersions for a fixed topological class. For orientable genus 0 surfaces, the preceding proposition provides a clear answer, which also applies to the class of non-orientable genus 1 surfaces.

Corollary 1.2.6. *The real projective plane cannot be minimally immersed into $\mathbb{S}^3$.*

Proof. We consider the real projective plane by its orientable double cover, that is, $\mathbb{R}P^2 = \mathbb{S}^2/\langle\sigma\rangle$ with the orientation-reversing, fixed-point-free involution $\sigma\colon \mathbb{S}^2 \to \mathbb{S}^2$, $x \mapsto -x$. In this sense, a minimal immersion of $\mathbb{R}P^2$ would correspond to a minimal immersion $\psi\colon \mathbb{S}^2 \to \mathbb{S}^3$ such that $\psi \circ \sigma = \psi$. By Proposition 1.2.5 (i), this would imply that the immersed surface ψ is a totally geodesic 2-sphere in $\mathbb{S}^3$ and therefore has a constant, global unit normal field N tangent to $\mathbb{S}^3 \subseteq \mathbb{R}^4$. But given a local orthonormal frame (E_1, E_2) at $x \in \mathbb{S}^2$, this yields

$$
\begin{aligned}
&\det\Big(\psi(x), \mathrm{d}\psi|_x\big(E_1(x)\big), \mathrm{d}\psi|_x\big(E_2(x)\big), N\Big) \\
&\qquad = \quad \det\Big(\psi\big(\sigma(x)\big), \mathrm{d}(\psi\circ\sigma)|_x\big(E_1(x)\big), \mathrm{d}(\psi\circ\sigma)|_x\big(E_2(x)\big), N\Big) \\
&\qquad = -\det\Big(\psi(x), \mathrm{d}\psi|_x\big(E_1(x)\big), \mathrm{d}\psi|_x\big(E_2(x)\big), N\Big)\,,
\end{aligned}
$$

where the first step uses that $\psi = \psi \circ \sigma$ and the second step that σ is orientation-reversing on $\mathbb{S}^2$. Hence, we would obtain

$$
\det\Big(\psi(x), \mathrm{d}\psi|_x\big(E_1(x)\big), \mathrm{d}\psi|_x\big(E_2(x)\big), N\Big) = 0\,,
$$

a contradiction. □

Remark 1.2.7. In fact, there exist examples of closed minimal surfaces in $\mathbb{S}^3$ for every other orientable and non-orientable topological classes. We will revisit this topic with more depth in Chapter 2.

Now, in order to proceed, we require the following relations.

Lemma 1.2.8 (cf. [37], Remark 1.3)**.** *For a conformal, minimal immersion $\psi\colon \Sigma \to \mathbb{S}^3$, the vector field*

$$\Phi := \frac{1}{2}(\sigma_{11} - i\sigma_{12}) = \varphi\nu$$

satisfies

$$\Phi = \lambda\partial\left(\frac{1}{\lambda}\partial\psi\right) \tag{1.20}$$

and

$$\overline{\partial}\Phi = -\frac{(1-K)\lambda}{2}\partial\psi\,. \tag{1.21}$$

Proof. To begin, we use (1.16) and compute

$$\begin{aligned}
\Phi &= \varphi\nu \\
&= -\frac{4}{\lambda^2}\det\left(\psi, \partial\psi, \overline{\partial}\psi, \partial^2\psi\right)\sum_{i=1}^{4}\det\left(\psi, \partial\psi, \overline{\partial}\psi, e_i\right)e_i \\
&= -\frac{1}{\lambda^2}\sum_{i=1}^{4}\det\begin{pmatrix}1 & 0 & 0 & \psi^i \\ 0 & 0 & \lambda & \partial\psi^i \\ 0 & \lambda & 0 & \overline{\partial}\psi^i \\ 0 & 0 & \partial\lambda & \partial^2\psi^i\end{pmatrix}e_i \\
&= -\frac{1}{\lambda}(\partial\lambda)\partial\psi + \partial^2\psi \\
&= \lambda\partial\left(\frac{1}{\lambda}\partial\psi\right).
\end{aligned}$$

Then, we use the minimal surface equation (1.13) and that, by (1.10),

$$K = \frac{2}{\lambda^3}(\partial\lambda)\big(\overline{\partial}\lambda\big) - \frac{2}{\lambda^2}\partial\overline{\partial}\lambda$$

to find

$$\begin{aligned}
\overline{\partial}\Phi &= \overline{\partial}\partial^2\psi - \overline{\partial}\left(\frac{1}{\lambda}(\partial\lambda)(\partial\psi)\right) \\
&= -\frac{1}{2}(\partial\lambda)\psi - \frac{\lambda}{2}\partial\psi + \frac{1}{\lambda^2}\big(\overline{\partial}\lambda\big)(\partial\lambda)(\partial\psi) - \frac{1}{\lambda}\big(\partial\overline{\partial}\lambda\big)(\partial\psi) + \frac{1}{2}(\partial\lambda)\psi \\
&= -\frac{(1-K)\lambda}{2}\partial\psi\,.
\end{aligned}$$

□

We conclude this section with a key insight on the existence of minimal surfaces in $\mathbb{S}^3$. To that end, we denote by $\psi^*\colon \Sigma \to \mathbb{S}^3$ the Gauss map of $\psi\colon \Sigma \to \mathbb{S}^3$ induced by the unit normal field ν (via a shift to the origin).

Proposition 1.2.9 (cf. [37], Proposition 10.1)**.** *Let $\psi\colon \Sigma \to \mathbb{S}^3$ be a minimally immersed, oriented surface with induced metric g and Gaussian curvature K. Then, its Gauss map $\psi^*\colon \Sigma \to \mathbb{S}^3$ is a possibly singular, minimally immersed surface with induced metric*

$$g^* = (1-K)g\,, \tag{1.22}$$

i.e., its singularities appear precisely at the isolated points where $K = 1$.

Definition 1.2.10 (Polar Variety of a Minimal Surface in $\mathbb{S}^3$)**.** Under the above conditions, $\psi^*(\Sigma)$ is called the *polar variety* of $\psi(\Sigma)$.

Proof of Proposition 1.2.9. Using first the minimal surface equation (1.13) and then (1.20), we have

$$\begin{aligned}
\partial\psi^* &= \partial\left(\frac{2}{i\lambda}\sum_{i=1}^{4}\det\left(\psi, \partial\psi, \overline{\partial}\psi, e_i\right)e_i\right) \\
&= \frac{2}{i}\sum_{i=1}^{4}\det\left(\psi, \partial\left(\frac{1}{\lambda}\partial\psi\right), \overline{\partial}\psi, e_i\right)e_i
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{i\lambda}\sum_{i=1}^{4}\det\Bigg(\psi,\Phi,\overline{\partial}\psi,e_i\Bigg)e_i && (1.23)\\
&= \frac{2\varphi}{i\lambda}\sum_{i=1}^{4}\det\Big(\psi,\psi^*,\overline{\partial}\psi,e_i\Big)e_i\\
&= \frac{\varphi}{i\sqrt{\lambda}}\Bigg(\sum_{i=1}^{4}\det\Big(\psi,\psi^*,\frac{1}{\sqrt{\lambda}}\partial_1\psi,e_i\Big)e_i + i\sum_{i=1}^{4}\det\Bigg(\psi,\psi^*,\frac{1}{\sqrt{\lambda}}\partial_2\psi,e_i\Bigg)e_i\Bigg)\\
&= \frac{\varphi}{i\lambda}(\partial_2\psi - i\partial_1\psi)\\
&= -\frac{2\varphi}{\lambda}\overline{\partial}\psi && (1.24)
\end{aligned}
$$

Thus, applying (1.19) in the first equation, we obtain

$$
\begin{aligned}
\big\langle\partial\psi^*,\overline{\partial}\psi^*\big\rangle &= \frac{2}{\lambda}|\varphi|^2 = \frac{(1-K)\lambda}{2}\,,\\
\big\langle\partial\psi^*,\partial\psi^*\big\rangle &= \big\langle\overline{\partial}\psi^*,\overline{\partial}\psi^*\big\rangle = 0
\end{aligned}
\tag{1.25}
$$

Therefore, in analogy to (1.9), we find

$$
g^* = (1-K)g
$$

and hence, by Corollary 1.2.4, ψ^* has singularities at the isolated points where $K = 1$.

To show that ψ^* is minimal, we use (1.23), (1.20) as well as (1.21) and compute

$$
\begin{aligned}
\partial\overline{\partial}\psi^* &= \overline{\partial}\Bigg(\frac{2}{i\lambda}\sum_{i=1}^{4}\det\Bigg(\psi,\Phi,\overline{\partial}\psi,e_i\Bigg)e_i\Bigg)\\
&= \frac{2}{i\lambda}\sum_{i=1}^{4}\det\Big(\psi,\overline{\partial}\Phi,\overline{\partial}\psi,e_i\Big)e_i + \frac{2}{i}\sum_{i=1}^{4}\det\Bigg(\psi,\Phi,\overline{\partial}\Bigg(\frac{1}{\lambda}\overline{\partial}\psi\Bigg),e_i\Bigg)e_i\\
&= -\frac{(1-K)}{i}\sum_{i=1}^{4}\det\Big(\psi,\partial\psi,\overline{\partial}\psi,e_i\Big)e_i + \frac{2}{i\lambda}\sum_{i=1}^{4}\det\Big(\psi,\Phi,\overline{\Phi},e_i\Big)e_i\\
&= -\frac{(1-K)\lambda}{2}\psi^*\,. && (1.26)
\end{aligned}
$$

So, comparing to (1.13), the assertion follows. □

On this base, Corollary 1.2.4 and Proposition 1.2.5 directly provide the following.

Corollary 1.2.11 (cf. [37], p. 361)**.** *The Gauss map* $\psi^*\colon \Sigma \to \mathbb{S}^3$ *of an oriented, minimally immersed surface* $\psi\colon \Sigma \to \mathbb{S}^3$ *is non-singular if and only if* Σ *covers a torus or a Klein bottle.*

At last, we would like to capture the geometry of the polar variety from the perspective of fundamental data as described in Remark 1.1.3. By the previous theorem, we already know that the metrics g and g^* are related by the (almost) conformal factor $1 - K$. Moreover, Definition 1.2.1 and (1.15) yield that $(\psi^*)^* = \psi$, away from the isolated singularities. Therefore, we consider the shape operator A^* of ψ^* into direction of the unit normal field associated with ψ.

Proposition 1.2.12. *Let* $\psi\colon \Sigma \to \mathbb{S}^3$ *be a minimally immersed, oriented surface with Gaussian curvature* K *and denote by* $S \subseteq \Sigma$ *the set of the isolated points where* $K = 1$*. Then, the shape operator* A^* *of the minimally immersed surface* $\psi^*\colon \Sigma \setminus S \to \mathbb{S}^3$ *is given by*

$$A^* = \frac{1}{1-K} A \,. \tag{1.27}$$

Proof. Let (E_1, E_2) be a local orthonormal frame on (Σ, g). Then, the Gauss formula (1.1) for ψ reads as

$$E_i E_j \psi + \langle E_i, E_j \rangle \psi = \nabla_{E_i} E_j + \sigma(E_i, E_j)\,, \qquad i, j \in \{1, 2\}\,,$$

and the analogue holds for ψ^* on $\Sigma \setminus S$. Together with their Weingarten equations (1.3), this implies

$$\begin{aligned}(1-K) g\big(A^*(E_i), E_j\big) &= g^*\big(A^*(E_i), E_j\big) \\ &= \langle \sigma^*(E_i, E_j), \psi \rangle \\ &= \langle E_i E_j \psi^*, \psi \rangle \\ &= \langle E_i E_j \psi, \psi^* \rangle \\ &= \langle \sigma(E_i, E_j), \psi \rangle\end{aligned}$$

$$= g\big(A(E_i), E_j\big) \qquad \text{for } i, j \in \{1, 2\}$$

and hence the assertion follows. □

Corollary 1.2.13 ([37], cf. Proposition 10.1)**.** *Under the conditions of* Proposition 1.2.12*, the Gaussian curvature K^* of the minimally immersed surface $\psi^* \colon \Sigma \setminus S \to \mathbb{S}^3$ is given by*

$$K^* = -\frac{K}{1-K}\,. \tag{1.28}$$

In particular, $\big|K^*\big|$ tends to infinity as a point in S is approached.

Proof. Let (E_1, E_2) be a local orthonormal frame on (Σ, g). Then, setting

$$E_i^* = \frac{1}{\sqrt{1-K}}\, E_i\,, \qquad i = 1, 2\,,$$

we obtain a local orthonormal frame $\big(E_1^*, E_2^*\big)$ on $\big(\Sigma \setminus S, g^*\big)$. Let now σ^* be the second fundamental form of $\psi^* \colon \Sigma \setminus S \to \mathbb{S}^3$. From Proposition 1.2.12 it follows that

$$\big|\sigma^*\big|_{g^*}^2 = \big|A^*\big|_{g^*} = \sum_{i=1}^{2} \big|A^*\big(E_i^*\big)\big|_{g^*}^2 = \frac{1}{(1-K)^2} \sum_{i=1}^{2} |A(E_i)|_g^2 = \frac{1}{(1-K)^2} |\sigma|_g^2\,.$$

Hence, the Gauss equation (1.8) for the considered surfaces yields

$$K^* = 1 - \frac{1}{2}\big|\sigma^*\big|_{g^*}^2 = 1 - \frac{1}{2}\frac{1}{(1-K)^2}|\sigma|_g^2 = 1 - \frac{1}{1-K} = -\frac{K}{1-K}\,.$$ □

1.3 Exterior Products of Vector Spaces

Moving forward, we first need to cover the algebraic concept of exterior products of vector spaces. In the brief overview below, we follow [1]. To maintain focus, we will skip the detailed proofs (which are mostly straight-forward computations in the setting of linear vector spaces). A more detailed characterization of exterior products of vector spaces can be found in [18].

In what follows, let $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ and V be a vector space over $\mathbb{F}$ of finite dimension n.

Definition 1.3.1 (k-fold Exterior Product of V)**.** Set

$$\Lambda^0 V := \mathbb{F}, \qquad \Lambda^1 V := V\,.$$

For $k \in \mathbb{N}_0$, the *k-fold exterior product* $\Lambda^k V$ *of* V is the $\binom{n}{k}$-dimensional vector space over $\mathbb{F}$ uniquely characterized by demanding the *wedge product*

$$\Lambda^k V \times \Lambda^m V \to \Lambda^{k+m} V\,, \quad (v, w) \mapsto v \wedge w$$

to be bilinear and

$$w \wedge v = (-1)^{mk}\, v \wedge w \qquad \text{for all } v \in \Lambda^k V,\, w \in \Lambda^m V\,.$$

The elements of $\Lambda^k V$ are called *k-vectors* (or also *bivectors* for $k = 2$).

From this definition, the following properties become evident.

Lemma 1.3.2. *If $(b_1, \ldots, b_n)$ is a basis of V, then the k-vectors*

$$b_{i_1} \wedge \ldots \wedge b_{i_k}\,, \qquad i_1, \ldots i_k \in \{1, \ldots, n\},\ i_1 < \ldots < i_k$$

form a basis of $\Lambda^k V$. The $(i_1, \ldots i_k)$-th component of $v_1 \wedge ... \wedge v_k \in \Lambda^k V$ with respect to this basis is given by

$$\big(v_1 \wedge ... \wedge v_k\big)^{(i_1, \ldots, i_k)} = \det\Big(\big(v_m^{i_l}\big)_{1 \le l, m \le k}\Big)\,.$$

For example if $k = 2$, we have

$$v \wedge w = \sum_{\substack{i, j \in \{1, \ldots, n\} \\ i < j}} \big(v^i w^j - v^j w^i\big)\, b_i \wedge b_j \in \Lambda^2 V\,.$$

Lemma 1.3.3. *If $v_1, \ldots v_k \in V$ are linearly dependent, then $v_1 \wedge \ldots \wedge v_k = 0$.*

Conversely, the wedge product of k linearly independent vectors in V determines a subspace of dimension k in V or, in other words, a point in the Grassmann manifold $G_k(V)$ of k-dimensional subspaces in V. In this sense, such type of k-vector is distinguished.

Definition 1.3.4. If $0 \neq v \in \Lambda^k V$ and there exist $v_1, \ldots, v_k \in V$ such that $v = v_1 \wedge \ldots \wedge v_k$, then v is called a *simple k-vector.*

Note that in general, k-vectors are linear combinations of simple k-vectors.

Lemma 1.3.5. *Two simple k-vectors v, $w \in \Lambda^k V$ determine the same point in $G_k(V)$ if and only if there exists $\lambda \in \mathbb{F} \setminus \{0\}$ such that $v = \lambda w$.*

Remark 1.3.6. If $F = \mathbb{R}$, a simple k-vector more precisely determines an *oriented*, k-dimensional subspace and therefore a point in the Grassmann manifold $G_k^+(V)$ of oriented, k-dimensional subspaces in V. In the light of the previous corollary, two simple k-vectors determine the same oriented subspace if and only if they differ by the multiplication with a positive scalar.

Now, in addition, suppose that V is equipped with a scalar product $\langle \cdot, \cdot \rangle$. This naturally induces a scalar product on $\Lambda^k V$.

Definition 1.3.7. For $v = v_1 \wedge \ldots \wedge v_k$, $w = w_1 \wedge \ldots \wedge w_k \in \Lambda^k V \setminus \{0\}$ we set

$$\langle v, w \rangle := \det \left(\left(\langle v_i, w_j \rangle \right)_{1 \leq i,j \leq k} \right) .$$

Then, by bilinear extension to the whole space, $\langle \cdot, \cdot \rangle$ defines a scalar product $\Lambda^k V$.

Lemma 1.3.8. *For $v_1 \wedge \ldots \wedge v_k \in \Lambda^k V$ we have*

$$|v_1 \wedge ... \wedge v_m| = 1 \qquad \Leftrightarrow \qquad |v_1| = ... = |v_m| = 1 .$$

We finish this interlude with the following duality of exterior products that will be essential later on in Chapter 3.

Definition 1.3.9. Let $v_1 \wedge \ldots \wedge v_k \in \Lambda^k V \setminus \{0\}$. Then, we define $*(v_1 \wedge ... \wedge v_k) \in \Lambda^{n-k} V$ to be the unique vector such that

$$\langle w_1 \wedge \ldots \wedge w_{n-k}, *(v_1 \wedge \ldots \wedge v_k)\rangle = \det(v_1, \ldots, v_k, w_1, \ldots, w_{n-k})$$

for all $w_1, \ldots, w_{n-k} \in V$. By linear extension, this determines a linear isomorphism

$$*\colon \Lambda^k V \to \Lambda^{n-k} V\,,$$

called the *Hodge isomorphism.*

Proposition 1.3.10. *With respect to the scalar products induced on $\Lambda^k V$ and Λ^{n-k}, the Hodge isomorphism is a linear isometry.*

Corollary 1.3.11. *Let $v_1, \ldots, v_n \in V$. Then, we have*

(i) $*(v_1 \wedge ... \wedge v_{n-1}) = \sum_{i=1}^n \det(v_1, ..., v_{n-1}, b_i)\, b_i$ *for every orthonormal basis* $(b_1, ..., b_n)$ *of* V*;*

(ii) $*(v_1 \wedge ... \wedge v_{n-1}) \in \operatorname{span}(v_1, ..., v_n)^\perp$*;*

(iii) *the family* $\big(v_1, ..., v_{n-1}, *(v_1 \wedge ... \wedge v_{n-1})\big)$ *is positively oriented;*

(iv) *Given* $v_1, ..., v_n \in V$*, then*

$$*(v_1 \wedge ... \wedge v_n) = \det(v_1 ..., v_n)\,.$$

1.4 Bipolar Minimal Surfaces in $\mathbb{S}^5$

At this point, we are prepared to complete the preliminary part with the central concept behind the main results of this thesis. More precisely, we will see that any minimal surface in $\mathbb{S}^3$ gives rise to a minimal surface in $\mathbb{S}^5$ – its so-called *bipolar surface*, introduced by H. Blaine Lawson in [37]. To that end, we view $\mathbb{R}^6$ as the linear space of bivectors $\Lambda^2\mathbb{R}^4 = \operatorname{span}\big\{v \wedge w :\, v,\, w \in \mathbb{R}^4\big\}$.

Theorem 1.4.1 ([37], p.361)**.** *Let $\psi\colon \Sigma \to \mathbb{S}^3$ be a minimally immersed, oriented surface with induced metric g, Gaussian curvature K and Gauss map $\psi^*\colon \Sigma \to \mathbb{S}^3$. Then,*

$$\widetilde{\psi}\colon \Sigma \to \mathbb{S}^5 \subseteq \mathbb{R}^6 \cong \Lambda^2\mathbb{R}^4\,, \quad \widetilde{\psi}(p) := \psi(p) \wedge \psi^*(p)$$

is a non-singular, minimally immersed surface in $\mathbb{S}^5$ with induced metric

$$\widetilde{g} = (2 - K)\, g\,.$$

Definition 1.4.2 (Bipolar Surface of a Minimal Surface in $\mathbb{S}^3$)**.** Under the above conditions, the immersed surface $\widetilde{\psi}\colon \Sigma \to \mathbb{S}^5$ is called the *bipolar surface* of $\psi\colon \Sigma \to \mathbb{S}^3$.

Proof of Theorem 1.4.1. We assume that ψ is a conformal immersion as in Section 1.2. Using the product rule for bilinear maps, we first have

$$\partial\widetilde{\psi} = \partial\psi \wedge \psi^* + \psi \wedge \partial\psi^*\,, \qquad \overline{\partial}\widetilde{\psi} = \overline{\partial}\psi \wedge \psi^* + \psi \wedge \overline{\partial}\psi^*\,.$$

Then, by Definition 1.3.7 and the application of (1.15), (1.17), (1.18), we obtain

$$\begin{aligned}
\left\langle \partial\widetilde{\psi}, \overline{\partial}\widetilde{\psi} \right\rangle &= \left\langle \partial\psi \wedge \psi^*, \overline{\partial}\psi \wedge \psi^* \right\rangle + \left\langle \partial\psi \wedge \psi^*, \psi \wedge \overline{\partial}\psi^* \right\rangle \\
&\qquad + \left\langle \psi \wedge \partial\psi^*, \overline{\partial}\psi \wedge \psi^* \right\rangle + \left\langle \psi \wedge \partial\psi^*, \psi \wedge \overline{\partial}\psi^* \right\rangle \\
&= \det\begin{pmatrix} \left\langle \partial\psi, \overline{\partial}\psi \right\rangle & \left\langle \partial\psi, \psi^* \right\rangle \\ \left\langle \psi^*, \overline{\partial}\psi \right\rangle & \left\langle \psi^*, \psi^* \right\rangle \end{pmatrix} + \det\begin{pmatrix} \left\langle \partial\psi, \psi \right\rangle & \left\langle \partial\psi, \overline{\partial}\psi^* \right\rangle \\ \left\langle \psi^*, \psi \right\rangle & \left\langle \psi^*, \overline{\partial}\psi^* \right\rangle \end{pmatrix} \\
&\qquad + \det\begin{pmatrix} \left\langle \psi, \overline{\partial}\psi \right\rangle & \left\langle \psi, \psi^* \right\rangle \\ \left\langle \partial\psi^*, \overline{\partial}\psi \right\rangle & \left\langle \partial\psi^*, \psi^* \right\rangle \end{pmatrix} + \det\begin{pmatrix} \left\langle \psi, \psi \right\rangle & \left\langle \psi, \overline{\partial}\psi^* \right\rangle \\ \left\langle \partial\psi^*, \psi \right\rangle & \left\langle \partial\psi^*, \overline{\partial}\psi^* \right\rangle \end{pmatrix} \\
&= \frac{(1-K)\lambda}{2}\,,
\end{aligned}$$

$$\begin{aligned}
\left\langle \partial\widetilde{\psi}, \partial\widetilde{\psi} \right\rangle &= \left\langle \partial\psi \wedge \psi^*, \partial\psi \wedge \psi^* \right\rangle + 2\left\langle \partial\psi \wedge \psi^*, \psi \wedge \partial\psi^* \right\rangle + \left\langle \psi \wedge \partial\psi^*, \psi \wedge \partial\psi^* \right\rangle \\
&= \det\begin{pmatrix} \left\langle \partial\psi, \partial\psi \right\rangle & \left\langle \partial\psi, \psi^* \right\rangle \\ \left\langle \psi^*, \partial\psi \right\rangle & \left\langle \psi^*, \psi^* \right\rangle \end{pmatrix} + 2 \cdot \det\begin{pmatrix} \left\langle \partial\psi, \psi \right\rangle & \left\langle \partial\psi, \partial\psi^* \right\rangle \\ \left\langle \psi^*, \psi \right\rangle & \left\langle \psi^*, \partial\psi^* \right\rangle \end{pmatrix}
\end{aligned}$$

$$+\det\begin{pmatrix}\langle\psi,\psi\rangle & \big\langle\psi,\partial\psi^*\big\rangle\\ \big\langle\partial\psi^*,\psi\big\rangle & \big\langle\partial\psi^*,\psi^*\big\rangle\end{pmatrix}$$

$$=0$$

and analogously

$$\Big\langle\overline{\partial}\widetilde{\psi},\overline{\partial}\widetilde{\psi}\Big\rangle=0\,.$$

Consequently, as $K\leq 1$ by Proposition 1.1.10, $\widetilde{\psi}$ is a conformal, non-singular immersion with conformal factor $(2-K)\lambda$. In particular, the metric $\widetilde{g}$ induced by $\widetilde{\psi}$ satisfies

$$\widetilde{g}=(2-K)g\,.$$

It remains to show that $\widetilde{\psi}$ satisfies the minimal surface equation in $\mathbb{S}^5$ (analogous to (1.13)). To this end, first note that combining the Weingarten equations (1.15), the Gauss equation (1.14) and the minimality of ψ yields

$$\big|\partial_i\psi\wedge\partial_i\psi^*\big|^2=\det\begin{pmatrix}\langle\partial_i\psi,\partial_i\psi\rangle & \big\langle\partial_i\psi,\partial_i\psi^*\big\rangle\\ \big\langle\partial_i\psi,\partial_i\psi^*\big\rangle & \big\langle\partial_i\psi^*,\partial_i\psi^*\big\rangle\end{pmatrix}=\beta_{12}^2\qquad\text{for } i=1,\,2$$

and moreover

$$\big\langle\partial_1\psi\wedge\partial_1\psi^*,\partial_2\psi\wedge\partial_2\psi^*\big\rangle=\det\begin{pmatrix}\langle\partial_1\psi,\partial_2\psi\rangle & \big\langle\partial_1\psi,\partial_2\psi^*\big\rangle\\ \big\langle\partial_1\psi^*,\partial_2\psi\big\rangle & \big\langle\partial_1\psi^*,\partial_2\psi^*\big\rangle\end{pmatrix}=-\beta_{12}^2\,.$$

Hence, it follows that

$$\begin{aligned}\big|\partial_1\psi\wedge\partial_1\psi^*+\partial_2\psi\wedge\partial_2\psi^*\big|^2&=\big|\partial_1\psi\wedge\partial_1\psi^*\big|^2+\big|\partial_2\psi\wedge\partial_2\psi^*\big|^2\\&\qquad+2\big\langle\partial_1\psi\wedge\partial_1\psi^*,\partial_2\psi\wedge\partial_2\psi^*\big\rangle\\&=0\,.\end{aligned}$$

In particular, we have

$$\overline{\partial}\psi \wedge \partial\psi^* + \partial\psi \wedge \overline{\partial}\psi^* = \frac{1}{2}\big(\partial_1\psi \wedge \partial_1\psi^* + \partial_2\psi \wedge \partial_2\psi^*\big) = 0\,.$$

Together with the minimal surface equations for ψ (1.13) and ψ^* (1.26), we finally deduce that

$$\partial\overline{\partial}\widetilde{\psi} = \partial\overline{\partial}\psi \wedge \psi^* + \overline{\partial}\psi \wedge \partial\psi^* + \partial\psi \wedge \overline{\partial}\psi^* + \psi \wedge \partial\overline{\partial}\psi^* = -\frac{(2-K)\lambda}{2}\widetilde{\psi}\,,$$

that is, $\widetilde{\psi}$ is minimal in $\mathbb{S}^5$. □

Remark 1.4.3. (i) Note that

$$\widetilde{\psi^*} = \psi^* \wedge \big(\psi^*\big)^* = \psi^* \wedge \psi = -\psi \wedge \psi^* = -\widetilde{\psi}\,.$$

So, up to an isometry of $\mathbb{S}^5$, a (non-singular) minimal surface in $\mathbb{S}^3$ and its polar variety lead to the same bipolar surface in $\mathbb{S}^5$.

(ii) Using a basis of $\Lambda^2\mathbb{R}^4$, it can be easily shown that the Hodge isomorphism, seen as $*\colon \mathbb{R}^6 \to \mathbb{R}^6$, is an isometry of $\mathbb{S}^5$. Therefore, $*\widetilde{\psi}\colon \Sigma \to \mathbb{S}^5$ is a minimally immersed surface in $\mathbb{S}^5$ isometric to $\widetilde{\psi}\colon \Sigma \to \mathbb{S}^5$.

(iii) As shown in [37], the components of a bipolar minimal immersion correspond to Jacobi fields of infinitesimal rotations of $\mathbb{S}^3$. This perspective allows to prove (see also [37]) that the image of a bipolar surface lies non-degenerately in a geodesic sphere $\mathbb{S}^{d-1} \subseteq \mathbb{S}^5$, where d (with $3 \leq d \leq 6$) is the dimension of the vector space of so-called Killing-Jacobi fields on the corresponding minimal surface in $\mathbb{S}^3$, also called its Killing nullity. Explicit values of d are determined in Theorem 10 in [24]. Thereby, it follows that $d = 3$ only for geodesic 2-spheres in $\mathbb{S}^3$ and $d = 4$ only for the Clifford torus $\mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2}) \subseteq \mathbb{S}^3$. Additionally, $d = 5$ is only possible for Euler characteristic 0. In particular, bipolar surfaces arising from minimal surfaces in $\mathbb{S}^3$ of higher orientable genera > 1 or non-orientable genera > 2 always lie non-degenerately in $\mathbb{S}^5$.

Chapter 2

Lawson's Bipolar Minimal Surfaces

In this chapter, we study the bipolar surfaces of closed minimal surfaces in $\mathbb{S}^3$ obtained from H. Blaine Lawson's construction method which was initially presented in [37]. Our primary result is the topological classification of the bipolar Lawson surfaces $\widetilde{\xi}_{m,k}$ and $\widetilde{\eta}_{m,k}$, accompanied by upper and lower area bounds. In addition, we show that these surfaces are not embedded for $m \geq 2$ or $k \geq 2$.

On the one hand, this result extends the findings for the bipolar surfaces $\widetilde{\tau}_{m,k}$ of Hugues Lapointe in [36] to the other families of Lawson surfaces in $\mathbb{S}^3$. On the other hand, it serves as a first reaction to the question whether bipolar surfaces of higher genera yield significant conclusions similar to those of the embedded Klein bottle $\widetilde{\tau}_{3,1}$ that appears in the context of several geometric variational problems. In [23], $\widetilde{\tau}_{3,1}$ was conjectured to be the unique minimizer of the Willmore energy among Klein bottles in $\mathbb{R}^4$. Before, the results of [26] and [15] showed that $\widetilde{\tau}_{3,1}$ realizes the unique maximal metric for the first eigenvalue of the Laplace-Beltrami operator on Klein bottles. Furthermore, due to the result of [11], this surface can be understood as the unique Hamiltonian stable Lagrangian minimal Klein bottle in $\mathbb{S}^2 \times \mathbb{S}^2$ (for this setup, we also refer to Chapter 3).

The constructions and results of this chapter can be found in the preprint [54].

2.1 Immersions for Lawson-type Surfaces in $\mathbb{S}^3$

In numerous ways, Lawson's examples of closed minimal surfaces in $\mathbb{S}^3$ from [37] have played a pioneering role and remain a central topic in current research. Their existence is rooted in the construction method outlined in Section 4 (and specifically Theorem 1) of the same article. In this section, we present a reformulation of Lawson's method specifically aimed at enabling an analysis of the corresponding bipolar surfaces later on. More precisely, we will define appropriate immersions of surfaces in $\mathbb{S}^3$ which are constructed in this way.

Lawson's approach starts from an embedded, minimal disk in $\mathbb{S}^3 \subseteq \mathbb{R}^4$, bounded by a geodesic polygon of specific type, and extends it to a complete, non-singular minimal surface by successive application of the Schwarz reflection principle. This principle states that a minimal surface containing a straight line remains invariant under the 180° rotation about that line. Similarly as in $\mathbb{R}^3$, where this is derived from the Schwarz reflection principle for harmonic functions, [37] also proves this concept for minimal surfaces in $\mathbb{S}^3$, using the minimal surface equation.

The key ingredient making the reflection process successful, is the considered class of geodesic polygons in $\mathbb{S}^3$. A polygon Γ in this class is defined by

(1) N vertices connected by geodesic arcs,

(2) interior angles of the form $\frac{\pi}{k_i}$ for positive integers $k_i \geq 2$, $i = 1, \ldots, N$,

(3) and the property $\Gamma \subseteq \partial\mathcal{C}(\Gamma)$, where

$$\mathcal{C}(\Gamma) := \bigcap \big\{H : H \text{ is a closed hemisphere in } \mathbb{S}^3 \text{ such that } \Gamma \subseteq H\big\}$$

denotes the convex hull of Γ in $\mathbb{S}^3$.

Considered as boundary values for the Plateau problem, such type of polygon in particular guarantees the existence of a unique, embedded, disk-like solution, providing the initial piece of the surface, as shown by William Meeks and Shing-Tung Yau in [47]. We remark that Lawson's original formulation in [37] uses the Plateau solution of Charles Morrey in [53].

Note that a striking point of Lawson's method is its ability to produce highly symmetric surfaces, because, by construction, they remain invariant (as point sets) under the group generated by the geodesic reflections across the boundary arcs of the initial polygon. Furthermore, the algebraic properties of this group directly allow to characterize the resulting surface. For example, a finite reflection group is a sufficient criterion leading to a *closed* surface.

We start our work by considering the framework of a closed, non-singular minimal surface $\mathcal{M}_\Gamma$ obtained from Lawson's construction method.

Definition 2.1.1 (Construction Data of a Lawson-type Surface)**.** Let

- $\Gamma \subseteq \mathbb{S}^3$ be a geodesic polygon associated to $\mathcal{M}_\Gamma$, which meets the specific requirements from above,
- $f\colon \Delta \to \mathbb{S}^3$ be the unique immersion of the initial piece of surface defined on the closed unit disk Δ,
- $\gamma_1, ..., \gamma_N \subseteq \mathbb{S}^3$ be the great circles containing the boundary arcs of Γ,
- $r_1, ..., r_N \in \mathrm{SO}(4)$ be the corresponding geodesic reflections (i.e., r_i is the reflection across the plane in $\mathbb{R}^4$ containing the great circle γ_i),
- $G = \langle r_{\gamma_1}, ..., r_{\gamma_N} \rangle$ be their freely generated group, which is finite in the considered case.

Note that we can assume, without loss of generality, that the subgroup $H \subseteq G$ which leaves Γ (and therewith $f(\Delta)$) invariant as a point set is trivial. Otherwise, we could simply descend to a smaller initial piece of surface. This assumption enters all the constructions presented below.

Remark 2.1.2. In general, the same surface $\mathcal{M}_\Gamma$ can result from different choices of the geodesic polygon Γ, as illustrated by the example of the Clifford torus, seen as the Lawson surfaces $\xi_{1,1} = \tau_{1,1}$ (cf. Section 2.3).

Now, as deduced in [37], the successive reflection process yields[1]

$$\mathcal{M}_\Gamma = \bigcup_{g \in G} (g \circ f)(\Delta)\,.$$

Motivated by the above notation, our goal is to define $\mathcal{M}_\Gamma$ as an immersed surface $\psi\colon S \to \mathbb{S}^3$, as concretely as possible. To compare with [37], this specification is not necessary to derive the existence of $\mathcal{M}_\Gamma$. But given Lawson's definition as presented in Theorem 1.4.1, this seems to be the direct approach to address the corresponding bipolar surface in later analyses.

In order to obtain a suitable domain S, the idea is to glue together the preimages of the minimal disks $g \circ f$ for $g \in G$. More precisely, this means to introduce an equivalence relation on the stack $G \times \Delta$ of labelled disks $\{g\} \times \Delta$ in accordance with Lawson's reflection process. Clearly, if two points (g,p), $(h,q) \in G \times \Delta$ are identified in this way, we must have

$$(g \circ f)(p) = (h \circ f)(q)\,.$$

But to define the gluing relation, this condition is not sufficient since it is also satisfied at points of self-intersection. To exclude the latter scenario, we have to put an additional condition on the group elements g and h, which is based on the following definition.

Definition 2.1.3. For each $p \in \Delta$, let G^p be the subgroup

$$G^p := \Big\langle \big\{ r_i : f(p) \in \gamma_i \big\} \Big\rangle \subseteq G\,.$$

Regarding the group structure of G^p, it is clear that only the following three cases can occur:

---

[1] In the general case where $\mathcal{M}_\Gamma$ is complete but not necessarily closed, this directly demonstrates that $\mathcal{M}_\Gamma$ is closed if and only if the group G is finite.

1. If $p \in \Delta^\circ$, then

$$G^p = \{\mathrm{id}_{\mathbb{R}^4}\}\,.$$

2. If $p \in \partial\Delta$ is not mapped onto a vertex of Γ by f, that is, $f(p) \in \gamma_i$ for only one $i \in \{1, ..., N\}$, then

$$G^p = \{\mathrm{id}_{\mathbb{R}^4}, r_i\} \cong \mathbb{Z}_2.$$

3. If $p \in \partial\Delta$ is mapped onto a vertex of Γ by f, that is, $f(p) \in \gamma_i \cap \gamma_{i+1}$ for some $i \in \{1, ..., N\}$ (with $N + 1 \equiv 1$), then

$$G^p = \langle r_i, r_{i+1}\rangle \cong D_n$$

   where n defines the interior angle $\frac{\pi}{n}$ of Γ at $f(p)$ and D_n denotes the dihedral group of order $2n$. This isomorphy follows immediately from the fact that $r_{i+1} \circ r_i$ is the rotation by $\frac{2\pi}{n}$ around the great circle through $f(p)$ which is orthogonal to both r_{i+1} and r_i, and from the fact that each r_i is self-inverse.

For the initial piece of surface f, the group G^p labels the different, neighbouring pieces of surface at the point $f(p)$, which yield an analytic, non-singular extension of f in a neighbourhood of p (cf. [37], Lemmata 4.2 and 4.3). As a consequence, $g \cdot G^p$ encodes the neighbouring pieces of surface at the point $(g \circ f)(p)$ for any $g \in G$. Thus, we conclude that two points (g, p), $(h, q) \in G \times \Delta$ that satisfy

$$(g \circ f)(p) = (h \circ f)(q)$$

belong to neighbouring pieces if and only if $g^{-1} \cdot h \in G^p$. In other words, this condition now ensures that we glue according to the local landscape provided by repeated application of Schwarz reflections. Concluded, we have the following construction.

Construction 2.1.4. On $G \times \Delta$, we introduce the equivalence relation

$$(g, p) \sim (h, q) \quad \Leftrightarrow \quad (g \circ f)(p) = (h \circ f)(q) \quad \text{and} \quad g^{-1} \cdot h \in G^p$$

and denote a corresponding equivalence class by $\big[(g,p)\big]$. Then, due to [37], the quotient

$$S := \big(G \times \Delta\big)/\sim$$

is a closed, two-dimensional manifold and

$$\psi\colon S \to \mathbb{S}^3\,, \quad \psi\big([(g,p)]\big) := (g \circ f)(p)$$

is a well-defined immersion of the closed minimal surface $\mathcal{M}_\Gamma$.

Remark 2.1.5. (i) Since we assumed that the subgroup of symmetries of Γ is trivial (which is, as already mentioned, no loss of generality), S is in fact a *smallest possible domain* to immerse $\mathcal{M}_\Gamma$. On page 345 in [37], Lawson refers to such (possibly non-orientable) manifold by $\mathcal{M}^*_\Gamma$. To clarify, this means that an immersion defined on such domain is one-to-one almost everywhere. In other words, it is not possible to define a smooth covering map on S, which leaves ψ (and thereby the induced metric of ψ) invariant. In that case, the immersion ψ would be well-defined on a quotient of S but still provide a full description of $\mathcal{M}_\Gamma$. From the viewpoint of the classification of closed, two-dimensional manifolds by genus and orientability, working with such smallest possible domains is preferable. It is exactly what characterizes expressions like "*the surface XY is an immersed Klein bottle*". This kind of information is also particularly relevant in further geometric variational problems, especially when searching for extremal surfaces of specific topological types.

(ii) For the theory of *covering manifolds*, we direct, for example, to [40], Chapter 21. More specifically, we will use Theorem 21.13 of that reference, which is a special version of the quotient manifold theorem, applicable to finite groups regarded as discrete Lie groups.

Suppose that $\mathcal{G}$ is a discrete Lie group acting smoothly, freely and properly on a connected, smooth manifold M. Then, the orbit space $M/\mathcal{G}$ is a topological manifold and has a unique smooth structure such that the projection $M \to M/\mathcal{G}$ is a smooth, normal covering map.

Note that *normal* covering maps provide the existence of a corresponding deck transformation group, ensuring uniformity in the fibers. In addition, note that an action $\mathcal{G} \times M\colon (g,p) \mapsto g \cdot p$ of a discrete Lie group $\mathcal{G}$ on M is smooth if and only if $p \mapsto g \cdot p$ is smooth for all $g \in \mathcal{G}$. Moreover, if $\mathcal{G}$ acts smoothly and freely, the action is proper if and only if the following conditions hold (cf. Lemma 21.11 in [40]):

(1) For all $p \in M$ there exists a neighborhood V such that for all $g \in \mathcal{G}$, $g \cdot V \cap V = \emptyset$ unless $g = e_{\mathcal{G}}$.

(2) If p and $p' \in M$ lie in different orbits, then there exist neighborhoods V of p and V' of p' such that for all $g \in \mathcal{G}$, $(g \cdot V) \cap V' = \emptyset$.

Now, by the preceding construction, we directly arrive at the following topological information about the domain S.

Corollary 2.1.6 (cf. [37], Proposition 4.4)**.** *The Euler characteristic $\chi(S)$ of S is fully determined by the polygon Γ: We have*

$$\chi(S) = |G| \cdot \left(1 - \sum_{i=1}^{N} \frac{k_i - 1}{2k_i}\right), \tag{2.1}$$

where $\frac{\pi}{k_1}.,...,\frac{\pi}{k_N}$, $k_i \in \mathbb{Z}_{\geq 2}$, denote the interior angles of Γ.

Remark 2.1.7. Note that the relation between the Euler characterisic and the (orientable or non-orientable) genus g of S is

$$\chi(S) = \begin{cases} 2 - 2g & \text{if } S \text{ is orientable,} \\ 2 - g & \text{if } S \text{ is non-orientable.} \end{cases}$$

Proof of Corollary 2.1.6. Let K_f and K_ψ be the Gaussian curvatures of f and ψ, respectively. Then, the local version of the Gauss-Bonnet theorem yields

$$\int_\Delta K_f \, \mathrm{d}\mu_f = 2\pi - \sum_{i=1}^{N} \left(\pi - \frac{\pi}{k_i} \right) = 2\pi \left(1 - \sum_{i=1}^{N} \frac{k_i - 1}{2k_i} \right).$$

Since

$$\int_S K_\psi \, \mathrm{d}\mu_\psi = |G| \cdot \int_\Delta K_f \, \mathrm{d}\mu_f \, ,$$

the global version of Gauss-Bonnet implies

$$2\pi\chi(S) = \int_S K_\psi \, \mathrm{d}\mu_\psi$$

and therefore the assertion follows. □

For a full topological classification of S, it remains to check whether S is orientable or not. In this regard, we consider a Gauss map $n : \Delta \to \mathbb{S}^3$ associated to the embedded piece of surface f. Under a Schwarz reflection across an arc of Γ, achieved by an 180° rotation about the corresponding great circle, the initial normal n can be extended by the same reflection as f but has to be combined with an additional flip for a continuous extension. Then, for an extension of the surface associated to a product of the generators $r_{\gamma_1}, ..., r_{\gamma_N}$, this generalizes in the sense that each generator contributes one flip. Consequently, S is non-orientable if and only if we find a sequence of the generators that starts and ends at the initial piece f but returns with the opposite normal $-n$. This situation would clearly imply the existence of a non-trivial path along which the direction of a unit normal reverses (see Figure 2.1).

In other words, we can conclude the following.

Proposition 2.1.8. *S is non-orientable if and only if the identity element $e_G = \mathrm{id}_{\mathbb{R}^4} \in G$ can be written as an odd number of the generators $r_1, ..., r_N$.*

Conversely, if S is orientable, any expression of a group element $g \in G$ as a

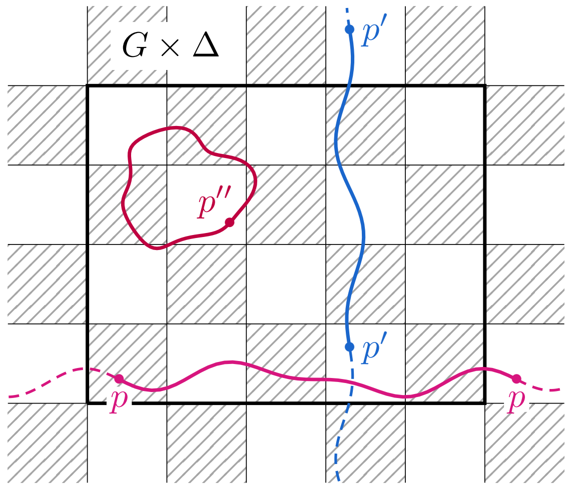

Figure 2.1: Counting the number of flips of an initial unit normal while extending the initial piece of surface by Schwarz reflections schematically induces a "chessboard pattern" on S. In this sense, a closed path starting in $\{e_G\} \times \Delta$ and returning to a field with the "wrong" color (assigned to an odd number of generators resulting in the identity element in G) implies that S is non-orientable. In the example above, this is indicated by the curve in the bottom containing the point p.

product of the generators either contains an even *or* an odd number of the latter. So, the following quantity is well-defined.

Definition 2.1.9. Let S be orientable. Then, we define the *parity* $\sigma(g)$ of $g \in G$ as follows: If $g = r_{i_1} \circ \ldots \circ r_{i_k}$ for $i_1, \ldots, i_k \in \{1, \ldots N\}$ and $k \in \mathbb{N}$,

$$\sigma(g) := \begin{cases} 0 & \\ & \text{if } k \text{ is} \begin{matrix} \text{even} \\ \text{odd} \end{matrix} \\ 1 & \end{cases}.$$

Remark 2.1.10. Note that $r_i \circ r_i = e_G$ for $i \in \{1, \ldots, N\}$ and hence $\sigma(e_G) = 0$. In particular, $\sigma\colon G \to (\mathbb{Z}_2, +)$ is a group homomorphism.

Accordingly, the extension of an initial unit normal n writes as follows.

Construction 2.1.11. Let $n\colon \Delta \to \mathbb{S}^3$ be a Gauss map of f. If S is orientable, then

$$\psi^*\colon S \to \mathbb{S}^3, \quad \psi^*\big([(g,p)]\big) := (-1)^{\sigma(g)} \cdot (g \circ n)(p)$$

is a Gauss map of the immersion $\psi\colon S \to \mathbb{S}^3$ from Construction 2.1.4.

Remark 2.1.12. If S is orientable, the choice of ψ^* induces an orientation form on S by the pullback of an orientation form Ω on $\mathbb{S}^3 \subseteq \mathbb{R}^4$. We choose

$$\omega|_x(v,w) := \det\Big(\psi(x), \mathrm{d}\psi|_x(v), \mathrm{d}\psi|_x(w), \psi^*(x)\Big) \tag{2.2}$$

for $x \in S$ and v, $w \in T_x S$.

Now, the concept of bipolar surfaces, which is the primary interest of this chapter, also exists for non-orientable domains in the sense that any non-orientable surface has a unique, orientable double cover, characterized by a smooth, orientation-reversing, fixed-point free involution. In the sense of Remark 2.1.5 (ii), this corresponds to a suitable $\mathbb{Z}_2$-action. Hence, to make sense of a Gauss map when S is non-orientable, we perform the transition to the orientable double cover of S , which can be, in a straightforward manner, constructed in similar way as S itself.

Construction 2.1.13. Suppose that S is non-orientable. Let $n\colon \Delta \to \mathbb{S}^3$ be a Gauß map of f. On $\mathbb{Z}_2 \times G \times \Delta$, we define the equivalence relation

$$(s,g,p) \sim_{\mathrm{dc}} (t,h,q) \quad :\Leftrightarrow \quad (g,p) \sim (h,q) \text{ and } (-1)^s \cdot (g \circ n)(p) = (-1)^t \cdot (h \circ n)(q)$$

and denote a corresponding equivalence class by $[(s,g,p)]$. Then, the orientable double cover of S is the two-dimensional manifold

$$\overline{S} := \big(\mathbb{Z}_2 \times G \times \Delta\big)/\sim_{\mathrm{dc}}$$

and the map

$$i\colon \overline{S} \to \overline{S}, \quad i\big([(s,g,p)]\big) := [(-s,g,p)]$$

is the corresponding orientation-reversing, fixed-point-free involution satisfying that $\overline{S}/\langle i \rangle = S$. In this case, we describe the surface $\mathcal{M}_\Gamma$ by the immersion

$$\overline{\psi}\colon \overline{S} \to \mathbb{S}^3, \quad \overline{\psi}\big([(s,g,p)]\big) := \psi\big([g,p]\big)$$

with Gauss map

$$\overline{\psi^*} : \overline{S} \to \mathbb{S}^3, \quad \overline{\psi^*}\big([(s,g,p)]\big) := (-1)^s \cdot (g \circ n)(p)\,.$$

Remark 2.1.14. (i) If S is orientable, then $\overline{S}$, as defined above, would consist of two components, each diffeomorphic to S. But as mentioned in Remark 2.1.5, for our purposes, we prefer S to be an orientable, smallest possible domain for $\mathcal{M}_\Gamma$ in that case.

(ii) The induced orientation form $\overline{\varpi}$ on $\overline{S}$ is defined analogously as in (2.2).

2.2 Topology of Lawson-type Bipolar Surfaces

In this section, we introduce a general indicator to identify certain topological characteristics and thereby area estimates of bipolar surfaces derived from Lawson-type, closed minimal surfaces in $\mathbb{S}^3$, as outlined in the previous section. (Not) surprisingly, this appears as an algebraic condition on the group G generated by the respective group of Schwarz reflections.

Recalling Remark 2.1.5, the notion of a smallest possible domain for an immersed surface crucially depends on the symmetries of the immersion, not only on the topological type of some possible domain. Therefore, it is not a priori clear whether the topological type of a closed minimal surface in $\mathbb{S}^3$ transmits to its bipolar surface in $\mathbb{S}^5$. One possible scenario is, for example, that a smallest possible domain for ψ admits covering maps of a higher degree that leave $\widetilde{\psi}$ invariant. Another phenomenon that can occur is that in the non-orientable case, where we define $\widetilde{\psi}$ on the orientable double cover of a smallest possible domain of ψ, we really have to stick with that orientable domain, i.e., loose symmetry in that case.

These types of topological differences are particularly important for the computation of the actual area of bipolar surfaces, that is, the area of a corresponding immersion on a smallest possible domain. More precisely, the following formula, which is an immediate consequence of Theorem 1.4.1 combined with the Gauss-Bonnet theorem, must be handled with caution when discussing the actual area. Recall that the latter equals the Willmore energy of the stereographically projected surfaces in $\mathbb{R}^5$, making it specifically significant the context of the Willmore problem.

Proposition 2.2.1. *Let $\psi\colon \Sigma \to \mathbb{S}^3$ be a closed, oriented minimal surface and $\widetilde{\psi}\colon \Sigma \to \mathbb{S}^5$ its bipolar surface. Then, we have*

$$\operatorname{area}\left(\widetilde{\psi}\right) = 2\operatorname{area}(\psi) - 2\pi\chi(\Sigma)\,. \tag{2.3}$$

If a smallest possible domain of $\widetilde{\psi}$ is multiply covered by an orientable, smallest possible domain of ψ, then the right-hand side of this formula has to be divided by the covering degree. Furthermore, in the case that the smallest possible domain of ψ is non-orientable, we have to start from twice the acutal area of ψ in the formula above.

The question that arises is how to detect these situations. In this context, we will now present a tool for the bipolar surfaces of closed minimal surfaces in $\mathbb{S}^3$ constructed by Lawson's method and therefore come back to the immersions from the previous section. In that setup, the bipolar surface $\widetilde{\mathcal{M}_\Gamma} \subseteq \mathbb{S}^5$ of $\mathcal{M}_\Gamma \subseteq \mathbb{S}^3$ is immersed by

$$\widetilde{\psi} := \begin{cases} \psi \wedge \psi^* & \text{if } S \text{ is orientable,} \\ \overline{\psi} \wedge \overline{\psi^*} & \text{if } S \text{ is non-orientable.} \end{cases}$$

perhaps not yet on a smallest possible domain. Based on the bilinearity of the wedge product, we can identify an indicator for $\widetilde{\psi}$ being a double cover, provided that $\mathcal{M}_\Gamma$ exhibits antipodal symmetry.

Note that in the estimates below, we denote by $\operatorname{area}(\mathcal{M}_\Gamma)$ and $\operatorname{area}\left(\widetilde{\mathcal{M}_\Gamma}\right)$ the actual area of the surfaces, obtained by the area in terms of an immersion defined on a smallest possible domain.

Theorem 2.2.2.

(i) *If S is orientable and $-\mathrm{id}_{\mathbb{R}^4} \in G$ with $\sigma(-\mathrm{id}_{\mathbb{R}^4}) = 0$, then the action*

$$\langle -\mathrm{id}_{\mathbb{R}^4}\rangle \times S \to S,\ \big(h, [(g,p)]\big) \mapsto [(hg,p)]$$

leaves the bipolar immersion $\widetilde{\psi}\colon S \to \mathbb{S}^5$ *invariant and induces a smooth covering map of degree 2 on* S *such that the corresponding quotient* $S/\langle -\mathrm{id}_{\mathbb{R}^4}\rangle$ *is orientable. In particular, we have*

$$\operatorname{area}\Big(\widetilde{\mathcal{M}}_\Gamma\Big) \leq \operatorname{area}(\mathcal{M}_\Gamma) - \pi\chi(S)\,.$$

(ii) *If* S *is non-orientable and* $-\mathrm{id}_{\mathbb{R}^4} \in G$*, then the action*

$$\langle -\mathrm{id}_{\mathbb{R}^4}\rangle \times \overline{S} \to \overline{S},\ \big(h, [(s,g,p)]\big) \mapsto [(s,hg,p)]$$

leaves the bipolar immersion $\widetilde{\psi}\colon \overline{S} \to \mathbb{S}^5$ *invariant and induces a smooth covering map of degree 2 on* $\overline{S}$ *such that the corresponding quotient* $\overline{S}/\langle -\mathrm{id}_{\mathbb{R}^4}\rangle$ *is orientable. In particular, we have*

$$\operatorname{area}\Big(\widetilde{\mathcal{M}}_\Gamma\Big) \leq 2\operatorname{area}(\mathcal{M}_\Gamma) - 2\pi\chi(S)\,.$$

Proof. We apply the quotient manifold theorem in the specific form of Theorem 21.13 from [40] (see also Remark 2.1.5 (ii)). As a finite group, G is a discrete (0-dimensional) Lie group. In this case, any G-action on a smooth manifold M is smooth if and only if $M \to M,\ x \mapsto h \cdot x$ is smooth for each $h \in G$. For the considered actions, this is clear by the definition of S and $\overline{S}$, respectively, as ψ and $\overline{\psi}$ are local embeddings and each $h \in G$ is smooth. Moreover, both actions are free and proper (cf. Lemma 21.11 in [40]) since $-\mathrm{id}_{\mathbb{R}^4}$ is not contained G_p, for all $p \in \Delta$. This follows from the specific cases of the group structure from Definition 2.1.3. Concluded, the corresponding orbit space projections yield smooth covering maps of degree two for both actions.

In addition, the bilinearity of the wedge product yields that $\widetilde{\psi}$ is in both cases invariant under the respective action. The same holds, by the definition of the Gauss maps ψ^* and $\overline{\psi^*}$, for the orientation forms ω and $\overline{\omega}$, respectively, implying that the quotient manifolds $S/\langle -\mathbb{1}_{\mathbb{R}^4}\rangle$ and $\overline{S}/\langle -\mathbb{1}_{\mathbb{R}^4}\rangle$ are orientable.

At last, with the detected covering maps, the area estimates are a direct consequence of the area formula (2.3). □

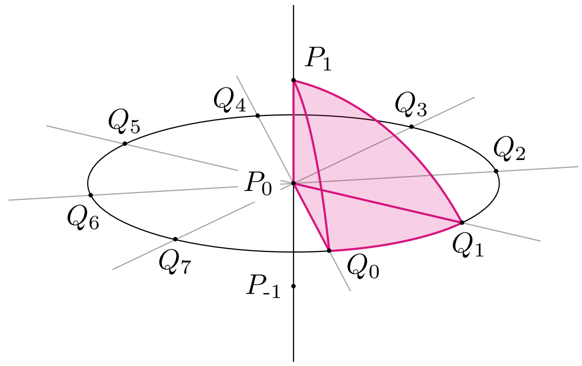

Figure 2.2: To develop his three families of closed minimal surfaces in $\mathbb{S}^3$, Lawson started from a tessellation by congruent, geodesic tetrahedra with edges defined by the set of geodesics connecting two families of equidistant points located on two polar great circles, with distances $\frac{\pi}{m}$ and $\frac{\pi}{k}$. In stereographic projection, the graphic above illustrates an exemplary excerpt.

2.3 Examples

In the context of the covering phenomenon described previously, this section treats the bipolar surfaces obtained from the three Lawson families of closed minimal surfaces in $\mathbb{S}^3$. The main insight is that the tool from Theorem 2.2.2 is actually sufficient to determine the topological classes of the $\widetilde{\xi}$- and $\widetilde{\eta}$-surfaces and thereby area estimates for these surfaces.

2.3.1 Construction of the Lawson Families in $\mathbb{S}^3$

Besides the general formulation, Lawson presents in [37] an explicit framework to apply his construction procedure for minimal surfaces in $\mathbb{S}^3$. The basic setting is described by positive integers $m,\ k \in \mathbb{Z}_{\geq 2}$ which specify a tessellation of $\mathbb{S}^3$ by congruent, geodesic tetrahedra[2]. More precisely, the corresponding 1-skeleton is given by the set of great circles connecting equidistant points $Q_0, ..., Q_{2m-1}$ and $P_0, ..., P_{2k-1}$ on two great circles $\gamma_Q,\ \gamma_P \subseteq \mathbb{S}^3$ that are orthogonal to each other.

[2]For a study of the application of the Schwarz reflection principle in that setting we direct the interested reader to [72].

On that kind of lattice, Lawson introduced three distinct types of explicit, geodesic 4-gons which serve as suitable candidates to bound the initial piece of a closed minimal surface as discussed in Section 2.1. In this way, he arrived at the three infinite families

$$(\xi_{m,k})_{m,\,k\in\mathbb{Z}_{\geq 1}}\,, \qquad (\tau_{m,k})_{m,\,k\in\mathbb{Z}_{\geq 1}}\,, \qquad (\eta_{m,k})_{m,\,k\in\mathbb{Z}_{\geq 1}}$$

of closed minimal surfaces in $\mathbb{S}^3$. Regarding Lawson's labeling within these families, note that we will often stick to the notation $\xi_{m-1,k-1}$ and $\eta_{m-1,k-1}$ for integers $m,\ k \geq 2$ as this refers to a distance $\frac{\pi}{m}$ between Q_j and Q_{j+1} and $\frac{\pi}{k}$ between P_i and P_{j+1} in the respective construction data. This kind of shift is not necessary for the τ-family.

Interestingly, the three families differ significantly regarding their topological properties (recall Remark 2.1.5) and regarding embeddedness:

- The surfaces $\xi_{m,k}$ provide embedded and thus orientable examples of genus mk (cf. [37], Proposition 6.1). In particular, together with the totally geodesic 2-spheres, the existence of the ξ-family demonstrates that among closed embedded minimal surfaces in $\mathbb{S}^3$, every orientable topolgogical class is represented.
- The surfaces $\tau_{m,k}$ are a family of immersed tori and Klein bottles, where τ_{mk} is non-orientable if and only if $2|mk$ (cf. [37], Theorem 3).
- If k is even, the surfaces $\eta_{m,k}$ are immersed, orientable surfaces with genus mk (cf. [37], Remark 8.1). In turn, if k is odd, $\eta_{m,k}$ is non-orientable and has (non-orientable) genus $mk+1$ (cf. [37], Theorem 4). Notably, except for non-orientable genus 1, this yields a complete list of (immersed) examples for the non-orientable topological classes (recall that by Corollary 1.2.6 minimal immersions of the real projective plane are forbidden).

The Clifford torus $\mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2}) \subseteq \mathbb{S}^3$ is realized by both $\xi_{1,1}$ and $\tau_{1,1}$. Within the τ-family, $\tau_{1,1}$ is the only embedded surface.

In order to analyze the corresponding families of bipolar surfaces in what follows, we start by a specification of the respective construction data for each family in $\mathbb{S}^3$

(cf. Definition 2.1.1). To that end, we establish the following: For m, $k \in \mathbb{Z}_{\geq 2}$, let

$$P_i := \begin{pmatrix} \cos\left(i \cdot \frac{\pi}{k}\right) \\ \sin\left(i \cdot \frac{\pi}{k}\right) \\ 0 \\ 0 \end{pmatrix} \quad \text{for } i \in \mathbb{Z}_{2k}\,, \qquad Q_j := \begin{pmatrix} 0 \\ 0 \\ \cos\left(j \cdot \frac{\pi}{m}\right) \\ \sin\left(j \cdot \frac{\pi}{m}\right) \end{pmatrix} \quad \text{for } j \in \mathbb{Z}_{2m}$$

represent the points from the tessellation of in [37], lying on the great circles

$$\gamma_P := \left\{x \in \mathbb{S}^3 : x_3 = x_4 = 0\right\}, \qquad \gamma_Q := \left\{x \in \mathbb{S}^3 : x_1 = x_2 = 0\right\}.$$

To determine G, note that the geodesic reflection r_γ across a great circle γ in $\mathbb{S}^3 \subseteq \mathbb{R}^4$ is the reflection at the 2-plane P_γ such that $P_\gamma \cap \mathbb{S}^3 = \gamma$, explicitly,

$$r_\gamma(x) = x^{||} - 2x^{\perp}$$

for all $x = x^{||} + x^{\perp} \in \mathbb{S}^3$, where $x^{||} \in P_\gamma$ and $x^{\perp} \in P_\gamma^{\perp}$. In what follows, we denote by r_{ij} the geodesic reflection across the great circle γ_{ij} through the points P_i and Q_j (represented by the corresponding matrix with respect to the standard basis in $\mathbb{R}^4$). We have

$$r_{00} = \begin{pmatrix} \mathbf{J}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2 \end{pmatrix}, \qquad \text{where} \quad \mathbf{J}_2 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Furthermore, setting

$$\mathbf{R}_\varphi := \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} \qquad \text{for } \varphi \in \mathbb{R}\,,$$

as well as

$$\mathbf{R}_\varphi^{(12)} := \begin{pmatrix} \mathbf{R}_\varphi & \mathbf{0} \\ \mathbf{0} & \mathbb{1}_2 \end{pmatrix}, \qquad \mathbf{R}_\varphi^{(34)} := \begin{pmatrix} \mathbb{1}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_\varphi \end{pmatrix},$$

a reflection r_{ij} reads as

$$r_{ij} = r_{00} \cdot \mathbf{R}^{(12)}_{\frac{2\pi}{k}\cdot i} \cdot \mathbf{R}^{(34)}_{\frac{2\pi}{m}\cdot j} \, .$$

Finally, note that in each case we have to determine the subgroup $H \subseteq G$ of symmetries of the polygon Γ (leaving it invariant as a set). If H is not trivial, we will need an adjustment in order to apply the methods from Sections 2.1 and 2.2.

2.3.2 The $\widetilde{\xi}$-Family

We consider the Lawson surface $\xi_{m-1,k-1}$ for m, $k \in \mathbb{Z}_{\geq 2}$ and assume that $m > 2$ or $k > 2$. Note that, later on, we will address the Clifford torus $\xi_{1,1} = \tau_{1,1}$ in the context of the τ-family.

The geodesic polygon defining $\xi_{m-1,k-1}$ is given by the circuit

$$\Gamma_{\xi_{m-1,k-1}} := P_0 Q_0 P_1 Q_1 \, ,$$

where the successive vertices are connected by shortest arcs. This polygon has two interior angles $\frac{\pi}{m}$ and two interior angles $\frac{\pi}{k}$, with all sides having a length $\frac{\pi}{2}$.

Looking at $\Gamma_{\xi_{m-1,k-1}}$, the group generated by Schwarz reflections of the surface $\xi_{m-1,k-1}$ is a priori given by

$$G_{\xi_{m-1,k-1}} = \langle r_{00}, r_{01}, r_{11}, r_{10} \rangle \, .$$

Using that

$$\begin{aligned}
r_{01} &= r_{00} \cdot \mathbf{R}^{(34)}_{\frac{2\pi}{m}} \, , \\
r_{11} &= r_{00} \cdot \mathbf{R}^{(12)}_{\frac{2\pi}{k}} \cdot \mathbf{R}^{(34)}_{\frac{2\pi}{m}} \, , \\
r_{10} &= r_{00} \cdot \mathbf{R}^{(12)}_{\frac{2\pi}{k}}
\end{aligned}$$

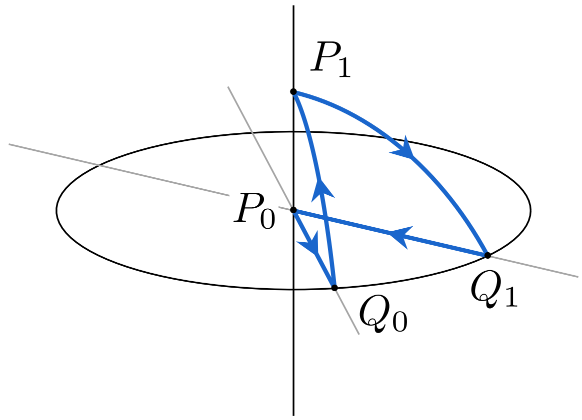

Figure 2.3: In stereographic projection, this graphic illustrates the type of geodesic polygon $\Gamma_\xi \subseteq \mathbb{S}^3$ used to construct a surface in the ξ-family.

and conversely, since r_{00} is self-inverse,

$$\begin{aligned} \mathbf{R}^{(12)}_{\frac{2\pi}{k}} &= r_{00} \cdot r_{10} \\ \mathbf{R}^{(34)}_{\frac{2\pi}{m}} &= r_{00} \cdot r_{01}\,, \end{aligned} \tag{2.4}$$

we find that

$$G_{\xi_{m-1,k-1}} = \left\langle \mathbf{R}^{(12)}_{\frac{2\pi}{k}}, \mathbf{R}^{(34)}_{\frac{2\pi}{m}}, r_{00} \right\rangle .$$

As the block matrices $\mathbf{R}^{(12)}_{\frac{2\pi}{k}}$ and $\mathbf{R}^{(34)}_{\frac{2\pi}{m}}$ commute and in addition

$$\mathbf{J}_2 \cdot \mathbf{R}_\varphi = \mathbf{R}_\varphi^{-1} \cdot \mathbf{J}_2 \qquad \text{for all } \varphi \in \mathbb{R}\,, \tag{2.5}$$

we can finally write

$$G_{\xi_{m-1,k-1}} = \left\{ \left(\mathbf{R}^{(12)}_{\frac{2\pi}{k}}\right)^{\alpha} \cdot \left(\mathbf{R}^{(34)}_{\frac{2\pi}{m}}\right)^{\beta} \cdot r_{00}^{\gamma} : \alpha \in \mathbb{Z}_k,\ \beta \in \mathbb{Z}_m,\ \gamma \in \mathbb{Z}_2 \right\} \tag{2.6}$$

and observe that

$$|G_{\xi_{m-1,k-1}}| = 2mk\,.$$

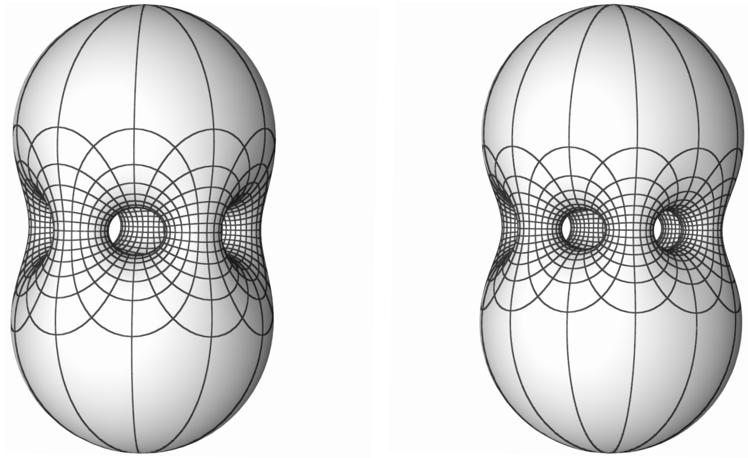

Figure 2.4: Stereographic projections of $\xi_{1,4}$ and $\xi_{1,5}$ (images from [21], produced by Nicholas Schmitt with xLab)

Therefore, (2.1) yields

$$\chi(\xi_{m-1,k-1}) = 2mk\left(1 - 2\left(\frac{k-1}{2k} + \frac{m-1}{2m}\right)\right) = 2 - 2(m-1)(k-1)\,. \qquad (2.7)$$

Furthermore, with the above considerations, the orientability of $\xi_{m-1,k-1}$ is evident by Proposition 2.1.8. Any odd number of the initial geodesic reflections carries a factor r_{00}. By comparing determinants, we see that the identity cannot be represented in such a way.

Concluded, Corollary 2.1.6 and Proposition 2.1.8 confirm the topological classification of $\xi_{m-1,k-1}$ from [37].

Now, that $\xi_{m-1,k-1}$ is orientable, the parity σ group elements is well-defined (cf. Definition 2.1.9). For a general group element

$$g = \left(\mathbf{R}^{(12)}_{\frac{2\pi}{k}}\right)^{\alpha} \cdot \left(\mathbf{R}^{(34)}_{\frac{2\pi}{m}}\right)^{\beta} \cdot r_{00}^{\gamma} \in G_{\xi_{m-1,k-1}}\,,$$

we have

$$\sigma(g) = \gamma\,, \qquad (2.8)$$

since $\mathbf{R}^{(12)}_{\frac{2\pi}{k}}$ and $\mathbf{R}^{(34)}_{\frac{2\pi}{m}}$ are given by an even number of Schwarz reflections by the representation (2.4).

Lemma 2.3.1. *Let $m, k \in \mathbb{Z}_{\geq 2}$ and assume that $m > 2$ or $k > 2$. Then, the subgroup $H_{\xi_{m-1,k-1}} \subseteq G_{\xi_{m-1,k-1}}$ leaving the polygon $\Gamma_{m-1,k-1}$ invariant is trivial, i.e.,*

$$H_{\xi_{m-1,k-1}} = \{\mathbb{1}_4\} .$$

Proof. To see this, consider for example the segment $P_0Q_0P_1$ of $\Gamma_{\xi_{m-1,k-1}}$ which determines the angle $\frac{\pi}{k}$ at Q_0. For $g \in G_{\xi_{m-1,k-1}}$ (denoted as above), this polygonal arc is mapped onto

$$g(P_0Q_0P_1) = \left(\left(\mathbf{R}^{(12)}_{\frac{2\pi}{k}}\right)^{\alpha} \cdot \left(\mathbf{R}^{(34)}_{\frac{2\pi}{m}}\right)^{\beta} \cdot r_{00}^{\gamma}\right)(P_0Q_0P_1)$$
$$= \begin{cases} P_{2\alpha}Q_{2\beta}P_{2\alpha+1} & \text{if } \gamma = 0\,, \\ P_{2\alpha}Q_{2\beta}P_{2\alpha-1} & \text{if } \gamma = 1\,. \end{cases} \tag{2.9}$$

Now, if $g \in H_{\xi_{m-1,k-1}}$, the prescribed angle yields that the candidates for the image of $P_0Q_0P_1$ under g are

$$P_0Q_0P_1\,, \quad P_1Q_0P_0\,, \quad P_1Q_1P_0\,, \quad P_0Q_1P_1\,.$$

But combined with (2.9), this implies that we must have $g = \mathbb{1}_4$. □

At this stage, the results from the preceding sections in this chapter lead to the following conclusions about the bipolar surface $\widetilde{\xi}_{m-1,k-1}$.

Theorem 2.3.2. *Let $m, k \in \mathbb{Z}_{\geq 2}$ such that $m > 2$ or $k > 2$. Then, the bipolar surface $\widetilde{\xi}_{m-1,k-1} \subseteq \mathbb{S}^5$ is orientable. Moreover,*

(i) *if both m and k are even, we have*

$$\chi\left(\widetilde{\xi}_{m-1,k-1}\right) = 1 - (m-1)(k-1)\,,$$

$$2\pi \max\{m,k\} \leq \operatorname{area}\Big(\widetilde{\xi}_{m-1,k-1}\Big) < 2\pi(mk+k-m)\,;$$

(ii) *if m or k is odd, we have*

$$\chi\Big(\widetilde{\xi}_{m-1,k-1}\Big) = 2 - 2(m-1)(k-1)\,,$$

$$4\pi \max\{m,k\} \leq \operatorname{area}\Big(\widetilde{\xi}_{m-1,k-1}\Big) < 4\pi(mk+k-m)\,.$$

Proof. We use the notation from Construction 2.1.4 and Construction 2.1.11. In addition, we define the points

$$\hat{P}_i := \begin{pmatrix} -\sin\left(i\cdot\frac{\pi}{k}\right) \\ \cos\left(i\cdot\frac{\pi}{k}\right) \\ 0 \\ 0 \end{pmatrix} \quad \text{for } i \in \mathbb{Z}_{2k}, \qquad \hat{Q}_j := \begin{pmatrix} 0 \\ 0 \\ -\sin\left(j\cdot\frac{\pi}{m}\right) \\ \cos\left(j\cdot\frac{\pi}{m}\right) \end{pmatrix} \quad \text{for } j \in \mathbb{Z}_{2m}\,.$$

Following the polygon $\Gamma_{\xi_{m-1,k-1}}$ as a monotonic curve, we determine that the initial Gauß map $n : \Delta \to \mathbb{S}^3$ can be chosen such that at the vertices of $\Gamma_{\xi_{m-1,k-1}}$ we have

$$n\big(f^{-1}(P_0)\big) = \hat{P}_0\,, \qquad n\big(f^{-1}(Q_0)\big) = -\hat{Q}_0\,,$$
$$n\big(f^{-1}(P_1)\big) = -\hat{P}_1\,, \qquad n\big(f^{-1}(Q_1)\big) = \hat{Q}_1\,.$$

Note that connecting these vertices by shortest arcs defines the so-called polar polygon $\Gamma^*_{\xi_{m-1,k-1}}$ of $\Gamma_{\xi_{m-1,k-1}}$ (cf. [37], Section 10), which can be understood to bound the initial piece of the corresponding polar variety $\xi^*_{m-1,k-1}$ from Proposition 1.2.9.

Now, as a first step, we determine the multiplicity of the point $P_0 \wedge \hat{P}_0 = e_1 \wedge e_2$ in the image of the bipolar immersion $\widetilde{\psi}\colon S \to \mathbb{S}^5$. Let therefore $p_0 \in \partial\Delta$ be the point such that $f(p_0) = P_0 = e_1$ and $n(p_0) = \hat{P}_0 = e_2$. We are looking for all $\big[(g,p)\big] \in S$ such that

$$\widetilde{\psi}\Big(\big[(g,p)\big]\Big) = \widetilde{\psi}\Big(\big[(e,p_0)\big]\Big).$$

By definition of the map $\widetilde{\psi} = \psi \wedge \psi^*$, these are all the points $\big[(g,p)\big] \in S$ such that

$$(-1)^{\sigma(g)} \cdot (g \circ f)(p) \wedge (g \circ n)(p) = e_1 \wedge e_2$$

or equivalently, by relabelling the group elements,

$$f(p) \wedge n(p) = (-1)^{\sigma(g)} \cdot g(e_1) \wedge g(e_2)\,. \tag{2.10}$$

Making use of (2.6) and (2.8), we observe that

$$g(e_1) \wedge g(e_2) = \begin{cases} \;\; e_1 \wedge e_2 & \text{if } \sigma(g) = 0\,, \\ -e_1 \wedge e_2 & \text{if } \sigma(g) = 1\,. \end{cases}$$

Hence, (2.10) reduces to

$$f(p) \wedge n(p) = e_1 \wedge e_2\,. \tag{2.11}$$

Clearly, (2.11) is satisfied by each point $\big[(g,p_0)\big]$ for arbitrary $g \in G_{\xi_{m-1,k-1}}$ and, in fact, these are already all solutions. Because, due to the properties of the wedge product (cf. Section 1.3), given any solution $\big[(g,p)\big]$ of (2.11), we must have

$$\begin{aligned} f(p) &= \quad \cos(\phi)\, e_1 + \sin(\phi)\, e_2\,, \\ n(p) &= -\sin(\phi)\, e_1 + \cos(\phi)\, e_2 \end{aligned}$$

for some $\phi \in [0, 2\pi)$. Looking at the initial pieces of surface described by f and n, which are both embedded in the convex hull of $\Gamma_{\xi_{m-1,k-1}}$ or $\Gamma^*_{\xi_{m-1,k-1}}$, we find that this is only satisfied for $\phi = 0$, that is, for $p = p_0$. Accordingly, we can conclude that

$$\widetilde{\psi}^{-1}\big(P_0 \wedge \hat{P}_0\big) = \Big\{ \big[(g,p_0)\big] : g \in G_{\xi_{m-1,k-1}} \Big\}\,.$$

Therefore, since

$$G^{p_0}_{\xi_{m-1,k-1}} = \langle r_{00}, r_{01} \rangle = \left\{ \left(\mathbf{R}^{(34)}_{\frac{2\pi}{m}} \right)^{\beta} \cdot r_{00}^{\gamma} : \beta \in \mathbb{Z}_m,\ \gamma \in \mathbb{Z}_2 \right\} \tag{2.12}$$

by Definition (2.1.3), the multiplicity of $P_0 \wedge \hat{P}_0$ is given by

$$\mu_{\widetilde{\psi}}\big(P_0 \wedge \hat{P}_0\big) = \Big|\Big\{\big[(g,p_0)\big] : g \in G_{\xi_{m-1,k-1}}\Big\}\Big| = \frac{|G_{\xi_{m-1,k-1}}|}{\big|G^{p_0}_{\xi_{m-1,k-1}}\big|} = \frac{2mk}{2m} = k\,.$$

Analogous steps lead to the conclusion that at the image point

$$Q_0 \wedge \big(-\hat{Q}_0\big) = \widetilde{\psi}\Big(\big[(e,q_0)\big]\Big)$$

the multiplicity is

$$\mu_{\widetilde{\psi}}\Big(Q_0 \wedge \big(-\hat{Q}_0\big)\Big) = \Big|\Big\{\big[(g,q_0)\big] : g \in G_{\xi_{m-1,k-1}}\Big\}\Big| = \frac{|G_{\xi_{m-1,k-1}}|}{\big|G^{q_0}_{\xi_{m-1,k-1}}\big|} = \frac{2mk}{2k} = m\,.$$

The question is whether these higher multiplicities result from a covering map on S under which $\widetilde{\psi}$ is invariant. Clarification is obtained by considering the associated tangent planes to the bipolar surface at $P_0 \wedge \hat{P}_0$ or $Q_0 \wedge \big(-\hat{Q}_0\big)$, which would coincide in that case.

Before we continue, note that since we assumed that $\chi(\xi_{m-1,k-1}) < 0$, the polar variety $\xi^*_{m-1,k-1}$ has singular points precisely at the vertices of $\Gamma_{\xi_{m-1,k-1}}$ with interior angles $< \frac{\pi}{2}$ and all their G-copies ([37], p. 361). At such points, we have $K = 1$ and thus the second term in

$$\mathrm{d}\widetilde{\psi}(v) = \mathrm{d}\psi(v) \wedge \psi^* + \psi \wedge \mathrm{d}\psi^*(v)$$

vanishes for all $v \in T\Sigma$ as $g^*(v,v) = 0$ (cf. (1.22)).

Now, suppose that $m > 2$ and denote by $P^{(0)}_{p_0}$ the tangent plane to the bipolar surface at $\widetilde{\psi}\big(\big[(e,p_0)\big]\big)$. Since ψ^* has a singular point at $\big[(e,p_0)\big]$ and the tangent plane to $\xi_{m-1,k-1}$ at $\psi\big(\big[(e,p_0)\big]\big)$ is spanned by e_3 and e_4, we find that

$$P^{(0)}_{p_0} = \operatorname{span}\big(e_2 \wedge e_3, e_2 \wedge e_4\big)\,.$$

Consequently, recalling (2.12), the k tangent planes at $\widetilde{\psi}\Big(\Big[\Big(\Big(\mathbf{R}^{(12)}_{\frac{2\pi}{k}}\Big)^{\alpha}, p_0\Big)\Big]\Big)$ are

$$P^{(\alpha)}_{p_0} := \operatorname{span}\left(\Big(\mathbf{R}^{(12)}_{\frac{2\pi}{k}}\Big)^{\alpha}(e_2) \wedge \Big(\mathbf{R}^{(12)}_{\frac{2\pi}{k}}\Big)^{\alpha}(e_3), \Big(\mathbf{R}^{(12)}_{\frac{2\pi}{k}}\Big)^{\alpha}(e_2) \wedge \Big(\mathbf{R}^{(12)}_{\frac{2\pi}{k}}\Big)^{\alpha}(e_4)\right)$$
$$\text{for } \alpha \in \mathbb{Z}_k\,,$$

and a computation shows that $P^{(\alpha)}_{p_0} = P^{(0)}_{p_0}$ if and only if $\alpha = 0$ or, when k is even, $\alpha = \frac{k}{2}$. Analogously, if $k > 2$, the m tangent planes to the bipolar surface at $Q_0 \wedge \big(-\hat{Q}_0\big) = f(q_0) \wedge n(q_0)$ are given by

$$P^{(\beta)}_{q_0} := \operatorname{span}\left(\Big(\mathbf{R}^{(34)}_{\frac{2\pi}{m}}\Big)^{\beta}(e_1) \wedge \Big(\mathbf{R}^{(34)}_{\frac{2\pi}{m}}\Big)^{\beta}(e_4), \Big(\mathbf{R}^{(34)}_{\frac{2\pi}{k}}\Big)^{\beta}(e_2) \wedge \Big(\mathbf{R}^{(34)}_{\frac{2\pi}{m}}\Big)^{\beta}(e_4)\right)$$
$$\text{for } \beta \in \mathbb{Z}_m\,,$$

with $P^{(\beta)}_{q_0} = P^{(0)}_{q_0}$ if and only if $\beta = 0$ or, when m is even, $\beta = \frac{m}{2}$.

Whenever m or k is odd, this shows that the bipolar surface has μ transversally intersecting tangent planes at some point of multiplicity $\mu > 1$. Therefore, that S must be a smallest possible domain of $\widetilde{\psi}$ and hence,

$$\chi\Big(\widetilde{\xi}_{m-1,k-1}\Big) = \chi(S) = 2 - 2(m-1)(k-1)\,,$$

where the latter is implied by (2.7). If both m and k are even, we have

$$-\mathbb{1}_4 = \Big(\mathbf{R}^{(12)}_{\frac{2\pi}{k}}\Big)^{\frac{k}{2}} \cdot \Big(\mathbf{R}^{(34)}_{\frac{2\pi}{m}}\Big)^{\frac{m}{2}} \in G_{\xi_{m-1,k-1}}$$

and in particular

$$\sigma(-\mathbb{1}_4) = 0\,.$$

In this light, the occurrence of the pairs of parallel tangent planes is due to the covering map from Theorem 2.2.2 (i). Since $m > 2$ or $k > 2$, at least for one of the considered image points from above, the planes corresponding to distinct pairs intersect transversally. So, also in this case, we deduce that $S/\langle -\mathbb{1}_4 \rangle$ is a smallest possible domain of $\widetilde{\psi}$ and we have

$$\chi\Big(\widetilde{\xi}_{m-1,k-1}\Big) = \chi\big(S/\langle -\mathbb{1}_4 \rangle\big) = \frac{\chi(S)}{2} = 1 - (m-1)(k-1)\,.$$

It remains to prove the area bounds. The lower bounds are obtained by the Li-Yau inequality (cf. Theorem 6 in [42], combined with Proposition 1.2.3 from [35]) applied to the vertex points we studied above. If both m and k are even, these are points of multiplicity $\frac{m}{2}$ and $\frac{k}{2}$. Otherwise, the detected multiplicities are m and k. Furthermore, together with the findings from above, the upper bounds on the area are a direct result of the area formula (2.3) for $\widetilde{\psi}$ defined on the smallest possible S (if m or k is odd) or $S/\langle -\mathbb{1}_4 \rangle$ (if both m and k are even) and the area bounds for the surface $\xi_{m-1,k-1}$ of Rob Kusner (cf. Proposition 3.2 in [34]), that is,

$$\operatorname{area}(\xi_{m-1,k-1}) < 4\pi k\,. \qquad \square$$

Finally, since we detected transversally intersecting tangent planes for the surface $\widetilde{\xi}_{m-1,k-1}$ in the preceding proof, the following conclusion is immediate.

Corollary 2.3.3. *Let m, $k \in \mathbb{Z}_{\geq 2}$ such that $m > 2$ or $k > 2$. Then, the bipolar surface $\widetilde{\xi}_{m-1,k-1} \subseteq \mathbb{S}^5$ is not embedded.*

Remark 2.3.4. (i) The example of the surfaces $\widetilde{\xi}_{m,k}$ demonstrates that, contrary to $\widetilde{\tau}_{3,1}$ (compare Section 2.3.4), which is embedded but originates from a surface in $\mathbb{S}^3$ with self-intersections, the opposite scenario can also happen.

(ii) Recall that by Remark 1.4.3 (iii) the surfaces $\widetilde{\xi}_{m,k}$ (except for $\widetilde{\xi}_{1,1}$, which is again a Clifford torus, as discussed in Remark 3.3.3 (ii)) lie non-degenerately in $\mathbb{S}^5$, i.e., they cannot be viewed as minimal surfaces in $\mathbb{S}^4$.

We conclude this section with some reflections on potential future developments that may be based on the preceding framework.

Remark 2.3.5. (i) Note that in [29], the symmetry groups of the ξ-family in $\mathbb{S}^3$ and corresponding subgroups were deeply studied. In particular, it was shown that any closed, embedded minimal surface in $\mathbb{S}^3$ with the symmetries as a Lawson surface $\xi_{m,k}$ has the same genus mk and is congruent to $\xi_{m,k}$. As a future perspective, it would be interesting to know whether this allows to classify the bipolar surfaces $\widetilde{\xi}_{m,k}$ by their symmetry group.

(ii) In [28], the index and the nullity of the Lawson surfaces $\xi_{g,1}$ was determined. According to [37] (Chapter 11), this seems closely related with the concept of bipolar minimal immersions, whose components are the restrictions of Jacobi fields of infinitesimal rotations in $\mathbb{S}^3$ (see also Remark 1.4.3 (iii)). In that context, it might be worth exploring how the aforementioned results can be interpreted.

(iii) In [21], new area estimates for the surfaces $\xi_{1,g}$ of genus g with $g \gg 1$ have been specified. These could also be considered in the context of area estimates for bipolar surfaces now.

2.3.3 The $\widetilde{\eta}$-Family

We proceed with the η-family. Again, let m, $k \geq 2$ such that $m > 2$ or $k > 2$. Hereby, we exclude the surface $\eta_{1,1} = \tau_{2,1}$, which we will treat in the context of the τ-family. The surface $\eta_{m-1,k-1}$ is based on the geodesic polygon

$$\Gamma_{\eta_{m-1,k-1}} := Q_0 P_1 Q_1 [P_0](-Q_1)\,,$$

where the arc of length π connecting Q_1 and $-Q_1$ passes through the point P_0. Note that $\Gamma_{\eta_{m-1,k-1}}$ has interior angles $\frac{\pi}{m}$, $\frac{\pi}{k}$ and two of $\frac{\pi}{2}$. Furthermore, it includes two arcs of length $\frac{\pi}{2}$, one arc of length π and one of length $\frac{(m-1)\pi}{m}$. Its group generated by the corresponding geodesic reflections is given by

$$G_{\eta_{m-1,k-1}} = \langle r_{10}, r_{11}, r_{01}, r_Q \rangle\,, \tag{2.13}$$

where r_Q denotes the geodesic reflection at the great circle γ_Q. To study $G_{\eta_{m-1,k-1}}$ in more detail, we have to distinguish between the cases where k is even and where k is odd.

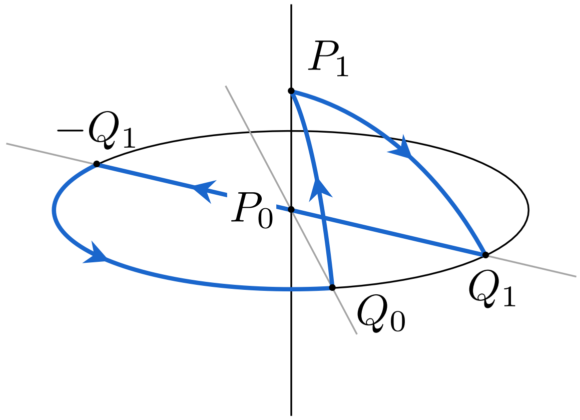

Figure 2.5: In stereographic projection, this graphic shows the type of geodesic polygon $\Gamma_\eta \subseteq \mathbb{S}^3$ used to construct a surface in the η-family.

At first, let k be even. Since

$$(r_{01} \cdot r_{11})^{\frac{k}{2}} = \left(\mathbf{R}^{(12)}_{\frac{2\pi}{k}}\right)^{\frac{k}{2}} = r_Q\,, \tag{2.14}$$

r_Q can be dropped as a generator. Moreover, this relation reads as

$$\mathbb{1}_4 = (r_{11} \cdot r_{01})^{\frac{k}{2}} \cdot r_Q$$

and hence, by Proposition 2.1.8, shows that $\eta_{m-1,k-1}$ is non-orientable.

Now, analogous to the case of the ξ-family, it follows that

$$\begin{aligned} G_{\eta_{m-1,k-1}} &= \left\langle \mathbf{R}^{(12)}_{\frac{2\pi}{k}}, \mathbf{R}^{(34)}_{\frac{2\pi}{m}}, r_{00} \right\rangle \\ &= \left\{ \left(\mathbf{R}^{(12)}_{\frac{2\pi}{k}}\right)^{\alpha} \cdot \left(\mathbf{R}^{(34)}_{\frac{2\pi}{m}}\right)^{\beta} \cdot r_{00}^{\gamma} : \alpha \in \mathbb{Z}_k,\ \beta \in \mathbb{Z}_m,\ \gamma \in \mathbb{Z}_2 \right\}. \end{aligned} \tag{2.15}$$

In particular, we have

$$|G_{\eta_{m-1,k-1}}| = 2mk \qquad \text{(for } k \text{ even)}$$

and therefore, by (2.1),

$$\chi(\eta_{m-1,k-1}) = 2mk\left(1 - \frac{m-1}{2m} - \frac{k-1}{2k} - \frac{1}{2}\right)$$
$$= 1 - (m-1)(k-1) \qquad (\text{for } k \text{ even})\,, \tag{2.16}$$

as given in [37].

Let now k be odd. In this case, the generator r_Q cannot be omitted, leading to the conclusion that $\eta_{m-1,k-1}$ is orientable due to Proposition 2.1.8. By the fact that r_Q commutes with all the other generators from (2.13), we have

$$G_{\eta_{m-1,k-1}} \cong \langle r_Q \rangle \times \langle r_{10}, r_{11}, r_{01} \rangle\,,$$

i.e.,

$$G_{\eta_{m-1,k-1}} = \left\{ r_Q^{\gamma} \cdot \left(\mathbf{R}^{(12)}_{\frac{2\pi}{k}}\right)^{\alpha} \cdot \left(\mathbf{R}^{(34)}_{\frac{2\pi}{m}}\right)^{\beta} \cdot r_{00}^{\delta} : \alpha \in \mathbb{Z}_k,\ \beta \in \mathbb{Z}_m,\ \gamma,\ \delta \in \mathbb{Z}_2 \right\}. \tag{2.17}$$

So here, we have

$$|G_{\eta_{m-1,k-1}}| = 4mk \qquad (\text{for } k \text{ odd})\,, \tag{2.18}$$

and (2.1) implies that

$$\chi(\eta_{m-1,k-1}) = 4mk\left(1 - \frac{m-1}{2m} - \frac{k-1}{2k} - \frac{1}{2}\right)$$
$$= 2 - 2(m-1)(k-1) \qquad (\text{for } k \text{ odd})\,, \tag{2.19}$$

as noted in [37]. Moreover, the parity $\sigma(g)$ of

$$g = r_Q^{\gamma} \cdot \left(\mathbf{R}^{(12)}_{\frac{2\pi}{k}}\right)^{\alpha} \cdot \left(\mathbf{R}^{(34)}_{\frac{2\pi}{m}}\right)^{\beta} \cdot r_{00}^{\delta} \in G_{\eta_{m-1,k-1}}\,,$$

is given by

$$\sigma(g) = \gamma + \delta\,. \tag{2.20}$$

We move to the next point on our checklist.

Lemma 2.3.6. *Let m, $k \in \mathbb{Z}_{\geq 2}$ and assume that $m > 2$ or $k > 2$. Then, the subgroup $H_{\eta_{m-1,k-1}} \subseteq G_{\eta_{m-1,k-1}}$ leaving the polygon $\Gamma_{m-1,k-1}$ invariant is trivial, i.e.,*

$$H_{\eta_{m-1,k-1}} = \{\mathbb{1}_4\}\,.$$

Proof. In fact, regardless of the parities of m and k, we have that the subgroup of $\mathrm{O}(4)$ which leaves

$$\Gamma_{\eta_{m-1,k-1}} = Q_0 P_1 Q_1 [P_0](-Q_1)$$

invariant as a point set is trivial. To see this, one can first consider the possible images under a symmetry of $Q_1[P_0](-Q_1)$, the only arc of length π, and then images of the piece $Q_0P_1Q_1$ if $m > 2$, or images of $P_1Q_1[P_0](-Q_1)$ if $k > 2$. □

We finally arrive at the topological classification of the bipolar $\widetilde{\eta}$-family.

Theorem 2.3.7. *Let m, $k \in \mathbb{Z}_{\geq 2}$ such that $m > 2$ or $k > 2$. Then, the bipolar surface $\widetilde{\eta}_{m-1,k-1} \subseteq \mathbb{S}^5$ is orientable. Moreover,*

(i) *if both m and k are even, we have*

$$\chi\Big(\widetilde{\eta}_{m-1,k-1}\Big) = 1 - (m-1)(k-1)\,,$$
$$2\pi \max\{m,k\} \leq \mathrm{area}\Big(\widetilde{\eta}_{m-1,k-1}\Big) < 2\pi(3mk - 3k - m)\,;$$

(ii) *if m or k is odd, we have*

$$\chi\Big(\widetilde{\eta}_{m-1,k-1}\Big) = 2\big(1 - (m-1)(k-1)\big)\,,$$
$$4\pi \max\{m,k\} \leq \mathrm{area}\Big(\widetilde{\eta}_{m-1,k-1}\Big) < 4\pi(3mk - 3k - m)\,.$$

Remark 2.3.8. Note that the whole $\widetilde{\eta}$-family (including the bipolar surface of $\eta_{1,1} = \tau_{2,1}$) is orientable. So, the (almost) complete list of closed, non-orientable minimal surfaces from $\mathbb{S}^3$ does not carry over to $\mathbb{S}^5$.

Proof of Theorem 2.3.7. We proceed analogously to the proof of Theorem 2.3.2. The initial Gauß map $n : \Delta \to \mathbb{S}^3$ can be chosen such that

$$n\big(f^{-1}(Q_0)\big) = \hat{P}_1\,, \qquad n\big(f^{-1}(P_1)\big) = -\hat{P}_1\,,$$
$$n\big(f^{-1}(Q_1)\big) = \hat{Q}_1\,, \qquad n\big(f^{-1}(-Q_1)\big) = \hat{P}_0\,,$$

at the vertices of $\Gamma_{\eta_{m-1,k-1}}$. Connected by shortest arcs, these values describe the polar polygon $\Gamma^*_{\eta_{m-1,k-1}}$ of $\Gamma_{\eta_{m-1,k-1}}$, where we note that the arc from $\hat{P}_1$ to $-\hat{P}_1$ runs across $-\hat{Q}_0$.

If k is even, then $\eta_{m-1,k-1}$ is non-orientable and we use the notation from Construction 2.1.13. In this setting, the multiplicity of an image point $\widetilde{\psi}\Big(\big[(0,e,p_0)\big]\Big)$ is determined by the numbers of solutions $\big[(s,g,p)\big] \in \widetilde{S}$ of

$$f(p) \wedge n(p) = (-1)^s g\big(f(p_0)\big) \wedge g\big(n(p_0)\big)\,. \tag{2.21}$$

Now, if $p_0 \in \partial\Delta$ is such that $f(p_0) = P_1$ or $f(p_0) = Q_1$, then the characterization of the group from (2.15) implies that (2.21) is equivalent to

$$f(p_0) \wedge n(p_0) = (-1)^{s+\gamma} f(p_0) \wedge n(p_0)\,.$$

Consequently, we have

$$\mu_{\widetilde{\psi}}\left(\widetilde{\psi}\Big(\big[(0,e,p_0)\big]\Big)\right) = \frac{1}{2}\cdot\frac{2\cdot|G_{\eta_{m-1,k-1}}|}{\Big|G^{p_0}_{\eta_{m-1,k-1}}\Big|}\,,$$

i.e.,

$$\mu_{\widetilde{\psi}}\Big(P_1 \wedge \big(-\hat{P}_1\big)\Big) = k\,, \qquad \mu_{\widetilde{\psi}}\Big(Q_1 \wedge Q_1\Big) = m\,.$$

Considering the different tangent planes at these points of higher multiplicity and, additionally, using Theorem 2.2.2 (ii) if m is even yields that smallest possible domains of $\widetilde{\psi}$ are given by $\widetilde{S}$ when m is odd, and by $\widetilde{S}/\langle -\mathbb{1}_4\rangle$ when m is even.

In turn, if k is odd, then $\eta_{m-1,k-1}$ is orientable and we are in the setting of Construction 2.1.4 and Construction 2.1.11. Looking at the image point

$$P_1 \wedge \big(-\hat{P}_1\big) = -e_1 \wedge e_2$$

and using (2.17) as well as (2.20), it follows that S is a fundamental domain for $\widetilde{\psi}$.

Thereby, in each of the above cases, the Euler characteristic of $\widetilde{\eta}_{m-1,k-1}$ can be computed by (2.16) and (2.19).

To finish the proof, the area bounds follow by the Li-Yau inequality, by the area formula from (2.3) (considered on the orientable double cover $\overline{S}$ in each of the non-orientable cases) and by Proposition 3.4 from [34], i.e.,

$$\operatorname{area}(\eta_{m-1,k-1}) < 2\pi(m-1)k \qquad \text{if } k \text{ is even},$$

which we completed by

$$\operatorname{area}(\eta_{m-1,k-1}) < 4\pi(m-1)k \qquad \text{if } k \text{ is odd}.$$

The latter follows analogously as in [34], that is, by the bound for the initial minimal disk, multiplied by the order of the group generated by Schwarz reflections (which is twice the order of the former case due to (2.18)). □

Ultimately, we arrive at the following.

Corollary 2.3.9. *Let m, $k \in \mathbb{Z}_{\geq 2}$ such that $m > 2$ or $k > 2$. Then, the bipolar surface $\widetilde{\eta}_{m-1,k-1} \subseteq \mathbb{S}^5$ is not embedded.*

We finish here with some remarks on future applications of the mechanism applied in the proofs of Theorem 2.3.2 and Theorem 2.3.7.

Remark 2.3.10. (i) Additional insights on topology and embeddedness of bipolar minimal surfaces could be gained by studying the bipolar surfaces of other closed minimal surfaces in $\mathbb{S}^3$ generated through reflection processes. It may be necessary to refine the above approach for such considerations. Potential candidates to start with include the surfaces of Karcher, Pinkall and Sterling (cf. [31]), the surfaces of Choe and Soret (cf. [13]) or, more recently, the surfaces of Bobenko, Heller and Schmitt (cf. [5]).

(ii) From Proposition 2.4 in [11], which can be understood in the context of Chapter 3, it follows that bipolar minimal immersions with a closed, orientable smallest possible domain of genus > 1 must necessarily have self-intersections. As we will address in the next section, note that the bipolar Lawson surface $\widetilde{\tau}_{3,1}$ is an embedded Klein bottle. It remains open whether there are closed, non-orientable bipolar minimal surfaces of higher genera and whether such examples could be embedded.

In fact, as demonstrated by the above characterization the $\widetilde{\eta}$-family and the characterization of the $\widetilde{\tau}$-family in [36], not even non-orientable minimally immersed surfaces in $\mathbb{S}^3$ necessarily lead to non-orientable bipolar surfaces. In this context, one could study the general characteristics that construction data must possess to generate non-orientable bipolar surfaces.

2.3.4 The $\widetilde{\tau}$-Family

We now focus on Lawson's τ-family and consider a surface $\tau_{m,k}$, where m, $k \in \mathbb{Z}_{\geq 1}$. Following Lawson's construction procedure, this surface is obtained by the right-angled geodesic polygon

$$\Gamma_{\tau_{m,k}} := P_0 P_1 Q_0 Q_1$$

including one arc of length $\frac{\pi}{m}$, one of length $\frac{\pi}{k}$ and two arcs of length $\frac{\pi}{2}$. For $m = 1$ or $k = 1$, the arc of length π between Q_0 and Q_1 or P_0 and P_1 lies on γ_Q or γ_P, respectively.

Note that, without loss of generality, we can assume that $\gcd(m, k) = 1$. Otherwise, congruent surfaces can be obtained using different pairs of m and k (a well-known fact that can be seen from the parametrizations at the final part of this

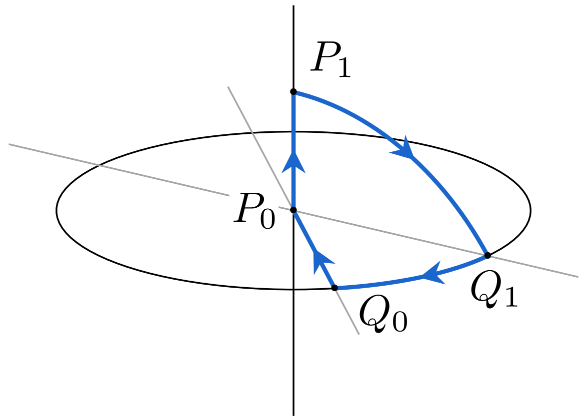

Figure 2.6: In stereographic projection, this graphic illustrates the type of geodesic polygon Γ_τ for the construction of a surface in the τ-family.

section). Except for $\tau_{1,1}$, which is the Clifford torus $\mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2}) \subseteq \mathbb{S}^3$, we can additionally assume that $m > k \geq 1$ for the same reason.

Now, (2.1) directly leads to

$$\chi(\tau_{m,k}) = 0\,,$$

so, depending on whether the surface is orientable or not, $\tau_{m,k}$ is an immersed torus or an immersed Klein bottle.

The group generated by the corresponding geodesic reflections is

$$G_{\tau_{m,k}} = \langle r_P, r_Q, r_{00}, r_{11} \rangle\,,$$

where r_P and r_Q are the reflections at the great circles γ_P and γ_Q. For an appropriate labeling of the group elements, similar to the ξ- and η-family, we have to distinguish between the case where both m and k are odd and the case where m or k is even. Before we start, recall that

$$r_{11} = r_{00} \cdot \mathbf{R}^{(12)}_{\frac{2\pi}{k}} \cdot \mathbf{R}^{(34)}_{\frac{2\pi}{m}}\,,$$

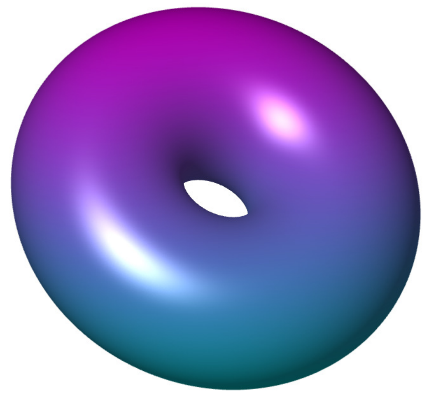

Figure 2.7: Stereographic projection of $\tau_{1,1}$, which is the Clifford torus $\mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2}) \subseteq \mathbb{S}^3$ (produced with MATLAB)

and, due to (2.5),

$$r_{11} = \mathbf{R}^{(12)}_{-\frac{2\pi}{k}} \cdot \mathbf{R}^{(34)}_{-\frac{2\pi}{m}} \cdot r_{00} = R_2^{-1} r_{00}\,, \tag{2.22}$$

where we define

$$R_2 := \mathbf{R}^{(12)}_{\frac{2\pi}{k}} \cdot \mathbf{R}^{(34)}_{\frac{2\pi}{m}} = r_{00} \cdot r_{11}\,.$$

Now, at first, suppose that both m and k are odd. In this case,

$$\left(\mathbf{R}^{(12)}_{\frac{2\pi}{k}}\right)^n \neq r_Q\,, \qquad \left(\mathbf{R}^{(34)}_{\frac{2\pi}{m}}\right)^n \neq r_P \qquad \text{for all } n \in \mathbb{Z}\,.$$

Hence, as $\operatorname{lcm}(m,k) = mk$, we derive that

$$G_{\tau_{m,k}} \cong \langle r_P, r_Q \rangle \times \left\{ R_2^n \cdot r_{00}^{\gamma} : n \in \mathbb{Z}_{mk},\ \gamma \in \mathbb{Z}_2 \right\} \tag{2.23}$$

and in this sense label a group element $g \in G_{\tau_{m,k}}$ by α, β, $\gamma \in \mathbb{Z}_2$ and $n \in \mathbb{Z}_{mk}$ such

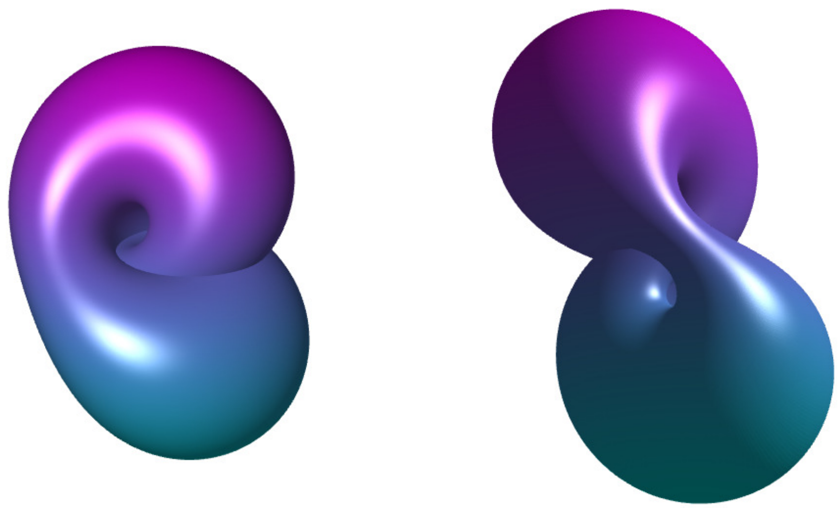

Figure 2.8: Examples of stereographically projected surfaces in the Lawson family $\tau_{m,k}$: The Klein bottle $\tau_{2,1}$ on the left and the torus $\tau_{3,1}$ on the right (produced with MATLAB)

that

$$g = r_P^{\alpha} \cdot r_Q^{\beta} \cdot R_2^n \cdot r_{00}^{\gamma} .$$

We observe that

$$|G_{\tau_{m,k}}| = 8mk \qquad (\text{for both } m \text{ and } k \text{ odd}) .$$

Next, we determine the subgroup $H_{\tau_{m,k}} \subseteq G_{\tau_{m,k}}$ which leaves the polygon $\Gamma_{\tau_{m,k}}$ invariant as a set.

Lemma 2.3.11. *Let m, $k \in \mathbb{Z}_{\geq 1}$ such that* $\gcd(m,k) = 1$*. Moreover, suppose that both m and k are odd. Then, we have*

$$H_{\tau_{m,k}} = \left\{ \mathbb{1}_4, r_P \cdot r_Q \cdot R_2^{\frac{mk+1}{2}} \cdot r_{00} \right\}$$

if $m > 1$ or $k > 1$ and

$$H_{\tau_{1,1}} = \left\{ \mathbb{1}_4, r_P \cdot r_Q, r_{00}, r_P \cdot r_Q \cdot r_{00} \right\}.$$

Remark 2.3.12. As a map, the group element

$$r_P \cdot r_Q \cdot R_2^{\frac{mk+1}{2}} \cdot r_{00} = \mathbf{R}^{(12)}_{\frac{\pi}{k}} \cdot \mathbf{R}^{(34)}_{\frac{\pi}{m}} \cdot r_{00}$$

corresponds to the geodesic reflection across the great circle through the points

$$P_{\frac{1}{2}} := \begin{pmatrix} \cos\left(\frac{\pi}{2k}\right) \\ \sin\left(\frac{\pi}{2k}\right) \\ 0 \\ 0 \end{pmatrix}, \qquad Q_{\frac{1}{2}} := \begin{pmatrix} 0 \\ 0 \\ \cos\left(\frac{\pi}{2m}\right) \\ \sin\left(\frac{\pi}{2m}\right) \end{pmatrix}.$$

Proof of Lemma 2.3.11. For the moment, suppose that $m > 1$ or $k > 1$. Then, without loss of generality, we can assume that $m > k \geq 1$.

Due to the block diagonal form of elements in $G_{\tau_{m,k}}$ and the included arcs, $g \in H_{\tau_{m,k}}$ if and only if g preserves the union of the arcs P_0Q_0 and P_1Q_1. Equivalently, this can be stated as

$$\text{either} \qquad g(P_0) = P_0\,, \quad g(Q_0) = Q_0\,, \quad g(P_1) = P_1\,, \quad g(Q_1) = Q_1 \tag{2.24}$$

$$\text{or} \qquad g(P_0) = P_1\,, \quad g(Q_0) = Q_1\,, \quad g(P_1) = P_0\,, \quad g(Q_1) = Q_0\,. \tag{2.25}$$

Now, suppose that $g \in H_{\tau_{m,k}}$. At first, we have

$$g(P_0) = \left(r_P^{\alpha} \cdot r_Q^{\beta} \cdot R_2^{n} \cdot r_{00}^{\gamma}\right)(P_0) = (-1)^{\beta} P_{2n}\,.$$

Then, the above conditions demand that

$$\text{either} \qquad (-1)^\beta P_{2n} = P_0 \qquad \text{or} \qquad (-1)^\beta P_{2n} = P_1\,.$$

The first case reads as

$$2n + \beta \cdot k \equiv 0 \mod 2k\,.$$

Since k is odd, we must have $\beta = 0$ and hence $n \in k\mathbb{Z}$. Moreover,

$$\big(r_P^\alpha \cdot R_2^n \cdot r_{00}^\gamma\big)(P_1) = r_{00}^\gamma(P_1)$$

implies that $\gamma = 0$. Otherwise, if

$$2n + \beta \cdot k \equiv 1 \mod 2k\,,$$

it follows that $\beta = 1$ and $n \in \frac{k+1}{2}\mathbb{Z}$. From

$$\big(r_P^\alpha \cdot r_Q^\beta \cdot R_2^n \cdot r_{00}^\gamma\big)(P_1) = -\big(R_2^n \cdot r_{00}^\gamma\big)(P_1)$$

we derive that $\gamma = 1$. Until now, we know that g could be of the form

$$g = r_P^\alpha \cdot R_2^{n_1} \qquad \text{for } n_1 \in k\mathbb{Z}$$

$$\text{or}$$

$$g = r_P^\alpha \cdot r_Q \cdot R_2^{n_2} \cdot r_{00} \qquad \text{for } n_2 \in \tfrac{k+1}{2}\mathbb{Z}\,.$$

In order to determine α and to restrict n_1, n_2 to values in $\mathbb{Z}_{mk}$, we consider the conditions (2.24) and (2.25) for Q_0 and Q_1. At first, suppose that

$$g = r_P^\alpha \cdot R_2^{n_1}\,, \quad n_1 \in k\mathbb{Z}\,.$$

Then,

$$g(Q_0) = \big(r_P^\alpha \cdot R_{2n}^{n_1}\big)(Q_0) = Q_0$$

implies that $\alpha = 0$ and $n_1 \in \mathbb{Z}_{mk}$ and hence $g = \mathbb{1}_4$. Otherwise, if

$$g(Q_0) = \big(r_P^\alpha \cdot R_2^{n_1}\big)(Q_0) = Q_1\,,$$

we have $\alpha = 1$ and

$$\begin{aligned} n_1 &\equiv 0 \mod k\,, \\ n_1 &\equiv \frac{m+1}{2} \mod m\,. \end{aligned}$$

However, as we assumed that $m > 1$, this would yield that

$$g(Q_1) = Q_2 \neq Q_0\,.$$

So, instead, we check the possibility that

$$g = r_P^\alpha \cdot r_Q \cdot R_2^{n_2}\,, \quad n_2 \in \tfrac{k+1}{2}\mathbb{Z}\,.$$

In this case,

$$g(Q_0) = \big(r_P^\alpha \cdot r_Q \cdot R_2^{n_2} \cdot r_{00}\big)(Q_0) = Q_0$$

yields $\alpha = 0$ and

$$\begin{aligned} n_2 &\equiv \frac{k+1}{2} \mod k\,, \\ n_2 &\equiv 0 \mod m\,. \end{aligned}$$

But since in addition

$$g(Q_1) = Q_{-1} \neq Q_1\,.$$

this form is not allowed due to (2.24) and (2.25). Finally, checking

$$g(Q_0) = \big(r_P^{\alpha} \cdot r_Q \cdot R_2^{n_2} \cdot r_{00}\big)(Q_0) = Q_1\,,$$

we obtain $\alpha = 1$ and

$$\begin{aligned} n_2 &\equiv \frac{k+1}{2} \mod k\,, \\ n_2 &\equiv \frac{m+1}{2} \mod m\,. \end{aligned}$$

This also yields that $g(Q_1) = Q_0$. And therefore,

$$g = r_P \cdot r_Q \cdot R_2^{n} \cdot r_{00} \in H_{\tau_{m,k}}\,,$$

for n such that

$$\begin{aligned} n &\equiv \frac{k+1}{2} \mod k\,, \\ n &\equiv \frac{m+1}{2} \mod m\,, \end{aligned}$$

which is only satisfied by $n = \frac{mk+1}{2} \in \mathbb{Z}_{mk}$.

For the case of $\tau_{1,1}$, note that $R_2 = \mathbb{1}$. Hence,

$$G_{\tau_{1,1}} \cong \langle r_P, r_Q \rangle \times \langle r_{00} \rangle\,.$$

In this case, we directly apply each group element to $\Gamma_{\tau_{1,1}} = P_0(-P_0)(-Q_0)Q_0$ and check when (2.24) or (2.25) is satisfied, proving the assertion. □

Now, by the above lemma, an analysis based on Section 2.1 and Section 2.2 is only possible if we choose a smaller initial piece of surface to continue, for example the one bounded by the geodesic polygon

$$\hat{\Gamma}_{\tau_{m,k}} = P_0 P_{\frac{1}{2}} Q_{\frac{1}{2}} Q_0\,.$$

Using analogous methods as above, we derive that the corresponding group generated by geodesic reflections is

$$\hat{G}_{\tau_{m,k}} \cong \langle r_P, r_Q \rangle \times \left\{ \hat{R}_2^n r_{00}^\gamma : n \in \mathbb{Z}_{2mk},\, \gamma \in \mathbb{Z}_2 \right\},$$

where we defined

$$\hat{R}_2 := \mathbf{R}^{(12)}_{\frac{\pi}{k}} \mathbf{R}^{(34)}_{\frac{\pi}{m}}.$$

In this sense, we label a group element $g \in \hat{G}_{\tau_{m,k}}$ by

$$g = r_P^\alpha \cdot r_Q^\beta \cdot \hat{R}_2^n \cdot r_{00}^\gamma \qquad \text{for } \alpha,\, \beta,\, \gamma \in \mathbb{Z}_2\,, n \in \mathbb{Z}_{2mk}\,.$$

Then, as orientability is ensured similarly as for the previous families, the parity of $g \in \hat{G}_{\tau_{m,k}}$ reads as

$$\sigma(g) = (-1)^{\alpha+\beta+\gamma}\,. \tag{2.26}$$

Before we continue, note that, purely as groups,

$$\hat{G}_{\tau_{m,k}} = G_{\tau_{m,k}}\,.$$

Expressed heuristically, changing the initial piece of surface as above, the number of group copies of the initial piece remains the same – but different pieces of the surface $\mathcal{M}_\Gamma$ are now uniquely described in terms of group elements.

At this stage, we are ready to draw the following conclusion. Note that the topology and area of the surfaces $\widetilde{\tau}_{m,k}$ were completely specified in [36], within the spectral geometric context of the normalized Laplacian eigenvalue functionals. So, our intention is to compare the characterization resulting from our framework with these results.

Theorem 2.3.13. *Let both m, $k \in \mathbb{Z}_{\geq 1}$ be odd with $m > k \geq 1$ and* $\gcd(m,k) = 1$.

Then, the area of the bipolar surface $\widetilde{\tau}_{m,k}$ satisfies

$$\operatorname{area}\Big(\widetilde{\tau}_{m,k}\Big) \leq 4\pi m\, E\left(\frac{\sqrt{m^2-k^2}}{m}\right),$$

where

$$E(\kappa) = \int_0^{\frac{\pi}{2}} \sqrt{1-\kappa^2\sin^2(\vartheta)}\,\mathrm{d}\vartheta$$

denotes the complete elliptic integral of the second kind.

Proof. We have

$$-\mathbb{1}_4 = r_P \cdot r_Q \in \hat{G}_{\tau_{m,k}}$$

implying that $\sigma(-\mathbb{1}_4) = 0$. Therefore, Theorem 2.2.2 is applicable and yields that the surface $\widetilde{\tau}_{m,k}$ is at least doubly covered by S which is a smallest possible domain for $\tau_{m,k}$. Therefore, we can use the area formula (2.3) for $\widetilde{\psi}\colon S/\langle -\mathbb{1}_4\rangle \to \mathbb{S}^5$. Since $S/\langle -\mathbb{1}_4\rangle$ is not necessarily a smallest possible domain for $\widetilde{\tau}_{m,k}$, we thereby only arrive at an upper bound for the actual area. Concluded, this leads to

$$\operatorname{area}\Big(\widetilde{\tau}_{m,k}\Big) \leq \operatorname{area}(\tau_{m,k}).$$

Then, the assertion follows from the main theorem in [60], providing that the metric induced by $\tau_{m,k}$ is extremal for the j-th Laplacian eigenvalue functional $\Lambda_j\big(\mathbb{T}^2, g\big) = \lambda_j\big(\mathbb{T}^2, g\big)\operatorname{area}\big(\mathbb{T}^2, g\big)$, where the order j is determined as

$$j = 2\left\lfloor \frac{\sqrt{m^2+k^2}}{2} \right\rfloor + m + k - 1\,.$$

This result in particular includes the explicit value of $\Lambda_j(\tau_{m,k})$. As $\lambda_j = 2$ for minimal surfaces in $\mathbb{S}^n$ (cf. 1.1.9 and note the sign convention of the Laplacian in the theory of Laplacian eigenvalues), dividing $\Lambda_j(\tau_{m,k})$ by 2 yields

$$\mathrm{area}(\tau_{m,k}) = 4\pi m\, E\left(\frac{\sqrt{m^2-k^2}}{m}\right) \tag{2.27}$$

and so the upper bound follows. $\square$

Remark 2.3.14. Note that a similar reasoning as in the proofs of Theorems 2.3.2 and 2.3.7 leading to smallest fundamental domains and a statement on embeddedness is not a priori clear in the case of the $\widetilde{\tau}$-family. This is due to the forms of the polygon $\Gamma_{\tau_{m,k}}$ and the polar polygon

$$\Gamma^*_{\tau_{m,k}} = \hat{Q}_0\hat{Q}_1\big(-\hat{P}_1\big)\big(-\hat{P}_0\big)\,.$$

Here, it cannot simply be excluded that interior points or points within the edges of group copies of the initial pieces of surface are not mapped to the same point as for example $P_0 \wedge \hat{Q}_0$. In the case of the ξ- and η-family pairs of vertex points of the surface in $\mathbb{S}^3$ and its polar surface are distinguished by either lying on γ_P or γ_Q.

For the $\widetilde{\tau}$-family, clarity is provided by [36]. In this paper, Hugues Lapointe showed that also the $\widetilde{\tau}$-family yield extremal metrics for the Laplacian eigenvalues. Among other things, he demonstrated that

- for $mk = 1 \bmod 4$, $\widetilde{\tau}_{m,k}$ is a torus and

$$\mathrm{area}\Big(\widetilde{\tau}_{m,k}\Big) = \frac{1}{2}\Lambda_{2m-2}\Big(\widetilde{\tau}_{m,k}\Big) = \frac{1}{2}\Lambda_{2m+2}\Big(\widetilde{\tau}_{m,k}\Big) = 4\pi m E\left(\frac{\sqrt{m^2-k^2}}{m}\right),$$

- for $mk = 3 \bmod 4$, $\widetilde{\tau}_{m,k}$ is a Klein bottle and

$$\mathrm{area}\Big(\widetilde{\tau}_{m,k}\Big) = \frac{1}{2}\Lambda_{m-2}\Big(\widetilde{\tau}_{m,k}\Big) = \frac{1}{2}\Lambda_{m+2}\Big(\widetilde{\tau}_{m,k}\Big) = 2\pi m E\left(\frac{\sqrt{m^2-k^2}}{m}\right).$$

To compare with the $\widetilde{\xi}$ and the $\widetilde{\eta}$-family, this in particular implies that in the case $mk \equiv 1 \bmod 4$, the detected covering map due to $-\mathbb{1}_4 \in G_{\tau_{m,k}}$ is also sufficient to arrive at a smallest possible domain for $\widetilde{\tau}_{m,k}$. By Theorem 2.2.2 (i), this domain is orientable, confirming the classification from [36] for that case.

The case $mk \equiv 3 \mod 4$ requires further methods concerning a topological classification and specification of the area. It contains the perhaps most prominent example among the considered surfaces, the Klein bottle $\widetilde{\tau}_{3,1}$, which plays a crucial role in several classical geometric variational problems. First, it is the only critical metric on a Klein bottle for the first Laplacian eigenvalue (cf. [15] and [26]). By [23], it is furthermore conjectured to be, after stereographic projection, the unique minimizer of the Willmore energy among Klein bottles in $\mathbb{R}^n$, $n \geq 4$. Moreover, due to [11], it can be seen as the only Hamiltonian stable, minimal Lagrangian Klein bottle in $\mathbb{S}^2 \times \mathbb{S}^2$. For $\widetilde{\tau}_{3,1}$, the above estimate reads as

$$\operatorname{area}\left(\widetilde{\tau}_{3,1}\right) \leq 12\pi E\left(\frac{2\sqrt{2}}{3}\right).$$

In fact, Theorem 1.3.1 and Theorem 1.4.1 in [26] yield that actually

$$\operatorname{area}\left(\widetilde{\tau}_{3,1}\right) = 6\pi E\left(\frac{2\sqrt{2}}{3}\right) \approx 6.682\,\pi\,.$$

For a more detailed background on the additional findings from [36] when $mk \equiv 3 \mod 4$ we refer to the end of this section.

We proceed with the case when m or k is even (and consequently, the other must be odd as $\gcd(m,k) = 1$). Here, the product mk is even, leading to

$$R_2^{\frac{mk}{2}} = \mathbf{R}^{(12)}_{m\pi} \cdot \mathbf{R}^{(34)}_{k\pi} = \begin{cases} r_P & \text{if } m \text{ is even}\,, \\ r_Q & \text{if } k \text{ is even}\,. \end{cases}$$

Due to (2.22), we specifically have that

$$\begin{aligned} (r_{00} \cdot r_{11})^{mk} \cdot r_P &= \mathbb{1}_4 \qquad \text{if } m \text{ is even}\,, \\ (r_{00} \cdot r_{11})^{mk} \cdot r_Q &= \mathbb{1}_4 \qquad \text{if } k \text{ is even}\,, \end{aligned}$$

confirming that $\tau_{m,k}$ is non-orientable in these cases. Moreover, similarly as in the preceding considerations, we thereby conclude that

$$G_{\tau_{m,k}} \cong \begin{cases} \langle r_Q \rangle \times \left\{ R_2^n \cdot r_{00}^\gamma : n \in \mathbb{Z}_{mk},\, \gamma \in \mathbb{Z}_2 \right\} & \text{if } m \text{ is even}\,, \\ \langle r_P \rangle \times \left\{ R_2^n \cdot r_{00}^\gamma : n \in \mathbb{Z}_{mk},\, \gamma \in \mathbb{Z}_2 \right\} & \text{if } k \text{ is even}\,. \end{cases}$$

Accordingly, we have

$$|G_{\tau_{m,k}}| = 4mk \qquad (\text{for } m \text{ or } k \text{ even})\,.$$

An analogous analysis as in the proof of Lemma 2.3.11 based on (2.24) and (2.25), moreover yields the following for the subgroup $H_{\tau_{m,k}}$ that leaves $\Gamma_{\tau_{m,k}}$ invariant.

Lemma 2.3.15. *Let $m,\, k \in \mathbb{Z}_{\geq 1}$ such that* $\gcd(m,k) = 1$*. Moreover, suppose that m or k is even. Then,*

$$H_{\tau_{m,k}} = \{\mathbb{1}_4\}\,.$$

Now, since

$$-\mathbb{1}_4 = r_P \cdot r_Q = \begin{cases} r_Q \cdot R_2^{\frac{mk}{2}} & \text{if } m \text{ is even}\,, \\ r_P \cdot R_2^{\frac{mk}{2}} & \text{if } k \text{ is even}\,, \end{cases}$$

we can apply Theorem 2.2.2 (ii) and arrive at the following.

Theorem 2.3.16. *Let both $m,\, k \in \mathbb{Z}_{\geq 1}$ with $m > k \geq 1$ and* $\gcd(m,k) = 1$*. If m or k is even, the surface $\widetilde{\tau}_{m,k}$ satisfies*

$$\text{area}\Big(\widetilde{\tau}_{m,k}\Big) \leq 8\pi m\, E\left(\frac{\sqrt{m^2 - k^2}}{m}\right).$$

Proof. Due to the covering map from Theorem 2.2.2 (ii), we use the area formula

(2.3) on the orientable double cover $\overline{S}$ and derive

$$\operatorname{area}\left(\widetilde{\tau}_{m,k}\right) \leq 2\operatorname{area}(\tau_{m,k}) = 8\pi m\, E\left(\frac{\sqrt{m^2-k^2}}{m}\right).$$

from (2.27). □

Remark 2.3.17. For the case that mk is even, [36] shows that $\widetilde{\tau}_{m,k}$ is a torus and

$$\operatorname{area}\left(\widetilde{\tau}_{m,k}\right) = \frac{1}{2}\Lambda_{4m-2}\left(\widetilde{\tau}_{m,k}\right) = \frac{1}{2}\Lambda_{4m+2}\left(\widetilde{\tau}_{m,k}\right) = 8\pi m\, E\left(\frac{\sqrt{m^2-k^2}}{m}\right),$$

So, we actually have equality above and hence, also here, the covering map from Section 2.2 is sufficient to determine the topology of a bipolar surface.

For a comprehensive understanding of the τ-family and $\widetilde{\tau}$-family as in [36] and, in general, the variety of results from spectral geometry mentioned above the crucial point is that Lawson additionally provided explicit parametrizations for the surfaces $\tau_{m,k}$ in [37], namely

$$\psi_{m,k}\colon \mathbb{R}^2 \to \mathbb{S}^3 \subseteq \mathbb{R}^4, \quad \psi_{m,k}(x,y) := \begin{pmatrix} \cos(mx)\cos(y) \\ \sin(mx)\cos(y) \\ \cos(kx)\,\sin(y) \\ \sin(kx)\,\sin(y) \end{pmatrix},$$

inducing the metric

$$g_{m,k}(x,y) = f_{m,k}(y)\,\mathrm{d}x^2 + \mathrm{d}y^2\,, \tag{2.28}$$
$$f_{m,k}(y) := \frac{1}{2}\Big(k^2 + m^2 + \left(m^2-k^2\right)\cos(2y)\Big)\,.$$

Note, that a Gauss map of $\psi_{m,k}$ is given by

$$\psi^*_{m,k}\colon \mathbb{R}^2 \to \mathbb{S}^3 \subseteq \mathbb{R}^4\,,$$

$$\psi^*_{m,k}(x,y) := \frac{1}{\sqrt{m^2\cos^2(y)+k^2\sin^2(y)}}\begin{pmatrix} k\sin(mx)\sin(y) \\ -k\cos(mx)\sin(y) \\ -m\sin(kx)\cos(y) \\ m\cos(kx)\cos(y)\end{pmatrix}.$$

Furthermore notice that viewing $\mathbb{S}^3 \subseteq \mathbb{R}^4 \cong \mathbb{C}^2$, the immersion can also be written as

$$\psi_{m,k}(x,y) = \begin{pmatrix} e^{imx}\cos(y) \\ e^{ikx}\sin(y)\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{ikx}\end{pmatrix}\begin{pmatrix} e^{imx} & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix}\cos(y) \\ \sin(y)\end{pmatrix}.$$

From this notation, we can derive that the surfaces $\tau_{m,k}$ are ruled surfaces, often referred to as *spherical helicoids* (for a comprehensive study see [10]). They are generated by rotating the geodesic $y \mapsto \big(\cos(y), \sin(y)\big)$ around the axis $s \mapsto \big(0, e^{is}\big)$ while simultaneously translating along this curve. A full rotation is related to a translation by $2\pi \cdot \frac{k}{m}$, indicating that the key parameter[3] to describe the immersed surface $\tau_{m,k}$ is the ratio $\frac{k}{m} \leq 1$.

Now, based on the the above picture, Lawson collects the properties of the τ-family in Theorem 3 in [37]. The parametrization directly allows to justify the assumption that $m > k \geq 1$ and $\gcd(m,k) = 1$ by applying appropriate reparametrizations or isometries of $\mathbb{S}^5$. Moreover, it can be easily shown that $\tau_{m,k} \subseteq \mathbb{S}^3$ is invariant under the one-parameter group

$$\mathbf{G}_{m,k} := \left\{\mathbf{R}^{(12)}_{mt} \cdot \mathbf{R}^{(34)}_{kt} : t \in \mathbb{R}\right\} \subseteq \mathrm{SO}(4)\,.$$

In this picture, the topological classification follows furthermore from the symmetries of the immersion $\psi_{m,k}$. Smallest possible domains are obtained from the actions on

[3] In fact, $\psi_{m,k}$ can be reparametrized as $\hat{\psi}_{\frac{k}{m}}(x,y) := \big(e^{ix}\cos(y), e^{i\frac{k}{m}x}\sin(y)\big)$, generalizing to a parametrization $\hat{\psi}_\alpha$ for parameters $\alpha > 0$. Among the immersed surfaces $\hat{\psi}_\alpha$ the closed ones are only given for $\alpha \in \mathbb{Q}$. In Proposition 7.2 in [37], Lawson shows that any ruled minimal surface in $\mathbb{S}^3$ corresponds to an open subset of a surface $\hat{\psi}_\alpha$.

$\mathbb{R}^2$ of the groups generated by the maps sending (x, y) to

$$
\begin{array}{llllr}
(x+\pi, -y) & \text{and} & (x, y+2\pi) & \text{if } m \text{ is even} & (\text{i.e., } \tau_{m,k} \text{ is a Klein bottle}),\\
(x+\pi, \pi - y) & \text{and} & (x, y+2\pi) & \text{if } k \text{ is even} & (\text{i.e., } \tau_{m,k} \text{ is a Klein bottle}),\\
(x+\pi, y+\pi) & \text{and} & (x, y+2\pi) & \text{if } m,\, k \text{ are odd} & (\text{i.e., } \tau_{m,k} \text{ is a torus}).
\end{array}
$$

Now, in more detail, we come back to the work of Hugues Lapointe in [36]. In Section 3 of that article, Lapointe computed parametrizations of the bipolar surfaces by considering the parametrizations

$$\widetilde{\psi}_{m,k} = \psi_{m,k} \wedge \psi^*_{m,k} \colon \mathbb{R}^2 \to \mathbb{S}^5 \subseteq \mathbb{R}^6\,,$$

or, more precisely, $\Phi_{m,k} := A \circ \widetilde{\psi}_{m,k}$ for an orthogonal transformation $A \in \mathrm{O}(6)$. Explicitly, these read as $\Phi_{m,k} \colon \mathbb{R}^2 \to \mathbb{S}^5 \subseteq \mathbb{R}^6$,

$$\Phi_{m,k}(x, y) := \frac{1}{a(y)} \begin{pmatrix} (m-k)\,\sin(2y) \\ (m+k)\,\sin(2y) \\ \big((m-k) + (m+k)\cos(2y)\big)\sin\big((m-k)x\big) \\ \big((m+k) + (m-k)\cos(2y)\big)\sin\big((m+k)x\big) \\ \big((m+k) + (m-k)\cos(2y)\big)\sin\big((m+k)x\big) \\ \big((m-k) + (m+k)\cos(2y)\big)\sin\big((m-k)x\big) \end{pmatrix}, \tag{2.29}$$

$$a(y) := \sqrt{8}\sqrt{m^2\cos^2(y) + k^2\sin^2(y)}$$

with induced metric

$$\widetilde{g}_{m,k}(x, y) = \frac{\Big(m^2 - \big(m^2 - k^2\big)\sin^2(y)\Big)^2 + m^2k^2}{m^2 - \big(m^2 - k^2\big)\sin^2(y)} \left(\mathrm{d}x^2 + \frac{\mathrm{d}y^2}{m^2 - \big(m^2 - k^2\big)\sin^2(y)}\right).$$

Note that for $N := (m + k, m - k, 0, 0, 0, 0)$, we have $\langle \Phi_{m,k}(x, y), N\rangle = 0$ for all $(x, y) \in \mathbb{R}^2$, implying that $\widetilde{\tau}_{m,k}$ lies in some $\mathbb{S}^4 \subseteq \mathbb{S}^5$ (cf. Remark 1.4.3 (iii)).

For the above immersions $\Phi_{m,k}(x, y)$, Lapointe determined smallest possible domains by analyzing their symmetries, i.e., by looking at the periods and parity of each component.

Smallest possible domains result from the action of the groups generated by the maps sending $(x, y) \in \mathbb{R}^2$ to

$$(x, y+\pi) \quad \text{and} \quad (x+2\pi, y) \qquad \text{if } mk \equiv 0 \bmod 2 \quad (\text{i.e., } \widetilde{\tau}_{m,k} \text{ is a torus})\,,$$

$$(x+\pi, y) \quad \text{and} \quad (x, y+\pi) \qquad \text{if } mk \equiv 1 \bmod 4 \quad (\text{i.e., } \widetilde{\tau}_{m,k} \text{ is a torus})\,.$$

To get a cleaner picture for the remaining odd cases where $mk \equiv 3 \bmod 4$, Lapointe considered a reparametrization $\mathbb{R}^2 \to \mathbb{R}^2\,,\ (u, v) \mapsto (x(u), y(v))$ with the property that the induced metric of $\psi_{m,k}$ is conformal (corresponding to the transformation H_1 on page 15 in [36]). Looking at (2.28), this is obtained by setting

$$u := x\,, \qquad v = \int \frac{1}{\sqrt{f}}\,\mathrm{d}y$$

and computing (for example using *Mathematica*)

$$y(v) = \operatorname{am}\left(mv, \sqrt{1-\tfrac{k^2}{m^2}}\right)\,,$$

where the latter denotes the Jacobi amplitude with elliptic modulus $\sqrt{1-\frac{k^2}{m^2}}$.

First, note that thereby, the Lawson surface $\tau_{m,k}$ is parametrized by

$$\phi_{m,k}\colon \mathbb{R}^2 \to \mathbb{S}^3 \subseteq \mathbb{C}^2\,, \quad \phi_{m,k}(u, v) = \left(e^{imu}\operatorname{cn}(mv), e^{iku}\operatorname{sn}(mv)\right)\,,$$

where cn and sn denote the Jacobi elliptic functions of the modulus $\sqrt{1-\frac{k^2}{m^2}}$. The metric h induced by $\phi_{m,k}$ is given by

$$h_{m,k}(u, v) = \lambda_{m,k}(u, v)\cdot\left(\mathrm{d}u^2+\mathrm{d}v^2\right)\,, \quad \lambda_{m,k}(u, v) := m^2\operatorname{dn}^2(mv)\,.$$

Now, for the immersed surfaces $\Phi_{m,k}$, this reparametrization yields a symmetry additional to the one given in the case $mk \equiv 1 \bmod 4$, which is given by

$$H_2(u, v) := \left(u+\frac{\pi}{2}, \frac{2}{m+k}\,K\left(\frac{m-k}{m+k}\right)-v\right)\,,$$

where

$$K(\kappa) = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\vartheta}{\sqrt{1 - \kappa^2 \sin^2(\vartheta)}}$$

denotes the complete elliptic integral of the second kind. Thereby, Lapointe concludes that if $mk \equiv 3 \bmod 4$, $\widetilde{\tau}_{m,k}$ is a Klein bottle (cf. Lemma 3.1.4 in [36]).

Remark 2.3.18. For future studies of bipolar surfaces, note that the τ-family in $\mathbb{S}^3$ belongs to a broader class of minimal surfaces analyzed in [24]. In this article, the surfaces $\tau_{m,k}$ are referred to as $T_{m,k,0}$. Surfaces in the aforementioned class arise from the study of closed geodesics on the orbit spaces $\mathbb{S}^3/\boldsymbol{G}_{m,k}$ and are obtained by taking the preimage of these geodesics under the orbit space projection.

Furthermore, note that the τ- family and $\widetilde{\tau}_{3,1}$ appear among the family $T_{a,b,c}$ of closed minimal surfaces in $\mathbb{S}^5$ from Theorem 15.1 in [61].

Chapter 3

Geometry of Bipolar Minimal Surfaces

Minimal surfaces in spheres are of significant interest, e.g., for the Willmore problem or in spectral geometry. Yet, known examples are sparse, particularly in higher codimensions. For this reason, one may ask:

"*When is a minimal surface in* $\mathbb{S}^5$ *the bipolar surface of a minimal surface in* $\mathbb{S}^3$?"

Towards an answer, the main result of this chapter reveals that, under certain conditions, a local geometric correspondence for a specific class of minimal surfaces to bipolar surfaces in $\mathbb{S}^5$. The result itself as well as many tools on the way to prove it were inspired by the work of Ildefonso Castro and Francisco Urbano in [11] about minimal Lagrangian surfaces in $\mathbb{S}^2 \times \mathbb{S}^2$, including the example of Gauss maps of minimal surfaces in $\mathbb{S}^3 \subseteq \mathbb{R}^4$. The latter, being closely related to bipolar minimal surfaces in $\mathbb{S}^5$, provide the link to our point of interest. Compared with the findings of Castro and Urbano, our result holds under a relaxed condition, tailored to the natural behavior of the considered class of surfaces. Furthermore, our proof explicitly handles the fundamental geometric data of the surfaces involved, which could potentially enhance future understanding of further geometric aspects.

3.1 Additional Structure on $\mathbb{S}^5$

We begin with a more detailed exploration of further structure available on $\mathbb{R}^6$ when we consider the latter as the space of bivectors $\Lambda^2\mathbb{R}^4$ (for our notation, we refer to Section 1.3). In this case, the Hodge isomorphism (cf. Definition 1.3.9) is an endomorphism

$$*\colon \Lambda^2\mathbb{R}^4 \to \Lambda^2\mathbb{R}^4 .$$

By Lemma 1.3.2, using an orthonormal basis (e_1, e_2, e_3, e_4) of $\mathbb{R}^4$ and setting

$$\begin{aligned} E_1^{\pm} &:= \frac{1}{\sqrt{2}}(e_1 \wedge e_2 \pm e_3 \wedge e_4) = \frac{1}{\sqrt{2}}\big(e_1 \wedge e_2 \pm *(e_1 \wedge e_2)\big) , \\ E_2^{\pm} &:= \frac{1}{\sqrt{2}}(e_1 \wedge e_3 \mp e_2 \wedge e_4) = \frac{1}{\sqrt{2}}\big(e_1 \wedge e_3 \pm *(e_1 \wedge e_3)\big) , \\ E_3^{\pm} &:= \frac{1}{\sqrt{2}}(e_1 \wedge e_4 \pm e_2 \wedge e_3) = \frac{1}{\sqrt{2}}\big(e_1 \wedge e_4 \pm *(e_1 \wedge e_4)\big) \end{aligned}$$

induces an orthonormal basis

$$\Big(E_1^+, E_2^+, E_3^+, E_1^-, E_2^-, E_3^-\Big) \tag{3.1}$$

of $\Lambda^2\mathbb{R}^4$ which satisfies

$$*E_i^{\pm} = \pm E_i^{\pm} \qquad \text{for } i = 1,\, 2,\, 3 .$$

This shows that spectrally, the Hodge isomorphism decomposes $\Lambda^2\mathbb{R}^4$ into a direct sum of two eigenspaces $\mathbb{R}^3_+$ and $\mathbb{R}^3_-$ of dimension 3, with eigenvalues $+1$ and -1. In particular, $*\colon \Lambda^2\mathbb{R}^4 \to \Lambda^2\mathbb{R}^4$ is a symmetric isometry.

Now, identifying $\Lambda^2\mathbb{R}^4 \cong \mathbb{R}^6$, any $p \in \mathbb{R}^6$ can be written as $p = p^+ + p^-$ with unique vectors $p^+ \in \mathbb{R}^3_+$, $p^- \in \mathbb{R}^3_-$. If $\langle p, *p\rangle = 0$, then we have

$$0 = \big\langle p^+ + p^-, p^+ - p^-\big\rangle = \big|p^+\big|^2 - \big|p^-\big|^2$$

and we are precisely in the situation that $\big|p^+\big| = \big|p^-\big|$. With this in mind, we consider in $\mathbb{S}^5$ the subset

$$\begin{aligned}\mathcal{M} &:= \left\{ v \in \mathbb{S}^5 : \langle v, *v \rangle = 0 \right\} \\ &= \left\{ v \in \mathbb{R}^6 : |v| = 1 \text{ and } \langle v, *v \rangle = 0 \right\} \\ &= \left\{ v \in \mathbb{R}^6 : v^{\pm} \in \mathbb{S}^2_{\pm}({}^1\!/\!\sqrt{2}) \right\}\end{aligned}$$

with $\mathbb{S}^2_{\pm}({}^1\!/\!\sqrt{2})$ denoting the spheres of radius $\frac{1}{\sqrt{2}}$ in the eigenspaces $\mathbb{R}^3_{\pm}$. From this, we directly deduce that $\mathcal{M}$ is a submanifold of $\mathbb{S}^5$, corresponding to a specific embedding of $\mathbb{S}^2({}^1\!/\!\sqrt{2}) \times \mathbb{S}^2({}^1\!/\!\sqrt{2})$ into $\mathbb{S}^5$. We have

$$T_p\mathcal{M} = \operatorname{span}(p, *p)^{\perp} \subseteq p^{\perp} = T_p\mathbb{S}^5 \qquad \text{at } p \in \mathcal{M} \,. \tag{3.2}$$

Remark 3.1.1. In fact, there are several ways to view the submanifold $\mathcal{M}$. Based on Definition 1.3.4 and Definition 1.3.9, it can be shown that $\mathcal{M}$ corresponds to the set of simple bivectors of length 1. In the light of Remark 1.3.6, the latter can be identified with the Grassmann manifold $G_2^+\big(\mathbb{R}^4\big)$. In conclusion, we have

$$\mathbb{S}^2\left({}^1\!/\!\sqrt{2}\right) \times \mathbb{S}^2\left({}^1\!/\!\sqrt{2}\right) \cong \mathcal{M} \cong \left\{ v \wedge w : \; v, \, w \in \mathbb{R}^4 \right\} \cong G_2^+\big(\mathbb{R}^4\big) \,.$$

Furthermore, since any oriented, two-dimensional subspace of $\mathbb{R}^4$ fixes a geodesic in $\mathbb{S}^3$ (as an oriented curve), $\mathcal{M}$ can also be seen as the space of geodesics in $\mathbb{S}^3$ (e.g., see [4]).

Now, the additional structure we want to consider on $\mathcal{M}$ is the following (for more details see for example [11, 70, 59]).

Proposition 3.1.2. *The standard almost complex structure on* $\mathbb{S}^2({}^1\!/\!\sqrt{2})$*,*

$$J_0|_x \colon T_x\mathbb{S}^2({}^1\!/\!\sqrt{2}) \to T_x\mathbb{S}^2({}^1\!/\!\sqrt{2}) \,, \; v \mapsto \sqrt{2} \cdot x \times v \qquad \text{for } x \in \mathbb{S}^2({}^1\!/\!\sqrt{2}) \,, \tag{3.3}$$

where $\times$ *denotes the cross product on* $\mathbb{R}^3$*, gives rise to two almost complex structures*

$$J^+ := J_0 \oplus J_0 \,, \qquad J^- := J_0 \oplus (-J_0) \qquad \text{on } \mathcal{M} \,.$$

For $p \in \mathcal{M}$,

$$J^{\pm}\big|_p \colon T_p\mathcal{M} \to T_p\mathcal{M}$$

are linear isometries with $J^{\pm}\big|_p^2 = -\mathrm{id}$, and therefore in particular bijective and skew-symmetric. Moreover, we have $* \circ J^{\pm}\big|_p = J^{\mp}\big|_p$ and $* \circ J^{\pm}\big|_p = J^{\pm}\big|_p \circ *$.

To get an idea of $J^{\pm}$ acting on 2-vectors, we consider the following example.

Example 3.1.3. Since $*(e_1 \wedge e_2) = e_3 \wedge e_4$, the tangent space to $\mathcal{M}$ at $e_1 \wedge e_2 \in \mathcal{M}$ is

$$\begin{aligned} T_{e_1\wedge e_2}\mathcal{M} &= \mathrm{span}(e_1 \wedge e_2, e_3 \wedge e_4)^{\perp} \\ &= \mathrm{span}(e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4) \\ &= \left\{ v \wedge w \in \Lambda^2\mathbb{R}^4 : v \in \mathrm{span}(e_1, e_2), w \in \mathrm{span}(e_3, e_4) \right\}. \end{aligned}$$

For $v \wedge w \in T_{e_1\wedge e_2}\mathcal{M}$, we have

$$\begin{aligned} J^{\pm}\big|_{e_1\wedge e_2}(v \wedge w) &= \sqrt{2}(e_1 \wedge e_2)^+ \times (v \wedge w)^+ \pm \sqrt{2}(e_1 \wedge e_2)^- \times (v \wedge w)^- \\ &= E_1^+ \times (v \wedge w)^+ \pm E_1^- \times (v \wedge w)^- \end{aligned}$$

and therefore, by an evaluation on the basis vectors from above,

$$\begin{aligned} J^{+}\big|_{e_1\wedge e_2}(v \wedge w) &= v \wedge R^*(w)\,, \\ J^{-}\big|_{e_1\wedge e_2}(v \wedge w) &= R(v) \wedge w\,, \end{aligned} \tag{3.4}$$

where R and R^* denote the rotation by $\frac{\pi}{2}$ on the oriented subspaces $\mathrm{span}(e_1, e_2)$ (associated to $e_1 \wedge e_2$) and $\mathrm{span}(e_3, e_4)$ (associated to $e_3 \wedge e_4$).

From now on, whenever we write J, it either refers to J^+ or J^-.

3.2 A Class of Minimal Surfaces in $\mathbb{S}^5$

In the following, we are interested in the specific properties of the class of minimally immersed surfaces in $\mathbb{S}^5$ that lie in the submanifold $\mathcal{M}$. According to our previous observations, an immersed surface $\Phi\colon \Sigma \to \mathbb{S}^5$ is part of this class if and only if $\langle \Phi, *\Phi \rangle = 0$. The latter implies that the unit vector field

$$\eta := *\Phi$$

is tangential to $\mathbb{S}^5$ and normal to the surface since

$$0 = \mathrm{d}(\langle \Phi, *\Phi \rangle)(v) = 2\langle \mathrm{d}\Phi(v), \eta \rangle \qquad \text{for all } v \in T_p\Sigma\,. \tag{3.5}$$

Therefore, the normal bundle over Σ can be expressed as

$$N\Sigma = \mathcal{N} \oplus \mathbb{R}\eta\,,$$

where $\mathcal{N} \oplus \{0\}$ is tangent to $\mathcal{M}$.

Below, we always work in the setting of a minimally immersed surface $\Phi\colon \Sigma \to \mathbb{S}^5$ with $\langle \Phi, *\Phi \rangle = 0$. By $\hat{g}$ and $\hat{\sigma}$ we denote its first and second fundamental form and by $\hat{A}_\xi$ its shape operator of a normal vector field $\xi \in \Gamma(N\Sigma)$. Furthermore, D, $\hat{\nabla}$ and $\hat{\nabla}^\perp$ are the connections on $\mathbb{R}^6$, $T\Sigma$ and $N\Sigma$, respectively. Note that we always work with oriented surfaces for technical reasons. As already mentioned in Section 1.2, a non-orientable manifold can always be considered by its orientable double cover.

The initial property we observe for this class is the significant role of the Hodge isomorphism in describing the geometry.

Lemma 3.2.1. *Let $\Phi\colon \Sigma \to \mathbb{S}^5$ be an oriented, minimally immersed surface such that $\langle \Phi, *\Phi \rangle = 0$. Then, we have*

$$*X = -\hat{A}_\eta(X) + \hat{\nabla}^\perp_X \eta \qquad \text{for all } X \in \mathfrak{X}(\Sigma) \tag{3.6}$$

and thereby

$$\langle E_1, *E_1 \rangle + \langle E_2, *E_2 \rangle = 0 \tag{3.7}$$

for any local orthonormal frame (E_1, E_2) on Σ.

Proof. Let X, $Y \in \mathfrak{X}(\Sigma)$. The Gauss formula (1.1) reads as

$$D_X Y + \langle X, Y \rangle \Phi = \hat{\nabla}_X Y + \hat{\sigma}(X, Y) \tag{3.8}$$

and hence

$$\langle \hat{\sigma}(X, Y), \eta \rangle = \langle D_X Y, \eta \rangle = X\left(\langle Y, \eta \rangle\right) - \langle Y, D_X \eta \rangle \overset{(3.5)}{=} -\langle X, *Y \rangle \,. \tag{3.9}$$

In particular, we find

$$(*X)^T = -\hat{A}_\eta(X) \,. \tag{3.10}$$

Moreover,

$$(*X)^N = (*X\Phi)^N = (X(*\Phi))^N = (D_X \eta)^N = \nabla^\perp_X \eta \,.$$

Finally, as (3.5) implies

$$\langle *X, \Phi \rangle = \langle X, *\Phi \rangle = 0 \,, \tag{3.11}$$

the first part of the assertion follows. Now, by definition, minimality in $\mathbb{S}^5$ necessitates that the trace of the second fundamental form $\hat{\sigma}$ vanishes. Specifically, this applies to its component into the direction of η. Therefore, the second part above is an immediate consequence of (3.10). □

Remarkably, this reveals a compatibility between the tangent and normal bundle for the considered surfaces.

Proposition 3.2.2. *If* $\Phi \colon \Sigma \to \mathbb{S}^5$ *is an oriented, minimaly immersed surface with* $\langle \Phi, *\Phi \rangle = 0$*, then*

$$\mathcal{N} \oplus \{0\} = J^+(T\Sigma) \quad \text{or} \quad \mathcal{N} \oplus \{0\} = J^-(T\Sigma) \,.$$

Proof. Let (E_1, E_2) be a local orthonormal frame on Σ. Due to (3.7), we have

$$\left|E_1^+\right|^2 + \left|E_2^+\right|^2 = \left|E_1^-\right|^2 + \left|E_2^-\right|^2 .$$

In particular, this yields

$$2 = \left|E_1\right|^2 + \left|E_2\right|^2 = \left|E_1^+\right|^2 + \left|E_2^+\right|^2 + \left|E_1^-\right|^2 + \left|E_2^-\right|^2 = 2\Big(\left|E_1^\pm\right|^2 + \left|E_2^\pm\right|^2\Big)$$

and hence

$$\left|E_1^\pm\right|^2 + \left|E_2^\pm\right|^2 = 1 . \tag{3.12}$$

Once again combined with the fact that $|E_1|^2 = |E_2|^2 = 1$, the latter shows that

$$\left|E_1^+\right| = \left|E_2^-\right| , \qquad \left|E_1^-\right| = \left|E_2^+\right| . \tag{3.13}$$

Moreover, as $\left|\Phi^\pm\right| \equiv \frac{1}{\sqrt{2}}$, we have

$$\left\langle \Phi^\pm, E_i^\pm \right\rangle = 0 \qquad \text{for } i = 1, 2 .$$

In sum, this implies

$$\begin{aligned}
\left|\det\left(\Phi^+, E_1^+, E_2^+\right)\right| &= \operatorname{vol}\big(\Phi^+, E_1^+, E_2^+\big) \\
&= \frac{1}{\sqrt{2}}\left|E_1^+\right|\left|E_2^+\right| \sin\left(\arccos\left(\frac{\left\langle E_1^+, E_2^+\right\rangle}{\left|E_1^+\right|\left|E_2^+\right|}\right)\right) \\
&= \frac{1}{\sqrt{2}}\left|E_1^+\right|\left|E_2^+\right|\sqrt{1 - \frac{\left\langle E_1^+, E_2^+\right\rangle^2}{\left|E_1^+\right|^2\left|E_2^+\right|^2}} \\
&= \frac{1}{\sqrt{2}}\sqrt{\left|E_1^+\right|^2\left|E_2^+\right|^2 - \left\langle E_1^+, E_2^+\right\rangle^2} .
\end{aligned}$$

Accordingly, using (3.13) and the fact that

$$\left\langle E_1^+, E_2^+\right\rangle = -\left\langle E_1^-, E_2^-\right\rangle$$

by $\langle E_1, E_2\rangle = 0$, we conclude that

$$\det\left(\Phi^+, E_1^+, E_2^+\right) = \pm\det\left(\Phi^-, E_1^-, E_2^-\right).$$

Now, in the „$-$“-case, we have

$$\begin{aligned}\langle J^+ E_1, E_2\rangle &= \sqrt{2}\Big(\big\langle \Phi^+ \times E_1^+, E_2^+\big\rangle + \big\langle \Phi^- \times E_1^-, E_2^-\big\rangle\Big)\\ &= \sqrt{2}\Big(\det\left(\Phi^+, E_1^+, E_2^+\right) + \det\left(\Phi^-, E_1^-, E_2^-\right)\Big)\\ &= 0\,.\end{aligned}$$

Analogously, the „$+$“-case implies

$$\begin{aligned}\langle J^- E_1, E_2\rangle &= \sqrt{2}\Big(\langle \Phi^+ \times E_1^+, E_2^+\rangle - \langle \Phi^- \times E_1^-, E_2^-\rangle\Big)\\ &= \sqrt{2}\Big(\det\left(\Phi^+, E_1^+, E_2^+\right) - \det\left(\Phi^-, E_1^-, E_2^-\right)\Big)\\ &= 0\,.\end{aligned}$$

In addition, we have

$$\langle J^\pm E_j, E_j\rangle = 0 \qquad \text{for } j \in \{1,2\}\,,$$

which follows directly from the definition of the cross product. Along with the fact that $J^\pm\big|_p$ are skew-symmetric, linear isomorphisms of $T_p\mathcal{M}$, we can finally infer that

$$\mathcal{N} \oplus \{0\} = J^+(T\Sigma) \quad \text{or} \quad \mathcal{N} \oplus \{0\} = J^-(T\Sigma)\,. \qquad \square$$

Remark 3.2.3. In other words, the above statement means that Φ is a minimal *Lagrangian* immersion into the manifold $\mathcal{M} \cong \mathbb{S}^2(1/\sqrt{2}) \times \mathbb{S}^2(1/\sqrt{2})$ with respect to J^+ or J^-. For a more detailed analysis from this viewpoint, we refer, for instance, to [11], [59] or [70]. To establish a link to our framework, note that for an immersed surface $\Phi\colon \Sigma \to \mathcal{M}$ with induced metric $\hat{g}$, we have

$$0 = \hat{\Delta}\langle \Phi, *\Phi\rangle = 2\big\langle \hat{\Delta}\Phi, *\Phi\big\rangle + 2\langle \mathrm{d}\Phi, \mathrm{d}(*\Phi)\rangle\,,$$

implying that

$$\left\langle \hat{\Delta}\Phi, *\Phi \right\rangle = -\langle \mathrm{d}\Phi, *\mathrm{d}\Phi\rangle = -\sum_{i=1}^{2}\langle E_i, *E_i\rangle = -\Big(\left|E_1^+\right|^2 + \left|E_2^+\right|^2 - \left|E_1^-\right|^2 - \left|E_2^-\right|^2\Big)$$

in terms of a local orthonormal frame on Σ. Reversing the steps in the proof above, we observe that

$$\left\langle \hat{\Delta}\Phi, *\Phi \right\rangle = 0$$

if and only if $\mathcal{N} \oplus \{0\} = J(T\Sigma)$ for $J \in \left\{J^-, J^+\right\}$. In this case, since also $|\Phi| \equiv 1$, we obtain

$$H^{\mathcal{M}} = -2\Phi\,,$$

similarly as in the proof of Theorem 1.1.9. In other words, the notions of minimal Lagrangian surfaces in $\mathcal{M}$ and minimal surfaces in $\mathbb{S}^5$, which lie in $\mathcal{M}$, coincide.

Proceeding with geometric properties, we furthermore notice that the components of the second fundamental form exhibit a high degree of symmetry.

Lemma 3.2.4. *For an oriented, minimally immersed surface $\Phi\colon \Sigma \to \mathbb{S}^5$ with $\langle \Phi, *\Phi\rangle = 0$,*

$$\langle \hat{\sigma}(X,Y), JZ\rangle \quad \text{and} \quad \langle \hat{\sigma}(X,Y), *Z\rangle$$

are totally symmetric in X, Y and $Z \in \mathfrak{X}(\Sigma)$.

Proof. Let X, Y and $Z \in \mathfrak{X}(\Sigma)$. Combined with the Gauss formula (3.8), Proposition 3.2.2 yields

$$\left\langle \hat{\sigma}(X,Y), JZ \right\rangle = \langle D_X Y, JZ\rangle\,.$$

Furthermore,

$$D_X JY = J D_X Y + \sqrt{2}\big(X^+ \times Y^+ - X^- \times Y^-\big)\,.$$

Then, the relation between the cross product and the determinant on $\mathbb{R}^3$ combined with the fact that $\left\langle \Phi^\pm, W^\pm \right\rangle = 0$ for all $W \in \mathfrak{X}(\Sigma)$ (as $\left|\Phi^\pm\right| \equiv \frac{1}{\sqrt{2}}$) yields the first symmetry. Therefrom, the second symmetry follows directly by a verification on a local orthonormal frame, using Lemma 3.2.1 and the fact that $\hat{\sigma}$ is trace-free. □

At this point, we introduce a quantity, which will ultimately prove to play a crucial role in the characterization of the regarded class of minimal surfaces in $\mathbb{S}^5$.

Proposition 3.2.5. *Let $\Phi\colon \Sigma \to \mathbb{S}^5$ be an oriented, minimally immersed surface with $\langle \Phi, *\Phi \rangle = 0$ and let $J \in \{J^+, J^-\}$ such that $\mathcal{N} = J(T\Sigma)$ (cf.* Proposition 3.2.2*). Moreover, let ω be the chosen orientation form on Σ and (E_1, E_2) be a local orthonormal frame. Then, the local expression*

$$C^\Phi := \frac{1}{\omega(E_1, E_2)} \cdot \frac{1}{2} \langle JE_1, *E_2 \rangle$$

defines a smooth function $C^\Phi\colon \Sigma \to \mathbb{R}$.

If the context is clear, we will simply denote C^Φ as C.

Remark 3.2.6. For $J = J^+$, the function C is deeply studied in [11], where it essentially appears as the so-called *Jacobian* of Φ^+, i.e., it satisfies $(\Phi^+)^*\omega_0 = C\omega$, where $\omega_0 = \langle J_0 \,\cdot\,, \,\cdot\, \rangle$ is the Kähler form on $\mathbb{S}^2$ induced by J_0.

Proof of Proposition 3.2.5. Obviously, the local expression is smooth. So, it remains to prove that it does not depend on the choice of the local orthonormal frame. On a common domain, consider two local orthonormal frames (E_1, E_2) and (E_1', E_2') on Σ. We write

$$E_i' = \langle E_i', E_1 \rangle E_1 + \langle E_i', E_2 \rangle E_2 \qquad \text{for } i = 1,\, 2\,.$$

Furthermore, we use that J is skew-symmetric, $*$ is symmetric and that

$$\langle JE_i, *E_i \rangle = 0 \qquad \text{for } i = 1,\, 2$$

due to the definition of J_0 in terms of the cross product. Thereby, the assertion follows as

$$\frac{1}{\omega(E_1, E_2)} \langle JE_1, *E_2 \rangle - \frac{1}{\omega(E_1', E_2')} \langle JE_1', *E_2' \rangle$$

$$
\begin{aligned}
&= \frac{1}{\omega(E_1,E_2)}\Big(\langle JE_1, *E_2\rangle \mp \big\langle E_1', E_1\big\rangle\big\langle E_2', E_2\big\rangle\langle JE_1, *E_2\rangle \\
&\qquad\qquad\qquad\qquad\qquad \mp \big\langle E_1', E_2\big\rangle\big\langle E_2', E_1\big\rangle\langle JE_2, *E_1\rangle\Big) \\
&= \frac{1}{\omega(E_1,E_2)}\Big[1 \mp \Big(\big\langle E_1', E_1\big\rangle\big\langle E_2', E_2\big\rangle - \big\langle E_2', E_1\big\rangle\big\langle E_1', E_2\big\rangle\Big)\Big]\langle JE_1, *E_2\rangle \\
&= \frac{1}{\omega(E_1,E_2)}(1-1)\langle JE_1, *E_2\rangle \\
&= 0\,,
\end{aligned}
$$

using that $\omega\big(E_1', E_2'\big) = \pm\omega(E_1,E_2)$ and that the linear change from (E_1,E_2) to $\big(E_1', E_2'\big)$ at each point has determinant ± 1. □

Remark 3.2.7. Due to the symmetry of $*$ and the skew-symmetry of J, the function C^Φ is invariant under a change of the orientation on Σ.

Example 3.2.8 (compare [11])**.** Suppose that $\Phi\colon \Sigma \to \mathbb{S}^5$ is a minimally immersed surface with $C^\Phi \equiv 0$. In terms of a positively oriented local orthonormal frame (E_1,E_2), this means locally $\langle JE_1, *E_2\rangle \equiv 0$. By tracing back through the proof of Proposition 3.2.2 to (3.12), we can infer that this implies

$$
\big|E_1^+\big|^2 + \big|E_2^+\big|^2 = \big|E_1^-\big|^2 + \big|E_2^-\big|^2 = 1
$$

at each point. As a consequence, we have

$$
\operatorname{rank}\big(\mathrm{d}\Phi^+\big) = \operatorname{rank}\big(\mathrm{d}\Phi^-\big) \equiv 1\,.
$$

In other words, Φ describes the product of two spherical curves in $\mathbb{S}^2 \times \mathbb{S}^2$. Hence, the immersed surface must be congruent to an open subset of the Clifford torus

$$
\boldsymbol{T} = \Big\{p = p^+ + p^- \in \mathbb{S}^5 : \big|p^\pm\big| = \tfrac{1}{\sqrt{2}}\,, \big(p^\pm\big)^1 = 0\Big\} \subseteq \mathbb{S}^5\,.
$$

This particularly shows that if we detect $C^\Phi \equiv 0$ on an open subset of Σ, then, by analytic continuation for minimal surfaces, we find that $C^\Phi \equiv 0$ on Σ.

Now, since the Hodge isomorphism $*$ and $-\mathrm{id}_{\mathbb{R}^6}$ are isometries of $\mathbb{S}^5$, they preserve minimality of immersed surfaces. Under these maps, the function C behaves as follows.

Corollary 3.2.9. *Let $\Phi\colon \Sigma \to \mathbb{S}^5$ be an oriented, minimally immersed surface with $\langle \Phi, *\Phi\rangle = 0$ and $\mathcal{N} = J(T\Sigma)$. Then, we have*

$$C^{*\Phi} = C^{\Phi} \qquad \text{and} \qquad C^{-\Phi} = -C^{\Phi}\,.$$

Proof. To treat both cases simultaneously, we write $\mathcal{N} = J^{\pm}(T\Sigma)$ for the immersion Φ. At first, given that

$$*J^{\pm}\big|_{\Phi(p)} = J^{\mp}\big|_{\Phi(p)} \quad \text{and} \quad J^{\pm}\big|_{*\Phi(p)} = *J^{\pm}\big|_{\Phi(p)} \qquad \text{for all } p \in \Sigma\,,$$

we obtain

$$\big\langle J^{\mp}\big|_{*\Phi(p)}(*v), *w\big\rangle = \big\langle J^{\pm}\big|_{\Phi(p)}(v), w\big\rangle = 0 \qquad \text{for all } p \in \Sigma,\ v,\ w \in T_p\Sigma\,.$$

Thus, for the surface $*\Phi$, we have $\mathcal{N} = J^{\mp}(T\Sigma)$ and $C^{*\Phi}$ is defined accordingly. Thereby, we have

$$\begin{aligned}
2C^{*\Phi}(p) &= \big\langle J^{\mp}\big|_{*\Phi(p)}(*E_1|_p), *(*E_2|_p)\big\rangle \\
&= \big\langle J^{\mp}\big|_{*\Phi(p)}(E_1|_p), *E_2|_p\big\rangle \\
&= \big\langle *\, J^{\pm}\big|_{*\Phi(p)}(E_1|_p), *E_2|_p\big\rangle \\
&= \big\langle J^{\pm}\big|_{\Phi(p)}(E_1|_p), *E_2|_p\big\rangle \\
&= 2C^{\Phi}(p)
\end{aligned}$$

for $p \in \Sigma$ and a positively oriented, local orthonormal frame (E_1, E_2) around p. Consequently, $C^{\Phi} = C^{*\Phi}$.

At second, we consider $-\mathrm{id}_{\mathbb{R}^6}$. For both Φ and $-\Phi$, we have $\mathcal{N} = J(T\Sigma)$. Hence,

$$\begin{aligned}
2C^{-\Phi}(p) &= \big\langle J|_{-\Phi(p)}(-E_1|_p), *(-E_2|_p)\big\rangle \\
&= \langle J|_{-\Phi(p)}E_1|_p, *E_2|_p\rangle \\
&= -\langle J|_{\Phi(p)}E_1|_p, *E_2|_p\rangle
\end{aligned}$$

$$= -2C^{\Phi}(p)$$

for all $p \in \Sigma$ and a positively oriented, local orthonormal frame (E_1, E_2) around p. Accordingly, $C^{-\Phi} = -C^{\Phi}$. □

To continue, now with the function C in place, we close the circle to Lemma 3.2.1.

Lemma 3.2.10. *For an oriented, minimally immersed surface $\Phi\colon \Sigma \to \mathbb{S}^5$ with $\langle \Phi, *\Phi \rangle = 0$, we have*

$$(*X)^N = -2C\big(J \circ R_{\frac{\pi}{2}}\big)X \qquad \text{for all } X \in \mathfrak{X}(\Sigma)\,, \tag{3.14}$$

where $R_{\frac{\pi}{2}}(p)$ denotes the rotation by $\frac{\pi}{2}$ on $T_p\Sigma$ for all $p \in \Sigma$.

Proof. Let (E_1, E_2) be a positively oriented, local orthonormal frame on Σ. Then, Proposition 3.2.2 implies

$$(*E_2)^N = \langle *E_2, JE_1 \rangle JE_1 + \langle *E_2, JE_2 \rangle JE_2 + \langle *E_2, \eta \rangle \eta\,.$$

Since

$$\big\langle E_2, JE_2 \big\rangle = 0\,, \quad \big\langle *E_2, \eta \big\rangle = \langle E_2, \Phi \rangle = 0\,,$$

it follows that

$$(*E_2)^N = \langle *E_2, JE_1 \rangle JE_1 = 2CJE_1$$

and analogously, as J is skew-symmetric,

$$(*E_1)^N = -2CJE_2\,. \tag{3.15}$$

This verifies the assertion for E_1, E_2 and consequently for all $X \in \mathfrak{X}(\Sigma)$. □

Remark 3.2.11. When $\mathcal{N} = J(T\Sigma)$, we have

$$\langle JE_i, E_j \rangle = 0 \quad \text{for } i,\, j \in \{1, 2\}$$

for any local orthonormal frame (E_1, E_2) on Σ. Therefore, the Lemma above yields

$$\langle *JE_i, E_j\rangle = -2C\langle E_i, R_{\frac{\pi}{2}}E_j\rangle \quad \text{for } i,\, j \in \{1, 2\}\,.$$

Unless $C \not\equiv 0$, the latter does clearly not vanish for $i \neq j$. In particular, we have

$$\text{either} \quad \mathcal{N} \oplus \{0\} = J^+(T\Sigma) \quad \text{or} \quad \mathcal{N} \oplus \{0\} = J^-(T\Sigma)$$

if the surface is not congruent to an open subset of the torus $\boldsymbol{T}$. In turn, up to isometries of $\mathbb{S}^5$, open subsets of $\boldsymbol{T}$ are the only surfaces that simultaneously meet both conditions.

We continue with several key insights based on the relation between C and the fundamental data of the corresponding surface (recall Remark 1.1.3). At first, we derive from the preceding Lemma that the possible values of C are limited.

Lemma 3.2.12. *If $\Phi\colon \Sigma \to \mathbb{S}^5$ is an oriented, minimally immersed surface such that $\langle \Phi, *\Phi\rangle = 0$, then*

$$4C^2 = 1 - \frac{1}{2}\big|\hat{A}_\eta\big|^2\,.$$

In particular, we have $C(\Sigma) \subseteq \left[-\frac{1}{2}, \frac{1}{2}\right]$.

Proof. Let (E_1, E_2) be a local orthonormal frame on $T\Sigma$. First, note that due to (3.9) and (3.7), we have

$$\big|\hat{A}_\eta\big|^2 = \sum_{i=1}^{2} \big|\hat{A}_\eta(E_i)\big|^2 = \sum_{i,j=1}^{2} \langle E_i, *E_j\rangle^2 = 2\big(\langle E_1, *E_1\rangle^2 + \langle E_1, *E_2\rangle^2\big)\,. \qquad (3.16)$$

Hence, (3.15) yields

$$\begin{aligned} 4C^2 &= \big\langle (*E_1)^N, (*E_1)^N \big\rangle \\ &= \big\langle *E_1 + \hat{A}_\eta E_1, *E_1 + \hat{A}_\eta E_1 \big\rangle \\ &= 1 + 2\big\langle \hat{A}_\eta E_1, *E_1 \big\rangle + \big|\hat{A}_\eta E_1\big|^2 \end{aligned}$$

$$
\begin{aligned}
&= 1 + 2\Big(\big\langle \hat{A}_\eta E_1, E_1\big\rangle\langle E_1, *E_1\rangle + \big\langle \hat{A}_\eta E_1, E_2\big\rangle\langle E_2, *E_1\rangle\Big) \\
&\qquad\qquad + \Big(\big\langle \hat{A}_\eta E_1, E_1\big\rangle^2 + \big\langle \hat{A}_\eta E_1, E_2\big\rangle^2\Big) \\
&= 1 + 2\big(- \langle E_1, *E_1\rangle^2 - \langle E_2, *E_1\rangle^2\big) + \big(\langle E_1, *E_1\rangle^2 + \langle E_1, *E_2\rangle^2\big) \\
&= 1 - \big(\langle E_1, *E_1\rangle^2 + \langle E_1, *E_2\rangle^2\big) \\
&= 1 - \frac{1}{2}\big|\hat{A}_\eta\big|^2 . \qquad \square
\end{aligned}
$$

Furthermore, we arrive at the following relation of the components of the second fundamental form into the direction of η and $\mathcal{N}$.

Corollary 3.2.13. *Let $\Phi\colon \Sigma \to \mathbb{S}^5$ be an oriented, minimally immersed surface such that $\langle \Phi, *\Phi\rangle = 0$. Then, we have*

$$
\hat{\nabla}_Z\big\langle \hat{\sigma}(X,Y), \eta\big\rangle = 6C\left\langle \hat{\sigma}(X,Y), \big(J \circ R_{\frac{\pi}{2}}\big)(Z)\right\rangle \tag{3.17}
$$

for all X, Y and $Z \in \mathfrak{X}(\Sigma)$.

Proof. Given X, Y and $Z \in \mathfrak{X}(\Sigma)$, we use (3.6), the Gauss formula (3.8), (3.9), (3.11), Lemma 3.2.4 and finally (3.14) in order to compute

$$
\begin{aligned}
&\hat{\nabla}_Z\big\langle \hat{\sigma}(X,Y), \eta\big\rangle \\
&\quad = Z\Big(\big\langle \hat{\sigma}(X,Y), \eta\big\rangle\Big) - \Big\langle \hat{\sigma}\big(\hat{\nabla}_Z X, Y\big), \eta\Big\rangle \\
&\qquad\qquad\qquad - \Big\langle \hat{\sigma}\big(X, \hat{\nabla}_Z Y\big), \eta\Big\rangle - \Big\langle \hat{\sigma}(X,Y), \hat{\nabla}^\perp_Z \eta\Big\rangle \\
&\quad = -Z\big(\langle X, *Y\rangle\big) - \big\langle \hat{\nabla}_Z X, *Y\big\rangle - \big\langle \hat{\nabla}_Z Y, *X\big\rangle - \big\langle \hat{\sigma}(X,Y), *Z\big\rangle \\
&\quad = -\big\langle \big(D_Z X - \hat{\nabla}_Z X\big), *Y\big\rangle - \big\langle \big(D_Z Y - \hat{\nabla}_Z Y\big), *X\big\rangle - \big\langle \hat{\sigma}(X,Y), *Z\big\rangle \\
&\quad = -\big\langle \hat{\sigma}(Z,X), *Y\big\rangle - \big\langle \hat{\sigma}(Z,Y), *X\big\rangle - \big\langle \hat{\sigma}(X,Y), *Z\big\rangle \\
&\quad = -3\big\langle \hat{\sigma}(X,Y), *Z\big\rangle \\
&\quad = 6C\left\langle \hat{\sigma}(X,Y), \big(J \circ R_{\frac{\pi}{2}}\big)(Z)\right\rangle . \qquad \square
\end{aligned}
$$

At this point, we can derive that the function C behaves according to the following equations (also stated in [11]), thereby concluding this section.

Lemma 3.2.14. *If* $\Phi\colon \Sigma \to \mathbb{S}^5$ *with* $\langle \Phi, *\Phi \rangle = 0$ *is an oriented, minimally immersed surface, then the function* C *satisfies*

$$\left|\hat{\nabla} C\right|^2 = \left(1 - 4C^2\right)\left(2C^2 - \frac{1}{2}\hat{K}\right) \tag{3.18}$$

and

$$\hat{\Delta} C = -2C\left(1 + 4C^2 - 2\hat{K}\right). \tag{3.19}$$

Proof. The strategy for this proof is to use the relation between C and $\left|\hat{A}_\eta\right|^2$ from Lemma 3.2.12. With this in mind, we will begin in each case by deriving an equation for $\left|\hat{A}_\eta\right|^2$. After having obtained the first order equation (3.18), we will finally use it to prove (3.19). Note that, within the scope of this proof, we work with normal geodesic coordinates $\left(x^1, x^2\right)$ with respect to $\hat{g}$ at an arbitrary point $p \in \Sigma$ (see for example [40]). Moreover, we remark that all summation indices always run in $\{1, 2\}$.

Regarding the first equation, we start with some preliminary steps. Let i, j, k, $l \in \{1, 2\}$. By the definition of normal geodesic coordinates, we have

$$\hat{g}_{ij} = \delta_{ij}\,, \quad \hat{g}^{ij} = \delta^{ij} \qquad \text{at } p \tag{3.20}$$

and the Christoffel symbols vanish, i.e.,

$$\hat{\Gamma}^k{}_{ij} = 0 \qquad \text{at } p\,. \tag{3.21}$$

As a consequence, we also have

$$\partial_k \hat{g}_{ij} = 0\,, \quad \partial_k \hat{g}^{ij} = 0 \qquad \text{at } p\,. \tag{3.22}$$

In this setup, the i-th component of the gradient $\hat{\nabla} f$ of a smooth function f on Σ is given by

$$\hat{\nabla}^i f = \sum_k \hat{g}^{ik} \partial_k f \overset{(3.20)}{=} \partial_i f \qquad \text{at } p$$

and so

$$\left|\hat{\nabla} f\right|^2 = \sum_k (\partial_k f)^2 \qquad \text{at } p\,.$$

Furthermore, we have

$$\big(\hat{A}_\eta\big)^j{}_i = \Big((*\partial_i\Phi)^T\Big)^j = \hat{g}^{jk}\langle *\partial_i\Phi, \partial_k\Phi\rangle \overset{(3.20)}{=} \langle \partial_i\Phi, *\partial_j\Phi\rangle \qquad \text{at } p$$

and therefore

$$\begin{aligned}
\big|\hat{A}_\eta\big|^2 &= \sum_{i,j,k,l} \hat{g}^{ij}\hat{g}_{kl}\big(\hat{A}_\eta\big)^k{}_i\big(\hat{A}_\eta\big)^l{}_j \\
&= \sum_{i,j,k,l} g^{ij}\hat{g}^{kl}\langle *\partial_i\Phi, \partial_k\Phi\rangle\langle *\partial_j\Phi, \partial_l\Phi\rangle \\
&\overset{(3.20)}{=} \sum_{i,j}\langle \partial_i\Phi, *\partial_j\Phi\rangle^2 \\
&\overset{\mathrm{tr}_{\hat{g}}=0}{=} 2\big(\langle \partial_1\Phi, *\partial_1\Phi\rangle^2 + \langle \partial_1\Phi, *\partial_2\Phi\rangle^2\big) \qquad \text{at } p\,.
\end{aligned} \tag{3.23}$$

Additionally, due to (3.21), the Gauss formula (3.8) has the form

$$\partial_i\partial_j\Phi = \hat{\sigma}_{ij} - \delta_{ij}\Phi \qquad \text{at } p\,. \tag{3.24}$$

At p, we thereby compute

$$\begin{aligned}
\left|\hat{\nabla}\big|\hat{A}_\eta\big|^2\right|^2 &= \sum_k \left(\partial_k\big|\hat{A}_\eta\big|^2\right)^2 \\
&\overset{(3.23)}{=} \sum_k \left(\sum_{i,j,m,n} \partial_k\big(\hat{g}^{im}\hat{g}^{jn}\langle \partial_i\Phi, *\partial_j\Phi\rangle\langle \partial_m\Phi, *\partial_n\Phi\rangle\big)\right)^2 \\
&\overset{(3.22)}{=} \sum_k \left(\sum_{i,j} \partial_k\big(\langle \partial_i\Phi, *\partial_j\Phi\rangle^2\big)\right)^2 \\
&= \sum_k \left(2\sum_{i,j}\langle \partial_i\Phi, *\partial_j\Phi\rangle\partial_k\big(\langle \partial_i\Phi, *\partial_j\Phi\rangle\big)\right)^2
\end{aligned}$$

$$
\begin{aligned}
&= 4\sum_k \left(\sum_{i,j}\langle\partial_i\Phi,*\partial_j\Phi\rangle\Big(\langle\partial_k\partial_i\Phi,*\partial_j\Phi\rangle+\langle\partial_k\partial_j\Phi,*\partial_i\Phi\rangle\Big)\right)^2\\
\overset{(3.24),(3.11)}&{=} 4\sum_k \left(\sum_{i,j}\langle\partial_i\Phi,*\partial_j\Phi\rangle\Big(\langle\hat\sigma_{ki},*\partial_j\Phi\rangle+\langle\hat\sigma_{kj},*\partial_i\Phi\rangle\Big)\right)^2\\
\overset{\text{Lemma 3.2.4}}&{=} 16\sum_k \left(\sum_{i,j}\langle\partial_i\Phi,*\partial_j\Phi\rangle\langle\hat\sigma_{ij},*\partial_k\Phi\rangle\right)^2\\
\overset{\mathrm{tr}_{\hat g}\hat\sigma=0}&{=} 16\sum_k \Big(2\Big[\langle\partial_1\Phi,*\partial_1\Phi\rangle\langle\hat\sigma_{11},*\partial_k\Phi\rangle+\langle\partial_1\Phi,*\partial_2\Phi\rangle\langle\hat\sigma_{12},*\partial_k\Phi\rangle\Big]\Big)^2\\
&= 64\sum_k \Big(\langle\partial_1\Phi,*\partial_1\Phi\rangle\langle\hat\sigma_{11},*\partial_k\Phi\rangle+\langle\partial_1\Phi,*\partial_2\Phi\rangle\langle\hat\sigma_{12},*\partial_k\Phi\rangle\Big)^2\\
&= 64\Big(\langle\partial_1\Phi,*\partial_1\Phi\rangle\langle\hat\sigma_{11},*\partial_1\Phi\rangle+\langle\partial_1\Phi,*\partial_2\Phi\rangle\langle\hat\sigma_{12},*\partial_1\Phi\rangle\Big)^2\\
&\qquad+64\Big(\langle\partial_1\Phi,*\partial_1\Phi\rangle\langle\hat\sigma_{11},*\partial_2\Phi\rangle+\langle\partial_1\Phi,*\partial_2\Phi\rangle\langle\hat\sigma_{12},*\partial_2\Phi\rangle\Big)^2\\
\overset{\text{Lemma 3.2.4},\,\mathrm{tr}_{\hat g}\hat\sigma=0}&{=} 64\Big(\langle\partial_1\Phi,*\partial_1\Phi\rangle\langle\hat\sigma_{11},*\partial_1\Phi\rangle+\langle\partial_1\Phi,*\partial_2\Phi\rangle\langle\hat\sigma_{11},*\partial_2\Phi\rangle\Big)^2\\
&\qquad+64\Big(\langle\partial_1\Phi,*\partial_1\Phi\rangle\langle\hat\sigma_{11},*\partial_2\Phi\rangle-\langle\partial_1\Phi,*\partial_2\Phi\rangle\langle\hat\sigma_{11},*\partial_1\Phi\rangle\Big)^2\\
\overset{(3.14)}&{=} 256C^2\Big(-\langle\partial_1\Phi,*\partial_1\Phi\rangle\langle\hat\sigma_{11},J\partial_2\Phi\rangle+\langle\partial_1\Phi,*\partial_2\Phi\rangle\langle\hat\sigma_{11},J\partial_1\Phi\rangle\Big)^2\\
&\qquad+256C^2\Big(\langle\partial_1\Phi,*\partial_1\Phi\rangle\langle\hat\sigma_{11},J\partial_1\Phi\rangle+\langle\partial_1\Phi,*\partial_2\Phi\rangle\langle\hat\sigma_{11},J\partial_2\Phi\rangle\Big)^2\\
&= 256C^2\Big(\langle\partial_1\Phi,*\partial_1\Phi\rangle^2\langle\hat\sigma_{11},J\partial_2\Phi\rangle^2+\langle\partial_1\Phi,*\partial_2\Phi\rangle^2\langle\hat\sigma_{11},J\partial_1\Phi\rangle^2\\
&\qquad+\langle\partial_1\Phi,*\partial_1\Phi\rangle^2\langle\hat\sigma_{11},J\partial_1\Phi\rangle^2+\langle\partial_1\Phi,*\partial_2\Phi\rangle^2\langle\hat\sigma_{11},J\partial_2\Phi\rangle^2\Big)\\
&= 256C^2\Big(\langle\partial_1\Phi,*\partial_1\Phi\rangle^2+\langle\partial_1\Phi,*\partial_2\Phi\rangle^2\Big)\Big(\langle\hat\sigma_{11},J\partial_1\Phi\rangle^2+\langle\hat\sigma_{11},J\partial_2\Phi\rangle^2\Big)\\
\overset{(3.23)}&{=} 256C^2\frac{\big|\hat A_\eta\big|^2}{2}\frac{\big|\hat\sigma^{\mathcal N}\big|^2}{4}\\
&= 32C^2\big|\hat A_\eta\big|^2\big|\hat\sigma^{\mathcal N}\big|^2, \qquad (3.25)
\end{aligned}
$$

where we also used that $\mathcal{N} = J(T\Sigma)$ by Proposition 3.2.2 and hence

$$\begin{aligned} \left|\hat{\sigma}^{\mathcal{N}}\right|^2 &= \sum_{i,j} \left|\hat{\sigma}^{\mathcal{N}}_{ij}\right|^2 \\ &\overset{(3.20)}{=} \sum_{i,j,k} \langle \hat{\sigma}_{ij}, J\partial_k \Phi \rangle^2 \\ &\overset{\text{Lemma 3.2.4}, \operatorname{tr}_{\hat{g}}\hat{\sigma}=0}{=} 4\Big(\langle \hat{\sigma}_{11}, J\partial_1 \Phi\rangle^2 + \langle \hat{\sigma}_{11}, J\partial_2 \Phi\rangle^2 \Big) \qquad \text{at } p\,. \end{aligned} \tag{3.26}$$

Now, by Lemma Lemma 3.2.12, we have

$$\left|\hat{A}_\eta\right|^2 = 2\big(1 - 4C^2\big)\,. \tag{3.27}$$

Since in addition the Gauss equation (1.8) reads as

$$1 - \hat{K} = \frac{1}{2}\left|\hat{\sigma}\right|^2 = \frac{1}{2}\Big(\left|\hat{A}_\eta\right|^2 + \left|\hat{\sigma}^{\mathcal{N}}\right|^2\Big)\,,$$

it follows that

$$\left|\hat{\sigma}^{\mathcal{N}}\right|^2 = 2\big(4C^2 - \hat{K}\big)\,. \tag{3.28}$$

Therefore, starting from (3.25), Lemma 3.2.12 leads to

$$C^2\left|\hat{\nabla} C\right|^2 = \frac{1}{256}\left|\hat{\nabla}\left|\hat{A}_\eta\right|^2\right|^2 = \frac{1}{8} C^2 \left|\hat{A}_\eta\right|^2 \left|\hat{\sigma}^{\mathcal{N}}\right|^2 = C^2\big(1 - 4C^2\big)\left(2C^2 - \frac{1}{2}\hat{K}\right).$$

Eventually, the equation holds globally and, due to continuity, also modulo C^2 on the (closed) support $\operatorname{supp}(C)$. Obviously, the same is true on the complement of $\operatorname{supp}(C)$. So, we can finally conclude that the function C satisfies

$$\left|\hat{\nabla} C\right|^2 = \big(1 - 4C^2\big)\left(2C^2 - \frac{1}{2}\hat{K}\right).$$

It remains to deduce the equation of second order. In this regard, we first collect some more identities that hold in the setting of normal geodesic coordinates at $p \in \Sigma$. Again, let $i, j, k, l \in \{1, 2\}$. First of all, (3.22) implies

$$\partial_i \hat{\Gamma}^l{}_{jk} = \frac{1}{2}\big(\partial_i \partial_j \hat{g}_{kl} + \partial_i \partial_k \hat{g}_{jl} - \partial_i \partial_l \hat{g}_{jk}\big) \qquad \text{at } p\,. \tag{3.29}$$

Moreover, it can be shown that the second order derivatives of the metric components satisfy

$$\partial_i^2 \hat{g}_{ij} = \partial_i^2 \hat{g}^{ij} = 0\,, \tag{3.30}$$

$$\partial_i \partial_j \hat{g}_{kl} = \partial_k \partial_l \hat{g}_{ij}\,, \qquad \partial_i \partial_j \hat{g}^{kl} = \partial_k \partial_l \hat{g}^{ij}\,, \tag{3.31}$$

$$\partial_i \partial_j \hat{g}_{kl} + \partial_i \partial_k \hat{g}_{jl} + \partial_i \partial_l \hat{g}_{jk} = \partial_i \partial_j \hat{g}^{kl} + \partial_i \partial_k \hat{g}^{jl} + \partial_i \partial_l \hat{g}^{jk} = 0\,, \tag{3.32}$$

$$\partial_i \partial_j g^{kl} = -\partial_i \partial_j g_{kl} \qquad \text{at } p\,. \tag{3.33}$$

Therefore, since $[\partial_i, \partial_k] \equiv 0$ for coordinate vector fields,

$$\begin{aligned}
\hat{R}_{ijkl} &= \big\langle \hat{\nabla}_i \hat{\nabla}_j \partial_k, \partial_l \big\rangle - \big\langle \hat{\nabla}_j \hat{\nabla}_i \partial_k, \partial_l \big\rangle \\
&= \partial_i \hat{\Gamma}^l{}_{jk} - \partial_j \hat{\Gamma}^l{}_{ik} \\
\overset{(3.29)}&{=} \frac{1}{2}\left(\partial_i \partial_j \hat{g}_{kl} + \partial_i \partial_k \hat{g}_{jl} - \partial_i \partial_l \hat{g}_{jk}\right) - \frac{1}{2}\left(\partial_j \partial_i \hat{g}_{kl} + \partial_j \partial_k \hat{g}_{il} - \partial_j \partial_l \hat{g}_{ik}\right) \\
&= \frac{1}{2}\left(\partial_i \partial_k \hat{g}_{jl} - \partial_i \partial_l \hat{g}_{jk} - \partial_j \partial_k \hat{g}_{il} + \partial_j \partial_l \hat{g}_{ik}\right) \\
\overset{(3.31)}&{=} \partial_i \partial_k \hat{g}_{jl} - \partial_i \partial_l \hat{g}_{jk} \qquad\qquad\qquad \text{at } p\,.
\end{aligned}$$

In particular,

$$\hat{K} = \hat{R}_{1221} = \partial_1 \partial_2 \hat{g}_{12} - \partial_1^2 g_{22} \overset{(3.32)}{=} \partial_1 \partial_2 \hat{g}_{12} - (-2\partial_1 \partial_2 \hat{g}_{12}) = 3\partial_1 \partial_2 \hat{g}_{12} \qquad \text{at } p\,. \tag{3.34}$$

Now, to complete the preparations, the Laplace-Beltrami operator of a smooth function f on Σ is

$$\hat{\Delta} f = \sum_{j,k,l=1}^{2} \hat{g}^{jk}\big(\partial_j\partial_k f - \hat{\Gamma}^l{}_{jk}\partial_l f\big) \overset{(3.21)}{=} \sum_{k=1}^{2} \partial_k^2 f \qquad \text{at p}\,. \tag{3.35}$$

At this point, we compute at p (note again that all summation indices run in $\{1,2\}$)

$$\begin{aligned}
\hat{\Delta}|\hat{A}_\eta|^2 &= \sum_{r,k,l,m,n} \partial_r^2 \left(\hat{g}^{kl}\hat{g}^{mn}\langle\partial_k\Phi, *\partial_m\Phi\rangle\langle\partial_l\Phi, *\partial_n\Phi\rangle\right) \\
&= \sum_{r,k,l,m,n} \partial_r \left(\left(\partial_r\hat{g}^{kl}\right)\hat{g}^{mn}\langle\partial_k\Phi, *\partial_m\Phi\rangle\langle\partial_l\Phi, *\partial_n\Phi\rangle\right) \\
&\qquad + \sum_{r,k,l,m,n} \partial_r \left(\hat{g}^{kl}\left(\partial_r\hat{g}^{mn}\right)\langle\partial_k\Phi, *\partial_m\Phi\rangle\langle\partial_l\Phi, *\partial_n\Phi\rangle\right) \\
&\qquad\qquad + \sum_{r,k,l,m,n} \partial_r \left(\hat{g}^{kl}\hat{g}^{mn}\partial_r\left(\langle\partial_k\Phi, *\partial_m\Phi\rangle\langle\partial_l\Phi, *\partial_n\Phi\rangle\right)\right) \\
&\overset{(3.22)}{=} \sum_{r,k,l,m,n} \left(\partial_r^2\hat{g}^{kl}\right)\hat{g}^{mn}\langle\partial_k\Phi, *\partial_m\Phi\rangle\langle\partial_l\Phi, *\partial_n\Phi\rangle \\
&\qquad + \sum_{r,k,l,m,n} \hat{g}^{kl}\left(\partial_r^2\hat{g}^{mn}\right)\langle\partial_k\Phi, *\partial_m\Phi\rangle\langle\partial_l\Phi, *\partial_n\Phi\rangle \\
&\qquad\qquad + \sum_{r,k,l,m,n} \hat{g}^{kl}\hat{g}^{mn}\partial_r^2\left(\langle\partial_k\Phi, *\partial_m\Phi\rangle\langle\partial_l\Phi, *\partial_n\Phi\rangle\right) \\
&= \sum_{r,k,l,m,n} \left(\partial_r^2\hat{g}^{kl}\right)\hat{g}^{mn}\langle\partial_k\Phi, *\partial_m\Phi\rangle\langle\partial_l\Phi, *\partial_n\Phi\rangle \\
&\qquad + \sum_{r,k,l,m,n} \hat{g}^{kl}\left(\partial_r^2\hat{g}^{mn}\right)\langle\partial_k\Phi, *\partial_m\Phi\rangle\langle\partial_l\Phi, *\partial_n\Phi\rangle \\
&\qquad\qquad + \sum_{r,k,l,m,n} \delta^{kl}\delta^{mn}\partial_r^2\left(\langle\partial_k\Phi, *\partial_m\Phi\rangle\langle\partial_l\Phi, *\partial_n\Phi\rangle\right) \\
&= 2\sum_{r,k,l,m} \left(\partial_r^2\hat{g}^{kl}\right)\langle\partial_k\Phi, *\partial_m\Phi\rangle\langle\partial_l\Phi, *\partial_m\Phi\rangle \\
&\qquad\qquad + \sum_{r,k,m} \partial_r^2\Big(\langle\partial_k\Phi, *\partial_m\Phi\rangle^2\Big)\,.
\end{aligned} \tag{3.36}$$

We consider the two terms in (3.36) separately. First, we have

$$
\begin{aligned}
2 \sum_{r,k,l,m} & \left(\partial_r^2 \hat{g}^{kl}\right) \langle \partial_k \Phi, *\partial_m \Phi\rangle \langle \partial_l \Phi, *\partial_m \Phi\rangle \\
&= 2 \sum_{r,l,m} \left(\partial_r^2 \hat{g}^{1l}\right) \langle \partial_1 \Phi, *\partial_m \Phi\rangle \langle \partial_l \Phi, *\partial_m \Phi\rangle \\
&\qquad + 2 \sum_{r,l,m} \left(\partial_r^2 \hat{g}^{2l}\right) \langle \partial_2 \Phi, *\partial_m \Phi\rangle \langle \partial_l \Phi, *\partial_m \Phi\rangle \\
&\overset{(3.30)}{=} 2 \sum_{m} \left(\partial_2^2 \hat{g}^{11}\right) \langle \partial_1 \Phi, *\partial_m \Phi\rangle^2 + 2 \sum_{m} \left(\partial_1^2 \hat{g}^{22}\right) \langle \partial_2 \Phi, *\partial_m \Phi\rangle^2 \\
&\overset{(3.31)}{=} 2 \left(\partial_2^2 \hat{g}^{11}\right) \left(\langle \partial_1 \Phi, *\partial_1 \Phi\rangle^2 + \langle \partial_1 \Phi, *\partial_2 \Phi\rangle^2 + \langle \partial_2 \Phi, *\partial_1 \Phi\rangle^2 + \langle \partial_2 \Phi, *\partial_2 \Phi\rangle^2\right) \\
&\overset{\mathrm{tr}_{\hat{g}}\hat{\sigma}=0}{=} 4 \left(\partial_2^2 \hat{g}^{11}\right) \left(\langle *\partial_1 \Phi, \partial_1 \Phi\rangle^2 + \langle *\partial_1 \Phi, \partial_2 \Phi\rangle^2\right) \\
&= 2 \left(\partial_2^2 \hat{g}^{11}\right) |\hat{A}_\eta|^2 \\
&\overset{(3.33)}{=} -2 \left(\partial_2^2 \hat{g}_{11}\right) |\hat{A}_\eta|^2 \\
&\overset{(3.32)}{=} 4 \left(\partial_1 \partial_2 \hat{g}_{12}\right) |\hat{A}_\eta|^2 \\
&\overset{(3.34)}{=} \frac{4}{3} \hat{K} |\hat{A}_\eta|^2 .
\end{aligned} \tag{3.37}
$$

Second, we have

$$
\begin{aligned}
\sum_{r,k,m} & \partial_r^2 \Big(\langle \partial_k \Phi, *\partial_m \Phi\rangle^2\Big) \\
&= \sum_{r,k,m} \partial_r \Big(2 \langle \partial_k \Phi, *\partial_m \Phi\rangle \left(\langle \partial_r \partial_k \Phi, *\partial_m \Phi\rangle + \langle \partial_r \partial_m \Phi, *\partial_k \Phi\rangle\right)\Big) \\
&\overset{\text{relabel}}{=} 4 \sum_{r,k,m} \partial_r \left(\langle \partial_k \Phi, *\partial_m \Phi\rangle \langle \partial_r \partial_k \Phi, *\partial_m \Phi\rangle\right) \\
&= 4 \sum_{r,k,m} \left(\partial_r \langle \partial_k \Phi, *\partial_m \Phi\rangle\right) \langle \partial_r \partial_k \Phi, *\partial_m \Phi\rangle \\
&\qquad + 4 \sum_{r,k,m} \langle \partial_k \Phi, *\partial_m \Phi\rangle \left(\partial_r \langle \partial_r \partial_k \Phi, *\partial_m \Phi\rangle\right) \\
&= 4 \sum_{r,k,m} \Big(\langle \partial_r \partial_k \Phi, *\partial_m \Phi\rangle + \langle \partial_r \partial_m \Phi, *\partial_k \Phi\rangle\Big) \langle \partial_r \partial_k \Phi, *\partial_m \Phi\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\qquad + 4\sum_{r,k,m}\langle\partial_k\Phi, *\partial_m\Phi\rangle\Big(\big\langle\partial_k\partial_r^2\Phi, *\partial_m\Phi\big\rangle + \langle\partial_r\partial_k\Phi, *\partial_r\partial_m\Phi\rangle\Big)\\
&= 4\sum_{r,k,m}\langle\partial_r\partial_k\Phi, *\partial_m\Phi\rangle^2\\
&\qquad + 4\sum_{r,k,m}\langle\partial_r\partial_m\Phi, *\partial_k\Phi\rangle\langle\partial_r\partial_k\Phi, *\partial_m\Phi\rangle\\
&\qquad\qquad + 4\sum_{r,k,m}\langle\partial_k\Phi, *\partial_m\Phi\rangle\big\langle\partial_k\partial_r^2\Phi, *\partial_m\Phi\big\rangle\\
&\qquad\qquad\qquad + 4\sum_{r,k,m}\langle\partial_k\Phi, *\partial_m\Phi\rangle\langle\partial_r\partial_k\Phi, *\partial_r\partial_m\Phi\rangle\\
&\overset{(3.20)}{=} 4\sum_{r,k,m}\langle\partial_r\partial_k\Phi, *\partial_m\Phi\rangle^2\\
&\qquad + 4\sum_{r,k,m}\langle\partial_r\partial_m\Phi, *\partial_k\Phi\rangle\langle\partial_r\partial_k\Phi, *\partial_m\Phi\rangle\\
&\qquad\qquad + 4\sum_{i,j,k,m}\langle\partial_k\Phi, *\partial_m\Phi\rangle\big\langle\partial_k\big(\hat{g}^{ij}\partial_i\partial_j\Phi\big), *\partial_m\Phi\big\rangle\\
&\qquad\qquad\qquad + 4\sum_{r,k,m}\langle\partial_k\Phi, *\partial_m\Phi\rangle\langle\partial_r\partial_k\Phi, *\partial_r\partial_m\Phi\rangle\\
&\overset{(3.35)}{=} 4\sum_{r,k,m}\langle\partial_r\partial_k\Phi, *\partial_m\Phi\rangle^2\\
&\qquad + 4\sum_{r,k,m}\langle\partial_r\partial_m\Phi, *\partial_k\Phi\rangle\langle\partial_r\partial_k\Phi, *\partial_m\Phi\rangle\\
&\qquad\qquad + 4\sum_{i,j,k,l,m}\langle\partial_k\Phi, *\partial_m\Phi\rangle\Big\langle\partial_k\left(\hat{\Delta}\Phi + \hat{g}^{ij}\hat{\Gamma}^l_{ij}\partial_l\Phi\right), *\partial_m\Phi\Big\rangle\\
&\qquad\qquad\qquad + 4\sum_{r,k,m}\langle\partial_k\Phi, *\partial_m\Phi\rangle\langle\partial_r\partial_k\Phi, *\partial_r\partial_m\Phi\rangle\\
&\overset{\text{Theorem 1.1.9}}{=} 4\sum_{r,k,m}\langle\partial_r\partial_k\Phi, *\partial_m\Phi\rangle^2\\
&\qquad + 4\sum_{r,k,m}\langle\partial_r\partial_m\Phi, *\partial_k\Phi\rangle\langle\partial_r\partial_k\Phi, *\partial_m\Phi\rangle\\
&\qquad\qquad - 8\sum_{k,m}\langle\partial_k\Phi, *\partial_m\Phi\rangle^2
\end{aligned}
$$

$$+ 4 \sum_{i,j,k,l,m} \langle \partial_k \Phi, *\partial_m \Phi \rangle \left\langle \partial_k \left(\hat{g}^{ij} \hat{\Gamma}^l_{ij} \partial_l \Phi \right), *\partial_m \Phi \right\rangle$$

$$+ 4 \sum_{r,k,m} \langle \partial_k \Phi, *\partial_m \Phi \rangle \langle \partial_r \partial_k \Phi, *\partial_r \partial_m \Phi \rangle . \tag{3.38}$$

Again, we treat the five terms in (3.38) separately. First, we have

$$\begin{aligned}
&4 \sum_{r,k,m} \langle \partial_r \partial_k \Phi, *\partial_m \Phi \rangle^2 \\
&\overset{(3.24)}{=} 4 \sum_{r,k,m} \left\langle \hat{\sigma}_{rk}, *\partial_m \Phi \right\rangle^2 \\
&\overset{\mathrm{tr}_{\hat{g}} \hat{\sigma}=0,\,\text{Lemma 3.2.4}}{=} 16 \Big(\langle \hat{\sigma}_{11}, *\partial_1 \Phi \rangle^2 + \langle \hat{\sigma}_{11}, *\partial_2 \Phi \rangle^2 \Big) \\
&\overset{(3.14)}{=} 64 C^2 \Big(\langle \hat{\sigma}_{11}, J\partial_1 \Phi \rangle^2 + \langle \hat{\sigma}_{11}, J\partial_2 \Phi \rangle^2 \Big) \\
&\overset{(3.26)}{=} 16 C^2 \left| \hat{\sigma}^{\mathcal{N}} \right|^2 .
\end{aligned} \tag{3.39}$$

The second term is

$$\begin{aligned}
&4 \sum_{r,k,m} \langle \partial_r \partial_m \Phi, *\partial_k \Phi \rangle \langle \partial_r \partial_k \Phi, *\partial_m \Phi \rangle \\
&= 8 \sum_{\substack{r,k,m \\ k<m}} \langle \partial_r \partial_m \Phi, *\partial_k \Phi \rangle \langle \partial_r \partial_k \Phi, *\partial_m \Phi \rangle + 4 \sum_{r,k} \langle \partial_r \partial_k \Phi, *\partial_k \Phi \rangle^2 \\
&= 8 \langle \partial_1 \partial_2 \Phi, *\partial_1 \Phi \rangle \langle \partial_1 \partial_1 \Phi, *\partial_2 \Phi \rangle + 8 \langle \partial_2 \partial_2 \Phi, *\partial_1 \Phi \rangle \langle \partial_2 \partial_1 \Phi, *\partial_2 \Phi \rangle \\
&\quad + 4 \langle \partial_1 \partial_1 \Phi, *\partial_1 \Phi \rangle^2 + 4 \langle \partial_2 \partial_1 \Phi, *\partial_1 \Phi \rangle^2 \\
&\quad + 4 \langle \partial_1 \partial_2 \Phi, *\partial_2 \Phi \rangle^2 + 4 \langle \partial_2 \partial_2 \Phi, *\partial_2 \Phi \rangle^2 \\
&\overset{(3.24),\,(3.11)}{=} 8 \left\langle \hat{\sigma}_{12}, *\partial_1 \Phi \right\rangle \left\langle \hat{\sigma}_{11}, *\partial_2 \Phi \right\rangle + 8 \left\langle \hat{\sigma}_{22}, *\partial_1 \Phi \right\rangle \left\langle \hat{\sigma}_{12} \Phi, *\partial_2 \Phi \right\rangle \\
&\quad + 4 \left\langle \hat{\sigma}_{11}, *\partial_1 \Phi \right\rangle^2 + 4 \left\langle \hat{\sigma}_{12}, *\partial_1 \Phi \right\rangle^2 \\
&\quad + 4 \left\langle \hat{\sigma}_{12} \Phi, *\partial_2 \Phi \right\rangle^2 + 4 \left\langle \hat{\sigma}_{22}, *\partial_2 \Phi \right\rangle^2 \\
&\overset{\text{Lemma 3.2.4}}{=} 16 \Big(\left\langle \hat{\sigma}_{11}, *\partial_1 \Phi \right\rangle^2 + \left\langle \hat{\sigma}_{11}, *\partial_2 \Phi \right\rangle^2 \Big) \\
&\overset{(3.14)}{=} 64 C^2 \Big(\left\langle \hat{\sigma}_{11}, J\partial_1 \Phi \right\rangle^2 + \left\langle \hat{\sigma}_{11}, J\partial_2 \Phi \right\rangle^2 \Big) \\
&\overset{(3.26)}{=} 16 C^2 \left| \hat{\sigma}^{\mathcal{N}} \right|^2 .
\end{aligned} \tag{3.40}$$

For the third term, we have

$$-8\sum_{k,m}\langle\partial_k\Phi,*\partial_m\Phi\rangle^2 \overset{(3.23)}{=} -8|\hat{A}_\eta|^2 \tag{3.41}$$

Moreover, the fourth term is given by

$$\begin{aligned}
&4\sum_{i,j,k,l,m}\langle\partial_k\Phi,*\partial_m\Phi\rangle\langle\partial_k\left(\hat{g}^{ij}\hat{\Gamma}^l_{ij}\partial_l\Phi\right),*\partial_m\Phi\rangle\\
\overset{(3.20),(3.22)}&{=} 4\sum_{j,k,l,m}\left(\partial_k\hat{\Gamma}^l_{jj}\right)\langle\partial_k\Phi,*\partial_m\Phi\rangle\langle\partial_l\Phi,*\partial_m\Phi\rangle\\
\overset{\mathrm{tr}_{\hat{g}}\hat{\sigma}=0}&{=} 4\sum_j\left(\partial_1\hat{\Gamma}^1_{jj}+\partial_2\hat{\Gamma}^2_{jj}\right)\left(\langle\partial_1\Phi,*\partial_1\Phi\rangle^2+\langle\partial_1\Phi,*\partial_2\Phi\rangle^2\right)\\
\overset{(3.23)}&{=} 2|\hat{A}_\eta|^2\sum_j\left(\partial_1\hat{\Gamma}^1_{jj}+\partial_2\hat{\Gamma}^2_{jj}\right)\\
\overset{(3.29)}&{=} |\hat{A}_\eta|^2\sum_j\left(2\partial_1\partial_j\hat{g}_{1j}-\partial_1^2\hat{g}_{jj}+2\partial_2\partial_j\hat{g}_{2j}-\partial_2^2\hat{g}_{jj}\right)\\
\overset{(3.30),(3.31)}&{=} |\hat{A}_\eta|^2\Big(4\partial_1\partial_2\hat{g}_{12}-2\partial_1^2\hat{g}_{22}\Big)\\
\overset{(3.32)}&{=} |\hat{A}_\eta|^2\Big(4\partial_1\partial_2\hat{g}_{12}-2\big(-2\partial_1\partial_2\hat{g}_{12}\big)\Big)\\
&= 8|\hat{A}_\eta|^2\partial_1\partial_2 g_{12}\\
\overset{(3.34)}&{=} \frac{8}{3}\hat{K}|\hat{A}_\eta|^2\,.
\end{aligned}\tag{3.42}$$

Finally, for the fifth term, we have

$$\begin{aligned}
&4\sum_{r,k,m}\langle\partial_k\Phi,*\partial_m\Phi\rangle\langle\partial_r\partial_k\Phi,*\partial_r\partial_m\Phi\rangle\\
&= 4\sum_{r,k}\langle*\partial_k\Phi,\partial_k\Phi\rangle\langle\partial_r\partial_k\Phi,*\partial_r\partial_k\Phi\rangle+8\sum_{\substack{r,k,m\\k<m}}\langle*\partial_k\Phi,\partial_m\Phi\rangle\langle\partial_r\partial_k\Phi,*\partial_r\partial_m\Phi\rangle\\
\overset{\mathrm{tr}_{\hat{g}}=0}&{=} 4\sum_r\langle\partial_1\Phi,*\partial_1\Phi\rangle\langle\partial_r\partial_1\Phi,*\partial_r\partial_1\Phi\rangle-4\sum_r\langle\partial_1\Phi,*\partial_1\Phi\rangle\langle\partial_r\partial_2\Phi,*\partial_r\partial_2\Phi\rangle\\
&\quad+8\sum_r\langle\partial_1\Phi,*\partial_2\Phi\rangle\langle\partial_r\partial_1\Phi,*\partial_r\partial_2\Phi\rangle
\end{aligned}$$

$$
\begin{aligned}
&= 4\langle \partial_1\Phi, *\partial_1\Phi\rangle\langle \partial_1^2\Phi, *\partial_1^2\Phi\rangle - 4\langle \partial_1\Phi, *\partial_1\Phi\rangle\langle \partial_2^2\Phi, *\partial_2^2\Phi\rangle \\
&\qquad\qquad + 8\langle \partial_1\Phi, *\partial_2\Phi\rangle\langle \partial_1^2\Phi, *\partial_1\partial_2\Phi\rangle + 8\langle \partial_1\Phi, *\partial_2\Phi\rangle\langle \partial_2^2\Phi, *\partial_1\partial_2\Phi\rangle \\
\overset{(3.35)}&{=} 4\langle \partial_1\Phi, *\partial_1\Phi\rangle\langle \partial_1^2\Phi, *\partial_1^2\Phi\rangle - 4\langle \partial_1\Phi, *\partial_1\Phi\rangle\langle \hat{\Delta}\Phi - \partial_1^2\Phi, *\hat{\Delta}\Phi - *\partial_1^2\Phi\rangle \\
&\qquad + 8\langle \partial_1\Phi, *\partial_2\Phi\rangle\langle \partial_1^2\Phi, *\partial_1\partial_2\Phi\rangle + 8\langle \partial_1\Phi, *\partial_2\Phi\rangle\langle \hat{\Delta}\Phi - \partial_1^2\Phi, *\partial_1\partial_2\Phi\rangle \\
\overset{\text{Theorem 1.1.9}}&{=} 4\langle \partial_1\Phi, *\partial_1\Phi\rangle\langle \partial_1^2\Phi, *\partial_1^2\Phi\rangle \\
&\quad - 4\langle \partial_1\Phi, *\partial_1\Phi\rangle\langle 2\Phi + \partial_1^2\Phi, 2\eta + *\partial_1^2\Phi\rangle \\
&\qquad + 8\langle \partial_1\Phi, *\partial_2\Phi\rangle\langle \partial_1^2\Phi, *\partial_1\partial_2\Phi\rangle \\
&\qquad\quad - 8\langle \partial_1\Phi, *\partial_2\Phi\rangle\langle 2\Phi + \partial_1^2\Phi, *\partial_1\partial_2\Phi\rangle \\
\overset{(3.24)}&{=} -16\Big(\langle \partial_1\Phi, *\partial_1\Phi\rangle\langle \hat{\sigma}_{11}, \eta\rangle + \langle \partial_1\Phi, *\partial_2\Phi\rangle\langle \hat{\sigma}_{12}, \eta\rangle\Big) \\
\overset{(3.9)}&{=} 16\left(\langle \partial_1\Phi, *\partial_1\Phi\rangle^2 + \langle \partial_1\Phi, *\partial_2\Phi\rangle^2\right) \\
\overset{(3.23)}&{=} 8\big|\hat{A}_\eta\big|^2 .
\end{aligned}
\tag{3.43}
$$

To conclude, taking the sum of (3.37) and (3.39) to (3.43), (3.36) yields

$$
\hat{\Delta}\big|\hat{A}_\eta\big|^2 = 4\hat{K}\big|\hat{A}_\eta\big|^2 + 32C^2\big|\hat{\sigma}^{\mathcal{N}}\big|^2 .
$$

Then, using Lemma 3.2.12, (3.18), (3.27) and (3.28), a straightforward computation shows that

$$
C\hat{\Delta}C = -\frac{1}{16}\hat{\Delta}\big|\hat{A}_\eta\big|^2 - \big|\hat{\nabla}C\big|^2 = -2C^2\big(1 + 4C^2 - 2\hat{K}\big) .
$$

Similarly as in the first part of the proof, we can therefore conclude that the function C satisfies

$$
\hat{\Delta}C = -2C\big(1 + 4C^2 - 2\hat{K}\big) . \qquad \square
$$

3.3 Fundamental Data of Bipolar Surfaces

Let $\psi\colon \Sigma \to \mathbb{S}^3$ be a conformal minimal immersion as in Section 1.2. Then, the corresponding bipolar surface $\widetilde{\psi}\colon \Sigma \to \mathbb{S}^5$ meets the prerequisites to be part of the class specified in the previous section: It is minimal by Theorem 1.4.1 and

$$\left\langle \widetilde{\psi}, *\widetilde{\psi} \right\rangle = 0$$

since

$$*\widetilde{\psi} = *\big(\psi \wedge \psi^*\big) = \frac{1}{\lambda}\, \partial_1 \psi \wedge \partial_2 \psi \,.$$

Considering bipolar surfaces as part of the class of surfaces from the previous section, we now present a more detailed characterization.

Proposition 3.3.1. *Let* $\widetilde{\psi} \colon \Sigma \to \mathbb{S}^5$ *be the bipolar surface of an oriented, minimally immersed surface* $\psi \colon \Sigma \to \mathbb{S}^3$*. Then, we have*

$$\mathcal{N} \oplus \{0\} = J^-(T\Sigma)\,.$$

Proof. As

$$\mathrm{d}\widetilde{\psi} = \mathrm{d}\psi \wedge \psi^* + \psi \wedge \mathrm{d}\psi^*\,.$$

we conclude from (3.4) that

$$J^- \mathrm{d}\widetilde{\psi} = -\mathrm{d}\psi \wedge \psi + \psi^* \wedge \mathrm{d}\psi^*\,. \tag{3.44}$$

Hence, using the Weingarten equation (1.15), it follows that

$$\big\langle J^- \partial_i \widetilde{\psi}, \partial_j \widetilde{\psi} \big\rangle = \big\langle \partial_i \psi, \partial_j \psi^* \big\rangle - \big\langle \partial_i \psi^*, \partial_j \psi \big\rangle = -\beta_{ji} + \beta_{ij} = 0 \qquad \text{for } i \in \{1, 2\}\,.$$

Since at each point $p \in \Sigma$, $J^-\big|_{\widetilde{\psi}(p)}$ is a linear automorphism of $T_{\widetilde{\psi}(p)}\mathcal{M}$, this is proves the assertion. □

Remark 3.3.2. Instead of bipolar surfaces, [11] treats a closely related type of surface, known as the *Gauss map* of ψ (not to be confused with the Gauss map in $\mathbb{S}^3$), defined by

$$\Sigma \to G_2^+\big(\mathbb{R}^4\big), \quad p \mapsto \mathrm{d}\psi|_p(T_p\Sigma)\,.$$

From the perspective within this chapter, the Gauss map is simply given by $*\widetilde{\psi}$ and therefore congruent to the bipolar surface $\widetilde{\psi}$.

Example 3.3.3. We consider the bipolar surfaces of the two simplest closed minimal surfaces in $\mathbb{S}^3$ from this perspective.

(i) For a geodesic 2-sphere in $\mathbb{S}^3$, written as

$$\boldsymbol{S} := \left\{ \big(0, x\big) \in \mathbb{R}^4 : x \in \mathbb{S}^2 \right\} \subseteq \mathbb{S}^3$$

we can choose the constant unit normal $n \equiv -e_1$. Using the frame (3.1), we compute for $\big(0, x^1, x^2, x^3\big) \in \boldsymbol{S}$

$$x \wedge n = x^1\, e_1 \wedge e_2 + x^2\, e_1 \wedge e_3 + x^3\, e_1 \wedge e_4 \tag{3.45}$$

$$= \frac{1}{\sqrt{2}}\Big(x^1 \left(E_1^+ + E_1^-\right) + x^2 \left(E_2^+ + E_2^-\right) + x^3 \left(E_3^+ + E_3^-\right) \Big) . \tag{3.46}$$

Consequently, by (3.45), the bipolar surface $\widetilde{\boldsymbol{S}}$ of $\boldsymbol{S}$ is a geodesic 2-sphere in $\mathbb{S}^5$ and, due to (3.46), reads as

$$\widetilde{\boldsymbol{S}} = \left\{ p^+ + p^- \in \mathcal{M} : p^+ = p^- = \frac{x}{\sqrt{2}},\ x \in \mathbb{S}^2 \right\} \subseteq \mathbb{S}^5 .$$

(ii) For the Clifford torus

$$\boldsymbol{C} := \mathbb{S}^1({}^1\!/\!\sqrt{2}) \times \mathbb{S}^1({}^1\!/\!\sqrt{2}) \subseteq \mathbb{S}^3 ,$$

the bipolar surface $\widetilde{\boldsymbol{C}}$ can be specified by the parametrization (2.29) with $m = k = 1$. It follows immediately that $\widetilde{\boldsymbol{C}}$ is congruent to the Clifford torus in $\mathbb{S}^5$, i.e., up to an isometry

$$\widetilde{\boldsymbol{C}} = \boldsymbol{T} .$$

To continue, we observe that for bipolar surfaces, the function C from the previous section is entirely determined by the Gaussian curvature of the surface in $\mathbb{S}^3$.

Lemma 3.3.4. *For the bipolar surface $\widetilde{\psi}\colon \Sigma \to \mathbb{S}^5$ of an oriented, minimally immersed surface $\psi\colon \Sigma \to \mathbb{S}^3$ with induced metric g and Gaussian curvature K, we have*

$$C = \frac{1}{2-K} - \frac{1}{2}\,. \tag{3.47}$$

Proof. Again, we consider local isothermal coordinates for g and $\widetilde{g}$ as in Section 1.2. Then, we use the definition of C, (3.44), (1.15) and finally (1.14) in order to compute

$$\begin{aligned}
C &= \frac{1}{2(2-K)\lambda}\left\langle J^-\left(\partial_1\widetilde{\psi}\right), *\partial_2\widetilde{\psi}\right\rangle \\
&= \frac{1}{2(2-K)\lambda}\Bigg\langle -\partial_1\psi\wedge\psi - \frac{\beta_{11}}{\lambda}\psi^*\wedge\partial_1\psi - \frac{\beta_{12}}{\lambda}\psi^*\wedge\partial_2\psi, \\
&\qquad\qquad\qquad \psi\wedge\partial_1\psi - \frac{\beta_{12}}{\lambda}\partial_2\psi\wedge\psi^* - \frac{\beta_{11}}{\lambda}\partial_1\psi\wedge\psi^*\Bigg\rangle \\
&= \frac{1}{2(2-K)}\left(1 - \frac{1}{\lambda^2}\left(\beta_{11}^2 + \beta_{12}^2\right)\right) \\
&= \frac{K}{2(2-K)} \\
&= \frac{1}{2-K} - \frac{1}{2}\,.
\end{aligned}$$

□

Remark 3.3.5. (i) Regarding the surfaces from Example 3.3.3, the above formula directly yields that $C^{\widetilde{\boldsymbol{S}}} \equiv \frac{1}{2}$ and $C^{\widetilde{\boldsymbol{C}}} \equiv 0$, where the latter is also clear by Example 3.2.8.

(ii) From Remark 1.4.3 (i), recall that $\widetilde{\psi^*} = -\widetilde{\psi}$. Using (1.28), we find

$$C^{\widetilde{\psi^*}} = \frac{1}{2-K^*} - \frac{1}{2} = \frac{1}{2+\frac{K}{1-K}} - \frac{1}{2} = -\left(\frac{1}{2-K} - \frac{1}{2}\right) = -C^{\widetilde{\psi}}\,,$$

which is consistent with Corollary 3.2.9.

Now, combining Proposition 1.2.5 and Lemma 3.2.12, we can draw the following conclusion about the image of C for bipolar surfaces.

Corollary 3.3.6. *Suppose that $\psi\colon \Sigma \to \mathbb{S}^3$ is an oriented, minimally immersed surface with Gaussian curvature K. Then, for the corresponding bipolar surface $\widetilde{\psi}\colon \Sigma \to \mathbb{S}^5$, we have*

$$C(\Sigma) \subseteq \big(-\tfrac{1}{2}, \tfrac{1}{2}\,\big]$$

and $C(p) = \frac{1}{2}$ only at the isolated points $p \in \Sigma$ where $K(p) = 1$.

In particular, if Σ is closed and has genus $g > 1$, the value $C = \frac{1}{2}$ is always attained.

Remark 3.3.7. Indeed, examples of closed minimal surfaces in $\mathbb{S}^3$ exist for every orientable genus $g > 1$ (cf. Section 2.3). So, recalling that $C^{-\Phi} = -C^{\Phi}$ by Corollary 3.2.9, C reaching one of the boundary values $-\frac{1}{2}$ or $\frac{1}{2}$ is a natural phenomenon for a minimal surface $\Phi\colon \Sigma \to \mathbb{S}^5$ with $\langle \Phi, *\Phi \rangle = 0$.

The established framework now permits a complete and precise specification of the fundamental data of a bipolar surface. In the light of Remark 1.1.3, these data fully determine the intrinsic and extrinsic geometry of a bipolar surface.

Theorem 3.3.8. *Let $\widetilde{\psi} = \psi \wedge \psi^*\colon \Sigma \to \mathbb{S}^5$ be the bipolar surface of an oriented, minimally immersed surface $\psi\colon \Sigma \to \mathbb{S}^3$ with induced metric g, Levi-Civita connection ∇ and shape operator A with respect to the unit normal field ν associated to ψ^*. Then, the fundamental data of $\widetilde{\psi}$ are given as follows:*

(i) *The induced metric reads as*

$$\widetilde{g} = \frac{2}{1 + 2C}\, g\,. \tag{3.48}$$

(ii) *The shape operator with respect to the normal field $\eta = *\widetilde{\psi}$ is given by*

$$\widetilde{A}_\eta = (1 + 2C)\, R_{\frac{\pi}{2}} \circ A\,, \tag{3.49}$$

where $R_{\frac{\pi}{2}}(p)$ denotes the rotation by $\frac{\pi}{2}$ on $T_p\Sigma$.

(iii) *Regarding the normal subbundle $\mathcal{N} = J^-(T\Sigma)$, the components of a shape operator $\widetilde{A}_{J^-Z}$ for $Z \in \mathfrak{X}(T\Sigma)$, are given by*

$$\left\langle \widetilde{A}_{J^-Z}(X), Y \right\rangle = -\big(\nabla\langle\sigma,\nu\rangle\big)(Z;X,Y) \qquad \text{for } X,\, Y \in \mathfrak{X}(\Sigma)\,.$$

Remark 3.3.9. Note that as an abstract bundle over Σ, the normal bundle of the bipolar surface can be seen as the sum of the tangent and normal bundle of the original surface in $\mathbb{S}^3$. This is clear as $J(T\Sigma) \cong T\Sigma$ and by the fact that $*\widetilde{\psi}(p)$ linked to an orthonormal basis of the orthogonal complement of $\psi^*(p)$ in $T_p\mathbb{S}^3$ at each point $p \in \Sigma$.

Proof of Theorem 3.3.8. At first, Theorem 1.4.1 together with (3.47) yields

$$\widetilde{g} = (2-K)\,g = \frac{2}{1+2C}\,g\,.$$

We verify the assertions regarding the shape operators by using local isothermal coordinates, employing the notation established in Section 1.2.

From the Gauss formula (3.8) for $\widetilde{\psi}$ and the Weingarten equations (1.15) for ψ we derive

$$\begin{aligned}
\widetilde{g}\left(\widetilde{A}_\eta(\partial_i),\partial_j\right) &= \langle\widetilde{\sigma}(\partial_i,\partial_j),\eta\rangle \\
&= \langle\partial_i\partial_j\widetilde{\psi},\eta\rangle \\
&= \Big\langle\big(\partial_i\partial_j\psi\wedge\psi^*\big) + \partial_i\psi\wedge\partial_j\psi^* \\
&\qquad\qquad + \partial_j\psi\wedge\partial_i\psi^* + \psi\wedge\big(\partial_i\partial_j\psi^*\big), \frac{1}{\lambda}\partial_1\psi\wedge\partial_2\psi\Big\rangle \\
&= \frac{1}{\lambda}\langle\partial_i\psi\wedge\partial_j\psi^* + \partial_j\psi\wedge\partial_i\psi^*, \partial_1\psi\wedge\partial_2\psi\rangle \\
&= \frac{1}{\lambda}\Big(\langle\partial_i\psi,\partial_1\psi\rangle\langle\partial_j\psi^*,\partial_2\psi\rangle - \langle\partial_i\psi,\partial_2\psi\rangle\langle\partial_j\psi^*,\partial_1\psi\rangle\Big) \\
&\qquad + \frac{1}{\lambda}\Big(\langle\partial_j\psi,\partial_1\psi\rangle\langle\partial_i\psi^*,\partial_2\psi\rangle - \langle\partial_j\psi,\partial_2\psi\rangle\langle\partial_i\psi^*,\partial_1\psi\rangle\Big) \\
&= -\beta_{j2}\delta_{i1} + \beta_{1j}\delta_{i2} - \beta_{i2}\delta_{1j} + \beta_{1i}\delta_{j2}\,.
\end{aligned}$$

From this, it follows that

$$\begin{aligned}
g\left(\widetilde{A}_\eta(\partial_1),\partial_1\right) &= \frac{1}{2-K}\,\widetilde{g}\left(\widetilde{A}_\eta(\partial_1),\partial_1\right) = -\frac{2\beta_{12}}{2-K} = -\frac{2}{2-K}\,g\left(A(\partial_1),\partial_2\right)\,,\\
g\left(\widetilde{A}_\eta(\partial_1),\partial_2\right) &= \frac{1}{2-K}\,\widetilde{g}\left(\widetilde{A}_\eta(\partial_1),\partial_2\right) = \frac{2\beta_{11}}{2-K} = \frac{2}{2-K}\,g\left(A(\partial_1),\partial_1\right)\,,\\
g\left(\widetilde{A}_\eta(\partial_2),\partial_2\right) &= \frac{1}{2-K}\,\widetilde{g}\left(\widetilde{A}_\eta(\partial_2),\partial_2\right) = \frac{2\beta_{12}}{2-K} = \frac{2}{2-K}\,g\left(A(\partial_1),\partial_2\right)\,,
\end{aligned}$$

or, in short,

$$\widetilde{A}_\eta = \frac{2}{2-K}R_{\frac{\pi}{2}}\circ A \overset{(3.47)}{=} (1+2C)R_{\frac{\pi}{2}}\circ A\,.$$

It remains to compute the components of a shape operator $\widetilde{A}_{JZ}$. To this end, first note that

$$\partial_1^2\widetilde{\psi} = \partial_1^2\psi\wedge\psi^* + 2\,\partial_1\psi\wedge\partial_1\psi^* + \psi\wedge\partial_1^2\psi^*$$

and

$$J^-\left(\partial_k\widetilde{\psi}\right) = -\partial_k\psi\wedge\psi + \psi^*\partial_k\psi^*\,.$$

Together with the Gauss formula and the Weingarten equations, we thereby find

$$\begin{aligned}
&\left\langle\widetilde{\sigma}(\partial_1,\partial_1), J^-\left(\partial_k\widetilde{\psi}\right)\right\rangle\\
&= \det\left(\begin{pmatrix}\langle\partial_1^2\psi,\psi^*\rangle & \langle\partial_1^2\psi,\partial_k\psi^*\rangle\\ 1 & 0\end{pmatrix}\right) - \det\left(\begin{pmatrix}0 & 1\\ \langle\partial_1^2\psi^*,\partial_k\psi\rangle & \langle\partial_1^2\psi^*,\psi\rangle\end{pmatrix}\right)\\
&= -\langle\partial_1^2\psi,\partial_k\psi^*\rangle + \langle\partial_1^2\psi^*,\partial_k\psi\rangle\\
&= \frac{1}{\lambda}\sum_{l=1}^{2}\left(\beta_{kl}\langle\partial_1^2\psi,\partial_l\psi\rangle - \beta_{1l}\langle\partial_1\partial_l\psi,\partial_k\psi\rangle\right) + \frac{\partial_1\lambda}{\lambda}\beta_{1k} - \partial_1\beta_{1k}
\end{aligned}\tag{3.50}$$

for $k\in\{1,2\}$. Now, we moreover use that

$$\langle\partial_1^2\psi,\partial_1\psi\rangle = \frac{1}{2}\partial_1\langle\partial_1\psi,\partial_1\psi\rangle = \frac{1}{2}\partial_1\lambda\,,$$

$$\left\langle \partial_1^2\psi, \partial_2\psi \right\rangle = -\left\langle \partial_2^2\psi, \partial_2\psi \right\rangle = -\frac{1}{2}\partial_2\langle \partial_2\psi, \partial_2\psi\rangle = -\frac{1}{2}\partial_2\lambda\,,$$

where the second equality is due to the minimal surface equation, written as

$$\partial_1^2\psi + \partial_2^2\psi = -2\lambda\psi\,.$$

Thus, after a straightforward computation, (3.50) more precisely reads as

$$\left\langle \widetilde{\sigma}(\partial_1, \partial_1), J^-\left(\partial_1\widetilde{\psi}\right)\right\rangle = \frac{1}{\lambda}\big((\partial_1\lambda)\beta_{11} - (\partial_2\lambda)\beta_{12}\big) - \partial_1\beta_{11}\,, \tag{3.51}$$

$$\left\langle \widetilde{\sigma}(\partial_1, \partial_1), J^-\left(\partial_2\widetilde{\psi}\right)\right\rangle = \frac{1}{\lambda}\big((\partial_2\lambda)\beta_{11} + (\partial_1\lambda)\beta_{12}\big) - \partial_1\beta_{12}\,. \tag{3.52}$$

Conversely, we now specify the components of $\nabla\sigma$. At first, note that with respect to the chosen local isothermal coordinates the Christoffel symbols of ∇ are given by

$$\Gamma^k{}_{ij} = \frac{1}{2\lambda}\Big((\partial_i\lambda)\delta_j^k + (\partial_j\lambda)\delta_i^k - (\partial_k\lambda)\delta_{ij}\Big) \qquad \text{for } i,\ j\,, k \in \{1, 2\}\,.$$

Furthermore, we have $\nabla^\perp_X\nu \equiv 0$ for all $X \in \mathfrak{X}(\Sigma)$ in codimension 1. Therefore,

$$\begin{aligned}
\nabla(\langle\sigma,\nu\rangle)(\partial_1;\partial_1,\partial_1) &= \partial_1\beta_{11} - 2\sigma(\nabla_{\partial_1}\partial_1, \partial_1)\\
&= \partial_1\beta_{11} - 2\sigma\big(\Gamma^1{}_{11}\partial_1 + \Gamma^2{}_{11}\partial_2, \partial_1\big)\\
&= -\frac{1}{\lambda}\big((\partial_1\lambda)\beta_{11} - (\partial_2\lambda)\beta_{12}\big) + \partial_1\beta_{11}\,, \qquad (3.53)\\
\nabla(\langle\sigma,\nu\rangle)(\partial_1;\partial_1,\partial_2) &= \partial_1\beta_{12} - \sigma(\nabla_{\partial_1}\partial_1, \partial_2) - \sigma(\nabla_{\partial_1}\partial_2, \partial_1)\\
&= \partial_1\beta_{12} - \sigma\big(\Gamma^1{}_{11}\partial_1 + \Gamma^2{}_{11}\partial_2, \partial_2\big) - \sigma\big(\Gamma^1{}_{12}\partial_1 + \Gamma^2{}_{12}\partial_2, \partial_1\big)\\
&= -\frac{1}{\lambda}\big((\partial_2\lambda)\beta_{11} + (\partial_1\lambda)\beta_{12}\big) + \partial_1\beta_{12}\,. \qquad (3.54)
\end{aligned}$$

Accordingly, comparing (3.53) and (3.54) with (3.51) and (3.52), we find

$$\begin{aligned}
\left\langle \widetilde{\sigma}(\partial_1, \partial_1), J^-\left(\partial_1\widetilde{\psi}\right)\right\rangle &= -\nabla_{\partial_1}\langle\sigma(\partial_1,\partial_1),\nu\rangle\,,\\
\left\langle \widetilde{\sigma}(\partial_1, \partial_1), J^-\left(\partial_2\widetilde{\psi}\right)\right\rangle &= -\nabla_{\partial_1}\langle\sigma(\partial_1,\partial_2),\nu\rangle\,.
\end{aligned}$$

By Lemma 3.2.4 and the minimality of $\widetilde{\psi}$ the components on the left-hand side are sufficient to determine $\widetilde{\sigma}^{\mathcal{N}}$. The same is holds for the components of $\nabla\langle\sigma,\nu\rangle$ on the right-hand side due to the symmetry of σ and the Codazzi equation in $\mathbb{S}^3$ (cf. (1.6)). From this we finally conclude that

$$\widetilde{g}\Big(\widetilde{A}_{JZ}(\partial_i),\partial_j\Big) = \Big\langle\widetilde{\sigma}(\partial_i,\partial_j),JZ\Big\rangle = -\big(\nabla\langle\sigma,\nu\rangle\big)(Z;\partial_i,\partial_j)$$

for i, $j \in \{1,2\}$ and $Z \in \mathfrak{X}(T\Sigma)$. □

Finally, we are prepared for the main theorem in this chapter.

Theorem 3.3.10. *Let Σ be an oriented, simply connected, two-dimensional manifold and let $\Phi\colon \Sigma \to \mathbb{S}^5$ be a minimal immersion with $\langle\Phi,*\Phi\rangle = 0$. Moreover, suppose that $C(p) \neq -\frac{1}{2}$ for all $p \in \Sigma$ or $C(p) \neq \frac{1}{2}$ for all $p \in \Sigma$. Then, there exists a minimal immersion $\psi\colon \Sigma \to \mathbb{S}^3$ such that up to an isometry of $\mathbb{S}^5$*

$$\Phi = \widetilde{\psi}\,.$$

Remark 3.3.11. (i) From Lemma 3.2.12 we already know that $-\frac{1}{2} \leq C(p) \leq \frac{1}{2}$ for all $p \in \Sigma$. So, the condition above demands that the domain is chosen such that not both of the potential boundary values are attained. In the light of Corollary 3.3.6, this criterion is non-exclusive regarding bipolar surfaces, for which C never reaches the value $-\frac{1}{2}$.

(ii) The above theorem can be translated to Theorem 4.4 in [11], where we relaxed the condition $-\frac{1}{2} < C(p) < \frac{1}{2}$ for all $p \in \Sigma$. In the context of Corollary 3.3.6 and especially Remark 3.3.7, we can say that the result above provides a completed picture.

Proof of Theorem 3.3.10. Our strategy is as follows: First, based on the fundamental data $\big(\hat{g},\hat{\sigma}\big)$ of Φ, we define fundamental data that satisfy the Gauss and the Codazzi equation in $\mathbb{S}^3$ and thereby guarantee the existence of an immersion $\psi\colon \Sigma \to \mathbb{S}^3$. Afterwards, checking that, conversely, the fundamental data of the bipolar surface $\widetilde{\psi}$ and Φ coincide, will finally prove the assertion.

Before we start, note that in the following, we can assume that $\mathrm{supp}(C)^c = \emptyset$. If that was the case, we would have $C \equiv 0$ and therefore congruence to $\boldsymbol{T} \cong \widetilde{\boldsymbol{C}}$ on an open subset (cf. Remarks 3.2.8 and 3.3.3 (ii)). By analytic continuation, this would in particular extend to the whole domain Σ.

Now first, possibly replacing Φ by $-\Phi$, we can assume that $C(p) \neq -\frac{1}{2}$ for all $p \in \Sigma$ (cf. Corollary 3.2.9). Define the pair (A, g) by

$$g := \frac{1+2C}{2}\hat{g}\,, \qquad A := -\frac{1}{1+2C}\, R_{\frac{\pi}{2}} \circ \hat{A}_\eta\,, \tag{3.55}$$

where $R_{\frac{\pi}{2}}(p)$ denotes the rotation by $\frac{\pi}{2}$ on $T_p\Sigma$. As $\frac{1+2C}{2}$ is a smooth function without zeroes under our assumption, g defines a Riemannian metric on Σ. Furthermore, A is well-defined since the function $\frac{1}{1+2C}$ is bounded. Notice that as C cannot attain $-\frac{1}{2}$, it is, by continuity, bounded away from that value.

Now, first, we verify that (g, A) satisfies the Gauss equation in $\mathbb{S}^3$. To that end, we start with the computation of the Gaussian curvature K_+ of (Σ, g). Using that g is conformal[1] to $\hat{g}$ and Lemma 3.2.14, we obtain

$$\begin{aligned}
K &= \frac{2}{1+2C}\left(\hat{K} - \frac{1}{2}\Delta_{\hat{g}}\log\left(\frac{1+2C}{2}\right)\right)\\
&= \frac{2}{1+2C}\left(\hat{K} - \frac{1}{2}\left[\frac{2}{1+2C}\hat{\Delta}C - \left(\frac{2}{1+2C}\right)^2 \left|\hat{\nabla}C\right|^2_{\hat{g}}\right]\right)\\
&\overset{(3.18),(3.19)}{=} \frac{2}{1+2C}\left(\hat{K} - \frac{1}{2}\left[\frac{2}{1+2C}\left(-2C\left(1+4C^2-2\hat{K}\right)\right)\right.\right.\\
&\qquad\qquad \left.\left. - \left(\frac{2}{1+2C}\right)^2\left(1-4C^2\right)\left(2C^2 - \frac{1}{2}\hat{K}\right)\right]\right)\\
&= \frac{2}{1+2C}\left(\hat{K} + \frac{2C}{1+2C}\left(1+4C^2-2\hat{K}\right) + \frac{2(1-2C)}{1+2C}\left(2C^2-\frac{1}{2}\hat{K}\right)\right)\\
&= \frac{2}{(1+2C)^2}\left(\hat{K} + 2C\hat{K} + 2C + 8C^3 - 4C\hat{K} + 4C^2 - \hat{K} - 8C^3 + 2C\hat{K}\right)
\end{aligned}$$

[1] For two conformally equivalent Riemannian metrics g and $h = e^{2u}g$ on a two-dimensional manifold Σ, the corresponding Gaussian curvatures are related by $K_h = e^{-2u}(K_g - \Delta_g u)$ (see for example [41]).

$$
\begin{aligned}
&= \frac{2}{(1+2C)^2}\left(2C+4C^2\right) \\
&= \frac{4C}{1+2C}\,.
\end{aligned} \tag{3.56}
$$

By the definition of A and Lemma 3.2.12, we also have

$$
\begin{aligned}
1-\frac{1}{2}|A|_g^2 &= 1-\frac{1}{2}\frac{1}{(1+2C)^2}\left|\hat{A}_\eta\right|_g^2 \\
&= 1-\frac{1}{2}\frac{1}{(1+2C)^2}\left|\hat{A}_\eta\right|_{\hat{g}}^2 \\
&= 1-\frac{(1-4C^2)}{(1+2C)^2} \\
&= 1-\frac{(1+2C)(1-2C)}{(1+2C)^2} \\
&= \frac{4C}{1+2C}\,.
\end{aligned}
$$

In combination, we therefore obtain

$$
K = 1-\frac{1}{2}|A|_g^2\,,
$$

i.e., the pair (g, A) satisfies the Gauss equation in $\mathbb{S}^3$ (cf. (1.8)).

To continue, the goal is to derive that (g, A) also satisfies the Codazzi equation in $\mathbb{S}^3$. For that purpose, let X, Y and $Z \in \mathfrak{X}(\Sigma)$. By the Codazzi equation of Φ with respect to the normal field η (cf. (1.5)), we have

$$
\begin{aligned}
&X\Big(\big\langle\hat{\sigma}(Y,Z),\eta\big\rangle\Big) - \big\langle\hat{\sigma}\big(\hat{\nabla}_X Y, Z\big),\eta\big\rangle - \big\langle\hat{\sigma}\big(Y,\hat{\nabla}_X Z\big),\eta\big\rangle - \big\langle\hat{\sigma}(Y,Z),\hat{\nabla}^\perp_X\eta\big\rangle \\
&\quad = Y\Big(\big\langle\hat{\sigma}(X,Z),\eta\big\rangle\Big) - \big\langle\hat{\sigma}\big(\hat{\nabla}_Y X, Z\big),\eta\big\rangle - \big\langle\hat{\sigma}\big(X,\hat{\nabla}_Y Z\big),\eta\big\rangle - \big\langle\hat{\sigma}(X,Z),\hat{\nabla}^\perp_Y\eta\big\rangle\,.
\end{aligned}
$$

By Lemma 3.2.4, $\big\langle\hat{\sigma}(Y,Z),\hat{\nabla}^\perp_X\eta\big\rangle$ is symmetric in X and Y. Hence, the equation above reduces to

$$
X\Big(\big\langle\hat{\sigma}(Y,Z),\eta\big\rangle\Big) - \big\langle\hat{\sigma}\big(\hat{\nabla}_X Y, Z\big),\eta\big\rangle - \big\langle\hat{\sigma}\big(Y,\hat{\nabla}_X Z\big),\eta\big\rangle
$$

$$= Y\Big(\langle\hat\sigma(X,Z),\eta\rangle\Big) - \big\langle\hat\sigma\big(\hat\nabla_Y X, Z\big),\eta\big\rangle - \big\langle\hat\sigma\big(X,\hat\nabla_Y Z\big),\eta\big\rangle\,.$$

In terms of $\hat{A}_\eta$, this means we have

$$\begin{aligned} &X\Big(\hat g\big(\hat A_\eta Y, Z\big)\Big) - \hat g\Big(\hat A_\eta\big(\nabla_X Y\big), Z\Big) - \hat g\big(\hat A_\eta Y, \hat\nabla_X Z\big) \\ &\quad = Y\Big(\hat g\big(\hat A_\eta X, Z\big)\Big) - \hat g\Big(\hat A_\eta\big(\hat\nabla_Y X\big), Z\Big) - \hat g\big(\hat A_\eta X, \hat\nabla_Y Z\big) \end{aligned}$$

Using the definition of g, A and that $R_{\frac{\pi}{2}}$ is unitary, this implies

$$\begin{aligned} &X\Big(g\big(AY, R_{\frac{\pi}{2}}Z\big)\Big) - g\big(A\big(\hat\nabla_X Y\big), R_{\frac{\pi}{2}}Z\big) - g\Big(AY, R_{\frac{\pi}{2}}\big(\hat\nabla_X Z\big)\Big) \\ &\quad = Y\Big(g\big(AX, R_{\frac{\pi}{2}}Z\big)\Big) - g\big(A\big(\hat\nabla_Y X\big), R_{\frac{\pi}{2}}Z\big) - g\Big(AX, R_{\frac{\pi}{2}}\big(\hat\nabla_Y Z\big)\Big) \end{aligned}$$

Since, moreover, $\hat\nabla$ commutes with $R_{\frac{\pi}{2}}$ (as $\big(\Sigma,\hat g\big)$ is a Kähler surface) and Z was chosen arbitrarily (so we can replace $R_{\frac{\pi}{2}}Z$ by Z), we have

$$\begin{aligned} &X\big(g(AY,Z)\big) - g\Big(A\big(\hat\nabla_X Y\big), Z\Big) - g\big(AY, \hat\nabla_X Z\big) \\ &\quad = Y\big(g(AX,Z)\big) - g\Big(A\big(\hat\nabla_Y X\big), Z\Big) - g\big(AX, \hat\nabla_Y Z\big)\,. \end{aligned}$$

Now, since $\hat g$ and g are conformally equivalent, we have

$$\hat\nabla_X Y = \nabla_X Y + X(f)Y + Y(f)X - g(X,Y)\nabla f\,,$$

where

$$f := \frac{1}{2}\log\left(\frac{2}{1+2C}\right)\,.$$

Therewith, after a straightforward canceling out of terms, we find

$$\begin{aligned} &X\big(g(AY,Z)\big) - g\Big(A\big(\nabla_X Y\big), Z\Big) - g\big(AY, \nabla_X Z\big) \\ &\qquad - X(f)g\big(AY,Z\big) + g(X,Z)g + \big(AY,\nabla f\big) \\ &\quad = Y\big(g(AX,Z)\big) - g\Big(A\big(\nabla_Y X\big), Z\Big) - g\big(AX, \nabla_Y Z\big) \end{aligned}$$

$$- Y(f)g\big(AX,Z\big) + g(Y,Z)g\big(AX,\nabla f\big)\,. \tag{3.57}$$

For a further simplification, we first notice that

$$\begin{aligned}
&- X(f)g\big(AY,Z\big) + g(X,Z)g\big(AY,\nabla f\big)\\
&\quad = -g\big(\nabla f, X\big)g\big(AY,Z\big) + g(X,Z)g\big(AY,\nabla f\big)\\
&\quad = \det\left(\begin{pmatrix} g(X,Z) & g\big(X,\nabla f\big)\\ g\big(AY,Z\big) & g\big(AY,\nabla f\big)\end{pmatrix}\right)\\
&\quad = g\Big(X \wedge (AY), \big(\nabla f\big) \wedge Z\Big)\,,
\end{aligned} \tag{3.58}$$

referring to Section 1.3. Using a local orthonormal frame (E_1, E_2) and the definition of A, it can be seen quickly that $E_1 \wedge (AE_2) = E_2 \wedge (AE_1)$, implying that $X \wedge (AY)$ and hence (3.58) is symmetric in X and Y. Therefore, (3.57) simplifies to

$$\begin{aligned}
&X\big(g(AY,Z)\big) - g\Big(A\big(\nabla_X Y\big), Z\Big) - g\big(AY, \nabla_X Z\big)\\
&\quad = Y\big(g(AX,Z)\big) - g\Big(A\big(\nabla_Y X\big), Z\Big) - g\big(AX, \nabla_Y Z\big)\,,
\end{aligned}$$

that is, (g, A) satisfies the Codazzi equation in $\mathbb{S}^3$ (cf. (1.6)). Consequently, there exists a non-singular immersion $\psi\colon \Sigma \to \mathbb{S}^3$ with fundamental data (g, A). Moreover, the minimality of Φ implies that $\det\big(\hat{A}_\eta\big) \equiv 0$. By the definition, the analogue holds for A and hence ψ is minimal.

Now, comparing with (3.48) and (3.49), we find that for the bipolar surface $\widetilde{\psi}$,

$$\widetilde{g} = \hat{g}\,, \qquad \widetilde{A}_\eta = \hat{A}_\eta\,.$$

Addionally, (3.47) and (3.56) yield that

$$C^{\widetilde{\psi}} = C^{\Phi} = C\,.$$

Putting together this information with the relation from (3.17), for which the left-hand side is now the same for both $\widetilde{\psi}$ and Φ, we find that

$$C \cdot \widetilde{A}_{J^-Z} = C \cdot \hat{A}_{JZ}$$

for all $Z \in \mathfrak{X}(\Sigma)$. By continuity, this holds modulo C on $\operatorname{supp}(C)$. Recalling our assumption that $\operatorname{supp}(C)^c = \emptyset$ from the very beginning, we hence conclude that

$$\widetilde{A}_{J^-Z} = \hat{A}_{JZ}$$

for all $Z \in \mathfrak{X}(\Sigma)$. Therefore, all the fundamental data of $\widetilde{\psi}$ and Φ coincide, showing that $\Phi = \widetilde{\psi}$ up to an isometry. □

Referring back to Example 3.3.3 (i), we can immediately derive the following result, which was also obtained in [11] using different methods.

Corollary 3.3.12. *If $\Phi \colon \mathbb{S}^2 \to \mathbb{S}^5$ is a minimal surface with $\langle \Phi, *\Phi \rangle = 0$, then it is congruent to the bipolar surface of a geodesic 2-sphere in $\mathbb{S}^3$.*

We finish this section with several remarks and future perspectives concerning the previous theorem.

Remark 3.3.13. (i) It remains open in how far the requirement of a simply connected domain in Theorem 3.3.10 can be generalized. To develop a comparable description for closed surfaces, a potential approach would be to examine symmetric surfaces as for example discussed in Chapter 2.

(ii) In the proof of Theorem 3.3.10, regardless of the specific values taken by the function C, both pairs

$$g_\pm := \frac{1 \pm 2C}{2}\,\hat{g}\,, \qquad A_\pm := -\frac{1}{1 \pm 2C}\, R_{\frac{\pi}{2}} \circ \hat{A}_\eta\,, \tag{3.59}$$

would have satisfied the Gauss and Codazzi equation in $\mathbb{S}^3$, one of them exhibits singular fundamental data. This phenomenon is related to the polar surface in $\mathbb{S}^3$ when comparing these singular data with (1.22) and (1.27). Thus, the question arises whether in the requirements of Theorem 3.3.10 the condition on the values of C can be dropped, i.e., if a similar result is obtained by a pair of singular minimal surfaces in $\mathbb{S}^3$, where one has singularities in $C^{-1}\big(\{\frac{1}{2}\}\big)$, the other in $C^{-1}\big(\{-\frac{1}{2}\}\big)$. Or, in other words, whether it is helpful to consider singular minimal surfaces in $\mathbb{S}^3$ to obtain a better picture of (non-singular) minimal surfaces in $\mathbb{S}^5$. In the context of the proof above, a first step towards an answer could be to check if there are conditions for singular surfaces under which the Gauss and Codazzi equation could be regarded as compatibility equations.

3.4 Generalization of Bipolar Surfaces

We finish this chapter with related work that can be understood in the context of the previous section and possibly provide future perspectives in the study of minimal surfaces in $\mathbb{S}^5$.

In [70], Francisco Torralbo and Francisco Urbano present a general study of minimal surfaces in $\mathbb{S}^2 \times \mathbb{S}^2$, extending from a framework similar to that of [11]. From the perspective of this chapter, their results provide information on minimal surfaces in $\mathcal{M}$. In the Lagrangian case, we have seen that these surfaces are also minimal in $\mathbb{S}^5$ (cf. Remark 3.2.3). In this context, one particular result stands out because it has the potential to provide additional examples of such surfaces. Particularly, as a special case, this construction includes bipolar surfaces (or, in the original formulation, Gauss maps of minimal surfaces in $\mathbb{S}^3$, as described in Remark 3.3.2). Within our setup, these findings of Torralbo and Urbano read as follows.

Theorem 3.4.1 (cf. [70], Theorem 2)**.** *Let ϕ, $\psi\colon \Sigma \to \mathbb{S}^3$ be two oriented, minimally immersed surfaces with conformally equivalent metrics g_ψ, g_ϕ and with the same Hopf differentials $\Theta_\psi = \Theta_\phi$. Then,*

$$\Phi_{\{\psi,\phi\}}\colon \Sigma \to \mathcal{M}\,, \qquad \Phi_{\{\psi,\phi\}} := \widetilde{\psi}^+ + \widetilde{\phi}^-$$

is an oriented minimal surface in $\mathcal{M}$ with induced metric

$$\hat{g}_{\{\psi,\phi\}} = \frac{1}{2}\Big((2 - K_\psi)\, g_\psi + (2 - K_\phi)\, g_\phi\Big)\,.$$

Additionally, $\Phi_{\{\psi,\phi\}}$ satisfies

$$\begin{aligned} \mathcal{N} = J^+(T\Sigma) \qquad &\Leftrightarrow \qquad K_\psi = K_\phi\,, \\ \mathcal{N} = J^-(T\Sigma) \qquad &\Leftrightarrow \qquad (1 - K_\psi)(1 - K_\phi) = 1\,, \end{aligned}$$

where K_ψ and K_ϕ denote the Gaussian curvatures of ϕ and ψ. In these situtations, $\Phi_{\{\phi,\psi\}}$ is also minimal in $\mathbb{S}^5$.

Remark 3.4.2. Recalling Proposition 1.2.3, the Hopf differential of an oriented, conformally parametrized minimal surface $\psi\colon \Sigma \to \mathbb{S}^3$ is locally given by

$$\varphi(z)\,\mathrm{d}z^2 = -\big\langle \partial\psi, \partial\psi^* \big\rangle\,\mathrm{d}z^2\,.$$

In particular, we can observe that ψ and ψ^* share the same Hopf differential so that the above Theorem is applicable for $\phi = \psi^*$. In this case, (1.28) yields

$$\begin{aligned} \hat{g}_{\{\psi,\psi^*\}} &= \frac{1}{2}\Big((2 - K_\psi)\, g_\psi + (2 - K_{\psi^*})\, g_{\psi^*}\Big) \\ &= \frac{1}{2}\left((2 - K_\psi) + \left(2 + \frac{K_\psi}{1 - K_\psi}\right)\right)\, g_\psi \\ &= (2 - K_\psi)\, g_\psi\,, \end{aligned}$$

which is consistent with Theorem 1.4.1. Furthermore, we always have

$$(1 - K_\psi)(1 - K_{\psi^*}) = (1 - K_\psi)\left(1 + \frac{K_\psi}{1 - K_\psi}\right) = 1$$

and

$$K_\psi = K_{\psi^*} \quad \Leftrightarrow \quad (2 - K_\psi)K_\psi = 0 \quad \Leftrightarrow \quad K \equiv 0\,,$$

as we would expect from Proposition 3.3.1 and Remark 3.2.11.

Besides bipolar surfaces, we now present another example where the above theorem can be applied, namely, within Lawson's τ-family in $\mathbb{S}^3$ as described in Section 2.3.4.

We parametrize a Lawson surface $\tau_{m,k} \subseteq \mathbb{S}^3 \subseteq \mathbb{R}^4 \cong \mathbb{C}^2$ for $m,\ k \in \mathbb{Z}_{\geq 1}$ and $\gcd(m,k) = 1$ by

$$\phi_{m,k}\colon \mathbb{R}^2 \to \mathbb{S}^3 \subseteq \mathbb{C}^2\,, \quad \phi_{m,k}(u,v) = \big(e^{imu}\mathrm{cn}(mv), e^{iku}\mathrm{sn}(mv)\big)\,,$$

with respect to the elliptic modulus $\sqrt{1-\frac{k^2}{m^2}}$, as explained in Section 2.3.4. The metric $h_{m,k}$ induced by $\phi_{m,k}$ is given by

$$h_{m,k}(u,v) = \lambda_{m,k}(v)\cdot\big(\mathrm{d}u^2 + \mathrm{d}v^2\big)\,, \quad \lambda_{m,k}(v) := m^2\mathrm{dn}^2(mv)\,.$$

In these terms, the Gauss map of $\phi_{m,k}$ reads as

$$\phi^*_{m,k}\colon \mathbb{R}^2 \to \mathbb{S}^3 \subseteq \mathbb{C}^2\,, \quad \phi^*_{m,k}(u,v) = q(v)\cdot\big(-ik\,e^{imu}\mathrm{sn}(mv), im\,e^{iku}\mathrm{cn}(mv)\big)\,,$$
$$q(v) := \frac{1}{\sqrt{m^2\mathrm{cn}^2(mv) + k^2\mathrm{sn}^2(mv)}}\,.$$

We have

$$\begin{aligned}
\partial_u\phi_{m,k}(u,v) &= \big(im\,e^{imu}\mathrm{cn}(mv), ik\,e^{iku}\mathrm{sn}(mv)\big)\,,\\
\partial_v\phi_{m,k}(u,v) &= m\,\mathrm{dn}(mv)\big(-e^{imu}\mathrm{sn}(mv), e^{iku}\mathrm{cn}(mv)\big)\,,
\end{aligned}$$

$$\begin{aligned}
\big(\partial_u\phi^*_{m,k}\big)^T(u,v) &= mk\,q(v)\cdot\big(e^{imu}\mathrm{sn}(mv), -e^{iku}\mathrm{cn}(mv)\big)\,,\\
\big(\partial_v\phi^*_{m,k}\big)^T(u,v) &= m\,q(v)\mathrm{dn}(mv)\big(-ike^{imu}\mathrm{cn}(mv), -ime^{iku}\mathrm{sn}(mv)\big)\,.
\end{aligned}$$

Therefore, writing $z = u + iv$, the Hopf differential $\Theta(z) = \varphi_{m,k}(z)\,\mathrm{d}z^2$ of $\phi_{m,k}$ (cf. Proposition 1.2.3) is given by

$$\begin{aligned}
\varphi_{m,k}(z) &= -\big\langle \partial\phi_{m,k}, \partial\phi^*_{m,k}\big\rangle\\
&= -\frac{1}{4}\big\langle \partial_u\phi_{m,k} - i\partial_v\phi_{m,k}, \partial_u\phi^*_{m,k} - i\partial_v\phi^*_{m,k}\big\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \frac{i}{4}\Big(\big\langle \partial_v \phi_{m,k}, \partial_u \phi^*_{m,k}\big\rangle + \big\langle \partial_u \phi_{m,k}, \partial_v \phi^*_{m,k}\big\rangle\Big) \\
&= \frac{i}{2} m^2 k\, q(v)\, \mathrm{dn}(mv) \\
&= \frac{i}{2} mk\,,
\end{aligned}
$$

where in the last step, we used the well-known identities

$$
\mathrm{cn}^2(mv) + \mathrm{sn}^2(mv) = 1\,, \quad \mathrm{dn}^2(mv) = 1 - \left(1 - \frac{k^2}{m^2}\right) \mathrm{sn}^2(mv)
$$

implying that

$$
m\, q(v)\, \mathrm{dn}(mv) = 1\,.
$$

So, the following is immediate.

Proposition 3.4.3. *Let m, k, $\hat{m}$, $\hat{k} \in \mathbb{Z}_{\geq 1}$ and $\gcd(m,k) = \gcd\big(\hat{m}, \hat{k}\big) = 1$. Then, the Hopf differentials of the Lawson surfaces $\tau_{m,k}$ and $\tau_{\hat{m},\hat{k}}$ coincide if and only if*

$$
mk = \hat{m}\hat{k}\,.
$$

Due to the above proposition, Theorem 3.4.1 applies to the surfaces $\phi_{m,k}$ and $\phi_{\hat{m},\hat{k}}$ if $mk = \hat{m}\hat{k}$. Non-trivial examples for such pairs (m,k) and $\big(\hat{m}, \hat{k}\big)$ are always obtained from a choice of three distinct prime numbers p_1, p_2 and p_3, arranged into products in different ways.

Concerning future work, a question that arises in the light of the previous sections is whether the surfaces constructed from such pairs satisfy the assumptions of Theorem 3.4.1 to be minimal in $\mathbb{S}^5$. In this context, note that the Gaussian curvature of $\phi_{m,k}$ is given by

$$
K_{\phi_{m,k}}(u,v) = -\frac{1}{2\lambda_{m,k}(v)} \partial_v^2 \log\big(\lambda_{m,k}(v)\big) = \frac{-m^2 \mathrm{dn}^4(mv) + k^2}{\mathrm{dn}^2(mv)}\,,
$$

where we used that

$$\mathrm{sn}'(t) = \mathrm{cn}(t)\mathrm{dn}(t)\,, \quad \mathrm{cn}'(t) = -\mathrm{sn}(t)\mathrm{dn}(t)\,, \quad \mathrm{dn}'(t) = -\left(1 - \frac{k^2}{m^2}\right)\mathrm{sn}(t)\mathrm{cn}(t)\,.$$

Furthermore, an interesting point to study is whether these surfaces can be defined on closed domains. Looking at the chosen parametrization, this could be realized by detecting common periods of appropriate surfaces $\phi_{m,k}$ and $\phi_{\hat{m},\hat{k}}$. In this regard, note that for the considered modulus, the Jacobi elliptic functions appearing in $\phi_{m,k}$ and $h_{m,k}$ have symmetries

$$\mathrm{cn}(t+2K) = -\mathrm{cn}(t)\,, \quad \mathrm{sn}(t+2K) = -\mathrm{sn}(t)\,, \quad \mathrm{cn}(t+2K) = \mathrm{dn}(t)$$

and

$$\mathrm{cn}(K-t) = -\mathrm{cn}(K+t)\,, \quad \mathrm{sn}(K-t) = \mathrm{sn}(K+t)\,, \quad \mathrm{dn}(K-t) = \mathrm{dn}(K+t)\,,$$

where

$$K = K\left(\sqrt{1 - \frac{k^2}{m^2}}\right) = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\vartheta}{\sqrt{1 - \left(1 - \frac{k^2}{m^2}\right)\sin^2(\vartheta)}}$$

is the complete elliptic integral of the first kind. Hence, a potential challenge could be to determine whether different elliptic integrals are rational multiples of each other.

Conclusion

In this thesis, we have topologically classified the closed bipolar minimal surfaces $\left(\widetilde{\xi}_{m,k}\right)$ and $\left(\widetilde{\eta}_{m,k}\right)$ in $\mathbb{S}^5$ for $m > 2$ or $k > 2$, induced by the two families $(\xi_{m,k})$ and $(\eta_{m,k})$ of Lawson surfaces in $\mathbb{S}^3$. More precisely, we found that all of them are orientable and have self-intersections. Defining specific immersions for bipolar surfaces arising from Lawson-type minimal surfaces in $\mathbb{S}^3$, we developed a mechanism to study such surfaces in terms of the algebraic properties of the symmetry group from the reflection process in $\mathbb{S}^3$. A future question is in how far this can be employed in further characterizations of bipolar surfaces, as for example their classification by symmetries or index computations.

Regarding the Willmore problem, the search for embedded, closed non-orientable minimal surfaces in spheres, requiring a higher codimension, continues. A perspective in that context is to apply our methods to other examples of closed minimal surfaces in $\mathbb{S}^3$ that result from a reflection process (cf. [31, 13, 5]). In addition, one could analyse in our setting how closed minimal surfaces in $\mathbb{S}^3$ need to behave in order to produce a non-orientable bipolar minimal surface, as given by the example $\widetilde{\tau}_{3,1}$.

Furthermore, we have examined bipolar surfaces within the broader class of minimal surfaces in $\mathbb{S}^5$ that lie in an embedding of $\mathbb{S}^2(1/\sqrt{2}) \times \mathbb{S}^2(1/\sqrt{2})$. This exploration has not only deepened our understanding of their extrinsic geometric properties but has also enabled us to demonstrate that, under certain conditions, locally any immersed surface of the specified class is congruent to a bipolar surface. Future research could further explore this class, particularly its distinction among minimal surfaces in $\mathbb{S}^5$. Additionally, appropriate examples from the construction in [70] could be considered.

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Wissenschaftlicher Werdegang

Okt. 2019 - Sept. 2024	Wissenschaftliche Mitarbeiterin am Fachbereich Mathematik der TU Darmstadt
Okt. 2018 - Sept. 2019	Studium der Physik mit Abschluss M.Sc. Physik an der Eberhard Karls Universität Tübingen
Okt. 2013 - Mai 2018	Studium der Physik mit Abschluss B.Sc. Physik an der Eberhard Karls Universität Tübingen
Juni 2013	Abitur am Gymnasium Balingen